Bilinear Littlewood-Paley Square Functions and Singular Integrals

By

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Submitted to the graduate degree program in Mathematics and
the Graduate Faculty of the University of Kansas in partial
fulfillment of the requirements for the degree of
Doctor of Philosophy

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Date Defended: April 17, 2013

The Dissertation Committee for Jarod Hart certifies
that this is the approved version of the following dissertation :

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Chairperson Rodolfo Torres

Date approved: April 17, 2013

Abstract

In this dissertation we further develop the bilinear theory of vector valued Calderón-Zygmund operators, Littlewood-Paley square functions, and singular integral operators. These areas of harmonic analysis are motivated by potential theory, boundary value problems in partial differential equations, harmonic and analytic extension problems in complex analysis, and many other classical problems in analysis. Multilinear operator theory addresses difficulties that arise from product type operations in harmonic analysis. We first introduce Banach valued Calderón-Zygmund operators in a bilinear setting, and prove weak endpoint estimates and interpolation results for them. By viewing Littlewood-Paley square functions as Calderón-Zygmund operators taking values in a particular Banach space, we are able to obtain bounds of the square functions on product Lebesgue spaces for a complete set of indices. We give an in depth analysis of Littlewood-Paley square functions, which includes estimates on some products of smooth function spaces as well as the estimates on product Lebesgue spaces that are needed to apply the vector valued Calderón-Zygmund results. Finally, we prove boundedness criteria for a certain class of bilinear singular integral operators on product Lebesgue spaces using Littlewood-Paley square function techniques. We provide a new proof of the bilinear T1 theorem that does not rely on the linear version of the result. We also prove a bilinear Tb theorem, a result missing in the theory so

far. The Littlewood-Paley square function techniques developed in this work are a powerful tool has potential to solve problems in areas like oscillatory integral operator theory, multiparameter operator theory, Fourier restriction, and non-linear partial differential equations.

Acknowledgements

I would like to express my sincere gratitude to my advisor, Professor Rodolfo Torres. Without his support and guidance, I would not have been able to complete this work. I also thank Professor Estela Gavosto for her support and guidance. They have been fantastic mentors for me over the last five years. My intellectual, personal, and professional development in recent years has been greatly influenced by them. I would also like to thank Professor Susan Kelly for her aid early my mathematical career and her recommendation to attend KU.

I am grateful to the Department of Mathematics at KU for their support over the last five years, and the members of my committee for their time and effort in helping me complete this work.

I thank my friends and collaborators, Lucas Chaffee, Lucas Oliviera, and Ana Grau de la Herrán for many enlightening and enjoyable conversations.

I would like to thank my parents, Gilbert and Annette, my brother, Bernie, and my sisters Rachel and Addy, for their continuing support and encouragement. They have always been there for me when I needed them. Finally, I thank my girlfriend, Rachel, for her unwavering patience, support, and companionship.

Contents

This work was completed with the support of

the **NSF grant DMS 1069015**

Introduction

The purpose of this work is to address the bilinear theory of vector valued Calderón-Zygmund theory, Littlewood-Paley techniques, and bilinear singular integral operators. Bilinear integral operators appear naturally in many contexts and have been a source of many challenging problems in harmonic analysis. Often in analysis, a non-linear problem can be analyzed or approximated by bilinear operators, much in the same way a Taylor polynomial can be used to approximate a smooth function. Solving problems in this area of harmonic analysis have led to the resolution of problems in other areas of analysis, including partial differential equations, complex analysis, signal analysis, among others.

Much of harmonic analysis is concerned with proving the boundedness of integral operators on appropriate functions spaces. That is, for given an operator T, the goal is often to prove that $||Tf||_Y \leq C||f||_X$ for some normed function spaces X and Y. In many situations, problems of this type can be solved by decomposing the action of the operator into basic components and controlling their interaction, which in the end implies boundedness of T. This is the the approach we take to analyze integral operators in this work.

During the 1950's, Calderón and Zygmund made great strides in developing the foundations for analyzing a large class singular integral operators (see e.g. [9, 11, 10]), which came to be known as Calderón-Zygmund operators. In subsequent years, the linear scalar valued Calderón-Zygmund theory was developed by many mathematicians, see e.g. Peetre [70], Spanne [75], Stein [77], Coifman-Meyer [20, 21], David-Journé [28], Christ [16], David-Journé-Semmes [29], among many others. Motivated by the Littlewood-Paley theory of Stein [76] (which will be briefly discussed later in this introduction), Benedek-Calderón-Panzone [2] and Rubio de Francia-Ruiz-Torrea [53] defined vector valued Calderón-Zygmund operators and developed the techniques of Calderón-Zygmund in this setting. Even though Littlewood-Paley theory was originally developed by Stein through different techniques, the vector valued Calderón-Zygmund theory of Benedek-Calderón-Panzone and Rubio de Francia-Ruiz-Torrea is now the standard way to prove many classical results in Littlewood-Paley theory.

In the last half of the 20$^{\text{th}}$ century, much of the work of Calderón-Zygmund was proved in a multilinear setting by Coifman-Meyer [22, 23], Christ-Journé [17], Kenig-Stein [57], Grafakos-Torres [45, 44], among others. In the work of Kenig-Stein and Grafakos-Torres, the authors found and proved the multilinear counterparts of many properties of linear Calderón-Zygmund operators, which included weak endpoint estimates, BMO endpoint estimates, and interpolation theory. More recently, the author of this work defined bilinear Calderón-Zygmund operators in a vector valued setting [49] (see also [51]). This work included a weak endpoint estimate and some interpolation results for bilinear vector valued operators. Further results for these operators were given by the author of this work in a collaboration with Grau de la Herrán-Oliviera [47], which included certain BMO endpoint estimates and more interpolation results.

The Littlewood-Paley theory developed by Stein [76] in the 1950's formed a characterization of $L^p(\mathbb{R}^n)$ using various decompositions involving harmonic extension. In the following years, there were many contributions to the study of Littlewood-Paley theory in this context, see e.g. Besov [5, 6], Taibleson [81, 82, 83], Peetre [71, 72], Triebel [85, 86], Lizorkin [61], Coifman-Meyer [21, 24], Kurtz [58], David-Journé [28], David-

Journé-Semmes [29], Duoandikoetxea-Rubio de Francia [31], Christ-Journé [17], Jones [56], Semmes [74], among others. These works included Lebesgue space bounds for the Littlewood-Paley square functions, as well as smooth function space estimates for certain modifications of the square functions. Over the years, Littlewood-Paley theory has developed into a very useful decomposition and estimation tool that can readily be interpreted as a frequency decomposition. In particular, it has been used in the analysis of Calderón-Zygmund theory in what is commonly called P-Q methods based on the works by Coifman-Meyer [21, 24], David-Journé [28], David-Journé-Semmes [29], Christ-Journé [17], Semmes [74], among others. Even today, linear Littlewood-Paley theory is still an active area of research, see e.g. Duoandikoetxea-Seijo [32], Cheng [15], Sato [73], Cruz-Uribe-Martell-Pérez [27], Grau [46], and Duoandikoetxea [30].

It was not until recently that this Littlewood-Paley theory was extended to the bilinear setting by, see e.g. Maldonado [62], Maldonado-Naibo [63], the current author [49, 51], Grafakos-Oliviera [41], the current author in collaboration with Grau de la Herrán-Oliviera [47], and Grafakos-Liu-Maldonado-Yang [40]. The first works in this area by Maldonado and Maldonado-Naibo achieved square function bounds on product Besov-Lebesgue spaces and in some particular cases on product Lebesgue spaces. In [49, 51], we prove bounds for the bilinear Littlewood-Paley square function on product Lebesgue spaces, making use of vector valued Calderón-Zygmund theory. Some of these results were obtained concurrently by Grafakos-Oliviera [41] and Grafakos-Liu-Maldonado-Yang [40]. In recent work, we have also proved Lebesgue space bounds for weaker local testing conditions in [47].

Singular integral operator theory was developed at the same time as the work listed above, and in fact, much of the work mention above was developed for the purpose of studying singular integral operators. In particular, David-Journé [28] developed and used

Littlewood-Paley theory to prove the T1 theorem, which provides a characterization for L^2 bounds of Calderón-Zygmund singular integral operators. The proof in [28] was later simplifies by Coifman-Meyer [24], again using Littelwood-Paley square function theory. Littlewood-Paley techniques were also used by David-Journé-Semmes [29] to prove a Tb theorem, which is a perturbation of the T1 theorem. Another version of the Tb theorem was proved by Christ [16], although his proof was not based on Littlewood-Paley theory.

A multilinear T1 theorem was proved by Christ-Journé [17] and Grafakos-Torres [45, 44], but their proofs do not rely directly on Littlewood-Paley theory in the same way as the proofs in the linear setting, [28, 24]. Instead they argue by freezing all but one function, and iteratively apply the a linear T1 theorem in some sense. A new proof of the bilinear T1 theorem was provided by the author of this work in [50], which uses the Littlewood-Paley theory constructed in [49, 51] to give a proof that parallels the ones of David-Journé [28] and Coifman-Meyer [24]. In this work, we will present this proof of the bilinear T1 theorem, and prove a bilinear analog of the Tb theorem of David-Journé-Semmes [29] and Christ [16]. Like the new proof of the bilinear T1 in [50], we argue using Littlewood-Paley theory to conclude operator bounds for bilinear singular integral operators, but now with perturbed cancellation conditions for Tb in place of T1.

The main results of this work are: (1) Weak endpoint estimates for bilinear vector valued Calderón-Zygmund operators, (2) product Lebesgue space bounds for both perturbed and unperturbed bilinear Littlewood-Paley square function operators, (3) a new proof of the bilinear T1 theorem via Littlewood-Paley theory, and (4) a bilinear Tb theorem via "para-accretive perturbed" Littlewood-Paley theory. Most of these results have been published or accepted for publication in articles by the current author in [49, 51, 50], and in a collaboration with Grau de la Herrán-Oliviera [47]. some of the results are new, and will be submitted for publication soon.

This work is organized in the following way: We provide some definitions, basic properties, and notation in Chapter 1. In Chapter 2, we define bilinear vector valued singular integral operators, and extend some results from the scalar valued theory to the vector valued setting. As mentioned above, the results in this section were originally proved in [49] and [47]. In Chapter 3, we prove some interpolation theorems for vector valued operators, most of which are natural extensions of scalar valued interpolation results. Again these interpolation results were published in [49] and [47]. In Chapters 4 and 5, we prove almost orthogonality estimates and convergence results in both linear and bilinear settings. Most of the convergence and linear almost orthogonality results presented are well established, but some of the bilinear ones are new. A few of the bilinear results were proved by Maldonado [62] and Maldonado-Naibo [63], others first appear in [49] (see also [51]), and a few are currently unpublished results of the author. In Chapter 6, we prove a number of square function bounds in what may be interpreted as vector valued T1 and Tb theorems. The bilinear results in Chapter 6 are an accumulation of the work of many people: Maldonado [62], Maldonado-Naibo [63], the author [49, 51], Grafakos-Oliviera [41], and Grafakos-Liu-Maldonado-Yange [40]. In Chapter 7, the Littlewood-Paley square function theory developed in Chapter 6 is applied to provide a new proof of the bilinear T1 theorem and prove a bilinear Tb theorem. The bilinear T1 theorem was originally proved in [17, 57, 45, 44], and we provide a new proof that was originally done in [50]. The bilinear Tb theorem is proved using similar Littlewood-Paley theory, and contains a slightly different way to prove the linear Tb theorem.

This work was completed with the support from the NSF grant DMS 1069015 as well as several graduate awards and scholarships from the University of Kansas Department of Mathematics.

Chapter 1

Preliminaries

In this section, we set notation, introduce some mathematical objects, and classical results that we use throughout the work. We will assume a background knowledge of real analysis, measure theory, functional analysis, and some elements of Fourier analysis.

1.1 Geometric Notation

Fix an integer dimension $n \in \mathbb{N}$, and the Euclidean space $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, define the Euclidean distance

$$|x| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}.$$

For a complex number $x + iy \in \mathbb{C}$, define the modulus $|x + iy| = |(x, y)|$. Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, define $|E|$ to be the Lebesgue measure of E. For $x \in \mathbb{R}^n$ and $R > 0$

define the ball centered at x of radius R

$$B(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\}.$$

A set $Q \subset \mathbb{R}^n$ is a cube with side length $R > 0$ (with sides parallel to the axes) if there exists $x \in \mathbb{R}^n$ such that $Q = \{x+y : y \in (0,R)^n\}$. Also define the cube centered at x of side length R

$$Q(x,R) = \{y \in \mathbb{R}^n : \max(|x_1 - y_1|, ..., |x_n - y_n|) < R/2\}.$$

Given a cube Q, we define $\ell(Q)$ to be the side length of Q. For $k \in \mathbb{Z}$ and $j = (j_1, ..., j_n) \in \mathbb{Z}^n$, define the dyadic cube

$$Q_{j,k} = \{x \in \mathbb{R}^n : j_i 2^k \leq x_i < (j_i + 1)2^k\},$$

the collection of dyadic cubes of scale 2^k to be $\mathscr{D}_k = \{Q_{j,k} : j \in \mathbb{Z}^n\}$, and the collection of all dyadic cubes $\mathscr{D} = \{Q_{j,k} : j \in \mathbb{Z}^n,\ k \in \mathbb{Z}\}$. Given a dyadic cube $Q_{j,k}$ for some $k \in \mathbb{Z}$ and $j \in \mathbb{Z}^n$, there exists a unique cube $R \in \mathscr{D}_{k+1}$ such that $Q_{j,k} \subset R$; we define this dyadic cube R to be the dyadic father of $Q_{j,k}$. Given a dyadic cube $Q_{j,k}$ for some $k \in \mathbb{Z}$ and $j \in \mathbb{Z}^n$, there exist $R_1, ..., R_{2^n} \in \mathscr{D}_{k-1}$ such that $R_i \subset Q_{j,k}$ for $i = 1, ..., 2^n$; we define these dyadic cubes $R_1, ..., R_{2^n}$ to be the dyadic children of $Q_{j,k}$.

1.2 Function and Operator Notations

For $k \in \mathbb{Z}$, $x, y \in \mathbb{R}^n$, $R > 0$, and $f : \mathbb{R}^n \to \mathbb{C}$, define

$$f^y(x) = f(x-y), \quad f^{y,R}(x) = f\left(\frac{x-y}{R}\right), \quad \text{and} \quad f_k(x) = 2^{nk} f(2^k x).$$

For any set $E \subset \mathbb{R}^n$, define $\chi_E : \mathbb{R}^n \to \mathbb{C}$ to be 1 on E and 0 on $\mathbb{R}^n \backslash E$. For any set E with positive measure and measurable function f, define the average of f over E,

$$Avg_E f = \frac{1}{|E|} \int_E f(x) dx.$$

For $k \in \mathbb{Z}$, $N > 0$, and $x \in \mathbb{R}^n$, define

$$\Phi_k^N(x) = \frac{2^{kn}}{(1+2^k|x|)^N}.$$

For $t_0 \in \mathbb{R}$, define the Dirac delta measure concentrated at t_0 as a measure on $\mathbb{R}$ by

$$\delta_{t_0}(E) = \begin{cases} 1 & t_0 \in E \\ 0 & t_0 \notin E \end{cases}.$$

A function b is called para-accretive if b and b^{-1} are uniformly bounded and there exists a constant $c_0 > 0$ such that for every cube $Q \subset \mathbb{R}^n$, there exists a sub-cube $R \subset Q$ such that

$$\frac{1}{|Q|} \left| \int_R b(x) dx \right| \geq c_0.$$

A function $a : \mathbb{R}^n \to \mathbb{C}$ is an H^1 atom if there exists a ball $B \subset \mathbb{R}^n$ such that $\text{supp}(a) \subset B$, a has integral zero, and $||a||_{L^\infty} \leq |B|^{-1}$.

Given functions $f, b : \mathbb{R}^n \to \mathbb{C}$, define the pointwise multiplication operator $M_b f(x) = b(x)f(x)$ for $x \in \mathbb{R}^n$. Define the Fourier transform $\widehat{f} = \mathscr{F}[f]$ of a function $f : \mathbb{R}^n \to \mathbb{C}$ for $\xi \in \mathbb{R}^n$

$$\mathscr{F}[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx$$

whenever the integral converges, and the inverse Fourier transform $\mathscr{F}^{-1}[f] = f^{\vee}$ for $x \in \mathbb{R}^n$

$$\mathscr{F}^{-1}[f](x) = f^{\vee}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi)e^{ix\xi}d\xi$$

whenever the integral converges. Given two functions $f, g : \mathbb{R}^n \to \mathbb{C}$, define for $x \in \mathbb{R}^n$

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

It follows that $\widehat{f * g} = \widehat{f}\,\widehat{g}$, when f and g are nice enough for these integrals to exist. For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, define the Hardy-Littlewood maximal function

$$\mathscr{M} f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ that contain x. Variations of the Hardly-Littlewood maximal function are formed by replacing the supremum over balls containing x with ball centered at x, cubes containing x, or cubes centered at x. There exists constants $c, C > 0$ such that for all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$, $x \in \mathbb{R}^n$, and any of these variation $\mathscr{M}'$ of $\mathscr{M}$, the pointwise inequality $c\mathscr{M}' f(x) \leq \mathscr{M} f(x) \leq C\mathscr{M}' f(x)$ holds. Since these operators are pointwise comparable, we will use these variants interchangeably throughout this work.

1.3 Function Spaces and Their Topologies

Given a topological vector space X, we define it's continuous dual, denoted X', to be the collection of all continuous linear functionals $W : X \to \mathbb{C}$. In this work, the only topological vector spaces we work with have topologies that are characterized by sequential convergence. So for the purpose of this work, we say that a linear functional $W : X \to \mathbb{C}$ is continuous if for $f_k, f \in X$

$$\lim_{k\to\infty} f_k = f \text{ in } X \text{ implies } \lim_{k\to\infty} W(f_k) = W(f)$$

where the second limit is a limit of complex numbers. We endow the dual space X' with the weak* topology, i.e. for $W, W_k \in X'$, we say that $W_k \to W$ in X' if for all $f \in X$,

$$\lim_{k\to\infty} W_k(f) \to W(f),$$

as a limit of complex numbers. Given a para-accretive function $b : \mathbb{R}^n \to \mathbb{C}$ and a topological vector space X made up of functions $f : \mathbb{R}^n \to \mathbb{C}$ (or Lebesgue almost everywhere equivalence classes of functions), define bX to be the collection of all functions f such that $b^{-1}f \in X$. Define the topology of bX via $f_k, f \in bX$ satisfies $f_k \to f$ in bX if and only if $b^{-1}f_k \to b^{-1}f$ in X.

Throughout this work, let $\mathscr{B}$ be a seperable Banach space with norm $|\cdot|_{\mathscr{B}}$, and dual pairing $\langle \cdot, \cdot \rangle_{\mathscr{B}', \mathscr{B}}$, i.e. for $f \in \mathscr{B}$ and $W \in \mathscr{B}'$, $W(f) = \langle W, f \rangle_{\mathscr{B}', \mathscr{B}}$. We will write $\langle W, f \rangle_{\mathscr{B}', \mathscr{B}} = \langle W, f \rangle$ when the meaning is clear by the context.

Define $C^\infty = C^\infty(\mathbb{R}^n)$ to be the collection of all infinitely differentiable functions from $\mathbb{R}^n$ into $\mathbb{C}$. Let C_0^∞ be the subspace of C^∞ of compactly supported functions. We also define

Schwartz semi-norms $\rho_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{N}_0^n$

$$\rho_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \varphi(x)|,$$

and the Schwartz class of rapidly decreasing smooth functions $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ to be the collection of all functions $\varphi \in C^\infty$ such that $\rho_{\alpha,\beta}(\varphi) < \infty$ for all $\alpha, \beta \in \mathbb{N}_0^n$. We endow $\mathscr{S}$ with the topology induced by the metric

$$d_{\mathscr{S}}(f,g) = \sum_{j \in \mathbb{N}} 2^{-j} \frac{\rho_j(f-g)}{1+\rho_j(f-g)},$$

where $\{\rho_j\}_{j \in \mathbb{N}}$ is an enumeration of the countable collection $\{\rho_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}_0^n}$. Also define subspace $\mathscr{S}_0 = \mathscr{S}_0(\mathbb{R}^n) \subset \mathscr{S}$ of all functions with infinite vanishing moment, i.e. the collection of functions $f \in \mathscr{S}$ that satisfy $\partial^\alpha \widehat{f}(0) = 0$ for all $\alpha \in \mathbb{N}_0^n$. Define the class of tempered distributions, $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$, the continuous dual of $\mathscr{S}$.

Given a real number $0 < p < \infty$, define $L^p(\mathbb{R}^n, \mathbb{C}) = L^p(\mathbb{R}^n) = L^p$ to be the collection of almost every equivalent classes of Lebesgue measurable functions f such that

$$||f||_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

with topology generated by $||\cdot||_{L^p}$, which is a norm for $1 \leq p < \infty$ and a quasi-norm for $0 < p < 1$. For $p = \infty$, define $L^\infty(\mathbb{R}^n, \mathbb{C}) = L^\infty(\mathbb{R}^n) = L^\infty$ to be the collection of equivalence classes of almost everywhere equivalent Lebesgue measurable functions such that

$$||f||_{L^\infty} = \inf\{C : |f(x)| \leq C \text{ for a.e. } x \in \mathbb{R}^n\},$$

with topology generated by $||\cdot||_{L^\infty}$. For $0 < p \leq \infty$, define the subspace L_c^p of L^p to be the collection of all functions $f \in L^p$ with compact support. For $0 < p < \infty$, define $L^{p,\infty}$ (which we call weak L^p) to be the collection of equivalence classes of almost everywhere equivalent Lebesgue measurable functions such that

$$||f||_{L^{p,\infty}} = \sup_{\lambda>0} \lambda |\{x : |f(x)| > \lambda\}|^{1/p}.$$

It follows that $|| \cdot ||_{L^{p,\infty}}$ is a quasi-norm on $L^{p,\infty}$. In the case $p = \infty$, we define $L^{\infty,\infty} = L^\infty$ with the same norm. Define $L^p_{loc} = L^p_{loc}(\mathbb{R}^n)$ for $0 < p < \infty$ to the collection of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $f\chi_K \in L^p$ for all compact sets $K \subset \mathbb{R}^n$. Also for $0 < p_1, p_2 \leq \infty$, define $L^{p_1} + L^{p_2} = \{f_1 + f_2 : f_i \in L^{p_i}\}$.

Define the class of functions with bounded mean oscillation, BMO, to be the collection of all $f \in L^1_{loc}$ such that

$$||f||_{BMO} = \sup_{\text{cubes } Q\subset\mathbb{R}^n} \frac{1}{|Q|} \int_Q |f(x) - Avg_Q f| dx < \infty.$$

Define $H^1(\mathbb{R}^n) = H^1$ to be the collection of all functions $f \in L^1$ such that $R_j f \in L^1$ for each $j = 1, .., n$, where R_j is the j^{th} Riesz transform. Also define the H^1 norm

$$||f||_{H^1} = ||f||_{L^1} + \sum_{j=1}^{n} ||R_j f||_{L^1}.$$

Later in this section we will state an atomic characterization of H^1, which will provide a more tractable way (at least in this work) of working with the space H^1.

For $\delta > 0$ and $f : \mathbb{R}^n \to \mathbb{C}$, define the δ-Hölder norm for $0 < \delta \leq 1$,

$$||f||_\delta = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\delta},$$

and the class of δ-Hölder continuous functions with compact support, written $C_0^\delta(\mathbb{R}^n,\mathbb{C}) = C_0^\delta(\mathbb{R}^n) = C_0^\delta$, to be the collection of compactly supported functions $f:\mathbb{R}^n \to \mathbb{C}$ such that $||f||_\delta < \infty$. Since we restrict to compactly supported functions, it follows that $||\cdot||_\delta$ is a norm on C_0^δ. Even though C^1 and C_0^1 are typically used to represent the class of continuously differentiable functions, we define C^1 and C_0^1 to be the Lipschitz continuous spaces in order to keep consistent notation.

Let $\varphi \in \mathscr{S}$ with integral 1. Define $\psi(x) = 2^n\varphi(2x) - \varphi(x)$ and $Q_k f = \psi_k * f$. Then we define the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ for $1 \le p,q \le \infty$ and $\alpha \in \mathbb{R}$ to be the completion of $\mathscr{S}_0$ with respect to its norm

$$||f||_{\dot{F}_p^{\alpha,q}} = \left|\left|\left(\sum_{k\in\mathbb{Z}} 2^{\alpha k q}|Q_k f|^q\right)^{\frac{1}{q}}\right|\right|_{L^p}.$$

Similarly define the homogeneous Besov space $\dot{B}_p^{\alpha,q}$ for $1 \le p,q \le \infty$ and $\alpha \in \mathbb{R}$ to be the completion of $\mathscr{S}_0$ with respect to its norm

$$||f||_{\dot{B}_p^{\alpha,q}} = \left(\sum_{k\in\mathbb{Z}} 2^{\alpha k q}\,||Q_k f||_{L^p}^q\right)^{\frac{1}{q}}.$$

Given different $\varphi_1,\varphi_2 \in \mathscr{S}$, each with integral 1, this construction defines the same collections $\dot{F}_p^{\alpha,q}$. Furthermore the norms generated by the two functions are equivalent up to a multiple depending on φ_1 and φ_2. Likewise for the Besov spaces $\dot{B}_p^{\alpha,q}$.

1.4 Operator Notations

Given topological vector spaces X, Y and a linear operator $T : X \to Y$, we define the transpose $T^* : Y' \to X'$ in the following way: For $W \in Y'$ and $x \in X$

$$\langle T^*W, x\rangle_{X',X} = \langle W, Tx\rangle_{Y',Y} \,.$$

Given topological vector spaces X_1, X_2, Y and a bilinear operator $T : X_1 \times X_2 \to Y$, we define the transposes $T^{1*} : Y' \times X_2 \to X_1'$ and $T^{2*} : X_1 \times Y' \to X_2'$ in the following way: For $W \in Y'$ and $x_i \in X_i$ for $i = 1, 2$

$$\left\langle T^{*1}(W, x_2), x_1\right\rangle_{X_1',X_1} = \left\langle T^{*2}(x_1, W), x_2\right\rangle_{X_2',X_2} = \langle W, T(x_1, x_2)\rangle_{Y',Y} \,.$$

For notation purposes, we will write $T^{*0} = T$.

Let T be a continuous operator from X into Y' for some topological vector spaces X and Y. Then T^* is defined from Y'' into X', but one can also define $T^*|_Y : Y \to X'$ as the restriction of T to Y via the embedding of Y into Y''. In this situation, it follows that T is continuous from X into Y' if and only if $T^*|_Y$ is continuous from Y into X'. This situation arises in Chapter 7. In Chapter 7, we will drop the notation $T^*|_Y$ and simply write T^* for the operator from Y into X'. We make a similar convention in the bilinear setting: For a bilinear operator T from $X_1 \times X_2$ into Y', in Chapter 7 we identify $T^{1*} : Y'' \times X_2 \to X_1'$ with the restriction to Y in the first spot and $T^{2*} : X_1 \times Y \to X_2'$ with the restriction to Y in the second spot.

Let X and Y be normed spaces, and T be an operator with domain X taking values in

Y. We say that T is bounded from X into Y if

$$||T||_{X,Y} = \sup_{f \in X,\, ||f||_X = 1} ||Tf||_Y < \infty.$$

Let X_1, X_2, and Y be normed spaces. An operator T is bounded from $X_1 \times X_2$ into Y if

$$||T||_{X_1,X_2,Y} = \sup_{f_i \in X_i,\, ||f_i||_{X_i} = 1} ||T(f_1, f_2)||_Y < \infty.$$

1.5 Integrating Banach Valued Functions

In this section, we give a very brief introduction to Banach space valued integrals. We will state the results without proof, and refer the reader to Yoshida [90] for the details of the proofs. Although this integration can be defined on more general measure spaces, we will only define it for integration on $\mathbb{R}^n$ with the Lebesgue measure, since that is all that is used in this work.

Given a seperable Banach space $\mathscr{B}$, a function $F : \mathbb{R}^n \to \mathscr{B}$ is a simple function if F takes only a finite number of values $v_1, ..., v_N \in \mathscr{B}$, and $E_i = F^{-1}(\{v_i\})$ has finite Lebesgue measure for $i = 1, ..., N$. In this case F can be written for $x \in \mathbb{R}^n$

$$F(x) = \sum_{i=1}^{N} v_i \chi_{E_i}(x) \in \mathscr{B}.$$

This simple function representation is not unique, but is consistent. Define the Banach valued integral of a simple function F as

$$\int_{\mathbb{R}^n} F(x)dx = \sum_{i=1}^{N} v_i |E_i| \in \mathscr{B}$$

which is consistent for different representations of the simple function F. A function $F : \mathbb{R}^n \to \mathscr{B}$ is measurable (or strongly measurable) if there exist $\mathscr{B}$-valued simple functions F_k such that $F_k \to F$ in $\mathscr{B}$ as $k \to \infty$. If in addition $|F - F_k|_{\mathscr{B}}$ is a real valued integrable function for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |F(x) - F_k(x)|_{\mathscr{B}} dx = 0,$$

then we say F is a $\mathscr{B}$-valued integrable function. If F is $\mathscr{B}$-valued integrable, then there exist simple functions F_k that converge to F in $\mathscr{B}$ and $v \in \mathscr{B}$ such that

$$\lim_{k\to\infty} \left| \int_{\mathbb{R}^n} F_k(x) dx - v \right|_{\mathscr{B}} = 0,$$

where this limit exists in the topology of $\mathscr{B}$. In this situation, we define

$$\int_{\mathbb{R}^n} F(x) dx = v.$$

If F is an integrable function, it follows that $|F|_{\mathscr{B}}$ is a real valued integrable function, and

$$\left| \int_{\mathbb{R}^n} F(x) dx \right|_{\mathscr{B}} \leq \int_{\mathbb{R}^n} |F(x)|_{\mathscr{B}} dx.$$

If $F : \mathbb{R}^n \to \mathscr{B}$ is an integrable function and $G \in \mathscr{B}'$, then $\langle F(x), G \rangle_{\mathscr{B}',\mathscr{B}}$ is a complex valued integrable function satisfying

$$\int_{\mathbb{R}^n} \langle G, F(x) \rangle_{\mathscr{B}',\mathscr{B}} dx = \left\langle G, \int_{\mathbb{R}^n} F(x) dx \right\rangle_{\mathscr{B}',\mathscr{B}}.$$

Given a Banach space $\mathscr{B}$, define $L^p(\mathbb{R}^n, \mathscr{B})$ to be the collection of Lebesgue almost everywhere equivalent classes of $\mathscr{B}$ valued measurable functions $F : \mathbb{R}^n \to \mathscr{B}$ such that

$|F|_{\mathscr{B}} \in L^p$. Also define $L^{p,\infty}(\mathbb{R}^n, \mathscr{B})$ to be the collection of equivalence classes of Lebesgue almost everywhere equivalent $\mathscr{B}$ valued measurable functions F such that $|F|_{\mathscr{B}} \in L^{p,\infty}$.

1.6 General Conventions and Classical Results

For $1 < p < \infty$, we define $p' = \frac{p}{p-1}$ to be the Hölder conjugate of p, $p' = \infty$ when $p = 1$, and $p' = 1$ when $p = \infty$. A triple of indices $0 < p, p_1, p_2 \leq \infty$ satisfies the Hölder relationship if

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{1.1}$$

where we use the convention that $\frac{1}{\infty} = 0$. Define the statement $A \lesssim B$ to mean that $A \leq CB$ for some constant C. The constant C will typically depends on the ambient dimension n, size and regularity parameters of operators, indices of functions spaces, etc. We specify the dependence of the constant when it is not clear in context. We will also write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Given a collection $\{a_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ and a real number t, we define

$$\sum_{2^k > t} a_k = \sum_{k=j}^{\infty} a_k$$

where $j \in \mathbb{Z}$ such that $2^{j-1} \leq t < 2^j$. We will use a similar interpretation with the obvious adaptations for the conditions $2^k \geq t$, $2^k < t$, and $2^k \leq t$.

To conclude this chapter, we state some classical results of measure theory and harmonic analysis that we will use throughout this work. The first result we will simply refer to as the Fefferman-Stein vector-valued maximal function bound.

Theorem 1.6.1 (Fefferman-Stein [33]) *For all $1 < p,q < \infty$ and Lebesgue measurable functions $\{f_k\}_{k\in\mathbb{Z}}$*

$$\left|\left|\left(\sum_{k\in\mathbb{Z}}(\mathscr{M}f_k)^q\right)^{\frac{1}{q}}\right|\right|_{L^p} \lesssim \left|\left|\left(\sum_{k\in\mathbb{Z}}|f_k|^q\right)^{\frac{1}{q}}\right|\right|_{L^p}$$

The next result is the atomic decomposition of H^1 that was alluded to earlier in this chapter. This characterization was proved by Coifman [18] for $H^1(\mathbb{R})$ and by Latter [60] for $H^1(\mathbb{R}^n)$.

Theorem 1.6.2 (Coifman [18], Latter [60]) *Define H^1_{at} to be the collection of all functions*

$$f = \sum_j \lambda_j a_j$$

where $\lambda_j \in \mathbb{C}$ form an ℓ^1 sequence, a_j are atoms, and this convergence holds pointwise almost everywhere. Define the norm

$$||f||_{H^1_{at}} = \inf\left\{\sum_j |\lambda_j| : \lambda_j \in \mathbb{C},\ a_j \text{ are atoms},\ f = \sum_j \lambda_j a_j\right\}.$$

Then $H^1_{at} = H^1$ and $||\cdot||_{H^1} \approx ||\cdot||_{H^1_{at}}$.

Chapter 2

Vector Valued Calderón-Zygmund Theory

The purpose of this chapter is to define vector valued (Banach space valued) standard kernels and singular integral operators, both linear and bilinear. Linear Calderón-Zygmund singular integral theory has been well developed in the vector valued setting by the work of Benedek-Calderón-Panzone [2], Rubio de Francia-Ruiz-Torrea [53], Marcinkiewicz-Zygmund [66], Calderón-Zygmund [9, 11, 10], Coifmain-Meyer [24, 20], Stein [79, 77], among others. Scalar valued binear Calderón-Zygmund theory was developed by Coifman-Meyer [22], Christ-Journé [17], Kenig-Stein [57], Grafakos-Torres [45, 44], among others. Bilinear vector valued theory was introduced in the work by H. [49] and used implicitly in Grau de la Herrán-H.-Oliveira [47]. In this chapter, we prove some analogues of classical Calderón-Zygmund theory in the bilinear vector-valued setting. These results are analogues of results in [45] and [44], but some of the proofs introduce new techniques to prove these results even in the scalar valued setting.

2.1 Vector-Valued Kernels and Operators

Fix a seperable Banach space $\mathscr{B}$, and we start by defining $\mathscr{B}$-valued standard Calderón-Zygmund kernels and operators. Linear vector valued Calderón-Zygmund theory was originally introduced by Benedek-Calderón-Panzone [2], and used to prove some results in Littlewood-Paley theory involving square functions.

Definition 2.1.1 *A function $K : \mathbb{R}^{2n} \backslash \{(x,x) : x \in \mathbb{R}^n\} \to \mathscr{B}$ is a standard $\mathscr{B}$-valued linear Calderón-Zygmund kernel (or just standard kernel) of class $CZK_{\mathscr{B}}(A,\gamma)$ for $A > 0$ and $0 < \gamma \leq 1$ if for all $x,y,x',y' \in \mathbb{R}^n$, $K(x,\cdot)$ and $K(\cdot,y)$ are $\mathscr{B}$ valued measurable functions, and*

$$
\begin{aligned}
&|K(x,y)|_{\mathscr{B}} \leq \frac{A}{|x-y|^n} \text{ whenever } |x-y| \neq 0 \\
&|K(x,y) - K(x',y)|_{\mathscr{B}} \leq A\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}} \text{ whenever } |x-x'| \leq |x-y|/2 \\
&|K(x,y) - K(x,y')|_{\mathscr{B}} \leq A\frac{|y-y'|^\gamma}{|x-y|^{n+\gamma}} \text{ whenever } |y-y'| \leq |x-y|/2.
\end{aligned}
$$

If K is a standard kernel of type $CZK_{\mathscr{B}}(A,\gamma)$ for some $A > 0$ and $0 < \gamma \leq 1$, then we simply write $K \in CZK_{\mathscr{B}}$. Assume for a moment that $\mathscr{B} \neq \mathbb{C}$. We say that T is a $\mathscr{B}$-valued linear singular integral operator associated to a standard kernel $K \in CZK_{\mathscr{B}}$ if for all $f_1 \in L^\infty_c$ and $F_0 \in L^\infty_c(\mathbb{R}^n, \mathscr{B}')$ such that $supp(f_1) \cap supp(F_0) = \emptyset$, we have that $\langle K(x,y), F_0(x)\rangle_{\mathscr{B}',\mathscr{B}}$ is a measurable function,

$$
\langle Tf_1, F_0\rangle_{\mathscr{B}',\mathscr{B}} = \int_{\mathbb{R}^{2n}} f_1(y) \langle K(x,y), F_0(x)\rangle_{\mathscr{B}',\mathscr{B}} \, dy\, dx,
$$

and for $x \notin supp(f_1)$

$$Tf_1(x) = \int_{\mathbb{R}^n} f_1(y)K(x,y)dy.$$

In the case that $\mathscr{B} = \mathbb{C}$, we require that the above holds for $f_1, F_0 \in \mathscr{S}$ instead of $f_1 \in L_c^\infty$ and $F_0 \in L_c^\infty(\mathbb{R}^n, \mathscr{B}')$, and realize the dual pairing $\langle w, z\rangle_{\mathbb{C}',\mathbb{C}} = wz$ as the standard product of complex numbers.

In this chapter, we will always make some continuity assumption on T in order to define Tf_1 for general $f_1 \in L_c^\infty$. More precisely, we will always at least assume that T is bounded from L^p into $L^{p,\infty}(\mathbb{R}^n, \mathscr{B})$ for some p. In this case Tf_1 is a $\mathscr{B}$-valued measurable function, and we are able to define the above quantities for more general f_1 and F_0.

Definition 2.1.2 *A function $K : \mathbb{R}^{3n} \backslash \{(x,x,x) \in \mathbb{R}^{3n} : x \in \mathbb{R}^n\} \to \mathscr{B}$ is a standard $\mathscr{B}$-valued bilinear Calderón-Zygmund kernel (or just standard kernel) of class $BCZK_{\mathscr{B}}(A,\gamma)$ for $A > 0$ and $0 < \gamma \leq 1$ if for all $x, y_1, y_2, x', y_1', y_2' \in \mathbb{R}^n$, we have that $K(x, y_1, \cdot)$, $K(x, \cdot, y_2)$, and $K(\cdot, y_1, y_2)$ are $\mathscr{B}$-valued measurable functions, and*

$$|K(x,y_1,y_2)|_{\mathscr{B}} \leq \frac{A}{(|x-y_1|+|x-y_2|)^{2n}} \text{ whenever } |x-y_1|+|x-y_2| \neq 0$$

$$|K(x,y_1,y_2) - K(x',y_1,y_2)|_{\mathscr{B}} \leq A\frac{|x-x'|^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}$$
$$\text{whenever } |x-x'| \leq \max(|x-y_1|,|x-y_2|)/2$$

$$|K(x,y_1,y_2) - K(x,y_1',y_2)|_{\mathscr{B}} \leq A\frac{|y_1-y_1'|^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}$$
$$\text{whenever } |y_1-y_1'| \leq \max(|x-y_1|,|x-y_2|)/2$$

$$|K(x,y_1,y_2) - K(x,y_1,y_2')|_{\mathscr{B}} \leq A\frac{|y_2-y_2'|^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}$$
$$\text{whenever } |y_2-y_2'| \leq \max(|x-y_1|,|x-y_2|)/2$$

If K is a standard kernel of type $BCZK_{\mathscr{B}}(A,\gamma)$ for some $A > 0$ and $0 < \gamma \leq 1$, then we write $K \in BCZK_{\mathscr{B}}$. Assume for a moment that $\mathscr{B} \neq \mathbb{C}$. We say that T is a $\mathscr{B}$-valued bilinear singular integral operator associated to a standard kernel $K \in BCZK_{\mathscr{B}}$ if for all $f_1, f_2 \in L_c^\infty$ and $F_0 \in L_c^\infty(\mathbb{R}^n, \mathscr{B}')$ such that supp$(f_1) \cap$ supp$(f_2) \cap$ supp$(F_0) = \emptyset$, we have that $\langle F_0(x), K(x,y_1,y_2)\rangle_{\mathscr{B}',\mathscr{B}}$ is a measurable function,

$$\langle T(f_1,f_2),F_0\rangle_{\mathscr{B}',\mathscr{B}} = \int_{\mathbb{R}^{3n}} f_1(y_1)f_2(y_2)\,\langle F_0(x),K(x,y_1,y_2)\rangle_{\mathscr{B}',\mathscr{B}}\,dy_1\,dy_2\,dx,$$

and for all $x \notin$ supp$(f_1) \cap$ supp(f_2)

$$T(f_1,f_2)(x) = \int_{\mathbb{R}^{2n}} f_1(y_1)f_2(y_2)K(x,y_1,y_2)dy_1\,dy_2.$$

Like in the linear case, when $\mathscr{B} = \mathbb{C}$ we require that the above holds for $f_1, f_2, F_0 \in \mathscr{S}$.

Remark 2.1.3 *In the definition of $\mathscr{B}$-valued singular integral operator (both linear and bilinear), the integral representation of $\langle T(f_1,f_2),F_0\rangle$ is absolutely convergent when supp$(f_1) \cap$ supp$(f_2) \cap$ supp$(F_0) = \emptyset$ (likewise for $\langle Tf_1, F_0\rangle$ under the appropriate support conditions on f_1, F_0). We verify this for the bilinear case here. The proof in the linear setting is analogous.*

Proof: Let $f_1, f_2 \in L_c^\infty$ and $F_0 \in L_c^\infty(\mathbb{R}^n, \mathscr{B}')$ with disjoint support. There exists $\delta, R > 0$ such that for all $x \in \text{supp}(F_0)$ and $y_i \in \text{supp}(f_i)$ for $i = 1,2$, $|x-y_1| + |x-y_2| \geq \delta$ and $\text{supp}(F_0) \cup \text{supp}(f_1) \cup \text{supp}(f_2) \subset B(0,R)$. By assumption $\langle K(x,y_1,y_2), F_0(x)\rangle_{\mathscr{B}',\mathscr{B}}$ is a measurable function. So it follows that

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} &|f_1(y_1)f_2(y_2)\langle K(x,y_1,y_2),F_0(x)\rangle_{\mathscr{B}',\mathscr{B}}|dx\,dy_1\,dy_2 \\
&\leq \int_{|x-y_1|+|x-y_2|\geq\delta} |f_1(y_1)f_2(y_2)|\,|F_0(x)|_{\mathscr{B}'}\,|K(x,y_1,y_2)|_{\mathscr{B}}|dx\,dy_1\,dy_2 \\
&\leq \int_{|x-y_1|+|x-y_2|\geq\delta} \frac{A}{(|x-y_1|+|x-y_2|)^{2n}}|f_1(y_1)f_2(y_2)|\,|F_0(x)|_{\mathscr{B}'}dx\,dy_1\,dy_2 \\
&\leq A\delta^{-2n}R^{3n}||F_0||_{L^\infty(\mathbb{R}^n,\mathscr{B}')}||f_1||_{L^\infty}||f_2||_{L^\infty}.
\end{aligned}$$

Then this integral representation is well defined, and by Fubini-Tonelli's Theorem we may switch the order of integration as we please. By a similar argument, it follows that for $x\notin \mathrm{supp}(f_1)\cap\mathrm{supp}(f_2)$

$$|T(f_1,f_2)(x)|_{\mathscr{B}} \leq \int_{\mathbb{R}^{2n}} |f_1(y_1)f_2(y_2)|\,|K(x,y_1,y_2)|_{\mathscr{B}}dy_1\,dy_2 \leq A\delta^{-2}R^{2n}||f_1||_{L^\infty}||f_2||_{L^\infty}.$$

So in this situation $T(f_1,f_2)$ can be realized as an absolutely convergent integral. □

2.2 A Weak Endpoint Estimate

In the linear setting, weak endpoint estimates have been proved in both the scalar valued setting by Calderón-Zygmund in [9] and the vector valued setting by Benedek-Calderón-Panzone [2] and Rubio de Francia-Ruiz-Torrea [53].

Theorem 2.2.1 ([9], [2], [53]) *Suppose T is a $\mathscr{B}$-valued Calderón-Zygmund operator with kernel $K\in CZK_{\mathscr{B}}$. If T is bounded from L^p into $L^p(\mathbb{R}^n,\mathscr{B})$ for some $1<p<\infty$, then T is bounded from L^1 into $L^{1,\infty}(\mathbb{R}^n,\mathscr{B})$.*

The bilinear scalar version of this result was proved by Grafakos-Torres [45, 44] and Maldonado-Naibo [64] from more general kernels. This proof of Theorem 2.2.4 is the same argument that was presented in [45, 44, 64] with obvious adaptations to replace modulus with Banach norm. We prove it here, but first we must state a result of Calderón-Zygmund and prove a short lemma.

Theorem 2.2.2 (Calderón-Zygmund [9]) *For $f \in L^1$ and $\lambda > 0$, there exists a collection of disjoint dyadic cubes $\{Q_j\}$ such that*

$$|f(x)| \leq \lambda, \quad \textit{for almost every } x \notin \Omega = \bigcup_j Q_j \tag{2.1}$$

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)| dx \leq 2^n \lambda, \quad \textit{for all } Q_j \tag{2.2}$$

$$|\Omega| \leq \frac{1}{\lambda} ||f||_{L^1}. \tag{2.3}$$

This result is known as the Calderón-Zygmund decomposition originally proved in [9]. It is a well-known classical result, so we do not provide the proof here.

