Steady-State Poisson-Nernst-Planck Systems: Asymptotic expansions and applications to ion channels

By

Nicole Abaid

Submitted to the Department of Mathematics and the
Faculty of the Graduate School of the University of Kansas
in partial fulfillment of the requirements for the degree of

Master of Arts

Weishi Liu, Chairperson

Committee members

Erik Van Vleck

Myunghyun Oh

Date defended:

The Dissertation Committee for Nicole Abaid certifies
that this is the approved version of the following thesis:

Steady-State Poisson-Nernst-Planck Systems: Asymptotic expansions and applications to ion channels

Committee:

____

Weishi Liu, Chairperson

____

Erik Van Vleck

____

Myunghyun Oh

Date approved: ______________________________

Chapter 1

Introduction

In this thesis, we will discuss the matched asymptotic expansions method of solving a singularly perturbed system and apply this method to find an approximate solution to the steady-state Poisson-Nernst-Planck system.

1.1 The problem and mathematical model

The Poisson-Nernst-Plank (PNP) system is related to the function of cells. Cells are protected from the world around them by their membranes. Communication with each other and their surroundings takes place by ions that enter the cell through ion channels embedded in this membrane. The ion channels regulate the movement of specific charged particles into and out of the cell by providing a path through the membrane. These channels are typically gated, opening for a stimulus like electric potential or particular chemical. We will consider the problem of the flow of ions through a channel which is open. As was stated, since the interaction between the ions and the channel is a function of the charges and concentrations of the particles, we can consider equations of electrodiffusion when examining their movement.

Important properties of ion channels can be described by a steady state Poisson-Nernst-Plank system for electrodiffusion. The system is comprised of two types of equations. The Nernst-Planck equations model the concentration of ions through the balance of concentration gradient and electric potential, and the Poisson equation determines the electric potential from the concentration of ions. The solution to the system gives a relation between the current and electric potential of the ions in the channel, called the I-V curve.

Limiting the steady-state system of the PNP system to only two types of ions gives the system below:

$$\begin{aligned}
&\varepsilon^2 \frac{\partial^2 \phi}{dx^2} = -(\alpha c_1 - \beta c_2 + Q(x)), \\
&\frac{dc_1}{dx} + \alpha c_1 \frac{d\phi}{dx} = -J_1, \\
&\frac{dc_2}{dx} - \beta c_2 \frac{d\phi}{dx} = -J_2, \\
&\frac{dJ_1}{dx} = 0, \frac{dJ_2}{dx} = 0,
\end{aligned} \tag{1.1}$$

for $x \in [0,1]$ in a channel of normalized length, where ϕ is the electric potential at a point in the channel, c_1 and c_2 are the concentrations of the two ions, $Q(x)$ is the permanent charge in the channel, J_1 and J_2 are the flux of the two ions and α and β are the charge valences of the ions.

We will consider the following boundary conditions:

$$\phi(0) = \nu_0,\ c_1(0) = L_1,\ c_2(0) = L_2,$$
$$\phi(1) = 0,\ c_1(1) = R_1,\ c_2(1) = R_2,$$

where L_1, L_2, R_1, R_2 are constants, and ν_0 is the initial electric potential in the channel. For simplicity, we accumulate the initial potential on the boundary at $x = 0$ which leaves zero initial potential at $x = 1$.

Notice the singular parameter ε is very small, since it is defined as the reciprocal of the large Debye number. Thus, this PNP system can be viewed as a singularly perturbed boundary value problem (BVP). Typically, singularly perturbed BVP's are characterized by a solution with two distinguished behaviors, called singular layers and regular layers. Singular layers can occur near the boundary, where they are called boundary layers, or at interior points, where they are called interior layers. Regular layers lie away from singular layers. To solve the system using asymptotic expansions, one must first find the solution on each layer separately by scaling time as a outer and inner variable using the singular parameter ε. Then one matches the layers together to get a smooth solution to the problem.

We want to approximate solutions to this system explicitly for the J_1 and J_2, which will give the flux. Using this result, we can find the current as the difference of the fluxes in the ion channel, and write a relationship between the current and electric potential, which we call ν_0. This relationship, called the I-V curve, is a function of $V = \nu_0$ for fixed L and R. The I-V curve is of interest in the field of biology. It should be noted that the I-V curve has been observed to be nonlinear.

1.2 Previous results

An early related treatment of the steady-state PNP system can be found in [2]. In this paper, the authors discuss the system with $Q = 0$ and obtain a zeroth-order approximation to a solution, then discretize the system and use numerics to characterize their approximation.

A response to this paper comes in [5], where the system is studied using a geometric approach. In this paper, the author tackles the same problem that will be studied in this thesis, with two types of ions and zero permanent charge. First, the author considers the slow and fast systems, known in this thesis as the inner and outer systems, and their "limiting" systems. These are the systems when the parameter is zero. The slow manifold, the solution set to the limiting slow system, turns out to be normally hyperbolic, so there are singular layer at both boundaries. With the help of a complete set of integrals for the limiting fast system, the boundary layer can be completely described. To find the singular orbit of the solution, the author first proves that a unique solution exists by appealing to the fact that these invariant manifolds at the boundaries intersect transversely [7]. By the Exchange Lemma, the existence and uniqueness of a solution near the singular orbit is established. The zeroth order solution that appears in this paper will be rederived in this thesis.

It should be noted that both the papers above calculate solutions while assuming the permanent charge in the channel to be identically zero. The last paper consulted, [3], appears after [5] and solves the system with a constant permanent charge inside the channel and a charge of zero at the endpoints $x = 0$ and $x = 1$. The authors then solve the PNP system using the same geometric approach as in [5]. In this case, there are 3 subintervals: near $x = 0$, between $x = 0$ and $x = 1$ and near $x = 1$. In each of these subintervals, one must consider the boundary value problem, with the slow orbit and two fast orbits that must be connected to find the singular orbit by the same method as in [5].

1.3 Thesis

The purpose of this thesis is to find a higher order I-V relation for the steady-state PNP system, assuming two types of ions and zero permanent charge in the channel. We will use the asymptotic expansion method to recover the known zeroth order solution for the system, then use the same technique to find the first order I-V relation. In the condition of electron-neutrality, we will derive the second and third order I-V relations.

Chapter 2

The method of matched asymptotic expansions: an example

In this chapter, we first describe the general procedure of matched asymptotic expansions and then demonstrate the procedure using a simple example from [4]. Techniques from [6] were also consulted.

2.1 Outer solution, Inner solution, and Matching

Consider a general singularly perturbed system:

$$\begin{aligned} \varepsilon\dot{x}(\tau) &= f(x,y;\varepsilon), \\ \dot{y}(\tau) &= g(x,y;\varepsilon). \end{aligned} \tag{2.1}$$

It is called the outer system, and its variable, outer. By scaling time $t = \frac{\tau}{\varepsilon}$, we obtain the so-called inner system

$$\begin{aligned} X'(t) &= f(X,Y;\varepsilon), \\ Y'(t) &= \varepsilon g(X,Y;\varepsilon), \end{aligned} \tag{2.2}$$

where $X(t) = x(\varepsilon t)$ and $Y(t) = y(\varepsilon t)$. For the zeroth and first order solutions of this system, we consider solutions of the form

$$x(\tau;\varepsilon) = x_0(\tau) + \varepsilon x_1(\tau),\ y(\tau;\varepsilon) = y_0(\tau) + \varepsilon y_1(\tau)$$

to the outer system (2.1), and of the form

$$X(t;\varepsilon) = X_0(t) + \varepsilon X_1(t),\ Y(t;\varepsilon) = Y_0(t) + \varepsilon Y_1(t)$$

to the inner system (2.2).

The method of matched asymptotic expansions generally involves three steps. First, substitute the outer expansion $(x(\tau;\varepsilon), y(\tau;\varepsilon))$ into the outer system (2.1). Then consider the terms of like order and solve. Next, substitute the inner expansion $(X(t;\varepsilon), Y(t;\varepsilon))$ into the inner system (2.2). Solve the like-ordered terms, and impose the boundary values on the inner solutions. When we solve both systems, we get the inner and outer solutions with undetermined constants. Lastly, we match the inner and outer solutions to solve for these undetermined constants. We do this by substituting εt for τ in the outer solution, and rewriting it as some new functions

$$x(\varepsilon t;\varepsilon) = \tilde{x}_0(t) + \varepsilon \tilde{x}_1(t) + \mathscr{O}(\varepsilon^2),\ y(\varepsilon t;\varepsilon) = \tilde{y}_0(t) + \varepsilon \tilde{y}_1(t) + \mathscr{O}(\varepsilon^2)$$

and match $(\tilde{x}_j(t), \tilde{y}_j(t))$ with $(X_j(t), Y_j(t))$ for $j = 1, 2$ to solve for the undetermined constants.

2.2 An example

Consider the singularly perturbed two-point boundary value problem from [4]:

$$\varepsilon \frac{d^2u}{dx^2} + \frac{du}{dx} - a - 2bx = 0, u(0) = 0,\ u(1) = 1.$$

The exact solution is

$$u(x;\varepsilon) = (1 - a - b + 2\varepsilon b)\frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} + ax + bx^2 - 2\varepsilon bx.$$

We will illustrate the procedure of finding asymptotic expansions, and compare our result to this exact solution.

The original system above is called the outer system. We will use this notation for clarity.

$$\begin{aligned} &\varepsilon \ddot{u}(x) = -\dot{u}(x) + a + 2bx, \\ &u(0) = 0,\ u(1) = 1. \end{aligned}$$

To put the second order equation into a system of first order equations, let's define $v(x) = \varepsilon \dot{u}(x) + u$. This is called a Lienard transformation. Notice this makes $\dot{v} = \varepsilon \ddot{u} + \dot{u}$. Let us also define $w(x) = x$ to parametrize the orbit of the solution in the independent variable x. The second-order differential equation becomes the system of first order differential equations below:

$$
\begin{aligned}
\varepsilon \dot{u} &= v - u, \\
\dot{v}(x) &= a + 2bw, \\
\dot{w} &= 1,
\end{aligned}
\tag{2.3}
$$

with boundary conditions $u(0) = w(0) = 0$ and $u(1) = w(1) = 1$.

In general, we are looking for solutions of the form

$$
u(x,\varepsilon) = \sum_{j=0} \varepsilon^j u_j(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots,
$$

$$
v(x,\varepsilon) = \sum_{j=0} \varepsilon^j v_j(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \cdots.
$$

2.2.1 The Outer Solution

Let's consider the zeroth and first order solutions $u(x;\varepsilon) = u_0(x) + \varepsilon u_1(x)$, $v(x;\varepsilon) = v_0(x) + \varepsilon v_1(x)$. Substituting these expansions into the outer system (2.3), we get

$$
\begin{aligned}
\varepsilon \dot{u}_0 + \varepsilon^2 \dot{u}_1 &= v_0 - u_0 + \varepsilon(v_1 - u_1), \\
\dot{v}_0 + \varepsilon \dot{v}_1 &= a + 2bw_0 + \varepsilon 2bw_1, \\
\dot{w}_0 + \varepsilon \dot{w}_1 &= 1.
\end{aligned}
$$

We then write the system of equations for the terms of zeroth order in ε.

$$
\begin{aligned}
0 &= v_0 - u_0, \\
\dot{v}_0 &= a + 2bw_0, \\
\dot{w}_0 &= 1,
\end{aligned}
$$

which has solutions $u_0 = v_0 = c_0 + ax + bx^2$ and $w_0 = x$, for some constant c_0.

The first order outer system is

$$
\begin{aligned}
\dot{u}_0 &= v_1 - u_1, \\
\dot{v}_1 &= 2bw_1, \\
\dot{w}_1 &= 0,
\end{aligned}
$$

which has solutions $u_1(x) = c_1 - a - 2bx$, $v_1(x) = c_1$, and $w_1 = 0$, for some constant c_1.

