PARTIAL DERIVATIVE OPERATORS APPLIED TO THE DERIVATION OF RELATIONS AMONG SUMS OF DETERMINANTS.

By

Ronald G. Smith
A.B.,1924, University of Kansas.

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Approved by:

E B Stouffer
Instructor in Charge.

C. H. Ashton
Chairman of Department.

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INTRODUCTION.

The object of this thesis has been to prove certain theorems on sums of determinants by means of partial derivative operators. These operators have greatly simplified the proofs of a number of known theorems and in many cases have shown that these theorems would permit of more general statements. Also a few new theorems have been discovered.

The first publication to suggest the derivation of relations between sums of determinants by means of partial derivative operators was by Stouffer[1] in December 1925. This paper, typifying the advantage of these operators , presented a very simple yet complete proof of "Kronecker's Theorem" in the brief space of a page. Previous proofs had been rather long or difficult.

If we expand the general determinant $A = |a_{rj}|$ of order n, with respect to the elements of the r th row it will take the form $a_{rn}A_{rn}+\cdots\cdots+a_{rj}A_{rj}+\cdots\cdots+a_{r2}A_{r2}+a_{r1}A_{r1} = A$ where A_{rj} is the cofactor of the element a_{rj}. It is immediately evident that the partial derivative of A with respect to the element a_{rj} is equal to the cofactor A_{rj}. It follows that the operator $\sum_{i=1}^{n} a_{ri}\frac{\partial}{\partial a_{ri}}$ when applied to A gives A. That is,

$$\left(a_{rn}\frac{\partial}{\partial a_{rn}} + \cdots\cdots\cdots + a_{r2}\frac{\partial}{\partial a_{r2}} + a_{r1}\frac{\partial}{\partial a_{r1}}\right)A = A.$$

Jacobi[2] used this operator as early as 1845 in connection with a function defined similar to a determinant, and Bellavitis[3] in 1857 published a text on determinants in which this operator occurred.

Again it is evident that the operator $\sum_{j=1}^{n} b_{rj} \frac{\partial}{\partial a_{rj}}$ acting on A will replace the elements of row r by the elements $b_{r1}, b_{r2}, \cdots, b_{rn}$; the operator $\left(1 - a_{rj} \frac{\partial}{\partial a_{rj}}\right)$ acting on A will replace the element a_{rj} by a zero element; and the operator $\left(1 - 2a_{rj} \frac{\partial}{\partial a_{rj}}\right)$ acting on A will change the sign of the element a_{rj}.

Definition: If M is a minor of order k of the determinant A of order n, then the minor of order n-k formed by striking out from A all the rows and columns represented in M is called the *complement* of M.

Definition: If M is the minor of order k of A in which the rows $r_1, r_2, \cdots, r_k$ and columns $j_1, \cdots, j_k$ are represented, then the *algebraic complement* of M is defined by the equation

Alg. Compl. of M = $(-1)^{r_1+\cdots+r_k+j_1+\cdots+j_k}$ [Compl. of M] .

This algebraic complement will be represented throughout this thesis by the notation $A_{r_1 r_2 \cdots r_k, j_1 j_2 \cdots j_k}$.

In the expansion of the determinant A of order n the coefficient of $a_{r_1 j_1} a_{r_2 j_2} \cdots a_{r_k j_k}$ where $k<n$, is the minor formed from A by striking out rows $r_1, \cdots, r_k$ and columns $j_1, \cdots, j_k$ the sign being $(-1)^{r_1+\cdots+r_k+j_1+\cdots+j_k}$. This coefficient is identically $A_{r_1 \cdots r_k, j_1 \cdots j_k}$ as just defined. It becomes evident at once that

$$\left(\frac{\partial}{\partial a_{\pi_k j_k}}\right) \cdots \left(\frac{\partial}{\partial a_{\pi_2 j_2}}\right)\left(\frac{\partial}{\partial a_{\pi_1 j_1}} A\right) = A_{\pi_1 \cdots \pi_k, i_1 \cdots i_k} .$$

To condense the writing of successive operators we will adopt a new symbol. If ψ_i is an operator then $\overset{K}{\underset{i=1}{\Pi}}(\psi_i)$ will represent the repeated operator $(\psi_K)\cdots\cdots(\psi_3)(\psi_2)(\psi_1)$. Where the order of operation is significant it will be understood that the order is the same as the order of subscripts of the operators ψ_i.

