

**A Simple Mixture Theory for Isothermal and Non-isothermal
Flows of ν Newtonian and Generalized Newtonian Constituents
including Interaction Effects**

By

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Abstract

Development of mathematical models based on conservation and balance laws including constitutive theories are presented for a saturated mixture of ν homogeneous, isotropic, and incompressible constituents for isothermal and non-isothermal flows. The constituents and the mixture are assumed to be Newtonian or generalized Newtonian fluids. Power law and Carreau-Yasuda models are considered for generalized Newtonian shear thinning fluids. The mathematical model is derived for a ν constituent mixture with volume fractions ϕ_α using principles of continuum mechanics: conservation of mass, balance of momenta, first and second laws of thermodynamics, and principles of mixture theory yielding continuity equations, momentum equations, energy equation, and constitutive theories for mechanical pressures, deviatoric Cauchy stress tensors, and heat vector in terms of the dependent variables related to the constituents and their material coefficients. In the derivation of the mathematical model effects of the interaction forces are accounted in the momentum and energy equations. In the development of the constitutive theories two approaches are considered. In the first approach we assume that the mixture stress is the sum of the constituent stresses. This approach requires derivation of the bulk properties of the constituents based on the constituent volume fractions and their properties which are then utilized in the constitutive theories for the constituents forming the mixture. In the second approach the mixture stress is assumed not to be the sum of the constituent stress. For a homogeneous isotropic mixture we begin with its own constitutive theory for the deviatoric mixture stress defined using mixture material coefficients and the symmetric part of the velocity gradient tensor for the mixture. Mixture material coefficients are derived

using volume and mole fractions of the constituents and a mixing rule. The mutual parameter in the mixing rule is described using arithmetic mean, geometric mean, and harmonic mean. The validity of the proposed models are demonstrated for degenerated cases of same constituents i.e., two of the constituents same etc. Dimensionless forms of the mathematical models are derived and used to present numerical studies for boundary value problems using finite element processes based on a residual functional, that is, least squares processes in which local approximations are considered in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces. Fully developed flow between parallel plates and 1:2 asymmetric backward facing step are used as model problems for a mixture of two constituents.

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Chapter 1

Introduction, literature review, and scope of work

The mixture theories for mixtures fall into two broad categories: volume averaging theories, and theories based upon continuum mechanics. Volume averaging theories [1–6] obtain “average” properties for the mixture while continuum mechanics based approaches assume that each point is simultaneously occupied by each constituent. It is also important to note that many models used in theories of volume averaging are not frame-indifferent thus violating basic principles in physics [7]. In this work we are concerned with theories based upon the principles and axioms of continuum mechanics. Theories with a continuum mechanics basis have grown from the fundamental assumption that the mixture is represented by constituents in which all constituents occupy regions of space simultaneously. This is not a physical assumption, however, it is necessary to allow the continuum to be both continuous and differentiable. Three “metaphysical principles” by Truesdell [8] provide the foundation of the mixture theories: (i) All properties of the mixture must be consequences of properties of the constituents. (ii) So as to describe the motion of a constituent, we may imagine it isolated from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it (iii) and the motion of the mixture is governed by the same equations as a single body.

Based on these principles, Truesdell presented a theory for the mechanical basis of diffusion [9] and derived Fick's equation of diffusion using the principles of continuum mechanics. Fick proposed that diffusion was linearly related to the density gradient between constituents based on an analogy to the flow of heat. Through the use of the principle of balance of momenta and constitutive equations, the author in [9] presented the derivation of Fick's law under isothermal conditions in which the total density is uniform and body forces are absent.

Green and Naghdi [10] introduce definitions for the mean velocity of the mixture, as well as total density, and bulk density of the fluid based on volume fractions. In this work the authors lay out these definitions for a binary mixture, though they can be applied to ν constituents. The authors are then able to construct an energy equation and an entropy production inequality as well as basic equations of mass and momenta balance. Using these equations they are able to show the specific case for a binary compressible Newtonian mixture. For this case the authors have deduced the full thermodynamical restrictions which must be satisfied by the constitutive equations for a compressible mixture of binary Newtonian fluids.

Mills [11] extends the work by Truesdell [9] and Green and Naghdi [10] to incompressible constituents. Mills uses the volume additivity constraint by assuming that the total volume of the mixture does not change, thus the sum of the volume fractions of the constituents must sum to unity. Mills then constructs constitutive equations for a binary incompressible mixture. This theory is then applied to deduce Fick's law for an incompressible mixture of ideal fluids. Mills also notes that there are no explicit solutions for fluids in motion. Green and Naghdi [10] introduced definitions of: mean velocity of mixtures, total density, bulk density based on volume fractions for binary mixtures as well as a mixture of ν constituents and derived a mathematical model based on conservation and balance laws. The authors present full thermodynamic restrictions to be satisfied by the constitutive equations for a compressible mixture of binary Newtonian fluids. Mills [11] extended the work in reference [9,10] to incompressible constituents by using the volume additivity constraint which states that as sum of the total volume of a mixture does not change, the sum of the volume fractions of the constituents must be unity. Constitutive equations for a binary mixture are

constructed in [11] and applied to deduce Fick's law of diffusion for an incompressible mixture of ideal fluids.

Following the works in [9, 10], Müller [12] presented a theory of mixtures for ν constituents consisting of the derivation of conservation and balance laws. In his work, Müller assigns one temperature θ to every point as each point of the mixture is occupied by ν constituents. This assumption is continued within this work. The author continues to create a linear theory of the mixtures of fluids, and explicitly shows the theory for binary mixtures. An important feature of Müller's work is that he was the first to include density gradients among the constitutive variables. The author also shows that based on this theory, a mixture of two ideal gases is still an ideal gas with properties that agree with the principle of partial pressures based on classical thermodynamics.

Bowen [13] presents a similar theory and then extends it for mixture theories for gases, as well as porous media models. Bowen [14] discusses the role of partial stresses. He shows that for an ideal mixture, a mixture in which the fluid response depends only on the motion of the fluid and the temperature of the mixture, the stress of the mixture is the sum of the partial stresses. He describes that for certain types of mixtures this may be too strong of an assumption.

Atkin and Craine [15] presented a review of the early mixture theory literature from 1957 through 1975. Here they describe the preliminary definitions of mixture velocity, mixture density, as well as development of the conservation and balance laws. The entropy inequality is derived and a constitutive theory for the mixture of two ideal gases is presented. The basic equations are compared with the published works. Bedford and Drumheller [16] also provide a review of the development of various mixture theories and present derivation of the volume fraction theories. The authors discuss preliminaries of the mixture theories of bubbly liquids, fluid-particle mixtures, fluid saturated porous media, and composite materials.

In [17], Rajagopal and Tao introduce the basic principles of mixture theory as well as a discussion on the solution of boundary value problems for mixture theories. The authors present kinematics and the derivation of the conservation and balance laws as well as a discussion regarding the volume additivity constraint. In their work, interaction effects between constituents are

considered and literature review regarding these effects are presented. The authors present constitutive theories through the theory of generators and invariants with regards to diffusion, wave propagation in solids infused with fluids, mixture of two Newtonian fluids, and mixtures of fluids and solids.

Chapter 2

Derivation of the mathematical model for a mixture of ν fluids

2.1 Introduction

In this section we present derivations of continuity equation, momentum equations, energy equation, entropy inequality, and the constitutive theory derived from the entropy inequality for a saturated mixture of ν Newtonian and generalized Newtonian fluids. Some basic definitions of bulk densities of constituents, mixture density, mixture velocities, etc. are introduced based on basic physical principles that are used in the development of the mathematical model for the mixture. To avoid confusion in the notation used here and those commonly used in continuum mechanics we adopt the following convention. Greek letters such as $\alpha, \beta, \gamma, \nu$, etc. used as subscripts, superscripts, or indices refer to a quantity associated with an individual constituent and have no implied summation when the index is repeated. Any index using English letters i, j, k , etc. imply standard continuum mechanics summation conventions, i.e. summation over repeated indices.

The derivation of the mathematical models presented in this section are based on principles of continuum mechanics and thermodynamics using two approaches. In the first approach we strictly

use the concepts of mixture theory and continuum mechanics principles to establish mathematical models for each of the constituents resulting in continuity, momentum and energy equations and entropy inequality. This approach requires constitutive theories for the Cauchy stress tensor for the constituents which in turn requires material coefficients of the constituents in the mixture. The mixture Cauchy stress tensor is assumed to be the sum of the constituents Cauchy stress tensors. Mixture deformation field is completely determined by the constituents deformation fields and the rules of mixture theory. In the derivations considered within this approach interaction forces between the constituents are included in the balance laws. If the mixture is homogeneous, isotropic and saturated then it must be possible to derive a single mathematical model for the mixture using mixture velocity, mixture Cauchy stress tensor, heat vector and the material coefficients of the mixture. The crucial elements in this approach are the mixture material coefficients which are established using volume and mole fractions of the constituents and a mixing rule. The mutual parameter in the mixing rule is described using arithmetic mean, geometric mean, and harmonic mean. We start by considering the first approach, ie the mathematical model based on the concepts of mixture theory and continuum mechanics principles.

2.2 Notations

We use an over bar to express quantities in the current configuration in Eulerian description, i.e. all quantities with overbars are functions of deformed coordinates \bar{x}_i and time t . Quantities without an over bar imply Lagrangian description of the quantities in the current configuration, i.e. these are functions of undeformed coordinates x_i and time t . Thus, $x_i ; i = 1, 2, 3$ and $\bar{x}_i ; i = 1, 2, 3$ are coordinates of a material point in the reference and current configurations, respectively, both measured in a fixed Cartesian x -frame. The present work only considers Lagrangian description, hence all measures are expressed in terms of coordinates of the material points in the undeformed configurations (same as reference configuration in the present work) $x_i ; i = 1, 2, 3$ and time t . We use $[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}} \right]$ to be the Jacobian of deformation. We denote ρ_0 to be the density in

the reference configuration, hence it is constant. Φ , θ and η denote the Helmholtz free energy density, temperature, and entropy density respectively. $\boldsymbol{\sigma}^{[0]}$ is the second Piola-Kirchoff stress tensor. Superscript ‘0’ is used to signify that it is rate of order zero and the upper case brackets distinguish it from the Cauchy stress tensor $\boldsymbol{\sigma}^{(0)}$ (in contravariant basis). $\boldsymbol{\epsilon}_{[0]}$ represents Green’s strain tensor, a measure of finite strain. $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{\epsilon}_{[0]}$ are a constitutive conjugate pair [18–20]. Dot on all quantities refers to material derivative. $\boldsymbol{\epsilon}$ refers to Green’s strain tensor and/or its material derivative up to orders n ($\boldsymbol{\epsilon}_{[i]}$; $i = 0, 1, \dots, n$).

2.3 Preliminary definitions

In this section we present basic definitions of bulk densities of constituents, mixture density, mixture velocity, material derivative for the constituents and the mixture etc. These are subsequently used in the conservation laws. We consider a saturated mixture of ν constituents with ϕ_α ; $\alpha = 1, 2, \dots, \nu$ volume fractions, and $\bar{\rho}^{(\alpha)}$; $\alpha = 1, 2, \dots, \nu$ constituent densities. Following Truesdell [9] we can give the following definitions:

2.3.1 Definitions of densities

Consider an elemental volume $d\bar{V}$ of the mixture of volume \bar{V} . Then $\bar{\rho}^{(\alpha)}\phi_\alpha d\bar{V}$ is the mass of each constituent in the volume $d\bar{V}$. If $\bar{\rho}_m$ is the total density of the mixture, then $\bar{\rho}_m d\bar{V}$ is also the total mass in the elemental volume $d\bar{V}$. Hence, for volume \bar{V} , we have

$$\int_{\bar{V}(t)} \bar{\rho}_m d\bar{V} = \sum_{\alpha=1}^{\nu} \int_{\bar{V}(t)} \bar{\rho}^{(\alpha)} \phi_\alpha d\bar{V} \quad (2.1)$$

or

$$\int_{\bar{V}(t)} \left(\bar{\rho}_m - \sum_{\alpha=1}^{\nu} \bar{\rho}^{(\alpha)} \phi_\alpha \right) d\bar{V} = 0 \quad (2.2)$$

Since $\bar{V}(t)$ is arbitrary, we have

$$\bar{\rho}_m = \sum_{\alpha=1}^{\nu} \bar{\rho}^{(\alpha)} \phi_{\alpha} \quad (2.3)$$

If we define bulk density of a constituent $\bar{\rho}_{\alpha}$ as

$$\bar{\rho}_{\alpha} = \bar{\rho}^{(\alpha)} \phi_{\alpha} \quad (2.4)$$

Then (2.3) can be written as

$$\bar{\rho}_m = \sum_{\alpha=1}^{\nu} \bar{\rho}_{\alpha} \quad (2.5)$$

Additionally, for a saturated incompressible mixture, the volume additivity constraint must hold, i.e.

$$\sum_{\alpha=1}^{\nu} \phi_{\alpha} = 1 \quad (2.6)$$

2.3.2 Mixture velocities

Let $\bar{\mathbf{v}}^{\alpha}$ be the velocities of the constituents at a material particle (simultaneously occupied by all constituents) and $\bar{\mathbf{v}}$ the velocity of the mixture, then using the principle of balance of momenta, i.e. the momenta of the mixture must be equal to the sum of the momenta of the constituents, we have

$$\bar{\rho}_m \bar{\mathbf{v}} = \sum_{\alpha=1}^{\nu} \bar{\rho}_{\alpha} \bar{\mathbf{v}}^{\alpha} \quad (2.7)$$

Equation (2.7) defines the mixture velocity at a material particle in terms of bulk densities of the constituents, their velocities, and the mixture density.

2.3.3 Material derivative for the constituents and the mixture

Since the material derivative $\frac{D(\cdot)}{Dt}$ in Eulerian description uses the velocity of a material particle, it needs to be defined for each constituent. The material derivative of a dependent variable \bar{Q} for

constituent α is defined as

$$\frac{D^\alpha \bar{Q}}{Dt} = \frac{\partial \bar{Q}}{\partial t} + \bar{\mathbf{v}}^\alpha \cdot \bar{\nabla} \bar{Q} \quad (2.8)$$

such that the material derivative of \bar{Q} for the mixture can be obtained from

$$\bar{\rho}_m \frac{D\bar{Q}}{Dt} = \sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \frac{D^\alpha \bar{Q}}{Dt} = \sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \left(\frac{\partial \bar{Q}}{\partial t} + \bar{\mathbf{v}}^\alpha \cdot \bar{\nabla} \bar{Q} \right)$$

or

$$\bar{\rho}_m \frac{D\bar{Q}}{Dt} = \left(\sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \right) \frac{\partial \bar{Q}}{\partial t} + \left(\sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \bar{\mathbf{v}}^\alpha \right) \cdot \bar{\nabla} \bar{Q}$$

$$\bar{\rho}_m \frac{D\bar{Q}}{Dt} = \bar{\rho}_m \frac{\partial \bar{Q}}{\partial t} + \bar{\rho}_m \bar{\mathbf{v}} \cdot \bar{\nabla} \bar{Q} \quad (2.9)$$

$$\therefore \frac{D\bar{Q}}{Dt} = \frac{\partial \bar{Q}}{\partial t} + \bar{\mathbf{v}} \cdot \bar{\nabla} \bar{Q} \quad (2.10)$$

2.4 Conservation and balance laws

We use the definitions presented in section 2.3 to derive details of the mathematical model for the mixture using conservation laws. We assume the constituents and the mixture to be incompressible. The constituents and the mixture are considered to be Newtonian and generalized Newtonian fluids. The viscosities of the constituents and the mixture are described using the power law and the Carreau-Yasuda model [21]. We present a general derivation which is made specific based on the assumptions stated above.

2.4.1 Conservation of mass

For volume \bar{V} of the mixture with bulk density $\bar{\rho}_\alpha$ of constituent α , its mass is given by

$$\bar{m}^\alpha = \int_{\bar{V}} \bar{\rho}_\alpha d\bar{V} \quad (2.11)$$

Conservation of mass implies that the rate of change of \bar{m}^α should be zero, ie

$$\frac{D^\alpha \bar{m}^\alpha}{Dt} = \frac{D^\alpha}{Dt} \left(\int_{\bar{V}} \bar{\rho}_\alpha d\bar{V} \right) = 0 \quad (2.12)$$

Using transport theorem in (2.12) we obtain

$$\int_{\bar{V}} \frac{\partial \bar{\rho}_\alpha}{\partial t} + \bar{\nabla} \cdot (\bar{\rho}_\alpha \bar{\mathbf{v}}^\alpha) = 0 \quad (2.13)$$

and since the volume \bar{V} is arbitrary we have

$$\frac{\partial \bar{\rho}_\alpha}{\partial t} + \bar{\nabla} \cdot (\bar{\rho}_\alpha \bar{\mathbf{v}}^\alpha) = 0 \quad (2.14)$$

A conservation of mass equation can be written for the entire mixture by summing over the constituents.

$$\sum_{\alpha=1}^{\nu} \frac{\partial \bar{\rho}_\alpha}{\partial t} + \sum_{\alpha=1}^{\nu} \bar{\nabla} \cdot (\bar{\rho}_\alpha \bar{\mathbf{v}}^\alpha) = 0 \quad (2.15)$$

or

$$\frac{\partial}{\partial t} \left(\sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \right) + \bar{\nabla} \cdot \left(\sum_{\alpha=1}^{\nu} (\bar{\rho}_\alpha \bar{\mathbf{v}}^\alpha) \right) = 0 \quad (2.16)$$

Using (2.5), and (2.16) can be written as

$$\frac{\partial \bar{\rho}_m}{\partial t} + \bar{\nabla} \cdot (\bar{\rho}_m \bar{\mathbf{v}}) = 0 \quad (2.17)$$

For incompressible constituents and mixture, $\bar{\rho}_m$ and $\bar{\rho}_\alpha$ are constant, hence (2.14) and (2.17) reduce to

$$\bar{\rho}_\alpha (\bar{\nabla} \cdot \bar{\mathbf{v}}^\alpha) = 0 \quad (2.18)$$

$$\bar{\rho}_m (\bar{\nabla} \cdot \bar{\mathbf{v}}) = 0 \quad (2.19)$$

2.4.2 Balance of linear momenta

Using principle of balance of linear momenta for a mixture volume \bar{V} , we can obtain the following momentum equations for constituent α in the absence of body forces.

$$\bar{\rho}_\alpha \frac{D^\alpha \bar{\mathbf{v}}^\alpha}{Dt} = \bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \right)^T + \bar{\boldsymbol{\pi}}^\alpha \quad (2.20)$$

$(\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ is the contravariant Cauchy stress tensor and $\bar{\boldsymbol{\pi}}^\alpha$ is the force exerted on the α^{th} constituent by each of the other constituents. These are referred to as interaction forces. In general the interaction force $\bar{\boldsymbol{\pi}}^\alpha$; $\alpha = 1, 2, \dots, \nu$ must satisfy

$$\sum_{\alpha=1}^{\nu} \bar{\boldsymbol{\pi}}^\alpha = 0 \quad (2.21)$$

In the case of a mixture with only two constituents (2.21) reduces to

$$\bar{\boldsymbol{\pi}}^1 = -\bar{\boldsymbol{\pi}}^2 \quad \text{or} \quad \bar{\boldsymbol{\pi}}^1 + \bar{\boldsymbol{\pi}}^2 = 0 \quad (2.22)$$

For a ternary mixture, we have

$$\bar{\boldsymbol{\pi}}^1 + \bar{\boldsymbol{\pi}}^2 + \bar{\boldsymbol{\pi}}^3 = 0 \quad (2.23)$$

We note that $\bar{\boldsymbol{\pi}}^\alpha$ for each constituent α consists of the sum of $(1 - \nu)$ interaction forces exerted on the α^{th} constituent by $(1 - \nu)$ constituents. We consider further details on these in a later section.

2.4.3 Energy equation

In the derivation of the energy equation we assume that the sum of the constituent energies is the total energy of the mixture. For a constituent α , the rate of change of the total energy must be equal to the rate of heat added and the rate of work done.

$$\frac{D^\alpha \bar{E}_t^\alpha}{Dt} = \frac{D^\alpha \bar{Q}^\alpha}{Dt} + \frac{D^\alpha \bar{W}^\alpha}{Dt} \quad (2.24)$$

and for the mixture

$$\sum_{\alpha=1}^{\nu} \frac{D^{\alpha} \bar{E}_t^{\alpha}}{Dt} = \sum_{\alpha=1}^{\nu} \frac{D^{\alpha} \bar{Q}^{\alpha}}{Dt} + \sum \frac{D^{\alpha} \bar{W}^{\alpha}}{Dt} \quad (2.25)$$

where (in the absence of body forces)

$$\bar{E}_t^{\alpha} = \int_{\bar{V}(t)} \bar{\rho}_{\alpha} \left(\bar{e}^{\alpha} + \frac{1}{2} \bar{\mathbf{v}}^{\alpha} \cdot \bar{\mathbf{v}}^{\alpha} \right) d\bar{V} \quad (2.26)$$

$$\sum_{\alpha=1}^{\nu} \frac{D^{\alpha} \bar{Q}^{\alpha}}{Dt} = - \int_{\partial \bar{V}(t)} \bar{\mathbf{q}} \cdot \bar{\mathbf{n}} d\bar{A} = - \int_{\bar{V}(t)} \bar{\nabla} \cdot \bar{\mathbf{q}} d\bar{V} \quad (2.27)$$

in which $\bar{\mathbf{q}}$ is total heat flux and $\bar{\mathbf{n}}$ is the outward unit normal to the boundary $\partial \bar{V}(t)$ of volume $\bar{V}(t)$ in the current configuration. Furthermore

$$\begin{aligned} \frac{D^{\alpha} \bar{W}^{\alpha}}{Dt} &= \int_{\partial \bar{V}(t)} \bar{\mathbf{P}} \cdot \bar{\mathbf{v}}^{\alpha} d\bar{A} = \int_{\partial \bar{V}(t)} \left(\left((\bar{\boldsymbol{\sigma}}^{(0)})^{\alpha} \right)^T \cdot \bar{\mathbf{n}} \right) \cdot \bar{\mathbf{v}}^{\alpha} d\bar{A} \\ &= \int_{\bar{V}(t)} \bar{\nabla} \cdot \left(\bar{\mathbf{v}}^{\alpha} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^{\alpha} \right)^T \right) d\bar{V} \end{aligned} \quad (2.28)$$

or

$$\frac{D^{\alpha} \bar{W}^{\alpha}}{Dt} = \int_{\bar{V}(t)} \left(\bar{\mathbf{v}}^{\alpha} \cdot \left(\bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^{\alpha} \right)^T + (\bar{\boldsymbol{\sigma}}^{(0)})_{ij}^{\alpha} \frac{\partial \bar{v}_i^{\alpha}}{\partial \bar{x}_j} \right) \right) d\bar{V} \quad (2.29)$$

and

$$\frac{D^{\alpha} \bar{E}_t^{\alpha}}{Dt} = \frac{D^{\alpha}}{Dt} \int_{\bar{V}(t)} \bar{\rho}_{\alpha} \left(\bar{e}^{\alpha} + \frac{1}{2} \bar{\mathbf{v}}^{\alpha} \cdot \bar{\mathbf{v}}^{\alpha} \right) d\bar{V} \quad (2.30)$$

For the α^{th} constituent

$$(\rho_{\alpha})_0 dV_0 = (\bar{\rho}_{\alpha}) d\bar{V} \quad (2.31)$$

$(\rho_{\alpha})_0$ and dV_0 are densities and volume in the reference configuration. Hence

$$\frac{D^{\alpha} E_t^{\alpha}}{Dt} = \int_{V_0} \frac{D^{\alpha}}{Dt} \left(\left(e^{\alpha} + \frac{1}{2} \mathbf{v}^{\alpha} \cdot \mathbf{v}^{\alpha} \right) (\rho_{\alpha})_0 \right) dV_0 \quad (2.32)$$

Since $\frac{D(\rho_\alpha)}{Dt} = 0$, (2.32) reduces to

$$\begin{aligned}\frac{D^\alpha E_t^\alpha}{Dt} &= \int_{V_0} \frac{D^\alpha}{Dt} \left(e^\alpha + \frac{1}{2} \mathbf{v}^\alpha \cdot \mathbf{v}^\alpha \right) (\rho_\alpha)_0 dV_0 \\ \frac{D^\alpha E_t^\alpha}{Dt} &= \int_{\bar{V}(t)} \frac{D^\alpha}{Dt} \left(\bar{e}^\alpha + \frac{1}{2} \bar{\mathbf{v}}^\alpha \cdot \bar{\mathbf{v}}^\alpha \right) \bar{\rho}_\alpha d\bar{V} \\ \frac{D^\alpha E_t^\alpha}{Dt} &= \int_{\bar{V}(t)} \left(\frac{D^\alpha \bar{e}^\alpha}{Dt} + \frac{1}{2} \frac{D^\alpha}{Dt} (\bar{\mathbf{v}}^\alpha \cdot \bar{\mathbf{v}}^\alpha) \right) \bar{\rho}_\alpha d\bar{V}\end{aligned}$$

or

$$\frac{D^\alpha \bar{E}_t^\alpha}{Dt} = \int_{\bar{V}(t)} \left(\frac{D^\alpha \bar{e}^\alpha}{Dt} + \bar{\mathbf{v}}^\alpha \cdot \frac{D^\alpha (\bar{\mathbf{v}}^\alpha)}{Dt} \right) \bar{\rho}_\alpha d\bar{V} \quad (2.33)$$