2.2.2 The Inner Solution

We will define an inner variable $\xi = \frac{x}{\varepsilon}$, and substitute it into the outer system (2.3). This gives us the inner system that determines the inner system that determines the limiting behavior of the singular layer at the boundary $x = 0$. More precisely, we will define

$$U(\xi) = u(\varepsilon\xi),\ V(\xi) = v(\varepsilon\xi),\ W(\xi) = w(\varepsilon\xi)$$

as the solutions to the inner system. Also, as $\frac{d}{dx}$ is denoted $\dot{}$, $\frac{d}{d\xi}$ is denoted '. The inner system is

$$\begin{aligned} U' &= V - U, \\ V' &= \varepsilon(a + 2bW), \\ W' &= \varepsilon. \end{aligned} \tag{2.4}$$

Expand as $U = U_0 + \varepsilon U_1$, $V = V_0 + \varepsilon V_1$, $W = W_0 + \varepsilon W_1$, and substitute into the inner system (2.4) to get

$$\begin{aligned} U_0' + \varepsilon U_1' &= V_0 - U_0 + \varepsilon(V_1 - U_1), \\ V_0' + \varepsilon V_1' &= \varepsilon(a + 2bW_0), \\ W_0' + \varepsilon W_1' &= \varepsilon. \end{aligned}$$

As mentioned above, the boundary conditions at $x = 0$ implies

$$U(0) = U_0(0) + \varepsilon U_1(0) = 0 \text{ and } W(0) = W_0(0) + \varepsilon W_1(0) = 0.$$

Therefore, $U_0(0) = U_1(0) = W_0(0) = W_1(0) = 0$.

The inner system of zeroth order terms is

$$\begin{aligned} U_0' &= V_0 - U_0, \\ V_0' &= 0, \\ W_0' &= 0. \end{aligned}$$

Since we defined $w = x$, we get $W = \varepsilon\xi$, so $W_0 = 0$. And $V_0 = k_{00}$, for some constant k_{00}. Using the variation of parameters formula, we get $U_0 = k_{00} + \alpha_0 e^{-\xi}$ for some constant α_0. But since $U_0(0) = 0$, this forces $\alpha_0 = -k_{00}$. At $x = 0$,

$$U_0 = k_{00}(1 - e^{-\xi}),\ V_0 = k_{00},\ W_0 = 0.$$

The inner system of first order terms is

$$
\begin{aligned}
U_1' &= V_1 - U_1, \\
V_1' &= a + 2bW_0, \\
W_1' &= 1.
\end{aligned}
$$

By the same reasoning as in the zeroth order case, $W_1 = \xi$. Notice $W_0(0) = 0$. By integrating, we can see $V_1' = a \implies V_1 = a\xi + k_{01}$. Then by again using the variation of parameters formula, we get $U_1 = \alpha_1 e^{-\xi} + a(\xi - 1) + k_{01}$. Notice $U_1(0) = \alpha_1 - a + k_{01} = 0$ so $\alpha_1 = a - k_{01}$. At $x = 0$,

$$U_1 = (a - k_{01})e^{-\xi} + a(\xi - 1) + k_{01},\ V_1 = a\xi + k_{01},\ W_1 = \xi.$$

Now consider the inner system at the boundary $x = 1$. We will use the inner variable $\eta = \frac{x-1}{\varepsilon}$. Notice that this new variable has the same derivative with respect to x as the inner variable at $x = 0$, so we can consider the same system we did at $x = 0$, but with different constants. Notice that $W_0 = 1$, and $V_0 = k_{10}$ for some constant k_{10}. Then $U_0 = k_{10} + \beta_0 e^{-\eta}$. In this case, the boundary condition is at $x = 1$ which makes $\eta = \frac{1-1}{\varepsilon} = 0$. This makes $\beta_0 = 1 - k_{10}$. At $x = 1$,

$$U_0 = k_{10} + (1 - k_{10})e^{-\eta},\ V_0 = k_{10},\ W_0 = 1.$$

In the first order system near $x = 1$, $W_0(1) = 1$. So $V_1' = a + 2b \implies V_1 = (a + 2b)\xi + k_{11}$ and $W_1 = \xi$. Now, $U_1 = \beta_1 e^{-\xi} + (a + 2b)(\xi - 1) + k_{11}$. So $U_1(0) = 0$ makes $\beta_1 = a + 2b - k_{11}$. At $x = 1$,

$$U_1 = (a + 2b - k_{11})e^{-\eta} + (a + 2b)(\eta - 1) + k_{11},\ V_1 = (a + 2b)\eta + k_{11},\ W_1 = \eta.$$

2.2.3 Matching

Let us first match the zeroth order solutions. We will do this by finding the constants c_0, k_{00}, k_{10} that make $u_0(0) = U(\infty)$ near $x = 0$ and $u_0(1) = U(-\infty)$ near $x = 1$.

Notice near $x = 1$, the solution for U_0 as $\eta \to -\infty$ blows up because of the $e^{-\eta}$ term. Thus, to make this solution satisfy the boundary condition, we must have $k_{10} = 1$. Then $U_0(-\infty) = 1 = c_0 + a + b = u_0(1)$ makes $c_0 = 1 - a - b$. Matching near $x = 0$ makes $U_0(\infty) = k_{00} = c_0 = u_0(0)$, so we get the following zeroth order solutions

$$
\begin{aligned}
&u_0(x) = 1 - a - b + ax + bx^2 \\
&U_0(\xi) = (1 - a - b)(1 - e^{-\xi}), && \text{near } x = 0, \\
&U_0(\eta) = 1, && \text{near } x = 1.
\end{aligned}
$$

To match the first order terms, we must consider the expansion of u and U in ε.

$$
\begin{aligned}
&u(x) = 1-a-b+ax+bx^2+\varepsilon(c_1-a-2bx) \\
&U(\xi) = (1-a-b)(1-e^{-\xi})+\varepsilon\left((a-k_{01})e^{-\xi}+a(\xi-1)+k_{01}\right), \quad \text{near } x=0, \\
&U(\eta) = 1+\varepsilon\left((a-k_{11})e^{-\eta}+a(\eta-1)+k_{11}\right), \qquad \text{near } x=1.
\end{aligned}
$$

Consider the outer solution near $x=1$. We need the ε^1 order terms of $u(x)$ to equal zero when $x=1$ to satisfy the boundary condition. Thus we get

$$c_1 = a+2b.$$

Now

$$u(x) = 1-a-b+\varepsilon(2b-2bx).$$

Consider the outer solution near $x=0$. Substitute the inner variable $x=\varepsilon\xi$ into $u(x)$ to get

$$u(\varepsilon\xi) = 1-a-b+\varepsilon(a\xi+2b).$$

Recall $U(\xi) = (1-a-b)(1-e^{-\xi})+\varepsilon\left((a-k_{01})e^{-\xi}+a(\xi-1)+k_{01}\right)$. Notice we recover the zeroth order matching here when we look at $x=0 \implies \xi=0$. We now want to match this with $U(\xi)$ near $x=0$ to find k_{01}. Notice that as $\xi\to\infty$, the terms $e^{-\xi}$ will vanish. These are called transcendental terms. Since they vanish near the boundary in the final solution, we do not need to consider them in the matching. At $x=0$, without the transcendental terms,

$$U(\xi) = 1-a-b+\varepsilon(a(\xi-1)+k_{01}).$$

This implies $a\xi+2b = a(\xi-1)+k_{01} \implies k_{01}=a+2b$. So we get the solution near $x=0$ to be

$$U(\xi) = (1-a-b)(1-e^{-\xi})+\varepsilon\left(2b(1-e^{-\xi})+a\xi\right).$$

Near $x=1$, we consider the outer system in the fast variable $x=\varepsilon\eta+1$,

$$u(\varepsilon\eta+1) = 1+\varepsilon(a+2b)\eta.$$

Recall $U(\eta) = 1+\varepsilon\left((a+2b-k_{11})e^{-\eta}+(a+2b)(\eta-1)+k_{11}\right)$ at $x=1$. Notice the $e^{-\eta}$ term goes to infinity when $\eta\to-\infty$. This requires $k_{11}=a+2b$. This completes the matching, since $k_{11}=a+2b$ makes the constant term equal to zero. So the solutions follow.

$$
\begin{aligned}
&u(x) = 1-a-b+ax+bx^2+\varepsilon(2b-2bx), && \\
&U(\xi) = (1-a-b)(1-e^{-\xi})+\varepsilon\left(2b(1-e^{-\xi})+a\xi\right), && \text{near } x=0, \\
&U(\eta) = 1+\varepsilon(a+2b)\eta; && \text{near } x=1.
\end{aligned}
$$

It can be checked that these expansions agree with the expansions of the known exact solution to the boundary value problem.

It should be noted that Lagerstrom, in [4], handles the problem similarly. He considers the inner and outer systems in the zeroth order in ε, then uses a limiting argument to match in the jth order of ε. He then combines the inner and outer solutions using a difference function.

Chapter 3

Solving the PNP system

We now apply the method of matched asymptotic expansions to the PNP system with $\alpha = \beta = 1$ and $Q(x) = 0$. Following the procedure, we will examine the outer dynamics in Section 3.1, the inner dynamics in Section 3.2 and the matching in Section 3.3. This chapter treats general boundary conditions with asymptotic expansions up to the first order. In Chapter 4, we study second and third order asymptotic expansions under the electron neutrality boundary conditions. Some of the results are reported in [1].

3.1 The Outer Dynamics

Take the simplified case of two ions with $\alpha = \beta = 1$ and zero permanent charge, $Q(x) = 0$.

The system now looks like:

$$\begin{aligned}
&\varepsilon^2 \ddot{\phi} - c_2 + c_1 = 0, \\
&\dot{c}_1 + c_1 \dot{\phi} = -J_1, \\
&\dot{c}_2 - c_2 \dot{\phi} = -J_2, \\
&\dot{J}_1 = \dot{J}_2 = 0,
\end{aligned} \tag{3.1}$$

with boundary conditions

$$\begin{aligned}
\phi(0) = \nu_0,\ c_1(0) = L_1,\ c_2(0) = L_2, \\
\phi(1) = 0,\ c_1(1) = R_1,\ c_2(1) = R_2.
\end{aligned} \tag{3.2}$$

We will define the reference to order to mean order in ε of a term in the asymptotic expansion of each of these functions. This is demonstrated below:

$$
\begin{aligned}
&\phi = \phi(x;\varepsilon) = \phi_0(x) + \varepsilon\phi_1(x) + \varepsilon^2\phi_2(x) + \cdots, \\
&c_1 = c_1(x;\varepsilon) = c_{10}(x) + \varepsilon c_{11}(x) + \cdots, \\
&c_2 = c_2(x;\varepsilon) = c_{20}(x) + \varepsilon c_{21}(x) + \cdots, \\
&J_1 = J_{10} + \varepsilon J_{11} + \cdots,\ J_2 = J_{20} + \varepsilon J_{21} + \cdots.
\end{aligned}
$$

Note that the current I is then given by

$$
I = I_1 + I_2 + \cdots = J_1 - J_2 = (J_{10} - J_{20}) + \varepsilon(J_{21} - J_{22}) + \cdots. \tag{3.3}
$$

This system has been studied using a geometric approach in [5] for the zeroth order approximation, but this method did not easily generalize to the higher order approximation, so we will apply the classical matched asymptotic expansions.

Rewriting the PNP system (3.1) with the expanded functions, we get the following system:

$$
\begin{aligned}
&\varepsilon^2\ddot{\phi}_0 + \varepsilon^3\ddot{\phi}_1 + \varepsilon^4\ddot{\phi}_2 + \cdots = c_{20} - c_{10} + \varepsilon(c_{21} - c_{11}) + \cdots, \\
&\dot{c}_{10} + \varepsilon\dot{c}_{11} + \cdots = -\dot{\phi}_0 c_{10} - \varepsilon(\dot{\phi}_0 c_{11} + \dot{\phi}_1 c_{10}) - J_{10} - \varepsilon J_{11} + \cdots, \\
&\dot{c}_{20} + \varepsilon\dot{c}_{21} + \cdots = \dot{\phi}_0 c_{20} + \varepsilon(\dot{\phi}_0 c_{21} + \dot{\phi}_1 c - 20) - J_{20} - \varepsilon J_{21} + \cdots.
\end{aligned} \tag{3.4}
$$

This is the outer system. It describes solutions between $x = 0$ and $x = 1$. This system is typically also counterintuitively called the "outer system". We can write the larger system as systems of like-ordered terms, that is, match all the coefficients of ε^0, ε^1, etc. to get "smaller" systems.

3.1.1 The Zeroth-Order Outer System

Considering the system given by the ε^0 order terms of (3.4), we get:

$$
\begin{aligned}
0 &= c_{20} - c_{10}, \\
\dot{c}_{10} &= -c_{10}\dot{\phi}_0 - J_{10}, \\
\dot{c}_{20} &= c_{20}\dot{\phi}_0 - J_{20}.
\end{aligned}
$$

This is the limiting outer system. Notice that the high-order term $\ddot{\phi}$ vanishes, so we get two ordinary differential equations coupled with an algebraic equation. In general, these solutions cannot satisfy the boundary conditions (3.2), so we will need to consider another system besides the outer system to solve the PNP system. First, let's attack the zeroth-order outer system.