Theorem 1. *Let $A = |a_{rj}|$ be a determinant of order n and D be a determinant of the same order, formed from A by substituting zero elements for the k elements $a_{r_1 j_1}, \cdots, a_{r_k j_k}$ no two of which are in the same row or column; then*

$$D = A - \sum a_{r_1 j_1} A_{r_1 j_1} + \sum a_{r_1 j_1} a_{r_2 j_2} A_{r_1 r_2, j_1 j_2} - \cdots + (-1)^k a_{r_1 j_1} \cdots a_{r_k j_k} A_{r_1 \cdots r_k, j_1 \cdots j_k} \tag{1}$$

where $\sum a_{r_1 j_1} \cdots a_{r_m j_m} A_{r_1 \cdots r_m, j_1 \cdots j_m}$ represents the sum of $\binom{k}{m}$ terms, each of which is formed by taking the product of m of the k deleted elements into the algebraic complement of the minor in which the m rows and m columns containing these m elements are represented.

By hypothesis the k indices $r_1, \cdots, r_k$ are all distinct, the indices $j_1, \cdots, j_k$ are all distinct, and $k \leq n$.

The operator $\left(1 - a_{r_1 j_1} \frac{\partial}{\partial a_{r_1 j_1}}\right)$ when applied to A replaces the element $a_{r_1 j_1}$ by zero; and the repeated operator $\prod_{m=1}^{k} \left(1 - a_{r_m j_m} \frac{\partial}{\partial a_{r_m j_m}}\right)$ when applied to A replaces the elements $a_{r_1 j_1}, \cdots, a_{r_k j_k}$ by zeros. Therefore,

$$D = \left[\prod_{m=1}^{k} \left(1 - a_{r_m j_m} \frac{\partial}{\partial a_{r_m j_m}}\right)\right] A \tag{2}$$

Upon expanding the right hand member of (2) the equality takes the form

$$D = \left[1 - \sum a_{r_1 j_1} \frac{\partial}{\partial a_{r_1 j_1}} + \sum a_{r_1 j_1} a_{r_2 j_2} \frac{\partial}{\partial a_{r_1 j_1}} \frac{\partial}{\partial a_{r_2 j_2}} - \cdots + (-1)^k a_{r_1 j_1} \cdots a_{r_k j_k} \frac{\partial}{\partial a_{r_1 j_1}} \cdots \frac{\partial}{\partial a_{r_k j_k}}\right] A \tag{3}$$

which is equivalent to the equality (1) given in the theorem.

If the k elements are chosen from the main diagonal, the theorem becomes the following corollary which was proved by Stetson[4] in 1904.

Corollary I. *If D is the determinant of order n formed by substituting zero elements for the first k elements of the main diagonal in A, then*

$$D = A - \Sigma a_{11}A_{11} + \Sigma a_{11}a_{22}A_{12,12} - \cdots + (-1)^K a_{11}\cdots a_{KK}A_{1\ldots K,1\ldots K} . \quad (4)$$

If in addition to the above condition we assume $k = n$, a second form results which is identical with the original theorem on this type of expansion proved by Muir[5] in 1898.

Corollary II. *If D is the invertebrate determinant of order n formed by substituting zero elements for the main diagonal elements in A, then*

$$D = A - \Sigma a_{11}A_{11} + \Sigma a_{11}a_{22}A_{12,12} - \cdots + (-1)^K a_{11}\cdots a_{nn} . \quad (5)$$

Theorem 2. *Let* $A = |a_{rj}|$ *be a determinant of order* n, *and* D *be a determinant of the same order formed from* A *by substituting zero elements for the* k *elements,* $a_{r_1 j_1}, \cdots, a_{r_k j_k}$ *no two of which are in the same row or column; then*

$$D = A - a_{r_1 j_1}|A_{r_1 j_1}|_0 - a_{r_2 j_2}|A_{r_2 j_2}|_0 - \cdots\cdots - a_{r_k j_k}|A_{r_k j_k}| \quad (6)$$

where $|A_{r_m j_m}|_0$ *represents the determinant formed from* $A_{r_m j_m}$, *the cofactor of* $a_{r_m j_m}$, *by substituting zeros for the elements* $a_{r_{m+1} j_{m+1}}, \cdots\cdots, a_{r_k j_k}$.

