Steady state solutions of the Ericksen-Leslie model

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Jointly with Weishi Liu and Yannan Shen

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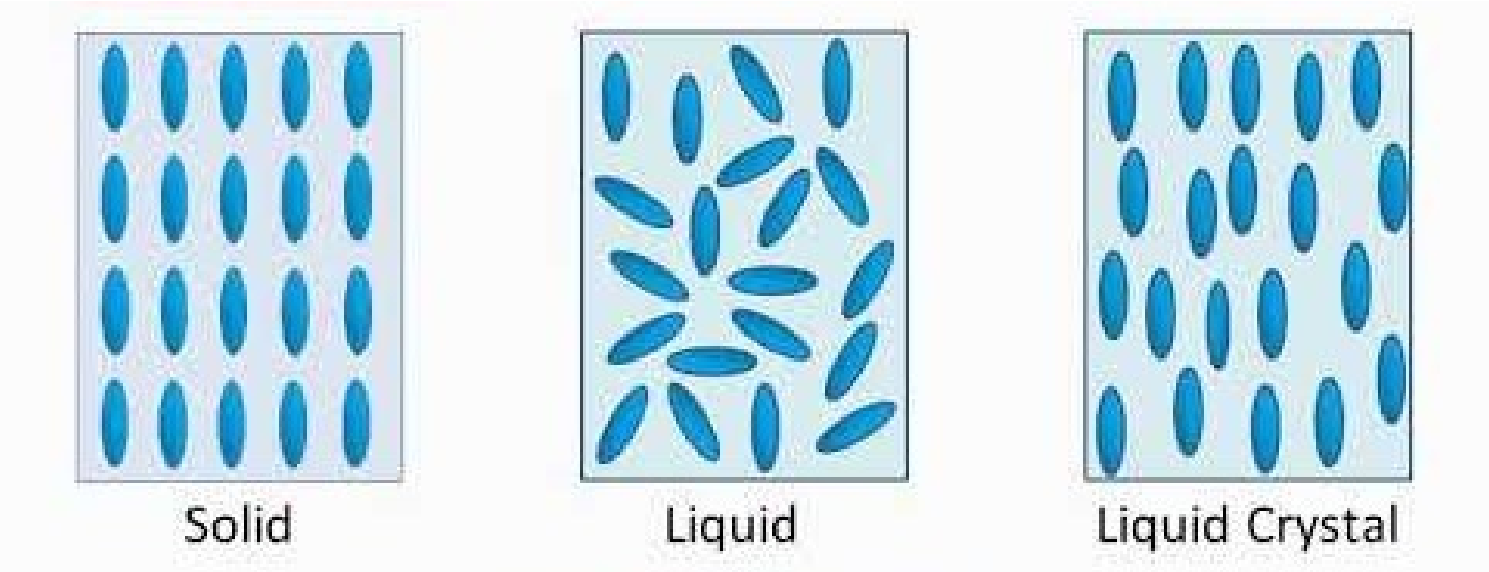
Background

What is a nematic liquid crystal?



Full Ericksen - Leslie Model

Oseen-Frank energy:

$$W(\mathrm{n}, \nabla \mathrm{n}) = K_1(div\, \mathrm{n})^2 + K_2(n \cdot curl\, \mathrm{n})^2 + K_3|\mathrm{n} \times curl\, \mathrm{n}|^2$$

Notation: $\dot{f} = f_t + \mathrm{v} \cdot \nabla f$.

$$\rho\, \dot{\mathbf{v}} + \nabla P = \nabla \cdot \sigma - \nabla \cdot \left(\frac{\partial W}{\partial \nabla \mathbf{n}} \otimes \nabla \mathbf{n} \right) \tag{1}$$

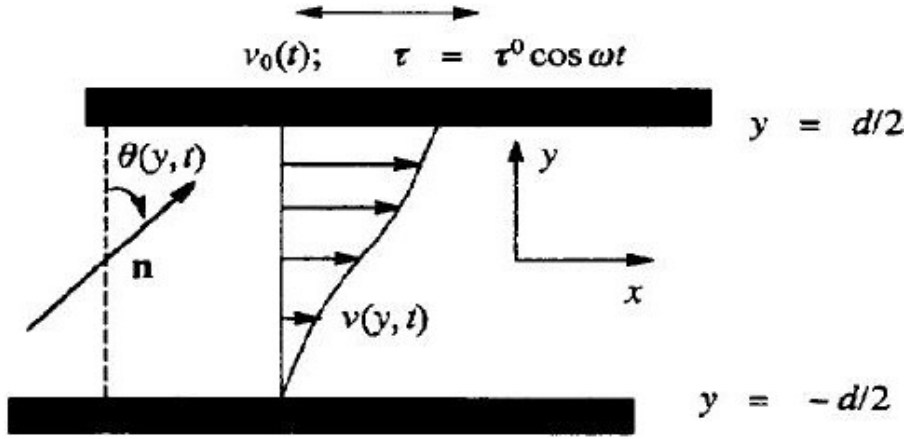
$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

$$\nu\, \ddot{\mathbf{n}} = \lambda\, \mathbf{n} - \frac{\partial W}{\partial \mathbf{n}} - \mathrm{G} + \nabla \cdot \left(\frac{\partial W}{\partial \nabla \mathbf{n}} \right) \tag{3}$$

$$\mathbf{n} \cdot \mathbf{n} = 1, \tag{4}$$

$$\sigma = \alpha_1(\mathbf{n}^T D\mathbf{n})\mathbf{n} \otimes \mathbf{n} + \alpha_2 N \otimes \mathbf{n} + \alpha_3 \mathbf{n} \otimes N + \alpha_4 D + \alpha_5 (D\mathbf{n}) \otimes \mathbf{n} + \alpha_6 \mathbf{n} \otimes (D\mathbf{n})$$

$$\mathrm{G} = \gamma_1 \dot{\mathrm{n}} - \gamma_1 \omega \mathrm{n} + \gamma_2 D\mathbf{n} \qquad D = \tfrac{1}{2}(\nabla v + \nabla^T v) \quad \omega = \tfrac{1}{2}(\nabla v - \nabla^T v)$$



As in Jiao, K. Huang, and W. Liu, 2022 for Poiseuille flow we have

$$\mathrm{n} = (\cos\phi(y,t), \sin\phi(y,t), 0)$$
$$\mathrm{v} = (v(y,t), 0, 0)$$

where ϕ is the angle of the director from the x-axis.

Model for Poiseuille Flow of Nematics

$$\rho\frac{\partial v}{\partial t} = \frac{\partial}{\partial y}\left(g(\phi)\frac{\partial v}{\partial y} + h(\phi)\frac{\partial \phi}{\partial t}\right),$$
$$\nu\frac{\partial^2\phi}{\partial t^2} + \gamma_1\frac{\partial \phi}{\partial t} = f(\phi)\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2}\frac{\partial f(\phi)}{\partial \phi}\left(\frac{\partial \phi}{\partial y}\right)^2 - h(\phi)\frac{\partial v}{\partial y}, \tag{5}$$

With boundary conditions,

$$\phi(0;t) = \phi_0 \qquad \phi(d;t) = \phi_d$$
$$v(0;t) = v_0 \qquad v(d;t) = v_d$$

Where f, g and h are periodic functions with period π and are given by

$$\begin{aligned} f(\phi) =& K_1\cos^2\phi + K_3\sin^2\phi, \\ g(\phi) =& \alpha_1\sin^2\phi\cos^2\phi + \frac{\alpha_5-\alpha_2}{2}\sin^2\phi + \frac{\alpha_6+\alpha_3}{2}\cos^2\phi + \frac{\alpha_4}{2}, \\ h(\phi) =& \alpha_3\cos^2\phi - \alpha_2\sin^2\phi = \frac{\gamma_1+\gamma_2\cos(2\phi)}{2}. \end{aligned} \tag{6}$$

Characteristics have been studied in a variety of cases in works by Wu, Xu, and C. Liu, 2013, Chen, T. Huang, and W. Liu, 2020, and many more

- (mid 90's-2015) Nonlinear Variational wave equation: Works on finite time singularity, global existence, uniqueness:
  J. Hunter, RT Glassey, A. Bressan, Y. Zheng, G. Chen, and Q. Zhang
  ...

- (2020 G. Chen, T. Huang, and W. Liu) Full Ericksen-Leslie (special case): Singularity formation and existence beyond singularity.