Since $c_{10} = c_{20} \equiv c_0$, we get the equations

$$\dot{c}_0 + c_0\dot{\phi}_0 = -J_{10},$$

$$\dot{c}_0 - c_0\dot{\phi}_0 = -J_{20}.$$

Add and subtract the equations to get

$$\dot{c}_0 = -\frac{J_{20}+J_{20}}{2}, \quad \dot{\phi}_0 = \frac{J_{20}-J_{10}}{2c_0}.$$

Integrating and substituting, we get

$$\begin{aligned} \phi_0(x) &= b_0 + \frac{I_0}{T_0}\ln|a_0 - T_0x|, \\ c_{10}(x) = c_{20}(x) &= \frac{a_0 - T_0x}{2}, \end{aligned} \tag{3.5}$$

where a_0 and b_0 are constants, and $I_0 = J_{10} - J_{20},\ T_0 = J_{10} + J_{20}$.

3.1.2 The First-Order Outer System

Let's now attack the first-order outer system. Group the ε^1 order terms of (3.4) to get:

$$\begin{aligned} 0 &= c_{21} - c_{11}, \\ \dot{c}_{11} &= -(\dot{\phi}_0 c_{11} + \dot{\phi}_1 c_{10}) - J_{11}, \\ \dot{c}_{21} &= (\dot{\phi}_0 c_{21} + \dot{\phi}_1 c_{20}) - J_{21}. \end{aligned}$$

Again, $c_{11} = c_{21} \equiv c_1$, so we get the equations

$$\dot{c}_1 = -c_0\dot{\phi}_1 - c_1\dot{\phi}_0 - J_{11},$$

$$\dot{c}_1 = c_0\dot{\phi}_1 + c_1\dot{\phi}_0 - J_{21}.$$

Add and subtract the equations to get

$$\dot{c}_1 = -\frac{J_{21}+J_{21}}{2},\ \dot{\phi}_1 = \frac{(J_{21}-J_{11}) - 2c_1\dot{\phi}_0}{2c_0}.$$

Integrating and substituting, we get

$$\begin{aligned} \phi_1(x) &= b_1 + \frac{T_0I_1 - I_0T_1}{T_0^2}\ln|a_0 - T_0x| + \frac{I_0(a_1T_0 - a_0T_1)}{T_0^2(a_0 - T_0x)}, \\ c_{11}(x) = c_{21}(x) &= \frac{a_1 - T_1x}{2}, \end{aligned} \tag{3.6}$$

where a_1, b_1 are constants, and $I_1 = J_{11} - J_{21}$, $T_1 = J_{11} + J_{21}$.

Now we can determine the constants a_0, b_0, a_1, b_1 by solving the system near the boundaries $x = 0$ and $x = 1$ and using these four degrees of freedom to match this outer solution with the solution found at the boundaries.

3.2 The Inner Dynamics

As previously stated, we have found solutions that satisfy the system (3.1) between $x = 0$ and $x = 1$, but don't satisfy the boundary conditions (3.2) in general. We can scale time in inner variable to determine a solution near the boundaries, then match the inner solution with the outer solution we just found to find the constants.

Define the inner variable ξ in x as $\xi = \frac{x}{\varepsilon}$. We will do a change of variables as follows. Let's consider the system in general, then look at the specific solutions near $x = 0$ and $x = 1$. Notice that these solutions need not be symmetric, if only because we chose the initial condition on ϕ to gather the potential v_0 at $x = 0$ and have zero potential at $x = 1$.

$$
\begin{aligned}
\Phi(\xi) &= \phi(\varepsilon\xi), \\
C_1(\xi) &= c_1(\varepsilon\xi), \\
C_2(\xi) &= c_2(\varepsilon\xi).
\end{aligned}
$$

In the inner variable, the PNP system becomes:

$$
\begin{aligned}
\Phi'' &= C_2 - C_1, \\
C_1' &= -\Phi' C_1 - \varepsilon J_1, \\
C_2' &= \Phi' C_2 + \varepsilon J_2.
\end{aligned}
\tag{3.7}
$$

Considering the following expansions

$$\Phi(\xi, \varepsilon) = \Phi_0(\xi) + \varepsilon \Phi_1(\xi) + \cdots,$$

$$
\begin{aligned}
C_1(\xi, \varepsilon) &= C_{10}(\xi) + \varepsilon \Phi_{11}(\xi) + \cdots, \\
C_2(\xi, \varepsilon) &= C_{20}(\xi) + \varepsilon \Phi_{21}(\xi) + \cdots,
\end{aligned}
$$

the system becomes

$$
\begin{aligned}
&\Phi_0'' + \varepsilon\Phi_1'' + \varepsilon^2\Phi_2'' = C_{20} - C_{10} + \varepsilon(C_{21} - C_{11}),\\
&C_{10}' + \varepsilon C_{11}' = -\Phi_0' C_{10} - \varepsilon(\Phi_0' C_{11} + \Phi_1' C_{10}) - \varepsilon J_{10} - \varepsilon^2 J_{11},\\
&C_{20}' + \varepsilon C_{21}' = \Phi_0' C_{20} + \varepsilon(\Phi_0' C_{21} + \Phi_1' C_{20}) - \varepsilon J_{20} - \varepsilon^2 J_{21}.
\end{aligned} \tag{3.8}
$$

We will call this the inner system. Notice that the inner system is of the same order as the original PNP system, so we will be able to find the parts of solutions associated with the higher order terms that were lost in the outer system when the $\ddot{\phi}$ term vanished. For convenience, let's introduce a new variable $U = \Phi'$. The inner system is then

$$
\begin{aligned}
&\Phi_0' + \varepsilon\Phi_1' + \varepsilon^2\Phi_2' = U_0 + \varepsilon U_1,\\
&U_0' + \varepsilon U_1' = C_{20} - C_{10} + \varepsilon(C_{21} - C_{11}),\\
&C_{10}' + \varepsilon C_{11}' = -U_0 C_{10} - \varepsilon(U_0 C_{11} + U_1 C_{10}) - \varepsilon J_{10} - \varepsilon^2 J_{11},\\
&C_{20}' + \varepsilon C_{21}' = U_0 C_{20} + \varepsilon(U_0 C_{21} + U_1 C_{20}) - \varepsilon J_{20} - \varepsilon^2 J_{21}.
\end{aligned}
$$

3.2.1 The Zeroth-Order Inner System

As we did in the outer system, compare the terms of like order in ε. Matching the ε^0 terms in (3.8), we get

$$
\begin{aligned}
&\Phi_0' = U_0,\\
&U_0' = C_{20} - C_{10},\\
&C_{10}' = -U_0 C_{10},\\
&C_{20}' = U_0 C_{20}.
\end{aligned} \tag{3.9}
$$

This is the limiting inner system.

It is known that this system has a complete set of first integrals; that is,

Lemma 1. *Lemma 1 The following functions*

$$
\begin{aligned}
&H_1 = C_{20} + C_{10} - \frac{1}{2}U_0^2,\\
&H_2 = e^{-\Phi_0} C_{20},\\
&H_3 = e^{\Phi_0} C_{10},
\end{aligned}
$$

are first integrals of the zeroth order inner system. This means if $\Phi_0(\xi), U_0(\xi), C_{10}(\xi), C_{20}(\xi))$ *is a solution, then* $\dfrac{dH_i}{d\xi} = 0$ *for* $i = 1, 2, 3$.

It can be verified that these functions are integrals for the system. Using the initial conditions in (3.2), we get $\Phi_0(0) = v_0$ and $C_{j0} = L_j,\ j = 1,2$, so $H_i,\ i = 1,2,3$ constant means

$$C_{10}e^{\Phi_0} = k_1 \Rightarrow C_{10}(\xi) = k_1 e^{-\Phi_0(\xi)}.$$

At $x = 0$, $C_{10} = L_1 = k_1 e^{-v_0}$ gives $k_1 = L_1 e^{v_0}$ and

$$C_{10} = L_1 e^{-(\Phi_0(\xi)-v_0)}. \tag{3.10}$$

Likewise,

$$C_{20} = L_2 e^{(\Phi_0(\xi)-v_0)}. \tag{3.11}$$

Substituting (3.10) and (3.11), the zeroth order system (3.9) becomes a Newtonian system

$$\Phi_0' = U_0, \quad U_0' = -L_1 e^{-(\Phi_0(\xi)-v_0)} + L_2 e^{(\Phi_0(\xi)-v_0)}.$$

$$\Rightarrow \Phi_0'' + L_1 e^{-(\Phi_0(\xi)-v_0)} - L_2 e^{(\Phi_0(\xi)-v_0)} = 0.$$

This implies

$$\frac{(\Phi_0')^2}{2} - L_1 e^{-(\Phi_0(\xi)-v_0)} - L_2 e^{\Phi_0(\xi)-v_0} = -M$$

for some constant M. Let $\Phi_0(\infty) = \Phi_0^L$. The above equation becomes

$$L_1 e^{-(\Phi_0^L - v_0)} - L_2 e^{\Phi_0^L - v_0} = 0.$$

Then

$$\Phi_0^L = v_0 + \frac{1}{2}\ln\frac{L_1}{L_2},\ M = 2\sqrt{L_1 L_2}.$$

We can solve for $\Phi_0(\xi)$, and plug into the earlier equations (3.10) and (3.11) for C_{10} and C_{20} to get

$$\begin{aligned} &\Phi_0(\xi) = v_0 + \frac{1}{2}\ln\left(\frac{L_1}{L_2}\right) + \ln\left(\frac{1+le^{-\sqrt{M}}\xi}{1-le^{-\sqrt{M}\xi}}\right)^2,\ U_0(\xi) = -\frac{4l\sqrt{M}e^{-\sqrt{M}\xi}}{1-l^2e^{-2\sqrt{M}\xi}}, \\ &C_{10}(\xi) = \sqrt{L_1L_2}\left(\frac{1-le^{-\sqrt{M}}\xi}{1+le^{-\sqrt{M}\xi}}\right)^2,\ C_{20}(\xi) = \sqrt{L_1L_2}\left(\frac{1+le^{-\sqrt{M}}\xi}{1-le^{-\sqrt{M}\xi}}\right)^2, \end{aligned} \tag{3.12}$$

for $l = \frac{L_2^{\frac{1}{4}} - L_1^{\frac{1}{4}}}{L_2^{\frac{1}{4}} + L_1^{\frac{1}{4}}}$ and $M = 2\sqrt{L_1 L_2}$. Notice this is the inner solution near $x = 0$.

Likewise, near $x = 1$, when we use the inner variable $x - 1 = \varepsilon\xi$, we get

$$\begin{aligned}
&\Psi_0(\xi) = \frac{1}{2}\ln\left(\frac{R_1}{R_2}\right) + \ln\left(\frac{1 + re^{-\sqrt{N}\xi}}{1 - re^{-\sqrt{N}\xi}}\right)^2, \; V_0(\xi) = \frac{4r\sqrt{N}e^{\sqrt{N}\xi}}{1 - r^2 e^{2\sqrt{N}\xi}}, \\
&D_{10}(\xi) = \sqrt{R_1 R_2}\left(\frac{1 - re^{\sqrt{N}\xi}}{1 + re^{\sqrt{N}\xi}}\right)^2, \; D_{20}(\xi) = \sqrt{R_1 R_2}\left(\frac{1 + re^{\sqrt{N}\xi}}{1 - re^{\sqrt{N}\xi}}\right)^2,
\end{aligned} \tag{3.13}$$

where $r = \frac{R_2^{\frac{1}{4}} - R_1^{\frac{1}{4}}}{R_2^{\frac{1}{4}} + R_1^{\frac{1}{4}}}$ and $N = 2\sqrt{R_1 R_2}$.

3.2.2 The First-Order Inner System

Matching the ε^1 order terms in (3.8), we get the system

$$\begin{aligned}
&\Phi_1' = U_1, \\
&U_1' = C_{21} - C_{11}, \\
&C_{11}' = -(C_{10}U_1 + C_{11}U_0) - J_{10}, \\
&C_{21}' = (C_{20}U_1 + C_{21}U_0) - J_{20}.
\end{aligned} \tag{3.14}$$

Notice that the homogeneous part of this system is the linearization of the ε^0 system (3.9). That is, we consider the zeroth order system $x' = f(x)$ where

$$f(x) = \begin{pmatrix} U_0 \\ C_{20} - C_{10} \\ -U_0 C_{10} \\ U_0 C_{20} \end{pmatrix}, D_x f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -C_{10} & -U_0 & 0 \\ 0 & C_{20} & 0 & U_0 \end{pmatrix}.$$

If we write the first order system as a matrix equation

$$\begin{pmatrix} \Phi_1 \\ U_1 \\ C_{11} \\ C_{21} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -C_{10} & -U_0 & 0 \\ 0 & C_{20} & 0 & U_0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ U_1 \\ C_{11} \\ C_{21} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -J_{10} \\ -J_{20} \end{pmatrix}.$$

To solve the system, we need the following lemma.

