

Nonlinear Integrals, Diffusion in Random Environments and Stochastic Partial Differential Equations

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Abstract

In this dissertation, we investigate various problems in the analysis of stochastic (partial) differential equations. A part of the dissertation introduces several notions of nonlinear integrations. Some differential equations associated with nonlinear integrations are investigated. Examples include transport differential equations in space-time random fields and parabolic equations with potentials of the type $\partial_t W$, where W is continuous in time variable and smooth in the spatial variables. Another part of the dissertation studies nonlinear stochastic convolution equations driven by a multiplicative Gaussian noise which is white in time and which has the covariance of a fractional Brownian motion with Hurst parameter $H \in (1/4, 1/2)$ in the spatial variable. The other part of the dissertation gives rigorous meaning to the Brox differential equation $X(t) = \mathcal{B}(t) - \frac{1}{2} \int_0^t \dot{W}(X(s)) ds$ where \mathcal{B} and W are independent Brownian motions. Furthermore, it is shown that the Brox differential equation has a unique strong solution which is a time-changed spatial transformation of a Brownian motion. Along the way, some appropriate tools are developed in order to solve these problems. In particular, we establish a multiparameter version of Garsia-Rodemich-Rumsey inequality which allows one to control rectangular increments in any dimensions of multivariate functions, definitions and compact criteria for some new functions spaces are developed. The methodologies employed form a combination of stochastic analysis, Malliavin calculus and functional analytic tools. Several parts of the dissertation are joint work of the author with Yaozhong Hu, Jingyu Huang, David Nualart, Leonid Mytnik and Samy Tindel.

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Chapter 1

Introduction

The dissertation collects some joint work of the author with others. Many parts of the dissertation are taken from the following manuscripts and papers:

- [54] *Stochastic differential equation for Brox diffusion*, with Yaozhong Hu and Leonid Mytnik, manuscript;
- [56] *Stochastic heat equation with rough dependence in space*, with Yaozhong Hu, Jingyu Huang, David Nualart and Samy Tindel, manuscript;
- [58] *A multiparameter Garsia-Rodemich-Rumsey inequality and some applications*, with Yaozhong Hu, *Stochastic Process. Appl.* **123** (2013);
- [59] *Nonlinear Young integrals and differential systems in Hölder media*, with Yaozhong Hu, arXiv preprint;
- [60] *Nonlinear Young integrals via fractional calculus*, with Yaozhong Hu, accepted to CAS SEFE Proceeding.

More precisely, Chapter 2 is from the paper [58]. Chapters 3, 4, 5 and 6 are from the preprints [60] and [59]. Chapter 7 is an original work of the author and will be published in the near future. Chapter 8 is part of the manuscript [56]. Chapter 9 is from the manuscript [54]. The content of each chapter is described as following.

Chapter 2 shows a multiparameter Garsia-Rodemich-Rumsey (GRR) inequality which allows one to control rectangular increments of a multivariate function. The Garsia-Rodemich-Rumsey inequality originated from [40] is a well-known tool in probability theory. A specific case of GRR inequality yields Sobolev-Morrey inequality and the Kolmogorov continuity criteria which is usually applied to obtain almost sure Hölder continuity for stochastic processes. To prove the multiparameter GRR inequality, we induct on the dimension and apply the classical GRR inequality. A special case of the two-parameter GRR inequality has been obtained earlier in [86] by a different method. We then apply the multiparameter GRR inequality to obtain sharp joint Hölder continuity for some Gaussian random fields.

Chapter 3 investigates nonlinear integrals of the form $\int_a^b W(ds, \phi)$, where W is a joint Hölder continuous function and ϕ is a Hölder continuous function. We discuss several ways to define the nonlinear integrals. More precisely, we provide three types of nonlinear integrations: Young type, which extends the Young integrals [98], Itô-Skorohod type, which extends the Itô-Skorohod integrals, and symmetric type, following the work of Russo and Vallois in [88]. We also study the relationship between these three types on nonlinear integrations.

In Chapter 4, we discuss some differential equations associated with the nonlinear Young integration. In particular, we study well-posedness of the differential equation $\phi_t = x + \int_0^t W(ds, \phi_s)$ and the transport differential equation $du(t, x) + \nabla u \cdot W(dt, x) = 0$. These equations describe the motion of a particle in the space-time vector field $W(dt, x)$. In the context of analysis, when $W(dt, x)$ takes the form $b(x)dt$, b is Lipschitz, these equations have been studied in details by the classical work [29] of DiPerna and Lions. The seminal work of Ambrosia [3] extends this situation to bounded variation vector fields b . The equations considered in this chapter do not fall under these treatments because of the irregularity and nonlinearity in the temporal variable.

In Chapter 5, we study the validity of Feynman-Kac formulae to some second order parabolic equation. An example is the equation $\partial_t u + \Delta u + u \partial_t W = 0$ with terminal condition $u(T, x) = u_T(x)$. Two separate conditions on W are considered. One condition is that W is function continuous in time and regular in the spatial variables, the other condition is that W is joint Hölder continuous. We show that in both sets of conditions, this equation has a continuous solution given by the Feynman-Kac formula. Our approach relies on a scale transformation and an Itô-Tanaka formula originated from [37].

Chapter 6 can be considered as a continuation of Chapter 2. We study the asymptotic growth of the sample paths of Gaussian random fields. Our goal is to understand the growth of the rectangular increments of a random field as the domain of parameters expands to infinity. These asymptotic growths are the assumed conditions in Chapter 4 and Chapter 5. The multiparameter GRR inequality considered Chapter 2 is no longer appropriate on unbounded domains. Thus, we employ a different method, majorizing measures. The results obtained are specifically for Gaussian random fields.

In Chapter 7, we consider the linear stochastic convolution equation (SCE) of the type

$$u(t, x) = w(t, x) + \int_0^t G_{t-s}(x - y)u(s, y)W(ds, dy)$$

where G is a Green kernel, w is given a priori and W is a centered Gaussian process with covariance

$$\mathbb{E}[W(s, x)W(t, y)] = \frac{1}{2} (|x|^{2H} + |y|^{2H} - |x - y|^{2H}) (s \wedge t). \quad (1.1)$$

The Hurst parameter H is assumed to be inside the region $(1/4, 1/2)$. The above equation is the mild formulation of several stochastic differential equations with multiplicative noise. Three main examples considered in the chapter are the stochastic heat equation (SHE), the stochastic wave equation (SWE) and the stochastic fractional heat equation (SFHE). It is shown that if the Green kernel belongs to some functions spaces, the above linear stochastic convolution equation has a unique random field solution with initial conditions

including bounded functions and Dirac delta masses. The methods used in this chapter are functional analytic, which are developed further in Chapter 8.