Set-Up

For $y \in (0, d)$,

$$\begin{aligned} \rho \frac{\partial v}{\partial t} &= \frac{\partial}{\partial y}\left(g(\phi)\frac{\partial v}{\partial y} + h(\phi)\frac{\partial \phi}{\partial t}\right), \\ \cancel{\nu \frac{\partial^2 \phi}{\partial t^2}} + \gamma_1 \frac{\partial \phi}{\partial t} &= f(\phi)\frac{\partial^2 \phi}{\partial y^2} + \frac{1}{2}\frac{\partial f(\phi)}{\partial \phi}\left(\frac{\partial \phi}{\partial y}\right)^2 - h(\phi)\frac{\partial v}{\partial y}, \end{aligned} \tag{7}$$

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With boundary conditions,

$$\begin{aligned} \phi(0;t) &= \phi_0 \qquad \phi(d;t) = \phi_d \\ v(0;t) &= v_0 \qquad v(d;t) = v_d \end{aligned}$$

Stationary System

We look for stationary shear flow of form $v(y,t) = \bar{v}(y),\ \phi(y,t) = \bar{\phi}(y)$, where from Ericksen-Leslie system (8), it satisfies, for $0 < y < d$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}y}\left(g(\bar{\phi})\frac{\mathrm{d}\bar{v}}{\mathrm{d}y}\right) = 0, \\ &f(\bar{\phi})\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}y^2} + \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}\phi}(\bar{\phi})\left(\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}y}\right)^2 - h(\bar{\phi})\frac{\mathrm{d}\bar{v}}{\mathrm{d}y} = 0, \end{aligned} \tag{9}$$

Stationary System

We may rewrite as a system of first-order equations. From the first equation in the system given by (9)

$$\bar{v}_y = \frac{a}{g(\bar{\phi})} \tag{10}$$

where a is a constant to be determined later

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From the second equation in the system given by (9) and introducing $\eta(y,t)$ with stationary representation $\bar{\eta}(y)$

$$\begin{aligned} \bar{\phi}_y &= \frac{\bar{\eta}}{f(\bar{\phi})} \\ \bar{\eta}_y &= \frac{f_\phi(\bar{\phi})}{2f^2(\bar{\phi})}\bar{\eta}^2 + a\frac{h(\bar{\phi})}{g(\bar{\phi})} \end{aligned} \tag{11}$$

Hamiltonian Structure

(11) is a Hamiltonian System

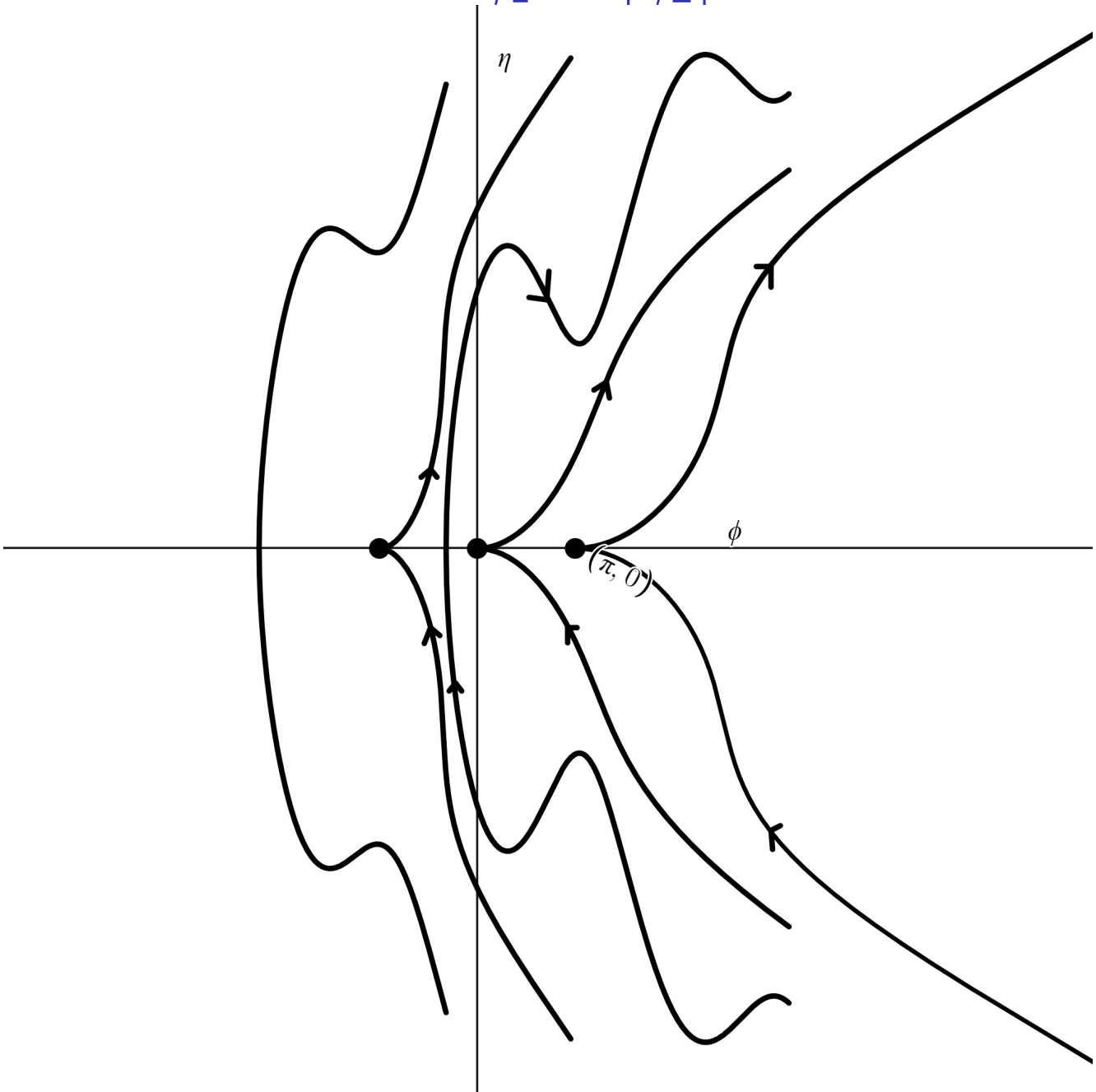
$$\phi' = \frac{\partial H}{\partial \eta}(\phi, \eta) \text{ and } \eta' = -\frac{\partial H}{\partial \phi}(\phi, \eta)$$

with

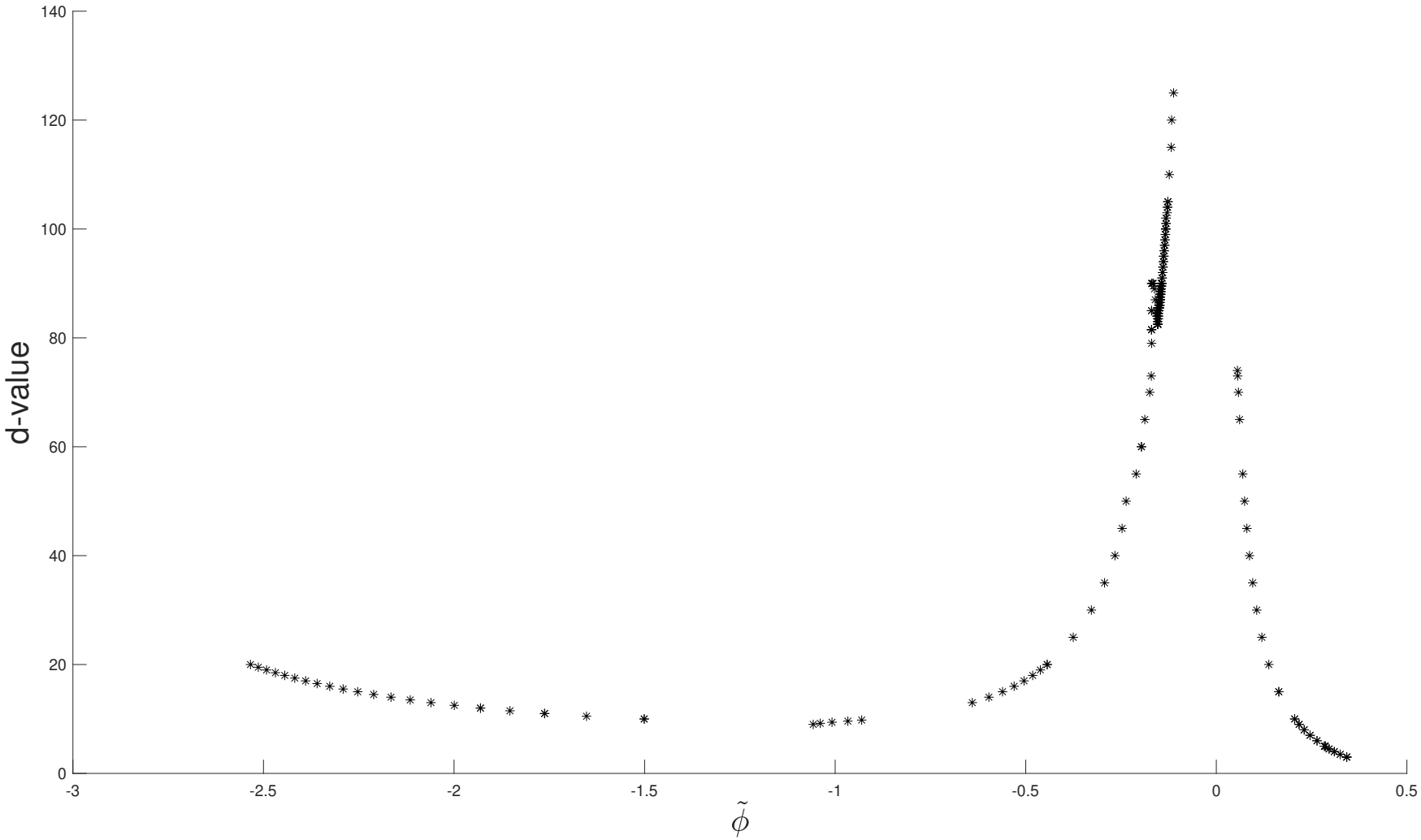
$$H(\phi, \eta) = \frac{\eta^2}{2f(\phi)} - \frac{a}{2}G(\phi),$$

$$\text{where } G(\phi) := \int_{\phi_0}^{\phi} \frac{2h(s)}{g(s)}\, ds$$

Phase Portrait for $\gamma_1 = |\gamma_2|$



Multiple Solutions & Parameter d $\gamma_1 = |\gamma_2|$



1 - 1 Correspondence for Stationary Solutions to $\tilde{\phi}$

Denoting

$$G(\phi) := \int_{\phi_0}^{\phi} \frac{2h(s)}{g(s)}\, ds \qquad \beta = G(\tilde{\phi})$$

And setting

$$D(\beta) = \beta^2 \int_0^1 \frac{f^{1/2}(Q(\beta t))Q'(\beta t)}{\sqrt{1-t}}\, dt \int_0^1 \frac{f^{1/2}(Q(\beta t))Q'(\beta t)}{g(Q(\beta t))\sqrt{1-t}}\, dt$$

There is a one-to-one correspondence between the solutions β of $D(\beta) = \dfrac{d(v_d - v_0)}{4}$ and the set of steady states of the BVP given in (9)

Spectral Stability

Now consider a perturbation of the stationary shear flow, $v(y,t) = \bar{v}(y) + \epsilon w(y,t)$, $\phi(y,t) = \bar{\phi}(y) + \epsilon\psi(y,t)$.