Consider the system:

$$x' = f(x), \; x \in \mathbb{R}^n. \tag{3.15}$$

Let $x_0(t)$ be a solution of (3.1), and

$$z' = Df(x_0(t))z, z \in \mathbb{R}^n \tag{3.16}$$

be the linearization of (3.15) about x_0.

Lemma 2. *Suppose $H(x)$, $H : \mathbb{R}^n \to \mathbb{R}$ is a first integral for system* (3.15). *Then, for any solution x_0, $\langle \bigtriangledown H(x_0(t)), z\rangle$ is a first integral for the linearization* (3.16).

Proof. Since $H(x(t))$ is a first integral of system (3.15), we know that

$$\begin{aligned}
0 = \langle \bigtriangledown H(x), f(x)\rangle &= \sum_{k=1}^{n} \frac{\partial H(x)}{x_k} f_k(x) \ \forall x \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left[\sum_{j=1}^{n} \frac{\partial H(x)}{\partial x_j} f_j(x)\right] \ \forall x, \forall k \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_k}\left[\frac{\partial H(x)}{\partial x_j} f_j(x)\right] \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} \left[\frac{\partial^2 H(x)}{\partial x_j \partial x_k} f_j(x) + \frac{\partial H(x)}{\partial x_k}\frac{f_j(x)}{\partial x_j}\right] \\
&= 0.
\end{aligned}$$

To see that $\langle \bigtriangledown H(x_0(t)), z\rangle$ is a first integral of (3.16), we need to verify that $\frac{d}{dt}\langle \bigtriangledown H(x_0(t)), z\rangle = 0$, i.e. that the inner product gives a constant. After calculation, we get

$$\frac{d}{dt}\langle \bigtriangledown H(x_0(t)), z\rangle = \sum_{j=1}^{n} z_j(t) \sum_{k=1}^{n}\left[\frac{\partial^2 H(x_0(t))}{\partial x_j \partial x_k} f_k(x_0(t)) + \frac{\partial H(x_0(t))}{\partial x_k}\frac{\partial f_k(x_0(t))}{\partial x_j}\right].$$

Since the sum over k has been shown to equal zero, the entire sum equals zero. Therefore,

$$\langle \bigtriangledown H(x_0(t)), z\rangle = 0,$$

which concludes the proof. □

Note that this is presumably not a new result, but no proof was found in the literature.

Using Lemma 2, we know

$$\nabla H_1 = \begin{pmatrix} \frac{\partial H_1}{\partial \phi_0} \\ \frac{\partial H_1}{\partial U_0} \\ \frac{\partial H_1}{\partial C_{10}} \\ \frac{\partial H_1}{\partial C_{20}} \end{pmatrix} = \begin{pmatrix} 0 \\ -U_0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \langle \nabla H_1, \begin{pmatrix} \Phi_1 \\ U_1 \\ C_{11} \\ C_{21} \end{pmatrix} \rangle = -U_0U_1 + C_{11} + C_{21} \equiv H_1^0.$$

Thus, we know this nonautonomous, homogeneous system has integrals given by

$$\begin{aligned} H_1^0 &= U_0U_1 - C_{11} - C_{21}, \\ H_2^0 &= -C_{20}e^{-\Phi_0}\Phi_1 + C_{21}e^{-\Phi_0}, \\ H_3^0 &= C_{10}e^{\Phi_0}\Phi_1 + C_{11}e^{\Phi_0}. \end{aligned}$$

So the inhomogeneous system (3.14) must have integrals as lemma 2 prescribes, along with some inhomogeneous term.

Take H_2 from the homogeneous first order system. We know

$$\frac{\partial H_2^0}{\partial \xi} = C_{10}'e^{\Phi_0}\Phi_1 + C_{10}e^{\Phi_0}\Phi_0'\Phi_1 + C_{10}e^{\Phi_0}\Phi_1' + C_{11}e^{\Phi_0}\Phi_0' + C_{11}'e^{\Phi_0} = 0.$$

Now take our nonhomogeneous equation: $C_{11}' = -(C_{10}U_1 + C_{11}U_0) - J_{10}$. To make H_2 into an integral for the nonhomogeneous system, we need to account for the extra term that comes from the C_{11}': $-J_{10}e^{\Phi_0}$. Add on a term $J_{10}F_1(\xi)$ for some function $F_1(\xi)$ that will make the integral constant in the system.

$$J_{10}F_1(\xi) = \int_0^\xi J_{10}e^{\Phi_0(s)}ds \Rightarrow F_1(\xi) = \int_0^\xi e^{\Phi_0(s)}ds.$$

Claim: The integrals of the nonhomogeneous ε^1 system are given by

$$\begin{aligned} H_1 &= U_0U_1 - C_{11} - C_{21} - (J_{10} + J_{20})\xi, \\ H_2 &= C_{10}e^{\Phi_0}\Phi_1 + C_{11}e^{\Phi_0} + J_{10}F_1(\xi), \\ H_3 &= -C_{20}e^{-\Phi_0}\Phi_1 + C_{21}e^{-\Phi_0} + J_{20}F_2(\xi), \end{aligned}$$

for some $F_1(\xi), F_2(\xi)$ found with $\Phi_1(0) = C_{11}(0) = C_{21}(0) = 0, U_1(0) = \rho$, for some initial condition ρ.

We get the following three equations:

$$
\begin{aligned}
&-U_0U_1 + C_{11} + C_{21} + (J_{10}+J_{20})\xi = U_0(0)\rho, \\
&C_{10}e^{\Phi_0}\Phi_1 + C_{11}e^{\Phi_0} + J_{10}F_1(\xi) = 0, \\
&-C_{20}e^{-\Phi_0}\Phi_1 + C_{21}e^{-\Phi_0} + J_{20}F_2(\xi) = 0,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&U_1 = \frac{U_0(0)\rho - (J_{10}+J_{20})\xi - C_{11} - C_{21}}{-U_0}, \\
&C_{11} = -C_{10}\Phi_1 - J_{10}e^{-\Phi_0}F_1(\xi), \\
&C_{21} = C_{20}\Phi_1 - J_{20}e^{\Phi_0}F_2(\xi).
\end{aligned}
$$

Thus,

$$
U_1 = \Phi_1' = \frac{C_{20}-C_{10}}{U_0}\Phi_1 + \frac{(J_{10}+J_{20})\xi - J_{10}F_1(\xi)e^{-\Phi_0} - J_{20}F_2(\xi)e^{\Phi_0} - \rho U_0(0)}{U_0}.
$$

Notice $C_{20} - C_{10} = U_0'$.

We need to find the expressions for $F_1(\xi), F_2(\xi)$ using these solutions. Since our initial conditions (3.2) are zeroth order in ε, $\Phi_1(0) = C_{11}(0) = C_{21}(0) = 0, U_1(0) = \rho$. So

$$
\begin{aligned}
F_1(\xi) &= \frac{-1}{\sqrt{M}}\sqrt{\frac{L_{10}}{L_{20}}}e^{v_0}\left(\frac{4}{1-le^{-\sqrt{M}\xi}} - \frac{4}{1-l} - \sqrt{M}\xi\right), \\
F_2(\xi) &= \frac{-1}{\sqrt{M}}\sqrt{\frac{L_{20}}{L_{10}}}e^{-v_0}\left(\frac{4}{1+le^{-\sqrt{M}\xi}} - \frac{4}{1+l} - \sqrt{M}\xi\right).
\end{aligned}
$$

Notice that, at $\xi = 0$, $H_1 = U_0(0)\rho$ and $H_2 = H_3 = 0$.

Plugging these functions into the equation for $U_1 = \Phi_1'$, we can use the variation of parameters formula to get

$$
\Phi_1(\xi) = \Phi_1(0)\frac{U_0(\xi)}{U_0(0)} + U_0(\xi)\int_0^{\xi}\frac{g(s)}{U_0(s)}ds,
$$

with $\Phi_1(0) = 0$ and

$$
g(s) = \frac{(J_{10}+J_{20})s - J_{10}F_1(s)e^{-\Phi_0(s)} - J_{20}F_2(s)e^{\Phi_0(s)} - \rho U_0(0)}{U_0(s)}.
$$

Therefore,

$$
\begin{aligned}
\Phi_1(\xi) &= -\frac{I_0}{M}\xi - \frac{4l(T_0+lI_0)}{M^{\frac{3}{2}}(1+l)(1-l)} + O(e^{-\sqrt{M}\xi}), \\
C_{21}(\xi) = C_{11}(\xi) &= -\frac{T_0}{2}\xi - \frac{2l(I_0+lT_0)}{M^{\frac{1}{2}}(1+l)(1-l)} + O(e^{-\sqrt{M}\xi}),
\end{aligned}
\tag{3.17}
$$

with $\rho = \dfrac{J_{20}}{M(1+l)} - \dfrac{J_{10}}{M(1-l)}$, $l = \frac{L_2^{\frac{1}{4}} - L_1^{\frac{1}{4}}}{L_2^{\frac{1}{4}} + L_1^{\frac{1}{4}}}$ and $M = 2\sqrt{L_1 L_2}$. Likewise, near $x = 1$, we get

$$
\begin{aligned}
\Psi_1(\xi) &= -\frac{I_0}{N}\xi - \frac{4r(T_0+rI_0)}{N^{\frac{3}{2}}(1+r)(1-r)} + O(e^{-\sqrt{N}\xi}), \\
D_{21}(\xi) = D_{11}(\xi) &= -\frac{T_0}{2}\xi - \frac{2r(I_0+rT_0)}{N^{\frac{1}{2}}(1+r)(1-r)} + O(e^{-\sqrt{N}\xi}),
\end{aligned}
\tag{3.18}
$$

where $r = \frac{R_2^{\frac{1}{4}} - R_1^{\frac{1}{4}}}{R_2^{\frac{1}{4}} + R_1^{\frac{1}{4}}}$ and $N = 2\sqrt{R_1 R_2}$.

3.3 Matching

3.3.1 Matching the zeroth order terms near x=0

Now match the inner solution $(\Phi_0(\xi), C_{10}(\xi), C_{20}(\xi))$ from (3.12) as $\xi \to \infty$ with the outer solution $(\phi_0(x), c_{10}(x), c_{20}(x))$ from (3.5) at $x = 0$:

$$
\begin{aligned}
&\frac{a_0}{2} = \sqrt{L_1 L_2} \quad \Rightarrow \quad a_0 = 2\sqrt{L_1 L_2}, \\
&b_0 + \frac{I_0}{T_0}\ln a_0 = \nu_0 + \frac{1}{2}\ln\frac{L_1}{L_2} \quad \Rightarrow \quad b_0 = \nu_0 + \frac{1}{2}\ln\frac{L_1}{L_2} - \frac{I_0}{T_0}\ln 2\sqrt{L_1 L_2}.
\end{aligned}
$$

3.3.2 Matching the zeroth order terms near x=1

Similarly, match the inner solution $(\Psi_0(\xi), D_{10}(\xi), D_{20}(\xi))$ from (3.13) as $\xi \to -\infty$ with the outer solution $(\phi_0(x), c_{10}(x), c_{20}(x))$ from (3.5) at $x = 1$:

$$a_0 = T_0 + 2\sqrt{R_1 R_2},$$
$$b_0 = \frac{1}{2}\ln\frac{R_1}{R_2} - \frac{I_0}{T_0}\ln 2\sqrt{R_1 R_2}.$$

Matching these a_0 and b_0 to the ones found at $x = 0$ gives two equations with which we can solve for J_{10} and J_{20}:

$$T_0 = 2\sqrt{L_1 L_2} - 2\sqrt{R_1 R_2},$$
$$I_0 = \frac{2(\sqrt{L_1 L_2} - \sqrt{R_1 R_2})(2\nu_0 + \ln L_1 R_2 - \ln R_1 L_2)}{\ln L_1 L_2 - \ln R_1 R_2}, \tag{3.19}$$

or

$$J_{10} = \frac{2(\sqrt{L_1 L_2} - \sqrt{R_1 R_2})(\ln L_1 - \ln R_1 + \nu_0)}{\ln L_1 L_2 - \ln R_1 R_2},$$
$$J_{20} = \frac{2(\sqrt{L_1 L_2} - \sqrt{R_1 R_2})(\ln L_2 - \ln R_2 - \nu_0)}{\ln L_1 L_2 - \ln R_1 R_2},$$

which gives I_0 from (3.3).