Lemma 2.2.3 *If $N > n$, then for all $t > 0$*

$$\int_{\mathbb{R}^n} \frac{dx}{(t+|x|)^N} \lesssim \frac{1}{t^{N-n}}$$

where the constant depends only on n and N.

Proof: This is a direct computation using polar coordinates

$$\int_{\mathbb{R}^n} \frac{dx}{(t+|x|)^N} \leq \int_{|x|\leq t} \frac{dx}{t^N} + \int_{|x|>t} \frac{dx}{|x|^N} \lesssim \frac{1}{t^{N-n}}.$$

□

Theorem 2.2.4 (Grafakos-Torres [45, 44], H. [49]) *Suppose T is an bilinear $\mathscr{B}$-valued singular integral operator with kernel $K \in BCZK_{\mathscr{B}}$. If T is bounded from $L^{p_1} \times L^{p_2}$ into $L^p(\mathbb{R}^n, \mathscr{B})$ for some $1 < p_1, p_2 < \infty$, then T is bounded from $L^1 \times L^1$ into $L^{1/2,\infty}(\mathbb{R}^n, \mathscr{B})$.*

Proof: Assume that T is as above, and let $f_1, f_2 \in L^1$ with norm 1 and $\lambda > 0$. Let $\{Q_{i,j}\}$ and Ω_i be the disjoint dyadic cubes and their union as defined in the Calderón-Zygmund decomposition at height $\lambda^{\frac{1}{2}}$ for f_i, $i = 1, 2$ as in Theorem 2.2.2. Define

$$g_i = f_i \chi_{\mathbb{R}^n \setminus \Omega} + \sum_j \chi_{Q_{i,j}} \left(\frac{1}{|Q_{i,j}|} \int_{Q_{i,j}} f_i(x) dx \right)$$
$$b_i = \sum_j b_{i,j} = \sum_j \chi_{Q_{i,j}} \left(f_i - \frac{1}{|Q_{i,j}|} \int_{Q_{i,j}} f_i(x) dx \right).$$

Note that $b_{i,j}$ has mean zero for all i, j and $||g_i||_{L^\infty} \leq 2^{n+1} \lambda^{1/2}$. It also follows that $f_i = g_i + b_i$ for each $i = 1, 2$ and so we can estimate

$$\begin{aligned} |\{x \in \mathbb{R}^n : |T(f_1, f_2)(x)|_{\mathscr{B}} > \lambda\}| &\leq |\Omega_1^*| + |\Omega_2^*| + |\{x \notin \Omega_1^* \cup \Omega_2^* : |T(f_1, f_2)(x)|_{\mathscr{B}} > \lambda\}| \\ &\leq \frac{2^{n+1}}{\lambda^{1/2}} + \sum_{h_i \in \{g_i, b_i\}} |\{x \notin \Omega_1^* \cup \Omega_2^* : |T(h_1, h_2)(x)|_{\mathscr{B}} > \lambda/4\}|. \end{aligned}$$

Here we use the notation $\Omega_i^* = \bigcup_j 2Q_{i,j}$ and for a cube Q, $2Q$ is the cube with the same center and twice the side length. We first estimate when $h_i = g_i$ for both $i = 1, 2$:

$$\begin{aligned} |\{x \notin \Omega_1^* \cup \Omega_2^* : |T(g_1, g_2)(x)|_{\mathscr{B}} > \lambda/4\}| &\leq \frac{2^{2p}}{\lambda^p} \int_{\mathbb{R}^n} |T(g_1, g_2)(x)|_{\mathscr{B}}^p dx \\ &\lesssim \frac{1}{\lambda^p} \prod_{i=1}^2 \left(\int_{\mathbb{R}^n} |g_i(y_i)|^{p_i} dy_i \right)^{\frac{p}{p_i}} \\ &\leq \frac{1}{\lambda^p} \prod_{i=1}^2 \left(\int_{\mathbb{R}^n} (2^n \lambda^{\frac{1}{2}})^{p_i - 1} |g_i(y_i)| dy_i \right)^{\frac{p}{p_i}} \\ &\leq \frac{1}{\lambda^p} 2^n \lambda^{p - 1/2} ||f_1||_{L^1}^{p/p_1} ||f_2||_{L^1}^{p/p_2} \lesssim \frac{1}{\lambda^{1/2}}. \end{aligned}$$

This completes the proof where $h_i = g_i$ for $i = 1,2$. Now assume that $h_1 = b_1$ and $h_2 = g_2$. Let $c_{i,j}$ denote the center of $Q_{i,j}$. Then

$$
\begin{aligned}
|\{x \notin \Omega_1^* \cup \Omega_2^* : |T(b_1,g_2)(x)|_{\mathscr{B}} > \lambda/4\}| &\le \frac{4}{\lambda}\int_{\mathbb{R}^n\setminus(\Omega_1^*\cup\Omega_2^*)} |T(b_1,g_2)(x)|_{\mathscr{B}}\,dx \\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus(\Omega_1^*\cup\Omega_2^*)} \sum_j \left|\int_{\mathbb{R}^{2n}} K(x,y_1,y_2)b_{1,j}(y_1)g_2(y_2)dy_2\right|_{\mathscr{B}} dx.
\end{aligned}
$$

Fix j, and for $x \notin \Omega_1^*$, it follows that $x \notin 2Q_{1,j}$ and

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^{2n}} (K(x,y_1,y_2) - K(x,c_{1,j},y_2))b_{1,j}(y_1)g_2(y_2)dy_1\,dy_2\right|_{\mathscr{B}} \\
&\qquad\lesssim \int_{\mathbb{R}^{2n}} \frac{|y_1 - c_{1,j}|^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}|b_{1,j}(y_1)g_2(y_2)|dy_1\,dy_2 \\
&\qquad\lesssim ||g_2||_{L^\infty}\int_{\mathbb{R}^n} \frac{\ell(Q_{1,j})^\gamma}{(|x-y_1|+|x-y_2|)^{n+\gamma}}|b_{1,j}(y_1)|dy_1 \\
&\qquad\lesssim \lambda^{\frac{1}{2}}\int_{\mathbb{R}^n} \frac{\ell(Q_{1,j})^\gamma}{|x-c_{i,j_i}|^{n+\gamma}}|b_{1,j}(y_1)|dy_1.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
&|\{x \notin \Omega_1^* \cup \Omega_2^* : |T(b_1,g_2)(x)|_{\mathscr{B}} > \lambda/4\}| \\
&\qquad\lesssim \frac{1}{\lambda^{1/2}}\int_{\mathbb{R}^n\setminus\Omega_1^*\cup\Omega_2^*} \sum_j \int_{\mathbb{R}^n} \frac{\ell(Q_{1,j})^\gamma}{|x-c_{1,j}|^{n+\gamma}}|b_{1,j}(y_1)|dy_1\,dx \\
&\qquad\le \frac{1}{\lambda^{1/2}}\sum_j \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus 2Q_{1,j}} \frac{\ell(Q_{1,j})^\gamma}{|x-c_{1,j}|^{n+\gamma}}dx\right)|b_{1,j}(y_1)|dy_1 \\
&\qquad\lesssim \frac{1}{\lambda^{1/2}}\sum_j \int_{Q_j} |b_{1,j}(y_1)|dy_1 \\
&\qquad\le \frac{1}{\lambda^{1/2}}.
\end{aligned}
$$

When $h_1 = b_1$ and $h_2 = b_2$, we have

$$
\begin{aligned}
|\{x \notin \Omega_1^* \cup \Omega_2^* : |T(b_1,b_2)(x)|_{\mathscr{B}} > \lambda/4\}| &\le \frac{2}{\lambda^{1/2}} \int_{\mathbb{R}^n \setminus \Omega_1^* \cup \Omega_2^*} |T(b_1,b_2)(x)|_{\mathscr{B}}^{\frac{1}{2}} dx \\
&\le \frac{1}{\lambda^{1/2}} \int_{\mathbb{R}^n \setminus \Omega^*} \left| \sum_{j_1,j_2} \int_{\mathbb{R}^{2n}} K(x,y_1,y_2) b_{1,j_1}(y_1) b_{2,j_2}(y_2) dy_1\, dy_2 \right|_{\mathscr{B}}^{1/2} dx.
\end{aligned}
$$

Fix j_1, j_2, and without loss of generality assume that $\ell(Q_{1,j_1}) \le \ell(Q_{2,j_2})$. For $x \notin \Omega_1^* \cup \Omega_2^*$, it follows that $x \notin 2Q_{1,j_1}$ and that

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^{2n}} (K(x,y_1,y_2) - K(x,c_{1,j_1},y_2)) b_{1,j_1}(y_1) b_{2,j_2}(y_2) dy_1\, dy_2 \right|_{\mathscr{B}} \\
&\qquad \lesssim \int_{\mathbb{R}^{2n}} \frac{|y_1 - c_{1,j_1}|^\gamma}{(|x-y_1| + |x-y_2|)^{2n+\gamma}} |b_{1,j_1}(y_1) b_{2,j_2}(y_2)| dy_1\, dy_2 \\
&\qquad \le \int_{\mathbb{R}^{2n}} \frac{\ell(Q_{1,j_1})^{\gamma/2} \ell(Q_{2,j_2})^{\gamma/2}}{|x-y_1|^{n+\gamma/2} |x-y_2|^{n+\gamma/2}} |b_{1,j_1}(y_1) b_{2,j_2}(y_2)| dy_1\, dy_2.
\end{aligned}
$$

Putting these together, we have that

$$
\begin{aligned}
&|\{x \notin \Omega_1^* \cup \Omega_2^* : |T(b_1,b_2)(x)|_{\mathscr{B}} > \lambda/4\}| \\
&\quad \lesssim \frac{1}{\lambda^{1/2}} \int_{\mathbb{R}^n \setminus \Omega_1^* \cup \Omega_2^*} \left[\int_{\mathbb{R}^{2n}} \left(\prod_{i=1}^{2} \sum_{j_i} \frac{\ell(Q_{i,j_i})^{\gamma/2}}{|x - c_{i,j_i}|^{n+\gamma/2}} |b_{i,j_i}(y_i)| \right) dy_1\, dy_2 \right]^{1/2} dx \\
&\quad = \frac{1}{\lambda^{1/2}} \int_{\mathbb{R}^n \setminus \Omega_1^* \cup \Omega_2^*} \left[\prod_{i=1}^{2} \left(\int_{\mathbb{R}^n} \sum_{j_i} \frac{\ell(Q_{i,j_i})^{\gamma/2}}{|x - c_{i,j_i}|^{n+\gamma/2}} |b_{i,j_i}(y_i)| dy_i \right) \right]^{1/2} dx \\
&\quad \le \frac{1}{\lambda^{1/2}} \left[\prod_{i=1}^{2} \int_{\mathbb{R}^n \setminus \Omega_1^* \cup \Omega_2^*} \sum_{j_i} \int_{\mathbb{R}^n} \left(\frac{\ell(Q_{i,j_i})^{\gamma/2}}{|x - c_{i,j_i}|^{n+\gamma/2}} |b_{i,j_i}(y_i)| \right) dy_i\, dx \right]^{1/2} \\
&\quad \le \frac{1}{\lambda^{1/2}} \left[\prod_{i=1}^{2} \sum_{j_i} \int_{\mathbb{R}^n} \left(\int_{|x - c_{i,j_i}| > \ell(Q_{i,j_i})} \frac{\ell(Q_{i,j_i})^{\gamma/2}}{|x - c_{i,j_i}|^{n+\gamma/2}} dx \right) |b_{i,j_i}(y_i)| dy_i \right]^{1/2} \\
&\quad \lesssim \frac{1}{\lambda^{1/2}} \left(\prod_{i=1}^{2} ||b_i||_{L^1} \right)^{1/2} \lesssim \frac{1}{\lambda^{1/2}}.
\end{aligned}
$$

By symmetry, this proves that T is is bounded from $L^1 \times L^1$ into $L^{1/2,\infty}(\mathbb{R}^n, \mathscr{B})$. $\square$

2.3 An L^∞-BMO Estimate

In the classical Calderón-Zygmund theory, a bounded Calderón-Zygmund operator can be defined on L^∞ and is bounded from L^∞ into BMO which was proved independently by Peetre [70], Spanne [75], and Stein [77] in the linear setting and by Grafakos-Torres [45] in the bilinear setting. When outside of the scalar value setting, it may be possible that T to extended to a continuous operator, but it would involve defining a Banach space valued BMO and verifying the use a variation of Fatou's lemma for Banach valued functions. In this work, we will not extend the definition of any Banach valued operators to a vector valued BMO. Instead we only prove an estimate for functions in L_c^∞, which will still be useful for interpolation.

We will state the linear and bilinear versions of this result, but only prove the bilinear one. The proof of the linear version is easily extracted from the bilinear one.

Theorem 2.3.1 (Peetre [70], Spanne [75], Stein [77]) *Suppose T is a $\mathscr{B}$-valued singular integral operator with kernel $K \in CZK_{\mathscr{B}}$. If T is bounded from L^p into $L^p(\mathbb{R}^n, \mathscr{B})$ for some $1 \le p < \infty$, then for all $f \in L_c^\infty$,*

$$|| |Tf|_{\mathscr{B}} ||_{BMO} \lesssim ||f||_{L^\infty},$$

where the constant is independent of f.

Theorem 2.3.2 (Grau de la Herrán-H.-Oliveira [47]) *Suppose T is a bilinear $\mathscr{B}$-valued singular integral operator with standard kernel $K \in BCZK_{\mathscr{B}}$. If T is bounded from $L^{p_1} \times$*

L^{p_2} into $L^p(\mathbb{R}^n,\mathscr{B})$ for some $1 \leq p_1, p_2 \leq \infty$ and $1 \leq p < \infty$ satisfying (1.1)*, then for all $f_1, f_2 \in L_c^\infty$,*

$$|||T(f_1,f_2)|_{\mathscr{B}}||_{BMO} \lesssim ||f_1||_{L^\infty}||f_2||_{L^\infty}$$

where the constant is independent of f_1, f_2.

This theorem was proved in a particular case where $\mathscr{B} = L^2(\mathbb{R}_+, dt/t)$ by Grau de la Herrán-H.-Oliveira in [47]. The proof is the same here, replacing the $L^2(\mathbb{R}_+, dt/t)$ norm with a general Banach space norm. Note that these estimates do not define Tf or $T(f_1, f_2)$ as elements of BMO for general $f, f_1, f_2 \in L^\infty$, nor do we claim any sort of continuity on L^∞ or $L^\infty \times L^\infty$. In the scalar valued case, this was done using the linearity and bilinearity of the operator and functional analysis results that rely on the operator being a complex valued function. In the Banach valued case, it is impossible to use the same argument since we can reduce the problem to a scalar valued sublinear or bi-sublinear operator, but not a linear scalar valued operator. That is, T is linear (respectivley bilinear) Banach valued operator, and $|T|_{\mathscr{B}}$ is a sublinear (respectively bi-sublinear) scalar valued operator. The definition of the scalar valued version of this is given in Chapter 7.

Proof: Let $f_1, f_2 \in L_c^\infty$ and $B = B(x_B, R) \subset \mathbb{R}^n$. Define

$$c_B = |T(f_1,f_2)(x_B) - T(f_1\chi_{2B}, f_2\chi_{2B})(x_B)|_{\mathscr{B}},$$

and we estimate

$$
\begin{aligned}
&\int_B \big|\, |T(f_1,f_2)(x)|_{\mathscr{B}} - c_B \big|\, dx \\
&\quad \le \int_B |T(f_1,f_2)(x) + T(f_1\chi_{2B}, f_2\chi_{2B})(x_B) - T(f_1,f_2)(x_B)|_{\mathscr{B}} dx \\
&\quad \le \int_B |T(f_1\chi_{2B}, f_2\chi_{2B})(x)|_{\mathscr{B}} dx \\
&\qquad\qquad + \int_B |T(f_1\chi_{(2B)^c}, f_2\chi_{2B})(x) - T(f_1\chi_{(2B)^c}, f_2\chi_{2B})(x_B)|_{\mathscr{B}} dx \\
&\qquad\qquad + \int_B |T(f_1\chi_{2B}, f_2\chi_{(2B)^c})(x) - T(f_1\chi_{2B}, f_2\chi_{(2B)^c})(x_B)|_{\mathscr{B}} dx \\
&\qquad\qquad + \int_B |T(f_1\chi_{(2B)^c}, f_2\chi_{(2B)^c})(x) - T(f_1\chi_{(2B)^c}, f_2\chi_{(2B)^c})(x_B)|_{\mathscr{B}} dx \\
&= I + II + III + IV,
\end{aligned}
$$

We bound I using the assumed bound for T

$$
\begin{aligned}
I \le |B|^{1-1/p} \left(\int_B |T(f_1\chi_{2B}, f_2\chi_{2B})(x)|_{\mathscr{B}}^p dx \right)^{\frac{1}{p}} &\lesssim |B|^{1-1/p} ||f_1\chi_{2B}||_{L^{p_1}} ||f_2\chi_{2B}||_{L^{p_2}} \\
&\lesssim |B|\, ||f_1||_{L^\infty} ||f_2||_{L^\infty}.
\end{aligned}
$$

To bound II, we use the kernel representation of T for $x \in B$

$$
\begin{aligned}
&\int_B |T(f_1\chi_{(2B)^c} f_2\chi_{2B})(x) - T(f_1\chi_{(2B)^c} f_2\chi_{2B})(x_B)|_{\mathscr{B}} dx \\
&\quad = \int_B \left| \int_{\mathbb{R}^{2n}} (K(x,y_1,y_2) - K(x_B,y_1,y_2)) f_1(y_1)\chi_{(2B)^c}(y_1) f_2(y_2)\chi_{2B}(y_2) dy_1\, dy_2 \right|_{\mathscr{B}} dx \\
&\quad \lesssim ||f_1||_{L^\infty} ||f_2||_{L^\infty} \int_B \int_{\mathbb{R}^{2n}} \frac{|x-x_B|^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}} \chi_{(2B)^c}(y_1)\chi_{2B}(y_2) dy_1\, dy_2\, dx \\
&\quad \le ||f_1||_{L^\infty} ||f_2||_{L^\infty} \int_B \int_{(2B)^c} \left(\int_{\mathbb{R}^n} \frac{R^\gamma dy_2}{(|x-y_1|+|x-y_2|)^{2n+\gamma}} dy_2 \right) dy_1\, dx \\
&\quad \lesssim ||f_1||_{L^\infty} ||f_2||_{L^\infty} \int_B \int_{|y_1-x_B|>2R} \frac{R^\gamma}{|x-y_1|^{n+\gamma}} dy_1\, dx \lesssim ||f_1||_{L^\infty} ||f_2||_{L^\infty} |B|.
\end{aligned}
$$

By symmetry III is bounded as well, and IV is bounded in a similar way

$$\begin{aligned}
&\int_B |T(f_1\chi_{(2B)^c} f_2\chi_{(2B)^c})(x) - T(f_1\chi_{(2B)^c} f_2\chi_{(2B)^c})(x_B)|_{\mathscr{B}} dx \\
&\qquad \le A||f_1||_{L^\infty}||f_2||_{L^\infty} \int_B \int_{\mathbb{R}^{2n}} \frac{|x-x_B|^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}} \chi_{(2B)^c}(y_1)\chi_{(2B)^c}(y_2) dy_1\, dy_2\, dx \\
&\qquad \lesssim ||f_1||_{L^\infty}||f_2||_{L^\infty} \int_B \int_{(2B)^c} \left(\int_{\mathbb{R}^n} \frac{R^\gamma dy_2}{(|x-y_1|+|x-y_2|)^{2n+\gamma}} dy_2 \right) dy_1\, dx \\
&\qquad \le ||f_1||_{L^\infty}||f_2||_{L^\infty}|B|.
\end{aligned}$$

This completes the proof. $\square$

Chapter 3

Interpolation

In this chapter, we prove a few interpolation results in the vector valued setting. Many of these results are analogs of the scalar valued versions. Since not all results can be directly extended to the vector valued setting, we reproduce the proofs even when they are essentially the same arguments as their scalar valued counterparts.

3.1 Marcinkiewicz Interpolation

First we state the linear version of Marcinkiewicz interpolation theorem, which was proved by Marcinkiewicz [65] in the scalar valued setting and can be extended to the Banach valued setting.

Theorem 3.1.1 (Marcinkiewicz [65]) *Let T be a linear operator that is bounded form L^{p_i} into $L^{p_i,\infty}(\mathbb{R}^n,\mathscr{B})$ for some $0 < p_1 < p_2 \leq \infty$. Then T is also bounded from L^p into $L^p(\mathbb{R}^n,\mathscr{B})$ for all $p_1 < p < p_2$.*

The proof the vector valued proof is essentially contained in the proof of the bilinear version Theorem 3.1.2, which we state and prove now.

Theorem 3.1.2 (H. [49]) *Suppose T is a bilinear operator that is bounded from $L^{p_{1,j}} \times L^{p_{2,j}}$ into $L^{p_j,\infty}(\mathbb{R}^n, \mathscr{B})$ for some $0 < p_j, p_{i,j} \leq \infty$ satisfying* (1.1) *for each $j = 1,2,3,4$. Then T is bounded from $L^{q_1} \times L^{q_2}$ into $L^q(\mathbb{R}^n, \mathscr{B})$ for any q, q_1, q_2 that satisfy* (1.1) *and for any $U \subset \{1,2\}$, there exists $j \in \{1,2,3,4\}$ with*

$$q_i > p_{i,j} \text{ for } i \in U$$
$$q_i < p_{i,j} \text{ for } i \in \{1,2\} \backslash U.$$

It should be noted that Theorem 3.1.2 is not a direct extension of the multilinear Marcinkiewicz interpolation theorem proved by Strichartz in [80]. In [80], a bilinear interpolation theorem is proved requiring that the operator is bounded on only three sets of indices, whereas Theorem 3.1.2 requires four sets. So in the scalar valued setting, Theorem 3.1.2 does not recover the interpolation theorem of Strichartz [80], but Theorem 3.1.2 holds in the Banach valued setting.

Proof: Let $f_i \in L^{q_i}$ for $i = 1,2$ with norm 1. Define for $U \subset \{1,2\}$

$$\vec{f}_{\lambda,U}(y_1,y_2) = \left(\prod_{j \in U} f_j(y_j) \chi_{|f_j| > \lambda^{q/q_j}}(y_j) \right) \left(\prod_{j \in \{1,2\} \backslash U} f_j(y_j) \chi_{|f_j| \leq \lambda^{q/q_j}}(y_j) \right),$$

and it follows that

$$f_1(y_1) f_2(y_2) = \sum_{U \subset \{1,2\}} \vec{f}_{\lambda,U}(y_1,y_2).$$

We use the convention here that $\prod_{i \in \emptyset} A_i = 1$ in the definition of $f_{\lambda,U}$. It also follows that $\vec{f}_{\lambda,U} \in L^{p_{1,j}} \times L^{p_{2,j}}$ for some $j \in \{1,2,3,4\}$. By hypothesis, T is bounded from $L^{p_{1,j}} \times L^{p_{2,j}}$ into $L^{p_j,\infty}$, so in particular $T\vec{f}_{\lambda,U}$ is a $\mathscr{B}$-valued measurable function for each $U \subset \{1,2\}$.

Therefore

$$T(f_1,f_2)=\sum_{U\subset\{1,2\}}T(\vec{f}_{\lambda,U}(y_1,y_2))$$

is a $\mathscr{B}$-valued measurable function since it is a finite sum of $\mathscr{B}$-valued measurable functions. Then

$$\begin{aligned}||T(f_1,f_2)||^q_{L^q(\mathbb{R}^n,\mathscr{B})}&=q\int_0^\infty\lambda^q|\{x\in\mathbb{R}^n:|T(f_1,f_2)(x)|_{\mathscr{B}}>\lambda\}|\frac{d\lambda}{\lambda}\\&\leq q\sum_{U\subset\{1,2\}}\int_0^\infty\lambda^q|\{x\in\mathbb{R}^n:|T(\vec{f}_{\lambda,U})(x)|_{\mathscr{B}}>\lambda/4\}|\frac{d\lambda}{\lambda}\\&\lesssim\sum_{U\subset\{1,2\}}\int_0^\infty\lambda^q|\{x\in\mathbb{R}^n:|T(\vec{f}_{\lambda,U})(x)|_{\mathscr{B}}>\lambda\}|\frac{d\lambda}{\lambda}.\end{aligned}$$

Now for each $U\subset\{1,2\}$, there exists $j=j_U$ such that $q_i>p_{i,j}$ for $i\in U$ and $q_i<p_{i,j}$ for $i\in\{1,2\}\backslash U$. Then we can estimate

$$\begin{aligned}\int_0^\infty\lambda^q|\{x:|T(\vec{f}_{\lambda,U})(x)|_{\mathscr{B}}>\lambda\}|\frac{d\lambda}{\lambda}&\lesssim\int_0^\infty\lambda^{q-p_j}\left(\prod_{i\in U}||f_i\chi_{|f_i|>\lambda^{q/q_i}}||^{p_j}_{L^{p_{i,j}}}\right)\\&\qquad\times\left(\prod_{i\in\{1,2\}\backslash U}||f_i\chi_{|f_i|\leq\lambda^{q/q_i}}||^{p_j}_{L^{p_{i,j}}}\right)\frac{d\lambda}{\lambda}\\&\leq\int_0^\infty\left(\prod_{i\in U}||f_i\chi_{|f_i|>\lambda^{q/q_i}}||^{p_j}_{L^{p_{i,j}}}\lambda^{\frac{qp_j}{p_{i,j}}\left(1-\frac{p_{i,j}}{q_i}\right)}\right)\\&\qquad\times\left(\prod_{i\in\{1,2\}\backslash U}||f_i\chi_{|f_i|\leq\lambda^{q/q_i}}||^{p_j}_{L^{p_{i,j}}}\lambda^{\frac{qp_j}{p_{i,j}}\left(1-\frac{p_{i,j}}{q_i}\right)}\right)\frac{d\lambda}{\lambda}\\&\leq\prod_{i\in U}\left(\int_0^\infty||f_i\chi_{|f_i|>\lambda^{q/q_i}}||^{p_{i,j}}_{L^{p_{i,j}}}\lambda^{q\left(1-\frac{p_{i,j}}{q_i}\right)}\frac{d\lambda}{\lambda}\right)^{\frac{p_j}{p_{i,j}}}\\&\qquad\times\prod_{i\in\{1,2\}\backslash U}\left(\int_0^\infty||f_i\chi_{|f_i|\leq\lambda^{q/q_i}}||^{p_{i,j}}_{L^{p_{i,j}}}\lambda^{q\left(1-\frac{p_{i,j}}{q_i}\right)}\frac{d\lambda}{\lambda}\right)^{\frac{p_j}{p_{i,j}}}\end{aligned}$$

$$= \prod_{i\in U}\left(\int_{\mathbb{R}^n}|f_i(y_i)|^{p_{i,j}}\int_0^{|f_i(y_i)|^{q_i/q}}\lambda^{q\left(1-\frac{p_{i,j}}{q_i}\right)}\frac{d\lambda}{\lambda}dy_i\right)^{\frac{p_j}{p_{i,j}}}$$

$$\times \prod_{i\in\{1,2\}\setminus U}\left(\int_{\mathbb{R}^n}|f_i(y_i)|^{p_{i,j}}\int_{|f_i(y_i)|^{q_i/q}}^{\infty}\lambda^{q\left(1-\frac{p_{i,j}}{q_i}\right)}\frac{d\lambda}{\lambda}dy_i\right)^{\frac{p_j}{p_{i,j}}}$$

$$= \prod_{i\in U}\left(\frac{q_i}{q(q_i-p_{i,j})}\int_{\mathbb{R}^n}|f_i(y_i)|^{q_i}dy_i\right)^{\frac{p_j}{p_{i,j}}}\prod_{i\in\{1,2\}\setminus U}\left(\frac{q_i}{q(p_{i,j}-q_i)}\int_{\mathbb{R}^n}|f_i(y_i)|^{q_i}dy_i\right)^{\frac{p_j}{p_{i,j}}}$$

$$= \prod_{i,j\in\{1,2\}}^{2}\left(\frac{q_i}{q|q_i-p_{i,j}|}\right)^{\frac{p_j}{p_{i,j}}}.$$

Therefore T is bounded from $L^{q_1}\times L^{q_2}$ into $L^q(\mathbb{R}^n,\mathscr{B})$. $\square$

We also state a slightly different version of these two theorems that will be useful when for interpolating with certain weak endpoints.

Theorem 3.1.3 (Marcinkiewicz [65]) *Let T be a sublinear operator that such that*

$$\sup_{f\in L^\infty_c,\,||f||_{L^{p_i}}=1}||Tf||_{L^{p_i}(\mathbb{R}^n,\mathscr{B})}<\infty$$

for some $0<p_1<p_2\le\infty$. Then

$$\sup_{f\in L^\infty_c,\,||f||_{L^{p}}=1}||Tf||_{L^{p}(\mathbb{R}^n,\mathscr{B})}<\infty$$

for all $p_1<p<p_2$.

Theorem 3.1.4 (H. [49]) *Suppose T is a bi-sublinear operator that such that*

$$\sup_{f_i\in L^\infty_c,\,||f_i||_{L^{p_{i,j}}}=1}||T(f_1,f_2)||_{L^{p_j}(\mathbb{R}^n,\mathscr{B})}<\infty$$

for some $0 < p_j, p_{i,j} \leq \infty$ *satisfying* (1.1) *for each* $j = 1,2,3,4$. *Then*

$$\sup_{f_i \in L_c^\infty,\, ||f_i||_{L^{q_i}}=1} ||T(f_1, f_2)||_{L^q(\mathbb{R}^n,\mathscr{B})} < \infty$$

for any q, q_1, q_2 *that satisfy* (1.1) *and for any* $U \subset \{1,2\}$*, there exists* $j \in \{1,2,3,4\}$ *with*

$$q_i > p_{i,j} \text{ for } i \in U$$
$$q_i < p_{i,j} \text{ for } i \in \{1,2\} \backslash U.$$

There are a few crucial differences between Theorems 3.1.2 and 3.1.4. One difference between the two is that in Theorem 3.1.4 the operator T need only be defined on L_c^∞ functions, and hence the conclusion only holds for L_c^∞ functions. Also Theorem 3.1.4 is applicable for bi-sublinear operators, whereas Theorem 3.1.2 is only applicable for bilinear operators.

Proof: Let q_1, q_2, q satisfy the hypotheses of the theorem and $f_1, f_2 \in L_c^\infty$ such that $||f_i||_{L^{q_i}} = 1$. Like in the proof of Theorem 3.1.2 define for $U \subset \{1,2\}$

$$\vec{f}_{\lambda,U}(y_1,y_2) = \left(\prod_{j\in U} f_j(y_j)\chi_{|f_j|>\lambda^{q/q_j}}(y_j)\right)\left(\prod_{j\in\{1,2\}\backslash U} f_j(y_j)\chi_{|f_j|\leq\lambda^{q/q_j}}(y_j)\right),$$

and it follows that

$$f_1(y_1)f_2(y_2) = \sum_{U\subset\{1,2\}} \vec{f}_{\lambda,U}(y_1,y_2).$$

We also know that $T(f_1,f_2)$ and $T\vec{f}_{\lambda,U}$ are $\mathscr{B}$-measurable since T is well defined on $L_c^\infty \times L_c^\infty$. Therefore

$$T(f_1,f_2) = \sum_{U\subset\{1,2\}} T(\vec{f}_{\lambda,U}(y_1,y_2))$$

is a $\mathscr{B}$-valued measurable function since it is a finite sum of $\mathscr{B}$ valued measurable functions. Then

$$\begin{aligned}||T(f_1,f_2)||^q_{L^q(\mathbb{R}^n,\mathscr{B})} &= q\int_0^\infty \lambda^q |\{x\in\mathbb{R}^n : |T(f_1,f_2)(x)|_{\mathscr{B}} > \lambda\}| \frac{d\lambda}{\lambda} \\ &\leq q \sum_{U\subset\{1,2\}} \int_0^\infty \lambda^q |\{x\in\mathbb{R}^n : |T(\vec{f}_{\lambda,U})(x)|_{\mathscr{B}} > \lambda/4\}| \frac{d\lambda}{\lambda}.\end{aligned}$$

From this point, the computation is reduced exactly to the one in Theorem 3.1.2. □

3.2 Interpolating with Weak Endpoint Esitmates

In this section, Theorem 3.1.2 is applied to some vector valued singular integral operators.

Corollary 3.2.1 *Let T be a bilinear operator taking values in a Banach space $\mathscr{B}$. If T is bounded from $L^1 \times L^1$ into $L^{1/2,\infty}(\mathbb{R}^n,\mathscr{B})$ and from $L^{p_1}\times L^{p_2}$ into $L^{1,\infty}(\mathbb{R}^n,\mathscr{B})$ for all $1 < p_1, p_2 < \infty$ satisfying* (1.1)*. Then T is bounded $L^{q_1}\times L^{q_2}$ into $L^q(\mathbb{R}^n,\mathscr{B})$ for all $1/2 < q < 1 < q_1, q_2 < \infty$ satisfying* (1.1).

Proof: Define $p_{1,1} = p_{1,2} = 1$, $p_1 = 1/2$, $p_{1,2} = \frac{q_1}{q}$, $p_{2,2} = \frac{q_2}{q}$, and $p_2 = 1$. Also choose $p_{1,3}$ such that $1 < p_{1,3} < \min(q_1, q_2')$ and $p_{1,4}$ such that $1 < p_{1,4} < \min(q_1', q_2)$. It follows that $p_{2,3} = p_{1,3}' > q_2$ and $p_{2,4} = p_{1,4}' > q_2$. Then by Theorem 3.1.2, it follows that T is bounded from $L^{q_1}\times L^{q_2}$ into $L^q(\mathbb{R}^n,\mathscr{B})$. □

Corollary 3.2.2 *Let T be an bilinear operator taking values in a Banach space $\mathscr{B}$. If T is bounded from $L^1 \times L^1$ into $L^{1/2,\infty}(\mathbb{R}^n, \mathscr{B})$ and from $L^{p_1} \times L^{p_2}$ into $L^p(\mathbb{R}^n, \mathscr{B})$ for all $1 < p, p_1, p_2 < \infty$ satisfying* (1.1). *Then T is bounded $L^{q_1} \times L^{q_2}$ into $L^q(\mathbb{R}^n, \mathscr{B})$ for all $1 < q_1, q_2 < \infty$ satisfying* (1.1) *with $1/2 < q < \infty$.*

Proof: Define $p_{1,1} = p_{1,2} = 1$, $p_1 = 1/2$, $p_{1,2} = \frac{2q_1}{q}$, $p_{2,2} = \frac{2q_2}{q}$, and $p_2 = \frac{2q_1q_2}{q(q_1+q_2)} > 1 + \frac{2q_1q_2}{q_1+q_2}$. Also choose $p_{1,3}$ such that $1 < p_{1,3} < \min(q_1, q_2')$ and $p_{1,4}$ such that $1 < p_{1,4} < \min(q_1', q_2)$. It follows that $p_{2,3} = p_{1,3}' + 1 > q_2$ and $p_{2,4} = p_{1,4}' + 1 > q_2$. Also $p_3 = \frac{p_{1,3}p_{2,3}}{p_{1,3}+p_{2,3}} > 1$ and similar for $p_4 = \frac{p_{1,4}p_{2,4}}{p_{1,4}+p_{2,4}} > 1$. Then by Theorem 3.1.2, it follows that T is bounded from $L^{q_1} \times L^{q_2}$ into L^q. $\square$

3.3 Interpolation with a weak *BMO* Endpoint

As we mentioned before, the L^∞_c-BMO type estimates do not necessarily imply continuity on L^∞, but they can be used as endpoints for interpolation. In this section, we state one such result for linear operators and prove one for bilinear operators, but we first define the sharp maximal function and state a result from Fefferman-Stein.

Definition 3.3.1 *For $f \in L^1_{loc}$, define the sharp maximal function*

$$\mathscr{M}^\# f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - Avg_Q f| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Note that $||f||_{BMO} = ||\mathscr{M}^\# f||_{L^\infty}$.

Theorem 3.3.2 (Fefferman-Stein [34]) *Let $0 < p_0 < \infty$. Then for any $p \in [p_0, \infty)$ and $f \in L^1_{loc}$ such that $\mathscr{M} f \in L^{p_0}$, it follows that $||f||_{L^p} \lesssim ||\mathscr{M}^\# f||_{L^p}$.*

Theorem 3.3.3 *Let T be a linear $\mathscr{B}$-valued singular integral operator with standard kernel $K \in CZK_{\mathscr{B}}$. Assume there exists $1 \le p < \infty$ such that T is bounded from L^p into $L^{p,\infty}(\mathbb{R}^n, \mathscr{B})$. Also assume that for any $p < q < \infty$ the following hold for $f \in L^q$: (1) Tf is a $\mathscr{B}$ measurable function, and (2) if $f_j \to f$ in L^q where $f_j \in L_c^\infty$, then*

$$|Tf(x)|_{\mathscr{B}} \le \liminf_{j\to\infty} |Tf_j(x)|_{\mathscr{B}} \quad a.e.\ x \in \mathbb{R}^n.$$

Then T is bounded from L^p into $L^p(\mathbb{R}^n, \mathscr{B})$ for all $q < p < \infty$.

Property (2) here is a replacement Fatou's lemma in the situation where $\mathscr{B} = L^p(\mathscr{X}, \mathbb{C})$ for some measure space $(\mathscr{X}, d\mu)$. In this situation, if $T = \{T_s\}_{s\in\mathscr{X}}$ is continuous from L^q into $L^p(\mathscr{X}, \mathbb{C})$ and $f_j \to f$ in L^q, then it follows from Fatou's lemma that

$$\begin{aligned} |Tf(x)|_{\mathscr{B}} = \left(\int_{\mathscr{X}} \lim_{j\to\infty} |T_s f_j(x)|^p d\mu(s)\right)^{\frac{1}{p}} &\le \liminf_{j\to\infty} \left(\int_{\mathscr{X}} |T_s f_j(x)|^p d\mu(s)\right)^{\frac{1}{p}} \\ &\le \liminf_{j\to\infty} |T_s f_j(x)|_{\mathscr{B}}. \end{aligned}$$

We will apply this result in Chapter 6 where $\mathscr{B} = \ell^2(\mathbb{Z})$ equipped with the counting measure. We now prove the bilinear version of this theorem.

Theorem 3.3.4 *Let T be a bilinear $\mathscr{B}$-valued singular integral operator with standard kernel $K \in BCZK_{\mathscr{B}}$. Assume there exists $1 < p < \infty$ such that T is bounded from $L^{p_1} \times L^{p_2}$ into $L^{p,\infty}(\mathbb{R}^n, \mathscr{B})$ for all $1 < p_1, p_2 < \infty$ satisfying* (1.1)*. Also assume that for any $1 < q_1, q_2 < \infty$ and $p < q < \infty$ satisfying* (1.1)*, the following hold for $f_i \in L^{q_i}$: (1) $T(f_1, f_2)$ is a $\mathscr{B}$ measurable function, and (2) if $f_{i,j} \to f_i$ in L^{q_i} for $f_{i,j} \in L_c^\infty$, then*

$$|T(f_1, f_2)(x)|_{\mathscr{B}} \le \liminf_{j\to\infty} |T(f_{1,j}, f_{2,j})(x)|_{\mathscr{B}} \quad a.e.\ x \in \mathbb{R}^n.$$

Then T is bounded from $L^{q_1} \times L^{q_2}$ into $L^q(\mathbb{R}^n, \mathscr{B})$ for all $1 < q_1, q_2 < \infty$ and $p < q < \infty$ satisfying (1.1).

Proof: Let T be as in the hypothesis and $1 < p < \infty$ such that T is bounded from $L^{p_1} \times L^{p_2}$ into $L^p(\mathbb{R}^n, \mathscr{B})$. By Theorem 2.3.1, it follows that $|| |T(f_1, f_2)|_{\mathscr{B}} ||_{BMO} \lesssim ||f||_{L^\infty}$ for all $f \in L_c^\infty$. Define $S(f_1, f_2) = \mathscr{M}^\#(|T(f_1, f_2)|_{\mathscr{B}})$, and we have that

$$||S(f_1, f_2)||_{L^p} \leq ||\mathscr{M}(|T(f_1, f_2)|_{\mathscr{B}})||_{L^p} \lesssim ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}},$$

$$||S(f_1, f_2)||_{L^\infty} = || |T(f_1, f_2)|_{\mathscr{B}} ||_{BMO} \lesssim ||f_1||_{L^\infty} ||f_2||_{L^\infty}$$

for all $f_1, f_2 \in L_c^\infty$ and $1 < p_1, p_2 < \infty$ satisfying (1.1). Fix $1 < q_1, q_2 < \infty$ and $p < q < \infty$ satisfying (1.1). Then by Theorem 3.1.4, it follows that $||S(f_1, f_2)||_{L^q} \lesssim ||f_1||_{L^{q_1}} ||f_2||_{L^{q_2}}$ for all $f_1, f_2 \in L_c^\infty$. Finally, we also have the pointwise bound

$$|| |T(f_1, f_2)|_{\mathscr{B}} ||_{L^q} \lesssim ||\mathscr{M}^\#(|T(f_1, f_2)|_{\mathscr{B}})||_{L^q},$$

whenever $f_1, f_2 \in L_c^\infty$ by Theorem 3.3.2 since $|T(f_1, f_2)|_{\mathscr{B}} \in L^p$ and $p < q < \infty$. Therefore, when $f_1, f_2 \in L_c^\infty$ and $p < q < \infty$

$$||T(f_1, f_2)||_{L^q(\mathbb{R}^n, \mathscr{B})} = || \, |T(f_1, f_2)|_{\mathscr{B}} ||_{L^q} \lesssim ||S(f_1, f_2)||_{L^q} \lesssim ||f_1||_{L^{q_1}} ||f_2||_{L^{q_2}}.$$

Finally, for an arbitrary $f_i \in L^{q_i}$ with $p < q < \infty$ and $1 < q_1, q_2 < \infty$ satisfy (1.1), there exists $f_{i,j} \in L_c^\infty$ such that $f_{i,j} \to f_i$ in L^q for $i = 1, 2$. Then by hypothesis (2) on T, Fatou's lemma, and the bound of S on L_c^∞ functions, we have that

$$\begin{aligned}||T(f_1,f_2)||_{L^q(\mathbb{R}^n,\mathscr{B})} &= |||T(f_1,f_2)|_{\mathscr{B}}||_{L^q} \le \liminf_{j\to\infty} |||T(f_{1,j},f_{2,j})|_{\mathscr{B}}||_{L^q} \\ &\lesssim \liminf_{j\to\infty} ||f_{1,j}||_{L^{q_1}}||f_{2,j}||_{L^{q_2}} = ||f_1||_{L^{q_1}}||f_2||_{L^{q_2}}.\end{aligned}$$

Note that we used hypthesis (1) on T to conclude that $T(f_1,f_2)$ is a measurable function. Therefore T is bounded from $L^{q_1}\times L^{q_2}$ into $L^q(\mathbb{R}^n,\mathscr{B})$. □

Chapter 4

Almost Orthogonality Estimates

In this chapter, we present a number of almost orthogonality results. Many of these estimates are classical (see e.g. Frazier-Han-Jawerth-Weiss [38], Frazier-Jawerth [39], Grafakos-Torres [43], or Maldonado-Naibo [63]), but we prove all estimates here for the sake of completeness. We also define Littlewood-Paley square function kernels and associated square functions in this chapter. Although we do not prove any results about square function bounds in this chapter, it is natural to define the kernels here to prove the estimates that we will use in the coming chapters.

4.1 Littlewood-Paley Square Function Kernels

Here we define linear and bilinear Littlewood-Paley square function kernels, which have been studied by many people in the past half century including Stein [79, 77], Semmes [74], Duoandikoetxea [30], Hofmann [52], Maldonado [62], Maldonado-Naibo [63], among others. Recall from Chapter 1 the definition for $N > 0$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$

$$\Phi_k^N(x) = \frac{2^{kn}}{(1+2^k|x|)^N}.$$

Definition 4.1.1 *Let θ_k be a function from $\mathbb{R}^{2n}$ into $\mathbb{C}$ for each $k \in \mathbb{Z}$. We call $\{\theta_k\}_{k\in\mathbb{Z}}$ a collection of Littlewood-Paley square function kernels of type $LPK(A,N,\gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \le 1$ if for all $x, y, y' \in \mathbb{R}^n$ and $k \in \mathbb{Z}$*

$$|\theta_k(x,y)| \le A\Phi_k^{N+\gamma}(x-y) \tag{4.1}$$

$$|\theta_k(x,y) - \theta_k(x,y')| \le A(2^k|y-y'|)^\gamma \left(\Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x-y')\right). \tag{4.2}$$

We also define for $k \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and $f \in L^1 + L^\infty$

$$\Theta_k f(x) = \int_{\mathbb{R}^n} \theta_k(x,y) f(y) dy.$$

We say that $\{\theta_k\}_{k\in\mathbb{Z}}$ is a collection of smooth Littlewood-Paley square function kernels of type $SLPK(A,N,\gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \le 1$ if it satisfies (4.1), (4.2), and for all $x, x', y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$

$$|\theta_k(x,y) - \theta_k(x',y)| \le A(2^k|x-x'|)^\gamma \left(\Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x'-y)\right). \tag{4.3}$$

If $\{\theta_k\}$ is a collection of Littlewood-Paley square function kernels of type $LPK(A,N,\gamma)$ (respectively $SLPK(A,N,\gamma)$) for some $A > 0$, $N > n$, and $0 < \gamma \le 1$, then we write $\{\theta_k\} \in LPK$ (respectively $\{\theta_k\} \in SLPK$).

