

INVERSION OF INTEGRALS OF THE FIRST KIND  
FOR HYPERELLIPTIC CURVES

by

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## INTRODUCTION.

The problem of inversion of integrals of the first kind for elliptic curves has been fully discussed by many mathematicians (Abel, Jacobi, Hermite, Weierstrass).<sup>(1)</sup> However, in the case of hyperelliptic curves, they have on the whole merely contented themselves with affirming that the integrals cannot be regularly inverted. It is the object of this paper to discuss, by means of investigations at the branch points of the integrals considered as functions of  $x$ , some of their peculiarities and thus to show why they cannot be inverted. The methods here are the extension of those used in the case of the elliptic curve by Professor Lefschetz in the course, Theory of Algebraic Functions.

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 (1)

Abel -- Oeuvres Complètes - I - 263.

Jacobi -- Gesammelte Werke - II - 7.

Hermite -- Oeuvres de Ch. Hermite - I - 10.

Weierstrass -- Mathematische Werke - IV - 403.

We shall first study the hyperelliptic integral  $u = \int_{\infty}^{(x,y)} \frac{A_n(x) dx}{y}$  [  $A_n(x)$  a polynomial of degree  $n$ . ]

where  $y^2 = (x - e_1)(x - e_2) \dots (x - e_g)$ .

After determining the degree of the polynomial  $A_n(x)$ , we shall investigate the irregularities of the integral by expressing  $u$  as a power series in a parameter  $t$  at regular points, at branch points, and at the infinite point. By inverting these series,  $t$  [hence  $(x,y)$ ] will be expressed as a function of  $u$ . Corresponding results will then be obtained for the hyperelliptic integral in which  $y^2$  is of order  $m$  [  $m$  even and  $\geq 6$  ]. Finally, certain definite integrals of the first kind will be represented by conformal mapping of the  $x$ -plane upon the  $u$ -plane.

I. INVERSION OF INTEGRALS FOR CURVE OF DEGREE 6

1) Let us take the hyperelliptic curve

$$y^2 = (x - e_1)(x - e_2) \dots (x - e_6) = \phi(x)$$

and the integral of the first kind as

$$u = \int_{\infty}^{(x,y)} \frac{A_n(x) dx}{y} \quad [A_n(x) \text{ a polynomial of degree } n]$$

and investigate the problem of expressing the coordinates  $(x,y)$  as functions of  $u$ .

The integral is finite everywhere at finite distance, for the branch points alone offer doubt, and there (say at  $e_i$ ), we have (1)

$$\frac{A_n(x)}{y} = \frac{a_0}{(x - e_i)^{1/2}} + a_1(x - e_i)^{1/2} + a_2(x - e_i)^{3/2} + \dots$$

$$\int \frac{A_n(x) dx}{y} = c + 2a_0(x - e_i)^{1/2} + \frac{2a_1}{3}(x - e_i)^{3/2} + \dots$$

which is finite for  $x = e_i$ .

It remains to find for what degree of the

- (1) P. Appell et Goursat: Théorie des Fonctions Algébriques et de leurs Intégrales. p. 62.

polynomial  $A_n(x)$ , that  $u$  is finite for  $x$  infinite.

At  $\infty$ , let  $x = \frac{1}{t}$ , and our integral becomes

$$u = \int \frac{-A_n\left(\frac{1}{t}\right) dt}{t^2 \sqrt{\left(\frac{1}{t} - e_1\right) \left(\frac{1}{t} - e_2\right) \dots \left(\frac{1}{t} - e_6\right)}}$$

$$= \int \frac{-A_n(t) dt}{t^{\lambda+2-3} \sqrt{1 + at^2 + \dots}}$$

which is finite at  $t = 0$ , if  $(n - 1) \leq 0$ ,  $n \leq 1$ .

$\therefore A_n =$  polynomial of the first degree  $= Ax + B$ .

Our integral is then of the form

$$u = \int_{\infty}^{(xy)} \frac{(Ax + B) dx}{(x - e_1) (x - e_2) \dots (x - e_6)}$$

which is finite everywhere.

2) LEMMA:  $u = at^{\lambda} + bt^{\lambda+1} + \dots$  can be inverted as a power series in  $u^{\frac{1}{\lambda}}$ .

Proof: (i)  $u = at^{\lambda} + bt^{\lambda+1} + \dots$ .

Then  $t^{\lambda + \alpha(u)} t^{\lambda+1 + \beta(u)} t^{\lambda+2 + \dots + \mu(u)} = 0$   
 [Implicit Function Theorem.]

where  $\alpha(u)$ ,  $\beta(u)$   $\dots$  are holomorphic for  $u$  near

$u_0$ .