3.3.3 Matching the first order terms near x=0

Near $\mathbf{x = 0}$ Recall that $\Phi(\xi) = \Phi_0(\xi) + \varepsilon\Phi_1(\xi) + \ldots$. So, by (3.5) and (3.12), near $x = 0$,

$$\Phi(\xi) = \nu_0 + \frac{1}{2}\ln\left|\frac{L_{10}}{L_{20}}\right| + \varepsilon\left[\frac{J_{20} - J_{10}}{M}\xi - \frac{4lJ_{10}}{M^{\frac{3}{2}}(1-l)} - \frac{4lJ_{20}}{M^{\frac{3}{2}}(1+l)}\right].$$

Notice we dropped the terms that would go to zero as $\xi \to \infty$. These are called transcendental terms, and vanish in the final solution.

Let's match this solution for $\Phi(\xi)$ with the outer solution $\phi(x)$. We can approximate $\phi(x)$ near $x = 0$ by its Taylor expansion $\phi(x) = \phi_0(0) + \phi_0'(0)x + \varepsilon[\phi_1(0) + \phi_1'(0)x]$, where ϕ_0 and ϕ_1 are as in (3.5) and (3.6).

We must scale the independent variable of ϕ before we can match the coefficients. Consider

$$\phi(\varepsilon\xi) = \phi_0(0) + \phi_0'(0)\varepsilon\xi + \varepsilon[\phi_1(0) + \phi_1'(0)\varepsilon\xi] = \phi_0(0) + \varepsilon[\phi_0'(0)\xi + \phi_1(0)]$$

when we drop the terms that are of a higher order than ε^1. This expansion gives

$$\phi(\varepsilon\xi) = b_0 + \frac{I_0}{T_0}\ln a_0 + \varepsilon\left(b_1 + \frac{I_0}{a_0}\xi + \frac{T_0I_1 - I_0T_1}{T_0^2}\ln a_0 + \frac{I_0(a_1T_0 - a_0T_1)}{T_0^2 a_0} + O(\varepsilon)\right).$$

Matching the coefficients of the ξ terms and the constant terms, we get

$$a_0 = 2\sqrt{L_1L_2}, \tag{3.20}$$

$$b_0 = -\frac{I_0}{T_0}\ln M + \frac{1}{2}\ln\left|\frac{L_1}{L_2}\right| + v_0. \tag{3.21}$$

Now, let's match the inner $C_i, i = 1,2$ equations near $x = 0$ with the Taylor expansions of the outer equation for $c_1 = c_2 = c$ to find a_1. It is worth noting that will be solving two equations for one unknown, which may be an overdetermined system.

Unlike the calculations with ϕ, the outer solution for c is linear and thus we need not do the Taylor expansion. So, from (3.5) and (3.6), get

$$c(x) = c_0(x) + \varepsilon c_1(x) = \frac{a_0}{2} - \frac{T_0}{2}x + \varepsilon\left[\frac{a_1}{2} - \frac{T_1}{2}x\right].$$

In the inner variable ξ, when we drop the high order terms, we get

$$c(\varepsilon\xi) = 2\sqrt{L_1L_2} + \varepsilon\left[\frac{a_1}{2} - \frac{T_0}{2}\xi\right].$$

Match this with the inner solutions for C_1 and C_2 from (3.12) and (3.17) and drop the transcendental terms we get when plugging Φ back into the first integrals of the inner system.

$$C_1 = C_{10} + \varepsilon C_{11} = M + \varepsilon\left[-\frac{J_{20}+J_{10}}{2}\xi + \frac{2lJ_{20}}{\sqrt{M}(1+l)} - \frac{2lJ_{10}}{\sqrt{M}(1-l)}\right] = C_{20} + \varepsilon C_{21} = C_2.$$

By this stroke of luck, our system is consistent, and we find

$$a_1 = \frac{4l}{\sqrt{M}}\left[\frac{J_{20}}{1+l} - \frac{J_{10}}{1-l}\right]. \tag{3.22}$$

We can now plug these three known constants (3.20), (3.21), and (3.22) into the equation for ϕ and match the constant ε terms of the inner and outer equations ϕ and Φ. We now have an equation for the last constant b_1:

$$b_1 = -\frac{I_0(a_1T_0 - a_0T_1)}{T_0^2 a_0} - \frac{T_0I_1 - I_0T_1}{T_0^2}\ln a_0 - \frac{2l(T_0 + lI_0)}{\sqrt{L_1L_2M}(1+l)(1-l)}. \tag{3.23}$$

We will find these same four constants near $x = 1$ and then match these equations with the ones near $x = 0$ to find the explicit solution for J_{11}, J_{21}.

3.3.4 Matching the first order terms near x=1

Near $\mathbf{x = 1}$ Consider the inner variable to now be $x - 1 = \varepsilon\xi$. From (3.12) and (3.17),

$$\Psi(\xi) = \frac{1}{2}\ln\left|\frac{R_{10}}{R_{20}}\right| + \varepsilon\left[\frac{J_{20} - J_{10}}{N}\xi - \frac{4rJ_{10}}{MN^{\frac{3}{2}}(1-r)} - \frac{4rJ_{20}}{N^{\frac{3}{2}}(1+r)}\right].$$

The Taylor expansion of the outer equation ϕ

$$\phi(x) = \phi_0(1) + \phi_0'(1)x + \varepsilon[\phi_1(1) + \phi_1'(1)x] \Rightarrow \phi(\varepsilon x) = \phi_0(0) + \varepsilon[\phi_0'(0)\xi + \phi_1(0)]$$

is again from (3.5) and (3.6). We can match this to the inner Ψ above and get

$$a_0 = 2\sqrt{R_1R_2} + T_0, \tag{3.24}$$

$$b_0 = \frac{1}{2}\ln\frac{R_{10}}{R_{20}} - \frac{I_0}{T_0}\ln 2\sqrt{R_1R_2}. \tag{3.25}$$

To find a_1, again take the the outer equation $c(x) = c_0(x) + \varepsilon c_1(x)$ and substitute the inner variable, which is now $x = 1 - \varepsilon\xi$. We get the same outer c as from the system at $x = 0$. Match this to the inner D_1 and D_2 equations from (3.13) and (3.18) which are again equal, and without the transcendental terms, look like

$$D(\xi) = \frac{N}{2} + \varepsilon\left[-\frac{J_{20} + J_{10}}{2}\xi + \frac{2rJ_{20}}{\sqrt{N}(1+r)} - \frac{2rJ_{10}}{\sqrt{N}(1-r)}\right].$$

Thus

$$a_1 = T_1 + \frac{4rJ_{20}}{\sqrt{N}(1+r)} - \frac{4rJ_{10}}{\sqrt{N}(1-r)}, \tag{3.26}$$

$$b_1 = -\frac{I_0(a_1T_0 - a_0T_1)}{2\sqrt{R_1R_2}T_0^2} - \frac{T_0I_1 - T_1I_0}{T_0^2}\ln 2\sqrt{R_1R_2} - \frac{2rJ_{20}}{\sqrt{R_1R_2N}(1+r)}\frac{2rJ_{10}}{\sqrt{R_1R_2N}(1-r)}. \tag{3.27}$$

Notice that the a_1 near $x = 0$ in (3.22) did not depend on the first order currents J_{11}, J_{21}, but the a_1 near $x = 1$ in (3.26) does. Since the regular layer should connect the two boundary layers, these constants should be equal. So, matching the two a_1's and b_1's give us two equations in the two unknowns J_{11}, J_{21} which we can solve explicitly.

$$
\begin{aligned}
T_1 =& \frac{4r(I_0+rT_0)}{\sqrt{N}(1+r)(1-r)} - \frac{4l(I_0+lT_0)}{\sqrt{M}(1+l)(1-l)}, \\
I_1 =& \frac{I_0T_1}{T_0} - \frac{I_0(a_1T_0-a_0T_1)}{T_0(\ln R_1R_2 - \ln L_1L_2)}\left(\frac{1}{\sqrt{R_1R_2}} - \frac{1}{\sqrt{L_1L_2}}\right) - \\
& \frac{4rT_0(T_0+rI_0)}{\sqrt{R_1R_2N}(1+r)(1-r)} + \frac{4lT_0(T_0+lI_0)}{\sqrt{L_1L_2M}(1+l)(1-l)}.
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
J_{21}+J_{11} =& \frac{4l}{\sqrt{M}}\left(\frac{J_{20}}{1+l} - \frac{J_{10}}{1-l}\right) - \frac{4r}{\sqrt{N}}\left(\frac{J_{20}}{1+r} - \frac{J_{10}}{1-r}\right), \\
J_{11}J_{20} - J_{10}J_{21} =& \frac{2(J_{20}+J_{10})^2}{\ln M - \ln N}\left[\frac{r}{N^{\frac{3}{2}}}\left(\frac{J_{10}}{1-r} + \frac{J_{20}}{1+r} - \frac{J_{20}-J_{10}}{J_{20}+J_{10}}\left(\frac{J_{20}}{1+r} - \frac{J_{10}}{1-r}\right)\right)\right. \\
& \left. - \frac{l}{M^{\frac{3}{2}}}\left(\frac{J_{10}}{1-l} + \frac{J_{20}}{1+l} - \frac{J_{20}-J_{10}}{J_{20}+J_{10}}\left(\frac{J_{20}}{1+l} - \frac{J_{10}}{1-l}\right)\right)\right].
\end{aligned}
$$

where $l = \frac{L_{20}^{\frac{1}{4}} - L_{10}^{\frac{1}{4}}}{L_{20}^{\frac{1}{4}} + L_{10}^{\frac{1}{4}}}$, $r = \frac{R_{20}^{\frac{1}{4}} - R_{10}^{\frac{1}{4}}}{R_{20}^{\frac{1}{4}} + R_{10}^{\frac{1}{4}}}$, $M = 2\sqrt{L_{10}L_{20}}$, and $N = 2\sqrt{R_{10}R_{20}}$. This solves explicitly for the first order I-V relation I_1 from (3.3).

Chapter 4

Higher Order Approximations in the condition of Electron Neutrality

Notice however, when $L_1 = L_2$ and $R_1 = R_2$, a situation that is called electron neutrality, l and r are zero so $I_1 = 0$. Since the I-V curve is empirically known to be nonlinear, we can find the higher order correction in this case. Notice that the calculation is simpler, because we can take $L_1 = L_2 \equiv L$ and $R_1 = R_2 \equiv R$ and because all the higher order systems have the same homogeneous part.

4.1 Second order approximation in electron neutrality

Consider the higher order expansion of the outer system below from (3.4).

$$
\begin{aligned}
&\varepsilon^2(\ddot{\phi}_0 + \varepsilon\ddot{\phi}_1 + \varepsilon^2\ddot{\phi}_2) - (c_{20} + \varepsilon c_{21} + \varepsilon^2 c_{22}) + (c_{10} + \varepsilon c_{11} + \varepsilon^2 c_{12}) = 0,\\
&(\dot{c}_{10} + \varepsilon\dot{c}_{11} + \varepsilon^2\dot{c}_{12}) = -(\dot{\phi}_0 + \varepsilon\dot{\phi}_1 + \varepsilon^2\dot{\phi}_2)(c_{10} + \varepsilon c_{11} + \varepsilon^2 c_{12}) - J_{10} - \varepsilon J_{11} - \varepsilon^2 J_{12},\\
&(\dot{c}_{20} + \varepsilon\dot{c}_{21} + \varepsilon^2\dot{c}_{22}) = (\dot{\phi}_0 + \varepsilon\dot{\phi}_1 + \varepsilon^2\dot{\phi}_2)(c_{20} + \varepsilon c_{21} + \varepsilon^2 c_{22}) - J_{20} - \varepsilon J_{21} - \varepsilon^2 J_{22}.
\end{aligned}
$$

From this system, consider the system of second order terms in ε.

$$
\begin{aligned}
&\ddot{\phi}_0 - c_{22} + c_{12} = 0,\\
&\dot{c}_{12} + c_{10}\dot{\phi}_2 + c_{11}\dot{\phi}_1 + c_{12}\dot{\phi}_0 = -J_{12},\\
&\dot{c}_{22} - c_{20}\dot{\phi}_2 - c_{21}\dot{\phi}_1 - c_{22}\dot{\phi}_0 = -J_{22}.
\end{aligned}
\tag{4.1}
$$