Definition 4.1.2 *Let θ_k be a function from $\mathbb{R}^{3n}$ into $\mathbb{C}$ for each $k \in \mathbb{Z}$. We call $\{\theta_k\}_{k\in\mathbb{Z}}$ a collection of bilinear Littlewood-Paley square function kernels of type $BLPK(A,N,\gamma)$ for*

$A>0$, $N>n$, and $0<\gamma\le 1$ if for all $x,y_1,y_2,y_1',y_2'\in\mathbb{R}^n$ and $k\in\mathbb{Z}$

$$|\theta_k(x,y_1,y_2)|\le A\Phi_k^{N+\gamma}(x-y_1)\Phi_k^{N+\gamma}(x-y_2) \tag{4.4}$$

$$|\theta_k(x,y_1,y_2)-\theta_k(x,y_1',y_2)|\le A(2^k|y_1-y_1'|)^{\gamma}\Phi_k^{N+\gamma}(x-y_2) \times\left(\Phi_k^{N+\gamma}(x-y_1)+\Phi_k^{N+\gamma}(x-y_1')\right) \tag{4.5}$$

$$|\theta_k(x,y_1,y_2)-\theta_k(x,y_1,y_2')|\le A(2^k|y_2-y_2'|)^{\gamma}\Phi_k^{N+\gamma}(x-y_1) \times\left(\Phi_k^{N+\gamma}(x-y_2)+\Phi_k^{N+\gamma}(x-y_2')\right). \tag{4.6}$$

We also define for $k\in\mathbb{Z}$, $x\in\mathbb{R}^n$, and $f_1,f_2\in L^1+L^\infty$

$$\Theta_k(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\theta_k(x,y_1,y_2)f_1(y_1)f_2(y_2)dy_1\,dy_2.$$

We say that $\{\theta_k\}_{k\in\mathbb{Z}}$ is a collection of smooth bilinear Littlewood-Paley square function kernels of type SBLPK(A,N,γ) for $A>0$, $N>n$, and $0<\gamma\le 1$ if it satisfies (4.1)-(4.3) *and for all $x,x',y_1,y_2\in\mathbb{R}^n$ and $k\in\mathbb{Z}$*

$$|\theta_k(x,y_1,y_2)-\theta_k(x',y_1,y_2)|\le A(2^k|x-x'|)^{\gamma}\left(\prod_{i=1}^{2}\Phi_k^{N+\gamma}(x-y_i)-\prod_{i=1}^{2}\Phi_k^{N+\gamma}(x'-y_i)\right). \tag{4.7}$$

If $\{\theta_k\}$ is a collection of bilinear Littlewood-Paley square function kernels of type BLPK(A,N,γ) (respectively of type SBLPK(A,N,γ)) for some $A>0$, $N>n$, and $0<\gamma\le 1$, then we write $\{\theta_k\}\in$ BLPK (respectively $\{\theta_k\}\in$ SBLPK).

Remark 4.1.3 *Let θ_k be a function from $\mathbb{R}^{3n}$ to $\mathbb{C}$ for each $k\in\mathbb{Z}$. Then $\{\theta_k\}$ is a collection of Littlewood-Paley square function kernels of type SBLPK(A_1,N_1,γ_1) if and only if there* *exists* $A_2 > 0$, $N_2 > n$, *and* $0 < \gamma_2 \leq 1$ *such that for all* $x, y_1, y_2, y_1', y_2 \in \mathbb{R}^n$ *and* $k \in \mathbb{Z}$

$$|\theta_k(x,y_1,y_2)| \leq A_2 \Phi_k^{N_2}(x-y_1)\Phi_k^{N_2}(x-y_2) \tag{4.8}$$

$$|\theta_k(x,y_1,y_2) - \theta_k(x,y_1',y_2)| \leq A_2 2^{2nk}(2^k|y_1-y_1'|)^{\gamma_2} \tag{4.9}$$

$$|\theta_k(x,y_1,y_2) - \theta_k(x,y_1,y_2')| \leq A_2 2^{2nk}(2^k|y_2-y_2'|)^{\gamma_2} \tag{4.10}$$

$$|\theta_k(x,y_1,y_2) - \theta_k(x',y_1,y_2)| \leq A_2 2^{2nk}(2^k|x-x'|)^{\gamma_2}. \tag{4.11}$$

A similar equivalence holds for square function kernels of type $BLPK(A,N,\gamma)$, $LPK(A,N,\gamma)$, *and* $SLPK(A,N,\gamma)$ *with the obvious modifications.*

In [49], we worked with Littlewood-Paley square function kernels of type *LPK*, *SLPK*, *BLPK*, and *SBLPK*, but they were not names as such. In [49], there is a gap in the proof of the equivalence of kernel conditions, which is rectified in the addendum [51]. The set of kernel conditions (4.4)-(4.6) are slightly different than the ones in [51], but are equivalent and simplify many of the computations in this work.

Proof: Assume that $\{\theta_k\} \in SBLPK(A_1,N_1,\gamma_1)$. Define $A_2 = 2A_1$, $N_2 = N_1 + \gamma_2$, and $\gamma_2 = \gamma_1$. It follows easily that (4.8) holds. Also

$$\begin{aligned}|\theta_k(x,y_1,y_2) - \theta_k(x,y_1',y_2)| &\leq A_1(2^k|y_1-y_1'|)^{\gamma_1}\Phi_k^{N_1+\gamma_1}(x-y_2) \\ &\qquad \times \left(\Phi_k^{N_1+\gamma_1}(x-y_1) + \Phi_k^{N_1+\gamma_1}(x-y_1')\right) \\ &\leq 2A_1 2^{2nk}(2^k|y_1-y_1'|)^{\gamma_2}.\end{aligned}$$

A similar argument holds for regularity in the y_2 and x spots. Then θ_k satisfies (4.8)-(4.11).

Conversely we assume that (4.8)-(4.11) hold. Define $\eta = \frac{N_2-n}{2(N_2+\gamma_2)}$, $A_1 = A_2$, $N_1 = N_2(1-\eta) - \eta\gamma_2$, and $\gamma_1 = \eta\gamma_2$. Estimate (4.1) easily follows since $N_1 + \gamma_1 < N_2$. Estimate

(4.2) also follows since

$$
\begin{aligned}
|\theta_k(x,y_1,y_2)-\theta_k(x,y_1',y_2)| &\le A_2(2^k|y_1-y_1'|)^{\eta\gamma_2}\Phi_k^{N_2(1-\eta)}(x-y_2)\\
&\qquad\times\left(\Phi_k^{N_2(1-\eta)}(x-y_1)+\Phi_k^{N_2(1-\eta)}(x-y_1')\right)\\
&\le A_1(2^k|y_1-y_1'|)^{\gamma_1}\Phi_k^{N_1+\gamma_1}(x-y_2)\\
&\qquad\times\left(\Phi_k^{N_1+\gamma_1}(x-y_1)+\Phi_k^{N_1+\gamma_1}(x-y_1')\right).
\end{aligned}
$$

Note that this selection satisfies

$$
N_1=N_2-\eta(N_2+\gamma_2)=N_2-\frac{N_2-n}{2}=\frac{N_2+n}{2}>n.
$$

Then (4.2) holds for this choice of A_1, N_1, and γ_1 as well. Estimates (4.3) and (4.7) follow with a similar argument, and hence $\{\theta_k\}$ is a collection of Littlewood-Paley square function kernel of type $BLPK(A_1,N_1,\gamma_1)$. The proofs of the other equivalences are contained in the proof of this one. □

Remark 4.1.4 *If $\{\lambda_k^i\}\in LPK$ for $i=1,2$, then $\{\theta_k\}\in BLPK$ where θ_k is defined for $x,y_1,y_2\in\mathbb{R}^n$, $\theta_k(x,y_1,y_2)=\lambda_k^1(x,y_1)\lambda_k^2(x,y_2)$.*

Proof: It easily follows that for all $x,y_1,y_2\in\mathbb{R}^n$

$$
|\theta_k(x,y_1,y_2)|=|\lambda_k^1(x,y_1)\lambda_k^2(x,y_2)|\lesssim\Phi_k^{N+\gamma}(x-y_1)\Phi_k^{N+\gamma}(x-y_2).
$$

It is also easy to see that for $x,y_1,y_1',y_2\in\mathbb{R}^n$

$$
\begin{aligned}
|\theta_k(x,y_1,y_2)-\theta_k(x,y_1',y_2)| &= |\lambda_k^1(x,y_1)-\lambda_k^1(x,y_1')|\,|\lambda_k^2(x,y_2)|\\
&\lesssim\left(\Phi_k^{N+\gamma}(x-y_1)+\Phi_k^{N+\gamma}(x-y_1')\right)\Phi_k^{N+\gamma}(x-y_2).
\end{aligned}
$$

By symmetry, the regularity in y_2 follows as well. Therefore $\{\theta_k\} \in BLPK$. $\square$

4.2 Almost Orthogonality Estimate for Non-negative Kernels

In this section, we prove some estimates for integrals with non-negative integrands. There will be no mention of cancellation conditions for square function kernels here, but these estimate will be used in conjuction with cancellation properties in later sections.

Proposition 4.2.1 *If $M,N > n$, then for all $j,k \in \mathbb{Z}$*

$$\int_{\mathbb{R}^n} \Phi_j^M(x-u)\Phi_k^N(u-y)du \lesssim \Phi_j^M(x-y) + \Phi_k^N(x-y).$$

Proof: Fix $x,y \in \mathbb{R}$ and $j,k \in \mathbb{Z}$, and it follows that $|x-y| \le |x-u| + |u-y|$ for all $u \in \mathbb{R}^n$. Then either $|x-u| \ge |x-y|/2$ or $|u-y| \ge |x-y|/2$ (since otherwise $|x-y| > |x-u| + |u-y|$), and so it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi_j^M(x-u)\Phi_k^N(u-y)du \\
&\le \int_{|x-u|\ge|x-y|/2} \Phi_j^M(x-u)\Phi_k^N(u-y)du + \int_{|u-y|\ge|x-y|/2} \Phi_j^M(x-u)\Phi_k^N(u-y)du = I + II.
\end{aligned}$$

Then we estimate

$$\begin{aligned}
I &\le \int_{|x-u|\ge|x-y|/2} \frac{2^{jn}}{(1+2^j|x-u|)^M} \frac{2^{kn}}{(1+2^k|u-y|)^N} du \\
&\le \frac{2^{jn}}{(1+2^j|x-y|/2)^M} \int_{\mathbb{R}^n} \frac{2^{kn}}{(1+2^k|u-y|)^N} du \lesssim \Phi_j^M(x-y).
\end{aligned}$$

Similarly we estimate II

$$II \leq \frac{2^{kn}}{(1+2^k|x-y|/2)^N} \int_{|u-y|\geq|x-y|/2} \frac{2^{jn}}{(1+2^j|x-u|)^M} du \lesssim \Phi_k^N(x-y).$$

This completes the proof of the estimate. □

Proposition 4.2.2 *If $\{\theta_k\}_{k\in\mathbb{Z}} \in LPK$, then for all $j,k\in\mathbb{Z}$, $x,y\in\mathbb{R}^n$*

$$\int_{\mathbb{R}^n} |\theta_j(x,y)-\theta_j(x,u)|\Phi_k^{N+\gamma}(u-y)du \lesssim 2^{\gamma(j-k)}\left(\Phi_j^N(x-y)+\Phi_k^N(x-y)\right).$$

Proof: Since $\{\theta_k\}_{k\in\mathbb{Z}}$ is of type $LPK(A,N,\gamma)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\theta_j(x,y)-\theta_j(x,u)|\Phi_k^{N+\gamma}(u-y)du \\
&\qquad \lesssim \int_{\mathbb{R}^n} (2^j|u-y|)^\gamma \left(\Phi_j^{N+\gamma}(x-y)+\Phi_j^{N+\gamma}(x-u)\right)\Phi_k^{N+\gamma}(u-y)du \\
&\qquad \leq 2^{\gamma(j-k)} \int_{\mathbb{R}^n} \left(\Phi_j^{N+\gamma}(x-y)+\Phi_j^{N+\gamma}(x-u)\right)\Phi_k^{N}(u-y)du \\
&\qquad \leq 2^{\gamma(j-k)} \left(\Phi_j^{N+\gamma}(x-y)\int_{\mathbb{R}^n}\Phi_k^N(u-y)du + \int_{\mathbb{R}^n}\Phi_j^{N+\gamma}(x-u)\Phi_k^N(u-y)du\right) \\
&\qquad \lesssim 2^{\gamma(j-k)}\left(\Phi_j^N(x-y)+\Phi_k^N(x-y)\right).
\end{aligned}$$

This completes the proof of the proposition. □

Proposition 4.2.3 *If $\{\theta_k\}_{k\in\mathbb{Z}} \in BLPK$, then for all $j,k\in\mathbb{Z}$, $x,y_1,y_2\in\mathbb{R}^n$*

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\theta_j(x,y_1,y_2)-\theta_j(x,u,y_2)|\Phi_k^{N+\gamma}(u-y_1)du \\
&\qquad\qquad \lesssim 2^{\gamma(j-k)}\left(\Phi_j^N(x-y_1)+\Phi_k^N(x-y_1)\right)\Phi_j^N(x-y_2), \\
&\int_{\mathbb{R}^n} |\theta_j(x,y_1,y_2)-\theta_j(x,y_1,u)|\Phi_k^{N+\gamma}(u-y_2)du \\
&\qquad\qquad \lesssim 2^{\gamma(j-k)}\Phi_j^N(x-y_1)\left(\Phi_j^N(x-y_2)+\Phi_k^N(x-y_2)\right),
\end{aligned}$$

and

$$\int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,y_2)-\theta_j(x,u_1,u_2)|\Phi_k^{N+\gamma}(u_1-y_1)\Phi_k^{N+\gamma}(u_2-y_2)du_1\,du_2$$
$$\lesssim 2^{\gamma(j-k)}\prod_{i=1}^{2}\left(\Phi_j^N(x-y_i)+\Phi_k^N(x-y_i)\right).$$

Proof: Since $\{\theta_k\}_{k\in\mathbb{Z}}$ is of type $BLPK(A,N,\gamma)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\theta_j(x,y_1,y_2)-\theta_j(x,u,y_2)|\Phi_k^{N+\gamma}(u-y_1)du \\
&\quad\lesssim \Phi_j^{N+\gamma}(x-y_2)\int_{\mathbb{R}^n}(2^j|u-y_1|)^{\gamma}\left(\Phi_j^{N+\gamma}(x-y_1)+\Phi_j^{N+\gamma}(x-u)\right)\Phi_k^{N+\gamma}(u-y_1)du \\
&\quad\le 2^{\gamma(j-k)}\Phi_j^{N+\gamma}(x-y_2)\int_{\mathbb{R}^n}\left(\Phi_j^{N+\gamma}(x-y_1)+\Phi_j^{N+\gamma}(x-u)\right)\Phi_k^{N}(u-y_1)du \\
&\quad\lesssim 2^{\gamma(j-k)}\left(\Phi_j^N(x-y_1)+\Phi_k^N(x-y_1)\right)\Phi_j^N(x-y_2).
\end{aligned}$$

By symmetry the second estimate holds as well. For the third estimate, we make a similar argument,

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,y_2)-\theta_j(x,u_1,u_2)|\prod_{i=1}^{2}\Phi_k^{N+\gamma}(u_i-y_i)du_i \\
&\le \int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,y_2)-\theta_j(x,y_1,u_2)|\prod_{i=1}^{2}\Phi_k^{N+\gamma}(u_i-y_i)du_i \\
&\quad+\int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,u_2)-\theta_j(x,u_1,u_2)|\prod_{i=1}^{2}\Phi_k^{N+\gamma}(u_i-y_i)du_i \\
&\lesssim 2^{\gamma(j-k)}\int_{\mathbb{R}^{2n}}\Phi_j^{N+\gamma}(x-y_1)\left(\Phi_j^{N+\gamma}(x-y_2)+\Phi_j^{N+\gamma}(x-u_2)\right)\prod_{i=1}^{2}\Phi_k^{N+\gamma}(u_i-y_i)du_i \\
&\quad+2^{\gamma(j-k)}\int_{\mathbb{R}^{2n}}\left(\Phi_j^{N+\gamma}(x-y_1)+\Phi_j^{N+\gamma}(x-u_1)\right)\Phi_j^{N+\gamma}(x-u_2)\prod_{i=1}^{2}\Phi_k^{N+\gamma}(u_i-y_i)du_i \\
&\lesssim 2^{\gamma(j-k)}\left(\Phi_j^N(x-y_1)+\Phi_k^N(x-y_1)\right)\left(\Phi_j^N(x-y_2)+\Phi_k^N(x-y_2)\right).
\end{aligned}$$

This completes the proof of the proposition. □

4.3 Operator Almost Orthogonality

In this section we prove almost orthogonality estimates for operators that have certain cancellation properties. We first note the following maximal average control properties for Φ_k^N.

Proposition 4.3.1 *If $N > n$, then for all $f \in L^1 + L^\infty$ and $k \in \mathbb{Z}$*

$$|\Phi_k^N * f(x)| \lesssim \mathscr{M} f(x). \tag{4.12}$$

Proof: This is verified by the computation.

$$\begin{aligned}
|\Phi_k^N * f(x)| &\leq \int_{|x-y|\leq 2^{-k}} 2^{kn}|f(y)|dy + \sum_{j=0}^{\infty}\int_{2^{j-k}<|x-y|\leq 2^{j+1-k}} \frac{2^{kn}|f(y)|dy}{(2^k|x-y|)^N}\\
&\lesssim \mathscr{M} f(x) + \sum_{j=0}^{\infty}\int_{|x-y|\leq 2^{j+1-k}} \frac{2^{kn}|f(y)|dy}{2^{jN}}\\
&\lesssim \mathscr{M} f(x) + \mathscr{M} f(x)\sum_{j=0}^{\infty} 2^{-(N-n)j} \lesssim \mathscr{M} f(x).
\end{aligned}$$

□

Now we state and prove the Littlewood-Paley square function operator almost orthogonality properties that will be used throughout this work.

Proposition 4.3.2 *If $\{\lambda_k\}, \{\theta_k\} \in LPK$ and there exists a para-accretive function b such that $\Lambda_k(b) = \Theta_k(b) = 0$ for all $k \in \mathbb{Z}$, then for all $f \in L^1 + L^\infty$ and $j, k \in \mathbb{Z}$*

$$|\Theta_j M_b \Lambda_k^* f(x)| \lesssim 2^{-\gamma|j-k|}\mathscr{M} f(x). \tag{4.13}$$

Here $\Lambda_k f(x)$ is defined by integrating $f(y)$ against the kernel $\lambda_k(x,y)$ in the same way Θ_k is defined through integration against θ_k. If $\{\lambda_k\} \in LPK$, $\{\theta_k\} \in SBLPK$ and there exists a para-accretive functions b such that $\Lambda_k(b) = 0$ and

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)b(x)dx = 0$$

for all $k \in \mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}^n$, then for all $f_1, f_2 \in L^1 + L^\infty$ and $j,k \in \mathbb{Z}$

$$|\Lambda_k M_b \Theta_j(f_1,f_2)(x)| \lesssim 2^{-\gamma|j-k|}\mathscr{M}(\mathscr{M}f_1 \cdot \mathscr{M}f_2)(x). \tag{4.14}$$

If $\{\lambda_k^1\}, \{\lambda_k^2\} \in LPK$, $\{\theta_k\} \in BLPK$ and there exist para-accretive functions b_1, b_2 such that $\Theta_k(b_1,b_2) = \Lambda_k^1(b_1)\cdot\Lambda_k^2(b_2) = 0$ for all $k \in \mathbb{Z}$, then for all $f_1, f_2 \in L^1 + L^\infty$ and $j,k \in \mathbb{Z}$

$$|\Theta_j(M_{b_1}\Lambda_k^{1*}f_1, M_{b_2}\Lambda_k^{2*}f_2)(x)| \lesssim 2^{-\gamma|j-k|}\mathscr{M}f_1(x)\mathscr{M}f_2(x). \tag{4.15}$$

If $\{\lambda_k^1\}, \{\lambda_k^2\} \in LPK$, $\{\theta_k\} \in BLPK$ and there exist para-accretive functions b_1, b_2 such that

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)b_1(y_1)dy_1 = \int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)b_2(y_2)dy_2 = 0$$

and $\Lambda_k^1(b_1) = \Lambda_k^2(b_2) = 0$ for all $x, y_1, y_2 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, then for all $f_1, f_2 \in L^1 + L^\infty$ and $j,k,\ell \in \mathbb{Z}$

$$|\Theta_j(M_{b_1}\Lambda_k^{1*}f_1, M_{b_2}\Lambda_\ell^{2*}f_2)(x)| \lesssim \min\left(2^{-\gamma|j-k|}, 2^{-\gamma|j-\ell|}\right)\mathscr{M}f_1(x)\mathscr{M}f_2(x). \tag{4.16}$$

If $\{\lambda_k\} \in LPK$, $\{\theta_k\} \in BLPK$ and there exist para-accretive functions b such that

$$\int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) b(y_1) dy_1 = 0$$

and $\Lambda_k(b) = 0$ for all $x, y_2 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, then for all $f_1, f_2 \in L^1 + L^\infty$ and $j, k, \ell \in \mathbb{Z}$

$$|\Theta_j(M_b \Lambda_k^* f_1, f_2)(x)| \lesssim 2^{-\gamma|j-k|} \mathscr{M} f_1(x) \mathscr{M} f_2(x). \tag{4.17}$$

In each of the statements above, we take γ to be the smallest of the smoothness parameters guaranteed by the definitions of LPK, SLPK, BLPK and SBLPK.

Proof: We first prove (4.13). By Proposition 4.2.2 and the hypothesis that $\Theta_k(b) = 0$,

$$\begin{aligned}
|\Theta_j M_b \Lambda_k^* f(x)| &\leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (\theta_j(x,u) - \theta_j(x,y)) b(u) \lambda_k(y,u) du \right| |f(y)| dy \\
&\lesssim \int_{\mathbb{R}^{2n}} |\theta_j(x,u) - \theta_j(x,y)| \Phi_k^{N+\gamma}(y-u) |f(y)| du\, dy \\
&\lesssim 2^{\gamma(j-k)} \left(\Phi_j^N * |f|(x) + \Phi_k^N * |f|(x) \right) \\
&\lesssim 2^{\gamma(j-k)} \mathscr{M} f(x).
\end{aligned}$$

With a symmetric argument, the same estimate holds replacing $2^{\gamma(j-k)}$ with $2^{\gamma(k-j)}$. Therefore (4.13) holds. Now we prove (4.14). We first use that $\Lambda_k(b) = 0$ and Proposition 4.2.3

to estimate

$$
\begin{aligned}
&|\Lambda_k M_b \Theta_j(f_1,f_2)(x)| \\
&\qquad \le \int_{\mathbb{R}^{2n}} \left| \int_{\mathbb{R}^n} \lambda_k(x,u) b(u) (\theta_j(u,y_1,y_2) - \theta_j(x,y_1,y_2)) du \right| |f_1(y_1) f_2(y_2)| dy_1\, dy_2 \\
&\qquad \lesssim 2^{\gamma(j-k)} \int_{\mathbb{R}^n} \Phi_k^N(x-u) \left(\prod_{i=1}^{2} \Phi_j^N * |f_i|(u) + \prod_{i=1}^{2} \Phi_j^N * |f_i|(x) \right) du \\
&\qquad \lesssim 2^{\gamma(j-k)} \mathscr{M}(\mathscr{M} f_1 \cdot \mathscr{M} f_2)(x).
\end{aligned}
$$

We also have using the cancellation assumed for Θ_j

$$
\begin{aligned}
&|\Lambda_k M_b \Theta_j(f_1,f_2)(x)| \\
&\qquad \le \int_{\mathbb{R}^{2n}} \left| \int_{\mathbb{R}^n} (\lambda_k(x,u) - \lambda_k(x,y_1))\, b(u) \theta_j(u,y_1,y_2) du \right| |f_1(y_1) f_2(y_2)| dy_1\, dy_2 \\
&\qquad \lesssim 2^{(k-j)\gamma} \int_{\mathbb{R}^{3n}} \left(\Phi_k^N(x-u) + \Phi_k^N(x-y_1) \right) \prod_{i=1}^{2} \Phi_j^N(u-y_i) |f_i(y_i)| dy_i\, du \\
&\qquad \lesssim 2^{(k-j)\gamma} \int_{\mathbb{R}^n} \Phi_k^N(x-u) \prod_{i=1}^{2} \Phi_j^N * |f_i|(u) du \\
&\qquad\qquad + 2^{(k-j)\gamma} \int_{|x-y_1| \ge |x-u|/2} \Phi_k^N(x-y_1) \prod_{i=1}^{2} \Phi_j^N(u-y_i) |f_i(y_i)| dy_i\, du \\
&\qquad\qquad\quad + 2^{(k-j)\gamma} \int_{|x-y_1| < |x-u|/2} \Phi_k^N(x-y_1) \prod_{i=1}^{2} \Phi_j^N(u-y_i) |f_i(y_i)| dy_i\, du \\
&= 2^{(k-j)\gamma}(I + II + III).
\end{aligned}
$$

Note that $I \lesssim \mathscr{M}(\mathscr{M} f_1 \cdot \mathscr{M} f_2)(x)$, so this term is fine. In II, we may replace $\Phi_k^N(x-y_1)$ with $\Phi_k^N((x-u)/2)$ since $|x-y_1| \ge |x-u|/2$, and it follows that $II \lesssim I$. So II is bounded appropriately as well. The final term, *III* is bounded by

$$\begin{aligned}
&\int_{|x-y_1|<|x-u|/2} \Phi_k^N(x-y_1)\frac{2^{jn}|f_1(y_1)|}{(1+2^j(|x-u|-|x-y_1|))^N}\Phi_j^N * |f_2|(u)dy_1\,du \\
&\quad\lesssim \int_{|x-y_1|<|x-u|/2} \Phi_k^N(x-y_1)\Phi_j^N(x-u)|f_1(y_1)|\Phi_j^N * |f_2|(u)dy_1\,du \\
&\quad\leq \left(\int_{\mathbb{R}^n} \Phi_k^N(x-y_1)|f_1(y_1)|dy_1\right)\left(\int_{\mathbb{R}^n} \Phi_j^N(x-u)\Phi_j^N * |f_2|(u)\,du\right) \lesssim \mathscr{M} f_1(x)\mathscr{M} f_2(x).
\end{aligned}$$

This verifies that (4.14) holds. We move on to prove (4.15). For the estimate when $j \leq k$, we argue similar to the other cases: Using that $\Lambda_k^1(b_1)\cdot\Lambda_k^2(b_2) = 0$ and Proposition 4.2.3

$$\begin{aligned}
&|\Theta_j(M_{b_1}\Lambda_k^{1*}f_1, M_{b_2}\Lambda_k^{2*}f_2)(x)| \\
&\qquad\qquad \leq \int_{\mathbb{R}^{4n}} |\theta_j(x,u_1,u_2) - \theta_j(x,y_1,y_2)| \prod_{i=1}^{2} |b_i(u)\lambda_k^i(y_i,u_i)f_i(y_i)|dy_i\,du_i
\end{aligned}$$

$$\begin{aligned}
&\lesssim 2^{\gamma(j-k)} \int_{\mathbb{R}^{4n}} \prod_{i=1}^{2} \left(\Phi_j^N(x-u_i) + \Phi_j^N(x-y_i)\right)\Phi_k^N(u_i-y_i)|f_i(y_i)|du_i\,dy_i \\
&= 2^{\gamma(j-k)} \prod_{i=1}^{2} \int_{\mathbb{R}^n} \left(\Phi_j^N(x-y_i) + \Phi_k^N(x-y_i)\right)|f_i(y_i)|dy_i \lesssim 2^{\gamma(j-k)} \mathscr{M} f_1(x)\mathscr{M} f_2(x).
\end{aligned}$$

Finally using that $\Theta_j(b_1,b_2) = 0$, it follows from Proposition 4.2.3 that

$$\begin{aligned}
&|\Theta_j(M_{b_1}\Lambda_k^{1*}f_1, M_{b_2}\Lambda_k^{2*}f_2)(x)| \\
&\quad\leq \int_{\mathbb{R}^{4n}} |\theta_j(x,u_1,u_2)| \left|\prod_{i=1}^{2}\lambda_k^i(y_i,u_i) - \prod_{i=1}^{2}\lambda_k^i(y_i,x)\right| \prod_{i=1}^{2}|b_i(u_i)f_i(y_i)|dy_i\,du_i \\
&\quad\lesssim \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^{2n}} \left|\prod_{i=1}^{2}\lambda_k^i(y_i,u_i) - \prod_{i=1}^{2}\lambda_k^i(y_i,x)\right| \prod_{i=1}^{2}\Phi_j^{N+\gamma}(u_i-y_i)du_i\right) \prod_{i=1}^{m}|f_i(y_i)|dy_i
\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{\gamma(k-j)} \left(\Phi_j^N * |f_1|(x) + \Phi_k^N * |f_1|(x)\right) \left(\Phi_j^N * |f_2|(x) + \Phi_k^N * |f_2|(x)\right) \\
&\lesssim 2^{\gamma(k-j)} \mathscr{M} f_1(x) \mathscr{M} f_2(x).
\end{aligned}
$$

Note that by Remark 4.1.4, $\{\lambda_k^i\} \in LPK$ for $i = 1,2$ implies $\{\lambda_k^1(x,y_1)\lambda_k^2(x,y_2)\} \in BLPK$. Then (4.15) holds as well. Now we prove (4.16). Using that $\Lambda_k^1(b_1) = 0$, it follows that

$$
\begin{aligned}
&|\Theta_j(M_{b_1}\Lambda_k^{1*} f_1, M_{b_2}\Lambda_\ell^{2*} f_2)(x)| \\
&\quad \leq \int_{\mathbb{R}^{4n}} |\theta_j(x,u_1,u_2) - \theta_j(x,y_1,u_2)||\lambda_k^2(y_1,u_1)\lambda_\ell^2(y_2,u_2) \prod_{i=1}^{2} |b_i(u) f_i(y_i)| dy_i\, du_i \\
&\quad \lesssim 2^{\gamma(j-k)} \int_{\mathbb{R}^{4n}} \Phi_k^N(u_1 - y_1)\Phi_\ell^{N+\gamma}(u_2 - y_2) \prod_{i=1}^{2} \left(\Phi_j^N(x-u_i) + \Phi_j^N(x-y_i)\right) |f_i(y_i)| du_i\, dy_i \\
&\quad \leq 2^{\gamma(j-k)} \prod_{i=1}^{2} \int_{\mathbb{R}^n} \left(\Phi_j^N(x-y_i) + \Phi_k^N(x-y_i) + \Phi_\ell^N(x-y_i)\right) |f_i(y_i)| dy_i \\
&\quad \lesssim 2^{\gamma(j-k)} \mathscr{M} f_1(x) \mathscr{M} f_2(x).
\end{aligned}
$$

By a symmetric argument, it follows that

$$
|\Theta_j(M_{b_1}\Lambda_k^{1*} f_1, M_{b_2}\Lambda_\ell^{2*} f_2)(x)| \lesssim 2^{\gamma(j-\ell)} \mathscr{M} f_1(x) \mathscr{M} f_2(x).
$$

Finally using that $\Theta_j(b_1, \cdot) = 0$, it follows that

$$
\begin{aligned}
&|\Theta_j(M_{b_1}\Lambda_k^{1*} f_1, M_{b_2}\Lambda_k^{2*} f_2)(x)| \\
&\quad \leq \int_{\mathbb{R}^{4n}} |\theta_j(x,u_1,u_2)| \left|\lambda_k^1(y_1,u_1) - \lambda_k^1(y_1,x)\right| \lambda_\ell^2(y_2,u_2) \prod_{i=1}^{2} |b_i(u_i) f_i(y_i)| dy_i\, du_i \\
&\quad \lesssim 2^{\gamma(k-j)} \int_{\mathbb{R}^{4n}} \left(\Phi_k^N(y_1 - u_1) + \Phi_k^N(y_1 - x)\right) \Phi_\ell^N(y_2 - u_2) \prod_{i=1}^{2} \Phi_j^N(x-u_i) |f_i(y_i)| dy_i\, du_i
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim 2^{\gamma(k-j)} \int_{\mathbb{R}^{2n}} \prod_{i=1}^{2} \left(\Phi_j^N(x-y_i) + \Phi_k^N(x-y_i) + \Phi_\ell^N(x-y_i)\right) |f_i(y_i)| dy_i \\
&\lesssim 2^{\gamma(k-j)} \mathscr{M} f_1(x) \mathscr{M} f_2(x).
\end{aligned}
$$

By symmetry, it follows that

$$
|\Theta_j(M_{b_1}\Lambda_k^{1\,*} f_1, M_{b_2}\Lambda_k^{2\,*} f_2)(x)| \lesssim 2^{\gamma(\ell-j)} \mathscr{M} f_1(x) \mathscr{M} f_2(x).
$$

Therefore estimate (4.16) holds as well. Finally (4.17) is a straight-forward argument: Using that $\Lambda_k(b) = 0$, it follows that

$$
\begin{aligned}
&|\Theta_j(M_b \Lambda_k^* f_1, f_2)(x)| \\
&\quad \leq \int_{\mathbb{R}^{3n}} |\theta_j(x,u,y_2) - \theta_j(x,y_1,y_2)| \, |b(y_1)\lambda_k(u,y_1)| du |f_1(u) f_2(y_2)| dy_1 \, dy_2 \\
&\quad \lesssim 2^{\gamma(j-k)} \int_{\mathbb{R}^{3n}} \left(\Phi_j^N(x-u) + \Phi_j^N(x-y_1)\right) \Phi_j^N(x-y_2) \Phi_k^N(u-y_1) du |f_1(u) f_2(y_2)| dy_1 \, dy_2 \\
&\quad \lesssim 2^{\gamma(j-k)} \mathscr{M} f_1(x) \mathscr{M} f_2(x),
\end{aligned}
$$

and using that $\Theta_j(b,\cdot) = 0$, it follows that

$$
\begin{aligned}
&|\Theta_j(M_b \Lambda_k^* f_1, f_2)(x)| \\
&\quad \leq \int_{\mathbb{R}^{3n}} |\theta_j(x,u,y_2) b(y_1)| \, |\lambda_k(u,y_1) - \lambda_k(x,y_1)| du |f_1(u) f_2(y_2)| dy_1 \, dy_2 \\
&\quad \lesssim 2^{\gamma(k-j)} \int_{\mathbb{R}^{3n}} \Phi_j^N(x-u) \Phi_j^N(x-y_2) \left(\Phi_k^N(u-y_1) + \Phi_k^N(x-y_1)\right) du |f_1(u) f_2(y_2)| dy_1 \, dy_2 \\
&\quad \lesssim 2^{\gamma(k-j)} \mathscr{M} f_1(x) \mathscr{M} f_2(x).
\end{aligned}
$$

This completes the proof. □

Chapter 5

Convergence Results

There is a natural trade-off between continuity assumptions of operators and convergence results necessary to approximate them. Namely, weaker continuity assumptions on an operator $T : X \to Y$ require stronger convergence of the input functions $f_N, f \in X$ in order to decompose the operator. For example, the linear Littlewood-Paley square function operators Θ_k are continuous from L^p into L^p for all $1 \leq p \leq \infty$, so one need only require $f_N \to f$ in L^p to conclude that $\Theta_k f_N \to \Theta_k f$ in L^p. On the other hand, a scalar valued Calderón-Zygmund singular integral operator T is only assumed to be continuous from $\mathscr{S}$ into $\mathscr{S}'$. So in order to pass a limit, $Tf_N \to Tf$ in $\mathscr{S}'$, we must have that $f_N \to f$ in $\mathscr{S}$, which is a much stronger type of convergence than only L^p.

In this chapter, we prove convergence in various spaces to suit the various operators that we work with. This chapter is organized in the following way: Approximation to identity operators tested on the function 1, reproducing formulas for operators tested on the function 1, approximation to identity operators tested on accretive functions, and reproducing formulas tested on accretive functions.

5.1 Approximations to the Identity

An approximation to the identity operator P_k is essentially operator that averages at scale 2^{-k}. Intuitively we expect $P_k f$ to approximate a function f well when averaged at very small scales (when $k >> 0$ is very large) given that f is a nice enough function. On the other hand averaging at very large scale (when $k << 0$ is very small), the operator $P_k f$ somehow indicates the assymptotic behavior of f. So if f has some sort of average decay (for example $f \in L^p$ for some p), it is reasonable to expect $P_k f$ to tend to zero as $k \to -\infty$. In this section, we make these concepts rigorous in the averaged L^p sense and in a much stronger $\mathscr{S}$ topology sense. These results are well known, but we provide a proof of them anyways for the sake of completeness.

Proposition 5.1.1 *Suppose $p_k : \mathbb{R}^{2n} \to \mathbb{C}$ for $k \in \mathbb{Z}$ satisfy $|p_k(x,y)| \lesssim \Phi_k^N(x-y)$, $N > n$, and define*

$$P_k f(x) = \int_{\mathbb{R}^n} p_k(x,y) f(y) dy$$

for $f \in L^1 + L^\infty$. If $P_k(1) = 1$ for all $k \in \mathbb{Z}$, then $P_k f \to f$ in L^p as $k \to \infty$ for all $f \in L^p$ when $1 \le p < \infty$ and $P_k f \to 0$ in L^p as $k \to -\infty$ for all $f \in L^p \cap L^q$ for $1 \le q < p < \infty$.

Proof: For $f \in L^p$ with $1 \le p < \infty$

$$\begin{aligned} ||P_k f - f||_{L^p} &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} p_k(x,y) f(y) dy - \int_{\mathbb{R}^n} p_k(x,y) f(x) dy \right|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} p_k(x, x - 2^{-k} y)(f(x - 2^{-k} y) - f(x)) 2^{-kn} dy \right|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

$$\lesssim \int_{\mathbb{R}^n} \Phi_0^N(y) \left(\int_{\mathbb{R}^n} |f(x-2^{-k}y) - f(x)|^p dx \right)^{\frac{1}{p}} dy$$
$$\lesssim \int_{\mathbb{R}^n} \Phi_0^N(y) ||f(\cdot - 2^{-k}y) - f||_{L^p} dy.$$

Note that $\Phi_0^N(y)||f(\cdot - 2^{-k}y) - f||_{L^p} \leq 2||f||_{L^p}\Phi_0^N(y)$ which is an $L^1(\mathbb{R}^n)$ function independent of k. So by dominated convergence and the continuity of translation in $||\cdot||_{L^p}$,

$$\lim_{k\to\infty} ||P_k f - f||_{L^p} \lesssim \int_{\mathbb{R}^n} \Phi_0^N(y) \lim_{k\to\infty} ||f(\cdot - 2^{-k}y) - f||_{L^p} dy = 0.$$

Next we compute

$$|P_k f(x)| \lesssim ||\Phi_k^N||_{L^{q'}} ||f||_{L^q} = 2^{kn/q} ||\Phi_0^N||_{L^{q'}} ||f||_{L^q}.$$

So $P_k f \to 0$ almost everywhere as $k \to -\infty$. Also by Proposition 4.3.1

$$|P_k f(x)| \lesssim \Phi_k^N * |f|(x) \lesssim \mathscr{M} f(x).$$

Since $f \in L^p$, it follows that $\mathscr{M} f \in L^p(\mathbb{R}^n)$ as well when $1 < p < \infty$. So by dominated convergence

$$\lim_{k\to-\infty} ||P_k f||_{L^p}^p = \int_{\mathbb{R}^n} \lim_{k\to\infty} |P_k f(x)|^p dx = 0.$$

This proves the proposition. □

Proposition 5.1.2 *Let $\varphi \in \mathscr{S}$ with $\widehat{\varphi}(0) = 1$, and define $P_k f = \varphi_k * f$. Then for any $f \in \mathscr{S}$, $P_k f \to f$ in $\mathscr{S}$ and for any $f \in \mathscr{S}_0$, $P_k f \to 0$ in $\mathscr{S}$ as $k \to -\infty$. Furthermore, for $f_1, f_2 \in \mathscr{S}$, $P_k f_1 \otimes P_k f_2 \to f_1 \otimes f_2$ in $\mathscr{S}(\mathbb{R}^{2n})$ and for $f_1, f_2 \in \mathscr{S}_0$, $P_k f_1 \otimes P_k f_2 \to 0$ as $k \to -\infty$ in*

$\mathscr{S}(\mathbb{R}^{2n})$.

Proof: Let $\alpha, \beta \in \mathbb{N}_0^n$, and since $\mathscr{F}$ is an isometry on $\mathscr{S}$, it is sufficient to show that $\widehat{P_k f} \to \widehat{f}$ as $k \to \infty$. So we consider for $k \in \mathbb{N}$

$$\rho_{\alpha,\beta}(\widehat{P_k f} - \widehat{f}) \leq \sum_{\mu+\nu=\beta} \sup_{\xi \in \mathbb{R}^n} \left| \xi^\alpha \partial^\mu (\widehat{\varphi}(2^{-k}\xi) - 1) \partial^\nu \widehat{f}(\xi) \right|.$$

When $\mu = 0$, we estimate

$$|\widehat{\varphi}(2^{-k}\xi) - 1| = |\widehat{\varphi}(2^{-k}\xi) - \widehat{\varphi}(0)| \leq ||\nabla\widehat{\varphi}||_{L^\infty}(2^{-k}|\xi|).$$

We also estimate for $|\mu| \geq 1$

$$|\partial^\mu(\widehat{\varphi}(2^{-k}\xi) - 1)| = 2^{-|\mu|k}|(\partial^\mu\widehat{\varphi})(2^{-k}\xi)| \leq 2^{-k}||\partial^\mu\widehat{\varphi}||_{L^\infty}.$$

Then we have for any $\mu \in \mathbb{N}_0^n$

$$|\xi^\alpha \partial^\mu(\widehat{\varphi}(2^{-k}\xi) - 1)\partial^\nu \widehat{f}(\xi)| \lesssim 2^{-k}(1 + |\xi|)|\xi^\alpha \partial^\nu \widehat{f}(\xi)| \leq 2^{-k}(\rho_{\alpha,\nu}(\widehat{f}) + \rho_{\alpha',\nu}(\widehat{f}))$$

where $\alpha' = \alpha + (1, .., 1)$. Therefore $P_k f \to f$ in $\mathscr{S}$ as $k \to \infty$ when $f \in \mathscr{S}$. Now assume that $f \in \mathscr{S}_0$, and we look at

$$\rho_{\alpha,\beta}(\widehat{P_{-k} f}) \leq \sum_{\mu+\nu=\beta} \sup_{\xi \in \mathbb{R}^n} \left| \xi^\alpha \partial^\mu \widehat{\varphi}(2^k\xi) \partial^\nu \widehat{f}(\xi) \right|.$$

With $\beta, \mu, \nu \in \mathbb{N}_0^n$ fixed such that $\mu + \nu = \beta$, we choose $\alpha' \in \mathbb{N}_0^n$ such that $|\alpha'| > |\beta| \geq |\mu|$ and $\alpha_i' > \alpha_i$ for each $i = 1, ..., n$. Then

$$\begin{aligned}
\left|\xi^\alpha \partial^\mu \widehat{\varphi}(2^k \xi) \partial^\nu \widehat{f}(\xi)\right| &= 2^{|\mu|k} \left|(\partial^\mu \widehat{\varphi})(2^k \xi) \xi^\alpha \partial^\nu \widehat{f}(\xi)\right| \\
&= 2^{|\mu|k} \frac{\rho_{\alpha',\mu}(\partial^\mu \widehat{\varphi})}{|2^k \xi^{\alpha'}|} |\xi^\alpha \partial^\nu \widehat{f}(\xi)| \\
&= 2^{k(|\mu|-|\alpha'|)} \rho_{\alpha',\mu}(\partial^\mu \widehat{\varphi}) \left|\frac{\xi^\alpha \partial^\nu \widehat{f}(\xi)}{\xi^{\alpha'}}\right| \\
&\leq 2^{k(|\mu|-|\alpha'|)} \rho_{\alpha',\mu}(\partial^\mu \widehat{\varphi})(\rho_0(\widehat{f}) + \rho_{\alpha'-\alpha,\nu}(\widehat{f})).
\end{aligned}$$

Since we chose $|\alpha'| > |\mu|$, it follows that $\rho_{\alpha,\beta}(P_k f) \to 0$ as $k \to -\infty$. Now if $f_1, f_2 \in \mathscr{S}$, it follows that for all $\alpha, \beta \in \mathbb{N}_0^{2n}$ where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ for $\alpha_i, \beta_i \in \mathbb{N}_0^n$,

$$\begin{aligned}
\rho_{\alpha,\beta}(P_k f_1 \otimes P_k f_2 - f_1 \otimes f_2) &\leq \rho_{\alpha,\beta}((P_k f_1 - f_1) \otimes P_k f_2) + \rho_{\alpha,\beta}(f_1 \otimes (P_k f_2 - f_2)) \\
&\leq \rho_{\alpha_1,\beta_1}(P_k f_1 - f_1) \rho_{\alpha_2,\beta_2}(P_k f_2) + \rho_{\alpha,\beta}(f_1) \rho_{\alpha,\beta}(P_k f_2 - f_2) \\
&\leq \rho_{\alpha_1,\beta_1}(P_k f_1 - f_1) \rho_{\alpha_2,\beta_2}(P_k f_2 - f_2) + \rho_{\alpha_1,\beta_1}(P_k f_1 - f_1) \rho_{\alpha_2,\beta_2}(f_2) \\
&\qquad + \rho_{\alpha,\beta}(f_1) \rho_{\alpha,\beta}(P_k f_2 - f_2).
\end{aligned}$$

Since $P_k f_1 \to f_1$ and $P_k f_2 \to f_2$ in $\mathscr{S}(\mathbb{R}^n)$ as $k \to \infty$, the above tends to zero as $k \to \infty$. A similar argument proves that $P_k f_1 \otimes P_k f_2 \to 0$ in $\mathscr{S}(\mathbb{R}^{2n})$ as $k \to -\infty$. $\square$

5.2 Reproducing Formulas

Reproducing formulas are a decomposition technique that breaks a function into many peices that "don't see each other" in some sense, typically quantified in terms of orthogonality or almost orthogonality. The formulas in this section are readily interpretted as a decomposition in the frequency domain, where each term is the frequency content of the function at scale 2^{-k} that do not interfere with each other. A little more precisely, these formulas can be constructed by telescoping an approximation to identity operator

$$P_N f - P_{-N} f = \sum_{k=-N}^{N-1} (P_{k+1} - P_k) f,$$

where the frequency content at scale 2^{-k} is given by the average at scale $2^{-(k+1)}$ minus the average at scale 2^{-k}. This roughly summarizes the technique we use for all the results in this section.

