Limit distributions for functionals of Gaussian processes

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Abstract

This thesis is devoted to the study of the convergence in distribution of functionals of Gaussian processes. Most of the problems that we present are addressed by using an approach based on Malliavin calculus techniques.

Our main contributions are the following:

First we study the asymptotic law of the approximate derivative of the self-intersection local time (SILT) in $[0,T]$ for the fractional Brownian motion. In order to do this, we describe the asymptotic behavior of the associated chaotic components and show that the first chaos approximates the SILT in L^2.

Secondly, we examine the asymptotic law of the approximate self-intersection local time process for the fractional Brownian motion. We achieve this in two steps: the first part consists on proving the convergence of the finite dimensional distributions by using the 'multidimensional fourth moment theorem'. The second part consists on proving the tightness property, for which we follow an approach based on Malliavin calculus techniques.

The third problem consists on proving a non-central limit theorem for the process of weak symmetric Riemann sums for a wide variety of self-similar Gaussian processes. We address this problem by using the so-called small blocks-big blocks methodology and a central limit theorem for the power variations of self-similar Gaussian processes.

Finally, we address the problem of determining conditions under which the eigenvalues of an Hermitian matrix-valued Gaussian process collide with positive probability.

The material we present is taken from the manuscripts [26], [27], [16], [28], which are a joint work between professors David Nualart, Daniel Harnett and myself. With the exception of [28], all of these papers have been accepted in peer reviewed journals.

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Introduction

The Malliavin calculus designates the theory and applications of a differential calculus, whose operators act on functionals of general Gaussian processes. It was initiated by Paul Malliavin and its motivation was the study of the regularity properties for the law of Wiener functionals, such as the solutions of stochastic differential equations. The range of its current applications, including density estimates, concentration inequalities, anticipative stochastic calculus and computations of “Greeks” in mathematical finance, has considerably broaden.

Our particular interest, is the relation of the theory of Malliavin calculus with limit theorems in the Wiener space. This relation was first investigated by Nualart and Peccati in a seminal paper of 2005, where a surprising central limit theorem for sequences of multiple stochastic integrals of a fixed order (nowadays referred to as “fourth moment theorem”) was proved: in this context, convergence in distribution to the standard normal law was proved to be equivalent to convergence of just the fourth moment.

There have been many refinements and applications of the fourth moment theorem. Among them is the work by Nourdin and Peccati in [37], where estimations of the distance in total variation between the law of multiple Itô integrals and the Gaussian distribution are obtained by combining Malliavin calculus techniques with the so-called Stein’s method, which can be roughly described as a collection of probabilistic techniques for estimating the distance between probability distributions by means of differential operators.

Since the publication of the aforementioned results, the interaction between the theory of Malliavin calculus and Stein's method, has played a major role in the study of limit theorems in the Wiener space, as it has led to some remarkable new results involving central and non-central limit theorems for functionals of infinite-dimensional Gaussian fields. One process for which this methodology has been particularly successful, is the fractional Brownian motion (fBm for short).

The fBm of Hurst parameter $H \in (0,1)$ is a self-similar Gaussian process with stationary increments and self-similarity exponent H, which generalizes the classical Brownian motion. It was first introduced by Kolmogorov for modeling turbulence in liquids, and was further studied by Mandelbrot and Van Ness. The behavior of the fBm is quite different as we vary the value for H: when $H > \frac{1}{2}$, its increments are positively correlated and for $H < \frac{1}{2}$, they are negatively correlated. Moreover, for $\beta \in (0,H)$, its sample paths are Hölder continuous with index β and if $H > \frac{1}{2}$, it is a long memory process. This flexibility on the behavior of the fBm, makes it very interesting for modeling purposes, since the value for H can be adjusted to accurately fit the observations of the random model we want to describe.

It is natural to ask if a stochastic calculus for fBm can be developed, which is not obvious since in general this process is not a semimartingale. For this reason, it is of great interest to investigate the theory of integration for the fractional Brownian motion as well as its associated local time and self-intersection local time. The self-intersection local time for the d-dimensional fractional Brownian motion (SILT), is a stochastic process that measures the amount of time that the trajectories of the fBm spend intersecting themselves. For the case $H = 1/2$, the SILT has been studied by many authors (see Albeverio, Hu and Zhou (1997), Calais and Yor (1987), He, Yang, Yao and Wang (1995), Hu (1996), Imkeller, Pérez Abreu and Vives (1995), Varadhan (1969), Yor (1985) and the references therein). The case $H \neq \frac{1}{2}$ was first studied by Rosen in [49] for the planar case ($d = 2$), and further investigated using techniques from Malliavin calculus by Hu and Nualart in [23].

One of the objectives of this dissertation, is to address the problem of determining the fluctuations of the approximations of the SILT, as well as those of the derivative of the SILT. We will show that, depending on the values of H and d, and after a suitable renormalization, the SILT converges in law to either a scalar multiple of a Brownian motion or a Rosenblatt process. We prove as well a central limit theorem for the derivative of the SILT and its chaotic components. Our approach relies heavily on the multivariated version of the fourth moment theorem and on techniques from Malliavin calculus. Proving a functional limit theorem for the approximations of the SILT represents a big challenge, due to the fact that the standard approach for proving tightness for a sequence of processes is hard to apply in this case. In order to overcome this difficulty, we developed a technique for proving tightness, based on Malliavin calculus and Meyer inequalities. This technique is new, and of independent interest in probability theory.

A second problem that we address concerns the integration with respect to self-similar Gaussian processes. It is well known that if $X = \{X_t\}_{t \geq 0}$ is a general Gaussian process and g is a real smooth function, the integral of $g(X)$ with respect to X doesn't exist in a general path-wise sense. Nevertheless, in [15], Gradinaru, Nourdin, Russo and Vallois proved that when X is a fBm with Hurst parameter H, this integral can be defined as the limit in probability of suitable ν-symmetric Riemann sums, for some symmetric measure ν in $[0,1]$, if the Hurst parameter is strictly bigger than a maximal threshold of the form $(4\ell(\nu)+2)^{-1}$, for some integer $\ell(\nu)>0$. In the case where the measure ν is given by $\nu(dx)=\frac{1}{2}(\delta_0+\delta_1)$, $\nu(dx)=\frac{1}{6}(\delta_0+4\delta_{1/2}+\delta_1)$ or $\nu(dx)=\frac{1}{90}(7\delta_0+32\delta_{1/4}+12\delta_{1/2}+32\delta_{3/4}+7\delta_1)$, the associated Riemann sums are the Trapezoidal rule, Simpson's rule and Milne's rule approximations respectively. The behavior at the critical value $H=(4\ell(\nu)+2)^{-1}$ was latter studied by Binotto and Nourdin in [5], where it was proved that the Symmetric Riemann sums converge in law to the stochastic integral of $g^{(2\ell(\nu))}(X_t)$ with respect to a standard Brownian motion independent of X.

It is natural to ask whether these results hold for more general Gaussian processes. Part of this thesis consists on determining the behavior of the ν-symmetric Riemann sums of X, in the case where X is self-similar of order β and has increment exponent α (which is defined by the property $E\left[(X_{t+s}-X_t)^2\right]=O(s^{\alpha})$). The results cover the cases where X is a fractional, bifractional and subfractional Brownian motion, as well the case where X is either the Gaussian process introduced by Durieu and Wang in [13] or those introduced by Swanson in [52]. It is worth mentioning that when X is a fractional Brownian motion of Hurst parameter H, and $H=(2\ell(\nu)+1)^{-1}$, our proof requires g to have only derivatives of order $8\ell(\nu)+1$, thus extending the results from [5], where g is required to have derivatives of order $20\ell(\nu)+4$ and moderate growth.

The approach we present here is based on the description of the asymptotic behavior of the Hermite variations of X, which is a topic with an interest on its own, and wasn't addressed before for general self-similar Gaussian processes (although it has been widely studied for the fractional Brownian motion in recent years). We prove that the process of Hermite variations of X, converges stably to a Gaussian process independent of X, satisfying the property of independent increments. In contrast with the case where X is a fractional Brownan motion (where the limit of the Hermite variations is a multiple of a standard Brownian motion), for a general self-similar X, the limit processes obtained from the Hermite variations might not be stationary. Surprisingly, the transition in the behavior of the symmetric Riemann sums doesn't occur necessarily when the self-similarity β reaches the critical value $(4\ell(\nu)+2)^{-1}$, but rather when the increment exponent α reaches $(2\ell(\nu)+1)^{-1}$. To be precise, we prove convergence in probability for the ν-symmetric Riemann sums of X in the case where $\alpha > (2\ell(\nu)+1)^{-1}$, while in the case $\alpha = (2\ell(\nu)+1)^{-1}$, we prove that the ν-symmetric Riemann sums converge to the integral of $g^{(2\ell(\nu))}(X_t)$ with respect to a suitable Gaussian martingale, independent of X.

The final topic we present is related to the study of the eigenvalues of matrix valued Gaussian processes. One big technical difficulty related to the study of this topic, is that the function Φ that associates a $d \times d$ symmetric matrix to its d-dimensional vector of ordered eigenvalues, is not smooth around matrices with at least one repeated eigenvalue. For this reason, it is of great interest to determine conditions under which the eigenvalues of a matrix-valued Gaussian process of dimension d, don't collide. The problem of collision of eigenvalues has been previously studied by Dyson in the Brownian motion case, and more recently by Nualart and Pérez-Abreu in [44] for the fBm with $H > \frac{1}{2}$.

In this thesis, we determine sharp conditions for general matrix-valued Hermitian Gaussian fields (including both the case of Hermitian complex matrices and symmetric real matrices), under which the associated eigenvalues collide. As an application, we show that the eigenvalues of a real symmetric matrix-valued fractional Brownian motion of Hurst parameter $H \in (0,1)$, collide when $H < \frac{1}{2}$ and don't collide when $H > \frac{1}{2}$, while those of a complex Hermitian fractional Brownian motion collide when $H < \frac{1}{3}$ and don't collide when $H > \frac{1}{3}$. Our approach is based on the relation between hitting probabilities for Gaussian processes with the capacity and Hausdorff dimension of measurable sets.

Chapter 1

Background

Our main goal for this chapter is to introduce the basic definitions and results related to Gaussian processes, with particular emphasis on the fractional Brownian motion. The random elements defined in the sequel will be assumed to be defined in a probability space $(\Omega, \mathscr{G}, \mathbb{P})$.

1.1 Fractional Brownian motion

Let $r \geq 2$. A random vector $G = (G_1, \ldots, G_r)$ defined in $(\Omega, \mathscr{G}, \mathbb{P})$ is said to have r-dimensional Gaussian distribution if, for every $\lambda_1, \ldots, \lambda_r$, the random variable $\sum_{k=1}^{r} G_k$ has Gaussian distribution. When G has r-dimensional Gaussian distribution we say that $G_1, \ldots, G_r$ are jointly Gaussian.

Notice that the distribution of any r-dimensional Gaussian distribution $G = (G_1, \ldots, G_r)$ is uniquely determined by its mean $\mathbb{E}[G] = (\mathbb{E}[G_1], \ldots, \mathbb{E}[G_r])$ and its covariance matrix $\text{Cov}[G] = \{\Sigma_{i,j}\}_{1 \leq i,j \leq r}$, which is given by

$$\Sigma_{i,j} = \text{Cov}[G_i, G_j].$$

Next we introduce the notion of Gaussian process

Definition 1.1.1. *A stochastic process $X = \{X_t\}_{t \geq 0}$ defined in $(\Omega, \mathscr{G}, \mathbb{P})$ is said to be Gaussian if, for all $r \geq 1$ $(X_{t_1}, \ldots, X_{t_r})$ is an r-dimensional Gaussian vector.*

The finite dimensional distributions of X are uniquely determined by the mean function $\mu : \mathbb{R}_+ \to \mathbb{R}$, defined by $\mu(t) := \mathbb{E}[X_t]$ and the covariance function $R : \mathbb{R}_+^2 \to \mathbb{R}$ given by $R(s,t) := \mathrm{Cov}[X_s, X_t]$. We will say that X is a centered Gaussian process if $\mu(t) = 0$ for all $t \geq 0$.

One of the most important examples of Gaussian processes is the classical Brownian motion, which is a centered Gaussian process $W = \{W_t\}_{t \geq 0}$ characterized by the property

$$R(s,t) = \mathbb{E}[W_s W_t] = s \wedge t.$$

The Brownian motion has been a powerful tool for mathematical modeling. It has been particularly useful for modeling of stock prices, thermal noise in electrical circuits, queuing and inventory systems, and random perturbations in a variety of other physical, biological, economic, and management systems. The existence of a Brownian motion with continuous trajectories can be easily obtained by means of the Kolmogorov existence theorem and Kolmogorov's continuity criterion. We refer the interested reader to [31] for the proof of this claims, as well as for a comprehensive presentation of other basic properties of the Brownian motion.

A closely related stochastic process is the fractional Brownian motion $B = \{B_t\}_{t \geq 0}$ of Hurst parameter $H \in (0,1)$, which is a centered Gaussian process with covariance function

$$\mathbb{E}[B_sB_t] = R(s,t) := \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

Notice that when $H = \frac{1}{2}$, B is a classical Brownian motion. The fractional Brownian motion was first introduced by Kolmogorov for modeling turbulence in liquids, and was further studied by Mandelbrot and Van Ness. The behavior of the fractional Brownian motion is quite different as we vary the value for H: when $H > \frac{1}{2}$, its increments are positively correlated and for $H < \frac{1}{2}$, they are negatively correlated. Moreover, for $\beta \in (0,H)$, its sample paths are Hölder continuous with index β, namely,

$$\sup_{0<s<t<T} \frac{|B_t - B_s|}{t-s} < \infty, \qquad \mathbb{P}\text{-a.e.}$$

for every $T > 0$. Moreover, if $H > \frac{1}{2}$, B is a long memory process, in the sense that

$$\sum_{k=1}^{\infty} |\mathrm{Cov}[B_{t+kh}, B_t]| = \infty$$

for all $h,t > 0$. This flexibility on the behavior of the fractional Brownian motion makes it very interesting for modeling purposes, since by adjusting the value for H we can accurately fit the observations of the random model we want to describe.

The fractional Brownian motion satisfies the following properties

1. **Selfsimilarity:** For all $c > 0$, $\{c^{-H}B_{ct}\}_{t\geq 0} \overset{Law}{=} \{B_t\}_{t\geq 0}$.
2. **Stationarity of increments:** For all $h > 0$, $(B_{t+h} - B_t) \overset{Law}{=} B_h$.
3. **Time inversion:** $\{t^{2H}B_{1/t}\}_{t>0} \overset{Law}{=} \{B_t\}_{t>0}$.

The fractional Brownian motion can be constructed as a Volterra process in the following manner

$$\{B_t\}_{t\geq 0} \overset{Law}{=} \int_0^t K_H(s,t)dW(t), \tag{1.1.1}$$

where $\{W_t\}_{t\geq 0}$ is a classical Brownian motion and

$$K_H(s,t) := c_H\left(\left(t/s\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-1/2)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right),$$

where $c_H := (2H)^{-\frac{1}{2}}(1-2H)\int_0^1(1-x)^{-2H}x^{H-\frac{1}{2}}dx$. The integration in (1.1.1) should be understood in the Itô sense (see [31] for details). We refer the interested reader to [35] for a proof of the identity (1.1.1), and for a detailed treatment of the basic properties of the fractional Brownian motion.

1.2 Some elements of Malliavin calculus

In the sequel, $\vec{X} = \{(X_t^{(1)},\dots,X_t^{(d)})\}_{t\geq 0}$ will denote a d-dimensional centered Gaussian process with covariance $R(s,t)$ defined in $(\Omega,\mathscr{G},\mathbb{P})$, namely, the components of $\vec{X}$ are independent and identically distributed centered Gaussian processes with covariance $R(s,t)$. In the case where $\vec{X}$ is a fractional Brownian motion of Hurst parameter $H\in(0,1)$, the notation $\vec{X}$ and $(X^{(1)},\dots,X^{(d)})$ will be replaced by $\vec{B}$ and $(B^{(1)},\dots,B^{(d)})$ respectively.

We will denote by $\mathscr{F}$ the σ-algebra generated by $\vec{X}$, by $L^2(\Omega)$ the space of real square integrable functions measurable with respect to $\mathscr{G}$ and by $L^2(\Omega;\mathscr{F})$ the space of real square integrable functions measurable with respect to $\mathscr{F}$.

Next we introduce the basic operators from the theory of Malliavin calculus and state some of their properties. The results we present in this section will be stated without proofs, and the reader will be refered to [43, Chapter 1] for a detailed treatment of these topics.

Denote by $\mathfrak{H}$ the Hilbert space obtained by taking the completion of the space of real step functions on $[0,\infty)$, endowed with the inner product

$$\left\langle \mathbb{1}_{[a,b]}, \mathbb{1}_{[c,d]} \right\rangle_{\mathfrak{H}} := \mathbb{E}\left[\left(X_b^{(1)} - X_a^{(1)} \right) \left(X_d^{(1)} - X_c^{(1)} \right) \right], \quad \text{for} \quad 0 \leq a \leq b, \text{ and } 0 \leq c \leq d.$$

For every $1 \leq j \leq d$ fixed, the mapping $\mathbb{1}_{[0,t]} \mapsto X_t^{(j)}$ can be extended to linear isometry between $\mathfrak{H}$ and the Gaussian subspace of $L^2(\Omega)$ generated by the process $X^{(j)}$. We will denote this isometry by $X^{(j)}(f)$, for $f \in \mathfrak{H}$. If $f \in \mathfrak{H}^d$ is of the form $f = (f_1, \dots, f_d)$, with $f_j \in \mathfrak{H}$, we set $\vec{X}(f) := \sum_{j=1}^{d} X^{(j)}(f_j)$. Then $f \mapsto \vec{X}(f)$ is a linear isometry between $\mathfrak{H}^d$ and the Gaussian subspace of $L^2(\Omega)$ generated by $\vec{X}$.

For any integer $q \geq 1$, we denote by $(\mathfrak{H}^d)^{\otimes q}$ and $(\mathfrak{H}^d)^{\odot q}$ the qth tensor product of $\mathfrak{H}^d$, and the qth symmetric tensor product of $\mathfrak{H}^d$, respectively. The qth Wiener chaos of $L^2(\Omega)$, denoted by $\mathscr{H}_q$, is the closed subspace of $L^2(\Omega)$ generated by the variables

$$\left\{ \prod_{j=1}^{d} H_{q_j}(X^{(j)}(f_j)) \mid \sum_{j=1}^{d} q_j = q, \text{ and } f_1, \dots, f_d \in \mathfrak{H}, \left\| f_j \right\|_{\mathfrak{H}} = 1 \right\},$$

where H_q is the qth Hermite polynomal, defined by

$$H_q(x) := (-1)^q e^{\frac{x^2}{2}} \frac{\mathrm{d}^q}{\mathrm{d}x^q} e^{-\frac{x^2}{2}}.$$

We observe that any monomial of the form $x^{2\ell+1}$, for $\ell \in \mathbb{N}$, can be expressed as a linear combination of odd Hermite polynomials with integer coefficients $c_{j,r}$, namely,

$$x^{2r+1} = \sum_{j=0}^{r} c_{j,r} H_{2(r-j)+1}(x). \tag{1.2.1}$$

For $q \in \mathbb{N}$, with $q \geq 1$, and $f \in \mathfrak{H}^d$ of the form $f = (f_1, \ldots, f_d)$, with $\|f_j\|_{\mathfrak{H}} = 1$, we can write

$$f^{\otimes q} = \sum_{i_1,\ldots,i_q=1}^{d} f_{i_1} \otimes \cdots \otimes f_{i_q}.$$

For such f, we define the mapping

$$I_q(f^{\otimes q}) := \sum_{i_1,\ldots,i_q=1}^{d} \prod_{j=1}^{d} H_{q_j(i_1,\ldots,i_q)}(X^{(j)}(f_j)),$$

where $q_j(i_1,\ldots,i_q)$ denotes the number of indices in $(i_1,\ldots,i_q)$ equal to j. The range of I_q is contained in $\mathscr{H}_q$. Furthermore, this mapping can be extended to a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the norm $\sqrt{q!}\,\|\cdot\|_{(\mathfrak{H}^d)^{\otimes q}}$) and $\mathscr{H}_q$ (equipped with the $L^2(\Omega)$-norm).

It is well known that every $\mathscr{F}$-measurable, square integrable random variable has a chaos decomposition of the type

$$F = \mathbb{E}[F] + \sum_{q=1}^{\infty} I_q(f_q),$$

for some $f_q \in (\mathfrak{H}^d)^{\odot q}$. In what follows, we will denote by $J_q(F)$, for $q \geq 1$, the projection of F over the qth Wiener chaos $\mathscr{H}_q$, and by $J_0(F)$ the expectation of F.

Let $\{e_n\}_{n\geq 1}$ be a complete orthonormal system in $\mathfrak{H}^d$. Given $f \in (\mathfrak{H}^d)^{\odot p}$, $g \in (\mathfrak{H}^d)^{\odot q}$ and $r \in \{0,\dots,p\wedge q\}$, the rth-order contraction of f and g is the element of $(\mathfrak{H}^d)^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{(\mathfrak{H}^d)^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{(\mathfrak{H}^d)^{\otimes r}},$$

where $f \otimes_0 g = f \otimes g$, and for $p = q$, $f \otimes_q g = \langle f, g \rangle_{(\mathfrak{H}^d)^{\otimes q}}$.

Let $\mathscr{S}$ denote the set of all cylindrical random variables of the form

$$F = g(\vec{X}(h_1),\dots,\vec{X}(h_n)),$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $h_j \in \mathfrak{H}^d$. In the sequel, for every Hilbert space V, we will denote by $L^2(\Omega;V)$ the set of square integrable V-valued random variables. The Malliavin derivative of F with respect to X, is the element of $L^2(\Omega;\mathfrak{H}^d)$, defined by

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\vec{X}(h_1),\dots,\vec{X}(h_n))h_i.$$

By iteration, one can define the rth derivative D^r for every $r \geq 2$, which is an element of $L^2(\Omega;(\mathfrak{H}^d)^{\otimes r})$.

For $p \geq 1$ and $r \geq 1$, the space $\mathbb{D}^{r,p}$ denotes the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{r,p}}$, defined by

$$\|F\|_{\mathbb{D}^{r,p}} := \left(\mathbb{E}\left[|F|^p\right] + \sum_{i=1}^{r} \mathbb{E}\left[\|D^i F\|^p_{(\mathfrak{H}^d)^{\otimes i}} \right] \right)^{\frac{1}{p}}.$$

The operator D^r can be consistently extended to the space $\mathbb{D}^{r,p}$. We denote by δ the adjoint of the operator D, also called the divergence operator. A random element $u \in L^2(\Omega;\mathfrak{H}^d)$ belongs to the domain of δ, denoted by $\mathrm{Dom}\,\delta$, if and only if satisfies

$$\left|\mathbb{E}\left[\langle DF,u\rangle_{\mathfrak{H}^d}\right]\right| \leq C_u \mathbb{E}\left[F^2\right]^{\frac{1}{2}}, \text{ for every } F \in \mathbb{D}^{1,2},$$

where C_u is a constant only depending on u. If $u \in \mathrm{Dom}\,\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF,u\rangle_{\mathfrak{H}^d}\right],$$

which holds for every $F \in \mathbb{D}^{1,2}$. The operator L is defined on the Wiener chaos by

$$LF := \sum_{q=1}^{\infty} -qJ_qF, \text{ for } F \in L^2(\Omega),$$

and coincides with the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{P_\theta\}_{\theta \geq 0}$, which is defined by

$$P_\theta := \sum_{q=0}^{\infty} e^{-q\theta} J_q.$$

A random variable F belongs to the domain of L if and only if $F \in \mathbb{D}^{1,2}$, and $DF \in \mathrm{Dom}\,\delta$, in which case

$$\delta DF = -LF.$$

We also define the operator L^{-1} as

$$L^{-1}F = \sum_{q=1}^{\infty} -\frac{1}{q} J_q F, \text{ for } F \in L^2(\Omega).$$

Notice that L^{-1} is a bounded operator and satisfies $LL^{-1}F = F - \mathbb{E}[F]$ for every $F \in L^2(\Omega)$, so that L^{-1} acts as a pseudo-inverse of L. The operator L^{-1} satisfies the following contraction property for every $F \in L^2(\Omega)$ with $\mathbb{E}[F] = 0$,

$$\mathbb{E}\left[\left\|DL^{-1}F\right\|_{\mathfrak{H}^d}^2\right] \leq \mathbb{E}\left[F^2\right].$$

Next we state Meyers inequalities (see [43, Theorem 1.5.1]), which is a fundamental result in the theory of Malliavin calculus, as it implies the continuity of the operator δ over the space $\Delta^{1,2}$. The most general version of Meyer's inequalities, sates that for every $p > 1$, there exists a constant $c_p > 0$ such that

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|Du\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})}. \tag{1.2.2}$$

Using (1.2.2), we can show that for every $F \in \mathbb{D}^{2,p}$, with $\mathbb{E}[F] = 0$,

$$\left\|\delta(DL^{-1}F)\right\|_{L^p(\Omega)} \leq c_p(\left\|D^2L^{-1}F\right\|_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})} + \left\|\mathbb{E}\left[DL^{-1}F\right]\right\|_{(\mathfrak{H})^d}). \tag{1.2.3}$$

The proof of this claim can be found in [43, Proposition 1.5.8].

Assume that $\widetilde{X}$ is an independent copy of $\vec{X}$, and such that $\vec{X}, \widetilde{X}$ are defined in the product space $(\Omega \times \widetilde{\Omega}, \mathscr{F} \otimes \widetilde{\mathscr{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}})$. Given a random variable $F \in L^2(\Omega)$, measurable with respect to the σ-algebra generated by X, we can write $F = \Psi_F(\vec{X})$, where Ψ_F is a measurable mapping from $\mathbb{R}^{\mathfrak{H}^d}$ to $\mathbb{R}$, determined $\mathbb{P}$-a.s. Then, for every $\theta \geq 0$ we have the Mehler formula

$$P_\theta F = \widetilde{\mathbb{E}}\left[\Psi_F(e^{-\theta}\vec{X} + \sqrt{1-e^{-2\theta}}\widetilde{X})\right], \tag{1.2.4}$$

where $\widetilde{\mathbb{E}}$ denotes the expectation with respect to $\widetilde{\mathbb{P}}$. The operator L^{-1} can be expressed in terms of P_θ, as follows

$$L^{-1}F = \int_0^\infty P_\theta F d\theta, \text{ for } F \text{ such that } \mathbb{E}[F] = 0. \tag{1.2.5}$$

We end this section by stating the following lemma, which has been proved in [38, Lemma 2.1]:

Lemma 1.2.1. *Let $q \geq 1$ be an integer. Suppose that $F \in \mathbb{D}^{q,2}$, and let u be a symmetric element in* $\operatorname{Dom}\delta^q$. *Assume that, for any $0 \leq r + j \leq q$, $\left\langle D^r F, \delta^j(u)\right\rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega; \mathfrak{H}^{\otimes q-r-j})$. Then, for any $r = 0, \ldots, q-1$, $\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}$ belongs to the domain of δ^{q-r} and we have*

$$F\delta^q(u) := \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}(\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}).$$

1.2.1 Hermite process

In this section we assume that $\vec{X} = \vec{B}$ is a d-dimensional fractional fractional Brownian motion with Hurst parameter $H \in (0,1)$. When $H > \frac{1}{2}$, the inner product in the space $\mathfrak{H}$ can be written, for every step functions φ, ϑ on $[0,\infty)$, as

$$\langle \varphi, \vartheta\rangle_{\mathfrak{H}} = H(2H-1)\int_{\mathbb{R}_+^2} \varphi(\xi)\vartheta(\nu)\,|\xi - \nu|^{2H-2} d\xi d\nu. \tag{1.2.6}$$

Following [36], we introduce the Hermite process $\{\mathscr{X}_T^j\}_{T\geq 0}$ of order 2, associated to the jth component of B, $\{B_t^{(j)}\}_{t\geq 0}$, and describe some of its properties. The family of kernels $\{\varphi_{j,T}^{\varepsilon} \mid T\geq 0, \varepsilon\in(0,1)\}\subset(\mathfrak{H}^d)^{\otimes 2}$, defined, for every multi-index $\mathbf{i}=(i_1,i_2)$, $1\leq i_1,i_2\leq d$, by

$$\varphi_{j,T}^{\varepsilon}(\mathbf{i},x_1,x_2):=\varepsilon^{-2}\int_0^T\delta_{j,i_1}\delta_{j,i_2}\mathbb{1}_{[s,s+\varepsilon]}(x_1)\mathbb{1}_{[s,s+\varepsilon]}(x_2)ds, \tag{1.2.7}$$

satisfies the following relation for every $H>\frac{3}{4}$, and $T\geq 0$

$$\lim_{\varepsilon,\eta\to 0}\left\langle\varphi_{j,T}^{\varepsilon},\varphi_{j,T}^{\eta}\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}}=H^2(2H-1)^2\int_{[0,T]^2}|s_1-s_2|^{4H-4}ds_1ds_2=c_HT^{4H-2}, \tag{1.2.8}$$

where $c_H:=\frac{H^2(2H-1)}{4H-3}$. This implies that $\varphi_{j,T}^{\varepsilon}$ converges, as $\varepsilon\to 0$, to an element of $(\mathfrak{H}^d)^{\otimes 2}$, denoted by π_T^j. In particular, for every $K>0$, $\left\|\varphi_{j,K}^{\varepsilon}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}$ is bounded by some constant $C_{K,H}$, only depending on K and H. On the other hand, by (1.2.6) and (1.2.7), we deduce that for every $T\in[0,K]$, it holds $\left\|\varphi_{j,T}^{\varepsilon}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}\leq\left\|\varphi_{j,K}^{\varepsilon}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}$, and hence

$$\begin{aligned}\sup_{\substack{T_1,T_2\in(0,K]\\ \varepsilon,\eta\in(0,1)}}\left|\left\langle\varphi_{j,T_1}^{\varepsilon},\varphi_{j,T_2}^{\eta}\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}}\right| &\leq \sup_{\substack{T_1,T_2\in(0,K]\\ \varepsilon,\eta\in(0,1)}}\left\|\varphi_{j,T_1}^{\varepsilon}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}\left\|\varphi_{j,T_2}^{\eta}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}\\ &\leq\sup_{\varepsilon\in(0,1)}\left\|\varphi_{j,K}^{\varepsilon}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\leq C_{K,H}.\end{aligned} \tag{1.2.9}$$

The element π_T^j, can be characterized as follows. For any vector of step functions with compact support $f_i = (f_i^{(1)}, \dots, f_i^{(d)}) \in \mathfrak{H}^d$, $i = 1, 2$, we have

$$\begin{aligned}\left\langle \pi_t^j, f_1 \otimes f_2 \right\rangle_{(\mathfrak{H}^d)^{\otimes 2}} &= \lim_{\varepsilon \to 0} \langle \varphi_{j,t}^{\varepsilon}, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}} \\ &= \lim_{\varepsilon \to 0} \varepsilon^{-2} H^2 (2H-1)^2 \\ &\times \int_0^T \prod_{i=1,2} \int_s^{s+\varepsilon} \int_0^T |\xi - \eta|^{2H-2} f_i^{(j)}(\eta) d\eta d\xi ds\end{aligned}$$

and hence

$$\left\langle \pi_t^j, f_1 \otimes f_2 \right\rangle_{(\mathfrak{H}^d)^{\otimes 2}} = H^2 (2H-1)^2 \int_0^T \prod_{i=1,2} \int_0^T |s - \eta|^{2H-2} f_i^{(j)}(\eta) d\eta ds. \quad (1.2.10)$$

We define the second order Hermite process $\{\mathscr{X}_T^j\}_{T \geq 0}$, with respect to $\{B_t^{(j)}\}_{t \geq 0}$, as $\mathscr{X}_T^j := I_2(\pi_T^j)$.

1.2.2 Central limit theorems via chaos expansion

In the seminal paper [44], Nualart and Peccati established a central limit theorem for sequences of multiple stochastic integrals of a fixed order. In this context, assuming that the variances converge, convergence in distribution to a centered Gaussian law is actually equivalent to convergence of just the fourth moment. Shortly afterwards, in [47], Peccati and Tudor gave a multidimensional version of this characterization. More recent developments on these type of results have been addressed by using Stein's method and Malliavin techniques (see the monograph by Nourdin and Peccati [37] and the references therein).

We will need the following modification of the Peccati-Tudor criterion, in which we will make use of the notation introduced in Chapter 1

Theorem 1.2.1. *Let $1 < q_1 < q_2 < \cdots < q_d$ be positive integers. Consider a sequence of stochastic processes $F_n^i = \{F_n^i(t)\}_{t\geq 0}$ of the form $F_n^i(t) = I_{q_i}(h_n^i(t))$, where each $h_n^i(t)$ is an element of $\mathfrak{H}^{\otimes q_i}$ and $1 \leq i \leq d$. Suppose in addition, that the following conditions hold for every $t \geq 0$ and $1 \leq i \leq d$:*

(i) There exist $c_1, \dots, c_d > 0$, such that for every $s, t \geq 0$

$$\lim_{n\to\infty} \left\langle h_n^i(s), h_n^i(t) \right\rangle_{\mathfrak{H}^{\otimes q_i}} = \frac{c_i^2}{q_i!} \Sigma(s,t). \tag{1.2.11}$$

(ii) For all $i = 1, \dots, d$ and $r = 1, \dots, q_i - 1$,

$$\lim_{n\to\infty} \left\| h_n^i(t) \otimes_r h_n^i(t) \right\|_{\mathfrak{H}^{\otimes 2(q_i - r)}} = 0. \tag{1.2.12}$$

Then the finite dimensional distributions of the process $\sum_{i=1}^d F_n^i$ converge stably to those of $\sum_{i=1}^d \sqrt{q_i!} c_i Y^i$.

We will use as well the following multivariate central limit theorem obtained by Peccati and Tudor in [47] (see also Theorems 6.2.3 and 6.3.1 in [37]).

Theorem 1.2.2. *For $r \in \mathbb{N}$ fixed, consider a sequence $\{F_n\}_{n\geq 1}$ of random vectors of the form $F_n = (F_n^{(1)}, \dots, F_n^{(r)})$. Suppose that for $i = 1, \dots, r$ and $n \in \mathbb{N}$, the random variables $F_n^{(i)}$ belong to $L^2(\Omega)$, and have chaos decomposition*

$$F_n^{(i)} = \sum_{q=1}^{\infty} I_q(f_{q,i,n}),$$

for some $f_{q,i,n} \in (\mathfrak{H}^d)^{\otimes q}$. Suppose, in addition, that for every $q \geq 1$, there is a real symmetric non negative definite matrix $C_q = \{C_q^{i,j} \mid 1 \leq i, j \leq r\}$, such that the following conditions hold:

(i) For every fixed $q \geq 1$, and $1 \leq i, j \leq r$, we have $q! \left\langle f_{q,i,n}, f_{q,j,n} \right\rangle_{(\mathfrak{H}^d)^{\otimes q}} \to C_q^{i,j}$ as $n \to \infty$.

(ii) There exists a real symmetric nonnegative definite matrix $C = \{C^{i,j} \mid 1 \leq i, j \leq r\}$, such that $C^{i,j} = \lim_{Q\to\infty} \sum_{q=1}^{Q} C_q^{i,j}$.

(iii) For all $q \geq 1$ and $i = 1, \ldots, r$, the sequence $\{I_q(f_{q,i,n})\}_{n\geq 1}$ converges in law to a centered Gaussian distribution as $n \to \infty$.

(iv) $\lim_{Q\to\infty} \sup_{n\geq 1} \sum_{q=Q}^{\infty} q! \left\| f_{q,i,n} \right\|^2_{(\mathfrak{H}^d)^{\otimes q}} = 0$, for all $i = 1, \ldots, r$.

Then, F_n converges in law as $n \to \infty$, to a centered Gaussian vector with covariance matrix C.

Chapter 2

Self-intersection local time for the fractional Brownian motion

Let $\vec{B} = \{\vec{B}_t\}_{t\geq 0}$ be a d-dimensional fractional Brownian motion of Hurst parameter $H \in (0,1)$. Fix $T > 0$. The self-intersection local time of $\vec{B}$ in the interval $[0,T]$ is formally defined by

$$I := \int_0^T \int_0^t \delta(\vec{B}_t - \vec{B}_s) ds dt,$$

where δ denotes the Dirac delta function. A rigorous definition of this random variable may be obtained by approximating the delta function by the heat kernel

$$p_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{d}{2}} \exp\left\{-\frac{1}{2\varepsilon}\|x\|^2\right\}, \quad x \in \mathbb{R}^d.$$

In the case $H = \frac{1}{2}$, $\vec{B}$ is a classical Brownian motion, and its self-intersection local time has been studied by many authors (see the work by Albeverio (1995), Hu (1996), Imkeller, Pérez-Abreu and Vives (1995), Varadhan (1969) and Yor (1985) in [1], [22], [25], [55], [58]). In the case $H \neq \frac{1}{2}$, the self-intersection local time for $\vec{B}$ was first studied by Rosen in [49] in the planar case and it was further investigated using techniques from Malliavin calculus by Hu and Nualart in [23]. In particular, it was proved that the approximation of the self-intersection local time of $\vec{B}$ in $[0,T]$, defined by

$$I_T^{\varepsilon} := \int_0^T \int_0^t p_{\varepsilon}(\vec{B}_t - \vec{B}_s) ds dt, \tag{2.0.1}$$

converges in $L^2(\Omega)$ when $H < \frac{1}{d}$. Furthermore, it was shown that when $\frac{1}{d} \leq H < \frac{3}{2d}$, $I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}]$ to converges in $L^2(\Omega)$, and for the case $\frac{3}{2d} < H < \frac{3}{4}$, the following limit theorem holds (see [23, Theorem 2]).

Theorem 2.0.1. *If $\frac{3}{2d} < H < \frac{3}{4}$, then $\varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])$ converges in law to a centered Gaussian distribution with variance $\sigma^2 T$, as $\varepsilon \to 0$, where the constant σ^2 is given by* (2.2.3).

The case $H = \frac{3}{2d}$ was addressed as well in [23], where it was shown that the sequence $(\log(1/\varepsilon))^{-\frac{1}{2}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])$ converges in law to a centered Gaussian distribution with variance σ_{log}^2, as $\varepsilon \to 0$, where σ_{log}^2 is the constant given by [23, Equation (42)].

The aim of this paper is to prove a functional version of Theorem 2.0.1, and extend it to the case $\frac{3}{4} \leq H < 1$. Our main results are Theorems 2.0.2, 2.0.3 and 2.0.4.

Theorem 2.0.2. *Let $\frac{3}{2d} < H < \frac{3}{4}$, $d \geq 2$ be fixed. Then,*

$$\{\varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])\}_{T \geq 0} \overset{Law}{\to} \{\sigma W_T\}_{T \geq 0}, \tag{2.0.2}$$

in the space $C[0,\infty)$, endowed with the topology of uniform convergence on compact sets, where W is a standard Brownian motion, and the constant σ^2 is given by (2.2.3).

We briefly outline the proof of (2.0.2). The proof of the convergence of the finite-dimensional distributions, is based on the application of a multivariate central limit theorem established by Peccati and Tudor in [47] (see Section 1.2.2), and follows ideas similar to those presented in [23]. On the other hand, proving the tightness property for the process

$$\widetilde{I}_T^{\varepsilon} := \varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}]),$$

presents a great technical difficulty. In fact, by the Billingsley criterion (see [4, Theorem 12.3]), the tightness property can be obtained by showing that there exists $p > 2$, such that for every $0 \leq T_1 \leq T_2$,

$$\mathbb{E}\left[\left|\widetilde{I}_{T_2}^{\varepsilon} - \widetilde{I}_{T_1}^{\varepsilon}\right|^p\right] \leq C|T_2 - T_1|^{\frac{p}{2}}, \tag{2.0.3}$$

for some constant $C > 0$ independent of T_1, T_2 and ε. The problem of finding a bound like (2.0.3) comes from the fact that the smallest even integer such that $p > 2$ is $p = 4$, and a direct computation of the moment of order four $\mathbb{E}\left[|\widetilde{I}_{T_2}^{\varepsilon} - \widetilde{I}_{T_1}^{\varepsilon}|^4\right]$ is too complicated to be handled. To overcome this difficulty, in this paper we introduce a new approach to prove tightness based on the techniques of Malliavin calculus. Let us describe the main ingredients of this approach.

First, we write the centered random variable $Z := \widetilde{I}_{T_2}^{\varepsilon} - \widetilde{I}_{T_1}^{\varepsilon}$ as

$$Z = -\delta D L^{-1} Z,$$

where δ, D and L are the basic operators in Malliavin calculus. Then, taking into consideration that $\mathbb{E}\left[DL^{-1}Z\right] = 0$ we apply Meyer's inequalities to obtain a bound of the type

$$\|Z\|_{L^p(\Omega)} \leq c_p \|D^2 L^{-1} Z\|_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})}, \tag{2.0.4}$$

for any $p > 1$, where the Hilbert space $\mathfrak{H}$ is defined in Section 1.2. Notice that

$$Z = \varepsilon^{\frac{d}{2}-\frac{3}{4H}} \int_{0\le s\le t, T_1\le t\le T_2} \left(p_\varepsilon(\vec{B}_t - \vec{B}_s) - \mathbb{E}\left[p_\varepsilon(\vec{B}_t - \vec{B}_s)\right]\right) dsdt.$$

Applying Minkowski's inequality and (2.0.4), we obtain

$$\|Z\|_{L^p(\Omega)} \le c_p \varepsilon^{\frac{d}{2}-\frac{3}{4H}} \int_{0\le s\le t, T_1\le t\le T_2} \|D^2 L^{-1} p_\varepsilon(\vec{B}_t - \vec{B}_s)\|_p dsdt.$$

Then, we get the desired estimate by choosing $p > 2$ close to 2, using the self-similarity of the fractional Brownian motion, the expression of the operator L^{-1} in terms of the Ornstein-Uhlenbeck semigroup, Mehler's formula and Gaussian computations. In this way, we reduce the problem to showing the finiteness of an integral (see Lemma 2.4.3), similar to the integral appearing in the proof of the convergence of the variances. It is worth mentioning that this approach for proving tightness has not been used before, and has its own interest.

In the case $H > \frac{3}{4}$, the process $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$ also converges in law, in the topology of $C[0,\infty)$, but the limit is no longer a multiple of a Brownian motion, but a multiple of a sum of independent Hermite processes of order two. More precisely, if $\{\mathscr{X}_T^j\}_{T\ge 0}$ denotes the second order Hermite process, with respect to $\{B_t^{(j)}\}_{t\ge 0}$, defined in Section 1.2, then $\{\widetilde{I}^\varepsilon\}_{\varepsilon\in(0,1)}$ satisfies the following limit theorem

Theorem 2.0.3. *Let $H > \frac{3}{4}$, and $d \ge 2$ be fixed. Then, for every $T > 0$,*

$$\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon]) \overset{L^2(\Omega)}{\to} -\Lambda \sum_{j=1}^{d} \mathscr{X}_T^j, \tag{2.0.5}$$

where the constant Λ *is defined by*

$$\Lambda := \frac{(2\pi)^{-\frac{d}{2}}}{2} \int_0^\infty (1+u^{2H})^{-\frac{d}{2}-1} u^2 du. \tag{2.0.6}$$

In addition,

$$\{\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])\}_{T\geq 0} \overset{Law}{\to} \{-\Lambda \sum_{j=1}^d \mathscr{X}_T^j\}_{T\geq 0}, \tag{2.0.7}$$

in the space $C[0,\infty)$*, endowed with the topology of uniform convergence on compact sets.*

We briefly outline the proof of Theorem 2.0.3. The convergence (2.0.5) is obtained from the chaotic decomposition of I_T^ε. It turns out that the chaos of order two completely determines the asymptotic behavior of $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$, and consequently, (2.0.5) can be obtained by the characterization of the Hermite processes presented in [36], applied to the second chaotic component of I_T^ε. Similarly to the case $\frac{3}{2d} < H < \frac{3}{4}$, we show that the sequence $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$ is tight, which proves the convergence in law (2.0.7).

The technique we use to prove tightness doesn't work for the case $Hd \leq \frac{3}{2}$, so the convergence in law of $\{\log(1/\varepsilon)^{-\frac{1}{2}}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])\}_{T\geq 0}$ to a scalar multiple of a Brownian motion for the case $Hd = \frac{3}{2}$ still remains open. Nevertheless, for the critical case $H = \frac{3}{4}$ and $d \geq 3$, the technique does work, and we prove the following limit theorem

Theorem 2.0.4. *Suppose* $H = \frac{3}{4}$ *and* $d \geq 3$*. Then,*

$$\{\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])\}_{T\geq 0} \overset{Law}{\to} \{\rho W_T\}_{T\geq 0}, \tag{2.0.8}$$

in the space $C[0,\infty)$*, endowed with the topology of uniform convergence on compact sets, where* W *is a standard Brownian motion, and the constant* ρ *is defined by* (2.2.52).

Remark

We impose the stronger condition $d \geq 3$ instead of $d \geq 2$, since the choice $H = \frac{3}{4}$, $d = 2$ gives $Hd = \frac{3}{2}$, and as mentioned before, it is not clear how to prove tightness for this case.

We briefly outline the proof of Theorem 2.0.4. The proof of the tightness property is analogous to the case $\frac{3}{2d} < H < \frac{3}{4}$. On the other hand, the proof of the convergence of the finite dimensional distributions requires a new approach. First we show that, as in the case $H > \frac{3}{4}$, the chaos of order two determines the asymptotic behavior of $\{I_T^\varepsilon\}_{T\geq 0}$. Then we describe the behavior of the second chaotic component of I_T^ε, which we denote by $J_2(I_T^\varepsilon)$, and is given by

$$J_2(I_T^\varepsilon) = -\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{\frac{2}{3}-\frac{d}{2}}}{2}\sum_{j=1}^{d}\int_0^T\int_0^{\varepsilon^{-\frac{2}{3}}(T-s)}\frac{u^{\frac{3}{2}}}{(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u}-B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}}\right)duds, \tag{2.0.9}$$

where H_2 denotes the Hermite polynomial of order 2. Then we show that we can replace the domain of integration of u by $[0,\infty)$, and this integral can be approximated by Riemann sums of the type

$$-\frac{1}{2^M}\sum_{k=2}^{M2^M}\frac{u(k)^{\frac{3}{2}}}{(1+u(k)^{\frac{2}{3}})^{\frac{d}{2}+1}}\int_0^T H_2\left(\frac{B^{(j)}_{s+\varepsilon^{2M}u(k)}-B^{(j)}_s}{\sqrt{\varepsilon}u(k)^{\frac{3}{4}}}\right)ds, \tag{2.0.10}$$

where $u(k) = \frac{k}{2^M}$, and M is some fixed positive number. By [11, Equation (1.4)], we have that, for k fixed, the random variable

$$\xi_k^\varepsilon(T) := \frac{\varepsilon^{-\frac{1}{3}}}{\sqrt{\log(1/\varepsilon)}} \int_0^T H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u(k)} - B_s^{(j)}}{\sqrt{\varepsilon}u(k)^{\frac{3}{4}}}\right) ds$$

converges in law to a Gaussian distribution as $\varepsilon \to 0$. Hence, after a suitable analysis of the covariances of the process $\{\xi_k^\varepsilon(T) \mid 2 \le k \le M2^M, \text{ and } T \ge 0\}$ and an application of the Peccati-Tudor criterion (see [47]), we obtain that the process (2.0.10) multiplied by the factor $\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{-\frac{1}{3}}}{2\sqrt{\log(1/\varepsilon)}}$ converges to a constant multiple of a Brownian motion $\rho_M W$, for some $\rho_M > 0$. The result then follows by proving that the approximations (2.0.10) to the integrals in the right-hand side of (2.0.9) are uniform over $\varepsilon \in (0, 1/e)$ as $M \to \infty$, and that $\rho_M \to \rho$ as $M \to \infty$.

This chapter is organized as follows. In Section 2 we present some preliminary results on the fractional Brownian motion and the chaotic decomposition of I_T^ε. In Section 3, we compute the asymptotic behavior of the variances of the chaotic components of I_T^ε as $\varepsilon \to 0$. The proofs of the main results are presented in Section 4. Finally, in Section 5 we prove some technical lemmas.

2.1 Chaos decomposition for the self-intersection local time

In this section we describe the chaos decomposition of the variable I_T^ε defined by (2.0.1). Let $\varepsilon \in (0,1)$, and $T \geq 0$ be fixed. Define the set

$$\mathscr{R} := \{(s,t) \in \mathbb{R}_+^2 \mid s \leq t \leq 1\}.$$

For every $\gamma > 0$, we will denote by $\gamma\mathscr{R}$ the set $\gamma\mathscr{R} := \{\gamma v \mid v \in \mathscr{R}\}$. First we write

$$I_T^\varepsilon = \int_{\mathbb{R}_+^2} \mathbb{1}_{T\mathscr{R}}(s,t) p_\varepsilon(\vec{B}_t - \vec{B}_s) ds dt. \tag{2.1.1}$$

We can determine the chaos decomposition of the random variable $p_\varepsilon(\vec{B}_t - \vec{B}_s)$ appearing in (2.1.1) as follows. Given a multi-index $\mathbf{i}_n = (i_1, \dots, i_n)$, $n \in \mathbb{N}$, $1 \leq i_j \leq d$, we set

$$\alpha(\mathbf{i}_n) := \mathbb{E}[\zeta_{i_1} \cdots \zeta_{i_n}],$$

where the ζ_i are independent standard Gaussian random variables. Notice that

$$\alpha(\mathbf{i}_{2q}) = \frac{(2q_1)! \cdots (2q_d)!}{(q_1)! \cdots (q_d)! 2^q}, \tag{2.1.2}$$

if $n = 2q$ is even and for each $k = 1, \dots, d$, the number of components of $\mathbf{i}_{2q}$ equal to k, denoted by $2q_k$, is also even, and $\alpha(\mathbf{i}_n) = 0$ otherwise. Proceeding as in [23, Lemma 7], we can prove that

$$p_\varepsilon(\vec{B}_t - \vec{B}_s) = \mathbb{E}\left[p_\varepsilon(\vec{B}_t - \vec{B}_s)\right] + \sum_{q=1}^{\infty} I_{2q}\left(f^{\varepsilon}_{2q,s,t}\right), \tag{2.1.3}$$

where $f^{\varepsilon}_{2q,s,t}$ is the element of $(\mathfrak{H}^d)^{\otimes 2q}$, given by

$$f^{\varepsilon}_{2q,s,t}(\mathbf{i}_{2q}, x_1, \dots, x_{2q}) := (-1)^q \frac{(2\pi)^{-\frac{d}{2}}\alpha(\mathbf{i}_{2q})}{(2q)!}(\varepsilon + (t-s)^{2H})^{-\frac{d}{2}-q} \prod_{j=1}^{2q} \mathbb{1}_{[s,t]}(x_j), \tag{2.1.4}$$

and

$$\mathbb{E}\left[p_\varepsilon(\vec{B}_t - \vec{B}_s)\right] = (2\pi)^{-\frac{d}{2}}(\varepsilon + (t-s)^{2H})^{-\frac{d}{2}}. \tag{2.1.5}$$

By (2.1.1), (2.1.3) and (2.1.5), it follows that the random variable I^{ε}_T has the chaos decomposition

$$I^{\varepsilon}_T = \mathbb{E}[I^{\varepsilon}_T] + \sum_{q=1}^{\infty} I_{2q}(h^{\varepsilon}_{2q,T}), \tag{2.1.6}$$

where

$$h^{\varepsilon}_{2q,T}(\mathbf{i}_{2q}, x_1, \dots, x_{2q}) := \int_{\mathbb{R}^2_+} \mathbb{1}_{T\mathscr{R}}(s,t) f^{\varepsilon}_{2q,s,t}(\mathbf{i}_{2q}, x_1, \dots, x_{2q}) ds dt, \tag{2.1.7}$$

and

$$\mathbb{E}[I^{\varepsilon}_T] = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^2_+} \mathbb{1}_{T\mathscr{R}}(s,t)(\varepsilon + (t-s)^{2H})^{-\frac{d}{2}} ds dt. \tag{2.1.8}$$

In Section 3, we will describe the behavior as $\varepsilon \to 0$ of the covariance function of the processes $\{I_T^{\varepsilon}\}_{T\geq 0}$ and $\{I_{2q}(h_{2q,T}^{\varepsilon})\}_{T\geq 0}$. In order to address this problem, we will first introduce some notation that will help us to describe the covariance function of the variables $p_{\varepsilon}(\vec{B}_t-\vec{B}_s)$ and its chaotic components, which ultimately will lead to an expresion for the covariance function of I_T^{ε}.

First we describe the inner product $\left\langle f_{2q,s_1,t_1}^{\varepsilon}, f_{2q,s_2,t_2}^{\varepsilon}\right\rangle_{(\mathfrak{H}^d)^{\otimes 2q}}$. From (2.1.4), we can prove that for every $0\leq s_1\leq t_1$ and $0\leq s_2\leq t_2$,

$$\left\langle f_{2q,s_1,t_1}^{\varepsilon}, f_{2q,s_2,t_2}^{\varepsilon}\right\rangle_{(\mathfrak{H}^d)^{\otimes 2q}} = \sum_{q_1+\cdots+q_d=q}(2q_1,\dots,2q_d)!\frac{(2\pi)^{-d}\alpha(\mathbf{i}_{2q})^2}{((2q)!)^2}(\varepsilon+(t_1-s_1)^{2H})^{-\frac{d}{2}-q}$$
$$\times(\varepsilon+(t_2-s_2)^{2H})^{-\frac{d}{2}-q}\left\langle \mathbb{1}_{[s_1,t_1]}^{\otimes 2q}, \mathbb{1}_{[s_2,t_2]}^{\otimes 2q}\right\rangle_{\mathfrak{H}^{\otimes 2q}}, \tag{2.1.9}$$

where $(2q_1,\dots,2q_d)!$ denotes the multinomial coefficient $(2q_1,\dots,2q_d)!=\frac{(2q)!}{(2q_1)!\cdots(2q_d)!}$. To compute the term $\left\langle \mathbb{1}_{[s_1,t_1]}^{\otimes 2q}, \mathbb{1}_{[s_2,t_2]}^{\otimes 2q}\right\rangle_{\mathfrak{H}^{\otimes 2q}}$ appearing in the previous expression, we will introduce the following notation. For every $x,u_1,u_2>0$, define

$$\mu(x,u_1,u_2):=\mathbb{E}\left[B_{u_1}^{(1)}\left(B_{x+u_2}^{(1)}-B_x^{(1)}\right)\right]. \tag{2.1.10}$$

Define as well $\mu(x,u_1,u_2)$, for $x<0$, by $\mu(x,u_1,u_2):=\mu(-x,u_2,u_1)$. Using the property of stationary increments of B, we can check that for every $s_1,s_2,t_1,t_2\geq 0$, such that $s_1\leq t_1$ and $s_2\leq t_2$, it holds

$$\mathbb{E}\left[\left(B_{t_1}^{(1)}-B_{s_1}^{(1)}\right)\left(B_{t_2}^{(1)}-B_{s_2}^{(1)}\right)\right]=\mu(s_2-s_1,t_1-s_1,t_2-s_2). \tag{2.1.11}$$

As a consequence, by (2.1.2) and (2.1.9),

$$\left\langle f^{\varepsilon}_{2q,s_1,t_1}, f^{\varepsilon}_{2q,s_2,t_2}\right\rangle_{(\mathfrak{H}^d)^{\otimes 2q}} = \frac{\alpha_q}{(2\pi)^d(2q)!2^{2q}}(\varepsilon+(t_1-s_1)^{2H})^{-\frac{d}{2}-q}(\varepsilon+(t_2-s_2)^{2H})^{-\frac{d}{2}-q} \times \mu(s_2-s_1,t_1-s_1,t_2-s_2)^{2q},$$

where the constant α_q is defined by

$$\alpha_q := \sum_{q_1+\cdots+q_d=q} \frac{(2q_1)!\cdots(2q_d)!}{(q_1!)^2\cdots(q_d!)^2}. \tag{2.1.12}$$

From here we can conclude that

$$\left\langle f^{\varepsilon}_{2q,s_1,t_1}, f^{\varepsilon}_{2q,s_2,t_2}\right\rangle_{(\mathfrak{H}^d)^{\otimes 2q}} = \frac{\alpha_q}{(2\pi)^d(2q)!2^{2q}} G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2), \tag{2.1.13}$$

where $G^{(q)}_{\varepsilon,x}(u_1,u_2)$ is defined by

$$G^{(q)}_{\varepsilon,x}(u_1,u_2) := \left(\varepsilon+u_1^{2H}\right)^{-\frac{d}{2}-q}\left(\varepsilon+u_2^{2H}\right)^{-\frac{d}{2}-q}\mu(x,u_1,u_2)^{2q}. \tag{2.1.14}$$

Now we describe the covariance $\mathrm{Cov}\left[p_\varepsilon\left(\vec{B}_{t_1}-\vec{B}_{s_1}\right), p_\varepsilon\left(\vec{B}_{t_2}-\vec{B}_{s_2}\right)\right]$. Using the chaos expansion (2.1.3) and (2.1.13), we obtain

$$\mathrm{Cov}\left[p_\varepsilon\left(\vec{B}_{t_1}-\vec{B}_{s_1}\right), p_\varepsilon\left(\vec{B}_{t_2}-\vec{B}_{s_2}\right)\right] = \sum_{q=1}^{\infty} \frac{\alpha_q}{(2\pi)^d 2^{2q}} G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2). \tag{2.1.15}$$

On the other hand, using once more the property of stationary increments of $\vec{B}$, we can prove that for every $s_1 \le t_1$, and $s_2 \le t_2$,

$$\mathrm{Cov}\left[p_\varepsilon\left(\vec{B}_{t_1}-\vec{B}_{s_1}\right), p_\varepsilon\left(\vec{B}_{t_2}-\vec{B}_{s_2}\right)\right] = F_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2), \tag{2.1.16}$$

where the function $F_{\varepsilon,x}(u_1,u_2)$, for $u_1,u_2>0$, is defined by

$$F_{\varepsilon,x}(u_1,u_2) := \mathrm{Cov}\left[p_\varepsilon(\vec{B}_{u_1}), p_\varepsilon\left(\vec{B}_{x+u_2}-\vec{B}_x\right)\right], \tag{2.1.17}$$

in the case $x>0$, and by $F_{\varepsilon,x}(u_1,u_2) := F_{\varepsilon,-x}(u_2,u_1)$ in the case $x<0$. Proceeding as in [23], equations (13)-(14), we can prove that for every $u_1,u_2\geq 0$, $x\in\mathbb{R}$,

$$\begin{aligned} F_{\varepsilon,x}(u_1,u_2) = (2\pi)^{-d}\Bigg[&\left((\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})-\mu(x,u_1,u_2)^2\right)^{-\frac{d}{2}} \\ &-(\varepsilon+u_1^{2H})^{-\frac{d}{2}}(\varepsilon+u_2^{2H})^{-\frac{d}{2}}\Bigg], \end{aligned} \tag{2.1.18}$$

and consequently,

$$\begin{aligned} F_{\varepsilon,x}(u_1,u_2) = (2\pi)^{-d}&(\varepsilon+u_1^{2H})^{-\frac{d}{2}}(\varepsilon+u_2^{2H})^{-\frac{d}{2}} \\ &\times\left(\left(1-\frac{\mu(x,u_1,u_2)^2}{(\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})}\right)^{-\frac{d}{2}}-1\right). \end{aligned} \tag{2.1.19}$$

From (2.1.15) and (2.1.16) it follows that the functions $G_{\varepsilon,x}^{(q)}(u_1,u_2)$ and $F_{\varepsilon,x}(u_1,u_2)$ appearing in (2.1.13) and (2.1.19) are related in the following manner:

$$F_{\varepsilon,x}(u_1,u_2) = \sum_{q=1}^{\infty}\beta_q G_{\varepsilon,x}^{(q)}(u_1,u_2), \tag{2.1.20}$$

where β_q is defined by

$$\beta_q := \frac{\alpha_q}{(2\pi)^d 2^{2q}}. \tag{2.1.21}$$

The functions $G_{1,x}^{(q)}(u_1,u_2)$ and $F_{1,x}(u_1,u_2)$ satisfy the following useful integrability condition, which was proved in [23, Lemma 13], .

Lemma 2.1.1. *Let $\frac{3}{2d} < H < \frac{3}{4}$, and $q \in \mathbb{N}$, $q \geq 1$ be fixed. Define $G^{(q)}_{1,x}(u_1,u_2)$ by* (2.1.14) *and β_q by* (2.1.21). *Then,*

$$\beta_q \int_{\mathbb{R}^3_+} G^{(q)}_{1,x}(u_1,u_2)dxdu_1du_2 \leq \int_{\mathbb{R}^3_+} F_{1,x}(u_1,u_2)dxdu_1du_2 < \infty.$$

Proof. By (2.1.20), it follows that $\beta_q G^{(q)}_{1,x}(u_1,u_2) \leq F_{1,x}(u_1,u_2)$. The integrability of the function $F_{1,x}(u_1,u_2)$ over $x,u_1,u_2 \geq 0$, written as in (2.1.18), is proved in [23, Lemma 13] (see equation (40) for notation reference). □

With the notation previously introduced, we can compute the covariance functions of the increments of the processes $\{I^\varepsilon_T\}_{T\geq 0}$ and $\{I_{2q}(h^\varepsilon_{2q,T})\}_{T\geq 0}$ as follows. Define the set $\mathscr{K}_{T_1,T_2}$ by

$$\mathscr{K}_{T_1,T_2} := \{(s,t) \in \mathbb{R}^2_+ \mid s \leq t, \text{ and } T_1 \leq t \leq T_2\}. \tag{2.1.22}$$

By (2.1.1) and (2.1.7), for every $T_1 < T_2$, we can write

$$I^\varepsilon_{T_2} - \mathbb{E}[I^\varepsilon_{T_2}] - \left(I^\varepsilon_{T_1} - \mathbb{E}[I^\varepsilon_{T_1}]\right) = \int_{\mathbb{R}^2_+} \mathbb{1}_{\mathscr{K}_{T_1,T_2}}(s,t)\left(p_\varepsilon(\vec{B}_t - \vec{B}_s) - \mathbb{E}\left[p_\varepsilon(\vec{B}_t - \vec{B}_s)\right]\right)dsdt,$$

and

$$I_{2q}(h^\varepsilon_{2q,T_2}) - I_{2q}(h^\varepsilon_{2q,T_1}) = \int_{\mathbb{R}^2_+} \mathbb{1}_{\mathscr{K}_{T_1,T_2}}(s,t) I_{2q}(f^\varepsilon_{2q,s,t})dsdt.$$

By (2.1.16), we deduce the following identity for every $T_1 \leq T_2$ and $\widetilde{T}_1 \leq \widetilde{T}_2$,

$$\operatorname{Cov}\left[I^\varepsilon_{T_2} - I^\varepsilon_{T_1}, I^\varepsilon_{\widetilde{T}_2} - I^\varepsilon_{\widetilde{T}_1}\right] = \int_{\mathbb{R}^4_+} \mathbb{1}_{\mathscr{K}_{T_1,T_2}}(s_1,t_1)\mathbb{1}_{\mathscr{K}_{\widetilde{T}_1,\widetilde{T}_2}}(s_2,t_2)F_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)ds_1ds_2dt_1dt_2. \tag{2.1.23}$$

Similarly, by (2.1.13),

$$\mathbb{E}\left[(I_{2q}(h^{\varepsilon}_{2q,T_2})-I_{2q}(h^{\varepsilon}_{2q,T_1}))(I_{2q}(h^{\varepsilon}_{2q,\widetilde{T}_2})-I_{2q}(h^{\varepsilon}_{2q,\widetilde{T}_1}))\right]$$
$$=\beta_q\int_{\mathbb{R}^4_+}\mathbb{1}_{\mathscr{K}_{T_1,T_2}}(s_1,t_1)\mathbb{1}_{\mathscr{K}_{\widetilde{T}_1,\widetilde{T}_2}}(s_2,t_2)G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)ds_1ds_2dt_1dt_2, \quad (2.1.24)$$

where β_q is defined by (2.1.21).

We end this section by introducing some notation, which will be used throughout the paper to describe expectations of the form $\mathbb{E}\left[p_\varepsilon(\vec{B}_{t_1}-\vec{B}_{s_1})p_\varepsilon(\vec{B}_{t_2}-\vec{B}_{s_2})\right]$. For every n-dimensional non-negative definite matrix A, we will denote by ϕ_A the density function of a Gaussian vector with mean zero and covariance A. In addition, we will denote by $|A|$ the determinant of A, and by I_n the identity matrix of dimension n.

Let Σ be the covariance matrix of the 2-dimensional random vector $(B^{(1)}_{t_1}-B^{(1)}_{s_1},B^{(1)}_{t_2}-B^{(1)}_{s_2})$. Then, the covariance matrix of the $2d$-dimensional random vector $(\vec{B}_{t_1}-\vec{B}_{s_1},\vec{B}_{t_2}-\vec{B}_{s_2})$ can be written as

$$\mathrm{Cov}(\vec{B}_{t_1}-\vec{B}_{s_1},\vec{B}_{t_2}-\vec{B}_{s_2})=I_d\otimes\Sigma,$$

where in the previous identity $\otimes$ denotes the Kronecker product of matrices. Consider the $2d$-dimensional Gaussian density $\phi_{\varepsilon I_{2d}}(x,y)=p_\varepsilon(x)p_\varepsilon(y)$, where $x,y\in\mathbb{R}^d$, and denote by $*$ the convolution operation. Then we have that

$$\mathbb{E}\left[p_\varepsilon(\vec{B}_{t_1}-\vec{B}_{s_1})p_\varepsilon(\vec{B}_{t_2}-\vec{B}_{s_2})\right]=\int_{\mathbb{R}^{2d}}\phi_{\varepsilon I_{2d}}(x,y)\phi_{I_d\otimes\Sigma}(-x,-y)dxdy$$
$$=\phi_{\varepsilon I_{2d}}*\phi_{I_d\otimes\Sigma}(0,0)=(2\pi)^{-d}\left|\varepsilon I_{2d}+I_d\otimes\Sigma\right|^{-\frac{1}{2}}.$$

From the previous equation it follows that

$$\mathbb{E}\left[p_\varepsilon(\vec{B}_{t_1}-\vec{B}_{s_1})p_\varepsilon(\vec{B}_{t_2}-\vec{B}_{s_2})\right]=(2\pi)^{-d}\left|\varepsilon I_2+\Sigma\right|^{-\frac{d}{2}}. \tag{2.1.25}$$

The right-hand side of the previous identity can be rewritten as follows. Define the function

$$\Theta_\varepsilon(x,u_1,u_2):=\varepsilon^2+\varepsilon(u_1^{2H}+u_2^{2H})+u_1^{2H}u_2^{2H}-\mu(x,u_1,u_2)^2. \tag{2.1.26}$$

Then, using (2.1.11), we can easily show that

$$|\varepsilon I_2+\Sigma|=\Theta_\varepsilon(s_2-s_1,t_1-s_1,t_2-s_2),$$

which, by (2.1.25), implies that

$$\mathbb{E}\left[p_\varepsilon(\vec{B}_{t_1}-\vec{B}_{s_1})p_\varepsilon(\vec{B}_{t_2}-\vec{B}_{s_2})\right]=(2\pi)^{-d}\Theta_\varepsilon(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{2}}. \tag{2.1.27}$$

Therefore, we can write $\mathbb{E}\left[(I_T^\varepsilon)^2\right]$, as

$$\mathbb{E}\left[(I_T^\varepsilon)^2\right]=(2\pi)^{-d}\int_{(T\mathscr{R})^2}\Theta_\varepsilon(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{2}}ds_1ds_2dt_1dt_2. \tag{2.1.28}$$

Finally, we prove the following inequality, which estimates the function $F_{\varepsilon,x}(u_1,u_2)$, defined in (2.1.17), in terms of $\Theta_\varepsilon(x,u_1,u_2)$

$$F_{\varepsilon,x}(u_1,u_2)\leq(2\pi)^{-d}\left(\frac{d}{2}+1\right)\frac{\mu(x,u_1,u_2)^2}{u_1^{2H}u_2^{2H}}\Theta_\varepsilon(x,u_1,u_2)^{-\frac{d}{2}}. \tag{2.1.29}$$

Indeed, using relation (2.1.19), as well as the binomial theorem, we deduce that

$$F_{\varepsilon,x}(u_1,u_2) = (2\pi)^{-d}(\varepsilon+u_1^{2H})^{-\frac{d}{2}-1}(\varepsilon+u_2^{2H})^{-\frac{d}{2}-1}\mu(x,u_1,u_2)^2 \times \sum_{q=0}^{\infty}\frac{(\frac{d}{2})^{\overline{q+1}}}{(q+1)!}\left(\frac{\mu(x,u_1,u_2)^2}{(\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})}\right)^q,$$

where $a^{\overline{n}}$ denotes the n-th raising factorial of a. Hence, using the fact that

$$\frac{(\frac{d}{2})^{\overline{q+1}}}{(q+1)!} = \frac{(\frac{d}{2}+q)}{q+1}\frac{(\frac{d}{2})^{\overline{q}}}{q!} \leq \left(\frac{d}{2}+1\right)\frac{(\frac{d}{2})^{\overline{q}}}{q!},$$

we deduce that

$$F_{\varepsilon,x}(u_1,u_2) \leq (2\pi)^{-d}\left(\frac{d}{2}+1\right)(1+u_1^{2H})^{-\frac{d}{2}}(1+u_2^{2H})^{-\frac{d}{2}}\frac{\mu(x,u_1,u_2)^2}{(\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})} \times \sum_{q=0}^{\infty}\frac{(\frac{d}{2})^{\overline{q}}}{q!}\left(\frac{\mu(x,u_1,u_2)^2}{(\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})}\right)^q,$$

which, by the binomial theorem, implies (2.1.29).