4.1.1 The Second Order Outer System

In the condition of electron neutrality, the information from the zeroth and first order outer systems is as follows:

$$
\begin{aligned}
&a_0 = 2L, \quad b_0 = \nu_0 - \frac{I_0}{T_0}, \quad c_{10} = c_{20} = \frac{a_0 - T_0 x}{2}, \quad \phi_0 = b_0 - \frac{I_0}{T_0} \ln|a_0 - T_0 x|, \\
&T_0 = 2(L-R), \quad I_0 = \frac{2\nu_0(L-R)}{\ln L - \ln R}, \quad a_1 = b_1 = c_{11} = c_{21} = \phi_1 = T_1 = I_1 = 0.
\end{aligned} \tag{4.2}
$$

So adding and subtracting the second and third equations in (4.1), we get

$$\dot{c}_{22} + \dot{c}_{12} = \ddot{\phi}_0 \dot{\phi}_0 - T_2,$$

$$\dot{c}_{22} - \dot{c}_{12} = 2c_{10}\dot{\phi}_2 + \dot{\phi}_0(c_{12} + c_{22}) + I_2,$$

where $T_2 = J_{12} + J_{22}$ and $I_2 = J_{12} - J_{22}$. Notice the first equation can be integrated, since ϕ_0 is known. So

$$c_{22} + c_{12} = \frac{1}{2}(\dot{\phi}_0)^2 + a_2 - T_2 x = \frac{I_0^2}{2(a_0 - T_0 x)^2} + a_2 - T_2 x,$$

for some constant a_2. Notice that $c_{22} - c_{12} = \ddot{\phi}_0 = \frac{-I_0 T_0}{(a_0 - T_0 x)^2}$, so we can find c_{12} and c_{22}. Also, since $c_{22} - c_{12} = \ddot{\phi}_0$, the difference equation of c_{12} and c_{22} can be solved explicitly for $\dot{\phi}_2$, and integrated to get

$$\dot{\phi}_2 = \frac{\dddot{\phi}_0 - \dot{\phi}_0 \left(\frac{I_0^2}{2(a_0 - T_0 x)^2} + a_2 - T_2 x \right) - I_2}{2c_{10}},$$

so we get the solutions

$$
\begin{aligned}
\phi_2 &= b_2 - \frac{4I_0 T_0^2 + I_0^3}{6T_0(a_0 - T_0 x)^3} + \frac{I_0(a_2 T_0 - a_0 T_2)}{T_0^2(a_0 - T_0 x)} + \frac{I_2 T_0 - I_0 T_2}{T_0^2} \ln|a_0 - T_0 x|, \\
c_{12} &= \frac{a_2 - T_2 x}{2} + \frac{I_0^2 + 2I_0 T_0}{4(a_0 - T_0 x)^2}, \\
c_{22} &= \frac{a_2 - T_2 x}{2} + \frac{I_0^2 - 2I_0 T_0}{4(a_0 - T_0 x)^2},
\end{aligned} \tag{4.3}
$$

for some constant b_2, where $T_2 = J_{12} + J_{22}$ and $I_2 = J_{12} - J_{22}$.

4.1.2 The Second Order Inner System

The second order inner system is

$$
\begin{aligned}
\Phi_2' &= U_2, \\
U_2' &= C_{22} - C_{12}, \\
C_{12}' &= -(U_0 C_{12} + U_1 C_{11} + U_2 C_{10}) - J_{11}, \\
C_{22}' &= U_0 C_{22} + U_1 C_{21} + U_2 C_{20} - J_{21}.
\end{aligned}
\tag{4.4}
$$

Write this system as a matrix equation.

$$
\begin{pmatrix} \Phi_2 \\ U_2 \\ C_{12} \\ C_{22} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -C_{10} & -U_0 & 0 \\ 0 & C_{20} & 0 & U_0 \end{pmatrix} \begin{pmatrix} \Phi_2 \\ U_2 \\ C_{12} \\ C_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -J_{11} - U_1 C_{11} \\ -J_{21} + u_1 C_{21} \end{pmatrix}.
$$

Using Lemma 2, we know that the homogeneous equation has the following integrals:

$$
\begin{aligned}
H_1^0 &= U_0 U_2 - C_{12} - C_{22}, \\
H_2^0 &= C_{22} e^{-\Phi_0} - \Phi_2 C_{20} e^{-\Phi_0}, \\
H_3^0 &= C_{12} e^{\Phi_0} + \Phi_2 C_{10} e^{\Phi_0}.
\end{aligned}
$$

By taking the derivative of these integrals with respect of ξ, we can find the inhomogeneous term contributed to the system (4.4) by C_{12} and C_{22}. Thus the integrals to the inhomogeneous system (4.4) are

$$
\begin{aligned}
H_1 &= U_0 U_2 - C_{12} - C_{22} - T_1 \xi - \frac{1}{2} U_1^2, \\
H_2 &= C_{22} e^{-\Phi_0} - \Phi_2 C_{20} e^{-\Phi_0} + J_{21} F_2(\xi) - F_{22}(\xi), \\
H_3 &= C_{12} e^{\Phi_0} + \Phi_2 C_{10} e^{\Phi_0} + J_{11} F_1(\xi) + F_{21}(\xi),
\end{aligned}
$$

where $T_1 = J_{11} + J_{21}$, $F_1(\xi) = \int_0^\xi e^{\Phi_0(s)} ds$, $F_2(\xi) = \int_0^\xi e^{-\Phi_0(s)} ds$ as in the integrals for the first order inner system. Also

$$
F_{21} = \int_0^\xi C_{11}(s) U_1(s) e^{\Phi_0(s)} ds, \; F_{22} = \int_0^\xi C_{21}(s) U_1(s) e^{-\Phi_0(s)} ds.
$$

In the condition of electron neutrality, the information from the zeroth and first order outer systems is as follows:

$$
\begin{aligned}
&\Phi_0 = v_0,\, U_0 = 0,\, C_{10} = C_{20} = L,\, F_1(\xi) = e^{v_0}\xi,\, F_2(\xi) = e^{-v_0}\xi, \\
&\Phi_1 = \frac{-I_0}{2L}\xi,\, U_1 = \frac{-I_0}{2L},\, C_{11} = C_{21} = \frac{-T_0}{2}\xi, \\
&F_{12} = \frac{I_0T_0}{8L}e^{v_0}\xi^2,\, F_{22} = \frac{I_0T_0}{8L}e^{-v_0}\xi^2,\, J_{11} = J_{21} = 0.
\end{aligned} \tag{4.5}
$$

With this information, the second and third integrals give that

$$
C_{12} = \frac{-I_0T_0}{8L}\xi^2 - L\Phi_2,\, C_{22} = \frac{I_0T_0}{8L}\xi^2 + L\Phi_2.
$$

The third order outer system tells us that $\Phi_2'' = C_{22} - C_{21}$, so we get the following differential equation:

$$
\Phi_2'' = \frac{I_0T_0}{4L}\xi^2 + 2L\Phi_2.
$$

Solving with the method of undetermined coefficients, we get

$$
\begin{aligned}
&\Phi_2 = \frac{I_0T_0}{8L^3}\left(e^{-\sqrt{2L}\xi} - 1\right) - \frac{I_0T_0}{8L^2}\xi^2, \\
&C_{12} = \frac{-I_0T_0}{8L^2}\left(e^{-\sqrt{2L}\xi} - 1\right),\, C_{22} = \frac{I_0T_0}{8L^2}\left(e^{-\sqrt{2L}\xi} - 1\right).
\end{aligned} \tag{4.6}
$$

Likewise, near $x = 1$, we get

$$
\begin{aligned}
&\Psi_2 = \frac{I_0T_0}{8R^3}\left(e^{\sqrt{2R}\xi} - 1\right) - \frac{I_0T_0}{8R^2}\xi^2, \\
&D_{12} = \frac{-I_0T_0}{8R^2}\left(e^{\sqrt{2R}\xi} - 1\right),\, D_{22} = \frac{I_0T_0}{8R^2}\left(e^{\sqrt{2R}\xi} - 1\right).
\end{aligned} \tag{4.7}
$$

4.1.3 Second Order Matching

Near x=0: As in the matching of the first order terms, we will match the Taylor expansion of $\phi(\varepsilon\xi)$ with $\Phi(\xi)$. Notice near $x=0$,

$$\begin{aligned}
\phi(\varepsilon\xi) =& \phi_0(0)+\phi_0'(0)(\varepsilon\xi)+\frac{\phi_0''(0)}{2}(\varepsilon\xi)^2+\varepsilon\left(\phi_1(0)+\phi_1'(0)(\varepsilon\xi)+\frac{\phi_1''(0)}{2}(\varepsilon\xi)^2\right)\\
&+\varepsilon^2\left(\phi_2(0)\right)\\
=&\phi_0(0)+\varepsilon\left(\phi_1(0)+\phi_0'(0)\xi\right)+\varepsilon^2\left(\phi_2(0)+\phi_1'(0)\xi+\phi_0''(0)\xi^2\right)\\
=&b_0-\frac{I_0}{T_0}\ln a_0+\varepsilon\left(\frac{I_0}{a_0}\xi\right)\\
&+\varepsilon^2\left(b_2-\frac{4I_0T_0^2+I_0^3}{6T_0a_0^3}+\frac{I_0(a_2T_0-a_0T_2)}{T_0^2a_0}+\frac{I_2T_0-I_0T_2}{T_0^2}\ln a_0-\frac{I_0T_0}{2a_0^2}\xi^2\right)
\end{aligned}$$

and

$$c_1(\varepsilon\xi)=\frac{a_0}{2}-\varepsilon\frac{T_0}{2}\xi+\varepsilon^2\left(\frac{a_2}{2}+\frac{I_0^2+2T_0I_0}{4a_0^2}\right),$$

$$c_2(\varepsilon\xi)=\frac{a_0}{2}-\varepsilon\frac{T_0}{2}\xi+\varepsilon^2\left(\frac{a_2}{2}+\frac{I_0^2-2T_0I_0}{4a_0^2}\right),$$

from (3.5), (3.6), and (4.3). Dropping the terms that go to 0 as $\xi\to\infty$, from (3.12), (3.17) and (4.6), we have

$$\Phi(\xi)=v_0-\varepsilon\left(\frac{I_0}{2L}\xi\right)-\varepsilon^2\left(\frac{I_0T_0}{8L^3}+\frac{I_0T_0}{8L^2}\xi^2\right),$$

$$C_1(\xi)=L-\varepsilon\frac{T_0}{2}\xi+\varepsilon^2\frac{I_0T_0}{8L^2},\quad C_2(\xi)=L-\varepsilon\frac{T_0}{2}\xi-\varepsilon^2\frac{I_0T_0}{8L^2}.$$

Near x=1: Using the Taylor expansion near $x=1$ with (3.5), (3.6), and (4.3) , we have $x=\varepsilon\xi+1$ and so find

$$\begin{aligned}
\phi(\varepsilon\xi+1)=&\phi_0(1)+\phi_0'(1)(\varepsilon\xi+1-1)+\frac{\phi_0''(1)}{2}(\varepsilon\xi+1-1)^2\\
&+\varepsilon(\phi_1(1)+\phi_1'(1)(\varepsilon\xi+1-1)+\frac{\phi_1''(1)}{2}(\varepsilon\xi+1-1)^2)+\varepsilon^2\left(\phi_2(1)\right)\\
=&\phi_0(1)+\varepsilon\left(\phi_1(1)+\phi_0'(1)\xi\right)+\varepsilon^2\left(\phi_2(1)+\phi_1'(1)\xi+\phi_0''(1)\xi^2\right)\\
=&b_0+\frac{I_0}{T_0}\ln(a_0-T_0)-\varepsilon\left(\frac{I_0}{a_0-T_0}\xi\right)\\
&+\varepsilon^2\left(b_2-\frac{4I_0T_0^2+I_0^3}{6T_0(a_0-T_0)^3}+\frac{I_0(a_2T_0-a_0T_2)}{T_0^2(a_0-T_0)}+\frac{I_2T_0-I_0T_2}{T_0^2}\ln(a_0-T_0)-\frac{I_0T_0}{2(a_0-T_0)^2}\xi^2\right),
\end{aligned}$$

and

$$c_1(\varepsilon\xi+1) = \frac{a_0-T_0}{2} - \varepsilon\frac{T_0}{2}\xi + \varepsilon^2\left(\frac{a_2-T_2}{2} + \frac{I_0^2+2I_0T_0}{4(a_0-T_0)^2}\right),$$

$$c_2(\varepsilon\xi+1) = \frac{a_0-T_0}{2} - \varepsilon\frac{T_0}{2}\xi + \varepsilon^2\left(\frac{a_2-T_2}{2} + \frac{I_0^2-2I_0T_0}{4(a_0-T_0)^2}\right).$$