The first reproducing formula we present contains what is know as Calderón's reproducing formula. The continuous version of this result is originally due to Calderón [8], but we state a well known discrete version of the formula. Again both of these results are well known, but we provide a quick proof for the convenience of the reader.

Proposition 5.2.1 *There exist convolution operators $Q_k f = \psi_k * f$ for $k \in \mathbb{Z}$ such that*

$$\sum_{k \in \mathbb{Z}} Q_k f = f \tag{5.1}$$

in L^p when $f \in L^p \cap L^q$ for some $1 \leq q < p < \infty$. Furthermore, $\psi \in \mathscr{S}(\mathbb{R}^n)$ and there exists $\widetilde{Q}_k f = \widetilde{\psi}_k * f$ where $\widetilde{\psi} \in \mathscr{S}$ has mean zero and $Q_k = \widetilde{Q}_k Q_k$ for all $k \in \mathbb{Z}$. Also if $\{\theta_k\} \in BLPK$, then for $f_i \in L^{p_i} \cap L^{q_i}$, $i = 1,2$ with $1 \leq q_i < p_i < \infty$, and all $j \in \mathbb{Z}$

$$\sum_{k \in \mathbb{Z}} \Theta_j \Pi_k^1(f_1, f_2) + \Theta_j \Pi_k^2(f_1, f_2) = \Theta_j(f_1, f_2) \tag{5.2}$$

where the convergence holds in $L^p(\mathbb{R}^n)$ when p, p_1, p_2 satisfy (1.1) *and*

$$\Pi_k^1(f_1, f_2) = Q_k f_1 \otimes P_{k+1} f_2,$$

$$\Pi_k^2(f_1, f_2) = P_k f_1 \otimes Q_k f_2.$$

Proof: Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\widehat{\varphi} \equiv 1$ on $B(0,1/2)$ and $\operatorname{supp}(\widehat{\varphi}) \subset B(0,1)$. Define $P_k f = \varphi_k * f$, $\psi(x) = 2^n \varphi(2x) - \varphi(x)$, $Q_k f = \psi_k * f$, $\widetilde{\psi}(x) = 2^{3n}\varphi(2^3 x) - 2^{-2n}\varphi(2^{-2}x)$, and $\widetilde{Q}_k f = \widetilde{\psi}_k * f$. Fix $1 < p < \infty$, and let $f \in L^p \cap L^q$ for some $1 \leq q < p$. By Proposition 5.1.1, we have

$$f = \lim_{N \to \infty} P_N f - P_{-N} f = \lim_{N \to \infty} \sum_{k=-N}^{N-1} P_{k+1} f - P_k f = \sum_{k \in \mathbb{Z}} Q_k f,$$

where the convergence holds in L^p. Note that

$$\operatorname{supp}(\widehat{\psi_k}) = \operatorname{supp}(\widehat{\psi}(2^{-k} \cdot)) = \operatorname{supp}\left(\widehat{\varphi}(2^{-(k+1)} \cdot) - \widehat{\varphi}(2^{-k} \cdot)\right) \subset B(0, 2^{k+1}) \backslash B(0, 2^{k-1})$$

and $\widehat{\widetilde{\psi}} \equiv 1$ on $B(0, 2^{k+1}) \backslash B(0, 2^{k-1})$ since

$$\widehat{\varphi}(2^{-(k+3)} \cdot) \equiv 1 \text{ and } \widehat{\varphi}(2^{-(k-2)} \cdot) \equiv 0 \text{ on } B(0, 2^{k+1}) \backslash B(0, 2^{k-1}).$$

It easily follows that $\widetilde{Q}_k Q_k f = Q_k f$ whenever $f \in \mathscr{S}$. For $f \in L^p$, take a sequence $f_j \in \mathscr{S}$

such that $f_j \to f$ in L^p as $j \to \infty$. Then

$$\begin{aligned}
||\widetilde{Q}_k Q_k f - Q_k f||_{L^p} &\le ||\widetilde{Q}_k Q_k f - Q_k f_j||_{L^p} + ||Q_k f_j - Q_k f||_{L^p} \\
&= ||\widetilde{Q}_k Q_k (f - f_j)||_{L^p} + ||Q_k (f_j - f)||_{L^p} \\
&\lesssim ||\mathscr{M}(f - f_j)||_{L^p} + ||\mathscr{M}(f_j - f)||_{L^p} \\
&\lesssim ||f_j - f||_{L^p},
\end{aligned}$$

which tends to zero as $j \to \infty$. Therefore $\widetilde{Q}_k Q_k f = Q_k f$ in L^p for any $1 \le p \le \infty$ and pointwise almost everywhere. Finally since $\widehat{\psi}_k$ and $\widehat{\widetilde{\psi}_k}$ are supported away from the origin, it follows that ψ_k and $\widetilde{\psi}_k$ have mean zero for all $k \in \mathbb{Z}$ as well. Now assume that $f_i \in L^{p_i} \cap L^{q_i}$ for $i = 1,2$ with $1 \le q_i < p_i < \infty$, and note that there exists a finite sum of Schwartz semi-norms ρ such that $|\varphi_k(x)| \le \rho(\varphi)\Phi_k^{n+1}(x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Then $|P_k f(x)| \lesssim \mathscr{M} f(x)$ for all $x \in \mathbb{R}^n$, and for $i = 1,2$, we have the uniform bounds $||P_k f_i||_{L^{p_i}} \lesssim ||f||_{L^{p_i}} < \infty$. Hence we have that

$$\begin{aligned}
&\left|\left|\left(\sum_{k=-N}^{N-1} \Theta_j \Pi_k(f_1, f_2)\right) - \Theta_j(f_1, f_2)\right|\right|_{L^p} \\
&\qquad = \left|\left|\left(\sum_{k=-N}^{N-1} \Theta_j(Q_k f_1, P_{k+1} f_2) + \Theta_j(P_k f_1, Q_k f_2)\right) - \Theta_j(f_1, f_2)\right|\right|_{L^p} \\
&\qquad = \left|\left|\left(\sum_{k=-N}^{N-1} \Theta_j(P_{k+1} f_1, P_{k+1} f_2) - \Theta_j(P_k f_1, P_k f_2)\right) - \Theta_j(f_1, f_2)\right|\right|_{L^p} \\
&\qquad \le ||\Theta_j(P_N f_1, P_N f_2) - \Theta_j(f_1, f_2)||_{L^p} + ||\Theta_j(P_{-N} f_1, P_{-N} f_2)||_{L^p} \\
&\qquad \lesssim ||P_N f_1 - f_1||_{L^{p_1}} ||f_2||_{L^{p_2}} + ||f_1||_{L^{p_1}} ||P_N f_2 - f_2||_{L^{p_2}} + ||P_{-N} f_1||_{L^{p_1}} ||P_{-N} f_2||_{L^{p_2}}.
\end{aligned}$$

All three terms above tend to zero as $N \to \infty$ by Proposition 5.1.1. This completes the proof. □

The next proposition is an extension of Calderón's reproducing formula to the stronger topology of the class of Schwartz functions. This was used in David-Journé [28] to prove the original T1 theorem. They use the stronger convergence of the reproducing formula work with the weak continuity assumptions on Calderón-Zygmund singular integral operators. The proof we present here is short, but the content of the proof was in Proposition 5.1.2.

Proposition 5.2.2 (David-Journé [28]) *Let Q_k be as in Proposition 5.2.1. Then formula* (5.1) *holds in the topology of $\mathscr{S}$ whenever $f \in \mathscr{S}_0$. Also formula* (5.2) *holds in the topology of $\mathscr{S}(\mathbb{R}^{2n})$ whenever $f_1, f_2 \in \mathscr{S}_0$.*

Proof: For $f \in \mathscr{S}_0$, consider

$$\sum_{|k|<N} Q_k f = \sum_{|k|<N} (P_{k+1} - P_k) f = P_N f - P_{-(N-1)} f.$$

It follows from Proposition 5.1.2 that the first term tends to f and the second to 0 in $\mathscr{S}$ as $N \to \infty$. It also easily follows that for $f_1, f_2 \in \mathscr{S}_0$

$$\begin{aligned}\sum_{|k|<N} Q_k f_1 \otimes P_{k+1} f_2 + P_k f_1 \otimes Q_k f_2 &= \sum_{|k|<N} (P_{k+1} - P_k) f_1 \otimes P_{k+1} f_2 + P_k f_1 \otimes (P_{k+1} - P_k) f_2 \\ &= \sum_{|k|<N} P_{k+1} f_1 \otimes P_{k+1} f_2 - P_k f_1 \otimes P_k f_2 \\ &= P_N f_1 \otimes P_N f_2 - P_{-(N-1)} f_1 \otimes P_{-(N-1)} f_2,\end{aligned}$$

which goes to zero in $\mathscr{S}(\mathbb{R}^{2n})$ as $N \to \infty$ by Proposition 5.1.2. □

5.3 Approximation to the Identity with Respect to Para-Accretive Functions

We also work with para-accretive perturbed versions of the convergence results we just proved. The approximation to identity formula in L^p follows immediately as a corollary of Proposition 5.1.1 since the convergence is in L^p, which "does not see" a pertubation by a para-accretive function. That is the convergence in Proposition 5.1.1 relies only on size estimate on the kernels $p_k(x,y)$, not on regularity estimates. So there is no harm in replacing $p_k(x,y)$ by $p_k(x,y)b(y)$. We make this precise in Corollary 5.3.1.

Corollary 5.3.1 *Let b be a para-accretive function. Suppose $s_k : \mathbb{R}^{2n} \to \mathbb{C}$ for $k \in \mathbb{Z}$ satisfy $|s_k(x,y)| \lesssim \Phi_k^N(x-y)$ for some $N > n$, and define S_k*

$$S_k f(x) = \int_{\mathbb{R}^n} s_k(x,y) f(y) dy$$

for $f \in L^1 + L^\infty$. If $S_k(b) = 1$ for all $k \in \mathbb{Z}$, then $S_k M_b f \to f$ in L^p as $k \to \infty$ for all $f \in L^p$ when $1 \leq p < \infty$ and $S_k M_b f \to 0$ in L^p as $k \to -\infty$ for all $f \in L^p \cap L^q$ when $1 \leq q < p < \infty$.

Proof: Define $P_k f = S_k M_b f$ with kernel p_k. It is obvious that $|p_k(x,y)| \lesssim ||b||_{L^\infty} \Phi_k^N(x-y)$, and $P_k(1) = S_k(b) = 1$. So by Proposition 5.1.1, $f \in L^p$ implies that $P_k f \to f$ in L^p when $f \in L^p$ and $1 \leq p < \infty$. Also when $f \in L^p \cap L^q$ fo $1 \leq q < p < \infty$, it follows that $P_k f \to 0$ as $k \to -\infty$. Therefore $S_k M_b f = P_k f \to f$ as $k \to \infty$ and $S_k f = P_k f \to 0$ as $k \to -\infty$ for appropriate f. □

We make a special definition of approximation to the identity operators with respect to a para-accretive function, which was defined David-Journé-Semes in [29] and further developed by Han in [48]. The convergence results we prove about these approximation to identity operators here (including Corollary 5.3.1) were first proved in [29] and [48].

Definition 5.3.2 *Let $b \in L^\infty$ be a para-accretive function. A collection of operators $\{S_k\}_{k\in\mathbb{Z}}$ defined by*

$$S_k f(x) = \int_{\mathbb{R}^n} s_k(x,y) f(y) dy$$

for kernel functions $s_k : \mathbb{R}^{2n} \to \mathbb{C}$ is an approximation to identity with respect to b if $\{s_k\} \in$ SLPK, $S_k(b) = S_k^*(b) = 1$, and

$$\begin{aligned}|s_k(x,y) - s_k(x',y) - s_k(x,y') + s_k(x',y')| &\le A 2^{kn} (2^k|x-x'|)^\gamma (2^k|y-y'|)^\gamma \\ &\times \left(\Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x'-y) + \Phi_k^{N+\gamma}(x-y') + \Phi_k^{N+\gamma}(x'-y')\right)\end{aligned} \tag{5.3}$$

We say that an approximation to identity with respect to b has compactly supported kernel if $s_k(x,y) = 0$ whenever $|x-y| > 2^{-k}$.

Remark 5.3.3 *Given a para-accretive function b, we define a particular approximation to the identity with respect to b. Let $\varphi \in \mathbb{C}_0^\infty$ be radial with integral 1 and supp$(\varphi) \subset B(0,1/8)$. Define $S_k^b = P_k M_{(P_k b)^{-1}} P_k$. To define the operators, one needs to know that $P_k b(x) \neq 0$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. In fact, it was shown in [29] that $|P_k f(x)| \ge c > 0$ uniformly for $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. It follows that the associated kernels $\{s_k^b\} \in$ SLPK, and it also follows that s_k^b satisfies* (5.3)

$$\begin{aligned}&|s_k(x,y) - s_k(x',y) - s_k(x,y') + s_k(x',y')| \\ &\quad = \left| \int_{\mathbb{R}^n} (\varphi_k(x-u) - \varphi_k(x'-u))(P_k b(u))^{-1} (\varphi_k(y-u) - \varphi_k(y'-u)) du \right| \\ &\quad \lesssim 2^{kn} (2^k|x-x'|)^\gamma (2^k|y-y'|)^\gamma \\ &\qquad \times \left(\Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x'-y) + \Phi_k^{N+\gamma}(x-y') + \Phi_k^{N+\gamma}(x'-y')\right).\end{aligned}$$

Hence S_k^b is an approximation to identity with respect to b. Furthermore, S_k^b is self-adjoint and has compactly supported kernel. These operators were originally defined in [29] and the condition (5.3) *was verified in [48].*

In the same way that we introduced stronger convergence results for singular integral operators in the "unperturbed" (T1 type testing condition setting), we introduce a stronger sense of convergence for the "perturbed" (Tb type testing condition setting) to work with the weak continuity of singular integral operators in the Tb setting. These stronger convergence results were used in the proof of the Tb theorem original by David-Journé-Semmes in [29].

Proposition 5.3.4 (David-Journé-Semmes [29]) *Let b be a para-accretive function, $\{S_k\}$ be the approximation to identity with respect to b that has compactly supported kernel, and $\delta_0 > 0$. Then $M_bS_NM_bf \to bf$ and $M_bS_{-N}M_bf \to 0$ in bC_0^{δ} as $N \to \infty$ for all $f \in C_0^{\delta_0}$ and $0 < \delta < \delta_0$. In particular these convergence results hold for the operators defined in Remark 5.3.3.*

Proof: Let $f \in C_0^{\delta_0}$ and $0 < \delta < \delta_0$. Without loss of generality assume that $\gamma = \delta$, where γ is the smoothness parameter of s_k. We must check that $||S_NM_bf - f||_{\delta} \to 0$ as $N \to \infty$. So we start by estimating

$$\begin{aligned}
&|(S_NM_bf(x) - f(x)) - (S_NM_bf(y) - f(y))| \\
&\qquad = \left| \int_{\mathbb{R}^n} (s_N(x,u)(f(u) - f(x))b(u)du - \int_{\mathbb{R}^n} (s_N(y,u)(f(u) - f(y))b(u)du \right| \\
&\qquad \leq ||b||_{L^\infty} \int_{\mathbb{R}^n} |F_N^x(u) - F_N^y(u)|du
\end{aligned}$$

where $F_N^x(u) = s_N(x,u)(f(u)-f(x))$. Consider $u \in B(y, 2^{-N})$ and it follows that

$$
\begin{aligned}
|F_N^x(u) - F_N^y(u)| &= |s_N(x,u)(f(u)-f(x)) - s_N(y,u)(f(u)-f(y))| \\
&\leq |s_N(x,u)|\,|f(y)-f(x)| + |s_N(x,u) - s_N(y,u)|\,|(f(u)-f(y))| \\
&\lesssim ||f||_{\delta_0} 2^{nN}|x-y|^{\delta_0} + ||f||_{\delta_0} 2^{nN}(2^N|x-y|)^{\delta_0}|y-u|^{\delta_0} \qquad (5.4) \\
&\lesssim ||f||_{\delta_0} 2^{nN}|x-y|^{\delta_0}
\end{aligned}
$$

With a similar argument, it follows that for $u \in B(x, 2^{-N})$, $|F_N^x(u) - F_N^y(u)| \lesssim ||f||_{\delta_0} 2^{nN}|x-y|^{\delta_0}$. Now we may also estimate $|F_N^x(u)|$ in the following way for $u \in B(x, 2^{-N})$,

$$
|F_N^x(u)| \lesssim 2^{nN}|f(u)-f(x)| \leq ||f||_{\delta_0} 2^{nN}|u-x|^{\delta_0} \leq ||f||_{\delta_0} 2^{nN} 2^{-\delta_0 N}. \qquad (5.5)
$$

Using the support properties of s_k, we have that $\text{supp}(F_N^x - F_N^y) \subset B(x, 2^{-N}) \cup B(y, 2^{-N})$. Then it follows from (5.4), (5.5) and $\frac{\delta}{\delta_0} \in (0,1)$ that

$$
\begin{aligned}
|F_N^x(u) - F_N^y(u)| &\lesssim \left(||f||_{\delta_0} 2^{nN}|x-y|^{\delta_0}\right)^{\frac{\delta}{\delta_0}} \left(||f||_{\delta_0} 2^{nN} 2^{-\delta_0 N}\right)^{1-\frac{\delta}{\delta_0}} \\
&\lesssim ||f||_{\delta_0} 2^{nN}|x-y|^{\delta} 2^{-(\delta_0-\delta)N}.
\end{aligned}
$$

Therefore $S_N M_b f \to f$ in $||\cdot||_\delta$ since

$$
\begin{aligned}
\frac{|(S_N M_b f(x) - f(x)) - (S_N M_b f(y) - f(y))|}{|x-y|^\delta} &\leq \frac{1}{|x-y|^\delta} \int_{\mathbb{R}^n} |F_N^x(u) - F_N^y(u)| du \\
&\lesssim ||f||_{\delta_0} 2^{-(\delta_0-\delta)N} \int_{B(x,2^{-N}) \cup B(y,2^{-N})} 2^{nN} du \\
&\lesssim ||f||_{\delta_0} 2^{-(\delta_0-\delta)N}.
\end{aligned}
$$

This proves that $S_N M_b f \to f$ in C_0^δ as $N \to \infty$. Now we consider $S_{-N} M_b f$ as $N \to \infty$. We also have

$$\begin{aligned}
\frac{|S_{-N}M_bf(x) - S_{-N}M_bf(y)|}{|x-y|^\delta} &\leq \frac{1}{|x-y|^\delta}\int_{\mathbb{R}^n} |s_{-N}(x,u) - s_{-N}(y,u)|\,|b(u)f(u)|du \\
&\lesssim \frac{||f||_{L^\infty}}{|x-y|^\delta}\left(\int_{|x-u|<2^N} + \int_{|y-u|<2^N}\right) 2^{-nN}(2^{-N}|x-y|)^\delta du \\
&\lesssim ||f||_{L^\infty} 2^{-\delta N}.
\end{aligned}$$

Note that $||f||_{L^\infty} < \infty$ since f is continuous and compactly supported. Therefore $S_N M_b f \to f$ and $S_{-N} M_b f \to 0$ as $N \to \infty$ in the topology of C_0^δ. □

5.4 Reproducing Formulas with Respect to Para-Accretive Functions

We will state a Calderón type reproducing formula for para-accretive functions, which was proved in [48]. This formula can roughly be though of as a perturbed version of Calderón's reproducing formula and may even seem intuitively obvious, but many non-trivial, technical details arrise in the proof. So we do not prove it here, and instead refer the reader to the work of Han [48].

Theorem 5.4.1 (Han [48]) *Let $b \in L^\infty$ be a para-accretive function and S_k for $k \in \mathbb{Z}$ be approximation to the identity operators with respect to b. Define $D_k = S_{k+1} - S_k$. There exist operators $\widetilde{D}_k$ such that*

$$\sum_{k\in\mathbb{Z}} \widetilde{D}_k M_b D_k M_b f = f \tag{5.6}$$

in L^p for all $1 < p < \infty$ whenever $f \in C_0^\delta$ for some $\delta > 0$ and bf has mean zero. Furthermore, $\widetilde{D}_k(b) = \widetilde{D}_k^*(b) = 0$ and $\widetilde{D}_k$ is defined by

$$\widetilde{D}_k f(x) = \int_{\mathbb{R}^n} \widetilde{d}_k(x,y) f(y) dy$$

where $\{\widetilde{d}_k^*\} \in LPK$, which is defined $\widetilde{d}_k^*(x,y) = \widetilde{d}_k(y,x)$.

We will use this L^p reproducing formula extensively for Littlewood-Paley square function operators, as the convergence in Theorem 5.4.1 is well suited for the continuity properties of these operators. Although we will avoid using this formula in the topology of C_0^δ, and get by only using Proposition 5.3.4 to decompose the para-accretive perturbed singular integral operators. We still need the perturbed reproducing formula in a slightly stronger version than Theorem 5.4.1. Namely we need that the formula converges in H^1. The remainder of this section is dedicated to extending the convergence of the para-accretive reproducing formula in Theorem 5.4.1 to convergence in H^1. We start with a lemma about H^1 functions.

Lemma 5.4.2 *If $f : \mathbb{R}^n \to \mathbb{C}$ has mean zero and*

$$|f(x)| \lesssim c_{j,k} \left(\Phi_j^N(x) + \Phi_k^N(x) \right)$$

for some $N > n$ and $j,k \in \mathbb{Z}$, then $f \in H^1$ and $||f||_{H^1} \lesssim c_{j,k} 2^{|j-k|(N-n)}$. The constant here is independent of j and k, but may depend on N.

The statement of this lemma is a bit counterintuitive in that given a function that satisfies the hypotheses for some N, the conclusion seems to be strengthened by taking $N > n$ as small as possible. This will make the term $2^{|j-k|(N-n)}$ smaller, but there is an implicit constant hidden by the $\lesssim$ symbol that blows up as $N \to n$. That is, the conclusion can be written of the form $||f||_{H^1} \le C_{N,n} c_{j,k} 2^{|j-k|(N-n)}$ where the constant $C_{N,n} \to \infty$ as N decreases to n. In fact, it can be shown that this constant satisfies $C_{N,n} \le C'_n (N-n)^{-2}$ for some C'_n depending only on the dimension.

The proof of this lemma is more or less a standard proof of other results for H^1, but precisely tracks a few more terms. The proof is essentially due to Uchiyama [87], but the one presented is closer to the proofs of Lemmas 1 and 2 of Wilson [89]. In these lemmas, Wilson proves that if $|f(x)| \lesssim \Phi_k^N(x)$ and f has mean zero, then $f \in H^1$ where the estimate for $||f||_{H^1}$ is independent of k. We make a few modifications to account for replacing Φ_k^N with $\Phi_j^N + \Phi_k^N$. The estimate on $||f||_{H^1}$ in Lemma 5.4.2 recovers the estimate from [89]: Assume that $j \approx k$ so that $\Phi_j^N(x) + \Phi_k^N(x) \approx \Phi_k^N(x)$. Then we are in a situation where both Lemma 5.4.2 and the result from [89] can be applied. It follows from Lemma 5.4.2 that $||f||_{H^1} \lesssim c_{j,k}$, which is the same as when we apply Lemma 2 in [89].

The impact of this lemma is subtle, but important to this work. As was shown by Wilson, if f has mean zero and $|f(x)| \lesssim \Phi_k^N(x)$, then $f \in H^1$. Then obviously if f_1, f_2 both have mean zero and satisfy $|f_1(x)| \lesssim \Phi_j^N(x)$, $|f_2(x)| \lesssim \Phi_k^N(x)$, then $f_1 + f_2 \in H^1$ with norm independent of j and k. On the othe hand if f satisfied the hypotheses of Lemma 5.4.2, it is not clear that f can be written $f = f_1 + f_2$ where both f_1, f_2 have mean zero and satisfy $|f_1(x)| \lesssim \Phi_j^N(x)$, $|f_2(x)| \lesssim \Phi_k^N(x)$. In this situation one cannot directly apply the results of Uchiyama [87] or Wilson [89], but Lemma 5.4.2 is still applicable. This is where the content of Lemma 5.4.2 lies.

Proof: Let $j,k \in \mathbb{Z}$ and without loss of generality assume that $j \geq k$. Define the ball $A_0 = B(0, 3\cdot 2^{-j})$, the annuli $A_\ell = B(0,3\cdot 2^{\ell-j}) \backslash B(0,3\cdot 2^{\ell-j-1})$, and the functions

$$h_0(x) = \chi_{A_0} f - g_0$$
$$h_\ell(x) = \chi_{A_\ell} f + g_{\ell-1} - g_\ell$$

for $\ell \in \mathbb{N}_0$ where

$$g_\ell(x) = \chi_{A_\ell} \frac{1}{|A_\ell|} \int_{|x|<3\cdot 2^{\ell-j}} f(y)dy.$$

Then it follows that

$$\begin{aligned}\sum_{\ell\in\mathbb{N}_0} h_\ell &= \chi_{A_0} f - g_0 + \sum_{\ell\in\mathbb{N}} \chi_{A_\ell} f - g_\ell + g_{\ell-1} \\ &= \chi_{A_0} f - g_0 + \sum_{\ell\in\mathbb{N}} \chi_{A_\ell} f + \sum_{\ell\in\mathbb{N}} g_{\ell-1} - g_\ell \\ &= f + \lim_{\ell\to\infty} g_\ell = f.\end{aligned}$$

This limit holds pointwise on $\mathbb{R}^n$: For $x \in \mathbb{R}^n$, take $\ell \in \mathbb{N}$ large enough so that $3\cdot 2^{\ell-j-1} > |x|$. Then $x \notin A_\ell$, $x \notin \text{supp}(g_\ell)$, and $g_\ell(x) = 0$. We check that h_ℓ are almost atoms (up to a constant depending on j,k,ℓ multiple which we will specify later) for each $\ell \in \mathbb{N}_0$: It is obvious that $\text{supp}(h_\ell) \subset B(0,3\cdot 2^{\ell-j})$. Also

$$\int_{\mathbb{R}^n} h_0(x)dx = \int_{B(0,3\cdot 2^{-j})} \left(f(x) - \frac{1}{|B(0,3\cdot 2^{-j})|} \int_{B(0,3\cdot 2^{-j})} f(y)dy \right) dx = 0,$$

and for $\ell \geq 1$

$$\begin{aligned}
\int_{\mathbb{R}^n} h_\ell(x)dx &= \int_{\mathbb{R}^n} \left(\chi_{A_\ell}(x) f(x) + g_{\ell-1}(x) - g_\ell(x)\right) dx \\
&= \int_{A_\ell} f(x)dx + |A_{\ell-1}| g_{\ell-1}(x) - |A_\ell| g_\ell(x) dx \\
&= \int_{A_\ell} f(x)dx + \int_{B(0,3\cdot 2^{\ell-j-1})} f(x)dx - \int_{B(0,3\cdot 2^{\ell-j})} f(x)dx = 0.
\end{aligned}$$

We also have for any $x \in \mathbb{R}^n$

$$\begin{aligned}
|h_0(x)| &\lesssim c_{j,k}\left(\Phi_j(x) + \Phi_k(x)\right) + \frac{c_{j,k}}{|B(0, 3\cdot 2^{-j})|} \int_{B(0,3\cdot 2^{-j})} \left(\Phi_j(x) + \Phi_k(x)\right) dy \\
&\lesssim c_{j,k}\left(2^{jn} + 2^{kn}\right) + \frac{c_{j,k}}{|B(0, 3\cdot 2^{-j})|} \\
&\lesssim c_{j,k} 2^{jn}.
\end{aligned}$$

Also for $\ell \geq 1$ and any $x \in \mathbb{R}^n$

$$\begin{aligned}
|h_\ell(x)| &\leq |f(x)|\chi_{A_\ell}(x) + \frac{1}{|A_\ell|}\left|\int_{|y|<3\cdot 2^{\ell-j}} f(y)dy\right| + \frac{1}{|A_{\ell-1}|}\left|\int_{|y|<3\cdot 2^{\ell-j-1}} f(y)dy\right| \\
&\lesssim c_{j,k}\left(\Phi_j^N(x) + \Phi_k^N(x)\right)\chi_{A_\ell}(x) + \frac{1}{2^{(\ell-j)n}}\left|\int_{|y|\geq 3\cdot 2^{\ell-j}} f(y)dy\right| \\
&\qquad + \frac{1}{2^{(\ell-j)n}}\left|\int_{|y|\geq 3\cdot 2^{\ell-j-1}} f(y)dy\right| \\
&\lesssim c_{j,k}\left(\frac{2^{jn}}{(2^j|x|)^N} + \frac{2^{kn}}{(2^k|x|)^N}\right)\chi_{A_j}(x) + \frac{c_{j,k}}{2^{(\ell-j)n}}\int_{|y|\geq 3\cdot 2^{\ell-j}}\left(\frac{2^{jn}}{(2^j|y|)^N} + \frac{2^{kn}}{(2^k|y|)^N}\right)dy \\
&\lesssim c_{j,k}\left(\frac{2^{jn}}{(2^j 2^{\ell-j})^N} + \frac{2^{kn}}{(2^k 2^{\ell-j})^N}\right) + \frac{c_{j,k}}{2^{(\ell-j)n}}\left(\frac{2^{jn}}{2^{jN}2^{(\ell-j)(N-n)}} + \frac{2^{kn}}{2^{kN}2^{(\ell-j)(N-n)}}\right) \\
&\lesssim \frac{c_{j,k}}{2^{(\ell-j)n}} 2^{(j-\ell)(N-n)}\left(2^{j(n-N)} + 2^{k(n-N)}\right)
\end{aligned}$$

Fix $C_0 > 0$ such that the above inequalities holds for h_ℓ, i.e.

$$\begin{aligned}
|h_0(x)| &\le C_0 c_{j,k} 2^{jn} \\
|h_\ell(x)| &\le \frac{C_0 c_{j,k}}{|B(0,2^{\ell-j})|} 2^{(j-\ell)(N-n)} \left(2^{j(n-N)} + 2^{k(n-N)}\right).
\end{aligned}$$

Now we modify h_ℓ to define our atoms to approximate f: Let $a_\ell(x) = \lambda_\ell^{-1} h_\ell$ for all $\ell \in \mathbb{N}_0$ where

$$\begin{aligned}
\lambda_0 &= C_0 c_{j,k} 2^{jn} |B(0,3\cdot 2^{-j})| \\
\lambda_\ell &= C_0 c_{j,k} 2^{(j-\ell)(N-n)} \left(2^{j(n-N)} + 2^{k(n-N)}\right), \qquad \text{when } \ell \ge 1.
\end{aligned}$$

It follows that for each $\ell \in \mathbb{N}_0$, $\operatorname{supp}(a_\ell) = \operatorname{supp}(h_\ell) \subset B(0,3\cdot 2^{\ell-j})$, a_ℓ has mean zero since h_ℓ does, and for all $x \in \mathbb{R}^n$

$$|a_0(x)| = \lambda_0^{-1} |h_0(x)| = \frac{1}{C_0 c_{j,k} 2^{jn} |B(0,3\cdot 2^{-j})|} |h_0(x)| \le \frac{1}{|B(0,3\cdot 2^{-j})|}$$

$$|a_\ell(x)| = \lambda_\ell^{-1} |h_\ell(x)| = \frac{\left(2^{j(n-N)} + 2^{k(n-N)}\right)^{-1}}{C_0 c_{j,k} 2^{(j-\ell)(N-n)}} |h_\ell(x)| \le \frac{1}{|B(0,2^{\ell-j})|} \quad \text{when } \ell \ge 1.$$

Then it follows that a_ℓ are atoms for $\ell \in \mathbb{N}_0$, and furthermore

$$\begin{aligned}
\sum_{\ell\in\mathbb{N}_0} |\lambda_\ell| &= \lambda_0 + \sum_{\ell\in\mathbb{N}} |\lambda_\ell| \\
&= C_0 c_{j,k} 2^{jn} |B(0,3\cdot 2^{-j})| + C_0 c_{j,k} \left(2^{j(n-N)} + 2^{k(n-N)}\right) \sum_{\ell\in\mathbb{N}} 2^{(j-\ell)(N-n)} \\
&= C_0 c_{j,k} 2^{jn} |B(0,3\cdot 2^{-j})| + C_0 c_{j,k} \left(2^{j(n-N)} + 2^{k(n-N)}\right) 2^{j(N-n)} \lesssim c_{j,k} 2^{|j-k|(N-n)}.
\end{aligned}$$

Note that in the last step we us that $j \geq k$ to conclude that $j-k=|j-k|$ and $1 \leq 2^{|j-k|(N-n)}$ since $N > n$. Then it follows that

$$\left|\left|f-\sum_{\ell=0}^{M}\lambda_\ell a_\ell\right|\right|_{H^1} \leq \sum_{\ell=M+1}^{\infty}\lambda_\ell||a_\ell||_{H^1} \lesssim c_{j,k}\left(2^{j(n-N)}+2^{k(n-N)}\right)\sum_{\ell=M+1}^{\infty}2^{(j-\ell)(N-n)}$$
$$\lesssim c_{j,k}\left(2^{j(n-N)}+2^{k(n-N)}\right)2^{(j-M)(N-n)},$$

which tends to zero as $M\to\infty$. Therefore (λ_ℓ, a_ℓ) is an H^1 atomic decomposition for f, and it follows from the atomic characterization of H^1 that $f \in H^1$ and $||f||_{H^1} \lesssim c_{j,k}2^{|j-k|(N-n)}$.
□

In the next result, we prove that one can extend certain reproducing formulas converging in L^p to convergence in H^1. Theorem 5.4.3 is stated for a general class of reproducing operator, and we apply this result in Corollary 5.4.4 to the operators D_k^b and $\widetilde{D}_k^b M_b D_k^b$ from Theorem 5.4.1. This result will be used for the construction of an accretive paraproduct operator, Theorem 7.3.3, which in turn is used to prove the bilinear Tb theorem, Theorem 7.5.2.

Theorem 5.4.3 *Let $b \in L^\infty$ be a para-accretive function and $\{\theta_k\} \in LPK$ such that $\Theta_k b = \Theta_k^* b = 0$ for all k and for any $f \in C_0^\delta$ such that bf has mean zero*

$$\sum_{k\in\mathbb{Z}} M_b\Theta_k M_b f = bf$$

where the convergence holds in L^p for some $1<p<\infty$. Then for any $\delta > 0$ and $f \in C_0^\delta$ where bf has mean zero, it follows that $bf \in H^1$ and

$$\sum_{k\in\mathbb{Z}} M_b\Theta_k M_b f = bf$$

in H^1.

Proof: Let $f \in C_0^\delta$ for some $0 < \delta \leq 1$ and $\{\theta_k\} \in LPK$ with $\Theta_k(b) = \Theta_k^*(b) = 0$ for all $k \in \mathbb{Z}$. Without loss of generality, assume that $\gamma = \delta$ and $N \leq n + \gamma/2$. Define $f_k(x) = M_b\Theta_k M_b f$ for $k \in \mathbb{Z}$, and it easily follows that

$$\int_{\mathbb{R}^n} f_k(x)dx = \int_{\mathbb{R}^n} M_b f(x)\Theta_k^* b(x)dx = 0.$$

Let R be large enough so that $\text{supp}(f) \subset B(0,R)$, and we estimate f_k using that bf has mean zero

$$\begin{aligned}
|f_k(x)| &\leq ||b||_{L^\infty} \left| \int_{\mathbb{R}^n} (\theta_k(x,y) - \theta_k(x,0))b(y)f(y)dy \right| \\
&\lesssim \int_{\mathbb{R}^n} (2^k|y|)^\gamma \left(\Phi_k^N(x-y) + \Phi_k^N(x)\right)|f(y)|dy \\
&\lesssim 2^{\gamma k}R^\gamma \left(\Phi_k^N * \Phi_0^{N'}(x) + \Phi_k^N(x)\right) \\
&\lesssim 2^{\gamma k}\left(\Phi_0^N(x) + \Phi_k^N(x)\right).
\end{aligned}$$

We also estimate f_k using that $\Theta_k(b) = 0$

$$\begin{aligned}
|f_k(x)| &\leq ||b||_{L^\infty} \left| \int_{\mathbb{R}^n} \theta_k(x,y)b(y)(f(y) - f(x))dy \right| \\
&\lesssim \int_{\mathbb{R}^n} \Phi_k^{N+\gamma}(x-y)|x-y|^\gamma(\Phi_0^N(y) + \Phi_0^N(x))dy \\
&\lesssim 2^{-\gamma k}\int_{\mathbb{R}^n} \Phi_k^N(x-y)(\Phi_0^N(y) + \Phi_0^N(x))dy \\
&\lesssim 2^{-\gamma k}\left(\Phi_0^N(x) + \Phi_k^N(x)\right).
\end{aligned}$$

So we have proved that $|f_k(x)| \lesssim 2^{-\gamma|k|}(\Phi_0^N(x) + \Phi_k^N(x))$. It follows from Lemma 5.4.2 that

$$||f_k||_{H^1} \lesssim 2^{-\gamma|k|}2^{|k|(N-n)} \leq 2^{-\gamma|k|/2},$$

and so

$$\left|\left|\sum_{|k|<M} f_k\right|\right|_{H^1} \leq \sum_{|k|<M} ||f_k||_{H^1} \lesssim \sum_{k\in\mathbb{Z}} 2^{-\gamma|k|/2} < \infty.$$

Hence $\sum_{|k|<M} f_k$ is a Cauchy sequence in H^1, and there exists $\widetilde{f} \in H^1$ such that

$$\widetilde{f} = \sum_{k\in\mathbb{Z}} f_k = \sum_{k\in\mathbb{Z}} M_b \Theta_k M_b f.$$

But since the reproducing formula holds for bf in L^p for some $1 < p < \infty$, it follows that $\widetilde{f} = bf$ and the reproducing formula holds for bf in H^1, which completes the proof. □

Corollary 5.4.4 *Let $b \in L^\infty$ be a para-accretive function, S_k^b, D_k^b, and $\widetilde{D}_k^b$ be approximation to identity and reproducing formula operator with respect to b as in Remark 5.3.3 and Theorem 5.4.1. Then for all $\delta > 0$ and $f \in C_0^\delta$ such that bf has mean zero,*

$$\sum_{k\in\mathbb{Z}} M_b \widetilde{D}_k^b M_b D_k^b M_b f = \sum_{k\in\mathbb{Z}} M_b D_k^b M_b f = bf$$

in H^1.

Proof: By Theorem 5.4.1, it follows that the kernels of $\widetilde{D}_k^b M_b D_k^b$ and D_k^b are Littlewood-Paley square function kernels of type LPK, that

$$\widetilde{D}_k^b M_b D_k^b(b) = (\widetilde{D}_k^b M_b D_k^b)^*(b) = D_k^b(b) = D_k^{b\,*}(b) = 0,$$

and finally that

$$\sum_{k\in\mathbb{Z}} M_b \widetilde{D}_k^b M_b D_k^b M_b f = \sum_{k\in\mathbb{Z}} M_b D_k^b M_b f = bf$$

in L^p for all $1 < p < \infty$ when $f \in C_0^\delta$ when bf has mean zero. Therefore by Theorem 5.4.3 it follows that the formula holds in H^1 as well. □

Chapter 6

Square Function Estimates

Given a function $\psi : \mathbb{R}^n \to \mathbb{C}$, the Littlewood-Paley-Stein type square function is defined

$$g_\psi(f) = \left(\sum_{k\in\mathbb{Z}} |\psi_k * f|^2 \right)^{\frac{1}{2}},$$

which are also known as the Littlewood-Paley g-functions. These convolution type square functions were introduced by Stein in the 1960's, see e.g. [76], [77], [78] or [79], and have since been studied extensively, including classical works Kurtz of [58], Duoandikoetxea-Rubio de Francia [31], and more recently Duoandikoetxea-Seijo [32], Cheng [15], Sato [73], Duoandikoetxea [30], Wilson [89], Lerner [59], and Cruz-Uribe-Martell-Perez [27]. Maybe the greatest impact of the Littlewood-Paley g-functions of Stein is for an appropriate $\psi \in \mathscr{S}$, one can obtain the Littlewood-Paley characterization of L^p for $1 < p < \infty$: For $f \in L^p$, it follows that $||g_\psi(f)||_{L^p} \approx ||f||_{L^p}$. A typical goal of Littlewood-Paley theory is to find size, regularity, and cancellation conditions on a function $\widetilde{\psi} : \mathbb{R}^n \to \mathbb{C}$ that guarantee $||g_{\widetilde{\psi}}(f)||_{L^p} \lesssim ||f||_{L^p}$ for appropriate p. In some situations, this corresponds to studying truncations of some operator $Tf(x) = K * f(x)$ of the form $\widetilde{\psi}_k(x) = K * \psi_k(x)$, and in some cases lead to L^p bounds for T itself. This is the general outline of how we prove bounds for the singular integral operators in Chapter 7.

Non-convolution variants of the kernels ψ_k were studied by Carleson [12], David-Journé-Semmes [29], Christ-Journé [17], Semmes [74], Hofmann [52], Auscher [1], and many others, where they replaced the convolution $\psi_k * f(x)$ with operators $\Theta_k f(x)$ which were defined in chapter 4. We also replace the Littlewood-Paley g-function, g_ψ, with a non-convolution variant:

Definition 6.0.5 *Given a collection of linear Littlewood-Paley square function kernels* $\{\theta_k\}_{k\in\mathbb{Z}}$ *of type LPK and associated integral operators* $\{\Theta_k\}_{k\in\mathbb{Z}}$*, define the Littlewood-Paley square function associated to* $\{\Theta_k\}_{k\in\mathbb{Z}}$

$$S_{\{\Theta_k\}}f(x) = \left(\sum_{k\in\mathbb{Z}} |\Theta_k f(x)|^2\right)^{\frac{1}{2}}.$$

Likewise, given a collection of bilinear Littlewood-Paley square function kernels $\{\theta_k\}_{k\in\mathbb{Z}}$ *of type BLPK and associated integral operators* $\{\Theta_k\}_{k\in\mathbb{Z}}$*, define the Littlewood-Paley square function associated to* $\{\Theta_k\}_{k\in\mathbb{Z}}$

$$S_{\{\Theta_k\}}(f_1, f_2)(x) = \left(\sum_{k\in\mathbb{Z}} |\Theta_k(f_1, f_2)(x)|^2\right)^{\frac{1}{2}}.$$

In this chapter, we start proving bounds of the Littlewood-Paley g_ψ and recap a historical development of the square function estimates for $S_{\{\Theta_k\}}$ in the linear, bilinear, perturbed, and unperturbed situations.

6.1 Classical Littlewood-Paley Square Function Bounds

Early in the study of Littlewood-Paley square functions Stein [76] proved L^p bounds for g_ψ. Later Benedek-Calderón-Panzone [2] and Rubio de Francia-Ruiz-Torrea [53] developed Littlewood-Paley theory in terms of Calderón-Zygmund theory. The unifying insight from [53] and [53] was to view the operator g_ψ as a vector-valued operator in order to apply the machinery of Calderón-Zygmund theory. In particular, instead of viewing g_ψ as a sublinear operator mapping to scalar valued functions, he shifted to view it as a linear operator $\{\psi_k\} : f \mapsto \{\psi_k * f\}$ which maps to $\ell^2(\mathbb{Z})$ valued functions. The trade-off here is obvious (non-linearity versus vector valued theory), but in certain situations it is preferable to work in a Banach space setting as long as it "linearizes" the operator. We provide the proof from [2] and [53] here for the sake of making this work self-contained as well as presenting the historical development of square function bounds. We first prove a lemma that demonstrates a fundamental connection between vector valued Calderón-Zygmund theory and the square function kernels defined in Chapter 4. In particular, given a collection of square function kernels $\{\theta_k\} \in SLPK$, one can define an associated standard kernel of type CZK_{ℓ^r} for $1 \le r \le \infty$.

Lemma 6.1.1 *Any $\{\theta_k\} \in SLPK$ is a standard kernel of type CZK_{ℓ^r} for all $1 \le r \le \infty$.*

Proof: For $x \neq y$, define $d = |x-y|$, and we estimate

$$||\{\theta_k(x,y)\}||_{\ell^1} \lesssim \sum_{2^k \le d^{-1}} 2^{kn} + \sum_{2^k > d^{-1}} \frac{2^{kn}}{(2^k d)^{N+\gamma}}$$
$$\lesssim d^{-n} + d^{-(N+\gamma)} \sum_{2^k > d^{-1}} 2^{k(n-(N+\gamma))} \lesssim d^{-n}.$$

When $|x-x'|<|x-y|/2$, it follows that

$$\begin{aligned}||\{\theta_k(x,y)\}-\{\theta_k(x',y)\}||_{\ell^1} &\lesssim |x-x'|^{\gamma}\sum_{2^k\leq d^{-1}}2^{k(n+\gamma)}\\ &\quad+|x-x'|^{\gamma}\sum_{2^k>d^{-1}}\frac{2^{k(n+\gamma)}}{(2^kd)^{N+\gamma}}+\frac{2^{k(n+\gamma)}}{(2^k(d-|x-x'|))^{N+\gamma}}\\ &\lesssim |x-x'|^{\gamma}d^{-(n+\gamma)}+|x-x'|^{\gamma}d^{-(N+\gamma)}\sum_{2^k>d^{-1}}2^{k(n-N)}\\ &\lesssim |x-x'|^{\gamma}d^{-(n+\gamma)}.\end{aligned}$$

A similar argument in the y variable proves that $\{\theta_k\}$ is a standard kernel of type CZK_{ℓ^1}. Note that for any $1\leq r\leq\infty$, we have $||\cdot||_{\ell^r}\leq||\cdot||_{\ell^1}$. Therefore $\{\theta_k\}$ is a standard kernel of type CZK_{ℓ^r} for any $1\leq r\leq\infty$. □

In the next result, we use Fourier analysis to prove an L^2 bound and extend to L^p for $p\neq 2$ using the vector valued Calderón-Zygmund theory and interpolation results from Chapters 2 and 3. This result is originally due to Stein [76], but the proof presented here is due to Benedek-Calderón-Panzone [2] and Rubio de Francia-Ruiz-Torrea [53].

