

NUMBER SCALES

by

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A. B., University of Kansas, June, 1928

Submitted to the Department of
Mathematics and the Faculty of
the Graduate School of the Un-
iversity of Kansas in partial
fulfillment of the requirements
for the degree of Master of Arts

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July 18, 1931

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I. INTRODUCTION

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The idea of number dates back to the earliest forms of society. It is such an elementary idea that it is difficult to define scientifically. The oral language of number arithmetic or numeration was developed first as the savage, whose livelihood was made by the art of fishing or hunting, was eager on his return home to count his possessions. Likewise, the leader of a troop was obliged to reckon up the slain and captives after a conflict. As long as the numbers were small, they could easily be represented by various emblems such as pebbles, shells, grains, knots in a string, notches on a stick, or by the fingers. To express the larger numbers it became necessary to place the objects or counters in groups or rows. From this grouping of counters developed the written language or notation of arithmetic. It is the purpose of this paper to determine, so far as possible from available literature, the notations which have actually been used, and some of the relations of one scale to another.

II. GENERAL METHOD OF NOTATION

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1. Place Value System. The general method of writing numbers, now adopted by all civilized nations, is the Hindu notation, most often spoken of as the Arabic notation. The fundamental principle
1. Throughout this paper "number" refers to positive integers only, unless otherwise specifically stated.
2. Leslie, page 13.

of this system is the idea of place value. Several characters are chosen to represent the smaller numbers and then they are employed to number the groups, the groups being numbered by the position of the character. Each character has a definite value when it stands alone, and a relative value when used in connection with the other characters.

2. Number of Characters. The number of characters is determined by the number of units in the group. The earliest system of numeration used by the savages was the Binary Scale, which had only two characters. The system that is universally used at the present time is the grouping by tens; hence, the number of characters is ten. The character known as naught or zero is necessary and used in the combination of characters to denote the absence of a group. The characters are called digits, from the Latin "digitus", a finger, the name commemorating the ancient custom of counting by fingers.

3. Origin of our Notation. The origin of our common notation is quite uncertain. The symbols are generally believed to have originated in India,³ to have been carried in an astronomical table to Bagdad in the 8th century, and from there found their way into Europe.⁴ This is not certain, for various authors believe that these numerals did not originate in India at all,⁵ but the evidence still seems much more favorable to the Hindu origin.

1. Tylor, page 247.

2. Smith, page 64.

3. Smith and Karpinski, state in a footnote (page 2) that "Maximus Planudes (c 1350) states that 'the nine symbols come from the Indians'".

4. Smith and Karpinski, pp. 63 and 99.

5. Cajori, page 459.

The decimal basis of our scale of notation is not essential but merely accidental. Various numbers have been used as bases of systems of notation.

III. SYSTEMS OF NOTATION

1. Binary Scale. The earliest and simplest method¹ of numeration was by pairs designated as the Binary Scale. It is still familiar among sportsmen who reckon by braces and couples. Some traces of the binary scale are found in the early monuments in China. Two centuries ago Leibnitz,² the celebrated philosopher, highly recommended the binary scale because it enables us to perform all the operations in symbolic arithmetic by addition and subtraction. Another reason for advocating this system was the theological idea associated with it, as unity was considered the symbol of Deity and all numbers were formed out of zero and unity. This scheme of notation was found at that time to be used by the tribes of Australia, which are classed among the lowest and the least intelligent of aboriginal races of the world. Traces of the binary scale are also present among the negro tribes of Africa, the Indian tribes of America, and the natives of South America. The binary scale inevitably suggests a low degree of mental development and is more pronounced among the Australians than any other extensive number of kindred races.

2. Ternary Scale. Another step in counting was by threes. This has been preserved by sportsmen under the term "leash,"³ meaning

1. Leslie, Page 2.

2. Conant, pp. 102-105.

3. Leslie, page 2.

a string by which three dogs and no more can be held at once in the hand. The Betoga¹ and Kamilaroi² scales are examples of the ternary scale. The latter is usually given as an example of a binary formation, but is partly ternary; for instance, the word for six, "guliba guliba", 3 - 3, is purely ternary. An occasional ternary trace is also found in number systems otherwise decimal or quinary vigesimal as found among the Haida Indians of British Columbia. In the Wakka dialect,³ found on the Burnett River, Australia, a single ternary numeral is found.