From (3.13), (3.18), and (4.7), get

$$\Psi(\xi) = -\varepsilon\left(\frac{I_0}{2R}\xi\right) - \varepsilon^2\left(\frac{I_0T_0}{8R^3} + I_0T_0 8R^2\xi^2\right).$$

$$D_1(\xi) = R - \varepsilon\frac{T_0}{2} + \varepsilon^2\frac{I_0T_0}{8R^2}, \quad D_2(\xi) = R - \varepsilon\frac{T_0}{2} - \varepsilon^2\frac{I_0T_0}{8R^2}.$$

Comparing like ordered terms, we get

$$\begin{aligned}
&a_2 = -\frac{I_0^2}{8L^2},\ T_2 = \frac{I_0^2(L^2-R^2)}{8L^2R^2},\\
&T_0I_2 - I_0T_2 = \frac{I_0T_0(2T_0^2+I_0^2)(L^3-R^3)}{48L^3R^3(\ln L - \ln R)} + \frac{I_0(a_2T_0-a_0T_2)(L-R)}{2LR(\ln L-\ln R)}.
\end{aligned}$$

as well as confirming the results from the zeroth and first order matchings. So

$$\begin{aligned}
T_2 =& \frac{(L-R)^3(L+R)}{2L^2R^2(\ln L-\ln R)^2},\\
I_2 =& \frac{(L-R)^4(L^2+LR+R^2)v_0}{3L^3R^3(\ln L-\ln R)^2} - \\
& \frac{(L-R)^3(L^3-R^3)v_0^3}{3L^3R^3(\ln L-\ln R)^4} + \frac{(L-R)^2(L^2-R^2)v_0^3}{2L^2R^2(\ln L-\ln R)^3}.
\end{aligned} \tag{4.8}$$

4.2 Third order approximation in electron neutrality

Consider the third order terms in ε from (3.4)

$$\begin{aligned}
&\ddot{\phi}_1 - c_{23} + c_{13} = 0,\\
&\dot{c}_{13} + c_{10}\dot{\phi}_3 + c_{13}\dot{\phi}_0 + c_{12}\dot{\phi}_1 + c_{11}\dot{\phi}_2 = -J_{13},\\
&\dot{c}_{23} - c_{20}\dot{\phi}_3 - c_{23}\dot{\phi}_0 - c_{22}\dot{\phi}_1 - c_{21}\dot{\phi}_2 = -J_{23}.
\end{aligned} \tag{4.9}$$

4.2.1 The Third Order Outer System

In the electron neutrality condition considered in (4.2), we have $c_{11} = c_{21} = \phi_1 = \dot{\phi}_1 = \ddot{\phi}_1 = 0$, so we can add the last two equations of (4.9) to get

$$\dot{c}_{13} + \dot{c}_{23} = \dot{\phi}_0(c_{23} - c_{13}) - T_3,$$

where $T_3 = J_{13} + J_{23}$. Then the first equation from the system tells us that

$$\dot{c}_{13} + \dot{c}_{23} = \dot{\phi}_0\ddot{\phi}_1 - T_3 \implies c_{13} + c_{23} = a_3 - T_3 x,$$

for some constant a_3. Notice that the first equation says that $c_{13} = c_{23}$, so we can find c_{13} and c_{23}.

Subtracting the last two equations of (4.9), we get $\dot{c}_{23} - \dot{c}_{13} = \dot{\phi}_3(c_{20} + c_{10}) + \dot{\phi}_0(c_{23} + c_{13}) - I_3$, where $I_3 = J_{13} - J_{23}$. And by the first equation of (4.9), this difference is $\ddot{\phi}_1 = 0$. So we get

$$\dot{\phi}_3 = \frac{0 - \dot{\phi}_0(c_{23} + c_{13}) - I_3}{c_{20} + c_{10}} = \frac{I_0(a_3 - T_3 x)}{(a_0 - T_0 x)^2} + \frac{I_3}{(a_0 - T_0 x)} = \frac{(a_3 I_0 - a_0 I_3) + (T_0 I_3 - T_3 I_0)x}{(a_0 - T_0 x)^2}.$$

Integrating, we get ϕ_3. The solutions to (4.9) are

$$\begin{aligned}
\phi_3 &= b_3 + \frac{a_3 I_0 - a_0 I_3}{T_0(a_0 - T_0 x)} + (T_0 I_3 - I_0 T_3)\left(\frac{x}{T_0(a_0 - T_0 x)} + \frac{1}{T_0^2}\ln|a_0 - T_0 x|\right), \\
c_{13} &= c_{23} = \frac{a_3 - T_3 x}{2},
\end{aligned} \tag{4.10}$$

where a_3 and b_3 are constants and $T_3 = J_{13} + J_{23}$ and $I_3 = J_{13} - J_{23}$.

4.2.2 The Third Order Inner System

Expanding the fast system, we get the following system of third order terms:

$$\begin{aligned}
\Phi_3'' &= C_{23} - C_{13}, \\
C_{13}' &= -C_{10}U_3 - C_{13}U_0 - C_{12}U_1 - C_{11}U_2 - J_{12}, \\
C_{23}' &= C_{20}U_3 + C_{23}U_0 + C_{22}U_1 + C_{21}U_2 - J_{22}.
\end{aligned}$$

As in the previous approximations, define a function $U_3 = \Phi_3'$. Consider the system

$$
\begin{aligned}
&\Phi_3' = U_3,\\
&U_3' = C_{23} - C_{13},\\
&C_{13}' = -C_{10}U_3 - C_{13}U_0 - C_{12}U_1 - C_{11}U_2 - J_{12},\\
&C_{23}' = C_{20}U_3 + C_{23}U_0 + C_{22}U_1 + C_{21}U_2 - J_{22}.
\end{aligned} \tag{4.11}
$$

Clearly, this system has the same homogeneous part as the inner system of first order terms, or the inner system of second order terms. Therefore, the homogeneous part of the system above has the previously defined integrals: $H_1^0, H_2^0,$ and H_3^0. Again, we have to find the inhomogeneous contributions from the C_{13} and C_{23} equations. We find the integrals to the inhomogeneous system :

$$
\begin{aligned}
&H_1 = U_0U_3 - C_{13} - C_{23} - T_2\xi + \frac{1}{2}U_1^2 + \frac{1}{2}U_2^2,\\
&H_2 = C_{13}e^{\Phi_0} + \Phi_3 C_{10}e^{\Phi_0} + J_{12}F_1(\xi) + F_{13}(\xi),\\
&H_3 = C_{23}e^{-\Phi_0} - \Phi_3 C_{20}e^{-\Phi_0} + J_{22}F_2(\xi) - F_{23}(\xi).
\end{aligned}
$$

Thus we find

$$
C_{13} = -L\Phi_3 - e^{-\nu_0}F_{13}(\xi),\; C_{23} = L\Phi_3 + e^{\nu_0}F_{23}(\xi),
$$

where

$$
F_{13}(\xi) = \int_0^{\xi} C_{11}(s)U_2(s)e^{\Phi_0(s)} + C_{12}(s)U_1(s)e^{\Phi_0(s)}ds,
$$

$$
F_{23}(\xi) = \int_0^{\xi} C_{21}(s)U_2(s)e^{-\Phi_0(s)} + C_{22}(s)U_1(s)e^{-\Phi_0(s)}ds.
$$

After integrating, we find

$$
F_{13} = e^{\nu_0}\left(\frac{I_0T_0^2}{24L^2}\xi^3 - \frac{I_0T_0}{16L^3}\xi\left(T_0e^{-\sqrt{2L}\xi} + I_0\right) - \frac{I_0T_0(T_0+I_0)}{16\sqrt{2}L^{\frac{7}{2}}}\left(e^{-\sqrt{2L}\xi} - 1\right)\right),
$$

$$
F_{23} = e^{-\nu_0}\left(\frac{I_0T_0^2}{24L^2}\xi^3 - \frac{I_0T_0}{16L^3}\xi\left(T_0e^{-\sqrt{2L}\xi} - I_0\right) - \frac{I_0T_0(T_0-I_0)}{16\sqrt{2}L^{\frac{7}{2}}}\left(e^{-\sqrt{2L}\xi} - 1\right)\right).
$$

We substitute the equations from the integrals to get the following second order differential equation:

$$
\Phi_3'' = C_{23} - C_{13} = 2L\Phi_3 + e^{\Phi_0}F_{23}(\xi) + e^{-\Phi_0}F_{13}(\xi).
$$

We can solve this differential equation using the method of undetermined coefficients to find

$$\Phi_3 = k_1 e^{\sqrt{2L}\xi} + k_2 e^{-\sqrt{2L}\xi} + I_0 T_0^2 \left(\frac{-1}{16\sqrt{2}L^{\frac{9}{2}}} - \frac{1}{8L^4}\xi - \frac{1}{24L^3}\xi^3 + \frac{3}{64L^4}\xi e^{-\sqrt{2L}\xi} + \frac{1}{32\sqrt{2}L^{\frac{7}{2}}}\xi^2 e^{-\sqrt{2L}\xi} \right),$$

for constants k_1 and k_2. To control the $e^{\sqrt{2L}\xi}$ term, $k_1 = 0$. Using the initial condition, we get $k_2 = \frac{I_0 T_0^2}{16\sqrt{2}L^{\frac{9}{2}}}$, Thus we get

$$\begin{aligned}
\Phi_3 &= \frac{-I_0 T_0^2}{16\sqrt{2}L^{\frac{9}{2}}} - \frac{I_0 T_0^2}{8L^4}\xi - \frac{I_0 T_0^2}{24L^3}\xi^3 + \frac{I_0 T_0^2}{16\sqrt{2}L^{\frac{9}{2}}} e^{-\sqrt{2L}\xi} \\
&\qquad + \frac{3 I_0 T_0^2}{64L^4}\xi e^{-\sqrt{2L}\xi} + \frac{I_0 T_0^2}{32\sqrt{2}L^{\frac{7}{2}}}\xi^2 e^{-\sqrt{2L}\xi}, \\
C_{13} &= \frac{-I_0^2 T_0}{16\sqrt{2}L^{\frac{7}{2}}} + \frac{I_0 T_0}{16L^3}(I_0 + 2T_0)\xi + \frac{I_0^2 T_0}{16\sqrt{2}L^{\frac{7}{2}}} e^{-\sqrt{2L}\xi} + \\
&\qquad \frac{I_0 T_0^2}{64L^3}\xi e^{-\sqrt{2L}\xi} - \frac{I_0 T_0^2}{32\sqrt{2}L^{\frac{5}{2}}}\xi^2 e^{-\sqrt{2L}\xi}, \\
C_{23} &= \frac{-I_0^2 T_0}{16\sqrt{2}L^{\frac{7}{2}}} + \frac{I_0 T_0}{16L^3}(I_0 - 2T_0)\xi + \frac{I_0^2 T_0}{16\sqrt{2}L^{\frac{7}{2}}} e^{-\sqrt{2L}\xi} - \\
&\qquad \frac{I_0 T_0^2}{64L^3}\xi e^{-\sqrt{2L}\xi} + \frac{I_0 T_0^2}{32\sqrt{2}L^{\frac{5}{2}}}\xi^2 e^{-\sqrt{2L}\xi}.
\end{aligned} \tag{4.12}$$

Likewise, at $x = 1$, with the initial condition $\Psi(1) = 0$, we get

$$
\begin{aligned}
\Psi_3 = & \frac{-I_0T_0^2}{16\sqrt{2}R^{\frac{9}{2}}} - \frac{I_0T_0^2}{8R^4}\xi - \frac{I_0T_0^2}{24R^3}\xi^3 + \frac{I_0T_0^2}{16\sqrt{2}R^{\frac{9}{2}}}e^{\sqrt{2R}\xi} \\
& + \frac{3I_0T_0^2}{64R^4}\xi e^{\sqrt{2R}\xi} + \frac{I_0T_0^2}{32\sqrt{2}R^{\frac{7}{2}}}\xi^2 e^{\sqrt{2R}\xi}, \\
D_{13} = & \frac{-I_0^2T_0}{16\sqrt{2}R^{\frac{7}{2}}} + \frac{I_0T_0}{16R^3}(I_0+2T_0)\xi + \frac{I_0^2T_0}{16\sqrt{2}R^{\frac{7}{2}}}e^{\sqrt{2R}\xi} + \\
& \frac{I_0T_0^2}{64R^3}\xi e^{\sqrt{2R}\xi} - \frac{I_0T_0^2}{32\sqrt{2}R^{\frac{5}{2}}}\xi^2 e^{\sqrt{2R}\xi}, \\
D_{23} = & \frac{-I_0^2T_0}{16\sqrt{2}R^{\frac{7}{2}}} + \frac{I_0T_0}{16R^3}(I_0-2T_0)\xi + \frac{I_0^2T_0}{16\sqrt{2}R^{\frac{7}{2}}}e^{\sqrt{2R}\xi} - \\
& \frac{I_0T_0^2}{64R^3}\xi e^{\sqrt{2R}\xi} + \frac{I_0T_0^2}{32\sqrt{2}R^{\frac{5}{2}}}\xi^2 e^{\sqrt{2R}\xi}.
\end{aligned}
\tag{4.13}
$$