If we assume

$$\bar{\mathbf{q}} = \sum_{\alpha=1}^{\nu} \bar{\mathbf{q}}^\alpha \quad (2.34)$$

then, the energy equation for the α^{th} constituent can be written as

$$\begin{aligned}\int_{\bar{V}(t)} \bar{\rho}_\alpha \left(\frac{D^\alpha \bar{e}^\alpha}{Dt} + \bar{\mathbf{v}}^\alpha \cdot \frac{D^\alpha (\bar{\mathbf{v}}^\alpha)}{Dt} \right) d\bar{V} &= - \int_{\bar{V}(t)} \bar{\nabla} \cdot \bar{\mathbf{q}}^\alpha d\bar{V} \\ &+ \int_{\bar{V}(t)} \left(\bar{\mathbf{v}}^\alpha \cdot \left(\bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \right)^T \right) + (\bar{\boldsymbol{\sigma}}^{(0)})_{ij}^\alpha \frac{\partial (\bar{v}_i^\alpha)}{\partial \bar{x}_j} \right) d\bar{V}\end{aligned} \quad (2.35)$$

Since the volume $\bar{V}(t)$ is arbitrary, (2.35) reduces to

$$\bar{\rho}_\alpha \frac{D^\alpha \bar{e}^\alpha}{Dt} + \bar{\rho}_\alpha \bar{\mathbf{v}}^\alpha \cdot \frac{D^\alpha (\bar{\mathbf{v}}^\alpha)}{Dt} + \bar{\nabla} \cdot \bar{\mathbf{q}}^\alpha - \left(\bar{\mathbf{v}}^\alpha \cdot \left(\bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \right)^T \right) + (\bar{\boldsymbol{\sigma}}^{(0)})_{ij}^\alpha \frac{\partial (\bar{v}_i^\alpha)}{\partial \bar{x}_j} \right) = 0 \quad (2.36)$$

Recall the momentum equation for the α^{th} constituent

$$\bar{\rho}_\alpha \frac{D^\alpha \bar{\mathbf{v}}^\alpha}{Dt} = \bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \right)^T + \bar{\boldsymbol{\pi}}^\alpha \quad (2.37)$$

Substituting from (2.37) into (2.36)

$$\begin{aligned} \bar{\rho}_\alpha \frac{D^\alpha \bar{e}^\alpha}{Dt} + \bar{\mathbf{v}}^\alpha \cdot \left(\bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \right)^T + \bar{\boldsymbol{\pi}}^\alpha \right) + \bar{\nabla} \cdot \bar{\mathbf{q}}^\alpha \\ - \left(\bar{\mathbf{v}}^\alpha \cdot \left(\bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \right)^T \right) + (\bar{\sigma}^{(0)})_{ij}^\alpha \frac{\partial(\bar{v}_i^\alpha)}{\partial \bar{x}_j} \right) = 0 \end{aligned} \quad (2.38)$$

or

$$\bar{\rho}_\alpha \frac{D^\alpha \bar{e}^\alpha}{Dt} + \bar{\mathbf{v}}^\alpha \cdot \bar{\boldsymbol{\pi}}^\alpha + \bar{\nabla} \cdot \bar{\mathbf{q}}^\alpha - (\bar{\sigma}^{(0)})_{ij}^\alpha \frac{\partial(\bar{v}_i^\alpha)}{\partial \bar{x}_j} = 0 \quad (2.39)$$

Summing (2.39) over the constituents and using (2.34)

$$\sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \frac{D^\alpha \bar{e}^\alpha}{Dt} + \sum_{\alpha=1}^{\nu} \bar{\mathbf{v}}^\alpha \cdot \bar{\boldsymbol{\pi}}^\alpha + \bar{\nabla} \cdot \bar{\mathbf{q}} - \sum_{\alpha=1}^{\nu} (\bar{\sigma}^{(0)})_{ij}^\alpha \frac{\partial(\bar{v}_i^\alpha)}{\partial \bar{x}_j} = 0 \quad (2.40)$$

If we assume that for the α^{th} constituent

$$\bar{e}^\alpha = \bar{c}_p^\alpha \bar{\theta} \quad (2.41)$$

and further assume constant \bar{c}_p^α , then (2.40) reduces to

$$\sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \bar{c}_p^\alpha \frac{D^\alpha \bar{\theta}}{Dt} + \sum_{\alpha=1}^{\nu} \bar{\mathbf{v}}^\alpha \cdot \bar{\boldsymbol{\pi}}^\alpha + \bar{\nabla} \cdot \bar{\mathbf{q}} - \sum_{\alpha=1}^{\nu} (\bar{\sigma}^{(0)})_{ij}^\alpha \frac{\partial(\bar{v}_i^\alpha)}{\partial \bar{x}_j} = 0 \quad (2.42)$$

This is the final form of the energy equation for a mixture of ν constituents. If we consider only two constituents then (2.42) becomes

$$\begin{aligned} \left(\bar{\rho}_1 \bar{c}_p^1 \frac{D^1 \bar{\theta}}{Dt} + \bar{\rho}_2 \bar{c}_p^2 \frac{D^2 \bar{\theta}}{Dt} \right) + (\bar{\mathbf{v}}^1 \cdot \bar{\boldsymbol{\pi}}^1 + \bar{\mathbf{v}}^2 \cdot \bar{\boldsymbol{\pi}}^2) + \bar{\nabla} \cdot \bar{\mathbf{q}} \\ - (\bar{\sigma}^{(0)})_{ij}^1 \frac{\partial(\bar{v}_i^1)}{\partial \bar{x}_j} - (\bar{\sigma}^{(0)})_{ij}^2 \frac{\partial(\bar{v}_i^2)}{\partial \bar{x}_j} = 0 \end{aligned} \quad (2.43)$$

2.5 Second law of thermodynamics and constitutive theories

We follow the derivations in reference [12, 13, 15, 22] based on the following notations

$$\begin{aligned} \bar{L}_{ij}^\alpha &= \frac{\partial \bar{v}_i^\alpha}{\partial \bar{x}_j} & \bar{\mathbf{D}}^\alpha &= \frac{1}{2} \left(\bar{\mathbf{L}}^\alpha + (\bar{\mathbf{L}}^\alpha)^T \right) & \bar{\mathbf{q}} &= \sum_{\alpha=1}^{\nu} \bar{\mathbf{q}}^\alpha \\ \bar{\eta} &= \frac{1}{\bar{\rho}_m} \sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \bar{\eta}^\alpha & \bar{\boldsymbol{\pi}} &= -\bar{\boldsymbol{\pi}}^1 = \bar{\boldsymbol{\pi}}^2 \quad (\text{for two constituents}) \end{aligned} \quad (2.44)$$

Recall the momentum and energy equation for a constituent α in a mixture of ν constituents

$$\bar{\rho}_\alpha \frac{D^\alpha \bar{\mathbf{v}}^\alpha}{Dt} = \bar{\nabla} \cdot \left((\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \right)^T + \bar{\boldsymbol{\pi}}^\alpha \quad (2.45)$$

$$\bar{\rho}_\alpha \frac{D^\alpha \bar{e}^\alpha}{Dt} + \bar{\mathbf{v}}^\alpha \cdot \bar{\boldsymbol{\pi}}^\alpha + \bar{\nabla} \cdot \bar{\mathbf{q}}^\alpha - (\bar{\sigma}^{(0)})_{ij}^\alpha \frac{\partial (\bar{v}_i^\alpha)}{\partial \bar{x}_j} = 0 \quad (2.46)$$

with the assumption that

$$\bar{\mathbf{q}} = \sum_{\alpha=1}^{\nu} \bar{\mathbf{q}}^\alpha \quad (2.47)$$

Using (2.45) and (2.46) we can derive entropy inequality for constituent α as [18] using Helmholtz free energy density $\bar{\Phi}^\alpha$.

$$\bar{\rho}_\alpha \left(\frac{D^\alpha \bar{\Phi}^\alpha}{Dt} + \bar{\eta}^\alpha \frac{D^\alpha \bar{\theta}}{Dt} \right) - \bar{\mathbf{v}}^\alpha \cdot \bar{\boldsymbol{\pi}}^\alpha + \frac{\bar{q}_i^\alpha \bar{g}_i}{\bar{\theta}} - (\bar{\sigma}^{(0)})_{ij}^\alpha \frac{\partial (\bar{v}_i^\alpha)}{\partial \bar{x}_j} \leq 0 \quad (2.48)$$

for the mixture of ν constituents (2.48) gives

$$\sum_{\alpha=1}^{\nu} \bar{\rho}_\alpha \left(\frac{D^\alpha \bar{\Phi}^\alpha}{Dt} + \bar{\eta}^\alpha \frac{D^\alpha \bar{\theta}}{Dt} \right) - \sum_{\alpha=1}^{\nu} \bar{\mathbf{v}}^\alpha \cdot \bar{\boldsymbol{\pi}}^\alpha + \sum_{\alpha=1}^{\nu} \frac{\bar{q}_i^\alpha \bar{g}_i}{\bar{\theta}} - \sum_{\alpha=1}^{\nu} (\bar{\sigma}^{(0)})_{ij}^\alpha \frac{\partial (\bar{v}_i^\alpha)}{\partial \bar{x}_j} \leq 0 \quad (2.49)$$

Constitutive theories for a constituent α can be derived using (2.48) alone without using (2.49) as (2.49) is a consequence of (2.48).

In deriving the constitutive theories, it is easier to consider a Lagrangian description of the entropy inequality (2.48). If we choose $(\boldsymbol{\sigma}^*)^\alpha$ (first Piola Kirchhoff stress tensor) and \mathbf{j}^α (material derivative of the Jacobian of deformation) then (2.48) in Lagrangian description can be written as

(for an incompressible medium)

$$(\rho_\alpha)_0(\dot{\Phi}^\alpha + \eta^\alpha \dot{\theta}) - \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} - (\sigma^*)_{ki}^\alpha \dot{J}_{ik}^\alpha \leq 0 \quad (2.50)$$

$(\rho_\alpha)_0$ is the density of constituent α in the reference configuration. For a constituent α , the choice of Φ^α , η^α , $(\sigma^*)^\alpha$ and \mathbf{q}^α as dependent variables in the constitutive theories is rather obvious. We do keep in mind that $\boldsymbol{\pi}^\alpha$ in (2.50) is unknown also. We consider choice of \mathbf{J} ; fundamental measure of deformation; $\dot{\mathbf{J}}$, because of fluids; \mathbf{g} due to \mathbf{q}^α and temperature θ as arguments of the dependent variables in the constitutive theories. All or some of these may be modified at a later stage if so warranted. Thus, we have

$$\begin{aligned} \Phi^\alpha &= \Phi^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \\ (\sigma^*)^\alpha &= (\sigma^*)^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \\ \mathbf{q}^\alpha &= \mathbf{q}^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \\ \eta^\alpha &= \eta^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \end{aligned} \quad (2.51)$$

Using Φ^α in (2.51)

$$\dot{\Phi}^\alpha = \frac{\partial \Phi^\alpha}{\partial J_{ik}^\alpha} \dot{J}_{ik}^\alpha + \frac{\partial \Phi^\alpha}{\partial \dot{J}_{ik}^\alpha} \ddot{J}_{ik}^\alpha + \frac{\partial \Phi^\alpha}{\partial g_i} \dot{g}_i + \frac{\partial \Phi^\alpha}{\partial \theta} \dot{\theta} \quad (2.52)$$

Substituting from (2.52) into (2.50) and rearranging

$$\left((\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial J_{ik}^\alpha} - (\sigma^*)_{ik}^\alpha \right) \dot{J}_{ik}^\alpha + (\rho_\alpha)_0 \left(\eta^\alpha + \frac{\partial \Phi^\alpha}{\partial \theta} \right) \dot{\theta} + (\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial g_i} \dot{g}_i + (\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial \dot{J}_{ik}^\alpha} \ddot{J}_{ik}^\alpha - \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} \leq 0 \quad (2.53)$$

For arbitrary but admissible value of $\dot{\theta}$, $\dot{\mathbf{g}}$, and $\dot{\mathbf{J}}$, (2.50) is satisfied if

$$(\rho_\alpha)_0 \left(\eta^\alpha + \frac{\partial \Phi^\alpha}{\partial \theta} \right) = 0 \quad \text{or} \quad \eta^\alpha + \frac{\partial \Phi^\alpha}{\partial \theta} = 0 \quad (2.54)$$

$$(\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial g_i} = 0 \quad \text{or} \quad \frac{\partial \Phi^\alpha}{\partial g_i} = 0 \quad (2.55)$$

$$(\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial \dot{\mathbf{J}}_{ik}^\alpha} = 0 \quad \text{or} \quad \frac{\partial \Phi^\alpha}{\partial \dot{\mathbf{J}}_{ik}^\alpha} = 0 \quad (2.56)$$

and

$$\left((\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial \mathbf{J}_{ik}^\alpha} - (\sigma^*)_{ki}^\alpha \right) - \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} \leq 0 \quad (2.57)$$

Equations (2.54) - (2.57) are fundamental relations from the entropy inequality.

Remarks

1. Equation (2.54) implies that η^α is deterministic using $\eta^\alpha = -\frac{\partial \Phi^\alpha}{\partial \theta}$, hence η^α can not be a dependent variable in the constitutive theory.
2. Equation (2.55) implies that Φ^α is not a function of \mathbf{g} .
3. Equation (2.56) implies that Φ^α is not a function of $\dot{\mathbf{J}}^\alpha$ either.
4. The inequality in the last equation (2.57) is essential in the form it is stated. For example

$$(\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial \mathbf{J}_{ik}^\alpha} - (\sigma^*)_{ki}^\alpha = 0 \quad \text{and} \quad -\mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} \leq 0 \quad (2.58)$$

are inappropriate due to the fact that these imply that $\boldsymbol{\sigma}^*$ is a function of $\dot{\mathbf{J}}^\alpha$ since Φ^α is not a function of $\dot{\mathbf{J}}^\alpha$ which is contrary to the original assumption in (2.51). We also note that (2.57) in its current form is unable to provide further details regarding the derivation of the constitutive theory for $(\boldsymbol{\sigma}^*)^\alpha$ and \mathbf{q}^α . At this stage we have

$$\begin{aligned} \Phi^\alpha &= \Phi^\alpha(\mathbf{J}^\alpha, \theta) \\ (\boldsymbol{\sigma}^*)^\alpha &= (\boldsymbol{\sigma}^*)^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \\ \mathbf{q}^\alpha &= \mathbf{q}^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \end{aligned} \quad (2.59)$$

2.5.1 Stress decomposition

In order to proceed further using (2.57) we consider decomposition of stress $(\boldsymbol{\sigma}^*)^\alpha$ into equilibrium stress $({}_e\boldsymbol{\sigma}^*)^\alpha$ and deviatoric stress $({}_d\boldsymbol{\sigma}^*)^\alpha$, ie

$$(\boldsymbol{\sigma}^*)^\alpha = ({}_e\boldsymbol{\sigma}^*)^\alpha + ({}_d\boldsymbol{\sigma}^*)^\alpha \quad (2.60)$$

In which we have the following

$$\begin{aligned} ({}_e\boldsymbol{\sigma}^*)^\alpha &= ({}_e\boldsymbol{\sigma}^*)^\alpha(\mathbf{J}^\alpha, 0, 0, \theta) \\ ({}_d\boldsymbol{\sigma}^*)^\alpha &= ({}_d\boldsymbol{\sigma}^*)^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \\ ({}_d\boldsymbol{\sigma}^*)^\alpha &= ({}_d\boldsymbol{\sigma}^*)^\alpha(\mathbf{J}^\alpha, 0, 0, \theta) = 0 \end{aligned} \quad (2.61)$$

That is, $({}_e\boldsymbol{\sigma}^*)^\alpha$ is not a function of $\dot{\mathbf{J}}^\alpha$ or \mathbf{g} and $({}_d\boldsymbol{\sigma}^*)^\alpha$ vanishes when $\dot{\mathbf{J}}^\alpha$ and \mathbf{g} are null. Substituting (2.60) in (2.57) gives

$$\left((\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial J_{ik}^\alpha} - ({}_e\boldsymbol{\sigma}^*)_{ki}^\alpha \right) \dot{J}_{ik}^\alpha - ({}_d\boldsymbol{\sigma}^*)_{ki}^\alpha \dot{J}_{ik}^\alpha - \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} \leq 0 \quad (2.62)$$

Since Φ^α is not a function of $\dot{\mathbf{J}}^\alpha$ and neither is $({}_e\boldsymbol{\sigma}^*)^\alpha$, then $({}_e\boldsymbol{\sigma}^*)^\alpha$ must be derivable from

$$({}_e\boldsymbol{\sigma}^*)_{ki}^\alpha = (\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial J_{ik}^\alpha} \quad \text{or} \quad [{}_e\boldsymbol{\sigma}^*]^T = (\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial [J]} \quad (2.63)$$

Using (2.63), the inequality (2.62) reduces to

$$- ({}_d\boldsymbol{\sigma}^*)_{ki}^\alpha \dot{J}_{ik}^\alpha - \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} \leq 0 \quad (2.64)$$

The inequality (2.64) is satisfied if

$$({}_d\boldsymbol{\sigma}^*)_{ki}^\alpha \dot{J}_{ik}^\alpha \geq 0 \quad \text{and} \quad \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha \geq 0 \quad (2.65)$$

Conditions (2.65) imply that work expanded due to deviatoric stress and interaction force $\boldsymbol{\pi}^\alpha$ must be positive. Thus, now we have

$$(\sigma_{ki}^*)^\alpha = (\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial J_{ik}} + ({}_d\sigma^*)_{ki}^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \quad (2.66)$$

and

$$\mathbf{q}^\alpha = \mathbf{q}^\alpha(\mathbf{J}^\alpha, \dot{\mathbf{J}}^\alpha, \mathbf{g}, \theta) \quad (2.67)$$

2.5.2 Constitutive theory for equilibrium stress tensor: incompressible matter

We use (2.63) to derive constitutive theory for $({}_e\boldsymbol{\sigma}^*)^\alpha$ first and then obtain the constitutive theory for Cauchy stress tensor $({}_e\boldsymbol{\sigma}^{(0)})^\alpha$ in Eulerian description by using the relationship between $({}_e\boldsymbol{\sigma}^*)^\alpha$ and $({}_e\boldsymbol{\sigma}^{\bar{(0)}})^\alpha$ [18]. In the following derivation we only consider incompressible medium for which $|J| = 1$ (constant density), hence in this case $\frac{\partial \Phi}{\partial |J|} = 0$, thus the constitutive theory for $({}_e\boldsymbol{\sigma}^*)^\alpha$ can not be derived using (2.63). Instead, the incompressibility $|J| = 1$ must be enforced. For incompressible matter (in Eulerian description)

$$\text{tr}[\bar{D}^\alpha] = \text{tr}[\bar{L}^\alpha] = \text{tr}\left([\dot{J}][J^\alpha]^{-1}\right) = \dot{J}_{ik}^\alpha \left((J^\alpha)^{-1}\right)_{ki} = 0 \quad (2.68)$$

We enforce (2.68) through the entropy inequality. If (2.68) holds then

$$\text{tr}[\bar{D}^\alpha] = \text{tr}[\bar{L}^\alpha] = p^\alpha(\theta) \dot{J}_{ik}^\alpha \left((J^\alpha)^{-1}\right)_{ki} = 0 \quad (2.69)$$

must hold, where p is a Lagrange multiplier. p can not be a function of \mathbf{J}^α but can depend upon θ , i.e. $p(\theta)$ is valid. We add (2.69) to the left side of the entropy inequality (2.62).

$$\left((\rho_\alpha)_0 \frac{\partial \Phi^\alpha}{\partial J_{ik}^\alpha} - ({}_e\boldsymbol{\sigma}^*)_{ki}^\alpha \right) \dot{J}_{ik}^\alpha - ({}_d\boldsymbol{\sigma}^*)_{ki}^\alpha \dot{J}_{ik}^\alpha - \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} + p^\alpha(\theta) \dot{J}_{ik}^\alpha \left((J^\alpha)^{-1}\right)_{ki} \leq 0 \quad (2.70)$$

Regrouping terms in (2.70) and using $\frac{\partial \Phi^\alpha}{\partial [J]} = 0$, we obtain

$$(p^\alpha(\theta) ((J^\alpha)^{-1})_{ki} - ({}_e\sigma^*)_{ik}^\alpha) \dot{J}_{ik}^\alpha - ({}_d\sigma^*)_{ki}^\alpha \dot{J}_{ik}^\alpha - \mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha + \frac{q_i^\alpha g_i}{\theta} \leq 0 \quad (2.71)$$

The inequality holds if

$$p^\alpha(\theta) ((J^\alpha)^{-1})_{ki} - ({}_e\sigma^*)_{ik}^\alpha = 0 \quad (2.72)$$

$$({}_d\sigma^*)_{ki}^\alpha \dot{J}_{ik}^\alpha \geq 0 \quad (2.73)$$

$$\mathbf{v}^\alpha \cdot \boldsymbol{\pi}^\alpha \geq 0 \quad (2.74)$$

and

$$q_i^\alpha g_i \leq 0 \quad ; \quad (\text{as } \theta > 0) \quad (2.75)$$

From (2.72)

$$[({}_e\sigma^*)^\alpha]^T = p^\alpha(\theta) [(J^\alpha)^T]^{-1} \quad (2.76)$$

For incompressible matter, the contravariant Cauchy stress tensor $(\boldsymbol{\sigma}^{(0)})^\alpha$ in Lagrangian description is given by

$$\left[(\boldsymbol{\sigma}^{(0)})^\alpha \right] = [({}_e\sigma^*)^\alpha]^T [J^\alpha]^T \quad (2.77)$$

Hence, using (2.77) for equilibrium stress tensor and substituting for (2.76)

$$\left[({}_e\sigma^{(0)})^\alpha \right] = p^\alpha(\theta) [I] \quad (2.78)$$

$({}_e\boldsymbol{\sigma}^{(0)})^\alpha$ is the equilibrium contravariant Cauchy stress tensor for constituent α in Lagrangian description, hence $\theta = \theta(\boldsymbol{x}, t)$. In the Eulerian description (needed for fluids) (2.78) becomes

$$\left[({}_e\bar{\sigma}^{(0)})^\alpha \right] = \bar{p}^\alpha(\bar{\theta}) [I] \quad (2.79)$$

$({}_e\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ is the equilibrium contravariant Cauchy stress tensor in Eulerian description. $\bar{p}^\alpha(\bar{\theta})$ is mechanical pressure of constituent α . If we assume the compressive pressure to be positive, then $\bar{p}^\alpha(\bar{\theta})$ and $p^\alpha(\theta)$ in (2.79) and (2.78) can be replaced by $-\bar{p}^\alpha(\bar{\theta})$ and $-p^\alpha(\theta)$.