As  $u$  repeatedly describes a small circle

in the  $u$ -plane around  $u_0$ , the  $t$ 's will vary, and  $t_1$  will become one of the other  $t$ 's  $[t_2, t_3, \dots, t_n]$ . These  $n$   $t$ 's will be permuted in one cycle, for if there were 2, 3, --- or  $(n - 1)$  cycles, each one would necessitate a distinct power series:

$$(v) \quad t = a u^{1/k} + b u^{2/k} + \dots$$

$$(vv) \quad t = a_1 u^{1/k} + b_1 u^{2/k} + \dots$$

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Equation (v) would give  $h$  determinations of the  $t$ 's, equation (vv)  $k$  determinations, ----- so that

$h + k \text{ -----} = n$  [Total number of determinations of  $t$ .]

From equation (v) we have  $t$  of order of magnitude  $u^{1/k}$ .  
 $\left[ \frac{t}{u^{1/k}} \rightarrow \text{finite value} \right]$

But from our original equation (i),  $t$  is of order of magnitude  $\frac{1}{n}$ .

$\therefore h = n$ , and  $k \text{ -----} = 0$ . [Since  $h + k + \text{-----} = n$ ].

The  $n$  values of the  $t$ 's are therefore permuted in one cycle and are expressible in the equation :

$$t = \alpha (u - u_0)^{1/n} + \beta (u - u_0)^{2/n} + \dots$$

Q. E. D.

3) Let us investigate the problem of expressing  $t$  [hence  $(x, y)$ ] as a power series in  $u$  at regular points, at branch points, and at infinity.

a) At any point  $(x, y)$  not a branch point, let  $x = x_0 + t$ , then

$$u = \frac{\int (A x_0 + B + A t) dt}{\sqrt{(x_0 - e_1 + t)(x_0 - e_2 + t) \dots (x_0 - e_n + t)}}$$

$$= \frac{\int (A x_0 + B + A t) dt}{\sqrt{F(t)}}$$

$$\sqrt{F(0)} = \sqrt{(x_0 - e_1)(x_0 - e_2) \dots (x_0 - e_n)} \neq 0$$

$\frac{1}{\sqrt{F(t)}} = \psi(t)$  is holomorphic for  $t = 0$ , and

$$\int (A x_0 + B + A t) \psi(t) dt = \int (a + bt + ct^2 + \dots) dt$$

$$= at + \frac{b}{2} t^2 + \frac{c}{3} t^3 + \dots$$

Investigating the coefficients of this series, we have

$$f(t) = (A x_0 + B + A t) \psi(t) = a + bt + ct^2 + \dots$$

Then  $a = (A x_0 + B) \psi(0)$  which vanishes when

$$x_0 = -\frac{B}{A}.$$

$$f'(t) = b + 2ct + \dots$$

$$f'(t) = (A x_0 + B + At) \psi'(t) + \psi(t)A$$

$$\text{Then } b = P'(t)_{t=0} = (A x_0 + B) \psi'(0) + \psi(0) A$$

For  $A x_0 + B = 0$  [ie.  $a = 0$ ] and  $A \neq 0$ , we have

$$b = A \psi(0) \neq 0; \text{ and}$$

$$u = b, t^2 + c, t^3 + \dots$$

THEOREM:  $t$  is a power series in  $u^{1/2}$ .

This follows directly from the lemma (2) where  $n = 2$ .

(b) At a branch point ( $e_i$ ), let

$$(x - e_i)^{1/2} = t; \quad x = e_i + t^2; \quad dx = 2t dt; \text{ then}$$

$$u = \int \frac{(A e_i + B + At^2) 2 dt}{\sqrt{(t^2 + e_1 - e_2) (t^2 + e_1 - e_3) \dots (t^2 + e_1 - e_6)}} \quad \boxed{\text{Let } i = 1}$$

$$\frac{1}{\sqrt{(t^2 + e_1 - e_2) \dots (t^2 + e_1 - e_6)}} \text{ is holomorphic}$$

for  $t^2 = 0$ , since the branch points are distinct, and

$$\begin{aligned} u &= \int (A e_i + B + At^2) \psi(t^2) dt \\ &= \int (a + bt^2 + ct^4 \dots) dt \\ &= at + b, t^3 + c, t^5 + \dots \end{aligned}$$

Investigating the coefficients of this series, we have

$$F(t^2) = (A e_1 + B + A t^2) \psi(t^2) = a + bt^2 + ct^4 + \dots$$

$$a = (A e_1 + B) \psi(0)$$

If  $(A x + B)$  does not vanish at the branch points, these points behave like ordinary points. If

$$A e_1 + B = 0,$$

$$a = 0, \quad b \neq 0, \quad \text{for}$$

$$b = F''(t)_{t=0} = 2A \psi(0) \neq 0, \quad \text{and}$$

$$u = b_1 t^3 + c_1 t^5 + \dots$$

THEOREM:  $t$  is a power series in  $u^{1/3}$ .