Chapter 8 investigates the nonlinear stochastic heat equation

$$u(t, x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y) \sigma(u(s, y)) W(ds, dy)$$

where u_0 is the initial function, $p_t(x)$ is the Gaussian density, W is a centered Gaussian process with covariance as in 1.1, and σ is a regular function. If $H \geq 1/2$, the above equation has been studied extensively (for instance, by Peszat and Zabczyk [82], and Dalang [22]). In the case considered here, the existing methods in literature can not be applied because the noise W is so irregular that the map $f \mapsto \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y) \sigma(f(s, y)) W(ds, dy)$ is no longer Lipschitz. Existence and uniqueness of the above equation are not transparent. Here, we develop some new functions spaces and study them in details, particularly compact criteria on these spaces. Once the functional framework is set, existence and uniqueness results for the above equation are obtained following the method originated from [47]. Comparing with Chapter 7, the assumptions on the initial condition in this chapter are more restrictive.

Chapter 9 studies the Brox diffusion and its stochastic differential equation. The Brox diffusion, after [13], is a continuum analog of Sinai's random walk in random environment first appeared in [92]. It is described by the stochastic differential equation $X(t) = \mathcal{B}(t) - \frac{1}{2} \int_0^t \dot{W}(X(s)) ds$, where \mathcal{B} is a free Brownian motion modeling the diffusion and \dot{W} is a white noise modeling the random environment, \mathcal{B} and W are independent. Due to its singular structure, the Brox equation has not been well-understood. In this chapter, we provide a rigorous meaning to this equation. We also show that the Brox stochastic differential equation has a unique strong solution given by the Itô-McKean representation, a time-changed spatial transformation of a Brownian motion. Our method relies on some estimates on the joint increments of local time of Brownian motions. We also obtain an Itô formula for the Brox diffusion.

Chapter 2

Multiparameter

Garsia-Rodemich-Rumsey inequality

Let the function $\Psi : [0, \infty) \rightarrow [0, \infty)$ be non decreasing with $\lim_{u \rightarrow \infty} \Psi(u) = \infty$ and let the function $p : [0, 1] \rightarrow [0, 1]$ be continuous and non decreasing with $p(0) = 0$. Set

$$\begin{cases} \Psi^{-1}(u) = \sup_{\Psi(v) \leq u} v & \text{if } \Psi(0) \leq u < \infty \\ p^{-1}(u) = \max_{p(v) \leq u} v & \text{if } 0 \leq u \leq p(1) \end{cases}$$

The celebrated Garsia-Rodemich-Rumsey inequality [40] takes the following form:

Lemma 2.0.1. *Let f be a continuous function on $[0, 1]$ and suppose that*

$$\int_0^1 \int_0^1 \Psi \left(\frac{|f(x) - f(y)|}{p(x-y)} \right) dx dy \leq B < \infty.$$

Then for all $s, t \in [0, 1]$ we have

$$|f(s) - f(t)| \leq 8 \int_0^{|s-t|} \Psi^{-1} \left(\frac{4B}{u^2} \right) dp(u). \quad (2.1)$$

This Garsia-Rodemich-Rumsey Lemma 2.0.1 is very powerful in the study of the sample path Hölder continuity of a stochastic process and in other occasions. For example if

$\Psi(u) = |u|^p$ and $p(u) = |u|^{\alpha+1/p}$, where $p\alpha > 1$, the inequality (2.1) implies the following Sobolev imbedding inequality

$$|f(s) - f(t)| \leq C_{\alpha,p} |t - s|^{\alpha-1/p} \left(\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 1}} dx dy \right)^{1/p}. \quad (2.2)$$

The Garsia-Rodemich-Rumsey lemma has been extended to several parameter or infinite many parameters. But the parameter space are assumed to have a distance (metric space) and the Garsia-Rodemich-Rumsey lemma is with respect to that distance. This method immediately yields the following result for a fractional Brownian field $W^H(x)$ of Hurst parameter $H = (H_1, \dots, H_d)$, then for any β_i with $\beta_i < H_i, i = 1, \dots, d$, one has

$$|W(y) - W(x)| \leq L \sum_{i=1}^d |y_i - x_i|^{\beta_i}, \quad (2.3)$$

where L is an integrable random variable. One can improve this result (see Remark 2.3.3) by our version of multiparameter Garsia-Rodemich-Rumsey inequality. We do not seek for a suitable metric but rather deal directly with the multidimensional nature of the parameter space.

Let us explain our motivation by considering the two parameter fractional Brownian field $\{W(x_1, x_2), (x_1, x_2) \in [0, 1]^2\}$ of Hurst parameter $H = (H_1, H_2)$. Given two points x and y in \mathbb{R}^2 , we consider the increment of W along with the rectangle determined by $x = (x_1, x_2)$ and $y = (y_1, y_2)$:

$$\square W := W(y_1, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, x_2). \quad (2.4)$$

In [86], using a two-parameter version (2.2), the author showed that for any β_1, β_2 with $\beta_1 < H_1$ and $\beta_2 < H_2$, there is an integrable random constant L_{β_1, β_2} such that

$$|\square W| \leq L_{\beta_1, \beta_2} |y_1 - x_1|^{\beta_1} |y_2 - x_2|^{\beta_2}. \quad (2.5)$$

The above result was also obtained in [5] based on a two-parameter version of Kolmogorov continuity theorem. Along the chapter (in Corollary 2.3.4), we shall see that the following sharper inequality than (2.5) holds

$$|\square W| \leq L_{H_1, H_2} |y_1 - x_1|^{H_1} |y_2 - x_2|^{H_2} \sqrt{|\log(|y_1 - x_1| |y_2 - x_2|)|}. \quad (2.6)$$

Consequently, this estimate implies

$$|W(x_1, x_2) - W(y_1, y_2)| \leq L_{H_1, H_2} \left(|x_1 - y_1|^{H_1} |x_2|^{H_2} \sqrt{|\log(|x_1 - y_1| |x_2|)|} \right. \\ \left. + |x_1|^{H_1} |x_2 - y_2|^{H_2} \sqrt{|\log(|x_2 - y_2| |x_1|)|} \right)$$

which improves (2.3). To our best knowledge, the estimate (2.6) is the first of its kind in literature. We shall call such property as in (2.6) or (2.5) joint Hölder continuity. It turns out that a large class of Gaussian fields enjoys sample path joint Hölder continuity (Theorem 2.3.1.)

Our method is first formulate and prove a multiparameter version of the classical Garsia-Rodemich-Rumsey inequality (2.1). The generalized inequality is then applied to obtain sample path joint Hölder continuity for random fields. Our result generalizes the results in [40], [86] and provides a different approach for sample path continuity problem of random fields (compare to the approach in [5], [6] and [96].)