If we add to $\left[\prod_{i=1}^{K}\left(1 - a_{\eta_i j_i}\frac{\partial}{\partial a_{\eta_i j_i}}\right)\right]A$, which is the right hand member of (2), the expression $a_{\eta_1 j_1}\frac{\partial}{\partial a_{\eta_1 j_1}}\left[\prod_{i=2}^{K}\left(1 - a_{\eta_i j_i}\frac{\partial}{\partial a_{\eta_i j_i}}\right)\right]A$ the result is $\left[\prod_{i=2}^{K}\left(1 - a_{\eta_i j_i}\frac{\partial}{\partial a_{\eta_i j_i}}\right)\right]A$. From K−1 similar additions will result $\left(1 - a_{\eta_K j_K}\frac{\partial}{\partial a_{\eta_K j_K}}\right)A$ which, if the term $a_{\eta_K j_K}\frac{\partial}{\partial a_{\eta_K j_K}}A$ be added will equal A. That is

$$\left[\prod_{i=1}^{K}\left(1 - a_{\eta_i j_i}\frac{\partial}{\partial a_{\eta_i j_i}}\right)\right]A + \sum_{m=1}^{K-1}\left[a_{\eta_m j_m}\frac{\partial}{\partial a_{\eta_m j_m}}\prod_{i=m+1}^{K}\left(1 - a_{\eta_i j_i}\frac{\partial}{\partial a_{\eta_i j_i}}\right)\right]A$$

$$+ a_{\eta_K j_K}\frac{\partial}{\partial a_{\eta_K j_K}}A = A \quad (7)$$

Upon substituting D for the first term of the equality and transposing the rest of the left hand member we have

$$D = A - \sum_{m=1}^{K-1}\left[a_{\eta_m j_m}\frac{\partial}{\partial a_{\eta_m j_m}}\prod_{i=m+1}^{K}\left(1 - a_{\eta_i j_i}\frac{\partial}{\partial a_{\eta_i j_i}}\right)\right]A$$

$$- a_{\eta_K j_K}\frac{\partial}{\partial a_{\eta_K j_K}}A \quad (8)$$

which equation is equivalent to (6).

We can at once state two important corollaries of this theorem.

Corollary I. *If D be the determinant of order n formed from A by substituting zero elements for the first k elements of the main diagonal, then*

$$D = A - a_{11}|A_{11}|_0 - a_{22}|A_{22}|_0 - \cdots\cdots - a_{kk}A_{kk} \quad . \qquad (9)$$

Corollary II. *If D be the determinant of order n formed from A by substituting zero elements for the n elements of the main diagonal, then*

$$D = A - a_{11}|A_{11}|_0 - a_{22}|A_{22}|_0 - \cdots\cdots - a_{nn}A_{nn} \quad . \qquad (10)$$

This last corollary was stated as a theorem by Horta[6] in 1890.

Theorem 3. *Let* $A = |a_{rj}|$ *be a determinant of order* n *and* Δ *be a determinant formed from* A *by deleting* k *elements* $a_{r_1 j_1}, \cdots, a_{r_k j_k}$, *no two of which are in the same row or column, and substituting respectively the elements* $b_{r_1 j_1}, \cdots, b_{r_k j_k}$ *where* $b_{rj} = a_{rj} - x_{rj}$; *then*

$$A = \Delta + \sum x_{r_1 j_1} \Delta_{r_1 j_1} + \sum x_{r_1 j_1} x_{r_2 j_2} \Delta_{r_1 r_2, j_1 j_2} + \cdots + x_{r_1 j_1} \cdots x_{r_k j_k} \Delta_{r_1 \cdots r_k, j_1 \cdots j_k} \tag{11}$$