Proposition 6.1.2 ([76], [2], [53]) *Let $Q_kf=\psi_k*f$ where $\psi\in\mathscr{S}$ has mean zero. Then the square function $S_{\{Q_k\}}$ associated to the kernels $\psi_k(x-y)$ is bounded from L^1 into $L^{1,\infty}$, from L^∞_c into BMO, and from L^p into L^p for all $1<p<\infty$. Note that in particular $S_{\{Q_k\}}$ and $S_{\{\widetilde{Q}_k\}}$ are bounded on L^p for all $1<p<\infty$ where Q_k and $\widetilde{Q}_k$ are defined in Proposition 5.2.1.*

Here we use the notation $S_{\{Q_k\}}$ to be the square function associated to Q_k, but this is exactly the same definition as g_ψ, i.e. $S_{\{Q_k\}}=g_\psi$.

Proof: We first consider the case $p = 2$, and compute by Plancherel's theorem

$$||S_{\{Q_k\}}f||_{L^2}^2 = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}|\widehat{\psi}(2^{-k}\xi)f(\xi)|^2 d\xi$$
$$\leq \int_{\mathbb{R}^n}\left(\sum_{2^k\leq|\xi|}|\widehat{\psi}(2^{-k}\xi)|^2 + \sum_{2^k>|\xi|}|\widehat{\psi}(2^{-k}\xi)|^2\right)|f(\xi)|^2 d\xi = I + II.$$

Since $\widehat{\psi} \in \mathscr{S}(\mathbb{R}^n)$ and $\widehat{\psi}(0) = 0$, there exists a finite linear combination of Schwartz seminorm, ρ, such that $|\widehat{\psi}(\xi)| \leq \rho(\psi)\min(|\xi|,|\xi|^{-1})$. Then we estimate I by

$$I \leq \rho(\psi)^2\int_{\mathbb{R}^n}\left(\sum_{2^k\leq|\xi|}(2^{-k}|\xi|)^{-2}\right)|f(\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^n}|f(\xi)|^2 d\xi \lesssim ||f||_{L^2}^2.$$

Likewise, we estimate II by

$$II \leq \rho(\psi)^2\int_{\mathbb{R}^n}\left(\sum_{2^k>|\xi|}(2^{-k}|\xi|)^{2}\right)|f(\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^n}|f(\xi)|^2 d\xi \lesssim ||f||_{L^2}^2.$$

Therefore $S_{\{Q_k\}}$ is bounded on L^2. By Lemma 6.1.1 the collection $\{\psi_k(x-y)\}_{k\in\mathbb{Z}}$ is an $\ell^2(\mathbb{Z})$-valued Calderón-Zygmund operator, so by Theorem 2.2.1, it follows that $\{Q_k\}_{k\in\mathbb{Z}}$ is bounded from L^1 into $L^{1,\infty}(\mathbb{R}^n,\ell^2(\mathbb{Z}))$, that is $S_{\{Q_k\}}$ is bounded from L^1 into $L^{1,\infty}$. Then by Theorem 3.1.1, it follows that $S_{\{Q_k\}}$ is bounded on L^p for $1 < p \leq 2$. By Theorem 2.3.1, $S_{\{Q_k\}}$ is bounded from L_c^∞ into BMO. Also by Theorem 3.3.3, it follows that $S_{\{Q_k\}}$ is bounded from L^p into L^p for all $2 < p < \infty$. It is not hard to see that $\{Q_k\}_{k\in\mathbb{Z}}$ as an ℓ^2 Calderón-Zygmund operator satisfies the extra assumptions (1) and (2) from Theorem 3.3.3: (1) For $f \in L^p$ with $p > 2$, $\{Q_k f\}_{k\in\mathbb{Z}}$ is an ℓ^2 measurable function since $Q_k f$ is measurable for each $k \in \mathbb{Z}$, and (2) if $f_j \to f$ in L^q where $f_j \in L_c^\infty$, then since $Q_k f_j \to Q_k f$

pointwise and by Fatou's lemma

$$|\{Q_k f\}(x)|_{\ell^2} = \left(\sum_{k\in\mathbb{Z}} |Q_k f(x)|^2\right)^{\frac{1}{2}} = \left(\sum_{k\in\mathbb{Z}} \lim_{j\to\infty} |Q_k f_j(x)|^2\right)^{\frac{1}{2}} \leq \liminf_{j\to\infty} |\{Q_k f_j\}(x)|_{\ell^2}.$$

This completes the proof. □

This proof is slightly different than the one that you may find in [2], [53], or the many other texted where this theory is developed. Typically one proves bounds for $1 < p \leq 2$ in the way we did above, and then continue by a duality argument. We have not developed some of the duality theory in the vector valued setting that is necessary for this argument. Instead we use the weak *BMO* interpolation argument since we have already proved the necessary *BMO* endpoint estimates in Chapter 2 and interpolation results in Chapter 3.

6.2 Square Function Bounds with T1 Type Testing Conditions

In the situation above, if $\psi \in \mathscr{S}$, then g_ψ is bounded on L^p for all $1 < p < \infty$ if and only if ψ has mean zero. Much of the afore mentioned work is in finding weaker size and regularity conditions than $\psi \in \mathscr{S}$. For the more general convolution and non-convolution type square functions $S_{\{\Theta_k\}}$, this mean zero condition is sufficient, but not necessary for L^p bounds. In the 1980's, mathematicians began to study these bounds of the square functions $S_{\{\Theta_k\}}$. There are Lebesgue space boundedness results for the square functions $S_{\{\Theta_k\}}$, but they can be set in the more general context of Trieble-Lizorkin and Besov space bounds. There is a rich history behind these results. Much of the theory was developed in the works of Besov [5, 6], Taibleson [81, 82, 83], Peetre [70, 71, 72], Triebel [85, 86], Lizorkin [61], among others that defined Triebel-Lizorkin and Besov spaces. Although Theorem 6.2.1 as stated here is closer to the results of David-Journé [28], Christ-Journé [17], and Semmes [74].

Theorem 6.2.1 *If $\{\theta_k\} \in LPK$ and $\Theta_k(1) = 0$ for all $k \in \mathbb{Z}$, then for all $f \in \dot{F}_p^{\alpha,q}$*

$$\left|\left|\left(\sum_{k\in\mathbb{Z}} 2^{\alpha qk}|\Theta_k f|^q\right)^{\frac{1}{q}}\right|\right|_{L^p} \lesssim ||f||_{\dot{F}_p^{\alpha,q}}$$

and $f \in \dot{B}_p^{\alpha,q}$

$$\left(\sum_{k\in\mathbb{Z}} 2^{\alpha qk}||\Theta_k f||_{L^p}^q\right)^{\frac{1}{q}} \lesssim ||f||_{\dot{B}_p^{\alpha,q}}$$

whenever $1 < p,q < \infty$ and $|\alpha| < \gamma$. Here γ is the regularity parameter guaranteed by $\{\theta_k\} \in LPK$.

In particular, for $q = 2$, $1 < p < \infty$, and $\alpha = 0$ the $\dot{F}_p^{0,2}$ bounds from Theorem 6.2.1 become L^p bounds for $S_{\{\Theta_k\}}$. We do not prove this result, but the proof can be easily extracted from any of the next three results, Theorems 6.2.2-6.2.4.

A bilinear version of Theorem 6.2.1 were proved by Maldonado [62] and Maldonado-Naibo [63]. In this work, we extend Theorem 6.2.1 in three different ways, Theorems 6.2.2-6.2.4. These theorems are presented from the weakest cancellation condition on Θ_k in Theorem 6.2.2 to the strongest in Theorem 6.2.4. The results of Maldonado and Maldonado-Naibo are the Besov space estimates in Theorem 6.2.3. Theorem 6.2.2 was proved by the author in [49, 51], and proved by different techniques by Grafakos-Oliviera in [41] and by Grafakos-Liu-Maldonado-Yang in [40].

Theorem 6.2.2 ([49], [41], [40]) *If $\{\theta_k\} \in BLPK$ and $\Theta_k(1,1)(x) = 0$ for all $x \in \mathbb{R}^n$ and*

$k \in \mathbb{Z}$, then for all $f_1 \in \dot{F}^{\alpha,q}_{p_{1,1}} \cap L^{p_{1,2}}$ and $f_2 \in L^{p_{2,1}} \cap \dot{F}^{\alpha,q}_{p_{2,1}}$

$$\left|\left|\left(\sum_{k\in\mathbb{Z}} 2^{\alpha kq}|\Theta_k(f_1,f_2)|^q\right)^{\frac{1}{q}}\right|\right|_{L^p} \lesssim ||f_1||_{\dot{F}^{\alpha,q}_{p_{1,1}}}\, ||f_2||_{L^{p_{2,1}}} + ||f_1||_{L^{p_{1,2}}}\, ||f_2||_{\dot{F}^{\alpha,q}_{p_{2,2}}}$$

and $f_1 \in \dot{B}^{\alpha,q}_{p_{1,1}} \cap L^{p_{1,2}}$ and $f_2 \in L^{p_{2,1}} \cap \dot{B}^{\alpha,q}_{p_{2,2}}$

$$\left(\sum_{k\in\mathbb{Z}} 2^{\alpha kq}||\Theta_k(f_1,f_2)||^q_{L^p}\right)^{\frac{1}{q}} \lesssim ||f_1||_{\dot{B}^{\alpha,q}_{p_{1,1}}}\, ||f_2||_{L^{p_{2,1}}} + ||f_1||_{L^{p_{1,2}}}\, ||f_2||_{\dot{B}^{\alpha,q}_{p_{2,2}}}$$

whenever $1 < p, p_{i,1}, p_{i,2}, q < \infty$, $\frac{1}{p} = \frac{1}{p_{i,1}} + \frac{1}{p_{i,2}}$ for $i = 1,2$, $\alpha \in \mathbb{R}$, and $|\alpha| < \gamma$. Here γ is the smoothness parameter for $\{\theta_k\} \in BLPK$.

Proof: Fix $1 < p, p_{i,1}, p_{i,2}, q < \infty$ satisfying (1.1), $\alpha \in \mathbb{R}$ with $|\alpha| < \gamma$, $f_1 \in L^1 \cap \dot{F}^{\alpha,q}_{p_{1,1}} \cap L^{p_{1,2}}$, $f_2 \in L^1 \cap L^{p_{2,1}} \cap \dot{F}^{\alpha,q}_{p_{2,2}}$, and $\{g_k\}_{k\in\mathbb{Z}}$ such that

$$\left|\left|\left(\sum_{k\in\mathbb{Z}} 2^{-k\alpha q'}|g_k|^{q'}\right)^{\frac{1}{q'}}\right|\right|_{L^{p'}} \leq 1.$$

Then we approximate the dual pairing

$$\left|\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \Theta_k(f_1,f_2)(x)g_k(x)dx\right| \leq \sum_{k\in\mathbb{Z}} \left|\int_{\mathbb{R}^n} \Theta_k(f_1,f_2)(x)g_k(x)dx\right| = \lim_{N\to\infty} \sum_{|k|<N} |\Omega_k|,$$

where we take the last equality to be the definition of $\Omega_k = \Omega_k(f_1, f_2, \{g_k\})$ to be the quantity inside the absolute value in the expression before it. Let Π^1_k and Π^2_k be defined as in Proposition 5.2.1, that is

$$\Pi_j^1(f_1,f_2)(y_1,y_2) = Q_k f_1(y_1) P_{k+1} f_2(y_2),$$
$$\Pi_j^2(f_1,f_2)(y_1,y_2) = P_k f_1(y_1) Q_k f_2(y_2).$$

Now for $M \in \mathbb{N}$

$$\left|\Omega_k - \int_{\mathbb{R}^n} \sum_{|j|<M} (\Theta_k \Pi_j^1(f_1,f_2)(x) + \Theta_k \Pi_j^2(f_1,f_2)(x)) g_k(x) dx\right| \\ \leq \left|\left|\Theta_k(f_1,f_2) - \sum_{|j|<M} \Theta_k \Pi_j^1(f_1,f_2) + \Theta_k \Pi_j^2(f_1,f_2)\right|\right|_{L^p} ||g_k||_{L^{p'}}.$$

By formula (5.2) in Proposition 5.2.1, the above tends to zero as $M \to \infty$, and hence we reduce the proof to bounding

$$\sum_{|k|<N} \sum_{j \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} (\Theta_k \Pi_j^1(f_1,f_2)(x) + \Theta_k \Pi_j^2(f_1,f_2)(x)) g_k(x) dx \right|$$

by the appropriate norms of f_1 and f_2 independent of N. By estimate (4.15) from Proposition 4.3.2 and recalling that $Q_k = \widetilde{Q}_k Q_k$, it follows that

$$|\Theta_k^1 \Pi_j(f_1,f_2)(x)| = |\Theta_k^1(\widetilde{Q}_j Q_j f_1, P_{j+1} f_2)(x)| \lesssim 2^{-\gamma|j-k|} \mathscr{M}(Q_j f_1)(x) \mathscr{M} f_2(x),$$
$$|\Theta_k^2 \Pi_j(f_1,f_2)(x)| = |\Theta_k^2(P_j f_1, \widetilde{Q}_j Q_j f_2)(x)| \lesssim 2^{-\gamma|j-k|} \mathscr{M} f_1(x) \mathscr{M}(Q_j f_2)(x).$$

Since $|\alpha| < \gamma$, there exists $0 < \lambda < 1$ such that $\lambda\gamma > |\alpha|$, for example we can take $\lambda = \frac{|\alpha|+\gamma}{2\gamma}$. Then we can estimate for any $N \in \mathbb{N}$

$$
\begin{aligned}
\sum_{|k|<N} |\Omega_k| &\leq \sum_{|k|<N} \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\Theta_k \Pi_j^1(f_1,f_2)(x) g_k(x) + \Theta_k \Pi_j^2(f_1,f_2)(x) g_k(x)| dx \\
&\lesssim \int_{\mathbb{R}^n} \sum_{j,k\in\mathbb{Z}} 2^{-\gamma\lambda|j-k|} 2^{-\gamma(1-\lambda)|j-k|} \left(\mathscr{M}(Q_j f_1)(x) \mathscr{M} f_2(x) + \mathscr{M} f_1(x) \mathscr{M}(Q_j f_2)(x) \right) \\
&\qquad \times |g_k(x)| dx \\
&\leq \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{\alpha q k} 2^{-q\lambda\gamma|j-k|} \left(\mathscr{M}(Q_j f_1) \mathscr{M} f_2 + \mathscr{M} f_1 \mathscr{M}(Q_j f_2) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
&\qquad \times \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{-\alpha q' k} 2^{-q'(1-\lambda)\gamma|j-k|} |g_k|^{q'} \right)^{\frac{1}{q'}} \right\|_{L^{p'}} \\
&\lesssim \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{q(|\alpha|-\lambda\gamma)|j-k|} 2^{\alpha j q} \left(\mathscr{M}(Q_j f_1) \mathscr{M} f_2 + \mathscr{M} f_1 \mathscr{M}(Q_j f_2) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
&\qquad \times \left\| \left(\sum_{k\in\mathbb{Z}} 2^{-\alpha q' k} |g_k|^{q'} \right)^{\frac{1}{q'}} \right\|_{L^{p'}} \\
&\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} 2^{\alpha j q} \left(\mathscr{M}(Q_j f_1) \right)^q \right)^{\frac{1}{q}} \mathscr{M} f_2 \right\|_{L^p} + \left\| \mathscr{M} f_1 \left(\sum_{j\in\mathbb{Z}} 2^{\alpha j q} \left(\mathscr{M}(Q_j f_2) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
&\leq \left\| \left(\sum_{j\in\mathbb{Z}} \left(\mathscr{M} Q_j (2^{\alpha j} f_1) \right)^q \right)^{\frac{1}{q}} \right\|_{L^{p_{1,1}}} ||\mathscr{M} f_2||_{L^{p_{2,1}}} \\
&\qquad + ||\mathscr{M} f_1||_{L^{p_{1,2}}} \left\| \left(\sum_{j\in\mathbb{Z}} \left(\mathscr{M} Q_j (2^{\alpha j} f_2) \right)^q \right)^{\frac{1}{q}} \right\|_{L^{p_{2,2}}} \\
&\lesssim ||f_1||_{\dot{F}^{\alpha,q}_{p_{1,1}}} ||f_2||_{L^{p_{2,1}}} + ||f_1||_{L^{p_{1,2}}} ||f_2||_{\dot{F}^{\alpha,q}_{p_{2,2}}}.
\end{aligned}
$$

In the last line, we use the Fefferman-Stein vector valued maximal function bound and the L^p bounds of $\mathscr{M}$ for $1 < p < \infty$. Then by duality and density, the Trieble-Lizorkin estimate follows. Fix $1 < p, p_{i,1}, p_{i,2}, q < \infty$ satisfying (1.1), $\alpha \in \mathbb{R}$ with $|\alpha| < \gamma$, $f_1 \in L^1 \cap \dot{B}^{\alpha,q}_{p_{1,1}} \cap L^{p_{1,2}}$ and $f_2 \in L^1 \cap L^{p_{2,1}} \cap \dot{B}^{\alpha,q}_{p_{2,2}}$, and $\{g_k\}_{k\in\mathbb{Z}}$ such that

$$\left(\sum_{k\in\mathbb{Z}} 2^{-k\alpha q'} ||g_k||_{L^{p'}}^{q'}\right)^{\frac{1}{q'}} \leq 1.$$

Take the same definition of λ as before and approximate the dual pairing in the same way, and it follows that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \Theta_k(f_1,f_2)(x) g_k(x)dx\right| \\
&\lesssim \sum_{j,k\in\mathbb{Z}} 2^{-\gamma\lambda|j-k|} 2^{-\gamma(1-\lambda)|j-k|} \int_{\mathbb{R}^n} \mathscr{M}Q_j f_1(x) \mathscr{M} f_2(x) |g_k(x)| dx \\
&\qquad + \sum_{j,k\in\mathbb{Z}} 2^{-\gamma\lambda|j-k|} 2^{-\gamma(1-\lambda)|j-k|} \int_{\mathbb{R}^n} \mathscr{M} f_1(x) \mathscr{M}Q_j f_2(x) |g_k(x)| dx \\
&\leq \left(\sum_{j,k\in\mathbb{Z}} 2^{-q\gamma\lambda|j-k|} 2^{q\alpha k} \left|\left|\mathscr{M}Q_j f_1\right|\right|_{L^{p_{1,1}}}^q ||\mathscr{M} f_2||_{L^{p_{2,1}}}^q\right)^{\frac{1}{q}} \\
&\qquad\qquad \times \left(\sum_{j,k\in\mathbb{Z}} 2^{-q'\gamma(1-\lambda)|j-k|} 2^{-q'\alpha k} ||g_k||_{L^{p'}}^{q'}\right)^{\frac{1}{q'}} \\
&\qquad + \left(\sum_{j,k\in\mathbb{Z}} 2^{-q\gamma\lambda|j-k|} 2^{q\alpha k} ||\mathscr{M} f_1||_{L^{p_{1,2}}}^q \left|\left|\mathscr{M}Q_j f_2\right|\right|_{L^{p_{2,2}}}^q\right)^{\frac{1}{q}} \\
&\qquad\qquad \times \left(\sum_{j,k\in\mathbb{Z}} 2^{-q'\gamma(1-\lambda)|j-k|} 2^{-q'\alpha k} ||g_k||_{L^{p'}}^{q'}\right)^{\frac{1}{q'}} \\
&\lesssim \left(\sum_{j,k\in\mathbb{Z}} 2^{(|\alpha|-\gamma\lambda)q|j-k|} 2^{\alpha jq} \left|\left|Q_j f_1\right|\right|_{L^{p_{1,1}}}^q ||f_2||_{L^{p_{2,1}}}^q\right)^{\frac{1}{q}} \left(\sum_{j,k\in\mathbb{Z}} 2^{-q'\alpha k} ||g_k||_{L^{p'}}^{q'}\right)^{\frac{1}{q'}} \\
&\qquad + \left(\sum_{j,k\in\mathbb{Z}} 2^{(|\alpha|-\gamma\lambda)q|j-k|} 2^{\alpha jq} ||f_1||_{L^{p_{1,2}}}^q \left|\left|Q_j f_2\right|\right|_{L^{p_{2,2}}}^q\right)^{\frac{1}{q}} \left(\sum_{j,k\in\mathbb{Z}} 2^{-q'\alpha k} ||g_k||_{L^{p'}}^{q'}\right)^{\frac{1}{q'}}
\end{aligned}$$

$$\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{q\alpha j}\,||Q_j f_1||_{L^{p_{1,1}}}^q\right)^{\frac{1}{q}} ||f_2||_{L^{p_{2,1}}} + ||f_1||_{L^{p_{1,2}}} \left(\sum_{j\in\mathbb{Z}} 2^{q\alpha j}\,||Q_j f_2||_{L^{p_{2,2}}}^q\right)^{\frac{1}{q}}$$

$$= ||f_1||_{\dot{B}^{\alpha,q}_{p_{1,1}}} ||f_2||_{L^{p_{2,1}}} + ||f_1||_{L^{p_{1,2}}} ||f_2||_{\dot{B}^{\alpha,q}_{p_{2,2}}}.$$

Then by duality and density, the Besov estimate follows as well. □

Next we state another bilinear version of Theorem 6.2.1 with a stronger cancellation condition imposed on Θ_k that proves a better estimate. In this version, we assume that $\Theta_k(1,\cdot) = 0$ in place of $\Theta_k(1) = 0$. Parts of this result have appeared in a number of works: Maldonado [62], Maldonado-Naibo [63], Grafakos-Oliveira [41], and Grafakos-Lui-Maldonado-Yang [40]. Even though this is not a result of the author, it is closely related to the work in [49], and it fits naturally with Theorems 6.2.2 and 6.2.4 as natural bilinear extensions of Theorem 6.2.1. So we prove it here as well.

Theorem 6.2.3 ([62], [63], [41], [40]) *Suppose $\{\theta_k\} \in BLPK$ and*

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)dy_1 = 0$$

for all $x, y_1 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, then for all $1 < p, p_1, p_2, q < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\alpha \in \mathbb{R}$ with $|\alpha| < \gamma$. Then for all $f_1 \in \dot{F}^{\alpha,q}_{p_1}$ and $f_2 \in L^{p_2}$

$$\left|\left|\left(\sum_{k\in\mathbb{Z}} 2^{\alpha k q}|\Theta_k(f_1,f_2)|^q\right)^{\frac{1}{q}}\right|\right|_{L^p} \lesssim ||f_1||_{\dot{F}^{\alpha,q}_{p_1}} ||f_2||_{L^{p_2}}$$

and for all $f_1 \in \dot{B}^{\alpha,q}_{p_1}$ and $f_2 \in L^{p_2}$

$$\left(\sum_{k\in\mathbb{Z}} 2^{\alpha k q}||\Theta_k(f_1,f_2)||_{L^p}^q\right)^{\frac{1}{q}} \lesssim ||f_1||_{\dot{B}^{\alpha,q}_{p_1}} ||f_2||_{L^{p_2}}.$$

Proof: Fix $1 < p, p_1, p_2, q < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\alpha \in \mathbb{R}$ with $|\alpha| < \gamma$, $f_1 \in \dot{F}_{p_1}^{\alpha,q} \cap L^{p_1} \cap L^1$, $f_2 \in L^{p_2} \cap L^1$, and $\{g_k\}_{k\in\mathbb{Z}}$ such that

$$\left\| \left(\sum_{k\in\mathbb{Z}} 2^{-k\alpha q'} |g_k|^{q'} \right)^{\frac{1}{q'}} \right\|_{L^{p'}} \leq 1.$$

Then we approximate the dual pairing similar to above using the L^{p_1} convergence of

$$\sum_{j\in\mathbb{Z}} Q_j f_1 = f_1$$

and the continuity of Θ_k: For any $N \in \mathbb{N}$

$$\begin{aligned}\sum_{|k|<N} \int_{\mathbb{R}^n} |\Theta_k(f_1, f_2)(x) g_k(x)| dx &= \sum_{|k|<N} \int_{\mathbb{R}^n} \left| \sum_{j\in\mathbb{Z}} \Theta_k(Q_j f_1, f_2)(x) g_k(x) \right| dx \\ &\leq \int_{\mathbb{R}^n} \sum_{j,k\in\mathbb{Z}} |\Theta_k(Q_j f_1, f_2)(x) g_k(x)| dx.\end{aligned}$$

Again, it is sufficient to prove a uniform estimate in N. It follows from estimate (4.17) from Proposition 4.3.2 and the fact that $Q_j = \widetilde{Q}_j Q_j$ that for all $x \in \mathbb{R}^n$ and $j, k \in \mathbb{Z}$

$$|\Theta_k(Q_j f_1, f_2)(x)| \lesssim 2^{-\gamma|j-k|} \mathscr{M}(Q_j f_1)(x) \mathscr{M} f_2(x).$$

Fix $0<\lambda<1$ such that $|\alpha|<\lambda\gamma<\gamma$. Then it follows that

$$\left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\Theta_k(f_1,f_2)(x)g_k(x)dx\right|\le\int_{\mathbb{R}^n}\sum_{j,k\in\mathbb{Z}}2^{-\gamma|j-k|}\mathscr{M}(Q_jf_1)(x)\mathscr{M}f_2(x)g_k(x)|dx$$

$$\le\left|\left|\left(\sum_{j,k\in\mathbb{Z}}2^{-q\gamma\lambda|j-k|}2^{q\alpha k}\left(\mathscr{M}(Q_jf_1)\mathscr{M}f_2\right)^q\right)^{\frac{1}{q}}\right|\right|_{L^p}$$

$$\times\left|\left|\left(\sum_{j,k\in\mathbb{Z}}2^{-q'\gamma(1-\lambda)|j-k|}2^{-q'\alpha k}|g_k|^{q'}\right)^{\frac{1}{q'}}\right|\right|_{L^{p'}}$$

$$\le\left|\left|\left(\sum_{j,k\in\mathbb{Z}}2^{(|\alpha|-\gamma\lambda)q|j-k|}2^{\alpha jq}\left(\mathscr{M}(Q_jf_1)\mathscr{M}f_2\right)^q\right)^{\frac{1}{q}}\right|\right|_{L^p}\left|\left|\left(\sum_{k\in\mathbb{Z}}2^{-q'\alpha k}|g_k|^{q'}\right)^{\frac{1}{q'}}\right|\right|_{L^{p'}}$$

$$\le\left|\left|\left(\sum_{j\in\mathbb{Z}}2^{\alpha jq}\left(\mathscr{M}Q_jf_1\right)^q\right)^{\frac{1}{q}}\right|\right|_{L^{p_1}}||\mathscr{M}f_2||_{L^{p_2}}$$

$$\lesssim||f_1||_{\dot{F}^{\alpha,q}_{p_1}}||f_2||_{L^{p_2}}.$$

By duality and density, the Trieble-Lizorkin estimate follows. Now let $f_i\in\dot{B}^{\alpha,q}_{p_1}\cap L^{p_1}\cap L^1$, $f_2\in L^{p_2}\cap L^1$, and g_k such that

$$\left(\sum_{k\in\mathbb{Z}}2^{-k\alpha q'}||g_k||^{q'}_{L^{p'}}\right)^{\frac{1}{q'}}\le 1.$$

Then using the above argument, we estimate

$$\left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\Theta_k(f_1,f_2)(x)g_k(x)dx\right|\le\sum_{j,k\in\mathbb{Z}}2^{-\gamma|j-k|}\int_{\mathbb{R}^n}\mathscr{M}(Q_jf_1)(x)\mathscr{M}f_2(x)g_k(x)|dx$$

$$\le\left(\sum_{j,k\in\mathbb{Z}}2^{-q\gamma\lambda|j-k|}2^{q\alpha k}\left|\left|\mathscr{M}(Q_jf_1)\mathscr{M}f_2\right|\right|^q_{L^p}\right)^{\frac{1}{q}}\left(\sum_{j,k\in\mathbb{Z}}2^{-q'\gamma(1-\lambda)|j-k|}2^{-q'\alpha k}||g_k||^{q'}_{L^{p'}}\right)^{\frac{1}{q'}}$$

$$\leq \left(\sum_{j,k\in\mathbb{Z}} 2^{(|\alpha|-\gamma\lambda)q|j-k|} 2^{q\alpha j} \left|\left|Q_j f_1\right|\right|_{L^{p_1}}^q ||f_2||_{L^{p_2}}^q \right)^{\frac{1}{q}} \left(\sum_{k\in\mathbb{Z}} 2^{-q'\alpha k} ||g_k||_{L^{p'}}^{q'} \right)^{\frac{1}{q'}}$$

$$\lesssim ||f_1||_{\dot{B}^{\alpha,q}_{p_1}} ||f_2||_{L^{p_2}}$$

By duality and density, the Besov estimate follows. □

In the last bilinear version of Theorem 6.2.1, we require an even stronger cancellation condition on Θ_k, that $\Theta_k(1,\cdot) = \Theta_k(\cdot,1) = 0$ for all $k \in \mathbb{Z}$.

Theorem 6.2.4 (H. [49], Grafakos-Lui-Maldonado-Yang [40]) *Suppose* $\{\theta_k\} \in BLPK$ *and*

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)dy_1 = \int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)dy_2 = 0$$

for all $x, y_1, y_2 \in \mathbb{R}^n$ *and* $k \in \mathbb{Z}$, *then for all* $1 < p, p_1, p_2, q, q_1, q_2 < \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, *and* $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ *with* $\alpha = \alpha_1 + \alpha_2$ *and* $|\alpha_1| + |\alpha_2| < \gamma$. *Then for all* $f_i \in \dot{F}^{\alpha_i,q_i}_{p_i}$

$$\left|\left| \left(\sum_{k\in\mathbb{Z}} 2^{\alpha k q} |\Theta_k(f_1,f_2)|^q \right)^{\frac{1}{q}} \right|\right|_{L^p} \lesssim ||f_1||_{\dot{F}^{\alpha_1,q_1}_{p_1}} ||f_2||_{\dot{F}^{\alpha_2,q_2}_{p_2}}$$

and for all $f_i \in \dot{B}^{\alpha_i,q_i}_{p_i}$

$$\left(\sum_{k\in\mathbb{Z}} 2^{\alpha k q} ||\Theta_k(f_1,f_2)||_{L^p}^q \right)^{\frac{1}{q}} \lesssim ||f_1||_{\dot{B}^{\alpha_1,q_1}_{p_1}} ||f_2||_{\dot{B}^{\alpha_2,q_2}_{p_2}}.$$

Proof: Fix $1 < p, p_1, p_2, q, q_1, q_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = \alpha$ such that $|\alpha_1| + |\alpha_2| < \gamma$, $f_i \in \dot{F}^{\alpha_i, q_i}_{p_i} \cap L^{p_i} \cap L^1$, and $\{g_k\}_{k\in\mathbb{Z}}$ such that

$$\left\| \left(\sum_{k\in\mathbb{Z}} 2^{-k\alpha q'} |g_k|^{q'} \right)^{\frac{1}{q'}} \right\|_{L^{p'}} \leq 1.$$

Then we approximate the dual pairing like in the proofs of Theorems 6.2.2 and 6.2.4: For $N \in \mathbb{N}$

$$\begin{aligned}\sum_{|k|<N} \int_{\mathbb{R}^n} |\Theta_k(f_1, f_2)(x) g_k(x)| dx &\leq \sum_{|k|<N} \int_{\mathbb{R}^n} \sum_{j,\ell\in\mathbb{Z}} |\Theta_k(Q_j f_1, Q_\ell f_2)(x) g_k(x)| dx \\ &\leq \int_{\mathbb{R}^n} \sum_{j,k,\ell\in\mathbb{Z}} |\Theta_k(Q_j f_1, Q_\ell f_2)(x) g_k(x)| dx.\end{aligned}$$

Again, it is sufficient to bound this quantity uniformly in N by the appropriate norms of f_1 and f_2. It follows from estimate (4.16) from Proposition 4.3.2 that for all $x \in \mathbb{R}^n$

$$|\Theta_k(Q_j f_1, Q_\ell f_2)(x)| = |\Theta_k(\widetilde{Q}_j Q_j f_1, \widetilde{Q}_\ell Q_\ell f_2)(x)| \lesssim 2^{-\gamma|j-k|} \mathscr{M}(Q_j f_1)(x) \mathscr{M}(Q_\ell f_2)(x) \tag{6.1}$$

$$|\Theta_k(Q_j f_1, Q_\ell f_2)(x)| = |\Theta_k(\widetilde{Q}_j Q_j f_1, \widetilde{Q}_\ell Q_\ell f_2)(x)| \lesssim 2^{-\gamma|\ell-k|} \mathscr{M}(Q_j f_1)(x) \mathscr{M}(Q_\ell f_2)(x). \tag{6.2}$$

Fix $\lambda \in (0,1)$ such that $|\alpha_1| + |\alpha_2| < \lambda\gamma < \gamma$ which is possible since $|\alpha_1| + |\alpha_2| < \gamma$. Also choose $\eta \in (0,1)$ such that $\frac{|\alpha_1|}{\gamma\lambda} < \eta < \frac{\gamma\lambda - |\alpha_2|}{\gamma\lambda}$. Note that this is possible since $|\alpha_1| <$

$\lambda\gamma - |\alpha_2| < \lambda\gamma$, it also follows that $|\alpha_1| < \lambda\gamma\eta$ and $|\alpha_2| < \lambda\gamma(1-\eta)$. Then we estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\Theta_k(f_1,f_2)(x)g_k(x)dx\right| \\
&\quad\leq \int_{\mathbb{R}^n}\sum_{j,k,\ell\in\mathbb{Z}} 2^{-\gamma(\eta|j-k|+(1-\eta)|\ell-k|)}\mathscr{M}(Q_jf_1)(x)\mathscr{M}(Q_\ell)f_2(x)g_k(x)|dx \\
&\quad\leq \left\|\left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{-q\gamma\lambda(\eta|j-k|+(1-\eta)|\ell-k|)}2^{q(\alpha_1+\alpha_2)k}\left(\mathscr{M}(Q_jf_1)\mathscr{M}(Q_\ell f_2)\right)^q\right)^{\frac{1}{q}}\right\|_{L^p} \\
&\qquad\times\left\|\left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{-q'\gamma(1-\lambda)(\eta|j-k|+(1-\eta)|\ell-k|)}2^{-q'\alpha k}|g_k|^{q'}\right)^{\frac{1}{q'}}\right\|_{L^{p'}} \\
&\quad\leq \left\|\left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{(|\alpha_1|-\gamma\lambda\eta)q|j-k|}2^{(|\alpha_2|-\gamma\lambda(1-\eta))q|\ell-k|}\right.\right. \\
&\qquad\qquad\left.\left.\times\left(2^{\alpha_1 j}\mathscr{M}(Q_jf_1)\cdot 2^{\alpha_2\ell}\mathscr{M}(Q_\ell f_2)\right)^q\right)^{\frac{1}{q}}\right\|_{L^p} \\
&\qquad\qquad\times\left\|\left(\sum_{k\in\mathbb{Z}} 2^{-q'\alpha k}|g_k|^{q'}\right)^{\frac{1}{q'}}\right\|_{L^{p'}} \\
&\quad\leq \left\|\left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{(|\alpha_1|-\gamma\lambda\eta)q_1|j-k|}2^{(|\alpha_2|-\gamma\lambda(1-\eta))q_1|\ell-k|}\left(2^{\alpha_1 j}\mathscr{M}(Q_jf_1)\right)^{q_1}\right)^{\frac{1}{q_1}}\right\|_{L^{p_1}} \\
&\qquad\times\left\|\left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{(|\alpha_1|-\gamma\lambda\eta)q_2|j-k|}2^{(|\alpha_2|-\gamma\lambda(1-\eta))q_2|\ell-k|}\left(2^{\alpha_2\ell}\mathscr{M}(Q_\ell f_2)\right)^{q_2}\right)^{\frac{1}{q_2}}\right\|_{L^{p_2}} \\
&\quad\lesssim \left\|\left(\sum_{j\in\mathbb{Z}} 2^{q_1\alpha_1 j}\left(\mathscr{M}(Q_jf_1)\right)^{q_1}\right)^{\frac{1}{q_1}}\right\|_{L^{p_1}}\left\|\left(\sum_{\ell\in\mathbb{Z}} 2^{q_2\alpha_2\ell}\left(\mathscr{M}(Q_\ell f_2)\right)^{q_2}\right)^{\frac{1}{q_2}}\right\|_{L^{p_2}} \\
&\quad\lesssim ||f_1||_{\dot{F}^{\alpha_1,q_1}_{p_1}}||f_2||_{\dot{F}^{\alpha_2,q_2}_{p_2}}.
\end{aligned}$$

Then the estimate involving Trieble-Lizorkin spaces follows by duality and density. Now let $f_i \in \dot{B}^{\alpha_i,q_i}_{p_i} \cap L^{p_i} \cap L^1$ and g_k such that

$$\left(\sum_{k\in\mathbb{Z}} 2^{-k\alpha q'} ||g_k||^{q'}_{L^{p'}}\right)^{\frac{1}{q'}} \leq 1.$$

Then using the above argument with the same definitions for λ and η, we estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \Theta_k(f_1,f_2)(x)g_k(x)dx\right| \\
&\quad \leq \sum_{j,k,\ell\in\mathbb{Z}} 2^{-\gamma(\eta|j-k|+(1-\eta)|\ell-k|)} \int_{\mathbb{R}^n} \mathscr{M}(Q_jf_1)(x)\mathscr{M}(Q_\ell f_2)(x)g_k(x)|dx \\
&\quad \leq \left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{-q\gamma\lambda(\eta|j-k|+(1-\eta)|\ell-k|)} 2^{q\alpha k} \left|\left|\mathscr{M}(Q_jf_1)\right|\right|^q_{L^{p_1}} ||\mathscr{M}(Q_\ell f_2)||^q_{L^{p_2}}\right)^{\frac{1}{q}} \\
&\qquad\qquad \times \left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{-q'\gamma(1-\lambda)(\eta|j-k|+(1-\eta)|\ell-k|)} 2^{-q'\alpha k} ||g_k||^{q'}_{L^{p'}}\right)^{\frac{1}{q'}} \\
&\quad \leq \left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{(|\alpha_1|-\gamma\lambda\eta)q|j-k|} 2^{(|\alpha_2|-\gamma\lambda(1-\eta))q|\ell-k|} 2^{\alpha_1 jq} 2^{\alpha_2 \ell q}\right. \\
&\qquad\qquad\qquad\qquad \left. \times \left|\left|\mathscr{M}(Q_jf_1)\right|\right|^q_{L^{p_1}} ||\mathscr{M}(Q_\ell f_2)||^q_{L^{p_2}}\right)^{\frac{1}{q}} \\
&\qquad\qquad\qquad\qquad\qquad \times \left(\sum_{k\in\mathbb{Z}} 2^{-q'\alpha k} ||g_k||^{q'}_{L^{p'}}\right)^{\frac{1}{q'}} \\
&\quad \lesssim \left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{(|\alpha_1|-\gamma\lambda\eta)q_1|j-k|} 2^{(|\alpha_2|-\gamma\lambda(1-\eta))q_1|\ell-k|} 2^{\alpha_1 jq_1} \left|\left|Q_jf_1\right|\right|^{q_1}_{L^{p_1}}\right)^{\frac{1}{q_1}} \\
&\qquad \times \left(\sum_{j,k,\ell\in\mathbb{Z}} 2^{(|\alpha_1|-\gamma\lambda\eta)q_2|j-k|} 2^{(|\alpha_2|-\gamma\lambda(1-\eta))q_2|\ell-k|} 2^{\alpha_2\ell q_2} ||Q_\ell f_2||^{q_2}_{L^{p_2}}\right)^{\frac{1}{q_2}}
\end{aligned}$$

$$\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{q_1\alpha_1 j} ||Q_j f_1||_{L^{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \left(\sum_{\ell\in\mathbb{Z}} 2^{q_2\alpha_2\ell} ||Q_\ell f_2||_{L^{p_2}}^{q_2} \right)^{\frac{1}{q_2}} = ||f_1||_{\dot{B}_{p_1}^{\alpha_1,q_1}} ||f_2||_{\dot{B}_{p_2}^{\alpha_2,q_2}}.$$

Then by density and duality the Besov type estimate holds as well. □

6.3 Carleson Measures and BMO

As discussed before, the $\Theta_k(1) = 0$ type conditions are sufficient for square function bounds, but not necessary. In this section, we develop some of the theory needed to quantify size conditions on $\Theta_k(1)$ to replace this mean zero condition. This can be thought of as a perturbation theory, and is crucial to proving square function bounds assuming testing conditions on para-accretive function in place of the function 1. First we define a Carleson measure.

Definition 6.3.1 *Let $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$ and $d\mu(x,t)$ be a non-negative measure on $\mathbb{R}^{n+1}_+$. Then $d\mu$ is a Carleson measure if*

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} d\mu(x,t) \lesssim 1,$$

where this supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

We will state and prove a result from Fefferman-Stein [34], Varopoulos [88], and Coifman-Meyer-Stein [26] which asserts that for appropriate operators Θ_k and any $\beta \in BMO$, we can generate a Carleson measure from $\Theta_k\beta(x)$ in a particular way. In fact, Fefferman-Stein [34] prove this only for $\Theta_k f = \psi_k * f$ as part of a full characterization of BMO in terms of Carleson measures. The work in [88] and [26] extends this implication to the non-convolution case involving the operators Θ_k, which we state now.

Theorem 6.3.2 ([34], [88], [26]) *Suppose $\theta_k : \mathbb{R}^{2n} \to \mathbb{C}$ satisfies (4.1) and $\Theta_k(1) = 0$ for all $k \in \mathbb{Z}$. If $S_{\{\Theta_k\}}$ is bounded on L^2, then*

$$d\mu(x,t) = \sum_{k \in \mathbb{Z}} |\Theta_t \beta(x)|^2 dx \delta_{t=2^{-k}}$$

is a Carleson measure for all $\beta \in BMO$.

Note that we require no regularity on the kernels θ_k here. In order to prove Theorem 6.3.2, we need a couple results about the space BMO. The first is a result of John-Nirenberg [55], and we state without proof.

Theorem 6.3.3 (John-Nirenberg [55]) *For $1 \leq p < \infty$ and $\beta \in L^1_{loc}$, define*

$$||\beta||_{BMO,p} = \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - Avg_Q f|^p dx \right)^{\frac{1}{p}}.$$

For $\beta \in L^1_{loc}$ and $1 \leq p < \infty$, $||\beta||_{BMO} < \infty$ if and only if $||\beta||_{BMO,p} < \infty$. Furthermore, $|| \cdot ||_{BMO,p}$ is an equivalent norm on BMO.

The next lemma contains a few classical estimates for BMO functions, but for the sake of completeness we prove them.

Lemma 6.3.4 *If $\beta \in BMO$, then the following hold:*

(i) For any cube $Q \subset \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$|Avg_Q \beta - Avg_{2^k Q} \beta| \lesssim k ||\beta||_{BMO},$$

where $2^k Q$ is the cube with the same center as Q and side length $2^k \ell(Q)$.

(ii) For any cube $Q \subset \mathbb{R}^n$ with center c_Q and $N > n$

$$\frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\beta(x)-Avg_Q\beta|}{(1+\ell(Q)^{-1}|x-c_Q|)^N}dx \lesssim ||\beta||_{BMO}$$

Proof: To prove *(i)* we first consider the case $k = 1$:

$$\begin{aligned}|Avg_Q\beta - Avg_{2Q}\beta| &= \left|\frac{1}{|Q|}\int_Q(\beta(x)-Avg_{2Q}\beta)dx\right| \\ &\leq \frac{2^n}{|2Q|}\int_{2Q}|\beta(x)-Avg_{2Q}|dx \leq 2^n||\beta||_{BMO}.\end{aligned}$$

Then it follows that for any $k \geq 1$, we apply this bound k times to

$$|Avg_Q\beta - Avg_{2^kQ}\beta| \leq \sum_{j=0}^{k-1}|Avg_{2^jQ}\beta - Avg_{2^{j+1}Q}\beta| \leq \sum_{j=0}^{k-1}2^n||\beta||_{BMO} \lesssim k||\beta||_{BMO}.$$

This proves *(i)*. Now for *(ii)* consider

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|\beta(x)-Avg_Q\beta|dx}{(1+\ell(Q)^{-1}|x-c_Q|)^N} &\leq \int_Q|\beta(x)-Avg_Q\beta|dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\frac{|\beta(x)-Avg_Q\beta|dx}{(\ell(Q)^{-1}|x-c_Q|)^N} \\ &\leq |Q|\;||\beta||_{BMO} + \sum_{k=0}^{\infty}2^{-kN}\int_{2^{k+1}Q\setminus 2^kQ}|\beta(x)-Avg_Q\beta|dx \\ &\leq |Q|\;||\beta||_{BMO} + |Q|\sum_{k=0}^{\infty}2^{k(n-N)}\frac{1}{|Q|2^{kn}}\int_{2^{k+1}Q}|\beta(x)-Avg_{2^{k+1}Q}\beta|dx \\ &\quad + |Q|\sum_{k=0}^{\infty}2^{k(n-N)}\frac{1}{|Q|2^{kn}}\int_{2^{k+1}Q}|Avg_{2^{k+1}Q}\beta - Avg_Q\beta|dx \\ &\lesssim |Q|\;||\beta||_{BMO} + |Q|\;||\beta||_{BMO}\sum_{k=0}^{\infty}2^{k(n-N)} + |Q|\;||\beta||_{BMO}\sum_{k=0}^{\infty}2^{k(n-N)}(k+1) \\ &\lesssim |Q|\;||\beta||_{BMO}.\end{aligned}$$

This completes the proof of *(ii)* □

Now we prove Theorem 6.3.2.