The chapter is structured as follows. In Section 2.1, we shall state and prove our multiparameter version of the Garsia-Rodemich-Rumsey lemma. The idea is to use induction on the dimension of the parameter space after some observations of the property of operator \square defined by (2.4). Some part of the proof is similar to the original proof of Garsia-Rodemich-Rumsey [40] with some modification. However, we feel it is more appropriate to give a detailed proof.

In Section 2.2, we introduce a multiparameter version of Kolmogorov continuity criteria

(Theorem 2.2.1). To our best knowledge, a two-parameter of Theorem 2.2.1 first appeared in [5].

Section 2.3 is devoted for the study of sample path continuity for Gaussian fields. We give a sufficient condition for a Gaussian field to possess sample path continuity (Theorem 2.3.1). We also derive the estimate (2.6) for fractional Gaussian field. In Section 2.4, we shall study the joint Hölder continuity of solution of a stochastic heat equation with additive space-time white noise.

2.1 The result

We state the following technical lemma which generalizes a crucial argument used in [40] in the proof of Lemma 2.0.1.

Lemma 2.1.1. *Let (Ω, \mathcal{F}) be a measurable space and let μ be a positive measure on (Ω, \mathcal{F}) . Let $g : \Omega \times [0, 1] \rightarrow \mathbb{R}^m$ be a measurable function such that*

$$\int_0^1 \int_0^1 \int_{\Omega} \Psi \left(\frac{|g(z, t) - g(z, s)|}{p(|t - s|)} \right) \mu(dz) ds dt \leq B < \infty.$$

Then there exist two decreasing sequences $\{t_k, k = 0, 1, \dots\}$ and $\{d_k, k = 0, 1, \dots\}$ with

$$t_k \leq d_{k-1} = p^{-1} \left(\frac{1}{2} p(t_{k-1}) \right), \quad k = 1, 2, \dots \quad (2.7)$$

such that the following inequality holds

$$\int_{\Omega} \Psi \left(\frac{|g(z, t_k) - g(z, t_{k-1})|}{p(|t_k - t_{k-1}|)} \right) \mu(dz) \leq \frac{4B}{d_{k-1}^2}. \quad (2.8)$$

Proof. We follow the argument in [40]. Let

$$I(t) = \int_0^1 \int_{\Omega} \Psi \left(\frac{|g(z, t) - g(z, s)|}{p(|t - s|)} \right) \mu(dz) ds.$$

From the assumption $\int_0^1 I(t)dt \leq B$ it follows that there is some $t_0 \in (0, 1)$ such that

$$I(t_0) \leq B.$$

Now we can describe how to obtain the sequences d_k and t_k recursively for $k = 1, 2, \dots$.

Given t_{k-1} , define

$$d_{k-1} = p^{-1} \left(\frac{1}{2} p(t_{k-1}) \right).$$

Then we choose $t_k \leq d_{k-1}$ such that

$$I(t_k) \leq \frac{2B}{d_{k-1}} \tag{2.9}$$

and

$$\int_{\Omega} \Psi \left(\frac{|g(z, t_k) - g(z, t_{k-1})|}{p(|t_k - t_{k-1}|)} \right) \mu(dz) \leq \frac{2I(t_{k-1})}{d_{k-1}}. \tag{2.10}$$

It is always possible to find t_k such that the inequalities (2.9) and (2.10) hold simultaneously, since each of the two inequalities can be violated only on a set of t_k 's of measure strictly less than $\frac{1}{2}d_{k-1}$. Now (2.9) and (2.10) gives

$$\int_{\Omega} \Psi \left(\frac{|g(z, t_k) - g(z, t_{k-1})|}{p(|t_k - t_{k-1}|)} \right) \mu(dz) \leq \frac{2I(t_{k-1})}{d_{k-1}} \leq \frac{4B}{d_{k-1}d_{k-2}} \leq \frac{4B}{d_{k-1}^2}.$$

This is (2.8). □

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be in \mathbb{R}^n . We denote $x' = (x_1, \dots, x_{n-1})$ and $y' = (y_1, \dots, y_{n-1})$. For each integer $k = 1, 2, \dots, n$, we define

$$V_{k,y}x = (x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n).$$

Let f be a function from \mathbb{R}^n to \mathbb{R}^m . We define the operator $V_{k,y}$ acting on f in the following

way:

$$V_{k,y}f(x) = f(V_{k,y}x)$$

or $V_{k,y}f = f \circ V_{k,y}$ in short. It is straight forward to verify that

$$V_{k,y}V_{k,y} = V_{k,y}$$

and

$$V_{k,y}V_{l,y} = V_{l,y}V_{k,y}$$

for $k \neq l$. Next, we define the joint increment or rectangular increment of a function f on an n -dimensional rectangle, namely

$$\square_y^n f(x) = \prod_{k=1}^n (I - V_{k,y})f(x)$$

where I denotes the identity operator.

Example 2.1.2. If $n = 2$, then it is easy to see that $\square_y^2 f(x) = f(y_1, y_2) - f(x_1, y_2) - f(y_1, x_2) + f(x_1, x_2)$, which is the increment of f over the rectangle containing the two points x and y with all sides parallel to the axis. In particular, if $f(x_1, x_2) = x_1 x_2$, then $\square_y^2 f(x) = (x_1 - y_1)(x_2 - y_2)$, which is the area of the rectangle. In a more general case, when f has the form $f(x) = \prod_{j=1}^n f_j(x_j)$, then

$$\square_y^n f(x) = \prod_{j=1}^n [f_j(x_j) - f_j(y_j)].$$

The following simple identity enable us to show our theorem by induction and plays

an essential role in our approach:

$$\begin{aligned}\square_y^n f(x) &= \prod_{k=1}^{n-1} (I - V_{k,y}) f(x) - V_{n,y} \prod_{k=1}^{n-1} (I - V_{k,y}) f(x) \\ &= \square_{y'}^{n-1} f(x', x_n) - \square_{y'}^{n-1} f(x', y_n).\end{aligned}\tag{2.11}$$

We are now in the position to state our general version of Lemma 2.0.1.