3. Quaternary Scale. Numbering by fours was evidently suggested by taking a pair of objects in each hand. The English fishermen used this method of counting their fish and spoke of every double pair of herring, for instance, as a "throw" or "cast".⁴ The term "warp," which originally meant to throw, is employed to denote four in various articles of trade. It is alleged that two of the lowest races of savages, the Guaranis and the Sulus,⁵ inhabiting the forests of South America, count only by fours; that is, they express five by four and one, six by four and two, etc. It has been inferred, from the passages of Aristotle, that a certain tribe of Thracians were accustomed to use only the quaternary scale. Instances of quaternary numeration are less rare than those of ternary. Quaternary traces⁶ are

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1. Conant, page 112, refers to Brinton, "Studies in South American Native Languages," page 67.
 2. Tylor, page 243.
 3. Curr, page 108.
 4. Leslie, page 19.
 5. Tylor, page 266.
 6. Conant, pages 113-114.

repeatedly to be found among the Indian languages of British Columbia. The preference for four is said to have existed in primitive times in the languages of Central Asia.

In the Hawaiian and a few other languages of the islands of the central Pacific, where in general the number systems employed are decimal, we find a most interesting case of the development, within number scales already well established, of both binary and quaternary systems. In Tahitian, Rarotongan, Mangarevan, and other dialects² found in neighboring islands, certain of the higher units which originally signified 10, 100, 1000, have become doubled in value and now stand for 20, 200, 2000. In Hawaiian and other dialects they have again doubled, and there they stand for 40, 400, 4000. The decimal-binary system appears more regularly on the southern islands, while the decimal-quaternary system is more prevalent on the northern or Nukuhivan islands of the Archipelago. The decimal-quaternary system, such as developed in Hawaiian dialect, is found nowhere else in the world except in neighboring regions. These Pacific island scales have been developed to a very high limit--in some cases, into the millions.

The Luli of Paraguay¹ show a decided preference for base four, which gives way only when they reach ten. All numbers above that point belong rather to decimal than to quaternary notation.

4. Quinary Scale. The quinary system has as its foundation the practice of counting the fingers of one hand. It has been found by travelers that this system has been adopted by various nations.

1. Conant, 118, refers to Peacock, "Arithmetic," Encyclopedia Metropolitana, page 479.

2. Conant, pages 113-114.

Certain tribes reckon by fives, which they call hands. Mungo
¹
 Park found this system practiced among the Zolofs and Foulahs,
 tribes of Africa, which designated ten by two hands, fifteen
 by three hands, etc. Some of the natives of Persia seemed to
 have used this system and it is even partially used in England
 among wholesale traders.

²
 If the quinary number system is extended, it usually
 merges into either the decimal or the vigesimal system, the
 result being a compound of two, and sometimes of three, systems
 in one scale. A pure quinary or vigesimal is exceedingly rare.
 The Eskimos of Point Barrow, though their systems may properly
 be classed as mixed systems, exhibit a decided preference for
 five as a base. One of the purest examples of quinary numera-
 tion ³ is that furnished by one of the Betoga dialects of South
 America, making no use of either "man" or "foot", but reckoning
 solely by fives or hands.

5. Senary Scale. The senary scale ⁴ is said to have been adopted
 at one time in China by the order of a capricious tyrant, who,
 having conceived an astrological fancy for the number six, com-
 manded its several combinations to be used in all concerns of
 business or learning throughout his vast empire. Pott remarks ⁵
 that the Bolans, of western Africa, appear to make some use of
 six as their number base, but their system taken as a whole is

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1. Brooks, page 122.
 2. Conant, page 137.
 3. Conant, page 57.
 4. Leslie, page 21.
 5. Conant, page 120.

really a quinary-decimal. The language of the Sundas,¹ or mountaineers of Java, contains traces of senary counting.

6. Septenary Scale. The number seven, according to Brooks, has been regarded as a kind of magic number. Its frequent use in the Bible has caused it to be regarded as a sacred number, the basis of a celestial system.