where $\sum x_{r_1 j_1} \cdots x_{r_m j_m} \Delta_{r_1 \cdots r_m, j_1 \cdots j_m}$ *represents the sum of* $\binom{k}{m}$ *terms, each consisting of the product of* m *of the* k *elements* $x_{r_1 j_1}, \cdots, x_{r_k j_k}$, *into the algebraic complement of the minor in which the rows* $r_1, \cdots, r_m$ *and columns* $j_1, \cdots, j_m$ *are represented. The elements* x_{rj} *are any* k *variables, which are mutually independent and independent of the elements* a_{rj}.

The single operator $\left(1 + X_{n_1 j_1} \frac{\partial}{\partial b_{n_1 j_1}}\right)$ applied to Δ will substitute the element $a_{r_1 j_1}$ for $b_{r_1 j_1}$; and the repeated operator $\prod_{m=1}^{K} \left(1 + X_{n_m j_m} \frac{\partial}{\partial b_{n_m j_m}}\right)$ applied to Δ will substitute "a" elements for the respective "b" elements hence:

$$A = \left[\prod_{m=1}^{K} \left(1 + X_{n_m j_m} \frac{\partial}{\partial b_{n_m j_m}}\right)\right] \Delta \tag{12}$$

Upon expanding, the right hand member of (12) takes the form

$$\left(1 + \sum X_{n_1 j_1} \frac{\partial}{\partial b_{n_1 j_1}} + \sum X_{n_1 j_1} X_{n_2 j_2} \frac{\partial}{\partial b_{n_1 j_1}} \frac{\partial}{\partial b_{n_2 j_2}} + \cdots + X_{n_1 j_1} \cdots X_{n_K j_K} \frac{\partial}{\partial b_{n_1 j_1}} \cdots \frac{\partial}{\partial b_{n_K j_K}}\right) \Delta \tag{13}$$

which is identically the right hand member of the equality (11).

Theorem 4. *If* A, Δ, *and* $\Delta_{r_1 \cdots r_m, j_1 \cdots j_m}$ *are as defined in the preceeding theorem, and* $D_{r_1 \cdots r_m, j_1 \cdots j_m}$ *is the determinant formed from* $\Delta_{r_1 \cdots r_m, j_1 \cdots j_m}$ *by substituting zeros for the elements* $b_{r_1 j_1}, \cdots, b_{r_k j_k}$ *; then*

$$A = D + \Sigma a_{r_1 j_1} D_{r_1 j_1} + \Sigma a_{r_1 j_1} a_{r_2 j_2} D_{r_1 r_2, j_1 j_2} + \cdots + a_{r_1 j_1} \cdots a_{r_k j_k} A_{r_1 \cdots r_k, j_1 \cdots j_k} . \tag{14}$$

It should be noted that this definition of D is equivalent to the definition used in the first two theorems.

This theorem follows at once if we consider the limiting form of the preceeding theorem when the variables $x_{r_m j_m}$ approach as limits the respective elements $a_{r_m j_m}$.

Stetson[4], in proving a generalization of Muir's Expansion Theorem (Th.1.,Cor.I.) , obtained zero elements in his determinants by employing a compound element similar to b_{rj}.

Corollary I. *If D be the determinant of order n formed from A by substituting zero elements for the first k elements of the main diagonal, then*

$$A = D + \Sigma a_{11} D_{11} + \Sigma a_{11} a_{22} D_{12\ 12} + \cdots + a_{11} \cdots a_{KK} A_{12 \cdots K, 12 \cdots K} . \tag{15}$$

Corollary II. *If D be the determinant of order n formed from A by substituting zero elements for the n elements of the main diagonal, then*

$$A = D + \Sigma a_{11} D_{11} + \Sigma a_{11} a_{22} D_{12\ 12} + \cdots + a_{11} \cdots a_{nn} . \tag{16}$$

Corollary II was first stated as a theorem by Cayley[7] in 1848 and is known as Cayley's Expansion Theorem. Cor.I which is more general than II was first stated as a theorem by Muir[8] in 1898.