Theorem 2.1.1. *Let $f(x)$ be a continuous function on $[0, 1]^n$ and suppose that*

$$\int_{[0,1]^n} \int_{[0,1]^n} \Psi \left(\frac{|\square_y^n f(x)|}{\prod_{k=1}^n p_k(|x_k - y_k|)} \right) dx dy \leq B < \infty.\tag{2.12}$$

Then for all $s, t \in [0, 1]^n$ we have

$$|\square_s^n f(t)| \leq 8^n \int_0^{|s_1-t_1|} \cdots \int_0^{|s_n-t_n|} \Psi^{-1} \left(\frac{4^n B}{u_1^2 \cdots u_n^2} \right) dp_1(u_1) \cdots dp_n(u_n).\tag{2.13}$$

Proof. We proceed by induction on n . For $n = 1$, it coincides with the original Garsia-Rodemich-Rumsey inequality (2.1). Suppose (2.13) holds for $n - 1$. Let f be a continuous function on $[0, 1]^n$. For any $x', y' \in \mathbb{R}^{n-1}$ and any $s \in [0, 1]$, put

$$g(x', y', s) = \frac{\square_{y'}^{n-1} f(x', s)}{\prod_{k=1}^{n-1} p_k(|x'_k - y'_k|)}.$$

Let $\Omega = [0, 1]^{n-1} \times [0, 1]^{n-1}$, $z = (x', y')$. By (2.11) we can rewrite (2.12) as

$$\int_0^1 \int_0^1 \int_{\Omega} \Psi \left(\frac{|g(z, s) - g(z, t)|}{p_n(|s - t|)} \right) dz ds dt \leq B < \infty.$$

Applying Lemma 2.1.1, we can find sequences $\{t_k\}$ and $\{d_k\}$ such that

$$t_k \leq d_{k-1} = p_n^{-1} \left(\frac{1}{2} p_n(t_{k-1}) \right)\tag{2.14}$$

and

$$\int_{\Omega} \Psi \left(\frac{|g(z, t_k) - g(z, t_{k-1})|}{p_n(t_k - t_{k-1})} \right) dz \leq \frac{4B}{d_{k-1}^2}. \quad (2.15)$$

For each $k \in \mathbb{N}$ and $x' \in [0, 1]^{n-1}$, let

$$h_k(x') = \frac{f(x', t_k) - f(x', t_{k-1})}{p_n(|t_k - t_{k-1}|)}.$$

Again from (2.11) it follows

$$\begin{aligned} \frac{\square_{y'}^{n-1} h_k(x')}{\prod_{i=1}^{n-1} p_i(|x'_i - y'_i|)} &= \frac{\square_{y'}^{n-1} f(x', t_k) - \square_{y'}^{n-1} f(x', t_{k-1})}{\prod_{i=1}^{n-1} p_i(|x'_i - y'_i|) p_n(|t_k - t_{k-1}|)} \\ &= \frac{g(x', y', t_k) - g(x', y', t_{k-1})}{p_n(|t_k - t_{k-1}|)}. \end{aligned}$$

Thus, the inequality (2.15) becomes

$$\int_{[0,1]^{n-1}} \int_{[0,1]^{n-1}} \Psi \left(\frac{\square_{y'}^{n-1} h_k(x')}{\prod_{i=1}^{n-1} p_i(|x'_i - y'_i|)} \right) dx' dy' \leq \frac{4B}{d_{k-1}^2}.$$

Now, by our induction hypothesis, for every $k \geq 1$, $a, b \in [0, 1]^{n-1}$, $a = (a_1, \dots, a_{n-1})$ and $b = (b_1, \dots, b_{n-1})$,

$$|\square_a^{n-1} h_k(b)| \leq 8^{n-1} \int_0^{|a_1-b_1|} \dots \int_0^{|a_{n-1}-b_{n-1}|} \Psi^{-1} \left(\frac{4^n B}{u_1^2 \dots u_{n-1}^2 d_{k-1}^2} \right) dp_1(u_1) \dots dp_{n-1}(u_{n-1}).$$

Denoting $A = [0, |a_1-b_1|] \times \dots \times [0, |a_{n-1}-b_{n-1}|]$ and $dp(u_1, \dots, u_{n-1}) = dp_1(u_1) \dots dp_{n-1}(u_{n-1})$, the above inequality can be rewritten as

$$|\square_a^{n-1} f(b, t_k) - \square_a^{n-1} f(b, t_{k-1})| \leq 8^{n-1} \int_A \Psi^{-1} \left(\frac{4^n B}{u_1^2 \dots u_{n-1}^2 d_{k-1}^2} \right) dp(u_1, \dots, u_{n-1}) p_n(t_{k-1} - t_k). \quad (2.16)$$

On the other hand, by (2.14), we have

$$p_n(t_{k-1} - t_k) \leq p_n(t_{k-1}) = 2p(d_{k-1}) \leq 4 [p(d_{k-1}) - p(d_k)].$$

Combining this inequality with (2.16) yields

$$\begin{aligned} & |\square_a^{n-1} f(b, t_0) - \square_a^{n-1} f(b, 0)| \\ & \leq \sum_{k=1}^{\infty} |\square_a^{n-1} f(b, t_k) - \square_a^{n-1} f(b, t_{k-1})| \\ & \leq 8^{n-1} \sum_{k=1}^{\infty} 4 [p(d_{k-1}) - p(d_k)] \int_A \Psi^{-1} \left(\frac{4^n B}{u_1^2 \cdots u_{n-1}^2 d_{k-1}^2} \right) dp(u_1, \dots, u_{n-1}) \\ & \leq 8^{n-1} \sum_{k=1}^{\infty} 4 \int_{d_k}^{d_{k-1}} \int_A \Psi^{-1} \left(\frac{4^n B}{u_1^2 \cdots u_{n-1}^2 u_n^2} \right) dp(u_1, \dots, u_{n-1}) dp_n(u_n) \\ & \leq 8^{n-1} 4 \int_0^1 \int_A \Psi^{-1} \left(\frac{4^n B}{u_1^2 \cdots u_n^2} \right) dp(u_1, \dots, u_{n-1}) dp_n(u_n). \end{aligned}$$

With $f(x', 1 - x_n)$ replaced $f(x', x_n)$ we can obtain the same bound for

$$|\square_a^{n-1} f(b, t_0) - \square_a^{n-1} f(b, 1)|.$$

Hence, for every $a, b \in [0, 1]^{n-1}$,

$$|\square_a^{n-1} f(b, 1) - \square_a^{n-1} f(b, 0)| \leq 8^n \int_0^1 \int_A \Psi^{-1} \left(\frac{4^n B}{u_1^2 \cdots u_n^2} \right) dp(u_1, \dots, u_{n-1}) dp_n(u_n). \quad (2.17)$$

To obtain (2.13) for general s, t in $[0, 1]^n$, we set

$$\bar{f}(t', \tau) = f(t', s_n + \tau(t_n - s_n)) \text{ for } \tau \in [0, 1]$$

and

$$\bar{p}_n(u) = p_n(u | s_n - t_n).$$

Upon restricting the range of the integration in (2.12) and carrying out a change of variables we get

$$\int_{[0,1]^n} \int_{[0,1]^n} \Psi \left(\frac{|\square_y^n \bar{f}(x)|}{\prod_{k=1}^{n-1} p_k(|x_k - y_k|) \bar{p}_n(|x_n - y_n|)} \right) dx dy \leq \frac{B}{|s_n - t_n|^2}.$$

Thus, by (2.17), we deduce

$$\begin{aligned} |\square_s^n f(t)| &= |\square_{s'}^{n-1} \bar{f}(t', 1) - \square_{s'}^{n-1} \bar{f}(t', 0)| \\ &\leq 8^n \int_0^1 \int_0^{|s_1 - t_1|} \cdots \int_0^{|s_{n-1} - t_{n-1}|} \Psi^{-1} \left(\frac{4^n B}{u_1^2 \cdots u_n^2 |s_n - t_n|^2} \right) dp(u_1, \dots, u_{n-1}) dp_n(u_n | s_n - t_n). \end{aligned}$$

Another change of variables yields (2.13). \square

2.2 Sample path Hölder continuity of random fields

In this section, given a random field W , we study sample path continuity property. The first application of Theorem 2.1.1 is the following criteria for joint continuity of sample paths which is similar to Kolmogorov continuity theorem, which we shall call joint Kolmogorov continuity theorem.