2.5.3 Constitutive theory for deviatoric stress tensor $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha_{ij}$ for constituent α

In Eulerian description with transport, the stress measure $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$, contravariant Cauchy stress tensor is preferred which upon decomposition gives

$$({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha = ({}_e\bar{\boldsymbol{\sigma}}^{(0)})^\alpha + ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \quad (2.80)$$

Constitutive theory for $({}_e\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ has already been derived. In Eulerian description $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ and $\bar{\mathbf{D}}^\alpha$ are a conjugate pair. We note that

$$\begin{aligned} [\dot{\mathbf{J}}^\alpha] &= [\bar{\mathbf{L}}^\alpha][\mathbf{J}^\alpha] & [\bar{\mathbf{D}}^\alpha] &= \frac{1}{2}([\bar{\mathbf{L}}^\alpha] + [\bar{\mathbf{L}}^\alpha]^T) & [\bar{\mathbf{W}}^\alpha] &= \frac{1}{2}([\bar{\mathbf{L}}^\alpha] - [\bar{\mathbf{L}}^\alpha]^T) \\ [\bar{\mathbf{L}}^\alpha] &= [\bar{\mathbf{D}}^\alpha] + [\bar{\mathbf{W}}^\alpha] & [\dot{\mathbf{J}}^\alpha] &= ([\bar{\mathbf{D}}^\alpha] + [\bar{\mathbf{W}}^\alpha])[\mathbf{J}^\alpha] \end{aligned} \quad (2.81)$$

Following (2.59), we can write the following in Eulerian description for fluids.

$$\begin{aligned} ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha &= ({}_e\bar{\boldsymbol{\sigma}}^{(0)})^\alpha + ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \\ ({}_e\bar{\boldsymbol{\sigma}}^{(0)})^\alpha &= \bar{p}^\alpha(\bar{\theta})\mathbf{I} \\ ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha &= ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha(\mathbf{J}^\alpha, \bar{\mathbf{D}}^\alpha, \bar{\mathbf{W}}^\alpha, \bar{\mathbf{g}}, \bar{\theta}) \\ \bar{\mathbf{q}}^\alpha &= \bar{\mathbf{q}}^\alpha(\mathbf{J}^\alpha, \bar{\mathbf{D}}^\alpha, \bar{\mathbf{W}}^\alpha, \bar{\mathbf{g}}, \bar{\theta}) \end{aligned} \quad (2.82)$$

In Eulerian description with transport i.e. fluids stresses do not depend upon \mathbf{J}^α (a measure of strain) or $\bar{\mathbf{W}}^\alpha$, pure rotation. Thus,

$$\begin{aligned}({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha &= ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha(\bar{\mathbf{D}}^\alpha, \bar{\mathbf{g}}, \bar{\theta}) \\ \bar{\mathbf{q}}^\alpha &= \bar{\mathbf{q}}^\alpha(\bar{\mathbf{D}}^\alpha, \bar{\mathbf{g}}, \bar{\theta})\end{aligned}\tag{2.83}$$

We note in (2.83), decision on the argument tensors has been primarily based on a single constituent α . However, in a mixture of ν constituents, the stress tensor $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ for constituent α may also show dependence on $\bar{\mathbf{D}}^\alpha$ of other constituents. Even though $\bar{\mathbf{W}}^\alpha$ for the constituent α is not an argument of $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$, but relative spin between the constituent α and others may also contribute to $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$. Lastly, the relative velocity between the constituent α and others can also be a contributing factor to $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$. Based on these considerations, (2.83) can be written as

$$\begin{aligned}({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha &= ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha(\bar{\mathbf{D}}^i; i = 1, 2, \dots, \alpha, \dots, \nu; \bar{\mathbf{W}}^{\alpha i}, i = 1, 2, \dots, \nu; \bar{\mathbf{v}}^{\alpha i}; i = 1, 2, \dots, \nu, \bar{\mathbf{g}}, \bar{\theta}) \\ \bar{\mathbf{q}}^\alpha &= \bar{\mathbf{q}}^\alpha(\bar{\mathbf{D}}^i; i = 1, 2, \dots, \alpha, \dots, \nu; \bar{\mathbf{W}}^{\alpha i}, i = 1, 2, \dots, \nu; \bar{\mathbf{v}}^{\alpha i}; i = 1, 2, \dots, \nu, \bar{\mathbf{g}}, \bar{\theta})\end{aligned}\tag{2.84}$$

For a mixture consisting of two constituents (2.84) reduces to

$$\begin{aligned}({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha &= ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha(\bar{\mathbf{D}}^1, \bar{\mathbf{D}}^2, \bar{\mathbf{W}}^{12}, \bar{\mathbf{v}}^{12}, \bar{\mathbf{g}}, \bar{\theta}) \\ \bar{\mathbf{q}}^\alpha &= \bar{\mathbf{q}}^\alpha(\bar{\mathbf{D}}^1, \bar{\mathbf{D}}^2, \bar{\mathbf{W}}^{12}, \bar{\mathbf{v}}^{12}, \bar{\mathbf{g}}, \bar{\theta})\end{aligned}\tag{2.85}$$

Using (2.85) constitutive theories for the deviatoric Cauchy stress tensor and heat vector can be derived for a two constituent mixture by using the theory of generators and invariants. For a mixture consisting of more than two constituents we can use (2.84) and the theory of generators and invariants. In the present work we consider simplified constitutive theories. Consider a constitutive theory for $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ that is linear in the components of $\bar{\mathbf{D}}^i$; $i = 1, 2, \dots, \nu$, does not include their product terms and neglects dependence of $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ on $\bar{\mathbf{W}}^{\alpha i}$; $i = 1, 2, \dots, \nu$, $\bar{\mathbf{v}}^{\alpha i}$; $i = 1, 2, \dots, \nu$, $\bar{\mathbf{g}}$

and $\bar{\theta}$. Such constitutive theory for $({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha$ can be written as (for compressible).

$$({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha = \sum_{i=1}^{\nu} 2\mu_{\alpha i} \bar{\mathbf{D}}^i + \sum_{i=1}^{\nu} \lambda_{\alpha i} \text{tr}(\bar{\mathbf{D}}^i) \quad ; \quad \alpha = 1, 2, \dots, \nu \quad (2.86)$$

and

$$({}_e\bar{\boldsymbol{\sigma}}^{(0)})^\alpha = \bar{p}^\alpha(\bar{\theta}) \mathbf{I} \quad (2.87)$$

For mixture

$${}_d\bar{\boldsymbol{\sigma}}^{(0)} = \sum_{\alpha=1}^{\nu} ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \quad (2.88)$$

$${}_e\bar{\boldsymbol{\sigma}}^{(0)} = \sum_{\alpha=1}^{\nu} \bar{p}^\alpha(\bar{\theta}) \mathbf{I} = \bar{p}(\bar{\theta}) \mathbf{I} \quad (2.89)$$

Hence,

$$\bar{p}(\bar{\theta}) = \sum_{\alpha=1}^{\nu} \bar{p}^\alpha(\bar{\theta}) \quad (2.90)$$

Equation (2.90) holds if

$$\bar{p}^\alpha(\bar{\theta}) = \phi_\alpha \bar{p}(\bar{\theta}) \quad (2.91)$$

$\bar{p}(\bar{\theta})$ is the total mechanical pressure for the mixture. For incompressible fluids $\text{tr}([\bar{D}^i]) = 0$, $i = 1, 2, \dots, \nu$, hence (2.86) reduces to

$$({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha = \sum_{i=1}^{\nu} 2\mu_{\alpha i} \bar{\mathbf{D}}^i \quad (2.92)$$

For a two constituent incompressible mixture of incompressible constituents we have the following

$$\begin{aligned} ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^1 &= 2\mu_1 \bar{\mathbf{D}}^1 + 2\mu_3 \bar{\mathbf{D}}^2 \\ ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^2 &= 2\mu_4 \bar{\mathbf{D}}^1 + 2\mu_2 \bar{\mathbf{D}}^2 \end{aligned} \quad (2.93)$$

In (2.93) we have redefined $\mu_{11} = \mu_1$, $\mu_{12} = \mu_3$, $\mu_{21} = \mu_4$ and $\mu_{22} = \mu_2$.

The general constitutive theory for $\bar{\mathbf{q}}^\alpha$ based on integrity can be derived using $\bar{\mathbf{q}}^\alpha$ in (2.84) and

the theory of generators and invariants. More simplified constitutive theories for $\bar{\mathbf{q}}^\alpha$ include

$$\bar{\mathbf{q}}^\alpha = -k_\alpha \bar{\mathbf{g}} - k_\alpha^1 \bar{\mathbf{v}}^{12} \quad (2.94)$$

or

$$\bar{\mathbf{q}}^\alpha = -k_\alpha \bar{\mathbf{g}} \quad (2.95)$$

As an example, using (2.95) and (2.34) we can write

$$\bar{\mathbf{q}} = \sum_{\alpha=1}^{\nu} \bar{\mathbf{q}}^\alpha = - \left(\sum_{\alpha=1}^{\nu} k_\alpha \right) \bar{\mathbf{g}} \quad (2.96)$$

If we assume that k_α follows

$$k_m = \sum_{\alpha=1}^{\nu} k_\alpha \quad (2.97)$$

Then

$$\bar{\mathbf{q}} = - \left(\sum_{\alpha=1}^{\nu} k_\alpha \right) \bar{\mathbf{g}} = -k_m \bar{\mathbf{g}} \quad (2.98)$$

In which k_m is the thermal conductivity for the mixture. Equation (2.95) and (2.98) are Fourier heat conduction law for the constituent α and the mixture. The material coefficients in (2.93), (2.95), and (2.98) can be function of the appropriate desired invariants of the argument tensors in (2.85) with some restrictions resulting from the entropy inequality.

2.5.4 Interaction forces $\bar{\boldsymbol{\pi}}^\alpha$

From the conditions resulting from the entropy inequality we note that $\bar{\boldsymbol{\theta}}^\alpha \cdot \bar{\boldsymbol{\pi}}^\alpha$ must be greater than or equal to zero but we do not have any mechanism to derive dependence of $\bar{\boldsymbol{\pi}}^\alpha$ on deformation. In this work we present some considerations based on physics of flow that permit us to define $\bar{\boldsymbol{\pi}}^\alpha$.

First, $\bar{\boldsymbol{\pi}}^\alpha$ must obviously be a function of the relative velocity between constituent α and the others. Even though the temperature gradient is same for all constituents occupying a position,

but their different molecular structure and weights create different levels of vibrational energy that produce interaction forces. Based on these criteria we can write

$$\bar{\boldsymbol{\pi}}^\alpha = \sum_{i=1}^{\nu} \beta_{\alpha i} \bar{\boldsymbol{v}}^{\alpha i} + \sum_{i=1}^{\nu} \gamma_{\alpha i} \bar{\boldsymbol{g}} \quad ; \quad \gamma_{ii} = 0 \quad ; \quad i = 1, 2, \dots, \nu \quad (2.99)$$

For a two constituent mixture we have

$$\begin{aligned} \bar{\boldsymbol{\pi}}^1 &= \beta_{12} \bar{\boldsymbol{v}}^{12} + \gamma_{12} \bar{\boldsymbol{g}} \\ \bar{\boldsymbol{\pi}}^2 &= \beta_{21} \bar{\boldsymbol{v}}^{21} + \gamma_{21} \bar{\boldsymbol{g}} \end{aligned} \quad (2.100)$$

Since $\bar{\boldsymbol{\pi}}^1 + \bar{\boldsymbol{\pi}}^2 = 0$ then $\gamma_{21} = -\gamma_{12}$ must also hold. In (2.99) the coefficients $\beta_{\alpha i}$ and $\gamma_{\alpha i}$ must obey

$$\beta_{ji} = \beta_{ij} \quad \text{and} \quad \gamma_{ji} = -\gamma_{ij} \quad (2.101)$$

2.6 Material coefficients

Each constituent in a mixture has its own ‘pure’ properties say $Q^{(\alpha)}$, a pure property Q for constituent α . When these are mixed with volume fractions ϕ_α to form a mixture, the mixture has (i) bulk property Q_α for each constituent α and (ii) the mixture property Q_m of property Q . If we consider the mixture stress to be the sum of the constituent stresses, then bulk properties of the constituents are required in the constitutive theories for the constituents. On the other hand if we consider the mixture to be homogenous and isotropic fluid, then its own mixture properties can be used in the constitutive theories for the mixture. We keep in mind that bulk properties of the constituents in a mixture must be related to the mixture properties.

2.6.1 Mixture material properties: mixing rules

Consider a property Q with mixture value Q_m . Let the mixture consist of ν constituents and let Q be the pure property of constituent α , then based on [23, 24] we have the following mixing

rule:

$$Q_m = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} y_i y_j Q_{ij} \quad (2.102)$$

where y_k ; $k = 1, 2, \dots, \nu$ could be volume fractions or molar fractions and Q_{ij} is called the mutual parameter, which depends on the pure properties $Q^{(k)}$; $k = 1, 2, \dots, \nu$. Following reference [23,24], the following three approaches for Q_{ij} are commonly used.

2.6.1.1 Arithmetic mean

In this case we assume Q_{ij} to be

$$Q_{ij} = \frac{Q^{(i)} + Q^{(j)}}{2} \quad (2.103)$$

Substituting (2.103) in (2.102)

$$\begin{aligned} Q_m &= \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} y_i y_j \left(\frac{Q^{(i)} + Q^{(j)}}{2} \right) \\ &= \sum_{i=1}^{\nu} \frac{y_i Q^{(i)}}{2} \left(\sum_{j=1}^{\nu} y_j \right) + \sum_{j=1}^{\nu} y_j Q^{(j)} \left(\sum_{i=1}^{\nu} y_i \right) \end{aligned} \quad (2.104)$$

But $\sum_{k=1}^{\nu} y_k = 1$, hence (2.104) reduce to

$$Q_m = \sum_{i=1}^{\nu} \frac{y_i Q^{(i)}}{2} + \sum_{j=1}^{\nu} \frac{y_j Q^{(j)}}{2} \quad (2.105)$$

or

$$Q_m = \sum_{\alpha=1}^{\nu} y_{\alpha} Q^{(\alpha)} \quad (2.106)$$

Based on this rule, mixture property Q_m is given by (2.106) in terms of pure properties $Q^{(\alpha)}$ of the constituents and the volume or molar fractions y_{α} .

2.6.1.2 Geometric mean

In this case

$$Q_{ij} = \sqrt{Q^{(i)}Q^{(j)}} \quad (2.107)$$

Substituting from (2.107) into (2.103)

$$Q_m = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} y_i y_j \sqrt{Q^{(i)}Q^{(j)}} \quad (2.108)$$

or

$$Q_m = \left(\sum_{i=1}^{\nu} y_i \sqrt{Q^{(i)}} \right) \left(\sum_{j=1}^{\nu} y_j \sqrt{Q^{(j)}} \right) \quad (2.109)$$

or

$$Q_m = \left(\sum_{i=1}^{\nu} y_i \sqrt{Q^{(i)}} \right)^2 \quad (2.110)$$

2.6.1.3 Harmonic mean

In this case we assume Q_{ij} to be

$$Q_{ij} = \frac{2}{\frac{1}{Q^{(i)}} + \frac{1}{Q^{(j)}}} \quad (2.111)$$

or

$$Q_{ij} = \frac{2Q^{(i)}Q^{(j)}}{Q^{(i)} + Q^{(j)}} \quad (2.112)$$

Substituting from (2.112) into (2.102) we obtain

$$Q_m = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} y_i y_j \left(\frac{2Q^{(i)}Q^{(j)}}{Q^{(i)} + Q^{(j)}} \right) \quad (2.113)$$

The expression in (2.113) has no obvious reducible form as in cases of arithmetic and geometric means.

Remarks

1. Using the mixing rule (2.102), with arithmetic, geometric and harmonic mean defining the mutual parameter, we can obtain the mixture properties.
2. Each of these three approaches will obviously give different mixture properties with some obvious overlap depending upon volume fractions and pure property variations.

2.6.2 Mixture thermal conductivity k_m

Since temperature gradient \bar{g} at a point is same for all constituents, it perhaps makes more sense to concentrate a little more on the specific details of the mixture conductivity k_m . There are many published works related to liquid mixture conductivity. It is noted [24] that the thermal conductivities of liquid mixtures are usually less than those predicted by mole or weight fraction average theories, though the deviations are often small. Some of these methods that are described in [23,24] are called Filippov Equation [25], Jamieson Correlation [26], Barconcini Correlation [27, 28], Method of Rowley [29], Power Law Method [29, 30], and Li's Method [31]. In this work we describe Li's method for multi-component or multi-constituent mixture. This method is based on two assumptions: (i) Energy transport in liquid state occurs by collision among molecules. This assumption implies that for this thermodynamic state the appropriate parameter for this collision process should be the volumetric fraction. (ii) The internal thermal conductivity can be approximated by a harmonic mean of the pure component values. Using volume fractions $\phi_i ; i = 1, 2, \dots, \nu$ we can express mixture conductivity k_m by using mutual conductivities k_{ij}

$$k_m = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \phi_i \phi_j k_{ij} \quad (2.114)$$

where

$$k_{ij} = 2 \left((k^{(i)})^{-1} + (k^{(j)})^{-1} \right)^{-1} = 2 \left(\frac{1}{k^{(i)}} + \frac{1}{k^{(j)}} \right)^{-1} \quad (2.115)$$

or

$$k_{ij} = 2 \left(\frac{k^{(i)} + k^{(j)}}{k^{(i)}k^{(j)}} \right)^{-1} = \frac{2k^{(i)}k^{(j)}}{k^{(i)} + k^{(j)}} \quad (2.116)$$

which is the same as (2.112), ie harmonic mean for k_{ij} .

For binary mixtures with $k^{(1)}$, $k^{(2)}$, we have

$$k_m = \sum_{i=1}^2 \sum_{j=1}^2 \phi_i \phi_j \left(\frac{2k^{(i)}k^{(j)}}{k^{(i)} + k^{(j)}} \right) \quad (2.117)$$

and upon expanding, we obtain

$$k_m = (\phi_1)^2 k^{(1)} + (\phi_2)^2 k^{(2)} + 4\phi_1 \phi_2 \frac{k^{(1)}k^{(2)}}{k^{(1)} + k^{(2)}} \quad (2.118)$$

2.6.3 Bulk properties of the constituents

Bulk properties of the constituents in the mixture can also be derived from the mixing rule by approximating the mutual parameter using arithmetic mean, geometric mean and the harmonic mean.

Consider rule (2.102) for a property Q

$$Q_m = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} y_i y_j Q_{ij} \quad (2.119)$$

Q_{ij} being the mutual parameter.

2.6.3.1 Arithmetic mean for Q_{ij}

Following the derivation presented earlier

$$Q_m = \sum_{\alpha=1}^{\nu} y_{\alpha} Q^{(\alpha)} = \sum_{\alpha=1}^{\nu} Q_{\alpha} \quad (2.120)$$

where Q_{α} is the bulk property of constituent α for property Q .

2.6.3.2 Geometric mean for Q_{ij}

Following the derivation in section 2.6.1 we have

$$Q_m = \left(\sum_{\alpha=1}^{\nu} y_{\alpha} \sqrt{Q^{(\alpha)}} \right)^2 \quad (2.121)$$

For a two constituent mixture:

$$Q_m = (y_1)^2 Q^{(1)} + (y_2)^2 Q^{(2)} + 2y_1 y_2 \sqrt{Q^{(1)} Q^{(2)}} \quad (2.122)$$

or

$$Q_m = \left((y_1)^2 Q^{(1)} + y_1 y_2 \sqrt{Q^{(1)} Q^{(2)}} \right) + \left((y_2)^2 Q^{(2)} + y_1 y_2 \sqrt{Q^{(1)} Q^{(2)}} \right) \quad (2.123)$$

or

$$Q_m = Q_1 + Q_2 \quad (2.124)$$

Q_1 and Q_2 being bulk properties of constituents 1 and 2 in the mixture.

For three constituent mixture:

$$Q_m = (y_1)^2 Q^{(1)} + (y_2)^2 Q^{(2)} + (y_3)^2 Q^{(3)} + 2y_1 y_2 \sqrt{Q^{(1)} Q^{(2)}} + 2y_2 y_3 \sqrt{Q^{(2)} Q^{(3)}} + 2y_3 y_1 \sqrt{Q^{(1)} Q^{(3)}} \quad (2.125)$$

or

$$\begin{aligned} Q_m = & \left((y_1)^2 Q^{(1)} + y_1 y_2 \sqrt{Q^{(1)} Q^{(2)}} + y_1 y_3 \sqrt{Q^{(1)} Q^{(3)}} \right) + \\ & \left((y_2)^2 Q^{(2)} + y_2 y_1 \sqrt{Q^{(1)} Q^{(2)}} + y_2 y_3 \sqrt{Q^{(2)} Q^{(3)}} \right) + \\ & \left((y_3)^2 Q^{(3)} + y_3 y_1 \sqrt{Q^{(1)} Q^{(3)}} + y_3 y_2 \sqrt{Q^{(2)} Q^{(3)}} \right) \end{aligned} \quad (2.126)$$

or

$$Q_m = Q_1 + Q_2 + Q_3 \quad (2.127)$$

Q_1, Q_2, Q_3 are the bulk properties of the three constituents in the mixture.

2.6.3.3 Generalization for ν constituents

The expression in (2.126) can be generalized for ν constituents.

$$Q_m = \sum_{\alpha=1}^{\nu} Q_{\alpha} \quad (2.128)$$

$$Q_{\alpha} = (y_{\alpha})^2 Q^{(\alpha)} + \sum_{\substack{j=1 \\ j \neq \alpha}}^{\nu} y_{\alpha} y_j \sqrt{Q^{(\alpha)} Q^{(j)}} \quad (2.129)$$

2.6.3.4 Limitations of geometric mean

We note that there are limitations to the use of the geometric mean to describe the bulk properties of the constituents. Let us consider the use of (2.129) to application of 2 constituents. Thus for constituent two

$$Q_2 = \phi_2^2 Q^{(2)} + \phi_1 \phi_2 \sqrt{Q^{(1)} Q^{(2)}} \quad (2.130)$$

For the bulk property to be valid, the bulk property must be less than or equal to the pure component property, that is

$$Q^{(\alpha)} \geq Q_{\alpha} \quad (2.131)$$

Applying (2.131) to (2.130) we obtain

$$Q^{(2)} \geq \phi_2^2 Q^{(2)} + \phi_1 \phi_2 \sqrt{Q^{(1)} Q^{(2)}} \quad (2.132)$$

If we consider the ratio between the pure properties $Q^{(1)}$ and $Q^{(2)}$ to be

$$\frac{Q^{(1)}}{Q^{(2)}} = R \quad (2.133)$$

and knowing that through the volume additivity constraint $\phi_1 = (1 - \phi_2)$ then from (2.132) we obtain

$$Q^{(2)} \geq \phi_2^2 Q^{(2)} + (\phi_2 - \phi_2^2) Q^{(2)} \sqrt{R} \quad (2.134)$$

or

$$1 \geq \phi_2^2 + (\phi_2 - \phi_2^2)\sqrt{R} \quad (2.135)$$

For this inequality to be valid for all ϕ_2 , the maximum value of $Q_2/Q^{(2)}$ occurs at

$$\frac{d}{d\phi_2} (1) = \frac{d}{d\phi_2} \left(\phi_2^2 + (\phi_2 - \phi_2^2)\sqrt{R} \right) \quad (2.136)$$

or

$$0 = 2\phi_2 \left(1 - \sqrt{R} \right) + \sqrt{R} \quad (2.137)$$

and solving for ϕ_2

$$\phi_2 = \frac{-\sqrt{R}}{2 - 2\sqrt{R}} \quad (2.138)$$

Substituting (2.138) into (2.135) and we obtain

$$1 \geq \frac{R}{4(\sqrt{R} - 1)} \quad (2.139)$$

and solving for R

$$R \leq 4 \quad (2.140)$$

Thus in order for bulk property not to exceed to value of the pure constituent for any value of ϕ_1 and ϕ_2 the ratio between $Q^{(1)}$ and $Q^{(2)}$ must be less than or equal to 4 which is expressed as

$$\frac{Q^{(1)}}{Q^{(2)}} \leq 4 \quad (2.141)$$

2.6.3.5 Harmonic mean for Q_{ij}

The expression for Q_m in (2.113) can be regrouped as follows

$$Q_m = \sum_{\alpha=1}^{\nu} Q_{\alpha} \quad (2.142)$$

where

$$Q_\alpha = (y_\alpha)^2 Q^{(\alpha)} + 2 \sum_{\substack{j=1 \\ j \neq \alpha}}^{\nu} \left(y_\alpha y_j \frac{Q^{(\alpha)} Q^{(j)}}{Q^{(\alpha)} + Q^{(j)}} \right) \quad (2.143)$$

2.6.4 Material coefficients in the constitutive theories for $({}_d\bar{\sigma}^{(0)})_{ij}^\alpha$

Consider simple constitutive theory for a two constituent mixture given by (2.93).

$$\begin{aligned} ({}_d\bar{\sigma}^{(0)})^1 &= 2\mu_1 \bar{\mathbf{D}}^1 + 2\mu_3 \bar{\mathbf{D}}^2 \\ ({}_d\bar{\sigma}^{(0)})^2 &= 2\mu_4 \bar{\mathbf{D}}^1 + 2\mu_2 \bar{\mathbf{D}}^2 \end{aligned} \quad (2.144)$$

If we assume that the mixture deviatoric stress ${}_d\bar{\sigma}^{(0)}$ is the sum of the constituent deviatoric stresses, then

$${}_d\bar{\sigma}^{(0)} = ({}_d\bar{\sigma}^{(0)})^1 + ({}_d\bar{\sigma}^{(0)})^2 \quad (2.145)$$

For a homogenous, isotropic and saturated mixture we can write

$${}_d\bar{\sigma}^{(0)} = 2\mu_m \bar{\mathbf{D}} \quad (2.146)$$

Then using (2.144) - (2.146) we can write

$$2\mu_m \bar{\mathbf{D}} = 2\mu_1 \bar{\mathbf{D}}^1 + 2\mu_2 \bar{\mathbf{D}}^2 + 2\mu_3 \bar{\mathbf{D}}^2 + 2\mu_4 \bar{\mathbf{D}}^1 \quad (2.147)$$

Using

$$\bar{\mathbf{D}} = \sum_{\alpha=1}^{\nu} \frac{\bar{\rho}_\alpha}{\bar{\rho}_m} \bar{\mathbf{D}}^\alpha \quad (2.148)$$

in (2.147) we obtain

$$\mu_m \left(\frac{\bar{\rho}_1}{\bar{\rho}_m} \bar{\mathbf{D}}^1 + \frac{\bar{\rho}_2}{\bar{\rho}_m} \bar{\mathbf{D}}^2 \right) = (\mu_1 + \mu_4) \bar{\mathbf{D}}^1 + (\mu_2 + \mu_3) \bar{\mathbf{D}}^2 \quad (2.149)$$

For the special case when the two constituents are the same fluids $\rho_1 + \rho_2 = \rho_m$, $\bar{\mathbf{D}}^1 = \bar{\mathbf{D}}^2 = \bar{\mathbf{D}}$, hence (2.147) reduces to

$$\mu_m = (\mu_1 + \mu_2 + \mu_3 + \mu_4) \quad (2.150)$$

Thus, the material coefficients in (2.144) and (2.146) must satisfy (2.150).

2.6.5 Material coefficients proposed in ref [32] for the constitutive equation

(2.144)

Williams and Sampaio [32] describe the material coefficients $\mu_1, \mu_2, \dots, \mu_4$ used in constitutive equation (2.144). This work follows on to the original paper by Dolezalek and Shultze [33]. Williams and Sampaio propose the following for a two constituent mixture

$$\mu_1 = (\phi_1)^2 \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \quad (2.151)$$

$$\mu_2 = (\phi_2)^2 \mu^{(2)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \quad (2.152)$$

$$\mu_3 = \mu_4 = \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \quad (2.153)$$

in which $\mu^{(1)}$ and $\mu^{(2)}$ are pure viscosities of the two constituents. These relations are purely empirical, however as we have seen (2.151) and (2.152) can be derived using mixing rule in which the mutual parameters are defined using geometric mean, but there is no basis for (2.153) except purely empirical.