Lemma (2).

c) At  $\infty$ , let  $x = \frac{1}{t}$ , then

$$u = \frac{\int -(At^{-1} + B) t^{-2} dt}{\sqrt{(t^{-1} - e_1)(t^{-1} - e_2) \dots (t^{-1} - e_6)}}$$

$$= \frac{\int -(A + Bt) dt}{\sqrt{(1 - e_1 t)(1 - e_2 t) \dots (1 - e_6 t)}}$$

$$= \int (A + Bt) \psi(t) dt = \int (a + bt + ct^2 + \dots) dt$$

$$= at + b_1 t^2 + c_1 t^3 + \dots$$

Investigating the coefficients of this series, we

have

$$f(t) = (A + Bt) \psi(t) = a + bt + \dots$$

$$a = -A$$

$$f'(t) = (A + Bt) \psi'(t) + \psi(t) B$$

$$b = f'(0) = A \psi'(0) + B \psi(0) \neq 0 \text{ if } A \neq 0.$$

$$u = -At + b, t^2 + c, t^3 + \dots$$

But if  $A = 0$ ,

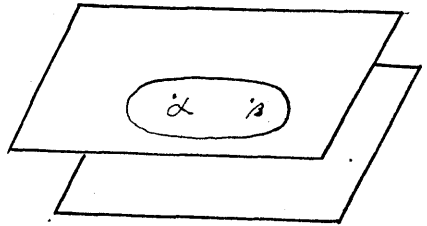
$$u = b, t^2 + c, t^3 + \dots$$

THEOREM:  $f$  is a power series in  $u^{1/2}$ .

Lemma (2).

4) Spreading the value  $(x, y)$  on the  $u$ -plane, we have two values at  $\alpha$  and at  $\beta$ . Let us investigate how these four values are permuted.

When  $u_0$  and  $-u_0$  [the two values corresponding to  $x = \frac{B}{A}$ ] come together, let  $A$  and  $B$  take on values, so



that

$$u = \frac{(x - e_1) dx}{\sqrt{(x - e_1)(x - e_2) \dots (x - e_6)}}$$

$$= \frac{2t^2 dt}{\sqrt{(t^2 + e_1 - e_2) \dots (t^2 + e_1 - e_6)}}$$

$$= \int t^2 (a + bt^2 + \dots) dt$$

$$= a_1 t^3 + b_1 t^5 + \dots$$

$\therefore$  3 values are permuted here, and we have at  $\alpha$  and  $\beta$  permutations of the form

$$\alpha = (12) ; \quad \beta = (13).$$

5) Assume none of the (e)'s equal to zero. [This can be obtained by making change of variables  $x' = x + k$ .] Then as a result of our discussion, we have:

THEOREM: If  $A \neq 0$ , and  $Ax + B$  does not vanish at the branch points, there are two points on the curve,  $(-\frac{B}{A}, y)$ ,  $(\frac{B}{A}, -y)$  where  $u$  has a branch point of order 2. If  $Ax + B$  vanishes at one of them, say  $e_i$ , these two points coincide with  $(e_i, 0)$  which is then a branch point of order 3 for the integral. If  $A = 0$  ( $B \neq 0$ ) the two branch points both go to infinity and remain distinct.

## II. EXTENSION TO CURVES OF DEGREE $> 6$ .

6) Extending our hyperelliptic curve to the form  $y^2 = (x - e_1)(x - e_2) \dots (x - e_n) = \phi(x)$  [m even and  $\leq 6$ ], let us take the integral of the first kind as

$$u = \int_{\infty}^{(x,y)} \frac{A_n(x) dx}{y}$$

and investigate the inversion problem.

The integral is finite everywhere at finite distance. (1). To find for what degree of the polynomial  $A_n(x)$  that  $u$  remains finite for  $x$  infinite, let  $x = \frac{1}{t}$  at  $\infty$  and our integral becomes

$$u = \int \frac{-2A_n(t) dt}{t^2 \sqrt{(t^{-1} - e_1)(t^{-1} - e_2) \dots (t^{-1} - e_n)}}$$

$$= \int \frac{-2A(t) dt}{t^{2+n-\frac{m}{2}} \sqrt{1 + kt + \dots}}$$

which is finite at  $t = 0$ , if  $(2 - n - \frac{m}{2}) \leq 0$ .