Proof: We first write

$$\Theta_k\beta = \Theta_k(Avg_Q\beta) + \Theta_k\left((\beta - Avg_Q\beta)\chi_{2Q}\right) + \Theta_k\left((\beta - Avg_Q\beta)\chi_{(2Q)^c}\right).$$

Since $\Theta_k(1) = 0$ and Θ_k is linear, it follows that $\Theta_k(Avg_Q\beta) = 0$. The second term satisfies the bound

$$\begin{aligned}
\int_Q \sum_{2^{-k}<\ell(Q)} |\Theta_k\left((\beta - Avg_Q\beta)\chi_{2Q}\right)(x)|^2 dx &\leq \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\Theta_k\left((\beta - Avg_Q\beta)\chi_{2Q}\right)(x)|^2 dx \\
&\lesssim \int_{2Q} |\beta(x) - Avg_Q\beta|^2 dx \\
&\leq \int_{2Q} |\beta(x) - Avg_{2Q}\beta|^2 dx + \int_{2Q} |Avg_{2Q}\beta - Avg_Q\beta|^2 dx \\
&\lesssim |Q|\,||\beta||_{BMO}^2.
\end{aligned}$$

Note that we used Theorem 6.3.3 here. Let c_Q be the center of Q, and we bound the integrand of the third term when $x \in Q$

$$\begin{aligned}
|\Theta_k\left((\beta - Avg_Q\beta)\chi_{(2Q)^c}\right)(x)| &\lesssim \int_{(2Q)^c} \Phi_k^N(x-y)|\beta(y) - Avg_Q\beta|dy \\
&\leq \int_{(2Q)^c} \frac{2^{kn}|\beta(y) - Avg_Q\beta|}{(1+2^k(|y-c_Q|-|x-c_Q|))^N} dy
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{(2Q)^c} \frac{2^{kn}|\beta(y) - Avg_Q\beta|}{2^{kN}(\ell(Q)+|y-c_Q|)^N} dy \\
&\leq 2^{k(n-N)}\ell(Q)^{n-N}\left(\frac{1}{\ell(Q)^n}\int_{\mathbb{R}^n} \frac{|\beta(y) - Avg_Q\beta|}{(1+\ell(Q)^{-1}|y-c_Q|)^N} dy\right) \\
&\lesssim 2^{k(n-N)}\ell(Q)^{n-N}||\beta||_{BMO}.
\end{aligned}$$

In the last line here, we have used Lemma 6.3.4. Then it follows that

$$\begin{aligned}\int_Q \sum_{2^{-k}<\ell(Q)} |\Theta_k\left((\beta - Avg_Q\beta)\chi_{(2Q)^c}\right)(x)|^2 dx &\lesssim |Q|\,||\beta||_{BMO}^2 \ell(Q)^{2(n-N)} \sum_{2^{-k}<\ell(Q)} 2^{2k(n-N)} \\ &\lesssim |Q|\,||\beta||_{BMO}^2.\end{aligned}$$

Therefore $d\mu$ is a Carleson measure. $\square$

Since Theorem 6.3.2 does not depend on any regularity of the kernel θ_k, like before we can replace $\theta_k(x,y)$ with $\theta_k(x,y)b(y)$ for a para-accretive function b and expect the same conclusion. More precisely we have the following:

Corollary 6.3.5 *Suppose* $\theta_k : \mathbb{R}^{2n} \to \mathbb{C}$ *satisfies* (4.1) *and there exists a para-accretive function b such that $\Theta_k(b) = 0$ for all $k \in \mathbb{Z}$. If $S_{\{\Theta_k\}}$ is bounded on L^2, then*

$$d\mu(x,t) = \sum_{k\in\mathbb{Z}} |\Theta_t M_b \beta(x)|^2 dx \delta_{t=2^{-k}}$$

is a Carleson measure for all $\beta \in BMO$.

Proof: This follows immediately by applying Theorem 6.3.2 to $\Theta_k M_b$. $\square$

In the context of square functions $S_{\{\Theta_k\}}$, we can use Carleson measure theory as a way of quantifying cancellation conditions for Θ_k. That is, instead of requiring $\Theta_k(1) = 0$ for all $k \in \mathbb{Z}$, we use Carleson measure estimates to require $\Theta_k(1)$ to be small in some sense. The next theorem is a result from Carleson [12] and Jones [56] that demonstrates the relationship between bounds for $S_{\{\Theta_k\}}$ and Carleson measures associated to $\Theta_k(1)$.

Theorem 6.3.6 (Carleson [12], Jones [56]) *A collection of non-negative measures $d\mu_k(x)$*

for $k \in \mathbb{Z}$ form a Carleson measure

$$d\mu(x,t) = \sum_{k\in\mathbb{Z}} d\mu_k(x)\delta_{t=2^{-k}}$$

if and only if for all $f \in L^p$

$$\left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\Phi_k^N * f(x)|^p d\mu_k(x)\right)^{1/p} \lesssim ||f||_{L^p},$$

whenever $N > n$ and $1 < p < \infty$.

Early versions of this result were proved by Carleson [12], and the equivalence as stated here was proved by Jones [56]. When $d\mu$ is a Carleson measure, Theorem 6.3.6 garauntees that $S_{\{\Theta_k\}}$ is bounded on L^2, we cannot say the same for the L^p bounds for $S_{\{\Theta_k\}}$ for any $p \neq 2$. These bounds must be addressed in a different way.

6.4 Accretive Type Square Functions

In this section, we address what we have called the perturbed cancellation conditions on Θ_k, which are testing conditions on a para-accretive functions instead of 1. This theory is closely related to Carleson measures and BMO, which we introduced in the last section. The next two results are due to Semmes [74] for $p = 2$. We only cite Semmes in the first result since we obtain the estimates for $p \neq 2$ in a familar way. More precisely, we use the vector valued Calderón-Zygmund theory like in the proof of Proposition 6.1.2, which was due to Benedek-Calderón-Panzone [2] and Rubio de Francia-Ruiz-Torrea [53].

Proposition 6.4.1 (Semmes [74]) *If $\{\theta_k\} \in SLPK$ and there exists a para-accretive function b such that $\Theta_k(b) = 0$ for all $k \in \mathbb{Z}$, then $S_{\{\Theta_k\}}$ is bounded from L^1 into $L^{1,\infty}$, from L_c^∞*

into BMO, and from L^p into L^p when $1 < p < \infty$.

Proof: Let P_k be a smooth approximation to identity operator with non-negative, compactly supported convolution kernel. Consider $R_k = \Theta_k - M_{\Theta_k 1} P_k$, whose kernels $\{r_k\}$ form a collection of Littlewood-Paley square function kernels, i.e. $\{r_k\} \in LPK$. Also $R_k(1) = \Theta_k(1) - \Theta_k(1)P_k(1) = 0$ for all $k \in \mathbb{Z}$, so it follows from Theorem 6.2.1 that $||S_{\{R_k\}}||_{L^p} \lesssim ||f||_{L^p}$ for all $f \in L^p$ and $1 < p < \infty$. Since

$$||S_{\{\Theta_k\}} f||_{L^2} \leq ||S_{\{R_k\}} f||_{L^2} + ||S_{\{M_{\Theta_k(1)} P_k\}} f||_{L^2}$$

by Theorem 6.3.6, it is sufficient for the L^2 bound to show that $\sum_k |\Theta_k 1(x)|^2 dx\, \delta_{t=2^{-k}}$ is a Carleson measure. Note first that since b is a para-accretive function, we have

$$c|\Theta_k 1(x)| \leq |P_k b(x) \Theta_k 1(x)| \leq |P_k b(x) \Theta_k 1(x) - \Theta_k P_k b(x)| + |\Theta_k P_k b(x) - \Theta_k b(x)|.$$

Here we have used a result from [29] that b is para-accretive implies $|P_k b(x)| \geq c > 0$ uniformly in $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Now looking at $R_k^1 f = M_{\Theta_k 1} P_k f - \Theta_k P_k f$ and $R_k^2 f = \Theta_k P_k f - \Theta_k f$, it easily follows that $\{r_k^i\} \in LPK$ for $i = 1, 2$. Furthermore $R_k^1(1) = M_{\Theta_k 1} P_k(1) - \Theta_k P_k(1) = 0$ and $R_k^2(1) = \Theta_k P_k(1) - \Theta_k(1) = 0$. Then the square functions $S_{\{R_k^1\}}$ and $S_{\{R_k^2\}}$ associated to R_k^1 and R_k^2 are bounded, and hence by Theorem 6.3.2 it follows that $\sum_k |R_k^1 b(x)|^2 dx\, \delta_{t=2^{-k}}$ and $\sum_k |R_k^2 b(x)|^2 dx\, \delta_{t=2^{-k}}$ are both Carleson measures. Therefore $\sum_k |\Theta_k 1(x)|^2 dx\, \delta_{t=2^{-k}}$ is a Carleson measure, and by Theorem 6.3.6 we have

$$\left|\left|S_{\{M_{\Theta_k 1} P_k\}} f\right|\right|_{L^2} \leq \left(\int_{\mathbb{R}^{n+1}_+} |P_k f(x)|^2 \sum_{k \in \mathbb{Z}} |\Theta_k 1(x)|^2 dx\, \delta_{t=2^{-k}} \right)^{\frac{1}{2}} \lesssim ||f||_{L^2}.$$

That is, $S_{\{\Theta_k\}}$ is bounded on L^2. By Lemma 6.1.1 $\{\theta_k\} \in CZK_{\ell^2(\mathbb{Z})}$, and so it follows that $S_{\{\Theta_k\}}$ is bounded from L^1 into $L^{1,\infty}$, from L^∞_c into BMO. Also by Theorems 3.1.1 and 3.3.3, using the argument from Proposition 6.1.2, it follows that $S_{\{\Theta_k\}}$ is bounded on L^p for all $1 < p < \infty$. $\square$

Theorem 6.4.2 (David-Journé-Semmes [29], Semmes [74]) *If $\{\theta_k\} \in LPK$ and there exists a para-accretive function $b \in L^\infty$ such that $\Theta_k(b) = 0$, then $S_{\{\Theta_k\}}$ is bounded from L^1 into $L^{1,\infty}$, from L^∞_c into BMO, and from L^p into L^p for $1 < p < \infty$.*

The difference between Proposition 6.4.1 and Theorem 6.4.2 is that in Proposition 6.4.1 it is required that $\{\theta_k\} \in SLPK$ and in Theorem 6.4.2 it is only require that $\{\theta_k\} \in LPK$. In fact, Proposition 6.4.1 is used in the proof of Theorem 6.4.2 (at least in the proof provided in this work). The proof of Theorem 6.4.2 is essentially contained in the proof of the following bilinear version of this result, so we omit the details of the linear case and refer the reader to the proof of Theorem 6.4.3.

The next result was proved by the author in [49], and for $p = 2$ it was proved by Grafakos-Oliveira [41].

Theorem 6.4.3 (H. [49], Grafakos-Oliviera [41]) *If $\{\theta_k\} \in BLPK$ and there exist para-accretive functions $b_1, b_2 \in L^\infty$ such that $\Theta_k(b_1, b_2) = 0$ for all $k \in \mathbb{Z}$, then $S_{\{\Theta_k\}}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p, p_1, p_2 < \infty$ satisfying* (1.1).

Proof: Fix $1 < p, p_1, p_2 < \infty$ satisfying (1.1), $f_1, f_2 \in C^1_0$ such that $b_i f_i$ have mean zero for $i = 1, 2$, and $g_k \in L^{p'}$ satisfying

$$\left\| \left(\sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^{p'}} \leq 1.$$

Let $S_k^{b_i}$, $D_k^{b_i}$, and $\widetilde{D}_k^{b_i}$ be the approximation to identity from Remark 5.3.3 and reproducing operators from Theorem 5.4.1 with respect to b_i for $i = 1,2$. Define

$$\Pi_j^1(f_1,f_2)(y_1,y_2) = M_{b_1}D_j^{b_1}M_{b_1}f_1(y_1)M_{b_2}S_{j+1}^{b_2}M_{b_2}f_2(y_2)$$
$$\Pi_j^2(f_1,f_2)(y_1,y_2) = M_{b_1}S_j^{b_1}M_{b_1}f_1(y_1)M_{b_2}D_j^{b_2}M_{b_2}f_2(y_2).$$

By Proposition 5.3.1, it follows that

$$\begin{aligned}
&\Theta_k(b_1f_1,b_2f_2)\\
&\quad= \lim_{N\to\infty}\Theta_k(M_{b_1}S_N^{b_1}M_{b_1}f_1,M_{b_2}S_N^{b_2}M_{b_2}f_2) - \Theta_k(M_{b_1}S_{-N}^{b_1}M_{b_1}f_1,M_{b_2}S_{-N}^{b_2}M_{b_2}f_2)\\
&\quad= \lim_{N\to\infty}\sum_{j=-N}^{N-1}\Theta_k(M_{b_1}S_{j+1}^{b_1}M_{b_1}f_1,M_{b_2}S_{j+1}^{b_2}M_{b_2}f_2) - \Theta_k(M_{b_1}S_j^{b_1}M_{b_1}f_1,M_{b_2}S_j^{b_2}M_{b_2}f_2)\\
&\quad= \sum_{j\in\mathbb{Z}}\Theta_k\Pi_j^1(f_1,f_2) + \Theta_k\Pi_j^2(f_1,f_2),
\end{aligned}$$

where this limit holds in L^p when $1 < p, p_1, p_2 < \infty$ that satisfy the Hölder relationship (1.1). Then we estimate $||S_{\{\Theta_k\}}(f_1,f_2)||_{L^p}$ by the dual pairing

$$\left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\Theta_k(b_1f_1,b_2f_2)(x)g_k(x)dx\right| \leq \sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}^n}\Theta_k(b_1f_1,b_2f_2)(x)g_k(x)dx\right| = \lim_{N\to\infty}\sum_{|k|<N}|\Omega_k|,$$

where we take the last equality to be the definition of $\Omega_k = \Omega_k(f_1,f_2,\{g_k\})$ to be the quantity inside the absolute value in the previous expression. Now for $M \in \mathbb{N}$

$$\begin{aligned}
&\sum_{|k|<N}\left|\Omega_k - \int_{\mathbb{R}^n}\sum_{|j|<M}(\Theta_k\Pi_j^1(f_1,f_2)(x) + \Theta_k\Pi_j^2(f_1,f_2)(x))g_k(x)dx\right|\\
&\qquad\qquad \leq \sum_{|k|<N}\left|\left|\Theta_k(b_1f_1,b_2f_2) - \sum_{|j|<M}(\Theta_k\Pi_j^1(f_1,f_2) + \Theta_k\Pi_j^2(f_1,f_2))\right|\right|_{L^p}||g_k||_{L^{p'}},
\end{aligned}$$

which tends to zero as $M \to \infty$. We make one final reduction before we prove the estimate: Since $d_j^{b_i}(x,\cdot) \in L^{p'}$ for all $x \in \mathbb{R}^n$ and $f_i \in C_0^\delta$ where $b_i f_i$ has mean zero, it follows that for $x \in \mathbb{R}^n$

$$\begin{aligned}
&\left| D_j^{b_i} M_{b_i} f_i(x) - \sum_{|\ell|<M} D_j^{b_i} M_{b_i} \widetilde{D}_\ell^{b_i} M_{b_i} D_\ell^{b_i} M_{b_i} f_i(x) \right| \\
&\qquad \lesssim \int_{\mathbb{R}^n} |d_j^{b_i}(x,y)| \left| f_i(y) - \sum_{|\ell|<M} \widetilde{D}_\ell^{b_i} M_{b_i} D_\ell^{b_i} M_{b_i} f_i(y) \right| dy \\
&\qquad \le \left(\int_{\mathbb{R}^n} \Phi_j^N(x-y)^{p'} dy \right)^{\frac{1}{p'}} \left\| f_i - \sum_{|\ell|<M} \widetilde{D}_\ell^{b_i} M_{b_i} D_\ell^{b_i} M_{b_i} f_i \right\|_{L^p},
\end{aligned}$$

which tends to zero as $M \to \infty$. This proves that the above convergence holds pointwise. Furthermore by estimate (4.13) from Proposition 4.3.2 we have

$$\left| \sum_{|\ell|<N} D_j^{b_i} M_{b_i} \widetilde{D}_\ell^{b_i} M_{b_i} D_\ell^{b_i} M_{b_i} f_i(x) \right| \lesssim \sum_{|\ell|<N} 2^{-\gamma|\ell-j|} \mathscr{M}(D_\ell^{b_i} f)(x) \lesssim \mathscr{M}^2 f(x).$$

Since $(\mathscr{M}^2 f)^p$ is an integrable function for $f \in C_0^1$, by dominated convergence, we have

$$\begin{aligned}
&\sum_{|j|,|k|<N} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1(f_1,f_2)(x) g_k(x) dx \right| \\
&\qquad \le \sum_{j,k,\ell \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1 \left(\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2 \right)(x) g_k(x) dx \right|.
\end{aligned}$$

As was done in Theorems 6.2.2-6.2.4, it is sufficient to prove that the term above is bounded uniformly in N. So we estimate $\Theta_k \Pi_j^1 \left(\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2 \right)$ pointwise. Note that Θ_k, $D_j^{b_1}$, and $S_j^{b_2}$ satisfy the hypotheses for estimate (4.15) of Proposition 4.3.2. So it follows that

$$\left|\Theta_k\Pi_j^1\left(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2\right)(x)\right| \lesssim 2^{-\gamma|j-k|}\mathscr{M}(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1)(x)\mathscr{M}f_2(x)$$
$$\lesssim 2^{-\gamma|j-k|}\mathscr{M}^2(D_\ell^{b_1}M_{b_1}f_1)(x)\mathscr{M}^2f_2(x) \qquad (6.3)$$

We also apply estimate (4.13) from Proposition 4.3.2 for $D_j^{b_1}$ and $D_\ell^{b_1}$ to estimate

$$\left|\Theta_k\Pi_j^1\left(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2\right)(x)\right| \lesssim \Phi_k^N*|D_j^{b_1}M_{b_1}\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1|(x)\Phi_k*\Phi_j*|f_2|(x)$$
$$\lesssim 2^{-\gamma|j-\ell|}\mathscr{M}^2(D_\ell^{b_1}M_{b_1}f_1)(x)\mathscr{M}^2f_2(x). \qquad (6.4)$$

Note here that we choose γ to be the smallest of the regularity parameters guaranteed for $\{\theta_k\}\in BLPK$ and $\{d_j^{b_i}\},\{\widetilde{d}_j^{b_i*}\}\in LPK$ for $i=1,2$. This is not of consequence in this proof, since all we need is that $\gamma>0$ and we are choosing the smallest of a finite number of positive parameters. Taking the geometric mean of the two estimates (6.3) and (6.4), it follows that for all $x\in\mathbb{R}^n$

$$\left|\Theta_k\left(M_{b_1}D_jM_{b_1}\widetilde{D}_\ell^{b_1}M_{b_1}f_1,M_{b_2}S_j^{b_2}M_{b_2}f_2\right)(x)\right| \lesssim 2^{-\gamma|j-k|/2}2^{-\gamma|j-\ell|/2}\mathscr{M}^2f_1(x)\mathscr{M}^2f_2(x).$$

Then it follows that

$$\int_{\mathbb{R}^n}|\Theta_k\Pi_j^1(f_1,f_2)(x)g_k(x)|dx$$
$$\leq \sum_{\ell\in\mathbb{Z}}\int_{\mathbb{R}^n}|\Theta_k\Pi_j^1(M_{b_1}D_jM_{b_1}\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2)(x)g_k(x)|dx$$
$$\lesssim \sum_{\ell\in\mathbb{Z}}2^{-\gamma\frac{|j-k|}{2}}2^{-\gamma\frac{|j-\ell|}{2}}\int_{\mathbb{R}^n}\mathscr{M}^2(D_\ell M_{b_1}f_1)(x)\mathscr{M}^2f_2(x)|g_k(x)|dx$$

The same estimate holds if we replace Π_j^1 with Π_j^2 and switch the roles of $D_\ell^{b_1} f_1$ and f_2 with f_1 and $D_\ell^{b_2} f_2$ respectively. Finally for any $N \in \mathbb{N}$, we use this to estimate

$$\begin{aligned}
\sum_{|k|<N} |\Omega_k| &\lesssim \int_{\mathbb{R}^n} \sum_{j,k,\ell \in \mathbb{Z}} 2^{-\gamma \frac{|j-k|+|j-\ell|}{2}} \\
&\qquad \times (\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1)(x) \mathscr{M}^2 f_2(x) + \mathscr{M}^2 f_1(x) \mathscr{M}^2(D_\ell^{b_2} M_{b_2} f_2)(x)) |g_k(x)| dx \\
&\leq \left|\left| \left(\sum_{j,k,\ell \in \mathbb{Z}} 2^{-\gamma \frac{|j-k|}{2}} 2^{-\gamma \frac{|j-\ell|}{2}} (\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1) \mathscr{M}^2 f_2 + \mathscr{M}^2 f_1 \mathscr{M}^2(D_\ell^{b_2} M_{b_2} f_2))^2 \right)^{\frac{1}{2}} \right|\right|_{L^p} \\
&\qquad \times \left|\left| \left(\sum_{j,k,\ell \in \mathbb{Z}} 2^{-\gamma \frac{|j-k|}{2}} 2^{-\gamma \frac{|j-\ell|}{2}} |g_k|^2 \right)^{\frac{1}{2}} \right|\right|_{L^{p'}} \\
&\lesssim \left|\left| \left(\sum_{\ell \in \mathbb{Z}} (\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1) \mathscr{M}^2 f_2 + \mathscr{M}^2 f_1 \mathscr{M}^2(D_\ell^{b_2} M_{b_2} f_2))^2 \right)^{\frac{1}{2}} \right|\right|_{L^p} \left|\left| \left(\sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{\frac{1}{2}} \right|\right|_{L^{p'}} \\
&\leq \left|\left| \left(\sum_{\ell \in \mathbb{Z}} (\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1))^2 \right)^{\frac{1}{2}} \right|\right|_{L^{p_1}} ||\mathscr{M}^2 f_2||_{L^{p_2}} \\
&\qquad + ||\mathscr{M}^2 f_1||_{L^{p_1}} \left|\left| \left(\sum_{\ell \in \mathbb{Z}} (\mathscr{M}^2(D_\ell^{b_2} M_{b_2} f_2))^2 \right)^{\frac{1}{2}} \right|\right|_{L^{p_2}} \\
&\lesssim ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}}.
\end{aligned}$$

Note here that we've used that $\{d_k^{b_i}\} \in SLPK$ to apply Proposition 6.4.1. Then it follows that

$$\left| \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \Theta_k(b_1 f_1, b_2 f_2) g_k(x) dx \right| \leq \lim_{N \to \infty} \sum_{|k|<N} |\Omega_k| \lesssim ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}},$$

and hence by duality and density, the estimate follows. □

Next we use this result to prove an accretive bilinear version of Theorem 6.3.6 when $d\mu$ is generated by $\Theta_k(b)$, generalized to testing Θ_k on para-accretive functions. We state the linear and bilinear versions of this this result, but again just prove the bilinear one. The proof of the linear version is easily extracted from the bilinear one. Parts of the bilinear result can be found in the work of Grafakos-Oliviera [41].

Theorem 6.4.4 (Semmes [74]) *Let $\{\theta_k\} \in LPK$ and $b \in L^\infty$ be para-accretive functions and define the non-negative measures $d\mu_k(x) = |\Theta_k b(x)|^2 dx\, \delta_{2^{-k}}(t)$ for $k \in \mathbb{Z}$. Then*

$$\sum_{k\in\mathbb{Z}} d\mu_k(x)\, \delta_{2^{-k}}(t).$$

is a Carleson measure if and only if $S_{\{\Theta_k\}}$ is bounded on L^2.

Theorem 6.4.5 (Grafakos-Oliviera [41], H.) *Let $\{\theta_k\} \in BLPK$ and $b_1, b_2 \in L^\infty$ be para-accretive functions. Define the non-negative measures $d\mu_k(x) = |\Theta_k(b_1, b_2)(x)|^2 dx\, \delta_{2^{-k}}(t)$ for $k \in \mathbb{Z}$. Then the following are equivalent:*

i. The measure $d\mu_k$ define a Carleson measure, i.e. the following is a Carleson measure

$$\sum_{k\in\mathbb{Z}} d\mu_k(x)\, \delta_{2^{-k}}(t). \tag{6.5}$$

ii. For all $1 < p_1, p_2 < \infty$ such that $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$, $S_{\{\Theta_k\}}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^2.

iii. There exist some $1 < p_1, p_2 < \infty$ such that $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$ and $S_{\{\Theta_k\}}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^2.

Proof: Fix para-accretive functions $b_1, b_2 \in L^\infty$, and assume that (6.5) defines a Carleson measure. Let $S_k^{b_i}$ be approximation to identity operators with respect to b_i for $i = 1, 2$. Also define $R_k(f_1,f_2) = \Theta_k(f_1,f_2) - M_{\Theta_k(b_1,b_2)} S_k^{b_1} f_1 \cdot S_k^{b_2} f_2$ and we approximate

$$\begin{aligned} &\left|\left|S_{\{\Theta_k\}}(f_1,f_2)\right|\right|_{L^2} \\ &\quad \le \left|\left|S_{\{R_k\}}(f_1,f_2)\right|\right|_{L^2} + \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |S_k^{b_1} f_1(x) S_k^{b_2} f_2(x)|^2 |\Theta_k(b_1,b_2)(x)|^2 dx\right)^{\frac{1}{2}} = I + II. \end{aligned}$$

The first term I is bounded appropriately by Theorem 6.4.3 since $R_k(b_1,b_2) = 0$ the kernels r_k of R_k staisfy $\{r_k\} \in BLPK$. Using Hölder's inequality and Theorem 6.3.6, II is bounded by

$$\begin{aligned} \prod_{i=1}^{2} \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |S_k^{b_i} f_i(x)|^{p_i} |\Theta_k(b_1,b_2)(x)|^2 dx\right)^{\frac{1}{p_i}} &\lesssim \prod_{i=1}^{2} \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\Phi_k^N * f_i(x)|^{p_i} d\mu_k(x)\right)^{\frac{1}{p_i}} \\ &\lesssim ||f_1||_{L^{p_1}} \, ||f_2||_{L^{p_2}}. \end{aligned}$$

Then $S_{\{\Theta_k\}}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^2 for appropriate p_1, p_2. This proves that i implies ii. It is trivial that ii implies iii. So we assume iii, and prove i. Fix a cube $Q \subset \mathbb{R}^n$

and we compute

$$\int_Q \int_0^{\ell(Q)} \sum_{k\in\mathbb{Z}} d\mu_k(x)\,\delta_{2^{-k}}(t) = \sum_{2^{-k}\leq \ell(Q)} \int_Q |\Theta_k(b_1,b_2)(x)|^2 dx$$

$$= \sum_{2^{-k}\leq \ell(Q)} \int_Q |\Theta_k(b_1\chi_{2Q}, b_2\chi_{2Q})(x)|^2 dx$$

$$+ \sum_{2^{-k}\leq \ell(Q)} \int_Q |\Theta_k(b_1\chi_{(2Q)^c}, b_2\chi_{2Q})(x)|^2 dx$$

$$+ \sum_{2^{-k}\leq \ell(Q)} \int_Q |\Theta_k(b_1\chi_{2Q}, b_2\chi_{(2Q)^c})(x)|^2 dx$$

$$+ \sum_{2^{-k}\leq \ell(Q)} \int_Q |\Theta_k(b_1\chi_{(2Q)^c}, b_2\chi_{(2Q)^c})(x)|^2 dx$$

$$= I + II + III + IV$$

Using hypothesis *(iii)*, it follows that I is bounded

$$I \leq \left|\left|S_{\{\Theta_k\}}(b_1\chi_{2Q}, b_2\chi_{2Q})\right|\right|_{L^2}^2 \lesssim ||b_1\chi_{2Q}||_{L^{p_1}}^2 ||b_2\chi_{2Q}||_{L^{p_2}}^2 \lesssim |Q|.$$

We bound II using the kernel conditions of θ_k:

$$|\Theta_k(b_1\chi_{(2Q)^c}, b_2\chi_{2Q})(x)| \leq ||b_1||_{L^\infty}||b_2||_{L^\infty} \left(\int_{(2Q)^c} \Phi_k^N(x-y_1)dy_1\right)\left(\int_{\mathbb{R}^n} \Phi_k^N(x-y_2)dy_2\right)$$

$$\lesssim \int_{|x-y_1|>\ell(Q)} \frac{2^{kn}}{(2^k|x-y_1|)^N} dy_1$$

$$\lesssim 2^{(n-N)k}\ell(Q)^{N-n}.$$

So it also follows that

$$II \lesssim \sum_{2^{-k}\leq \ell(Q)} \int_Q 2^{2(n-N)k}\ell(Q)^{2(N-n)} dx \lesssim |Q|.$$

By symmetry, $III \lesssim |Q|$ as well. Also it is clear that IV is bounded using the same argument since we only use the support properties of $\chi_{(2Q)^c}$, which is in the first spot of Θ_k in IV as well. $\square$

6.5 Extending Square Function Bound for $1/2 < p \leq 1$

Just like in the linear case, we can apply the vector valued Calderón-Zygmund theory from Chapter 2 to the bilinear square functions $S_{\{\Theta_k\}}$ when $\{\theta_k\} \in SBLPK$. We prove the bilinear analogue of Lemma 6.1.1. This again demonstrates a connection between the vector valued Calderón-Zygmund and Littlewood-Paley theory, this time in the bilinear setting.

Lemma 6.5.1 *If $\{\theta_k\} \in SLPK$ with size index $N > 2n$, then $\{\theta_k\}_{k\in\mathbb{Z}}$ is an $\ell^r(\mathbb{Z})$-valued Calderón-Zygmund operator for $1 \leq r \leq \infty$.*

Proof: Consider first $r = 1$ and let $x, y_1, y_2 \in \mathbb{R}^n$. To prove the size estimate, we take $d = |x-y_1| + |x-y_2| \neq 0$ and compute

$$\begin{aligned} ||\{\theta_k(x,y_1,y_2)\}||_{\ell^1(\mathbb{Z})} &\lesssim \sum_{k\in\mathbb{Z}} \frac{2^{2kn}}{(1+2^k|x-y_1|)^{N+\gamma}(1+2^k|x-y_2|)^{N+\gamma}} \\ &\lesssim \sum_{2^k \leq d^{-1}} 2^{2kn} + \sum_{2^k > d^{-1}} \frac{2^{2kn}}{(2^k d)^{N+\gamma}} \lesssim d^{-2n}. \end{aligned}$$

For the regularity in x, we take $x, x', y_1, y_2 \in \mathbb{R}^n$ with $|x-x'| < \max(|x-y_1|, |x-y_2|)/2$ and define $d' = |x'-y_1| + |x'-y_2|$. Then

$$\begin{aligned} ||\theta_k(x,y_1,y_2) - \theta_k(x',y_1,y_2)||_{\ell^1(\mathbb{Z})} &\lesssim \sum_{k\in\mathbb{Z}} \frac{(2^k|x-x'|)^\gamma 2^{2kn}}{(1+2^k|x-y_1|)^{N+\gamma}(1+2^k|x-y_2|)^{N+\gamma}} \\ &\quad + \sum_{k\in\mathbb{Z}} \frac{(2^k|x-x'|)^\gamma 2^{2kn}}{(1+2^k|x'-y_1|)^{N+\gamma}(1+2^k|x'-y_2|)^{N+\gamma}} \\ &= I + II. \end{aligned}$$

We first bound I by $|x-x'|^\gamma$ times

$$\sum_{2^k \le d^{-1}} 2^{k(2n+\gamma)} + \sum_{2^k > d^{-1}} \frac{2^{k(2n+\gamma)}}{(2^k d)^{N+\gamma}} \lesssim d^{-(2n+\gamma)} + d^{-(N+\gamma)} \sum_{2^k > d^{-1}} 2^{k(2n-N)} \lesssim d^{-(2n+\gamma)}.$$

By symmetry, it follows that $II \lesssim |x-x'|^\gamma d'^{-(2n+\gamma)}$, but since $|x-x'| < \max(|x-y_1|, |x-y_2|)/2$, without loss of generality say $|x-y_1| \ge |x-y_2|$ it follows that

$$II \lesssim \frac{|x-x'|^\gamma}{(|x'-y_1|+|x'-y_2|)^{2n+\gamma}} \le \frac{|x-x'|^\gamma}{(|x-y_1|-|x-x'|)^{2n+\gamma}} \lesssim \frac{|x-x'|^\gamma}{|x-y_1|^{2n+\gamma}} \lesssim \frac{|x-x'|^\gamma}{d^{2n+\gamma}}$$

With a similar computation for y_1, y_2, it follows that $\{\theta_k\}_{k\in\mathbb{Z}}$ is a standard kernel of type $BCZK_{\ell^1}$. For any $r > 1$ (including $r = \infty$), it follows that $\{\theta_k\}_{k\in\mathbb{Z}}$ is a standard kernel of type $BCZK_{\ell^r}$ as well since $||\cdot||_{\ell^r} \le ||\cdot||_{\ell^1}$ for $r > 1$. □

Just like in the linear case, we can use the connection between square functions and vector valued Calderón-Theory to extend the square function bounds to a larger range of indices. This result was originally obtained in [49].

Corollary 6.5.2 (H. [49]) *If $\{\theta_k\} \in SBLPK(A,N,\gamma)$ where $N > 2n$ and there exists para-accretive functions b_1, b_2 such that $\Theta_k(b_1,b_2) = 0$ for all $k \in \mathbb{Z}$, then S_Θ is bounded from*

$L^1 \times L^1$ *into* $L^{1/2,\infty}$*, from* $L_c^\infty \times L_c^\infty$ *into* BMO*, and from* $L^{p_1} \times L^{p_2}$ *into* L^p *for all* $1 < p_1, p_2 < \infty$ *satisfying* (1.1).

Proof: By Theorem 6.4.3, it follows that $S_{\{\Theta_k\}}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^p when $1 < p, p_1, p_2 < \infty$ satisfy (1.1). By Lemma 6.5.1, $\{\theta_k\}_{k\in\mathbb{Z}} \in BCZK_{\ell^2}$. Since $S_{\{\Theta_k\}}$ is bounded for $p > 1$, it follows from Theorems 2.2.4 and 2.3.2 that $S_{\{\Theta\}}$ is bounded from $L^1 \times L^1$ into $L^{1/2,\infty}$ and from $L_c^\infty \times L_c^\infty$ into BMO. Finally by Corollary 3.2.2, it follows that $S_{\{\Theta_k\}}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p_1, p_2 < \infty$ satisfying (1.1) without restriction on p. □

6.6 Estimates for Smooth Littlewood-Paley Operators with Extra Cancellation

In this section we prove an estimate assuming additional cancellation and regularity assumptions on the Littlewood-Paley square function kernels. This estimate will be particularly useful in the next chapter since the truncated singular integrals we will construct satisfy the hypotheses of this theorem.

Theorem 6.6.1 *If* $\{\theta_k\} \in SLPK$ *and there exist para-accretive functions* b_0, b_1, b_2 *such that* $\Theta_k(b_1, b_2) = 0$ *and*

$$\int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) b_0(x)\,dx = 0$$

for all $x, y_1, y_2 \in \mathbb{R}^n$ *and* $k \in \mathbb{Z}$*, then*

$$\sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k(f_1, f_2)(x) f_0(x)\,dx \right| \lesssim \prod_{i=0}^{2} ||f_i||_{L^{p_i}} \tag{6.6}$$

for all $f_i \in L^{p_i}$ *when* $1 < p, p_1, p_2 < \infty$ *satisfying* (1.1) *and* $p_0 = p'$.

Proof: Fix $1 < p, p_1, p_2 < \infty$ satisfying (1.1), $f_0, f_1, f_2 \in C_0^1$ such that $b_i f_i$ has mean zero for $i = 0, 1, 2$, and we will approximate (6.6) replacing the integrand with $\Theta_k(b_1 f_1, b_2 f_2)(x) b_0(x) f_0(x)$. Since b_0, b_1, b_2 are para-accretive functions, it follows that $||f||_{L^p} \approx ||b_i f||_{L^p}$:

$$||b_i^{-1}||_{L^\infty} ||f||_{L^p} \leq ||b_i f||_{L^p} \leq ||b_i||_{L^\infty} ||f||_{L^p}$$

for i=0,1,2. Furthermore, since $\{f \in C_0^1 : b_i f \text{ has mean zero}\}$ is dense in L^p for $1 < p < \infty$, this is an acceptable reduction to prove the theorem. Define Π_j^1 and Π_j^2 as in Theorem 6.4.3, i.e.

$$\Pi_j^1(f_1, f_2)(y_1, y_2) = M_{b_1} D_j^{b_1} M_{b_1} f_1(y_1) M_{b_2} S_{j+1}^{b_2} M_{b_2} f_2(y_2)$$
$$\Pi_j^2(f_1, f_2)(y_1, y_2) = M_{b_1} S_j^{b_1} M_{b_1} f_1(y_1) M_{b_2} D_j^{b_2} M_{b_2} f_2(y_2)$$

where $S_j^{b_i}$ and $D_j^{b_i}$ are as in Remark 5.3.3 and Theorem 5.4.1 respectively. Then we approximate (6.6) using the following in the same way as in the proof of Theorem 6.4.3: It is sufficient to prove that

$$\sum_{|k|<N} \left| \int_{\mathbb{R}^n} \Theta_k(b_1 f_1, b_2 f_2)(x) b_0(x) f_0(x) dx \right| \lesssim ||f_0||_{L^{p'}} ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}}.$$

uniformly for $N \in \mathbb{N}$. Since we have truncated to a finite sum it follows that

$$\sum_{|k|<N} \left| \int_{\mathbb{R}^n} \Theta_k(b_1 f_1, b_2 f_2)(x) b_0(x) f_0(x) dx \right| \leq \sum_{j,k,\ell \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \left(\Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x) \right. \right.$$
$$\left. \left. + \Theta_k \Pi_j^2 (f_1, \widetilde{D}_\ell^{b_2} M_{b_2} D_\ell^{b_2} M_{b_2} f_2)(x) \right) b_0(x) f_0(x) dx \right|.$$

These two terms are symmetric, so we only prove the estimates for the first one. So by symmetry and the same reduction in the proof of Theorem 6.4.3, it is sufficient to prove that

$$\sum_{|j|,|k|,|\ell|<N} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x) b_0(x) f_0(x) dx \right| \lesssim ||f_0||_{L^{p'}} ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}}$$

independent of N. Once again, since this is a finite sum, it follows from the continuity of Θ_k and the convergence in Theorem 5.4.1 that

$$\sum_{|j|,|k|,|\ell|<N} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x) b_0(x) f_0(x) dx \right|$$
$$\leq \sum_{|j|,|k|,|\ell|<N} \sum_{m \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x) M_{b_0} \widetilde{D}_m^{b_0} M_{b_0} D_m^{b_0} M_{b_0} f_0(x) dx \right|$$
$$\lesssim \sum_{j,k,\ell,m \in \mathbb{Z}} \int_{\mathbb{R}^n} |\widetilde{D}_m^{b_0 *} M_{b_0} \Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x) D_m^{b_0} M_{b_0} f_0(x)| dx.$$

Note that estimates (6.3) and (6.4) hold for $\Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)$, so we have

$$|\widetilde{D}_m^{b_0 *} M_{b_0} \Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x)| \lesssim \Phi_m^N * \left| \Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2) \right| (x)$$
$$\lesssim \min\left(2^{-\gamma|j-k|}, 2^{-\gamma|j-\ell|}\right) \mathscr{M}\left(\mathscr{M}^2 (D_\ell^{b_1} M_{b_1} f_1) \cdot \mathscr{M}^2 f_2\right)(x)$$

In addition, by estiamte (4.14) from Proposition 4.3.2 we have

$$
\begin{aligned}
&|\widetilde{D}_m^{b_0 *} M_{b_0} \Theta_k \Pi_j^i (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x)| \\
&\qquad\qquad \lesssim 2^{-\gamma|m-k|} \mathscr{M}\left(\mathscr{M}(D_j^{b_0} M_{b_0} \widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1) \cdot \mathscr{M}(S_{j+1} M_{b_2} f_2)\right)(x) \\
&\qquad\qquad \le 2^{-\gamma|m-k|} \mathscr{M}\left(\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1) \cdot \mathscr{M}^2 f_2\right)(x).
\end{aligned}
$$

Then it follows that for all $N \in \mathbb{N}$

$$
\begin{aligned}
&\sum_{|j|,|k|,|\ell|<N} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1 (\widetilde{D}_\ell^{b_1} M_{b_1} D_\ell^{b_1} M_{b_1} f_1, f_2)(x) b_0 f_0(x) dx \right| \\
&\quad \lesssim \int_{\mathbb{R}^n} \sum_{j,k,\ell,m \in \mathbb{Z}} 2^{-\gamma\left(\frac{|m-k|}{3} + \frac{|k-j|}{3} + \frac{|j-\ell|}{3}\right)} \mathscr{M}\left(\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1) \mathscr{M}^2 f_2\right)(x) |D_m^{b_0} f_0(x)| dx \\
&\quad \le \left|\left| \left(\sum_{j,k,\ell,m\in\mathbb{Z}} 2^{-\gamma\left(\frac{|m-k|}{3} + \frac{|k-j|}{3} + \frac{|j-\ell|}{3}\right)} \mathscr{M}\left(\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1) \mathscr{M}^2 f_2\right)^2 \right)^{\frac{1}{2}} \right|\right|_{L^p} \\
&\qquad\qquad\qquad \times \left|\left| \left(\sum_{j,k,\ell,m\in\mathbb{Z}} 2^{-\gamma\left(\frac{|m-k|}{3} + \frac{|k-j|}{3} + \frac{|j-\ell|}{3}\right)} |D_m^{b_0} f_0|^2 \right)^{\frac{1}{2}} \right|\right|_{L^{p'}} \\
&\quad \lesssim \left|\left| \left(\sum_{\ell\in\mathbb{Z}} \mathscr{M}\left(\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1) \mathscr{M}^2 f_2\right)^2 \right)^{\frac{1}{2}} \right|\right|_{L^p} \left|\left| \left(\sum_{m\in\mathbb{Z}} |\widetilde{D}_m^{b_0 *} f_0|^2 \right)^{\frac{1}{2}} \right|\right|_{L^{p'}} \\
&\quad \lesssim \left|\left| \left(\sum_{\ell\in\mathbb{Z}} (\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1) \mathscr{M}^2 f_2)^2 \right)^{\frac{1}{2}} \right|\right|_{L^p} ||f_0||_{L^{p'}} \\
&\quad \le \left|\left| \left(\sum_{\ell\in\mathbb{Z}} \left(\mathscr{M}^2(D_\ell^{b_1} M_{b_1} f_1)\right)^2 \right)^{\frac{1}{2}} \right|\right|_{L^{p_1}} ||\mathscr{M}^2 f_2||_{L^{p_2}} ||f_0||_{L^{p'}} \lesssim \prod_{i=0}^{2} ||f_i||_{L^{p_i}}.
\end{aligned}
$$

In the last line we use the Fefferman-Stein vector valued maximal inequality twice and the square function bound $S_{\{D_\ell^{b_1}\}}$ on L^{p_1} which holds by Theorem 6.4.2. By symmetry and density this completes the proof. □

We do not state the linear version of this result, but it is not hard to extract the statement and proof of the linear version from Theorem 6.6.1.

Chapter 7

Singular Integral Operators

In this chapter, we prove some results for bilinear Calderón-Zygmund operators. We provide a new proof of the bilinear T1 theorem that was originally proved by Christ-Journé [17] and Grafakos-Torres [45]. The proof we present was originally presented by the author of this work in [50]. We also prove the a natural extension of the Tb theorem of David-Journé-Semmes to bilinear operators, which is a new result. This result will be submitted for publication soon.

We start by introducing definitions, notation, and basic properties that will be used throughout this chapter. From there, the proof for both theorems roughly follows the approach of the proof of the linear T1 theorem from [28]. We start by proving a reduced theorem which assumes $T1 = 0$ and $Tb = 0$ conditions. We then use the weak continuity of the operator to form smooth truncations, and apply the square function theory we've developed to bound these operators. Then we construct a para-product operator to verify that we can write operators satisfying the $T1 \in BMO$ and $Tb \in BMO$ conditions as a perturbation of the reduced $T1 = 0$ and $Tb = 0$ conditions. The weak continuity of these two types of singular integral operators are different, and hence require different defintiions and convergence results. So we will develop the pertinent definitions and prove the reduced T1 and Tb theorems separately. On the other hand, the para-product construction in the accretive setting is more general that in the unpurterbed setting, so we only construct the accretive version of the para-product and apply it to both T1 and Tb theorems.

7.1 A Reduced T1 Theorem

We now define the weak boundedness property for linear and bilinear singular integral operators. We will not use the linear weak boundedness property in this work, but it is worth stating to compare with the bilinear weak boundedness property, as there have been a number of different definitions used. The linear version we state here is the one from [28]. For examples of other definitions of the bilinear weak boundedness property in [17] and [3].