2.6.5.1 Validity of μ_3, μ_4 in (2.153)

In the constitutive equations (2.144) we consider (2.151) - (2.153) and their validity. Consider a mixture in which the two constituents are the same, say constituent one. Thus, for this case

$\mu^{(2)} = \mu^{(1)}$ and we can write

$$\begin{aligned}\mu_1 &= \mu^{(1)} (\phi_1^2 + \phi_1\phi_2) \\ \mu_2 &= \mu^{(1)} (\phi_2^2 + \phi_1\phi_2) \\ \mu_3 &= \mu_4 = \phi_1\phi_2\mu^{(1)}\end{aligned}\tag{2.154}$$

Therefore

$$\begin{aligned}({}_d\bar{\sigma}^{(0)})^1 &= 2\mu^{(1)} (\phi_1^2 + \phi_1\phi_2) \bar{\mathbf{D}}^1 + 2\phi_1\phi_2\mu^{(1)} \bar{\mathbf{D}}^2 \\ ({}_d\bar{\sigma}^{(0)})^2 &= 2\mu^{(1)} (\phi_2^2 + \phi_1\phi_2) \bar{\mathbf{D}}^2 + 2\phi_1\phi_2\mu^{(1)} \bar{\mathbf{D}}^1\end{aligned}\tag{2.155}$$

Since both constituents are one

$$\rho^{(2)} = \rho^{(1)} \quad ; \quad \bar{\rho}_1 = \phi_1\rho^{(1)} \quad ; \quad \bar{\rho}_2 = \phi_2\rho^{(1)}\tag{2.156}$$

Since

$$\bar{\rho}_m\bar{\mathbf{v}} = \bar{\rho}_1\bar{\mathbf{v}}^1 + \bar{\rho}_2\bar{\mathbf{v}}^2 \quad \text{and} \quad \bar{\rho}_m = \rho^{(1)}\tag{2.157}$$

$$\rho^{(1)}\bar{\mathbf{v}} = \phi_1\rho^{(1)}\bar{\mathbf{v}}^1 + \phi_2\rho^{(1)}\bar{\mathbf{v}}^2\tag{2.158}$$

$$\therefore \quad \bar{\mathbf{v}} = \phi_1\bar{\mathbf{v}}^1 + \phi_2\bar{\mathbf{v}}^2\tag{2.159}$$

and we also have

$$\bar{\mathbf{D}} = \phi_1\bar{\mathbf{D}}^1 + \phi_2\bar{\mathbf{D}}^2\tag{2.160}$$

Since $\phi_1 + \phi_2 = 1$, using (2.155) we can write

$${}_d\bar{\sigma}^{(0)} = ({}_d\bar{\sigma}^{(0)})^1 + ({}_d\bar{\sigma}^{(0)})^2 = 2\mu^{(1)} (\phi_1\bar{\mathbf{D}}^1 + \phi_2\bar{\mathbf{D}}^2) + 2\mu^{(1)}\phi_1\phi_2 (\bar{\mathbf{D}}^1 + \bar{\mathbf{D}}^2)\tag{2.161}$$

Using (2.160), (2.161) can be written as

$${}_d\bar{\boldsymbol{\sigma}}^{(0)} = 2\mu^{(1)}\bar{\mathbf{D}} + 2\mu^{(1)}\phi_1\phi_2\left(\bar{\mathbf{D}}^1 + \bar{\mathbf{D}}^2\right) \quad (2.162)$$

But ${}_d\bar{\boldsymbol{\sigma}}^{(0)} = 2\mu^{(1)}\bar{\mathbf{D}}$ regardless of ϕ_1 and ϕ_2 , hence the second term in (2.162) must be zero which is only possible if $\mu_3 = \mu_4 = 0$.

Thus for a saturated mixture of two incompressible Newtonian constituents and generalized Newtonian fluids we have the following constitutive equations

$$\begin{aligned} ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^1 &= 2\mu_1\bar{\mathbf{D}}^1 \\ ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^2 &= 2\mu_2\bar{\mathbf{D}}^2 \end{aligned} \quad (2.163)$$

where μ_1 and μ_2 are bulk viscosities of the two constituents. In general for ν constituent mixture we shall use the following constitutive equation

$$({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha = 2\mu_\alpha\bar{\mathbf{D}}^\alpha \quad (2.164)$$

and

$$\{\bar{q}\} = -[k_m]\{\bar{g}\} \quad (2.165)$$

$$\bar{q} = \sum_{\alpha=1}^{\nu} \bar{q}^\alpha \quad ; \quad \{\bar{q}^\alpha\} = -[k_\alpha]\{g\} \quad (2.166)$$

choice of k_α can be described using mixing rule with mutual parameter described by arithmetic, geometric, and harmonic means which satisfy (2.165) and (2.166).

In numerical studies presented in the following sections we use (2.164) - (2.166) as constitutive theories. We keep in mind that there are at least three different ways to determine bulk and mixture material coefficients.

2.7 Complete mathematical model for ν constituents in \mathbb{R}^2

If we consider a liquid mixture consisting of incompressible, homogenous, isotropic constituents with a saturated mixture which is also consider to be incompressible

Continuity equations

Here we have ν equations describing the continuity of the constituents of the mixture.

$$\bar{\rho}_\alpha (\nabla \cdot \bar{\mathbf{v}}^\alpha) = 0 \quad ; \quad \alpha = 1, \dots, \nu \quad (2.167)$$

Momentum equations

Here we have 2ν equations to describe the momentum of the mixture constituents.

$$\begin{aligned} \bar{\rho}_\alpha \left(\frac{\partial v_1^\alpha}{\partial t} + \bar{v}_1^\alpha \frac{\partial \bar{v}_1^\alpha}{\partial \bar{x}_1} + \bar{v}_2^\alpha \frac{\partial \bar{v}_2^\alpha}{\partial \bar{x}_2} \right) + \frac{\partial \bar{p}^\alpha}{\partial \bar{x}_1} - \frac{\partial ({}_d \bar{\sigma}^{(0)})_{11}^\alpha}{\partial \bar{x}_1} - \frac{\partial ({}_d \bar{\sigma}^{(0)})_{12}^\alpha}{\partial \bar{x}_2} - \sum_{i=1}^{\nu} (\beta_{\alpha i} \bar{v}_1^{\alpha i} + \gamma_{\alpha i} g_1) &= 0 \\ \bar{\rho}_\alpha \left(\frac{\partial \bar{v}_2^\alpha}{\partial t} + \bar{v}_1^\alpha \frac{\partial \bar{v}_1^\alpha}{\partial \bar{x}_1} + \bar{v}_2^\alpha \frac{\partial \bar{v}_2^\alpha}{\partial \bar{x}_2} \right) + \frac{\partial \bar{p}^\alpha}{\partial \bar{x}_2} - \frac{\partial ({}_d \bar{\sigma}^{(0)})_{21}^\alpha}{\partial \bar{x}_1} - \frac{\partial ({}_d \bar{\sigma}^{(0)})_{22}^\alpha}{\partial \bar{x}_2} - \sum_{i=1}^{\nu} (\beta_{\alpha i} \bar{v}_2^{\alpha i} + \gamma_{\alpha i} g_2) &= 0 \\ \alpha &= 1, \dots, \nu \end{aligned} \quad (2.168)$$

Energy equation

Here we have 1 equation to describe the energy of the mixture in terms of the properties of the mixture and the constituents.

$$\begin{aligned} \bar{\rho}_m \bar{c}_p \left(\frac{\partial \bar{\theta}}{\partial t} + \bar{v}_1 \frac{\partial \bar{\theta}}{\partial \bar{x}_1} + \bar{v}_2 \frac{\partial \bar{\theta}}{\partial \bar{x}_2} \right) + \sum_{\alpha=1}^{\nu} (\bar{v}_1^\alpha \bar{\pi}_1^\alpha + \bar{v}_2^\alpha \bar{\pi}_2^\alpha) + \frac{\partial \bar{q}_1}{\partial \bar{x}_1} + \frac{\partial \bar{q}_2}{\partial \bar{x}_2} \\ - \sum_{\alpha=1}^2 \left(({}_{\bar{\sigma}}^{(0)})_{11}^\alpha \frac{\partial \bar{v}_1^\alpha}{\partial \bar{x}_1} + ({}_{\bar{\sigma}}^{(0)})_{12}^\alpha \frac{\partial \bar{v}_1^\alpha}{\partial \bar{x}_2} + ({}_{\bar{\sigma}}^{(0)})_{21}^\alpha \frac{\partial \bar{v}_2^\alpha}{\partial \bar{x}_1} + ({}_{\bar{\sigma}}^{(0)})_{22}^\alpha \frac{\partial \bar{v}_2^\alpha}{\partial \bar{x}_2} \right) = 0 \end{aligned} \quad (2.169)$$

2.8 Dimensionless form of the mathematical models in \mathbb{R}^2 for a binary mixture

For convenience, we introduce more familiar notation. Let

$$\begin{aligned} u^\alpha &= \bar{v}_1^\alpha \quad , \quad v^\alpha = \bar{v}_2^\alpha \quad , \quad x = \bar{x}_1 \quad , \quad y = \bar{x}_2 \\ \boldsymbol{\pi}^\alpha &= \bar{\boldsymbol{\pi}}^\alpha \quad , \quad {}_d\boldsymbol{\sigma}^\alpha = ({}_d\bar{\boldsymbol{\sigma}}^{(0)})^\alpha \quad , \quad \rho_\alpha = \bar{\rho}_\alpha \quad , \quad p^\alpha = \bar{p}^\alpha \\ \rho_m &= \bar{\rho}_m \quad , \quad c_p = \bar{c}_p \quad , \quad \mathbf{q} = \bar{\mathbf{q}} \quad , \quad \theta = \bar{\theta} \end{aligned}$$

in which $({}_d\sigma^\alpha)_{ij}$; $i, j = 1, 2$ correspond to x_1 and x_2 (replaced by x and y). Velocities u and v are x and y components of \mathbf{v} . Likewise, \mathbf{v}^α has components u^α and v^α in the x and y directions.

Using this notation, the mathematical model in \mathbb{R}^2 for a two constituent, saturated, incompressible mixture of Newtonian or generalized Newtonian fluids can be written as

$$\begin{aligned} \rho_\alpha &= \phi_\alpha \rho^{(\alpha)} \\ \rho_m &= \sum_{\alpha=1}^2 \rho_\alpha \\ \sum_{\alpha=1}^2 \phi_\alpha &= 1 \\ \rho_m \mathbf{v} &= \sum_{\alpha=1}^2 \rho_\alpha \mathbf{v}^\alpha \end{aligned} \tag{2.170}$$

Continuity equations

$$\rho_\alpha \left(\frac{\partial u^\alpha}{\partial x} + \frac{\partial v^\alpha}{\partial y} \right) = 0 \quad ; \quad \alpha = 1, 2 \tag{2.171}$$

Momentum equations

$$\begin{aligned} \rho_\alpha \left(\frac{\partial u^\alpha}{\partial t} + u^\alpha \frac{\partial u^\alpha}{\partial x} + v^\alpha \frac{\partial u^\alpha}{\partial y} \right) + \frac{\partial p^\alpha}{\partial x} - \frac{\partial {}_d\sigma_{xx}^\alpha}{\partial x} - \frac{\partial {}_d\sigma_{xy}^\alpha}{\partial y} - \pi_x^\alpha &= 0 \quad ; \quad \alpha = 1, 2 \\ \rho_\alpha \left(\frac{\partial v^\alpha}{\partial t} + u^\alpha \frac{\partial v^\alpha}{\partial x} + v^\alpha \frac{\partial v^\alpha}{\partial y} \right) + \frac{\partial p^\alpha}{\partial y} - \frac{\partial {}_d\sigma_{xy}^\alpha}{\partial x} - \frac{\partial {}_d\sigma_{yy}^\alpha}{\partial y} - \pi_y^\alpha &= 0 \quad ; \quad \alpha = 1, 2 \end{aligned} \quad (2.172)$$

Energy equation

$$\begin{aligned} \sum_{\alpha=1}^2 \rho_\alpha c_p^\alpha \left(\frac{\partial \theta}{\partial t} + u^\alpha \frac{\partial \theta}{\partial x} + v^\alpha \frac{\partial \theta}{\partial y} \right) + \sum_{\alpha=1}^2 (u^\alpha \pi_x^\alpha + v^\alpha \pi_y^\alpha) + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \\ - \sum_{\alpha=1}^2 \left(\sigma_{xx}^\alpha \frac{\partial u^\alpha}{\partial x} + \sigma_{xy}^\alpha \frac{\partial u^\alpha}{\partial y} + \sigma_{yx}^\alpha \frac{\partial v^\alpha}{\partial x} + \sigma_{yy}^\alpha \frac{\partial v^\alpha}{\partial y} \right) = 0 \end{aligned} \quad (2.173)$$

where

$$\mathbf{q} = \sum_{\alpha=1}^2 \mathbf{q}^\alpha \quad (2.174)$$

Constitutive equations

$${}_d\boldsymbol{\sigma}^\alpha = 2\mu_\alpha \mathbf{D}^\alpha \quad ; \quad \alpha = 1, 2 \quad (2.175)$$

where bulk viscosities μ_α can be described by mixing rules with arithmetic, geometric, or harmonic means.

Arithmetic mean:

$$\mu_1 = \phi_1 \mu^{(1)} \quad ; \quad \mu_2 = \phi_2 \mu^{(2)} \quad (2.176)$$

Geometric mean:

$$\mu_1 = \phi_1^2 \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \quad ; \quad \mu_2 = \phi_2^2 \mu^{(2)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \quad (2.177)$$

Harmonic mean:

$$\mu_1 = \phi_1^2 \mu^{(1)} + 2\phi_1\phi_2 \frac{\mu^{(1)}\mu^{(2)}}{\mu^{(1)} + \mu^{(2)}} \quad ; \quad \mu_2 = \phi_2^2 \mu^{(2)} + 2\phi_1\phi_2 \frac{\mu^{(1)}\mu^{(2)}}{\mu^{(1)} + \mu^{(2)}} \quad (2.178)$$

where $\mu^{(1)}$ and $\mu^{(2)}$ are the ‘pure’ viscosities of the two constituents.

Power law model:

$$\begin{aligned} \mu^{(\alpha)} &= \mu_0^{(\alpha)} (I_2^\alpha)^{\frac{n_\alpha-1}{2}} \quad ; \quad \alpha = 1, 2 \\ I_2^\alpha &= 2 \left(\frac{\partial u^\alpha}{\partial x} \right)^2 + 2 \left(\frac{\partial v^\alpha}{\partial y} \right)^2 + \left(\frac{\partial u^\alpha}{\partial y} + \frac{\partial v^\alpha}{\partial x} \right)^2 \quad ; \quad \alpha = 1, 2 \end{aligned} \quad (2.179)$$

Carreau-Yasuda model:

$$\mu^{(\alpha)} = \mu_\infty^{(\alpha)} + \left(\mu_0^{(\alpha)} + \mu_\infty^{(\alpha)} \right) \left(1 + \lambda_\alpha^2 I_2^\alpha \right)^{\frac{m_\alpha-1}{2}} \quad ; \quad \alpha = 1, 2 \quad (2.180)$$

2.8.1 Dimensionless form

First we introduce ‘ $\hat{}$ ’, hat on all quantities in (2.170) – (2.180) indicating that the quantities have their usual dimensions or units and use the following reference quantities and dimensionless variables.

$$\begin{aligned} \hat{x} &= xL_0, \quad \hat{y} = yL_0, \quad \hat{u}^\alpha = u^\alpha u_0, \quad \hat{v}^\alpha = v^\alpha u_0 \\ \hat{\mu}^{(\alpha)} &= \mu^{(\alpha)} \mu_0, \quad \hat{p}^\alpha = p^\alpha p_0, \quad \hat{\sigma}^\alpha = \sigma^\alpha \tau_0, \quad \hat{\rho}_\alpha = \rho_\alpha \rho_0 \\ \hat{\theta} &= \theta \theta_0, \quad \hat{k} = k k_0, \quad \hat{c}_p = c_p c_{p_0}, \quad \hat{\rho}_m = \rho_m \rho_0 \end{aligned} \quad (2.181)$$

In which L_0 is the reference length, u_0 is the reference velocity, μ_0 is the reference viscosity, p_0 is the reference pressure, τ_0 is the reference stress, and ρ_0 is reference density. For consistency we must us $p_0 = \tau_0$. We can use either characteristic kinetic energy or characteristic viscous stress to choose reference value τ_0 .

The reference time t_0 is given by

$$t_0 = \frac{L_0}{u_0} \quad (2.182)$$

Using (2.170) – (2.180) with ' ^ ' (hat) on all quantities and using (2.181) and (2.182), we can obtain the following dimensionless form of the GDEs for the two constituent mathematical model in \mathbb{R}^2 .

Equations (2.170) and the continuity equations remain unchanged.

$$\rho_\alpha = \phi_\alpha \rho^{(\alpha)} \quad ; \quad \rho_m = \sum_{\alpha=1}^2 \rho_\alpha \quad ; \quad \sum_{\alpha=1}^2 \phi_\alpha = 1 \quad ; \quad \rho_m \mathbf{v} = \sum_{\alpha=1}^2 \rho_\alpha \mathbf{v}_\alpha \quad (2.183)$$

Continuity equations

$$\rho_\alpha \left(\frac{\partial u^\alpha}{\partial x} + \frac{\partial v^\alpha}{\partial y} \right) = 0 \quad ; \quad \alpha = 1, 2 \quad (2.184)$$

Momentum equations

$$\begin{aligned} & \rho_\alpha \left(\frac{\partial u^\alpha}{\partial t} + u^\alpha \frac{\partial u^\alpha}{\partial x} + v^\alpha \frac{\partial u^\alpha}{\partial y} \right) + \left(\frac{p_0}{\rho_0 u_0^2} \right) \frac{\partial p^\alpha}{\partial x} \\ & - \left(\frac{\tau_0}{\rho_0 u_0^2} \right) \left(\frac{\partial_d \sigma_{xx}^\alpha}{\partial x} + \frac{\partial_d \sigma_{xy}^\alpha}{\partial y} \right) - \left(\frac{1}{\rho_0 u_0} \right) \sum_{i=1}^2 \beta_{\alpha i} u^{\alpha i} - \left(\frac{\theta_0}{L_0 \rho_0 u_0^2} \right) \sum_{i=1}^2 \gamma_{\alpha i} g_x = 0 \\ & \rho_\alpha \left(\frac{\partial v^\alpha}{\partial t} + u^\alpha \frac{\partial v^\alpha}{\partial x} + v^\alpha \frac{\partial v^\alpha}{\partial y} \right) + \left(\frac{p_0}{\rho_0 u_0^2} \right) \frac{\partial p^\alpha}{\partial y} \\ & - \left(\frac{\tau_0}{\rho_0 u_0^2} \right) \left(\frac{\partial_d \sigma_{xy}^\alpha}{\partial x} + \frac{\partial_d \sigma_{yy}^\alpha}{\partial y} \right) - \left(\frac{1}{\rho_0 u_0} \right) \sum_{i=1}^2 \beta_{\alpha i} v^{\alpha i} - \left(\frac{\theta_0}{L_0 \rho_0 u_0^2} \right) \sum_{i=1}^2 \gamma_{\alpha i} g_y = 0 \end{aligned} \quad (2.185)$$

Energy equation

$$\begin{aligned}
& \frac{1}{Ec} \sum_{\alpha=1}^2 \rho_{\alpha} c_p^{\alpha} \left(\frac{\partial \theta}{\partial t} + u^{\alpha} \frac{\partial \theta}{\partial x} + v^{\alpha} \frac{\partial \theta}{\partial y} \right) + \frac{1}{Re Br} \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) \\
& + \frac{L_0}{\rho_0 u_0^2} \sum_{\alpha=1}^{\nu} \left\{ u^{\alpha} \left(u_0 \sum_{i=1}^2 \beta_{\alpha i} u^{\alpha i} + \frac{\theta_0}{L_0} \sum_{i=1}^2 \gamma_{\alpha i} g_x \right) + v^{\alpha} \left(u_0 \sum_{i=1}^2 \beta_{\alpha i} v^{\alpha i} + \frac{\theta_0}{L_0} \sum_{i=1}^2 \gamma_{\alpha i} g_y \right) \right\} \\
& - \frac{1}{Re} \sum_{\alpha=1}^2 \left(\sigma_{xx}^{\alpha} \frac{\partial u^{\alpha}}{\partial x} + \sigma_{xy}^{\alpha} \frac{\partial u^{\alpha}}{\partial y} + \sigma_{yx}^{\alpha} \frac{\partial v^{\alpha}}{\partial x} + \sigma_{yy}^{\alpha} \frac{\partial v^{\alpha}}{\partial y} \right) = 0
\end{aligned} \tag{2.186}$$

where

$$\begin{aligned}
Re &= \frac{u_0 \rho_0 L_0}{\mu_0} \quad ; \text{ Reynolds number} \\
Br &= \frac{\mu_0 u_0^2}{k_0 \theta_0} \quad ; \text{ Brinkman number} \\
Ec &= \frac{u_0^2}{c_{p0} \theta_0} \quad ; \text{ Eckerts number}
\end{aligned} \tag{2.187}$$

2.8.2 Power Law for constituents and mixture using geometric mean as the mutual parameter in the mixing rule

$$\hat{\mu}^{(\alpha)} = \hat{\mu}_0^{(\alpha)} \left(\hat{I}_2^{\alpha} \right)^{\frac{n_{\alpha}-1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.188}$$

where $\hat{\mu}^{(\alpha)}$ are the viscosities of the constituents. $\hat{\mu}_0^{(\alpha)}$, \hat{I}_2^{α} , and n_{α} are zero shear rate viscosity, second invariant of the strain rate tensor, and power law index for constituent α . Using (2.181), we can write (2.188) as

$$\hat{\mu}^{(\alpha)} = \mu_0 \mu_0^{(\alpha)} \left(\frac{u_0}{L_0} \right)^{n_{\alpha}-1} \left(I_2^{\alpha} \right)^{\frac{n_{\alpha}-1}{2}} = \left(\mu_0 \left(\frac{u_0}{L_0} \right)^{n_{\alpha}-1} \right) \mu_0^{(\alpha)} \left(I_2^{\alpha} \right)^{\frac{n_{\alpha}-1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.189}$$

$\mu_0^{(\alpha)}$ is dimensionless zero shear rate viscosity and I_2^{α} is the dimensionless second invariant of the strain rate tensor for constituent α .

or

$$\hat{\mu}^{(\alpha)} = \left(\mu_0 \left(\frac{u_0}{L_0} \right)^{n_\alpha - 1} \right) \mu^{(\alpha)} \quad ; \quad \mu^{(\alpha)} = \mu_0^{(\alpha)} (I_2^\alpha)^{\frac{n_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \quad (2.190)$$

in which $\mu^{(\alpha)}$ is the dimensionless viscosity of constituent α . Using (2.190), we can define $\hat{\mu}_1$ and $\hat{\mu}_2$ in (2.176 - 2.178).

$$\begin{aligned} \hat{\mu}_1 &= \phi_1^2 \hat{\mu}^{(1)} + \phi_1 \phi_2 \sqrt{\hat{\mu}^{(1)} \hat{\mu}^{(2)}} \\ \hat{\mu}_2 &= \phi_2^2 \hat{\mu}^{(2)} + \phi_1 \phi_2 \sqrt{\hat{\mu}^{(1)} \hat{\mu}^{(2)}} \end{aligned} \quad (2.191)$$

Consider $\hat{\mu}_1$. Substituting from (2.190).

$$\hat{\mu}_1 = \phi_1^2 \mu_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \mu_0 \left(\frac{u_0}{L_0} \right)^{n_2 - 1} \mu^{(1)} \mu^{(2)}} \quad (2.192)$$

Consider ${}_d \hat{\sigma}_{xx}^1$ in (2.175). Substituting from (2.192) and non-dimensionalizing gives

$$\tau_0 {}_d \sigma_{xx}^1 = 2 \left(\phi_1^2 \mu_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \mu_0 \left(\frac{u_0}{L_0} \right)^{n_2 - 1} \mu^{(1)} \mu^{(2)}} \right) \frac{u_0}{L_0} \frac{\partial u^1}{\partial x}$$

or

$$\begin{aligned} {}_d \sigma_{xx}^1 &= 2 \left(\phi_1^2 \left(\frac{u_0}{\tau_0 L_0} \right) \mu_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \right. \\ &\quad \left. + \phi_1 \phi_2 \sqrt{\left(\frac{u_0}{\tau_0 L_0} \mu_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \right) \left(\frac{u_0}{\tau_0 L_0} \mu_0 \left(\frac{u_0}{L_0} \right)^{n_2 - 1} \right) \mu^{(1)} \mu^{(2)}} \right) \frac{\partial u^1}{\partial x} \end{aligned} \quad (2.193)$$

If we use $\tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy), then

$$\frac{u_0}{\tau_0 L_0} \left(\mu_0 \left(\frac{u_0}{L_0} \right)^{n_1-1} \right) = \frac{\mu_0 u_0}{\rho_0 u_0^2 L_0} \left(\mu_0 \left(\frac{u_0}{L_0} \right)^{n_1-1} \right) = \frac{\mu_0}{\rho_0 (L_0)^{n_1} (u_0)^{2-n_1}} = \frac{1}{(R_{en})_1} \quad (2.194)$$

where $(R_{en})_1$ is the Reynolds number for constituent one. Similarly

$$\frac{u_0}{\tau_0 L_0} \left(\mu_0 \left(\frac{u_0}{L_0} \right)^{n_2-1} \right) = \frac{\mu_0}{\rho_0 (L_0)^{n_2} (u_0)^{2-n_2}} = \frac{1}{(R_{en})_2} \quad (2.195)$$

Hence, we can write the following for $d\sigma_{xx}^1$

$$d\sigma_{xx}^1 = 2 \left(\phi_1^2 \frac{\mu^{(1)}}{(R_{en})_1} + \phi_1 \phi_2 \sqrt{\frac{1}{(R_{en})_1 (R_{en})_2} \mu^{(1)} \mu^{(2)}} \right) \frac{\partial u^1}{\partial x} \quad (2.196)$$

or

$$d\sigma_{xx}^1 = 2\tilde{\mu}_1 \frac{\partial u^1}{\partial x} \quad (2.197)$$

where

$$\tilde{\mu}_1 = \phi_1^2 \frac{\mu^{(1)}}{(R_{en})_1} + \phi_1 \phi_2 \sqrt{\frac{1}{(R_{en})_1 (R_{en})_2} \mu^{(1)} \mu^{(2)}} \quad (2.198)$$

Similarly for $d\sigma_{xx}^2$, we have

$$d\sigma_{xx}^2 = 2\tilde{\mu}_2 \frac{\partial u^2}{\partial x} \quad (2.199)$$

where

$$\tilde{\mu}_2 = \phi_2^2 \frac{\mu^{(2)}}{(R_{en})_2} + \phi_1 \phi_2 \sqrt{\frac{1}{(R_{en})_1 (R_{en})_2} \mu^{(1)} \mu^{(2)}} \quad (2.200)$$

Similar derivation holds for the other components of the deviatoric Cauchy stress components.