4.2.3 Third Order Matching

Near x=0: Using the process we are now familiar with, we will match the Taylor expansion of $\phi(\varepsilon\xi)$ with $\Phi(\xi)$. Notice near $x = 0$,

$$
\begin{aligned}
\phi(\varepsilon\xi) = & \phi_0(0) + \phi_0'(0)(\varepsilon\xi) + \frac{\phi_0''(0)}{2}(\varepsilon\xi)^2 + \frac{\phi_0'''(0)}{6}(\varepsilon\xi)^3 + \\
& \varepsilon\left(\phi_1(0) + \phi_1'(0)(\varepsilon\xi) + \frac{\phi_1''(0)}{2}(\varepsilon\xi)^2\right) + \varepsilon^2\left(\phi_2(0) + \phi_2'(0)(\varepsilon\xi)\right) + \varepsilon^3(\phi_3(0)) \\
= & \phi_0(0) + \varepsilon\left(\phi_1(0) + \phi_0'(0)\xi\right) + \varepsilon^2\left(\phi_2(0) + \phi_1'(0)\xi + \phi_0''(0)\xi^2\right) + \\
& \varepsilon^3\left(\phi_3(0) + \phi_2'(0)\xi + \frac{\phi_1''(0)}{2}\xi^2 + \frac{\phi_0'''(0)}{6}\xi^3\right)
\end{aligned}
$$

So the coefficient of the ε^3 term in $\phi_3(\varepsilon\xi)$ is

$$
\begin{aligned}
\phi_3(0) + \phi_2'(0)\xi + \frac{\phi_1''(0)}{2}\xi^2 + \frac{\phi_0'''(0)}{6}\xi^3 = & -\frac{I_0T_0^2}{3a_0^3}\xi^3 + \left(\frac{2I_0(a_1T_0 - a_0T_1) - a_0(T_0I_1 - I_0T_1)}{a_0^2T_0}\right)\xi^2 + \\
& \left(\frac{I_0(a_2T_0 - a_0T_2) + a_0(I_2T_0 - I_0T_2)}{a_0^2T_0} - \frac{4I_0T_0^2}{2a_0^4}\right)\xi + b_3 + \frac{a_3I_0 - a_0I_3}{a_0T_0} + \frac{T_0I_3 - I_0T_3}{T_0^2}\ln|a_0|.
\end{aligned}
$$

And $\Phi(\xi)$ from (3.12), (3.17), (4.6), and (4.12)with transcendental terms omitted is

$$
\Phi(\xi) = v_0 - \varepsilon\left(\frac{I_0}{2L}\xi\right) - \varepsilon^2\left(\frac{I_0T_0}{8L^3} + \frac{I_0T_0}{8L^2}\xi^2\right) - \varepsilon^3\left(\frac{I_0T_0^2}{16\sqrt{2}L^{\frac{9}{2}}} + \frac{I_0T_0^2}{8L^4}\xi + \frac{I_0T_0^2}{24L^3}\xi^3\right).
$$

We also have the equations

$$c_1(\varepsilon\xi) = \frac{a_0}{2} - \varepsilon\frac{T_0}{2}\xi + \varepsilon^2\left(\frac{a_2}{2} + \frac{I_0^2 + 2T_0I_0}{4a_0^2}\right) + \varepsilon^3\left(\frac{a_3}{2} + \left(\frac{T_0(I_0^2+2I_0T_0)}{2a_0^3} - \frac{T_2}{2}\right)\xi\right),$$

$$c_2(\varepsilon\xi) = \frac{a_0}{2} - \varepsilon\frac{T_0}{2}\xi + \varepsilon^2\left(\frac{a_2}{2} + \frac{I_0^2 - 2T_0I_0}{4a_0^2}\right) + \varepsilon^3\left(\frac{a_3}{2} + \left(\frac{T_0(I_0^2-2I_0T_0)}{2a_0^3} - \frac{T_2}{2}\right)\xi\right),$$

to compare with

$$C_1(\xi) = L - \varepsilon\frac{T_0}{2}\xi + \varepsilon^2\frac{I_0T_0}{8L^2} + \varepsilon^3\left(\frac{-I_0^2T_0}{16\sqrt{2}L^{\frac{7}{2}}} + \frac{I_0T_0}{16L^3}(I_0+2T_0)\xi\right),$$
$$C_2(\xi) = L - \varepsilon\frac{T_0}{2}\xi - \varepsilon^2\frac{I_0T_0}{8L^2} + \varepsilon^3\left(\frac{-I_0^2T_0}{16\sqrt{2}L^{\frac{7}{2}}} + \frac{I_0T_0}{16L^3}(I_0+2T_0)\xi\right).$$

Matching the C equations, we find the consistent result

$$a_3 = \frac{-I_0^2T_0}{8\sqrt{2}L^{\frac{7}{2}}}. \tag{4.14}$$

Near x=1: We first match the C equations:

$$c(\varepsilon\xi+1) = c_{10}(1) + \varepsilon(c_{11}(1) + c_{10}'(1)\xi) + \varepsilon^2\left(c_{12}(1) + c_{11}'(1)\xi + c_{10}''(1)\xi^2\right)$$
$$+\varepsilon^3\left(c_{13}(1) + c_{12}'(1)\xi + c_{11}''(1)\xi^2 + c_{10}'''(1)\xi^3\right)).$$

Then the coefficient of the ε^3 term is $\frac{a_3-T_3}{2} + \left(\frac{T_0I_0(I_0+2T_0)}{2(a_0-T_0)} - \frac{T_2}{2}\right)\xi$, which we match with

$$D(\xi) = \frac{-I_0^2T_0}{16\sqrt{2}R^{\frac{7}{2}}} + \frac{I_0T_0}{16R^3}(I_0+2T_0)\xi$$

to find $T_3 = a_3 + \frac{I_0^2T_0}{8\sqrt{2}R^{\frac{7}{2}}} = \frac{I_0^2T_0}{8\sqrt{2}}\left(\frac{1}{R^{\frac{7}{2}}} - \frac{1}{L^{\frac{7}{2}}}\right)$. Now consider

$$\phi(\varepsilon\xi+1) = \phi_0(1) + \varepsilon\left(\phi_1(1) + \phi_0'(1)\xi\right) + \varepsilon^2\left(\phi_2(1) + \phi_1'(1)\xi + \phi_0''(1)\xi^2\right) +$$
$$\varepsilon^3\left(\phi_3(1) + \phi_2'(1)\xi + \frac{\phi_1''(1)}{2}\xi^2 + \frac{\phi_0'''(1)}{6}\xi^3\right).$$

The coefficient of ε^3 in $\phi(\varepsilon\xi+1)$ is

$$\phi_3(1) + \phi_2'(1)\xi + \frac{\phi_1''(1)}{2}\xi^2 + \frac{\phi_0'''(1)}{6}\xi^3$$

as in (3.5), (3.6), (4.3), and (4.10). And, with the terms that go to zero as $\xi \to -\infty$ omitted,

$$\Psi(\xi) = -\varepsilon\left(\frac{I_0}{2R}\xi\right) - \varepsilon^2\left(\frac{I_0T_0}{8R^3} + \frac{I_0T_0}{8R^2}\xi^2\right) - \varepsilon^3\left(\frac{I_0T_0^2}{16\sqrt{2}R^{\frac{9}{2}}} + \frac{I_0T_0^2}{8R^4}\xi + \frac{I_0T_0^2}{24R^3}\xi^3\right).$$

Notice first that the previous matchings of zeroth, first and second order terms can be recovered. By matching at $x = 0$ and $x = 1$, we get

$$\begin{aligned} T_3 &= \frac{I_0^2T_0}{8\sqrt{2}}\left(\frac{1}{R^{\frac{7}{2}}} - \frac{1}{L^{\frac{7}{2}}}\right), \quad I_3 = \frac{(L-R)^4}{\sqrt{2}\left(\ln\left(\frac{L}{R}\right)\right)^2}\left(\frac{1}{R^{\frac{9}{2}}} - \frac{1}{L^{\frac{9}{2}}}\right)v_0 \\ &+ \frac{(L-R)^3}{\sqrt{2}\left(\ln\left(\frac{L}{R}\right)\right)^3}\left(\left(\frac{1}{R^{\frac{9}{2}}} - \frac{1}{L^{\frac{9}{2}}}\right) - \frac{(L-R)}{\ln\left(\frac{L}{R}\right)}\left(\frac{1-\frac{R^{\frac{9}{2}}}{L^{\frac{9}{2}}}}{R^{\frac{9}{2}}}\right) + \frac{(L-R)}{L^{\frac{9}{2}}\ln\left(\frac{L}{R}\right)}\right)v_0^3. \end{aligned} \tag{4.15}$$

Chapter 5

Remarks and Conclusions

In general, for nonlinear systems, it is impossible to obtain any reasonable representations of solutions. For the Poisson-Nernst-Planck system studied in this thesis, it is the special structure of the system that allows one to get an asymptotic expansion of the boundary value problem. More precisely, due to the presence of the singular parameter ε, we can treat the problem as a singularly perturbed problem, and by considering the inner and outer systems, we get systems of lower order. Most importantly, this system with specific nonlinearity has special structures described in Lemmas 1 and 2 that are crucial for the explicit higher order asymptotic expansions of the solutions.

Recall that $I(v_0;\varepsilon) = I_0 + I_1 + I_2 + I_3 + \cdots$. From our calculations the I-V relation has been found to be nonlinear, even in the condition of electron neutrality. This corroborates empirical observations. Notice that I_0 is linear in v_0 in (3.19), and I_1 quadratic in (3.28). Then, under electron neutrality, I_2 and I_3 are both cubic in v_0 in (4.8), (4.15). It is expected that the higher order terms will depend on a higher order of v_0.

Applications of this thesis to ion channels are limited, since we considered a simplified model with two species of ions and a zero permanent charge in the channel. To generalize to n ions should not be difficult because of the symmetry in the Nernst-Planck equations. However, even a piecewise constant non-zero permanent charge makes the inner system nonhomogeneous. Then the system can be solved geometrically in the zeroth order, but not with asymptotic expansions. More work on this problem could be done in finding the integrals of the inner system with non-zero permanent charge.

The results of this thesis are related to the I-V relation in the Hodgkins-Huxley model. Using the FitzHugh-Nagumo simplification of the Hodgkins-Huxley model, the I-V relation of the transmembrane current over multiple cells has three roots. Although the ion channel problem considers only one cell, the I-V relation is consistent, since I is cubic in v_0 in higher orders.

Most likely, due to the special structures of the problem, the method could be applied to find any order approximation. One natural approach would be an automation of the method for computers.

Bibliography

[1] Nicole Abaid, Robert Eisenberg, and Weishi Liu, *Asymptotic Expansions of I-V relations via a Poisson-Nernst-Planck system*, Submitted to SIAM Journal of Applied Dynamical Systems. Cited on 13

[2] V. Barcilon, D.-P. Chen, R.S. Eisenberg, and J.W. Jerome, *Qualitative Properties of Steady-State Poisson-Nernst-Planck Systems: Perturbation and Simulation Study*, SIAM Journal of Applied Mathematics **57** (1997) 631–648. Cited on 5

[3] R.S. Eisenberg and Weishi Liu, *Poisson-Nernst-Planck Systems for Ion Channels with Permanent Charges*, SIAM Journal of Math. Anal. **38** (2007) 1932–1966. Cited on 5

[4] P.A. Lagerstrom, *Matched Asymptotic Expansions*, 1988. Cited on 6, 7, 12

[5] Weishi Liu, *Geometric Singular Perturbation Approach to Steady-State Poisson-Nernst-Planck Systems*, SIAM J. Appl. Math. **65** (2005) 754–766, 2005. Cited on 5, 14

[6] Robert E. O'Malley, Jr., *Singular Perturbation Methods for Ordinary Differential Equations*, 1991. Cited on 6

[7] Lawrence Perko, *Differential Equations and Dynamical Systems*, 2001. Cited on 5