Due to relations (2.1.24) and (2.1.27), the integrals

$$\int_{[0,T]^3} G_{\varepsilon}^{(q)}(x,u_1,u_2)dxdu_1du_2 \quad \text{and} \quad \int_{[0,T]^3} F_{\varepsilon}(x,u_1,u_2)dxdu_1du_2 \qquad (2.1.30)$$

will frequently appear throughout the paper, and their asymptotic behavior as $\varepsilon \to 0$ will depend on the value Hurst parameter H. In order to simplify the study of such integrals, we introduce the following sets

$$\begin{aligned}
\mathscr{S}_1 &:= \{(x,u_1,u_2) \in \mathbb{R}^3_+ \mid x+u_2-u_1 \geq 0, u_1 - x \geq 0\}, \\
\mathscr{S}_2 &:= \{(x,u_1,u_2) \in \mathbb{R}^3_+ \mid u_1 - x - u_2 \geq 0\}, \\
\mathscr{S}_3 &:= \{(x,u_1,u_2) \in \mathbb{R}^3_+ \mid x - u_1 \geq 0\}. \qquad (2.1.31)
\end{aligned}$$

The sets $\mathscr{S}_1, \mathscr{S}_2$ and $\mathscr{S}_3$ satisfy $\mathbb{R}^3_+ = \cup_{i=1}^3 \mathscr{S}_i$, and $|\mathscr{S}_i \cap \mathscr{S}_j| = 0$ for $i \neq j$. In addition, they satisfy the property that the integrals of $G_\varepsilon^{(q)}$ and F_ε over $[0,T]^3 \cap \mathscr{S}_i$ are considerably simpler to handle than the integrals (2.1.30). This phenomenon arises from the local nondeterminism property of the factional Brownian motion (see Lemma 2.4.1).

2.2 Behavior of the covariances of the approximate self-intersection local time and its chaotic components

In this section we describe the behavior as $\varepsilon \to 0$ of the covariance of $I_{T_1}^\varepsilon$ and $I_{T_2}^\varepsilon$, as well as the covariance of $I_{2q}(h_{2q,T_1}^\varepsilon)$ and $I_{2q}(h_{2q,T_2}^\varepsilon)$, for $0 \leq T_1 \leq T_2$.

Theorem 2.2.1. *Let $T_1, T_2 \geq 0$ be fixed. Then, if $\frac{3}{2d} < H < \frac{3}{4}$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{d-\frac{3}{2H}} \mathbb{E}\left[I_{2q}(h_{2q,T_1}^\varepsilon) I_{2q}(h_{2q,T_2}^\varepsilon)\right] = \sigma_q^2 (T_1 \wedge T_2),$$

where

$$\sigma_q^2 := 2\beta_q \int_{\mathbb{R}^3_+} G_{1,x}^{(q)}(u_1,u_2)\,dx\,du_1\,du_2, \qquad (2.2.1)$$

β_q is defined by (2.1.21) *and $G^{(q)}_{1,x}(u_1,u_2)$ by* (2.1.14)*. Moreover, we have*

$$\sum_{q=1}^{\infty} \sigma_q^2 = \sigma^2, \tag{2.2.2}$$

where σ^2 is a finite constant given by

$$\sigma^2 := 2\int_{\mathbb{R}_+^3} F_{1,x}(u_1,u_2)dxdu_1du_2, \tag{2.2.3}$$

and $F_{1,x}(u_1,u_2)$ is defined in (2.1.17).

Proof. To prove the result, it suffices to show that for each $a<b<\alpha<\beta$,

$$\lim_{\varepsilon\to 0}\varepsilon^{d-\frac{3}{2H}}\mathbb{E}\left[(I_{2q}(h^{\varepsilon}_{2q,b})-I_{2q}(h^{\varepsilon}_{2q,a}))(I_{2q}(h^{\varepsilon}_{2q,\beta})-I_{2q}(h^{\varepsilon}_{2q,\alpha}))\right]=0, \tag{2.2.4}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^{d-\frac{3}{2H}}\mathbb{E}\left[(I_{2q}(h^{\varepsilon}_{2q,b})-I_{2q}(h^{\varepsilon}_{2q,a}))^2\right]=\sigma_q^2(b-a). \tag{2.2.5}$$

First we prove (2.2.4). Set

$$\Phi^{\varepsilon}=\mathbb{E}\left[(I_{2q}(h^{\varepsilon}_{2q,b})-I_{2q}(h^{\varepsilon}_{2q,a}))(I_{2q}(h^{\varepsilon}_{2q,\beta})-I_{2q}(h^{\varepsilon}_{2q,\alpha}))\right].$$

Define the set $\mathscr{K}_{T_1,T_2}$ by (2.1.22), and $\gamma := \frac{\alpha-b}{2}>0$. We can easily check that for every $(s_1,t_1)\in\mathscr{K}_{a,b}$, and $(s_2,t_2)\in\mathscr{K}_{\alpha,\beta}$, it holds that either $t_2-s_2>\gamma$, or $s_2-s_1\geq\gamma$, and hence, by taking $T_1 = a$, $T_2 = b$, $\widetilde{T}_1 = \alpha$, $\widetilde{T}_2 = \beta$ in (2.1.24), we get

$$\begin{aligned}|\Phi^\varepsilon| \;\le\;& \beta_q \int_{[0,\beta]^4} \mathbb{1}_{(\gamma,\infty)}(t_2 - s_2) G^{(q)}_{\varepsilon,s_2-s_1}(t_1 - s_1, t_2 - s_2) ds_1 ds_2 dt_1 dt_2 \\ &+\beta_q \int_{[0,\beta]^4} \mathbb{1}_{(\gamma,\infty)}(s_2 - s_1) G^{(q)}_{\varepsilon,s_2-s_1}(t_1 - s_1, t_2 - s_2) ds_1 ds_2 dt_1 dt_2. \quad (2.2.6)\end{aligned}$$

Changing the coordinates (s_1, s_2, t_1, t_2) by $(s := s_1, x := s_2 - s_1, u_1 := t_1 - s_1, u_2 := t_2 - s_2)$ for $s_2 \geq s_2$, and by $(s := s_2, x := s_1 - s_2, u_1 := t_1 - s_1, u_2 := t_2 - s_2)$ for $s_2 \leq s_1$, in (2.2.6), using the fact that $G^{(q)}_{\varepsilon,-x}(u_1, u_2) = G^{(q)}_{\varepsilon,x}(u_2, u_1)$, and integrating the s_1 variable, we can prove that

$$|\Phi^\varepsilon| \le \beta_q \beta \int_{[0,\beta]^3} \left(\mathbb{1}_{(\gamma,\infty)}(u_1) + \mathbb{1}_{(\gamma,\infty)}(u_2) + \mathbb{1}_{(\gamma,\infty)}(x)\right) G^{(q)}_{\varepsilon,x}(u_1, u_2) dx du_1 du_2.$$

Next, changing the coordinates (x, u_1, u_2) by $(\varepsilon^{-\frac{1}{2H}} x, \varepsilon^{-\frac{1}{2H}} u_1, \varepsilon^{-\frac{1}{2H}} u_2)$, and using the fact that $G^{(q)}_{\varepsilon,\varepsilon^{\frac{1}{2H}} x}(\varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2) = \varepsilon^{-d} G^{(q)}_{1,x}(u_1, u_2)$, we get

$$\begin{aligned}|\Phi^\varepsilon| \le& \varepsilon^{\frac{3}{2H}-d} \beta_q \beta \int_{[0,\varepsilon^{-\frac{1}{2H}}\beta]^3} \left(\mathbb{1}_{(\gamma,\infty)}(\varepsilon^{\frac{1}{2H}} u_1) + \mathbb{1}_{(\gamma,\infty)}(\varepsilon^{\frac{1}{2H}} u_2) + \mathbb{1}_{(\gamma,\infty)}(\varepsilon^{\frac{1}{2H}} x)\right) \\ &\times G^{(q)}_{1,x}(u_1, u_2) dx du_1 du_2.\end{aligned}$$

Since $\gamma > 0$, the arguments in the previous integrals converge to zero pointwise, and are dominated by the function $3\beta_q \beta G^{(q)}_{1,x}(u_1, u_2)$, which is integrable by Lemma 2.1.1 due to the condition $\frac{3}{2d} < H < \frac{3}{4}$. Hence, by the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} \varepsilon^{d-\frac{3}{2H}} |\Phi^\varepsilon| = 0,$$

as required. Next we prove (2.2.5). By taking $T_1 = \widetilde{T}_1 = a$, and $T_2 = \widetilde{T}_2 = b$ in (2.1.24), we deduce that

$$\mathbb{E}\left[(I_{2q}(h^{\varepsilon}_{2q,b}) - I_{2q}(h^{\varepsilon}_{2q,a}))^2\right] = 2\beta_q \int_{[0,b]^4} \mathbb{1}_{\{s_1 \le s_2\}} \mathbb{1}_{\mathscr{K}_{a,b}}(s_1,t_1) \mathbb{1}_{\mathscr{K}_{a,b}}(s_2,t_2) \times G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2) ds_1 ds_2 dt_1 dt_2.$$

Changing the coordinates (s_1,s_2,t_1,t_2) by $(s_1, x := s_2 - s_1, u_1 := t_1 - s_1, u_2 := t_2 - s_2)$, we get

$$\begin{aligned}
&\mathbb{E}\left[(I_{2q}(h^{\varepsilon}_{2q,b}) - I_{2q}(h^{\varepsilon}_{2q,a}))^2\right] \\
&= 2\beta_q \int_{[0,b]^4} \mathbb{1}_{\mathscr{K}_{a,b}}(s_1, s_1+u_1) \mathbb{1}_{\mathscr{K}_{a,b}}(s_1+x, s_1+x+u_2) G^{(q)}_{\varepsilon,x}(u_1,u_2) ds_1 dx du_1 du_2 \\
&= 2\beta_q \int_{[0,b]^3} \int_{(a-u_1)_+ \vee (a-x-u_2)_+}^{(b-u_1)_+ \wedge (b-x-u_2)_+} ds_1 G^{(q)}_{\varepsilon,x}(u_1,u_2) dx du_1 du_2.
\end{aligned} \tag{2.2.7}$$

Notice that $G^{(q)}_{\varepsilon,\varepsilon^{\frac{1}{2H}}x}(\varepsilon^{\frac{1}{2H}}u_1, \varepsilon^{\frac{1}{2H}}u_2) = \varepsilon^{-d} G_{1,x}(u_1,u_2)$. Therefore, integrating the variable s_1, and changing the coordinates (x,u_1,u_2) by $(\varepsilon^{-\frac{1}{2H}}x, \varepsilon^{-\frac{1}{2H}}u_1, \varepsilon^{-\frac{1}{2H}}u_2)$ in (2.2.7), we conclude that

$$\begin{aligned}
\varepsilon^{d-\frac{3}{2H}} \mathbb{E}\left[(I_{2q}(h^{\varepsilon}_{2q,b}) - I_{2q}(h^{\varepsilon}_{2q,a}))^2\right] &= 2\beta_q \int_{[0,\varepsilon^{-\frac{1}{2H}}b]^3} G^{(q)}_{1,x}(u_1,u_2) \\
&\quad \times \Big[(b - \varepsilon^{\frac{1}{2H}}u_1)_+ \wedge (b - \varepsilon^{\frac{1}{2H}}(x+u_2))_+ \\
&\quad - (a - \varepsilon^{\frac{1}{2H}}u_1)_+ \vee (a - \varepsilon^{\frac{1}{2H}}(x+u_2))_+\Big] dx du_1 du_2.
\end{aligned} \tag{2.2.8}$$

The integrand in (2.2.8) converges increasingly to $2(b-a)G^{(q)}_{1,x}(u_1,u_2)$ as $\varepsilon \to 0$, which is integrable by Lemma 2.1.1. Identity (2.2.5) then follows by applying the dominated convergence theorem in (2.2.8).

Relation (2.2.2) is obtained by integrating both sides of relation (2.1.20) over the variables $x, u_1, u_2 \geq 0$, for $\varepsilon = 1$, and then using the monotone convergence theorem. The constant σ^2 is finite by Lemma 2.1.1. The proof is now complete. $\square$

In order to determine the behavior of the covariances of I_T^{ε} for the case $H = \frac{3}{4}$, we will first prove that the second chaotic component $I_2(h_{2,T}^{\varepsilon})$ characterizes the asymptotic behavior of $I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}]$ as $\varepsilon \to \infty$, for every $H \geq \frac{3}{4}$.

We start by showing that, after a suitable rescaling, the sequence $I_2(h_{2,T}^{\varepsilon})$ approximates $I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}]$ in $L^2(\Omega)$ for $H > \frac{3}{4}$. This result will be latter used in the proof of Theorem 2.0.3.

Lemma 2.2.2. *Let $\frac{3}{4} < H < 1$ be fixed. Then,*

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{d}{2} - \frac{3}{2H} + 1} \| I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}] - J_2(I_T^{\varepsilon}) \|_{L^2(\Omega)} = 0.$$

Proof. For $T > 0$ fixed, define the quantity

$$Q_{\varepsilon} := \| I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}] - J_2(I_T^{\varepsilon}) \|_{L^2(\Omega)}^2 .$$

From the chaos decomposition (2.1.6), we get

$$\begin{aligned} Q_{\varepsilon} &= \mathbb{E}\left[(I_T^{\varepsilon})^2\right] - \mathbb{E}[I_T^{\varepsilon}]^2 - \mathbb{E}\left[J_2(I_T^{\varepsilon})^2\right] \\ &= \mathbb{E}\left[(I_T^{\varepsilon})^2\right] - \mathbb{E}[I_T^{\varepsilon}]^2 - 2 \left\| h_{2,T}^{\varepsilon} \right\|_{(\mathfrak{H}^d)^{\otimes 2}}^2 \\ &= \mathbb{E}\left[(I_T^{\varepsilon})^2\right] - \mathbb{E}[I_T^{\varepsilon}]^2 - 2 \left\| \int_{T\mathscr{R}} f_{2,s,t}^{\varepsilon} ds dt \right\|_{(\mathfrak{H}^d)^{\otimes 2}}^2 . \end{aligned} \tag{2.2.9}$$

By (2.1.8) and (2.1.28), the first two terms in the right-hand side of the previous identity can be written as

$$\mathbb{E}\left[(I_T^{\varepsilon})^2\right] = (2\pi)^{-d}\int_{(T\mathscr{R})^2}\Theta_{\varepsilon}(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{2}}ds_1ds_2dt_1dt_2, \tag{2.2.10}$$

and

$$\mathbb{E}\,[I_T^{\varepsilon}]^2 = (2\pi)^{-d}\int_{(T\mathscr{R})^2}G^{(0)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)ds_1ds_2dt_1dt_2, \tag{2.2.11}$$

where $G^{(q)}_{\varepsilon,x}(u_1,u_2)$ and $\Theta_{\varepsilon}(x,u_1,u_2)$ are given by (2.1.14) and (2.1.26), respectively. To handle the third term in (2.2.9), recall that the constants α_q are given by (2.1.12), and notice that $\alpha_1 = 2d$. Hence, from (2.1.13), we deduce that

$$\left\|\int_{T\mathscr{R}}f^{\varepsilon}_{2,s,t}dsdt\right\|^2_{(\mathfrak{H}^d)^{\otimes 2}} = \frac{d(2\pi)^{-d}}{4}\int_{(T\mathscr{R})^2}G^{(1)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)ds_1ds_2dt_1dt_2. \tag{2.2.12}$$

From equations (2.2.9)-(2.2.12), we conclude that

$$\begin{aligned}Q_{\varepsilon} = (2\pi)^{-d}\int_{(T\mathscr{R})^2}\bigg(&\Theta_{\varepsilon}(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{2}}\\ &-G^{(0)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)-\frac{d}{2}G^{(1)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)\bigg)ds_1ds_2dt_1dt_2.\end{aligned} \tag{2.2.13}$$

The integrand appearing in the right-hand side is positive. Indeed, if we define

$$\rho_{\varepsilon}(x,u_1,u_2) := \mu(x,u_1,u_2)^2(\varepsilon+u_1^{2H})^{-1}(\varepsilon+u_2^{2H})^{-1},$$

then, applying relations (2.1.14), (2.1.26) we obtain

$$\Theta_\varepsilon(x,u_1,u_2)^{-\frac{d}{2}} - G^{(0)}_{\varepsilon,x}(u_1,u_2) - \frac{d}{2}G^{(1)}_{\varepsilon,x}(u_1,u_1) = 2(2\pi)^{-d}(\varepsilon+u_1^{2H})^{-\frac{d}{2}}(\varepsilon+u_2^{2H})^{-\frac{d}{2}} \times \left((1-\rho_\varepsilon(x,u_1,u_2))^{-\frac{d}{2}} - 1 - \frac{d}{2}\rho_\varepsilon(x,u_1,u_2)\right) \tag{2.2.14}$$

and the right-hand side of the previous identity is positive by the binomial theorem. As a consequence, by changing the coordinates (s_1,s_2,t_1,t_2) by $(s_1, x := s_2 - s_1, u_1 := t_1 - s_1, u_2 := t_2 - s_2)$, and integrating the variable s_1 in (2.2.13), we get

$$Q_\varepsilon \leq 2(2\pi)^{-d} T \int_{[0,T]^3} \left(\Theta_\varepsilon(x,u_1,u_2)^{-\frac{d}{2}} - G^{(0)}_{\varepsilon,x}(u_1,u_2) - \frac{d}{2}G^{(1)}_{\varepsilon,x}(u_1,u_2)\right) dx du_1 du_2.$$

In addition, by the binomial theorem, we have that for every $0 < y < 1$,

$$(1-y)^{-\frac{d}{2}} - 1 - \frac{d}{2}y = \sum_{q=2}^{\infty}(-1)^q \begin{pmatrix} -\frac{d}{2} \\ q \end{pmatrix} y^q = y^2 \sum_{q=0}^{\infty} \frac{(\frac{d}{2})^{\overline{q+2}}}{(q+2)!} y^q,$$

where $(x)^{\overline{q}}$ denotes the raising factorial $(x)^{\overline{q}} := x(x+1)\dots(x+q-1)$. Hence, by (2.2.14),

$$Q_\varepsilon \leq 2(2\pi)^{-d} T \int_{[0,T]^3} (\varepsilon+u_1^{2H})^{-\frac{d}{2}}(\varepsilon+u_2^{2H})^{-\frac{d}{2}} \times \rho_\varepsilon(x,u_1,u_2)^2 \sum_{q=0}^{\infty} \frac{(\frac{d}{2})^{\overline{q+2}}}{(q+2)!} \rho_\varepsilon(x,u_1,u_2)^q dx du_1 du_2. \tag{2.2.15}$$

Since

$$\frac{(\frac{d}{2})^{\overline{q+2}}}{(q+2)!} = \frac{(\frac{d}{2})^{\overline{q}}}{q!} \frac{(\frac{d}{2}+q)(\frac{d}{2}+q+1)}{(q+1)(q+2)} \leq \left(\frac{d}{2}+1\right)^2 \frac{(\frac{d}{2})^{\overline{q}}}{q!},$$

then, by (2.2.15),

$$Q_\varepsilon \leq 2(2\pi)^{-d} T \left(\frac{d}{2}+1\right)^2 \int_{[0,T]^3} (\varepsilon + u_1^{2H})^{-\frac{d}{2}} (\varepsilon + u_2^{2H})^{-\frac{d}{2}}$$
$$\times \rho_\varepsilon(x,u_1,u_2)^2 \sum_{q=0}^{\infty} \frac{(\frac{d}{2})^{\overline{q}}}{q!} \rho_\varepsilon(x,u_1,u_2)^q dx du_1 du_2,$$

which, by the binomial theorem, implies that there exists a constant $C>0$ only depending on T and d, such that

$$Q_\varepsilon \leq C \int_{[0,T]^3} \frac{\mu(x,u_1,u_2)^4}{(\varepsilon+u_1^{2H})^2(\varepsilon+u_2^{2H})^2} \Theta_\varepsilon(x,u_1,u_2)^{-\frac{d}{2}} dx du_1 du_2. \tag{2.2.16}$$

Hence, to prove the lemma it suffices to show that

$$\lim_{\varepsilon \to 0} \varepsilon^{d-\frac{3}{H}+2} \int_{[0,T]^3} \Psi_\varepsilon(x,u_1,u_2) dx du_1 du_2 = 0, \tag{2.2.17}$$

where

$$\Psi_\varepsilon(x,u_1,u_2) := \frac{\mu(x,u_1,u_2)^4}{(\varepsilon+u_1^{2H})^2(\varepsilon+u_2^{2H})^2} \Theta_\varepsilon(x,u_1,u_2)^{-\frac{d}{2}}. \tag{2.2.18}$$

In order to prove (2.2.17), we proceed as follows. First we decompose the domain of integration of (2.2.17) as $[0,T]^3 = \widetilde{\mathscr{S}_1} \cup \widetilde{\mathscr{S}_2} \cup \widetilde{\mathscr{S}_3}$, where

$$\widetilde{\mathscr{S}_1} := \{(x,u_1,u_2) \in [0,T]^3 \mid x+u_2-u_1 \geq 0, u_1 - x \geq 0\},$$
$$\widetilde{\mathscr{S}_2} := \{(x,u_1,u_2) \in [0,T]^3 \mid u_1 - x - u_2 \geq 0\},$$
$$\widetilde{\mathscr{S}_3} := \{(x,u_1,u_2) \in [0,T]^3 \mid x - u_1 \geq 0\}. \tag{2.2.19}$$

Then, it suffices to show that

$$\lim_{\varepsilon\to 0}\varepsilon^{d-\frac{3}{H}+2}\int_{\widetilde{\mathscr{S}_i}}\Psi_\varepsilon(x,u_1,u_2)dxdu_1du_2=0, \tag{2.2.20}$$

for $i=1,2,3$.

First prove (2.2.20) in the cases $i=1,2$. Changing the coordinates (x,u_1,u_2) by $(\varepsilon^{-\frac{1}{2H}}x,\varepsilon^{-\frac{1}{2H}}u_1,\varepsilon^{-\frac{1}{2H}}u_2)$, and using the fact that $\Psi_\varepsilon(\varepsilon^{\frac{1}{2H}}x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2)=\varepsilon^{-d}\Psi_1(x,u_1,u_2)$, we get

$$\varepsilon^{d-\frac{3}{H}+2}\int_{\widetilde{\mathscr{S}_i}}\Psi_\varepsilon(x,u_1,u_2)dxdu_1du_2\leq\varepsilon^{2-\frac{3}{2H}}\int_{\mathscr{S}_i}\Psi_1(x,u_1,u_2)dxdu_1du_2,$$

where the sets $\mathscr{S}_i$ are defined by (2.1.31). Therefore, using the inequality $\mu(x,u_1,u_2)^2\leq(u_1u_2)^{2H}$, we obtain

$$\varepsilon^{d-\frac{3}{H}+2}\int_{\widetilde{\mathscr{S}_i}}\Psi_\varepsilon(x,u_1,u_2)dxdu_1du_2\leq\varepsilon^{2-\frac{3}{2H}}\int_{\mathscr{S}_i}\frac{\mu(x,u_1,u_2)^2}{(u_1u_2)^{2H}}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}}dxdu_1du_2. \tag{2.2.21}$$

The integral appearing in the right-hand side of the previous inequality is finite by Lemma 2.4.3 (see equation (2.4.6) for $p=2$ and $i=1,2$). Relation (2.2.20) for $i=1,2$ is then obtained by taking $\varepsilon\to 0$ in (2.2.21).

It then remains to prove (2.2.20) for $i=3$. Changing the coordinates (x,u_1,u_2) by $(a:=u_1,b:=x-u_1,c:=u_2)$, we get

$$\int_{\widetilde{\mathscr{S}_3}}\Psi_\varepsilon(x,u_1,u_2)dxdu_1du_2\leq\int_{[0,T]^3}\Psi_\varepsilon(a+b,a,c)dadbdc. \tag{2.2.22}$$

We bound the right-hand side of the previous inequality as follows. First we write

$$\begin{aligned}\mu(a+b,a,c) &= \frac{1}{2}((a+b+c)^{2H}+b^{2H}-(b+c)^{2H}-(a+b)^{2H}) \\ &= H(2H-1)ac\int_{[0,1]^2}(b+av_1+cv_2)^{2H-2}dv_1dv_2. \end{aligned} \tag{2.2.23}$$

Notice that if $a>c$, then $b+av_1+cv_2 \geq v_1(b+a) \geq v_1(b+\frac{a}{2}+\frac{c}{2})$, and if $c>a$, then $b+av_1+cv_2 \geq v_2(b+c) \geq v_2(b+\frac{a}{2}+\frac{c}{2})$. Therefore, since $H>\frac{3}{4}$, by (2.2.23) we deduce that there exists a constant $K>0$, such that

$$\mu(a+b,a,c) \leq Kac(a+b+c)^{2H-2}. \tag{2.2.24}$$

On the other hand, if Σ denotes the covariance matrix of $(B_a, B_{a+b+c}-B_{a+b})$, we can write

$$\Theta_\varepsilon(a+b,a,c) = \varepsilon^2+\varepsilon(a^{2H}+c^{2H})+|\Sigma|.$$

As a consequence, by part (3) of Lemma 2.4.1, we deduce that $\Theta_\varepsilon(a+b,a,c) \geq \varepsilon^2 + \delta(ac)^{2H}$ for some constant $\delta \in (0,1)$. Hence, by (2.2.18) and (2.2.24), that there exists a constant $C>0$, such that

$$\Psi_\varepsilon(a+b,a,c) \leq C(ac)^{4-4H}(a+b+c)^{8H-8}(\varepsilon^2+(ac)^{2H})^{-\frac{d}{2}}. \tag{2.2.25}$$

Next we bound the right-hand side of (2.2.25) by using Young's inequality. Since $H>\frac{3}{4}$ and $Hd>\frac{3}{2}$, then

$$0<\frac{3-2H}{Hd}<\frac{3}{2Hd}<1. \tag{2.2.26}$$

Using the relation (2.2.26), as well as the fact that $\frac{3}{4} < H < 1$, we deduce that there exists a constant $y > 0$, such that

$$4H - 4 + 4Hdy < 0, \tag{2.2.27}$$

$$4H - 3 - 4Hdy > 0, \tag{2.2.28}$$

$$\frac{3-2H}{Hd} + y < 1. \tag{2.2.29}$$

By (2.2.29), the constant $\gamma := \frac{3-2H}{Hd} + y$ belongs to $(0,1)$, and hence, by Young's inequality, we have

$$(1-\gamma)\varepsilon^2 + \gamma(ac)^{2H} \geq \varepsilon^{2(1-\gamma)}(ac)^{2H\gamma}. \tag{2.2.30}$$

In addition, by (2.2.27), we have

$$\begin{aligned}(a+b+c)^{8H-8} &= (a+b+c)^{4H-4-4Hdy}(a+b+c)^{4H-4+4Hdy} \\ &\leq b^{4H-4-4Hdy}(a+c)^{4H-4+4Hdy} \\ &\leq b^{4H-4-4Hdy}(2\sqrt{ac})^{4H-4+4Hdy}, \end{aligned} \tag{2.2.31}$$

where the last inequality follows from the arithmetic mean-geometric mean inequality. Hence, by (2.2.25), (2.2.30) and (2.2.31), we obtain

$$\begin{aligned}\varepsilon^{d-\frac{3}{H}+2}\int_{[0,T]^3}\Psi_\varepsilon(a+b,a,c) &\leq \varepsilon^{d-\frac{3}{H}+2-d(1-\gamma)}C\int_{[0,T]^3} b^{4H-4-4Hdy}(ac)^{2-2H+2Hdy-Hd\gamma}dadbdc \\ &= \varepsilon^{dy}C\int_{[0,T]^3} b^{4H-4-4Hdy}(ac)^{-1+Hdy}dadbdc. \end{aligned} \tag{2.2.32}$$

The integral in the right-hand side is finite by (2.2.28). Relation (2.2.20) for $i = 3$ then follows from (2.2.22) and (2.2.32). $\square$

The next result extends Lemma 2.2.2 to the case $H = \frac{3}{4}$.

Lemma 2.2.3. *Let $d \geq 3$ be fixed. Then, if $H = \frac{3}{4}$,*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} \|I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon] - J_2(I_T^\varepsilon)\|_{L^2(\Omega)} = 0. \tag{2.2.33}$$

Proof. For $T > 0$ fixed, define the quantity

$$Q_\varepsilon := \|I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon] - J_2(I_T^\varepsilon)\|^2_{L^2(\Omega)}.$$

As in the proof of equation (2.2.16) in Lemma 2.2.2, we can show that there exists a constant $C > 0$ such that

$$Q_\varepsilon \leq C \int_{[0,T]^3} \Psi_\varepsilon(x, u_1, u_2) dx du_1 du_2, \tag{2.2.34}$$

where

$$\Psi_\varepsilon(x, u_1, u_2) := \frac{\mu(x, u_1, u_2)^4}{(\varepsilon + u_1^{\frac{3}{2}})^2 (\varepsilon + u_2^{\frac{3}{2}})^2} \Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{2}}. \tag{2.2.35}$$

Hence, by splitting the domain of integration in (2.2.34) as $[0,T]^3 = \bigcup_{i=1}^3 \widetilde{\mathscr{S}_i}$, where the sets $\widetilde{\mathscr{S}_i}$ are defined by (2.2.19), we deduce that the relation (2.2.33) holds, provided that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \int_{\widetilde{\mathscr{S}_i}} \Psi_\varepsilon(x, u_1, u_2) dx du_1 du_2 = 0, \tag{2.2.36}$$

for $i = 1, 2, 3$. To prove (2.2.36) for $i = 1, 2$, we change the coordinates (x, u_1, u_2) by $(\varepsilon^{-\frac{2}{3}} x, \varepsilon^{-\frac{2}{3}} u_1, \varepsilon^{-\frac{2}{3}} u_2)$ and use the fact that $\Psi_\varepsilon(\varepsilon^{\frac{2}{3}} x, \varepsilon^{\frac{2}{3}} u_1, \varepsilon^{\frac{2}{3}} u_2) = \varepsilon^{-d} \Psi_1(x, u_2, u_2)$, in order to get

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\int_{\widetilde{\mathscr{S}_i}}\Psi_\varepsilon(x,u_1,u_2)dxdu_1du_2 \le \frac{1}{\log(1/\varepsilon)}\int_{\mathscr{S}_i}\Psi_1(x,u_1,u_2)dxdu_1du_2, \quad (2.2.37)$$

where the sets $\mathscr{S}_i$ are defined by (2.1.31). As a consequence, by applying the inequality $\mu(x,u_1,u_2)^2 \le (u_1u_2)^{\frac{3}{2}}$, we get

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\int_{\widetilde{\mathscr{S}_i}}\Psi_\varepsilon(x,u_1,u_2)dxdu_1du_2 \le \frac{1}{\log(1/\varepsilon)}\int_{\mathscr{S}_i}\frac{\mu(x,u_1,u_2)^2}{(u_1u_2)^{\frac{3}{2}}}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}}dxdu_1du_2. \quad (2.2.38)$$

The integral appearing the right-hand side of the previous inequality is finite for $i=1,2$ by Lemma 2.4.3 (see equation (2.4.6) for $p=2$). Relation (2.2.36) for $i=1,2$ is then obtained by taking $\varepsilon\to 0$ in (2.2.38).

It then suffices to handle the case $i=3$. Define the function $K(x,u_1,u_2)$ by

$$K(x,u_1,u_2) := \frac{\mu(x,u_1,u_2)^4}{(u_1u_2)^3}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}}. \quad (2.2.39)$$

Notice that

$$\frac{1}{\log(1/\varepsilon)}\int_{\mathscr{S}_3}\Psi_1(x,u_1,u_2)dxdu_1du_2 \le \frac{1}{\log(1/\varepsilon)}\int_{\mathscr{S}_3}K(x,u_1,u_2)dxdu_1du_2. \quad (2.2.40)$$

Using the representation

$$\mu(a+b,a,c) = H(2H-1)ac\int_{[0,1]^2}(b+a\xi+c\eta)^{2H-2}d\xi d\eta,$$

we get

$$\mu(a+b,a,c) \leq \frac{3ac}{8}\int_{[0,1]^2}(b\xi\eta+a\xi\eta+c\xi\eta)^{-\frac{1}{2}}d\xi d\eta = \frac{3ac}{2}(a+b+c)^{-\frac{1}{2}}.$$

As a consequence,

$$K(a+b,a,c) \leq \frac{3^4}{2^4}ac(a+b+c)^{-2}\Theta_1(a+b,a,c)^{-\frac{d}{2}}.$$

Notice that $\Theta_1(a+b,a,c) = 1+a^{\frac{3}{2}}+c^{\frac{3}{2}}+|\Sigma|$, where Σ denotes the covariance matrix of $(B_a, B_{a+b+c}-B_{a+b})$. Therefore, by part (3) of Lemma 2.4.1, we deduce that

$$\Theta_1(a+b,a,c) \geq 1+a^{\frac{3}{2}}+c^{\frac{3}{2}}+\delta(ac)^{\frac{3}{2}},$$

for some constant $\delta \in (0,1)$. From here, it follows that there exists a constant $C>0$, such that

$$K(a+b,a,c) \leq Cac(a+b+c)^{-2}\left(1+a^{\frac{3}{2}}+c^{\frac{3}{2}}+a^{\frac{3}{2}}c^{\frac{3}{2}}\right)^{-\frac{d}{2}}.$$

From here it follows that there exists a constant $C>0$ such that the following inequalities hold

$$\begin{aligned}
K(a+b,a,c) &\leq Cac^{-1}\left(1+c^{\frac{3}{2}}+a^{\frac{3}{2}}c^{\frac{3}{2}}\right)^{-\frac{d}{2}} && \text{if} \quad a \leq b \leq c,\\
K(a+b,a,c) &\leq Ca^{-1}c\left(1+a^{\frac{3}{2}}+a^{\frac{3}{2}}c^{\frac{3}{2}}\right)^{-\frac{d}{2}} && \text{if} \quad c \leq b \leq a,\\
K(a+b,a,c) &\leq Cacb^{-2}\left(1+(a\vee c)^{\frac{3}{2}}+a^{\frac{3}{2}}c^{\frac{3}{2}}\right)^{-\frac{d}{2}} && \text{if} \quad a,c \leq b,\\
K(a+b,a,c) &\leq Cac^{-1}\left(1+c^{\frac{3}{2}}+a^{\frac{3}{2}}c^{\frac{3}{2}}\right)^{-\frac{d}{2}} && \text{if} \quad b \leq a \leq c,\\
K(a+b,a,c) &\leq Ca^{-1}c\left(1+a^{\frac{3}{2}}+a^{\frac{3}{2}}c^{\frac{3}{2}}\right)^{-\frac{d}{2}} && \text{if} \quad b \leq c \leq a.
\end{aligned}$$

Using the previous inequalities, as well as the condition $d \geq 3$, we can easily check that $K(a+b,a,c)$ is integrable in $\mathbb{R}_+^3$, which in turn implies that $K(x,u_1,u_2)$ is integrable in $\mathscr{S}_3$. Using this observation, as well as relations (2.2.37) and (2.2.40), we obtain

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\int_{\widetilde{\mathscr{S}_3}}\Psi_\varepsilon(x,u_1,u_2)dxdu_1du_2 = 0,$$

as required. The proof is now complete. □

The next result provides a useful approximation for $I_2(h_{2,T}^\varepsilon)$.

Lemma 2.2.4. *Assume that $H=\frac{3}{4}$ and $d\geq 3$. Let $h_{2,T}^\varepsilon$ be defined as in* (2.1.7) *and consider the following approximation of $I_2(h_{2,T}^\varepsilon)$*

$$\widetilde{J}_T^\varepsilon := -\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{-\frac{d}{2}+1}}{2}\sum_{j=1}^d\int_0^T\int_0^\infty\frac{u^{\frac{3}{2}}}{\varepsilon^{\frac{1}{3}}(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\left(\frac{B_{s+\varepsilon^{\frac{2}{3}}u}^{(j)}-B_s^{(j)}}{\sqrt{\varepsilon}u^{\frac{3}{2}}}\right)duds. \tag{2.2.41}$$

Then we have that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left\|I_2(h_{2,T}^\varepsilon)-\widetilde{J}_T^\varepsilon\right\|_{L^2(\Omega)} = 0.$$

Proof. Using (2.1.4), we can easily check that

$$I_2(h_{2,T}^\varepsilon) = -\frac{(2\pi)^{-\frac{d}{2}}}{2}\sum_{j=1}^d\int_0^T\int_0^{T-u}\frac{u^{\frac{3}{2}}}{(\varepsilon+u^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\left(\frac{B_{s+u}^{(j)}-B_s^{(j)}}{u^{\frac{3}{4}}}\right)dsdu.$$

Making the change of variables $v := \varepsilon^{-\frac{2}{3}} u$, we get

$$I_2(h^{\varepsilon}_{2,T}) = -\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{-\frac{d}{2}+\frac{2}{3}}}{2}\sum_{j=1}^{d}\int_0^T\int_0^{\varepsilon^{-\frac{2}{3}}(T-s)}\frac{v^{\frac{3}{2}}}{(1+v^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}v}-B^{(j)}_s}{\sqrt{\varepsilon}v^{\frac{3}{4}}}\right)dvds,$$

and hence,

$$\widetilde{J}^{\varepsilon}_T - I_2(h^{\varepsilon}_{2,T}) = -\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{-\frac{d}{2}+\frac{2}{3}}}{2}\sum_{j=1}^{d}\int_0^T\int_{\varepsilon^{-\frac{2}{3}}(T-s)}^{\infty}\frac{v^{\frac{3}{2}}}{(1+v^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u}-B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}}\right)duds. \tag{2.2.42}$$

Set

$$\Phi^{\varepsilon} = \varepsilon^{d-2}\left\|\widetilde{J}^{\varepsilon}_T - I_2(h^{\varepsilon}_{2,T})\right\|^2_{L^2(\Omega)}.$$

Using (2.2.42), as well as the fact that

$$\mathbb{E}\left[H_2\left(\frac{B^{(j)}_{s_1+v_1}-B^{(j)}_{s_1}}{v_1^H}\right)H_2\left(\frac{B^{(j)}_{s_2+v_2}-B^{(j)}_{s_2}}{v_2^H}\right)\right] = 2(v_1v_2)^{-2H}\mu(s_2-s_1,v_1,v_2)^2, \tag{2.2.43}$$

for all $s_1, s_2, v_1, v_2 \geq 0$, we can easily check that

$$\Phi^{\varepsilon} = \frac{d(2\pi)^{-d}}{2}\int_{[0,T]^2}\int_{\mathbb{R}^2_+}\mathbb{1}_{[T,\infty)}(s_1+\varepsilon^{\frac{2}{3}}u_1)\mathbb{1}_{[T,\infty)}(s_2+\varepsilon^{\frac{2}{3}}u_2)V_{\varepsilon,s_2-s_1}(u_1,u_2)du_1du_2ds_1ds_2,$$

where

$$V_{\varepsilon,x}(u_1,u_2) := \varepsilon^{-\frac{8}{3}}\psi(u_1,u_2)\mu(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)^2,$$

and

$$\psi(u_1,u_2) := (1+u_1^{\frac{3}{2}})^{-\frac{d}{2}-1}(1+u_2^{\frac{3}{2}})^{-\frac{d}{2}-1}. \tag{2.2.44}$$

Hence, using the fact that $\mu(x,v_1,v_2) = \mu(-x,v_2,v_1)$, we can write

$$\Phi^\varepsilon = d(2\pi)^{-d}\int_0^T\int_0^{s_2}\int_{\mathbb{R}_+^2}\mathbb{1}_{[T,\infty)}(s_1+\varepsilon^{\frac{2}{3}}u_1)\mathbb{1}_{[T,\infty)}(s_2+\varepsilon^{\frac{2}{3}}u_2)V_{\varepsilon,s_2-s_1}(u_1,u_2)du_1du_2ds_1ds_2. \tag{2.2.45}$$

Changing the coordinates (s_1,s_2,u_1,u_2) by $(s := s_1, x := s_2-s_1, u_1, u_2)$ in the expression (2.2.45), and then integrating the variable s, we obtain

$$|\Phi^\varepsilon| = d(2\pi)^{-d}\int_0^T\int_{\mathbb{R}_+^2}(T-(T-\varepsilon^{\frac{2}{3}}u_1)_+\vee(T-x-\varepsilon^{\frac{2}{3}}u_2)_+)V_{\varepsilon,x}(u_1,u_2)du_1du_2dx,$$

and consequently, there exists a constant $C>0$ such that

$$|\Phi^\varepsilon| \leq C\int_0^T\int_{\mathbb{R}_+^2} r_{\varepsilon^{\frac{2}{3}}}(u_1)V_{\varepsilon,x}(u_1,u_2)du_1du_2dx, \tag{2.2.46}$$

where $r_\delta(u_1) := T-(T-\delta u_1)_+$. Making the change of variable $v := \varepsilon^{-\frac{2}{3}}x$ in (2.2.46) and using the fact that $V_{\varepsilon,\varepsilon^{\frac{2}{3}}v}(u_1,u_2) = \varepsilon^{-\frac{2}{3}}G_{1,v}^{(1)}(u_1,u_2)$, we get

$$|\Phi^\varepsilon| \leq C\int_0^{\varepsilon^{-\frac{2}{3}}T}\int_{\mathbb{R}_+^2} r_{\varepsilon^{\frac{2}{3}}}(u_1)G_{1,v}^{(1)}(u_1,u_2)du_1du_2dv.$$

Therefore, defining $N := \varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon) = \frac{3\log N}{2}$, we obtain

$$\frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \frac{2C}{3\log N}\int_0^{NT}\int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx.$$

To bound the right-hand side of the previous relation we split the domain of integration as follows. Define the sets $\mathscr{S}_i$, for $i = 1,2,3$, by (2.1.31). Then

$$\begin{aligned}\limsup_{\varepsilon\to 0}\frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} &\leq \limsup_{N\to\infty}\frac{2C}{3\log N}\int_0^{NT}\int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(x,u_1)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx \quad (2.2.47)\\ &\leq \frac{2C}{3}\sum_{i=1}^{3}\limsup_{N\to\infty}\frac{1}{\log N}\int_0^{NT}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)r_{\frac{1}{N}}(u_1)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx.\end{aligned}$$

By relations (2.1.20) and (2.1.29), there exists a constant $C > 0$, such that

$$G_{1,x}^{(1)}(u_1,u_2) \leq C\frac{\mu^2(x,u_1,u_2)}{(u_1u_2)^{\frac{3}{2}}}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}}. \quad (2.2.48)$$

Hence, by Lemma 2.4.3, the terms with $i = 1$ and $i = 2$ in the sum in the right-hand side of (2.2.47) converge to zero. From this observation, we conclude that there exists a constant $C > 0$, such that

$$\limsup_{\varepsilon\to 0}\frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \limsup_{N\to\infty}\frac{C}{\log N}\int_0^{NT}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)r_{\frac{1}{N}}(u_1)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx. \quad (2.2.49)$$

Using Lemma 2.4.2, we can easily show that there exists a constant $C > 0$, such for every $(x,u_1,u_2) \in \mathscr{S}_3$, the following inequality holds

$$G_{1,x}^{(1)}(u_1,u_2) = \psi(u_1,u_2)\mu(x,u_1,u_2)^2 \leq C\psi(u_1,u_2)(x+u_1+u_2)^{-1}(u_1u_2)^2, \quad (2.2.50)$$

where $\psi(u_1,u_2)$ is defined in (2.2.44). From (2.2.49) and (2.2.50), it follows that

$$\limsup_{\varepsilon\to 0}\frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \limsup_{N\to\infty}\frac{C}{\log N}\int_0^{NT}\int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1)(x+u_1+u_2)^{-1}(u_1u_2)^2\psi(u_1,u_2)du_1du_2dx.$$

In addition, we have that

$$\limsup_{N\to\infty} \frac{1}{\log N} \int_0^1 \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1)(x+u_1+u_2)^{-1}(u_1u_2)^2\psi(u_1,u_2)du_1du_2dx$$
$$\leq \limsup_{N\to\infty} \frac{T}{\log N} \int_0^1 \int_{\mathbb{R}_+^2} (u_1+u_2)^{-1}(u_1u_2)^2\psi(u_1,u_2)du_1du_2 = 0,$$

and consequently,

$$\limsup_{\varepsilon\to 0} \frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \limsup_{N\to\infty} \frac{C}{\log N} \int_1^{NT} \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1)x^{-1}(u_1u_2)^2\psi(u_1,u_2)du_1du_2dx.$$

For $\delta > 0$ fixed, let $M > 1$ be such that

$$\int_M^\infty \int_0^\infty (u_1u_2)^2\psi(u_1,u_2)du_2du_1 < \delta. \tag{2.2.51}$$

Using (2.2.51), as well as the fact that $r_{\frac{1}{N}}(u)$ is increasing on u, we obtain

$$\frac{1}{\log N} \int_1^{NT} \int_M^\infty \int_0^\infty x^{-1}(u_1u_2)^2\psi(u_1,u_2)du_1du_2dx \leq \delta\left(1+\frac{\log(T)}{\log N}\right),$$

and

$$\limsup_{N\to\infty} \frac{1}{\log N} \int_1^{NT} \int_0^M \int_0^\infty r_{\frac{1}{N}}(u_1)x^{-1}(u_1u_2)^2\psi(u_1,u_2)du_1du_2dx$$
$$\leq \limsup_{N\to\infty} \left(1+\frac{\log(T)}{\log N}\right) \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(M)(u_1u_2)^2\psi(u_1,u_2)du_1du_2 = 0.$$

As a consequence,

$$\limsup_{\varepsilon\to 0} \frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq C\delta.$$

Hence, taking $\delta \to 0$, we get

$$\lim_{\varepsilon \to 0} \frac{\Phi^\varepsilon}{\log(1/\varepsilon)} = 0,$$

as required. $\square$

Finally, we describe the behavior of the covariance function of $I_2(h^\varepsilon_{2,T})$ for the case $H = \frac{3}{4}$.

Theorem 2.2.5. *Let $T_1, T_2 \geq 0$ be fixed. Then, if $d \geq 3$ and $H = \frac{3}{4}$,*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\left[I_2(h^\varepsilon_{2,T_1}) I_2(h^\varepsilon_{2,T_2})\right] = \rho^2 (T_1 \wedge T_2),$$

where ρ is a finite constant defined by

$$\rho := \frac{\sqrt{3d}}{2^{\frac{d+5}{2}} \pi^{\frac{d}{2}}} \int_0^\infty (1 + u^{\frac{3}{2}})^{-\frac{d}{2}-1} u^2 du. \tag{2.2.52}$$

Proof. Consider the approximation $\widetilde{J}^\varepsilon_T$ of $I_2(h^\varepsilon_{2,T})$, introduced in (2.2.41). By Lemma 2.2.4,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \left\| \widetilde{J}^\varepsilon_T - I_2(h^\varepsilon_{2,T}) \right\|^2_{L^2(\Omega)} \to 0.$$

Therefore, it suffices to show that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\left[\widetilde{J}^\varepsilon_{T_1} \widetilde{J}^\varepsilon_{T_2}\right] = \rho^2 (T_1 \wedge T_2). \tag{2.2.53}$$

As in Lemma 2.2, to prove (2.2.53), it suffices to show that for each $a < b < \alpha < \beta$,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\left[(\widetilde{J}^\varepsilon_b - \widetilde{J}^\varepsilon_a)(\widetilde{J}^\varepsilon_\beta - \widetilde{J}^\varepsilon_\alpha)\right] = 0, \tag{2.2.54}$$

and

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(\widetilde{J}^{\varepsilon}_{b}-\widetilde{J}^{\varepsilon}_{a})^2\right]=\rho^2(b-a). \tag{2.2.55}$$

First we prove (2.2.54). Set

$$\Phi^{\varepsilon}=\varepsilon^{d-2}\mathbb{E}\left[(\widetilde{J}^{\varepsilon}_{b}-\widetilde{J}^{\varepsilon}_{a})(\widetilde{J}^{\varepsilon}_{\beta}-\widetilde{J}^{\varepsilon}_{\alpha})\right].$$

Using (2.2.43) and (2.2.41), we can easily check that

$$\Phi^{\varepsilon}=\frac{d(2\pi)^{-d}}{2}\int_{\alpha}^{\beta}\int_{a}^{b}\int_{\mathbb{R}^2_+}V_{\varepsilon,s_2-s_1}(u_1,u_2)du_1du_2ds_1ds_2, \tag{2.2.56}$$

where

$$V_{\varepsilon,x}(u_1,u_2):=\varepsilon^{-\frac{8}{3}}\psi(u_1,u_2)\mu(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)^2,$$

and $\psi(u_1,u_2)$ is defined by (2.2.39). Changing the coordinates (s_1,s_2,u_1,u_2) by $(s:=s_1,x:=s_2-s_1,u_1,u_2)$ in (2.2.56), and then integrating the variable s, we can show that

$$|\Phi^{\varepsilon}|\leq d(2\pi)^{-d}\beta\int_{\gamma}^{\beta}\int_{\mathbb{R}^2_+}V_{\varepsilon,x}(u_1,u_2)du_1du_2dx, \tag{2.2.57}$$

where the constant γ is defined by $\gamma:=\alpha-b$. Making the change of variable $v:=\varepsilon^{-\frac{2}{3}}x$ and using the fact that

$$V_{\varepsilon,\varepsilon^{\frac{2}{3}}v}(u_1,u_2)=\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,v}(u_1,u_2),$$

we get

$$|\Phi^{\varepsilon}| \leq d(2\pi)^{-d}\beta \int_{\varepsilon^{-\frac{2}{3}}\gamma}^{\varepsilon^{-\frac{2}{3}}\beta} \int_{\mathbb{R}_+^2} G_{1,v}^{(1)}(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}v.$$

Therefore, defining $N := \varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon) = \frac{3\log N}{2}$, we obtain

$$\frac{|\Phi^{\varepsilon}|}{\log(1/\varepsilon)} \leq \frac{2d(2\pi)^{-d}\beta}{3\log N} \int_{N\gamma}^{N\beta} \int_{\mathbb{R}_+^2} G_{1,x}^{(1)}(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}x.$$

To bound the right-hand side of the previous relation we split the domain of integration as follows. Define the sets $\mathscr{S}_i$, for $i = 1,2,3$, by (2.1.31). Then, there exists a constant $C > 0$, such that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{|\Phi^{\varepsilon}|}{\log(1/\varepsilon)} &\leq \limsup_{N\to\infty} \frac{C}{\log N} \int_{N\gamma}^{N\beta} \int_{\mathbb{R}_+^2} G_{1,x}^{(1)}(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}x \\ &\leq \sum_{i=1}^{3} \limsup_{N\to\infty} \frac{C}{\log N} \int_{N\gamma}^{N\beta} \int_{\mathbb{R}_+^2} \mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2) G_{1,x}^{(1)}(u_1,u_2) du_1\mathrm{d}u_2 dx.\end{aligned} \tag{2.2.58}$$

Taking into account (2.2.48), by Lemma 2.4.3, the terms with $i = 1$ and $i = 2$ in the sum in the right-hand side of (2.2.58) converge to zero. From this observation, we conclude that

$$\limsup_{\varepsilon\to 0} \frac{|\Phi^{\varepsilon}|}{\log(1/\varepsilon)} \leq \limsup_{N\to\infty} \frac{C}{\log N} \int_{N\gamma}^{N\beta} \int_{\mathbb{R}_+^2} \mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2) G_{1,x}^{(1)}(u_1,u_2)\mathrm{d}u_1 du_2 dx. \tag{2.2.59}$$

By Lemma 2.4.2, there exists a constant $C>0$, such for every $(x,u_1,u_2)\in\mathscr{S}_3$, the following inequality holds

$$G^{(1)}_{1,x}(u_1,u_2)=\psi(u_1,u_2)\mu(x,u_1,u_2)^2\le C\psi(u_1,u_2)x^{-1}(u_1u_2)^2. \tag{2.2.60}$$

From (2.2.59) and (2.2.60), we obtain

$$\limsup_{\varepsilon\to0}\frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)}\le C\limsup_{N\to\infty}\frac{\log(N\beta)-\log(N\gamma)}{\log N}\int_{\mathbb{R}^2_+}\psi(u_1,u_2)(u_1u_2)^2du_1du_2,$$

for some constant $C>0$. The function $(1+u^{\frac{3}{2}})^{-\frac{d}{2}-1}u^2$ is integrable for u in $\mathbb{R}_+$ due to the condition $d\ge3$, and hence, from the previous inequality we conclude that

$$\limsup_{\varepsilon\to0}\frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)}=0. \tag{2.2.61}$$

Relation (2.2.54) then follows from (2.2.61).

Next we prove (2.2.55). By taking $\alpha=a$ and $\beta=b$ in relation (2.2.56), we obtain

$$\varepsilon^{d-2}\mathbb{E}\left[(\widetilde{J}^\varepsilon_b-\widetilde{J}^\varepsilon_a)^2\right]=d(2\pi)^{-d}\int_a^b\int_a^{s_2}\int_{\mathbb{R}^2_+}V_{\varepsilon,s_2-s_1}(u_1,u_2)du_1du_2ds_1ds_2.$$

Changing the coordinates (s_1,s_2,t_1,t_2) by $(s_1,x:=\varepsilon^{-\frac{2}{3}}(s_2-s_1),u_1:=t_1-s_1,u_2:=t_2-s_2)$, integrating the variable s_1 and using the fact that $V_{\varepsilon,\varepsilon^{\frac{2}{3}}\hat{x}}(u_1,u_2)=\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\hat{x}}(u_1,u_2)$, we deduce that

$$\begin{aligned}\varepsilon^{d-2}\mathbb{E}\left[(\widetilde{J}^\varepsilon_b-\widetilde{J}^\varepsilon_a)^2\right]&=d(2\pi)^{-d}\int_0^{\varepsilon^{-\frac{2}{3}}(b-a)}\int_{\mathbb{R}^2_+}(b-\varepsilon^{\frac{2}{3}}x-a)\varepsilon^{\frac{2}{3}}V_{\varepsilon,\varepsilon^{\frac{2}{3}}x}(u_1,u_2)du_1du_2dx\\&=d(2\pi)^{-d}\int_0^{\varepsilon^{-\frac{2}{3}}(b-a)}\int_{\mathbb{R}^2_+}(b-\varepsilon^{\frac{2}{3}}x-a)G^{(1)}_{1,x}(u_1,u_2)du_1du_2dx.\end{aligned}$$

Therefore, defining $N := \varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon) = \frac{3\log N}{2}$, we obtain

$$
\begin{aligned}
&\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(\widetilde{J}_b^{\varepsilon} - \widetilde{J}_a^{\varepsilon})^2\right] \\
&\quad = \frac{2d(2\pi)^{-d}}{3\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}(b-\frac{x}{N}-a)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx \\
&\quad = \frac{2d(2\pi)^{-d}}{3\log N}\sum_{i=1}^{3}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}(b-\frac{x}{N}-a)\mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx.
\end{aligned}
\tag{2.2.62}
$$

By inequality (2.2.48) and Lemma 2.4.3, the terms with $i=1$ and $i=2$ in the sum in the right-hand side of (2.2.62) converge to zero. From this observation, it follows that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(\widetilde{J}_b^{\varepsilon} - \widetilde{J}_a^{\varepsilon})^2\right] \\
&\quad = \lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}(b-\frac{x}{N}-a)\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx \\
&\quad = \lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}(b-a)\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx \\
&\quad - \lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)xG_{1,x}^{(1)}(u_1,u_2)du_1du_2dx,
\end{aligned}
\tag{2.2.63}
$$

provided that the limits in the right-hand side exist. By (2.2.60), there exists a constant $C>0$ such that

$$
\begin{aligned}
&\frac{1}{N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)xG_{1,x}^{(1)}(u_1,u_2)du_1du_2dx \\
&\qquad \leq \frac{C}{N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}\psi(u_1,u_2)(u_1u_2)^2du_1du_2dx \\
&\qquad = \frac{C(b-a)}{\log N}\int_{\mathbb{R}_+^2}\psi(u_1,u_2)(u_1u_2)^2du_1du_2.
\end{aligned}
$$

Since $d \geq 3$, the integral in the right-hand side is finite, and hence

$$\lim_{N\to\infty} \frac{1}{N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2} \mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)xG_{1,x}^{(1)}(u_1,u_2)du_1du_2dx = 0.$$

Therefore, by (2.2.63),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(\widetilde{J}_b^{\varepsilon}-\widetilde{J}_a^{\varepsilon})^2\right] \\
&= \lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}(b-a)\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)G_{1,x}^{(1)}(u_1,u_2)du_1du_2dx. \qquad (2.2.64)
\end{aligned}$$

Applying L'Ĥopital's rule in (2.2.64), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(\widetilde{J}_b^{\varepsilon}-\widetilde{J}_a^{\varepsilon})^2\right] \\
&= \lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3}\int_{\mathbb{R}_+^2}N(b-a)^2\mathbb{1}_{\mathscr{S}_3}(N(b-a),u_1,u_2)G_{1,N(b-a)}^{(1)}(u_1,u_2)du_1du_2dx. \qquad (2.2.65)
\end{aligned}$$

By (2.2.60), the integrand in the right-hand side is bounded by the function

$$C\psi(u_1,u_2)(u_1u_2)^2$$

for some constant $C > 0$. On the other hand, using (1.2.6), we can easily check that

$$\begin{aligned}
|\mu(x,v_1,v_2)| &= \left|\left\langle \mathbb{1}_{[0,v_1]}, \mathbb{1}_{[x,x+v_2]}\right\rangle_{\mathfrak{H}}\right| \\
&= H(2H-1)v_1v_2\int_{[0,1]^2}|x+v_2w_2-v_1w_1|^{2H-2}dw_1dw_2 \\
&= \frac{3v_1v_2}{8}\int_{[0,1]^2}|x+v_2w_2-v_1w_1|^{-\frac{1}{2}}dw_1dw_2,
\end{aligned}$$

so that

$$\lim_{N\to\infty} N(b-a)\mu(N(b-a),u_1,u_2)^2 = \frac{3^2(u_1u_2)^2}{2^6},$$

and hence,

$$\lim_{N\to\infty} N(b-a)\mathbb{1}_{\mathscr{S}_3}(N(b-a),u_1,u_2)G^{(1)}_{1,N(b-a)}(u_1,u_2) = \frac{3^2}{2^6}\psi(u_1,u_2)(u_1u_2)^2.$$

Therefore, by applying the dominated convergence theorem to (2.2.65), we get

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(\widetilde{J}^{\varepsilon}_b-\widetilde{J}^{\varepsilon}_a)^2\right] = (b-a)\frac{3d}{2^{d+5}\pi^d}\left(\int_{\mathbb{R}_+}(1+u^{\frac{3}{2}})^{-\frac{d}{2}-1}u^2du\right)^2.$$

□

Relation (2.2.55) follows from the previous inequality. The proof is now complete.

2.3 Proof of Theorems 2.0.2, 2.0.3 and 2.0.4

In the sequel, $W=\{W_t\}_{t\geq 0}$ will denote a standard one-dimensional Brownian motion independent of B, and $\mathscr{X}^j=\{\mathscr{X}^j_t\}_{t\geq 0}$ will denote the second order Hermite process introduced in Section 1.2.

Proof of Theorem 2.0.2

We start with the proof of Theorem 2.0.2, which will be done in two steps.

Step 1. First we prove the convergence of the finite dimensional distributions, namely, we will show that for every $r \in \mathbb{N}$, and $T_1, \ldots, T_r \geq 0$ fixed, it holds

$$\varepsilon^{\frac{d}{2} - \frac{3}{4H}} \left((I_{T_1}^{\varepsilon}, \ldots, I_{T_r}^{\varepsilon}) - \mathbb{E}\left[(I_{T_1}^{\varepsilon}, \ldots, I_{T_r}^{\varepsilon}) \right] \right) \stackrel{Law}{\to} \sigma(W_{T_1}, \ldots, W_{T_r}), \tag{2.3.1}$$

as $\varepsilon \to 0$, where σ is the finite constant defined by (2.2.3). To this end, define the kernels h_{2q,T_i}^{ε} by (2.1.7), and the constants σ_q^2 by (2.2.1), for $q \in \mathbb{N}$. Notice that the constants σ_q^2 are well defined due to the condition $\frac{3}{2d} < H < \frac{3}{4}$. Define as well the matrices $C_q = \{C_q^{i,j} \mid 1 \leq i,j \leq r\}$ and $C = \{C^{i,j} \mid 1 \leq i,j \leq r\}$, by $C_q^{i,j} := \sigma_q^2(T_i \wedge T_j)$, and $C^{i,j} := \sigma^2(T_i \wedge T_j)$. Since $I_{T_i}^{\varepsilon}$ has chaos decomposition (2.1.6), by Theorem 1.2.2, we deduce that in order to prove the convergence (2.3.1), it suffices to show the following properties:

(i) For every fixed $q \geq 1$, and $1 \leq i,j \leq r$, we have

$$\varepsilon^{d - \frac{3}{2H}} (2q)! \left\langle h_{2q,T_i}^{\varepsilon}, h_{2q,T_j}^{\varepsilon} \right\rangle_{(\mathfrak{H}^d)^{\otimes 2q}} \to \sigma_q^2 (T_i \wedge T_j), \quad \text{as} \quad \varepsilon \to 0.$$

(ii) The constants σ_q^2 satisfy $\sum_{q=1}^{\infty} \sigma_q^2 = \sigma^2$. In particular, $C^{i,j} = \lim_{Q \to \infty} \sum_{q=1}^{Q} C_q^{i,j}$,

(iii) For all $q \geq 1$ and $i = 1, \ldots, r$, the random variables $\varepsilon^{\frac{d}{2} - \frac{3}{4H}} I_{2q}(h_{2q,T_i}^{\varepsilon})$ converge in law to a centered Gaussian distribution as $\varepsilon \to 0$,

(iv) $\lim_{Q \to \infty} \sup_{\varepsilon \in (0,1)} \varepsilon^{d - \frac{3}{2H}} \sum_{q=Q}^{\infty} (2q)! \left\| h_{2q,T_i}^{\varepsilon} \right\|_{(\mathfrak{H}^d)^{\otimes 2q}}^2 = 0$, for every $i = 1, \ldots, r$.

Part (i) follows from Theorem 2.2.1. Condition (ii) follows from equation (2.2.2). In [23, Theorem 2], it was proved that for $T > 0$ fixed, $\varepsilon^{\frac{d}{2} - \frac{3}{4H}} I_{2q}(h_{2q,T}^{\varepsilon})$ converges in law to a centered Gaussian random variable when $\varepsilon \to 0$, and

$$\lim_{Q \to \infty} \sup_{\varepsilon \in (0,1)} \varepsilon^{d - \frac{3}{2H}} \sum_{q=Q}^{\infty} (2q)! \left\| h_{2q,T}^{\varepsilon} \right\|_{(\mathfrak{H}^d)^{\otimes 2q}}^2 = 0,$$

which proves conditions (iii) and (iv). This finishes the proof of (2.3.1).

Step 2. We are going to show the tightness of the sequence of processes $\{\varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])\}_{T\geq 0}$. To this end, we will prove that there exists a sufficiently small $p > 2$, depending only on d and H, such that for every $0 \leq T_1 \leq T_2$, it holds

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\left[\left|\varepsilon^{\frac{d}{2}-\frac{3}{4H}}\left(I_{T_2}^{\varepsilon} - \mathbb{E}\left[I_{T_2}^{\varepsilon}\right] - \left(I_{T_1}^{\varepsilon} - \mathbb{E}\left[I_{T_1}^{\varepsilon}\right]\right)\right)\right|^p\right] \leq C\left|T_2 - T_1\right|^{\frac{p}{2}}, \tag{2.3.2}$$

for some constant $C > 0$ only depending on d, p and H. The tightness property for $\{\varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])\}_{T\geq 0}$ then follows from the Billingsley criterion (see [4, Theorem 12.3]).