Let  $m = 2m'$

$$n \leq m' - 2.$$

$\therefore A_n(x)$  is a polynomial of degree  $\leq m' - 2$ , and our

integral ds of the form

$$u = \int_{\infty}^{(x,y)} \frac{(A x^{\mu-2} + B x^{\mu-3} + \dots + k) dx}{\sqrt{(x - e_1)(x - e_2) \dots (x - e_n)}}$$

which is finite everywhere.

7) Let us investigate the problem of expressing  $t$  [hence  $(x,y)$ ] as a function of  $u$ .

a) At any point  $x_0$ , not a branch point,

let  $x - x_0 = t$ , then

$$\begin{aligned} u &= \int \frac{[A(x_0 + t)^{\mu-2} + B(x_0 + t)^{\mu-3} + \dots]}{\sqrt{(x_0 - e_1 + t)(x_0 - e_2 + t) \dots (x_0 - e_n + t)}} dt \\ &= \int (a + bt + ct^2 + \dots) dt = \int f(t) dt \\ &= at + b_1 t^2 + \dots \end{aligned}$$

Investigating this series, we have

$$a = \frac{A x_0^{\mu-2} + B x_0^{\mu-3} + \dots}{\sqrt{\phi(x_0)}} = P(x_0) \psi(x_0)$$

$$b = f'(t)_{t=0} = P(x_0) \psi'(x_0) + \psi(x_0) P'(x_0)$$

If  $P(x_0) = 0$ ,  $P'(x_0) \neq 0$

$$u = b_1 t^2 + c_1 t^3 + \dots$$

THEOREM:  $t$  is a power series in  $u^{1/2}$ .

Lemma (2).

If  $P(x_0) = P'(x_0) = 0$

$$u = c_1 t^3 + d_1 t^4 + \dots$$

THEOREM: t is a power series in  $u^{1/3}$ .

Lemma (2).

b) At a branch point  $e_i$ , let  $(x - e_i)^{1/2} = t$   
then

$$\begin{aligned} u &= \int \frac{[A(e_i + t^2)^{m-2} + B(e_i + t^2)^{m-3} + \dots] 2 dt}{(t^2 + e_1 - e_2)(t^2 + e_1 - e_3) \dots (t^2 + e_1 - e_m)} \\ &= \int (a + bt^2 + ct^4 + \dots) dt = \int f(t) dt \\ &= at + bt^3 + ct^5 + \dots \end{aligned}$$

Investigating this series, we have

$$a = \frac{2(Ae_1^{m-2} + Be_1^{m-3} + \dots)}{\sqrt{(e_1 - e_2) \dots (e_1 - e_m)}} = 2P(e_1) \psi(o)$$

$$f'(t^2)_{t=0} = 0$$

$$f''(t^2)_{t=0} = b = 2P'(e_1) \psi(o) \neq 0$$

If  $P(e_1) = 0$ ,

$$u = b_1 t^3 + c_1 t^5 + \dots$$

THEOREM: t is a power series in  $u^{1/3}$ .

Lemma (2).

c) At  $\infty$ , let  $x = \frac{1}{t}$ , and

$$\begin{aligned} u &= \int \frac{-(At^{-(m-2)} + Bt^{-(m-3)} + \dots) dt}{t^2 \sqrt{(t^{-1} - e_1)(t^{-1} - e_2) \dots (t^{-1} - e_m)}} \\ &= \int \frac{-(A + Bt + \dots) dt}{\sqrt{(1 - e_1 t)(1 - e_2 t) \dots (1 - e_m t)}} \end{aligned}$$

$$\begin{aligned}
 &= \int (a + bt + \dots) \psi(t) dt = \int f(t) dt \\
 &= at + b_1 t^2 + \dots
 \end{aligned}$$

Investigating this series, we have

$$a = -A$$

$$f'(t) = P(t)\psi'(t) + \psi(t)P'(t) \quad [(A + Bt + \dots) = P(t)]$$

$$b = f'(0) \neq 0 \text{ if } A = 0 \text{ and}$$

$$u = b_1 t^2 + c_1 t^3 + \dots$$

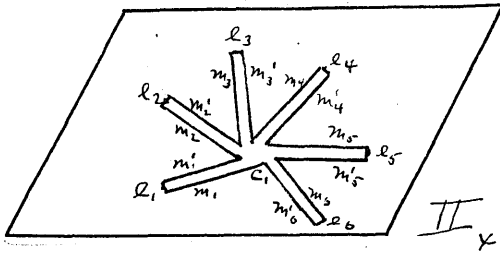
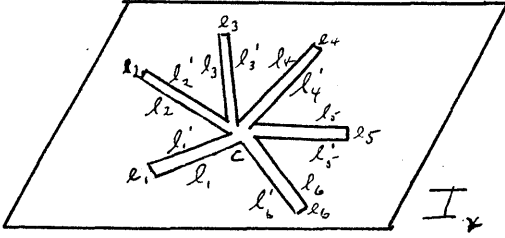
THEOREM: t is a power series in u<sup>1/2</sup>.