Theorem 5. *Let* $A = |a_{rj}|$ *be a determinant of order* n *and* δ *be a determinant of the same order, formed from* A *by changing the signs of the* k *elements* $a_{r_1 j_1}, \cdots, a_{r_k j_k}$ *no two of which are in the same row or column; then*

$$\delta = A - 2\Sigma\, a_{r_1 j_1} A_{r_1 j_1} + 2^2 \Sigma\, a_{r_1 j_1} a_{r_2 j_2} A_{r_1 r_2, j_1 j_2} - \cdots + (-2)^k a_{r_1 j_1} \cdots a_{r_k j_k} A_{r_1 \cdots r_k, j_1 \cdots j_k} \tag{17}$$

where the terms in the expansion are as defined in the first theorem.

The single operator $\left(1 - 2\, a_{r_1 j_1} \frac{\partial}{\partial a_{r_1 j_1}}\right)$ when applied to A changes the sign of $a_{r_1 j_1}$; and the repeated operator $\prod_{m=1}^{k} \left(1 - 2\, a_{r_m j_m} \frac{\partial}{\partial a_{r_m j_m}}\right)$ when applied to A changes the signs of the k elements $a_{r_1 j_1}, \cdots, a_{r_k j_k}$. Hence

$$\delta = \left[\prod_{m=1}^{k} \left(1 - 2\, a_{r_m j_m} \frac{\partial}{\partial a_{r_m j_m}}\right)\right] A \tag{18}$$

Upon expanding the right hand member of (18) the equality takes the form

$$\delta = \left[1 - 2\sum a_{r_1 j_1} \frac{\partial}{\partial a_{r_1 j_1}} + 2^2 \sum a_{r_1 j_1} a_{r_2 j_2} \frac{\partial}{\partial a_{r_1 j_1}} \frac{\partial}{\partial a_{r_2 j_2}} - \cdots + (-2)^k a_{r_1 j_1} \cdots a_{r_k j_k} \frac{\partial}{\partial a_{r_1 j_1}} \cdots \frac{\partial}{\partial a_{r_k j_k}}\right] A \tag{19}$$

which is equivalent to the equality of 17.

Theorem 6. *If A is a determinant of order n and B,C,···,L are p other determinants of the same order, the sum of the* $[(n)(n-1)\cdots(n-p+1)]$ *determinants composed of* $n-p$ *rows from A and one row from each of the p determinants B,C,···,L is equal to that of the same number of determinants composed of* $n-p$ *columns from A and one column from each of the p determinants B,C,···,L. The columns and rows are to occupy the same places in the new determinants as in the originals, and* $p \leqq n-1$.

The elements of the determinants A,B,C,···,L will be designated by $a_{rj}, b_{rj}, c_{rj}, \cdots, l_{rj}$. Also in the formation of any one of the new determinants let $r_1, r_2, \cdots, r_p$ or $j_1, \cdots, j_p$ designate the p rows or columns of A into which rows or columns from the p given determinants are to be substituted.

Case I. The elements $a_{rj}, \cdots, l_{rj}$ are assumed to be mutually independent.

Since the order of partial differentiation is immaterial it is at once evident that

$$\left[\sum_{\eta_p=1}^{n}\sum_{j_p=1}^{n}\cdots\cdots\sum_{\eta_1=1}^{n}\sum_{j_1=1}^{n} l_{\eta_p j_p}\cdots\cdots C_{\eta_2 j_2}\, a_{\eta_1 j_1}\frac{\partial}{\partial a_{\eta_p j_p}}\cdots\cdots\frac{\partial}{\partial a_{\eta_1 j_1}}\right]A$$

$$=\left[\sum_{j_p=1}^{n}\sum_{\eta_p=1}^{n}\cdots\sum_{j_1=1}^{n}\sum_{\eta_1=1}^{n} l_{\eta_p j_p}\cdots\cdots C_{\eta_2 j_2}\, a_{\eta_1 j_1}\frac{\partial}{\partial a_{\eta_p j_p}}\cdots\cdots\frac{\partial}{\partial a_{\eta_1 j_1}}\right]A. \tag{20}$$

Both members of the equality sum the same $[(n)(n-1)\cdots(n-p+1)]^2$ terms, each term consisting of the product $b_{r_1 j_1}, c_{r_2 j_2}, \cdots, l_{r_p j_p}$ into the algebraic complement of the minor in which the rows $r_1, \cdots, r_p$ and columns $j_1, \cdots, j_p$ are represented.