In order to prove (2.3.2) we proceed as follows. Define, for $0 \leq T_1 \leq T_2$ fixed, the random variable $Z_\varepsilon = Z_\varepsilon(T_1, T_2)$, by

$$Z_\varepsilon := I_{T_2}^{\varepsilon} - \mathbb{E}\left[I_{T_2}^{\varepsilon}\right] - \left(I_{T_1}^{\varepsilon} - \mathbb{E}\left[I_{T_1}^{\varepsilon}\right]\right). \tag{2.3.3}$$

From the chaos decomposition (2.1.6), we can easily check that $J_0(L^{-1}Z_\varepsilon) = J_1(L^{-1}Z_\varepsilon) = 0$, which in turn implies that

$$\mathbb{E}\left[DL^{-1}Z_\varepsilon\right] = J_0(DL^{-1}Z_\varepsilon) = DJ_1(L^{-1}Z_\varepsilon) = 0.$$

Hence, by (1.2.3), there exists a constant $c_p > 0$ such that

$$\left\|Z_\varepsilon\right\|_{L^p(\Omega)} \leq c_p \left\|D^2L^{-1}Z_\varepsilon\right\|_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})}. \tag{2.3.4}$$

The right-hand side of the previous inequality can be estimated as follows. From (1.2.5), we can easily check that

$$D^2L^{-1}Z_\varepsilon = \int_0^\infty \int_{\mathscr{K}_{T_1,T_2}} D^2P_\theta[p_\varepsilon(\vec{B}_t - \vec{B}_s)]dsdtd\theta, \tag{2.3.5}$$

where $\mathscr{K}_{T_1,T_2}$ is defined by (2.1.22). Let $\widetilde{B}$ be an independent copy of $\vec{B}$. Using Mehler's formula (1.2.4) and the semigroup property of the heat kernel, we obtain

$$\begin{aligned} P_\theta[p_\varepsilon(\vec{B}_t - \vec{B}_s)] &= \widetilde{\mathbb{E}}\left[p_\varepsilon(e^{-\theta}(\vec{B}_t - \vec{B}_s) + \sqrt{1-e^{-2\theta}}(\widetilde{B}_t - \widetilde{B}_s))\right] \\ &= p_{\lambda_\varepsilon(\theta,s,t)}(e^{-\theta}(\vec{B}_t - \vec{B}_s)), \end{aligned} \tag{2.3.6}$$

where the function $\lambda_\varepsilon = \lambda_\varepsilon(\theta,s,t)$ is defined by

$$\lambda_\varepsilon(\theta,s,t) := \varepsilon + (1-e^{-2\theta})(t-s)^{2H}. \tag{2.3.7}$$

This implies that for every multi-index $\mathbf{i} = (i_1,i_2)$, with $1 \le i_1, i_2 \le d$, we have

$$\begin{aligned} D^2P_\theta[p_\varepsilon(\vec{B}_t - \vec{B}_s)](\mathbf{i},x_1,x_2) &= e^{-2\theta}\mathbb{1}_{[s,t]}(x_1)\mathbb{1}_{[s,t]}(x_2) \\ &\times\lambda_\varepsilon(\theta,s,t)^{-1}p_{\lambda_\varepsilon(\theta,s,t)}(e^{-\theta}(\vec{B}_t - \vec{B}_s))g_{\mathbf{i},\lambda_\varepsilon(\theta,s,t)}(e^{-\theta}(\vec{B}_t - \vec{B}_s)), \end{aligned} \tag{2.3.8}$$

where the function $g_{\mathbf{i},\lambda}$, for $\lambda > 0$, is defined by

$$g_{\mathbf{i},\lambda}(x_1,\dots,x_d) = \begin{cases} \lambda^{-1}x_{i_1}^2 - 1 & \text{if } i_1 = i_2 \\ \lambda^{-1}x_{i_1}x_{i_2} & \text{if } i_1 \neq i_2. \end{cases}$$

From (2.3.5) and (2.3.8), we deduce that

$$\begin{aligned}\left\|D^2L^{-1}Z_\varepsilon\right\|^2_{(\mathfrak{H}^d)^{\otimes 2}} = &\int_{\mathbb{R}^2_+}\int_{\mathscr{K}^2_{T_1,T_2}} e^{-2\theta-2\beta}\mu(s_2-s_1,t_1-s_1,t_2-s_2)^2\\ &\times(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2))^{-1}p_{\lambda_\varepsilon(\theta,s_1,t_1)}(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1}))\\ &\times p_{\lambda_\varepsilon(\beta,s_2,t_2)}(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2}))\sum_{\mathbf{i}} g_{\mathbf{i},\lambda_\varepsilon(\theta,s_1,t_1)}\left(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1})\right)\\ &\times g_{\mathbf{i},\lambda_\varepsilon(\beta,s_2,t_2)}\left(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2})\right)ds_1dt_1ds_2dt_2d\theta d\beta, \qquad (2.3.9)\end{aligned}$$

where the sum runs over all the possible muti-indices $\mathbf{i}=(i_1,i_2)$, with $1\leq i_1,i_2\leq d$. Using Minkowski inequality, as well as (2.3.4) and (2.3.9), we deduce that

$$\begin{aligned}\|Z_\varepsilon\|^2_{L^p(\Omega)} &\leq c_p^2\left\|D^2L^{-1}Z_\varepsilon\right\|^2_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})} = c_p^2\left\|\left\|D^2L^{-1}Z_\varepsilon\right\|^2_{(\mathfrak{H}^d)^{\otimes 2}}\right\|_{L^{\frac{p}{2}}(\Omega)}\\ &\leq c_p^2\int_{\mathbb{R}^2_+}\int_{\mathscr{K}^2_{T_1,T_2}} e^{-2\theta-2\beta}\mu(s_2-s_1,t_1-s_1,t_2-s_2)^2\\ &\times(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2))^{-1}\big\|p_{\lambda_\varepsilon(\theta,s_1,t_1)}(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1}))\\ &\times p_{\lambda_\varepsilon(\beta,s_2,t_2)}(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2}))\sum_{\mathbf{i}} g_{\mathbf{i},\lambda_\varepsilon(\theta,s_1,t_1)}\left(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1})\right)\\ &\times g_{\mathbf{i},\lambda_\varepsilon(\beta,s_2,t_2)}\left(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2})\right)\big\|_{L^{\frac{p}{2}}(\Omega)}ds_1dt_1ds_2dt_2d\theta d\beta. \qquad (2.3.10)\end{aligned}$$

Next we bound the $L^{\frac{p}{2}}(\Omega)$-norm in the right-hand side of the previous inequality. Let $y\in(0,1)$ be fixed. We can easily check that there exists a constant $C>0$ only depending on y, such that for every $\lambda_1,\lambda_2>0$ and $\eta,\xi\in\mathbb{R}^d$, and every multi-index $\mathbf{i}=(i_1,i_2)$, with $1\leq i_1,i_2\leq d$,

$$\left|g_{\mathbf{i},\lambda_1}(\eta)g_{\mathbf{i},\lambda_2}(\xi)\right| \leq (1+\lambda_1^{-1}\|\eta\|^2)(1+\lambda_2^{-1}\|\xi\|^2) \leq Ce^{\frac{y}{2}(\lambda_1^{-1}\|\eta\|^2+\lambda_2^{-1}\|\xi\|^2)}. \qquad (2.3.11)$$

From (2.3.10) and (2.3.11), it follows that there exists a constant $C > 0$, not depending on ε, T_1, T_2, such that

$$\begin{aligned}
&\|Z_\varepsilon\|^2_{L^p(\Omega)} \\
&\leq C \int_{\mathbb{R}^2_+} \int_{\mathscr{K}^2_{T_1,T_2}} e^{-2\theta-2\beta} \mu(s_2-s_1, t_1-s_1, t_2-s_2)^2 \\
&\times (\lambda_\varepsilon(\theta, s_1, t_1)\lambda_\varepsilon(\beta, s_2, t_2))^{-1} \\
&\times \left\| p_{\frac{\lambda_\varepsilon(\theta,s_1,t_1)}{1-y}}(e^{-\theta}(\vec{B}_{t_1} - \vec{B}_{s_1})) p_{\frac{\lambda_\varepsilon(\beta,s_2,t_2)}{1-y}}(e^{-\beta}(\vec{B}_{t_2} - \vec{B}_{s_2})) \right\|_{L^{\frac{p}{2}}(\Omega)} ds_1 dt_1 ds_2 dt_2 d\theta d\beta.
\end{aligned} \tag{2.3.12}$$

Proceeding as in the proof of (2.1.25), we can easily check that

$$\begin{aligned}
&\mathbb{E}\left[p_{\frac{\lambda_\varepsilon(\theta,s_1,t_1)}{1-y}}(e^{-\theta}(\vec{B}_{t_1} - \vec{B}_{s_1}))^{\frac{p}{2}} p_{\frac{\lambda_\varepsilon(\beta,s_2,t_2)}{1-y}}(e^{-\beta}(\vec{B}_{t_2} - \vec{B}_{s_2}))^{\frac{p}{2}} \right] \\
&\qquad = (2\pi)^{-\frac{d(p-2)}{2}} \left(\frac{\lambda_\varepsilon(\theta, s_1, t_1)\lambda_\varepsilon(\beta, s_2, t_2)}{(1-y)^2} \right)^{-\frac{dp}{4}+\frac{d}{2}} \frac{2^d}{p^d} e^{d(\theta+\beta)} \\
&\qquad \times \mathbb{E}\left[p_{\frac{2\lambda_\varepsilon(\theta,s_1,t_1)e^{2\theta}}{p(1-y)}}(\vec{B}_{t_1} - \vec{B}_{s_1}) p_{\frac{2\lambda_\varepsilon(\beta,s_2,t_2)e^{2\beta}}{p(1-y)}}(\vec{B}_{t_2} - \vec{B}_{s_2}) \right] \\
&\qquad = (2\pi)^{-\frac{d(p-2)}{2}} \left(\frac{\lambda_\varepsilon(\theta, s_1, t_1)\lambda_\varepsilon(\beta, s_2, t_2)}{(1-y)^2} \right)^{-\frac{dp}{4}+\frac{d}{2}} \frac{2^d}{p^d} e^{d(\theta+\beta)} \\
&\qquad \times \left| \frac{2}{p(1-y)} \begin{pmatrix} \lambda_\varepsilon(\theta, s_1, t_1)e^{2\theta} & 0 \\ 0 & \lambda_\varepsilon(\beta, s_2, t_2)e^{2\beta} \end{pmatrix} + \Sigma \right|^{-\frac{d}{2}},
\end{aligned}$$

where $\Sigma = \{\Sigma_{i,j}\}_{1\leq i,j\leq 2}$, denotes the covariance matrix of $(B^{(1)}_{t_1} - B^{(1)}_{s_1}, B^{(1)}_{t_2} - B^{(1)}_{s_2})$, whose components are given by $\Sigma_{1,1} = (t_1-s_1)^{2H}$, $\Sigma_{1,2} = \Sigma_{2,1} = \mu(s_2-s_1, t_1-s_1, t_2-s_2)$, and $\Sigma_{2,2} = (t_2-s_2)^{2H}$. Therefore, there exists a constant $C > 0$ only depending on

p and d, such that

$$\mathbb{E}\left[p_{\frac{\lambda(\theta,s_1,t_1)}{1-y}}(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1}))^{\frac{p}{2}}p_{\frac{\lambda(\beta,s_2,t_2)}{1-y}}(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2}))^{\frac{p}{2}}\right]$$
$$\leq C(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2))^{-\frac{dp}{4}+\frac{d}{2}}$$
$$\times e^{d(\theta+\beta)}\left|\frac{2}{p(1-y)}\begin{pmatrix}\lambda_\varepsilon(\theta,s_1,t_1)e^{2\theta} & 0\\ 0 & \lambda_\varepsilon(\beta,s_2,t_2)e^{2\beta}\end{pmatrix}+\Sigma\right|^{-\frac{d}{2}}.$$

Choosing $y<1-\frac{2}{p}$, so that $\frac{p(1-y)}{2}\Sigma\geq\Sigma$, we deduce that there exists a constant $C>0$ only depending on p,y and d, such that

$$\mathbb{E}\left[p_{\frac{\lambda_\varepsilon(\theta,s_1,t_1)}{1-y}}(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1}))^{\frac{p}{2}}p_{\frac{\lambda_\varepsilon(\beta,s_2,t_2)}{1-y}}(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2}))^{\frac{p}{2}}\right]$$
$$\leq C(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2))^{-\frac{dp}{4}+\frac{d}{2}}$$
$$\times e^{d(\theta+\beta)}\left|\begin{pmatrix}\lambda_\varepsilon(\theta,s_1,t_1)e^{2\theta}+(t_1-s_1)^{2H} & \mu(s_2-s_1,t_1-s_2,t_2-s_2)\\ \mu(s_2-s_1,t_1-s_2,t_2-s_2) & \lambda_\varepsilon(\beta,s_2,t_2)e^{2\beta}+(t_2-s_2)^{2H}\end{pmatrix}\right|^{-\frac{d}{2}}.$$

Hence, by the multilinearity of the determinant function,

$$\mathbb{E}\left[p_{\frac{\lambda_\varepsilon(\theta,s_1,t_1)}{1-y}}(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1}))^{\frac{p}{2}}p_{\frac{\lambda_\varepsilon(\beta,s_2,t_2)}{1-y}}(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2}))^{\frac{p}{2}}\right]$$
$$\leq C(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2))^{-\frac{dp}{4}+\frac{d}{2}}$$
$$\times\left|\begin{pmatrix}\lambda_\varepsilon(\theta,s_1,t_1)+e^{-2\theta}(t_1-s_1)^{2H} & e^{-2\beta}\mu(s_2-s_1,t_1-s_2,t_2-s_2)\\ e^{-2\theta}\mu(s_2-s_1,t_1-s_2,t_2-s_2) & \lambda_\varepsilon(\beta,s_2,t_2)+e^{-2\beta}(t_2-s_2)^{2H}\end{pmatrix}\right|^{-\frac{d}{2}}. \tag{2.3.13}$$

By relation (2.3.7), we have that $\lambda_\varepsilon(\theta,s,t)+e^{-2\theta}(t-s)^{2H}=\varepsilon+(t-s)^{2H}$ for every $\theta,s,t>0$. As a consequence, relation (2.3.13) can be written as

$$\begin{aligned}
&\mathbb{E}\left[p_{\frac{\lambda_\varepsilon(\theta,s_1,t_1)}{1-y}}(e^{-\theta}(\vec{B}_{t_1}-\vec{B}_{s_1}))^{\frac{p}{2}}p_{\frac{\lambda_\varepsilon(\beta,s_2,t_2)}{1-y}}(e^{-\beta}(\vec{B}_{t_2}-\vec{B}_{s_2}))^{\frac{p}{2}}\right]\\
&\quad\le C(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2))^{-\frac{dp}{4}+\frac{d}{2}}\\
&\quad\times\left(\varepsilon^2+\varepsilon((t_1-s_1)^{2H}+(t_2-s_2)^{2H})+(t_1-s_1)^{2H}(t_2-s_2)^{2H}-e^{-2\beta-2\theta}\mu^2\right)^{-\frac{d}{2}}\\
&\quad\le C(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2))^{-\frac{dp}{4}+\frac{d}{2}}\Theta_\varepsilon(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{2}},
\end{aligned}\tag{2.3.14}$$

where $\Theta_\varepsilon(x,u_1,u_2)$ is defined by (2.1.26). From (2.3.7), (2.3.12) and (2.3.14), it follows that

$$\begin{aligned}
\|Z_\varepsilon\|_{L^p(\Omega)}^2\le C&\int_{\mathbb{R}_+^2}\int_{\mathscr{S}_{T_1,T_2}^2}e^{-2\theta-2\beta}\mu(s_2-s_1,t_1-s_1,t_2-s_2)^2\\
&\times((\varepsilon+(1-e^{-2\theta})(t_1-s_1)^{2H})(\varepsilon+(1-e^{-2\beta})(t_2-s_2)^{2H}))^{-1-\frac{d}{2}+\frac{d}{p}}\\
&\times\Theta_\varepsilon(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{p}}ds_1dt_1ds_2dt_2d\theta d\beta.
\end{aligned}\tag{2.3.15}$$

Changing the coordinates (s_1,t_1,s_2,t_2) by $(s_1,x:=s_2-s_1,u_1:=t_1-s_1,u_2:=t_2-s_2)$ in (2.3.15), we get

$$\begin{aligned}
\|Z_\varepsilon\|_{L^p(\Omega)}^2\le 2C&\int_{\mathbb{R}_+^2}e^{-2\theta-2\beta}\int_{[0,T_2]^3}\int_{(T_1-u_1)_+\vee(T_1-x-u_2)_+}^{(T_2-u_1)_+\wedge(T_2-x-u_2)_+}ds_1\\
&\times\mu(x,u_1,u_2)^2((\varepsilon+(1-e^{-2\theta})u_1^{2H})(\varepsilon+(1-e^{-2\beta})u_2^{2H}))^{-1-\frac{d}{2}+\frac{d}{p}}\\
&\times\Theta_\varepsilon(x,u_1,u_2)^{-\frac{d}{p}}dxdu_1du_2d\theta d\beta.
\end{aligned}$$

Integrating the variable s_1, and making the change of variables $\eta := 1 - e^{-2\theta}$, and $\xi := 1 - e^{-2\beta}$, we deduce that there exists a constant $C > 0$, such that

$$\|Z_\varepsilon\|^2_{L^p(\Omega)} \le C(T_2 - T_1) \int_{[0,T_2]^3} \mu(x, u_1, u_2)^2 \Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{p}} \times \int_{[0,1]^2} ((\varepsilon + \eta u_1^{2H})(\varepsilon + \xi u_2^{2H}))^{-1-\frac{d}{2}+\frac{d}{p}} d\eta d\xi dx du_1 du_2. \quad (2.3.16)$$

Changing the coordinates (x, u_1, u_2) by $(\varepsilon^{-\frac{1}{2H}} x, \varepsilon^{-\frac{1}{2H}} u_1, \varepsilon^{-\frac{1}{2H}} u_2)$ in (2.3.16), and using the fact that $\Theta_\varepsilon(\varepsilon^{-\frac{1}{2H}} x, \varepsilon^{-\frac{1}{2H}} u_1, \varepsilon^{-\frac{1}{2H}} u_2) = \varepsilon^2 \Theta_1(x, u_1, u_2)$, we get

$$\left\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\right\|^2_{L^p(\Omega)} \le C(T_2 - T_1) \int_{\mathbb{R}^3_+} \mu(x, u_1, u_2)^2 \Theta_1(x, u_1, u_2)^{-\frac{d}{p}} \times \int_{[0,1]^2} ((1 + \eta u_1^{2H})(1 + \xi u_2^{2H}))^{-1-\frac{d}{2}+\frac{d}{p}} d\eta d\xi dx du_1 du_2.$$

Integrating the variables η and ξ, we obtain

$$\left\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\right\|^2_{L^p(\Omega)} \le C(1 + \frac{d}{2} - \frac{d}{p})(T_2 - T_1) \int_{\mathbb{R}^3_+} \frac{\mu(x, u_1, u_2)^2}{u_1^{2H} u_2^{2H}} \Theta_1(x, u_1, u_2)^{-\frac{d}{p}} \times (1 - (1 + u_1^{2H})^{-\frac{d}{2}+\frac{d}{p}})(1 - (1 + u_2^{2H})^{-1-\frac{d}{2}+\frac{d}{p}}) dx du_1 du_2.$$

Hence, choosing $p > 2$, we deduce that there exists a constant C only depending on H, d and p, such that

$$\left\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\right\|^2_{L^p(\Omega)} \le C(T_2 - T_1) \int_{\mathbb{R}^3_+} \frac{\mu(x, u_1, u_2)^2}{u_1^{2H} u_2^{2H}} \Theta_1(x, u_1, u_2)^{-\frac{d}{p}} dx du_1 du_2. \quad (2.3.17)$$

Since $Hd > \frac{3}{2}$, we can choose p so that $2 < p < \frac{4Hd}{3}$. For this choice of p, the integral in the right-hand side of (2.3.17) is finite by Lemma 2.4.3. Therefore, from (2.3.17), it follows that there exists a constant $C > 0$, independent of T_1, T_2 and ε, such that

$\left\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}}Z_\varepsilon\right\|^2_{L^p(\Omega)} \le C(T_2-T_1)$, which in turn implies that

$$\mathbb{E}\left[\left|\varepsilon^{\frac{d}{2}-\frac{3}{4H}}Z_\varepsilon\right|^p\right] \le C(T_2-T_1)^{\frac{p}{2}}. \tag{2.3.18}$$

Relation (2.3.2) then follows from (2.3.18). This finishes the proof of Theorem 2.0.2.

Proof of Theorem 2.0.3

Now we proceed with the proof of Theorem 2.0.3, in which we will prove (2.0.5) and (2.0.7) in the case $H > \frac{3}{4}$. In order to prove (2.0.5), it suffices to show that for every $T > 0$,

$$\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon] - J_2(I_T^\varepsilon)) \overset{L^2(\Omega)}{\to} 0, \tag{2.3.19}$$

and

$$\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}J_2(I_T^\varepsilon) \overset{L^2(\Omega)}{\to} -\Lambda\sum_{j=1}^d \mathscr{X}_T^j, \tag{2.3.20}$$

as $\varepsilon \to 0$. Relation (2.3.19) follows from Lemma 2.2.2. In order to prove the convergence (2.3.20) we proceed as follows. Using (2.1.4), we can easily check that

$$J_2(I_T^\varepsilon) = -\frac{(2\pi)^{-\frac{d}{2}}}{2}\sum_{j=1}^d\int_0^T\int_0^{T-u}(\varepsilon+u^{2H})^{-\frac{d}{2}-1}u^{2H}H_2\left(\frac{B_{s+u}^{(j)}-B_s^{(j)}}{u^H}\right)dsdu.$$

Making the change of variable $v := \varepsilon^{-\frac{1}{2H}} u$, we get

$$\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1} J_2(I_T^\varepsilon) = -\frac{(2\pi)^{-\frac{d}{2}}}{2} \sum_{j=1}^{d} \int_0^{\varepsilon^{-\frac{1}{2H}} T} \int_0^{T-\varepsilon^{\frac{1}{2H}} v} (1+v^{2H})^{-\frac{d}{2}-1} v^{2H} \varepsilon^{1-\frac{1}{H}} H_2 \left(\frac{B^{(j)}_{s+\varepsilon^{\frac{1}{2H}} v} - B^{(j)}_s}{\sqrt{\varepsilon} v^H} \right) dv$$

$$= -\frac{(2\pi)^{-\frac{d}{2}}}{2} \sum_{j=1}^{d} \int_0^{\varepsilon^{-\frac{1}{2H}} T} (1+u^{2H})^{-\frac{d}{2}-1} u^2 I_2(\varphi^{\varepsilon^{\frac{1}{2H}} u}_{j, T-\varepsilon^{\frac{1}{2H}} u}) du, \quad (2.3.21)$$

where the kernel $\varphi^{\varepsilon^{\frac{1}{2H}} u}_{j, T-\varepsilon^{\frac{1}{2H}} u}$ is defined by (1.2.7). From (2.3.21), it follows that for every $\varepsilon, \eta > 0$,

$$\begin{aligned} &\mathbb{E}\left[\varepsilon^{\frac{d}{2}-\frac{3}{2H}-1} J_2(I_T^\varepsilon) \eta^{\frac{d}{2}-\frac{3}{2H}-1} J_2(I_T^\eta) \right] \\ &\quad = \frac{(2\pi)^{-d}}{2} \sum_{j=1}^{d} \int_0^{\varepsilon^{-\frac{1}{2H}} T} \int_0^{\eta^{-\frac{1}{2H}} T} (1+u_1^{2H})^{-\frac{d}{2}-1} (1+u_2^{2H})^{-\frac{d}{2}-1} \\ &\quad \times (u_1 u_2)^2 \left\langle \varphi^{\varepsilon^{\frac{1}{2H}} u_1}_{j, T-\varepsilon^{\frac{1}{2H}} u_1}, \varphi^{\eta^{\frac{1}{2H}} u_2}_{j, T-\eta^{\frac{1}{2H}} u_1} \right\rangle_{(\mathfrak{H}^d)^{\otimes 2}} du_1 du_2. \end{aligned} \quad (2.3.22)$$

By (1.2.8),

$$\begin{aligned} \lim_{\varepsilon \to 0} \left\langle \varphi^{\varepsilon^{\frac{1}{2H}} u_1}_{j, T-\varepsilon^{\frac{1}{2H}} u_1}, \varphi^{\eta^{\frac{1}{2H}} u_2}_{j, T-\eta^{\frac{1}{2H}} u_1} \right\rangle_{(\mathfrak{H}^d)^{\otimes 2}} &= H^2(2H-1)^2 \int_{[0,T]^2} |s_1 - s_2|^{4H-4} ds_1 ds_2 \\ &= \frac{H^2(2H-1)}{4H-3} T^{4H-2}. \end{aligned} \quad (2.3.23)$$

On the other hand, by (1.2.9), there exists a constant $C_{H,T} > 0$, only depending on H and T, such that

$$0 \leq \left\langle \varphi^{\varepsilon^{\frac{1}{2H}} u_1}_{j, T-\varepsilon^{\frac{1}{2H}} u_1}, \varphi^{\eta^{\frac{1}{2H}} u_2}_{j, T-\eta^{\frac{1}{2H}} u_1} \right\rangle_{(\mathfrak{H}^d)^{\otimes 2}} \leq C_{H,K}.$$

Hence, using the pointwise convergence (2.3.23), we can apply the dominated convergence theorem to (2.3.22), in order to obtain

$$\lim_{\varepsilon,\nu\to 0}\mathbb{E}\left[\varepsilon^{\frac{d}{2}-\frac{3}{2H}-1}J_2(I_T^{\varepsilon})\eta^{\frac{d}{2}-\frac{3}{2H}-1}J_2(I_T^{\eta})\right]=\frac{d(2\pi)^{-d}\Lambda^2H^2(2H-1)T^{4H-2}}{2(4H-3)},$$

where the constant Λ is defined by (2.0.6). From the previous identity, it follows that $\varepsilon^{\frac{d}{2}-\frac{3}{2H}-1}J_2(I_T^{\varepsilon})$ converges to some $\widetilde{h}_T\in(\mathfrak{H}^d)^{\otimes 2}$, as $\varepsilon\to 0$.

Recall that the element $\pi_T^j\in(\mathfrak{H}^d)^{\otimes d}$, is defined as the limit in $(\mathfrak{H}^d)^{\otimes 2}$, as $\varepsilon\to 0$, of $\varphi_{j,T}^{\varepsilon}$, and is characterized by relation (1.2.10). In order to prove (2.3.20), it suffices to show that $\widetilde{h}_T=\Lambda\sum_{j=1}^d\pi_T^j$, or equivalently, that

$$\left\langle\widetilde{h}_T,f_1\otimes f_2\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}}=-\Lambda\sum_{j=1}^d\left\langle\pi_T^j,f_1\otimes f_2\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}},$$

for vectors of step functions with compact support $f_i=(f_i^{(1)},\dots,f_i^{(d)})\in\mathfrak{H}^d$, $i=1,2$. By (2.3.21),

$$\lim_{\varepsilon\to 0}\left\langle\widetilde{h}_T,f_1\otimes f_2\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}}=\lim_{\varepsilon\to 0}-\frac{(2\pi)^{-\frac{d}{2}}}{2}\int_0^{\varepsilon^{-\frac{1}{2H}}T}(1+u^{2H})^{-\frac{d}{2}}u^2\left\langle\varphi_{j,T-\varepsilon^{\frac{1}{2H}}u}^{\varepsilon^{\frac{1}{2H}}u},f_1\otimes f_2\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}}du. \tag{2.3.24}$$

Proceeding as in the proof of (2.3.23), we can easily check that

$$\lim_{\varepsilon\to 0}\left\langle\varphi_{j,T-\varepsilon^{\frac{1}{2H}}u}^{\varepsilon^{\frac{1}{2H}}u},f_1\otimes f_2\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}}=-H^2(2H-1)^2\sum_{j=1}^d\int_0^T\prod_{i=1,2}\int_0^T|s-\eta|^{2H-2}f_i^{(j)}(\eta)d\eta ds.$$

Moreover, by (1.2.9),

$$\left|\left\langle \varphi^{\varepsilon^{\frac{1}{2H}}u}_{j,T-\varepsilon^{\frac{1}{2H}}u}, f_1\otimes f_2\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}}\right| \le \left\|\varphi^{\varepsilon^{\frac{1}{2H}}u}_{j,T-\varepsilon^{\frac{1}{2H}}u}\right\|_{(\mathfrak{H}^d)^{\otimes 2}} \|f_1\|_{\mathfrak{H}^d}\|f_2\|_{\mathfrak{H}^d} \le C_{H,T}\|f_1\|_{\mathfrak{H}^d}\|f_2\|_{\mathfrak{H}^d},$$

for some constant $C_{H,T} > 0$ only depending on T and H. Therefore, applying the dominated convergence theorem in (2.3.24), we get

$$\lim_{\varepsilon\to 0}\left\langle \widetilde{h}_T, f_1\otimes f_2\right\rangle_{(\mathfrak{H}^d)^{\otimes 2}} = -\Lambda H^2(2H-1)^2\sum_{j=1}^{d}\int_0^T\prod_{i=1,2}\int_0^T |s-\eta|^{2H-2} f_i^{(j)}(\eta)d\eta ds, \tag{2.3.25}$$

and from the characterization (1.2.10), we conclude that $\widetilde{h}_T = -\Lambda\sum_{j=1}^d \pi_T^j$, as required. This finishes the proof of (2.3.20), which, by (2.3.19), implies that the convergence (2.0.5).

It only remains to prove (2.0.7). By (2.0.5), it suffices to show the tightness property for $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$, which, as in the proof of (2.0.2), can be reduced to proving that there exists $p > 2$, such that for every $0\le T_1\le T_2\le K$,

$$\mathbb{E}\left[\left|\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}Z_\varepsilon\right|^p\right] \le C(T_2-T_1)^{\frac{p}{2}}, \tag{2.3.26}$$

where Z_ε is defined by (2.3.3), and C is some constant only depending on d, H, K and p. Changing the coordinates (x,u_1,u_2) by $(x,\varepsilon^{-\frac{1}{2H}}u_1,\varepsilon^{-\frac{1}{2H}}u_2)$ in (2.3.16), and using the fact that

$$\Theta_\varepsilon(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2) = \varepsilon^2\Theta_1(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2),$$

we can easily check that

$$\left\|\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}Z_\varepsilon\right\|_{L^p(\Omega)}^2 \leq C(T_2-T_1)\int_{\mathbb{R}_+^2}\int_0^{T_2}\varepsilon^{-\frac{2}{H}}\mu(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2)^2$$
$$\times\Theta_1(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2)^{-\frac{d}{p}}\int_{[0,1]^2}((1+\eta u_1^{2H})(1+\xi u_2^{2H}))^{-1-\frac{d}{2}+\frac{d}{p}}d\eta d\xi dx du_1 du_2,$$

and hence, if $p>2$, we obtain

$$\left\|\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}Z_\varepsilon\right\|_{L^p(\Omega)}^2 \leq C(T_2-T_1)\int_{\mathbb{R}_+^2}\int_0^{T_2}\varepsilon^{-\frac{2}{H}}\mu(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2)^2(u_1u_2)^{-2H}$$
$$\times\Theta_1(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2)^{-\frac{d}{p}}dx du_1 du_2. \tag{2.3.27}$$

By Lemma 2.4.4, if $T_1,T_2\in[0,K]$, for some $K>0$, the integral in the right-hand side of the previous inequality is bounded by a constant only depending on H,d,p and K. Relation (2.3.26) then follows from (2.3.27). This finishes the proof of the tightness property for $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon-\mathbb{E}[I_T^\varepsilon])$ in the case $H>\frac{3}{4}$.

Proof of Theorem 2.0.4

Finally we prove Theorem 2.0.4. First we show the convergence of the finite dimensional distributions, namely, that for every $r\in\mathbb{N}$ and $T_1,\dots,T_r\geq 0$ fixed, it holds

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left((I_{T_1}^\varepsilon,\dots,I_{T_r}^\varepsilon)-\mathbb{E}\left[(I_{T_1}^\varepsilon,\dots,I_{T_r}^\varepsilon)\right]\right)\overset{Law}{\to}\rho(W_{T_1},\dots,W_{T_r}), \tag{2.3.28}$$

where ρ is defined by (2.2.52). Consider the random variable $\widetilde{J}_T^\varepsilon$ introduced in (2.2.41). By Lemma 2.2.3, we have

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left\|I_T^\varepsilon-\mathbb{E}[I_T^\varepsilon]-I_2(h_{2,T}^\varepsilon)\right\|_{L^2(\Omega)}=0, \tag{2.3.29}$$

and by Lemma 2.2.4

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left\|I_2(h^{\varepsilon}_{2,T})-\widetilde{J}^{\varepsilon}_T\right\|_{L^2(\Omega)}=0. \tag{2.3.30}$$

Consequently,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left\|I^{\varepsilon}_T-\mathbb{E}\left[I^{\varepsilon}_T\right]-\widetilde{J}^{\varepsilon}_T\right\|_{L^2(\Omega)}=0,$$

and hence, relation (2.3.28) is equivalent to

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left(\widetilde{J}^{\varepsilon}_{T_1},\dots,\widetilde{J}^{\varepsilon}_{T_r}\right)\overset{Law}{\to}\rho(W_{T_1},\dots,W_{T_r}). \tag{2.3.31}$$

By the Peccati-Tudor criterion, the convergence (2.3.31) holds provided that $\widetilde{J}^{\varepsilon}_t$ satisfies the following conditions:

(i) For every $1\leq i,j\leq r$,

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[\widetilde{J}^{\varepsilon}_{T_i}\widetilde{J}^{\varepsilon}_{T_j}\right]\to\rho^2(T_i\wedge T_j),\qquad\text{as}\quad\varepsilon\to 0.$$

(ii) For all $i=1,\dots,r$, the random variables $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\widetilde{J}^{\varepsilon}_{T_i}$ converge in law to a centered Gaussian distribution as $\varepsilon\to 0$.

Relation (i) follows from relation (2.3.30), as well as Theorem 2.2.5. Hence, it suffices to check (ii). To this end, consider the following Riemann sum approximation for $\widetilde{J}^{\varepsilon}_T$

$$R^{\varepsilon}_{T,M}:=-\frac{c_{log}\varepsilon^{\frac{2}{3}-\frac{d}{2}}}{2^M}\sum_{k=2}^{M2^M}\int_0^T\sum_{j=1}^{d}\frac{u(k)^{\frac{3}{2}}}{(1+u(k)^{\frac{3}{2}}}H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u(k)}-B^{(j)}_s}{\sqrt{\varepsilon}u(k)^{\frac{3}{4}}}\right)ds, \tag{2.3.32}$$

where $c_{log} := \frac{(2\pi)^{-\frac{d}{2}}}{2}$ and $u(k) := \frac{k}{2^M}$, for $k = 2, \ldots, M2^M$. We will prove that

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}(R^{\varepsilon}_{T,M} - \widetilde{J}^{\varepsilon}_{T})$$

converges to zero uniformly in $\varepsilon \in (0, 1/e)$, and $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} R^{\varepsilon}_{T,M} \overset{Law}{\to} T\mathcal{N}(0, \widetilde{\rho}^2_M)$ as $\varepsilon \to 0$ for some constant $\widetilde{\rho}^2_M$ satisfying $\widetilde{\rho}^2_M \to \rho^2$ as $M \to \infty$. The result will then follow by a standard approximation argument. We will separate the argument in the following steps.

Step I

We prove that $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}(R^{\varepsilon}_{T,M} - \widetilde{J}^{\varepsilon}_{T}) \to 0$ in $L^2(\Omega)$ as $M \to \infty$ uniformly in $\varepsilon \in (0, 1/e)$, namely,

$$\lim_{M \to \infty} \sup_{\varepsilon \in (0,1/e)} \frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} \left\| R^{\varepsilon}_{T,M} - \widetilde{J}^{\varepsilon}_{T} \right\|_{L^2(\Omega)} = 0. \tag{2.3.33}$$

For $\varepsilon \in (0, 1/e)$ fixed, we decompose the term $\widetilde{J}^{\varepsilon}_{T}$ as

$$\widetilde{J}^{\varepsilon}_{T} = \widetilde{J}^{\varepsilon,M}_{T,1} + \widetilde{J}^{\varepsilon,M}_{T,2}, \tag{2.3.34}$$

where

$$\widetilde{J}^{\varepsilon,M}_{T,1} := -c_{log}\varepsilon^{\frac{3}{2}-\frac{d}{2}} \int_0^T \int_{2^{-M}}^{M} \sum_{j=1}^{d} \frac{u^{\frac{3}{2}}}{(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u} - B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}} \right) duds$$

and

$$\widetilde{J}_{T,2}^{\varepsilon,M} := -c_{log}\varepsilon^{\frac{3}{2}-\frac{d}{2}}\int_0^T\int_0^\infty \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u)\sum_{j=1}^d \frac{u^{\frac{3}{2}}}{(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u}-B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}}\right)duds.$$

From (2.3.34), we deduce that the relation (2.3.33) is equivalent to

$$\lim_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left\|R^{\varepsilon}_{T,M}-\widetilde{J}^{\varepsilon,M}_{T,1}\right\|_{L^2(\Omega)}=0, \tag{2.3.35}$$

provided that

$$\lim_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\left\|\widetilde{J}^{\varepsilon,M}_{T,2}\right\|_{L^2(\Omega)}=0. \tag{2.3.36}$$

To prove (2.3.36) we proceed as follows. First we use the relation (2.2.43) to write

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,2}\right\|^2_{L^2(\Omega)}=\frac{2dc^2_{log}}{\log(1/\varepsilon)}\int_{[0,T]^2}\int_{[0,\varepsilon^{-\frac{2}{3}}T]}\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)$$
$$\times\psi(u_1,u_2)\varepsilon^{-8/3}\mu(s_2-s_1,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_1)^2ds_1ds_2du_1du_2,$$

where $\psi(u_1,u_2)$ is defined by (2.2.39). Changing the coordinates (s_1,s_2,u_1,u_2) by $(s := s_1, x := \varepsilon^{-\frac{2}{3}}(s_2-s_1), u_1, u_2)$ when $s_1\le s_2$, and by $(s := s_2, x := \varepsilon^{-\frac{2}{3}}(s_1-s_2), u_1, u_2)$ when $s_1\ge s_2$, integrating the variable s, and using the identity $\mu(\varepsilon^{\frac{2}{3}}x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)^2 = \varepsilon^2\mu(x,u_1,u_2)$, we get

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,2}\right\|^2_{L^2(\Omega)}\le\frac{4Tdc^2_{log}}{\log(1/\varepsilon)}\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)G^{(1)}_{1,x}(u_1,u_2)dxdu_1du_2, \tag{2.3.37}$$

where the function $G^{(1)}_{1,x}(u_1,u_2)$ is defined by (2.1.14). Define the regions $\mathscr{S}_i$ by (2.1.31). Splitting the domain of integration of the right-hand side of (2.3.37) into $[0,T]^3 = \bigcup_{i=1}^{3}([0,\varepsilon^{-\frac{2}{3}}T]^3 \cap \mathscr{S}_i)$, we obtain

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,2}\right\|^2_{L^2(\Omega)} \leq \frac{4Tdc^2_{log}}{\log(1/\varepsilon)}\sum_{i=1}^{3}\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2) \times \prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)G^{(1)}_{1,x}(u_1,u_2)dxdu_1du_2,$$

and hence, dropping the normalization term $\frac{1}{\log(1/\varepsilon)}$ in the regions $\mathscr{S}_1,\mathscr{S}_2$, we obtain

$$\begin{aligned}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,2}\right\|^2_{L^2(\Omega)} \leq{}& \frac{4Tdc^2_{log}}{\log(1/\varepsilon)}\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2)\\ &\times \prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)G^{(1)}_{1,x}(u_1,u_2)dxdu_1du_2\\ &+4Tdc^2_{log}\sum_{i=1}^{2}\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)\\ &\times \prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)G^{(1)}_{1,x}(u_1,u_2)dxdu_1du_2.\end{aligned}$$

The integrands corresponding to $i=1,2$ converge pointwise to zero as $M\to\infty$, and are bounded by the functions $\mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)G^{(1)}_{1,x}(u_1,u_2)$, which, by relations (2.1.20) and (2.1.29), are in turn bounded by

$$\mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)C\frac{\mu(x,u_1,u_2)^2}{(u_1u_2)^{2H}}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}}, \tag{2.3.38}$$

for some constant $C>0$. In addition, by Lemma 2.4.3, the function (2.3.38) is integrable for $i=1,2$, and hence, by the dominated convergence theorem,

$$\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}_{T,2}^{\varepsilon,M}\right\|_{L^2(\Omega)}^2\leq\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{4Tdc_{log}^2}{\log(1/\varepsilon)}\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\mathbb{1}_{\mathscr{S}_3}(x,u_1,u_2) \tag{2.3.39}$$

$$\times\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)G_{1,x}^{(1)}(u_1,u_2)dxdu_1du_2.$$

On the other hand, by equation (2.4.5) in Lemma 2.4.2, we deduce that there exists a constant $C>0$, such that for every $(x,u_1,u_2)\in\mathscr{S}_3$,

$$G_{1,x}^{(1)}(u_1,u_2)\leq C(x+u_1+u_2)^{-1}(u_1u_2)^2\psi(u_1,u_2). \tag{2.3.40}$$

Therefore, from (2.3.39) we deduce that

$$\begin{aligned}\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}&\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}_{T,2}^{\varepsilon,M}\right\|_{L^2(\Omega)}^2\\ &\leq\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{4Cdc_{log}^2T}{\log(1/\varepsilon)}\int_0^{\varepsilon^{-\frac{2}{3}}T}\int_{\mathbb{R}_+^2}\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)\\ &\times(x+u_1+u_2)^{-1}(u_1u_2)^2\psi(u_1,u_2)du_1du_2dx,\end{aligned}$$

so that there exists a constant $C>0$ such that

$$\begin{aligned}\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}&\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}_{T,2}^{\varepsilon,M}\right\|_{L^2(\Omega)}^2\\ &\leq\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}CT\int_{\mathbb{R}_+^2}\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)\psi(u_1,u_2)\\ &\times\left(\frac{\log(\varepsilon^{-\frac{2}{3}}T+u_1+u_2)-\log(u_1+u_2)}{\log(1/\varepsilon)}\right)(u_1u_2)^2du_1du_2=0,\end{aligned}$$

where the last equality easily follows from the dominated convergence theorem. This finishes the proof of (2.3.36).

To prove (2.3.35) we proceed as follows. Define the intervals $I_k := (\frac{k-1}{2^M}, \frac{k}{2^M}]$. Then, we can write $R^{\varepsilon}_{T,M}$ and $\widetilde{J}^{M}_{T,1}$, as

$$R^{\varepsilon}_{T,M} = -\sum_{k=2}^{M2^M} c_{log}\varepsilon^{\frac{3}{2}-\frac{d}{2}} \int_0^T \int_{\mathbb{R}_+} \sum_{j=1}^{d} \mathbb{1}_{I_k}(u) \frac{u(k)^{\frac{3}{2}}}{(1+u(k)^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u(k)} - B^{(j)}_s}{\sqrt{\varepsilon}u(k)^{\frac{3}{4}}}\right) duds,$$

and

$$\widetilde{J}^{\varepsilon,M}_{T,1} = -\sum_{k=2}^{M2^M} c_{log}\varepsilon^{\frac{3}{2}-\frac{d}{2}} \int_0^T \int_{\mathbb{R}_+} \sum_{j=1}^{d} \mathbb{1}_{I_k}(u) \frac{u(k)^{\frac{3}{2}}}{(1+u(k)^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u} - B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}}\right) duds.$$

Notice that by (2.2.43),

$$\mathbb{E}\left[H_2\left(\frac{B_{s_1+\varepsilon^{\frac{2}{3}}v_1} - B_{s_1}}{\sqrt{\varepsilon}v_1^{\frac{3}{4}}}\right) H_2\left(\frac{B_{s_1+\varepsilon^{\frac{2}{3}}v_2} - B_{s_2}}{\sqrt{\varepsilon}v_2^{\frac{3}{4}}}\right)\right] = 2(v_1v_2)^{-\frac{3}{2}}\mu(\varepsilon^{-\frac{2}{3}}(s_2-s_1), v_1, v_2)^2,$$

and hence,

$$\begin{aligned}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)} = \frac{2dc_{log}^2}{\log(1/\varepsilon)} \int_{[0,T]^2}\int_{\mathbb{R}^2_+} \sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1)\mathbb{1}_{I_{k_2}}(u_2) \\ \times \varepsilon^{-\frac{2}{3}} A^M_{k_1,k_2}(\varepsilon^{-\frac{2}{3}}(s_2-s_1), u_1, u_2) ds_1 ds_2 du_1 du_2,\end{aligned}$$

where the function $A^M_{k_1,k_2}(x,u_1,u_2)$ is defined by

$$\begin{aligned}A^M_{k_1,k_2}(x,u_1,u_2) := \big(G^{(1)}_{1,x}(u_1,u_2) - G^{(1)}_{1,x}(u(k_1),u_2) \\ - G^{(1)}_{1,x}(u_1,u(k_2)) + G^{(1)}_{1,x}(u(k_1),u(k_2))\big).\end{aligned}$$

Changing the coordinates (s_1, s_2, u_1, u_2) by $(s := s_1, x := \varepsilon^{-\frac{2}{3}}(s_2 - s_1), u_1, u_1)$ in the case $s_2 \geq s_1$ and by $(s := s_2, x := \varepsilon^{-\frac{2}{3}}(s_1 - s_2), u_1, u_1)$ in the case $s_1 \geq s_2$, and integrating the variable s, we deduce that there exists a constant $C > 0$, such that

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \left\| \widetilde{J}_{T,1}^{\varepsilon,M} - R_{T,M}^{\varepsilon} \right\|_{L^2(\Omega)}^2 \leq \frac{CT}{\log(1/\varepsilon)} \int_0^{\varepsilon^{-\frac{2}{3}}T} \int_{\mathbb{R}_+^2} \sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \times \left| A_{k_1,k_2}^M(x, u_1, u_2) \right| du_1 du_2 dx. \tag{2.3.41}$$

In order to bound the term $\left| A_{k_1,k_2}^M(x, u_1, u_2) \right|$ we proceed as follows. Consider the function

$$D_x^M(u_1, u_2) := \psi(u_1 - 2^{-M}, u_2 - 2^{-M}) \mu(x, u_1 + 2^{-M})^2 - \psi(u_1 + 2^{-M}, u_2 + 2^{-M}) \mu(x, u_1 - 2^{-M})^2,$$

where $\psi(u_1, u_2)$ is defined by (2.2.44). By relation (1.2.6), we have that

$$\begin{aligned} \mu(x, u_1, u_2) &= \frac{3}{8} \int_0^{u_1} \int_x^{x+u_2} |v_1 - v_2|^{-\frac{1}{2}} dv_1 dv_2 \\ &= \frac{3u_1u_2}{8} \int_{[0,1]^2} |x + v_2u_2 - v_1u_1|^{-\frac{1}{2}} dv_1 dv_2, \end{aligned} \tag{2.3.42}$$

and consequently, $\mu(x, u_1, u_2) \leq \mu(x, v_1, v_2)$ for every $u_1 \leq v_1$ and $u_2 \leq v_2$. Using this observation, we can easily show that for every $v_1 \in [u_1 - 2^{-M}, u_1 + 2^{-M}]$ and $v_2 \in [u_2 - 2^{-M}, u_2 + 2^{-M}]$, the following inequality holds

$$\psi(u_1 + 2^{-M}, u_2 + 2^{-M})^{-\frac{d}{2}} \mu(x, u_1 - 2^{-M})^2 \leq G_{1,x}^{(1)}(v_1, v_2) \leq \psi(u_1 - 2^{-M}, u_2 - 2^{-M})^{-\frac{d}{2}} \mu(x, u_1 + 2^{-M})^2.$$

Hence, for every $u_1 \in I_{k_1}$ and $u_2 \in I_{k_2}$,

$$\left|A^M_{k_1,k_2}(u_1,u_2)\right| \le 2D^M_x(u_1,u_2). \tag{2.3.43}$$

Using relations (2.3.41) and (2.3.43), as well as the fact that

$$\sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1)\mathbb{1}_{I_{k_2}}(u_2) = \mathbb{1}_{[2^{-M},M]^2}(u_1,u_2),$$

we obtain

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)} \le \frac{CT}{\log(1/\varepsilon)}\int_0^{\varepsilon^{-\frac{2}{3}}T}\int_{\mathbb{R}^2_+}\mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)D^M_x(u_1,u_2)du_1du_2dx. \tag{2.3.44}$$

To bound the integral in the right-hand side we proceed as follows. Define $N := \varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon) = \frac{3\log N}{2}$. Then, applying L'Hôpital's rule in (2.3.44), we deduce that there is a constant $C > 0$, such that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)}\\
&\qquad\le \limsup_{N\to\infty}\frac{CT}{\log N}\int_0^{NT}\int_{\mathbb{R}^2_+}\mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)D^M_x(u_1,u_2)du_1du_2dx\\
&\qquad= \limsup_{N\to\infty} CT\int_{\mathbb{R}^2_+}\mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)NTD^M_{NT}(u_1,u_2)du_1du_2.
\end{aligned} \tag{2.3.45}$$

On the other hand, using (2.3.42) and equation (2.4.5) in Lemma 2.4.2, we get that for every $(x,u_1,u_2) \in \mathscr{S}_3$,

$$\lim_{x\to\infty} x\mu(x,u_1,u_2)^2 = \frac{3^2u_1^2u_2^2}{2^6}, \tag{2.3.46}$$

and

$$x\mu(x,u_1,u_2)^2 \le x(x+u_1+u_2)^{-1}(u_1u_2)^2 \le (u_1u_2)^2.$$

Hence, by applying the dominated convergence theorem in (2.3.45), we deduce that there is a constant $C>0$, such that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,1}-R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)}\\
&\quad\le CT\int_{\mathbb{R}^2_+}\mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)\Big(\psi(u_1-2^{-M},u_2-2^{-M})((u_1+2^{-M})(u_1+2^{-M}))^2\\
&\qquad\qquad -\psi(u_1+2^{-M},u_2+2^{-M})((u_1-2^{-M})(u_1-2^{-M}))^2\Big)du_1du_2. \quad (2.3.47)
\end{aligned}$$

Let $M_0\in\mathbb{N}$ and $\delta>0$ be fixed. Using the fact that integrands in (2.3.47) are decreasing on M and

$$\sum_{k_1,k_2=2}^{M_0 2^{M_0}}\mathbb{1}_{I_{k_1}}(x_1)\mathbb{1}_{I_{k_2}}(x_2)=\mathbb{1}_{[2^{-M_0},M_0]}(x_1)\mathbb{1}_{[2^{-M_0},M_0]}(x_2)\le 1,$$

we can easily check from the definition of the convergence (2.3.47), that there exists $\gamma=\gamma(M_0,\delta)>0$ such that for every $M>M_0$, the following inequality holds

$$\begin{aligned}
&\sup_{\varepsilon\in(0,\gamma)}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\left\|\widetilde{J}^{\varepsilon,M}_{T,1}-R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)}\\
&\quad\le \delta+CT\int_{\mathbb{R}^2_+}\Big(\psi(u_1-2^{-M_0},u_2-2^{-M_0})((u_1+2^{-M_0})(u_1+2^{-M_0}))^2\\
&\qquad\qquad -\psi(u_1+2^{-M_0},u_2+2^{-M_0})((u_1-2^{-M_0})(u_1-2^{-M_0}))^2\Big). \quad (2.3.48)
\end{aligned}$$

To handle the term $\sup_{\varepsilon\in(\gamma,1/e)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \left\|\widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)}$, we use (2.3.44) to get

$$\sup_{\varepsilon\in(\gamma,1/e)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \left\|\widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)} \leq CT \int_0^{\gamma^{-\frac{2}{3}}T} \int_{\mathbb{R}^2_+} \mathbb{1}_{[2^{-M},M]^2}(u_1,u_2) D^M_x(u_1,u_2) du_1 du_2 dx. \tag{2.3.49}$$

From (2.3.48) and (2.3.49), we conclude that there exists a constant $C > 0$, only depending on T, such that for every $M > M_0$,

$$\begin{aligned}
&\sup_{\varepsilon\in(0,1/e)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \left\|\widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)} \\
&\leq \delta + CT \int_{\mathbb{R}^2_+} \sum_{k_1,k_2=2}^{M_0 2^{M_0}} \Big(\psi(u_1 - 2^{-M_0}, u_2 - 2^{-M_0})((u_1 + 2^{-M_0})(u_1 + 2^{-M_0}))^2 \\
&- \psi(u_1 + 2^{-M_0}, u_2 + 2^{-M_0})((u_1 - 2^{-M_0})(u_1 - 2^{-M_0}))^2 \Big) \\
&+ CT \int_0^{\gamma^{-\frac{2}{3}}T} \int_{\mathbb{R}^2_+} \mathbb{1}_{[2^{-M},M]^2}(u_1,u_2) D^M_x(u_1,u_2) du_1 du_2 dx.
\end{aligned} \tag{2.3.50}$$

Taking first the limit as $M \to \infty$ and then as $M_0 \to \infty$ in (2.3.50), and applying the dominated convergence theorem, we get

$$\limsup_{M\to\infty} \sup_{\varepsilon\in(0,1)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \left\|\widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M}\right\|^2_{L^2(\Omega)} \leq \delta.$$

Relation (2.3.35) is then obtained by taking $\delta \to 0$ in the previous inequality.

Step II

Next we prove that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\left[(R^{\varepsilon}_{T,M})^2\right] = T\widetilde{\rho}^2_M, \tag{2.3.51}$$

where $\widetilde{\rho}_M$ is given by

$$\widetilde{\rho}_M = \frac{\sqrt{3d}}{2^{\frac{d+5}{2}}\pi^{\frac{d}{2}}2^M}\sum_{k=2}^{M2^M}(1+u(k)^{\frac{3}{2}})^{-\frac{d}{2}-1}u(k)^2, \tag{2.3.52}$$

and $u(k)=\frac{k}{2^M}$. Notice that in particular, $\widetilde{\rho}_M^2$ satisfies

$$\lim_{M\to\infty}\widetilde{\rho}_M^2=\rho^2,$$

where ρ^2 is defined by (2.2.52). To prove (2.3.52) we proceed as follows. Recall that the constant c_{log} is defined by $c_{log}=\frac{(2\pi)^{-\frac{d}{2}}}{2}$. Then, from the definition of $R^{\varepsilon}_{T,M}$ (see equation (2.3.32)), it easily follows that

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(R^{\varepsilon}_{T,M})^2\right]=\frac{2dc_{log}^2}{\log(1/\varepsilon)2^{2M}}\int_{[0,T]^2}\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}(s_2-s_1)}(u(k_1),u(k_2))ds_1ds_2.$$

Changing the coordinates (s_1,s_2) by $(s_1,x:=s_2-s_1)$, and then integrating the variable s_1, we get

$$\begin{aligned}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(R^{\varepsilon}_{T,M})^2\right]&=\frac{4dc_{log}^2}{\log(1/\varepsilon)2^{2M}}\int_0^T T\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}(\varepsilon^{\frac{2}{3}}u(k_1),\varepsilon^{\frac{2}{3}}u(k_2))dx\\&\quad-\frac{4dc_{log}^2}{\log(1/\varepsilon)2^{2M}}\int_0^T x\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}(\varepsilon^{\frac{2}{3}}u(k_1),\varepsilon^{\frac{2}{3}}u(k_2))dx.\end{aligned}$$

Using relation (2.2.60) as well as the Cauchy-Schwarz inequality $\mu(x,u_1,u_2)\leq(u_1u_2)^{\frac{3}{4}}$, we can easily deduce that there exists a constant $C>0$, depending on $u_1,\dots,u_{M2^M}$, but not on x or ε, such that

$$G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}(u(k_1),u(k_2))\leq C\varepsilon^{\frac{2}{3}}x^{-1},$$

and hence,

$$\lim_{\varepsilon\to 0}\frac{1}{\log(1/\varepsilon)}\int_0^T x\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}(u(k_1),u(k_2))dx=0,$$

which implies that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(R^{\varepsilon}_{T,M})^2\right]&=\lim_{\varepsilon\to 0}\frac{4dc^2_{log}T}{\log(1/\varepsilon)2^{2M}}\int_0^T\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G_{1,\varepsilon^{-\frac{2}{3}}x}(u(k_1),u(k_2))dx\\&=\lim_{\varepsilon\to 0}\frac{4dc^2_{log}T}{\log(1/\varepsilon)2^{2M}}\int_0^{\varepsilon^{-\frac{2}{3}}T}\sum_{k_1,k_2=2}^{M2^M}G_{1,x}(u(k_1),u(k_2))dx,\end{aligned}$$

where the last equality follows by making the change of variables $\widetilde{x}:=\varepsilon^{-\frac{2}{3}}x$. Hence, writing $N:=\varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon)=\frac{2\log N}{3}$, and using L'Hôpital's rule, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\left[(R^{\varepsilon}_{T,M})^2\right]&=\lim_{N\to\infty}\frac{8dc^2_{log}T}{3\log N2^{2M}}\int_0^{NT}\sum_{k_1,k_2=2}^{M2^M}G^{(1)}_{1,x}(u(k_1),u(k_2))dx\\&=\lim_{N\to\infty}\frac{8dc^2_{log}T}{3\cdot 2^{2M}}\sum_{k_1,k_2=2}^{M2^M}NTG^{(1)}_{1,NT}(u(k_1),u(k_2))dx=\widetilde{\rho}^2_M,\end{aligned}\tag{2.3.53}$$

where the last identity follows from (2.1.14) and (2.3.46). This finishes the proof of (2.3.51).

Step III

Next we prove the convergence in law of $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\widetilde{J}^{\varepsilon}_T$ to a Gaussian random variable with variance ρ^2. From Steps I and II, it suffices to show that

$$R^{\varepsilon}_{T,M}\overset{Law}{\to}\mathscr{N}(0,\widetilde{\rho}^2_M),\qquad \text{as }\varepsilon\to 0,\tag{2.3.54}$$

In order to prove (2.3.54) we proceed as follows. Define the random vector

$$D^{\varepsilon} = (D^{\varepsilon}_k)_{k=2}^{M2^M},$$

where

$$D^{\varepsilon}_k := -\frac{c_{log} u(k)^{\frac{3}{2}}}{2^M (1+u(k)^{\frac{3}{2}})^{\frac{d}{2}+1}} \sum_{j=1}^{d} \frac{1}{\varepsilon^{\frac{1}{3}} \sqrt{\log(1/\varepsilon)}} \int_0^T H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}} u(k)} - B^{(j)}_s}{\sqrt{\varepsilon} u(k)^{\frac{3}{4}}} \right) ds,$$

and $c_{log} = \frac{(2\pi)^{-\frac{d}{2}}}{2}$. Notice that

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(\varepsilon)}} R^{\varepsilon}_{T,M} = \sum_{k=2}^{M2^M} D^{\varepsilon}_k.$$

We will prove that D^{ε} converges to a centered Gaussian vector. By the Peccati-Tudor criterion (see [47]), it suffices to prove that the components of the vector D^{ε} converge to a Gaussian distribution, and the covariance matrix of D^{ε} is convergent. To prove the former statement, define

$$\Psi^{j}_{k_1,k_2}(\varepsilon) := \mathbb{E}\left[\int_0^T H_2\left(\frac{B^{(j)}_{s_1+\varepsilon^{\frac{2}{3}} u(k_1)} - B^{(j)}_{s_1}}{\sqrt{\varepsilon} u(k_1)^{\frac{3}{4}}} \right) ds_1 \int_0^T H_2\left(\frac{B^{(j)}_{s_2+\varepsilon^{\frac{2}{3}} u(k_2)} - B^{(j)}_{s_2}}{\sqrt{\varepsilon} u(k_2)^{\frac{3}{4}}} \right) ds_2 \right].$$

Proceeding as in the proof of (2.3.53), we can show that for $2 \leq k_1, k_2 \leq M2^M$,

$$\begin{aligned} \Psi^{j}_{k_1,k_2}(\varepsilon) &= \frac{2(u(k_1)u(k_2))^{-\frac{3}{2}}}{\varepsilon^{\frac{8}{3}} \log(1/\varepsilon)} \int_{[0,T]^2} \mu(s_2 - s_1, \varepsilon^{\frac{2}{3}} u(k_1), \varepsilon^{\frac{2}{3}} u(k_2))^2 ds_1 ds_2 \\ &= \frac{8(u(k_1)u(k_2))^{-\frac{3}{2}}}{3\log(\varepsilon^{-\frac{2}{3}})} \int_0^{\varepsilon^{-\frac{2}{3}} T} \int_0^{T-\varepsilon^{\frac{2}{3}} x} \mu(x, u(k_1), u(k_2))^2 ds dx. \end{aligned}$$

As in the proof of (2.3.53), we can use L'Hôpital's rule, (2.3.46) and the previous identity, to get

$$\lim_{\varepsilon\to 0}\Psi_n^{i,j} = \lim_{\varepsilon\to 0}\frac{8(u(k_1)u(k_2))^{-\frac{3}{2}}T}{3\log(\varepsilon^{-\frac{2}{3}})}\int_0^{\varepsilon^{-\frac{2}{3}}T}\mu(x,u(k_1),u(k_2))^2 dsdx = \frac{3}{2^3}T\sqrt{u(k)u(j)}.$$

From here, it follows that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[D_{k_1}^{\varepsilon}D_{k_2}^{\varepsilon}\right] = \Sigma_{i,j} := \frac{3dT}{2^{d+5}\pi^d 2^{2M}}\psi(u(k_1),u(k_2))(u(k_1)u(k_2))^2,$$

namely, the covariance matrix of D^{ε} converges to the matrix $\Sigma = (\Sigma_{k,j})_{2\leq k,j\leq M2^M}$. In addition, by [11, Equation(1.4)] , for $2 \leq k \leq M2^M$ fixed, the sequence of random variables D_k^{ε} converges to a Gaussian random variable as $\varepsilon \to 0$. Therefore, by the Peccati-tudor criterion, the random vector D converges to a jointly Gaussian vector $Z = (Z_k)_{k=2}^{M2^M}$, with mean zero and covariance Σ. In particular, we have

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(\varepsilon)}}R_{T,M}^{\varepsilon} = \sum_{k=2}^{M2^M} D_k^{\varepsilon} \overset{Law}{\to} \mathscr{N}\left(0,\sum_{j,k=2}^{M2^M}\Sigma_{k,j}\right) \qquad \text{as} \quad \varepsilon \to 0.$$

Relation (2.3.54) easily follows from the previous identity.

Since (2.3.28) holds, in order to finish the proof of Theorem 2.0.4 it suffices to prove tightness. As before, we define, for $T_1 \leq T_2$ belonging to a compact interval $[0,K]$ the random variable Z_{ε} by the formula (2.3.3). Then, by the Billingsley criterion, it suffices to prove that there exist constants $C > 0$ and $p > 2$, only depending on K, such that

$$\mathbb{E}\left[\left|\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}Z_{\varepsilon}\right|^p\right] \leq C(T_2-T_1)^{\frac{p}{2}}. \tag{2.3.55}$$

Using relation (2.3.27) with $H = \frac{3}{4}$, we can easily check that

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \|Z_\varepsilon\|^2_{L^p(\Omega)} \leq \frac{C(T_2-T_1)}{\log(1/\varepsilon)} \int_{\mathbb{R}^2_+} \int_0^{T_2} \varepsilon^{-\frac{8}{3}} \mu(x, \varepsilon^{\frac{2}{3}} u_1, \varepsilon^{\frac{2}{3}} u_2)^2 (u_1 u_2)^{-2H}$$
$$\times \Theta_1(\varepsilon^{-\frac{2}{3}} x, u_1, u_2)^{-\frac{d}{p}} dx du_1 du_2 \tag{2.3.56}$$
$$\leq \sup_{\varepsilon \in (0,1/e)} \frac{C(T_2-T_1)}{\log(1/\varepsilon)} \int_{\mathbb{R}^2_+} \int_0^{T_2} \varepsilon^{-\frac{8}{3}} \mu(x, \varepsilon^{\frac{2}{3}} u_1, \varepsilon^{\frac{2}{3}} u_2)^2 (u_1 u_2)^{-2H}$$
$$\times \Theta_1(\varepsilon^{-\frac{2}{3}} x, u_1, u_2)^{-\frac{d}{p}} dx du_1 du_2.$$

The right-hand side in the previous identity is finite for $p > 2$ sufficiently small by Lemma 2.4.5, and hence, there exists a constant $p > 2$ such that

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\left[|Z_\varepsilon|^p\right] \leq C(T_2-T_1)^{\frac{p}{2}}.$$

This finishes the proof of the tightness property for $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}(I_T^\varepsilon - \mathbb{E}\left[I_{T_1}^\varepsilon\right])$. The proof of Theorem 2.0.4 is now complete.

2.4 Technical lemmas

In this section we prove some technical lemmas, which where used in the proof of Theorems 2.0.2, 2.0.3 and 2.0.4.

Lemma 2.4.1. *Let $s_1, s_2, t_1, t_2 \in \mathbb{R}_+$ be such that $s_1 \leq s_2$, and $s_i \leq t_i$ for $i = 1, 2$. Denote by Σ the covariance matrix of $(B^{(1)}_{t_1} - B^{(1)}_{s_1}, B^{(1)}_{t_2} - B^{(1)}_{s_2})$. Then, there exists a constants $0 < \delta < 1$ and $k > 0$, such that the following inequalities hold*

1. *If* $s_1 < s_2 < t_1 < t_2$,

$$|\Sigma| \geq \delta((a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}), \tag{2.4.1}$$

where $a := s_2 - s_1$, $b := t_1 - s_2$ *and* $c := t_2 - t_1$.

2. *If* $s_1 < s_2 < t_2 < t_1$,

$$|\Sigma| \geq \delta b^{2H}(a^{2H} + c^{2H}), \tag{2.4.2}$$

where $a := s_2 - s_1$, $b := t_2 - s_2$ *and* $c := t_1 - t_2$.

3. *If* $s_1 < t_1 < s_2 < t_2$,

$$|\Sigma| \geq \delta a^{2H}c^{2H}, \tag{2.4.3}$$

where $a := t_1 - s_1$ *and* $c := t_2 - s_2$.

Proof. Relations (2.4.1)-(2.4.3) follow from Lemma B.1. in [29]. The inequalities (2.4.1) and (2.4.3) where also proved in [23, Lemma 9], but the lower bound given in this lemma for the case $s_1 < s_2 < t_2 < t_1$ is not correct. □

Lemma 2.4.2. *There exists a constant* $k > 0$*, such that for every* $s_1 < t_1 < s_2 < t_2$,

$$\mu(a+b, a, c) \leq kb^{2H-2}ac, \tag{2.4.4}$$

where $a := t_1 - s_1$, $b := s_2 - t_1$ *and* $c := t_2 - s_2$. *In addition, if* $H > \frac{1}{2}$,

$$\mu(x, u_1, u_2) \leq k(x + u_1 + u_2)^{2H-2}u_1u_2, \tag{2.4.5}$$

where $x := s_2 - s_1$, $u_1 := t_1 - s_1$ *and* $u_2 := t_2 - s_2$.

Proof. We can easily check that

$$\mu(a+b,a,c) = \frac{1}{2}((a+b+c)^{2H} + b^{2H} - (b+c)^{2H} - (a+b)^{2H}),$$

and hence,

$$\mu(a+b,a,c) = H(2H-1)ac \int_{[0,1]^2} |b + av_1 + cv_2|^{2H-2} dv_1 dv_2,$$

Relation (2.4.4) follows by dropping the term $av_1 + cv_2$ in the previous integral, while (2.4.5) follows from the following computation, which is valid for every $H > \frac{1}{2}$,

$$\begin{aligned}
\mu(a+b,a,c) &= H(2H-1)ac \int_{[0,1]^2} |b + av_1 + cv_2|^{2H-2} dv_1 dv_2 \\
&\leq H(2H-1)ac \int_0^1 |(a \vee b \vee c)v|^{2H-2} dv \\
&= Hac\, |a \vee b \vee c|^{2H-2} \leq H4^{2H-2} ac\, |2a + b + c|^{2H-2} \\
&= 4^{2H-2} H (x + u_1 + u_2)^{2H-2} u_1 u_2.
\end{aligned}$$

□

Lemma 2.4.3. *Define the functions* μ *and* Θ_1 *by* (2.1.10) *and* (2.1.26) *respectively. Let* $\frac{3}{2d} < H < 1$, *and* $0 < p < \frac{4Hd}{3}$ *be fixed. Then, the following integral is convergent*

$$\int_{\mathscr{S}_i} \frac{\mu(x,u_1,u_2)^2}{u_1^{2H} u_2^{2H}} \Theta_1(x,u_1,u_2)^{-\frac{d}{p}} dx du_1 du_2 < \infty, \tag{2.4.6}$$

for $i = 1,2$, where the sets $\mathscr{S}_i$ are defined by (2.1.31). *Moreover, if $H < \frac{3}{4}$, then*

$$\int_{\mathbb{R}^3_+} \frac{\mu(x,u_1,u_2)^2}{u_1^{2H}u_2^{2H}} \Theta_1(x,u_1,u_2)^{-\frac{d}{p}} dxdu_1du_2 < \infty. \tag{2.4.7}$$

Proof. Denote the integrand in (2.4.7) and (2.4.6) by $\Psi(x,u_1,u_2)$, namely,

$$\Psi(x,u_1,u_2) = \mu(x,u_1,u_2)^2(u_1u_2)^{-2H}\Theta_1(x,u_1,u_2)^{-\frac{d}{p}}. \tag{2.4.8}$$

We can decompose the domain of integration of (2.4.7), as $\mathbb{R}^3_+ = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3$, where $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ are defined by (2.1.31). Then, it suffices to show that

$$\int_{\mathscr{S}_i} \Psi(x,u_1,u_2)dxdu_1du_2 < \infty, \tag{2.4.9}$$

for $i = 1,2$ provided that $0 < p < \frac{4Hd}{3}$, and for $i = 3$, provided that $0 < p < \frac{4Hd}{3}$ and $H < \frac{3}{4}$. First consider the case $i = 1$. Changing the coordinates (x,u_1,u_2) by $(a := x, b := u_1 - x, c := x + u_2 - u_1)$ in (2.4.9) for $i = 1$, we get

$$\int_{\mathscr{S}_1} \Psi(x,u_1,u_2)dxdu_1du_2 = \int_{\mathbb{R}^3_+} \Psi(a,a+b,b+c)dadbdc.$$

To bound the integral in the right-hand side we proceed as follows. First we notice that the term $\mu(a,a+b,b+c)$ is given by

$$\mu(a,a+b,b+c) = \frac{1}{2}((a+b+c)^{2H} + b^{2H} - c^{2H} - a^{2H}).$$

By the Cauchy-Schwarz inequality, $|\mu(a,a+b,b+c)| \leq (a+b)^H(b+c)^H$. In addition, by (2.4.1) there exists a constant $\delta > 0$ such that

$$(a+b)^{2H}(b+c)^{2H} - \mu(a,a+b,b+c)^2 \geq \delta((a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}). \quad (2.4.10)$$

As a consequence,

$$\Psi(a,a+b,b+c) \leq \left(1+(a+b)^{2H}+(b+c)^{2H}+\delta((a+b)^{2H}c^{2H}+(b+c)^{2H}a^{2H})\right)^{-\frac{d}{p}}.$$

Hence, we deduce that there exists a constant $K > 0$ such that the following inequalities hold

$$\begin{aligned}
\Psi(a,a+b,b+c) &\leq K\left(1+c^{2H}+c^{2H}b^{2H}\right)^{-\frac{d}{p}} && \text{if } a \leq b \leq c,\\
\Psi(a,a+b,b+c) &\leq K\left(1+c^{2H}+c^{2H}a^{2H}\right)^{-\frac{d}{p}} && \text{if } b \leq a \leq c,\\
\Psi(a,a+b,b+c) &\leq K\left(1+b^{2H}+c^{2H}b^{2H}\right)^{-\frac{d}{p}} && \text{if } a \leq c \leq b,\\
\Psi(a,a+b,b+c) &\leq K\left(1+b^{2H}+b^{2H}a^{2H}\right)^{-\frac{d}{p}} && \text{if } c \leq a \leq b,\\
\Psi(a,a+b,b+c) &\leq K\left(1+a^{2H}+c^{2H}a^{2H}\right)^{-\frac{d}{p}} && \text{if } b \leq c \leq a,\\
\Psi(a,a+b,b+c) &\leq K\left(1+a^{2H}+b^{2H}a^{2H}\right)^{-\frac{d}{p}} && \text{if } c \leq b \leq a.
\end{aligned}$$

Using the condition $p < \frac{4Hd}{3}$, as well as the previous inequalities, we can easily check that $\Psi(a,a+b,b+c)$ is integrable in $\mathbb{R}_+^3$, which in turn implies that $\Psi(x,u_1,u_2)$ is integrable in $\mathscr{S}_1$, as required.

Next we consider the case $i = 2$. Changing the coordinates (x,u_1,u_2) by $(a := x, b :=$

$u_2, c := u_1 - x - u_2$) in (2.4.9) for $i = 2$, we get

$$\int_{\mathscr{S}_2} \Psi(x, u_1, u_2) dx du_1 du_2 = \int_{\mathbb{R}_+^3} \Psi(a, a+b+c, b) da db dc.$$

To bound the integral in the right-hand side we proceed as follows. First notice that the term $\mu(a, a+b+c, b)$ is given by

$$\mu(a, a+b+c, b) = \frac{1}{2}((b+c)^{2H} + (a+b)^{2H} - c^{2H} - a^{2H}). \tag{2.4.11}$$

By the Cauchy-Schwarz inequality, $|\mu(a, a+b+c, b)| \leq b^H (a+b+c)^H$. In addition, by (2.4.2), there exists a constant $\delta > 0$ such that

$$b^{2H}(a+b+c)^{2H} - \mu(a, a+b+c, b)^2 \geq \delta b^{2H}(a^{2H} + c^{2H}).$$

As a consequence,

$$\Psi(a, a+b+c, b) \leq \left(1 + b^{2H} + (a+b+c)^{2H} + \delta b^{2H}(a^{2H} + c^{2H})\right)^{-\frac{d}{p}}.$$

From here it follows that there exists a constant $K > 0$ such that the following inequalities hold

$$\begin{aligned}
\Psi(a, a+b+c, b) &\leq K\left(1 + c^{2H} + b^{2H}c^{2H}\right)^{-\frac{d}{p}} && \text{if } \ a \leq b \leq c, \\
\Psi(a, a+b+c, b) &\leq K\left(1 + b^{2H} + b^{2H}c^{2H}\right)^{-\frac{d}{p}} && \text{if } \ a \leq c \leq b, \\
\Psi(a, a+b+c, b) &\leq K\left(1 + b^{2H} + b^{2H}a^{2H}\right)^{-\frac{d}{p}} && \text{if } \ c \leq a \leq b, \\
\Psi(a, a+b+c, b) &\leq K\left(1 + a^{2H} + b^{2H}a^{2H}\right)^{-\frac{d}{p}} && \text{if } \ c \leq b \leq a.
\end{aligned} \tag{2.4.12}$$

Using the condition $p < \frac{4Hd}{3}$, as well as the previous inequalities, we can easily check that $\Psi(a, a+b+c, b)$ is integrable in the region $\{(a,b,c) \in \mathbb{R}^3_+ \mid b \geq a \wedge c\}$.

Next we check the integrability of $\Psi(a, a+b+c, b)$ in $\{(a,b,c) \in \mathbb{R}^3_+ \mid b \leq a \wedge c\}$. Applying the mean value theorem in (2.4.11), we can easily check that

$$\mu(a, a+b+c, b) = \frac{1}{2}(2H(a+\xi_1)^{2H-1}b + 2H(c+\xi_2)^{2H-1}b), \tag{2.4.13}$$

for some ξ_1, ξ_2 between 0 and b. Therefore, if $H < \frac{1}{2}$, we obtain

$$\mu(a, a+b+c, b) \leq H(a^{2H-1} + c^{2H-1})b, \tag{2.4.14}$$

which in turn implies that

$$\begin{aligned}\Psi(a, a+b+c, b) \leq\ & H^2(a^{2H-1} + c^{2H-1})^2 b^{2-2H}(a+b+c)^{-2H} \\ & \left(1 + b^{2H} + (a+b+c)^{2H} + \delta b^{2H}(a^{2H} + c^{2H})\right)^{-\frac{d}{p}}.\end{aligned} \tag{2.4.15}$$

For the case $H \geq \frac{1}{2}$, we use (2.4.13), in order to obtain

$$\mu(a, a+b+c, b) \leq H((a+b)^{2H-1} + (c+b)^{2H-1})b,$$

which in turn implies that

$$\begin{aligned}\Psi(a, a+b+c, b) \leq\ & H^2((a+b)^{2H-1} + (c+b)^{2H-1})^2 b^{2-2H}(a+b+c)^{-2H} \\ & \left(1 + b^{2H} + (a+b+c)^{2H} + \delta b^{2H}(a^{2H} + c^{2H})\right)^{-\frac{d}{p}}.\end{aligned} \tag{2.4.16}$$

From (2.4.15), we deduce that, if $H < \frac{1}{2}$, there exists a constant $K > 0$ such that

$$\Psi(a, a+b+c, b) \leq K a^{4H-2} b^{2-2H} c^{-2H} \left(1 + c^{2H} + b^{2H} c^{2H}\right)^{-\frac{d}{p}} \qquad \text{if} \quad b \leq a \leq c,$$
$$\Psi(a, a+b+c, b) \leq K c^{4H-2} b^{2-2H} a^{-2H} \left(1 + a^{2H} + b^{2H} a^{2H}\right)^{-\frac{d}{p}} \qquad \text{if} \quad b \leq c \leq a. \tag{2.4.17}$$

In turn, from (2.4.16), it follows that if $H \geq \frac{1}{2}$, there exists a constant $K > 0$, such that

$$\Psi(a, a+b+c, b) \leq K c^{4H-2} b^{2-2H} \left(1 + c^{2H} + b^{2H} c^{2H}\right)^{-\frac{d}{p}} \qquad \text{if} \quad b \leq a \leq c,$$
$$\Psi(a, a+b+c, b) \leq K a^{4H-2} b^{2-2H} \left(1 + a^{2H} + b^{2H} a^{2H}\right)^{-\frac{d}{p}} \qquad \text{if} \quad b \leq c \leq a. \tag{2.4.18}$$

Using the conditions $H < \frac{3}{4}$ and $p < \frac{4Hd}{3}$, we can easily check that $2H < \frac{Hd}{2p}$, which, by (2.4.17) and (2.4.18), implies that $\Psi(a, a+b+c, b)$ is integrable in $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \leq a \wedge c\}$. From here it follows that $\Psi(a, a+b+c, b)$ is integrable in $\mathbb{R}_+^3$, and hence $\Psi(x, u_1, u_2)$ is integrable in $\mathscr{S}_2$, as required.