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At order of $O(\epsilon)$, $w(y,t)$, $\psi(y,t)$ satisfies

$$\begin{aligned}
\rho w_t &= \partial_y\left[g_\phi(\bar{\phi})\bar{v}_y\psi + g(\bar{\phi})w_y + h(\bar{\phi})\psi_t\right], \\
\gamma_1\psi_t &= f(\bar{\phi})\psi_{yy} + f_\phi(\bar{\phi})\bar{\phi}_{yy}\psi + f_\phi(\bar{\phi})\bar{\phi}_y\psi_y + \frac{1}{2}f_{\phi\phi}(\bar{\phi})\bar{\phi}_y^2\psi \\
&\quad - h(\bar{\phi})w_y - h_\phi(\bar{\phi})\bar{v}_y\psi.
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with $w(y,t) = e^{\lambda t}\tilde{w}(y)$ and $\psi(y,t) = e^{\lambda t}\tilde{\psi}(y)$, we find the spectral stability satisfies

$$\begin{aligned}
\rho\lambda\tilde{w} &= \partial_y\left[g_\phi(\bar{\phi})\bar{v}_y\tilde{\psi} + g(\bar{\phi})\tilde{w}_y + \lambda h(\bar{\phi})\tilde{\psi}\right],\\
\gamma_1\lambda\tilde{\psi} &= f(\bar{\phi})\tilde{\psi}_{yy} + f_\phi(\bar{\phi})\bar{\phi}_{yy}\tilde{\psi} + f_\phi(\bar{\phi})\bar{\phi}_y\tilde{\psi}_y + \frac{1}{2}f_{\phi\phi}(\bar{\phi})\bar{\phi}_y^2\tilde{\psi} \\
&\quad - h(\bar{\phi})\tilde{w}_y - h_\phi(\bar{\phi})\bar{v}_y\tilde{\psi}.
\end{aligned} \tag{13}$$

Eigenvalue Problem

Letting $V = \begin{bmatrix} \tilde{w} & \tilde{\psi} \end{bmatrix}^T$,

$$\lambda \mathcal{L} V = \mathcal{R} V \tag{14}$$

with

$$\mathcal{L} = \begin{bmatrix} \rho & -(h_\phi(\bar{\phi})\bar{\phi}_y + h(\bar{\phi})\partial_y) \\ 0 & \gamma_1 \end{bmatrix},$$

$$\mathcal{R} = \begin{bmatrix} g_\phi(\bar{\phi})\bar{\phi}_y\partial_y + g_\phi(\bar{\phi})\partial_y^2 & \left(g_\phi(\bar{\phi})\bar{v}_y\right)_y + g_\phi(\bar{\phi})\bar{v}_y\partial_y \\ -h(\bar{\phi})\partial_y & f(\bar{\phi})\partial_y^2 + f_\phi(\bar{\phi})\bar{\phi}_{yy} + f_\phi(\bar{\phi})\bar{\phi}_y\partial_y + \frac{1}{2}f_{\phi\phi}(\bar{\phi})\bar{\phi}_y^2 - h_\phi(\bar{\phi})\bar{v}_y \end{bmatrix}$$

Comments on Spectrum for Case $\gamma_1 = |\gamma_2|$

For the case where the solution ϕ is constant we see

- if ϕ is an equilibrium point (πn for $-\gamma_2 = \gamma_1$ or $\frac{\pi}{2} + \pi n$ for $\gamma_2 = \gamma_1$) then $\lambda_0 = 0, \lambda_k = \dfrac{-4K_1}{\gamma_1 \Delta y} \sin^2\left(\dfrac{k\pi}{2(n+1)}\right)$

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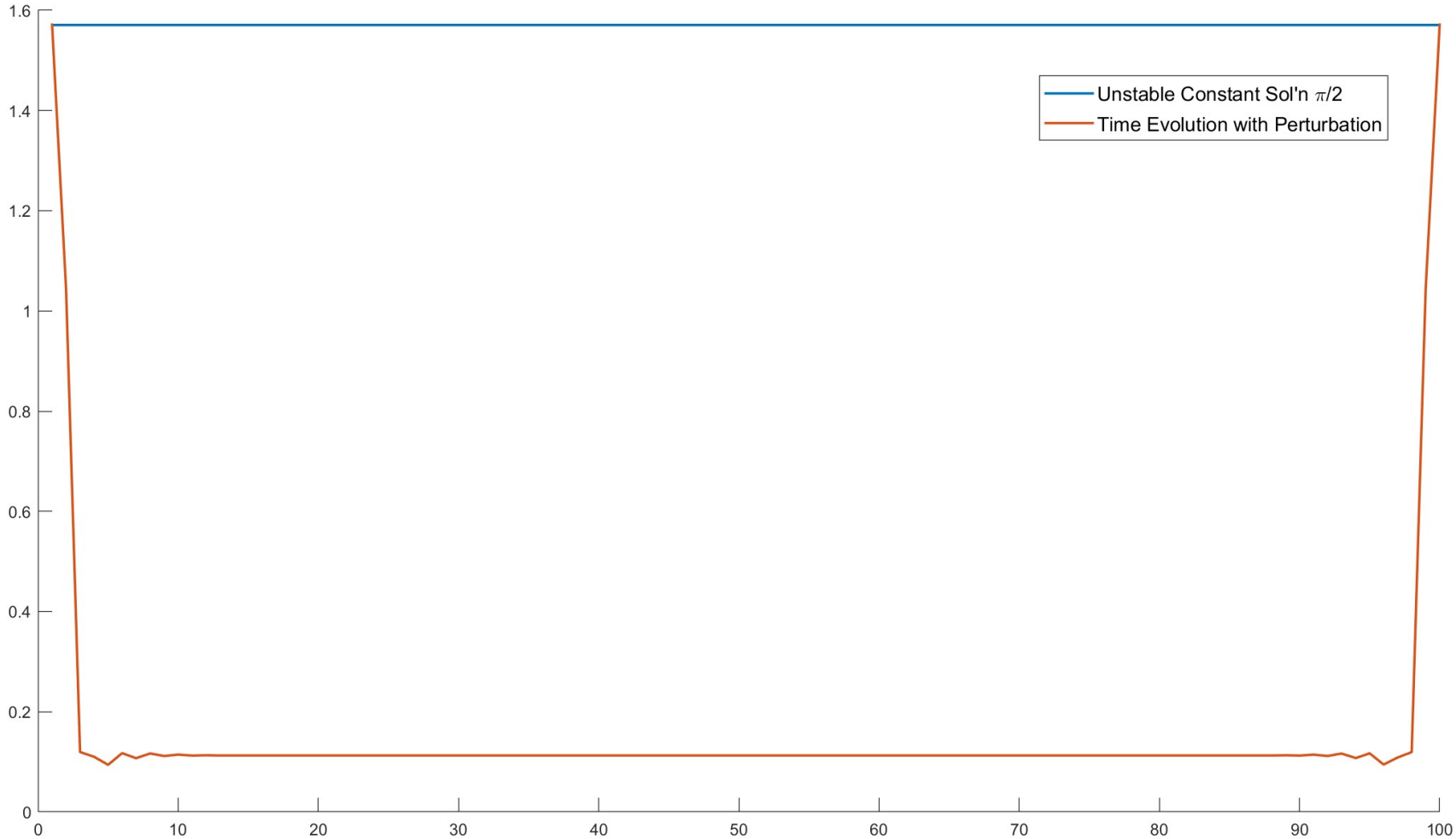

Unstable Constant Sol'n π/2
Time Evolution with Perturbation
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100