Upon rewriting the equality of (20) we obtain

$$\left[\left(\sum_{r_p=1}^{n}\sum_{j_p=1}^{n} l_{r_p j_p}\frac{\partial}{\partial a_{r_p j_p}}\right)\cdots\cdots\left(\sum_{r_1=1}^{n}\sum_{j_1=1}^{n} b_{r_1 j_1}\frac{\partial}{\partial a_{r_1 j_1}}\right)\right]A$$

$$=\left[\left(\sum_{j_p=1}^{n}\sum_{r_p=1}^{n} l_{r_p j_p}\frac{\partial}{\partial a_{r_p j_p}}\right)\cdots\left(\sum_{j_1=1}^{n}\sum_{r_1=1}^{n} b_{r_1 j_1}\frac{\partial}{\partial a_{r_1 j_1}}\right)\right]A \qquad (21)$$

The operator $\left(\sum_{j_1=1}^{n} b_{r_1 j_1}\frac{\partial}{\partial a_{r_1 j_1}}\right)$ applied to A replaces the elements of row r_1 by the elements $b_{r_1 1}, b_{r_1 2}, \cdots, b_{r_1 n}$, and the operator $\left(\sum_{r_1=1}^{n}\sum_{j_1=1}^{n} b_{r_1 j_1}\frac{\partial}{\partial a_{r_1 j_1}}\right)$ applied to A results in n distinct determinants each having one row deleted and the same row of elements from B substituted.

Again the operator $\left(\sum_{r_2=1}^{n}\sum_{j_2=1}^{n} c_{r_2 j_2}\frac{\partial}{\partial a_{r_2 j_2}}\right)\left(\sum_{r_1=1}^{n}\sum_{j_1=1}^{n} b_{r_1 j_1}\frac{\partial}{\partial a_{r_1 j_1}}\right)$ applied to A results in (n)(n-1) distinct determinants each having two rows deleted and the same rows of elements, one from each B and C, substituted for the deleted rows.

From this interpretation it follows that the left hand member of (21) is the sum of [(n)(n-1)$\cdots$(n-p+1)] distinct determinants each composed of n-p rows from A and one row from each of the p determinants B,C,$\cdots$,L, while the right hand member is the sum of [(n)(n-1)$\cdots$(n-p+1)] distinct determinants each composed of n-p columns from A and one column from each of the p determinants B,C,$\cdots$,L.

Case II. The elements $b_{rj}, c_{rj}, \cdots, l_{rj}$ are not assumed to be mutually independent or independent of the elements a_{rj}. The elements a_{rj} are assumed to be mutually independent.

It is evident that the fundamental identity of (20) still holds since the multipliers $b_{rj}, c_{rj}, \cdots, l_{rj}$ occur after the partial derivatives have operated on A.

In the preceeding case the assumptions as to independence permitted any of the elements $b_{rj}, \cdots, l_{rj}$ to occur before any partial derivative operation with respect to a_{rj}, hence the validity of equation (21).

To rewrite (20) in this instance we must first arbitrarily decree that no partial derivative is to operate on any element which has been substituted into the determinant by a previous operation, else the equality of (21) will not be the fundamental identity in a different form but probably some false statement. After placing these restrictions on the operators the rest of the proof follows just as in the preceeding case.

An interesting example of the theorem occurs when p+1 = n in which case one row is taken from each of the n given determinants for the construction of each new one.

Another example of interest is obtained if p = 1 in which instance one row is taken from B and the remaining rows from A in the construction of each new determinant. This example was stated as a theorem by Muir[9] in 1888.