7. Octary Scale. The octary scale² is said to have been used by the ancient Saxons, but how, or for what purpose, is not stated. In the history of the Aryan race, this scale was regarded as the predecessor of the decimal scale. The use and importance of the number eight as a base in China, India, central Asia, among some of the islands of the Pacific, and in central America, has led Conant to believe that there was a time long before the beginning of recorded history, when eight was the common number base of the world.

8. Nonary Scale. The Senary, Septenary, Octary, and Nonary Scales³ (formed on the numbers six, seven, eight, and nine, respectively) so far as can be learned have never been used excessively by any nation or tribe. No natural reason exists for the choice of any of these numbers as a basis.

Many years ago a statement⁴ appeared which at once attracted much attention and curiosity. It was to the effect

1. Conant, 120, refers to Crawford, History of Indian Archipelago, Vol. 1, page 256.

2. Lappenberg, page 82.

3. Conant, page 119; Brooks, page 123.

4. Conant, page 123. Footnote--Information upon which above statements are based was obtained from Mr. U. L. Williams of Gisborne, N.Z.

that the number eleven was used as the basis of the numeral system of the Maoris, the aboriginal inhabitants of New Zealand. For a long time the Maori scale was looked upon as something quite exceptional and outside all ordinary rules of number system formation. Upon a closer investigation of their language and customs it was found that the system was a simple decimal system, and that the error arose from the following habit. When counting, the Maoris would put aside one to represent each ten, and then those so set aside would be afterward counted to ascertain the number in the heap. Early observers among this people, seeing them count ten and then set one aside, at the same time pronouncing a word, imagined that the word meant eleven. This misconception found its way into the early New Zealand dictionary, but was later corrected.

9. Duodecimal Scale. The scale based on twelve¹ may be supposed to have had its origin from celestial phenomena, there being twelve lunar months in a solar year. This scale appeared in the adoption of subdivisions of units of measure and weight and is employed in wholesale business to some extent. An example of the use of the duodecimal was long sought for in vain among the primitive races. Humboldt, in commenting on the number systems of various peoples he had visited during his travels, remarks that no race had ever used exclusively the best of bases, twelve. But, it was announced²

1. Leslie, page 25.

2. Conant, page 133, where reference is made to Schubert, H., quoting Robert Fliegel, In Neumayer's Anleitung zu Wissenschaftlichen Beobachtung auf Reisen, Vol. 11, page 290.

that the discovery of such a tribe had actually been made, and that the Aphos of Benue¹¹, an African tribe, count to twelve by simple words, and then for thirteen say 12 and 1, for fourteen, 12 and 2, etc.

10. Vigesimal System. The vigesimal system based on twenty, like the quinary and decimal scales, probably was formed by counting the toes in addition to the fingers. The language of many tribes indicates that this method has been used. It is believed to have been used by the inhabitants of the peninsula of Kamschatka,¹ the natives of Barbadoes, and by tribes on other islands of the Caribbean Sea. Some of the European countries have made use of this system. Reckoning by scores² seems to have prevailed among the Scandinavian nations, and the descendants of the ancient Celts. The inhabitants of Biscay and of Armorica are said to reckon like the Mexicans, by powers of twenty, or the terms of progressive scores.

11. Denary Scale. The denary scale is the system which has prevailed among all civilized nations. Its universal use and adoption manifest the existence of some common principle of numbering familiar in earlier periods of society, namely, the practice of reckoning by counting the fingers of both hands. Among the various tribes and nations³ which employed this system were the Muysca Indians of Bogota and other tribes of South America. The Peruvian

1. Brooks, page 125.

2. Leslie, page 27.

3. Leslie, pages 23-24.

system¹ was a pure decimal system.