Theorem 2.2.1. *Let W be a random field on \mathbb{R}^n . Suppose there exist positive constants α, β_k ($1 \leq k \leq n$) and K such that for every x, y in $[0, 1]^n$,*

$$\mathbb{E} \left[|\square_y^n W(x)|^\alpha \right] \leq K \prod_{k=1}^n |x_k - y_k|^{1+\beta_k}.$$

Then, for every $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with $0 < \epsilon_k \alpha < \beta_k$ ($1 \leq k \leq n$), there exist a random variable η with $\mathbb{E} \eta^\alpha \leq K$, such that the following inequality holds almost surely

$$|\square_t^n W(s)| \leq C \eta(\omega) \prod_{k=1}^n |t_k - s_k|^{\beta_k \alpha^{-1} - \epsilon_k}$$

for all s, t in $[0, 1]^n$, where C is a constant defined by

$$C = 8^n 4^{n/\alpha} \prod_{k=1}^n \left(1 + \frac{2}{\beta_k - \alpha \epsilon_k} \right).$$

Proof. Let $\Psi(u) = |u|^\alpha$, $p_k(u) = |u|^{\gamma_k}$ where $\gamma_k \in (\frac{2}{\alpha}, \frac{2+\beta_k}{\alpha})$, $1 \leq k \leq n$. A direct application of Theorem 2.1.1 gives that for all s, t in $[0, 1]^n$

$$|\square_s^n W(t)| \leq 8^n \prod_{k=1}^n \frac{\gamma_k |t_k - s_k|^{\gamma_k - \frac{2}{\alpha}}}{\gamma_k - \frac{2}{\alpha}} \left(4^n \iint_{[0,1]^{2n}} \frac{|\square_y^n W(x)|^\alpha}{\prod_{k=1}^n |x_k - y_k|^{\alpha \gamma_k}} dx dy \right)^{\frac{1}{\alpha}}. \quad (2.18)$$

Let

$$B(\omega) = \iint_{[0,1]^{2n}} \Psi \left(\frac{|\square_y^n W(x)|}{\prod_{k=1}^n p_k(x_k - y_k)} \right) dx dy.$$

From our assumption and Fubini-Tonelli's theorem,

$$\begin{aligned} \mathbb{E}B &= \iint_{[0,1]^{2n}} \frac{\mathbb{E}|\square_y^n W(x)|^\alpha}{\prod_{k=1}^n |x_k - y_k|^{\alpha \gamma_k}} dx dy \\ &\leq K \iint_{[0,1]^{2n}} \prod_{k=1}^n |x_k - y_k|^{1+\beta_k - \alpha \gamma_k} dx dy < \infty. \end{aligned}$$

Hence, the event $\Omega^* = \{\omega : B(\omega) < \infty\}$ has probability one. Therefore for each ω in Ω^* , the inequality (2.18) gives

$$|\square_s^n W(t, \omega)| \leq 8^n \prod_{k=1}^n \frac{\gamma_k |t_k - s_k|^{\gamma_k - \frac{2}{\alpha}}}{\gamma_k - \frac{2}{\alpha}} (4^n B(\omega))^{\frac{1}{\alpha}}$$

for every s, t in $[0, 1]^n$. For each k , the power $\gamma_k - \frac{2}{\alpha}$ can be made arbitrarily close to $\frac{\beta_k}{\alpha}$.

This completes the proof with $\eta = B^{1/\alpha}$. \square

Remark 2.2.1. The result obtained by Ral'chenko [86] was the inequality (2.18) in the case $n = 2$.

2.3 Sample path continuity of Gaussian fields

We now focus on sample path continuity of Gaussian random fields. In case of Gaussian processes ($n = 1$), one of the first sufficient and necessary conditions for sample path continuity was given by Fernique [33]. Namely, let $p(u)$ be an increasing positive function such that

$$\mathbb{E}|W(x) - W(y)|^2 \leq p^2(|x - y|)$$

for any pair (x, y) in $[0, 1]^2$. Then Fernique [33] showed that a sufficient condition for almost sure continuity of the process $(W(x), 0 \leq x \leq 1)$ is

$$\int_0^1 \frac{p(u)}{u \sqrt{\log \frac{1}{u}}} du < \infty.$$

In the original paper of Garsia-Rodemich-Rumsey [40], the authors also observed that the above condition is equivalent to the condition (by integration by part)

$$\int_0^1 \sqrt{\log \frac{1}{u}} dp(u) < \infty.$$

Later, it was shown that the above condition is also necessary [34, 75]. In case of Gaussian fields, recent progress on modulus of continuity of Gaussian random fields has been reported in [6, 78, 96].

Let W be a centered Gaussian random field with covariance function

$$\mathbb{E} [W(x)W(y)] = Q(x, y). \tag{2.19}$$

We will always assume that Q is a continuous function of x and y . For any fixed x, y , the random variable $\square_y^n W(x)$ is also Gaussian with mean zero. In the following proposition, we compute its variance.