Finally we consider the case $i = 3$ for $H < \frac{3}{4}$. Changing the coordinates (x, u_1, u_2) by $(a := u_1, b := x - u_1, c := u_2)$ in (2.4.9) for $i = 3$, we get

$$\int_{\mathscr{S}_3} \Psi(x, u_1, u_2) dx du_1 du_2 = \int_{\mathbb{R}^3} \Psi(a+b, a, c) da db dc.$$

To bound the integral in the right-hand side we proceed as follows. First we notice that the term $\mu(a+b, a, c)$ is given by

$$\mu(a+b, a, c) = \frac{1}{2}((a+b+c)^{2H} + b^{2H} - (b+c)^{2H} - (a+b)^{2H}). \tag{2.4.19}$$

By the Cauchy-Schwarz inequality, $\mu(a+b,a,c) \leq a^H c^H$. In addition, by (2.4.3), there exist constants $k, \delta > 0$ such that

$$a^{2H}c^{2H} - \mu(a+b,a,c)^2 \geq \delta a^{2H}c^{2H}, \tag{2.4.20}$$

and

$$\mu(a+b,a,c) \leq k b^{2H-2} ac. \tag{2.4.21}$$

From (2.4.20)-(2.4.21), we deduce the following bounds for Ψ

$$\Psi(a+b,a,c) \leq \left(1+a^{2H}+c^{2H}+\delta a^{2H}c^{2H}\right)^{-\frac{d}{p}}, \tag{2.4.22}$$

$$\Psi(a+b,a,c) \leq 2Hb^{4H-4}(ac)^{-2H+2}\left(1+a^{2H}+c^{2H}+\delta a^{2H}c^{2H}\right)^{-\frac{d}{p}}. \tag{2.4.23}$$

Using (2.4.22), as well as the condition $p < \frac{4Hd}{3}$, we can easily check that $\Psi(a+b,a,c)$ is integrable in the region $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \leq a \wedge c\}$.

Next we check the integrability of $\Psi(a+b,a,c)$ in the region $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \geq a \vee c\}$. Since $H < \frac{3}{4}$, from (2.4.23) it follows that there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{(a\vee c)}^{\infty} \Psi(a+b,a,c)db &\leq C(ac)^{-2H+2}(a\vee c)^{4H-3}\left(1+a^{2H}+c^{2H}+a^{2H}c^{2H}\right)^{-\frac{d}{p}} \\ &\leq C(ac)^{\frac{1}{2}}\left(1+a^{2H}+c^{2H}+a^{2H}c^{2H}\right)^{-\frac{d}{p}}.\end{aligned}$$

The integrability of $\Psi(a+b,a,c)$ in the region $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \geq a \vee c\}$ then follows from condition the $p < \frac{4Hd}{3}$.

Finally, we prove the integrability of $\Psi(a+b,a,c)$ in the regions $\{(a,b,c) \in \mathbb{R}_+^3 \mid a \leq b \leq c\}$ and $\{(a,b,c) \in \mathbb{R}_+^3 \mid c \leq b \leq a\}$. Let $a,b,c \geq 0$ be such that $a \leq b \leq c$. Applying the mean value theorem to (2.4.19), we can easily show that

$$\mu(a+b,a,c) = \frac{1}{2}(\xi_1^{2H-1}a - \xi_2^{2H-1}a),$$

for some ξ_1 between $c+b$ and $a+b+c$, and ξ_2 between b and $a+b$. Hence, if $H \leq \frac{1}{2}$, it follows that

$$\begin{aligned} |\mu(a+b,a,c)| &\leq \frac{1}{2}(|\xi_1|^{2H-1}a + |\xi_2|^{2H-1}a) \\ &\leq \frac{1}{2}((c+b)^{2H-1}a + b^{2H-1}a). \end{aligned}$$

From here it follows that there exists a constant $C > 0$, only depending on H such that

$$|\mu(a+b,a,c)| \leq Cb^{2H-1}a. \tag{2.4.24}$$

Using inequalities (2.4.20) and (2.4.24), we deduce that there exists a constant $K > 0$ such that

$$\Psi(a+b,a,c) \leq Kb^{4H-2}a^{2-2H}c^{-2H}(1+a^{2H}+c^{2H}+a^{2H}c^{2H})^{-\frac{d}{p}}.$$

From here, it follows that

$$\Psi(a+b,a,c) \leq Kb^{4H-2}a^{2-2H}c^{-2H}(1+a^{2H}+c^{2H}+a^{2H}c^{2H})^{-\frac{d}{p}}. \tag{2.4.25}$$

Using the condition $H \leq \frac{3}{4}$, we can easily show that $2H - \frac{2Hd}{p} \leq \frac{3}{2} - \frac{2Hd}{p} < 0$. Hence, from (2.4.25), we deduce that $\Psi(a+b,a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}_+^3 \mid a \leq b \leq c\}$.

The integrability of $\Psi(a+b,a,c)$ over the region $\{(a,b,c) \in \mathbb{R}^3_+ \mid c \leq b \leq a\}$ in the case $H \leq \frac{1}{2}$, follows from a similar argument. To handle the case $H > \frac{1}{2}$, we proceed as follows. From (2.4.19), we can easily show that for every $a,b,c \geq 0$ such that $a \leq b \leq c$,

$$\begin{aligned}\mu(a+b,a,c) &= H(2H-1)ac\int_{[0,1]^2}(b+a\xi+c\eta)^{2H-2}d\xi d\eta \\ &\leq H(2H-1)ac\int_0^1 (c\eta)^{2H-2}d\eta,\end{aligned}$$

and hence

$$\mu(a+b,a,c) \leq Hac^{2H-1}.$$

From here it follows that

$$\Psi(a+b,a,c) \leq a^{2-2H}c^{2H-2}(1+a^{2H}+c^{2H}+a^{2H}c^{2H})^{-\frac{d}{p}}.$$

Using the condition $p < \frac{4Hd}{3}$, we deduce that $\Psi(a+b,a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}^3_+ \mid a \leq b \leq c\}$. The integrability of $\Psi(a+b,a,c)$ over the region $\{(a,b,c) \in \mathbb{R}^3_+ \mid c \leq b \leq a\}$ in the case $H > \frac{1}{2}$, follows from a similar argument. From the previous analysis it follows that $\Psi(a+b,a,c)$ is integrable in $\mathbb{R}^3_+$, and hence $\Psi(x,u_1,u_2)$ is integrable in $\mathscr{S}_3$, as required. The proof is now complete.

$\square$

Following similar arguments to those presented in the proof of Lemma 2.4.3, we can prove the following result

Lemma 2.4.4. *Let the functions* μ *and* Θ_1 *be defined by* (2.1.10) *and* (2.1.26) *respectively. Then, for every* $\frac{3}{4} < H < 1$ *and* $0 < p < \frac{4Hd}{3}$*,*

$$\sup_{\varepsilon \in (0,1)} \int_{\mathbb{R}_+^2} \int_0^T \varepsilon^{-\frac{2}{H}} \frac{\mu(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2)^2}{u_1^{2H} u_2^{2H}} \Theta_1(\varepsilon^{-\frac{1}{2H}} x, u_1, u_2)^{-\frac{d}{p}} dx du_1 du_2 < \infty. \quad (2.4.26)$$

Proof. Denote by $\kappa_\varepsilon(x, u_1, u_2)$ the function

$$\kappa_\varepsilon(x, u_1, u_2) := \varepsilon^{-\frac{2}{H}} \mu(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2)^2 (u_1 u_2)^{-2H} \Theta_1(\varepsilon^{-\frac{1}{2H}} x, u_1, u_2)^{-\frac{d}{p}}.$$

To prove (2.4.26), it suffices to show that

$$\sup_{\varepsilon \in (0,1)} \int_{\mathbb{R}_+^2} \int_0^T \mathbb{1}_{\mathscr{S}_i}(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2) \kappa_\varepsilon(x, u_1, u_2) dx du_1 du_2 < \infty, \quad (2.4.27)$$

for $i = 1, 2, 3$. To prove (2.4.27) in the case $i = 1, 2$, we make the change of variable $\widehat{x} := \varepsilon^{-\frac{1}{2H}} x$, in order to get

$$\int_{\mathbb{R}_+^2} \int_0^T \mathbb{1}_{\mathscr{S}_i}(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2) \kappa_\varepsilon(x, u_1, u_2) dx du_1 du_2$$
$$= \varepsilon^{-\frac{3}{2H}+2} \int_{\mathbb{R}_+^2} \int_0^{\varepsilon^{-\frac{1}{2H}} T} \mathbb{1}_{\mathscr{S}_i}(\widehat{x}, u_1, u_2) \Psi(\widehat{x}, u_1, u_2) d\widehat{x} du_1 du_2,$$

where Ψ is defined by (2.4.8). Hence,

$$\int_{\mathbb{R}_+^2} \int_0^T \mathbb{1}_{\mathscr{S}_i}(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2) \kappa_\varepsilon(x, u_1, u_2) dx du_1 du_2 \leq \int_{\mathscr{S}_i} \Psi(x, u_1, u_2) dx du_1 du_2. \quad (2.4.28)$$

In Lemma 2.4.3, we proved that $\int_{\mathscr{S}_1} \Psi(x, u_1, u_2) dx du_1 du_2 < \infty$, provided that $p < \frac{4Hd}{3}$. To handle the case $i = 2$, we change the coordinates (x, u_1, u_2) by $(a := x, b := u_2, c :=$

$u_1 - x - u_2$), in order to get

$$\int_{\mathscr{S}_2} \Psi(x,u_1,u_2)dxdu_1du_2 = \int_{\mathbb{R}^3_+} \Psi(a,a+b+c,b)dadbdc.$$

By (2.4.12), $\Psi(a,a+b+c,b)$ is integrable in $\{(a,b,c) \in \mathbb{R}^3_+ \mid b \geq a \wedge c\}$. In addition, since $2H - \frac{1}{2} \leq \frac{3}{2} < Hd$, by (2.4.18), $\Psi(a,a+b+c,b)$ is integrable in $\{(a,b,c) \in \mathbb{R}^3_+ \mid b \leq a \wedge c\}$, and hence, $\Psi(x,u_1,u_2)$ is integrable in $\mathscr{S}_2$, as required. It then remains to prove (2.4.27) in the case $i = 3$. Using (1.2.6), we can easily check that for every $(x,v_1,v_2) \in \mathscr{S}_3$,

$$\begin{aligned}
|\mu(x,v_1,v_2)| &= \left| \left\langle \mathbb{1}_{[0,v_1]}, \mathbb{1}_{[x,x+v_2]} \right\rangle_{\mathfrak{H}^d} \right| \\
&= H(2H-1)v_1v_2 \int_{[0,1]^2} |x + v_2w_2 - v_1w_1|^{2H-2} dw_1dw_2 \\
&\leq H(2H-1)v_1v_2 \int_{[0,1]^2} |x - xw_1|^{2H-2} dw_1dw_2,
\end{aligned}$$

and hence, there exists a constant $C > 0$ only depending on H, such that for every $(x,v_1,v_2) \in \mathscr{S}_3$,

$$|\mu(x,v_1,v_2)| \leq Cv_1v_2x^{2H-2}. \tag{2.4.29}$$

On the other hand, for every $(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2) \in \mathscr{S}_3$, it holds $(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2) \in \mathscr{S}_3$, and hence, by (2.4.20),

$$\Theta_1(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2) \geq \delta u_1^{2H}u_2^{2H} \tag{2.4.30}$$

By (2.4.29) and (2.4.30), we obtain

$$\kappa_\varepsilon(x,u_1,u_2) \le C(u_1u_2)^{2-2H}x^{4H-4}(1+u_1^{2H}+u_2^{2H}+u_1^{2H}u_2^{2H})^{-\frac{d}{p}}, \tag{2.4.31}$$

for some constant $C>0$, and hence,

$$\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathscr{S}_3}(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2)\kappa_\varepsilon(x,u_1,u_2)dxdu_1du_2$$
$$\le \int_{\mathbb{R}_+^2}\int_0^T (u_1u_2)^{2-2H}x^{4H-4}(1+u_1^{2H}+u_2^{2H}+u_1^{2H}u_2^{2H})^{-\frac{d}{p}}dxdu_1du_2.$$

Since $H>\frac{3}{4}$, then $3-2H<\frac{3}{2}<Hd$, and hence, the integral in the right-hand side of the previous identity is finite, which implies that (2.4.27) holds for $i=3$, as required. The proof is now complete. □

Lemma 2.4.5. *Let $d\ge 3$, and $T>0$ be fixed. Let the functions μ and Θ_ε be defined by* (2.1.10) *and* (2.1.26) *respectively and and assume that $H=\frac{3}{4}$. Then, for every $0<p<d$,*

$$\sup_{\varepsilon\in(0,1/e)}\frac{\varepsilon^{-8/3}}{\log(1/\varepsilon)}\int_{\mathbb{R}_+^2}\int_0^T \frac{\mu(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)^2}{(u_1u_2)^{\frac{3}{2}}}\Theta_1(\varepsilon^{-\frac{2}{3}}x,u_1,u_2)^{-\frac{d}{p}}dxdu_1du_2<\infty.$$

Proof. Denote by $\kappa_\varepsilon(x,u_1,u_2)$ the function

$$\kappa_\varepsilon(x,u_1,u_2):=\frac{\varepsilon^{-8/3}}{\log(1/\varepsilon)}\mu(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)^2(u_1u_2)^{-\frac{3}{2}}\Theta_1(\varepsilon^{-\frac{2}{3}}x,u_1,u_2)^{-\frac{d}{p}}.$$

As in Lemma 2.4.4, it suffices to show that

$$\sup_{\varepsilon\in(0,1)}\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathscr{S}_i}(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)\kappa_\varepsilon(x,u_1,u_2)dxdu_1du_2<\infty, \tag{2.4.32}$$

for $i = 1,2,3$, where the regions $\mathscr{S}_i$ are defined by (2.1.31). The cases $i = 1,2$ are handled similarly to Lemma 2.4.4, so it suffices to prove (2.4.27) in the case $i = 3$.

Suppose $(x, \varepsilon^{\frac{2}{3}} u_1, \varepsilon^{\frac{2}{3}} u_2) \in \mathscr{S}_3$. Then, by Lemma 2.4.2, there exists a constant $C > 0$, such that

$$\begin{aligned}
\left|\mu(x, \varepsilon^{\frac{2}{3}} u_1, \varepsilon^{\frac{2}{3}} u_2)\right| &\leq C\varepsilon^{4/3}(x + \varepsilon^{\frac{2}{3}} u_1 + \varepsilon^{\frac{2}{3}} u_2)^{-\frac{1}{2}} u_1 u_2 \\
&= C\varepsilon(\varepsilon^{-\frac{2}{3}} x + u_1 + u_2)^{-\frac{1}{2}} u_1 u_2
\end{aligned}$$

In addition, by Lemma 2.4.1 we have that $u_1^{\frac{3}{2}} u_2^{\frac{3}{2}} - \mu(\varepsilon^{-\frac{2}{3}} x, u_1, u_2)^2 \geq \delta(u_1 u_2)^{\frac{3}{2}}$, for some $\delta > 0$. Therefore, we conclude that there exists a constant $C > 0$, such that

$$\begin{aligned}
\kappa_\varepsilon(x, u_1, u_2) &\leq \frac{\varepsilon^{-\frac{2}{3}} C^2}{\log(1/\varepsilon)} (\varepsilon^{-\frac{2}{3}} x + u_1 + u_2)^{-1} \sqrt{u_1 u_2} \left(1 + u_1^{\frac{3}{2}} + u_2^{\frac{3}{2}} + u_1^{\frac{3}{2}} u_2^{\frac{3}{2}} - \mu(x, u_1, u_2)^2\right)^{-\frac{d}{p}} \\
&\leq \frac{\varepsilon^{-\frac{2}{3}} C^2 \delta^{-\frac{d}{p}}}{\log(1/\varepsilon)} (\varepsilon^{-\frac{2}{3}} x + u_1 + u_2)^{-1} \sqrt{u_1 u_2} \left(1 + u_1^{\frac{3}{2}} + u_2^{\frac{3}{2}} + u_1^{\frac{3}{2}} u_2^{\frac{3}{2}}\right)^{-\frac{d}{p}}.
\end{aligned}$$

Consequently, there exists a constant $C > 0$, such that

$$\begin{aligned}
&\int_{\mathbb{R}_+^2} \int_0^T \mathbb{1}_{\mathscr{S}_i}(x, u_1, u_2) \kappa_\varepsilon(x, u_1, u_2) dx du_1 du_2 \\
&\qquad \leq \frac{C\varepsilon^{-\frac{2}{3}}}{\log(1/\varepsilon)} \int_0^T \int_{\mathbb{R}_+^2} (\varepsilon^{-\frac{2}{3}} x + u_1 + u_2)^{-1} \sqrt{u_1 u_2} \left(1 + u_1^{\frac{3}{2}} + u_2^{\frac{3}{2}} + u_1^{\frac{3}{2}} u_2^{\frac{3}{2}}\right)^{-\frac{d}{p}}.
\end{aligned}$$

Hence, making the change of variable $\widetilde{x} := \varepsilon^{-\frac{2}{3}}x$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)\kappa_\varepsilon(x,u_1,u_2)dxdu_1du_2 \\
&\leq \frac{C}{\log(1/\varepsilon)}\int_{\mathbb{R}_+^2}\int_0^{\varepsilon^{-\frac{2}{3}}T}(x+u_1+u_2)^{-1}\sqrt{u_1u_2}\left(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\right)^{-\frac{d}{p}}dxdu_1du_2 \\
&= \frac{C}{\log(1/\varepsilon)}\int_{\mathbb{R}_+^2}\int_0^1(x+u_1+u_2)^{-1}\sqrt{u_1u_2}\left(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\right)^{-\frac{d}{p}}dxdu_1du_2 \\
&+ \frac{C}{\log(1/\varepsilon)}\int_{\mathbb{R}_+^2}\int_1^{\varepsilon^{-\frac{2}{3}}T}(x+u_1+u_2)^{-1}\sqrt{u_1u_2}\left(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\right)^{-\frac{d}{p}}dxdu_1du_2.
\end{aligned} \tag{2.4.33}$$

Applying the inequalities $(x+u_1+u_2)^{-1} \leq (u_1+u_2)^{-1} \leq \frac{1}{2}(u_1u_2)^{-\frac{1}{2}}$ for $x \in [0,1]$, and $(x+u_1+u_2)^{-1} \leq x^{-1}$ for $x \geq 1$, in the first and second terms in the right-hand side of (2.4.33), and then integrating the variable x, we can show that

$$\begin{aligned}
\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)\kappa_\varepsilon(x,u_1,u_2)dxdu_1du_2 \leq C\int_{\mathbb{R}_+^2}\left(\frac{(u_1u_2)^{-\frac{1}{2}}+\frac{2}{3}\log(1/\varepsilon)+\log(T)}{\log(1/\varepsilon)}\right) \\
\times\sqrt{u_1u_2}\left(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\right)^{-\frac{d}{p}}dxdu_1du_2,
\end{aligned}$$

and consequently, for every $\varepsilon < 1/e$,

$$\begin{aligned}
&\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathscr{S}_i}(x,u_1,u_2)\kappa_\varepsilon(x,u_1,u_2)dxdu_1du_2 \\
&\qquad \leq C\int_{\mathbb{R}_+^2}\left((u_1u_2)^{-\frac{1}{2}}+\log(T)\right)\sqrt{u_1u_2}\left(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\right)^{-\frac{d}{p}}dxdu_1du_2.
\end{aligned}$$

The right-hand side of the previous inequality is finite due to the condition $0 < p < d$. This finishes the proof of (2.4.32). □

Chapter 3

Derivative self-intersection local time for the fractional Brownian motion

Let $B = \{B_t\}_{t\geq 0}$ be a one-dimensional fractional Brownian motion of Hurst parameter $H \in (0,1)$. Fix $T > 0$. The self-intersection local time of B, formally defined by

$$I(y) := \int_0^T \int_0^t \delta(B_t - B_s - y)\mathrm{d}s\mathrm{d}t,$$

was first studied by Rosen in [49] in the planar case and it was further investigated using techniques from Malliavin calculus by Hu and Nualart in [23]. In particular, in [23] it is proved that for a d-dimensional fractional Brownian motion, $I(0)$ exists in L^2 whenever the Hurst parameter H satisfies $H < \frac{1}{d}$.

Motivated by spatial integrals with respect to local time, developed by Rogers and Walsh in [48], Rosen introduced in [50] a formal derivative of $I(y)$, in the one-dimensional Brownian case, denoted by

$$\alpha(y) := \frac{\mathrm{d}I}{\mathrm{d}y}(y) = -\int_0^T \int_0^t \delta'(B_t - B_s - y)\mathrm{d}s\mathrm{d}t.$$

The random variable $\alpha := \alpha(0)$ is called the derivative of the self-intersection local time at zero, and is equal to the limit in L^2 of

$$\alpha_\varepsilon := \int_0^T \int_0^t p'_\varepsilon(B_t - B_s)\mathrm{d}s\mathrm{d}t, \tag{3.0.1}$$

where $p_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{1}{2}} e^{-\frac{x^2}{2\varepsilon}}$. This random variable was subsequently used by Hu and Nualart [24], to study the asymptotic properties of the third spacial moment of the Brownian local time. In [34], Markowsky gave an alternative proof of the existence of such limit by using Wiener chaos expansion.

Jung and Markowsky extended this result in [29] to the case $0 < H < \frac{2}{3}$ and conjectured that for the case $H > \frac{2}{3}$, $\varepsilon^{-\gamma(H)}\alpha_\varepsilon$ converges in law to a Gaussian distribution for some suitable constant $\gamma(H) > 0$, and at the critical point $H = \frac{2}{3}$, the variable $\log(1/\varepsilon)^{-\gamma}\alpha_\varepsilon$ converges in law to a Gaussian distribution for some $\gamma > 0$.

Let $\mathscr{N}(0,\sigma^2)$ denote a centered Gaussian random variable with variance σ^2. The primary goal of this paper is to analyze the behavior of the law of α_ε as $\varepsilon \to 0$, when $\frac{2}{3} < H < 1$. We will prove that when $\frac{2}{3} < H < 1$,

$$\varepsilon^{\frac{3}{2}-\frac{1}{H}}\alpha_\varepsilon \overset{Law}{\to} \mathscr{N}(0,\sigma^2), \qquad \text{when } \varepsilon \to 0,$$

for some constant σ^2 that will be specified later (see Theorem 3.3.1). Moreover, we will prove that for every $q \geq 2$ and $\frac{2}{3} < H < \frac{3}{4}$, $\lim_{\varepsilon\to 0} J_q[\alpha_\varepsilon]$ exists in L^2, where J_q denotes the projection on the q-th Wiener chaos (see Theorem 3.3.2), while in the case $\frac{3}{4} < H < \frac{4q-3}{4q-2}$, the chaotic components $J_q[\alpha_\varepsilon]$ of α_ε satisfy

$$\varepsilon^{1-\frac{3}{4H}} J_q[\alpha_\varepsilon] \overset{Law}{\to} \mathscr{N}(0,\sigma_q^2), \qquad \text{when } \varepsilon \to 0,$$

for some constant σ_q^2 that will be specified later (see Theorem 3.3.3). The proof of the central limit theorem for $\varepsilon^{\frac{3}{2}-\frac{1}{H}}\alpha_\varepsilon$ follows easily from estimations of the L^2-norm of the chaotic components of α_ε, while the proof of the central limit theorem for $\varepsilon^{1-\frac{3}{4H}}J_q[\alpha_\varepsilon]$ relies on the multivariate version of the fourth moment theorem (see [44, 47]), as well as on a continuous version of the Breuer-Major theorem ([7]) proved in [11]. The behavior of α_ε in the critical case $H=\frac{2}{3}$, and the behavior of $J_q[\alpha_\varepsilon]$ in the critical cases $H=\frac{2}{3}$, $H=\frac{3}{4}$ and $H=\frac{4q-3}{4q-2}$ seems more involved and will not be discussed in this paper.

It is surprising to remark that the limit behavior of the chaotic components of α_ε is different from that of the whole sequence. This phenomenon was observed, for instance, in the central limit theorem for the second spatial moment of Brownian local time increments (see [12]). However, in this case the limit of the whole sequence is a mixture of Gaussian distributions, whereas in the present paper the normalization of α_ε converges to a Gaussian law. In our case, the projection on the first chaos of α_ε is the leading term and is responsible for the Gaussian limit of the whole sequence.

The chapter is organized as follows. In Section 3.1 we present some preliminary results on the fractional Brownian motion and the chaotic decomposition of α_ε. In Section 2.2 we compute the asymptotic behavior of the variances of the normalizations of the chaotic components of α_ε as $\varepsilon \to 0$. The asymptotic behavior of the law of α_ε and its chaotic components is presented in section 2.3. Finally, some technical lemmas are proved in Section 5.

3.1 Chaos decomposition for the approcimate derivative self-intersection local time

Proceeding as in [29] (also see [23]), we can determine the chaos decomposition of the random variable α_ε defined in (3.0.1) as follows. First we write

$$\alpha_\varepsilon = \int_0^T \int_0^t \alpha_{\varepsilon,s,t} \mathrm{d}s\mathrm{d}t, \tag{3.1.1}$$

where $\alpha_{\varepsilon,s,t} := p'_\varepsilon(B_t - B_s)$. We know that

$$\alpha_{\varepsilon,s,t} = \sum_{q=1}^{\infty} I_{2q-1}\left(f_{2q-1,\varepsilon,s,t}\right), \tag{3.1.2}$$

where

$$f_{2q-1,\varepsilon,s,t}(x_1,\ldots,x_{2q-1}) := (-1)^q \beta_q (\varepsilon + (t-s)^{2H})^{-q-\frac{1}{2}} \prod_{j=1}^{2q-1} \mathbb{1}_{[s,t]}(x_j), \tag{3.1.3}$$

and

$$\beta_q := \frac{1}{2^{q-\frac{1}{2}}(q-1)!\sqrt{\pi}}. \tag{3.1.4}$$

As a consequence, the random variable α_ε has the chaos decomposition

$$\alpha_\varepsilon = \sum_{q=1}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon}), \tag{3.1.5}$$

where

$$f_{2q-1,\varepsilon}(x_1,\ldots,x_{2q-1}) := \int_{\mathscr{R}} f_{2q-1,\varepsilon,s,t}(x_1,\ldots,x_{2q-1})\mathrm{d}s\mathrm{d}t, \tag{3.1.6}$$

and

$$\mathscr{R} := \{(s,t) \in \mathbb{R}_+^2 \mid s \le t \le T\}. \tag{3.1.7}$$

Let $T, \varepsilon > 0$, $\frac{2}{3} < H < 1$, and $q \in \mathbb{N}$ be fixed. Our first goal is to find the behavior as $\varepsilon \to 0$ of the variances of α_ε and $I_{2q-1}\left(f_{2q-1,\varepsilon}\right)$. Before addressing this problem, we will introduce some notation. First notice that

$$\begin{aligned}\mathbb{E}\left[I_{2q-1}\left(f_{2q-1,\varepsilon}\right)^2\right] &= (2q-1)!\left\|f_{2q-1,\varepsilon}\right\|^2_{\mathfrak{H}^{\otimes(2q-1)}} \\ &= (2q-1)!\left\langle \int_{\mathscr{R}} f_{2q-1,\varepsilon,s_1,t_1}\mathrm{d}s_1\mathrm{d}t_1, \int_{\mathscr{R}} f_{2q-1,\varepsilon,s_2,t_2}\mathrm{d}s_2\mathrm{d}t_2\right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} \\ &= 2(2q-1)!\int_{\mathscr{S}}\left\langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2}\right\rangle_{\mathfrak{H}^{\otimes(2q-1)}}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2,\end{aligned} \tag{3.1.8}$$

where the set $\mathscr{S}$ is defined by

$$\mathscr{S} := \{(s_1,s_2,t_1,t_2) \in [0,T]^4 \mid s_1 \le t_1, \;\; s_2 \le t_2, \;\; \text{and} \;\; s_1 \le s_2\}. \tag{3.1.9}$$

We can write the set $\mathscr{S}$ as the union of the sets $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ defined by

$$\mathscr{S}_1 := \{(s_1,s_2,t_1,t_2) \in [0,T]^4 \mid s_1 \le s_2 \le t_1 \le t_2\}, \tag{3.1.10}$$

$$\mathscr{S}_2 := \{(s_1,s_2,t_1,t_2) \in [0,T]^4 \mid s_1 \le s_2 \le t_2 \le t_1\}, \tag{3.1.11}$$

$$\mathscr{S}_3 := \{(s_1,s_2,t_1,t_2) \in [0,T]^4 \mid s_1 \le t_1 \le s_2 \le t_2\}. \tag{3.1.12}$$

Then, by (3.1.1),

$$\begin{aligned}\mathbb{E}\left[\alpha_\varepsilon^2\right] &= \mathbb{E}\left[\left(\int_{\mathscr{R}} \alpha_{\varepsilon,s,t}\mathrm{d}s\mathrm{d}t\right)^2\right]\\ &= 2\int_{\mathscr{S}} \mathbb{E}\left[\alpha_{\varepsilon,s_1,t_1}\alpha_{\varepsilon,s_2,t_2}\right]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2\\ &= V_1(\varepsilon)+V_2(\varepsilon)+V_3(\varepsilon), \end{aligned} \tag{3.1.13}$$

where

$$V_i(\varepsilon) := 2\int_{\mathscr{S}_i} \mathbb{E}\left[\alpha_{\varepsilon,s_1,t_1}\alpha_{\varepsilon,s_2,t_2}\right]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2, \qquad i=1,2,3. \tag{3.1.14}$$

Similarly, from (3.1.6) and (3.1.8), taking $q=1$, we get

$$\mathbb{E}\left[I_1\left(f_{1,\varepsilon}\right)^2\right] = V_1^{(1)}(\varepsilon)+V_2^{(1)}(\varepsilon)+V_3^{(1)}(\varepsilon), \tag{3.1.15}$$

where

$$V_i^{(1)}(\varepsilon) := 2\int_{\mathscr{S}_i} \left\langle f_{1,\varepsilon,s_1,t_1}, f_{1,\varepsilon,s_2,t_2}\right\rangle_{\mathfrak{H}}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2, \qquad i=1,2,3. \tag{3.1.16}$$

As a consequence of (3.1.13) and (3.1.15), to determine the behavior of the variances of α_ε and $I_1\left(f_{1,\varepsilon}\right)$ as $\varepsilon\to 0$, it suffices to determine the behavior of $V_i(\varepsilon)$ and $V_i^{(1)}(\varepsilon)$ respectively, for $i=1,2,3$.

In order to describe the terms $\left\langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2}\right\rangle_{\mathfrak{H}^{\otimes(2q-1)}}$, we will introduce the following notation. For every $x,u_1,u_2>0$ define

$$\mu(x,u_1,u_2) := \mathbb{E}\left[B_{u_1}(B_{x+u_2}-B_x)\right]. \tag{3.1.17}$$

We can easily prove that for every $s_1, s_2, t_1, t_2 \geq 0$, such that $s_1 \leq t_1$, $s_2 \leq t_2$ and $s_1 \leq s_2$,

$$\mathbb{E}\left[(B_{t_1} - B_{s_1})(B_{t_2} - B_{s_2})\right] = \mu(s_2 - s_1, t_1 - s_1, t_2 - s_2). \tag{3.1.18}$$

Using (3.1.3) and (3.1.18), for every $0 \leq s_1 \leq t_1$, $0 \leq s_2 \leq t_2$ such that $s_1 \leq s_2$, we can write

$$\begin{aligned}
\left\langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2} \right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} &= \beta_q^2 (\varepsilon + (t_1 - s_1)^{2H})^{-\frac{1}{2}-q} (\varepsilon + (t_2 - s_2)^{2H})^{-\frac{1}{2}-q} \\
&\quad \times \left\langle \mathbb{1}_{[s_1,t_1]}^{\otimes(2q-1)}, \mathbb{1}_{[s_2,t_2]}^{\otimes(2q-1)} \right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} \\
&= \beta_q^2 (\varepsilon + (t_1 - s_1)^{2H})^{-\frac{1}{2}-q} (\varepsilon + (t_2 - s_2)^{2H})^{-\frac{1}{2}-q} \\
&\quad \times \mu(s_2 - s_1, t_1 - s_1, t_2 - s_2)^{2q-1}.
\end{aligned}$$

Therefore,

$$\left\langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2} \right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} = \beta_q^2 G_{\varepsilon,s_2-s_1}^{(q)}(t_1 - s_1, t_2 - s_2), \tag{3.1.19}$$

where $G_{\varepsilon,x}^{(q)}(u_1, u_2)$ is defined by

$$G_{\varepsilon,x}^{(q)}(u_1, u_2) := \left(\varepsilon + u_1^{2H}\right)^{-\frac{1}{2}-q} \left(\varepsilon + u_2^{2H}\right)^{-\frac{1}{2}-q} \mu(x, u_1, u_2)^{2q-1}. \tag{3.1.20}$$

Next we present some useful properties of the functions $\mu(x, u_1, u_2)$ and $G_{\varepsilon,x}^{(q)}(u_1, u_2)$. Taking into account that $H > \frac{2}{3}$, we can write the covariance of B as

$$\mathbb{E}\left[B_t B_s\right] = H(2H - 1) \int_0^t \int_0^s |v_1 - v_2|^{2H-2} \mathrm{d}v_1 \mathrm{d}v_2. \tag{3.1.21}$$

In particular, this leads to

$$\mu(x,u_1,u_2) = H(2H-1)\int_0^{u_1}\int_x^{x+u_2} |v_2 - v_1|^{2H-2}\,\mathrm{d}v_1\mathrm{d}v_2, \tag{3.1.22}$$

which implies

$$G^{(q)}_{\varepsilon,x}(u_1,u_2) \geq 0 \qquad \text{for every } \ \varepsilon \geq 0. \tag{3.1.23}$$

Using the chaos decomposition (3.1.2), as well as (3.1.19) and (3.1.23), we can check that for $i = 1,2,3$, the terms $V_i(\varepsilon), V_i^{(1)}(\varepsilon)$, defined by (3.1.14), (3.1.16), satisfy

$$0 \leq V_i^{(1)}(\varepsilon) \leq V_i(\varepsilon). \tag{3.1.24}$$

Further properties for the function $G^{(q)}_{\varepsilon,x}(u_1,u_2)$ are described in the following lemma.

Lemma 3.1.1. *Let $G^{(q)}_{1,x}(u_1,u_2)$ be defined by* (3.1.20)*. There exists a constant $K > 0$, depending on H and q, such that for all $x > 0$, and $0 < v_1 \leq w_1$, $0 < v_2 \leq w_2$ satisfying $|v_i - w_i| \leq 1$,*

$$G^{(q)}_{1,x}(v_1,v_2) \leq K G^{(q)}_{1,x}(w_1,w_2).$$

Proof. From (3.1.22) it follows that

$$\mu(x,v_1,v_2) \leq \mu(x,w_1,w_2).$$

As a consequence,

$$\begin{aligned}G_{1,x}^{(q)}(v_1,v_2) &= (1+v_1^{2H})^{-\frac{1}{2}-q}(1+v_2^{2H})^{-\frac{1}{2}-q}\mu(x,v_1,v_2)^{2q-1}\\ &\leq (1+v_1^{2H})^{-\frac{1}{2}-q}(1+v_2^{2H})^{-\frac{1}{2}-q}\mu(x,w_1,w_2)^{2q-1}\\ &= G_{1,x}^{(q)}(w_1,w_2)\left(\frac{(1+w_1^{2H})(1+w_2^{2H})}{(1+v_1^{2H})(1+v_2^{2H})}\right)^{q+\frac{1}{2}}.\end{aligned}$$

Using condition $|v_i - w_i| \leq 1$, $i = 1, 2$, we get

$$G_{1,x}^{(q)}(v_1,v_2) \leq G_{1,x}^{(q)}(w_1,w_2)\left(\frac{(1+(v_1+1)^{2H})(1+(v_2+1)^{2H})}{(1+v_1^{2H})(1+v_2^{2H})}\right)^{q+\frac{1}{2}}. \tag{3.1.25}$$

The second factor in the right-hand side of (3.1.25) is uniformly bounded for $v_1, v_2 \geq 0$, which implies the desired result. □

3.2 Behavior of the variances of the approximate derivative self-intersection local time and its chaotic components

The behavior of the variance of α_ε is described in the following lemma.

Lemma 3.2.1. *Let $T > 0$ and $\frac{2}{3} < H < 1$ be fixed. Then,*

$$\lim_{\varepsilon\to 0} \varepsilon^{3-\frac{2}{H}}\mathbb{E}\left[\alpha_\varepsilon^2\right] = \sigma^2, \tag{3.2.1}$$

where σ^2 is defined by

$$\sigma^2 := \frac{T^{2H}(2H-1)}{4H\pi} B\left(\frac{1}{H}, \frac{3H-2}{2H}\right)^2 B(2, 2H-1), \tag{3.2.2}$$

and $B(\cdot,\cdot)$ denotes the Beta function.

Proof. From (3.1.13) we have

$$\varepsilon^{3-\frac{2}{H}} \mathbb{E}\left[\alpha_\varepsilon^2\right] = \varepsilon^{3-\frac{2}{H}} V_1(\varepsilon) + \varepsilon^{3-\frac{2}{H}} V_2(\varepsilon) + \varepsilon^{3-\frac{2}{H}} V_3(\varepsilon),$$

where $V_1(\varepsilon)$, $V_2(\varepsilon)$ and $V_3(\varepsilon)$ are defined by (3.1.14). By Lemmas 3.4.3 and 3.4.4, we have $\lim_{\varepsilon\to 0} \varepsilon^{3-\frac{2}{H}} V_1(\varepsilon) = 0$ and $\varepsilon^{3-\frac{2}{H}} V_2(\varepsilon) = 0$, respectively. In addition, from Lemma 3.4.6 we have $\lim_{\varepsilon\to 0} \varepsilon^{3-\frac{2}{H}} V_3(\varepsilon) = \sigma^2$, where σ^2 is defined by (3.2.2). This completes the proof of equation (3.2.1). □

The behavior of the variance of the first chaotic component of α_ε is described by the following lemma.

Lemma 3.2.2. *Let $T > 0$ be fixed. Define $f_{1,\varepsilon}$ as in equation* (3.1.6)*. Then, for every $\frac{2}{3} < H < 1$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{3-\frac{2}{H}} \mathbb{E}\left[I_1\left(f_{1,\varepsilon}\right)^2\right] = \sigma^2, \tag{3.2.3}$$

where σ^2 is given by (3.2.2)*.*

Proof. From (3.1.15) we have

$$\varepsilon^{3-\frac{2}{H}} \mathbb{E}\left[I_1(f_{1,\varepsilon})^2\right] = \varepsilon^{3-\frac{2}{H}} V_1^{(1)}(\varepsilon) + \varepsilon^{3-\frac{2}{H}} V_2^{(1)}(\varepsilon) + \varepsilon^{3-\frac{2}{H}} V_3^{(1)}(\varepsilon),$$

where $V_1^{(1)}(\varepsilon)$, $V_2^{(1)}(\varepsilon)$ and $V_3^{(1)}(\varepsilon)$ are defined by (3.1.16). By Lemmas 3.4.3 and 3.4.4, we have $\lim_{\varepsilon\to 0}\varepsilon^{3-\frac{2}{H}}V_1(\varepsilon)=0$ and $\varepsilon^{3-\frac{2}{H}}V_2(\varepsilon)=0$, respectively. Consequently, by (3.1.24) we get $\lim_{\varepsilon\to 0}\varepsilon^{3-\frac{2}{H}}V_1^{(1)}(\varepsilon)=0$ and $\lim_{\varepsilon\to 0}\varepsilon^{3-\frac{2}{H}}V_2^{(1)}(\varepsilon)=0$. In addition, from Lemma 3.4.7, the term $V_3^{(1)}(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0}\varepsilon^{3-\frac{2}{H}}V_3^{(1)}(\varepsilon)=\sigma^2$, where σ^2 is given by (3.2.2). This completes the proof of equation (3.2.3). □

The behavior of the variance of the chaotic components of α_ε of order greater than or equal to two and is described by the following lemma.

Lemma 3.2.3. *Let $T,\varepsilon>0$, $\frac{2}{3}<H<1$ and $q\in\mathbb{N}$, $q\geq 2$ be fixed. Define $\beta_q, f_{2q-1,\varepsilon}$, and $G_{\varepsilon,x}^{(q)}(u_1,u_2)$ by* (3.1.4), (3.1.6) *and* (3.1.20) *respectively. Then,*

1. *If $\frac{3}{4}<H<\frac{4q-3}{4q-2}$,*

$$\lim_{\varepsilon\to 0}\varepsilon^{2-\frac{3}{2H}}\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right]=\sigma_q^2, \tag{3.2.4}$$

where σ_q^2 is a finite constant given by

$$\sigma_q^2:=2(2q-1)!\beta_q^2T\int_{\mathbb{R}_+^3}G_{1,x}^{(q)}(u_1,u_2)dxdu_1du_2. \tag{3.2.5}$$

2. *In the case $\frac{2}{3}<H<\frac{3}{4}$,*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right]=\overline{\sigma}_q^2, \tag{3.2.6}$$

where $\overline{\sigma}_{q,d}^2$ is a finite constant given by

$$\overline{\sigma}_{q,d}^2:=2(2q-1)!\beta_q^2\int_{\mathscr{S}}G_{0,s_2-s_1}^{(q)}(t_1-s_1,t_2-s_2)ds_1ds_2dt_1dt_2, \tag{3.2.7}$$

and $\mathscr{S}$ is defined by (3.1.9).

Proof. First we prove (3.2.4) in the case $\frac{3}{4} < H < \frac{4q-3}{4q-2}$. By (3.1.8) and (3.1.19),

$$\varepsilon^{2-\frac{3}{2H}} \mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right] = 2(2q-1)!\beta_q^2 \varepsilon^{2-\frac{3}{2H}} \int_{\mathscr{S}} G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2,$$

where $\mathscr{S}$ is defined by (3.1.9). Therefore, changing the coordinates (s_1,s_2,t_1,t_2) by $(\varepsilon^{-\frac{1}{2H}}s_1, x := \varepsilon^{-\frac{1}{2H}}(s_2-s_1), u_1 := \varepsilon^{-\frac{1}{2H}}(t_1-s_1), u_2 := \varepsilon^{-\frac{1}{2H}}(t_2-s_2))$, and using the fact that $G_{\varepsilon,\varepsilon^{\frac{1}{2H}}x}(\varepsilon^{\frac{1}{2H}}u_1, \varepsilon^{\frac{1}{2H}}u_2) = \varepsilon^{-2}G_{1,x}(u_1,u_2)$, we get

$$\begin{aligned}\varepsilon^{2-\frac{3}{2H}} \mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right] = 2(2q-1)!\beta_q^2 \varepsilon^{\frac{1}{2H}} \int_{\mathbb{R}_+^4} &\mathbb{1}_{(0,\varepsilon^{-\frac{1}{2H}}T)}(s_1+u_1)\\ &\times \mathbb{1}_{(0,\varepsilon^{-\frac{1}{2H}}T)}(s_1+x+u_2) G^{(q)}_{1,x}(u_1,u_2)\mathrm{d}s_1\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2.\end{aligned}$$

Integrating with respect to the variable s_1 we get

$$\begin{aligned}\varepsilon^{2-\frac{3}{2H}} \mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right] = 2(2q-1)!\beta_q^2 \int_{\mathbb{R}_+^3} &(T-\varepsilon^{\frac{1}{2H}}(u_1 \vee (x+u_2)))\mathbb{1}_{(0,\varepsilon^{-\frac{1}{2H}}T)}(u_1)\\ &\times \mathbb{1}_{(0,\varepsilon^{-\frac{1}{2H}}T)}(s_1+x+u_2) G^{(q)}_{1,x}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2. \qquad (3.2.8)\end{aligned}$$

From (3.1.23) we deduce that the integrand in the right-hand side of (3.2.8) is positive and increasing as ε decreases to zero. Therefore, applying the monotone convergence theorem in relation (3.2.8) we obtain (3.2.4). The constant σ_q^2 is finite by Lemma 3.4.8.

To prove relation (3.2.6), notice that equations (3.1.8) and (3.1.19) imply that

$$\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right] = 2(2q-1)!\beta_q^2 \int_{\mathscr{S}} G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2. \quad (3.2.9)$$

Relation (3.2.6) follows by applying the monotone convergence theorem to (3.2.9). To prove that $\overline{\sigma}_q$ is finite we change the coordinates (s_1,s_2,t_1,t_2) by $(s_1, x := s_2-s_1, u_1 :=$

$t_1 - s_1, u_2 := t_2 - s_2)$ in the integral of the right-hand side of (3.2.7), to get

$$\int_{\mathscr{S}} G^{(q)}_{0,s_2-s_1}(t_1-s_1,t_2-s_2)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2 \leq \int_{[0,T]^4} G^{(q)}_{0,x}(u_1,u_2)\mathrm{d}s_1\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2$$
$$= T\int_{[0,T]^3} G^{(q)}_{0,x}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2.$$

The latter integral is finite by Lemma 3.4.9. Therefore, the constant $\overline{\sigma}_q^2$ is finite. □

3.3 Limit behavior of the approximate derivative self-intersection local time and its chaotic components

The next result is a central limit theorem for α_ε in case $\frac{2}{3} < H < 1$.

Theorem 3.3.1. *Let $T, \varepsilon > 0$ and $\frac{2}{3} < H < 1$ be fixed. Then*

$$\varepsilon^{\frac{3}{2}-\frac{1}{H}}\alpha_\varepsilon \stackrel{Law}{\to} \mathscr{N}(0,\sigma^2), \qquad \textit{when } \ \varepsilon \to 0, \tag{3.3.1}$$

where σ^2 is defined by (3.2.2).

Proof. Let $f_{2q-1,\varepsilon}$ be defined by (3.1.6). By equation (3.1.5),

$$\varepsilon^{\frac{3}{2}-\frac{1}{H}}\alpha_\varepsilon = \varepsilon^{\frac{3}{2}-\frac{1}{H}}I_1\left(f_{1,\varepsilon}\right) + \varepsilon^{\frac{3}{2}-\frac{1}{H}}\sum_{q=2}^{\infty} I_{2q-1}\left(f_{2q-1,\varepsilon}\right).$$

By Lemma 3.2.2, the variance of $\varepsilon^{\frac{3}{2}-\frac{1}{H}}I_1\left(f_{1,\varepsilon}\right)$ converges to σ^2, where σ^2 is defined by (3.2.2). In addition, combining Lemmas 3.2.1 and 3.2.2, it follows that the term

$$\varepsilon^{\frac{3}{2}-\frac{1}{H}}\sum_{q=2}^{\infty} I_{2q-1}\left(f_{2q-1,\varepsilon}\right)$$

converges to zero in L^2. Then (3.3.1) follows from the fact that $\varepsilon^{\frac{3}{2}-\frac{1}{H}} I_1(f_{1,\varepsilon})$ is Gaussian and its variance converges to σ^2. $\square$

In the next result we describe the asymptotic behavior of the chaotic components of α_ε in the case $\frac{2}{3} < H < 1$.

Theorem 3.3.2. *Let $T, \varepsilon > 0$ and $q \in \mathbb{N}$, $q \geq 2$ be fixed. Define $f_{2q-1,\varepsilon}$ by* (3.1.6)*. If $\frac{2}{3} < H < \frac{3}{4}$, then $I_{2q-1}(f_{2q-1,\varepsilon})$ converges in L^2 when $\varepsilon \to 0$.*

Proof. Define $f_{2q-1,\varepsilon,s,t}$ by (3.1.3). For every $\varepsilon, \eta > 0$ we have

$$\begin{aligned}\mathbb{E}\left[\left(I_{2q-1}(f_{2q-1,\varepsilon}) - I_{2q-1}(f_{2q-1,\eta})\right)^2\right] &= \mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right] + \mathbb{E}\left[I_{2q-1}(f_{2q-1,\eta})^2\right] \\ &\quad - 2\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon}) I_{2q-1}(f_{2q-1,\eta})\right].\end{aligned}$$

Define $\mathscr{R}$ and $\mathscr{S}$ by (3.1.7) and (3.1.9), respectively. Then we have

$$\begin{aligned}&\mathbb{E}\left[I_{2q-1}\left(f_{2q-1,\varepsilon}\right) I_{2q-1}\left(f_{2q-1,\eta}\right)\right] \\ &\quad = (2q-1)! \left\langle f_{2q-1,\varepsilon}, f_{2q-1,\eta}\right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} \\ &\quad = (2q-1)! \left\langle \int_{\mathscr{R}} f_{2q-1,\varepsilon,s,t} \mathrm{d}s\mathrm{d}t, \int_{\mathscr{R}} f_{2q-1,\eta,s,t} \mathrm{d}s\mathrm{d}t \right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} \\ &\quad = 2(2q-1)! \int_{\mathscr{S}} \left\langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\eta,s_2,t_2}\right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}t_1 \mathrm{d}t_2.\end{aligned} \tag{3.3.2}$$

Substituting (3.1.19) into (3.3.2), yields

$$\begin{aligned}&\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon}) I_{2q-1}(f_{2q-1,\eta})\right] \\ &\quad = 2(2q-1)!\beta_q^2 \int_{\mathscr{S}} (\varepsilon + (t_1-s_1)^{2H})^{-\frac{1}{2}-q} (\eta + (t_2-s_2)^{2H})^{-\frac{1}{2}-q} \\ &\quad \times \mu(s_2-s_1, t_1-s_1, t_2-s_2)^{2q-1} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}t_1 \mathrm{d}t_2,\end{aligned} \tag{3.3.3}$$

The integrand in the right-hand side is nonnegative, decreasing on the variables ε and η, and converges pointwise to $G^{(q)}_{0,x}(u_1,u_2)$ as $\varepsilon,\eta\to 0$, where $G^{(q)}_{0,x}(u_1,u_2)$ is defined by (3.1.23). Hence, by the monotone convergence theorem, as $\varepsilon,\eta\to 0$, the terms $\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})I_{2q-1}(f_{2q-1,\eta})\right]$, $\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right]$ and $\mathbb{E}\left[I_{2q-1}(f_{2q-1,\eta})^2\right]$ converge to

$$2(2q-1)!\beta_q^2\int_{\mathscr{S}} G^{(q)}_{0,s_2-s_1}(t_1-s_1,t_2-s_2)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2. \tag{3.3.4}$$

The previous quantity is finite thanks to Lemma 3.2.3. From the previous analysis we conclude that the sequence $\{I_{2q-1}(f_{2q-1,\varepsilon_n})\}_{n\in\mathbb{N}}$ is Cauchy in L^2, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}\subset[0,1]$ such that $\varepsilon_n\to 0$ as $n\to\infty$, which implies the desired result. $\square$

The next result is a central limit theorem for $I_{2q-1}(f_{2q-1,\varepsilon})$ in the case $\frac{3}{4}<H<\frac{4q-3}{4q-2}$.

Theorem 3.3.3. *Let $T,\varepsilon>0$ and $q\in\mathbb{N}$, $q\geq 2$ be fixed. Define $f_{2q-1,\varepsilon}$ by* (3.1.6)*. Then, for every $\frac{3}{4}<H<\frac{4q-3}{4q-2}$ we have*

$$\varepsilon^{1-\frac{3}{4H}}I_{2q-1}(f_{2q-1,\varepsilon})\overset{Law}{\to}\mathscr{N}(0,\sigma_q^2),\qquad \textit{when }\ \varepsilon\to 0, \tag{3.3.5}$$

where σ_q^2 is the finite constant defined by (3.2.5)*.*

Proof. Define $f_{2q-1,\varepsilon,s,t}$, for $0\leq s\leq t$, by (3.1.3) and $\mathscr{R}$ by (3.1.7). By (3.1.6),

$$\varepsilon^{1-\frac{3}{4H}}I_{2q-1}(f_{2q-1,\varepsilon})=(-1)^q\varepsilon^{1-\frac{3}{4H}}\int_{\mathscr{R}}\beta_q(\varepsilon+(t-s)^{2H})^{-\frac{1}{2}-q}I_{2q-1}\left(\mathbb{1}^{\otimes(2q-1)}_{[s,t]}\right)\mathrm{d}s\mathrm{d}t.$$

Then, using the self-similarity of the fractional Brownian motion we get

$$\varepsilon^{1-\frac{3}{4H}} I_{2q-1}(f_{2q-1,\varepsilon})$$
$$\stackrel{Law}{=} (-1)^q \varepsilon^{1-\frac{3}{4H}} \int_{\mathscr{R}} \beta_q (\varepsilon + (t-s)^{2H})^{-\frac{1}{2}-q} I_{2q-1}\left(\left(\sqrt{\varepsilon} \mathbb{1}_{\varepsilon^{-\frac{1}{2H}}[s,t]} \right)^{\otimes(2q-1)} \right) \mathrm{d}s\mathrm{d}t.$$

Therefore, changing the coordinates (s,t) by $(\varepsilon^{-\frac{1}{2H}} s, \varepsilon^{-\frac{1}{2H}} t)$ we get

$$\begin{aligned} &\varepsilon^{1-\frac{3}{4H}} I_{2q-1}(f_{2q-1,\varepsilon}) \\ &\qquad \stackrel{Law}{=} (-1)^q \varepsilon^{\frac{1}{4H}} \int_{\varepsilon^{-\frac{1}{2H}} \mathscr{R}} \beta_q \left(1 + (t-s)^{2H}\right)^{-\frac{1}{2}-q} I_{2q-1}\left(\mathbb{1}_{[s,t]}^{\otimes(2q-1)} \right) \mathrm{d}s\mathrm{d}t \\ &\qquad = \varepsilon^{\frac{1}{4H}} \int_{\varepsilon^{-\frac{1}{2H}} \mathscr{R}} I_{2q-1}\left(f_{2q-1,1,s,t} \right) \mathrm{d}s\mathrm{d}t. \end{aligned} \tag{3.3.6}$$

Changing the coordinates (s,t) by $(s, u := t-s)$ in (3.3.6), and defining $N := \varepsilon^{-\frac{1}{2H}}$, we obtain

$$\varepsilon^{1-\frac{3}{4H}} I_{2q-1}(f_{2q-1,\varepsilon}) \stackrel{Law}{=} \frac{1}{\sqrt{N}} \int_0^{NT} \int_0^{NT-s} I_{2q-1}\left(f_{2q-1,1,s,s+u} \right) \mathrm{d}u\mathrm{d}s. \tag{3.3.7}$$

From (3.3.7) it follows that the convergence (3.3.5) is equivalent to

$$\frac{1}{\sqrt{N}} \int_0^{NT} \int_0^{NT-s} I_{2q-1}\left(f_{2q-1,1,s,s+u} \right) \mathrm{d}u\mathrm{d}s \stackrel{Law}{\to} \mathscr{N}(0, \sigma_q^2), \qquad \text{as} \quad N \to \infty. \tag{3.3.8}$$

The proof of (3.3.8) will be done in several steps.

Step I

Define the random variable

$$Y_N := \frac{1}{\sqrt{N}} \int_0^{NT} \int_{NT-s}^{\infty} I_{2q-1}\left(f_{2q-1,1,s,s+u} \right) \mathrm{d}u\mathrm{d}s.$$

First we show that Y_N converges to zero in L^2 as $N \to \infty$. Notice that

$$\mathbb{E}\left[Y_N^2\right] = \frac{2}{N}\int_0^{NT}\int_0^{NT}\int_{NT-s_2}^{\infty}\int_{NT-s_1}^{\infty}\mathbb{1}_{s_1\leq s_2} \tag{3.3.9}$$
$$\times\,\mathbb{E}\left[I_{2q-1}\left(f_{2q-1,1,s_1,s_1+u_1}\right)I_{2q-1}\left(f_{2q-1,1,s_2,s_2+u_2}\right)\right]\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}s_1\mathrm{d}s_2$$
$$= \frac{2(2q-1)!}{N}\int_0^{NT}\int_0^{NT}\int_{NT-s_2}^{\infty}\int_{NT-s_1}^{\infty}\mathbb{1}_{s_1\leq s_2}$$
$$\times\left\langle f_{2q-1,1,s_1,s_1+u_1}, f_{2q-1,1,s_2,s_2+u_2}\right\rangle_{\mathfrak{H}^{\otimes(2q-1)}}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}s_1\mathrm{d}s_2. \tag{3.3.10}$$

Define the function $G_{1,x}^{(q)}(v_1,v_2)$, $x,v_1,v_2\geq 0$, as in (3.1.20). Substituting equation (3.1.19) in (3.3.10), and changing the order of integration, we get

$$\mathbb{E}\left[Y_N^2\right] = \frac{2(2q-1)!\beta_q^2}{N}\int_0^{\infty}\int_0^{\infty}\int_{0\vee(NT-u_2)}^{NT}\int_{0\vee(NT-u_1)}^{NT}\mathbb{1}_{s_1\leq s_2}$$
$$\times\, G_{1,s_2-s_1}^{(q)}(u_1,u_2)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}u_1\mathrm{d}u_2. \tag{3.3.11}$$

Changing the coordinates (s_1,s_2,u_1,u_2) by $(s_1, x := s_2-s_1, u_1, u_2)$ in the right hand side of (3.3.11), we get

$$\mathbb{E}\left[Y_N^2\right] \leq \frac{2(2q-1)!\beta_q^2}{N}\int_{\mathbb{R}_+^3}\int_{0\vee(NT-u_1)}^{NT} G_{1,x}^{(q)}(u_1,u_2)\mathrm{d}s_1\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2,$$

and then integrating the s_1 variable,

$$\mathbb{E}\left[Y_N^2\right] \leq 2(2q-1)!\beta_q^2\int_{\mathbb{R}_+^3}\left(T-\frac{0\vee(NT-u_1)}{N}\right)G_{1,x}^{(q)}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2. \tag{3.3.12}$$

The integrand in (3.3.12) converges to zero pointwise, and is dominated by the function

$$2(2q-1)!\beta_q^2 T G_{1,x}^{(q)}(u_1,u_2).$$

By condition $H < \frac{4q-3}{4q-2}$ and Lemma 3.4.8, the function $G^{(q)}_{1,x}(u_1,u_2)$ is integrable in $\mathbb{R}^3_+$. Hence, applying the dominated convergence theorem to (3.3.12), we obtain $\mathbb{E}\left[Y_N^2\right] \to 0$, as $N \to \infty$ as required.

Step II

Since $Y_N \to 0$ in L^2 as $N \to \infty$, to prove the convergence (3.3.8) it suffices to show that the random variable

$$J_{2q-1,N} := \frac{1}{\sqrt{N}} \int_0^{NT} \int_0^{\infty} I_{2q-1}\left(f_{2q-1,1,s,s+u}\right) \mathrm{d}u \mathrm{d}s,$$

converges in law to a Gaussian distribution with variance σ_q^2 as $N \to \infty$. For $M \in \mathbb{N}$, $M \geq 1$ fixed, consider the following Riemann sum approximation for $J_{2q-1,N}$

$$\widetilde{J}_{2q-1,M,N} := \frac{1}{2^M} \sum_{k=2}^{M2^M} \frac{1}{\sqrt{N}} \int_0^{NT} I_{2q-1}\left(f_{2q-1,1,s,s+u(k)}\right) \mathrm{d}s,$$

where $u(k) := \frac{k}{2^M}$, for $k = 2, \ldots, M2^M$. We will prove that $\widetilde{J}_{2q-1,M,N} \to J_{2q-1,N}$ in L^2 as $M \to \infty$ uniformly in $N > 1$, and $\widetilde{J}_{2q-1,M,N} \to \mathscr{N}(0, \widetilde{\sigma}^2_{q,M})$ as $N \to \infty$ for some constant $\widetilde{\sigma}^2_{q,M}$ satisfying $\widetilde{\sigma}^2_{q,M} \to \sigma_q^2$ as $M \to \infty$. The result will then follow by a standard approximation argument. We will separate the argument in the following steps.

Step III

Next we prove that $\widetilde{J}_{2q-1,M,N} \to J_{2q-1,N}$ in L^2 as $M \to \infty$ uniformly in $N > 1$, namely,

$$\lim_{M\to\infty} \sup_{N>1} \left\| J_{2q-1,N} - \widetilde{J}_{2q-1,M,N} \right\|_{L^2} = 0. \tag{3.3.13}$$

For $M \in \mathbb{N}$ fixed, we decompose the term $J_{2q-1,N}$ as

$$J_{2q-1,N} = J^{(1)}_{2q-1,M,N} + J^{(2)}_{2q-1,M,N}, \tag{3.3.14}$$

where

$$J^{(1)}_{2q-1,M,N} := \frac{1}{\sqrt{N}} \int_0^{NT} \int_{2^{-M}}^{M} I_{2q-1}\left(f_{2q-1,1,s,s+u}\right) \mathrm{d}u\mathrm{d}s$$

and

$$J^{(2)}_{2q-1,M,N} := \frac{1}{\sqrt{N}} \int_0^{NT} \int_0^{\infty} \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u) I_{2q-1}\left(f_{2q-1,1,s,s+u}\right) \mathrm{d}u\mathrm{d}s.$$

From (3.3.14) we deduce that relation (3.3.13) is equivalent to

$$\lim_{M\to\infty} \sup_{N>1} \left\| J^{(1)}_{2q-1,M,N} - \widetilde{J}_{2q-1,M,N} \right\|_{L^2} = 0, \tag{3.3.15}$$

provided that

$$\lim_{M\to\infty} \sup_{N>1} \left\| J^{(2)}_{2q-1,M,N} \right\|_{L^2} = 0. \tag{3.3.16}$$

To prove (3.3.16) we proceed as follows. First we write

$$\begin{aligned} \left\| J^{(2)}_{2q-1,M,N} \right\|^2_{L^2} = \frac{2(2q-1)!}{N} \int_{\mathbb{R}^2_+} \int_{[0,NT]^2} \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_1) \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_2) \\ \times \mathbb{1}_{\{s_1 \le s_2\}} \left\langle f_{2q-1,1,s_1,s_1+u_1}, f_{2q-1,1,s_2,s_2+u_2} \right\rangle_{\mathfrak{H}^{\otimes(2q-1)}} \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}u_1\mathrm{d}u_2. \end{aligned} \tag{3.3.17}$$

Let $G^{(q)}_{1,x}(v_1,v_2)$, $x,v_1,v_2\in\mathbb{R}_+$ be defined by (3.1.20). Applying identity (3.1.19) in (3.3.17), and then changing the coordinates (s_1,s_2,u_1,u_2) by $(s_1,x:=s_2-s_1,u_1,u_2)$ in (3.3.17), we get

$$\left\|J^{(2)}_{2q-1,M,N}\right\|^2_{L^2}\leq\frac{2(2q-1)!\beta_q^2}{N}\int_{\mathbb{R}^3_+}\int_0^{NT}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_1)$$
$$\times\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_2)G^{(q)}_{1,x}(u_1,u_2)\mathrm{d}s_1\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2. \tag{3.3.18}$$

Integrating the variable s_1 in (3.3.18) we obtain

$$\left\|J^{(2)}_{2q-1,M,N}\right\|^2_{L^2}\leq 2T(2q-1)!\beta_q^2\int_{\mathbb{R}^3_+}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_2)$$
$$\times\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_2)G^{(q)}_{1,x}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2. \tag{3.3.19}$$

The integrand is dominated by the function $2(2q-1)!\beta_q^2TG^{(q)}_{1,x}(u_1,u_2)$, which is integrable by the condition $H<\frac{2q-3}{4q-2}$, and Lemma 3.4.8. Hence, applying the dominated convergence theorem to (3.3.19), we get (3.3.16).

To prove (3.3.15) we proceed as follows. For $k=2,\dots,M2^M$ define the interval $I_k:=\left(\frac{k-1}{2^M},\frac{k}{2^M}\right]$. Notice that $J^{(1)}_{2q-1,M,N}$ and $\widetilde{J}_{2q-1,M,N}$ can be written, respectively, as

$$J^{(1)}_{2q-1,M,N}=\frac{1}{\sqrt{N}}\int_0^{NT}\int_{\mathbb{R}_+}\sum_{k=2}^{M2^M}I_{2q-1}\left(f_{2q-1,1,s,s+u}\right)\mathbb{1}_{I_k}(u)\mathrm{d}u\mathrm{d}s, \tag{3.3.20}$$

and

$$\widetilde{J}_{2q-1,M,N}=\frac{1}{\sqrt{N}}\int_0^{NT}\int_{\mathbb{R}_+}\sum_{k=2}^{M2^M}I_{2q-1}\left(f_{2q-1,1,s,s+u(k)}\right)\mathbb{1}_{I_k}(u)\mathrm{d}u\mathrm{d}s. \tag{3.3.21}$$

Applying (3.1.19), we can prove that

$$\left\| J^{(1)}_{2q-1,M,N} - \widetilde{J}_{2q-1,M,N} \right\|^2_{L^2} = \frac{2(2q-1)!\beta_q^2}{N} \int_{\mathbb{R}^2_+} \int_{[0,NT]^2} \sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \times \mathbb{1}_{\{s_1 \le s_2\}} \Theta^{(q)}_{k_1,k_2}(s_2 - s_1, u_1, u_2) \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}u_1 \mathrm{d}u_2, \tag{3.3.22}$$

where the function $\Theta^{(q)}_{k_1,k_2}$ is defined by

$$\Theta^{(q)}_{k_1,k_2}(x, u_1, u_2) := \Big(G^{(q)}_{1,x}(u_1, u_2) - G^{(q)}_{1,x}(u(k_1), u_2) - G^{(q)}_{1,x}(u_1, u(k_2)) + G^{(q)}_{1,x}(u(k_1), u(k_2)) \Big).$$

Changing the coordinates (s_1, s_2, u_1, u_2) by $(s_1, x := s_2 - s_1, u_1, u_2)$, and then integrating the s_1 variable in (3.3.22), we obtain

$$\left\| J^{(1)}_{2q-1,M,N} - \widetilde{J}_{2q-1,M,N} \right\|^2_{L^2} = 2(2q-1)!\beta_q^2 \int_{\mathbb{R}^2_+} \int_0^{NT} \sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \times \left(T - \frac{x}{N} \right) \Theta^{(q)}_{k_1,k_2}(x, u_1, u_2) \mathrm{d}x \mathrm{d}u_1 \mathrm{d}u_2.$$

As a consequence,

$$\left\| J^{(1)}_{2q-1,M,N} - \widetilde{J}_{2q-1,M,N} \right\|^2_{L^2} \le 2(2q-1)!\beta_q^2 T \int_{\mathbb{R}^3_+} \sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \times \Theta^{(q)}_{k_1,k_2}(x, u_1, u_2) \mathrm{d}x \mathrm{d}u_1 \mathrm{d}u_2.$$

By the continuity of $G_{1,x}(u_1, u_2)$, the term

$$\sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \Theta^{(q)}_{k_1,k_2}(x, u_1, u_2)$$

converges to zero as $M \to \infty$. Next we prove that this term is dominated by an integrable function. Let $u_1 \in I_{k_1}, u_2 \in I_{k_2}$ be fixed. Notice that $u_i, u(k_i) \leq u_i + 2^{-M} \leq u_i + 1$ for $i = 1, 2$. Hence, applying Lemma 3.1.1, we deduce that the terms $G^{(q)}_{1,x}(u_1, u_2)$, $G^{(q)}_{1,x}(u(k_1), u_2)$, $G^{(q)}_{1,x}(u_1, u(k_2))$ and $G^{(q)}_{1,x}(u(k_1), u(k_2))$ are bounded by $KG^{(q)}_{1,x}(u_1 + 1, u_2 + 1)$, for some constant $K > 0$ only depending on H and q. As a consequence,

$$\sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \Theta^{(q)}_{k_1,k_2}(x, u_1, u_2) \leq 4KG^{(q)}_{1,x}(u_1 + 1, u_2 + 1),$$

for some constant K only depending on H and q. Therefore, the right-hand side of the previous identity is integrable over $x, u_1, u_2 > 0$ due to Lemma 3.4.8, since

$$\begin{aligned}\int_{\mathbb{R}^3_+} G^{(q)}_{1,x}(u_1 + 1, u_2 + 1) \mathrm{d}x \mathrm{d}u_1 \mathrm{d}u_2 &= \int_{[1,\infty)^2} \int_{\mathbb{R}_+} G^{(q)}_{1,x}(u_1, u_2) \mathrm{d}x \mathrm{d}u_1 \mathrm{d}u_2 \\ &\leq \int_{\mathbb{R}^3_+} G^{(q)}_{1,x}(u_1, u_2) \mathrm{d}x \mathrm{d}u_1 \mathrm{d}u_2 < \infty. \qquad (3.3.23)\end{aligned}$$

This finishes the proof of (3.3.15).