The universal adoption and use of the decimal system among all civilized nations might lead some persons to regard this basis as the perfection of simplicity and utility. It has been shown that this base is entirely arbitrary. If a new basis were to be selected by mathematicians, familiar with the properties of numbers, several considerations would undoubtedly lead them to adopt some scale other than the decimal system. Some of the disadvantages² of the decimal basis are the following:

First, the decimal scale is unnatural. It has been urged that it is the most natural that could have been chosen. However, there is nothing natural about it except the fingers and they are grouped by fours instead of fives. In nature and art things seldom if ever are seen to exist in tens. Nature usually groups in pairs, in threes, in fours, in fives, and in sixes. Man doubles, triples, and quadruples his units; he divides them into halves, thirds, and quarters, but where does he estimate by tens, or tenths?

Second, the decimal method is unscientific. The confused idea of the relation of the base of the scale to the mode of notation has led some to suppose that the decimal system is

1. Conant, page 206.

2. Brooks, pages 114-115.

scientific. The decimal scale originated by mere chance and has no relation whatsoever to the Hindu notation of place value by tens, hundreds, etc.

Third, the decimal scale is inconvenient. One of the essentials for a base is the property of being divisible into a number of smaller numbers, and the other is that the number be neither too large nor too small. The number ten permits only two such divisions, the half and the fifth. The third, fourth, and sixth are not exact parts of the denary base; in consequence of which it is inconvenient to express fractions in the scale. If twelve were the basis, the half, third, fourth, and sixth could all be expressed in a single place; whereas the fourth now requires two places (.25), and the third and sixth cannot be expressed exactly in the decimal scale.

IV. PROPOSALS FOR CHANGE OF BASIS

1 - 2. Binary and Octary Scales. Several other bases have been recommended as preferable to the decimal; the most important of which are the Binary, the Octary, and the Duodecimal. The binary scale¹ was advocated by Leibnitz. He even constructed an arithmetic upon this basis, called the Binary Arithmetic. The objection to this base is that it requires too many names and becomes very cumbersome in writing large numbers. The octary scale has also been strongly upheld.

1. Conant, 102-103.

3. Duodecimal Scale. When everything is considered, it is probable that the duodecimal scale would be the most suitable. The number twelve complies with both essentials of a base. Its susceptibility of division into halves, thirds, fourths, and sixths, is an especial recommendation and is also one of the advantages over the decimal scale. The following table will help to note the advantages of the latter system:

DECIMAL SCALE		DUODECIMAL SCALE	
$1/2 = .5$	$1/6 = .166+$	$1/2 = .6$	$1/6 = .2$
$1/3 = .333+$	$1/7 = .142857+$	$1/3 = .4$	$1/7 = .186135+$
$1/4 = .25$	$1/8 = .125$	$1/4 = .3$	$1/8 = .16$
$1/5 = .2$	$1/9 = .111+$	$1/5 = .2457+$	$1/9 = .14$

It is seen that in the decimal scale all the simple fractions used in practice except $1/2$ and $1/5$ give circulates or require three places while in the duodecimal scale the most common fractions used in business transactions are expressed in a single place, and even $1/8$ and $1/9$ require only two places. A very interesting phenomenon which shows here is that $1/5$ and $1/7$ are perfect repetends in the duodecimal scale.

A natural tendency seems to be to lean toward the duodecimal scale since large numbers are reckoned by the dozen, the gross and great gross, that is, by powers of twelve. The multiplication tables include twelve times, the division of the year into months, the circle into twelve signs, the foot into twelve inches, and the pound into twelve ounces are other attracting factors to the duodecimal scale.

The various objections to the decimal scale have led scientific men to advocate a change in our scale of numeration and notation. Such a change would be of great advantage to science and art, yet practical difficulties are so great that a revision seems almost impossible. Two systems would be necessary but such was the situation in Europe when the transition from the Roman to the Arabic system was made. The history of several different nations has taken in the changes of notation, some nations having changed two or three times. The Greeks changed theirs, first for the alphabetic, and afterwards, with the rest of the civilized world, for the Arabic system. The Arabs first adopted the Greek and later changed it for the Hindo method. The people of Europe changed from the Roman to the Arabic system even as late as the fourteenth century, though it took one or two centuries to make the transition. A writer¹ for one of the American periodicals says, when speaking of a change in notation, "The probability is that it will be done. The question is one of time rather than of fact, and there is plenty of time. The diffusion of education will ultimately cause it to be demanded."