8) CONCLUSION: For curves of order  $2m' > 6$  the behavior is in general the same as for the degree 6, only there are  $2(m' - 2)$  branch points for the integral.

### III. CONFORMAL MAPPING.

9) The integral  $u = \frac{(Ax + B) dx}{(x - e_1)(x - e_2)\dots(x - e_6)}$

may be represented by a two sheeted surface in the x-plane, sheet I for the value y and sheet II for the value -y. In each of these sheets, let us join



the branch points ( $e_1, e_2 \dots e_6$ ) to any regular point (say  $c$ ) and cut along these lines. Superposing the two sheets and matching corresponding cuts, we have a point for point representation of our integral in the  $x$ -plane. Let us map this  $x$ -plane upon the  $u$ -plane. The

angles at  $c$  will be preserved, for the integral is holomorphic there. The angles of  $360^\circ$  at  $e_1, e_2, \dots e_6$  will become  $180^\circ$  on the  $u$ -plane, since  $e_1, e_2, \dots e_6$  are the branch points; thus the boundary  $l, l' [i = 1, 2, \dots 6]$  of a cut in the  $x$ -plane will become one line in the  $u$ -plane. Our figure in sheet I of the  $u$ -plane will then be composed of 6 lines, and it will be closed, since a path in the  $x$ -plane starting at  $c$  and following the cuts around

all the branch points may be reduced to a point.

[For a circuit around all the branch points is equal to one around the one point at infinity, which has been shown to be a regular point.]

THEOREM: The polygon of sheet II (in the u-plane) is obtained from that of sheet I by a translation plus a rotation of  $180^\circ$ .

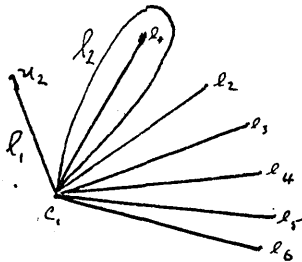
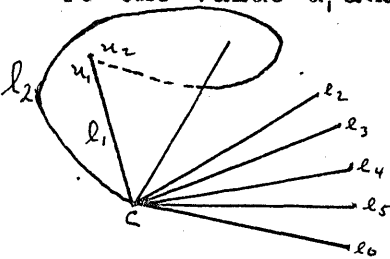
Proof: Let the path  $l_1$ , in the first sheet lead to the value  $u_1$ , and  $l_2$  in the second sheet to  $u_2$ ;

then  $u_1 =$

$$\int_{l_1} \frac{(Ax + B) dx}{\sqrt{(x - e_1)(x - e_2)\dots(x - e_6)}}$$

Now the path  $l_2$  can be deformed into the loop  $ce_1$ , plus the path lying under  $l_1$ , in the second sheet.

The integral along the loop is a constant, say  $K$  (the value to which the integral is returned at  $c$  after describing the loop). Along the remaining path lying under  $l_1$ , the radical takes the negative value of that which



it had along the previous path  $l_1$ .

$$\therefore u_2 = K - u_1.$$

To see how  $u_1$  may be obtained from  $u_2$  let

$$K = 0 \text{ and } u_2 = -u_1$$

$$\text{that is } u_2 = u_1 \cdot e^{\pi i}$$

$$\text{or } \rho_2 e^{i\theta_2} = \rho_1 e^{i(\theta_1 + \pi)}$$

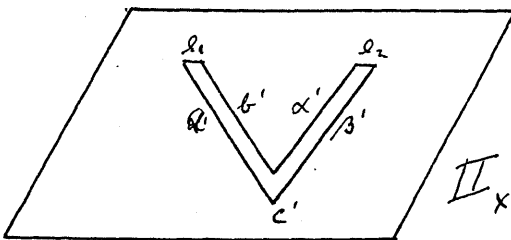
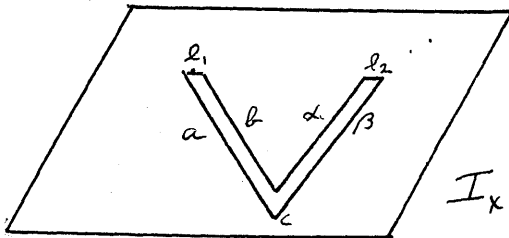
$$\theta_2 = \theta_1 + \pi \quad [\text{Since } \rho_2 = \rho_1]$$

$\therefore u_2 = K$  plus a rotation of  $180^\circ$  of  $u_1$ .