Step IV

Next we prove that

$$\lim_{N \to \infty} \mathbb{E}\left[\widetilde{J}^2_{2q-1,M,N}\right] = \widetilde{\sigma}^2_{q,M}, \qquad (3.3.24)$$

where $\widetilde{\sigma}^2_{q,M}$ is the finite constant defined by

$$\widetilde{\sigma}^2_{q,M} := (2q-1)! \beta_q^2 2^{1-2M} T \sum_{k_1,k_2=2}^{M2^M} \int_0^\infty G^{(q)}_{1,x}(u(k_1), u(k_2)) \mathrm{d}x. \qquad (3.3.25)$$

In addition, we will prove that $\widetilde{\sigma}^2_{q,M}$ satisfies

$$\lim_{M\to\infty} \widetilde{\sigma}^2_{q,M} = \sigma^2_q, \tag{3.3.26}$$

where σ^2_q is defined by (3.2.5). In order to prove (3.3.24) and (3.3.26) we proceed as follows. From (3.3.21), we can prove that

$$\mathbb{E}\left[\widetilde{J}^2_{2q-1,M,N}\right] = \int_{\mathbb{R}^3_+} Q_{M,N}(x,u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2,$$

where

$$\begin{aligned} Q_{M,N}(x,u_1,u_2) := 2(2q-1)!\mathbb{1}_{[0,NT]}(x)\beta_q^2 \sum_{k_1,k_2=2}^{M2^M} \left(T - \frac{x}{N}\right) \\ \times G^{(q)}_{1,x}(u(k_1),u(k_2))\mathbb{1}_{I_{k_1}}(u_1)\mathbb{1}_{I_{k_2}}(u_2). \end{aligned}$$

Notice that $Q_{M,N}$ satisfies

$$\lim_{N\to\infty} Q_{M,N}(x,u_1,u_2) = Q_M(x,u_1,u_2), \tag{3.3.27}$$

where Q_M is defined by

$$Q_M(x,u_1,u_2) := 2(2q-1)!\beta_q^2 T \sum_{k_1,k_2=2}^{M2^M} G^{(q)}_{1,x}(u(k_1),u(k_2))\mathbb{1}_{I_{k_1}}(u_1)\mathbb{1}_{I_{k_2}}(u_2).$$

In turn, Q_M satisfies

$$\lim_{M\to\infty} Q_M(x,u_1,u_2) = Q(x,u_1,u_2), \tag{3.3.28}$$

where Q is defined by

$$Q(x,u_1,u_2) := 2(2q-1)!\beta_q^2 T G_{1,x}^{(q)}(u_1,u_2).$$

Let $x > 0$ and $2 \le k_1, k_2 \le M2^M$ be fixed, and take $u_i \in I_{k_i}$, $i = 1,2$. Since $u(k_i) \le u_i + 2^{-M} \le u_i + 1$, by Lemma 3.1.1, there exists a constant $K > 0$, only depending on q and H, such that

$$G_{1,x}^{(q)}(u(k_1),u(k_2)) \le K G_{1,x}^{(q)}(u_1+1,u_2+1),$$

As a consequence, there exists a constant K only depending on q,H and T such that

$$Q_{M,N}(x,u_1,u_2) \le K G_{1,x}^{(q)}(u_1+1,u_2+1), \tag{3.3.29}$$

and, hence,

$$Q_M(x,u_1,u_2) \le K G_{1,x}^{(q)}(u_1+1,u_2+1). \tag{3.3.30}$$

The function $G_{1,x}^{(q)}(u_1+1,u_2+1)$ is integrable with respect to the variables $x,u_1,u_2 > 0$ thanks to (3.3.23). Hence, taking into account (3.3.27) and (3.3.28), as well as the estimates (3.3.29) and (3.3.30), we can apply the dominated convergence theorem twice, to obtain

$$\begin{aligned} \lim_{M\to\infty}\lim_{N\to\infty} \mathbb{E}\left[\widetilde{J}^2_{2q-1,M,N}\right] &= \lim_{M\to\infty}\int_{\mathbb{R}^3_+} Q_M(x,u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2 \\ &= \int_{\mathbb{R}^3_+} Q(x,u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2. \end{aligned} \tag{3.3.31}$$

Equations (3.3.24) and (3.3.26) then follow from (3.3.31).

Step V

Next we prove the convergence in law of $J_{2q-1,N}$ to a Gaussian random variable with variance σ_q^2, which we will denote by $\mathscr{N}(0,\sigma_q^2)$. Let $y \in \mathbb{R}$ be fixed. Notice that

$$\begin{aligned}\left|\mathbb{P}[J_{2q-1,N} \leq y] - \mathbb{P}[\mathscr{N}\left(0,\sigma_q^2\right) \leq y]\right| \leq \sup_{N>1}\left|\mathbb{P}\left[J_{2q-1,N} \leq y\right] - \mathbb{P}\left[\widetilde{J}_{2q-1,M,N} \leq y\right]\right| \\ + \left|\mathbb{P}\left[\widetilde{J}_{2q-1,M,N} \leq y\right] - \mathbb{P}\left[\mathscr{N}(0,\widetilde{\sigma}_{q,M}^2) \leq y\right]\right| \\ + \left|\mathbb{P}\left[\mathscr{N}(0,\widetilde{\sigma}_{q,M}^2) \leq y\right] - \mathbb{P}\left[\mathscr{N}(0,\sigma_q^2) \leq y\right]\right|.\end{aligned} \tag{3.3.32}$$

Therefore, if we prove that for $M > 0$ fixed

$$\widetilde{J}_{2q-1,M,N} \overset{Law}{\to} \mathscr{N}\left(0,\widetilde{\sigma}_{q,M}^2\right) \qquad \text{as} \quad N \to \infty, \tag{3.3.33}$$

then from (3.3.32) we get

$$\begin{aligned}\limsup_{N\to\infty}\left|\mathbb{P}[J_{2q-1,N} \leq y] - \mathbb{P}[\mathscr{N}(0,\sigma_q^2) \leq y]\right| \leq \sup_{N>1}\left|\mathbb{P}\left[J_{2q-1,N} \leq y\right] - \mathbb{P}\left[\widetilde{J}_{2q-1,M,N} \leq y\right]\right| \\ + \left|\mathbb{P}\left[\mathscr{N}\left(0,\widetilde{\sigma}_{q,M}^2\right) \leq y\right] - \mathbb{P}\left[\mathscr{N}\left(0,\sigma_q^2\right) \leq y\right]\right|,\end{aligned} \tag{3.3.34}$$

and hence, from relations (3.3.13), (3.3.26) and (3.3.34), we conclude that

$$\limsup_{N\to\infty}\left|\mathbb{P}[J_{2q-1,N}^2 \leq y] - \mathbb{P}[\mathscr{N}(0,\sigma_q^2) \leq y]\right| = 0, \tag{3.3.35}$$

and the proof will then be complete. Therefore, it suffices to show (3.3.33) for M fixed. To prove this first we show that the random vector

$$Z^{(N)} = \left(Z_k^{(N)}\right)_{k=2}^{M2^M} := \left(\frac{1}{\sqrt{N}}\int_0^{NT} I_{2q-1}\left(f_{2q-1,1,s,s+u(k)}\right)\mathrm{d}s\right)_{k=2}^{M2^M}$$

converges to a multivariate Gaussian distribution. By the Peccati-Tudor criterion (see [47]), it suffices to prove that the components of the vector $Z^{(N)}$ converge to a Gaussian distribution, and the covariance matrix of $Z^{(N)}$ is convergent.

In order to prove that the covariance matrix of $Z^{(N)}$ is convergent we proceed as follows. First, for $2 \leq j,k \leq M2^M$, we write

$$\mathbb{E}\left[Z_k^{(N)}Z_j^{(N)}\right] = \frac{1}{N}\int_{[0,NT]^2}\mathbb{E}\left[I_{2q-1}\left(f_{2q-1,1,s_1,s_1+u(k)}\right)I_{2q-1}\left(f_{2q-1,1,s_2,s_2+u(j)}\right)\right]\mathrm{d}s_1\mathrm{d}s_2.$$

Then, using (3.1.19) we get

$$\mathbb{E}\left[Z_k^{(N)}Z_j^{(N)}\right] = \frac{(2q-1)!\beta_q^2}{N}\int_{[0,NT]^2} G^{(q)}_{1,s_2-s_1}(u(k),u(j))\mathrm{d}s_1\mathrm{d}s_2, \tag{3.3.36}$$

where in the last equality we used the notation $G_{1,-y}(v_1,v_2) := G_{1,y}(v_2,v_1)$, for $y,v_1,v_2 > 0$. Changing the coordinates (s_1,s_2) by $(s_1, x := s_2 - s_1)$ in relation (3.3.36) and integrating the s_1, yields

$$\mathbb{E}\left[Z_k^{(N)}Z_j^{(N)}\right] = (2q-1)!\beta_q^2\int_{-NT}^{NT}\left(T - \frac{|x|}{N}\right)G^{(q)}_{1,x}(u(k),u(j))\mathrm{d}x. \tag{3.3.37}$$

Finally, applying the monotone convergence theorem in (3.3.37), we get

$$\lim_{N\to\infty}\mathbb{E}\left[Z_k^{(N)}Z_j^{(N)}\right] = (2q-1)!\beta_q^2 T\int_{\mathbb{R}} G^{(q)}_{1,x}(u(k),u(j))\mathrm{d}x,$$

which is clearly finite. Thus, we have proved that the covariance matrix of $Z^{(N)}$ converges to the matrix $\Sigma = (\Sigma_{k,j})_{2\le k,j\le M2^M}$, where

$$\Sigma_{k,j} := T(2q-1)!\beta_q^2 \int_{\mathbb{R}} G_{1,x}^{(q)}(u(k),u(j))\mathrm{d}x.$$

Next, for $2 \le k \le M2^M$ fixed, we prove the convergence of $Z_k^{(N)}$ to a Gaussian law. By (3.1.3),

$$Z_k^{(N)} = \frac{C_{q,k}}{\sqrt{N}} \int_0^{NT} I_{2q-1}\left(\mathbb{1}_{[s,s+u_k]}^{\otimes(2q-1)}\right)\mathrm{d}s,$$

where $C_{q,k} = (-1)^q \beta_q (1+u_k^{2H})^{-\frac{1}{2}-q}$. Hence, by the self-similarity of the fractional Brownian motion we can write

$$Z_k^{(N)} \overset{Law}{=} \frac{C_{q,k}}{\sqrt{N}} \int_0^{NT} I_{2q-1}\left(\left(u_k^H N^H \mathbb{1}_{[\frac{s}{Nu_k},\frac{s}{Nu_k}+\frac{1}{N}]}\right)^{\otimes(2q-1)}\right)\mathrm{d}s. \tag{3.3.38}$$

Making the change of variables $r := \frac{s}{Nu_k}$ in the right hand side of (3.3.38), we get

$$\begin{aligned} Z_k^{(N)} &\overset{Law}{=} C_{q,k} u_k^{H(2q-1)+1} \sqrt{N} \int_0^{\frac{T}{u_k}} I_{2q-1}\left(\left(N^H \mathbb{1}_{[r,r+\frac{1}{N}]}\right)^{\otimes(2q-1)}\right)\mathrm{d}r \\ &= C_{q,k} u_k^{H(2q-1)+1} \sqrt{N} \int_0^{\frac{T}{u_k}} H_{2q-1}\left(N^H (B_{r+\frac{1}{N}} - B_r)\right)\mathrm{d}r. \end{aligned} \tag{3.3.39}$$

where H_{2q-1} denotes the Hermite polynomial of degree $2q-1$. The convergence in law of the right-hand side of (3.3.39) to a centered Gaussian distribution as $N \to \infty$ is proven in [11], equation (1.3). As a consequence, the components of $Z^{(N)}$ converge to a Gaussian random variable as $N \to \infty$. Therefore, by the Peccati-Tudor criterion, $Z^{(N)}$

converges in law to a centered Gaussian distribution with covariance Σ. Hence,

$$\widetilde{J}_{2q-1,M,N} = \frac{1}{2^{2M}} \sum_{k=2}^{M2^M} Z_k^{(N)} \overset{Law}{\to} \mathcal{N}\left(0, \frac{1}{2^{2M}} \sum_{j,k=2}^{M2^M} \Sigma_{k,j}\right) \qquad \text{as} \quad N \to \infty. \tag{3.3.40}$$

The convergence (3.3.33) follows from (3.3.40) by using the fact that

$$\frac{1}{2^{2M}} \sum_{k,j=2}^{M2^M} \Sigma_{k,j} = T(2q-1)!\beta_q^2 2^{-2M} \sum_{j,k=2}^{M2^M} \int_{\mathbb{R}} G_{1,x}^{(q)}(u(k), u(j)) \mathrm{d}x = \widetilde{\sigma}_{q,M}.$$

The proof is now complete. □

3.4 Technical lemmas

In this section we prove several technical results that were used to determine the asymptotic behavior of the variance of $I_{2q-1}\left(f_{2q-1,\varepsilon}\right)$ and α_ε. In Lemma 3.4.1 we provide an alternative expression for the terms $V_i(\varepsilon)$, $i = 1, 2, 3$ defined in (3.1.14). In Lemma 3.4.2 we prove some useful bounds that we will use later to estimate the covariance of $p_\varepsilon(B_{t_1} - B_{s_1})$ and $p_\varepsilon(B_{t_2} - B_{s_2})$, $s_1 \leq t_1$, $s_2 \leq t_2$ and $s_1 \leq s_2$. In Lemmas 3.4.3 and 3.4.4 we estimate the order of $V_1(\varepsilon)$ and $V_2(\varepsilon)$ when ε is small, while in Lemmas 3.4.6 and 3.4.7 we determine the exact behavior of $V_3(\varepsilon)$ and $V_3^{(1)}(\varepsilon)$ as $\varepsilon \to 0$. Finally, we prove Lemmas 3.4.9 and 3.4.8, which were used in Lemma 3.2.3 to determine the behavior of the variance of $I_{2q-1}\left(f_{2q-1,\varepsilon}\right)$ for $q \geq 2$.

In what follows, I will denote the identity matrix of dimension 2. In addition, for every square matrix A of dimension 2, we will denote by $|A|$ its determinant.

Lemma 3.4.1. *Let $\varepsilon > 0$ be fixed. Define $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_3$ by* (3.1.10), (3.1.11), (3.1.12) *respectively, and $V_1(\varepsilon)$, $V_2(\varepsilon)$, $V_3(\varepsilon)$ by* (3.1.14). *Then, for $i = 1,2,3$, we have*

$$V_i(\varepsilon) = \frac{1}{\pi}\int_{\mathscr{S}_i} |\varepsilon I + \Sigma|^{-\frac{3}{2}}\,\Sigma_{1,2} ds_1 ds_2 dt_1 dt_2, \tag{3.4.1}$$

where $\Sigma = (\Sigma_{i,j})_{i,j=1,2}$ is the covariance matrix of $(B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2})$.

Proof. Let (X,Y) be a jointly Gaussian vector with mean zero, covariance $\Sigma = (\Sigma_{i,j})_{i,j=1,2}$, and density $f_\Sigma(x,y)$. First we prove that for every $\theta > 0$,

$$\mathbb{E}\left[XY p_\theta(X) p_\theta(Y)\right] = (2\pi)^{-1}\theta^2\,|\theta I + \Sigma|^{-\frac{3}{2}}\,\Sigma_{1,2}. \tag{3.4.2}$$

To prove this, notice that

$$\begin{aligned}\mathbb{E}\left[XY p_\theta(X) p_\theta(Y)\right] &= \int_{\mathbb{R}^2} xy p_\theta(x) p_\theta(y) f_\Sigma(x,y)\mathrm{d}x\mathrm{d}y \\ &= (2\pi)^{-2}\theta^{-1}|\Sigma|^{-\frac{1}{2}}\int_{\mathbb{R}^2} xy\exp\left\{-\frac{1}{2}(x,y)\left(\theta^{-1}I + \Sigma^{-1}\right)(x,y)^T\right\}\mathrm{d}x\mathrm{d}y \\ &= (2\pi)^{-1}\theta^{-1}|\Sigma|^{-\frac{1}{2}}\left|\theta^{-1}I + \Sigma^{-1}\right|^{-\frac{1}{2}}\int_{\mathbb{R}^2} xy f_{\widetilde{\Sigma}}(x,y)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{3.4.3}$$

where $\widetilde{\Sigma} := \left(\theta^{-1}I + \Sigma^{-1}\right)^{-1}$ and $f_{\widetilde{\Sigma}}(x,y)$ denotes the density of a Gaussian vector with mean zero and covariance $\widetilde{\Sigma}$. Clearly, $\theta^{-1}|\Sigma|^{-\frac{1}{2}}|\theta^{-1}I + \widetilde{\Sigma}^{-1}|^{-\frac{1}{2}} = |\theta I + \Sigma|^{-\frac{1}{2}}$. Then, substituting this identity in (3.4.3), we get

$$\begin{aligned}\mathbb{E}\left[XY p_\theta(X) p_\theta(Y)\right] &= (2\pi)^{-1}\,|\theta I + \Sigma|^{-\frac{1}{2}}\int_{\mathbb{R}^2} xy f_{\widetilde{\Sigma}}(x,y)\mathrm{d}x\mathrm{d}y \\ &= (2\pi)^{-1}\,|\theta I + \Sigma|^{-\frac{1}{2}}\,\widetilde{\Sigma}_{1,2}.\end{aligned}$$

Taking into account that $\widetilde{\Sigma}_{1,2}$ is given by

$$\widetilde{\Sigma}_{1,2} = \theta^2 |\theta I + \Sigma|^{-1} \Sigma_{1,2},$$

we conclude that

$$\mathbb{E}\left[XY p_\theta(X) p_\theta(Y)\right] = (2\pi)^{-1} \theta^2 \, |\theta I + \Sigma|^{-\frac{3}{2}} \, \Sigma_{1,2},$$

as required. From (3.4.2), we can write

$$\begin{aligned}
V_i(\varepsilon) &= 2 \int_{\mathscr{S}_i} \mathbb{E}\left[p'_\varepsilon(B_{t_1} - B_{s_1}) p'_\varepsilon(B_{t_2} - B_{s_2})\right] \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}t_1 \mathrm{d}t_2 \\
&= \frac{2}{\varepsilon^2} \int_{\mathscr{S}_i} \mathbb{E}\left[(B_{t_1} - B_{s_1})(B_{t_2} - B_{s_2}) p_\varepsilon(B_{t_1} - B_{s_1}) p_\varepsilon(B_{t_2} - B_{s_2})\right] \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}t_1 \mathrm{d}t_2 \\
&= \frac{1}{\pi} \int_{\mathscr{S}_i} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \, \Sigma_{1,2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}t_1 \mathrm{d}t_2.
\end{aligned}$$

This finishes the proof of (3.4.1). □

Lemma 3.4.2. *Let $s_1, s_2, t_1, t_2 \in \mathbb{R}_+$ be such that $s_1 \leq s_2$, and $s_i \leq t_i$ for $i = 1, 2$. Denote by Σ the covariance matrix of $(B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2})$. Then, there exists a constant $0 < \delta < 1$, such that the following inequalities hold*

1. *If $s_1 < s_2 < t_1 < t_2$,*

$$|\Sigma| \geq \delta\left((a+b)^{2H} c^{2H} + (b+c)^{2H} a^{2H}\right), \tag{3.4.4}$$

   where $a := s_2 - s_1$, $b := t_1 - s_2$, and $c := t_2 - t_1$.

2. *If* $s_1 < s_2 < t_2 < t_1$,

$$|\Sigma| \geq \delta b^{2H}(a^{2H} + c^{2H}), \tag{3.4.5}$$

where $a := s_2 - s_1$, $b := t_2 - s_2$, *and* $c := t_1 - t_2$.

3. *If* $s_1 < t_1 < s_2 < t_2$,

$$|\Sigma| \geq \delta(t_1 - s_1)^{2H}(t_2 - s_2)^{2H}. \tag{3.4.6}$$

Proof. The result follows from Lemma B.1. in [29]. The inequalities (3.4.4) and (3.4.6) where also proved in Lemma 9 in [23], but the lower bound given in this lemma for the case $s_1 < s_2 < t_2 < t_1$ is not correct. □

Lemma 3.4.3. *Let* $\varepsilon > 0$ *and define* $V_1(\varepsilon)$ *by* (3.1.14). *Then, for every* $\frac{2}{3} < H < 1$ *we have*

$$\lim_{\varepsilon \to 0} \varepsilon^{3-\frac{2}{H}} V_1(\varepsilon) = 0. \tag{3.4.7}$$

Proof. Changing the coordinates (s_1, s_2, t_1, t_2) by $(s_1, a := s_2 - s_1, b := t_1 - s_2, c := t_2 - t_1)$ in (3.4.1), we get

$$V_1(\varepsilon) \leq \frac{1}{\pi} \int_{[0,T]^4} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} \mathrm{d}s_1 \mathrm{d}a \mathrm{d}b \mathrm{d}c, \tag{3.4.8}$$

where Σ denotes the covariance matrix of $(B_{a+b}, B_{a+b+c} - B_a)$, namely,

$$\Sigma_{1,1} = (a+b)^{2H}, \tag{3.4.9}$$

$$\Sigma_{2,2} = (c+b)^{2H}, \tag{3.4.10}$$

$$\Sigma_{1,2} = \frac{1}{2}((a+b+c)^{2H} + b^{2H} - c^{2H} - a^{2H}). \tag{3.4.11}$$

Integrating the s_1 variable in (3.4.8) we obtain

$$V_1(\varepsilon) \leq \frac{T}{\pi} \int_{[0,T]^3} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} \mathrm{d}a\mathrm{d}b\mathrm{d}c. \tag{3.4.12}$$

Next we bound the right-hand side of (3.4.12). Applying (3.4.4), (3.4.9), (3.4.10) and (3.4.11), we get

$$\begin{aligned} |\varepsilon I + \Sigma| &= (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 = \varepsilon^2 + \varepsilon\Sigma_{1,1} + \varepsilon\Sigma_{2,2} + |\Sigma| \\ &\geq \delta(\varepsilon^2 + \varepsilon(a+b)^{2H} + \varepsilon(b+c)^{2H} + (a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}), \end{aligned} \tag{3.4.13}$$

for some $\delta > 0$ only depending on H. Using the inequality $\Sigma_{1,2} \leq (a+b)^H (b+c)^H$, as well as (3.4.12) and (3.4.13), we deduce that there exists a constant K only depending on T, H such that

$$V_1(\varepsilon) \leq K \int_{[0,T]^3} \frac{(a+b)^H (b+c)^H}{\Theta_\varepsilon(a,b,c)^{\frac{3}{2}}} \mathrm{d}a\mathrm{d}b\mathrm{d}c, \tag{3.4.14}$$

where the function Θ_ε is defined by

$$\Theta_\varepsilon(a,b,c) := \varepsilon^2 + \varepsilon(a+b)^{2H} + \varepsilon(b+c)^{2H} + c^{2H}(a+b)^{2H} + a^{2H}(b+c)^{2H}. \tag{3.4.15}$$

By the arithmetic mean-geometric mean inequality, we have

$$\frac{1}{2}((a+b)^{2H}+(b+c)^{2H}) \geq (a+b)^H(b+c)^H,$$

and

$$\frac{1}{2}(c^{2H}(a+b)^{2H}+a^{2H}(b+c)^{2H}) \geq (a+b)^H(b+c)^H(ac)^H.$$

Consequently,

$$\Theta_\varepsilon \geq 2(a+b)^H(b+c)^H(\varepsilon+(ac)^H).$$

Therefore, by (3.4.14) there exists a constant $K>0$ only depending on T and H such that

$$\begin{aligned} V_1(\varepsilon) &\leq K\int_{[0,T]^3}(a+b)^{-\frac{H}{2}}(b+c)^{-\frac{H}{2}}(\varepsilon+(ac)^H)^{-\frac{3}{2}}\mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &\leq K\int_{[0,T]^3}b^{-H}(\varepsilon+(ac)^H)^{-\frac{3}{2}}\mathrm{d}a\mathrm{d}b\mathrm{d}c. \end{aligned} \tag{3.4.16}$$

Let $0<y<\frac{3H}{2}-1$ be fixed, and define $\gamma:=\frac{2y}{3H}+1-\frac{2}{3H}$. By the weighted arithmetic mean-geometric mean inequality, we have

$$\gamma\varepsilon+(1-\gamma)(ac)^H \geq \varepsilon^\gamma(ac)^{(1-\gamma)H}.$$

Hence, by (3.4.16), we get

$$\begin{aligned} \varepsilon^{3-\frac{2}{H}}V_1(\varepsilon) &\leq K\varepsilon^{3-\frac{2}{H}-\frac{3\gamma}{2}}\int_{[0,T]^3}b^{-H}(ac)^{-\frac{3}{2}(1-\gamma)H}\mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &= K\varepsilon^{\frac{3}{2}-\frac{1}{H}-\frac{y}{H}}\left(\int_0^T b^{-H}\mathrm{d}b\right)\left(\int_{[0,T]^2}(ac)^{-1+y}\mathrm{d}a\mathrm{d}c\right). \end{aligned}$$

This implies that (3.4.7) holds and the proof of the lemma is complete. □

Lemma 3.4.4. *Let $\varepsilon > 0$ be fixed. Define $V_2(\varepsilon)$ by* (3.1.14). *Then, for every $\frac{2}{3} < H < 1$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{3-\frac{2}{H}} V_2(\varepsilon) = 0. \tag{3.4.17}$$

Proof. Changing the coordinates (s_1, s_2, t_1, t_2) by $(s_1, a := s_2 - s_1, b := t_2 - s_2, c := t_1 - t_2)$ in (3.4.1) for $i = 2$, and integrating s_1, we obtain, as before

$$V_2(\varepsilon) \le \frac{T}{\pi} \int_{[0,T]^3} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} \mathrm{d}a \mathrm{d}b \mathrm{d}c, \tag{3.4.18}$$

where the matrix Σ is given by

$$\Sigma_{1,1} = (a+b+c)^{2H}, \tag{3.4.19}$$

$$\Sigma_{2,2} = b^{2H}, \tag{3.4.20}$$

$$\Sigma_{1,2} = \frac{1}{2}((a+b)^{2H} + (b+c)^{2H} - c^{2H} - a^{2H}). \tag{3.4.21}$$

Using relation (3.4.5) in Lemma 3.4.2, as well as (3.4.19) and (3.4.20), we get

$$\begin{aligned} |\varepsilon I + \Sigma| &= (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 = \varepsilon^2 + \varepsilon(\Sigma_{1,1} + \Sigma_{2,2}) + |\Sigma| \\ &\ge \varepsilon^2 + \varepsilon((a+b+c)^{2H} + b^{2H}) + \delta b^{2H}(a^{2H} + c^{2H}). \end{aligned} \tag{3.4.22}$$

From (3.4.18) and (3.4.22) we deduce that there exists a constant $K > 0$, only depending on T and H, such that

$$V_2(\varepsilon) \le K \int_{[0,T]^3} \frac{\Sigma_{1,2}}{(\varepsilon^2 + \varepsilon(b^{2H} + (a+b+c)^{2H}) + b^{2H}(a^{2H} + c^{2H}))^{\frac{3}{2}}} \mathrm{d}a \mathrm{d}b \mathrm{d}c. \tag{3.4.23}$$

The term $\Sigma_{1,2}$ can be written as

$$\begin{aligned}\Sigma_{1,2} &= \frac{1}{2}\left((a+b)^{2H}+(b+c)^{2H}-a^{2H}-c^{2H}\right)\\ &= Hb\int_0^1\left((a+bv)^{2H-1}+(c+bv)^{2H-1}\right)\mathrm{d}v,\end{aligned}$$

which implies

$$\Sigma_{1,2} \leq 2Hb(a+b+c)^{2H-1}. \tag{3.4.24}$$

From (3.4.23) and (3.4.24), we deduce that there exists a constant $K>0$ only depending on T and H, such that

$$V_2(\varepsilon) \leq K\int_{[0,T]^3}\Phi_\varepsilon(a,b,c)\mathrm{d}a\mathrm{d}b\mathrm{d}c, \tag{3.4.25}$$

where the function $\Phi_\varepsilon:\mathbb{R}_+^3\to\mathbb{R}_+$ is defined by

$$\Phi_\varepsilon(a,b,c) := \frac{b(a+b+c)^{2H-1}}{(\varepsilon^2+\varepsilon(b^{2H}+(a+b+c)^{2H})+b^{2H}(a^{2H}+c^{2H}))^{\frac{3}{2}}}.$$

We split the domain of integration in the right hand side of (3.4.25) as $[0,T]^3=\mathscr{C}_1\cup\mathscr{C}_2$, where the sets $\mathscr{C}_1$ and $\mathscr{C}_2$ are defined by

$$\begin{aligned}\mathscr{C}_1 &:= \{(a,b,c)\in[0,T]^3 \mid b\leq a\vee c\},\\ \mathscr{C}_2 &:= \{(a,b,c)\in[0,T]^3 \mid b\geq a\vee c\}.\end{aligned}$$

Then, to prove that $\lim_{\varepsilon\to 0}V_2(\varepsilon)=0$, it suffices to show that

$$\lim_{\varepsilon\to 0}\int_{\mathscr{C}_i}\Phi_\varepsilon(a,b,c)\mathrm{d}a\mathrm{d}b\mathrm{d}c = 0, \tag{3.4.26}$$

for $i=1,2$. First we prove (3.4.26) in the case $i=1$. Notice that for every $(a,b,c)\in\mathscr{C}_1$, it holds that $a+b+c\leq 3(a\vee c)$, which, in addition to

$$\varepsilon^2+\varepsilon(b^{2H}+(a+b+c)^{2H})+b^{2H}(a^{2H}+c^{2H})\geq \varepsilon(a\vee c)^{2H}+b^{2H}(a\vee c)^{2H},$$

leads to

$$\Phi_\varepsilon(a,b,c)\leq \frac{3^{2H-1}b(a\vee c)^{-H-1}}{(\varepsilon+b^{2H})^{\frac{3}{2}}}.$$

Therefore, by (3.4.25), we deduce that there exists a constant $K>0$ such that

$$V_2(\varepsilon)\leq K\left(\int_{[0,T]^2}(a\vee c)^{-(H+1)}\mathrm{d}a\mathrm{d}c\right)\left(\int_0^T b(\varepsilon+b^{2H})^{-\frac{3}{2}}\mathrm{d}b\right). \tag{3.4.27}$$

The term $(a\vee c)^{-(H+1)}$ is clearly integrable over the region $0\leq a,c\leq T$. To bound the integral over $[0,T]$ of $b(\varepsilon+b^{2H})^{-\frac{3}{2}}$ we proceed as follows. Define $y:=\frac{3}{2}-\frac{1}{H}$. Notice that $0<y<1$ due to the condition $\frac{2}{3}<H<1$. Therefore, by the weighted arithmetic mean-geometric mean inequality, we have

$$y\varepsilon+(1-y)b^{2H}\geq \varepsilon^y b^{2H(1-y)}. \tag{3.4.28}$$

From (3.4.27) and (3.4.28), it follows that there exists a constant $K>0$, only depending on H and T, such that

$$\begin{aligned}\varepsilon^{3-\frac{2}{H}}V_2(\varepsilon)&\leq K\varepsilon^{3-\frac{2}{H}-\frac{3y}{2}}\int_0^T b^{1-3H(1-y)}\mathrm{d}b\\&=K\varepsilon^{\frac{3}{4}-\frac{1}{2H}}\int_0^T b^{\frac{3H}{2}-2}\mathrm{d}b.\end{aligned} \tag{3.4.29}$$

The integral in the right-hand side of (3.4.29) is finite thanks to the condition $H > \frac{2}{3}$. Relation (3.4.26), for $i=1$, follows by taking limit as $\varepsilon \to 0$ in (3.4.29). To prove (3.4.26) for $i=2$ we proceed as follows. Notice that for every $(a,b,c) \in \mathscr{C}_2$, it holds $a+b+c \leq 3b$, which, in addition to

$$\varepsilon^2 + \varepsilon(b^{2H} + (a+b+c)^{2H}) + b^{2H}(a^{2H} + c^{2H}) \geq \varepsilon b^{2H} + b^{2H}(a \vee c)^{2H},$$

leads to

$$\Phi_\varepsilon(a,b,c) \leq 3^{2H-1} b^{-H}(\varepsilon + (a \vee c)^{2H})^{-\frac{3}{2}}.$$

Therefore, by (3.4.25), we deduce that there exists a constant $K > 0$ such that

$$\begin{aligned} V_2(\varepsilon) &\leq K \int_{[0,T]^3} b^{-H}(\varepsilon + (a \vee c)^{2H})^{-\frac{3}{2}} \mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &= \frac{KT^{1-H}}{1-H} \int_{[0,T]^2} (\varepsilon + (a \vee c)^{2H})^{-\frac{3}{2}} \mathrm{d}a\mathrm{d}c. \end{aligned} \tag{3.4.30}$$

To bound the integral over $[0,T]^2$ of $(\varepsilon + (a \vee c)^{2H})^{-\frac{3}{2}}$ we proceed as follows. Define $y := \frac{3}{2} - \frac{1}{H}$. Notice that $0 < y < 1$ due to the condition $\frac{2}{3} < H < 1$. Therefore, by the weighted arithmetic mean-geometric mean inequality, we have

$$y\varepsilon + (1-y)(a \vee c)^{2H} \geq \varepsilon^y (a \vee c)^{2H(1-y)}. \tag{3.4.31}$$

From (3.4.30) and (3.4.31), it follows that there exists a constant $K > 0$, only depending on H and T, such that

$$\varepsilon^{3-\frac{2}{H}} V_2(\varepsilon) \leq K \varepsilon^{3-\frac{2}{H}-\frac{3y}{2}} \int_{[0,T]^2} (a \vee c)^{-3H(1-y)} \mathrm{d}b.$$

Hence, changing the coordinates (a,c) by $(w := a \wedge c, z := a \vee c)$, we get

$$\begin{aligned}\varepsilon^{3-\frac{2}{H}} V_2(\varepsilon) &\leq 2K\varepsilon^{3-\frac{2}{H}-\frac{3y}{2}} \int_0^T z^{1-3H(1-y)} \mathrm{d}z \\ &= 2K\varepsilon^{\frac{3}{4}-\frac{1}{2H}} \int_0^T z^{\frac{3H}{2}-2} \mathrm{d}z. \qquad (3.4.32)\end{aligned}$$

The integral in the right-hand side of (3.4.32) is finite thanks to the condition $H > \frac{2}{3}$. Relation (3.4.26), for $i = 2$, follows by taking limit as $\varepsilon \to 0$ in (3.4.32). The proof is now complete. □

Lemma 3.4.5. *Let c, β, α and γ be real numbers such that c, $\beta > 0$, $\alpha > -1$ and $1 + \alpha + \gamma\beta < 0$. Then we have*

$$\int_0^\infty a^\alpha (c + a^\beta)^\gamma da = \beta^{-1} c^{\frac{\alpha+1+\beta\gamma}{\beta}} B\left(\frac{\alpha+1}{\beta}, -\frac{1+\alpha+\gamma\beta}{\beta}\right), \qquad (3.4.33)$$

where $B(\cdot,\cdot)$ denotes the Beta function.

Proof. Making the change of variables $x = a^\beta$ in the left-hand side of (3.4.33) we obtain

$$\int_0^\infty a^\alpha (c + a^\beta)^\gamma \mathrm{d}a = \beta^{-1} \int_0^\infty x^{\frac{\alpha+1-\beta}{\beta}} (c + x)^\gamma \mathrm{d}x. \qquad (3.4.34)$$

Hence, making the change of variables $a = \frac{x}{c}$ in the right hand side of (3.4.34) we get

$$\int_0^\infty a^\alpha (c + a^\beta)^\gamma \mathrm{d}a = \beta^{-1} c^{\frac{\alpha+1+\beta\gamma}{\beta}} \int_0^\infty a^{\frac{\alpha+1-\beta}{\beta}} (1 + a)^\gamma \mathrm{d}a. \qquad (3.4.35)$$

Finally, the change of variables $x = \frac{a}{1+a}$ in the right hand side of (3.4.35) leads to

$$\int_0^\infty a^\alpha (c + a^\beta)^\gamma \mathrm{d}a = \beta^{-1} c^{\frac{\alpha+1+\beta\gamma}{\beta}} \int_0^1 x^{\frac{\alpha+1-\beta}{\beta}} (1 - x)^{-\frac{\beta+1+\alpha+\gamma\beta}{\beta}} \mathrm{d}x, \qquad (3.4.36)$$

which implies the desired result. $\square$

Lemma 3.4.6. *Let $\varepsilon, T > 0$, and define $V_3(\varepsilon)$ by* (3.1.14)*. Then, for every $\frac{2}{3} < H < 1$ we have*

$$\lim_{\varepsilon \to 0} \varepsilon^{3-\frac{2}{H}} V_3(\varepsilon) = \sigma^2, \tag{3.4.37}$$

where σ^2 is given by (3.2.2)*.*

Proof. Changing the coordinates (x, u_1, u_2) by $(a := u_1, b := x - u_1, c := u_2)$ in (3.4.1) for $i = 3$, we obtain

$$V_3(\varepsilon) = \frac{1}{\pi} \int_{[0,T]^3} \mathbb{1}_{(0,T)}(a+b+c)(T-(a+b+c)) \, |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} \mathrm{d}a \mathrm{d}b \mathrm{d}c, \tag{3.4.38}$$

where the matrix Σ is given by

$$\Sigma_{1,1} = a^{2H},$$
$$\Sigma_{2,2} = c^{2H},$$
$$\Sigma_{1,2} = \frac{1}{2}((a+b+c)^{2H} + b^{2H} - (b+c)^{2H} - (a+b)^{2H}).$$

We can easily check, as before, that

$$\begin{aligned} |\varepsilon I + \Sigma| &= (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 = \varepsilon^2 + \varepsilon(\Sigma_{1,1} + \Sigma_{2,2}) + |\Sigma| \\ &= \varepsilon^2 + \varepsilon(a^{2H} + c^{2H}) + a^{2H} c^{2H} - \mu(a+b, a, c)^2, \end{aligned} \tag{3.4.39}$$

where μ is defined by (3.1.17). Changing the coordinates (a,b,c) by $(\varepsilon^{-\frac{1}{2H}}a, b, \varepsilon^{-\frac{1}{2H}}c)$ in (3.4.38) and using (3.4.39), we obtain

$$\varepsilon^{3-\frac{2}{H}}V_3(\varepsilon) = \frac{1}{\pi}\int_{\mathbb{R}_+^3} \mathbb{1}_{(0,T)}(\varepsilon^{\frac{1}{2H}}(a+c)+b)\Psi_\varepsilon(a,b,c)\mathrm{d}a\mathrm{d}b\mathrm{d}c, \tag{3.4.40}$$

where

$$\Psi_\varepsilon(a,b,c) := \frac{(T-b-\varepsilon^{\frac{1}{2H}}(a+c))\varepsilon^{-\frac{1}{H}}\mu(\varepsilon^{\frac{1}{2H}}a+b, \varepsilon^{\frac{1}{2H}}a, \varepsilon^{\frac{1}{2H}}c)}{\left(1+a^{2H}+c^{2H}+a^{2H}c^{2H}-\varepsilon^{-2}\mu(\varepsilon^{\frac{1}{2H}}a+b, \varepsilon^{\frac{1}{2H}}a, \varepsilon^{\frac{1}{2H}}c)^2\right)^{\frac{3}{2}}}.$$

The term $\mu(x+y,x,z)$ can be written as

$$\mu(x+y,x,z) = H(2H-1)xz\int_{[0,1]^2}(y+xv_1+zv_2)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2, \tag{3.4.41}$$

which implies

$$\begin{aligned}\lim_{\varepsilon\to 0}\Psi_\varepsilon(a,b,c) &= \frac{H(2H-1)(T-b)acb^{2H-2}}{(1+a^{2H}+c^{2H}+a^{2H}c^{2H})^{\frac{3}{2}}} \\ &= H(2H-1)(T-b)b^{2H-2}ac(1+a^{2H})^{-\frac{3}{2}}(1+c^{2H})^{-\frac{3}{2}}. \end{aligned}\tag{3.4.42}$$

Therefore, provided we show that $\mathbb{1}_{(0,T)}(\varepsilon^{\frac{1}{2H}}(a+c)+b)\Psi_\varepsilon(a,b,c)$ is dominated by a function integrable in $\mathbb{R}_+^3$, we obtain the following identity by applying the dominated convergence theorem in (3.4.40)

$$\lim_{\varepsilon\to 0}\varepsilon^{3-\frac{2}{H}}V_3(\varepsilon) = \frac{H(2H-1)}{\pi}\int_{\mathbb{R}_+^3}\mathbb{1}_{(0,T)}(b)(T-b)b^{2H-2}ac((1+a^{2H})(1+c^{2H}))^{-\frac{3}{2}}\mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

Making the change of variables $x=\frac{b}{T}$, and using Lemma 3.4.5 we obtain (3.4.37). Next we show that $\mathbb{1}_{(0,T)}(\varepsilon^{\frac{1}{2H}}(a+c)+b)\Psi_\varepsilon(a,b,c)$ is dominated by a function integrable in

$\mathbb{R}^3_+$. Using (3.4.41), we deduce that there exists a constant $K > 0$ only depending on T and H such that

$$\begin{aligned}\Psi_\varepsilon(a,b,c) &\le K\frac{acb^{2H-2}}{(1+a^{2H}+c^{2H}+a^{2H}c^{2H})^{\frac{3}{2}}}\\ &= Kb^{2H-2}ac(1+a^{2H})^{-\frac{3}{2}}(1+c^{2H})^{-\frac{3}{2}}.\end{aligned}$$

The right-hand side in the previous relation is integrable in $\mathbb{R}^3_+$ thanks to condition $H > \frac{2}{3}$. The proof is now complete. □

Lemma 3.4.7. *Let $T, \varepsilon > 0$ be fixed. Define $V_3^{(1)}(\varepsilon)$ by* (3.1.16)*. Then, for every $\frac{2}{3} < H < 1$ it holds*

$$\lim_{\varepsilon\to 0} \varepsilon^{3-\frac{2}{H}} V_3^{(1)}(\varepsilon) = \sigma^2, \tag{3.4.43}$$

where σ^2 is given by (3.2.2)*.*

Proof. By (3.1.16) and (3.1.19),

$$V_3^{(1)}(\varepsilon) = (2q-1)!\beta_q^2 \int_{\mathscr{S}_3} G^{(q)}_{\varepsilon, s_2-s_1}(t_1-s_1, t_2-s_2), \tag{3.4.44}$$

where $\mathscr{S}_3$ is defined by (3.1.12). Changing the coordinates (s_1,s_2,t_1,t_2) by $(a := t_1 - s_1, b := s_2 - t_1, c := t_2 - s_2)$ in (3.4.44), and using (3.1.20), we obtain

$$\begin{aligned}V_3^{(1)}(\varepsilon) \;=\; & \frac{1}{\pi}\int_{\mathbb{R}^3_+}\int_0^{T-(a+b+c)} \mathbb{1}_{(0,T)}(a+b+c)\left(\varepsilon+a^{2H}\right)^{-\frac{3}{2}}\left(\varepsilon+c^{2H}\right)^{-\frac{3}{2}}\\ & \times\mu(a+b,a,c)\mathrm{d}s_1\mathrm{d}a\mathrm{d}b\mathrm{d}c.\end{aligned} \tag{3.4.45}$$

Then, changing the coordinates (a,b,c) by $(\varepsilon^{-\frac{1}{2H}}a, b, \varepsilon^{-\frac{1}{2H}}c)$, and integrating s_1 in equation (3.4.45), we get

$$V_3^{(1)}(\varepsilon) = \frac{\varepsilon^{\frac{1}{H}-3}}{\pi}\int_{\mathbb{R}_+^3}(T-b-\varepsilon^{\frac{1}{2H}}(a+c))\mathbb{1}_{(0,\varepsilon^{-\frac{1}{2H}}(T-b))}(a+c)$$
$$\times\left(1+a^{2H}\right)^{-\frac{3}{2}}\left(1+c^{2H}\right)^{-\frac{3}{2}}\mu(\varepsilon^{\frac{1}{2H}}a+b, \varepsilon^{\frac{1}{2H}}a, \varepsilon^{\frac{1}{2H}}c)\mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

Next, using the identity

$$\mu(x+y,x,z) = H(2H-1)xz\int_{[0,1]^2}(y+xv_1+zv_2)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2,$$

we get

$$\varepsilon^{3-\frac{2}{H}}V_3^{(1)}(\varepsilon) = \frac{H(2H-1)}{\pi}\int_0^T\int_{\mathbb{R}_+^2}\int_{[0,1]^2}\mathbb{1}_{(0,\varepsilon^{-\frac{1}{2H}}(T-b))}(a+c)(T-b-\varepsilon^{\frac{1}{2H}}(a+c))$$
$$\times(1+a^{2H})^{-\frac{3}{2}}(1+c^{2H})^{-\frac{3}{2}}ac(b+\varepsilon^{\frac{1}{2H}}(av_1+cv_2))^{2H-2}\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}a\mathrm{d}c\mathrm{d}b. \tag{3.4.46}$$

Notice that the argument of the integral in the right-hand side of (3.4.46) is dominated by the function

$$\Theta(a,b,c,v_1,v_2) := \frac{TH(2H-1)}{\pi}(1+a^{2H})^{-\frac{3}{2}}(1+c^{2H})^{-\frac{3}{2}}acb^{2H-2}.$$

The integral $\int_0^T\int_{\mathbb{R}_+^2}\int_{[0,1]^2}\Theta(a,b,c,v_1,v_2)\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}a\mathrm{d}c\mathrm{d}b$ is finite thanks to condition $H > \frac{2}{3}$. Therefore, applying the dominated convergence theorem to (3.4.46), we get

$$\lim_{\varepsilon\to 0}\varepsilon^{3-\frac{2}{H}}V_3^{(1)}(\varepsilon) = \frac{H(2H-1)}{\pi}\int_0^T\int_{\mathbb{R}_+^2}(T-b)(1+a^{2H})^{-\frac{3}{2}}(1+c^{2H})^{-\frac{3}{2}}acb^{2H-2}\mathrm{d}a\mathrm{d}c\mathrm{d}b$$
$$= \frac{H(2H-1)}{\pi}\left(\int_0^T(T-b)b^{2H-2}\mathrm{d}b\right)\left(\int_0^\infty a(1+a^{2H})^{-\frac{3}{2}}\mathrm{d}a\right)^2.$$

Making the change of variables $x = \frac{b}{T}$, and using Lemma 3.4.5 we obtain (3.4.43). $\square$

Lemma 3.4.8. *Let $T > 0$ and $q \in \mathbb{N}$, $q \geq 2$ be fixed. Define $G^{(q)}_{1,x}(u_1,u_2)$ by* (3.1.20). *Then, for every $\frac{3}{4} < H < \frac{4q-3}{4q-2}$, it holds that*

$$\int_{\mathbb{R}^3_+} G^{(q)}_{1,x}(u_1,u_2)dxdu_1du_2 < \infty. \tag{3.4.47}$$

Proof. Let $T > 0$, and $q \in \mathbb{N}$ be fixed, and define the sets

$$\begin{aligned}
\mathscr{T}_1 &:= \{(x,u_1,u_2) \in \mathbb{R}^3_+ \mid u_1 - x \geq 0,\ x + u_2 - u_1 \geq 0\},\\
\mathscr{T}_2 &:= \{(x,u_1,u_2) \in \mathbb{R}^3_+ \mid u_1 - x - u_2 \geq 0\},\\
\mathscr{T}_3 &:= \{(x,u_1,u_2) \in \mathbb{R}^3_+ \mid x - u_1 \geq 0\}.
\end{aligned}$$

Since $\mathbb{R}^3_+ = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$, it suffices to prove that $G^{(q)}_{1,x}(u_1,u_2)$ is integrable in $\mathscr{T}_i$, for $i = 1,2,3$.

To prove the integrability of $G^{(q)}_{1,x}(u_1,u_2)$ in $\mathscr{T}_1$ we change the coordinates (x,u_1,u_2) by $(a := x, b := u_1 - x, c := x + u_2 - u_1)$. Then,

$$\int_{\mathscr{T}_1} G^{(q)}_{1,x}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2 = \int_{\mathbb{R}^3_+} G^{(q)}_{1,a}(a+b,b+c)\mathrm{d}a\mathrm{d}b\mathrm{d}c. \tag{3.4.48}$$

Next we prove that the right hand of (3.4.48) is finite. Notice that

$$G^{(q)}_{1,a}(a+b,b+c) = (1+(a+b)^{2H})^{-\frac{1}{2}-q}(1+(b+c)^{2H})^{-\frac{1}{2}-q}\mu(a,a+b,b+c)^{2q-1}.$$

By the Cauchy-Schwarz inequality, we get

$$\mu(a,a+b,b+c) \leq (a+b)^H(b+c)^H \leq \sqrt{(1+(a+b)^{2H})(1+(b+c)^{2H})},$$

and consequently,

$$G_{1,a}^{(q)}(a+b,b+c) \leq (1+(a+b)^{2H})^{-1}(1+(b+c)^{2H})^{-1}.$$

Hence, using the inequalities $\frac{2}{3}a+\frac{1}{3}b \geq a^{\frac{2}{3}}b^{\frac{1}{3}}$ and $\frac{2}{3}c+\frac{1}{3}b \geq c^{\frac{2}{3}}b^{\frac{1}{3}}$, we deduce that there exists a constant K only depending on T and H such that the following bounds hold

$$\begin{aligned}
G_{1,a}^{(q)}(a+b,b+c) &\leq K(abc)^{-\frac{4H}{3}} && \text{if} \quad a,b,c \geq 1,\\
G_{1,a}^{(q)}(a+b,b+c) &\leq K(1+b^{2H})^{-1}(1+c^{2H})^{-1} && \text{if} \quad a \leq 1,\\
G_{1,a}^{(q)}(a+b,b+c) &\leq K(1+b^{2H})^{-1}(1+a^{2H})^{-1} && \text{if} \quad c \leq 1,\\
G_{1,a}^{(q)}(a+b,b+c) &\leq K(1+a^{2H})^{-1}(1+c^{2H})^{-1} && \text{if} \quad b \leq 1.
\end{aligned}$$

Using the previous bounds, as well as condition $H > \frac{3}{4}$, we deduce that $G_{1,a}^{(q)}(a+b,b+c)$ is integrable in the variables $a,b,c \geq 0$.

To prove the integrability of $G_{1,x}^{(q)}(u_1,u_2)$ in $\mathscr{T}_2$ we change the coordinates (x,u_1,u_2) by $(a := x, b := u_2, c := u_1 - x - u_2)$. Then,

$$\int_{\mathscr{T}_2} G_{1,x}^{(q)}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2 = \int_{\mathbb{R}_+^3} G_{1,a}^{(q)}(a+b+c,b)\mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

Next we prove that $G_{1,a}^{(q)}(a+b+c,b)$ is integrable in the variables $a,b,c \geq 0$. Notice that

$$G_{1,a}^{(q)}(a+b+c,b) = (1+(a+b+c)^{2H})^{-\frac{1}{2}-q}(1+b^{2H})^{-\frac{1}{2}-q}\mu(a,a+b+c,b)^{2q-1}. \tag{3.4.49}$$

Using inequality

$$\mu(a,a+b+c,b) \leq (a+b+c)^H b^H \leq \sqrt{(1+(a+b+c)^{2H})(1+b^{2H})}, \qquad (3.4.50)$$

as well as the condition $q \geq 2$, we obtain

$$\begin{aligned}\mu(a,a+b+c,b)^{2q-1} &= \mu(a,a+b+c,b)^3 \mu(a,a+b+c,b)^{2(q-2)} \\ &\leq \mu(a,a+b+c,b)^3 (1+(a+b+c)^{2H})^{q-2}(1+b^{2H})^{q-2},\end{aligned}$$

which, by (3.4.49), leads to

$$\begin{aligned}G_{1,a}^{(q)}(a+b+c,b) &\leq (1+(a+b+c)^{2H})^{-\frac{5}{2}}(1+b^{2H})^{-\frac{5}{2}}\mu(a,a+b+c,b)^3 \\ &\leq (1\vee a\vee b\vee c)^{-5H}(1\vee b)^{-5H}\mu(a,a+b+c,b)^3. \qquad (3.4.51)\end{aligned}$$

Similarly, by (3.4.50),

$$\mu(a,a+b+c,b)^{2q-1} \leq (1+(a+b+c)^{2H})^{q-\frac{1}{2}}(1+b^{2H})^{q-\frac{1}{2}},$$

which, by (3.4.49), leads to

$$\begin{aligned}G_{1,a}^{(q)}(a+b+c,b) &\leq (1+(a+b+c)^{2H})^{-1}(1+b^{2H})^{-1} \\ &\leq (1\vee a\vee b\vee c)^{-2H}(1\vee b)^{-2H}. \qquad (3.4.52)\end{aligned}$$

In addition, using the representation

$$\begin{aligned}\mu(a,a+b+c,b) &= \frac{1}{2}\left((a+b)^{2H}+(b+c)^{2H}-a^{2H}-c^{2H}\right)\\ &= Hb\int_0^1 \left((a+bu)^{2H-1}+(c+bu)^{2H-1}\right)\mathrm{d}u,\end{aligned}$$

we deduce that there exist constants K,K' only depending on H such that

$$\begin{aligned}\mu(a,a+b+c,b)\mathbb{1}_{(0,a\wedge c)}(b) &\leq K\mathbb{1}_{(0,a\wedge c)}(b)b((a+b)^{2H-1}+(c+b)^{2H-1})\\ &\leq K'\mathbb{1}_{(0,a\wedge c)}(b)b(a\vee c)^{2H-1}\\ &\leq K'(1\vee b)(1\vee a\vee c)^{2H-1}. \end{aligned} \tag{3.4.53}$$

Combining the inequalities (3.4.51) and (3.4.53), we deduce that there exists a constant $K>0$ such that

$$\begin{aligned}G_{1,a}^{(q)}(a+b+c,b)\mathbb{1}_{(0,a\wedge c)}(b) &\leq K\mathbb{1}_{(0,a\wedge c)}(b)(1\vee a\vee b\vee c)^{-5H}(1\vee b)^{-5H+3}(1\vee a\vee c)^{6H-3}\\ &\leq K(1\vee a\vee c)^{H-3}(1\vee b)^{-5H+3}.\end{aligned}$$

Using the previous inequality, as well as the condition $H>\frac{3}{4}$, we deduce that $G_{1,a}^{(q)}(a+b+c,b)$ is integrable in $\{(a,b,c)\in\mathbb{R}_+^3 \mid b\leq a\wedge c\}$. In addition, from (3.4.52) we obtain

$$G_{1,a}^{(q)}(a+b+c,b)\mathbb{1}_{(0,b\wedge c)}(a)\leq(1\vee b)^{-2H}(1\vee b\vee c)^{-2H}.$$

Therefore, using condition $H>\frac{3}{4}$, we deduce that $G_{1,a}^{(q)}(a+b+c,b)$ is integrable in $\{(a,b,c)\in\mathbb{R}_+^3 \mid a\leq b\wedge c\}$. By symmetry $G_{1,a}^{(q)}(a+b+c,b)$ is integrable in $\{(a,b,c)\in$

$\mathbb{R}^3_+ \mid c \leq a \wedge b\}$. From the previous analysis we conclude that $G^{(q)}_{1,x}(u_1,u_2)$ is integrable in $\mathscr{T}_2$.

To prove the integrability of $G^{(q)}_{1,x}(u_1,u_2)$ in $\mathscr{T}_3$, we change the coordinates (x,u_1,u_2) by $(a := u_1, b := x-u_1, c := u_2)$. Then,

$$\int_{\mathscr{T}_3} G^{(q)}_{1,x}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2 = \int_{\mathbb{R}^3_+} G^{(q)}_{1,a+b}(a,c)\mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

To bound $G^{(q)}_{1,a+b}(a,c)$ we proceed as follows. We first notice that

$$G^{(q)}_{1,a+b}(a,c) = (1+a^{2H})^{-\frac{1}{2}-q}(1+c^{2H})^{-\frac{1}{2}-q}\mu(a+b,a,c)^{2q-1}.$$

Hence, using inequality $\mu(a+b,a,c) \leq a^H c^H \leq \sqrt{(1+a^{2H})(1+c^{2H})}$, we deduce that

$$G^{(q)}_{1,a+b}(a,c) \leq \left(1+a^{2H}\right)^{-1}\left(1+c^{2H}\right)^{-1} \leq (1\vee a)^{-2H}(1\vee c)^{-2H}. \tag{3.4.54}$$

As a consequence, $G^{(q)}_{1,a+b}(a,c)$ is integrable in $\{(a,b,c)\in\mathbb{R}^3_+ \mid b \leq a\wedge c\}$. In addition, from relation

$$\mu(x+y,x,z) = H(2H-1)xz\int_{[0,1]^2}(y+xv_1+zv_2)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2, \tag{3.4.55}$$

we can prove that

$$\mu(x+y,x,z) \leq H(2H-1)xzy^{2H-2}. \tag{3.4.56}$$

Using (3.4.56), we deduce that there exists a constant $K > 0$, only depending on H and q, such that

$$\begin{aligned} G^{(q)}_{1,a+b}(a,c) &\leq K\left(\left(1+a^{2H}\right)\left(1+c^{2H}\right)\right)^{-\frac{1}{2}-q}(ac)^{2q-1}b^{2(2q-1)(H-1)} \\ &\leq K\left((1\vee a)(1\vee c)\right)^{-H-2qH+2q-1}b^{2(2q-1)(H-1)}. \end{aligned}$$

Taking into account that $H < \frac{4q-3}{4q-2}$, we get $2(2q-1)(H-1) < -1$, and hence

$$\begin{aligned} \int_{1\vee a\vee c}^{\infty} G^{(q)}_{1,a+b}(a,c)\mathrm{d}b &\leq K\left((1\vee a)(1\vee c)\right)^{-H-2qH+2q-1}(1\vee a\vee c)^{2(2q-1)(H-1)+1} \\ &\leq K(1\vee a)^{-2H+\frac{1}{2}}(1\vee c)^{-2H+\frac{1}{2}}, \end{aligned} \tag{3.4.57}$$

where in the last inequality we used the relation

$$(1\vee a\vee c)^{2(2q-1)(H-1)+1} \leq (1\vee a)^{(2q-1)(H-1)+\frac{1}{2}}(1\vee c)^{(2q-1)(H-1)+\frac{1}{2}}.$$

Using relation (3.4.57) as well as condition $H > \frac{3}{4}$, we conclude that $G^{(q)}_{1,a+b}(a,c)$ is integrable in $\{(a,b,c)\in\mathbb{R}^3_+ \mid 1\vee a\vee c\leq b\}$. In addition, from (3.4.55) we obtain

$$\begin{aligned} \mu(x+y,x,z) &\leq H(2H-1)xz\int_{[0,1]^2}(xv_1+zv_2)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2 \\ &\leq H(2H-1)xz\int_0^1((x\vee z)w)^{2H-2}\mathrm{d}w \\ &= Hxz(x\vee z)^{2H-2} = H(x\wedge z)(x\vee z)^{2H-1}. \end{aligned}$$

Hence, there exist constants $K, \widetilde{K} \geq 0$ such that

$$\begin{aligned} &G^{(q)}_{1,a+b}(a,c)\mathbb{1}_{(a\wedge c, a\vee c)}(b) \\ &\qquad = \left(\left(1+a^{2H}\right)\left(1+c^{2H}\right)\right)^{-\frac{1}{2}-q}\mu(a+b,a,c)^{2q-1} \\ &\qquad \leq K\left((1\vee a)(1\vee c)\right)^{-H-2qH}(a\wedge c)^{2q-1}(a\vee c)^{(2q-1)(2H-1)} \\ &\qquad \leq K\left((1\vee a)(1\vee c)\right)^{-H-2qH}(1\vee(a\wedge c))^{2q-1}(1\vee a\vee c)^{(2q-1)(2H-1)} \\ &\qquad = K(1\vee(a\wedge c))^{-H(2q+1)+2q-1}(1\vee a\vee c)^{-3H-2q+2qH+1}. \end{aligned} \tag{3.4.58}$$

Using relation (3.4.58) as well as condition $H > \frac{3}{4}$, we obtain that $G^{(q)}_{1,a+b}(a,c)$ is integrable in the region $\{(a,b,c) \in \mathbb{R}^3_+ \mid a\wedge c \leq b \leq a \vee c\}$. From the previous analysis we conclude that $G^{(q)}_{1,a+b}(a,c)$ is integrable in the variables $a,b,c \geq 0$, which in turn implies that $G^{(q)}_{1,x}(u_1,u_2)$ is integrable in $\mathscr{T}_3$ as required. □

Lemma 3.4.9. *Let $T > 0$ and $q \in \mathbb{N}$, $q \geq 2$ be fixed, and define $G^{(q)}_{0,x}(u_1,u_2)$ by* (3.1.20). *Then, for every $\frac{2}{3} < H < \frac{3}{4}$, we have*

$$\int_{[0,T]^3} G^{(q)}_{0,x}(u_1,u_2)dxdu_1du_2 < \infty.$$

Proof. Let $T > 0$, and $q \in \mathbb{N}$, and define the sets

$$\begin{aligned} \widetilde{\mathscr{T}_1} &:= \{(x,u_1,u_2) \in [0,T]^3 \mid u_1 - x \geq 0,\ x+u_2-u_1 \geq 0\}, \\ \widetilde{\mathscr{T}_2} &:= \{(x,u_1,u_2) \in [0,T]^3 \mid u_1 - x - u_2 \geq 0\}, \\ \widetilde{\mathscr{T}_3} &:= \{(x,u_1,u_2) \in [0,T]^3 \mid x - u_1 \geq 0\}. \end{aligned}$$

Since $[0,T]^3 = \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2} \cup \widetilde{\mathscr{T}_3}$, it suffices to check the integrability of $G^{(q)}_{0,x}(u_1,u_2)$ in $\widetilde{\mathscr{T}_i}$, for $i = 1,2,3$. To prove integrability in $\widetilde{\mathscr{T}_1}$ change the coordinates (x,u_1,u_2) by

$(a := x, b := u_1 - x, c := x + u_2 - u_1)$. Then,

$$\int_{\widetilde{\mathscr{T}_1}} G^{(q)}_{0,x}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2 \le \int_{[0,T]^3} G^{(q)}_{0,a}(a+b,b+c)\mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

By the inequality $\mu(a, a+b, b+c) \le (a+b)^H (b+c)^H$, we can write

$$G^{(q)}_{0,a}(a+b,b+c) \le (a+b)^{-2H}(b+c)^{-2H}. \tag{3.4.59}$$

Therefore, using $\frac{2a}{3} + \frac{b}{3} \ge a^{\frac{2}{3}} b^{\frac{1}{3}}$ and $\frac{2c}{3} + \frac{b}{3} \ge c^{\frac{2}{3}} b^{\frac{1}{3}}$, as well as (3.4.59), we deduce that there exists a universal constant K such that

$$G^{(q)}_{0,a}(a+b,b+c) \le K(abc)^{-\frac{4H}{3}}.$$

The right hand side in the previous inequality is integrable in $[0,T]^3$ thanks to the condition $H < \frac{3}{4}$. Therefore, $G^{(q)}_{0,x}(u_1,u_2)$ is integrable in $\widetilde{\mathscr{T}_1}$.

To prove the integrability of $G^{(q)}_{0,x}(u_1,u_2)$ in $\widetilde{\mathscr{T}_2}$ we change the coordinates (x, u_1, u_2) by $(a := x, b := u_2, c := u_1 - x - u_2)$. Then,

$$\int_{\widetilde{\mathscr{T}_2}} G^{(q)}_{0,x}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2 \le \int_{[0,T]^3} G^{(q)}_{0,a}(a+b+c,b)\mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

In order to bound the term $G^{(q)}_{0,a}(a+b+c,b)$ we proceed as follows. Applying the inequality $\mu(a, a+b+c, b) \le (a+b+c)^H b^H$, as well as the condition $q \ge 2$, we obtain

$$\begin{aligned} G^{(q)}_{0,a}(a+b+c,b) &= (a+b+c)^{-5H} b^{-5H} \mu(a, a+b+c, b)^3 \\ &\quad \times \left(\frac{\mu(b, a+b+c, b)}{b^H (a+b+b)^H} \right)^{2(q-2)} \\ &\le (a+b+c)^{-5H} b^{-5H} \mu(a, a+b+c, b)^3. \end{aligned} \tag{3.4.60}$$

On the other hand, by the relation

$$\mu(a,a+b+c,b) = \frac{1}{2}\left((a+b)^{2H} + (b+c)^{2H} - a^{2H} - c^{2H}\right)$$
$$= Hb\int_0^1 \left((a+bw)^{2H-1} + (c+bw)^{2H-1}\right) \mathrm{d}w,$$

we deduce that there exists a constant $K > 0$ such that

$$\mu(a,a+b+c,b)\mathbb{1}_{(0,a\wedge c)}(b) \leq \mathbb{1}_{(0,a\wedge c)}(b)Hb\int_0^1 \left((a+bw)^{2H-1} + (c+bw)^{2H-1}\right) \mathrm{d}w$$
$$= Kb(a\vee c)^{2H-1}. \tag{3.4.61}$$

Using (3.4.60) and (3.4.61) we get

$$G_{0,a}^{(q)}(a+b+c,b)\mathbb{1}_{(0,a\wedge c)}(b) \leq Kb^{-5H+3}(a+b+c)^{-5H}(a\vee c)^{6H-3}$$
$$\leq Kb^{-5H+3}(a\vee c)^{H-3}. \tag{3.4.62}$$

From (3.4.62) as well as the condition $H < \frac{3}{4}$, we deduce that $G_{0,a}^{(q)}(a+b+c,b)$ is integrable in $\{(a,b,c) \in [0,T]^3 \mid b \leq a\wedge c\}$. In addition, using the relation $\mu(a,a+b+c,b) \leq (a+b+c)^H b^H$, we can prove that

$$G_{0,a}^{(q)}(a+b+c,b) \leq b^{-2H}c^{-2H}.$$

Therefore, by the condition $H < \frac{3}{4}$, we deduce that $G_{0,a}^{(q)}(a+b+c,b)$ is integrable in $\{(a,b,c) \in [0,T]^3 \mid a \leq b\wedge c\}$. Similarly, we can prove that

$$G_{0,a}^{(q)}(a+b+c,b) \leq b^{-2H}a^{-2H},$$

and hence, since $H < \frac{3}{4}$ we conclude that $G_{0,a}^{(q)}(a+b+c,b)$ is integrable in $\{(a,b,c) \in [0,T]^3 \mid c \leq b \wedge a\}$. From the analysis we conclude that $G_{0,a}^{(q)}(a+b+c,b)$ is integrable in $[0,T]^3$.

To prove the integrability of $G_{0,x}^{(q)}(u_1,u_2)$ in $\widetilde{\mathscr{T}_3}$ we change the coordinates (x,u_1,u_2) by $(a := u_1, b := x-u_1, c := u_2)$ to get

$$\int_{\widetilde{\mathscr{T}_3}} G_{0,x}^{(q)}(u_1,u_2)\mathrm{d}x\mathrm{d}u_1\mathrm{d}u_2 \leq \int_{[0,T]^3} G_{0,a+b}^{(q)}(a,c)\mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

In order to bound the term $G_{0,a+b}^{(q)}(a,c)$ we proceed as follows. From relation

$$\mu(x+y,x,z) = H(2H-1)xz\int_{[0,1]^2}(y+xv_1+zv_2)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2, \tag{3.4.63}$$

we can deduce that

$$\mu(x+y,x,z) \leq H(2H-1)xzy^{2H-2}.$$

Hence, since

$$G_{0,a+b}^{(q)}(a,c) = a^{-H-2qH}c^{-H-2qH}\mu(a+b,a,c)^{2q-1}, \tag{3.4.64}$$

we deduce that there exists a constant $K>0$ only depending on H such that

$$G_{0,a+b}^{(q)}(a,c)\mathbb{1}_{(a\vee c,T)}(b) \leq a^{-H-2qH+2q-1}c^{-H-2qH+2q-1}b^{2(2q-1)(H-1)}\mathbb{1}_{(a\vee c,T)}(b). \tag{3.4.65}$$

Since $q \geq 2$, we have that $H < \frac{3}{4} < \frac{4}{5} \leq \frac{2q}{1+2q}$. As a consequence, from (3.4.65) we deduce that $G_{0,a+b}^{(q)}(a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \geq a,c\}$. In addition, by

(3.4.63) we get

$$\mu(x+y,x,z) \leq H(2H-1)xz\int_{[0,1]^2}((x\vee z)w_1)^{2H-2}\mathrm{d}w_1\mathrm{d}w_2$$
$$= Hxz(x\vee z)^{2H-2} = H(x\wedge z)(x\vee z)^{2H-1}.$$

Therefore,

$$G^{(q)}_{0,a+b}(a,c)\mathbb{1}_{(a\wedge c,a\vee c)}(b)$$
$$\leq (a\wedge c)^{-H(2q+1)+2q-1}(a\vee c)^{-3H-2q+2qH+1}\mathbb{1}_{(a\wedge c,a\vee c)}(b). \qquad (3.4.66)$$

From (3.4.66), and $H < \frac{3}{4} < \frac{4}{5} \leq \frac{2q}{1+2q}$, it follows that $G^{(q)}_{0,a+b}(a,c)$ is integrable in $\{(a,b,c)\in[0,T]^3 \mid a\wedge c \leq b \leq a\vee c\}$. Finally, by inequalities $\mu \leq a^H c^H$ and (3.4.64), we get

$$G^{(q)}_{0,a+b}(a,c)\mathbb{1}_{(0,a\wedge c)}(b) \leq a^{-2H}c^{-2H}. \qquad (3.4.67)$$

Using (3.4.67) as well as condition $H < \frac{3}{4}$, we deduce that $G^{(q)}_{0,a+b}(a,c)$ is integrable in $\{(a,b,c)\in[0,T]^3 \mid b \leq a\wedge c\}$. From the previous analysis it follows that $G^{(q)}_{0,x}(u_1,u_2)$ is integrable in $\widetilde{\mathscr{T}}$ as required. $\square$

Chapter 4

Symmetric stochastic integrals with respect to a class of self-similar Gaussian processes.

Consider a centered self-similar Gaussian process $X := \{X_t\}_{t\geq 0}$ with self-similarity exponent $\beta \in (0,1)$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. That is, X is a centered Gaussian process such that $\{c^{-\beta} X_{ct}\}_{t\geq 0}$ has the same law as X, for every $c > 0$. We also assume that $X_0 = 0$. The covariance of X is characterized by the values of the function $\phi : [1,\infty) \to \mathbb{R}$, defined by

$$\phi(x) := \mathbb{E}\left[X_1 X_x\right]. \tag{4.0.1}$$

Indeed, for $0 < s \leq t$,

$$R(s,t) := \mathbb{E}\left[X_s X_t\right] = s^{2\beta} \phi(t/s). \tag{4.0.2}$$

The idea of describing a self-similar Guassian process in terms of the function ϕ was first used by Harnett and Nualart in [18], and the concept was further developed in [20].

The purpose of this paper is to study the behavior as $n \to \infty$ of ν-*symmetric Riemann sums* with respect to X, defined by

$$S_n^{\nu}(g,t) := \sum_{j=0}^{\lfloor nt \rfloor - 1} \int_0^1 g(X_{\frac{j}{n}} + y\Delta X_{\frac{j}{n}})\Delta X_{\frac{j}{n}} \nu(dy), \tag{4.0.3}$$

where $\Delta X_{\frac{j}{n}} = X_{\frac{j+1}{n}} - X_{\frac{j}{n}}$, $g : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function and ν is a symmetric probability measure on $[0,1]$, meaning that $\nu(A) = \nu(1-A)$ for any Borel set $A \subset [0,1]$.