A short time before his death, Charles XII of Sweden,² while lying in the trenches before a Norwegian fortress, seriously deliberated on a scheme of introducing the duodecimal system of numeration into his domain.

1. Brooks, page 119.

2. Conant, page 132.

V. RELATIONS WHICH EXIST BETWEEN SCALES

Theorem I.¹ Any positive integer N can be expressed in a number scale of given radix s as

$$N = r_0 + r_1s + r_2s^2 + \dots + r_ns^n$$

where $r_0, r_1, r_2, \dots, r_n$ is a complete residue system modulo s.

Proof:

If $s = 1$, the theorem is obvious.

If $s = 2$, there exists an n such that

$$s^{n-1} > N \geq s^n$$

where n could be zero if $N = 1$ (Axiom of Archimedes).

Then

$$\begin{aligned} N &= r_ns^n + c_1 && \text{where } c_1 < s^n \\ c_1 &= r_{n-1}s^{n-1} + c_2 && \text{where } c_2 < s^{n-1} \\ c_2 &= r_{n-2}s^{n-2} + c_3 && \text{where } c_3 < s^{n-2} \\ &\dots\dots\dots \\ &\dots\dots\dots \\ c_n &= r_1s + c_n && \text{where } c_n < s \end{aligned}$$

As there exists a finite number of remainders $c_1 < s^{n-1+1}$ this process must come to a close; that is, finally

$$c_n = r_0$$

$$\therefore N = r_0 + r_1s + r_2s^2 + \dots + r_ns^n \quad (1)$$

1. In Chrystal's Algebra, Part I, page 163, a more general theorem is proved from which Theorem I may be obtained as a corollary.

Corollary 1. The representation in (1) above is unique.

Corollary 2. The number of places necessary to express a number in a system of given radix is $n + 1$ where n is the highest power of the base less than or equal to the given number.

Examples:

1. Express 221 of the decimal scale in the binary, ternary and duodecimal scales.

The highest power of 2 contained in 221 is 7, \therefore the number of digits required to express 221 in the binary scale is 8.

$$221 = 1 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1.$$

Hence,

$$221 \text{ (decimal scale)} = 11011101 \text{ (binary scale)}$$

The highest power of 3 contained in 221 is 4, and the number of places necessary to express 221 in the ternary scale is 5.

$$221 \text{ (decimal scale)} = 22012 \text{ (ternary scale)}$$

Since 12^2 is the highest power of 12 contained in 221, it takes 3 places to express 221 in the duodecimal scale.

$$221 \text{ (decimal scale)} = 165 \text{ (duodecimal scale)}$$

2. Express 40 of the duodecimal scale in the ternary and quinary scales.

$$40 \text{ (duodecimal scale)} = 2020 \text{ (ternary scale)}$$

$$40 \text{ (duodecimal scale)} = 220 \text{ (quinary scale)}$$

Theorem II. Any number

$$N = f(s) = r_0 + r_1s + r_2s^2 + \dots + r_ns^n$$

is divisible by $s-k$, where $k < s$, if $f(k)$ is divisible by $s-k$.

Expand s as $(s-k+k)$.

Hence,

$$\begin{aligned}
 N = f(s) &= r_0 + r_1(s-k+k) + r_2(s-k+k)^2 + \dots + \\
 & r_n(s-k+k)^n = r_0 + r_1 k + r_2 k^2 + \dots + r_n k^n + \\
 & r_1(s-k) + 2r_2 k(s-k) + r_2(s-k)^2 + r_3(s-k)^3 + \\
 & 3r_3 k(s-k)^2 + 3r_3 k^2(s-k) + r_4(s-k)^4 + 4r_4 k(s-k)^3 + \\
 & 6r_4 k^2(s-k)^2 + 4r_4 k^3(s-k) + \dots + r_n(s-k)^n + \\
 & nr_n k(s-k)^{n-1} + \dots + \frac{n(n-1)\dots\cdot 2\cdot 1}{1(n-1)} r_n k^{n-1}(s-k).
 \end{aligned}$$