Definition 7.1.1 *A function $\phi \in C_0^\infty$ is a normalized bump of order $M \in \mathbb{N}$ if supp$(\phi) \subset B(0,1)$ and*

$$\sup_{|\alpha| \leq M} ||\partial^\alpha \phi||_{L^\infty} \leq 1.$$

Let T be a linear operator from $\mathscr{S}$ into $\mathscr{S}'$. We say that T satisfies the weak boundedness property (written $T \in WBP$) if there exists an $M \in \mathbb{N}$ such that for all normalized bumps $\phi_0, \phi_1 \in C_0^\infty$ of order M, T satisfies for all $x \in \mathbb{R}^n$ and $R > 0$

$$\left|\left\langle T\phi_1^{x,R}, \phi_0^{x,R}\right\rangle\right| \lesssim R^n$$

where $\phi^{x,R}(u) = \phi(\frac{u-x}{R})$.

Definition 7.1.2 *Let T be a bilinear operator from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$. We say that T satisfies the weak boundedness property (written $T \in WBP$) if there exists an $M \in \mathbb{N}$ such that for all normalized bumps $\phi_0, \phi_1, \phi_2 \in C_0^\infty$ of order M, T satisfies for all $x \in \mathbb{R}^n$ and $R > 0$*

$$\left|\left\langle T(\phi_1^{x,R}, \phi_2^{x,R}), \phi_0^{x,R} \right\rangle\right| \lesssim R^n.$$

The weak boundedness property essentially says that T behaves like a translation invariant bounded operator on normalized bumps. In the next lemma, we extend the weak boundedness property to a slightly more general formulation. We verify that we can relax the requirement that the bumps be centered at the same point.

Lemma 7.1.3 *Assume that T is a bilinear singular integral operator that satisfies the weak boundedness property for normalized bumps of order M. Then for all normalized bumps ϕ_0, ϕ_1, ϕ_2, $R > 0$ of order M, and $y_0, y_1, y_2 \in \mathbb{R}^n$ such that $|y_0 - y_i| \leq tR$*

$$\left|\left\langle T(\phi_1^{y_1,R}, \phi_2^{y_2,R}), \phi_0^{y_0,R} \right\rangle\right| \lesssim (1+t)^{3M+n} R^n.$$

Proof: Let $y_0, y_1, y_2 \in \mathbb{R}^n$, $R > 0$, and define $D = 1 + 2t$. Then it follows that

$$\left|\left\langle T(\phi_1^{y_1,R}, \phi_2^{y_2,R}), \phi_0^{y_0,R} \right\rangle\right| = \left|\left\langle T(\widetilde{\phi}_1^{y_0,DR}, \widetilde{\phi}_2^{y_0,DR}), \widetilde{\phi}_0^{y_0,DR} \right\rangle\right|.$$

where $\widetilde{\phi}_0(u) = \phi_0(Du)$ and $\widetilde{\phi}_i(u) = \phi_i\left(Du + \frac{y_0 - y_i}{R}\right)$ for $i = 1, 2$. If $|u| > 1$, then clearly $D|u| > 1$, and

$$\left|Du + \frac{y_0 - y_i}{R}\right| \geq D|u| - \frac{|y_0 - y_i|}{R} \geq (1+2t)|u| - t \geq 1.$$

Therefore $\operatorname{supp}(\widetilde{\phi_i}) \subset B(0,1)$. Also for $\alpha \in \mathbb{N}_0$ with $|\alpha| \leq M$

$$|\partial_u^\alpha \widetilde{\phi}_0(u)| = D^{|\alpha|} |\partial_u^\alpha \phi_0(Du)| \leq D^M.$$

Therefore $D^{-M}\widetilde{\phi}_0$ is a normalized bump. As similar argument proves that $D^{-M}\widetilde{\phi}_i$ for $i = 1,2$ are normalized bumps as well. Then it follows that

$$\left|\left\langle T(\widetilde{\phi}_1^{y_0,DR}, \widetilde{\phi}_2^{y_0,DR}), \widetilde{\phi}_0^{y_0,DR} \right\rangle\right| \lesssim D^{3M}(DR)^n \lesssim (1+t)^{3M+n} R^n.$$

This completes the proof. □

Definition 7.1.4 *Let T be a bilinear singular integral operator and $f_1, f_2 \in C^\infty \cap L^\infty$. Also fix a function $\eta_R \in C_0^\infty$ for $R > 0$ such that $\eta_R \equiv 1$ on $B(0,R)$ and $supp(\eta_R) \subset B(0,2R)$. We define*

$$T(f_1, f_2) = \lim_{R\to\infty} T(\eta_R f_1, \eta_R f_2) - \int_{|y_1|,|y_2|>1} K(0, y_1, y_2) \prod_{i=1}^{2} f(y_i) \eta_R(y_i) dy, \tag{7.1}$$

where this limit is taken in the topology of $(C_0^\infty)'$. Furthermore, when $f_1, f_2, f_0 \in C_0^\infty$ where f_0 has mean zero, the a priori definition of $\langle T(f_1, f_2), f_0 \rangle$ agrees with limit definition above. We prove that his definition is well defined and agrees with T.

This is the same definition that was given by Torres [84] in the linear setting and by Grafakos-Torres [45] in the multilinear setting.

Proof: Let $f_0 \in C_0^\infty$ and take $R_0 > 1$ large enough so that $\operatorname{supp}(f_0) \subset B(0,R_0)$. Then

$$\begin{aligned}
\langle T(\eta_R f_1, \eta_R f_2), f_0\rangle &= \langle T(\eta_{R_0} f_1, \eta_{R_0} f_2), f_0\rangle \\
&\qquad + \langle T(\eta_{R_0} f_1, (\eta_R - \eta_{R_0}) f_2), f_0\rangle \\
&\qquad + \langle T((\eta_R - \eta_{R_0}) f_1, \eta_{R_0} f_2), f_0\rangle \\
&\qquad + \langle T((\eta_R - \eta_{R_0}) f_1, (\eta_R - \eta_{R_0}) f_2), f_0\rangle \\
&= \langle T(\eta_{R_0} f_1, \eta_{R_0} f_2), f_0\rangle \\
&\quad + \int_{\mathbb{R}^{3n}} K(x,y_1,y_2) f_0(x) \eta_{R_0}(y_1)(\eta_R(y_2) - \eta_{R_0}(y_2)) \prod_{i=1} f_i(y_i)\, dx\, dy_1\, dy_2 \\
&\quad + \int_{\mathbb{R}^{3n}} K(x,y_1,y_2) f_0(x) (\eta_R(y_1) - \eta_{R_0}(y_1)) \eta_{R_0}(y_2) \prod_{i=1}^{2} f_i(y_i)\, dx\, dy_1\, dy_2 \\
&\quad + \langle T((\eta_R - \eta_{R_0}) f_1, (\eta_R - \eta_{R_0}) f_2), f_0\rangle \\
&= I + II + III + IV.
\end{aligned}$$

The first term I is well defined since $\eta_{R_0} f_i \in C_0^\infty$ for a fixed R_0 (depending on f_0). We check that the first integral term II is absolutely convergent by bounding the integrand by $||f_1||_{L^\infty} ||f_2||_{L^\infty}$ times

$$\begin{aligned}
|K(x,y_1,y_2)\eta_{R_0}(y_1)(\eta_R(y_2) - \eta_{R_0}(y_2)) f_0(x)| &\lesssim \frac{|\eta_{R_0}(y_1)(\eta_R(y_2) - \eta_{R_0}(y_2)) f_0(x)|}{(|x-y_1| + |x-y_2|)^{2n}} \\
&\le \frac{|\eta_{R_0}(y_1)(\eta_R(y_2) - \eta_{R_0}(y_2)) f_0(x)|}{(|x-y_1| + |x-y_2|/2 + (R_0 - R_0/2)/2)^{2n}} \\
&\lesssim \frac{|\eta_{R_0}(y_1) f_0(x)|}{(R_0 + |x-y_2|)^{2n}}.
\end{aligned}$$

This is an $L^1(\mathbb{R}^{3n})$ function that is independent of R (as long as $R > 4R_0$),

$$\int_{\mathbb{R}^{3n}} \frac{|\eta_{R_0}(y_1) f_0(x)|}{(R_0 + |x-y_2|)^{2n}} dx\, dy_1\, dy_2 \lesssim \int_{\mathbb{R}^{2n}} \frac{|\eta_{R_0}(y_1) f_0(x)|}{R_0^n} dx\, dy_1 \lesssim ||f_0||_{L^\infty} R_0^n.$$

Since $\eta_R \to 1$ pointwise, it follows by dominated convergence that the following limit exists:

$$\lim_{R\to\infty}\int_{\mathbb{R}^{3n}} K(x,y_1,y_2)f_0(x)\eta_{R_0}(y_1)(\eta_R(y_2)-\eta_{R_0}(y_2))\prod_{i=1}^{2} f_i(y_i)dx\,dy_1\,dy_2$$
$$= \int_{\mathbb{R}^{3n}} K(x,y_1,y_2)f_0(x)\eta_{R_0}(y_1)(1-\eta_{R_0}(y_2))\prod_{i=1}^{2} f_i(y_i)dx\,dy_1\,dy_2.$$

By symmetry, III is well defined as well. Finally, we consider IV minus the extra integral term from (7.1)

$$IV - \left\langle \int_{|y_1|,|y_2|>1} K(0,y_1,y_2)\prod_{i=1}^{2} f(y_i)\eta_R(y_i)dy_1\,dy_2, b_0 f_0 \right\rangle$$
$$= \int_{\mathbb{R}^{3n}} (K(x,y_1,y_2)-K(0,y_1,y_2))f_0(x)\prod_{i=1}^{2}(\eta_R(y_i)-\eta_{R_0}(y_i))f(y_i)dy_1\,dy_2\,dx.$$

Again we bound the integrand by $||f_1||_{L^\infty}||f_2||_{L^\infty}$ times

$$|K(x,y_1,y_2)-K(0,y_1,y_2)|\,|f_0(x)|(\eta_R(y_1)-\eta_{R_0}(y_1) \lesssim \frac{|x|^\gamma|\eta_R(y_1)-\eta_{R_0}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}|f_0(x)|$$
$$\lesssim \frac{|x|^\gamma|\eta_R(y_1)-\eta_{R_0}(y_1)|}{(|x-y_1|/2+R_0/4+|x-y_2|)^{2n+\gamma}}|f_0(x)|$$
$$\lesssim \frac{R_0^\gamma|f_0(x)|}{(R_0+|x-y_1|+|x-y_2|)^{2n+\gamma}},$$

which is an $L^1(\mathbb{R}^{3n})$ function:

$$\int_{\mathbb{R}^{3n}} \frac{R_0^\gamma|f_0(x)|}{(R_0+|x-y_1|+|x-y_2|)^{2n+\gamma}}dy_1\,dy_2\,dx \lesssim \int_{\mathbb{R}^{2n}} \frac{R_0^\gamma|f_0(x)|}{(R_0+|x-y_1|)^{n+\gamma}}dy_1\,dx$$
$$\lesssim \int_{\mathbb{R}^n} |f_0(x)|dx \lesssim ||f_0||_{L^\infty}R_0^n.$$

Then it follows again by dominated convergence that

$$\lim_{R\to\infty}\langle T((\eta_R-\eta_{R_0})f_1,(\eta_R-\eta_{R_0})f_2),f_0\rangle \\ -\left\langle \int_{|y_1|,|y_2|>1} K(0,y_1,y_2)\prod_{i=1}^{2} f(y_i)\eta_R(y_i)dy_1\,dy_2, b_0 f_0\right\rangle \\ = \int_{\mathbb{R}^{3n}} (K(x,y_1,y_2)-K(0,y_1,y_2))f_0(x)\prod_{i=1}^{2}(1-\eta_{R_0}(y_i))f(y_i)dy_1\,dy_2\,dx,$$

which is an absolutely convergent integral. Therefore $T(f_1,f_2)$ is well defined as an element of $(C_0^\infty)'$. Furthermore if $f_0\in C_0^\infty$ has mean zero and $f_1,f_2\in C_0^\infty$, then this definition of T is consistent with the a priori definition of T since

$$\lim_{R\to\infty}\left\langle \int_{|y_1|,|y_2|>1} K(0,y_1,y_2)\prod_{i=1}^{2}\eta_R(y_i)f_i(y_i)dy_1\,dy_2, f_0\right\rangle \\ = \left(\int_{|y_1|,|y_2|>1} K(0,y_1,y_2)\prod_{i=1}^{2} f_i(y_i)dy_1\,dy_2\right)\left(\int_{\mathbb{R}^n} f_0(x)dx\right)=0,$$

since both of these integrals are absolutely convergent. Also, when f_0 has mean zero in this way, the definition of $\langle T(f_1,f_2),f_0\rangle$ is independent of the choice of η_R. We will also use the notation $T(1,1)\in BMO$ or $T(1,1)=\beta$ for $\beta\in BMO$ to mean that for all $f_0\in C_0^\infty$ such that f_0 has mean zero

$$\langle T(1,1),f_0\rangle = \langle \beta, f_0\rangle\,.$$

Note that the left hand side makes sense sine $T(1,1)$ is defined in $(C_0^\infty)'$. Also the right hand side makes sense since $f_0\in H^1$ for $f_0\in C_0^\infty$ with mean zero. Whenever we write $T(1,1)=0$, we mean that $T(1,1)=0$ as an element of BMO in the sense above, not the zero element of $(C_0^\infty)'$. □

Remark 7.1.5 *The assumptions $T^{j*}(1,1) \in BMO$ and $T \in WBP$ are symmetric for the transposes of T in the following sense: $T^{j*}(1,1) \in BMO$ and $T \in WBP$ if and only if the same $(T^{1*})^{j*}(1,1) \in BMO$ for $j = 0,1,2$ and $T^{1*} \in WBP$ if and only if the same $(T^{2*})^{j*}(1,1) \in BMO$ for $j = 0,1,2$ and $T^{2*} \in WBP$. Also by the discussion in Section 1.4 in Chapter 1, it follows that T is continuous from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$ if and only if T^{1*} is continuous from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$ if and only if T^{2*} is continuous from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$. So the assumptions of Theorem 7.1.7 are symmetric for the transposes of T.*

The next proposition is a bilinear version of a result of David-Journé in [28], and is important to the connection between the square function bounds of Chapter 6 and the singular integral operator T. It essentially states that the square functions we defined in Proposition 7.1.6 are the type of square functions that serve as smooth truncations of our Calderón-Zygmund operators. More precisely, it says that the smooth truncations satisfy the hypotheses of Theorem 6.6.1.

Proposition 7.1.6 *Suppose T is a bilinear singular integral operator with standard kernel K of class $BCZK_{\mathbb{C}}$ that satisfies the weak boundedness property, and $\psi, \varphi \in C_0^\infty(\mathbb{R}^n)$ where ψ has mean zero. Then*

$$\theta_k(x, y_1, y_2) = \left\langle T(\varphi_k^{y_1}, \varphi_k^{y_2}), \psi_k^x \right\rangle$$

is a collection of smooth bilinear Littlewood-Paley square function kernels, i.e. $\{\theta_k\} \in SBLPK$. Furthermore θ_k satisfies

$$\int_{\mathbb{R}^n} \theta_k(x, y_1, y_2)dx = 0$$

for all $y_1, y_2 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

Proof: Without loss of generality, we assume that $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi) \subset B(0,1)$. Fix $x, y_1, y_2 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. We split estimate (4.1) into two cases: $|x-y_1|+|x-y_2| \leq 2^{3-k}$ and $|x-y_1|+|x-y_2| > 2^{3-k}$. In the first case, we use that $T \in WBP$ and Lemma 7.1.3 to estimate

$$\begin{aligned}|\theta_k(x,y_1,y_2)| &= 2^{3nk} \left| \left\langle T(\varphi^{y_1,2^{-k}}, \varphi^{y_2,2^{-k}}), \psi_k^{x,2^{-k}} \right\rangle \right| \\ &\lesssim 2^{2nk} \lesssim \Phi_k^{n+\gamma/2}(x-y_1)\Phi_k^{n+\gamma/2}(x-y_2).\end{aligned}$$

which is sufficient to prove (4.1). Note that included in the constant of this estimate is a normalizing factor to make φ and ψ normalized bumps. We disregard this factor since it only depends on φ, ψ, and the order of normalized bumps specified by $T \in WBP$. Now if we assume that $|x-y_1|+|x-y_2| > 2^{3-k}$, then it follows that $|x-y_{i_0}| > 2^{2-k}$ for either $i_0 = 1$ or $i_0 = 2$ and hence

$$\operatorname{supp}(\psi_k^x) \cap \operatorname{supp}(\varphi_k^{y_1}) \cap \operatorname{supp}(\varphi_k^{y_2}) \subset B(x,2^{-k}) \cap B(y_{i_0},2^{-k}) = \emptyset.$$

Therefore, we can estimate θ_k the kernel representation of T in the following way

$$\begin{aligned}|\theta_k(x,y_1,y_2)| &= \left| \int_{\mathbb{R}^{3n}} (K(u_0,u_1,u_2) - K(x,u_1,u_2))\varphi_k^{y_1}(u_1)\varphi_k^{y_2}(u_2)\psi_k^x(u_0)\,du_0\,du_1\,du_2 \right| \\ &\lesssim \int_{|x-u_0|<2^{-k}} \int_{|y_1-u_1|<2^{-k}} \int_{|y_2-u_2|<2^{-k}} \frac{|u_0-x|^\gamma\, 2^{3nk} du}{(|x-u_1|+|x-u_2|)^{2n+\gamma}} du_0\,du_1\,du_2 \\ &\lesssim \frac{2^{-\gamma k} 2^{3nk}}{(2^{-k}+|x-y_1|+|x-y_2|)^{2n+\gamma}} \int_{|x-u_0|<2^{-k}} \int_{|y_1-u_1|<2^{-k}} \int_{|y_2-u_2|<2^{-k}} du_0\,du_1\,du_2 \\ &\lesssim \frac{2^{-\gamma k}}{(2^{-k}+|x-y_1|+|x-y_2|)^{2n+\gamma}} \\ &\leq \Phi_k^{n+\gamma/2}(x-y_1)\Phi_k^{n+\gamma/2}(x-y_2).\end{aligned}$$

Here we have used the following: Let $i_0 \in \{1,2\}$ such that $|x-y_{i_0}| = \max(|x-y_1|, |x-y_2|)$, and it follows that $|x-y_{i_0}| \geq 2^{2-k}$. Then for $u_{i_0} \in B(y_{i_0}, 2^{-k})$

$$|x-u_{i_0}| \geq |x-y_{i_0}| - |y_{i_0} - u_{i_0}| \geq 2^{-k} + |x-y_{i_0}|/2 \geq 2^{-k} + (|x-y_1| + |x-y_2|)/4.$$

This proves that θ_k satisfies (4.1). Now for (4.2), note that by the continuity of T from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$

$$\partial_{y_1}^{\alpha} \theta_k(x,y_1,y_2) = \left\langle T(\partial_{y_1}^{\alpha}\left(\varphi_k^{y_1}\right), \varphi_k^{y_2}), \psi_k^x \right\rangle = -2^k \left\langle T\left(\left(\partial_{y_1}^{\alpha}\varphi\right)_k^{y_1}, \varphi_k^{y_2}\right), \psi_k^x \right\rangle.$$

Then θ_k satisfies (4.2) as well: By the weak continuity of T from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$, we have

$$|\theta_k(x,y_1,y_2) - \theta_k(x,y_1',y_2)| \leq |y_1 - y_1'| \sup_{\xi \in \mathbb{R}^n} |\nabla_{\xi} \theta_k(x,\xi,y_2)| \lesssim 2^{2nk}(2^k|y_1 - y_1'|).$$

So by Remark 4.1.3 and symmetric arguments for x, y_2, it follows that $\{\theta_k\} \in SBLPK$. Now we verify that θ_k has integral 0 in the x spot: Again by the continuity of T from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)dx = \lim_{R\to\infty} \left\langle T(\varphi_k^{y_1}, \varphi_k^{y_2}), \int_{|x|<R} \psi_k^x dx \right\rangle = \lim_{R\to\infty} \left\langle T(\varphi_k^{y_1}, \varphi_k^{y_2}), \lambda_R \right\rangle$$

where we take this to be the definition of λ_R. Now if $R > 2 \cdot 2^{-k}$, then for $|u| < R - 2^{-k}$ it follows that

$$\text{supp}(\psi_k(u-\cdot)) \subset B(u, 2^{-k}) \subset B(0, |u| + 2^{-k}) \subset B(0,R),$$

and hence for $|u| < R - 2^{-k}$ we have that

$$\lambda_R(u) = \int_{|x|<R} \psi_k^x(u)dx = \int_{\text{supp}(\psi_k^x)} \psi_k^x(u)dx = 0.$$

Also when $|u| > R + 2^{-k}$, it follows that $\text{supp}(\psi_k(u - \cdot)) \cap B(0,R) = \emptyset$, and hence that $\lambda_R(u) = 0$. It is also easy to see that $||\lambda_R||_{L^\infty} \leq ||\psi||_{L^1} \lesssim 1$. Then for $R > 2(2^{1-k} + |y_1|)$, $\varphi_k^{y_i}$ and λ_R have disjoint support for $i = 1, 2$. So it follows that

$$\begin{aligned}
\left|\left\langle T(\varphi_k^{y_1}, \varphi_k^{y_2}), \lambda_R \right\rangle\right| &\leq \int_{\mathbb{R}^{3n}} |K(u_0, u_1, u_2)\varphi_k^{y_1}(u_1)\varphi_k^{y_2}(u_2)\lambda_R(u_0)| du_0\, du_1\, du_2 \\
&\lesssim \int_{\mathbb{R}^{3n}} \frac{|\lambda_R(u_0)\varphi_k^{y_1}(u_1)\varphi_k^{y_2}(u_2)|}{(|u_0 - u_1| + |u_0 - u_2|)^{2n}} du_0\, du_1\, du_2 \\
&\lesssim \int_{\mathbb{R}^{3n}} \frac{|\lambda_R(u_0)\varphi_k^{y_1}(u_1)\varphi_k^{y_2}(u_2)|}{(|u_0| - (|u_1 - y_1| + |y_1|))^{2n}} du_0\, du_1\, du_2 \\
&\lesssim \int_{\mathbb{R}^{3n}} \frac{|\lambda_R(u_0)\varphi_k^{y_1}(u_1)\varphi_k^{y_2}(u_2)|}{(R/2 + (2^{1-k} + |y_1| - 2^{-k}) - (2^{-k} + |y_1|))^{2n}} du_0\, du_1\, du_2 \\
&\lesssim \frac{1}{R^{2n}} \left(\int_{\mathbb{R}^n} |\lambda_R(u_0)| du_0 \right) \prod_{i=1}^{2} \left(\int_{\mathbb{R}^n} |\varphi_k^{y_i}(u_i)| du_i \right) \\
&\lesssim \frac{1}{R^{2n}} \left((R + 2^{-k})^n - (R - 2^{-k})^n \right) \\
&\lesssim \frac{1}{R^{n+1}}.
\end{aligned}$$

The above tends to 0 as $R \to \infty$, and so θ_k has zero integral in the x variable. This completes the proof. □

Now we are prepared to state and prove the reduced T1 theorem.

Theorem 7.1.7 (Christ-Journé [17], Grafakos-Torres [45]) *Assume that T is a bilinear singular integral operator with standard kernel of type $CZK_{\mathbb{C}}$ and $T \in WBP$. If $T^{*i}(1,1) = 0$ for $i = 0, 1, 2$, then T can be extended to a bounded operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p, p_1, p_2 < \infty$ satisfying* (1.1).

Proof: Let T be as in the hypothesis, $1 < p, p_1, p_2 < \infty$ satisfy (1.1), and $f_0, f_1, f_2 \in \mathscr{S}_0$. Also fix $\varphi \in C_0^\infty$ with integral 1 and $\text{supp}(\varphi) \subset B(0,1)$, $\psi(x) = 2^n\varphi(2x) - \varphi(x)$, $P_k f = \varphi_k * f$, and $Q_k = P_{k+1} - P_k = \psi_k * f$. Then it follows from the continuity of T from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$ and Proposition 5.1.2 that

$$\begin{aligned}
|\langle T(f_1,f_2),f_0\rangle| &= \lim_{N\to\infty} |\langle T(P_N f_1, P_N f_2), P_N f_0\rangle - \langle T(P_{-N} f_1, P_{-N} f_2), P_{-N} f_0\rangle| \\
&\leq \sum_{k\in\mathbb{Z}} |\langle T(P_{k+1} f_1, P_{k+1} f_2), P_{k+1} f_0\rangle - \langle T(P_k f_1, P_k f_2), P_k f_0\rangle| \\
&= \sum_{k\in\mathbb{Z}} |\langle T(Q_k f_1, P_k f_2, P_k f_0\rangle + \langle T(P_{k+1} f_1, Q_k f_2, P_k f_0\rangle + \langle T(P_{k+1} f_1, P_{k+1} f_2, Q_k f_0\rangle|
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \left| \left\langle Q_k^* T^{*1}(P_k f_0, P_k f_2), f_1 \right\rangle + \left\langle Q_k^* T^{*2}(P_{k+1} f_1, P_k f_0), f_2 \right\rangle + \langle Q_k^* T(P_k f_1, P_k f_2), f_0\rangle \right| \\
&\leq \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k^1(f_0,f_2)(x) f_1(x)\,dx \right| + \left| \int_{\mathbb{R}^n} \Theta_k^2(f_1,f_0)(x) f_2(x)\,dx \right| \\
&\qquad\qquad + \left| \int_{\mathbb{R}^n} \Theta_k^0(f_1,f_2)(x) f_0(x)\,dx \right|,
\end{aligned}$$

where we take the last inequality to give the definition of Θ_k^i for $i = 0,1,2$. Now we simplify notation $\Theta_k(f_1,f_2) = \Theta_k^0(f_1,f_2) = Q_k^* T(P_{k+1} f_1, P_{k+1} f_2)$. By Proposition 7.1.6, the kernel of Θ_k, which is given by

$$\theta_k(x,y_1,y_2) = \left\langle T(\varphi_{k+1}^{y_1}, \varphi_{k+1}^{y_2}), \psi_k^x \right\rangle,$$

is a collection Littlewood-Paley square function kernels of type *SBLPK* that has integral zero in the x variable. Also

$$\int_{\mathbb{R}^{2n}} \theta_k(x,y_1,y_2)dy_1\,dy_2 = \lim_{R\to\infty}\left\langle T\left(\int_{|y_1|<R}\varphi_{k+1}^{y_1}dy_1,\int_{|y_2|<R}\varphi_{k+1}^{y_2}dy_2\right),\psi_k^x\right\rangle$$
$$= \lim_{R\to\infty}\langle T(\eta_R,\eta_R),\psi_k^x\rangle,$$

where the last line is taken to be the definition of η_R. Now for $R>2^{3-k}$ and $|u|<R-2^{-k}$ it follows that

$$\eta_R(u) = \int_{|y|<R}\varphi_{k+1}^y(u)dy = \int_{\mathbb{R}^n}\varphi_{k+1}^y(u)dy = 1.$$

Also, when $|u|>R+2^{-k}$, it follows that $\eta_R(u)=0$ and clearly $\eta_R\in C_0^\infty$ for all $R>0$. Since $\psi_k^x\in C_0^\infty$ has mean zero, it follows that from that assumption $T(1,1)=0$ that

$$\int_{\mathbb{R}^{2n}}\theta_k(x,y_1,y_2)dy = \lim_{R\to\infty}\langle T(\eta_R,\eta_R),\psi_k^x\rangle = \langle T(1,1),\psi_k^x\rangle = 0.$$

Then by Theorem 6.6.1, it follows that

$$\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}^n}\Theta_k(f_1,f_2)(x)f_0(x)dx\right| \lesssim ||f_0||_{L^{p'}}||f_1||_{L^{p_1}}||f_2||_{L^{p_2}}$$

With a similar argument for the other terms, we have

$$\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}^n}\Theta_k^1(f_0,f_2)(x)f_1(x)dx\right| \lesssim ||f_0||_{L^{p'}}||f_1||_{L^{p_1}}||f_2||_{L^{p_2}},\text{ and}$$
$$\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}^n}\Theta_k^2(f_1,f_0)(x)f_2(x)dx\right| \lesssim ||f_0||_{L^{p'}}||f_1||_{L^{p_1}}||f_2||_{L^{p_2}}.$$

using the bound from Theorem 6.6.1 for indices $1 < p, p', p_1, p_1', p_2, p_2' < \infty$ which satisfy

$$\frac{1}{p'} + \frac{1}{p_2} = \frac{1}{p_1'} \text{ and } \frac{1}{p_1} + \frac{1}{p'} = \frac{1}{p_2'}.$$

Therefore

$$|\langle T(f_1, f_2), f_0\rangle| \lesssim ||f_0||_{L^{p'}} ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}},$$

and so by the density of C_0^∞ functions with mean zero in L^q for $1 < q < \infty$ and the embedding of L^q into $\mathscr{S}'$ for $1 < q < \infty$, T can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into L^p whenever $1 < p, p_1, p_2 < \infty$ satisfy (1.1). □

7.2 A Reduced Bilinear Tb Theorem

We now move on to the reduced Tb theorem. We will essentially reproduce the previous section making the appropriate modifications for working in the accretive setting. We now define the weak boundedness property for linear and bilinear singular integral operators.

Definition 7.2.1 *Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions. A bilinear operator T that is continuous from $b_1 C_0^\delta \times b_2 C_0^\delta$ into $(b_0 C_0^\delta)'$ for some $\delta > 0$ is called a bilinear singular integral operator associated to b_0, b_1, b_2 if there exists a standard Calderón-Zygmund kernel K of type $BCZK_{\mathbb{C}}$ such that for $f_i \in C_0^\delta$ with disjoint support*

$$\left\langle T(M_{b_1} f_1, M_{b_2} f_2), M_{b_0} f_0 \right\rangle = \int_{\mathbb{R}^{3n}} K(x, y_1, y_2) \prod_{i=0}^{2} f_i(y_i) b_i(y_i) dy_i.$$

Definition 7.2.2 *Let b_0, b_1, b_2 be para-accretive functions and T be a bilinear singular integral operator associated to b_0, b_1, b_2. We say that $M_{b_0} T(M_{b_1}, M_{b_2})$ satisfies the weak* *boundedness property (written $M_{b_0}T(M_{b_1}, M_{b_2}) \in WBP$) if there exists an $M \in \mathbb{N}$ such that for all normalized bumps $\phi_0, \phi_1, \phi_2 \in C_0^\infty$ of order M, T satisfies for all $x \in \mathbb{R}^n$ and $R > 0$*

$$\left|\left\langle T(b_1\phi_1^{x,R}, b_2\phi_2^{x,R}), b_0\phi_0^{x,R}\right\rangle\right| \lesssim R^n.$$

Lemma 7.2.3 *Assume that T is a bilinear singular integral operator associated to para-accretive functions b_0, b_1, b_2 that satisfies $M_{b_0}T(M_{b_1}, M_{b_2}) \in WBP$ for normalized bumps of order M. Then for all normalized bumps ϕ_0, ϕ_1, ϕ_2, $R > 0$ of order M, and $y_0, y_1, y_2 \in \mathbb{R}^n$ such that $|y_0 - y_i| \le tR$*

$$\left|\left\langle T(b_1\phi_1^{y_1,R}, b_2\phi_2^{y_2,R}), b_0\phi_0^{y_0,R}\right\rangle\right| \lesssim (1+t)^{3M+n}R^n.$$

Proof: Let $y_0, y_1, y_2 \in \mathbb{R}^n$, $R > 0$, and define $D = 1 + 2t$. Then it follows that

$$\left|\left\langle T(b_1\phi_1^{y_1,R}, b_1\phi_2^{y_2,R}), b_0\phi_0^{y_0,R}\right\rangle\right| = \left|\left\langle T(b_1\widetilde{\phi}_1^{y_0,DR}, b_2\widetilde{\phi}_2^{y_0,DR}), b_0\widetilde{\phi}_0^{y_0,DR}\right\rangle\right|.$$

where $\widetilde{\phi}_0(u) = \phi_0(Du)$ and $\widetilde{\phi}_i(u) = \phi_i\left(Du + \frac{y_0 - y_i}{R}\right)$ for $i = 1, 2$. Like in Lemma 7.1.3, it follows that $D^{-M}\widetilde{\phi}_i$ is a normalized bump for $i = 0, 1, 2$, and hence

$$\left|\left\langle T(b_1\widetilde{\phi}_1^{y_0,DR}, b_2\widetilde{\phi}_2^{y_0,DR}), b_0\widetilde{\phi}_0^{y_0,DR}\right\rangle\right| \lesssim D^{3M}(DR)^n \lesssim (1+t)^{3M+n}R^n.$$

This completes the proof. □

Definition 7.2.4 *Let b_0, b_1, b_2 be para-accretive function, T be a bilinear singular integral operator associated to b_0, b_1, b_2 that is continuous from $b_1C_0^\delta \times b_2C_0^\delta$ into $(b_0C_0^\delta)'$, and $f_1, f_2 \in C^\infty \cap L^\infty$. Also fix function $\eta_R^i \in C_0^\infty$ for $R > 0$, $i = 1, 2$ such that $\eta_R^i \equiv 1$ on $B(0,R)$*

and supp$(\eta_R^i) \subset B(0,2R)$*. Then we define*

$$T(b_1f_1,b_2f_2) = \lim_{R\to\infty} T(\eta_R^1 b_1f_1, \eta_R^2 b_2f_2) \\ - \int_{|y_1|,|y_2|>1} K(0,y_1,y_2) \prod_{i=1}^{2} f(y_i)\eta_R^i(y_i)b_i(y_i)dy_1\,dy_2, \qquad (7.2)$$

where this limit is taken in the weak$*$ topology of $(b_0C_0^\delta)'$*. Note that for this definition we allow for two different functions* η_R^1 *and* η_R^2 *to compute* $T(b_1f_1,b_2f_2)$*. We could have done this in the unperturbed Definition 7.1.4 as well, but it was not necessary for the proof of the reduced T1 theorem. It turns out that it is necessary for the reduced Tb theorem. For* $f_0 \in C_0^\delta$*, there exists* $R_0 > 1$ *such that supp*$(f_0) \subset B(0,R_0/2)$*. When* $R > 2R_0$*, we have*

$$\begin{aligned}
\langle T(\eta_R^1 b_1f_1,\eta_R^2 b_2f_2), b_0f_0\rangle &= \langle T(\eta_{R_0}^1 b_1f_1,\eta_{R_0}^2 b_2f_2), b_0f_0\rangle \\
&\quad + \langle T(\eta_{R_0}^1 b_1f_1,(\eta_R^2-\eta_{R_0}^2) b_2f_2), b_0f_0\rangle \\
&\quad + \langle T((\eta_R^1-\eta_{R_0}^1) b_1f_1,\eta_{R_0}^2 b_2f_2), b_0f_0\rangle \\
&\quad + \langle T((\eta_R^1-\eta_{R_0}^1) b_1f_1,(\eta_R^2-\eta_{R_0}^2) b_2f_2), b_0f_0\rangle \\
&= \langle T(\eta_{R_0}^1 b_1f_1,\eta_{R_0}^2 b_2f_2), b_0f_0\rangle \\
&\quad + \int_{\mathbb{R}^{3n}} K(x,y_1,y_2)b_0(x)f_0(x)\eta_{R_0}^1(y_1)(\eta_R^2(y_2)-\eta_{R_0}^2(y_2))\prod_{i=1}^{2} b_i(y_i)f_i(y_i)dx\,dy_1\,dy_2 \\
&\quad + \int_{\mathbb{R}^{3n}} K(x,y_1,y_2)b_0(x)f_0(x)(\eta_R^1(y_1)-\eta_{R_0}^1(y_1))\eta_{R_0}^2(y_2)\prod_{i=1}^{2} b_i(y_i)f_i(y_i)dx\,dy_1\,dy_2 \\
&\quad + \langle T((\eta_R^1-\eta_{R_0}^1) b_1f_1,(\eta_R^2-\eta_{R_0}^2) b_2f_2), b_0f_0\rangle \\
&= I + II + III + IV.
\end{aligned}$$

The first term I is well defined since $\eta_{R_0}^i b_if_i \in b_iC_0^\delta$ *for a fixed* R_0 *(depending on* f_0*). We check that the first integral term II is absolutely convergent: The integrand of II is bounded* *by*

$||b_0||_{L^\infty}\prod_{i=1}^2||b_i||_{L^\infty}||f_i||_{L^\infty}$ *times*

$$\begin{aligned}|K(x,y_1,y_2)\eta^1_{R_0}(y_1)(\eta^2_R(y_2)-\eta^2_{R_0}(y_2))f_0(x)| &\lesssim \frac{|\eta^1_{R_0}(y_1)(\eta^2_R(y_2)-\eta^2_{R_0}(y_2))f_0(x)|}{(|x-y_1|+|x-y_2|)^{2n}}\\ &\leq \frac{|\eta^1_{R_0}(y_1)(\eta^2_R(y_2)-\eta^2_{R_0}(y_2))f_0(x)|}{(|x-y_1|+|x-y_2|/2+(R_0-R_0/2)/2)^{2n}}\\ &\lesssim \frac{|\eta^1_{R_0}(y_1)f_0(x)|}{(R_0+|x-y_2|)^{2n}}.\end{aligned}$$

This is an $L^1(\mathbb{R}^{3n})$ *function that is independent of* R *(as long as* $R>4R_0$*),*

$$\int_{\mathbb{R}^{3n}}\frac{|\eta^1_{R_0}(y_1)f_0(x)|}{(R_0+|x-y_2|)^{2n}}dx\,dy_1\,dy_2\lesssim\int_{\mathbb{R}^{2n}}\frac{|\eta^1_{R_0}(y_1)f_0(x)|}{R_0^n}dx\,dy_1\lesssim||f_0||_{L^\infty}R_0^n.$$

Since $\eta_R\to 1$ *pointwise, by dominated convergence the following limit exists:*

$$\begin{aligned}&\lim_{R\to\infty}\int_{\mathbb{R}^{3n}}K(x,y_1,y_2)b_0(x)f_0(x)\eta^1_{R_0}(y_1)(\eta^2_R(y_2)-\eta^2_{R_0}(y_2))\prod_{i=1}^2 b_i(y_i)f_i(y_i)dx\,dy_1\,dy_2\\ &\qquad=\int_{\mathbb{R}^{3n}}K(x,y_1,y_2)b_0(x)f_0(x)\eta^1_{R_0}(y_1)(1-\eta^2_{R_0}(y_2))\prod_{i=1}^2 b_i(y_i)f_i(y_i)dx\,dy_1\,dy_2.\end{aligned}$$

A symmetric argument holds for III. Finally, we consider IV minus the integral term from

(7.2)

$$\begin{aligned}&IV-\left\langle\int_{|y_1|,|y_2|>1}K(0,y_1,y_2)\prod_{i=1}^2 f(y_i)\eta^i_R(y_i)b_i(y_i)dy_1\,dy_2,b_0f_0\right\rangle\\ &=\int_{\mathbb{R}^{3n}}(K(x,y_1,y_2)-K(0,y_1,y_2))b_0(x)f_0(x)\prod_{i=1}^2(\eta^i_R(y_i)-\eta^i_{R_0}(y_i))f(y_i)b_i(y_i)dy_1\,dy_2\,dx.\end{aligned}$$

Again we bound the integrand by $||b_0||_{L^\infty}\prod_{i=1}^2 ||b_i||_{L^\infty}||f_i||_{L^\infty}$ *times*

$$\begin{aligned}|K(x,y_1,y_2)-K(0,y_1,y_2)|\,|f_0(x)|(\eta_R^1(y_1)-\eta_{R_0}^1(y_1) &\lesssim \frac{|x|^\gamma|\eta_R^1(y_1)-\eta_{R_0}^1(y_1)|}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}|f_0(x)|\\ &\lesssim \frac{|x|^\gamma|\eta_R^1(y_1)-\eta_{R_0}^1(y_1)|}{(|x-y_1|/2+R_0/4+|x-y_2|)^{2n+\gamma}}|f_0(x)|\\ &\lesssim \frac{R_0^\gamma|f_0(x)|}{(R_0+|x-y_1|+|x-y_2|)^{2n+\gamma}},\end{aligned}$$

which is an $L^1(\mathbb{R}^{3n})$ *function:*

$$\begin{aligned}\int_{\mathbb{R}^{3n}}\frac{R_0^\gamma|f_0(x)|}{(R_0+|x-y_1|+|x-y_2|)^{2n+\gamma}}dy_1\,dy_2\,dx &\lesssim \int_{\mathbb{R}^{2n}}\frac{R_0^\gamma|f_0(x)|}{(R_0+|x-y_1|)^{n+\gamma}}dy_1\,dx\\ &\lesssim \int_{\mathbb{R}^n}|f_0(x)|dx \lesssim ||f_0||_{L^\infty}R_0^n.\end{aligned}$$

Then it follows again by dominated convergence that

$$\begin{aligned}\lim_{R\to\infty}&\langle T((\eta_R^1-\eta_{R_0}^1)b_1f_1,(\eta_R^2-\eta_{R_0}^2)b_2f_2),b_0f_0\rangle\\ &\qquad-\left\langle\int_{|y_1f|,|y_2|>1}K(0,y_1,y_2)\prod_{i=1}^2 f(y_i)\eta_R^i(y_i)b_i(y_i)dy_1\,dy_2,b_0f_0\right\rangle\\ &=\int_{\mathbb{R}^{3n}}(K(x,y_1,y_2)-K(0,y_1,y_2))b_0(x)f_0(x)\prod_{i=1}^2(1-\eta_{R_0}^i(y_i))f(y_i)b_i(y_i)dy_1\,dy_2\,dx,\end{aligned}$$

which is an absolutely convergent integral. Therefore $T(b_1,b_2)$ *is well defined as an element of* $(b_0C_0^\delta)'$*. Furthermore if* $f_0,f_1,f_2\in C_0^\delta$ *and* b_0f_0 *has mean zero, then this definition* *of T is consistent with the a priori definition of T since*

$$\lim_{R\to\infty}\left\langle \int_{|y_1|,|y_2|>1} K(0,y_1,y_2)\prod_{i=1}^{2}\eta_R(y_i)b_i(y_i)f_i(y_i)dy_1\,dy_2, b_0f_0\right\rangle$$
$$= \left(\int_{|y_1|,|y_2|>1} K(0,y_1,y_2)\prod_{i=1}^{2}b_i(y_i)f_i(y_i)dy_1\,dy_2\right)\left(\int_{\mathbb{R}^n} b_0(x)f_0(x)dx\right) = 0,$$

since both of these integrals are absolutely convergent. Also, when b_0f_0 has mean zero in this way, the definition of $\langle T(b_1,b_2), b_0f_0\rangle$ is independent of the choice of η_R^1 and η_R^2. We will also use the notation $M_{b_0}T(b_1,b_2)\in BMO$ or $M_{b_0}T(b_1,b_2)=\beta$ for $\beta\in BMO$ to mean that for all $f_0\in C_0^\delta$ such that b_0f_0 has mean zero the following holds

$$\langle T(b_1,b_2), b_0f_0\rangle = \langle \beta, b_0f_0\rangle\,.$$

Note that the left hand side makes sense since $T(b_1,b_2)$ is defined in $(b_0C_0^\delta)'$. The right hand side also makes sense since $b_0f_0\in H^1$ for $f_0\in C_0^\delta$ and b_0f_0 has mean zero. The condition $M_{b_0}T(b_1,b_2)\in BMO$ defined here is weaker than (possibly equivalent to) $T(b_1,b_2)\in BMO$ when we can make sense of $T(b_1,b_2)$ as a locally integrable function. This is because our definition of $M_{b_0}T(b_1,b_2)\in BMO$ only requires this equality to hold when paired with a subset of the predual space of BMO, namely we require this to hold for $\{b_0f : f\in C_0^\delta$ and b_0f has mean zero$\}\subsetneq H^1$. It is possible that this is equivalent through some sort of density argument, but that is not of consequence here. So we do not pursue it any further, and use the definition of $M_{b_0}T(b_1,b_2)\in BMO$ that we have provided. Furthermore, if T is bounded then T can be defined on $L^\infty\times L^\infty$, and by Theorem 2.3.2 satisfies a uniform BMO estimate. These two facts imply that if T is bounded, then $T(b_1,b_2)\in BMO$. This is result is due to Peetre [70], Spanne [75], and Stein [77] in the linear case and Grafakos-Torres [45] in the bilinear case.

Remark 7.2.5 *The assumptions of Theorem 7.2.7 are symmetric in the same sense as was described in Remark 7.1.5 replacing $\mathscr{S}$ with the appropriate $b_iC_0^\delta$ spaces.*

Next we prove the analogue of Proposition 7.1.6. The proof is almost identical, but we reproduce it in full to verify that it holds in the accretive setting.

Proposition 7.2.6 *Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions and suppose T is a bilinear singular integral operator associated to b_0, b_1, b_2 that satisfies $M_{b_0}T(M_{b_1}\ , M_{b_2}\ ) \in WBP$. Also let $S_k^{b_i}$ be approximations to the identity with respect to b_i and $D_k^{b_i} = S_{k+1}^{b_i} - S_k^{b_i}$ with compactly supported kernels $s_k^{b_i}$ and $d_k^{b_i}$ for $k \in \mathbb{Z}$. Then*

$$\theta_k(x, y_1, y_2) = \left\langle T\left(b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2)\right), b_0 d_k^{b_0}(x, \cdot)\right\rangle$$

is a collection of Littlewood-Paley square function kernels of type SBLPK. Furthermore θ_k satisfies

$$\int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) b_0(x) dx = 0$$

for all $x, x', y_1, y_2 \in \mathbb{R}^n$. In particular, this holds for $s_k^{b_i}$ defined in Remark 5.3.3 and $d_k^{b_i} = s_{k+1}^{b_i} - s_k^{b_i}$.