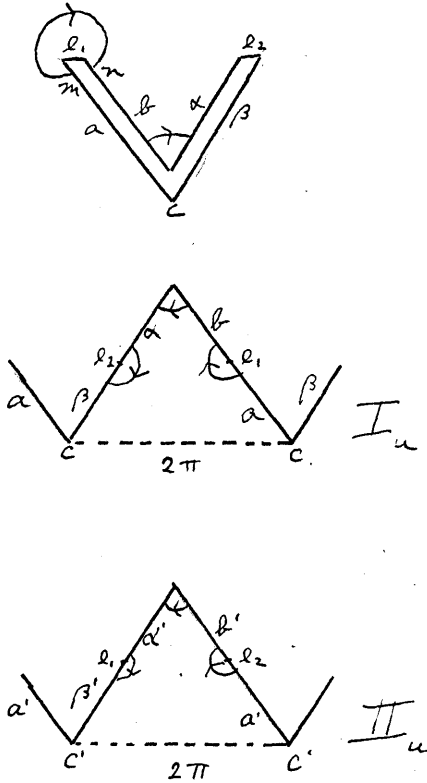
Q. E. D.

10) As a preliminary study of our hyperelliptic integrals, let us consider the integral

$$u = \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin x.$$



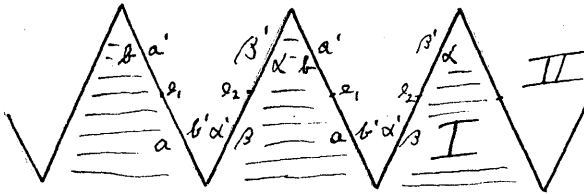
Let us join the branch points in the  $x$ -plane to the point  $c$ , making cuts as described (9). In mapping this  $x$  plane on the  $u$ -plane, let us start at  $c$ , and the boundary  $(a, b)$  of



the first cut will become the line  $(a + b)$ , since the  $360^\circ$  angle at  $e_2$  becomes  $180^\circ$ . The angle between  $b$  and  $\alpha$  is preserved, since  $c$  is a regular point. The boundary  $(\alpha, \beta)$  becomes the line  $(\alpha + \beta)$  which brings us back to the value  $c$  at a distance of  $2\pi$  from the point  $c$  since  $u = \arccos \sin x$  is of period  $2\pi$ .

Let us consider how sheets I and II of the  $u$ -plane may be fitted to gether on one sheet. In turning in the  $x$ -plane around  $e$ , from  $m$  on  $a$  to  $n$  on  $b$  we follow a clockwise direction. If at  $n$  we go into the second sheet, the boundary  $b$  must be joined to  $a'$  in order that the clockwise direction be continued. Similarly  $a$  must be joined to  $b'$ ,

$\alpha$  to  $\beta'$ ,  $\beta$  to  $\alpha'$ . We then have a point for point representation of the x-plane on the u-plane.



11) Let us next consider the elliptic integral

$$u = \int \frac{dx}{\sqrt{(x - e_1)(x - e_2)(x - e_3)(x - e_4)}}$$

Let us join the

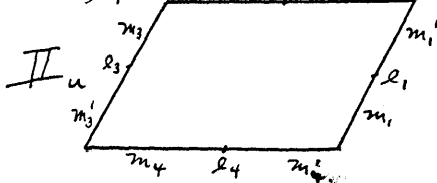
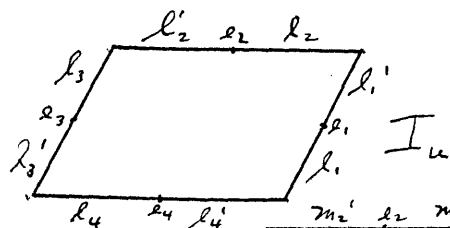
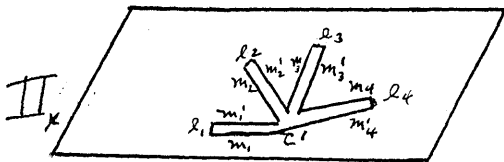
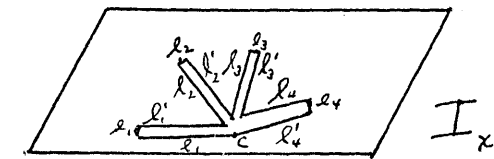
branch points  $(e_1, e_2, e_3, e_4)$  to  $c$  (as 9).