If B=C=D=$\cdots$=L the theorem reduces at once to Corollary I. *If A and B are two determinants of order n, the sum of the* $\binom{n}{p}$ *determinants of the same order composed of n−p*

rows from A and of p rows from B is equal to that of the $\binom{n}{p}$ *determinants composed of* $n-p$ *columns from A and of p columns from B.*

This corollary was given as a lemma by J. Deruyts[10] in 1883.

A second corollary is obtained if certain groups of the determinants are identical. Suppose that k_1 determinants are identical with B, k_2 determinants are identical with C, and so on, where $\Sigma k_i \leq n$.

Corollary II. *If A,B,C,⋯⋯,L are p+1 determinants of order n, the sum of the* $\dfrac{(n)(n-1)\cdots\cdots(n-p+1)}{k_1!\,k_2!\cdots\cdots k_p!}$ *determinants each composed of* k_1 *rows from B,* k_2 *rows from C,* ⋯⋯⋯⋯, k_p *rows from L , and the remaining* $n - \Sigma k_i$ *rows from A, is equal to the sum of the same number of determinants each composed of* k_1 *columns from B,* k_2 *columns from C,* ⋯⋯, k_p *columns from L, and the remaining* $n - \Sigma k_i$ *columns from A. The columns and rows are to occupy the same places in the new determinants as in the originals and* $\Sigma k_i \leq n$.

Theorem 7. *If* $\delta_{k_i} = |a_{gh}|$, $g = 1, 2, \cdots, n$, $h = k_i, k_i+1, \cdots, n, n+1, 1, 2, \cdots, k_i-2$, *and if* $k_1, k_2, \cdots, k_p$ *are* p *distinct numbers different from unity, then the sum of the* $p!\binom{n}{p}$ *determinants of order* n, *composed of* $n-p$ *rows from* A *and one row from each of the* p *determinants* $\delta_{k_1}, \delta_{k_2}, \cdots, \delta_{k_p}$ *is equal to* $(-1)^{\Sigma k}\, p!\, \Delta_{n+p+1-\Sigma k}$, *where* Δ_r *denotes the determinant obtained by deleting the* r *th column of the* n *by* $n+1$ *array* $\|a_{gh}\|$, $g = 1, \cdots, n$, $h = 1, \cdots, n+1$. *If the number* $n+p+1-\Sigma k$ *is not between* 1 *and* $n+1$ *then a convenient multiple of* $n+1$ *is to be added to it.*

If theorem 6 be applied to the conditions of this theorem it is evident that all of the $p!\binom{n}{p}$ determinants so obtained are zero due to duplicate columns except for those $p!$ determinants whose columns upon substitution satisfy the conditions

$$\begin{aligned}
&j_1 + k_1 - 1 = n + 1 \\
&j_2 + k_2 - 1 = j_1 = n + 2 - k_1 \\
&j_3 + k_3 - 1 = j_2 = n + 3 - k_1 - k_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&j_p + k_p - 1 = n + p - \Sigma k \;;\; j_p = n + (p+1) - \Sigma k\,.
\end{aligned}$$

Each of these $p!$ determinants then is equal to $\Delta_{n+p+1-\Sigma k}$ except for the sign which is determined by $(-1)^{\Sigma k}$.

If in the statement of the theorem we assume $k_1 = k_2 = \cdots = k_p$ a corollary follows at once.

Corollary I. *The sum of the* $\binom{n}{p}$ *determinants of order* n *composed of* $n-p$ *rows from* A *and of* p *rows from* δ_k *is equal to* $(-1)^{kp}\, \Delta_{n+p+1-pk}$. This theorem[10] and corollary[11] were theorems of Deruyts published in 1883 and 1881. Le Paige[12] proved the corollary for the special case p=1, k=2 in 1880.