Proposition 2.3.1. *Let W be a centered Gaussian random field with covariance function given by*

(2.19). Then

$$\mathbb{E} \left[|\square_y^n W(x)|^2 \right] = \square_{(y,y)}^{2n} Q(x, x). \quad (2.20)$$

Furthermore, if the covariance function Q has the following product form

$$Q(x, y) = \prod_{k=1}^n Q_k(x_k, y_k) \quad (2.21)$$

then (2.20) is simplified as

$$\mathbb{E} \left[|\square_y^n W(x)|^2 \right] = \prod_{k=1}^n [Q_k(x_k, x_k) - Q_k(x_k, y_k) - Q_k(y_k, x_k) + Q_k(y_k, y_k)]. \quad (2.22)$$

Proof. We calculate the variance directly as follows

$$\begin{aligned} \mathbb{E} \left[|\square_y^n W(x)|^2 \right] &= \mathbb{E} \left[\square_y^n W(x) \cdot \square_y^n W(x) \right] \\ &= \mathbb{E} \left[\square_{(y,y)}^{2n} W(x) W(x) \right] \\ &= \square_{(y,y)}^{2n} \mathbb{E} [W(x) W(x)] \\ &= \square_{(y,y)}^{2n} Q(x, x). \end{aligned}$$

The identity (2.20) follows. To prove (2.22), we notice that the pair of operators $(I - V_{k,(y,y)})(I - V_{n+k,(y,y)})$ transforms the k -th factor of Q in (2.21) to

$$Q_k(x_k, x_k) - Q_k(x_k, y_k) - Q_k(y_k, x_k) + Q_k(y_k, y_k).$$

Since the operators $I - V_{k,(y,y)}$, $(1 \leq k \leq 2n)$ are commutative, we can write

$$\begin{aligned} \square_{(y,y)}^{2n} Q(x, x) &= \prod_{k=1}^n (I - V_{k,(y,y)})(I - V_{n+k,(y,y)}) Q(x, x) \\ &= \prod_{k=1}^n [Q_k(x_k, x_k) - Q_k(x_k, y_k) - Q_k(y_k, x_k) + Q_k(y_k, y_k)]. \end{aligned}$$

Hence, the identity (2.22) follows. \square

Definition 2.3.2. Let f be a continuous function on \mathbb{R}^n . We call a set of non-negative even functions $\{p_1, \dots, p_n\}$ joint modulus of continuity of f if

- (i) For each $1 \leq k \leq n$, $p_k(0) = 0$, and p_k is non-decreasing and continuous.
- (ii) For every pair (s, t) in \mathbb{R}^{2n} , the following inequality holds

$$|\square_s^m f(t)| \leq \prod_{k=1}^n p_k(|t_k - s_k|).$$

In view of Theorem 2.1.1 and Theorem 2.2.1, the continuity of sample paths is governed by the joint modulus of continuity of $\square_{(y,y)}^{2n} Q(x, x)$. Such modulus of continuity always exists. For instance, we can define a joint modulus of continuity for $\square_{(y,y)}^{2n} Q(x, x)$ as follows.

We set

$$p_1(u) = \sup_{x, y \in [0,1]^n: |x_1 - y_1| \leq u} \left[\square_{(y,y)}^{2n} Q(x, x) \right]^{\frac{1}{2}}.$$

Given p_1, \dots, p_{k-1} , define

$$p_k(u) = \sup_{x, y \in [0,1]^n: |x_k - y_k| \leq u} \frac{\left[\square_{(y,y)}^{2n} Q(x, x) \right]^{\frac{1}{2}}}{\prod_{j=1}^{k-1} p_j(|x_j - y_j|)},$$

in which we have adopted the convention $0/0 = 0$. It follows immediately that p_k 's are non-decreasing and continuous. Furthermore, we have $p_k(0) = 0$ and

$$\square_{(y,y)}^{2n} Q(x, x) \leq \prod_{k=1}^n p_k^2(|x_k - y_k|). \quad (2.23)$$

Namely, $\{p_1, p_1, p_2, p_2, \dots, p_n, p_n\}$ is a modulus of continuity for $\square_{(y,y)}^{2n} Q(x, x)$. We also call $\{p_1, \dots, p_n\}$ a modulus of continuity for $\square_{(y,y)}^{2n} Q(x, x)$.

In the following theorem, we give a sufficient condition for almost sure continuity of a Gaussian random field.

Theorem 2.3.1. Let W be a centered Gaussian random field with covariance function given by (2.19), and p_k ($1 \leq k \leq n$) be a modulus of continuity for $\square_{(y,y)}^{2n} Q(x, x)$, namely the inequality (2.23) is satisfied. Suppose that

$$\sum_{k=1}^n \int_0^1 \left(\log \frac{1}{u} \right)^{\frac{1}{2}} dp_k(u) < \infty. \quad (2.24)$$

Then, with probability one W has continuous sample path. Furthermore, we have almost surely

$$\lim_{\delta \rightarrow 0^+} \sup_{0 \leq |x-y| \leq \delta} \frac{|\square_y^n W(x)|}{h(x, y)} \leq c_n \quad (2.25)$$

where $h(x, y)$ is the function

$$h(x, y) = \prod_{k=1}^n p_k(|x_k - y_k|) \sqrt{\log \prod_{j=1}^n \frac{1}{|x_j - y_j|}} \quad (2.26)$$

and c_n is some constant depends on n .

Proof. We set $\Psi(x) = e^{x^2/4}$ and

$$B(\omega) = \iint_{[0,1]^{2n}} \exp \left[\frac{|\square_y^n W(x)|^2}{4 \prod_{k=1}^n p_k^2(|x_k - y_k|)} \right] dx dy.$$

Theorem 2.1.1 immediately gives us

$$\begin{aligned} |\square_y^n W(x)| &\leq 2 \cdot 8^n \int_0^{|x_1 - y_1|} \cdots \int_0^{|x_n - y_n|} \left(\log \frac{1}{u_1^2 \cdots u_n^2} \right)^{\frac{1}{2}} dp_1(u_1) \cdots dp_n(u_n) \\ &\quad + \sqrt{\log(4^n B(\omega))} \prod_{k=1}^n p_k(|x_k - y_k|) \end{aligned} \quad (2.27)$$

for ω such that $B(\omega)$ is finite. It is elementary to see that

$$\lim_{|x-y| \rightarrow 0} \frac{1}{h(x, y)} \int_0^{|x_n - y_n|} \left(\log \frac{1}{u_1^2 \cdots u_n^2} \right)^{\frac{1}{2}} dp_1(u_1) \cdots dp_n(u_n) = c_n$$

for some constant c_n and

$$\lim_{|x-y| \rightarrow 0} \frac{\prod_{k=1}^n p_k(|x_k - y_k|)}{h(x, y)} = 0.$$

Thus (2.25) follows by passing through a limit in (2.27). To see (2.27) indeed holds for almost every ω , it is sufficient to show that B has finite expectation. We notice that the random variable

$$N = \frac{\square_y^n W(x)}{\prod_{k=1}^n p_k(|x_k - y_k|)}$$

is Gaussian, has mean zero and variance less than or equal to one. Thus, an application of Stirling's formula gives

$$\begin{aligned} \mathbb{E} \exp\left(\frac{N^2}{4}\right) &= \sum_{k=0}^{\infty} \frac{\mathbb{E} N^{2k}}{4^k k!} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(2k)!}{8^k (k!)^2} (\mathbb{E} N^2)^k \\ &\leq 1 + \frac{1}{2} \sum_{k=1}^{\infty} 8^{-k} = \frac{15}{14}. \end{aligned}$$