Proof: Fix $x, y_1, y_2 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. We split estimate (4.1) into two cases: $|x-y_1|+|x-y_2| \le 2^{3-k}$ and $|x-y_1|+|x-y_2| > 2^{3-k}$. Assume $|x-y_1|+|x-y_2| \le 2^{3-k}$, and note that

$$\phi_1(u) = 2^{-kn} s_k^{b_1}(2^{-k}u + y_1, y_1)$$

is a normalized bump up to a constant multiple and $s_k^{b_1}(\cdot, y_1) = 2^{kn}\phi_1^{y_1, 2^{-k}}$. Likewise $s_k^{b_2}(\cdot, y_2) = 2^{-kn}\phi_2^{y_2, 2^{-k}}$ and $d_k^{b_0}(x, \cdot) = 2^{-kn}\phi_0^{x, 2^{-k}}$ where ϕ_0 and ϕ_2 are normalize bumps up to a constant multiple. Then by Lemma 7.2.3

$$\begin{aligned}|\theta_k(x,y_1,y_2)| &= \left|\left\langle T\left(b_1 s_k^{b_1}(\cdot,y_1), b_2 s_k^{b_2}(\cdot,y_2)\right), b_0 d_k^{b_0}(x,\cdot)\right\rangle\right| \\ &= 2^{3kn}\left|\left\langle T\left(b_1\phi_1^{y_1,2^{-k}}, \phi_2^{y_2,2^{-k}}\right), \phi_0^{x,2^{-k}}\right\rangle\right| \lesssim 2^{2kn}\end{aligned}$$

Now if we assume that $|x-y_1|+|x-y_2|>2^{3-k}$, then it follows that $|x-y_{i_0}|>2^{2-k}$ for either $i_0=1$ or $i_0=2$ and hence

$$\operatorname{supp}(d_k^{b_0}(x,\cdot))\cap\operatorname{supp}(s_k^{b_i}(\cdot,y_1))\cap\operatorname{supp}(s_k^{b_i}(\cdot,y_2))\subset B(x,2^{-k})\cap B(y_{i_0},2^{-k})=\emptyset.$$

Therefore, we can estimate θ_k via the kernel representation of T in the following way

$$\begin{aligned}|\theta_k(x,y_1,y_2)| &= \bigg|\int_{\mathbb{R}^{3n}}(K(u_0,u_1,u_2)-K(x,u_1,u_2))b_1(u_1)s_k^{b_1}(u_1,y_1)b_2(u_2)s_k^{b_2}(u_2,y_2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \times b_0(u_0)d_k^{b_0}(x,u_0)du_0du_1du_2\bigg| \\ &\lesssim \int_{|x-u_0|<2^{-k}}\int_{|y_1-u_1|<2^{-k}}\int_{|y_2-u_2|<2^{-k}}\frac{|u_0-x|^\gamma\, 2^{3nk}du_0du_1du_2}{(|x-u_1|+|x-u_2|)^{2n+\gamma}} \\ &\lesssim \int_{|x-u_0|<2^{-k}}\int_{|y_1-u_1|<2^{-k}}\int_{|y_2-u_2|<2^{-k}}\frac{2^{-\gamma k}\, 2^{3nk}du_0du_1du_2}{(2^{-k}+|x-y_1|+|x-y_2|)^{2n+\gamma}} \\ &\lesssim \frac{2^{-\gamma k}}{(2^{-k}+|x-y_1|+|x-y_2|)^{2n+\gamma}} \\ &\lesssim \Phi_k^{n+\gamma/2}(x-y_1)\Phi_k^{n+\gamma/2}(x-y_2).\end{aligned}$$

To verify (4.5), note that by the continuity from $b_1C_0^\delta\times b_2C_0^\delta$ into $(b_0C_0^\delta)'$, we have for $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=1$

$$|\partial_x^\alpha\theta_k(x,y,z)| = \left|\left\langle T\left(b_1 s_k^{b_1}(\cdot,y_1), b_2 s_k^{b_2}(\cdot,y_2)\right), b_0\partial_x^\alpha(d_k^{b_0}(x,\cdot))\right\rangle\right| \lesssim 2^k 2^{2kn}.$$

Estimate (4.5) easily follows. By symmetry, it follows that $\{\theta_k\}$ is a collection of smooth bilinear Littlewood-Paley square function kernels. Now we verify that θ_k has integral 0 in the x spot: By the continuity of T from $b_1C_0^\delta \times b_2C_0^\delta$ into $(b_0C_0^\delta)'$

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)b_0(x)dx = \lim_{R\to\infty} \left\langle T(b_1 s_k^{b_1}(\cdot,y_1), b_2 s_k^{b_2}(\cdot,y_2)), b_0 \int_{|x|<R} d_k^{b_0}(x,\cdot)b_0(x)dx \right\rangle$$
$$= \lim_{R\to\infty} \left\langle T(b_1 s_k^{b_1}(\cdot,y_1), b_2 s_k^{b_2}(\cdot,y_2)), \lambda_R \right\rangle$$

where we take this to be the definition of λ_R. Now if without loss of generality we take $R > 2\cdot 2^{-k}$, then for $|u| < R-2^{-k}$ it follows that

$$\text{supp}(d_k^{b_0}(\cdot,u)) \subset B(u,2^{-k}) \subset B(0,|u|+2^{-k}) \subset B(0,R),$$

and hence for $|u| < R-2^{-k}$ we have that

$$\lambda_R(u) = b_0(u)\int_{|x|<R} d_k^{b_0}(x,u)b_0(x)dx = b_0(u)D_k^{b_0\,*}b_0(u) = 0.$$

Also when $|u| > R+2^{-k}$, it follows that $\text{supp}(d_k^{b_0}(\cdot,u)) \cap B(0,R) = \emptyset$, and hence that $\lambda_R(u) = 0$. So we have $\lambda_R(x) = 0$ for $|x| \le R-2^{-k}$ or $|x| \ge R+2^{-k}$. Finally $||\lambda_R||_{L^\infty} \le \sup_u ||d_k^{b_0}(\cdot,u)||_{L^1} \lesssim 1$. Since $\text{supp}(d_k^{b_0}(x,\cdot)) \subset B(0,R+2^{-k})\backslash B(0,R-2^{-k})$, it follows that for $R > 4(2^{-k}+|y_1|)$, we may use the integral representation

$$\left|\left\langle T(b_1 s_k^{b_1}(\cdot,y_1), b_2 s_k^{b_2}(\cdot,y_2)), \lambda_R \right\rangle\right|$$
$$\le \int_{\mathbb{R}^{3n}} |K(u,v_1,v_2)b_1(v_1)s_k^{b_1}(v_1,y_1)b_2(v_2)s_k^{b_2}(v_2,y_2)\lambda_R(u)|du\,dv_1\,dv_2$$
$$\le \int_{|v_2-y_2|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{\text{supp}(\lambda_R)} \frac{2^{2kn}}{(|u|-|v_1-y_1|-|y_1|)^{2n}} du\,dv_1\,dv_2$$

$$\leq \int_{|v_2-y_2|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{\text{supp}(\lambda_R)} \frac{2^{2kn}}{(R-2^{-k}-|v_1-y_1|-|y_1|)^{2n}} du\, dv_1\, dv_2$$
$$\leq \int_{|v_2-y_2|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{\text{supp}(\lambda_R)} \frac{2^{2kn}}{R^{2n}} du\, dv_1\, dv_2$$
$$\lesssim |\text{supp}(\lambda_R)| R^{-2n}$$
$$\lesssim 2^{-k} R^{-(n+1)}.$$

This tends to zero as $R \to \infty$. Hence $\theta_k(x, y_1, y_2)$ has integral zero in the x variable. □

Now we prove the reduced Tb theorem, which again follows the same argument as the proof of the T1 version Theorem 7.1.7.

Theorem 7.2.7 *Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions, and T be a bilinear singular integral operator associated to b_0, b_1, b_2 such that $M_{b_0} T(M_{b_1}\ , M_{b_2}\ ) \in WBP$. If $M_{b_0} T(b_1, b_2) = M_{b_1} T^{*1}(b_0, b_2) = M_{b_2} T^{*2}(b_1, b_0) = 0$, then T can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p, p_1, p_2 < \infty$ satisfying* (1.1).

Note that in the hypothesis of Theorem 7.2.7, we take $M_{b_0} T(b_1, b_2) = 0$ in the sense of Definition 7.2.4: For appropriate η_R^1, η_R^2 and all $\phi \in C_0^\delta$ such that $b_0\phi$ has mean zero

$$\lim_{R\to\infty} \left\langle T(\eta_R^1 b_1, \eta_R^2 b_2), b_0\phi \right\rangle = 0.$$

The meaning of $M_{b_1} T^{*1}(b_0, b_2) = M_{b_2} T^{*2}(b_1, b_0) = 0$ are expressed in a similar way interchanging the roles of b_0, b_1, b_2.

Proof: Let T be as in the hypothesis, $1 < p, p_1, p_2 < \infty$ satisfy (1.1), and $f_0, f_1, f_2 \in C_0^1$ such that $b_i f_i$ have mean zero. Then by Proposition 5.3.4 and the continuity of T from

$b_1C_0^\delta \times b_2C_0^\delta$ into $(b_0C_0^\delta)'$, it follows that

$$\begin{aligned}
|\langle T(b_1f_1,b_2f_2),b_0f_0\rangle| &= \lim_{N\to\infty}\Big|\Big\langle T(M_{b_1}S_N^{b_1}M_{b_1}f_1,M_{b_2}S_N^{b_2}M_{b_2}f_2),M_{b_0}S_N^{b_0}M_{b_0}f_0\Big\rangle \\
&\qquad - \Big\langle T(M_{b_1}S_{-N}^{b_1}M_{b_1}f_1,M_{b_2}S_{-N}^{b_2}M_{b_2}f_2),M_{b_0}S_{-N}^{b_0}M_{b_0}f_0\Big\rangle\Big| \\
&= \lim_{N\to\infty}\Big|\sum_{k=-N}^{N-1}\Big\langle T(M_{b_1}S_{k+1}^{b_1}M_{b_1}f_1,M_{b_2}S_{k+1}^{b_2}M_{b_2}f_2),M_{b_0}S_{k+1}^{b_0}M_{b_0}f_0\Big\rangle \\
&\qquad - \Big\langle T(M_{b_1}S_k^{b_1}M_{b_1}f_1,M_{b_2}S_k^{b_2}M_{b_2}f_2),M_{b_0}S_k^{b_0}M_{b_0}f_0\Big\rangle\Big| \\
&\le \sum_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}^n}\Theta_k^0(b_1f_1,b_2f_2)b_0(x)f_0(x)dx\Big| \\
&\qquad + \Big|\int_{\mathbb{R}^n}\Theta_k^1(b_0f_0,b_2f_2)b_1(x)f_1(x)dx\Big| \\
&\qquad + \Big|\int_{\mathbb{R}^n}\Theta_k^2(b_1f_1,b_0f_0)b_2(x)f_2(x)dx\Big|.
\end{aligned}$$

where

$$\begin{aligned}
\Theta_k^0(f_1,f_2) &= D_k^{b_0}M_{b_0}T(M_{b_1}S_{k+1}^{b_1}f_1,M_{b_2}S_{k+1}^{b_2}f_2), \\
\Theta_k^1(f_1,f_2) &= D_k^{b_1}M_{b_1}T^{*1}(M_{b_0}S_k^{b_0}f_1,M_{b_2}S_k^{b_2}f_2), \\
\Theta_k^2(f_1,f_2) &= D_k^{b_2}M_{b_2}T^{*2}(M_{b_1}S_{k+1}^{b_1}f_1,M_{b_0}S_k^{b_0}f_2).
\end{aligned}$$

We focus on $\Theta_k^0 = \Theta_k$ to simplify notation; the other terms are handled in the same way. Since $M_{b_0}T(M_{b_1}\ ,M_{b_2}\ ) \in WBP$ and T has a standard kernel, it follows by Proposition 7.2.6 that $\{\theta_k\} \in SBLPK$ and $\theta_k(x,y_1,y_2)b_0(x)$ has mean zero in the x variable for all $y_1,y_2 \in \mathbb{R}^n$. Now we show that $\Theta_k(b_1,b_2) = 0$, which follows from the assumption that

$M_{b_0}T(b_1,b_2)=0$:

$$\begin{aligned}\Theta_k(b_1,b_2)(x) &= \int_{\mathbb{R}^{2n}} \left\langle M_{b_0}T\left(M_{b_1}s_k^{b_1}(\cdot,y_1)b_1(y_1), M_{b_2}s_k^{b_2}(\cdot,y_2)b_2(y_2)\right), d_k^{b_0}(x,\cdot)\right\rangle dy \\ &= \lim_{R\to\infty}\left\langle T\left(b_1\eta_R^1, b_2\eta_R^2\right), b_0 d_k^{b_0}(x,\cdot)\right\rangle = 0,\end{aligned}$$

where

$$\eta_R^i(u) = \int_{|y|<R} s_k^{b_i}(u,y)b_i(y)dy.$$

It follows that $\eta_R^i \in C^\infty$, $\eta_R^i \equiv 1$ on $B(0,R)$, and $\text{supp}(\eta_R^i) \subset B(0,2R)$ for R sufficiently large. Then by Theorem 6.6.1, it follows that

$$\sum_{k\in\mathbb{Z}} \left|\left\langle \Theta_k^0(M_{b_1}f_1, M_{b_2}f_2), M_{b_0}f_0\right\rangle\right| \lesssim ||f_0||_{L^{p'}}||f_1||_{L^{p_1}}||f_2||_{L^{p_2}}.$$

A similar argument holds for Θ_k^i with $i=1,2$ again taking advantage of the facts $\frac{1}{p'}+\frac{1}{p_2}=\frac{1}{p_1'}$ and $\frac{1}{p_1}+\frac{1}{p'}=\frac{1}{p_2'}$. Therefore T can be extended to a bounded operator from $L^{p_1}\times L^{p_2}$ into L^p for all $1<p,p_1,p_2<\infty$ satisfying (1.1). □

7.3 A Paraproduct Construction

In [28], David-Journé reduced the original T1 theorem by using a para-product operator. The construction of this operator goes back to the work of Bony [7] and was further developed by Coifman-Meyer [21]. For a nice discussion of the Bony para product in the context of Littlewood-Paley operators, see the work by Bényi-Maldonaado-Naibo [4]. In [29], David-Journé-Semmes extended this paraproduct to the perturbed, para-accretive setting.

Theorem 7.3.1 (Bony [7], Coifman-Meyer [21], David-Journé-Semmes [29]) *Given para-accretive functions $b_0, b_1 \in L^\infty$ and $\beta \in BMO$, there exists a bilinear Calderón-Zygmund operator L that is bounded from L^p into L^p for all $1 < p < \infty$ such that*
$M_{b_0} L(b_1) = \beta$ *and* $M_{b_1} T^* b_0 = 0$.

A new proof of this results is easily extracted from the bilinear version, which we state and prove here. First we prove a short lemma that emphasizes the connection between Littlewood-Paley square function operators and singular integral operators with standard kernels.

Lemma 7.3.2 *Suppose $\{\theta_k\} \in SBLPK$ with decay parameter $N > 2n$, and define $K : \mathbb{R}^{3n} \backslash \{(x,x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$*

$$K(x,y_1,y_2) = \sum_{k\in\mathbb{Z}} \theta_k(x,y_1,y_2).$$

Then $K \in BCZK_{\mathbb{C}}$.

Proof: By Lemma 6.5.1 with $r = 1$, it follows that $\{\theta_k\}$ is an ℓ^1-valued standard kernel. Then

$$|K(x,y_1,y_2)| \leq ||\{\theta_k(x,y_1,y_2)\}||_{\ell^1(\mathbb{Z})} \lesssim \frac{1}{(|x-y_1|+|x-y_2|)^{2n}}.$$

Similarly when $|x-x'| < \max(|x-y_1|,|x-y_2|)/2$

$$\begin{aligned} |K(x,y_1,y_2) - K(x',y_1,y_2)| &\leq ||\{\theta_k(x,y_1,y_2) - \theta_k(x',y_1,y_2)\}||_{\ell^1(\mathbb{Z})} \\ &\lesssim \frac{|x-x'|^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}. \end{aligned}$$

A similar argument holds for the regularity in y_1, y_2, it follows that K is a standard kernel of type $BCZK_{\mathbb{C}}$. □

Theorem 7.3.3 *Given para-accretive functions $b_0, b_1, b_2 \in L^\infty$ and $\beta \in BMO$, there exists a bilinear Calderón-Zygmund operator L that is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p_1, p_2 < \infty$ satisfying* (1.1) *with $p = 2$ such that $M_{b_0}T(b_1, b_2) = \beta$, $M_{b_1}T^{*1}(b_0, b_2) = M_{b_2}T^{*2}(b_1, b_0) = 0$.*

Proof: Let b_0, b_1, b_2 be para-accretive functions, and $S_k^{b_i}$, $D_k^{b_i}$, and $\widetilde{D}_k^{b_i}$ be the approximation to identity and reproducing formula operators with respect to b_i for $i = 0, 1, 2$ defined in Remark 5.3.3 and Theorem 5.4.1. Define

$$L(f_1, f_2) = \sum_{k\in\mathbb{Z}} L_k(f_1, f_2) = \sum_{k\in\mathbb{Z}} D_k^{b_0} M_{b_0}\left((\widetilde{D}_k^{b_0*} M_{b_0}\beta)(S_k^{b_1} f_1)(S_k^{b_2} f_2)\right)$$
$$\ell(x, y) = \sum_{k\in\mathbb{Z}} \ell_k(x, y) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} d_k^{b_0}(x, u) b_0(u) \widetilde{D}_k^{b_0*} M_{b_0}\beta(u) s_k^{b_1}(u, y_1) s_k^{b_2}(u, y_2) du.$$

It follows that L is bounded from $L^{p_1} \times L^{p_2}$ into L^2 for all $1 < p_1, p_2 < \infty$ satisfying $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$:

$$\left|\int_{\mathbb{R}^n} L(f_1, f_2)(x) f_0(x) dx\right| \leq \sum_{k\in\mathbb{Z}} \left|\int_{\mathbb{R}^n} \widetilde{D}_k^{b_0*} M_{b_0}\beta(x) S_k^{b_1} f_1(x) S_k^{b_2} f_2(x) D_k^{b_0} f_0(x) b_0(x) dx\right|$$
$$\lesssim \left|\left|\left(\sum_{k\in\mathbb{Z}} |M_{\widetilde{D}_k^{b_0*} M_{b_0}\beta} S_k^{b_1} f_1 S_k^{b_2} f_2|^2\right)^{\frac{1}{2}}\right|\right|_{L^2} \left|\left|\left(\sum_{k\in\mathbb{Z}} |D_k^{b_0} f_0(x)|^2\right)^{\frac{1}{2}}\right|\right|_{L^2}$$
$$\lesssim \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left[\Phi_k^N * |f_1|(x) \Phi_k^N * |f_2|(x)\right]^2 |\widetilde{D}_k^{b_0*} M_b\beta(x)|^2\right)^{\frac{1}{2}} ||f_0||_{L^2}$$

$$\lesssim \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left[\Phi_k^N * |f_1|(x)\right]^{p_1} |\widetilde{D}_k^{b_0*} M_b \beta(x)|^2 \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left[\Phi_k^N * |f_2|(x)\right]^{p_2} |\widetilde{D}_k^{b_0*} M_b \beta(x)|^2 \right)^{\frac{1}{p_2}} ||f_0||_{L^2}$$

$$\lesssim ||f_0||_{L^2} ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}}.$$

Note that in the last line we apply Theorem 6.3.6 with the operator $\widetilde{D}_k^{b_0*} M_{b_0}$. It is easy to check that $D_k^{b_0} M_{b_0}$ generates a Carleson measure in the sense of Theorem 6.3.6: We apply Corrollary 6.3.5 to $\widetilde{D}_k^{b_0*}$. We know that $\widetilde{D}_k^{b_0*}(b_0) = 0$, and by Theorem 6.4.3, $\{\widetilde{d}_k^{b_0*}\} \in LPK$ and $\widetilde{D}_k^{b_0*} b_0 = 0$ implies that

$$\left|\left| \left(\sum_{k\in\mathbb{Z}} |\widetilde{D}_k^{b_0*} M_{b_0} f|^2 \right)^{\frac{1}{2}} \right|\right|_{L^2} = \left|\left| \left(\sum_{k\in\mathbb{Z}} |\widetilde{D}_k^{b_0*} (b_0 f)|^2 \right)^{\frac{1}{2}} \right|\right|_{L^2} \lesssim ||b_0 f||_{L^2} \lesssim ||f||_{L^2}$$

for all $f \in L^2$. This proves that L is bounded from $L^{p_1} \times L^{p_2}$ into L^2 for all $1 < p_1, p_2 < \infty$ satisfying (1.1) with $p = 2$. It is easy to check that $\{\ell_k\} \in SLPK$, so by Lemma 7.3.2, we know that L has a standard Calderón-Zygmund kernel $\ell \in KCZ_{\mathbb{C}}$. It follows from a result of Grafakos-Torres [45, 44] that L is bounded from $L^{p_1} \times L^{p_2}$ into L^p where $1 < p_1, p_2 < \infty$ satisfy (1.1). Next we compute $M_{b_0} L(b_1, b_2)$: Let $\delta > 0$, $\phi \in C_0^\delta$ such that $\mathrm{supp}(\phi) \subset B(0,N)$ and $b_0\phi$ has mean zero. Let $\eta_R(x) = \eta(x/R)$ where $\eta \in C_0^\infty$ satisfies $\eta \equiv 1$ on $B(0,1)$, and $\mathrm{supp}(\eta) \subset B(0,2)$. Then

$$\begin{aligned}
&\langle L(b_1,b_2), b_0\phi \rangle \\
&\quad = \lim_{R\to\infty} \sum_{2^{-k}<R/4} \int_{\mathbb{R}^n} \widetilde{D}_k^{b_0{}^*} M_{b_0} \beta(x) S_k^{b_1} M_{b_1} \eta_R(x) S_k^{b_2} M_{b_2} \eta_R(x) M_{b_0} D_k^{b_0}(b_0\phi)(x) dx \\
&\qquad + \lim_{R\to\infty} \sum_{2^{-k}\geq R/4} \int_{\mathbb{R}^n} \widetilde{D}_k^{b_0{}^*} M_{b_0} \beta(x) S_k^{b_1} M_{b_1} \eta_R(x) S_k^{b_2} M_{b_2} \eta_R(x) M_{b_0} D_k^{b_0}(b_0\phi)(x) dx,
\end{aligned}$$

where we may write this only if the two limits on the right hand side of the equation exist. As we are taking $R \to \infty$ and N is a fixed quantity determined by ϕ, without loss of generality assume that $R > 2N$. Note that for $2^{-k} \leq R/4$ and $|x| < N + 2^{-k}$,

$$\operatorname{supp}(s_k^{b_i}(x,\cdot)) \subset B(x, 2^{-k}) \subset B(0, N + 2^{1-k}) \subset B(0,R).$$

Since $\eta_R \equiv 1$ on $B(0,R)$, it follows that $S_k^{b_i} M_{b_i} \eta_R(x) = 1$ for all $|x| < N + 2^{-k}$ when $2^{-k} \leq R/4$. Therefore

$$\begin{aligned}\lim_{R\to\infty} \sum_{2^{-k}<R/4} \int_{\mathbb{R}^n} \widetilde{D}_k^{b_0\,*} M_{b_0} \beta(x) M_{b_0} D_k^{b_0}(b_0\phi)(x)dx &= \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} M_{b_0} \widetilde{D}_k^{b_0} M_{b_0} D_k M_{b_0} \phi(x)\beta(x)dx \\ &= \langle \beta, b_0\phi \rangle .\end{aligned}$$

Here we use the convergence of the accretive type reproducing formula in H^1 from Corollary 5.4.4. For any $k \in \mathbb{Z}$, we have the estimates

$$||S_k^{b_i} M_{b_i} \eta_R||_{L^1} \lesssim ||\eta_R||_{L^1} \lesssim R^n, \tag{7.3}$$

$$||S_k^{b_i} M_{b_i} \eta_R||_{L^\infty} \lesssim ||\eta_R||_{L^\infty} = 1, \tag{7.4}$$

and for any $x \in \mathbb{R}^n$

$$\begin{aligned}|D_k^{b_0} M_{b_0} \phi(x)| &\leq \int_{\mathbb{R}^n} |d_k^{b_0}(x,y) - d_k^{b_0}(x,0)|\,|b_0(y)\phi(y)|dy \\ &\lesssim \int_{\mathbb{R}^n} (2^k|y|)^\gamma |\phi(y)|dy \lesssim N^\gamma ||\phi||_{L^1} 2^{k(n+\gamma)}.\end{aligned}$$

Here we know that $\{d_k^{b_0}\} \in LPK$, so without loss of generality we take the corresponding smoothness parameter $\gamma \leq \delta$. Later we will use that $\gamma \leq \delta$ implies $\phi \in C_0^\delta \subset C_0^\gamma$, so we have that $|\phi(x)-\phi(y)| \lesssim |x-y|^{\gamma}$. Therefore

$$\begin{aligned}
&\sum_{2^{-k}>R/4}\int_{\mathbb{R}^n} |\widetilde{D}_k^{b_0\,*} M_{b_0}\beta(x) S_k^{b_1} M_{b_1}\eta_R(x) S_k^{b_2} M_{b_2}\eta_R(x) M_{b_0} D_k^{b_0}(b_0\phi)(x)|dx \\
&\qquad \leq \sum_{2^{-k}>R/4} ||\widetilde{D}_k^{b_0\,*} M_{b_0}\beta||_{L^\infty} ||S_k^{b_1} M_{b_1}\eta_R||_{L^1} ||S_k^{b_2} M_{b_2}\eta_R||_{L^\infty} ||M_{b_0} D_k^{b_0}(b_0\phi)||_{L^\infty} \\
&\qquad \lesssim R^n \sum_{2^{-k}>R/4} 2^{k(n+\gamma)} \lesssim R^{-\gamma}. \qquad (7.5)
\end{aligned}$$

Hence the second limit above exists and tends to 0 as $R\to\infty$. Then $\langle L(b_1,b_2), b_0\phi\rangle = \langle \beta, b_0\phi\rangle$ for all $\phi \in C_0^\delta$ such that $b_0\phi$ has mean zero and hence $M_{b_0}L(b_1,b_0) = \beta$ as an element of BMO. Again for any $\phi \in C_0^\delta$ such that $b_1\phi$ has mean zero and $\text{supp}(\phi) \subset B(0,N)$, we have

$$\begin{aligned}
&\left|\left\langle L^{1*}(b_0,b_2), b_1\phi\right\rangle\right| \\
&\quad = \lim_{R\to\infty}\left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n} \widetilde{D}_k^{b_0\,*} M_{b_0}\beta(x) S_k^{b_1} M_{b_1}\phi(x) S_k^{b_2} M_{b_2}\eta_R(x) D_k^{b_0} M_{b_0}\eta_R(x) b_0(x)dx\right| \\
&\quad \lesssim \lim_{R\to\infty}\sum_{k\in\mathbb{Z}} ||\widetilde{D}_k^{b_0\,*} M_{b_0}\beta||_{L^\infty} ||S_k^{b_1} M_{b_1}\phi||_{L^1} ||S_k^{b_2} M_{b_2}\eta_R||_{L^\infty} ||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty} \\
&\quad \lesssim \lim_{R\to\infty}\sum_{k\in\mathbb{Z}} ||S_k^{b_1} M_{b_1}\phi||_{L^1} ||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty}.
\end{aligned}$$

We will now show that $||S_k^{b_1} M_{b_1}\phi||_{L^1} ||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty}$ bounded by a in integrable function in k (i.e. summable) independent of R, so that we can bring the limit in R inside the sum. To do this we start by estimating

$$\begin{aligned}
|S_k^{b_1} M_{b_1}\phi(x)| &\leq \int_{\mathbb{R}^n} |s_k^{b_1}(x,y) - s_k^{b_1}(x,0)|\,|\phi(y) b_1(y)|dy \\
&\leq N^{\gamma} ||\phi||_{L^1} ||b_1||_{L^\infty} 2^{\gamma k}\left(\Phi_0^N(x) + \Phi_k^N(x)\right)
\end{aligned}$$

and so $||S_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim 2^{\gamma k}$. We also have that $||S_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim ||\phi||_{L^1} \lesssim 1$, so $||S_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim \min(1, 2^{\gamma k})$. Also

$$\begin{aligned}|D_k^{b_0} M_{b_0} \eta_R(x)| &\leq \int_{\mathbb{R}^n} |d_k^{b_0}(x,y)| \, |\eta_R(y) - \eta_R(x)| \, |b_0(y)| dy \\ &\lesssim 2^{-\gamma k} R^{-\gamma} \int_{\mathbb{R}^n} \Phi_k^{N+\gamma}(x-y)(2^k|x-y|)^{\gamma} dy \lesssim 2^{-\gamma k} R^{-\gamma}.\end{aligned}$$

It follows that $||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \lesssim ||\eta_R||_{L^\infty} \lesssim 1$, and hence $||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \lesssim \min(1, 2^{-\gamma k})$. So when $R > 1$, we have

$$||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} ||S_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim \min(2^{-\gamma k} R^{-\gamma}, 2^{\gamma k}) \leq \min(2^{-\gamma k}, 2^{\gamma k}),$$

and hence by dominated convergence

$$\left| \left\langle L^{1*}(b_0, b_2), b_1 \phi \right\rangle \right| \lesssim \sum_{k \in \mathbb{Z}} \lim_{R \to \infty} ||S_k^{b_1} M_{b_1} \phi||_{L^1} ||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \lesssim \sum_{k \in \mathbb{Z}} \lim_{R \to \infty} 2^{-k\gamma} R^{-\gamma} = 0$$

Then $M_{b_1} L^{*1}(b_0, b_2) = 0$ and a similar argument shows that $M_{b_2} L^{*2}(b_1, b_0) = 0$, which concludes the proof. □

7.4 Full Bilinear T1 Theorem

The linear T1 theorem of David-Journé provided a complete characterization of Lebesgue space bounds for Calderón-Zygmund singular integral operators. Later Coifman-Meyer [24] give a quick proof of the T1 theorem. These results highlight the the intrinsic connection between operator bounds and the cancellation of their kernels through the T1 testing condition.

Theorem 7.4.1 (David-Journé [28]) *Assume that T is a linear singular integral operator that is continuous from $\mathscr{S}$ into $\mathscr{S}'$. Then T can be extended to a bounded operator on L^p for all $1 < p < \infty$ if and only if T satisfies the weak boundedness property and $T1, T^*1 \in BMO$.*

In recent years, a multilinear version of this result was proved by Christ-Journé [17] and Grafakos-Torres [45, 44]. The proofs presented in these two works somehow involved iterative applications of linear version of the T1 theorem. Here we present an alternate proof of the bilinear T1 theorem based on the results developed in this work that depend on bilinear square function bounds.

Theorem 7.4.2 (Christ-Journé [17], Grafakos-Torres [45, 44]) *Assume that T is a bilinear singular integral operator that is continuous from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$. Then T can be extended to a bounded operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p_1, p_2 < \infty$ satisfying* (1.1) *if and only if T satisfies the weak boundedness property and $T^{*i}(1,1) \in BMO$ for $i = 0, 1, 2$.*

Proof: Assume that T satisfies the weak boundedness property and $T^{*i}(1,1) \in BMO$ for $i = 0, 1, 2$. By Theorem 7.3.3, there exist bounded bilinear Calderón-Zygmund operators L_i such that

$$L_i^{*i}(1,1) = T^{*i}(1,1) \text{ for } i = 0, 1, 2$$
$$L_j^{*i}(1,1) = 0 \text{ for } j \neq i$$

Now define the operator

$$S = T - \sum_{i=0}^{2} L_i,$$

which is continuous from $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}'$. Also S satisfies the weak boundedness property since T and L_i for $i = 0,1,2$ do. Also we have

$$S^{*i}(1,1) = T^{*i}(1,1) - \sum_{j=0}^{2} L_j^{*i}(1,1) = 0.$$

Then by Theorem 7.2.7, S can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p, p_1, p_2 < \infty$ satisfying (1.1). By Theorems 2.2.4, 2.3.2 and Corollary 3.2.2, it follows that T can be extends to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p_1, p_2 < \infty$ satisfying (1.1) without restriction on p. Conversely, if T is bounded from $L^{p_1} \times L^{p_2}$ into L^p for $1 < p_1, p_2 < \infty$ satisfying (1.1), then by Fatou's lemma and Theorem 2.3.2, it follows that for all $\phi \in C_0^\infty$ with mean zero and appropriate η_R

$$\begin{aligned} |\langle T(1,1), \phi \rangle| = \lim_{R\to\infty} |\langle T(\eta_R, \eta_R), \phi \rangle| &\leq \limsup_{R\to\infty} ||T(\eta_R, \eta_R)||_{BMO} ||\phi||_{H^1} \\ &\lesssim \limsup_{R\to\infty} ||\eta_R||_{L^\infty}^2 ||\phi||_{H^1} \leq ||\phi||_{H^1}. \end{aligned}$$

therefore $T(1,1) \in BMO$. Also for any normalized bumps ϕ_0, ϕ_1, ϕ_2, $x \in \mathbb{R}^n$, and $R > 0$ we have

$$\left| \left\langle T(\phi_1^{x,R}, \phi_2^{x,R}), \phi_0^{x,R} \right\rangle \right| \lesssim ||\phi_0^R||_{L^2} ||\phi_1^R||_{L^4} ||\phi_2^R||_{L^4} \lesssim R^n.$$

So $T \in WBP$ as well. □

7.5 Full Bilinear Tb Theorem

In the 1980's, David-Journé-Semmes [29] and McIntosh-Meyer [67] proved the linear Tb theorem, which we state now.

Theorem 7.5.1 (David-Journé-Semmes [29], McIntosh-Meyer [67]) *Let $b_0, b_1 \in L^\infty$ be para-accretive functions. Assume that T is a bilinear singular integral operator associated to b_0, b_1. Then T can be extended to a bounded operator on L^p for all $1 < p < \infty$ if and only if $M_{b_0} T M_{b_1}$ satisfies the weak boundedness property and $M_{b_0} T b_1$, $M_{b_1} T^* b_0 \in BMO$.*

Although there are multilinear versions of the T1 theorem, up to this point there has been no multilinear version of the Tb theorem for Calderón-Zygmund singular integral operators, which we state and prove now.

Theorem 7.5.2 *Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions. Assume that T is a bilinear singular integral operator associated to b_0, b_1, b_2. Then T can be extended to a bounded operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p_1, p_2 < \infty$ satisfying* (1.1) *if and only if $M_{b_0} T(M_{b_1}, M_{b_2})$ satisfies the weak boundednes property and $M_{b_0} T(b_1, b_2)$, $M_{b_1} T^{*1}(b_0, b_2)$, $M_{b_2} T^{*2}(b_1, b_0) \in BMO$.*

Proof: Assume that $M_{b_0} T(M_{b_1}, M_{b_2})$ satisfies the weak boundedness property and $M_{b_0} T(b_1, b_2)$, $M_{b_1} T^{*1}(b_0, b_2)$, $M_{b_2} T^{*2}(b_1, b_0) \in BMO$ for $i = 0, 1, 2$. By Theorem 7.3.3, there exist bounded bilinear Calderón-Zygmund operators L_i such that

$$
\begin{aligned}
M_{b_0} L_0(b_1, b_2) &= M_{b_0} T(b_1, b_2) \qquad & M_{b_1} L_0^{*1}(b_0, b_2) = M_{b_2} L_0^{*2}(b_1, b_0) = 0 \\
M_{b_1} L_1^{*1}(b_0, b_2) &= M_{b_1} T^{*1}(b_0, b_2) \qquad & M_{b_0} L_1(b_1, b_2) = M_{b_2} L_1^{*2}(b_1, b_0) = 0 \\
M_{b_2} L_2^{*2}(b_1, b_0) &= M_{b_2} T^{*2}(b_1, b_0) \qquad & M_{b_1} L_2^{*1}(b_0, b_2) = M_{b_0} L_2(b_1, b_2) = 0
\end{aligned}
$$

Now define the operator

$$S = T - \sum_{i=0}^{2} L_i,$$

which is continuous from $b_1C_0^\delta \times b_2C_0^\delta$ into $(b_0C_0^\delta)'$. Also $M_{b_0}S(M_{b_1}, M_{b_2})$ satisfies the weak boundedness property since $M_{b_0}T(M_{b_1}, M_{b_2})$ and $M_{b_0}L_i(M_{b_1}, M_{b_2})$ for $i = 0, 1, 2$ do. Also we have

$$M_{b_0}S(b_1, b_2) = M_{b_0}T(b_1, b_2) - \sum_{i=0}^{2} M_{b_0}L_i(b_1, b_2) = 0$$
$$M_{b_1}S^{*1}(b_0, b_2) = M_{b_1}T^{*1}(b_0, b_2) - \sum_{i=0}^{2} M_{b_1}L_i^{*1}(b_0, b_2) = 0$$
$$M_{b_2}S^{*2}(b_1, b_0) = M_{b_2}T^{*2}(b_1, b_0) - \sum_{i=0}^{2} M_{b_2}L_i^{*2}(b_1, b_0) = 0$$

Then by Theorem 7.2.7, S can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p, p_1, p_2 < \infty$ satisfying (1.1). In exactly the same way as in the proof of Theorem 7.4.2, Theorems 2.2.4, 2.3.2 and Corollary 3.2.2 imply that T can be extends to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p_1, p_2 < \infty$ satisfying (1.1) without restriction on p. The converse is almost exactly the same as in Theorem 7.4.2 as well. If T is bounded from $L^{p_1} \times L^{p_2}$ into L^p for $1 < p_1, p_2 < \infty$ satisfying (1.1), then by Fatou's lemma and Theorem 2.3.2, it follows that for all $\phi \in C_0^\delta$ such that $b_0\phi$ has mean zero and appropriate η_R

$$|\langle T(b_1, b_2), b_0\phi\rangle| \lesssim \limsup_{R\to\infty} ||b_1\eta_R||_{L^\infty} ||b_2\eta_R||_{L^\infty} ||\phi||_{H^1} \lesssim ||b_1||_{L^\infty} ||b_2||_{L^\infty} ||b_0\phi||_{H^1}.$$

therefore $M_{b_0}T(b_1, b_2) \in BMO$. The proof that $M_{b_0}T(M_{b_1}, M_{b_2}) \in WBP$ is exactly the same as the one from the proof of Theorem 7.4.2. $\square$

Chapter 8

Closing Remarks

8.1 Conclusions

In Chapter 2, we presented a theory of vector-valued Calderón-Zygmund operators that naturally parallels the linear vector-valued Calderón-Zygmund theory developed in [2, 53]. Some of the proof techniques in this chapter are natural analogues in as the ones in scalar setting in [17, 57, 45, 44, 63], but other results required new techniques that provide new proof in the scalar valued case.

In Chapter 3, we presented a number of interpolation results for vector valued operators. The contributions we make here are to identify and address the difficulties that arise in extending these results to Banach valued ones. The difficulties addressed are issues of well-definedness, density, and the failure of certain measure theory results in more general vector integration settings (e.g. Fatou's lemma).

Chapter 4 was dedicated to prove a number of almost orthogonality results. These almost orthogonality estimates strike a balance between oscillation, regularity, and decay properties for functions and operators. The estimates in this chapter are are crucial to the square function estimates proved in Chapter 6, and hence to our approach to proving singular integral bounds in Chapter 7. The linear version of all the results in this chapter are well established (see e.g. [5, 6, 81, 82, 83, 70, 71, 72, 85, 86, 61]), and some of the bilinear estimates had been proved in [62, 63]. Other estimates in this section were developed in the work of the author, including [49] and work that will be submitted for publication soon.

The convergence results in Chapter 5 are largely dealing with technical issues. In practice, one can often prove an estimate on some convenient class of functions (like C_0^∞ or bC_0^δ) and pass to density arguments to prove operator bounds. The appropriate type of convergence for the arguments are determined by the topologies involved in the continuity assumptions of the operators. In our case, we needed to prove convergence results in many settings due to the number of different operators that we worked with. We needed the Lebesgue space convergence results to work with the continuity Littlewood-Paley square function operators, the Schwartz space convergence results to work with the unperturbed singular integral operators, and the perturbed Hölder space convergence results to work with the continuity of the perturbed singular integral operators. Most of these results are well known, and have been used in [28, 29, 48, 50] among others. The main contribution of Chapter 5 is the extension of certain reproducing formulas for L^p to formulas for H^1 in the perturbed setting.

The square function theory presented in Chapter 6 reconstructs a particular section of Littlewood-Paley theory from the ground up (which included results from [79, 77, 12, 56, 28, 38, 39, 29, 74] among others), and extends these results to many analogous ones in the bilinear setting. Some of the first results in the bilinear settings were proved by Maldonado [62] and Maldonado-Naibo [63], which are contained in Theorem 6.2.3. Many of the bilinear results from this section were proved by the author in [49, 51], some of which were proved concurrently by Grafakos-Oliviera [41] and Grafakos-Liu-Maldonado-Yang [40]. Theorem 6.6.1 is currently unpublished, and we developed with the intent to be applied to truncations of singular integral operators as in Theorems 7.1.7 and 7.2.7, and is part of a work that will be submitted for publication soon.

In Chapter 7, we applied the results from the previous chapters to prove estimates for bilinear singular integral operators. The primary results of this section were Theorems 7.4.2, 7.5.2, and 7.3.3. Much of Chapter 7 was dedicated to address the technical issues in proving that singular integral operators can be decomposed into Littlewood-Paley smooth truncation operators via weak continuity assumptions. These ideas were used in many works including [28, 29, 24]. The bilinear T1 theorem (Theorem 7.4.2) was originally proved in [17, 45, 44]. The proof we gave in Chapter 7 is the one that was proved by the author in [50], which is a different proof of the theorem. In particular, we provide an constructive bilinear proof through smooth truncation operators independent of the linear T1 theorem, whereas the arguments in [17, 45, 44] depended in some way on iterative applications of the linear version of the result. One benefit of developing this theory is that we were able to extend the techniques to the perturbed singular integral operator for a bilinear Tb theorem, which is a new result that will be submitted for publication soon. In fact, this work provides a new proof of the linear Tb theorem that is in the same spirit as the original proof by [29], but provides a slightly different argument. In particular, we avoid the need for a Cotlar-Knapp-Stein lemma by directly approximating the operator norms via the embedding of Lebesgue spaces in tempered distributions and concluding bound by the density of the Schwartz class in Lebesgue spaces.

The main ideas of this work, vector-valued Calderón-Zygmund thoery, Littlewood-Paley square functions, and singular integral with standard kernel, are related at a fundamental level. There is a sort of weak correspondence between these objects: Littlewood-

Paley square functions form ℓ^r valued Calderón-Zygmund operators in the sense of Lemma 6.5.1, Littlewood-Paley square functions define singular integral operators in the sense of Lemma 7.3.2, and singular integrals define ℓ^2 valued Calderón-Zygmund operators in the sense of Propositions 7.1.6 and 7.2.6. These relationships permit us to use the advantages of each one when convenient. Ultimately, these topics are manifestations of the same concepts of the interaction of oscillatory and regularity.

8.2 Future Work

There are number of directions that this work can lead. Much of this further research is ongoing, and most of it is collaborative work.

Square Functions on Weighted Spaces

The pointwise estimates of the operators in terms of the Hardy-Littlewood maximal function from Chapter 4 lead to a natural application to estimate in Lebesgue spaces with Muckenhoupt weights. These can be directly extended to the weighted version when one assumes a $\Theta(1) = 0$, but this mean zero condition is not a necessary one. We have developed some of the Carleson measure theory introduced in Chapter 6 to obtain weaker sufficient conditions for square function bounds in the multilinear weighted setting. There has been a lot of work done in the area of linear, convolution type square function operators along these lines, see e.g. the work of Kurtz [58], Duoandikoetxea [30], Duoandikoetxea-Seijo [32], and Cruz-Uribe-Martell-Pérez[27], but there is little progress for the non-convolution and mutilinear Littlewood-Paley square function operators. This project is a joint work with Lucas Chaffee and Lucas Oliviera.

Biparamater Square Functions and Singular Integrals

Many of the bilinear techniques developed in this work can be applied in a biparameter setting. There is a natural analogy that can be drawn between these two types of problems, and many of the technique used for bilinear problems can be readily adapted for bilinear problems. In fact, Theorem 6.2.4 is essentially an estimate for biparameter operators on a tensor product. Also many of the general strategies of proving singular integral bounds by reducing to Littlewood-Paley square functions hold in the biparameter setting. Thus this smooth truncation to singular integral approach through Littlewood-Paley theory may be extended to the biparameter setting. Multiparameter harmonic analysis has been studied by Jessen-Marcinkiewicz-Zygmund [54], R. Fefferman [35, 36], Chang [13], R. Fefferman-Stein [37], Chang-Ciesielski [14], among others. More recently there has been interest in this topic, see e.g. Muscalu-Pipher-Tao-Thiele [68, 69]. This is and ongoing joint work with Rodolfo Torres.

Bilinear Fourier Integral Operators

Like in the last section, the smooth truncation techniques formed via the Littlewood-Paley square function operators can be adapted in the situation of many bilinear Fourier integrals. By adapting these Littelwood-Paley techniques to some types of Fourier integrals, the analogous square function theory may apply to prove various estimates for the operators. Some bilinear Fourier integral operators were introduced by Grafakos-Peloso [42], where they prove Fourier integral operator bounds using Littlewood-Paley smooth truncation techniques. This is an ongoing joint work with Rodolfo Torres.

Non-Pointwise Square Function Estimates for a Local T(b) Theorem

In some work with Grau de la Hérran and Oliveira [47], we proved a multilinear version of the local Tb theorem for square functions result of Hofmann [52]. The estimates assumed for the square function kernels in [47] are the same as the ones listed in Chapter 4 for kernels of type *SBLPK*. These pointwise estimate were relaxed to integral estimates by Grau [46] in the linear setting. There are various complications that arise in the multilinear case that we plan to address. This will be a joint work with Ana Grau de la Hérran and Lucas Oliviera.

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