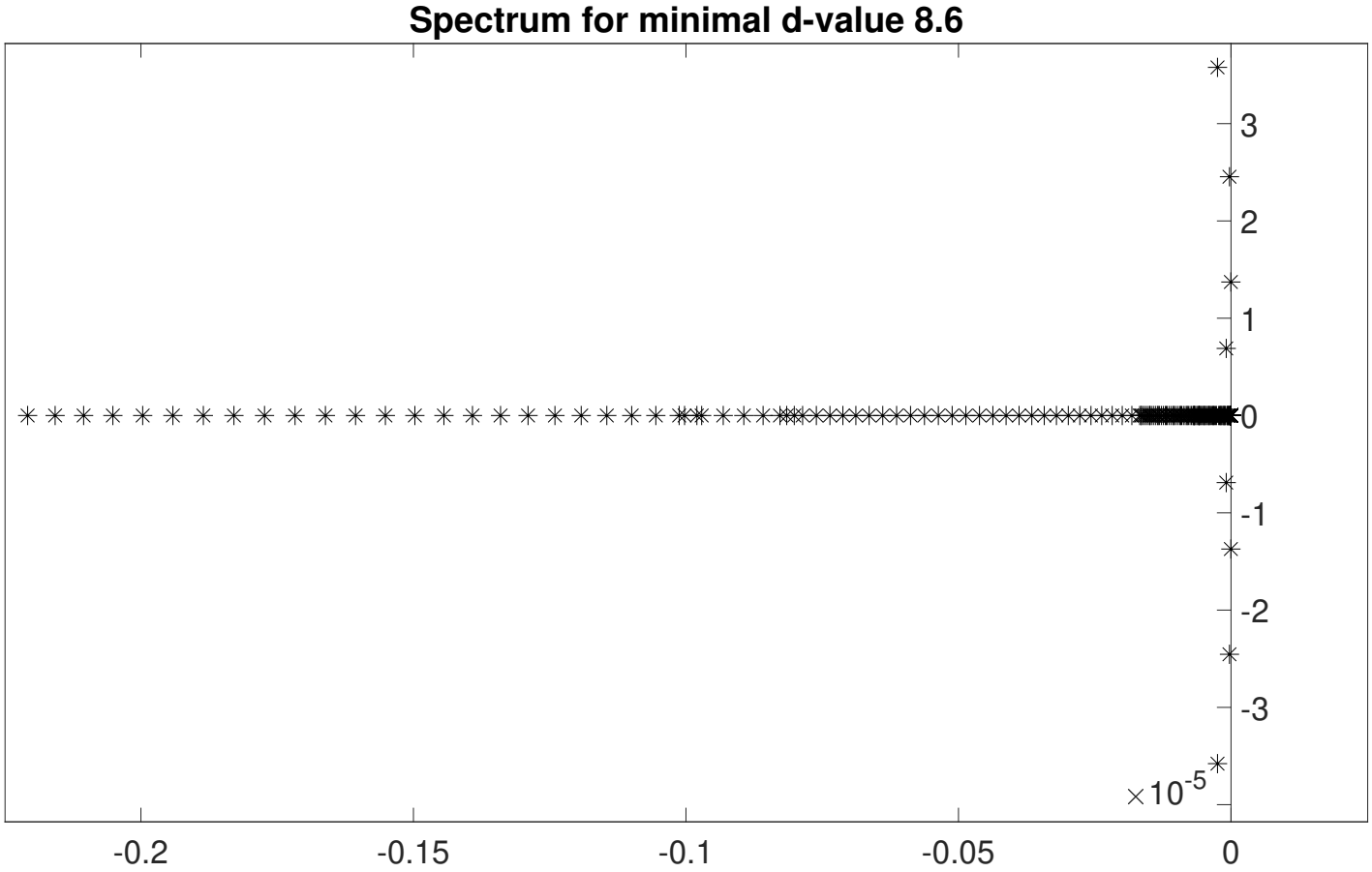


Figure: The spectrum associated to minimal d - value that can be found in the range of $\tilde{\phi}$ values between $-\pi$ and 0.

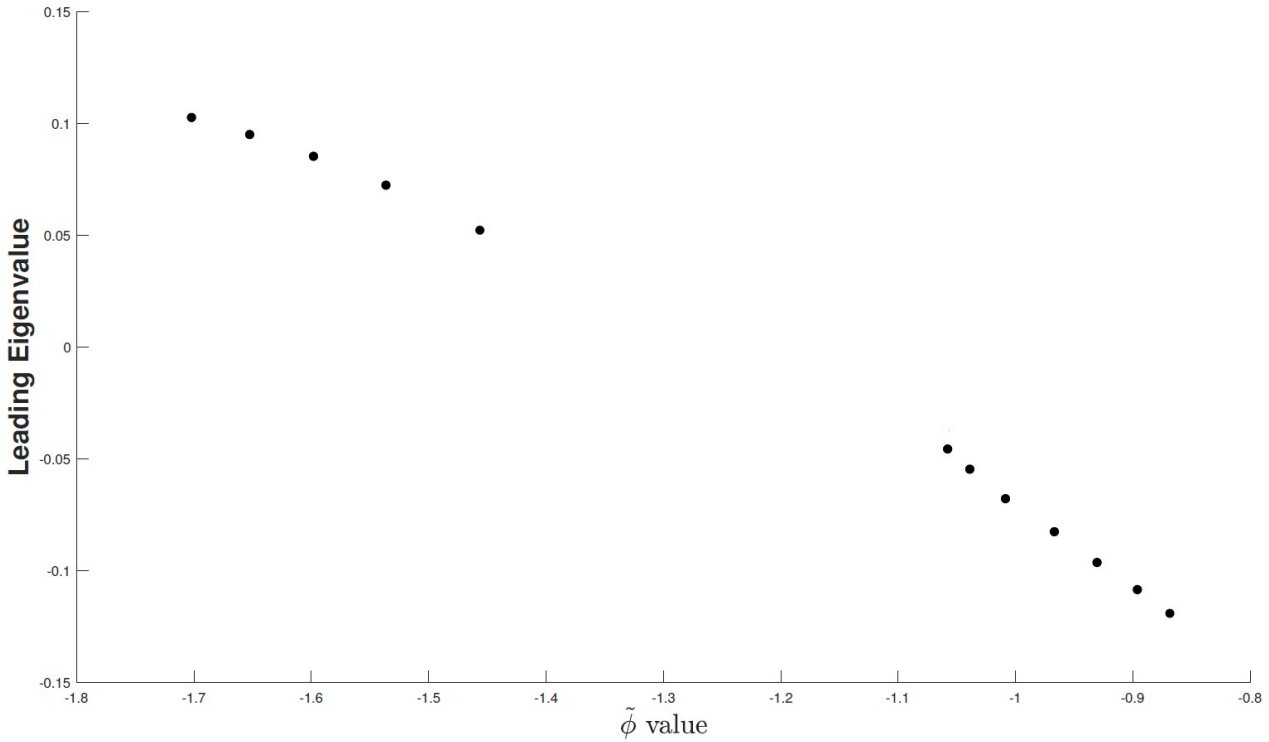


Figure: $\tilde{\phi}$ values versus the leading eigenvalue. We can see as we cross $\tilde{\phi}_c$ that we go from positive to negative