The best known self-similar centered Gaussian process is the fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$, whose covariance is given by

$$R(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{4.0.4}$$

The ν-symmetric Riemann sums $S_n^{\nu}(g,t)$ given in (4.0.3) were investigated in the seminal paper by Gradinaru, Nourdin, Russo and Vallois [15], when X is a fBm with Hurst parameter H. In this case, if g is a function of the form $g = f'$ with $f \in \mathscr{C}^{4\ell(\nu)+2}(\mathbb{R})$ and $\ell = \ell(\nu) \geq 1$ denotes the largest integer such that

$$\int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j+1}, \quad \text{for} \quad j = 1, \ldots, \ell - 1,$$

then, provided that $H > \frac{1}{4\ell+2}$, there exists a random variable $\int_0^t g(X_s)d^{\nu}X_s$ such that

$$S_n^{\nu}(g,t) \xrightarrow{\mathbb{P}} \int_0^t g(X_s)d^{\nu}X_s \qquad \text{as } n \to \infty.$$

The limit in the right-hand side is called the ν-*symmetric integral* of g with respect to X, and satisfies the chain rule

$$f(X_t) = f(0) + \int_0^t f'(X_s) d^\nu X_s.$$

The results from [15] provided a method for constructing Stratonovich-type integrals in the rough-path case where $H < 1/2$. Some well-known examples of measures ν and their corresponding ν-symmetric Riemann sums are:

1. Trapezoidal rule ($\ell = 1$): $\nu = \frac{1}{2}(\delta_0 + \delta_1)$,
2. Simpson's rule ($\ell = 2$): $\nu = \frac{1}{6}(\delta_0 + 4\delta_{1/2} + \delta_1)$,
3. Milne's rule ($\ell = 3$): $\nu = \frac{1}{90}(7\delta_0 + 32\delta_{1/4} + 12\delta_{1/2} + 32\delta_{3/4} + 7\delta_1)$,

where δ_x is the Dirac function. For example, if $\nu = \frac{1}{2}(\delta_0 + \delta_1)$, then (4.0.3) is the sum

$$S_n^\nu(g,t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{g(X_{\frac{j}{n}}) + g(X_{\frac{j+1}{n}})}{2} \Delta X_{\frac{j}{n}},$$

which is the standard Trapezoidal rule from elementary Calculus. If X is fBm with Hurst parameter $H > \frac{1}{6}$, then the Trapezoidal rule sum converges in probability as n tends to infinity (see [9, 15]), but in general the limit does not exist if $H \leq \frac{1}{6}$.

More generally, it is known that $S_n^\nu(g,t)$ does not necessarily converge in probability if $H \leq \frac{1}{4\ell+2}$. Nevertheless, in certain instances of the case $H = \frac{1}{4\ell+2}$, it has been found that $S_n^\nu(g,t)$ converges in law to a random variable with a conditional Gaussian distribution. Cases $\ell = 1$ and $\ell = 2$ were studied in [41] and [19], respectively. More recently, Binotto, Nourdin and Nualart have obtained the following general result for $H = \frac{1}{4\ell+2}$:

Theorem 4.0.1 ([5]). *Assume X is a fBm of Hurst parameter $H = \frac{1}{4\ell+4}$. Consider a function $f \in \mathscr{C}^{20\ell+5}(\mathbb{R})$ such that f and its derivatives up to the order $20\ell+5$ have moderate growth (they are bounded by $Ae^{B|x|^{\alpha}}$, with $\alpha < 2$). Then,*

$$S_n^{\nu}(f',t) \xrightarrow{\mathscr{L}} f(X_t) - f(0) - c_{\nu} \int_0^t f^{(2\ell+1)}(X_s)dW_s \quad \text{as } n \to \infty, \tag{4.0.5}$$

where c_{ν} is some positive constant, W is a Brownian motion independent of X and the convergence holds in the topology of the Skorohod space $\mathbf{D}[0,\infty)$.

The previous convergence can be written as the following change of variables formula in law:

$$f(X_t) = f(0) + \int_0^t f'(X_s)d^{\nu}X_s + c_{\nu} \int_0^t f^{(2\ell+1)}(X_s)dW_s.$$

When extending these results to self-similar processes, surprisingly the critical value is not the scaling parameter β but the *increment exponent* α which controls the variance of the increments of X and is defined below.

Definition 4.0.1. *We say that α is the increment exponent for X if for any $0 < \varepsilon < T < \infty$ there are positive constants $0 < c_1 \leq c_2$ and $\delta > 0$, such that*

$$c_1 s^{\alpha} \leq \mathbb{E}\left[(X_{t+s} - X_t)^2\right] \leq c_2 s^{\alpha}, \tag{4.0.6}$$

for every $t \in [\varepsilon, T]$ and $s \in [0, \delta)$.

The extension of stochastic integration to nonstationary Gaussian processes has been studied in the papers [53, 17, 18]. Each of these papers considered critical values of α, for which particular ν-symmetric Riemann sums $S_n^{\nu}(g,t)$ converge in distribution (but not necessarily in probability) to a limit which has a Gaussian distribution given the process X. For the fBm, $\alpha = 2H$ and the critical value for α coincides with $H = \frac{1}{4\ell+2}$. Papers [53, 18] were both based on the Midpoint integral, and show that the corresponding critical value is $\alpha = \frac{1}{2}$. Because of the structure of the measure ν, the Midpoint rule integral is not covered in our present paper. Harnett and Nualart considered in [17] a Trapezoidal integral with $\alpha = \frac{1}{3}$ and the results in this paper can be expressed as a special case of Theorem 4.1.2 below.

4.1 Main results

Our goal for this paper is to extend the results of [5] and [15] to a general class of self-similar Gaussian processes X, and a wider class of functions g. In the particular case where X is a fBm, we extend Theorem 4.0.1 to the class of functions f with continuous derivatives up to order $8\ell+2$. The idea of the proof is similar to the one presented in [5], but there are technical challenges that arise because in general X is not a stationary process.

Our analysis of the asymptotic distribution of $S_n^{\nu}(f',t)$ relies heavily on a central limit theorem for the odd variations of X, which we establish in Theorem 4.1.1. The study of the fluctuations of the variations of X has an interest on its own, and has been extensively studied for the case where X is a fBm (see for instance [40] and [10]). Nevertheless, Theorem 4.1.1 is the first one to prove a result of this type for an extended class of self-similar Gaussian process that are not necessarily stationary.

For most of the stochastic processes that we consider, such as the fBm and its variants, the self-similarity exponent β and the increment exponent α satisfy $\alpha = 2\beta$, but there are examples where $\alpha < 2\beta$. In the sequel, we will assume that the parameters α

and β satisfy $0<\alpha<1$, $\beta\leq 1/2$ and $\alpha\leq 2\beta$. Following [20], we assume as well that the function ϕ introduced in (4.0.1), satisfies the following conditions:

(H.1) ϕ is twice continuously differentiable in $(1,\infty)$ and for some $\lambda>0$ and $\alpha\in(0,1)$, the function

$$\psi(x)=\phi(x)+\lambda(x-1)^{\alpha} \tag{4.1.1}$$

has a bounded derivative in $(1,2]$.

(H.2) There are constants $C_1, C_2>0$ and $1<\nu\leq 2$ such that

$$|\phi''(x)|\leq C_1\mathbb{1}_{(1,2]}(x)(x-1)^{\alpha-2}+C_2\mathbb{1}_{(2,\infty)}(x)x^{-\nu-1}. \tag{4.1.2}$$

Although the formulation is slightly different, these hypotheses are equivalent to conditions (H.1) and (H.2) in [20], with the restrictions $\alpha<1$ and $2\beta\leq 1$. In particular, they imply that

$$|\phi'(x)|\leq C_1'\mathbb{1}_{(1,2]}(x)(x-1)^{\alpha-1}+C_2'\mathbb{1}_{(2,\infty)}(x)x^{-\nu}, \tag{4.1.3}$$

for some constants C_1' and C_2'. Notice that by Lemma 4.5.1 in the Appendix, Hypothesis (H.1) implies that α is the increment exponent of X. Moreover the upper bound in (4.0.6) holds for any $t\in[0,T]$.

The following are examples of self-similar processes satisfying the above hypotheses (see [20]):

(i) *Fractional Brownian motion*. This is a centered Gaussian process with covariance function given by (4.0.4). Here (H.1) and (H.2) hold if $H<\frac{1}{2}$. In this case,

$$\phi(x)=\frac{1}{2}(1+x^{2H}-(x-1)^{2H}),$$

$\alpha = 2\beta = 2H$ and $\nu = 2 - 2H$.

(ii) *Bifractional Brownian motion*. This is a generalization of the fBm, with covariance given by

$$R(s,t) = \frac{1}{2^K}\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right)$$

for constants $H \in (0,1)$ and $K \in (0,1]$. See [21, 33, 51] for properties, and note that $K = 1$ gives the classic fBm case. Here (H.1) and (H.2) hold if $HK < 1$. For this process we have

$$\phi(x) = \frac{1}{2^K}\left((1 + x^{2H})^K - (x-1)^{2HK}\right)$$

with $\lambda = 2^{-K}$, $\alpha = 2\beta = 2HK$ and $\nu = (2 + 2H - 2HK) \wedge (3 - 2HK) - 1$.

(iii) *Subfractional Brownian motion*. This Gaussian process has been studied in [6, 8] and it has a covariance given by

$$R(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left((s+t)^{2H} + |s-t|^{2H}\right),$$

with parameter $H \in (0,1)$. Here (H.1) and (H.2) hold if $H < \frac{1}{2}$, in which case $\lambda = 1/2$, $\alpha = 2\beta = 2H$, and

$$\phi(x) = 1 + x^{2H} - \frac{1}{2}\left((x+1)^{2H} + (x-1)^{2H}\right).$$

(iv) *Two processes in a recent paper by Durieu and Wang*. For $0 < \alpha < 1$, we consider the centered Gaussian processes $Z_1(t),\ Z_2(t)$, with covariances given by:

$$\mathbb{E}\left[Z_1(s)Z_1(t)\right] = \Gamma(1-\alpha)\left((s+t)^{\alpha} - \max(s,t)^{\alpha}\right)$$
$$\mathbb{E}\left[Z_2(s)Z_2(t)\right] = \Gamma(1-\alpha)\left(s^{\alpha} + t^{\alpha} - (s+t)^{\alpha}\right),$$

where $\Gamma(y)$ denotes the Gamma function. These processes are discussed in a recent paper by Durieu and Wang [13], where it is shown that the process $Z = Z_1 + Z_2$ (where $Z_1,\ Z_2$ are independent) is the limit in law of a discrete process studied by Karlin. The process Z_2, with a different scaling constant, was first described in Lei and Nualart [33]. The corresponding functions ϕ of these self-similar processes are:

$$\phi_1(x) = -\Gamma(1-\alpha)(x-1)^{\alpha} + \Gamma(1-\alpha)\left((x-1)^{\alpha} + (x+1)^{\alpha} - x^{\alpha}\right)$$

and

$$\begin{aligned}\phi_2(x) &= \Gamma(1-\alpha)(1+x^{\alpha} - (x+1)^{\alpha}) \\ &= -\Gamma(1-\alpha)(x-1)^{\alpha} + \Gamma(1-\alpha)\left(1 + x^{\alpha} + (x-1)^{\alpha} - (x+1)^{\alpha}\right).\end{aligned}$$

It is shown in [20] that both ϕ_1 and ϕ_2 satisfy (H.1) and (H.2), with $2\beta = \alpha$ and $\nu = 2 - \alpha$.

(v) *Gaussian process in a paper by Swanson*. This process was introduced in [52], and arises as the limit of normalized empirical quantiles of a system of independent Brownian motions. The covariance is given by

$$R(s,t) = \sqrt{st}\sin^{-1}\left(\frac{s\wedge t}{\sqrt{st}}\right),$$

and the corresponding function ϕ is given by

$$\phi(x) = \sqrt{x}\sin^{-1}\left(\frac{1}{\sqrt{x}}\right).$$

This process has $\alpha = \beta = 1/2$ and $\nu = 2$, so is an example of the case $\alpha < 2\beta$.

It is interesting to remark the differences on the asymptotic behavior of both the power variations and the ν-symmetric integrals of X, depending on whether $\alpha = 2\beta$ or $\alpha < 2\beta$. As we show in Theorem 4.1.1, the process of variations of X satisfies an asymptotic nonstationarity property when $\alpha < 2\beta$, which differs from the case $\alpha = 2\beta$, where the limit process is a scalar multiple of a Brownian motion. To better describe this phenomena, we denote by $Y = \{Y_t\}_{t\geq 0}$ a continuous centered Gaussian process independent of X, with covariance function

$$\mathbb{E}[Y_sY_t] = \Sigma(s,t) := (t\wedge s)^{\frac{2\beta}{\alpha}}, \tag{4.1.4}$$

defined on an enlarged probability space $(\Omega, \mathscr{G}, \mathbb{P})$. The process Y is characterized by the property of independent increments, and

$$\mathbb{E}\left[(Y_{t+s} - Y_s)^2\right] = t^{\frac{2\beta}{\alpha}} - s^{\frac{2\beta}{\alpha}} \quad \text{for} \quad 0 \leq s \leq t.$$

Notice that for $\alpha < 2\beta$, the increments of Y are not stationary and when $\alpha = 2\beta$, Y is a standard Brownian motion. We need the following definition of stable convergence.

Definition 4.1.1. *Assume ξ_n is a sequence random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$ with values on a complete and separable metric space S and ξ is an S-valued random variable defined on the enlarged probability space $(\Omega, \mathscr{G}, \mathbb{P})$. We say that ξ_n* converges stably *to ξ as $n \to \infty$, if for any continuous and bounded function $f : S \to \mathbb{R}$ and any $\mathbb{R}$-valued, $\mathscr{F}$-measurable bounded random variable M, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[f(\xi_n)M\right] = \mathbb{E}\left[f(\xi)M\right].$$

Next we present a central limit theorem for the odd power variations of X, which is a key ingredient for proving Theorem 4.1.2 and illustrates the asymptotic nonstationarity property that we mentioned before.

Theorem 4.1.1. *Fix an integer $\ell \geq 1$. Define the functional*

$$V_n(t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \Delta X_{\frac{j}{n}}^{2\ell+1}, \quad t \geq 0. \tag{4.1.5}$$

If $\alpha = \frac{1}{2\ell+1}$ and the process X satisfies (H.1) and (H.2), then for every $0 \leq t_1, \ldots, t_m < \infty$, $m \geq 1$, the vector $(V_n(t_1), \ldots, V_n(t_m))$ converges stably to $\sigma_\ell(Y_{t_1}, \ldots, Y_{t_m})$, where

$$\sigma_\ell^2 = \frac{\alpha}{2\beta} \sum_{r=0}^{\ell-1} K_{r,\ell} \sum_{p\in\mathbb{Z}} \left(|p+1|^\alpha + |p-1|^\alpha - 2|p|^\alpha\right)^{2(\ell-r)+1}, \tag{4.1.6}$$

and $K_{r,\ell} = c_{r,\ell}^2 2^{2r} \lambda^{2\ell+1} (2(\ell-r)+1)!$, where λ is the constant appearing in Hypothesis (H.1) and $c_{r,\ell}$ are the coefficients introduced in (1.2.1).

Our main results are Theorems 4.1.2 and 4.1.3 below.

Theorem 4.1.2. *Assume $f \in \mathscr{C}^{8\ell+2}(\mathbb{R})$. For a given symmetric probability measure ν and associated integer $\ell(\nu)$, assume the process X satisfies (H.1) and (H.2) with*

$2\beta \geq \alpha = \frac{1}{2\ell+1}$. *Then, as n tends to infinity,*

$$\{S_n^\nu(f',t)\}_{t\geq 0} \overset{Stably}{\longrightarrow} \{f(X_t) - f(0) - \kappa_{\nu,\ell}\sigma_\ell \int_0^t f^{(2\ell+1)}(X_s)\, dY_s\}_{t\geq 0},$$

where σ_ℓ *and* $\kappa_{\nu,\ell}$ *are the constants given by* (4.1.6) *and* (4.4.2)*, respectively, and the convergence is in the Skorohod space* $\mathbf{D}[0,\infty)$*. Consequently, we have the Itô-like formula in law*

$$f(X_t) \overset{\mathscr{L}}{=} f(0) + \int_0^t f'(X_s)\, d^\nu X_s + \kappa_{\nu,\ell}\sigma_\ell \int_0^t f^{(2\ell+1)}(X_s)\, dY_s.$$

The proof of Theorem 4.1.2 follows the same path as the proof of Theorem 1.1 of Binotto, Nourdin and Nualart [5], but there are technical challenges that arise because in general X is not stationary. The next generalization of the result in [15] easily follows from the proof of Theorem 4.1.2.

Theorem 4.1.3. *Under the assumptions of Theorem 4.1.2, if* $\alpha > \frac{1}{2\ell+1}$*, then the* ν*-symmetric integral* $\int_0^t f'(X_s)\, d^\nu X_s$ *exists as the limit in probability of the* ν*-symmetric Riemann sums* $S_n^\nu(f',t)$ *and for all* $t \geq 0$*, we have*

$$f(X_t) = f(0) + \int_0^t f'(X_s)\, d^\nu X_s.$$

The important new developments compared to previous work are:

- A system for constructing stochastic integrals with respect to rough-path processes, originally developed in [5, 15, 19, 41] for the fBm, is now extended to a wider class of processes that are not necessarily stationary.
- We prove a central limit theorem for the power variations of general self-similar Gaussian processes.

- We present a more efficient proof of tightness, which allows for less restrictions on the integrand function f compared with [5].

The chapter is organized as follows. In Section 4.3 we prove the convergence of the variations of the process X. Section 4.4 is devoted to the proofs of Theorems 4.1.2 and Theorem 4.1.3. Finally, in Section 4.5 we prove some technical lemmas.

4.2 Notation

For $n \geq 2$ we consider the discretization of $[0,\infty)$ by the points $\{\frac{j}{n}, j \geq 0\}$. For $t \geq 0$, $j \geq 0$ and $n \geq 2$, we define:

$$\varepsilon_t = \mathbb{1}_{[0,t)}, \quad \widetilde{\varepsilon}_{\frac{j}{n}} = \frac{1}{2}\left(\varepsilon_{\frac{j}{n}} + \varepsilon_{\frac{j+1}{n}}\right) \text{ and } \partial_{\frac{j}{n}} = \mathbb{1}_{[\frac{j}{n}, \frac{j+1}{n})}.$$

For the process X, we introduce the notation:

$$\Delta X_{\frac{t}{n}} = X_{\frac{t+1}{n}} - X_{\frac{t}{n}}; \quad \widetilde{X}_{\frac{t}{n}} = \frac{1}{2}\left(X_{\frac{t+1}{n}} + X_{\frac{t}{n}}\right) \text{ and } \xi_{t,n} = \|\Delta X_{\frac{t}{n}}\|_{L^2(\Omega)}.$$

When not otherwise defined, the symbol C denotes a generic positive constant, which may change from line to line. The value of C may depend on the parameters of the process X and the length of the time interval $[0,t]$ or $[0,T]$ we are considering.

4.3 Asymptotic behavior of the power variations

This section is devoted to the proof of Theorem 4.1.1. Define $V_n(t)$ by (4.1.5) and recall that $\alpha = \frac{1}{2\ell+1}$. By the Hermite polynomial expansion of $x^{2\ell+1}$ (see (1.2.1)), we can write

$$\frac{\Delta X_{\frac{j}{n}}^{2\ell+1}}{\xi_{j,n}^{2\ell+1}} = \sum_{r=0}^{\ell} c_{r,\ell}\, H_{2(\ell-r)+1}\left(\frac{\Delta X_{\frac{j}{n}}}{\xi_{j,n}}\right) = \sum_{r=0}^{\ell} c_{r,\ell}\, I_{2(\ell-r)+1}\left(\frac{\partial_{\frac{j}{n}}^{\otimes 2(\ell-r)+1}}{\xi_{j,n}^{2(\ell-r)+1}}\right),$$

where each $c_{r,\ell}$ is an integer with $c_{0,\ell} = 1$. It follows that

$$\Delta X_{\frac{j}{n}}^{2\ell+1} = \sum_{r=0}^{\ell} c_{r,\ell}\, \xi_{j,n}^{2r} I_{2(\ell-r)+1}\left(\partial_{\frac{j}{n}}^{\otimes 2(\ell-r)+1}\right).$$

Define $q_r = 2(\ell-r)+1$ and notice that $q_\ell = 1$ and $3 = q_{\ell-1} < \cdots < q_0 = 2\ell+1$. We can write for $t \geq 0$

$$V_n(t) = \sum_{r=0}^{\ell} c_{r,\ell} V_n^r(t), \tag{4.3.1}$$

where

$$V_n^r(t) = \sum_{j=0}^{\lfloor nt \rfloor -1} \xi_{j,n}^{2r} I_{q_r}(\partial_{\frac{j}{n}}^{\otimes q_r}) = I_{q_r}(h_n^r(t)),$$

and

$$h_n^r(t) = \sum_{j=0}^{\lfloor nt \rfloor -1} \xi_{j,n}^{2r} \partial_{\frac{j}{n}}^{\otimes q_r}.$$

In the next lemma, we show that the term $V_n^\ell(t)$ does not contribute to the limit of $V_n(t)$ as n tends to infinity.

Lemma 4.3.1. *The term*

$$V_n^\ell(t) = \sum_{j=0}^{\lfloor nt \rfloor -1} \xi_{j,n}^{2\ell} I_1(\partial_{\frac{j}{n}}) = \sum_{j=0}^{\lfloor nt \rfloor -1} \xi_{j,n}^{2\ell} \Delta X_{\frac{j}{n}}$$

tends to zero in $L^2(\Omega)$ as n tends to infinity.

Proof. Recalling that $X_0 = 0$ and $\Delta X_{j/n} = X_{(j+1)/n} - X_{j/n}$, we can rewrite the sum as

$$V_n^\ell(t) = X_{\frac{\lfloor nt \rfloor}{n}} \xi_{\lfloor nt \rfloor -1,n}^{2\ell} - X_{\frac{1}{n}} \left(\xi_{1,n}^{2\ell} - \xi_{0,n}^{2\ell} \right) - \sum_{j=2}^{\lfloor nt \rfloor -1} X_{\frac{j}{n}} \left(\xi_{j,n}^{2\ell} - \xi_{j-1,n}^{2\ell} \right).$$

We have, for any integer $j \geq 1$,

$$\begin{aligned} \xi_{j,n}^2 &= \left(\frac{j+1}{n} \right)^{2\beta} \phi(1) + \left(\frac{j}{n} \right)^{2\beta} \phi(1) - 2 \left(\frac{j}{n} \right)^{2\beta} \phi(\frac{j+1}{j}) \\ &= \frac{\phi(1)}{n^{2\beta}} \left((j+1)^{2\beta} - j^{2\beta} \right) - \frac{2j^{2\beta}}{n^{2\beta}} \left(\phi(1+\frac{1}{j}) - \phi(1) \right). \end{aligned}$$

By (H.1), we can write this as

$$\xi_{j,n}^2 = \frac{2\beta\phi(1)}{n^{2\beta}} \int_0^1 (j+y)^{2\beta-1} dy - \frac{2j^{2\beta}}{n^{2\beta}} \left(-\lambda j^{-\alpha} + \psi(1+\frac{1}{j}) - \psi(1) \right) := a_n(j).$$

By the previous formula, we can extend the function a_n to all reals $x \geq 1$. Using the fact that $\psi(x)$ has a bounded derivative in $(1,2]$, we can find positive constants C, C' such that for all $x \geq 1$,

$$\left| a_n'(x) \right| \leq Cn^{-2\beta} \left(x^{2\beta-2} + x^{2\beta-\alpha-1} \right) \leq C' n^{-2\beta} x^{2\beta-\alpha-1}.$$

Hence, by (4.5.2), it follows that for integers $2 \leq j \leq \lfloor nt \rfloor$,

$$\begin{aligned} \left| a_n^\ell(j) - a_n^\ell(j-1) \right| \;\; &\leq \;\; C \sup_{2 \leq j \leq \lfloor nt \rfloor} |a_n(j)|^{\ell-1} \\ &\quad \times \int_0^1 \left| a_n'(j-1+y) \right| \, dy \leq Cn^{-(\ell-1)\alpha-2\beta} (j-1)^{2\beta-\alpha-1}. \end{aligned}$$

As a consequence, using again inequality (4.5.2), we can write

$$\mathbb{E}\left[\left(X_{\frac{\lfloor nt\rfloor}{n}}\xi^{2\ell}_{\lfloor nt\rfloor-1,n}-X_{\frac{1}{n}}\left(\xi^{2\ell}_{1,n}-\xi^{2\ell}_{0,n}\right)-\sum_{j=2}^{\lfloor nt\rfloor-1}X_{\frac{j}{n}}\left(\xi^{2\ell}_{j,n}-\xi^{2\ell}_{j-1,n}\right)\right)^2\right]^{\frac{1}{2}}$$
$$\leq Cn^{-\ell\alpha}+C\sum_{j=2}^{\lfloor nt\rfloor-1}\left|a^{\ell}_n(j)-a^{\ell}_n(j-1)\right|\leq Cn^{-\ell\alpha},$$

which tends to zero as n tends to infinity. □

Then, Theorem 4.1.1 will be a consequence of Theorem 1.2.1, if we show that the remaining terms $h^r_n(t)$, $0\leq r\leq \ell-1$, $t\geq 0$, satisfy conditions (1.2.11) and (1.2.12). This will be done in the next two lemmas.

Lemma 4.3.2. *Let* $1\leq p\leq q_r-1$ *be an integer. Then,*

$$\lim_{n\to\infty}\left\|h^r_n(t)\otimes_p h^r_n(t)\right\|^2_{\mathfrak{H}^{\otimes(2q_r-2p)}}=0.$$

Proof. We have for each $n\geq 2$

$$\left\|h^r_n(t)\otimes_p h^r_n(t)\right\|^2_{\mathfrak{H}^{\otimes(2q_r-2p)}}$$
$$=\sum_{j_1,j_2,k_1,k_2=0}^{\lfloor nt\rfloor-1}\xi^{2r}_{j_1,n}\xi^{2r}_{j_2,n}\xi^{2r}_{k_1,n}\xi^{2r}_{k_2,n}\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_2}{n}}\right\rangle^p_{\mathfrak{H}}\left\langle\partial_{\frac{k_1}{n}},\partial_{\frac{k_2}{n}}\right\rangle^p_{\mathfrak{H}}\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{k_1}{n}}\right\rangle^{q_r-p}_{\mathfrak{H}}\left\langle\partial_{\frac{j_2}{n}},\partial_{\frac{k_2}{n}}\right\rangle^{q_r-p}_{\mathfrak{H}}.$$

Note that for applicable values of q_r and p we always have $p\geq 1$ and $q_r-p\geq 1$. By (4.5.2) and Cauchy-Schwarz inequality, we have

$$\sup_{0\leq j,k\leq\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\leq Cn^{-\alpha}.$$

As a consequence,

$$\begin{aligned}\left\|h_n^r(t)\otimes_p h_n^r(t)\right\|^2_{\mathfrak{H}^{\otimes(2q_r-2p)}} &\leq \left(\sup_{0\le j\le \lfloor nt\rfloor-1}\left|\xi^{2r}_{j,n}\right|\right)^4 \sup_{0\le j,k\le\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^{2q_r-3}\\ &\quad\times\sum_{j_1,j_2,k_1,k_2=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j_1}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}\left\langle \partial_{\frac{j_1}{n}},\partial_{\frac{k_1}{n}}\right\rangle_{\mathfrak{H}}\left\langle \partial_{\frac{j_2}{n}},\partial_{\frac{k_2}{n}}\right\rangle_{\mathfrak{H}}\right|\\ &\leq Cn^{-\alpha(4r+2q_r-3)+1}\left(\sup_{0\le j\le\lfloor nt\rfloor-1}\sum_{k=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\right)^3.\end{aligned}$$

We now apply Lemma 4.5.1 and noting that $4r+2q_r=4\ell+2=\frac{2}{\alpha}$, we have up to a constant C,

$$\left\|h_n^r(t)\otimes_p h_n^r(t)\right\|^2_{\mathfrak{H}^{\otimes(2q_r-2p)}}\le Cn^{-1},$$

which tends to zero as n tends to infinity. This completes the proof of the lemma. □

In the next lemma we show that the functions h_n^r, $0\le r\le \ell-1$, satisfy condition (1.2.11) of Theorem 1.2.1, with some constants c_r to be defined below.

Lemma 4.3.3. *Under above notation, let $s,t\ge 0$. Then for each integer $0\le r\le\ell-1$,*

$$\lim_{n\to\infty}\langle h_n^r(t),h_n^r(s)\rangle_{\mathfrak{H}^{\otimes q_r}}=\frac{\alpha}{2\beta}(s\wedge t)^{\frac{2\beta}{\alpha}}2^{2r}\lambda^{2\ell+1}\sum_{m\in\mathbb{Z}}(\rho_\alpha(m))^{q_r}, \tag{4.3.2}$$

where $\rho_\alpha(m)=|m+1|^\alpha+|m-1|^\alpha-2|m|^\alpha$.

Proof. We can easily check that

$$\langle h_n^r(t),h_n^r(s)\rangle_{\mathfrak{H}^{\otimes q_r}}=\sum_{j=0}^{\lfloor nt\rfloor-1}\sum_{k=0}^{\lfloor ns\rfloor-1}G_n(j,k),$$

where the function $G_n(j,k)$ is defined by

$$G_n(j,k) = \xi_{j,n}^{2r}\xi_{k,n}^{2r}\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{q_r}. \tag{4.3.3}$$

Then the convergence (4.3.2) will be a consequence of the following two facts:

(i) For every $0 < s_1 < t_1 < s_2 < t_2$,

$$\lim_{n\to\infty} \sum_{j=\lfloor ns_1 \rfloor}^{\lfloor nt_1 \rfloor -1} \sum_{k=\lfloor ns_2 \rfloor}^{\lfloor nt_2 \rfloor -1} |G_n(j,k)| = 0. \tag{4.3.4}$$

(ii) For every $t > 0$,

$$\lim_{n\to\infty} \sum_{j,k=0}^{\lfloor nt \rfloor -1} G_n(j,k) = \frac{\alpha}{2\beta} t^{\frac{2\beta}{\alpha}} 2^{2r} \lambda^{2\ell+1} \sum_{m\in\mathbb{Z}} (\rho_\alpha(m))^{q_r}. \tag{4.3.5}$$

First we prove (4.3.4). We can assume that $n \geq 6$, $\lfloor ns_1 \rfloor \geq 1$ and $\lfloor nt_1 \rfloor + 2 < \lfloor ns_2 \rfloor$, which is true if n is large enough. This implies that $j + 3 \leq k$ for each k and j such that $\lfloor ns_1 \rfloor \leq j \leq \lfloor nt_1 \rfloor - 1$ and $\lfloor ns_2 \rfloor \leq k \leq \lfloor nt_2 \rfloor - 1$. As a consequence, applying inequalities (4.5.1) and (4.5.3), we obtain the estimate

$$\begin{aligned} \sum_{j=\lfloor ns_1 \rfloor}^{\lfloor nt_1 \rfloor -1} \sum_{k=\lfloor ns_2 \rfloor}^{(2j+2)\wedge(\lfloor nt_2 \rfloor -1)} |G_n(j,k)| &\leq C \sum_{j=\lfloor ns_1 \rfloor}^{\lfloor nt_1 \rfloor -1} \sum_{k=\lfloor ns_2 \rfloor}^{\lfloor nt_2 \rfloor -1} n^{-4\beta r} k^{(2\beta-\alpha)r} j^{(2\beta-\alpha)r} n^{-2\beta q_r} j^{(2\beta-\alpha)q_r} k^{(\alpha-2)q_r} \\ &\leq C n^{2-2(\alpha r + q_r)}, \end{aligned}$$

which converges to zero as n tends to infinity due to the fact that $\alpha > 0$ and $q_r \geq 1$. On the other hand, applying inequalities (4.5.1) and (4.5.4) we obtain the estimate

$$\sum_{j=\lfloor ns_1 \rfloor}^{\lfloor nt_1 \rfloor -1} \sum_{k=(2j+2)\vee\lfloor ns_2 \rfloor}^{\lfloor nt_2 \rfloor -1} |G_n(j,k)| \leq C \sum_{j=\lfloor ns_1 \rfloor}^{\lfloor nt_1 \rfloor -1} \sum_{k=\lfloor ns_2 \rfloor}^{\lfloor nt_2 \rfloor -1} n^{-4\beta r} j^{(2\beta-\alpha)r} k^{(2\beta-\alpha)r} n^{-2\beta q_r} j^{(2\beta+\nu-2)q_r} k^{-\nu q_r}$$
$$\leq C n^{2-2(\alpha r + q_r)}.$$

The exponent of n is the above estimate is always negative, so this term converges to zero as n tends to infinity.

Next we prove (4.3.5). We can write

$$\sum_{j,k=0}^{\lfloor nt \rfloor -1} G_n(j,k) = \sum_{x=0}^{\lfloor nt \rfloor -1} \sum_{j=0}^{\lfloor nt \rfloor -1-x} (2-\delta_{x,0}) G_n(j, j+x), \tag{4.3.6}$$

where $\delta_{x,0}$ denotes the Kronecker delta. First we will show that there exist constants $C, \delta > 0$, such that for $3 \leq x \leq \lfloor nt \rfloor - 1$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1-x} (2-\delta_{x,0}) |G_n(j, j+x)| \leq C x^{-1-\delta}. \tag{4.3.7}$$

To show (4.3.7) we consider three cases:

Case 1: For $j = 0$, we have, using (4.5.1) and (4.1.3),

$$\begin{aligned} |G(0,x)| &\leq Cn^{-4\beta r} x^{(2\beta-\alpha)r} |n^{-2\beta}(\phi(x+1)-\phi(x))|^{q_r} \leq Cn^{-2\beta(2\ell+1)} x^{(2\beta-\alpha)r-\nu q_r} \\ &\leq Cx^{-2\beta(2\ell+1)+(2\beta-\alpha)r-\nu q_r} \end{aligned} \tag{4.3.8}$$

which provides the desired estimate, because the largest value of the exponent $-2\beta(2\ell+1)+(2\beta-\alpha)r-\nu q_r$ is obtained for $r=\ell-1$, and in this case this exponent becomes

$$-2\beta(\ell+2)-(\ell-1)\alpha-3\nu \leq -\alpha(2\ell+1)-3\nu = -1-3\nu.$$

Case 2: Applying (4.5.3), yields

$$\begin{aligned}\sum_{j=x-2}^{\lfloor nt\rfloor-1-x}|G_n(j,j+x)| &\leq C\sum_{j=x-2}^{\lfloor nt\rfloor-1-x} n^{-2\beta(2\ell+1)}j^{(2\beta-\alpha)(r+q_r)}(j+x)^{(2\beta-\alpha)r+(\alpha-2)q_r}\\ &\leq C\sum_{j=x-2}^{\lfloor nt\rfloor-1-x} n^{-2\beta(2\ell+1)}(j+x)^{(2\beta-\alpha)(2\ell+1)+(\alpha-2)q_r}\\ &\leq C'\sum_{j=x-2}^{\lfloor nt\rfloor-1-x}(j+x)^{-\alpha(2\ell+1)+(\alpha-2)q_r}.\end{aligned}$$

Hence, using the bound $(j+x)^{(\alpha-2)q_r} \leq j^{(\alpha-2)(q_r-1)}x^{\alpha-2}$, and the condition $\alpha=\frac{1}{2\ell+1}$, we get

$$\sum_{j=x-2}^{\lfloor nt\rfloor-1-x}|G_n(j,j+x)| \leq Cx^{\alpha-2}\sum_{j=1}^{\infty} j^{-1+(\alpha-2)(q_r-1)}. \tag{4.3.9}$$

The sum in the right hand side is finite due to the fact that $q_r \geq 3$ and $\alpha<1$.

Case 3: By (4.5.4),

$$\sum_{j=0}^{x-2}|G_n(j,j+x)| \leq C\sum_{j=0}^{x-2} n^{-2\beta(2\ell+1)}j^{2\beta(r+q_r)-\alpha r+(\nu-2)q_r}(j+x)^{(2\beta-\alpha)r-\nu q_r}. \tag{4.3.10}$$

Notice that

$$n^{-2\beta(2\ell+1)}j^{2\beta(r+q_r)}(j+x)^{2\beta r} \leq n^{-2\beta(2\ell+1)}(j+x)^{2\beta(2\ell+1)} \leq C$$

and

$$(j+x)^{-\nu q_r} \le j^{-\nu(q_r-1)} x^{-\nu}.$$

Hence, by (4.3.10),

$$\sum_{j=0}^{x-2} |G_n(j,j+x)| \le \sum_{j=0}^{x-2} j^{-\alpha r+(\nu-2)q_r} (j+x)^{-\alpha r-\nu q_r} \le x^{-\nu} \sum_{j=0}^{x-2} j^{-\alpha r-2q_r+\nu} (j+x)^{-\alpha r}$$
$$\le C x^{-\nu} \sum_{j=0}^{x-2} j^{-2\alpha r-2q_r+\nu}. \tag{4.3.11}$$

The sum in the right hand side is finite due to the conditions $q_r \ge 3$ and $\nu \le 2$.

Relation (4.3.7) follows from (4.3.8), (4.3.9) and (4.3.11). As a consequence, provided that we prove the pointwise convergence

$$\lim_{n\to\infty} \sum_{j=0}^{\lfloor nt \rfloor -1-x} G_n(j,j+x) = \frac{\alpha}{2\beta} 2^{2\ell+1-q_r} \lambda^{2\ell+1} t^{\frac{2\beta}{\alpha}} (\rho_\alpha(x))^{q_r}, \tag{4.3.12}$$

for any $x \ge 0$, by applying the dominated convergence theorem in (4.3.6), we obtain (4.3.5). The proof of (4.3.12) will be done in three steps.

Step 1. Since $\phi(y) = -\lambda(y-1)^\alpha + \psi(y)$, for every $x \ge 1$ we can write

$$\begin{aligned}
&\mathbb{E}\left[(X_{j+1}-X_j)(X_{j+x+1}-X_{j+x})\right] \\
&\quad = (j+1)^{2\beta}(\phi(1+\frac{x}{j+1}) - \phi(1+\frac{x-1}{j+1})) + j^{2\beta}(\phi(1+\frac{x}{j}) - \phi(1+\frac{x+1}{j})) \\
&\quad = -\lambda(j+1)^{2\beta-\alpha}(x^\alpha - (x-1)^\alpha) - \lambda j^{2\beta-\alpha}(x^\alpha - (x+1)^\alpha) \\
&\quad + (j+1)^{2\beta}(\psi(1+\frac{x}{j+1}) - \psi(1+\frac{x-1}{j+1})) + j^{2\beta}(\psi(1+\frac{x}{j}) - \psi(1+\frac{x+1}{j})).
\end{aligned}$$

Hence, using the Mean Value Theorem for ψ, as well as (H.1), we deduce that for every $x \geq 1$, there exist constants γ_1 and $\gamma_2 > 0$, such that

$$\begin{aligned}
&\mathbb{E}\left[(X_{j+1}-X_j)(X_{j+x+1}-X_{j+x})\right] \\
&\qquad = -\lambda (j+1)^{2\beta-\alpha}(x^{\alpha}-(x-1)^{\alpha}) - \lambda j^{2\beta-\alpha}(x^{\alpha}-(x+1)^{\alpha}) \\
&\qquad + (j+1)^{2\beta-1}\psi'(1+\gamma_1) - j^{2\beta-1}\psi'(1+\gamma_2).
\end{aligned}$$

As a consequence, taking into account that ψ' is bounded and $\alpha < 1$,

$$\lim_{j\to\infty}(j+1)^{\alpha-2\beta}\mathbb{E}\left[(X_{j+1}-X_j)(X_{j+x+1}-X_{j+x})\right] = -\lambda(2x^{\alpha}-(x-1)^{\alpha}-(x+1)^{\alpha}). \tag{4.3.13}$$

In addition, from Lemma 4.5.1, it follows that

$$\lim_{j\to\infty}(j+1)^{\alpha-2\beta}\mathbb{E}\left[(X_{j+1}-X_j)^2\right] = \lim_{j\to\infty}(j+1)^{\alpha-2\beta}\mathbb{E}\left[(X_{j+x+1}-X_{j+x})^2\right] = 2\lambda. \tag{4.3.14}$$

Using (4.3.13) and (4.3.14), we get

$$\lim_{j\to\infty}\xi_{j,1}^{-1}\xi_{j+x,1}^{-1}\mathbb{E}\left[(X_{j+1}-X_j)(X_{j+x+1}-X_{j+x})\right] = \frac{1}{2}(|x-1|^{\alpha}+|x+1|^{\alpha}-2|x|^{\alpha}). \tag{4.3.15}$$

Notice that the previous relation is also true for $x=0$. Therefore, we deduce that for every $\varepsilon > 0$, there exists $M > 0$, such that for every $j \geq M$,

$$\left|\xi_{j,1}^{-q_r}\xi_{j+x,1}^{-q_r}\mathbb{E}\left[(X_{j+1}-X_j)(X_{j+x+1}-X_{j+x})\right]^{q_r} - 2^{-q_r}(\rho_{\alpha}(x))^{q_r}\right| < \varepsilon. \tag{4.3.16}$$

Step 2. Provided that we prove that

$$\lim_{n\to\infty} n^{-\frac{2\beta}{\alpha}} \sum_{j=0}^{\lfloor nt\rfloor-1-x} \xi_{j,1}^{2\ell+1}\xi_{j+x,1}^{2\ell+1} = \frac{\alpha}{2\beta}(2\lambda)^{2\ell+1} t^{\frac{2\beta}{\alpha}}, \tag{4.3.17}$$

taking into account the self-similarity of the process X, and the fact that $\alpha = \frac{1}{2\ell+1}$, the proof of (4.3.12) will follow from

$$\lim_{n\to\infty} \sum_{j=0}^{\lfloor nt\rfloor-1-x} \left| \xi_{j,n}^{2\ell+1-q_r}\xi_{j+x,n}^{2\ell+1-q_r} \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j+x}{n}} \right\rangle_{\mathfrak{H}}^{q_r} - 2^{-q_r}\xi_{j,n}^{2\ell+1}\xi_{j+x,n}^{2\ell+1}(\rho_\alpha(x))^{q_r} \right| = 0. \tag{4.3.18}$$

Using (4.5.2) we can easily prove that

$$\limsup_{n\to\infty} \sum_{j=0}^{M} \left| \xi_{j,n}^{2\ell+1-q_r}\xi_{j+x,n}^{2\ell+1-q_r} \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j+x}{n}} \right\rangle_{\mathfrak{H}}^{q_r} - 2^{-q_r}\xi_{j,n}^{2\ell+1}\xi_{j+x,n}^{2\ell+1}(\rho_\alpha(x))^{q_r} \right| = 0. \tag{4.3.19}$$

Applying the estimate (4.3.16) and the limit (4.3.17), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{j=M}^{\lfloor nt\rfloor-1-x} \left| \xi_{j,n}^{2\ell+1-q_r}\xi_{j+x,n}^{2\ell+1-q_r} \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j+x}{n}} \right\rangle_{\mathfrak{H}}^{q_r} - 2^{-q_r}\xi_{j,n}^{2\ell+1}\xi_{j+x,n}^{2\ell+1}(\rho_\alpha(x))^{q_r} \right| \\
&\quad = \limsup_{n\to\infty} n^{-\frac{2\beta}{\alpha}} \sum_{j=M}^{\lfloor nt\rfloor-1-x} \xi_{j,1}^{2\ell+1}\xi_{j+x,1}^{2\ell+1} \\
&\qquad \times \left| \xi_{j,1}^{-q_r}\xi_{j+x,1}^{-q_r}\mathbb{E}\left[(X_{j+1}-X_j)(X_{j+x+1}-X_{j+x})\right]^{q_r} - 2^{-q_r}(\rho_\alpha(x))^{q_r} \right| \\
&\qquad \leq \varepsilon \frac{\alpha}{2\beta}(2\lambda)^{2\ell+1} t^{\frac{2\beta}{\alpha}}.
\end{aligned} \tag{4.3.20}$$

Therefore, (4.3.19) and (4.3.20) imply (4.3.18).

Step 3. In order to prove (4.3.17) we proceed as follows. Using Lemma 4.5.1, as well as the condition $\alpha < 1$, we deduce that for every $\varepsilon > 0$, there exists $M \in \mathbb{N}$, such that for every $j \geq M$,

$$\left| (j^{-(2\beta-\alpha)}\xi_{j,1}\xi_{j+x,1})^{2\ell+1} - (2\lambda)^{2\ell+1} \right| < \varepsilon,$$

and hence, since $\alpha = (2\ell+1)^{-1}$,

$$n^{-\frac{2\beta}{\alpha}} \sum_{j=M}^{\lfloor nt \rfloor -1-x} \left| \xi_{j,1}^{2\ell+1} \xi_{j+x,1}^{2\ell+1} - (2\lambda)^{2\ell+1} j^{(2\beta-\alpha)(2\ell+1)} \right|$$

$$= n^{-\frac{2\beta}{\alpha}} \sum_{j=M}^{\lfloor nt \rfloor -1-x} j^{(2\beta-\alpha)(2\ell+1)} \left| \xi_{j,1}^{2\ell+1} \xi_{j+x,1}^{2\ell+1} j^{-(2\beta-\alpha)(2\ell+1)} - (2\lambda)^{2\ell+1} \right|$$

$$\leq \varepsilon n^{-\frac{2\beta}{\alpha}} \sum_{j=M}^{\lfloor nt \rfloor -1-x} j^{\frac{2\beta-\alpha}{\alpha}}.$$

Therefore, since

$$\lim_{n\to\infty} n^{-\frac{2\beta}{\alpha}} \sum_{j=0}^{\lfloor nt \rfloor -1-x} j^{\frac{2\beta-\alpha}{\alpha}} = \frac{\alpha}{2\beta} t^{\frac{2\beta}{\alpha}}, \tag{4.3.21}$$

we conclude that there exists a constant $C > 0$ depending on t and x, such that

$$\limsup_{n\to\infty} n^{-\frac{2\beta}{\alpha}} \sum_{j=M}^{\lfloor nt \rfloor -1-x} \left| \xi_{j,1}^{2\ell+1} \xi_{j+x,1}^{2\ell+1} - (2\lambda)^{2\ell+1} j^{((2\beta-\alpha)(2l+1)} \right| < C\varepsilon,$$

and hence, by relation (4.3.21) and condition $\alpha = (2\ell+1)^{-1}$, we conclude that

$$\lim_{n\to\infty} n^{-2\beta(2\ell+1)} \sum_{j=0}^{\lfloor nt \rfloor -1-x} \xi_{j,1}^{2\ell+1} \xi_{j+x,1}^{2\ell+1}$$

$$= (2\lambda)^{2\ell+1} \lim_{n\to\infty} n^{-2\beta(2\ell+1)} \sum_{j=0}^{\lfloor nt \rfloor -1-x} j^{(2\beta-\alpha)(2\ell+1)} = \frac{\alpha}{2\beta} (2\lambda)^{2\ell+1} t^{\frac{2\beta}{\alpha}},$$

as required. The proof of Lemma 4.3.3 is now complete. □

4.4 Asymptotic behavior of weak symmetric Riemann sums

In this section we prove the main results, Theorems 4.1.2 and 4.1.3. We follow arguments similar to those used in the proof of Theorem 1.1 of Binotto, Nourdin and Nualart [5], which was originally used in [15]. For $f \in \mathscr{C}^{8\ell+2}(\mathbb{R})$ and $a<b$, we consider the approximation (4.4.1) below, which was proved in [15, Theorem 3.6] using Taylor's formula and the properties of ν

$$\begin{aligned} f(b) &= f(a)+(b-a)\int_0^1 f'(a+y(b-a))\,\nu(dy)+\sum_{h=\ell}^{2\ell}\kappa_{\nu,h}f^{(2h+1)}\left(\frac{a+b}{2}\right)(b-a)^{2h+1} \\ &\quad +C(a,b)(b-a)^{4\ell+2}, \end{aligned} \tag{4.4.1}$$

where $C(a,b)$ is a continuous function with $C(a,a)=0$, and the $\kappa_{\nu,h}$ are the constants given in [15, Theorem 3.6]. In particular,

$$\kappa_{\nu,\ell}=\frac{1}{(2\ell)!}\left(\frac{1}{(2\ell+1)2^{2\ell}}-\int_0^1\left(y-\frac{1}{2}\right)^{2\ell}\nu(dy)\right). \tag{4.4.2}$$

Recall the notation $\widetilde{X}_{\frac{t}{n}}$ and $\Delta X_{\frac{t}{n}}$ introduced in Section 4.2. From (4.4.1), it follows that for $n \geq 2$,

$$f(X_t)-f(0)=S_n^{\nu}(f',t)+\sum_{h=\ell}^{2\ell}\sum_{j=0}^{\lfloor nt\rfloor-1}\kappa_{\nu,h}f^{(2h+1)}(\widetilde{X}_{\frac{j}{n}})(\Delta X_{\frac{j}{n}})^{2h+1}+R_n(t), \tag{4.4.3}$$

where

$$R_n(t)=\sum_{j=0}^{\lfloor nt\rfloor-1}C(X_{\frac{j}{n}},X_{\frac{j+1}{n}})(\Delta X_{\frac{j}{n}})^{4\ell+2}.$$

Then, we can write

$$f(X_t) - f(0) = S_n^{\nu}(f', t) + \sum_{h=\ell}^{2\ell} \Phi_n^h(t) + R_n(t), \tag{4.4.4}$$

where

$$\Phi_n^h(t) = \kappa_{\nu,h} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(2h+1)}(\widetilde{X}_{\frac{j}{n}})(\Delta X_{\frac{j}{n}})^{2h+1}. \tag{4.4.5}$$

The term R_n converges to zero in probability, uniformly in compact sets. Indeed, for every $T, K, \varepsilon > 0$, we can write

$$\mathbb{P}\left[\sup_{0 \le t \le T} |R_n(t)| > \varepsilon\right] \le \mathbb{P}\left[\sup_{\substack{s,t \in [0,T] \\ |t-s| \le \frac{1}{n}}} |C(X_s, X_t)| > \frac{1}{K}\right] + \mathbb{P}\left[\sum_{j=0}^{\lfloor nT \rfloor - 1} (\Delta X_{\frac{j}{n}})^{4\ell+2} > K\varepsilon\right]. \tag{4.4.6}$$

Since $\Delta X_{\frac{j}{n}}$ is a centered Gaussian variable, by (4.5.2), for all even integer r

$$\sup_{1 \le j \le \lfloor nT \rfloor - 1} \mathbb{E}\left[\left|\Delta X_{\frac{j}{n}}\right|^r\right] \le (r-1)!! \sup_{1 \le j \le \lfloor nT \rfloor - 1} \mathbb{E}\left[\left|\Delta X_{\frac{j}{n}}\right|^2\right]^{\frac{r}{2}} \le C(r-1)!! n^{-\frac{r\alpha}{2}},$$

where $(r-1)!!$ denotes the double factorial $(r-1)!! = \prod_{k=0}^{r-1}(r-1-2k)$. As a consequence, using the Chebychev inequality and the condition $\alpha = \frac{1}{2\ell+1}$, we get

$$\mathbb{P}\left[\sum_{j=0}^{\lfloor nT \rfloor - 1} (\Delta X_{\frac{j}{n}})^{4\ell+2} > K\varepsilon\right] \le \frac{C}{K\varepsilon} \frac{\lfloor nT \rfloor}{n} \le \frac{C}{K\varepsilon}. \tag{4.4.7}$$

The convergence to zero in probability, uniformly in compact sets, of $R_n(t)$ is obtained from (4.4.6) and (4.4.7), by letting first $n \to \infty$, and then $K \to \infty$.

The previous analysis shows that the term R_n appearing in right hand side (4.4.4), does not contribute to the limit as n goes to infinity, so the asymptotic behavior of $S_n^{\nu}(f',t)$ is completely determined by $\sum_{h=\ell}^{2\ell}\Phi_n^h(t)$. The study of the stochastic process $\sum_{h=\ell}^{2\ell}\Phi_n^h$ can be decomposed in the following steps: first, we reduce the problem of proving Theorems 4.1.2 and 4.1.3, to the case where f is compactly supported, by means of a localization argument. Then we prove that the processes $\Phi_n^h(t)$, with $h = \ell,\dots,2\ell$ are tight in the Skorohod topology, and only contribute to the limit as n goes to infinity, when $h=\ell$.

Finally, we determine the behavior of Φ_n^{ℓ} by splitting into the cases $\alpha = \frac{1}{2\ell+1}$ and $\alpha > \frac{1}{2\ell+1}$. In the case $\alpha > \frac{1}{2\ell+1}$, we show that $\Phi_n^{\ell} \to 0$ in probability, which proves Theorem 4.1.3. For the case $\alpha = \frac{1}{2\ell+1}$, we use the small blocks-big blocks methodology (see [5] and [10]) and Theorem 4.1.1, to prove that Φ_n^{ℓ} converges stably to $\{\kappa_{\nu,\ell}\sigma_\ell\int_0^t f^{(2\ell+1)}(X_s)dY_s\}_{t\geq 0}$, which proves Theorem 4.1.2.

We start reducing the problem of proving Theorems 4.1.2 and 4.1.3, to the case where f is compactly supported. Define the process $Z=\{Z_t\}_{t\geq 0}$, by

$$Z_t = \kappa_{\nu,\ell}\sigma_\ell\int_0^t f^{(2\ell+1)}(X_s)dY_s. \tag{4.4.8}$$

By (4.4.4), it suffices to show that for all $f\in\mathscr{C}^{8\ell+2}(\mathbb{R})$, the following claims hold:

1. If $\alpha = \frac{1}{2\ell+1}$,

$$\{\sum_{h=\ell}^{2\ell}\Phi_n^h(t)\}_{t\geq 0} \overset{stably}{\to} \{Z_t\}_{t\geq 0} \qquad \text{as } n\to\infty, \tag{4.4.9}$$

in the topology of $\mathbf{D}[0,\infty)$.

2. If $\alpha > \frac{1}{2\ell+1}$, then for every $t \geq 0$

$$\sum_{h=\ell}^{2\ell} \Phi_n^h(t) \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty. \tag{4.4.10}$$

Notice that the convergences (4.4.9) and (4.4.10) hold, provided that:

1. If $\alpha = \frac{1}{2\ell+1}$, then,

   a) For every $h = \ell, \dots, 2\ell$, the sequence Φ_n^h is tight in $\mathbf{D}[0,\infty)$.

   b) The finite dimensional distributions of Φ_n^ℓ converge stably to those of Z.

   c) For every $h = \ell+1, \dots, 2\ell$ and $t \geq 0$, the sequence $\Phi_n^h(t)$ converges to zero in probability.

2. If $\alpha > \frac{1}{2\ell+1}$, then $\Phi_n^h(t)$ converges to zero in probability for every $h = \ell, \dots, 2\ell$ and $t > 0$.

In turn, these conditions are a consequence of the following claims:

(i) For every $\varepsilon, T > 0$ and $h = \ell, \dots, 2\ell$, there is a compact set $K \subset \mathbf{D}[0,T]$, such that

$$\sup_{n \geq 1} \mathbb{P}\left[\Phi_n^h \in K^c\right] < \varepsilon.$$

(ii) For every $\varepsilon, \delta > 0$, $t \geq 0$ and $h = \ell+1, \dots, 2\ell$, there exists $N \in \mathbb{N}$, such that for every $n \geq N$,

$$\mathbb{P}\left[\left|\Phi_n^h(t)\right| > \delta\right] < \varepsilon.$$

(iii) Let $\varepsilon > 0$ and $0 \leq t_1 \leq \cdots \leq t_d \leq T$ be fixed. If $\alpha = \frac{1}{2\ell+1}$, then for every compactly supported function $\phi \in \mathscr{C}^1(\mathbb{R}^d, \mathbb{R})$, and every event $B \in \sigma(X)$, there exists $N \in \mathbb{N}$, such that for $n \geq N$,

$$\left|\mathbb{E}\left[(\phi(\Phi_n^\ell(t_1),\ldots,\Phi_n^\ell(t_d)) - \phi(Z_{t_1},\ldots,Z_{t_d}))\mathbb{1}_B\right]\right| < \varepsilon. \tag{4.4.11}$$

1. If $\alpha > \frac{1}{2\ell+1}$, then for every $\varepsilon, \delta > 0$, $t \geq 0$ there exists $N \in \mathbb{N}$, such that for every $n \geq N$,

$$\mathbb{P}\left[\left|\Phi_n^\ell(t)\right| > \delta\right] < \varepsilon.$$

Recall that Φ_n^h depends on f via (4.4.5). We claim that it suffices to show conditions (i)-(iv) for f compactly supported. Suppose that (i)-(iv) hold for every function in $\mathscr{C}^{8\ell+2}(\mathbb{R})$ with compact support, and take a general element $g \in \mathscr{C}^{8\ell+2}(\mathbb{R})$. Fix $L \geq 1$ and let $g_L : \mathbb{R} \to \mathbb{R}$ be a compactly supported function, with derivatives up to order $8\ell+2$, such that $g_L(x) = g(x)$ for every $x \in [-L, L]$, and define the processes $\widetilde{\Phi}_n^{h,L} = \{\widetilde{\Phi}_n^{h,L}(t)\}_{t\geq 0}$, $h = \ell,\ldots,2\ell$ and $\widetilde{Z}^L = \{\widetilde{Z}_t^L\}_{t\geq 0}$, by

$$\widetilde{\Phi}_n^{h,L} = k_{\nu,h} \sum_{j=0}^{\lfloor nt \rfloor - 1} g_L^{(2h+1)}(\widetilde{X}_{\frac{j}{n}})(\Delta X_{\frac{j}{n}})^{2h+1},$$

and

$$\widetilde{Z}_t^L = \kappa_{\nu,\ell}\sigma_\ell \int_0^t g_L^{(2\ell+1)}(X_s)dY_s.$$

Fix $T > 0$ and define as well the events $A_{L,T} = \{\sup_{0\leq s\leq T} |X_s| \leq L\}$. Then, for every $\varepsilon > 0$, there exists a compact set $K_L \subset \mathbf{D}[0,T]$ such that for all $h = \ell,\ldots,2\ell$

$$\sup_{n\geq 1} \mathbb{P}\left[\widetilde{\Phi}_n^{h,L} \in K_L^c\right] < \frac{\varepsilon}{2}. \tag{4.4.12}$$

Since $\Phi_n^h = \widetilde{\Phi}_n^{h,L}$ in $A_{L,T}$, we have

$$\begin{aligned}\mathbb{P}\left[\Phi_n^h \in K_L^c\right] &\leq \mathbb{P}\left[\Phi_n^h \in K_L^c, A_{L,T}\right] + \mathbb{P}\left[A_{L,T}^c\right] = \mathbb{P}\left[\widetilde{\Phi}_n^{h,L} \in K_L^c, A_{L,T}\right] + \mathbb{P}\left[A_{L,T}^c\right] \\ &\leq \frac{\varepsilon}{2} + \mathbb{P}\left[A_{L,T}^c\right] \leq \varepsilon,\end{aligned}$$

if L is large enough. This proves property (i) for g.

Given $t \in [0,T]$, for every $\varepsilon > 0$ there exists a constant $N_L > 0$, such that for every $n \geq N_L$ and for every $h = \ell+1, \ldots, 2\ell$,

$$\sup_{n\geq 1} \mathbb{P}\left[\left|\widetilde{\Phi}_n^{h,L}(t)\right| > \delta\right] < \frac{\varepsilon}{2}. \tag{4.4.13}$$

Again, this implies that

$$\begin{aligned}\mathbb{P}\left[\left|\Phi_n^h(t)\right| > \delta\right] &\leq \mathbb{P}\left[\left|\Phi_n^h(t)\right| > \delta, A_{L,T}\right] + \mathbb{P}\left[A_{L,T}^c\right] = \mathbb{P}\left[\left|\widetilde{\Phi}_n^{h,L}(t)\right| > \delta, A_{L,T}\right] + \mathbb{P}\left[A_{L,T}^c\right] \\ &\leq \frac{\varepsilon}{2} + \mathbb{P}\left[A_{L,T}^c\right] \leq \varepsilon,\end{aligned}$$

if L is large enough, which proves property (ii) for g.

Moreover, if $\alpha = \frac{1}{2\ell+1}$, then for every $0 \leq t_1 \leq \cdots \leq t_d \leq T$ there exists $M_L \in \mathbb{N}$, such that for all $n \geq M_L$,

$$\left|\mathbb{E}\left[(\phi(\widetilde{\Phi}_n^{\ell,L}(t_1), \ldots, \widetilde{\Phi}_n^{\ell,L}(t_d)) - \phi(\widetilde{Z}_{t_1}^L, \ldots, \widetilde{Z}_{t_d}^L))\mathbb{1}_{B\cap A_{L,T}}\right]\right| < \frac{\varepsilon}{2}, \tag{4.4.14}$$

and if $t \in [0,T]$ and $\alpha > \frac{1}{2\ell+1}$, there exists $R_L \in \mathbb{N}$, such that for all $n \geq R_L$,

$$\sup_{n\geq 1} \mathbb{P}\left[\left|\widetilde{\Phi}_n^{\ell,L}(t)\right| > \delta\right] < \frac{\varepsilon}{2}. \tag{4.4.15}$$

Similarly, we have

$$\begin{aligned}
&\left|\mathbb{E}\left[(\phi(\Phi_n^\ell(t_1),\ldots,\Phi_n^\ell(t_d))-\phi(Z_{t_1},\ldots,Z_{t_d}))\mathbb{1}_B\right]\right| \\
&\le \left|\mathbb{E}\left[(\phi(\widetilde{\Phi}_n^{\ell,L}(t_1),\ldots,\widetilde{\Phi}_n^{\ell,L}(t_d))-\phi(\widetilde{Z}_{t_1}^L,\ldots,\widetilde{Z}_{t_d}^L))\mathbb{1}_B\mathbb{1}_{A_{L,T}}\right]\right| + 2\sup_{x\in\mathbb{R}^d}|\phi(x)|\,\mathbb{P}\left[A_{L,T}^c\right] \\
&\le \frac{\varepsilon}{2} + 2\sup_{x\in\mathbb{R}^d}|\phi(x)|\,\mathbb{P}\left[A_{L,T}^c\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}\left[\left|\Phi_n^h(t)\right|>\delta\right] &\le \mathbb{P}\left[\left|\Phi_n^h(t)\right|>\delta, A_{L,T}\right]+\mathbb{P}\left[A_{L,T}^c\right] = \mathbb{P}\left[\left|\widetilde{\Phi}_n^{h,L}(t)\right|>\delta, A_L\right]+\mathbb{P}\left[A_{L,T}^c\right] \\
&\le \frac{\varepsilon}{2}+\mathbb{P}\left[A_{L,T}^c\right].
\end{aligned}$$

Taking L large enough we conclude that properties (iii) and (iv) hold for g.

Therefore, we can assume without loss of generality that f has compact support. Relations (i), (ii) and (iv), for f compactly supported follow from Lemma 4.4.1, while relation (iii) follows from Lemma 4.4.2. Modulo these two lemmas, which we state below, the proof of Theorem 4.1.2 is now complete.

Lemma 4.4.1. *Assume that $\alpha \ge \frac{1}{2h+1}$. Consider the process Φ_n^h, $h=\ell,\ldots,2\ell$ defined in* (4.4.5), *for $f\in\mathscr{C}^{8\ell+2}(\mathbb{R})$ with compact support. Then,*

1. *The sequence of processes $\{\Phi_n^h\}_{n\ge1}$, is tight in $\boldsymbol{D}[0,\infty)$, for $h=\ell,\ldots,2\ell$.*

2. *If $h\ge\ell+1$, then $\Phi_n^h \xrightarrow{\mathbb{P}} 0$, in the topology of $\boldsymbol{D}[0,\infty)$, as $n\to\infty$.*

3. *If $\alpha>\frac{1}{2\ell+1}$, then $\Phi_n^\ell \xrightarrow{\mathbb{P}} 0$, in the topology of $\boldsymbol{D}[0,\infty)$, as $n\to\infty$.*

Proof. Fix h, $\ell \le h \le 2\ell$. As in Section 4.3, $c_{0,h}, \dots, c_{h,h}$ will denote the coefficients of the Hermite expansion of x^{2h+1}, namely,

$$x^{2h+1} = \sum_{u=0}^{h} c_{u,h} H_{2(h-u)+1}(x).$$

Then, we can write

$$\frac{\Delta X_{\frac{j}{n}}^{2h+1}}{\xi_{j,n}^{2h+1}} = \sum_{u=0}^{h} c_{u,h}\, H_{2(h-u)+1}\left(\frac{\Delta X_{\frac{j}{n}}}{\xi_{j,n}}\right) = \sum_{u=0}^{h} c_{u,h}\, \delta\left(\frac{\partial_{\frac{j}{n}}^{\otimes 2(h-u)+1}}{\xi_{j,n}^{2(h-u)+1}}\right).$$

To prove the result, we use the above relation to write the process Φ_n^h as a sum of multiple Skorohod integrals plus a remainder term that converges uniformly to zero on compact intervals. Indeed, we can write, for $h = \ell, \dots, 2\ell$,

$$\Phi_n^h(t) = \kappa_{\nu,h} \sum_{j=0}^{\lfloor nt \rfloor - 1} \sum_{u=0}^{h} c_{u,h} \xi_{j,n}^{2u} f^{(2h+1)}(\widetilde{X}_{\frac{j}{n}}) \delta^{2h+1-2u}(\partial_{\frac{j}{n}}^{\otimes 2h+1-2u}).$$

Hence, applying Lemma 1.2.1 with $F = f^{(2h+1)}(\widetilde{X}_{\frac{j}{n}})$, $q = 2h+1-2u$ and $u = \partial_{\frac{j}{n}}^{\otimes 2h+1-2u}$, we obtain

$$\Phi_n^h(t) = \kappa_{\nu,h} \sum_{u=0}^{h} \sum_{r=0}^{2h+1-2u} \binom{2h+1-2u}{r} c_{u,h} \Theta_{u,r}^n(t), \tag{4.4.16}$$

where the random variable $\Theta_{u,r}^n(t)$, for $h = \ell, \dots, 2\ell$ fixed, is defined by

$$\Theta_{u,r}^n(t) = \delta^{2h+1-2u-r}\left(\sum_{j=0}^{\lfloor nt \rfloor - 1} \xi_{j,n}^{2u} f^{(2h+1+r)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 2h+1-2u-r} \left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}^{r} \right).$$

By (4.4.16), we can decompose the process $\Phi_n^h(t)$, as

$$\Phi_n^h(t) = \Psi_n^h(t) + R_n^h(t), \tag{4.4.17}$$

where

$$\Psi_n^h(t) = \kappa_{\nu,h} \sum_{u=0}^{h} \sum_{r=0}^{2h-2u} \binom{2h+1-2u}{r} c_{u,h} \Theta_{u,r}^n(t), \tag{4.4.18}$$

and

$$R_n^h(t) = \kappa_{\nu,h} \sum_{u=0}^{h} c_{u,h} \sum_{j=0}^{\lfloor nt \rfloor -1} \xi_{j,n}^{2u} f^{(4h+2-2u)}(\widetilde{X}_{\frac{j}{n}}) \left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}^{2h+1-2u}.$$

Therefore, to prove the lemma, it suffices to show the following four claims:

(a) The process $R_n^h = \{R_n^h(t)\}_{t\geq 0}$ converges uniformly to zero in $L^1(\Omega)$ on compact intervals, namely, for each $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| R_n^h(t) \right| \right] \to 0$$

(b) The process $\Psi_n^h = \{\Psi_n^h(t)\}_{t\geq 0}$ is tight in $\mathbf{D}[0,\infty)$ for all $\ell \leq h \leq 2\ell$.

(c) The process $\Psi_n^h = \{\Psi_n^h(t)\}_{t\geq 0}$ converges to zero in $\mathbf{D}[0,\infty)$ for $\ell + 1 \leq h \leq 2\ell$.

(d) If $\alpha > \frac{1}{2\ell+1}$, then the process $\Psi_n^\ell = \{\Psi_n^\ell(t)\}_{t\geq 0}$ converges to zero in probability in $\mathbf{D}[0,\infty)$.

Proof of claim (a): Using inequality (4.5.2), as well as the fact that f has compact support, we deduce that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|R_n^h(t)\right|\right] \le C\sum_{u=0}^{h}\sum_{j=0}^{\lfloor nT\rfloor-1}\mathbb{E}\left[\left|f^{(4h+2-2u)}(\widetilde{X}_{\frac{j}{n}})\right|\right]\xi_{j,n}^{2u}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|^{2h+1-2u}$$
$$\le C\sum_{u=0}^{h}\sum_{j=0}^{\lfloor nT\rfloor-1} n^{-\alpha u}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|^{2h+1-2u}.$$

Hence, by inequality (4.5.6), there exists a constant $C>0$, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|R_n^{h,m}(t)\right|\right] \le C\sum_{u=0}^{h} n^{-\alpha u-4\beta(h-u)}$$
$$= C(n^{-\alpha h}+\sum_{u=0}^{h-1} n^{-\alpha u}n^{-4\beta(h-u)}) \le C(n^{-\alpha h}+hn^{-4\beta}), \quad (4.4.19)$$

which implies that $\sup_{t\in[0,T]} R_n^h$ converges to zero in $L^1(\Omega)$, as required.

Proof of claims (b), (c) and (d): Since $h\ge \ell$ and $\alpha \ge (2\ell+1)^{-1}$, by the 'Billingsley criterion' (see [4, Theorem 13.5]), it suffices to show that for every $0\le s\le t\le T$, and $p>2$, there exists a constant $C>0$, such that

$$\mathbb{E}\left[\left|\Psi_n^h(t)-\Psi_n^h(s)\right|^p\right] \le Cn^{\frac{p}{2}(1-\alpha(2h+1))}\left|\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right|^{\frac{p}{2}}. \quad (4.4.20)$$

Indeed, relation (4.4.20) implies that

$$\mathbb{E}\left[\left|\Psi_n^h(t)-\Psi_n^h(s)\right|^p\right] \le C\left|\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right|^{\frac{p}{2}},$$

so that Ψ_n^h is tight. Moreover, if $\ell+1\ge h$ or $\alpha>\frac{1}{2\ell+1}$, then $\mathbb{E}\left[\left|\Psi_n^h(t)-\Psi_n^h(w)\right|^p\right]\to 0$ as $n\to\infty$, which implies conditions (c) and (d).