In summary we have the following for the constitutive equations

$$d\boldsymbol{\sigma}^\alpha = 2\mu_\alpha \mathbf{D}^\alpha \quad ; \quad \alpha = 1, 2 \quad (2.201)$$

and

$$d\boldsymbol{\sigma}_m = 2\mu_m \mathbf{D} \quad (2.202)$$

Equations (2.183)–(2.185), (2.201), (2.198), (2.200) and (2.190) constitute the dimensionless form of the complete mathematical model in \mathbb{R}^2 for a mixture of two power law constituents.

2.8.3 Carreau model for constituents and mixture using geometric mean as the mutual parameter in the mixing rule

In the case of the Carreau model, the definitions of μ_1 and μ_2 change compared to power law. We consider details in the following.

Using (2.180)

$$\hat{\mu}^{(\alpha)} = \hat{\mu}_0^{(\alpha)} + \left(\hat{\mu}_0^{(\alpha)} - \hat{\mu}_\infty^{(\alpha)} \right) \left(1 + \lambda_\alpha^2 \hat{I}_2^\alpha \right)^{\frac{m_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \quad (2.203)$$

Using (2.181) we can write the following for (2.203)

$$\hat{\mu}^{(\alpha)} = \mu_0 \left(\mu_0^{(\alpha)} + \left(\mu_0^{(\alpha)} - \mu_\infty^{(\alpha)} \right) \left(1 + \lambda_\alpha^2 \left(\frac{u_0}{L_0} \right)^2 I_2^\alpha \right)^{\frac{m_\alpha - 1}{2}} \right) \quad ; \quad \alpha = 1, 2 \quad (2.204)$$

Let $\frac{\lambda_1 u_0}{L_0} = c_{u\alpha}$ be the Carreau number for constituent α .

$$\therefore \hat{\mu}^{(\alpha)} = \mu_0 \left(\mu_0^{(\alpha)} + \left(\mu_0^{(\alpha)} - \eta_\infty^{(\alpha)} \right) \left(1 + (c_{u\alpha})^2 I_2^\alpha \right)^{\frac{m_\alpha - 1}{2}} \right) = \mu_0 \mu^{(\alpha)} \quad ; \quad \alpha = 1, 2 \quad (2.205)$$

where

$$\mu^{(\alpha)} = \mu_0^{(\alpha)} + \left(\mu_0^{(\alpha)} - \mu_\infty^{(\alpha)} \right) \left(1 + (c_{u\alpha})^2 I_2^\alpha \right)^{\frac{m_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \quad (2.206)$$

Using (2.205) we can define $\hat{\mu}_1$ and $\hat{\mu}_2$.

$$\begin{aligned}\hat{\mu}_1 &= \phi_1^2 \hat{\mu}^{(1)} + \phi_1 \phi_2 \sqrt{\hat{\mu}^{(1)} \hat{\mu}^{(2)}} \\ \hat{\mu}_2 &= \phi_2^2 \hat{\mu}^{(2)} + \phi_1 \phi_2 \sqrt{\hat{\mu}^{(1)} \hat{\mu}^{(2)}}\end{aligned}\quad (2.207)$$

Consider $\hat{\mu}_1$. Substituting from (2.205) we obtain

$$\hat{\mu}_1 = \phi_1^2 \mu_0 \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu_0 \mu^{(1)} \mu_0 \mu^{(2)}} \quad (2.208)$$

Consider $(d\bar{\sigma}^1)_{xx}$ in (2.175). Substituting from (2.208) and nondimensionalizing gives

$$\tau_0 d\bar{\sigma}_{xx}^1 = 2 \left(\phi_1^2 \mu_0 \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu_0 \mu^{(1)} \mu_0 \mu^{(2)}} \right) \frac{u_0}{L_0} \frac{\partial u^1}{\partial x} \quad (2.209)$$

using $\tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy)

$$d\sigma_{xx}^1 = 2 \left(\phi_1^2 \left(\frac{\mu_0}{L_0 \rho_0 u_0} \right) \mu^{(1)} + \phi_1 \phi_2 \sqrt{\left(\frac{\mu_0}{L_0 \rho_0 u_0} \right) \mu^{(1)} \left(\frac{\mu_0}{L_0 \rho_0 u_0} \right) \mu^{(2)}} \right) \frac{\partial u^1}{\partial x} \quad (2.210)$$

or

$$d\sigma_{xx}^1 = 2 \left(\frac{1}{Re} \phi_1^2 \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \right) \frac{\partial u^1}{\partial x} = 2\tilde{\mu}_1 \frac{\partial u^1}{\partial x} \quad (2.211)$$

where $Re = \frac{L_0 \rho_0 u_0}{\mu_0}$; Reynolds number

Similarly for constituent two we have

$$d\sigma_{xx}^2 = 2 \left(\frac{1}{Re} \phi_2^2 \mu^{(2)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \right) \frac{\partial u^2}{\partial x} = 2\tilde{\mu}_2 \frac{\partial u^2}{\partial x} \quad (2.212)$$

In summary, we have the following for the constitutive equations

$$d\boldsymbol{\sigma}^\alpha = 2\tilde{\mu}_\alpha \mathbf{D}^\alpha \quad ; \quad \alpha = 1, 2 \quad (2.213)$$

Clearly, $\tilde{\mu}_1 = \frac{\mu_1}{Re}$ and $\tilde{\mu}_2 = \frac{\mu_2}{Re}$.

2.8.4 Newtonian constituents and mixture using arithmetic, geometric, and harmonic means to describe the mutual parameter in the mixing rule

For this case $\hat{\mu}^{(\alpha)}$; $\alpha = 1, 2$ are constant, hence we have

$$\hat{\mu}_1 = \mu_0 \mu_1 \quad (2.214)$$

$$\hat{\mu}_2 = \mu_0 \mu_2$$

where μ_1 and μ_2 can be shown using the mixing rule and mutual parameters in (2.176) - (2.178)

Consider ${}_d\sigma_{xx}^1$. Using (2.214) and nondimensionalizing ${}_d\sigma_{xx}^1$

$$\tau_0 {}_d\sigma_{xx}^1 = 2\mu_0 \mu_1 \frac{u_0}{L_0} \frac{\partial u^1}{\partial x} \quad (2.215)$$

or

$${}_d\sigma_{xx}^1 = 2\mu_1 \left(\frac{\mu_0 u_0}{\tau_0 L_0} \right) \frac{\partial u^1}{\partial x} \quad (2.216)$$

when $\tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy), we have

$${}_d\sigma_{xx}^1 = 2\mu_1 \left(\frac{\mu_0}{\rho_0 u_0 L_0} \right) \frac{\partial u^1}{\partial x} = 2 \frac{\mu_1}{Re} \frac{\partial u^1}{\partial x} = 2\tilde{\mu}_1 \frac{\partial u^1}{\partial x} \quad (2.217)$$

In summary, we have the following constitutive equations in the dimensionless form when the constituents and the mixture are Newtonian fluids

$${}_d\boldsymbol{\sigma}^\alpha = 2\tilde{\mu}_\alpha \mathbf{D}^\alpha \quad ; \quad \alpha = 1, 2 \quad (2.218)$$

and

$${}_d\boldsymbol{\sigma}_m = 2\tilde{\mu}_m \mathbf{D} \quad (2.219)$$

2.8.5 Remarks

1. If the constituents are Newtonian fluids and the mixture is also a Newtonian fluid and if we neglect π_x^1 , π_x^2 , π_y^1 , and π_y^2 , then the mathematical model for the constituents is decoupled. In this case we can use the continuity equation, momentum equations, and the constitutive equations for each constituent to obtain deformation fields and then use (2.183) to obtain the mixture deformation field. The combined model will also function properly in the least squares computational process. In the following we present details of the decoupled mathematical models in \mathbb{R}^2 for constituents one and two. We assume the partial pressures of the constituents p^1 and p^2 to be

$$\begin{aligned} p^\alpha &= \phi^\alpha p \\ \frac{\partial p^\alpha}{\partial x_i} &= \phi_\alpha \frac{\partial p}{\partial x_i} \end{aligned} \tag{2.220}$$

2. However, when the constituents are generalized Newtonian fluids and when the mixture is also a generalized Newtonian fluid, decoupling is not possible due to the fact that μ_1 and μ_2 are functions of deformation fields of both constituents.
3. Following these remarks, we derive the mathematical model for fully developed flow between parallel plates. This model reveals some features that are not obvious in the mathematical model in \mathbb{R}^2 .

2.9 Mathematical model for fully developed flow between parallel plates: mixture of two constituents

In this case the mathematical model describes a BVP. For fully developed flow between parallel plates we only need to consider the one dimensional case i.e. a typical section A–A (Figure 2.1) where the flow is fully developed. In this case

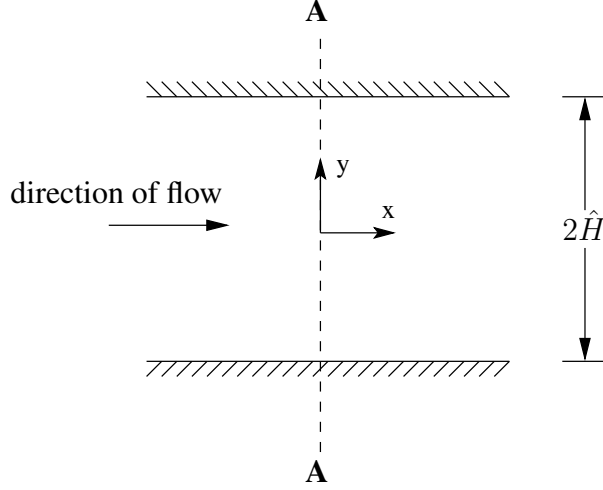


Figure 2.1: Flow between parallel plates

$$\begin{aligned}
 v^1 &= 0 \quad , \quad u^1 \neq 0 \\
 \frac{\partial u^1}{\partial x} &= 0 \quad , \quad \frac{\partial v^1}{\partial x} = 0 \quad , \quad \frac{\partial v^1}{\partial y} = 0 \quad , \quad \frac{\partial u^1}{\partial y} \neq 0 \quad , \quad d\sigma_{xy}^1 \neq 0 \quad , \\
 \frac{\partial p^1}{\partial x} &\neq 0 \quad , \quad \frac{\partial p^1}{\partial y} = 0 \quad , \quad d\sigma_{xx}^1 = 0 \quad , \quad d\sigma_{yy}^1 = 0
 \end{aligned}$$

similarly

$$\begin{aligned}
 v^2 &= 0 \quad , \quad u^2 \neq 0 \\
 \frac{\partial u^2}{\partial x} &= 0 \quad , \quad \frac{\partial v^2}{\partial x} = 0 \quad , \quad \frac{\partial v^2}{\partial y} = 0 \quad , \quad \frac{\partial u^2}{\partial y} \neq 0 \quad , \quad d\sigma_{xy}^2 \neq 0 \quad , \\
 \frac{\partial p^2}{\partial x} &\neq 0 \quad , \quad \frac{\partial p^2}{\partial y} = 0 \quad , \quad d\sigma_{xx}^2 = 0 \quad , \quad d\sigma_{yy}^2 = 0
 \end{aligned} \tag{2.221}$$

Hence, continuity equations are identically satisfied. Using (2.221), the dimensionless forms of the momentum equations and the constitutive equations reduce to

$$\begin{aligned}
 \phi_1 \left(\frac{p_0}{\rho_0 u_0^2} \right) \frac{\partial p}{\partial x} - \left(\frac{\tau_0}{\rho_0 u_0^2} \right) \frac{\partial d\sigma_{xy}^1}{\partial y} + \left(\frac{\beta_1}{\rho_0 u_0} \right) (u^1 - u^2) &= 0 \\
 d\sigma_{xy}^1 &= \underline{\mu}_1 (\mu^{(1)}, \mu^{(2)}, \phi_1, \phi_2) \frac{\partial u^1}{\partial y}
 \end{aligned} \tag{2.222}$$

$$\phi_2 \left(\frac{p_0}{\rho_0 u_0^2} \right) \frac{\partial p}{\partial x} - \left(\frac{\tau_0}{\rho_0 u_0^2} \right) \frac{\partial d\sigma_{xy}^2}{\partial y} + \left(\frac{\beta_1}{\rho_0 u_0} \right) (u^2 - u^1) = 0 \quad (2.223)$$

$$d\sigma_{xy}^2 = \underline{\mu}_2 (\mu^{(1)}, \mu^{(2)}, \phi_1, \phi_2) \frac{\partial u^2}{\partial y}$$

with energy equation

$$-\frac{\beta_1 L_0}{\rho_0 u_0} (u^1 - u^2)^2 - \sum_{\alpha=1}^2 \frac{k_\alpha}{Re Br} \frac{\partial^2 \theta}{\partial y^2} - \frac{1}{Re} \left(\sigma_{xy}^1 \frac{\partial u^1}{\partial y} + \sigma_{xy}^2 \frac{\partial u^2}{\partial y} \right) = 0 \quad (2.224)$$

Details of $\underline{\mu}_1$ and $\underline{\mu}_2$ are given in the following.

Newtonian

$$d\sigma_{xy}^1 = \underline{\mu}_1 \frac{\partial u^1}{\partial y}$$

$$d\sigma_{xy}^2 = \underline{\mu}_2 \frac{\partial u^2}{\partial y} \quad (2.225)$$

$$\underline{\mu}_1 = \frac{\mu_1}{Re}$$

$$\underline{\mu}_2 = \frac{\mu_2}{Re}$$

μ_1 and μ_2 are dimensionless viscosities of the constituents (constant) and can be described using mixing rule with choice of arithmetic, geometric or harmonic means for the mutual parameter in (2.176) - (2.178).

If we assume the mixture to be a Newtonian fluid, then

$$d\sigma_{xy} = \underline{\mu}_m \frac{\partial u_m}{\partial y} \quad (2.226)$$

In which $d\sigma_{xy} = d\sigma_{xy}^1 + d\sigma_{xy}^2$ and u_m is the mixture velocity in the x -direction. Using (2.226) we can determine $\underline{\mu}_m$ for the mixture. However, since $\frac{\partial u}{\partial y} = 0$ at the centerline it is better to use

$$\underline{\mu}_m = \left(\frac{\frac{\partial d\sigma_{xy}}{\partial y}}{\frac{\partial^2 u}{\partial y^2}} \right) \quad (2.227)$$

to determine $\underline{\mu}_m$.

Power law model for constituents and mixture using geometric mean for the mutual parameter in the mixing rule

$$\begin{aligned} d\sigma_{xy}^1 &= \underline{\mu}_1 \frac{\partial u^1}{\partial y} \\ d\sigma_{xy}^2 &= \underline{\mu}_2 \frac{\partial u^2}{\partial y} \end{aligned} \quad (2.228)$$

where

$$\begin{aligned} \underline{\mu}_1 &= \phi_1^2 \frac{\mu^{(1)}}{(R_{en})_1} + \phi_1 \phi_2 \sqrt{\frac{1}{(R_{en})_1 (R_{en})_2} \mu^{(1)} \mu^{(2)}} \\ \underline{\mu}_2 &= \phi_2^2 \frac{\mu^{(2)}}{(R_{en})_1} + \phi_1 \phi_2 \sqrt{\frac{1}{(R_{en})_1 (R_{en})_2} \mu^{(1)} \mu^{(2)}} \end{aligned} \quad (2.229)$$

and

$$\begin{aligned} \mu^{(1)} &= \mu_0^{(1)} (I_2^1)^{\frac{n_1-1}{2}} ; \quad I_2^1 = \left(\frac{\partial u^1}{\partial y} \right)^2 \\ \mu^{(2)} &= \mu_0^{(2)} (I_2^2)^{\frac{n_2-1}{2}} ; \quad I_2^2 = \left(\frac{\partial u^2}{\partial y} \right)^2 \end{aligned} \quad (2.230)$$

For the mixture we can write

$$d\sigma_{xy} = \underline{\mu}_m \frac{\partial u_m}{\partial y} \quad (2.231)$$

Using (2.231) we can determine $\underline{\mu}_m$ for the mixture.

**Carreau model for constituents and mixture using geometric mean as the mutual parameter
in the mixing rule**

$$\begin{aligned} d\sigma_{xy}^1 &= \underline{\mu}_1 \frac{\partial u^1}{\partial y} \\ d\sigma_{xy}^2 &= \underline{\mu}_1 \frac{\partial u^2}{\partial y} \end{aligned} \quad (2.232)$$

$$\begin{aligned} \underline{\mu}_1 &= \frac{1}{Re} \left(\phi_1^2 \mu^{(1)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \right) = \frac{\mu_1}{Re} \\ \underline{\mu}_2 &= \frac{1}{Re} \left(\phi_2^2 \mu^{(2)} + \phi_1 \phi_2 \sqrt{\mu^{(1)} \mu^{(2)}} \right) = \frac{\mu_1}{Re} \end{aligned} \quad (2.233)$$

and

$$\begin{aligned} \mu^{(1)} &= \mu_0^{(1)} + \left(\mu_0^{(1)} - \mu_\infty^{(1)} \right) \left(1 + (c_{u1})^2 I_2^1 \right)^{\frac{m_1-1}{2}} ; \quad I_2^1 = \left(\frac{\partial u^1}{\partial y} \right)^2 \\ \mu^{(2)} &= \mu_0^{(2)} + \left(\mu_0^{(2)} - \mu_\infty^{(2)} \right) \left(1 + (c_{u2})^2 I_2^2 \right)^{\frac{m_2-1}{2}} ; \quad I_2^2 = \left(\frac{\partial u^2}{\partial y} \right)^2 \end{aligned} \quad (2.234)$$

For the mixture we can write

$$d\sigma_{xy} = \underline{\mu}_m \frac{\partial u_m}{\partial y} \quad (2.235)$$

In this case also we can determine $\underline{\mu}_m$ for the mixture using (2.235).

Remarks

1. We note that the mathematical model consists of four PDEs, (2.222) and (2.223) in u^1 , u^2 , $d\sigma_{xy}^1$, $d\sigma_{xy}^2$, p^1 , and p^2 . Thus, the mathematical model does not have closure. However, for this case (fully developed flow), if we assume the flow to be pressure driven, then $\frac{\partial p^1}{\partial x}$ and $\frac{\partial p^2}{\partial x}$ are known. p^1 and p^2 are partial pressures of the constituents are given in (2.91).

Chapter 3

Numerical studies

3.1 Introduction

The mathematical models presented in the previous chapter are a system of non-linear partial differential equations describing boundary value problems. Based on references [34–37] the finite element processes derived using the residual functional (least squares process) yield variationally consistent integral forms when the second variation of the residuals are neglected in the second variation of the residual functional. Justifications for doing so are given in the references by the authors. Variationally consistent integral forms yield unconditionally stable computations. Hence, in the present work we use this approach for obtaining numerical solutions of the mixtures of Newtonian fluids. The local approximations are considered in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces in which k is the order of the space defining global differentiability of approximations and p is the degree of local approximations for all dependent variables. With this choice the least squares processes remain convergent [38].

We consider two model problems consisting of fully developed flow between parallel plates and an asymmetric backward facing step. In both model problems we only consider a saturated mixture of two fluids. Only Newtonian fluids are considered. In all numerical studies (both \mathbb{R}^1 and \mathbb{R}^2) $p_0 = \tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy) is used to choose reference pressure and

reference stress.

3.2 Fully developed flow between parallel plates

In this model problem we consider fully developed flow between parallel plates. Figure 2.1 shows a schematic. We only need to consider a typical section **A–A**. Furthermore, due to symmetry considerations we only need one half of the domain **A–A** (consider $0 < y < 1$ at **A–A**). We consider distances between the plates to $2\hat{H} = 2$ cm and if we choose $L_0 = 0.01$ m then the dimensionless distance H between the plates is 1 and our computational domain is $0 \leq y \leq 1$ at **A–A**. We consider saturated mixtures of two constituents. The properties of the constituents are given in the following.

Newtonian constituents [39–41]

Fluid 1 (or constituent 1)

$$\hat{\rho}^{(1)} = 900 \quad , \quad \hat{\mu}^{(1)} = 0.0267 \quad , \quad \hat{k}^{(1)} = 0.140 \quad , \quad \hat{c}_p^{(1)} = 1880$$

Fluid 2 (or constituent 2)

$$\hat{\rho}^{(2)} = 1000 \quad , \quad \hat{\mu}^{(2)} = 0.0018 \quad , \quad \hat{k}^{(2)} = 0.412 \quad , \quad \hat{c}_p^{(2)} = 3724$$

We consider a 10 uniform element discretization of the domain $0 \leq y \leq 1$ (at **A–A**) using 3-node p -version elements with local approximation in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces.

3.2.1 Newtonian constituents and Newtonian mixture

In this section we present a number of different numerical studies to demonstrate

1. Effects of bulk material coefficients and the choice of the mixing rule
2. Effect of the interaction coefficient β_1 on the velocity, and stress profiles of the constituents in isothermal flows

3. Interaction of the two constituents in non-isothermal flows

In these numerical studies we choose the following reference quantities

$$\begin{aligned} L_0 = 0.01 \quad u_0 = 0.19491 \quad \rho_0 = 1000 \quad \mu_0 = 0.0267 \\ \theta_0 = 0.03623 \quad k_0 = 0.140 \end{aligned} \quad (3.1)$$

and choose to set the pressure gradient

$$\frac{\partial \hat{p}}{\partial \hat{x}} = -1.0 \quad (3.2)$$

We note that the choice of the reference quantities (3.1) gives dimensionless parameters

$$Re = 73 \quad Br = 0.2 \quad (3.3)$$

Consideration of bulk material viscosity

In the previous sections it was shown that the bulk viscosity of the constituents can be determined using a mixing rule. Selection of the mutual parameter can be done by using the arithmetic, geometric, or harmonic mean. It is obvious that the selection of the mutual parameter will affect the bulk viscosity. In Figure 3.1 and Figure 3.2 the bulk viscosities $\hat{\mu}_1$ and $\hat{\mu}_2$ for constituent 1 and constituent 2 are shown as a function of ϕ_1 and ϕ_2 , keeping in mind that $\phi_2 = 1 - \phi_1$.

We note that when at $\phi_1 = 0$ or $\phi_2 = 0$, $\hat{\mu}_1 = 0$ or $\hat{\mu}_2 = 0$ respectively ie, when the volume fraction of a constituent is zero, its bulk viscosity is also zero. It should also be noted that when $\phi_1 = 1$ or $\phi_2 = 1$ then $\hat{\mu}_1 = \hat{\mu}^{(1)}$ or $\hat{\mu}_2 = \hat{\mu}^{(2)}$. If the mixture consists of only one constituent then the bulk viscosity of that constituent will be equal to the pure viscosity of the constituent. Arithmetic mean is a linear function of the volume fraction. In this case the bulk viscosity will linearly increase with increasing volume fraction of the constituent. However, this is not true in the cases for geometric and harmonic means. As these involve the computation of a mutual parameter, they exhibit nonlinear behavior. We note that constituent 1 has higher viscosity than

constituent 2. Figure 3.1 shows that the geometric and harmonic mean for constituent 1 will be lower than the arithmetic mean. However, in Figure 3.2 this behavior is opposite of Figure 3.1 as the contribution of the mutual parameter is higher. We note that in Figure 3.2 for geometric mean the bulk viscosity can be larger than the viscosity of the constituents. We note that arithmetic, geometric, and harmonic means for the mutual parameter are possible alternatives. In a given application one may be more suitable than the others.