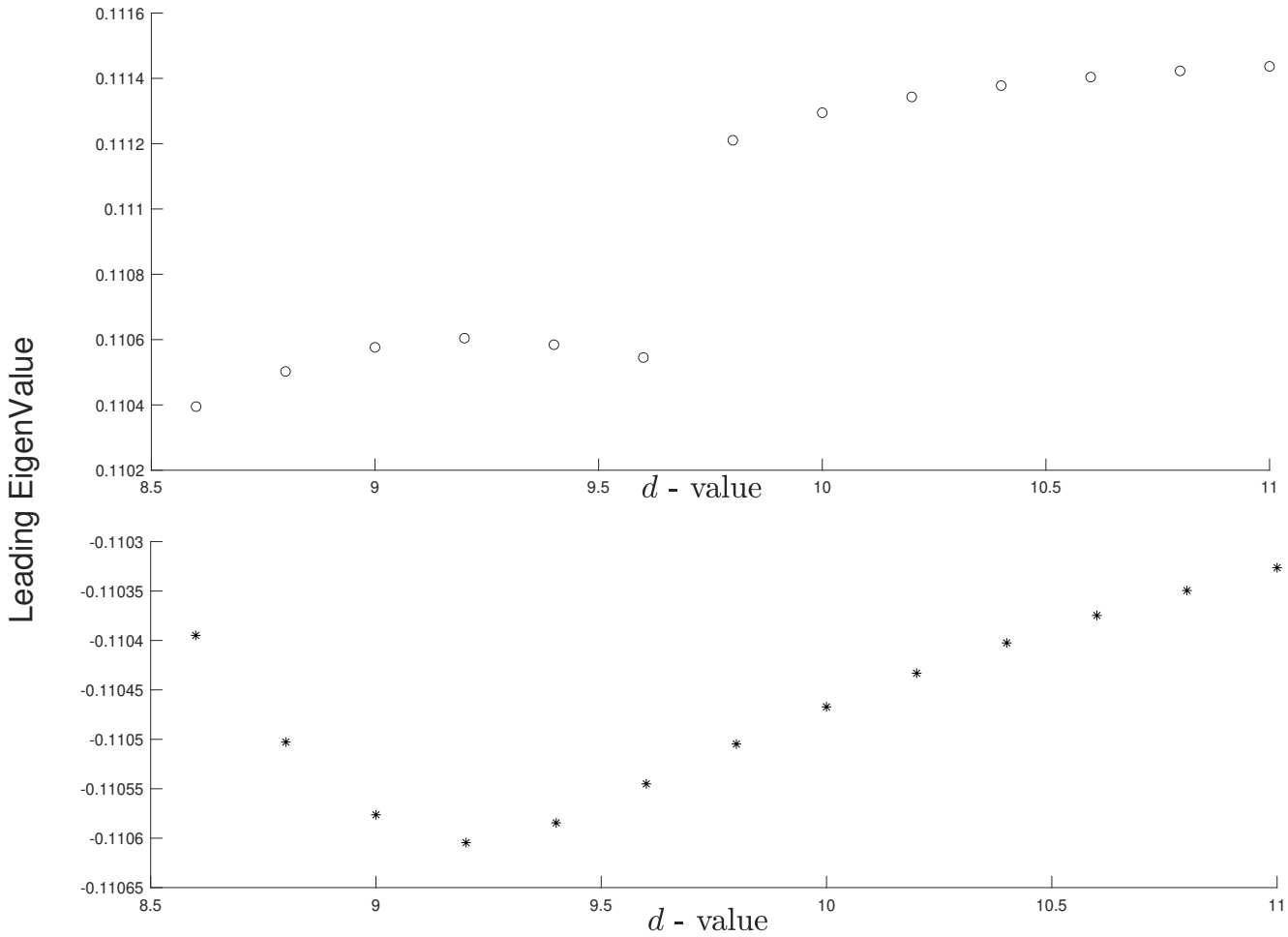
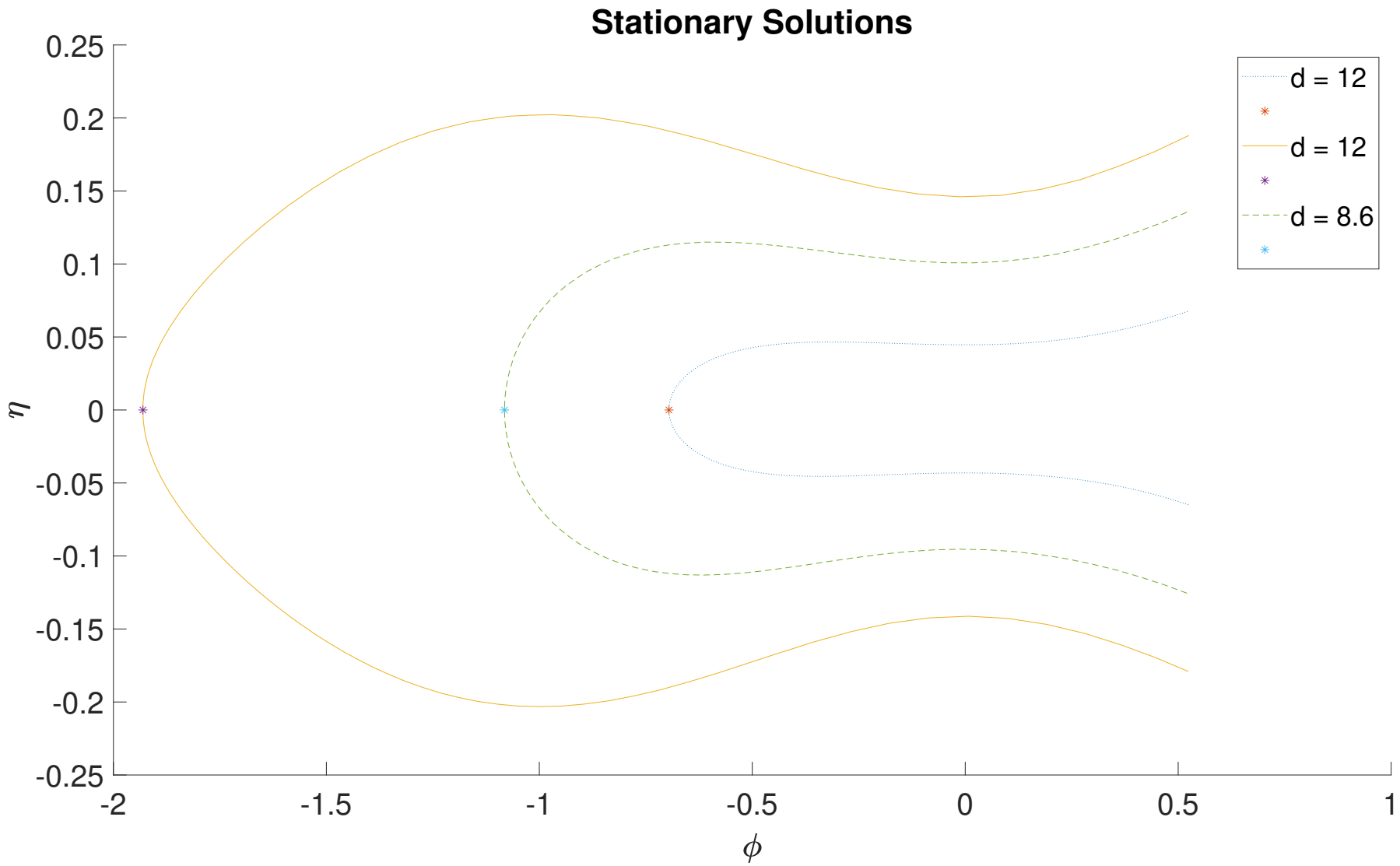


Figure: The leading eigenvalues as we vary the parameter d where we can see a positive eigenvalue for one stationary solution and a negative for the other.

Perturbation of d



Time Evolution

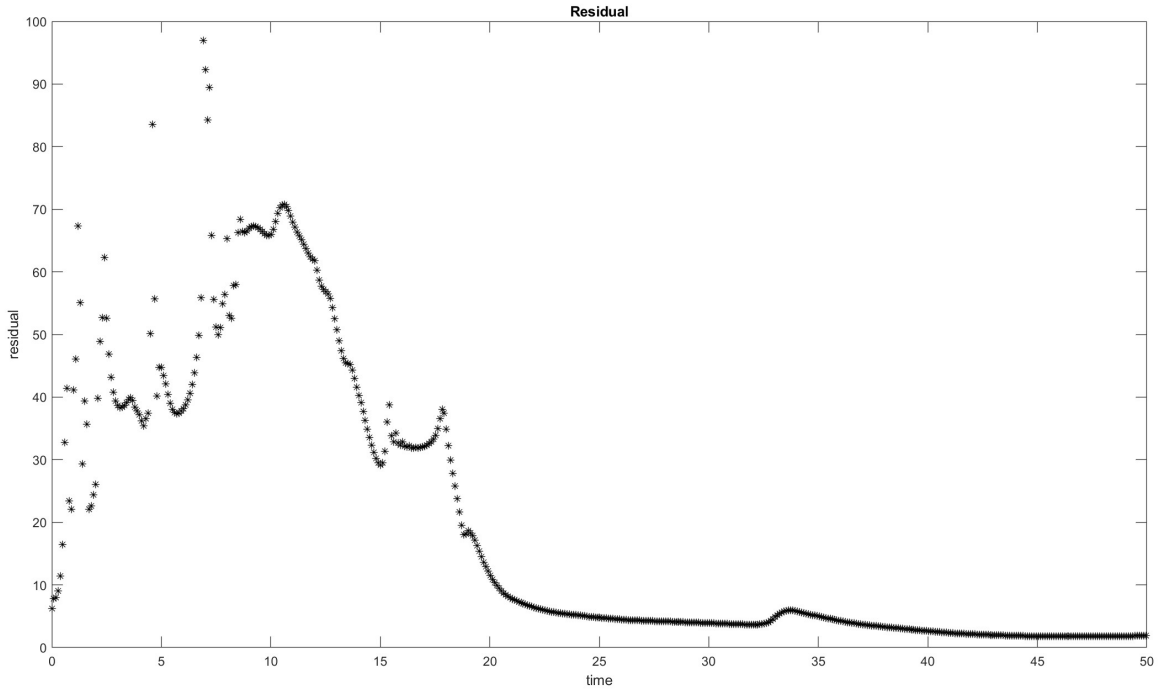


Figure: The two-norm measuring the difference between the time evolution of the unstable solution and the stable solution.