Mapping this x-plane upon the u-plane [method

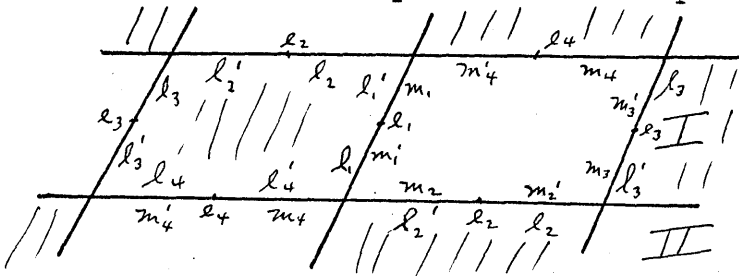
here for curve of order 4, same as that for order 6 in (9)], we obtain a

parallelogram  $I_u$ ; similarly the parallelogram

$II_u$ . The parallelograms may be



fitted together by joining  $l'_i$  to  $m_i$ , and  $l_i$  to  $m'_i$  ( $i = 1, 2, 3, 4$ ) as in (10), that is by rotating  $I_u$  through an angle of  $180^\circ$ , we may obtain  $II_u$  [Theorem section (9)]. Thus we have a point for point representation of the  $x$ -plane on the  $u$ -plane.

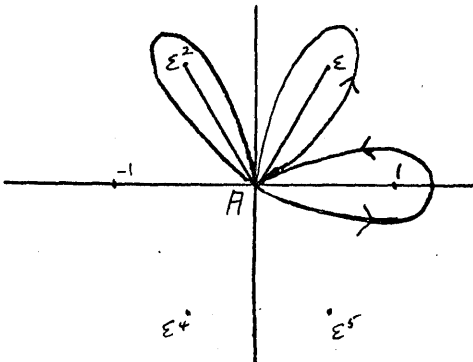


IV. A SPECIAL CASE OF MAPPING.

12) The hyperelliptic integral

$$u = \int \frac{x \, dx}{\sqrt{x^6 - 1}}$$

has its six branch points ( $\pm 1$  and the imaginary 6<sup>th</sup> roots of unity) as the vertices of a regular hexagon in the  $x$ -plane. To map this upon the  $u$ -



plane, let us consider the loop paths around the branch

points. The value of the integral along the loop path A 1 is

$$\int_0^1 \frac{x dx}{\sqrt{x^6 - 1}} + \int_1^0 \frac{x dx}{-\sqrt{x^6 - 1}} = -2 \int_0^1 \frac{x dx}{\sqrt{x^6 - 1}}$$

which may be mapped as a line (a) on the u-plane.

[For we may consider the loop A 1 a curve of such a nature that it maps as a straight line on the u-plane.]

Having started with the positive determination of the radical, we return after following the loop A 1 with the negative value. Therefore the value of the integral along the loop A $\epsilon$  is

$$\int_0^\epsilon \frac{x dx}{-\sqrt{x^6 - 1}} + \int_\epsilon^0 \frac{x dx}{\sqrt{x^6 - 1}} = -2 \int_0^\epsilon \frac{x dx}{\sqrt{x^6 - 1}} .$$

Making the change of variable

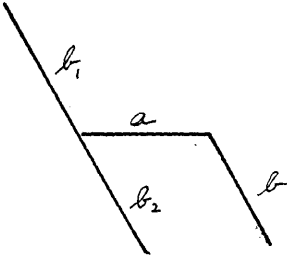
$$x = \epsilon x' ; x' = \epsilon^{-1} x = e^{-\frac{2\pi i}{6}} x,$$

we have

$$-2 \int_0^\epsilon \frac{x dx}{\sqrt{x^6 - 1}} = -2 \epsilon^2 \int_0^1 \frac{x' dx'}{\sqrt{x'^6 - 1}}$$

so that the value of the integral on the loop A $\epsilon$

is  $u_2 = K - \epsilon^2 u_1$ , [ $u_1$  = value on loop A 1.] This is rep-



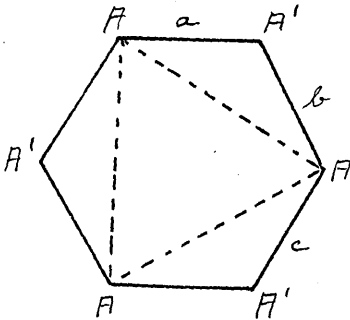
represented first by the line  $b_1$ , on the  $u$ -plane at an angle of  $120^\circ$  to the line  $a$  [For  $\varepsilon^2 = \frac{2\pi i}{3}$ ]; since we have the negative value  $b_1$ , is rotated to  $b_2$ , and by the constant  $K$  is translated to  $b$ .