Taking the derivative of $f(s)$ we have

$$\begin{aligned}
 f'(s) &= r_1 + 2r_2(s-k) + 2r_2 k + 3r_3(s-k)^2 + 2\cdot 3r_3 k(s-k) + \\
 & 3r_3 k^2 + 4r_4(s-k)^3 + 3\cdot 4r_4 k(s-k)^2 + 2\cdot 3\cdot 2r_4 k^2(s-k) + \\
 & 4r_4 k^3 + \dots + nr_n(s-k)^{n-1} + (n-1)nr_n k(s-k)^{n-2} + \\
 & \dots + nr_n k^{n-1}.
 \end{aligned}$$

$$\begin{aligned}
 f''(s) &= 2r_2 + 2\cdot 3r_3(s-k) + 2\cdot 3r_3 k + 3\cdot 4r_4(s-k)^2 + 2\cdot 3\cdot 4r_4 k(s-k) + \\
 & 2\cdot 2\cdot 3r_4 k^2 + \dots + (n-1)nr_n(s-k)^{n-2} + (n-2)(n-1)\cdot \\
 & nr_n k(s-k)^{n-3} + \dots
 \end{aligned}$$

$$\begin{aligned}
 f'''(s) &= 2\cdot 3r_3 + 2\cdot 3\cdot 4r_4(s-k) + 2\cdot 3\cdot 4r_4 k + \dots + (n-2)(n-1)\cdot \\
 & nr_n(s-k)^{n-3} + (n-3)(n-2)(n-1)nr_n k(s-k)^{n-4} + \dots
 \end{aligned}$$

.....

$$f^n(s) = nr_n k^n$$

Consider $s = k$, then

$$N = f(s) = f(k) + (r_1 + 2r_2 k + 3r_3 k^2 + 4r_4 k^3 + \dots)(s-k) +$$

$$(2r_2 + 2 \cdot 3r_3 k + 2 \cdot 2 \cdot 3r_4 k^2 + \dots)(s-k)^2 + (2 \cdot 3r_3 +$$

$$2 \cdot 3 \cdot 4r_4 k + \dots)(s-k)^3 + \dots + (\dots)(s-k)^n.$$

We have, when substituting the derivatives for the polynomials, the following:

$$N = f(s) = f(k) + f'(k)(s-k) + \frac{f''(k)}{2!}(s-k)^2 + \frac{f'''(k)}{3!}(s-k)^3 + \dots$$

$$\dots + \frac{f^{n-1}(k)}{(n-1)!}(s-k)^{n-1} + \frac{f^n(k)}{n!}(s-k)^n$$

which is Taylor's series.

Thus, N is divisible by $s-k$ if $f(k)$ is divisible by $s-k$.

Theorem III. Any number,

$$N = f(s) = r_0 + r_1 s + r_2 s^2 + \dots + r_n s^n,$$

is divisible by $s+k$, where $k \neq s$, if $f(-k)$ is divisible by $s+k$.

$$f(s) = N = r_0 + r_1 s + r_2 s^2 + \dots + r_n s^n$$

Expand s as $(s+k-k)$.

$$f(s) = r_0 + r_1(s+k-k) + r_2(s+k-k)^2 + r_3(s+k-k)^3 +$$

$$\begin{aligned}
& r_4(s+k-k)^4 + \dots + r_n(s+k-k)^n \\
& = r_0 - r_1k + r_2k^2 - r_3k^3 + \dots + r_n(-k)^n + r_1(s+k) + \\
& r_2(s+k)^2 - 2r_2k(s+k) + r_3(s+k)^3 - 3r_3k(s+k)^2 + \\
& 3r_3k^2(s+k) + r_4(s+k)^4 - 3r_4k(s+k)^3 + 6r_4k^2(s+k)^2 - \\
& 3r_4k^3(s+k) + \dots + r_n(s+k)^n - nr_nk(s+k)^{n-1} + \dots
\end{aligned}$$

Then

$$\begin{aligned}
f(s) = N = f(-k) + \frac{f'(-k)(s+k)}{1!} + \frac{f''(-k)(s+k)^2}{2!} + \frac{f'''(-k)(s+k)^3}{3!} + \\
\dots + \frac{f^{(n)}(-k)(s+k)^n}{n!}
\end{aligned}$$

Hence,

N is divisible by $s+k$ if $f(-k)$ is divisible by $s+k$.