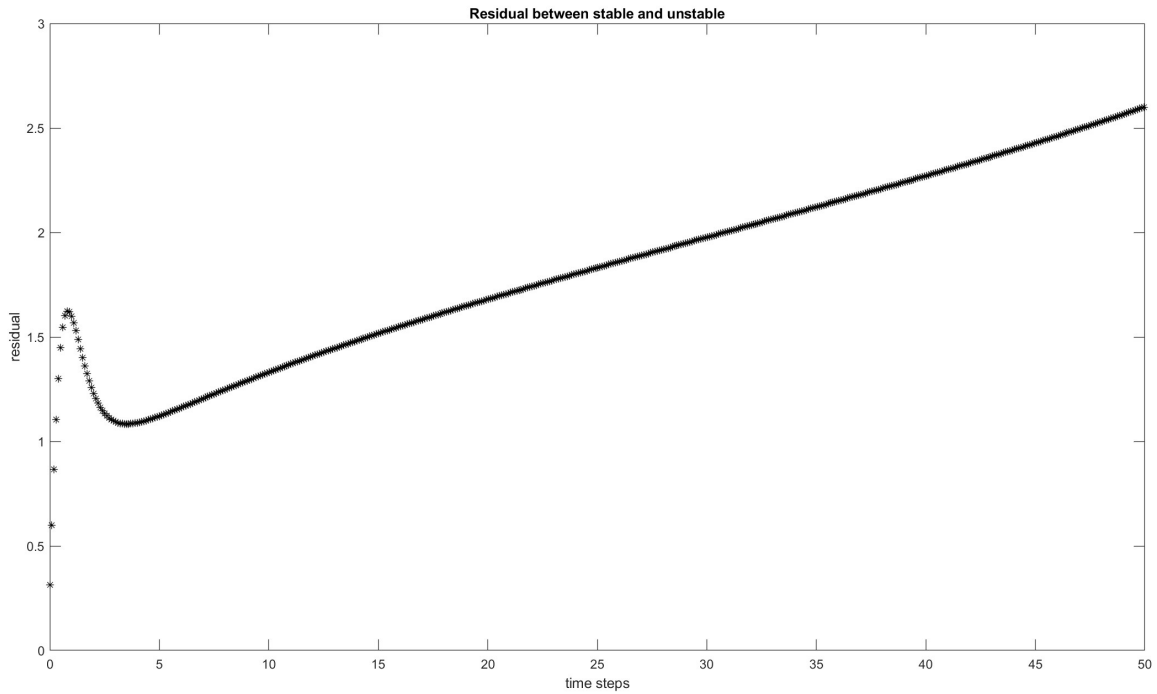
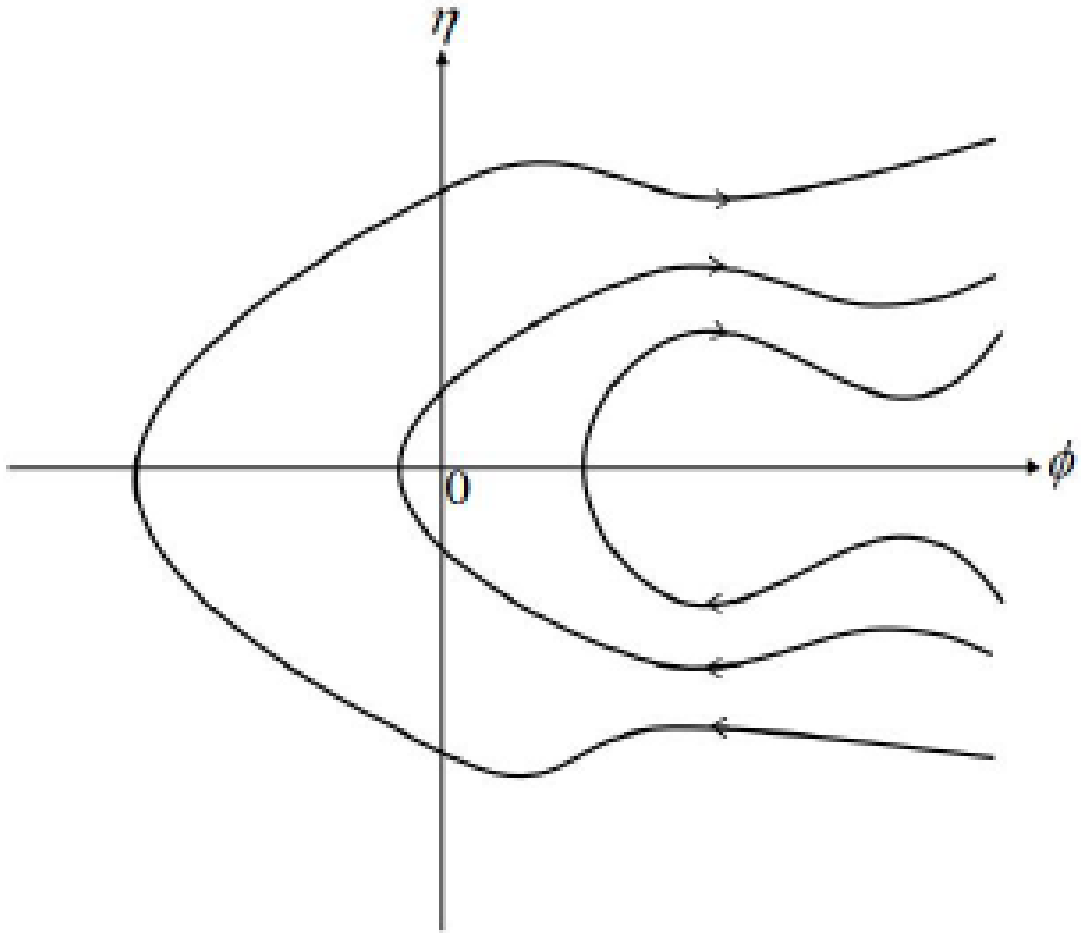


Figure: The two-norm measuring the difference between the time evolution of the unstable solution and the stable solution with the perturbation in the opposing direction.