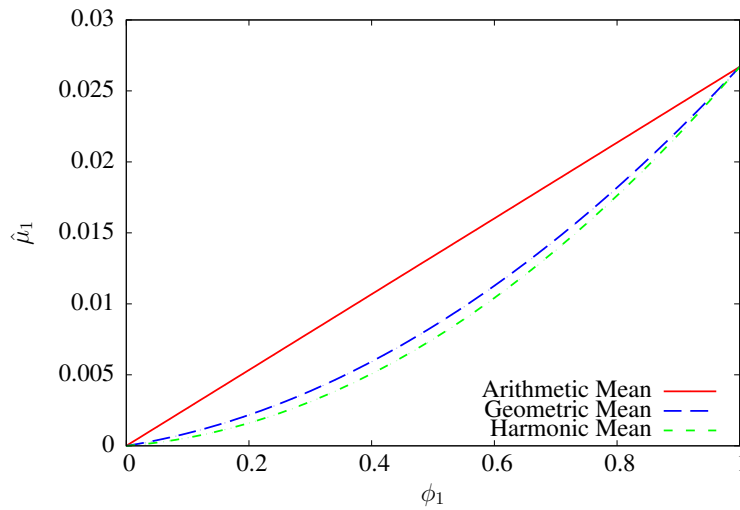


Figure 3.1: Bulk viscosity $\hat{\mu}_1$ as a function of volume fraction

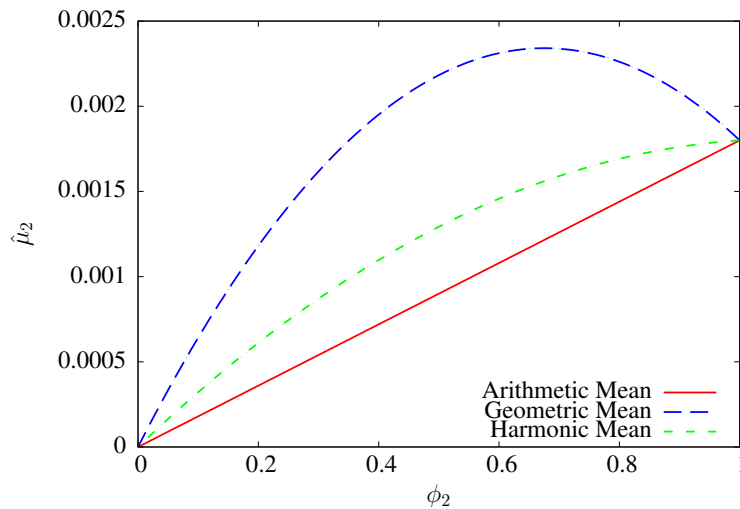


Figure 3.2: Bulk viscosity $\hat{\mu}_2$ as a function of volume fraction

Case (a) isothermal flow of a mixture of two constituents with interaction forces

In this study we consider a saturated mixture of two constituents (same as fluids 1 and 2) for different volume fractions. Here we choose $\phi_1 = 0.1, 0.5, 0.9$ and $\beta_1 = 0, 0.10$ and 1.00 for the numerical studies.

Graphs of axial velocities vs distance y are shown in Figure 3.3 - Figure 3.5. Velocity for constituent 1 is always less than the velocity for constituent 2. This is a result of the difference in viscosities of the constituents. As expected the velocity of the mixture is always be between the values of velocity of the constituents. These figures clearly show that changes in mixture composition through change in volume fractions ϕ_1 and ϕ_2 result in changes in the velocity field for the mixture. For $\beta_1 = 0$ in Figure 3.3 arithmetic mean results are independent of volume fraction. For fully developed flow we note that without interaction forces the velocity profile is

$$u^1 = -\frac{1}{2\mu_1} \frac{\partial p^1}{\partial x} (H^2 - y^2) \quad (3.4)$$

For arithmetic mean $\mu_1 = \phi_1 \mu^{(1)}$ and $\frac{\partial p^1}{\partial x} = \phi_1 \frac{\partial p}{\partial x}$ thus (3.4) for arithmetic mean reduces to

$$u^1 = -\frac{1}{2\mu^{(1)}} \frac{\partial p}{\partial x} (H^2 - y^2) \quad (3.5)$$

Thus we note that in the case in the absence of interaction forces, for arithmetic mean the velocity profile is not a function of volume fractions. However, harmonic and geometric means velocity profiles are functions of volume fractions.

From Figure 3.3 - Figure 3.5 as β_1 increases the velocity profiles begin to move towards one another. Here we observe that the interaction coefficient β_1 is creating a lower mixture velocity profile. In the limit that β approaches ∞ then the velocity profiles of constituent one and constituent two are equal to that of the mixture ($u^1 = u^2 = u$).

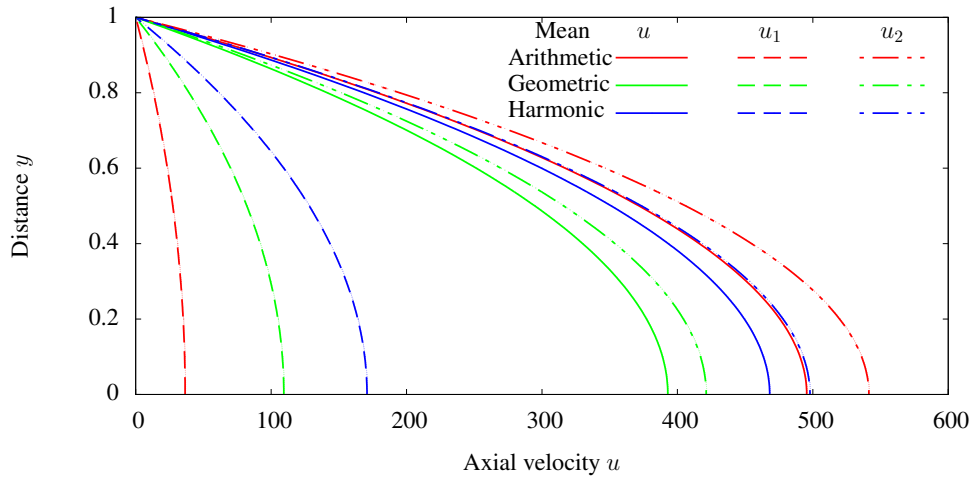
Results for the deviatoric cauchy stress ${}_d\sigma_{xy}^\alpha$ are shown in Figure 3.6 - Figure 3.8. Here, ${}_d\sigma_{xy}$ is the sum of the constituent stresses. For $\beta_1 = 0$, without interaction forces, results are those typical of a Newtonian fluid in which the stress is linear with respect to the distance y . The sum of

the constituent stress does not change with volume fraction as it can be shown that the sum of the constituent stresses is not a function of volume fraction or viscosity. For non zero β_1 , the results are shown in Figures 3.7 and 3.8. The constituent stress begins to deviate from linear behavior. As β_1 increases the velocity profiles deviate from parabolic behavior. However, the sum of the constituent stresses is still independent of both volume fraction and viscosity. Figure 3.8 further shows this deviation from linear behavior of the stress.

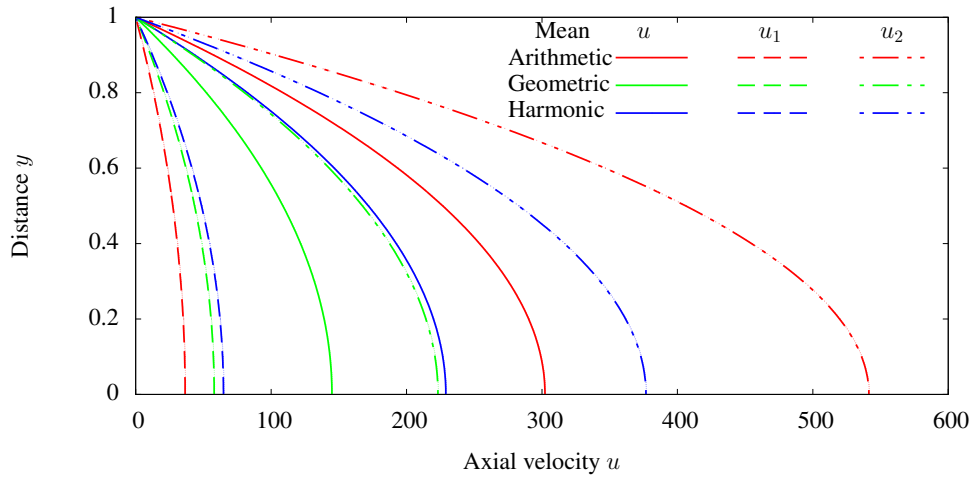
Case (b) non-isothermal flow of two constituent mixture with interaction forces

In the case of non-isothermal mixture of constituent 1 and constituent 2 we need not show results for velocity and stress. Velocity and temperature fields are uncoupled thus the velocity results for non-isothermal case also hold for isothermal flow. Thus results for the velocity and stress profiles for the non-isothermal case are also shown in Figure 3.3 - Figure 3.5 and Figure 3.6 - Figure 3.8.

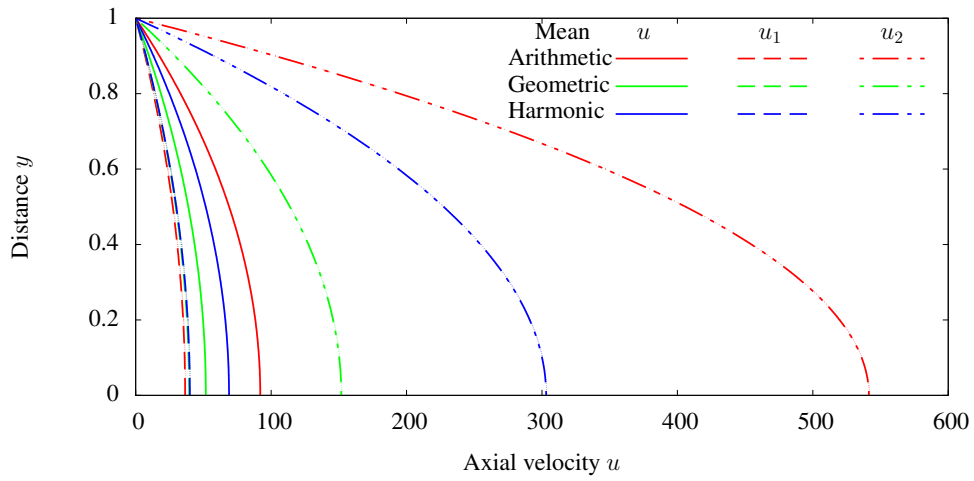
Thus in Figure 3.9 - Figure 3.11 we present results for the temperature profile for the non-isothermal case. In this model problem the temperature field is purely due to dissipation. Here the temperature is held constant at the wall at $\theta = 0$ and no-slip boundary conditions are also imposed at the wall.



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

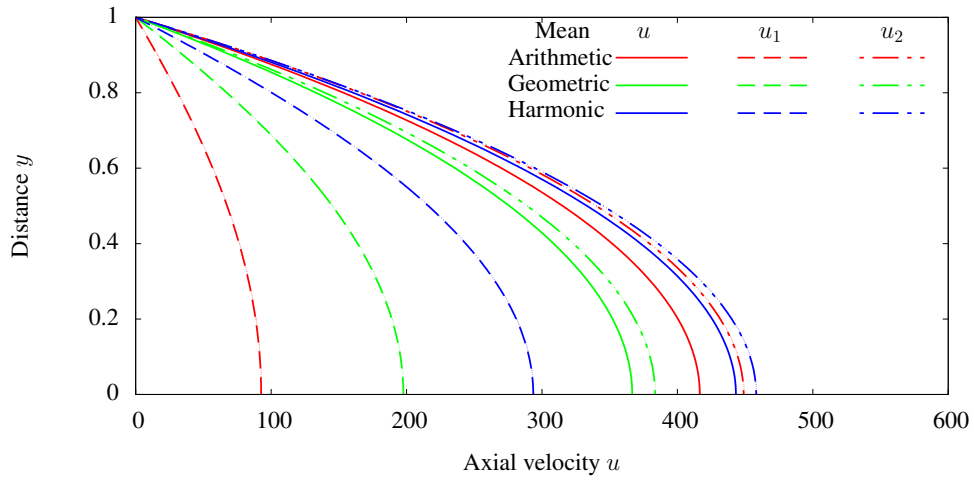


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

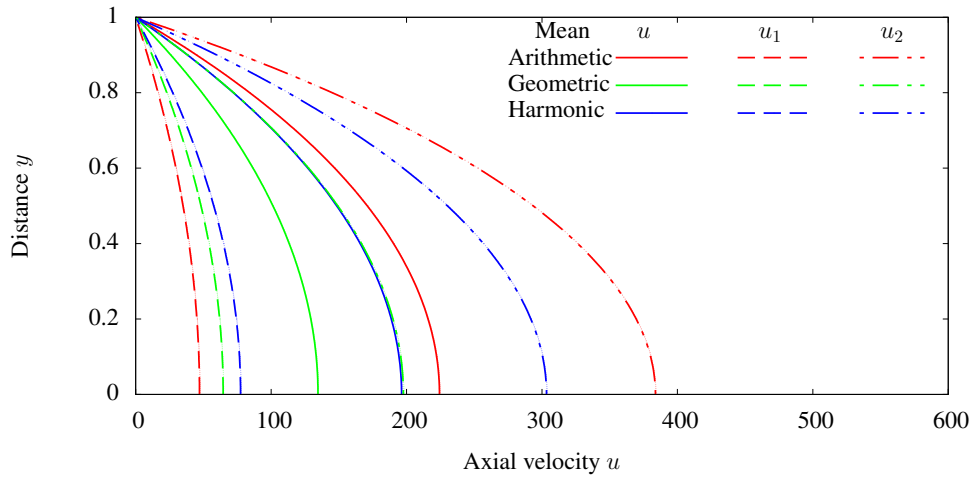


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

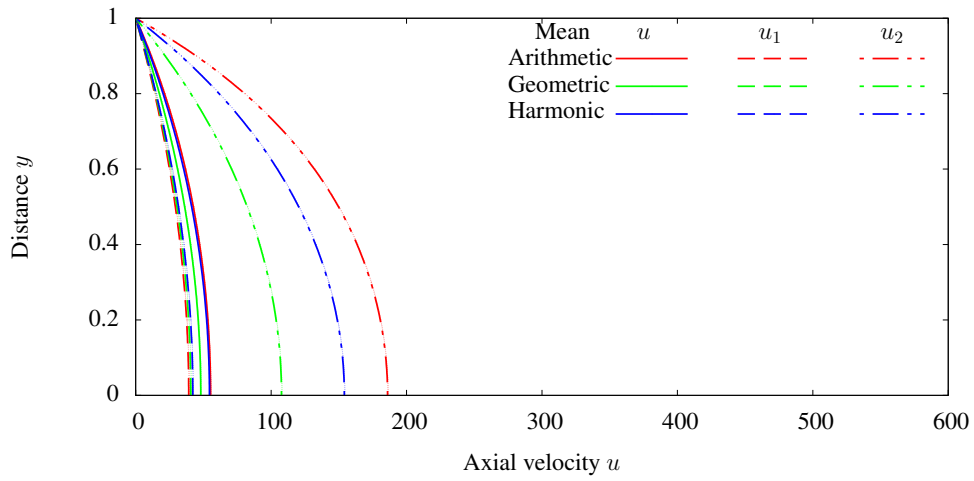
Figure 3.3: Axial velocity vs distance y for different volume fractions and mixing rules ; isothermal, $\beta_1 = 0.00$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

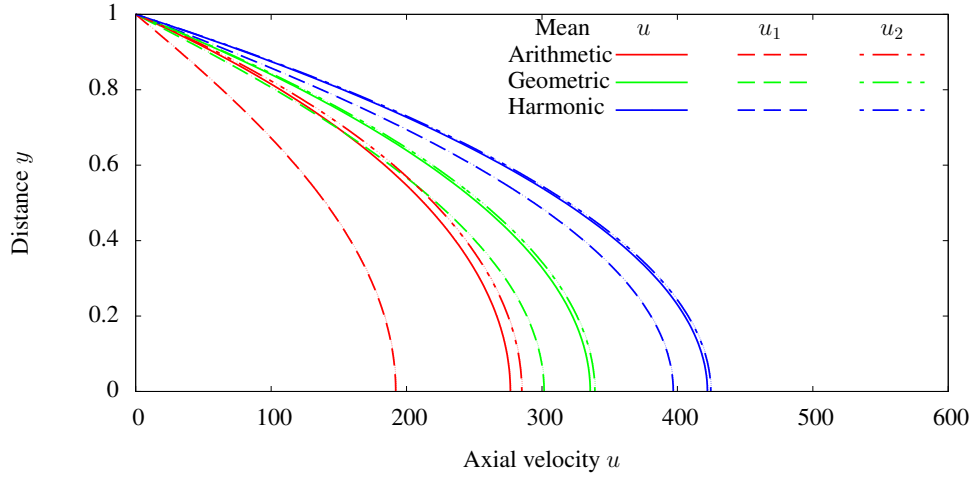


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

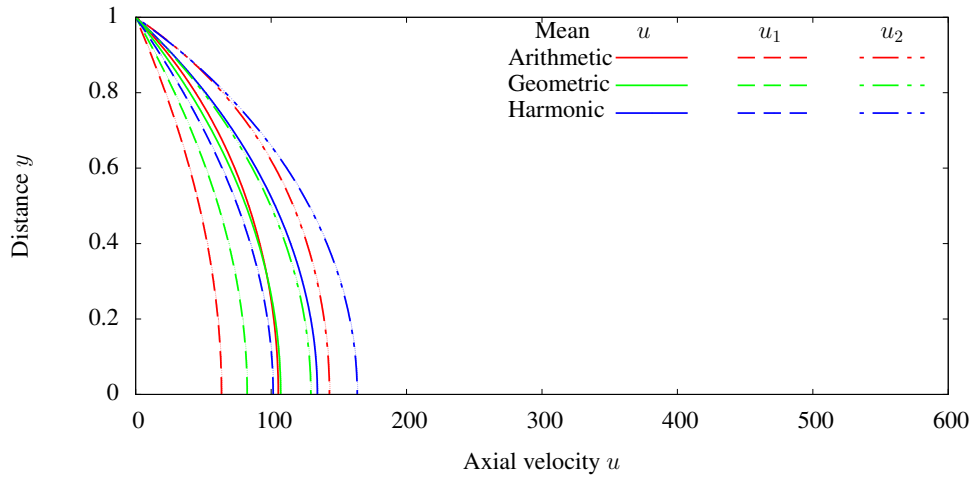


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

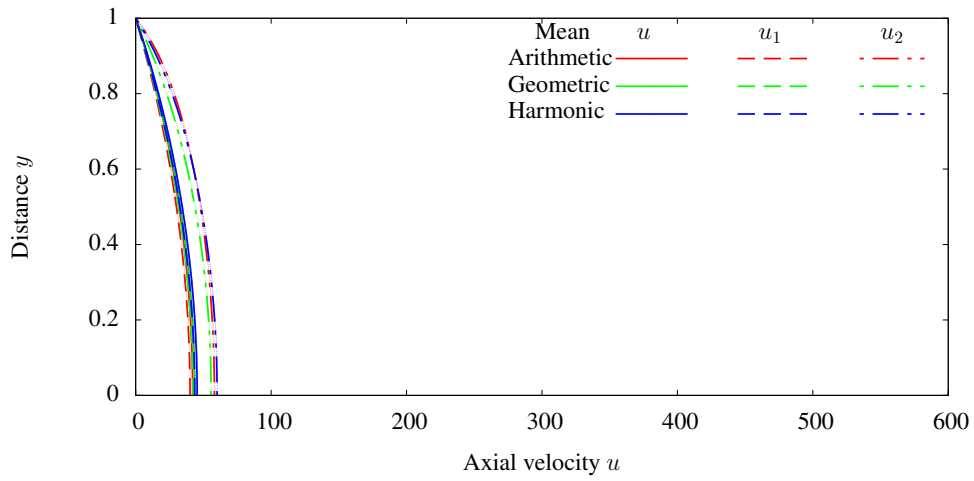
Figure 3.4: Axial velocity vs distance y for different volume fractions and mixing rules ; isothermal, $\beta_1 = 0.10$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

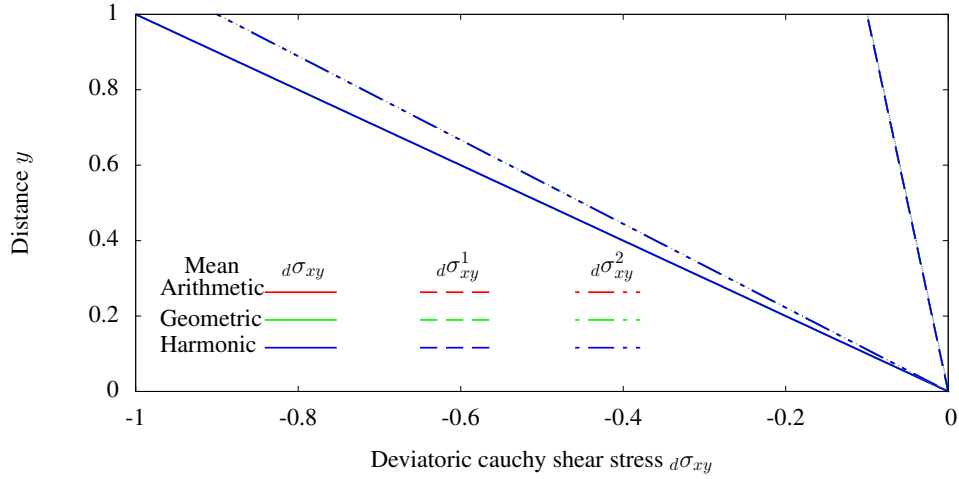


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

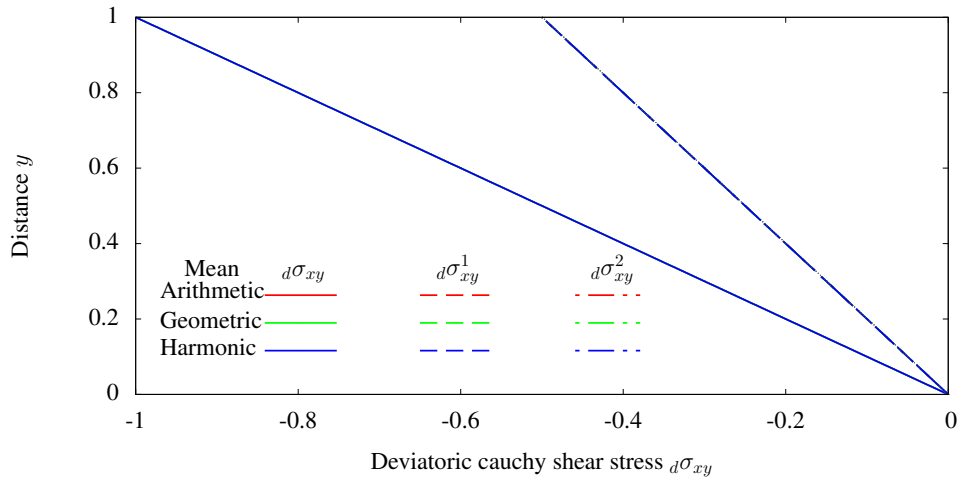


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

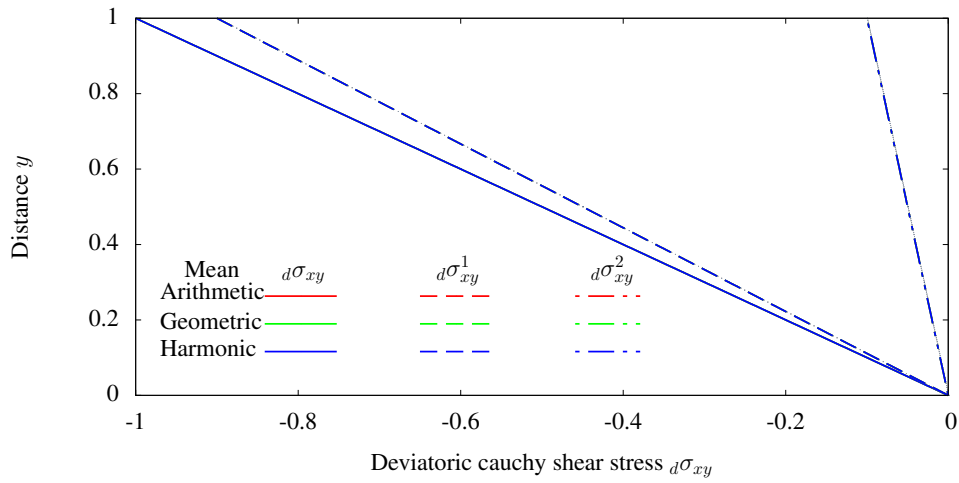
Figure 3.5: Axial velocity vs distance y for different volume fractions and mixing rules ; isothermal, $\beta_1 = 1.00$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