This brings us back to the original point  $A$ .

Similarly the value  $u$  of the integral on the loop  $A\varepsilon^2$  is

$$2 \int_0^{\varepsilon^2} \frac{x dx}{\sqrt{x^6 - 1}} = K - \varepsilon^2 u_1 + \varepsilon^4 u_2$$

which gives us the line  $c$  in the  $u$ -plane; and this



brings us back to the negative determination [A'] of the radical. We thus see that two consecutive loops represent a period and that

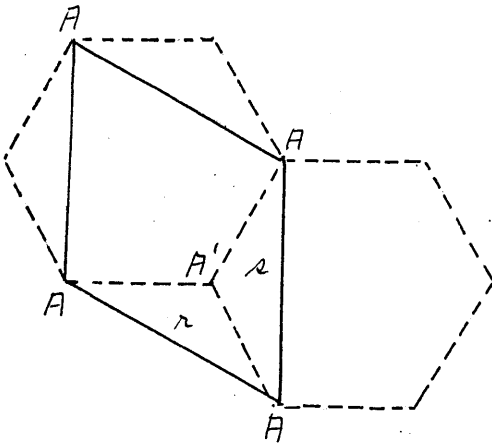
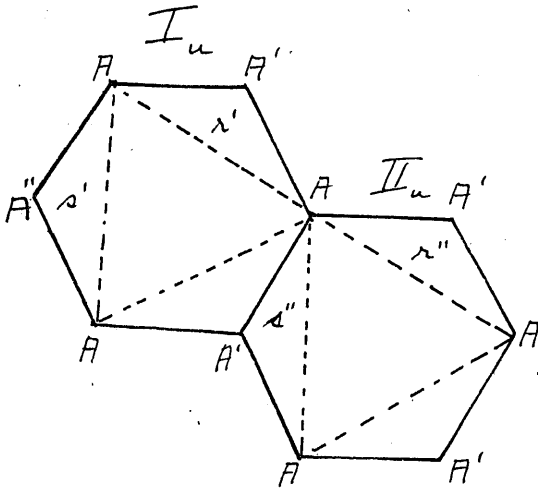
we have a regular hexagon in the  $u$ -plane with alternating vertices representing the same point.

By a rotation of  $180^\circ$  of  $I_u$  we add the hexagon of the second sheet.

By a translation of a period the area  $r$  may be replaced by its congruent area  $r'$  of  $I_u$  or by  $r''$  of  $II_u$ , and similarly  $s$  by  $s'$  of  $I_u$  or  $s''$  of  $II_u$ . We have

thus formed in the  $u$ -plane the parallelogram  $A$  which is doubly covered since it may be obtained by translations from both hexagons

$[I_u \text{ and } II_u]$ . We therefore have a point for point representation of

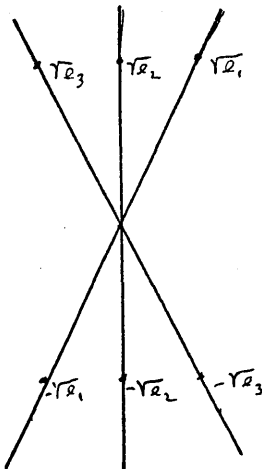


the  $x$ -plane upon the  $u$ -plane. The vertices [A] of the parallelogram and also the point A' within it are branch points, where the two sheets hang together.

13) This representation is possible not because the hexagons are regular but because of the congruent points in the polygons due to the periods. We are therefore able to say that this method may be extended to the hyperelliptic integral

$$u = \int \frac{2x \, dx}{\sqrt{4x^6 - g_2 x^2 - g_3}} = \int \frac{dx^2}{\sqrt{4(x^2)^3 - g_2 x^2 - g_3}}$$

which is that of the Weierstrass elliptic function for  $(x^2, y)$ . Its six branch points are of the form  $\pm\sqrt{e_1}$ ,  $\pm\sqrt{e_2}$ ,  $\pm\sqrt{e_3}$  in the  $x$ -plane.



We infer that they will be mapped upon the  $u$ -plane with slight deformations by parallelograms as those in the case of the integral

$$u = \int \frac{x \, dx}{\sqrt{x^6 - 1}}$$

though the effective construction will not be carried out here.