Phase Portrait for $\gamma_1 > |\gamma_2|$



Continuation

Recall we started with the case

$$\gamma_1 = |\gamma_2|$$

and then moved to the case

$$\gamma_1 > |\gamma_2|$$

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and then moved to the case

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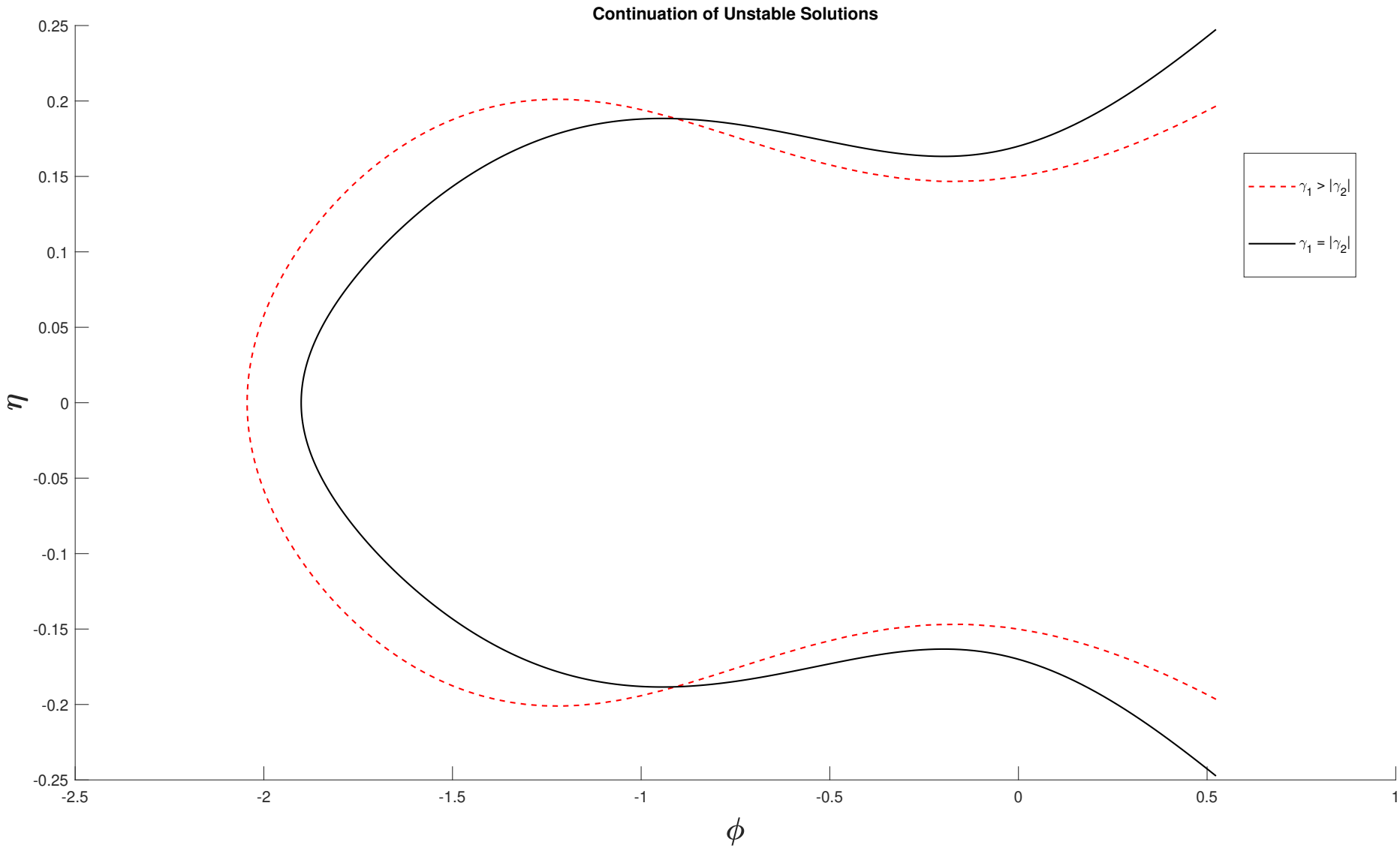
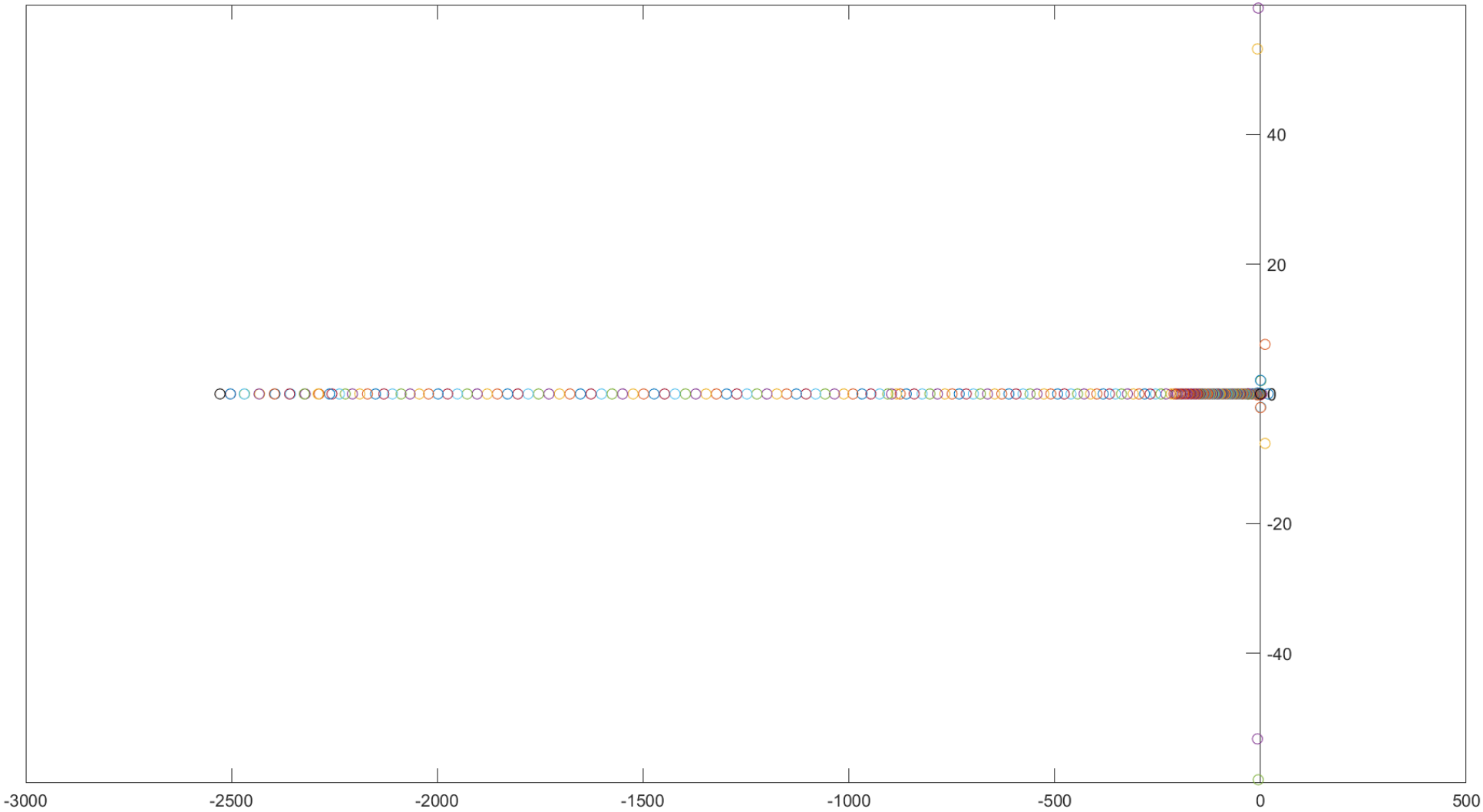
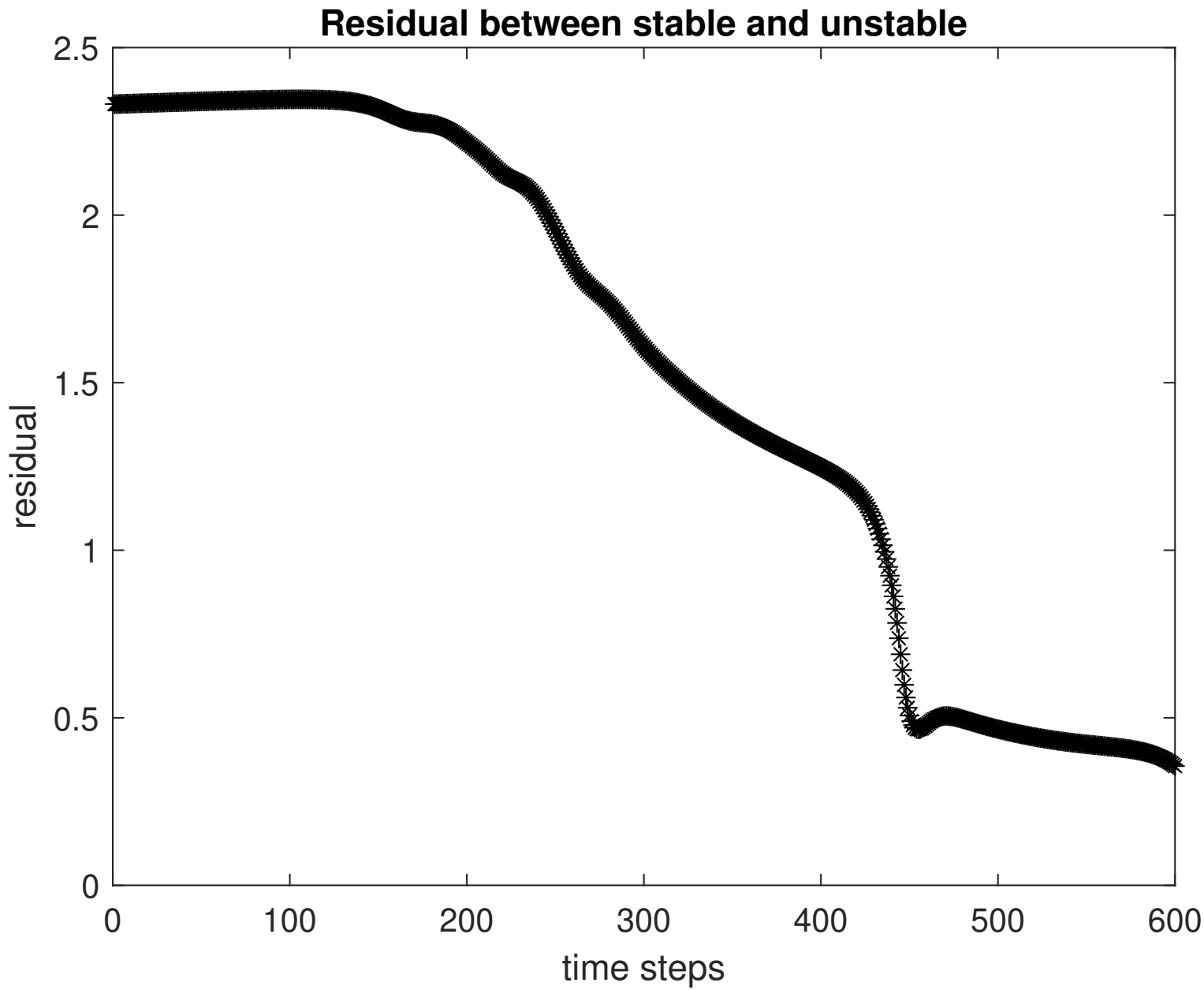
The way we move between cases is through a continuation over the parameters γ_1 and γ_2