To prove (4.4.20) we proceed as follows. By (4.4.18), there exists a constant $C>0$, only depending on h,ν and T, such that

$$\left\|\Psi_n^h(t)-\Psi_n^h(s)\right\|_{L^p(\Omega)} \leq C \max_{\substack{0\leq u\leq h\\ 0\leq r\leq 2h-2u}} \left\|\Theta_{u,r}^n(t)-\Theta_{u,r}^n(s)\right\|_{L^p(\Omega)}. \tag{4.4.21}$$

For $0\leq u\leq h$ and $0\leq r\leq 2h-2u$, define the constant $w=2h+1-2u-r\geq 1$. By Meyer's inequality (1.2.2), we have the following bound for the L^p-norm appearing in the right-hand side of (4.4.21).

$$\begin{aligned}
&\left\|\Theta_{u,r}^n(t)-\Theta_{u,r}^n(s)\right\|_{L^p(\Omega)}^2\\
&=\left\|\delta^w\left(\sum_{j=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1}\xi_{j,n}^{2u}f^{(2h+1+r)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes w}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^r\right)\right\|_{L^p(\Omega)}^2\\
&\leq C\sum_{i=0}^{w}\left\|\sum_{j=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1}\xi_{j,n}^{2u}f^{(2h+1+r+i)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes w}\otimes\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes i}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^r\right\|_{L^p(\Omega,\mathfrak{H}^{\otimes(w+i)})}^2 \qquad (4.4.22)\\
&=C\sum_{i=0}^{w}\left\|\left\|\sum_{j=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1}\xi_{j,n}^{2u}f^{(2h+1+r+i)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes w}\otimes\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes i}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^r\right\|_{\mathfrak{H}^{\otimes(w+i)}}^2\right\|_{L^{\frac{p}{2}}(\Omega)}.
\end{aligned}$$

From the previous relation, it follows that there exists a constant $C>0$, such that

$$\begin{aligned}
\left\|\Theta_{u,r}^n(t)-\Theta_{u,r}^n(s)\right\|_{L^p(\Omega)}^2 \leq\ & C\sum_{i=0}^{w}\Bigg\|\sum_{j,k=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1}\xi_{j,n}^{2u}\xi_{k,n}^{2u}f^{(2h+1+r+i)}(\widetilde{X}_{\frac{j}{n}})f^{(2h+1+r+i)}(\widetilde{X}_{\frac{k}{n}})\\
&\times\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^w\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^i\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^r\left\langle\widetilde{\varepsilon}_{\frac{k}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^r\Bigg\|_{L^{\frac{p}{2}}(\Omega)} \qquad (4.4.23)
\end{aligned}$$

Since f has compact support, by applying Minkowski inequality and Cauchy-Schwarz inequality in (4.4.23), we deduce that

$$\left\|\Theta_{u,r}^n(t)-\Theta_{u,r}^n(s)\right\|_{L^p(\Omega)}^2 \leq C\sum_{i=0}^{w}\sum_{j,k=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1}\xi_{j,n}^{2u+r}\xi_{k,n}^{2u+r}\left\|\widetilde{\varepsilon}_{\frac{j}{n}}\right\|_{\mathfrak{H}}^{i+r}\left\|\widetilde{\varepsilon}_{\frac{k}{n}}\right\|_{\mathfrak{H}}^{i+r}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^w.$$

From here, using the Cauchy Schwarz inequality, it follows that

$$\left\|\Theta^n_{u,r}(t)-\Theta^n_{u,r}(s)\right\|^2_{L^p(\Omega)} \le C\sum_{j,k=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1} \xi^{2u+r}_{j,n}\xi^{2u+r}_{k,n}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^w$$
$$\le C\sum_{j,k=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1} \xi^{2h}_{j,n}\xi^{2h}_{k,n}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|.$$

Consequently, we get

$$\left\|\Theta^n_{u,r}(t)-\Theta^n_{u,r}(s)\right\|^2_{L^p(\Omega)} \le C\sum_{x=0}^{\lfloor nt\rfloor-\lfloor ns\rfloor-1}\sum_{j=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1-x} \xi^{2h}_{j,n}\xi^{2h}_{j+x,n}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{j+x}{n}}\right\rangle_{\mathfrak{H}}\right|. \quad (4.4.24)$$

Then the estimate (4.4.20) will follow from

$$\xi^{2h}_{j,n}\xi^{2h}_{j+x,n}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{j+x}{n}}\right\rangle_{\mathfrak{H}}\right| \le Cn^{-\alpha(2h+1)}x^{-1-\delta}, \quad (4.4.25)$$

for some $\delta>0$ and for all $x\ge 3$ and $\lfloor ns\rfloor \le j\le \lfloor nt\rfloor-1$. Set

$$\widehat{G}(j,j+x)=\xi^{2h}_{j,n}\xi^{2h}_{j+x,n}|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{j+x}{n}}\right\rangle_{\mathfrak{H}}|.$$

By considering the cases $j=0$, $j\ge x+2$ and $1\le j\le x+2$, for $x\ge 3$, we obtain the following bounds:

Case $j=0$: Using (4.1.3) and (4.5.2), we get

$$\widehat{G}(0,x)\le Cn^{-(2\alpha h+2\beta)}|\phi(x+1)-\phi(x)|$$
$$\le Cn^{-\alpha(2h+1)}x^{-\nu}.$$

Case $j \geq x+2$: Using (4.5.3), we deduce that for every $j \geq x-2$,

$$\begin{aligned}\widehat{G}(j,x) &\leq Cn^{-2\beta(2h+1)} j^{(2\beta-\alpha)(h+1)}(j+x)^{(2\beta-\alpha)h+\alpha-2} \\ &\leq Cn^{-2\beta(2h+1)} j^{(2\beta-\alpha)(h+1)}(j+x)^{(2\beta-\alpha)h} x^{\alpha-2} \\ &\leq Cn^{-2\beta(2h+1)}(j+x)^{(2\beta-\alpha)(2h+1)} x^{\alpha-2} = Cn^{-\alpha(2h+1)} x^{\alpha-2}.\end{aligned}$$

Case $j \leq x+2$: Using (4.5.4), we deduce that for all $j \leq x-2$,

$$\widehat{G}(j,x) \leq Cn^{-2\beta(2h+1)} j^{(2\beta-\alpha)h+2\beta+\nu-2}(j+x)^{(2\beta-\alpha)h-\nu}. \tag{4.4.26}$$

If $\nu \geq 2-\alpha$, then

$$(j+x)^{-\nu} = (j+x)^{\alpha-2}(j+x)^{2-\alpha-\nu} \leq x^{\alpha-2} j^{2-\alpha-\nu},$$

and thus, by (4.4.26),

$$\hat{G}(j,x) \leq Cn^{-2\beta(2h+1)} j^{(2\beta-\alpha)(h+1)}(j+x)^{(2\beta-\alpha)h} x^{\alpha-2} \leq Cn^{-\alpha(2h+1)} x^{\alpha-2}. \tag{4.4.27}$$

On the other hand, if $\nu \leq 2-\alpha$, then by (4.4.26),

$$\begin{aligned}\hat{G}(j,x) &\leq Cn^{-2\beta(2h+1)} j^{(2\beta-\alpha)h+2\beta-\alpha}(j+x)^{(2\beta-\alpha)h-\nu} \\ &\leq Cn^{-2\beta(2h+1)} j^{(2\beta-\alpha)(h+1)}(j+x)^{(2\beta-\alpha)h} x^{-\nu} \\ &\leq Cn^{-\alpha(2h+1)} x^{-\nu}. \end{aligned}\tag{4.4.28}$$

The proof of the lemma is now complete. □

Lemma 4.4.2. *Assume that* $\alpha = \frac{1}{2\ell+1}$ *and let* $0 \leq t_1 \leq \cdots \leq t_d \leq T$ *be fixed. Define* Φ_n^ℓ *and* Z *by* (4.4.5) *and* (4.4.8) *respectively, for some function* f *with compact support.*

Then,

$$(\Phi_n^\ell(t_1),\dots,\Phi_n^\ell(t_d)) \stackrel{stably}{\to} (Z_{t_1},\dots,Z_{t_d}). \tag{4.4.29}$$

Proof. We follow the small blocks-big blocks methodology (see [5] and [10]). Let $2 \le p < n$. For $k \ge 0$, define the set

$$I_k = \{j \in \{0,\dots,\lfloor nt \rfloor - 1\} \mid \frac{k}{p} \le \frac{j}{n} < \frac{k+1}{p}\}.$$

The basic idea of the proof of (4.4.29), consists on approximating $(\Phi_n^\ell(t_1),\dots,\Phi_n^\ell(t_d))$ by the random vector $(\widetilde{\Phi}_{n,p}(t_1),\dots,\widetilde{\Phi}_{n,p}(t_d))$, where

$$\widetilde{\Phi}_{n,p}(t) = \kappa_{\nu,\ell} \sum_{k=0}^{\lfloor pt \rfloor} \sum_{j \in I_k} f^{(2\ell+1)}(X_{\frac{k}{p}})(\Delta X_{\frac{j}{n}})^{2\ell+1}.$$

By Proposition 4.1.1, for every $\mathscr{F}$-measurable and bounded random variable η, the vector $(\widetilde{\Phi}_{n,p}(t_1),\dots,\widetilde{\Phi}_{n,p}(t_d),\eta)$ converges in law, as n tends to infinity, to the vector $(\Xi_p^1,\dots,\Xi_p^d,\eta)$, where

$$\Xi_p^i = \kappa_{\nu,\ell}\sigma_\ell \sum_{k=0}^{\lfloor pt_i \rfloor} f^{(2\ell+1)}(X_{\frac{k}{p}})(Y_{\frac{k+1}{p}} - Y_{\frac{k}{p}}), \qquad \text{for } i = 1,\dots,d.$$

In turn, when $p \to \infty$, the random vector $(\Xi_p^1,\dots,\Xi_p^d,\eta)$ converges in probability to a random vector with the same law as $(Z_{t_1},\dots,Z_{t_d},\eta)$, which implies (4.4.29), provided that

$$\lim_{p\to\infty} \limsup_{n\to\infty} \sum_{i=0}^{d} \left\| \Phi_n^\ell(t_i) - \widetilde{\Phi}_{n,p}(t_i) \right\|_{L^2(\Omega)} = 0. \tag{4.4.30}$$

Indeed, if (4.4.30) holds, then for all $g:\mathbb{R}^{d+1}\to\mathbb{R}$ differentiable with compact support, and every $p\geq 1$,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\mathbb{E}\left[g(\Phi_n^\ell(t_1),\ldots,\Phi_n^\ell(t_d),\eta)-g(Z_{t_1},\ldots,Z_{t_d},\eta)\right]\right| \\
&\qquad\leq \limsup_{n\to\infty}\left|\mathbb{E}\left[g(\Phi_n^\ell(t_1),\ldots,\Phi_n^\ell(t_d),\eta)-g(\widetilde{\Phi}_{n,p}(t_1),\ldots,\widetilde{\Phi}_{n,p}(t_d),\eta)\right]\right| \\
&\qquad+\limsup_{n\to\infty}\left|\mathbb{E}\left[g(\widetilde{\Phi}_{n,p}(t_1),\ldots,\widetilde{\Phi}_{n,p}(t_d),\eta)-g(Z_{t_1},\ldots,Z_{t_d},\eta)\right]\right| \\
&\qquad= \limsup_{n\to\infty}\left|\mathbb{E}\left[g(\Phi_n^\ell(t_1),\ldots,\Phi_n^\ell(t_d),\eta)-g(\widetilde{\Phi}_{n,p}(t_1),\ldots,\widetilde{\Phi}_{n,p}(t_d),\eta)\right]\right| \\
&\qquad+\left|\mathbb{E}\left[g(\Xi_p^1,\ldots,\Xi_p^d,\eta)-g(Z_{t_1},\ldots,Z_{t_d},\eta)\right]\right|.
\end{aligned}$$

Then, taking $p\to\infty$, we get

$$\lim_{n\to\infty}\left|\mathbb{E}\left[g(\Phi_n^\ell(t_1),\ldots,\Phi_n^\ell(t_d),\eta)\right]-\mathbb{E}\left[g(z_{t_1},\ldots,Z_{t_d},\eta)\right]\right|=0,$$

as required.

In order to prove (4.4.30) we proceed as follows. Following the proof of (4.4.16), we can show that

$$\Phi_n^\ell(t_i)=\kappa_{\nu,\ell}\sum_{u=0}^{\ell}\sum_{r=0}^{2\ell+1-2u}\binom{2\ell+1-2u}{r}c_{u,\ell}\Theta_{u,r}^n(t_i), \tag{4.4.31}$$

$$\widetilde{\Phi}_{n,p}(t_i)=\kappa_{\nu,\ell}\sum_{u=0}^{\ell}\sum_{r=0}^{2\ell+1-2u}\binom{2\ell+1-2u}{r}c_{u,\ell}\widetilde{\Theta}_{u,r}^{n,p}(t_i), \tag{4.4.32}$$

where $\Theta^n_{u,r}(t)$ and $\widetilde{\Theta}^{n,p}_{u,r}(t)$ are defined, for $0 \le u \le \ell$ and $0 \le r \le 2\ell+1-2u$, by

$$\Theta^n_{u,r}(t) = \delta^{2\ell+1-2u-r}\left(\sum_{k=0}^{\lfloor pt_i \rfloor}\sum_{j\in I_k}\xi^{2u}_{j,n} f^{(2\ell+1+r)}(\widetilde{X}_{\frac{j}{n}})\partial^{\otimes 2\ell+1-2u-r}_{\frac{j}{n}}\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}}\right\rangle^r\right),$$

$$\widetilde{\Theta}^{n,p}_{u,r}(t) = \delta^{2\ell+1-2u-r}\left(\sum_{k=0}^{\lfloor pt_i \rfloor}\sum_{j\in I_k}\xi^{2u}_{j,n} f^{(2\ell+1+r)}(X_{\frac{k}{p}})\partial^{\otimes 2\ell+1-2u-r}_{\frac{j}{n}}\left\langle \varepsilon_{\frac{k}{p}}, \partial_{\frac{j}{n}}\right\rangle^r\right).$$

In view of (4.4.31) and (4.4.32), relation (4.4.30) holds true, provided that we show that for every $t \ge 0$

$$\lim_{p\to\infty}\limsup_{n\to\infty}\left\|\Theta^n_{u,r}(t) - \widetilde{\Theta}^{n,p}_{u,r}(t)\right\|_{L^2(\Omega)} = 0. \tag{4.4.33}$$

We divide the proof of (4.4.33) in several steps.

Step 1. First we prove (4.4.33) in the case $r = 2\ell+1-2u$. To this end, it suffices to show that for every p fixed,

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{\lfloor pt \rfloor}\sum_{j\in I_k}\xi^{2u}_{j,n} f^{(4\ell+2-2u)}(X_{\frac{k}{p}})\left\langle \varepsilon_{\frac{k}{p}}, \partial_{\frac{j}{n}}\right\rangle^{2\ell+1-2u}_{\mathfrak{H}}\right\|_{L^2(\Omega)} = 0, \tag{4.4.34}$$

and

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{\lfloor pt \rfloor}\sum_{j\in I_k}\xi^{2u}_{j,n} f^{(4\ell+2-2u)}(\widetilde{X}_{\frac{j}{n}})\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}}\right\rangle^{2\ell+1-2u}_{\mathfrak{H}}\right\|_{L^2(\Omega)} = 0. \tag{4.4.35}$$

Relation (4.4.35) was already proved in Lemma 4.4.1 (see inequality (4.4.19)). In order to prove (4.4.34) we proceed as follows. Since f has compact support, there exists a constant $C>0$, such that for every $u=0,\dots,\ell$, we have

$$\left\|\sum_{k=0}^{\lfloor pt\rfloor}\sum_{j\in I_k}\xi_{j,n}^{2u}f^{(4\ell+2-2u)}(X_{\frac{k}{p}})\left\langle \varepsilon_{\frac{k}{p}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{2\ell+1-2u}\right\|_{L^2(\Omega)}$$
$$\leq C\sum_{k=0}^{\lfloor pt\rfloor}\sum_{j\in I_k}\xi_{j,n}^{2u}\left|\left\langle \varepsilon_{\frac{k}{p}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|^{2\ell+1-2u}$$
$$\leq C\left(\frac{k}{p}\right)^{2\beta(\ell-u)}\phi(1)^{\ell-u}\sum_{k=0}^{\lfloor pt\rfloor}\sum_{j\in I_k}\xi_{j,n}^{2\ell}\left|\left\langle \varepsilon_{\frac{k}{p}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|,$$

where the last inequality follows from Cauchy-Schwarz inequality and (4.0.2). Therefore, by relation (4.5.2) there exist a constant $C_{k,p}>0$, such that

$$\left\|\sum_{k=0}^{\lfloor pt\rfloor}\sum_{j\in I_k}\xi_{j,n}^{2u}f^{(4\ell+2-2u)}(X_{\frac{k}{p}})\left\langle \varepsilon_{\frac{k}{p}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{2\ell+1-2u}\right\|_{L^2(\Omega)}$$
$$\leq C_{k,p}n^{-\alpha\ell}\sum_{k=1}^{\lfloor pt\rfloor}\sum_{j\in I_k}\left(\frac{k}{p}\right)^{2\beta}\left|\phi\left(\frac{(j+1)p}{nk}\right)-\phi\left(\frac{jp}{nk}\right)\right|.$$

Using the decomposition (4.1.1) we get

$$\begin{aligned}\left|\phi\left(\frac{(j+1)p}{nk}\right)-\phi\left(\frac{jp}{nk}\right)\right| &\leq \lambda\left[\left(\frac{(j+1)p}{nk}-1\right)^{\alpha}-\left(\frac{jp}{nk}-1\right)^{\alpha}\right]\\ &\quad+\left|\psi\left(\frac{(j+1)p}{nk}\right)-\psi\left(\frac{jp}{nk}\right)\right|\\ &\leq \lambda\left[\left(\frac{(j+1)p}{nk}-1\right)^{\alpha}-\left(\frac{jp}{nk}-1\right)^{\alpha}\right]+\sup_{x\geq 1}|\psi'(x)|\frac{p}{nk}.\end{aligned}$$

The sum in $j\in I_k$ of this expression is bounded by a constant not depending on n because the first term produces a telescopic sum and the second term is bounded by a constant times $1/n$. This completes the proof of the convergence (4.4.34).

Step 2. Next we show (4.4.33) for $0 \leq r \leq 2\ell - 2u$. To this end, define the variables

$$F^{n,p}_{k,j,r} = f^{(2\ell+1+r)}(\widetilde{X}_{\frac{j}{n}}) \left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}} \right\rangle^r - f^{(2\ell+1+r)}(X_{\frac{k}{p}}) \left\langle \varepsilon_{\frac{k}{p}}, \partial_{\frac{j}{n}} \right\rangle^r.$$

We aim to show that for every $u = 0, \ldots, \ell$, and $0 \leq r \leq 2\ell - 2u$,

$$\lim_{p\to\infty} \limsup_{n\to\infty} \left\| \delta^{2\ell+1-2u-r} \left(\sum_{k=0}^{\lfloor pt \rfloor} \sum_{j\in I_k} \xi^{2u}_{j,n} F^{n,p}_{k,j,r} \partial^{\otimes 2\ell+1-2u-r}_{\frac{j}{n}} \right) \right\|_{L^2(\Omega)} = 0. \tag{4.4.36}$$

Define $w = 2\ell + 1 - 2u - r$. By Meyer's inequality (1.2.2), we have

$$\begin{aligned}
&\left\| \delta^w \left(\sum_{k=0}^{\lfloor pt \rfloor} \sum_{j\in I_k} \xi^{2u}_{j,n} F^{n,p}_{k,j,r} \partial^{\otimes w}_{\frac{j}{n}} \right) \right\|^2_{L^2(\Omega)} \\
&\quad \leq C \sum_{i=0}^{w} \left\| \sum_{k=0}^{\lfloor pt \rfloor} \sum_{j\in I_k} \xi^{2u}_{j,n} D^i F^{n,p}_{k,j,r} \otimes \partial^{\otimes w}_{\frac{j}{n}} \right\|^2_{L^2(\Omega;\mathfrak{H}^{\otimes(w+i)})} \\
&\quad = C \sum_{i=0}^{w} \sum_{k_1,k_2=0}^{\lfloor pt \rfloor} \sum_{\substack{j_1\in I_{k_1} \\ j_2\in I_{k_2}}} \xi^{2u}_{j_1,n} \xi^{2u}_{j_2,n} \mathbb{E}\left[\left\langle D^i F^{n,p}_{k_1,j_1,r}, D^i F^{n,p}_{k_2,j_2,r} \right\rangle_{\mathfrak{H}^{\otimes i}} \right] \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle^w_{\mathfrak{H}}.
\end{aligned} \tag{4.4.37}$$

By the Cauchy-Schwarz inequality, we have $\left| \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}} \right| \leq \xi_{j_1,n} \xi_{j_2,n}$, and hence,

$$\left| \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}} \right|^w \leq (\xi_{j_1,n} \xi_{j_2,n})^{2\ell-2u-r} \left| \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}} \right|,$$

which, by (4.4.37), implies that

$$
\begin{aligned}
&\left\|\delta^w\left(\sum_{k=0}^{\lfloor pt\rfloor}\sum_{j\in I_k}\xi_{j,n}^{2u}F_{k,j,r}^{n,p}\partial_{\frac{j}{n}}^{\otimes q}\right)\right\|_{L^2(\Omega)}^2\\
&\le\sum_{i=0}^{w}\sum_{k_1,k_2=0}^{\lfloor pt\rfloor}\sum_{\substack{j_1\in I_{k_1}\\ j_2\in I_{k_2}}}\xi_{j_1,n}^{2\ell-r}\xi_{j_2,n}^{2\ell-r}\left\|D^iF_{k_1,j_1,r}^{n,p}\right\|_{L^2(\Omega,\mathfrak{H}^{\otimes i})}\left\|D^iF_{k_2,j_2,r}^{n,p}\right\|_{L^2(\Omega,\mathfrak{H}^{\otimes i})}\left|\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}\right|\\
&\le\sum_{i=0}^{w}\max_{(k,j)\in J_{n,p}}\left\|D^iF_{k,j,r}^{n,p}\right\|_{L^2(\Omega,\mathfrak{H}^{\otimes i})}^2\sum_{k_1,k_2=0}^{\lfloor pt\rfloor}\sum_{\substack{j_1\in I_{k_1}\\ j_2\in I_{k_2}}}\xi_{j_1,n}^{2\ell-r}\xi_{j_2,n}^{2\ell-r}\left|\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}\right|,
\end{aligned}
\tag{4.4.38}
$$

where $J_{n,p}$ denotes the set of indices

$$
J_{n,p}=\{(k,j)\in\mathbb{N}\mid 0\le k\le\lfloor pt\rfloor+1\ \text{ and }\ \frac{k}{p}\le\frac{j}{n}\le\frac{k+1}{p}\}.
$$

We can easily check that

$$
\begin{aligned}
F_{k,j,r}^{n,p}&=f^{(2\ell+1+r)}(\widetilde{X}_{\frac{j}{n}})\left\langle\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes r}-\varepsilon_{\frac{k}{p}}^{\otimes r},\partial_{\frac{j}{n}}^{\otimes r}\right\rangle_{\mathfrak{H}^{\otimes r}}\\
&\quad+\left(f^{(2\ell+1+r)}(\widetilde{X}_{\frac{j}{n}})-f^{(2\ell+1+r)}(X_{\frac{k}{p}})\right)\left\langle\varepsilon_{\frac{k}{p}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^r,
\end{aligned}
$$

and hence, we have

$$
\begin{aligned}
D^iF_{k,j,r}^{n,p}&=f^{(2\ell+1+r+i)}(\widetilde{X}_{\frac{j}{n}})\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes i}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes r}-\varepsilon_{\frac{k}{p}}^{\otimes r},\partial_{\frac{j}{n}}^{\otimes r}\right\rangle_{\mathfrak{H}^{\otimes r}}\\
&\quad+f^{(2\ell+1+r+i)}(\widetilde{X}_{\frac{j}{n}})\left(\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes i}-\varepsilon_{\frac{k}{p}}^{\otimes i}\right)\left\langle\varepsilon_{\frac{k}{p}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^r\\
&\quad+\left(f^{(2\ell+1+r+i)}(\widetilde{X}_{\frac{j}{n}})-f^{(2\ell+1+r+i)}(X_{\frac{k}{p}})\right)\varepsilon_{\frac{k}{p}}^{\otimes i}\left\langle\varepsilon_{\frac{k}{p}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^r.
\end{aligned}
$$

From the previous equality, and the compact support condition of f, we deduce that there exists a constant $C>0$, such that

$$\begin{aligned}
&\left\| D^i F^{n,p}_{k,j,r} \right\|_{L^2(\Omega,\mathfrak{H}^{\otimes i})} \\
&\qquad \leq C \left\| \widetilde{\varepsilon}^{\otimes i}_{\frac{j}{n}} \right\|_{\mathfrak{H}^{\otimes i}} \left| \left\langle \widetilde{\varepsilon}^{\otimes r}_{\frac{j}{n}} - \varepsilon^{\otimes r}_{\frac{k}{p}}, \partial^{\otimes r}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}^{\otimes r}} \right| + C \left\| \widetilde{\varepsilon}^{\otimes i}_{\frac{j}{n}} - \varepsilon^{\otimes i}_{\frac{k}{p}} \right\|_{\mathfrak{H}^{\otimes i}} \left| \left\langle \varepsilon_{\frac{k}{p}}, \partial_{\frac{j}{n}} \right\rangle^r_{\mathfrak{H}} \right| \\
&\qquad + \left\| f^{(2\ell+1+r+i)}(\widetilde{X}_{\frac{j}{n}}) - f^{(2\ell+1+r+i)}(X_{\frac{k}{p}}) \right\|_{L^2(\Omega)} \left\| \varepsilon^{\otimes i}_{\frac{k}{p}} \right\|_{\mathfrak{H}^{\otimes i}} \left| \left\langle \varepsilon_{\frac{k}{p}}, \partial_{\frac{j}{n}} \right\rangle^r_{\mathfrak{H}} \right|,
\end{aligned}$$

and hence,

$$\begin{aligned}
\left\| D^i F^{n,p}_{k,j,r} \right\|_{L^2(\Omega,\mathfrak{H}^{\otimes i})} &\leq C \left\| \widetilde{\varepsilon}_{\frac{j}{n}} \right\|^i_{\mathfrak{H}} \left\| \widetilde{\varepsilon}^{\otimes r}_{\frac{j}{n}} - \varepsilon^{\otimes r}_{\frac{k}{p}} \right\|_{\mathfrak{H}^{\otimes r}} \left\| \partial_{\frac{j}{n}} \right\|^r_{\mathfrak{H}} \\
&\quad + C \left\| \widetilde{\varepsilon}^{\otimes i}_{\frac{j}{n}} - \varepsilon^{\otimes i}_{\frac{k}{p}} \right\|_{\mathfrak{H}^{\otimes i}} \left\| \varepsilon_{\frac{k}{p}} \right\|^r_{\mathfrak{H}} \left\| \partial_{\frac{j}{n}} \right\|^r_{\mathfrak{H}} \\
&\quad + \left\| f^{(2\ell+1+r+i)}(\widetilde{X}_{\frac{j}{n}}) - f^{(2\ell+1+r+i)}(X_{\frac{k}{p}}) \right\|_{L^2(\Omega)} \left\| \varepsilon_{\frac{k}{p}} \right\|^{r+i}_{\mathfrak{H}} \left\| \partial_{\frac{j}{n}} \right\|^r_{\mathfrak{H}}.
\end{aligned} \tag{4.4.39}$$

Using the Cauchy-Schwarz inequality, as well as (4.0.2), we have that for every $\gamma \in \mathbb{N}$, $\gamma \geq 1$, there exists a constant $C>0$ such that

$$\left\| \widetilde{\varepsilon}^{\otimes \gamma}_{\frac{j}{n}} - \varepsilon^{\otimes \gamma}_{\frac{k}{p}} \right\|_{\mathfrak{H}^{\otimes \gamma}} \leq \left\| \widetilde{\varepsilon}_{\frac{j}{n}} - \varepsilon_{\frac{k}{p}} \right\|_{\mathfrak{H}} \sum_{i=0}^{\gamma-1} \left\| \widetilde{\varepsilon}_{\frac{j}{n}} \right\|^i_{\mathfrak{H}} \left\| \varepsilon_{\frac{k}{p}} \right\|^{\gamma-1-i}_{\mathfrak{H}} \leq C \left\| \widetilde{\varepsilon}_{\frac{j}{n}} - \varepsilon_{\frac{k}{p}} \right\|_{\mathfrak{H}}.$$

As a consequence, by (4.4.39), there exists a constant $C>0$ such that

$$\begin{aligned}
&\left\| D^i F^{n,p}_{k,j,r} \right\|_{L^2(\Omega,\mathfrak{H}^{\otimes i})} \\
&\leq C \xi^r_{j,n} \left(\left\| \widetilde{\varepsilon}_{\frac{j}{n}} - \varepsilon_{\frac{k}{p}} \right\|_{\mathfrak{H}} + \left\| f^{(2\ell+1+r+i)}(\widetilde{X}_{\frac{j}{n}}) - f^{(2\ell+1+r+i)}(X_{\frac{k}{p}}) \right\|_{L^2(\Omega)} \right) \\
&\leq C \xi^r_{j,n} \left(\mathbb{E}\left[\sup_{|t-s| \leq \frac{1}{p}} \left| \widetilde{X}_t - X_s \right|^2 \right]^{\frac{1}{2}} + \mathbb{E}\left[\sup_{|t-s| \leq \frac{1}{p}} \left| f^{(2\ell+1+r+i)}(\widetilde{X}_t) - f^{(2\ell+1+r+i)}(X_s) \right|^2 \right]^{\frac{1}{2}} \right).
\end{aligned}$$

From the previous inequality, we deduce that the function

$$Q_p = \sup_{n\geq 1} \xi_{j,n}^{-2r} \sum_{i=0}^{q} \max_{(k,j)\in J_{n,p}} \left\| D^i F_{k,j,r}^{n,p} \right\|^2_{L^2(\Omega,\mathfrak{H}^{\otimes i})},$$

satisfies $\lim_{p\to\infty} Q_p = 0$. Hence, by (4.5.2) and (4.4.38),

$$\begin{aligned} \left\| \delta^q \left(\sum_{k=0}^{\lfloor pt \rfloor} \sum_{j\in I_k} \xi_{j,n}^{2u} F_{k,j,r}^{n,p} \partial_{\frac{j}{n}}^{\otimes q} \right) \right\|^2_{L^2(\Omega)} &\leq C Q_p \sum_{k_1,k_2=0}^{\lfloor pt \rfloor} \sum_{\substack{j_1\in I_{k_1} \\ j_2 \in I_{k_2}}} \xi_{j_1,n}^{2\ell} \xi_{j_2,n}^{2\ell} \left| \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}} \right| \\ &= C Q_p \sum_{i_1,i_2=0}^{\lfloor nt \rfloor} \xi_{i_1,n}^{2\ell} \xi_{i_2,n}^{2\ell} \left| \left\langle \partial_{\frac{i_1}{n}}, \partial_{\frac{i_2}{n}} \right\rangle_{\mathfrak{H}} \right| \\ &\leq C Q_p \sum_{x=0}^{\lfloor nt \rfloor -1} \sum_{j=0}^{\lfloor nt \rfloor -1-x} \xi_{j,n}^{2\ell} \xi_{j+x,n}^{2\ell} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j+x}{n}} \right\rangle_{\mathfrak{H}} \right|. \end{aligned} \tag{4.4.40}$$

Using the previous inequality, as well as (4.4.25), we deduce that

$$\left\| \delta^q \left(\sum_{k=0}^{\lfloor pt \rfloor} \sum_{j\in I_k} \xi_{j,n}^{2u} F_{k,j,r}^{n,p} \partial_{\frac{j}{n}}^{\otimes q} \right) \right\|^2_{L^2(\Omega)} \leq Ct Q_p \sum_{x=0}^{\infty} n^{1-\alpha(2\ell+1)} (1+x)^{-1-\delta}, \tag{4.4.41}$$

for some $\delta > 0$. Since, $\alpha = \frac{1}{2\ell+1}$, relation (4.4.41) implies that

$$\left\| \delta^q \left(\sum_{k=0}^{\lfloor pt \rfloor} \sum_{j\in I_k} \xi_{j,n}^{2u} F_{k,j,r}^{n,p} \partial_{\frac{j}{n}}^{\otimes q} \right) \right\|^2_{L^2(\Omega)} \leq Ct Q_p. \tag{4.4.42}$$

Relation (4.4.36) then follows from (4.4.42) since $\lim_{p\to\infty} Q_p = 0$. The proof is now complete. □

4.5 Appendix

The following lemmas are estimations on the covariances of increments of X. The proof of these results relies on some technical lemmas proved by Nualart and Harnett in [20]. In what follows C is a generic constant depending only on the covariance of the process X.

Lemma 4.5.1. *Under (H.1), for* $0 < s \le t$ *we have*

$$\mathbb{E}\left[(X_{t+s} - X_t)^2\right] = 2\lambda t^{2\beta-\alpha} s^{\alpha} + g_1(t,s),$$

where $|g_1(t,s)| \le Cst^{2\beta-1}$.

Proof. See [20, Lemma 3.1] and notice that the proof only uses that $|\psi'|$ is bounded in $(1,2]$. □

Remark 4.5.2. *Notice that* $g_1(t,s)$ *satisfies* $|g_1(t,s)| \le Cs^{\alpha}t^{2\beta-\alpha}$, *because* $\alpha < 1$ *and* $\alpha \le 2\beta$. *Therefore, for any* $0 < s \le t$, *we obtain*

$$\mathbb{E}\left[(X_{t+s} - X_t)^2\right] \le Cs^{\alpha}t^{2\beta-\alpha}.$$

With the notation of Section 2.3, this implies

$$\xi_{j,n}^2 \le Cn^{-2\beta} j^{2\beta-\alpha}. \tag{4.5.1}$$

On the other hand, we deduce that for every $T > 0$, *there exists* $C > 0$, *which depends on* T *and the covariance of* X, *such that*

$$\sup_{0 \le t \le \lfloor nT \rfloor} \mathbb{E}\left[\Delta X_{\frac{t}{n}}^2\right] \le Cn^{-\alpha}. \tag{4.5.2}$$

Lemma 4.5.3. *Let j,k,n be integers with $n \geq 6$ and $1 \leq j \leq k$. Under (H.1)-(H.2), we have the following estimates:*

(a) If $j+3 \leq k \leq 2j+2$, then

$$\left|\mathbb{E}\left[\Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}}\right]\right| \leq C n^{-2\beta} j^{2\beta-\alpha} k^{\alpha-2}. \tag{4.5.3}$$

(b) If $k \geq 2j+2$, then

$$\left|\mathbb{E}\left[\Delta X_{\frac{j}{n}} \Delta X_{\frac{k}{n}}\right]\right| \leq C n^{-2\beta} j^{2\beta+\nu-2} k^{-\nu}. \tag{4.5.4}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[\Delta X_{\frac{k}{n}} \Delta X_{\frac{j}{n}}\right] &= n^{-2\beta}(j+1)^{2\beta}\left(\phi\left(\frac{k+1}{j+1}\right)-\phi\left(\frac{k}{j+1}\right)\right) \\
&\quad - n^{-2\beta} j^{2\beta}\left(\phi\left(\frac{k+1}{j}\right)-\phi\left(\frac{k}{j}\right)\right) \\
&= n^{-2\beta}\left((j+1)^{2\beta}-j^{2\beta}\right)\left(\phi\left(\frac{k+1}{j+1}\right)-\phi\left(\frac{k}{j+1}\right)\right) \\
&\quad + n^{-2\beta} j^{2\beta}\left[\phi\left(\frac{k+1}{j+1}\right)-\phi\left(\frac{k}{j+1}\right)-\phi\left(\frac{k+1}{j}\right)+\phi\left(\frac{k}{j}\right)\right].
\end{aligned}$$

We first show (4.5.3). Condition $j+3 \leq k \leq 2j+2$ implies that the interval $\left[\frac{k}{j+1}, \frac{k+1}{j}\right]$ is included in the interval $[1,5]$. Therefore, using (4.1.2) and (4.1.3), we deduce that there exists a constant $C > 0$ such that for all $x \in \left[\frac{k}{j+1}, \frac{k+1}{j}\right]$,

$$\left|\phi'(x)\right| \leq C(k/j)^{\alpha-1}.$$

and

$$\left|\phi''(x)\right| \leq C(k/j)^{\alpha-2}.$$

The estimate (4.5.3) follows easily from the Mean Value Theorem.

On the other hand $k \geq 2j+2$ implies that the interval $\left[\frac{k}{j+1}, \frac{k+1}{j}\right]$ is included in the interval $[2,\infty]$. Therefore, using (4.1.2) and (4.1.3), we deduce that there exists a constant $C > 0$ such that for all $x \in \left[\frac{k}{j+1}, \frac{k+1}{j}\right]$,

$$\left|\phi'(x)\right| \leq C(k/j)^{-\nu}.$$

and

$$\left|\phi''(x)\right| \leq C(k/j)^{-\nu-1}.$$

Therefore, estimate (4.5.4) follows easily from the Mean Value Theorem. The proof of the lemma is now complete. □

Last, we have two technical results that have been used in the proofs of Theorems 4.1.2 and 4.1.3. For a fixed integer n and nonnegative real t_1, t_2, note that the notation of Section 4.2 gives

$$\mathbb{E}[\Delta X_{\frac{t_1}{n}} \Delta X_{\frac{t_2}{n}}] = \left\langle \partial_{\frac{t_1}{n}}, \partial_{\frac{t_2}{n}} \right\rangle_{\mathfrak{H}}.$$

Lemma 4.5.1. *Assume X satisfies (H.1) and (H.2). Then for any integer $n \geq 2$ and real $T > 0$, there is a constant C is a constant which depends on T and the covariance of X, such that*

$$\sup_{0 \leq k \leq \lfloor nT \rfloor - 1} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \leq C n^{-\alpha}. \tag{4.5.5}$$

Proof. In view of the estimate (4.5.2), we can assume that $n \geq 6$ and $4 \leq j+3 \leq k$ or $4 \leq k+3 \leq j$. If $4 \leq j+3 \leq k$, from the estimates (4.5.3) and (4.5.4), we deduce

$$\left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \leq C n^{-2\beta} j^{2\beta-2}.$$

Summing in the index j we get the desired result, because $2\beta-1\leq 0$ and $2\beta\geq\alpha$. On the other hand, if $4\leq k+3\leq j\leq 2k+2$, the estimates (4.5.3) yields

$$\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\leq Cn^{-2\beta}k^{2\beta-\alpha}j^{\alpha-2}\leq Cn^{-\alpha}j^{\alpha-2},$$

which gives the desired estimate. Finally, if $4\leq k+3$ and $2k+2\leq j$, the estimate (4.5.4) yields

$$\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\leq Cn^{-2\beta}k^{2\beta+\nu-2}j^{-\nu}.$$

If $\alpha+\nu-2\leq 0$, then summing the above estimate in j we obtain the bound

$$Cn^{-2\beta}k^{2\beta-\alpha+(\alpha+\nu-2)}\leq Cn^{-\alpha}.$$

On the other hand, if $\alpha+\nu-2>0$, then

$$Cn^{-2\beta}k^{2\beta+\nu-2}j^{-\nu}\leq Cn^{-2\beta}k^{2\beta-\alpha}\left(\frac{k}{j}\right)^{\alpha+\nu-2}j^{\alpha-2}\leq Cn^{-\alpha}j^{\alpha-2}$$

and summing in j we get the desired bound. □

Lemma 4.5.4. *Assume that* $0<\alpha<1$ *and let* $n\geq 1$ *be an integer. Then, for every* $r\in\mathbb{N}$ *and* $T\geq 0$*,*

$$\sum_{j=0}^{\lfloor nT\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|^{r}\leq Cn^{-2\beta(r-1)}. \tag{4.5.6}$$

Proof. By (4.0.2),

$$\left\langle \partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}=\frac{1}{2}\mathbb{E}\left[(X_{\frac{j+1}{n}}-X_{\frac{j}{n}})(X_{\frac{j+1}{n}}+X_{\frac{j}{n}})\right]=\frac{1}{2}\mathbb{E}\left[X^2_{\frac{j+1}{n}}-X^2_{\frac{j}{n}}\right]=\phi(1)\Psi_n(j),$$

where

$$\Psi_n(j) = \left(\left(\frac{j+1}{n} \right)^{2\beta} - \left(\frac{j}{n} \right)^{2\beta} \right).$$

We can easily show that $\Psi_n(j) \le Cn^{-2\beta}$, and hence,

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|^r = \phi(1)^r \sum_{j=0}^{\lfloor nT \rfloor - 1} \Psi_n(j)^r \le Cn^{-2\beta(r-1)} \sum_{j=0}^{\lfloor nT \rfloor - 1} \Psi_n(j).$$

Since the right-hand side of the last inequality is a telescopic sum, we get

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|^r \le Cn^{-2\beta(r-1)} \left(\frac{\lfloor nT \rfloor}{n} \right)^{2\beta}.$$

Relation (4.5.6) follows from the previous inequality. □

Chapter 5

Collision of eigenvalues for matrix-valued processes.

5.1 Introduction

For $r \in \mathbb{N}$ fixed, consider a centered Gaussian random field $\xi = \{\xi(t); r \in \mathbb{R}_+^r\}$, defined in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with covariance function given by

$$\mathbb{E}\big[\xi(s)\xi(t)\big] = R(s,t),$$

for some non-negative definite function $R : (\mathbb{R}_+^r)^2 \to \mathbb{R}$. Let $\{\xi_{i,j}, \eta_{i,j}; i, j \in \mathbb{N}\}$, be a family of independent copies of ξ. For $\beta \in \{1,2\}$ and $d \in \mathbb{N}$, with $d \geq 2$ fixed, consider the matrix-valued process $X^{\beta} = \{X_{i,j}^{\beta}(t); t \in \mathbb{R}_+^r,\ 1 \leq i, j \leq d\}$, defined by

$$X_{i,j}^{\beta}(t) = \begin{cases} \xi_{i,j}(t) + \mathbf{i}\mathbb{1}_{\{\beta=2\}}\eta_{i,j}(t) & \text{if } i < j \\ (\mathbb{1}_{\{\beta=1\}}\sqrt{2} + \mathbb{1}_{\{\beta=2\}})\xi_{i,i}(t) + \mathbf{i}\mathbb{1}_{\{\beta=2\}}\eta_{i,i}(t) & \text{if } i = j \\ \xi_{i,j}(t) - \mathbf{i}\mathbb{1}_{\{\beta=2\}}\eta_{i,j}(t) & \text{if } j < i. \end{cases} \tag{5.1.1}$$

In accordance to the type of symmetry of $X^{\beta}(t)$, we will refer to X^1 and X^2 as the Gaussian orthogonal ensemble process (GOE) and Gaussian unitary ensemble process

(GUE), respectively. Let A^β be a fixed Hermitian deterministic matrix, such that A^β has real entries in the case $\beta = 1$, and complex entries in the case $\beta = 2$.

Consider the set of the ordered eigenvalues $\lambda_1^\beta(t) \geq \cdots \geq \lambda_d^\beta(t)$ of

$$Y^\beta(t) := A^\beta + X^\beta(t). \tag{5.1.2}$$

The purpose of this paper is to determine necessary and sufficient conditions under which, with probability one, we have $\lambda_1^\beta(t) > \cdots > \lambda_d^\beta(t)$ for all t belonging to a suitable rectangle of $\mathbb{R}_+^r$.

The matrix-valued process Y^β was first studied by Dyson for $\beta = r = 1$, in the case where ξ is a standard Brownian. In particular, he proved that the processes $\lambda_1^1, \ldots, \lambda_d^1$ satisfy a system of stochastic differential equations with non-smooth diffusion coefficients, as well as the non-collision property

$$\mathbb{P}\left[\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t > 0 \text{ and } 1 \leq i < j \leq n\right] = 0. \tag{5.1.3}$$

For a more recent treatment of these results, see [2, Section 4.3].

Afterwards, Nualart and Pérez-Abreu used Young's theory of integration, to prove that in the case where $\beta = r = 1$ and ξ is a Gaussian process with Hölder continuous parths larger than $\frac{1}{2}$, relation (5.1.3) holds. This result can be applied to the case where X^1 is a fractional Brownian matrix with Hurst parameter $\frac{1}{2} < H < 1$. Namely, when $\xi = \{\xi(t); t \geq 0\}$ is centered Gaussian processes with covariance

$$R(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{5.1.4}$$

In this manuscript we prove, among other things, that the results presented in [45] are sharp, in the sense that for $H < 1/2$, the eigenvalues $\lambda_1^1, \dots, \lambda_d^1$ collide with positive probability, and with probability one if $A^1 = 0$. We also give an alternative proof of the results obtained by Nualart and Pérez-Abreu in [45]. On the other hand, we obtain the surprising results that for the fractional Hermitian matrix X^2, the eigenvalues $\lambda_1^2, \dots, \lambda_d^2$ do not collide when $H > \frac{1}{3}$ and collide with positive probability (or with probability one if $A^2 = 0$), when $H < \frac{1}{3}$. The case $H = \frac{1}{3}$ cannot be handled with the techniques used in this paper and remains an open problem.

When $\psi(s,t)$ is of the form (5.1.4) and $\beta = 1$, the non-collision property is of great interest, since it is a necessary condition for characterizing $(\lambda_1^1, \dots, \lambda_d^1)$ as the unique solution of a Young integral equation (in the case where $H > \frac{1}{2}$), or as an Itô stochastic differential equation (in the case $H = \frac{1}{2}$). We refer the reader to [2] and [46] for a proof of such characterizations.

The goal of this manuscript is to investigate the probability of collision of the eigenvalues $\lambda_1^\beta, \dots, \lambda_d^\beta$, for ξ belonging to a class of processes that includes the complex Hermitian and real symmmetric fractional Brownian motion of Hurst parameter $H \neq \frac{1}{2}$. The proofs of our main results are based on estimations of hitting probabilities for Gaussian processes, as well as some geometric properties of the set of degenerate matrices. This approach is different from the methodology used in [45] and [2], where the process $(\lambda_1^1, \dots, \lambda_d^1)$ is studied by means of stochastic integral techniques.

5.2 Main results

As mentioned before, the ideas presented in this manuscript rely heavily on the the hitting probability estimations presented in [3]. In order to apply such results, we will assume that the there exists a multiparameter index $(H_1,\dots,H_r)\in(0,1)^r$, and an interval

$$I=[a,b]:=\prod_{j=1}^{r}[a_j,b_j]\subset\mathbb{R}_+^r, \tag{5.2.1}$$

with $a=(a_1,\dots,a_r), b=(b_1,\dots,b_r)\in\mathbb{R}_+^r$ satisfying $a_i\le b_i$ for $1\le i\le r$, such that the following technical conditions hold:

(H1) There exist strictly positive and finite constants $c_{2,1}, c_{2,2}$ and $c_{2,3}$ such that $\mathbb{E}\left[\xi(t)^2\right]\ge c_{2,1}$ for all $t\in I$ and

$$c_{2,2}\sum_{j=1}^{r}\left|s_j-t_j\right|^{2H_j}\le\mathbb{E}\left[(\xi(s)-\xi(t))^2\right]\le c_{2,3}\sum_{j=1}^{r}\left|s_j-t_j\right|^{2H_j},$$

for $s,t\in I$ of the form $s=(s_1,\dots,s_r)$ and $t=(t_1,\dots,t_r)$.

(H2) There exists a constant $c_{2,4}>0$ such that for all $s=(s_1,\dots,s_r), t=(t_1,\dots,t_r)\in I$,

$$\operatorname{Var}[\xi(t)\mid\xi(s)]\ge c_{2,4}\sum_{j=1}^{r}\left|s_j-t_j\right|^{2H_j},$$

where $\operatorname{Var}[\xi(t)\mid\xi(s)]$ denotes the conditional variance of $\xi(t)$ given $\xi(s)$.

The collection of random fields satisfying conditions **(H1)** and **(H2)** includes, among others, the fractional Brownian sheet and the solutions to the stochastic heat equation driven by space-time white noise. Our main results are Theorem 5.2.1 and Corollary 5.2.2 below. The proofs will be presented in Section 5.5.

Theorem 5.2.1. *Define $Q:=\sum_{j=1}^{r}\frac{1}{H_j}$. Then, for $\beta=1,2$, we have the following results:*

(i) If $Q < \beta + 1$,

$$\mathbb{P}\left[\lambda_i^\beta(t) = \lambda_j^\beta(t) \ \textit{for some} \ t \in I \textit{ and } 1 \le i < j \le n\right] = 0. \tag{5.2.2}$$

(ii) If $Q > \beta + 1$,

$$\mathbb{P}\left[\lambda_i^\beta(t) = \lambda_j^\beta(t) \ \textit{for some} \ t \in I \textit{ and } 1 \le i < j \le n\right] > 0. \tag{5.2.3}$$

In particular, when ξ is a one-parameter fractional Brownian motion with Hurst parameter $H \in (0,1)$, we obtain the following result.

Corollary 5.2.2. *If* $\xi = \{\xi(t); t \ge 0\}$ *is a fractional Brownian motion with Hurst parameter* $0 < H < 1$ *and* $I = [a,b]$*, where* $0 < a < b$*. we have the following results:*

(i) If $\frac{1}{1+\beta} < H < 1$,

$$\mathbb{P}\left[\lambda_i^\beta(t) = \lambda_j^\beta(t) \ \textit{for some} \ t \in I \textit{ and } 1 \le i, j \le n\right] = 0. \tag{5.2.4}$$

(ii) If $0 < H < \frac{1}{1+\beta}$,

$$\mathbb{P}\left[\lambda_i^\beta(t) = \lambda_j^\beta(t) \ \textit{for some} \ t \in I \textit{ and } 1 \le i, j \le n\right] > 0. \tag{5.2.5}$$

Moreover, if either $A^\beta = 0$ *or the spectrum of* A^β *has cardinality* $d-1$*, then*

$$\mathbb{P}\left[\lambda_i^\beta(t) = \lambda_j^\beta(t) \ \textit{for some} \ t > 0 \textit{ and } 1 \le i, j \le n\right] = 1. \tag{5.2.6}$$

Remark 5.2.3. *Combining Corollary 5.2.2 with [2, Section 4.3], we conclude that the condition* $H \ge \frac{1}{2}$ *is necessary and sufficient for the non-collision property of real* *symmetric fractional Brownian matrices. On the other hand, the critical value for the collision property for the fractional GUE is $H = \frac{1}{3}$. Nevertheless, our proof of Corollary 5.2.2 is not valid for the critical value $H = \frac{1}{1+\beta}$. Thus, if $\beta = 2$ and $H = \frac{1}{3}$, the non-collision property for $\lambda_1^2, \ldots, \lambda_d^2$ is still an open problem.*

The rest of the paper is organized as follows. Section 3 contains the results from hitting probabilities for Gaussian fields that we will use throughout the paper. In Section 4, we describe some geometric properties of the set of degenerate Hermitian matrices of dimension d; namely, the Hermitian matrices with at least one repeated eigenvalue. Finally, in Section 5 we prove Theorem 5.2.2 and Corollary 5.2.2.

5.3 Hitting probabilities

In this section we present some results on hitting probabilities for Gaussian fields and their relation to the capacity and Hausdorff dimension of Borel sets. We will closely follow the work by Biermé, Lacaux and Xiao presented in [3], and we refer the interested reader to [3, 56, 57] for a detailed treatment of the theory of hitting probabilities.

For $n \in \mathbb{N}$, let $W = \{(W_1(t), \ldots, W_n(t)); t \in \mathbb{R}_+^r\}$ be an n-dimensional Gaussian field, whose entries are independent copies of ξ. In the sequel, for every $q > 0$ and any Borel set $F \subset \mathbb{R}^n$, $\mathscr{H}_q(F)$ will denote the q-dimensional Hausdorff measure of F and $\mathscr{C}_\alpha(F)$ will denote the Bessel-Riesz capacity of order α of F, defined by

$$\mathscr{C}_\alpha(F) := \left(\inf_{\mu \in \mathscr{P}(F)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_\alpha(\|x-y\|)\mu(dx)\mu(dy) \right)^{-1}, \tag{5.3.1}$$

where $\mathscr{P}(F)$ is the family of probability measures supported in F and the function $f_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$f_\alpha(r) := \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \log\left(\frac{e}{r \wedge 1}\right) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \tag{5.3.2}$$

Define as well the Hausdorff dimension $\dim_H(F)$, by

$$\dim_H(F) := \inf\{q > 0 \mid \mathscr{H}_q(F) = 0\}.$$

We refer the reader to [14, 30] for basic properties of the Hausdorff measure and capacity of Borel sets. The following results, presented in [3, Theorem 2.1], will be used to prove Theorem 5.2.1.

Theorem 5.3.1 (Biermé, Lacaux and Xiao)**.** *Consider an interval I of the form* (5.2.1)*. If $F \subset \mathbb{R}^n$ is a Borel set, then there exist constants $c_1, c_2 > 0$, such that*

$$c_1 \mathscr{C}_{n-Q}(F) \leq \mathbb{P}\left[W^{-1}(F) \cap I \neq \emptyset\right] \leq c_2 \mathscr{H}_{n-Q}(F),$$

where $Q = \sum_{j=1}^{r} \frac{1}{H_j}$.

As a consequence, we have the following result.

Corollary 5.3.2. *Let $F \subset \mathbb{R}^n$ be a Borel set. Then*

1. *If $\dim_H(F) < n - Q$, the set $W^{-1}(F) \cap I$ is empty with probability one.*
2. *If $\dim_H(F) > n - Q$, the set $W^{-1}(F) \cap I$ is non-empty with positive probability.*

5.4 Geometric properties of degenerate Hermitian matrices

Let $\mathscr{S}(d)$ and $\mathscr{H}(d)$ denote the set of real symmetric matrices and complex Hermitian matrices, respectively. Define

$$n_\beta(d) := \begin{cases} d(d+1)/2 & \text{if } \beta = 1 \\ d^2 & \text{if } \beta = 2. \end{cases}$$

In the sequel, we will identify an element $x \in \mathbb{R}^{n_1(d)}$ with the unique $\hat{x} = \{\hat{x}_{i,j}\}_{1 \le i,j \le d} \in \mathscr{S}(d)$ satisfying $\hat{x}_{i,j} = x_{\frac{1}{2}i(1+2d-i)-d+j}$, for $1 \le i \le j \le d$. In a similar way, we can identify an element $x \in \mathbb{R}^{n_2(d)}$ with the unique $\hat{x} \in \mathscr{H}(d)$ given by

$$\hat{x}_{i,j} = \begin{cases} x_{\frac{1}{2}i(1+2d-i)-d} & \text{if } i = j \\ x_{\frac{1}{2}i(1+2d-i)-d+j} + \mathbf{i} x_{n_1(d)+\frac{1}{2}i(2d-i-1)-d+j} & \text{if } i < j. \end{cases}$$

We will denote by $\Phi_i(x)$ the i-th largest eigenvalue of $\hat{x}$. Notice that since $(\Phi_1(x), \dots \Phi_d(x))$ are the ordered roots of the characteristic polynomial of $\hat{x}$, it follows that $\Phi_i(x)$ is continuous over x for every $1 \le i \le d$.

Define the sets $\mathscr{H}^d_{deg}$ and $\mathscr{S}^d_{deg}$ by

$$\mathscr{H}^d_{deg} := \{x \in \mathbb{R}^{n_2(d)} \mid \Phi_i(x) = \Phi_j(x), \text{ for some } 1 \le i < j \le d\}, \tag{5.4.1}$$

$$\mathscr{S}^d_{deg} := \{x \in \mathbb{R}^{n_1(d)} \mid \Phi_i(x) = \Phi_j(x), \text{ for some } 1 \le i < j \le d\}. \tag{5.4.2}$$

We are interested in describing the size of the sets $\mathscr{H}^d_{deg}$ and $\mathscr{S}^d_{deg}$. The main results of this section are Propositions 5.4.5, 5.4.6, 5.4.7 and 5.4.8 which, roughly speaking, state that there exist measurable sets $\mathscr{S}^d_{in}, \mathscr{S}^d_{out} \subset \mathbb{R}^{n_1(d)}$ and $\mathscr{H}^d_{in}, \mathscr{H}^d_{out} \subset \mathbb{R}^{n_2(d)}$, satisfying

$$\mathscr{S}^d_{in} \subset \mathscr{S}^d_{deg} \subset \mathscr{S}^d_{out} \qquad \text{and} \qquad \mathscr{H}^d_{in} \subset \mathscr{H}^d_{deg} \subset \mathscr{H}^d_{out},$$

as well as the following properties:

1. $\mathscr{S}^d_{in}$ and $\mathscr{H}^d_{in}$ are manifolds of dimensions $n_1(d)-2$ and $n_2(d)-3$, respectively.
2. $\mathscr{S}^d_{out}$ is the image of a smooth function defined in an open subset of $\mathbb{R}^{n_1(d)-2}$ with values in $\mathbb{R}^{n_1(d)}$ and $\mathscr{H}^d_{in}$ is the image of a smooth function defined in an open subset of $\mathbb{R}^{n_2(d)-3}$ with values in $\mathbb{R}^{n_2(d)}$.

In Section 5.5, we will use these properties to show that $\mathscr{S}^d_{deg}$ and $\mathscr{H}^d_{deg}$ have Hausdorff dimension $n_1(d)-2$ and $n_2(d)-3$ respectively, which will be an important ingredient in the proof of Theorem 5.2.1. Notice that after identifying the random matrix $Y^{\beta}(t)$ defined in (5.1.2) as a random vector with values in $\mathbb{R}^{n_\beta(d)}$, we have that

$$\{\lambda^1_i(t) = \lambda^1_j(t) \ \text{ for some } \ t \in I \text{ and } 1 \leq i < j \leq n\} = \{Y^1(t) \in \mathscr{S}^d_{deg} \ \text{ for some } \ t \in I\},$$

and

$$\{\lambda^2_i(t) = \lambda^2_j(t) \ \text{ for some } \ t \in I \text{ and } 1 \leq i < j \leq n\} = \{Y^2(t) \in \mathscr{H}^d_{deg} \ \text{ for some } \ t \in I\}.$$

Thus, in order to prove Theorem 5.2.1, it suffices to study the hitting probability of $Y^1(t)$ to $\mathscr{S}^d_{deg}$ and $Y^2(t)$ to $\mathscr{H}^d_{deg}$.

To prove the main results of this section, we will require the following terminology from differential geometry. In the sequel, for every $n \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $\delta > 0$, we will denote by $B_\delta(x)$ the open ball of radius δ and center x. In addition, we will say that an

$\mathbb{R}^n$-valued function, defined over an open subset of $\mathbb{R}^m$ with $m \in \mathbb{N}$, is smooth, if it is infinitely differentiable.

Definition 5.4.1. *Let $m, n \in \mathbb{N}$ be such that $m \leq n$. A set $M \subset \mathbb{R}^n$ is a smooth submanifold of $\mathbb{R}^n$, with dimension m, if for every $x_0 \in M$, there exists $\varepsilon > 0$, an open neighborhood of zero $U \subset \mathbb{R}^m$ and a smooth mapping*

$$F : U \to M,$$

satisfying $F(0) = x_0$, as well as the following properties:

- *F is a homeomorphism from U to $M \cap B_\varepsilon(x_0)$.*
- *For every $p \in U$, the derivative of F at p, denoted by DF_p, is an injective mapping.*

If such mapping F exists, we call it a local chart for M covering x_0.

If M is a smooth submanifold of $\mathbb{R}^n$, we define its tangent plane at a given point $x \in M$, denoted by TM_x, as the set of vectors of the form $\alpha'(0)$, where $\alpha : (-1, 1) \to M$ is a smooth curve satisfying $\alpha(0) = x$.

Let M and N be smooth manifolds. We say that $f : M \to N$ is smooth if for every $x \in M$ and all charts F and G, covering x and $f(x)$ respectively, the function $G^{-1} \circ f \circ F$ is smooth. In this case, we can define the derivative of f at a given point $x \in M$, as the function $Df_x : TM_x \to TN_{f(x)}$, that maps every vector $v \in TM_x$ of the form $v = \alpha'(0)$, to the vector $Df_x(v) := \frac{d}{dt} f(\alpha(t))|_{t=0}$.

Let $f : M \to N$ be a smooth mapping between manifolds $M, N \subset \mathbb{R}^n$. We say that a point $y \in N$ is a regular value for f, if for all $x \in f^{-1}\{y\}$, the derivative $Df_x : TM_x \to TN_y$ is surjective. The following result allows us to identify the level curves of a smooth function, as smooth manifolds. Its proof can be found, for instance, in [54, Theorem 9.9].

Theorem 5.4.2 (Preimage theorem)**.** *Consider a smooth mapping $f : M \to N$, where M and N are smooth submanifolds of $\mathbb{R}^n$ of dimensions m_M and m_N respectively, with $m_N \leq m_M \leq n$. If $y \in N$ is a regular value for f, then $f^{-1}\{y\}$ is a smooth submanifold of $\mathbb{R}^n$ of dimension $m_M - m_N$.*

Along the paper we will denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^N$ and by $\langle\cdot,\cdot\rangle$ the corresponding inner product. We will use the same notation for the norm and inner product in $\mathbb{C}^N$.

For $d,h \in \mathbb{N}$, let $\mathbb{R}^{d\times h}$ denote the set of real matrices of dimensions $d\times h$ and let I_d be the identity element of $\mathbb{R}^{d\times d}$. For every integer $0 \leq i \leq d$, we define the sets

$$\mathscr{O}(d;i) := \{A \in \mathbb{R}^{d\times(d-i)} : A^*A = I_{d-i}\}, \tag{5.4.3}$$

where A^* is the transpose of A. In the case where $i = 0$, the set $\mathscr{O}(d;i)$ is the orthogonal group of dimension d, which will be denoted simply by $\mathscr{O}(d) := \mathscr{O}(d;0)$. Using the preimage theorem, we can show that $\mathscr{O}(d;i)$ is a submanifold of $\mathbb{R}^{d\times(d-i)} \cong \mathbb{R}^{d(d-i)}$, of dimension $\frac{d(d-1)-i(i-1)}{2}$. This result can be proved in the following manner. Consider the mapping $f : \mathbb{R}^{d\times(d-i)} \to \mathscr{S}(d-i)$, defined by

$$f(X) := X^*X - I_{d-i}.$$

Then, for every $A \in f^{-1}\{0\}$, the derivative of f at A, denoted by Df_A, satisfies

$$Df_A B = A^*B + B^*A, \qquad \text{for every } B \in \mathbb{R}^{d\times(d-i)}. \tag{5.4.4}$$

In particular, for every $C \in \mathscr{S}(d-i)$, the matrix $B := \frac{1}{2}AC$ satisfies $Df_A B = C$, so that Df_A is surjective for every $A \in f^{-1}\{0\}$. Consequently, zero is a regular value for f, and by the preimage theorem, $\mathscr{O}(d;i) = f^{-1}\{0\}$ is a smooth submanifold of $\mathbb{R}^{d\times(d-i)}$ of dimension $\dim(\mathbb{R}^{d(d-i)}) - \dim(\mathscr{S}(d-i)) = \frac{d(d-1)-i(i-1)}{2}$.

Similarly, for $d,h \in \mathbb{N}$ we denote by $\mathbb{C}^{d\times h}$ the set of complex matrices of dimensions $d \times h$, and define

$$\mathscr{U}(d;i) := \{A \in \mathbb{C}^{d\times(d-i)} : A^*A = I_{d-i}\}, \tag{5.4.5}$$

where A^* denotes the conjugate of the transpose of A. Proceeding as before, we can show that $\mathscr{U}(d;i)$ is a smooth submanifold of $\mathbb{C}^{d\times(d-i)} \cong \mathbb{R}^{2d(d-i)}$, of dimension $d^2 - i^2$. In particular, the unitary group $\mathscr{U}(d) := \mathscr{U}(d;0)$ has dimension d^2.

In the sequel, for every $A \in \mathbb{C}^{d\times h}$, we will denote by $A_{*,j}$ the j-th column of A, where $1 \le j \le h$. Next we will show the following technical result.

Lemma 5.4.1. *For every $R \in \mathscr{U}(d;2)$, there exists $\gamma > 0$, such that the set*

$$\mathscr{V}_\gamma^R := \{A \in \mathscr{U}(d;2) \cap B_\gamma(R) : \langle A_{*,j}, R_{*,j}\rangle = |\langle A_{*,j}, R_{*,j}\rangle| \quad \text{for } 1 \le j \le d-2\}, \tag{5.4.6}$$

is a (d^2-d-2)-dimensional submanifold of $\mathscr{U}(d;2) \cap B_\gamma(R)$.

Proof. Consider the manifold

$$\mathbb{T}^{d-2} := \{(e^{\mathbf{i}\theta_1}, \ldots, e^{\mathbf{i}\theta_{d-2}}) \in \mathbb{C}^{d-2} : \theta_i \in [-\pi/2, \pi/2)\}.$$

We will prove that if $\gamma > 0$ is sufficiently small, the point $\vec{1} := (1, \ldots, 1)$ is a regular value for the smooth function $f : \mathscr{U}(d;2) \cap B_\gamma(R) \to \mathbb{T}^{d-2}$, defined by

$$f(A) := (|\langle A_{*,1}, R_{*,1}\rangle|^{-1}\langle A_{*,1}, R_{*,1}\rangle, \ldots, |\langle A_{*,d-2}, R_{*,d-2}\rangle|^{-1}\langle A_{*,d-2}, R_{*,d-2}\rangle). \tag{5.4.7}$$

Notice that $\mathscr{U}(d;2)$ is a (d^2-4)-dimensional manifold. This implies, by Theorem 5.4.2, that the set $\mathscr{V}_\gamma^R = f^{-1}\{\vec{1}\}$ is a (d^2-d-2)-dimensional manifold. To check that $\vec{1}$ is a regular value for f, notice that the tangent plane to $\mathbb{T}^{d-2}$ at $\vec{1}$, consists of the the set of vectors $\eta \in \mathbb{C}^{d-2}$ of the form $\eta = (\mathbf{i}\eta_1, \ldots, \mathbf{i}\eta_{d-2})$, for $\eta_i \in \mathbb{R}$. For such η, there exists $\delta > 0$, such that the mapping $A : (-\delta, \delta) \to \mathscr{V}_\gamma^R$, given by

$$A_{i,j}(t) = e^{\mathbf{i}\eta_j t} R_{i,j},$$

is a curve inside of $\mathscr{U}(d;2) \cap B_\gamma(R)$, satisfying $Df_R(\frac{d}{dt} f(A(t))\big|_{t=0}) = \eta$. This proves that $\vec{1}$ is indeed a regular value of f. $\square$

The next lemma is a refinement of the well-known continuity property for the eigen-projections of real symmetric matrices. In the sequel, $\mathscr{D}(d)$ will denote the set of diagonal real matrices of dimension d. In addition, for every $A \in \mathbb{C}^{d\times d}$, the set $\mathbf{Sp}(A)$ will denote the spectrum of A and for $\lambda \in \mathbf{Sp}(A)$, $\mathbf{E}_\lambda^A$ will denote the eigenspace associated to λ. For every $w^1, \ldots w^h \in \mathbb{C}^d$, with $h \in \mathbb{N}$, we will denote by $[w^1, \ldots, w^h]$ the element of $\mathbb{C}^{d\times h}$, whose j-th column is equal to w^j for all $1 \le j \le h$.

Lemma 5.4.3. *Let A be a $d \times d$ real symmetric matrix, with $|\boldsymbol{Sp}(A)| = d-1$, such that*

$$A = PDP^*,$$

for some $P \in \mathscr{O}(d)$ *and* $D \in \mathscr{D}(d)$*. Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$*, such that for all* $B \in \mathscr{S}^d_{deg}$ *satisfying*

$$\max_{1 \le i,j \le d} \left| A_{i,j} - B_{i,j} \right| < \delta, \tag{5.4.8}$$

there exists a spectral decomposition of the form $B = Q \Delta Q^*$*, where* $Q \in \mathscr{O}(d)$ *and* $\Delta \in \mathscr{D}(d)$ *satisfy*

$$\max_{1 \le i,j \le d} \left| Q_{i,j} - P_{i,j} \right| < \varepsilon \tag{5.4.9}$$

and

$$\max_{1 \le i \le d} |D_{i,i} - \Delta_{i,i}| < \varepsilon. \tag{5.4.10}$$

Proof. The existence of a matrix Δ satisfying (5.4.10) follows from the continuity of Φ, so it suffices to prove (5.4.9). The idea for proving this relation is the following: first we express the eigenprojections of the degenerate symmetric matrices lying within a small neighborhood U around A, as matrix-valued Cauchy integrals. This representation allows us to prove that the mapping that sends an element $B \in U$, to the eigenprojection of B over its i-th largest eigenvalue, is continuous with respect to the entries of B. Finally, we will choose a set of eigenvectors for B by applying the (continuous) eigenprojections of B to the eigenvectors of A. The matrix Q, with columns given by the renormalization of such eigenvectors will then satisfy (5.4.9).

The detailed proof is as follows. Define $\lambda_i := D_{i,i}$ for $1 \le i \le d$, and assume without loss of generality that $\lambda_1 \le \cdots \le \lambda_{d-1} = \lambda_d$. Using the fact that $|\mathbf{Sp}(A)| = d-1$, we get

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{d-2} < \lambda_{d-1} = \lambda_d. \tag{5.4.11}$$

For $i = 1, \ldots, d$, let $\mathscr{C}_A^i \subset \mathbb{C} \backslash \mathbf{Sp}(A)$ be any smooth closed curve around λ_i and denote by $\mathscr{I}_A^i$ the closure of the interior of $\mathscr{C}_A^i$. Assume that $\mathscr{C}_A^{d-1} = \mathscr{C}_A^d$ and that the diameter of $\mathscr{C}_A^i$ is sufficiently small, so that $\mathscr{I}_A^1, \ldots, \mathscr{I}_A^{d-1}$ are disjoint. For $\delta > 0$, define the set

$$V_\delta := \{B \in \mathscr{S}_{deg}^d \mid \max_{1 \le i,j \le d} \left| A_{i,j} - B_{i,j} \right| < \delta \}.$$

Using (5.4.11), as well as the continuity of $\Phi_1, \ldots, \Phi_d$ and the fact that $V_\delta \subset \mathscr{S}_{deg}^d$, we can easily show that there exists $\delta > 0$, such that for all $B \in V_\delta$,

$$\Phi_1(B) < \Phi_2(B) < \cdots < \Phi_{d-2}(B) < \Phi_{d-1}(B) = \Phi_d(B), \tag{5.4.12}$$

and

$$\Phi_i(B) \in \mathscr{I}_A^i \quad \text{for all } \ B \in V_\delta \ \text{ and } \ 1 \le i \le d. \tag{5.4.13}$$

For such δ, define the mapping $\kappa_A^i : V_\delta \to \mathscr{S}(d)$, by

$$\kappa_A^i(B) := \frac{1}{2\pi \mathbf{i}} \int_{\mathscr{C}_A^i} (\xi I_d - B)^{-1} d\xi. \tag{5.4.14}$$

The matrix $\kappa_A^i(B)$ is the projection over the sum of the eigenspaces associated to eigenvalues of B inside of $\mathscr{I}_A^i$ (see [32, page 200, Theorem 6]). Thus, using (5.4.12), (5.4.13) and the fact that $\mathscr{I}_A^1, \ldots, \mathscr{I}_A^{d-1}$ are disjoint, we conclude that $\kappa_A^i(B)$ is the projection over $\mathbf{E}_{\Phi_i(B)}^B$, for all $1 \le i \le d$.