Hence

$$\mathbb{E} B = \iint_{[0,1]^{2n}} \mathbb{E} \exp\left(\frac{N^2}{4}\right) dx dy \leq \frac{15}{14}$$

and the proof is complete. \square

Remark 2.3.3. The inequality (2.25) implies the following estimate which usually appears in literature

$$\lim_{\delta \rightarrow 0^+} \sup_{|x-y| \leq \delta} \frac{W(x) - W(y)}{\sigma(x, y)} \leq c_n \quad (2.28)$$

where $\sigma(x, y)$ is the function

$$\sigma(x, y) = \sum_{k=1}^n \left(\prod_{j \neq k} p_j(|x_j|) \right) \left| \log \prod_{j \neq k} |x_j| \right|^{\frac{1}{2}} p_k(|x_k - y_k|) \log |x_k - y_k|^{\frac{1}{2}}. \quad (2.29)$$

Indeed, fix ω such that (2.25) holds and δ sufficiently small, for every x, y in $[0, \delta]^n$, with x and $(0, 0, \dots, 0, y_n)$, the estimate (2.25) gives the following estimate for the increment along an edge of the n -dimensional rectangle $[x_1, y_1] \times \dots \times [x_n, y_n]$

$$\begin{aligned} & \left| W(x_1, \dots, x_n) - W^H(x_1, \dots, x_{n-1}, y_n) \right| \\ & \leq c_n \left(\prod_{k=1}^{n-1} p_k(|x_k|) \right) \left| \log \prod_{k=1}^{n-1} x_k \right|^{1/2} p_n(|x_n - y_n|) |\log |x_n - y_n||^{1/2}. \end{aligned}$$

Similarly, we can obtain analogue estimates along any edge of the n -dimensional rectangle $[x_1, y_1] \times \dots \times [x_n, y_n]$. The increment along the diagonal is majorized by the total increments along all the edges connecting x and y . Hence, this argument yields the following estimate

$$|W(x) - W(y)| \leq c_n \sigma(x, y) \quad (2.30)$$

which implies (2.28).

As an application of the above theorem, we obtain joint continuity for sample paths of fractional Brownian field, as mentioned in (2.6).

Corollary 2.3.4. *Let W^H be a fractional Brownian field on \mathbb{R}^n with Hurst parameter $H = (H_1, \dots, H_n)$. Then, the following inequality holds almost surely*

$$\lim_{\delta \rightarrow 0^+} \sup_{|x-y| \leq \delta} \frac{|\square_y^n W^H(x)|}{h^H(x, y)} \leq c_n \quad (2.31)$$

where $h^H(x, y)$ is the function

$$h^H(x, y) = \left| \log \prod_{k=1}^n |x_k - y_k| \right|^{\frac{1}{2}} \prod_{j=1}^n |x_j - y_j|^{H_k}$$

for some constant c_n depending on n .

Proof. The covariance function of a fractional Brownian field is given by

$$\mathbb{E} \left[W^H(x) W^H(y) \right] = \prod_{k=1}^n R_k(x_k, y_k),$$

where

$$R_k(s, t) = \frac{1}{2} \left[|s|^{2H_k} + |t|^{2H_k} - |s - t|^{2H_k} \right], \quad \forall s, t \in \mathbb{R}.$$

By Proposition 2.3.1, we obtain the second moment for $\square_y^n W^H(x)$

$$\mathbb{E} |\square_y^n W^H(x)|^2 = \prod_{k=1}^n |x_k - y_k|^{2H_k}.$$

This means that $p_i(u) = u^{H_i}$, $i = 1, 2, \dots, n$ are the modulus of continuity of $\square_{(y,y)}^{2n} Q(x, x)$.

Now the corollary is a direct consequence of Theorem 2.3.1. \square

Remark 2.3.5. As in Remark 2.3.3, the previous result implies the following

$$\lim_{\delta \rightarrow 0^+} \sup_{|x-y| \leq \delta} \frac{W^H(x) - W^H(y)}{\sigma^H(x, y)} \leq c_n \quad (2.32)$$

where $\sigma(x, y)$ is the function

$$\sigma(x, y) = \left(\prod_{k=1}^{n-1} s_k^{H_k} \right) \left| \log \prod_{k=1}^{n-1} s_k \right|^{1/2} |s_n - t_n|^{H_n} |\log |s_n - t_n||^{1/2}. \quad (2.33)$$

2.4 Stochastic heat equations with additive space time white noise

In this section let us consider the following one dimensional stochastic differential equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \dot{W} & 0 < t \leq T, y \in \mathbb{R} \\ u(0, y) = 0 & y \in \mathbb{R}, \end{cases} \quad (2.34)$$

where $\Delta u = \frac{\partial^2}{\partial y^2} u$, W is space time standard Brownian sheet, and $\dot{W} = \frac{\partial^2}{\partial t \partial y} W$. Let $p_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}$. Then the (mild) solution of the above equation is given by

$$u(t, y) = \int_0^t \int_{\mathbb{R}} p_{t-r}(y-z) W(dr, dz),$$

where the above integral is the usual (Itô) stochastic integral (however, the integrand is simple. It is a deterministic function). The solution $u(t, y)$ is a Gaussian random field. It is known that $u(t, y)$ is Hölder continuous of exponent $\frac{1}{4}-$ for time parameter and $\frac{1}{2}-$ for space parameter. Namely, for any $\alpha < 1/4$ and any $\beta < 1/2$, there is a random constant $C_{\alpha, \beta}$ such that

$$|u(t, y) - u(s, x)| \leq C_{\alpha, \beta} (|t - s|^\alpha + |x - y|^\beta). \quad (2.35)$$

We are interested in the joint Hölder continuity of the solution $u(t, y)$. We need the following simple technical lemma.

Lemma 2.4.1. *Let a, b, δ be some positive numbers, where $a < b$, and let I, J be the integrations*

$$I = \int_a^b \frac{1}{\sqrt{r}} \left(1 - e^{-\frac{\delta^2}{2r}}\right) dr,$$

$$J = \int_0^a \frac{1}{\sqrt{r}} \left(1 - e^{-\frac{\delta^2}{2r}}\right) dr.$$

Then for every $\alpha \in [0, 1/2]$,

$$2(\sqrt{b} - \sqrt{a}) \left(1 - e^{-\frac{\delta^2}{2b}}\right) \leq I \leq 2(\sqrt{b} - \sqrt{a}) \left(1 - e^{-\frac{\delta^2}{2a}}\right)$$

and

$$J \leq c_\alpha \delta^{2\alpha} a^{1/2-\alpha}.$$

Proof. On the interval $a \leq r \leq b$, we have $1 - e^{-\frac{\delta^2}{2b}} \leq 1 - e^{-\frac{\delta^2}{2r}} \leq 1 - e^{-\frac{\delta^2}{2a}}$. The estimate for I is then a straightforward consequence. To estimate J , we first use integration by part to

obtain

$$\begin{aligned} J &= 2\sqrt{r} \left(1 - e^{-\frac{\delta^2}{2r}}\right) \Big|_{r=0}^{r=a} + \int_0^a \left(\frac{d}{dr} e^{-\frac{\delta^2}{2r}}\right) 2\sqrt{r} dr \\ &= 2\sqrt{a} \left(1 - e^{-\frac{\delta^2}{2a}}\right) + \delta^2 \int_0^a e^{-\frac{\delta^2}{2r}} r^{-3/2} dr. \end{aligned}$$