Theorem 8. *If* $A = |a_{rj}|$ *is a determinant of order n, and if from A a new determinant be formed by altering the order of the elements in the first row, a second determinant by making the same alteration in the elements of the second row, and so on, the sum of the n determinants thus derived is m times the original, where m is the number of elements of a row that after the change of order occupy the same positions as before.*

To prove this theorem first construct a new determinant B from A by altering the order of the columns. Then the sum of the n determinants each composed of one row from B and the remaining rows from A is equal to the sum of the n determinants each composed of one column from B and the remaining columns from A. But all the n determinants composed of one column from B and the remaining columns from A are zero due to duplicate columns except for those m determinants in which the column taken from B is identical with the one it replaces in A.

This theorem is equivalent to a theorem given by Deruyts[1,3] in 1885.

Theorem 9. *If* $A = |a_{rj}|$ *is a determinant of order* n *and*

$$B = \begin{vmatrix} \lambda_1 a_{1n} & \lambda_2 a_{11} & \lambda_3 a_{12} & \cdots & \lambda_n a_{1n-1} \\ \lambda_1 a_{2n} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1 a_{nn} & \lambda_2 a_{n1} & \lambda_3 a_{n2} & \cdots & \lambda_n a_{nn-1} \end{vmatrix}$$

then the sum of the $\binom{n}{p}$ *determinants, each composed of* p *rows from* B *and* $n-p$ *rows from* A, *is equal to zero. The rows are to occupy the same positions in the new determinants as in the originals, and* $p < n$.

Applying the statement of theorem 8 corollary I it is evident that the determinants obtained by inserting columns from B into A are all zero due to duplicate columns.

This theorem was stated as a theorem for the special case of p=1, n=4 by Hammond[14] in 1881.

BIBLIOGRAPHY

1. Stouffer, E.B.," A simple derivation of Kronecker's relations among the minors of a symmetric determinant.", *Proceedings of the National Academy of Sciences., Vol.XII, No.I, pp.63-64, (1926).*

2. Jacobi, . .," Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi.", *Crelle's Journal., Vol.XXVII, pp.199-268, Vol.XXIX, pp.213-279, pp.333-376, (1845).*

3. Bellavitis, G.,"Sposizione elementare della teorica dei determinante.", *Memorie del Reale Istituto Veneto di Scienze, Lettere, ed Arti., Vol.VII, pp.67-144, (1857).*

4. Stetson, O.S.,"Note on the expansion of devertebrate determinants.",*American Mathematical Monthly, Vol.XI, pp.166-168, (1904).*

5. Muir, T.,"The relations between the coaxial minors of a determinant of the fourth order.", *Transactions of the Royal Society of Edinburgh, Vol.XXXIX, pp.323-339,(1898).*

6. Horta, F. Da P., "Nota sobre os determinantes.", *Jornal de Sciencias Mathematicas, Physicas e Naturaes (Lisboa)., (2) Vol.V, pp.67-73, (1890).*

7. Cayley, A., "Sur les determinants Cauches.", *Collected Works., Vol.I, p.410, (1848).*

8. Muir, T., "The expression of any bordered skew determinant as a sum of products of Pfaffians.", *Royal Society of Edinburgh, Proceedings., Vol.XXI, pp.342-359, (1896).*

9. Muir, T., "On vanishing aggregates of determinants.", *Royal Society of Edinburgh, Proceedings., Vol.XV,pp.96-105, (1888).*

10. Deruyts, J., "Sur certaines sommes de determinants.", *Memoires de la Societe des Sciences de Liege., (2) Vol.X, No. 4, (1883).*

11. Deruyts, J., "Note sur quelques proprietes des determinants multiples.", *Memoires de la Societe des Sciences de Liege., (2) Vol.IX, No.7, (1881).*

12. Paige, C. Le, "Sur quelques points de la theorie des formes algebrique.", *Memoires de la Societe des Sciences de Liege., (2) Vol.IX, No.4, (1880).*

13. Deruyts, J., "Sur L'analyse combinatoire des determinants.", *Memoires de la Societe des Sciences de Liege, (2) Vol.XI,(1885).*

14. Hammond, J.,"Question 5815." *Educational Times, Vol.XXXIV, p.218, Mathematics from Educational Times , Vol.XXXIV, pp.56, (1881).*