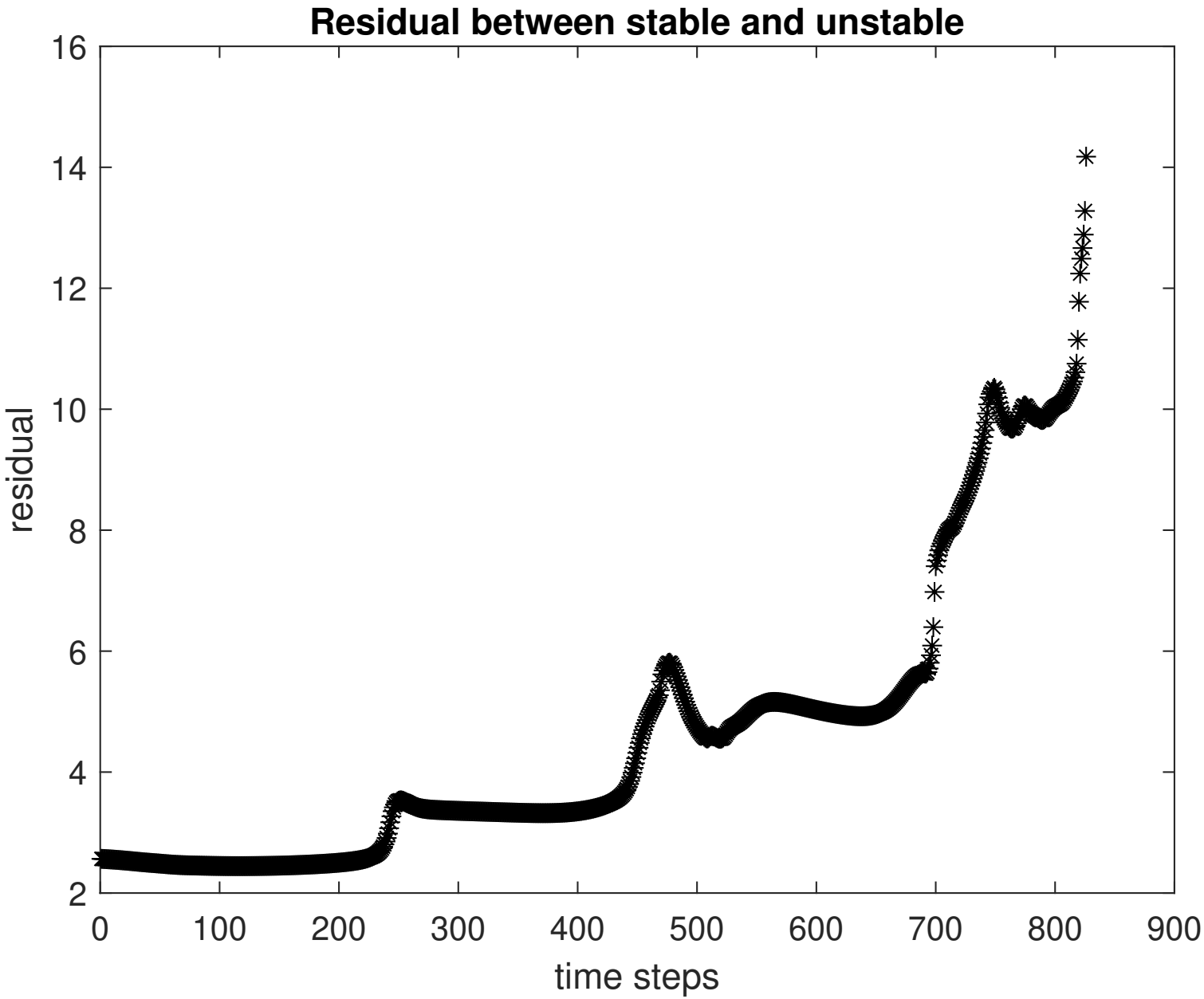
Continuation



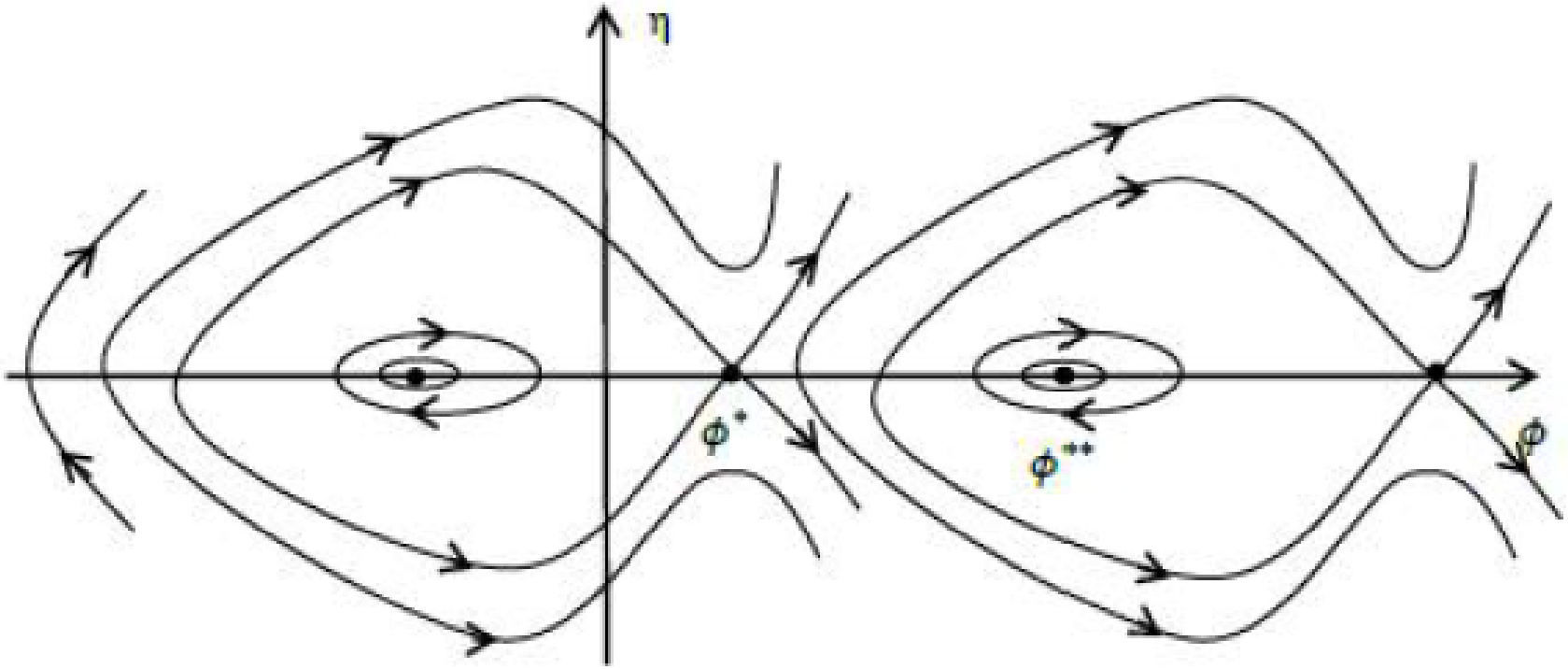

40
20
0
-20
-40
-3000
-2500
-2000
-1500
-1000
-500
0
500

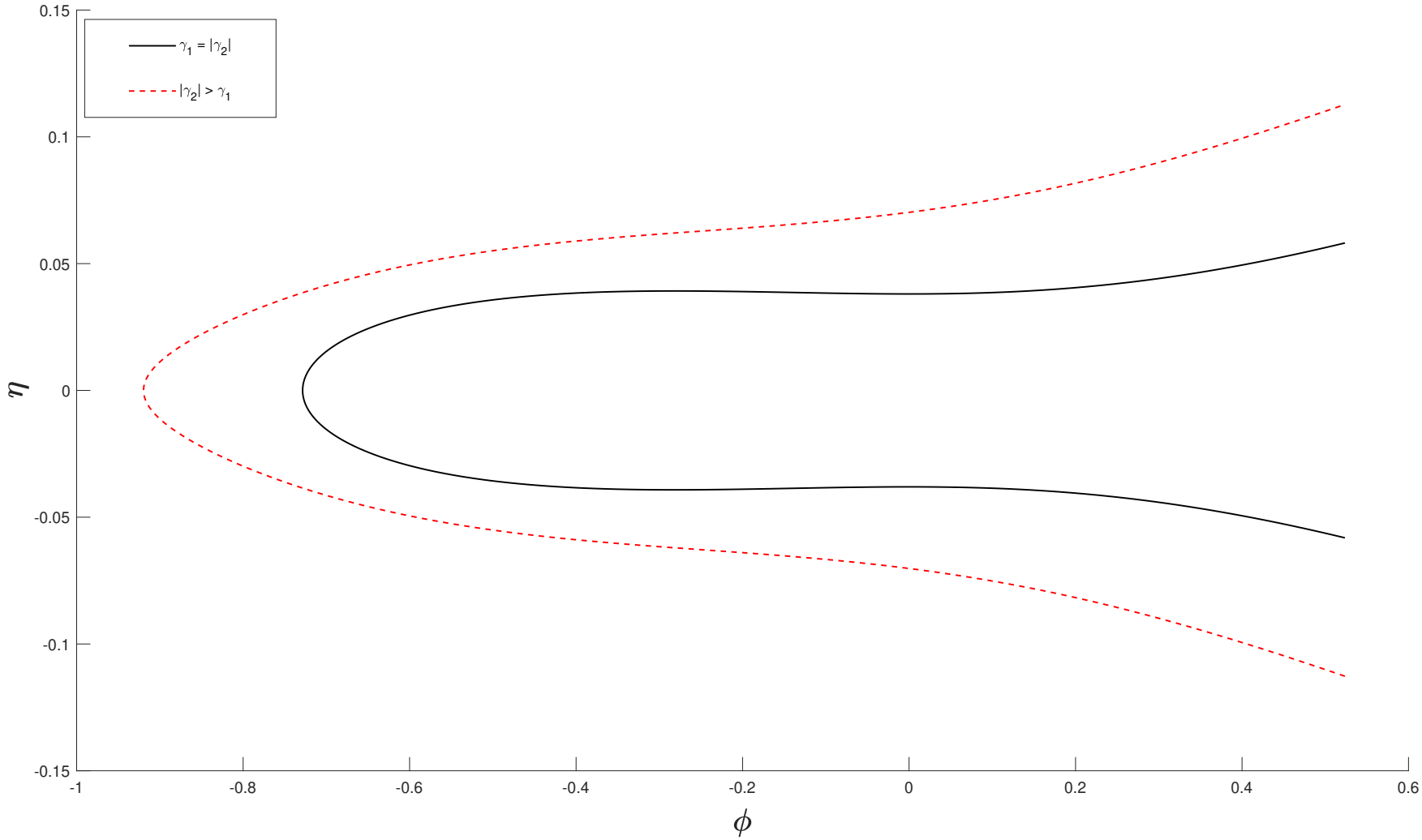
Time Evolution




Residual between stable and unstable
residual
time steps
16
14
12
10
8
6
4
2
0
100
200
300
400
500
600
700
800
900

Phase Portrait for $|\gamma_2| > \gamma_1$




$\gamma_1 = |\gamma_2|$
$|\gamma_2| > \gamma_1$
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
η
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
ϕ

Thank you for your attention

Thank you for your attention
Any questions?

- Chen, Geng, Tao Huang, and Weishi Liu (2020). "Poiseuille flow of nematic liquid crystals via the full Ericksen–Leslie model". In: *Archive for Rational Mechanics and Analysis* 236.2, pp. 839–891.
- Jiao, Jia, Kaiyin Huang, and Weishi Liu (2022). "Stationary Shear Flows of Nematic Liquid Crystals: A Comprehensive Study via Ericksen–Leslie Model". In: *Journal of Dynamics and Differential Equations* 34.1, pp. 239–269.
- Wu, Hao, Xiang Xu, and Chun Liu (2013). "On the general Ericksen–Leslie system: Parodi's relation, well-posedness and stability". In: *Archive for Rational Mechanics and Analysis* 208.1, pp. 59–107.