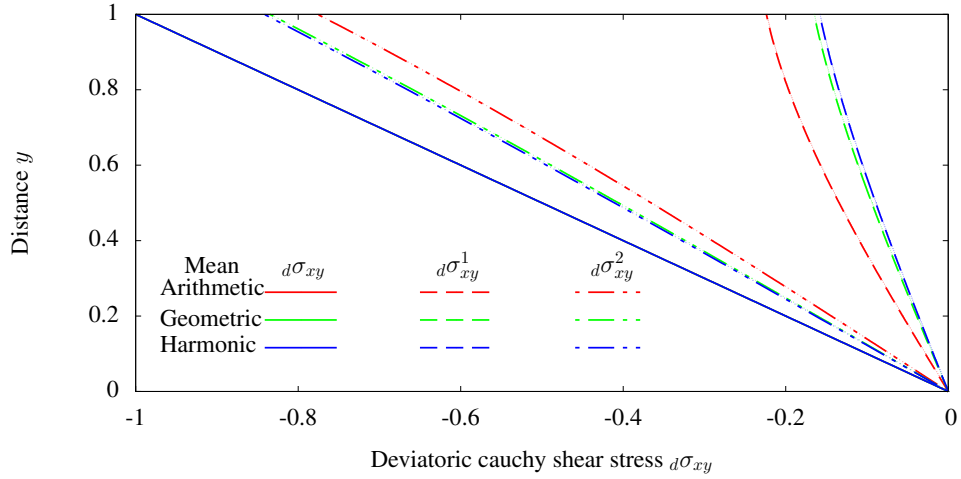


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

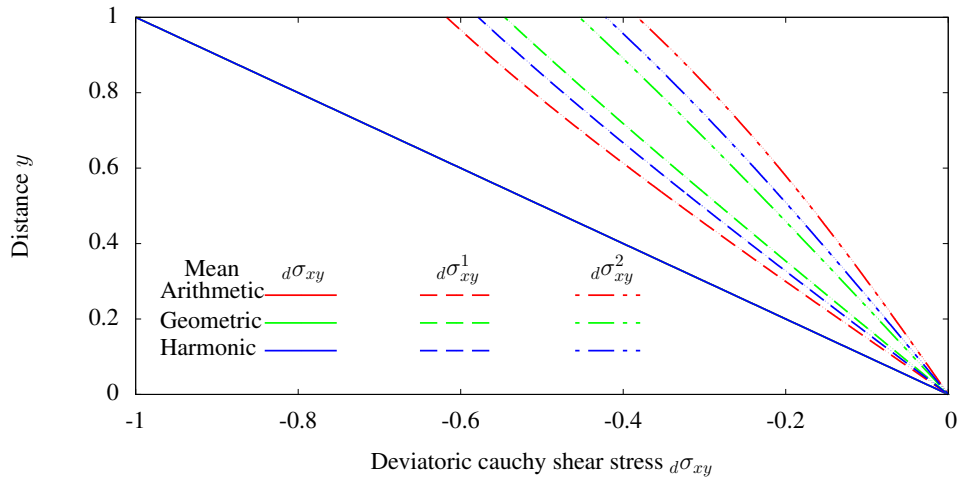


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

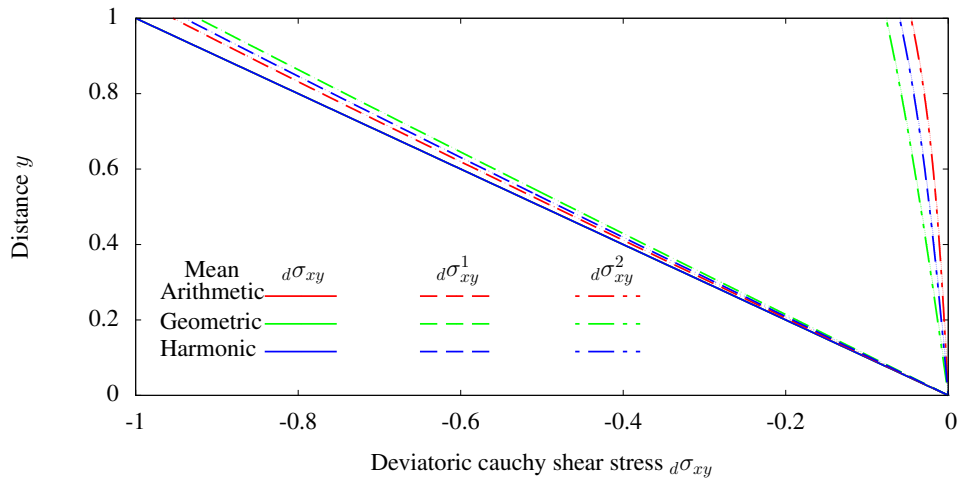
Figure 3.6: Cauchy shear stress vs distance y for different volume fractions and mixing rules ; isothermal, $\beta_1 = 0.00$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

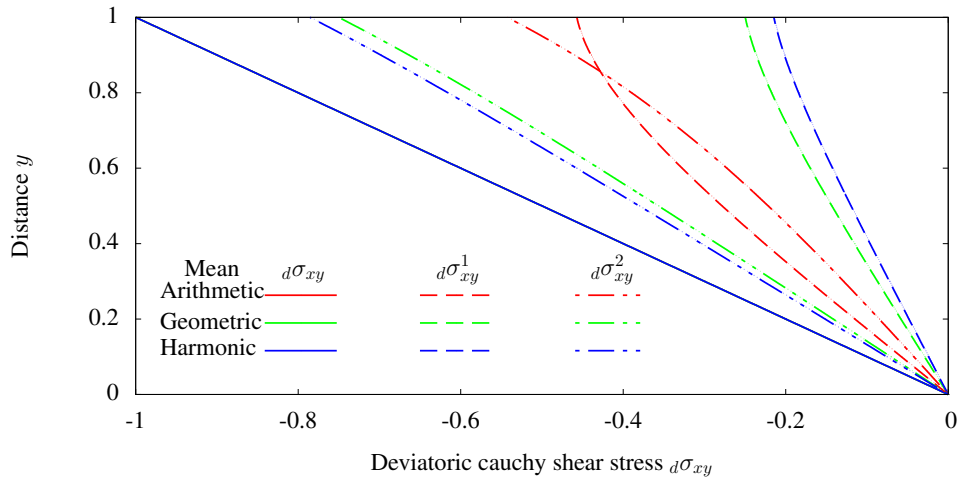


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

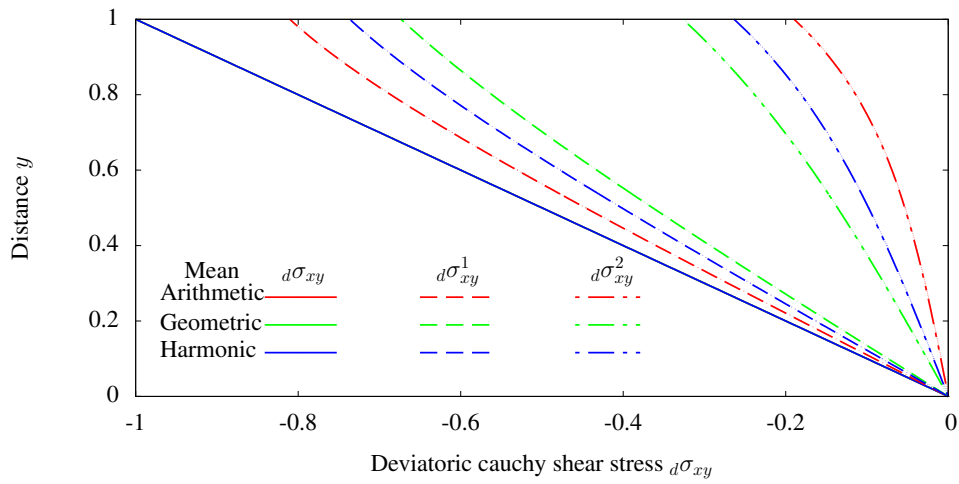


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

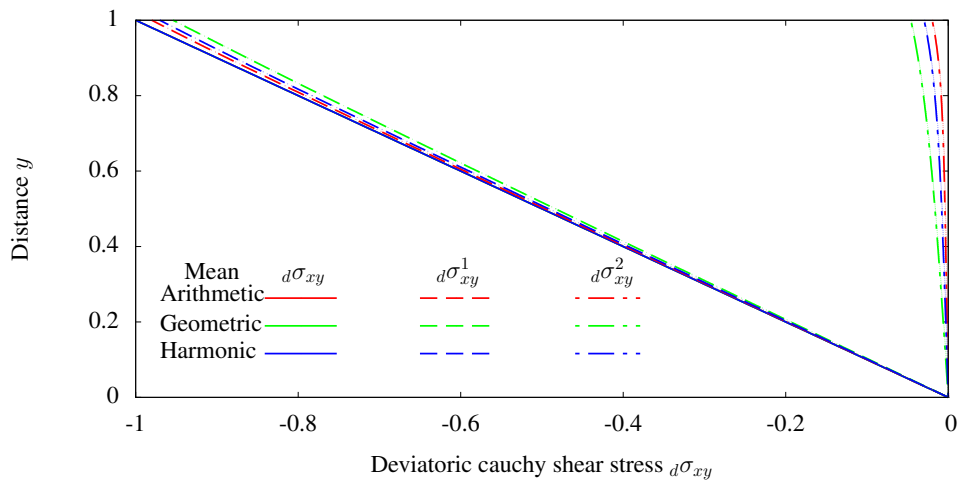
Figure 3.7: Cauchy shear stress vs distance y for different volume fractions and mixing rules ; isothermal, $\beta_1 = 0.10$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

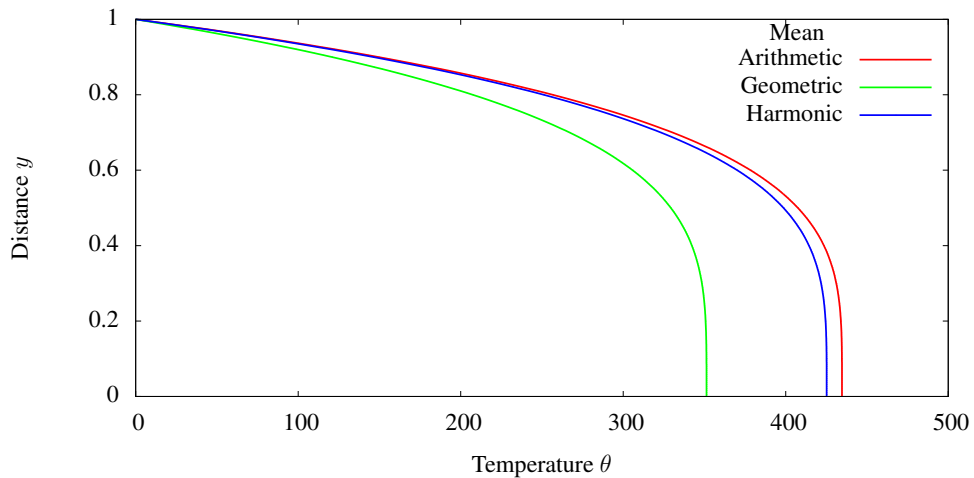


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

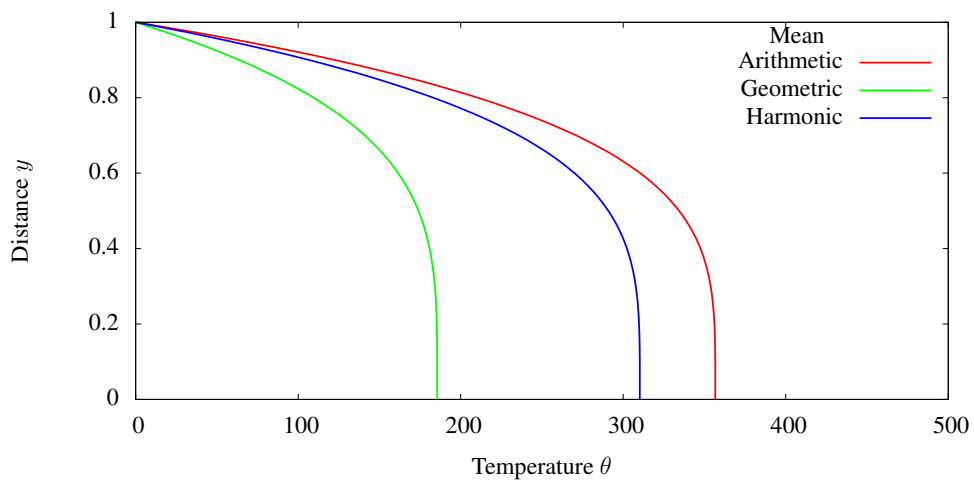


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

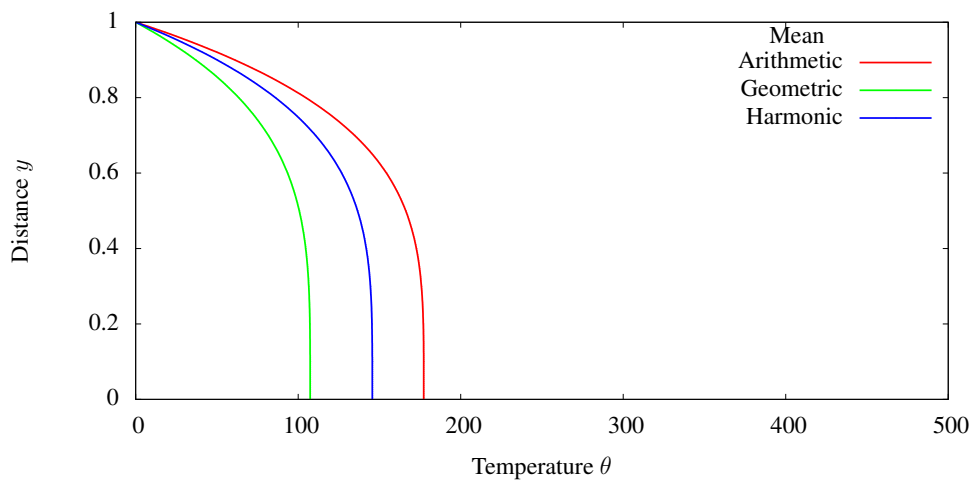
Figure 3.8: Cauchy shear stress vs distance y for different volume fractions and mixing rules ; isothermal, $\beta_1 = 1.00$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

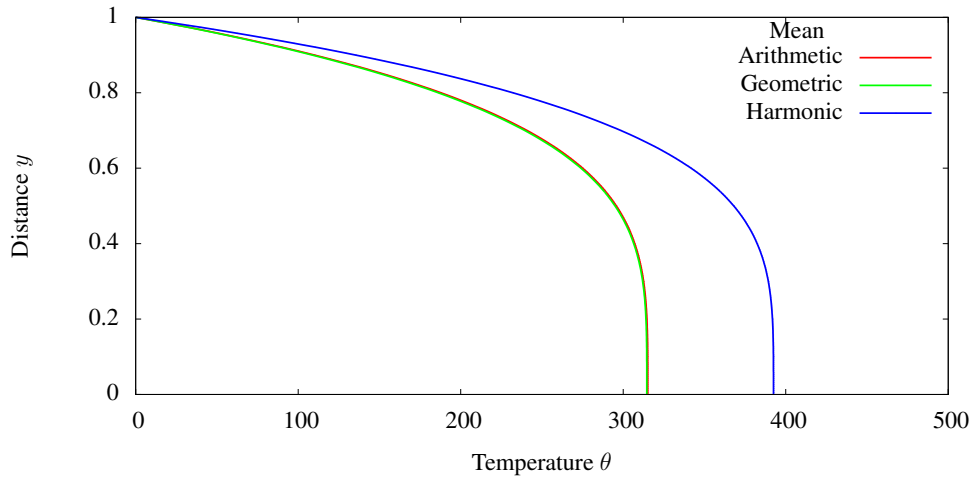


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

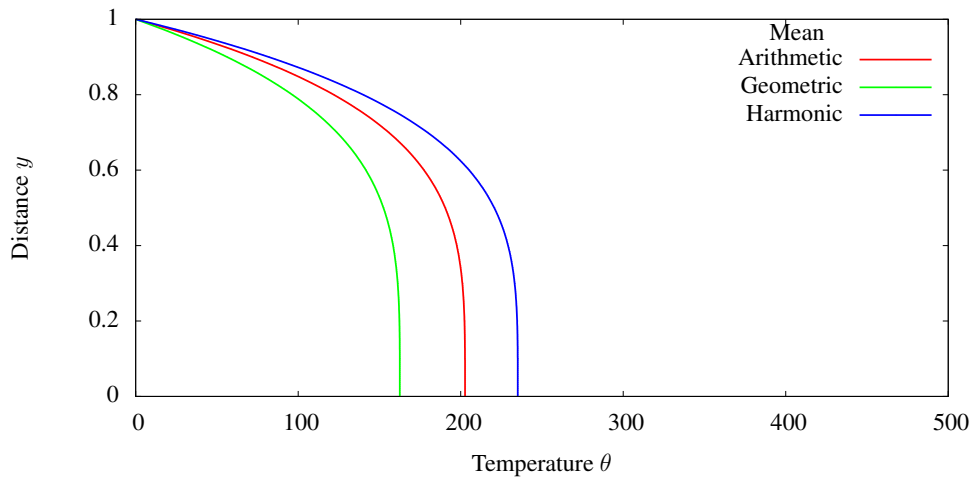


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

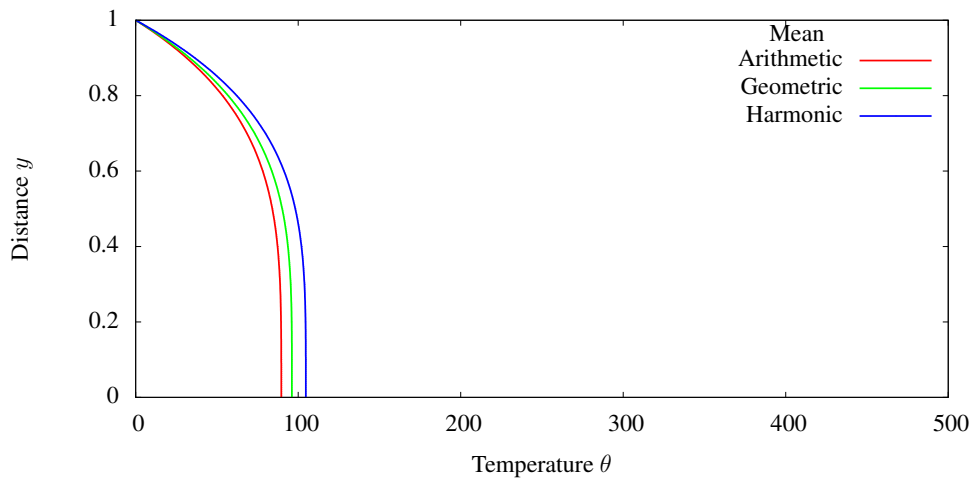
Figure 3.9: Temperature vs distance y for different volume fractions and mixing rules ; nonisothermal, $\beta_1 = 0.00$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$

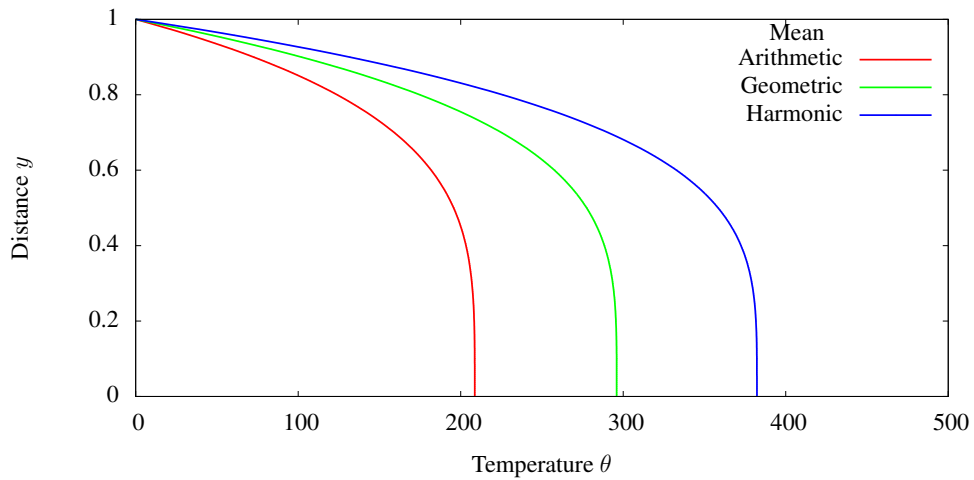


(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$

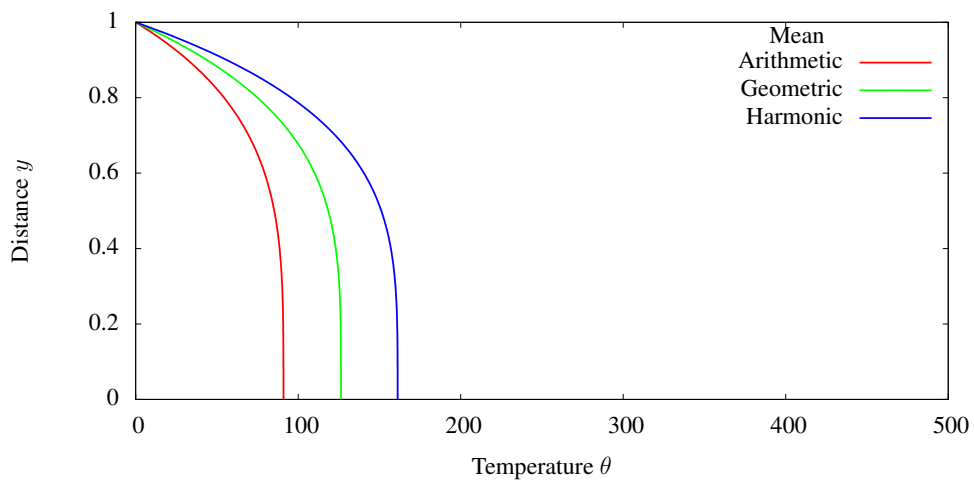


(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

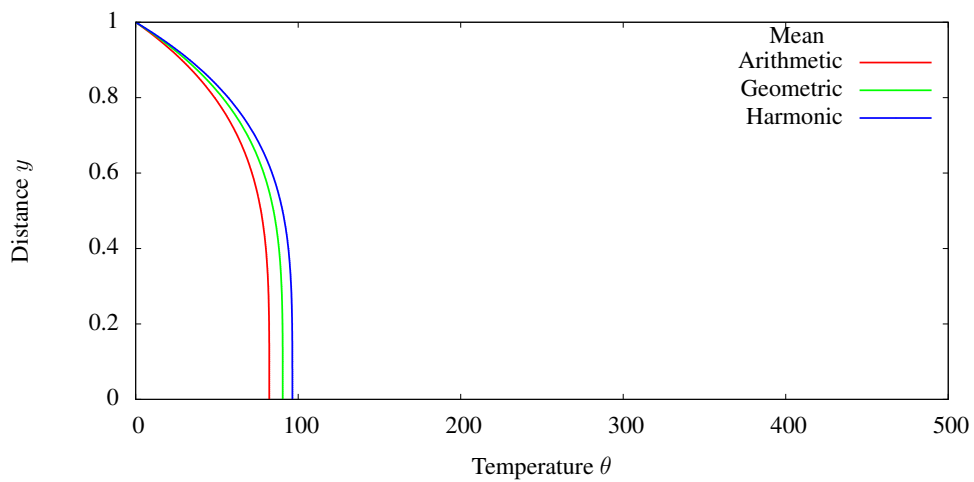
Figure 3.10: Temperature vs distance y for different volume fractions and mixing rules ; non-isothermal, $\beta_1 = 0.10$



(a) Volume fractions $\phi_1 = 0.1, \phi_2 = 0.9$



(b) Volume fractions $\phi_1 = 0.5, \phi_2 = 0.5$



(c) Volume fractions $\phi_1 = 0.9, \phi_2 = 0.1$

Figure 3.11: Temperature vs distance y for different volume fractions and mixing rules ; non-isothermal, $\beta_1 = 1.00$

3.3 Isothermal 1:2 backward facing asymmetric expansion

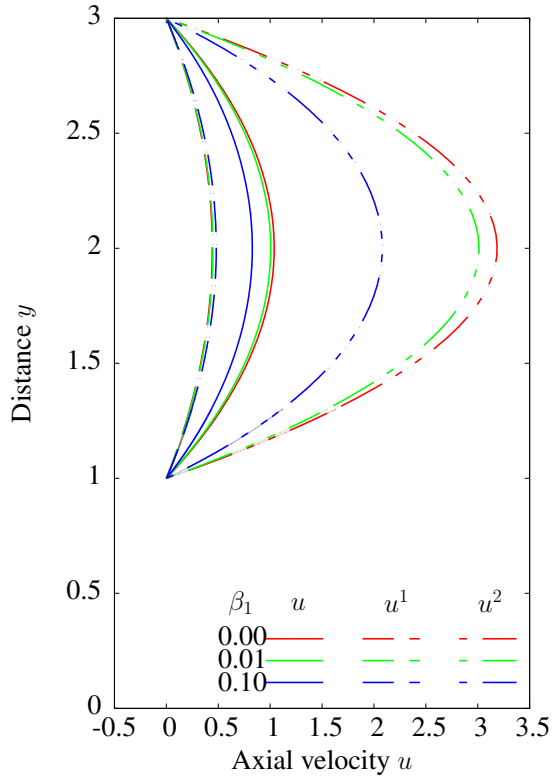
We consider a 1:2 backward facing asymmetric expansion. This problem has been experimentally investigated by Patrick and Denham [42]. More recently Winterscheidt and Surana [37] presented numerical simulations using p -version least squares finite element method. Here we consider imposition of no-slip boundary conditions along all walls. At the inlet we impose fully developed flow conditions corresponding to $\frac{\partial \hat{p}}{\partial x} = 0.01$. Care must be taken when imposing these boundary conditions as we note for fully developed flow $u^1 \neq u^2$ except in the cases when fluid one is same as fluid two and as β_1 approaches infinity. And at the outlet boundary we impose no boundary conditions to simulate a free outlet. Here we use the same dimensionless quantities as in the 1-d studies as shown in (3.1). A schematic of the boundary value problem with boundary conditions is shown in Figure 3.16.

In the numerical studies we only consider the constituents and the mixture to be Newtonian and use the same properties as listed for the Newtonian constituents for fully developed flow between parallel plates (section 3.2). For this study we select harmonic mean to be used in the mixing rule and volume fractions $\phi_1 = 0.8$, $\phi_2 = 0.2$. C^{11} local approximations at p -level 7 are used for all variables. For this choice, I values are $O(10^{-7})$ or lower confirming good accuracy of the solution. Figure 3.17 shows a graded three hundred thirty-two element discretization using nine node p -version elements. Characteristic kinetic energy is used for reference pressure and reference stress.