From (5.4.14), it follows that the mapping $B \mapsto \kappa_A^i(B)$, defined on V_δ, is a continuous function of the entries of B. Let $v^1, \ldots, v^d$ denote the columns of P and define

$$w^j := \frac{\kappa_A^j(B)v^j}{\|\kappa_A^j(B)v^j\|}, \tag{5.4.15}$$

for $1 \leq j \leq d-1$ and

$$w^d := \frac{\kappa_A^d(B)v^d}{\|\kappa_A^d(B)v^d\|} - \frac{\langle \kappa_A^d(B)v^d, \kappa_A^{d-1}(B)v^{d-1}\rangle}{\|\kappa_A^d(B)v^d\|\|\kappa_A^{d-1}(B)v^{d-1}\|^2}\kappa_A^{d-1}(B)v^{d-1}. \tag{5.4.16}$$

Since $\kappa_A^j(B)$ is the projection over $\mathbf{E}^B_{\Phi_i(B)}$, for all $1 \leq j \leq d$ and $B \in V_\delta$, we can easily check that $w^1, \ldots, w^d$ are orthonormal eigenvectors for B. Thus, using the continuity of κ_A^j and the fact that $\kappa_A^i(A)v^j = v^j$ for all $1 \leq j \leq d$, we deduce that there exists $\delta' > 0$, such that for all $B \in V_{\delta'}$, the vectors $w^1, \ldots, w^d$ given by (5.4.15) and (5.4.16), form an orthonormal base of eigenvectors for B satisfying

$$\max_{1 \leq i,j \leq d} \left| v_i^j - w_i^j \right| < \varepsilon,$$

where

$$v^j = (v_1^j, \ldots, v_d^j), \quad \text{and} \quad w^j = (w_1^j, \ldots w_d^j).$$

Thus, the matrix $Q = [w^1, \ldots, w^d]$ satisfies $B = Q\Delta Q^*$ and (5.4.9), as required. □

The next result is the complex version of Lemma 5.4.3, where the sets $\mathscr{S}(d)$ and $\mathscr{O}(d)$ are replaced by $\mathscr{H}(d)$ and $\mathscr{U}(d)$, respectively.

Lemma 5.4.4. *Let A be a $d \times d$ complex Hermitian matrix, with $|\boldsymbol{Sp}(A)| = d-1$, such that*

$$A = PDP^*,$$

for some $P \in \mathscr{U}(d)$ and $D \in \mathscr{D}(d)$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $B \in \mathscr{H}^d_{deg}$ satisfying

$$\max_{1 \le i,j \le d} \left| A_{i,j} - B_{i,j} \right| < \delta,$$

there exist a spectral decomposition of the form $B = Q\Delta Q^*$, where $Q \in \mathscr{U}(d)$ and $\Delta \in \mathscr{D}(d)$ satisfy the relations

$$\max_{1 \le i,j \le d} \left| Q_{i,j} - P_{i,j} \right| < \varepsilon \qquad \text{and} \qquad \max_{1 \le i \le d} |D_{i,i} - \Delta_{i,i}| < \varepsilon.$$

Proof. It follows from arguments similar to those used in the proof of Lemma 5.4.3. □

Define the function $\Lambda : \mathbb{R}^{d-1} \to \mathscr{D}(d)$, that maps the vector $\beta = (\beta_1, \ldots, \beta_{d-1}) \in \mathbb{R}^{d-1}$, to the matrix $\Lambda(\beta) = \{\Lambda_{i,j}(\beta); 1 \le i,j \le d\}$, given by

$$\Lambda_{i,j}(\beta) := \begin{cases} \delta_{i,j}\beta_i & \text{if} \quad 1 \le i \le d-2 \\ \delta_{i,j}\beta_{d-1} & \text{if} \quad i = d-1, d. \end{cases} \tag{5.4.17}$$

In the next proposition, we bound from above the set $\mathscr{S}^d_{deg}$.

Proposition 5.4.5. *There exists a compactly supported smooth function* $\Pi : \mathbb{R}^{\frac{d(d-1)}{2}-1} \to \mathbb{R}^{d\times d}$*, such that the mapping* $F : \mathbb{R}^{\frac{d(d-1)}{2}-1} \times \mathbb{R}^{d-1} \to \mathscr{S}(d)$*, defined by*

$$F(\alpha,\beta) := \Pi(\alpha)\Lambda(\beta)\Pi(\alpha)^*, \tag{5.4.18}$$

for $\alpha \in \mathbb{R}^{\frac{d(d-1)}{2}-1}$ *and* $\beta \in \mathbb{R}^{d-1}$*, satisfies*

$$\mathscr{S}^d_{deg} \subset \mathscr{S}^d_{out} := \{x \in \mathbb{R}^{n_1(d)} : \hat{x} \in \mathrm{Im}(F)\}. \tag{5.4.19}$$

Proof. For $\varepsilon > 0$, define the interval $J_\varepsilon := (-\varepsilon,\varepsilon)^{\frac{d(d-1)}{2}-1}$. First we reduce the problem, to proving that there exist $L \in \mathbb{N}$ and smooth functions $\Pi^1,\dots,\Pi^L : \mathbb{R}^{\frac{d(d-1)}{2}-1} \to \mathbb{R}^{d\times d}$, supported in J_ε, such that the mappings $F^l : J_\varepsilon \times \mathbb{R}^{d-1} \to \mathscr{S}(d)$, defined by

$$F^l(\alpha,\beta) := \Pi^l(\alpha)\Lambda(\beta)\Pi^l(\alpha)^*, \tag{5.4.20}$$

for $1 \le l \le L$, $\alpha \in J_\varepsilon$ and $\beta \in \mathbb{R}^{d-1}$, satisfy

$$\mathscr{S}^d_{deg} \subset \{x \in \mathbb{R}^{n_1(d)} : \hat{x} \in \bigcup_{l=1}^{L} \mathrm{Im}(F^l)\}. \tag{5.4.21}$$

To show this reduction, notice that if (5.4.21) holds, then any smooth function Π, supported in $J_{3\varepsilon L}$, satisfying

$$\Pi(x) := \Pi^l(x - 3l\varepsilon, 0,\dots,0)) \qquad \text{if} \qquad x \in B_\varepsilon(3l\varepsilon,0,\dots,0) \subset \mathbb{R}^{\frac{d(d-1)}{2}-1},$$

is such that the mapping (5.4.18) satisfies (5.4.19).

Therefore, it suffices to find $\Pi^1,\dots,\Pi^L$. The heuristics for constructing such functions is the following: every matrix $X \in \mathscr{S}^d_{deg}$ can be expressed in the form

$$X = PDP^*,$$

with $D \in \mathscr{D}(d)$ and $P \in \mathscr{O}(d)$. Since X is degenerate, we have some flexibility for choosing P, due to the fact that if X has eigenvalues $\mu_1,\dots,\mu_d$, and $\mu_h = \mu_{h+1}$, then the eigenspaces $\mathbf{E}^X_{\mu_j}$, with $\mu_j \neq \mu_h$, completely determine $\mathbf{E}^X_{\mu_h}$. This allows us to construct P by describing only the eigenvectors associated to $\mathbf{E}^X_{\mu_j}$, with $\mu_j \neq \mu_h$. We can show that these spaces can be locally embedded into the set $\mathscr{O}(d;2)$, which has dimension $\frac{d(d-1)}{2} - 1$. Then we extend such local embeddings to compactly supported $\mathbb{R}^{d\times d}$-valued functions, and apply a compactness argument to obtain the existence of $\Pi^1,\dots,\Pi^L$.

The detailed construction is as follows. For each matrix $R \in \mathscr{O}(d;2)$, we have that $R^*R = I_{d-2}$, and thus, the columns of R are orthonormal. As a consequence, by completing $\{R_{*,1},\dots,R_{*,d-2}\}$ to an orthonormal basis of $\mathbb{R}^d$, we can choose an element $P \in \mathscr{O}(d)$, such that $P_{*,j} = R_{*,j}$ for all $1 \le j \le d-2$. Since $\mathscr{O}(d;2)$ is a smooth manifold of dimension $\frac{d(d-1)}{2} - 1$, we have that if $\gamma > 0$ is sufficiently small, the set $\mathscr{O}(d;2)\cap B_\gamma(R)$ can be parametrized with a chart φ, defined on J_ε, for some $\varepsilon > 0$. Namely, the mapping

$$\varphi : J_\varepsilon \to \mathscr{O}(d;2)\cap B_\gamma(R)$$

is a diffeomorphism satisfying $\varphi(0) = R$. Denote by $\varphi_{*,j}$ the j-th column vector of φ. By construction, every matrix $S \in \mathscr{O}(d;2)$ of the form $S = \varphi(\alpha)$, with $\alpha \in J_\varepsilon$, satisfies

$\left\|P_{*,j} - S_{*,j}\right\| < \gamma$ for all $1 \leq j \leq d-2$, and thus, for γ sufficiently small,

$$\left| \|P_{*,d-1} - \sum_{j=1}^{d-2} \left\langle S_{*,j}, P_{*,d-1} \right\rangle S_{*,j}\| - 1 \right|$$
$$= \left| \|P_{*,d-1} - \sum_{j=1}^{d-2} \left\langle S_{*,j}, P_{*,d-1} \right\rangle S_{*,j}\| - \|P_{*,d-1} - \sum_{j=1}^{d-2} \left\langle P_{*,j}, P_{*,d-1} \right\rangle P_{*,j}\| \right| < \frac{1}{2}.$$

As a consequence, $\left\|P_{*,d-1} - \sum_{j=1}^{d-2} \left\langle \varphi_{*,j}(\alpha), P_{*,j} \right\rangle \varphi_{*,j}(\alpha)\right\|$ is bounded away from zero for all $\alpha \in J_\varepsilon$, and hence, the mapping $\alpha \mapsto \psi_1(\alpha)$, with

$$\psi_1(\alpha) := \frac{P_{*,d-1} - \sum_{j=1}^{d-2} \left\langle \varphi_{*,j}(\alpha), P_{*,d-1} \right\rangle \varphi_{*,j}(\alpha)}{\left\|P_{*,d-1} - \sum_{j=1}^{d-2} \left\langle \varphi_{*,j}(\alpha), P_{*,d-1} \right\rangle \varphi_{*,j}(\alpha)\right\|} \tag{5.4.22}$$

is smooth. Proceeding similarly, we can show that for γ sufficiently small, the mapping $\alpha \mapsto \psi_2(\alpha)$, with

$$\psi_2(\alpha) := \frac{P_{*,d} - \left\langle \psi_1(\alpha), P_{*,d} \right\rangle \psi_1(\alpha) - \sum_{j=1}^{d-2} \left\langle \varphi_{*,j}(\alpha), P_{*,d} \right\rangle \varphi_{*,j}(\alpha)}{\left\|P_{*,d} - \left\langle \psi_1(\alpha), P_{*,d} \right\rangle \psi_1(\alpha) - \sum_{j=1}^{d-2} \left\langle \varphi_{*,j}(\alpha), P_{*,d} \right\rangle \varphi_{*,j}(\alpha)\right\|} \tag{5.4.23}$$

is smooth as well. Let $\Pi : \mathbb{R}^{\frac{d(d-1)}{2}-1} \to \mathbb{R}^{d \times d}$ be any smooth function, supported in J_ε, such that for all $\alpha \in J_{\varepsilon/2}$,

$$\Pi_{*,j}(\alpha) := \begin{cases} \varphi_{*,j}(\alpha) & \text{if } 1 \leq j \leq d-2 \\ \psi_1(\alpha) & \text{if } j = d-1 \\ \psi_2(\alpha) & \text{if } j = d. \end{cases} \tag{5.4.24}$$

By construction, Π has the property that

$$V_\Pi^R := \{[\Pi_{*,1}(\alpha), \ldots, \Pi_{*,d-2}(\alpha)] : \alpha \in J_{\varepsilon/2}\} = \varphi(J_{\varepsilon/2}), \tag{5.4.25}$$

is an open subset of $\mathscr{O}(d;2)$ containing R. Therefore, since $\mathscr{O}(d;2)$ is compact and the collection of sets $\{V_\Pi^R : R \in \mathscr{O}(d;2)\}$ is an open cover for $\mathscr{O}(d;2)$, we deduce that there exist $L \in \mathbb{N}$ and smooth $\mathbb{R}^{d\times d}$-valued functions $\Pi^1,\ldots,\Pi^L$ of the form (5.4.24), supported in intervals of the form J_{ε_l}, with $\varepsilon_l > 0$, such that the sets

$$V_l = \{[\Pi^l_{*,1}(\alpha),\ldots,\Pi^l_{*,d-2}(\alpha)] : \alpha \in J_{\varepsilon/2}\},$$

satisfy

$$\mathscr{O}(d;2) = V_1 \cup \cdots \cup V_L. \tag{5.4.26}$$

In the sequel, we will assume without loss of the generality that there exists $\varepsilon > 0$, such that $\varepsilon_l = \varepsilon$ for all $l = 1,\ldots,L$.

By construction, the functions $\Pi^1,\ldots,\Pi^L$ are smooth and compactly supported, so it suffices to show that

$$\mathscr{S}^d_{deg} \subset \bigcup_{1\le l\le L} \{x \in \mathbb{R}^{n_1(d)} : \hat{x} \in \mathrm{Im}(F^l)\},$$

where $F^1,\ldots F^L$ are defined by (5.4.20). To this end, take $x \in \mathscr{S}_{deg}$ and let $Q \in \mathscr{O}(d)$ and $\Delta \in \mathscr{D}(d)$ be such that $\hat{x} = Q\Delta Q^*$. By permuting the diagonal of Δ and the columns of Q if necessary, we can assume that $\Delta_{d-1,d-1} = \Delta_{d,d}$. Applying (5.4.26) to $[Q_{*,1},\ldots,Q_{*,d-2}] \in \mathscr{O}(d;2)$, we deduce that there exist $1 \le l \le L$ and $\alpha \in J_\varepsilon$, such that $[Q_{*,1},\ldots,Q_{*,d-2}] = [\Pi^l_{*,1}(\alpha),\ldots,\Pi^l_{*,d-2}(\alpha)]$.

Let $\Delta = \Lambda(\beta)$ for $\beta \in \mathbb{R}^{d-1}$. To finish the proof, it suffices to show that $\hat{x} = \Pi^l(\alpha)\Lambda(\beta)\Pi^l(\alpha)^*$. By construction,

$$\{\Pi^l_{*,1}(\alpha), \ldots, \Pi^l_{*,d}(\alpha)\} \quad \text{and} \quad \{Q_{*,1}, \ldots, Q_{*,d}\}$$

are orthonormal bases of $\mathbb{R}^d$ satisfying

$$\{\Pi^l_{*,1}(\alpha), \ldots, \Pi^l_{*,d-2}(\alpha)\} = \{Q_{*,1}, \ldots, Q_{*,d-2}\}.$$

Thus, $\operatorname{span}\{\Pi^l_{*,d-1}(\alpha), \Pi^l_{*,d}(\alpha)\} = \operatorname{span}\{Q_{*,d-1}, Q_{*,d}\}$. In particular, $\operatorname{span}\{\Pi^l_{*,d-1}(\alpha), \Pi^l_{*,d}(\alpha)\}$ is contained in the eigenspace associated to $\Delta_{d-1,d-1}$, which implies that $\Pi^l_{*,d-1}(\alpha), \Pi^l_{*,d}(\alpha)$ are orthonormal eigenvectors of $\hat{x}$ with eigenvalue $\Delta_{d-1,d-1}$. From here we conclude that $\{\Pi^l_{*,1}(\alpha), \ldots, \Pi^l_{*,d}(\alpha)\}$ is a basis of eigenvectors for $\hat{x}$, hence implying that

$$\hat{x} = \Pi^l(\alpha)\Lambda(\beta)\Pi^l(\alpha)^*,$$

as required. □

In the next proposition, we bound from above the set $\mathscr{H}^d_{deg}$.

Proposition 5.4.6. *There exists a compactly supported smooth function* $\widetilde{\Pi} : \mathbb{R}^{d^2-d-2} \to \mathbb{C}^{d\times d}$*, such that the mapping* $\widetilde{F} : \mathbb{R}^{d^2-d-2} \to \mathscr{H}(d)$*, defined by*

$$\widetilde{F}(\alpha, \beta) := \widetilde{\Pi}(\alpha)\Lambda(\beta)\widetilde{\Pi}(\alpha)^*, \tag{5.4.27}$$

for $\alpha \in \mathbb{R}^{d^2-d-2}$ *and* $\beta \in \mathbb{R}^{d-1}$*, satisfies*

$$\mathscr{H}^d_{deg} \subset \{x \in \mathbb{R}^{n_2(d)} : \hat{x} \in \operatorname{Im}(\widetilde{F})\}. \tag{5.4.28}$$

Proof. For $\varepsilon > 0$, set $\widetilde{J}_\varepsilon := (-\varepsilon,\varepsilon)^{d^2-d-2}$. Similarly to the proof of Proposition 5.4.5, it suffices to show that there exist $M \in \mathbb{N}$ and smooth $\mathbb{C}^{d\times d}$-valued functions $\widetilde{\Pi}^l$, with $1 \le l \le M$, supported in $\widetilde{J}_\varepsilon$, with $\varepsilon > 0$, such that the mappings $\widetilde{F}^l : \widetilde{J}_\varepsilon \times \mathbb{R}^{d-1} \to \mathscr{H}(d)$, defined by

$$\widetilde{F}^l(\alpha,\beta) \quad := \widetilde{\Pi}^l(\alpha)\Lambda(\beta)\widetilde{\Pi}^l(\alpha)^*, \tag{5.4.29}$$

satisfy

$$\mathscr{H}^d_{deg} \subset \mathscr{H}^d_{out} := \{x \in \mathbb{R}^{n_2(d)} : \hat{x} \in \bigcup_{l=1}^{M} \operatorname{Im}(\widetilde{F}^l)\}. \tag{5.4.30}$$

For each $R \in \mathscr{U}(d;2)$, choose a unitary matrix $P \in \mathscr{U}(d)$, such that $P_{i,j} = R_{i,j}$ for all $1 \le i \le d$ and $1 \le j \le d-2$. Using the fact that the set $\mathscr{V}^R_\nu$, defined by (5.4.6), is a smooth manifold of dimension d^2-d-2 for ν sufficiently small, it follows that there exist $\varepsilon,\gamma > 0$, and a smooth diffeomorphism $\widetilde{\varphi} : \widetilde{J}_\varepsilon \to \mathscr{V}^R_\gamma$, such that $\widetilde{\varphi}(0) = R$. Moreover, as in the proof of Proposition 5.4.5, if γ is sufficiently small, the mappings $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ defined as in (5.4.22) and (5.4.23) (when φ is replaced by $\widetilde{\varphi}$), are smooth. Let $\widetilde{\Pi} : \mathbb{R}^{d^2-d-2} \to \mathbb{C}^{d\times d}$ be any smooth function, supported in $\widetilde{J}^d_\varepsilon$, such that for all $\alpha \in \widetilde{J}_{\varepsilon/2}$,

$$\widetilde{\Pi}_{*,j}(\alpha) := \begin{cases} \widetilde{\varphi}_{*,j}(\alpha) & \text{if } 1 \le j \le d-2 \\ \widetilde{\psi}_1(\alpha) & \text{if } j = d-1 \\ \widetilde{\psi}_2(\alpha) & \text{if } j = d. \end{cases} \tag{5.4.31}$$

Define the function $\zeta^R : \mathscr{U}(d;2) \cap B_\gamma(R) \to \mathscr{U}(d;2)$ by $\zeta^R(A) = \{\zeta^R_{i,j}(A); 1 \le i \le d \text{ and } 1 \le j \le d-2\}$, where

$$\zeta^R_{*,j}(A) := \langle A_{*,j}, R_{*,j} \rangle^{-1} | \langle A_{*,j}, R_{*,j} \rangle | A_{*,j},$$

and the set

$$V^R_{\widetilde{\Pi},\delta} := \{[\widetilde{\Pi}_{*,1}(\alpha), \ldots, \widetilde{\Pi}_{*,d-2}(\alpha)] : \alpha \in \widetilde{J}_\delta\} = \widetilde{\varphi}(\widetilde{J}_\delta),$$

for $0 < \delta < \varepsilon$. By the continuity of the inner product in $\mathbb{C}^d$, there exists $0 < \varepsilon' < \varepsilon/2$, such that

$$\zeta^R(\widetilde{\varphi}(\widetilde{J}_{\varepsilon'})) \subset \widetilde{\varphi}(\widetilde{J}_\varepsilon).$$

By construction, $\widetilde{\Pi}(0) = P$ and $V^R_{\widetilde{\Pi},\varepsilon'}$ is an open subset of $\mathscr{U}(d;2)$ containing R, such that

$$\zeta^R(V^R_{\widetilde{\Pi},\varepsilon'}) \subset V^R_{\widetilde{\Pi},\varepsilon}.$$

Therefore, since $\mathscr{U}(d;2)$ is compact and the collection $\{V^R_{\widetilde{\Pi},\varepsilon'} : R \in \mathscr{U}(d;2)\}$ is an open cover for $\mathscr{U}(d;2)$, we deduce that there exist $M \in \mathbb{N}$, $\varepsilon'_1, \varepsilon_1, \ldots, \varepsilon'_M, \varepsilon_M > 0$ and smooth $\mathbb{C}^{d\times d}$-valued functions $\widetilde{\Pi}^1, \ldots, \widetilde{\Pi}^M$, supported in intervals of the form $\widetilde{J}_{\varepsilon_l}$, with $\varepsilon'_l < \varepsilon_l/2$, such that the sets

$$\widetilde{V}_l := \{[\widetilde{\Pi}^l_{*,1}(\alpha), \ldots, \widetilde{\Pi}^l_{*,d-2}(\alpha)] : \alpha \in \widetilde{J}_{\varepsilon_l/2}\},$$

satisfy

$$\mathscr{U}(d;2) = \widetilde{V}_1 \cup \cdots \cup \widetilde{V}_M, \tag{5.4.32}$$

and the matrices $R_l := [\widetilde{\Pi}^l_{*,1}(0), \ldots, \widetilde{\Pi}^l_{*,d-2}(0)]$, with $1 \leq l \leq M$, satisfy

$$\zeta^{R_l}(V^R_{\widetilde{\Pi}^l, \varepsilon'_l}) \subset V^R_{\widetilde{\Pi}^l, \frac{\varepsilon_l}{2}}. \tag{5.4.33}$$

In the sequel, we will assume without loss of the generality that there exist $\varepsilon, \varepsilon' > 0$, such that $\varepsilon_l = \varepsilon$ and $\varepsilon'_l = \varepsilon'$ for all $l = 1, \ldots, M$.

By construction, the functions $\widetilde{\Pi}^1, \ldots, \widetilde{\Pi}^M$ are smooth and supported in $\widetilde{J}_\varepsilon$, so it suffices to show relation (5.4.30). To this end, take $x \in \mathscr{H}^d_{deg}$ and let $\Delta \in \mathscr{D}(d)$, $Q \in \mathscr{U}(d)$ be such that

$$\hat{x} = Q \Delta Q^*. \tag{5.4.34}$$

As in the proof of Proposition 5.4.5, we can assume that $\Delta_{d-1,d-1} = \Delta_{d,d}$ and thus there exists $\beta \in \mathbb{R}^{d-1}$ such that $\Delta = \Lambda(\beta)$. Let $B \in \mathbb{C}^{d \times (d-2)}$ be given by $B_{i,j} = Q_{i,j}$, for $1 \leq i \leq d$ and $1 \leq j \leq d-2$. By (5.4.32), there exists $1 \leq l_0 \leq M$, such that $B \in \widetilde{\Pi}^{l_0}(\widetilde{J}_{\varepsilon'})$. Define $P := \widetilde{\Pi}^{l_0}(0)$ and $R \in \mathbb{C}^{d \times (d-2)}$ by $R_{i,j} := P_{i,j}$ for all $1 \leq i \leq d$ and $1 \leq j \leq d-2$. Notice that the decomposition (5.4.34) still holds if the columns of Q are multiplied by any complex number of unit length. Moreover, by (5.4.33), $\zeta^R(B)$ belongs to $V^R_{\widetilde{\Pi}^{l_0}, \frac{\varepsilon_{l_0}}{2}}$, and thus, since the columns of $[Q_{*,1}, \ldots, Q_{*,d-2}]$ are scalar multiples of $\zeta^R(B)$, by replacing the first $d-2$ columns of Q by those of the matrix $\zeta^R(B)$ in relation (5.4.34), we can assume that

$$[Q_{*,1}, \ldots, Q_{*,d-2}] = [\widetilde{\Pi}^{l_0}_{*,1}(\alpha), \ldots, \widetilde{\Pi}^{l_0}_{*,d-2}(\alpha)],$$

for some $\alpha \in \widetilde{J}_{\varepsilon/2}$. To finish the proof, it suffices to show that $\hat{x} = \widetilde{\Pi}^{l_0}(\alpha)\Lambda(\beta)\widetilde{\Pi}^{l_0}(\alpha)^*$. By construction,

$$\{Q_{*,1} = \widetilde{\Pi}^{l_0}_{*,1}(\alpha), \ldots, Q_{*,d-2} = \widetilde{\Pi}^{l_0}_{*,d-2}(\alpha), \widetilde{\Pi}^{l_0}_{*,d-1}(\alpha), \widetilde{\Pi}^{l_0}_{*,d}(\alpha)\}$$

and

$$\{Q_{*,1}, \ldots, Q_{*,d}\}$$

are orthonormal basis of $\mathbb{C}^d$, and thus, $\operatorname{span}\{\widetilde{\Pi}^{l_0}_{*,d-1}(\alpha), \widetilde{\Pi}^{l_0}_{*,d}(\alpha)\} = \operatorname{span}\{Q_{*,d-1}, Q_{*,d}\}$. In particular, $\operatorname{span}\{\widetilde{\Pi}^{l_0}_{*,d-1}(\alpha), \widetilde{\Pi}^{l_0}_{*,d}(\alpha)\}$ is contained in the eigenspace associated to $\Delta_{d-1,d-1} = \Delta_{d,d}$, which implies that $\widetilde{\Pi}^{l_0}_{*,d-1}(\alpha), \widetilde{\Pi}^{l_0}_{*,d}(\alpha)$ are orthonormal eigenvectors of $\hat{x}$ with eigenvalue $\Lambda_{d-1,d-1}(\beta)$. From here we conclude that

$$\{\widetilde{\Pi}^{l_0}_{*,1}(\alpha), \ldots, \widetilde{\Pi}^{l_0}_{*,d}(\alpha)\},$$

forms a base of eigenvectors for $\hat{x}$, hence implying that

$$\hat{x} = \widetilde{\Pi}(\alpha)\Lambda(\beta)\widetilde{\Pi}(\alpha)^*,$$

as required. The proof is now complete. $\square$

The following result gives sufficient conditions for points $x_0 \in \mathscr{S}_{deg}$ to have a neighborhood diffeomorphic to $\mathbb{R}^{n_1(d)-2}$.

Proposition 5.4.7. *Let $x_0 \in \mathscr{S}^d_{deg}$ be such that $|\boldsymbol{Sp}(\hat{x}_0)| = d-1$. Then there exists $\gamma > 0$ such that $\mathscr{S}^d_{deg} \cap B_\gamma(x_0)$ is an $(n_1(d)-2)$-dimensional manifold.*

Proof. The ideas of the proof are similar to those used in Proposition 5.4.5, but in this case, the compactness argument that leads to (5.4.26), is replaced by a localization argument for the matrix of eigenvectors of $\hat{x}_0$.

Let $P \in \mathscr{O}(d)$ and $D \in \mathscr{D}(d)$ be such that

$$\hat{x}_0 = PDP^*.$$

Since $|\mathbf{Sp}(\hat{x}_0)| = d-1$, only one of the eigenvalues $D_{1,1}, \ldots, D_{d,d}$ of $\hat{x}_0$ is repeated. We will assume without loss of generality that $D_{d-1,d-1} = D_{d,d}$. Define J_ε, for $\varepsilon > 0$, by $J_\varepsilon := (-\varepsilon, \varepsilon)^{\frac{d(d-1)}{2}-1}$, and let $R \in \mathscr{O}(d;2)$ be the matrix $R = \{R_{i,j}; 1 \leq i \leq d,\ \ 1 \leq j \leq d-2\}$, with $R_{i,j} = P_{i,j}$ for all $1 \leq i \leq d$ and $1 \leq j \leq d-2$. Since $\mathscr{O}(d;2)$ is a manifold of dimension $\frac{d(d-1)}{2} - 1$, there exists $\gamma > 0$ and a smooth diffeomorphism

$$\varphi : J_\varepsilon \to \mathscr{O}(d;2) \cap B_\gamma(R),$$

with $\varphi(0) = R$. Denote by $\varphi_{*,j}$ the j-th column vector of φ. Proceeding as in the proof of Proposition 5.4.5, we can show that if γ is sufficiently small, the functions ψ_1 and ψ_2 defined in (5.4.22) and (5.4.23) are smooth. Define $\Pi : J_\varepsilon \to \mathscr{O}(d)$ by

$$\Pi_{*,j}(\alpha) := \begin{cases} \varphi_{*,j}(\alpha) & \text{if } 1 \leq j \leq d-2 \\ \psi_1(\alpha) & \text{if } j = d-1 \\ \psi_2(\alpha) & \text{if } j = d, \end{cases}$$

and $F : J_\varepsilon \times \mathbb{R}^{d-1} \to \mathscr{S}^d_{deg}$ by

$$F(\alpha, \beta) := \Pi(\alpha)\Lambda(\beta)\Pi(\alpha)^*.$$

In order to show that $\mathscr{S}^d_{deg} \cap B_\gamma(x_0)$ is an $(n_1(d)-2)$-dimensional manifold, we will prove that there exist open subsets $U \subset J_\varepsilon$ and $V \subset \mathscr{S}^d_{deg} \cap B_\gamma(\hat{x}_0)$, such that the mapping

$$\begin{array}{ccc} U \times \mathbb{R}^{d-1} & \to & V \\ (\alpha, \beta) & \longmapsto & F(\alpha, \beta) \end{array} \tag{5.4.35}$$

is a diffeomorphism. To this end, define

$$r := \frac{1}{2} \min_{\substack{\mu, \nu \in \mathbf{Sp}(\hat{x}_0) \\ \mu \neq \nu}} |\mu - \nu|. \tag{5.4.36}$$

Notice that by Lemma 5.4.3, there exists $\delta > 0$ satisfying that for all $x \in \mathscr{S}^d_{deg} \cap B_\delta(\hat{x}_0)$, there exist $Q \in \mathscr{O}(d)$ and $\Delta \in \mathscr{D}(d)$, such that $\hat{x} = Q\Delta Q^*$,

$$Q \in \mathscr{O}(d) \cap B_{\gamma/2}(P), \tag{5.4.37}$$

and

$$\Delta \in \mathscr{D}(d) \cap B_r(D). \tag{5.4.38}$$

By (5.4.37), there exists $\alpha \in J_\varepsilon$ such that $\varphi(\alpha) = [Q_{*,1}, \ldots, Q_{*,d-2}]$. As a consequence, since

$$\{\Pi_{*,1}(\alpha), \ldots, \Pi_{*,d}(\alpha)\} \qquad \text{and} \qquad \{Q_{*,1}, \ldots, Q_{*,d}\}$$

are orthonormal bases of $\mathbb{R}^d$ satisfying

$$\{\Pi_{*,1}(\alpha), \ldots, \Pi_{*,d-2}(\alpha)\} = \{Q_{*,1}, \ldots, Q_{*,d-2}\},$$

we have that $\text{span}\{\Pi_{*,d-1}(\alpha),\Pi_{*,d}(\alpha)\} = \text{span}\{Q_{*,d-1},Q_{*,d}\}$. On the other hand, by (5.4.38), we have that $\Delta_{1,1} < \cdots < \Delta_{d-1,d-1} = \Delta_{d,d}$, and thus, we conclude that $\Pi_{*,d-1}(\alpha),\Pi_{*,d}(\alpha)$ are eigenvectors of $\hat{x}$ with eigenvalue $\Delta_{d-1,d-1}$, hence implying that

$$\{\Pi_{*,1}(\alpha),\ldots,\Pi_{*,d}(\alpha)\}$$

is a basis of eigenvectors for $\hat{x}$ and

$$\hat{x} = \Pi(\alpha)\Lambda(\beta)\Pi(\alpha)^*.$$

From here it follows that if $U \subset \mathbb{R}^{n_1(d)-2}$ and $V \subset \mathscr{S}^d_{deg}$ are given by $V := B_\delta(\hat{x}_0)$ and $U := F^{-1}(V)$, the mapping (5.4.35) is onto. Therefore, in order to show that the mapping F defined in (5.4.35) is a diffeomorphism, it suffices to show that the following conditions hold:

(i) The restriction of F to U is injective,

(ii) The function F^{-1} is continuous over V,

(iii) D_pF is injective for every $p \in J_\varepsilon \times \mathbb{R}^{d-1}$.

Notice that condition (iii) implies that F is locally injective, which gives condition (i) for $\delta > 0$ sufficiently small. Hence, it suffices to show that F^{-1} is continuous and D_pF is injective for every $p \in J_\varepsilon \times \mathbb{R}^{d-1}$. We split the proof of these claims into the following two steps:

Step 1. First we show that F^{-1} is continuous. Consider a sequence $\{y_n\}_{n\geq 1} \subset \mathscr{S}^d_{deg} \cap B_\delta(\hat{x}_0)$ such that $\lim_n y_n = y$ for some $y \in \mathscr{S}^d_{deg} \cap B_\delta(\hat{x}_0)$. Consider the elements $(\alpha_n,\beta_n),(\alpha,\beta) \in$

$J_\varepsilon \times \mathbb{R}^{d-1}$, defined by $(\alpha_n, \beta_n) = F^{-1}(y_n)$ and $(\alpha, \beta) := F^{-1}(y)$, that satisfy

$$y_n = \Pi(\alpha_n)\Lambda(\beta_n)\Pi(\alpha_n)$$

and

$$y = \Pi(\alpha)\Lambda(\beta)\Pi(\alpha). \tag{5.4.39}$$

Our aim is to show that $\lim_n \alpha_n = \alpha$ and $\lim_n \beta_n = \beta$. Condition $\lim_n \beta_n = \beta$ follows from the continuity of $\Phi_1, \ldots, \Phi_d$. To show that

$$\lim_n \alpha_n = \alpha, \tag{5.4.40}$$

we proceed as follows. By construction, for all $n \in \mathbb{N}$, $\Pi(\alpha_n) \in \mathscr{O}(d) \cap B_{\gamma/2}(P)$, and thus $\varphi(\alpha_n) \in \mathscr{O}(d;2) \cap B_{\gamma/2}(R)$. As a consequence, the sequence $\{\alpha_n\}_{n\geq 1}$ is contained in the compact set $K := \varphi^{-1}(\mathscr{O}(d;2) \cap \overline{B_{\gamma/2}(R)})$. Therefore, it suffices to show that every convergent subsequence $\{\alpha_{m_n}\}_{n\geq 1} \subset \{\alpha_n\}_{n\geq 1}$, satisfies $\lim_n \alpha_{m_n} = \alpha$.

By taking limit as $n \to \infty$ in the relation $y_{m_n} = \Pi(\alpha_{m_n})\Lambda(\beta_{m_n})\Pi(\alpha_{m_n})^*$, we get

$$y = \Pi(\lim_n \alpha_{m_n})\Lambda(\beta)\Pi(\lim_n \alpha_{m_n})^*. \tag{5.4.41}$$

Assume that $\Lambda(\beta) = (\mu_1, \ldots, \mu_d)$ for some $\mu_1, \ldots, \mu_d$ such that $\mu_{d-1} = \mu_d$. Since $K \subset J_\varepsilon$, then $\lim_n \alpha_{m_n}$ belongs to the domain of Π. Moreover, by (5.4.41), we have that

$$\Pi_{*,j}(\lim_n \alpha_{m_n}) \in \mathbf{E}^{\hat{y}}_{\mu_j} \qquad \text{for all} \quad 1 \leq j \leq d-2. \tag{5.4.42}$$

On the other hand, since $\Lambda(\beta) \in B_r(D)$, we have that $\mu_1 > \mu_2 > \cdots > \mu_{d-1}$, and consequently, $\mathbf{E}^{\hat{y}}_{\mu_j}$ is one-dimensional for $1 \leq j \leq d-2$. Therefore, using (5.4.42) as well as the fact that $|\Pi_{*,j}(\lim_n \alpha_{m_n})| = 1$ for all $1 \le j \le d$, it follows that

$$\Pi_{*,j}(\lim_n \alpha_{m_n}) \in \{\Pi_{*,j}(\alpha), -\Pi_{*,j}(\alpha)\}, \tag{5.4.43}$$

for all $1 \le j \le d-2$. Since the image of $\Pi_{*,j}$ is contained in $B_{\frac{1}{2}}(\Pi_{*,j}(\alpha))$, we conclude that $\Pi_{*,j}(\lim_n \alpha_{m_n}) = \Pi_{*,j}(\alpha)$, which implies that $\varphi(\lim_n \alpha_{m_n}) = \varphi(\alpha)$. Therefore, using the fact that φ is a diffeomorphism, we conclude that $\lim_n \alpha_{m_n} = \alpha$, as required.

Step 2. Next we prove that DF_p is injective for all $p \in J_\varepsilon$. Consider an element $(a,b) \in \mathbb{R}^{\frac{d(d-1)}{2}-1} \times \mathbb{R}^{d-1}$ satisfying $DF_{\hat{x}_0}(a,b) = 0$. Then, for $\varepsilon > 0$ sufficiently small, the curve $M : (-\varepsilon, \varepsilon) \to \mathscr{S}_{deg} \cap B_\delta(\hat{x}_0)$ given by $M(t) := F(ta,tb)$, satisfies $M(0) = \hat{x}_0$ and $\dot{M}(0) = DF_{\hat{x}_0}(a,b) = 0$. Denote by $v^1(t), \dots, v^d(t)$ the columns of $\Pi(ta)$ and define $\mu_i(t) := \Lambda_{i,i}(tb)$. Then, we have

$$M(t)v^i(t) = \mu_i(t)v^i(t). \tag{5.4.44}$$

By taking derivative with respect to t in (5.4.44), we get

$$\dot{M}(t)v^i(t) + M(t)\dot{v}^i(t) = \dot{\mu}_i(t)v^i(t) + \mu_i(t)\dot{v}^i(t), \qquad \text{for all } 1 \le i \le d,$$

which, by the condition $\dot{M}(0) = 0$, implies that

$$M(0)\dot{v}^i(0) = \dot{\mu}_i(0)v^i(0) + \mu_i(0)\dot{v}^i(0), \qquad \text{for all } 1 \le i \le d. \tag{5.4.45}$$

By taking the inner product with $v^j(0)$ in (5.4.45), for $j \ne i$, we get

$$\left\langle v^j(0), \dot{v}^i(0) \right\rangle (\mu_j(0) - \mu_i(0)) = 0.$$

In particular, since $\mu_{d-1} = \mu_d$ is the only repeated eigenvalue for $\hat{x}_0$, we deduce that for $1 \le i, j \le d-1$ satisfying $i \neq j$,

$$\left\langle v^j(0), \dot{v}^i(0) \right\rangle = 0. \tag{5.4.46}$$

On the other hand, the condition $\left\| v^i(t) \right\|^2 = 1$ implies that

$$\left\langle \dot{v}^i(0), v^i(0) \right\rangle = 0, \tag{5.4.47}$$

which by (5.4.46) leads to $\dot{v}^i(0) = 0$ for all $1 \le i \le d-1$. Since the last two columns of Π are smooth functions of the first $d-2$ (see equations (5.4.22) and (5.4.23)), from the equations $\dot{v}^1(0) = \cdots = \dot{v}^{d-1}(0) = 0$, we conclude that $\frac{d}{dt}\Pi(ta)\big|_{t=0} = 0$. On the other hand, since Π is a local chart for the manifold $\mathscr{O}(d;2)$ around $\Pi(0)$, the derivative $\dot{\Pi}(0)$ is injective, and thus the equation $\frac{d}{dt}\Pi(ta)\big|_{t=0} = 0$ implies that $a = 0$.

Finally we prove that $b = 0$. By definition, $M(t) = \Pi(\alpha t)\Lambda(\beta t)\Pi(\alpha t)^*$, and hence

$$\dot{M}(t) = \big(\frac{d}{dt}\Pi(\alpha t)\big)\Lambda(\beta t)\Pi(\alpha t)^* + \Pi(\alpha t)\frac{d}{dt}\Lambda(\beta t)\Pi(\alpha t)^* + \Pi(\alpha t)\Lambda(\beta t)\big(\frac{d}{dt}\Pi(\alpha t)\big).$$

Since $a = 0$, by evaluating the previous identity at $t = 0$, we get

$$0 = \Pi(0)(\dot{\Lambda}(0)\beta)\Pi(0)^*,$$

which implies that $b = 0$. From here we conclude that the only solution to $DF_{x_0}(a,b) = 0$ is $(a,b) = 0$. This finishes the proof of the injectivity for DF_{x_0}. The proof is now complete. $\square$

The next result gives a sufficient condition for points $x_0 \in \mathscr{H}_{deg}$ to have a neighborhood diffeomorphic to $\mathbb{R}^{n_2(d)-3}$.

Proposition 5.4.8. *Let $x_0 \in \mathscr{H}_{deg}$ be such that $|\boldsymbol{Sp}(\hat{x}_0)| = d-1$. Then, there exists $\gamma > 0$, such that $\mathscr{H}_{deg}^d \cap B_\gamma(x_0)$ is an $(n_2(d)-3)$-dimensional manifold.*

Proof. Let $P \in \mathscr{H}(d)$ and $D \in \mathscr{D}(d)$ be such that

$$\hat{x}_0 = PDP^*.$$

Since $|\mathbf{Sp}(\hat{x}_0)| = d-1$, only one of the eigenvalues $D_{1,1}, \ldots, D_{d,d}$ of $\hat{x}_0$ is repeated. We will assume without loss of generality that $D_{d-1,d-1} = D_{d,d}$. Define $\widetilde{J}_\varepsilon$, for $\varepsilon > 0$, by $\widetilde{J}_\varepsilon := (-\varepsilon, \varepsilon)^{d^2-d-2}$, and let $R \in \mathscr{U}(d;2)$ be the matrix $R = \{R_{i,j}; 1 \leq i \leq d, \ \ 1 \leq j \leq d-2\}$, with $R_{i,j} = P_{i,j}$ for all $1 \leq i \leq d$ and $1 \leq j \leq d-2$. Using the fact that for $\nu > 0$ sufficiently small the set $\mathscr{V}_\nu^R$ given by (5.4.6) is a manifold, we deduce that there exist $\varepsilon, \gamma > 0$ and a diffeomorphism

$$\widetilde{\varphi} : \widetilde{J}_\varepsilon \to \mathscr{V}_\gamma^R,$$

such that $\widetilde{\varphi}(0) = R$. As in the proof of Proposition 5.4.7, we can construct a smooth function $\Pi : \widetilde{J}_\varepsilon \to \mathscr{U}(d)$ with entries $\Pi_{i,j}$, such that $\Pi_{i,j}(\alpha) = \widetilde{\varphi}_{i,j}(\alpha)$ for all $\alpha \in \widetilde{J}_\varepsilon$ and $1 \leq i \leq d$ and $1 \leq j \leq d-2$.

Define $\widetilde{F} : \widetilde{J}_\varepsilon \times \mathbb{R}^{d-1} \to \mathscr{H}_{deg}^d$ by

$$\widetilde{F}(\alpha, \beta) := \widetilde{\Pi}(\alpha)\Lambda(\beta)\widetilde{\Pi}(\alpha)^*.$$

By Lemma 5.4.4, there exists $\delta > 0$ such that for all $x \in \mathscr{H}^d_{deg} \cap B_\delta(\hat{x}_0)$, there exist $Q \in \mathscr{U}(d)$ and $\Delta \in \mathscr{D}(d)$, satisfying

$$\hat{x} = Q\Delta Q^*, \tag{5.4.48}$$

as well as

$$Q \in \mathscr{U}(d) \cap B_{\gamma/2}(P) \quad \text{and} \quad \Delta \in \mathscr{D}(d) \cap B_r(\Delta),$$

where r is given by (5.4.36). Notice that relation (5.4.48) still holds if we multiply the j-th column of Q, for $1 \leq j \leq d-2$, by $\langle P_{*,j}, R_{*,j}\rangle / |\langle P_{*,j}, R_{*,j}\rangle|$, so we can assume without loss of generality that $[Q_{*,1}, \ldots, Q_{*,d-2}] \in \mathscr{V}^R_\gamma$. In particular, there exists $\alpha \in \widetilde{J}_\varepsilon$ such that $\widetilde{\varphi}(\alpha) = [Q_{*,1}, \ldots, Q_{*,d-2}]$. Then, by proceeding as in the proof of Proposition 5.4.7, we can show that

$$\hat{x} = \widetilde{\Pi}(\alpha)\Lambda(\beta)\widetilde{\Pi}(\alpha)^*$$

for some $\beta \in \mathbb{R}^{d-1}$. As a consequence, if we define $\widetilde{V} := B_\delta(\hat{x}_0)$ and $\widetilde{U} := F^{-1}(\widetilde{V})$, then the mapping

$$\begin{array}{ccc} \widetilde{U} \times \mathbb{R}^{d-1} & \rightarrow & \widetilde{V} \\ (\alpha, \beta) & \longmapsto & \widetilde{F}(\alpha, \beta) \end{array} \tag{5.4.49}$$

is onto. As in the proof of Proposition 5.4.7, provided that we show the conditions

(ii) $\widetilde{F}^{-1}$ is continuous over $\tilde{U}$

(iii) $D\widetilde{F}_p$ is injective for every $\tilde{J}_\varepsilon$,

then the mapping (5.4.49) is a diffeomorphism. The proof of the continuity of $\widetilde{F}^{-1}$ follows ideas similar to those from the GOE case. The only argument that needs to be modified is the proof of (5.4.40), since equation (5.4.43) is not necessarily true when $\beta = 2$. To fix this problem, we replace equation (5.4.43) by

$$\widetilde{\Pi}_{*,i}(\lim_n \alpha_{m_n}) = \eta \widetilde{\Pi}_{*,i}(\alpha), \quad \text{for} \quad 1 \leq i \leq d-2,$$

which holds for some $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then, by using the fact that $[\Pi_{*,1}(\alpha), \ldots, \Pi_{*,d-2}(\alpha)]$ belongs to $\mathscr{V}_\gamma^R$, we conclude that $\widetilde{\Pi}(\lim_n \alpha_{m_n}) = \widetilde{\Pi}(\alpha)$, which in turn implies that $\varphi(\lim_n \alpha_{m_n}) = \varphi(\alpha)$. Then, since φ is a diffeomorphism we conclude that $\lim_n \alpha_{m_n} = \alpha$, as required.

The proof of the injectivity of DF_p , for $p \in \widetilde{J}_\varepsilon$, follows the same arguments as in the GOE case, with the exception that identity (5.4.47) must be replaced by

$$\mathrm{Re}(\langle \dot{v}^i(t), v^i(t) \rangle) = 0. \tag{5.4.50}$$

Then, since $\langle v^i(t), v^i(0) \rangle = |\langle v^i(t), v^i(0) \rangle|$, we conclude that $\langle v^i(t), v^i(0) \rangle_{\mathbb{C}^d}$ is real. This relation can be combined with (5.4.50), in order to get (5.4.47). The rest of the proof is analogous to Proposition 5.4.7. □

5.5 Proof of the main results

This section is devoted to the proofs of Theorem 5.2.1 and Corollary 5.2.2.

Proof of Theorem 5.2.1. The cases $\beta = 1$ and $\beta = 2$ can be handled similarly, so it suffices to prove the result for $\beta = 1$. First suppose that $Q < 2$. By Proposition 5.4.5, there exists an infinitely differentiable mapping $F : \mathbb{R}^{n_1(d)-2} \to \mathscr{S}(d)$, such that $\mathscr{S}_{deg}^d -$

$A^1 \subset \mathrm{Im}(F)$. As a consequence,

$$\begin{aligned}\mathbb{P}\left[\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t \in I \text{ and } 1 \le i < j \le d\right] \\ = \mathbb{P}\left[X^1(t) \in \mathscr{S}_{deg}^d - A^1 \text{ for some } t \in I\right] \\ \le \mathbb{P}\left[X^1(t) \in \mathrm{Im}(F) \text{ for some } t \in I\right].\end{aligned} \tag{5.5.1}$$

Since the smooth mapping F is defined over $\mathbb{R}^{n_1(d)-2}$, it follows that the set $\mathrm{Im}(F)$ has Hausdorff dimension at most $n_1(d)-2$. Thus, since $Q < 2$, by Corollary 5.3.2,

$$\mathbb{P}\left[X^1(t) \in \mathrm{Im}(F) \text{ for some } t \in I\right] = 0.$$

Therefore, by (5.5.1) we get that

$$\mathbb{P}\left[\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t \in I \text{ and } 1 \le i < j \le n\right] = 0,$$

as required. To prove (5.2.3) in the case $Q > 2$, choose any $x_0 \in \mathscr{S}_{deg}^d$ satisfying $|\mathbf{Sp}(\hat{x}_0)| = d-1$. By Lemma 5.4.7, there exists $\delta > 0$, such that $\mathscr{S}_{deg}^d \cap B_\delta(x_0)$ is an $n_1(d)$-dimensional manifold. In particular, the Hausdorff dimension of $\mathscr{S}_{deg}^d$ is at least $n_1(d)-2$. The Hausdorff dimension of the shifted manifold $\mathscr{S}_{deg}^d - A^2$ is also larger than or equal to $n_1(d)-2$. Relation (5.2.3) then follows by Corollary 5.3.2. This finishes the proof of Theorem 5.2.1. □

Proof of Corollary 5.2.2. The cases $\beta = 1$ and $\beta = 2$ can be handled similarly, so we will assume without loss of generality, that $\beta = 1$. Our goal is to prove that with strictly positive probability, the eigenvalues of $Y^1(t)$ collide for values of t arbitrarily close to zero. Corollary 5.2.2 then follows from the representation of the fractional Brownian motion as a Volterra process and Blumenthal's zero-one law.

Suppose that the process ξ is a one dimensional fractional Brownian motion of Hurst parameter $0 < H < 1$. If $H > \frac{1}{2}$, relation (5.2.4) follows from equation (5.2.2) in Theorem 5.2.1. Moreover, if $H < \frac{1}{2}$, then relation (5.2.5) follows from equation (5.2.3). Therefore, it suffices to show relation (5.2.6) in the case where $H < \frac{1}{2}$ and $A^1 \in \mathscr{S}^d_{deg}$ satisfies either $|\mathbf{Sp}(A^1)| = d-1$ or $A^1 = 0$.

The proof of (5.2.6) will be done in several steps.

Step 1. We will show first that there exists $\delta' > 0$ such that for any $0 < T < 1$,

$$\mathbb{P}\left[\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t \in (0,T] \text{ and } 1 \le i < j \le n\right] \ge \delta' > 0. \tag{5.5.2}$$

We will split the proof of (5.5.2) into the cases $A^1 = 0$ and $|\mathbf{Sp}(A^1)| = d-1$.

(i) Suppose $|\mathbf{Sp}(A^1)| = d-1$. Then A^1 has exactly one repeated eigenvalue. We will assume without loss of generality that $\Phi_{d-1}(A^1) = \Phi_d(A^1)$. Fix $T < 1$. By the self-similarity of $X^1(t)$, we can write

$$\begin{aligned}
&\mathbb{P}\left[\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t \in (0,T] \text{ and } 1 \le i < j \le n\right] \\
&= \mathbb{P}\left[X^1(t) \in (\mathscr{S}^d_{deg} - A^1) \text{ for some } t \in (0,T] \text{ and } 1 \le i < j \le n\right] \\
&= \mathbb{P}\left[X^1(s) \in T^{-H}(\mathscr{S}^d_{deg} - A^1) \text{ for some } s \in (0,1] \text{ and } 1 \le i < j \le n\right] \\
&\ge \mathbb{P}\left[X^1(s) \in T^{-H}(\mathscr{S}^d_{deg} - A^1) \text{ for some } s \in (1/2,1] \text{ and } 1 \le i < j \le n\right].
\end{aligned} \tag{5.5.3}$$

By Theorem 5.3.1, there exists $c_1 > 0$, such that

$$\mathbb{P}\left[X(s) \in T^{-H}(\mathscr{S}^d_{deg} - A^1) \text{ for some } s \in (1/2,1] \text{ and } 1 \le i < j \le n\right] \\
\ge c_1 \mathscr{C}_{n_1(d) - \frac{1}{H}}(T^{-H}(\mathscr{S}^d_{deg} - A^1)). \tag{5.5.4}$$

Let $G:(-1,1)^{n_1(d)-2}\to \mathscr{S}^d_{deg}-A^1$ be a parametrization of the manifold $\mathscr{S}^d_{deg}-A^1$ around zero. Consider the probability measure $m_\varepsilon(dx):=(2\varepsilon)^{2-n_1(d)}\mathbb{1}_{[-\varepsilon,\varepsilon]^{n_1(d)-2}}(x)dx$ and let $\nu_\varepsilon(dx)$ be the pullback measure of m_ε under the map $x\mapsto \varepsilon^{-1}G(x)$. Define f_α by (5.3.2). Since $\nu_{T^H}(dx)$ is a probability measure with support in $T^{-H}(\mathscr{S}^d_{deg}-A^1)$, we have

$$\begin{aligned}
\mathscr{C}_{n_1(d)-\frac{1}{H}}(T^{-H}(\mathscr{S}^d_{deg}-A^1)) &\geq \left(\int_{T^{-H}(\mathscr{S}^d_{deg}-A^1)} f_{n_1(d)-\frac{1}{H}}(\|u-v\|)\nu_{T^H}(du)\nu_{T^H}(dv)\right)^{-1}\\
&\geq \left((2T^H)^{2(2-n_1(d))}\int_{(-T^H,T^H)^{2(n_1(d)-2)}} f_{n_1(d)-\frac{1}{H}}(T^{-H}\|G(x)-G(y)\|)dxdy\right)^{-1}\\
&= 2^{2(n_1(d)-2)}\left(\int_{(-1,1)^{2(n_1(d)-2)}} f_{n_1(d)-\frac{1}{H}}(T^{-H}\|G(T^Hx)-G(T^Hy)\|)dxdy\right)^{-1}.
\end{aligned} \tag{5.5.5}$$

By the mean value theorem, there exists $\tau\in(0,1)$, depending on T, such that the vector $v(\tau):=\tau(1-x)+\tau y$ satisfies

$$T^{-H}(G(T^Hx)-G(T^Hy)) = T^{-H}\frac{d}{d\tau}G(T^H(\tau(1-x)+\tau y)) = DG_{v(\tau)}[x-y]. \tag{5.5.6}$$

Consider the mapping $(w,v)\mapsto \|DG_v[w]\|$, defined over the compact set $K:=\{(w,v):\|w\|=1,\quad\text{and}\quad v\in[-T^H,T^H]^{n_1(d)-2}\}$. By the smoothness of G, this mapping has a minimizer (w_0,τ_0). Moreover, since DG_v is injective for v near zero, we have that $\delta:=\|DG_{v_0}[w_0]\|>0$. Using this observation as well as relation (5.5.6), we get that

$$\begin{aligned}
T^{-H}\|G(T^Hx)-G(T^Hy)\| &= \|x-y\|\|DG_{v(\tau)}[\|x-y\|^{-1}(x-y)]\|\\
&\geq \delta\|y-x\|.
\end{aligned}$$

Therefore, by (5.5.5), it follows that if $n_1(d) > \frac{1}{H}$,

$$\mathscr{C}_{n_1(d)-\frac{1}{H}}(T^{-H}(\mathscr{S}^d_{deg} - A^1)) \geq (2\delta)^{2(n_1(d)-\frac{1}{H})}\left(\int_{(-1,1)^{2(n_1(d)-2)}} \|x-y\|^{\frac{1}{H}-n_1(d)} dxdy\right)^{-1}.$$

The integral in the right-hand side is finite due to the condition $\frac{1}{H} > 2$, and thus, there exists a constant $\delta' > 0$, such that

$$\mathscr{C}_{n_1(d)-\frac{1}{H}}(T^{-H}(\mathscr{S}^d_{deg} - A^1)) \geq \delta' > 0. \tag{5.5.7}$$

By following a similar approach, we can show that (5.5.7) also holds for the case $n_1(d) = \frac{1}{H}$, while in the case $n_1(d) < \frac{1}{H}$, identity (5.5.7) follows from the fact that $f_\alpha = 1$ for all $\alpha > 0$. Combining (5.5.3), (5.5.4) and (5.5.7), we conclude that there exists $\delta' > 0$ such that for all $T \in (0,1)$, (5.5.2) holds.

(ii) Next we show that relation (5.5.2) holds as well in the case $A = 0$, if $\delta' > 0$ is sufficiently small. Notice that if $A = 0$, by the self-similarity of ξ and the homogenity of the function $(\Phi_1, \ldots, \Phi_d)$, we have

$$\begin{aligned}
&\mathbb{P}\left[\lambda^1_i(t) = \lambda^1_j(t) \text{ for some } t \in (0,T] \text{ and } 1 \leq i < j \leq n\right] \\
&\quad = \mathbb{P}\left[\Phi_i(X^1(t)) = \Phi_j(X^1(t)) \text{ for some } t \in (0,T] \text{ and } 1 \leq i < j \leq n\right] \\
&\quad = \mathbb{P}\left[\Phi_i(T^H X^1(t)) = \Phi_j(T^H X^1(t)) \text{ for some } t \in (0,1] \text{ and } 1 \leq i < j \leq n\right] \\
&\quad = \mathbb{P}\left[\lambda^1_i(t) = \lambda^1_j(t) \text{ for some } t \in (0,1] \text{ and } 1 \leq i < j \leq n\right] \\
&\quad \geq \mathbb{P}\left[\lambda^1_i(t) = \lambda^1_j(t) \text{ for some } t \in [1/2,1] \text{ and } 1 \leq i < j \leq n\right].
\end{aligned}$$

Relation (5.5.2) for $A = 0$ then follows from Theorem 5.2.1.

Step 2. By taking $T \to 0$ in the left hand side of (5.5.2), we get

$$\mathbb{P}\left[\bigcap_{T\in(0,1)} \{\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t \in (0,T] \text{ and } 1 \le i < j \le n\}\right] \ge \delta' > 0. \quad (5.5.8)$$

Finally, for $i \le j$, we write $\xi_{i,j}$ as a Volterra process of the form $\xi_{i,j}(t) = \int_0^t K_H(s,t)dW_{i,j}(t)$, where the $\{W_{i,j}(t);t \ge 0\}$ are independent standard Brownian motions and

$$K_H(s,t) := c_H\left(\left(t/s\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-1/2)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right),$$

where $c_H := (2H)^{-\frac{1}{2}}(1-2H)\int_0^1(1-x)^{-2H}x^{H-\frac{1}{2}}dx$. We can easily check that

$$\bigcap_{T\in(0,1)} \{\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t \in (0,T] \text{ and } 1 \le i < j \le n\}$$

belongs to the germ σ-algebra $\mathscr{F}_{0+} := \bigcap_{s>0}\sigma\{W_{i,j}(u); 0 < u \le s, 0 \le i \le j \le d\}$. Thus, combining (5.5.8) with Blumenthal's zero-one law, we conclude that

$$\mathbb{P}\left[\bigcap_{T\in(0,1)} \{\lambda_i^1(t) = \lambda_j^1(t) \text{ for some } t \in (0,T] \text{ and } 1 \le i < j \le n\}\right] = 1.$$

The proof is now complete. □

Bibliography

[1] Albeverio, S. and Hu, Y. (1997). A remark on non-smoothness of the self-intersection local time of planar Brownian motion. *Statist. Probab. Lett.* **32** 57–65. Cited on 22

[2] Anderson, G., Guionnet, A. and Zeitouni, O. (2010). An introduction to random matrices. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge. Cited on 210, 211, 213

[3] Biermé, H., Lacaux, C. and Xiao, Y. (2009). Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. *Bull. Lond. Math. Soc.*. **41** 253–273. Cited on 211, 214, 215

[4] Billingsley, P. (1999). Convergence of probability measures (2nd ed.). John Wiley & Sons, Inc., New York. Cited on 24, 65, 192

[5] Binotto G., Nourdin I. and Nualart D. (2016). Weak symmetric integrals with respect to the fractional Brownian motion. *Ann. Probab.* to appear. Cited on 5, 163, 164, 170, 171, 183, 185, 196

[6] Bojdecki T., Gorostiza L. and Talarczyk A. (2004). Sub-fractional Brownian motion and its relation to occupation times. *Statist. Probab. Letters.* **69** 405-419. Cited on 166

[7] Breuer, P. and Major, P. (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** 425–441. Cited on 109

[8] Chavez J. and Tudor C. (2009). A decomposition of sub-fractional Brownian motion. *Math. Reports.* **11** 67-74. Cited on 166

[9] Cheridito P. and Nualart D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in (0, 1/2). *Ann. Inst. Henri Poincaré Probab. Statist.* **41** 1049-1081. Cited on 162

[10] Corcuera J.M., Nualart D., and Woerner J.H.C. (2006). Power variation of some integral fractional processes. *Bernoulli* **12** 713-735. Cited on 164, 185, 196

[11] Darses S., Nourdin I. and Nualart D. (2010). Limit theorems for nonlinear functionals of Volterra processes via white noise analysis. *Bernoulli* **16** 1262–1293. Cited on 28, 90, 109, 133

[12] Deya A., Nualart, D. and Tindel, S. (2015). On L^2 modulus of continuity of Brownian local times and Riesz potentials. *Ann. Probab.* **43** 1493–1534. Cited on 109

[13] Durieu O. and Wang Y. (2015). From infinite urn schemes to decompositions of self-similar Gaussian processes. *Electron. J. Probab.* **21** 1-23. Cited on 5, 167

[14] Falconer, K. (2014). Fractal geometry. John Wiley & Sons, Ltd., Chichester. Cited on 215

[15] Gradinaru M., Nourdin I., Russo F. and Vallois P. (2005). *m*-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. Henri Poincaré Probab. Statist.* **41** 781-806. Cited on 4, 161, 162, 164, 170, 183

[16] D. Harnett, A. Jaramillo, D. Nualart. Symmetric stochastic integrals with respect to a class of self-similar Gaussian processes. To appear in the Journal of Theoretical Probability. Arxiv Cited on iv

[17] Harnett D. and Nualart D. (2012). Weak convergence of the Stratonovich integral with respect to a class of Gaussian processes. *Stoch. Proc. Appl.* 122(10): 3460 - 3505. Cited on 163, 164

[18] Harnett D. and Nualart D. (2013). Central limit theorem for a Stratonovich integral with Malliavin calculus. *Ann. Probab.* **41** 2820-2879. Cited on 160, 163, 164

[19] Harnett D. and Nualart D. (2015). On Simpson's rule and fractional Brownian motion with H = 1/10. *J. Theoret. Probab.* **28** 1651-1688. Cited on 162, 170

[20] Harnett D. and Nualart D. (2015). Central limit theorem for a generalized self-similar Gaussian process. *Stochastic Process. Appl.* **128** 404–425. Cited on 160, 165, 167, 204

[21] Houdré C. and Villa J. (2003). An example of infinite dimensional quasi-helix. *Contemp. Math.* **336** 3-39. Cited on 166

[22] Hu, Y. (1996). On the self-intersection local time of Brownian motion-via chaos expansion. *Publ. Mat.* **40** 337–350. Cited on 22

[23] Hu, Y. and Nualart, D. (2005). Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.* **33** 948–983. Cited on 4, 22, 23, 29, 33, 34, 64, 92, 107, 110, 137

[24] Hu, Y. and Nualart, D. (2010). Central limit theorem for the third moment in space of the Brownian local time increments. *Electron. Commun. Probab.* **15** 396–410. Cited on 108

[25] Imkeller P., Pérez-Abreu V. and Vives J. (1995). Chaos expansions of double intersection local time of Brownian motion in $\mathbb{R}^d$ and renormalization. *Stochastic Process. Appl.* **56** 1–34. Cited on 22

[26] A. Jaramillo, D. Nualart. Asymptotic properties of the derivative self-intersection local time of fractional Brownian motion. Stochastic Processes and Their Applications (2017). 669 - 700. Arxiv Cited on iv

[27] A. Jaramillo, D. Nualart. Functional limit theorem for the self-intersection local time of the fractional Brownian motion. To appear in Annals of l'institut Henri Poincaré. Arxiv Cited on iv

[28] A. Jaramillo, D. Nualart. Collision of eigenvalues for matrix-valued processes. Arxiv Cited on iv

[29] Jung P. and Markowsky G. (2014). On the Tanaka formula for the derivative of self-intersection local time of fractional Brownian motion. *Stochastic Process. Appl.* **124** 3846–3868. Cited on 92, 108, 110, 137

[30] Kahane, J-P. Some random series of functions (1985). In *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. Cited on 215

[31] Karatzas, I. and Shreve, S. (1991). Brownian motion and stochastic calculus 2nd ed. Graduate Texts in Mathematic. Springer-Verlag, New York. Cited on 9, 11

[32] Lax P. (2002). *Functional Analysis*. Wiley-Interscience, New York. Cited on 223

[33] Lei P. and Nualart D. (2008). A decomposition of the bifractional Brownian motion and some applications. *Statist. Probab. Lett.* **75** 619-624. Cited on 166, 167

[34] Markowsky, G. (2012). The derivative of the intersection local time of Brownian motion through Wiener chaos. *Séminaire de Probabilités XLIV*. **2046** 141–148. Cited on 108

[35] Nourdin, I. (2012). Selected aspects of fractional Brownian motion. Bocconi & Springer Series. Springer, Milan; Bocconi University Press, Milan. Cited on 11

[36] Nourdin, I., Nualart D. and Tudor, C. (2010). Central and non-central limit theorems for weighted power variations of fractional Brownian motion, *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 1055–1079. Cited on 18, 26

[37] Nourdin, I. and Peccati, G. (2012). Normal approximations with Malliavin calculus. In *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge. Cited on 2, 19, 20

[38] Nourdin I. and Nualart D. (2010). Central limit theorems for multiple Skorokhod integrals. *J. Theor. Probab.* **23** 39-64. Cited on 17

[39] Nourdin I. and Peccati. G. (2012). *Normal approximations with Malliavin calculus: from Stein's method to universality*. Cambridge U. Press, 2012. Cited on

[40] Nourdin I., Nualart D. and Tudor C.A. (2010). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Statist.* **46** 1055–1079. Cited on 164

[41] Nourdin I., Révaillac A. and Swanson J. (2010). The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter 1/6. *Electron. J. Probab.* **15** 2087-2116. Cited on 162, 170

[42] Nualart D. and Ortiz-Latorre S. (2008). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stoch. Proc. Appl.* **118** 614-628. Cited on

[43] Nualart, D. (2006). The Malliavin calculus and related topics. In *Probability and its Applications.* Springer-Verlag, Berlin. Cited on 12, 16

[44] Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. Cited on 6, 19, 109

[45] Nualart, D. and Pérez-Abreu, V. (2014). On the eigenvalue process of a matrix fractional Brownian motion. *Stochastic Processes and their Applications*. **124** 4266–4282. Cited on 211

[46] Pardo, J.P., Pérez, J.L., and Pérez-Abreu V. (2016). A random matrix approximation for the non-commutative fractional Brownian motion. *Journal of Theoretical Probability.* **29** 1581–1598. Cited on 211

[47] Peccati, G. and Tudor, C. (2005). Gaussian limits for vector-valued multiple stochastic integrals. *Séminaire de Probabilités XXXVIII*. **1857** 247–262. Cited on 19, 20, 23, 28, 89, 109, 132

[48] Rogers, L.C.G. and Walsh, J.B. (1991). Local time and stochastic area integrals. *Ann. Probab.* **19** 457–482. Cited on 107

[49] Rosen, J. (1987). The intersection local time of fractional Brownian motion in the plane. *J. Multivar. Anal.* **23** 37–46. Cited on 4, 22, 107

[50] Rosen, J. (2005). Derivatives of self-intersection local times. *Lecture Notes in Math.* **1857** 263–281. Cited on 107

[51] Russo F. and Tudor C.A. (2006). On bi-fractional Brownian motion. *Stoch. Proc. Appl.* **116** 830-856. Cited on 166

[52] Swanson J. (2007). Weak convergence of the scaled median of independent Brownian motions. *Probab. Theory Relat. Fields* **138** 269-304. Cited on 5, 167

[53] Swanson J. (2011). Fluctuations of the empirical quantiles of independent Brownian motions. *Stochastic Process Appl.*, **121** 479-514. Cited on 163, 164

[54] Tu, L. (2011). An introduction to manifolds. Springer, New York. Cited on 219

[55] Varadhan, S. (1969). Appendix to Euclidean quantum field theory, by K. Symanzik. In *Local Quantum Theory* (R. Jost, ed.). Academic Press, New York. Cited on 22

[56] Xiao, Y. (1999). Hitting probabilities and polar sets for fractional Brownian motion *Stochastics and Stochastics Reports*. **66** 121–151. Cited on 214

[57] Xiao, Y. (2009). Sample path properties of anisotropic Gaussian random fields, "A minicourse on stochastic partial differential equations". In *Lecture Notes in Mathematics*. **1962** 145–212. Springer-Verlag, Berlin. Cited on 214

[58] Yor, M. (1985). Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement brownien dans $\mathbb{R}^3$. *Séminaire de probabilités, XIX. Lecture Notes in Math.* **1123** 350–365. Springer, Berlin. Cited on 22