Divisibility. The laws of divisibility of any number N in our system, (that is, where $s = 10$) by integers are special cases of the above theorems.

Examples considering $s = k$:

1. If $k = 1$,

N is divisible by 9 if $r_0 + r_1 + r_2 + \dots + r_n$ is divisible by 9. Hence, a number is divisible by 9 if the sum of its digits is divisible by 9.

2. If $k = 2$,

N is divisible by 8 if $k^2 r_2 + k r_1 + r_0$ or the last three digits to the right express a number which is divisible by 8 or are all zeros.

3. If $k = 5$,

N is divisible by 5 if the last digit to the right (r_0) is divisible by 5, that is, if it is zero or 5.

4. If $k = 6$,

N is divisible by 4 if $r_1k + r_0$ or the last two digits to the right are divisible by 4 or are both zeros.

5. If $k = 8$,

N is divisible by 2 if r_0 or the last digit to the right is zero or an even number.

Example~~s~~ considering $s + k$:

1. If $k = 1$,

N is divisible by 11, if $(-1)^n r_n + (-1)^{n-1} r_{n-1} + \dots + (-1)^2 r_2 + (-1) r_1 + r_0$ or the difference between the digits in the odd places and even places is divisible by 11.

Theorem IV. The decimal value of a fraction k/s is $.k$, where s is the radix. ($k = 1, 2, \dots, s-1$).

Proof:

$$\text{Since } \frac{1}{s} = .1$$

$$\text{Then } \frac{2}{s} = .2$$

$$\frac{3}{s} = .3$$

.....

.....

.....

$$\text{and } \frac{s-1}{s} = .(s-1)$$

In general $k/s = .k$

Theorem V. The decimal value of a fraction $k/s - 1$ is $.k$
where s is the radix. ($k = 1, 2, \dots, s-2$).

Proof:

$$\text{Since } \frac{1}{s-1} = .1 \frac{1}{s-1} = .11 \frac{1}{s-1} = .1111\dots = .\dot{1}$$

$$\text{Then } \frac{2}{s-1} = .222 = .\dot{2}$$

.....

.....

$$\frac{s-2}{s-1} = .(s-2)$$

In general $k/s - 1 = .k$

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100
Ternary	0	1	2	10	11	12	20	21	22	100	101	102	110
Quaternary	0	1	2	3	10	11	12	13	20	21	22	23	30
Quinary	0	1	2	3	4	10	11	12	13	14	20	21	22
Senary	0	1	2	3	4	5	10	11	12	13	14	15	20
Septenary	0	1	2	3	4	5	6	10	11	12	13	14	15
Octary	0	1	2	3	4	5	6	7	10	11	12	13	14
Nonary	0	1	2	3	4	5	6	7	8	10	11	12	13
Un decimal	0	1	2	3	4	5	6	7	8	9	ϑ	10	11
Duodecimal	0	1	2	3	4	5	6	7	8	9	ϑ	ϕ	10

Decimal	13	14	15	16	17	18	19	20	21	22	23	24	25
Binary	1101	1110	1111	10000	10001	10010	10011	10100	10101	10110	10111	11000	11001
Ternary	111	112	120	121	122	200	201	202	210	211	212	220	221
Quaternary	31	32	33	100	101	102	103	110	111	112	113	120	121
Quinary	23	24	30	31	32	33	34	40	41	42	43	44	100
Senary	21	22	23	24	25	30	31	32	33	34	35	40	41
Septenary	16	20	21	22	23	24	25	26	30	31	32	33	34
Octary	15	16	17	20	21	22	23	24	25	26	27	30	31
Nonary	14	15	16	17	18	20	21	22	23	24	25	26	27
Unodecimal	12	13	14	15	16	17	18	19	1 ^ϑ	20	21	22	23
Duodecimal	11	12	13	14	15	16	17	18	19	1 ^ϑ	1 ^ϕ	20	21

Table which shows the numbers from 0 to 25 in 11 systems of notation with the numbers from 2 to 12 as bases. (ϑ and ϕ are characters introduced to represent 10 and 11, respectively).

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