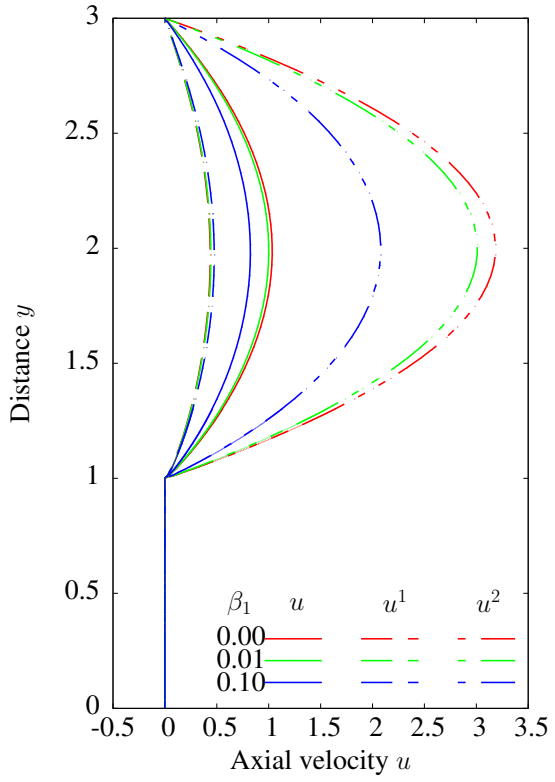
Results are presented in the following figures

1. Figure 3.12 axial velocity at various x locations
2. Figure 3.13 vertical velocity v at various x locations
3. Figure 3.14 first derivative of velocity at various x locations
4. Figure 3.15 deviatoric cauchy stress at various x locations

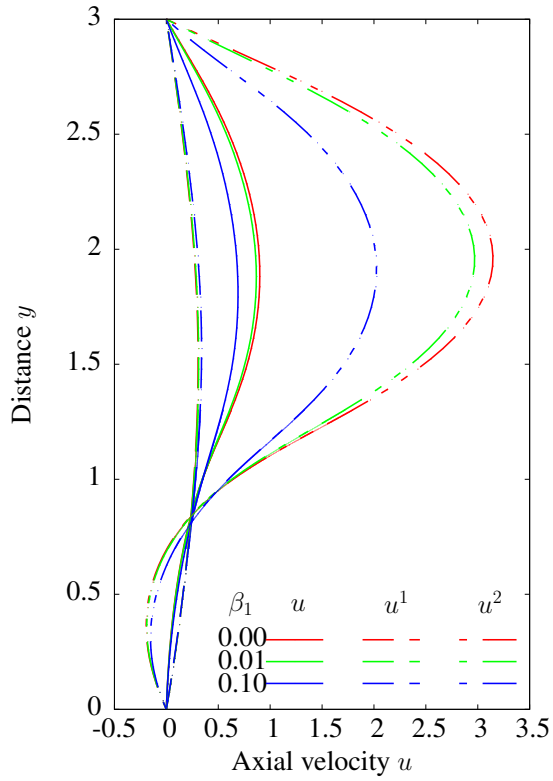
at the following x -locations: (a.) Inlet at $x = -1$; (b.) Expansion point $x = 0$; (c.) Distance past expansion $x = 6$; (d.) Outlet at $x = 28$.



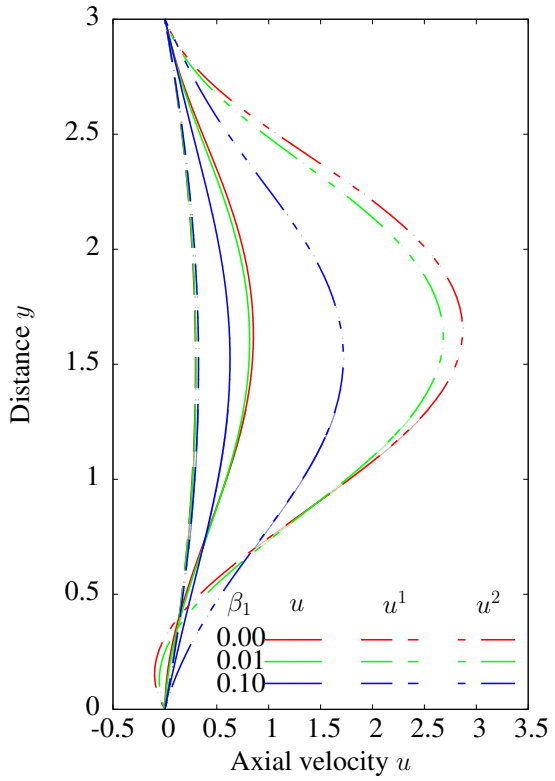
(a) Velocity u at inlet



(b) Velocity u at $x = 0$

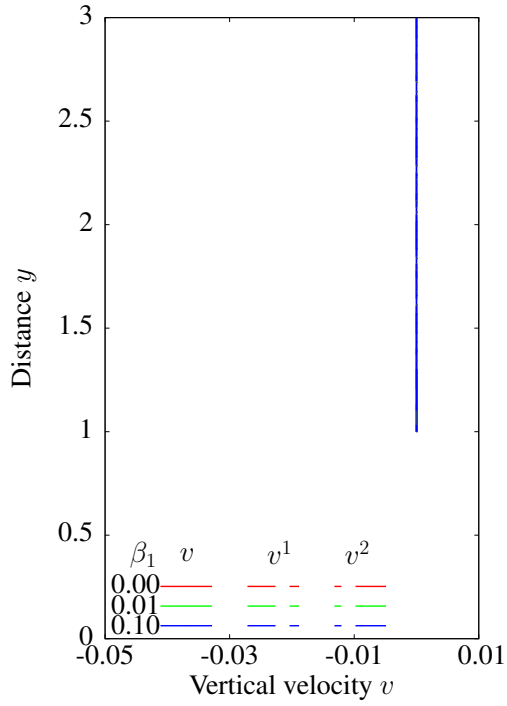


(c) Velocity u at $x = 6$

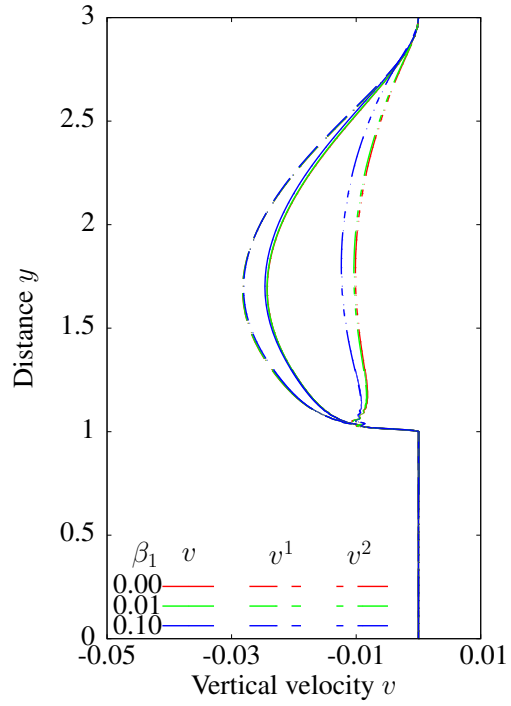


(d) Velocity u at $x = 28$

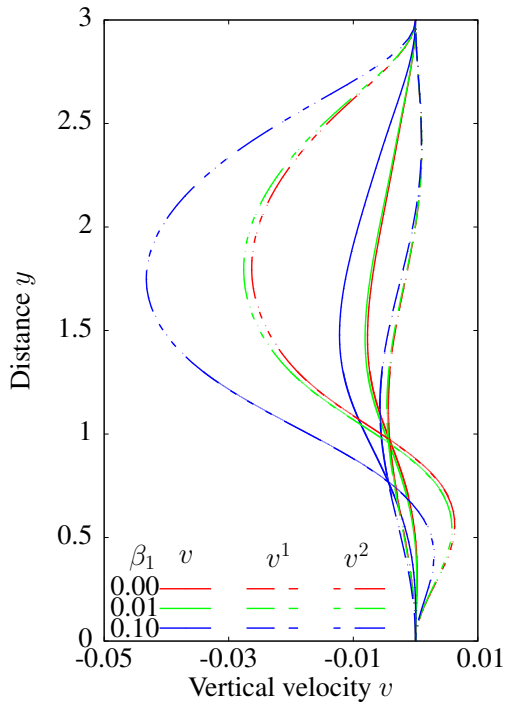
Figure 3.12: Axial velocity at various x locations



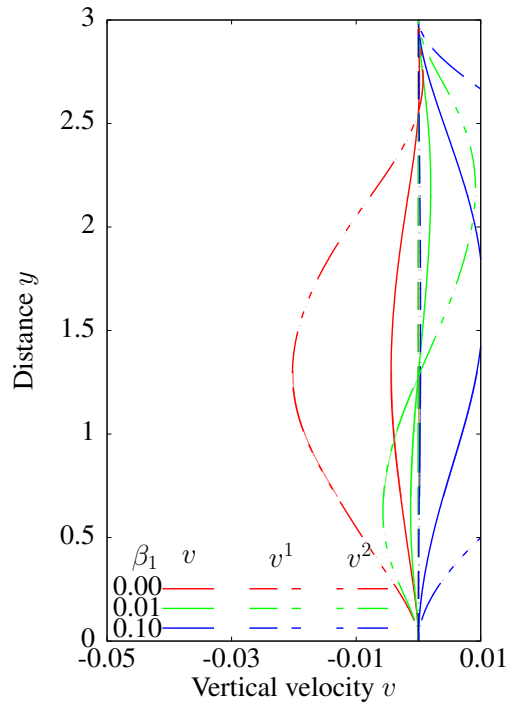
(a) Velocity v at inlet



(b) Velocity v at $x = 0$



(c) Velocity v at $x = 6$



(d) Velocity v at $x = 28$

Figure 3.13: Vertical velocity v at various x locations

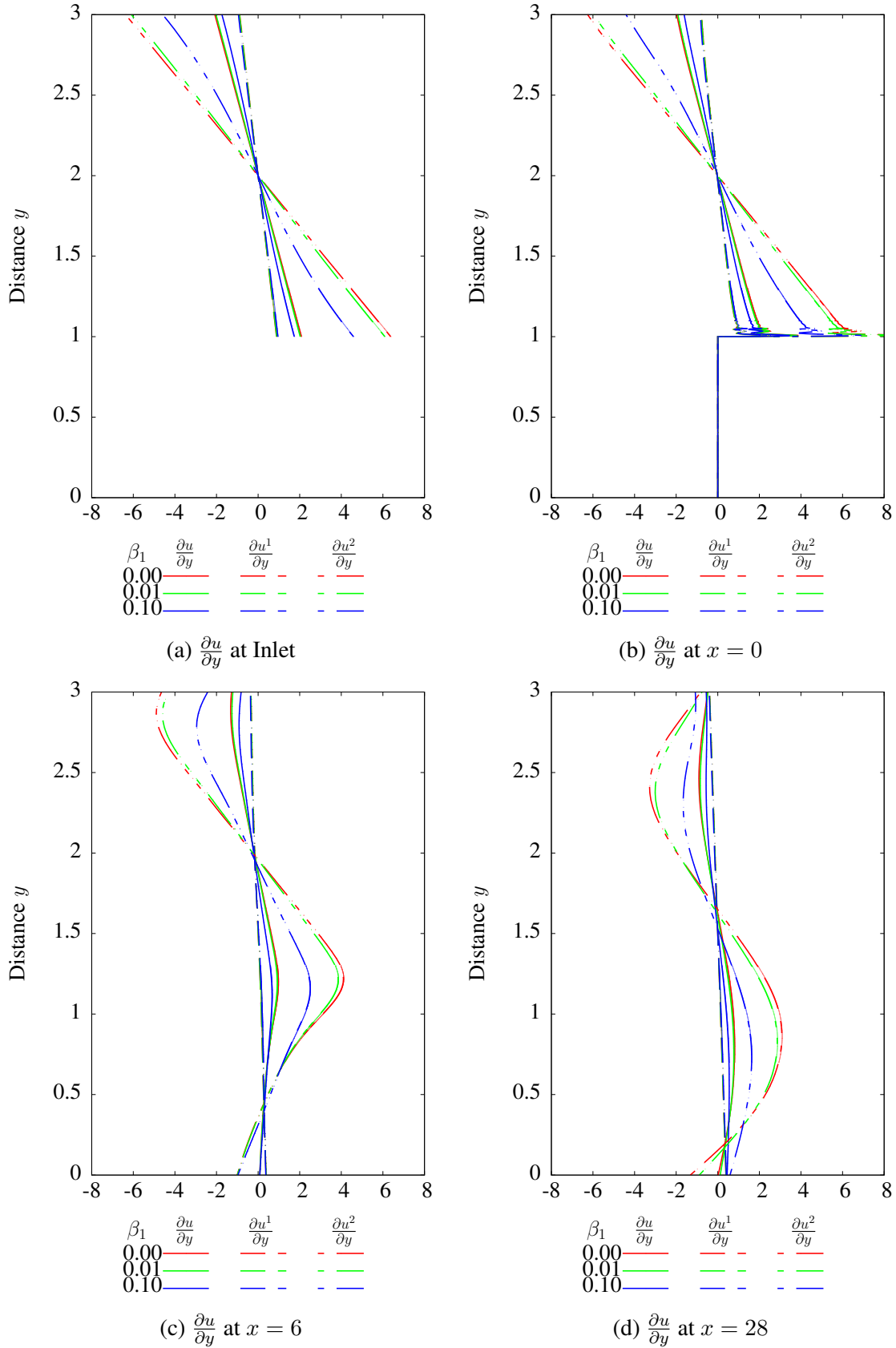
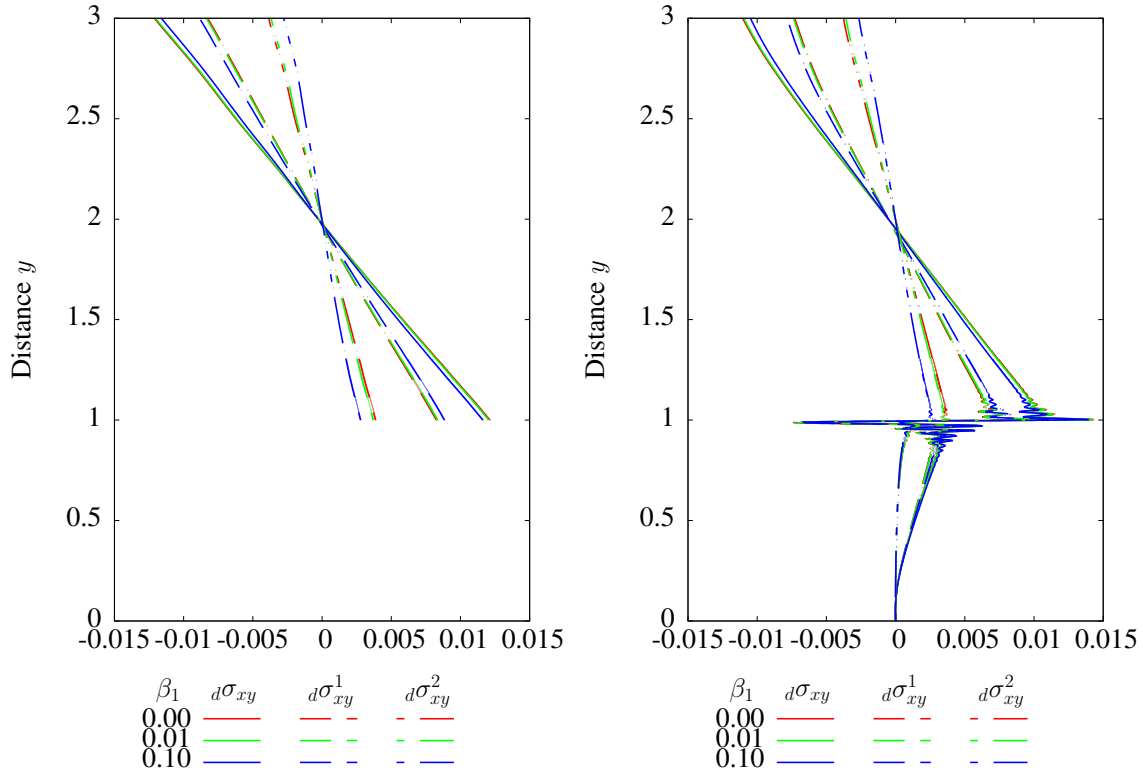
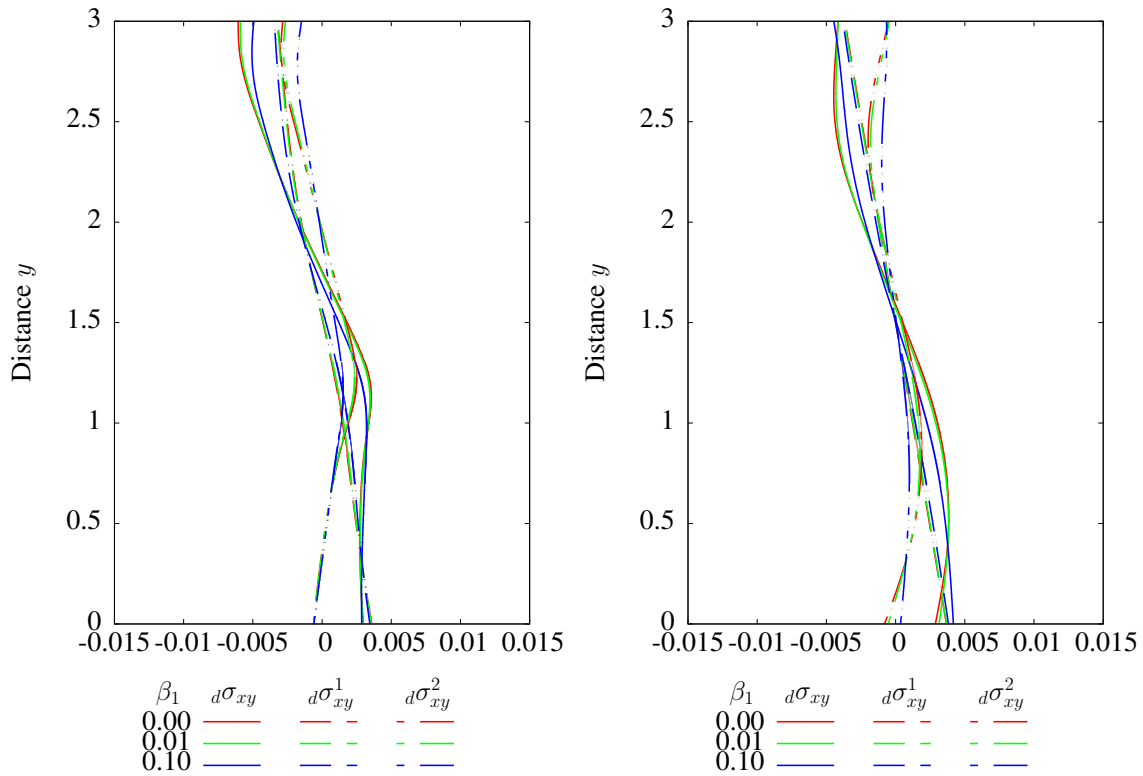


Figure 3.14: First derivative of axial velocity with respect to y at various x locations



(a) Deviatoric Cauchy stress $d\sigma_{xy}$ at Inlet

(b) Deviatoric Cauchy stress $d\sigma_{xy}$ at $x = 0$



(c) Deviatoric Cauchy stress $d\sigma_{xy}$ at $x = 6$

(d) Deviatoric Cauchy stress $d\sigma_{xy}$ at $x = 28$

Figure 3.15: Deviatoric Cauchy stress $d\sigma_{xy}$ at various x locations

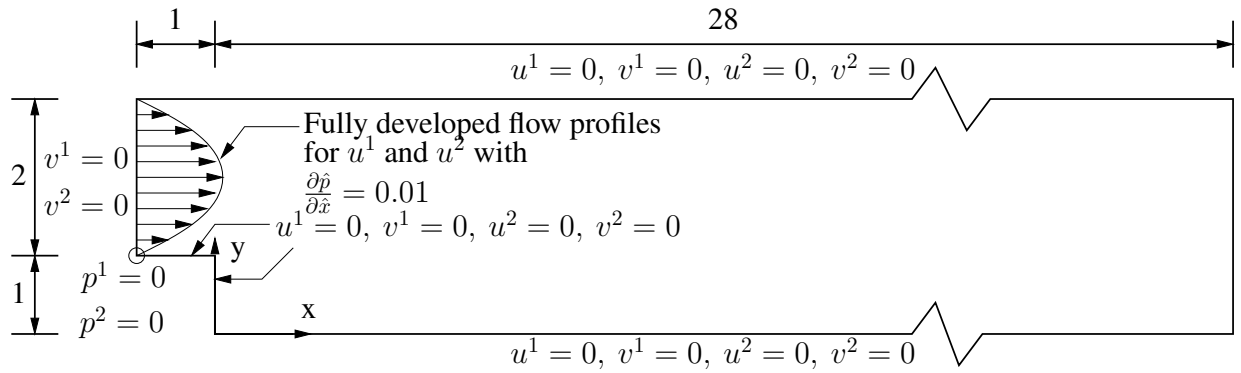


Figure 3.16: Schematic of 2D Domain and Boundary Conditions

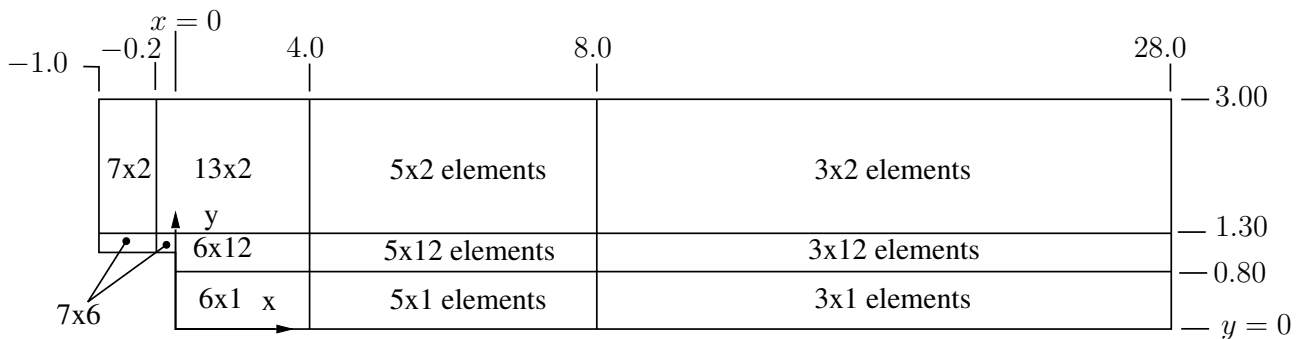


Figure 3.17: Graded Discretization: 322 nine-node p-version elements

Results show similar behavior as in the case of parallel plates. We note that at the inlet in Figure 3.12 the results are consistent with that of parallel plates. As the velocity approaches the outlet it is not quite fully developed but is becoming fully developed flow. An increase in the interaction coefficient β_1 makes the velocity fields of u^1 and u^2 closer to one another.

In Figure 3.13 we note that at the expansion point and after the expansion the velocity in the vertical direction is non-zero. However towards the outlet the velocity in the vertical direction is becoming zero indicating that the flow is moving towards fully developed.

In Figure 3.14 and Figure 3.15 we note at the inlet that stress has a fully developed profile and is becoming fully developed as it moves towards the outlet.

Chapter 4

Summary and conclusions

Development of mathematical models based on conservation and balance laws including constitutive theories are presented for a saturated mixture of ν homogeneous, isotropic, and incompressible constituents for isothermal and non-isothermal flows. The constituents and the mixture are assumed to be Newtonian or generalized Newtonian fluids. Power law and Carreau-Yasuda models are considered for generalized Newtonian shear thinning fluids. The mathematical model is derived for a ν constituent mixture with volume fractions ϕ_α using principles of continuum mechanics: conservation of mass, balance of momenta, first and second laws of thermodynamics, and principles of mixture theory yielding continuity equations, momentum equations, energy equation, and constitutive theories for mechanical pressures, deviatoric Cauchy stress tensors, and heat vector in terms of the dependent variables related to the constituents and their material coefficients. In the derivation of the mathematical model effects of the interaction forces are accounted in the momentum and energy equations.

In the development of the constitutive theories two approaches are considered. In the first approach we assume that the mixture stress is the sum of the constituent stresses. This approach requires derivation of the bulk properties of the constituents based on the constituent volume fractions and their properties which are then utilized in the constitutive theories for the constituents forming the mixture. In the second approach the mixture stress is assumed not to be the sum of the

constituent stress. For a homogenous isotropic mixture we begin with its own constitutive theory for the deviatoric mixture stress defined using mixture material coefficients and the symmetric part of the velocity gradient tensor for the mixture. Mixture material coefficients are derived using volume and mole fractions of the constituents and a mixing rule. The mutual parameter in the mixing rule is described using arithmetic mean, geometric mean, and harmonic mean. The validity of the proposed models are demonstrated for degenerated cases of same constituents i.e., two of the constituents same etc. Dimensionless forms of the mathematical models are derived and used to present numerical studies for boundary value problems using finite element processes based on a residual functional, that is, least squares processes in which local approximations are considered in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces. Fully developed flow between parallel plates and 1:2 asymmetric backward facing step are used as model problems for a mixture of two constituents.

The numerical studies demonstrate the influence of interaction parameter on the flow physics for fully developed flow between parallel plates and the asymmetric expansion.

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