By a change of variable $x = \frac{\delta}{\sqrt{2r}}$, we see that

$$J = 2\sqrt{a} \left(1 - e^{-\frac{\delta^2}{2a}}\right) + 2\sqrt{2}\delta \int_{\frac{\delta}{\sqrt{2a}}}^{\infty} e^{-x^2} dx.$$

If $\frac{\delta}{\sqrt{2a}} \geq 1$, since $\lim_{t \rightarrow \infty} \frac{\int_t^{\infty} e^{-x^2} dx}{(1-e^{-t^2})^{t^{-1}}} = 0$, J is majorized by

$$J \leq c\sqrt{a} \left(1 - e^{-\frac{\delta^2}{2a}}\right).$$

If $\frac{\delta}{\sqrt{2a}} \leq 1$, the integration $\int_{\frac{\delta}{\sqrt{2a}}}^{\infty} e^{-x^2} dx$ is bounded by $\sqrt{\pi}/2$, thus J is majorized by

$$J \leq 2\sqrt{a} \left(1 - e^{-\frac{\delta^2}{2a}}\right) + c\delta.$$

Therefore, for any $0 \leq \alpha \leq 1/2$, employing the elementary inequality $1 - e^{-x} \leq c_{\alpha} x^{\alpha}$, we obtain

$$J \leq c_{\alpha} \delta^{2\alpha} a^{1/2-\alpha}$$

and the lemma follows. □

Theorem 2.4.2. *For every α in $[0, 1/4]$, there is a constant c_{α} depending on α such that*

$$\lim_{\delta \rightarrow 0^+} \sup_{|t-s| \leq \delta, |x-y| \leq \delta} \frac{|u(t, y) - u(t, x) - u(s, y) + u(s, x)|}{|t-s|^{\frac{1}{4}-\alpha} |x-y|^{2\alpha} |\log(|t-s||x-y|)|^{\frac{1}{2}}} \leq c_{\alpha} \quad (2.36)$$

$$|u(t, y) - u(t, x) - u(s, y) + u(s, x)| \leq C_{\alpha} |t-s|^{\frac{1}{4}-\alpha} |x-y|^{2\alpha} \sqrt{|\log(|t-s||x-y|)|}. \quad (2.37)$$

Proof. $u(t, y)$ is a mean zero Gaussian field. The covariance of $u(t, y)$ and $u(s, x)$ is given by

$$\begin{aligned}\mathbb{E}[u(s, x)u(t, y)] &= \int_{\mathbb{R}^2} \chi_{[0,s]}(r)\chi_{[0,t]}(r)p_{s-r}(x-z)p_{t-r}(y-z)drdz \\ &= \int_{\mathbb{R}^2} f(s, x)f(t, y)drdz,\end{aligned}$$

where $f(s, x) = \chi_{[0,s]}(r)p_{s-r}(x-z)$.

We calculate the second moment of $\square_{(s,x)}^2 u(t, y)$ as follows

$$\begin{aligned}\mathbb{E} \left[\square_{(s,x)}^2 u(t, y) \right]^2 &= \mathbb{E} \square_{(s,x)}^2 u(t, y) \square_{(s,x)}^2 u(t, y) \\ &= \mathbb{E} \square_{(s,x,s,x)}^4 u(t, y) u(t, y) \\ &= \square_{(s,x,s,x)}^4 \mathbb{E} [u(t, y) u(t, y)] \\ &= \square_{(s,x,s,x)}^4 \int_{\mathbb{R}^2} f(t, y)^2 drdz \\ &= \int_{\mathbb{R}^2} \square_{(s,x,s,x)}^4 [f(t, y) f(t, y)] drdz \\ &= \int_{\mathbb{R}^2} \left[\square_{(s,x)}^2 f(t, y) \right] \left[\square_{(s,x)}^2 f(t, y) \right] drdz \\ &= \int_{\mathbb{R}^2} \left[\square_{(s,x)}^2 f(t, y) \right]^2 drdz,\end{aligned}$$

where

$$\begin{aligned}\left[\square_{(s,x)}^2 f(t, y) \right]^2 &= [f(s, x) - f(t, x) - f(s, y) + f(t, y)]^2 \\ &= f(s, x)^2 + f(t, x)^2 + f(s, y)^2 + f(t, y)^2 \\ &\quad - 2f(s, x)f(t, x) - 2f(s, x)f(s, y) + 2f(s, x)f(t, y) \\ &\quad + 2f(t, x)f(s, y) - 2f(t, x)f(t, y) - 2f(s, y)f(t, y).\end{aligned}$$

Taking the integration with respect to z and using the following identity

$$\int_{\mathbb{R}} p_a(z-x)p_b(z-y)dz = p_{a+b}(x-y)$$

we obtain

$$\begin{aligned} & \mathbb{E} \left[\square_{(s,x)}^2 u(t,y) \right]^2 \\ &= \int_{\mathbb{R}} \left[2\chi_{[0,s]}(r)p_{2s-2r}(0) + 2\chi_{[0,t]}(r)p_{2t-2r}(0) \right] dr \\ &+ \int_{\mathbb{R}} \left[-2\chi_{[0,s \wedge t]}p_{s+t-2r}(0) - 2\chi_{[0,s]}p_{2s-2r}(x-y) + 2\chi_{[0,s \wedge t]}p_{s+t-2r}(x-y) \right] dr \\ &+ \int_{\mathbb{R}} \left[2\chi_{[0,s \wedge t]}p_{s+t-2r}(x-y) - 2\chi_{[0,t]}p_{2t-2r}(x-y) - 2\chi_{[0,s \wedge t]}p_{s+t-2r}(0) \right] dr \\ &= 2 \int_0^s [p_{2s-2r}(0) - p_{2s-2r}(x-y)] dr + 2 \int_0^t [p_{2t-2r}(0) - p_{2t-2r}(x-y)] dr \\ &- 4 \int_0^{s \wedge t} [p_{s+t-2r}(0) - p_{s+t-2r}(x-y)] dr. \end{aligned}$$

By change of variables $u = 2s - 2r$, $v = 2t - 2r$ and $w = s + t - 2r$ in the above corresponding integrals respectively and noticing that $s + t - 2(s \wedge t) = |t - s|$, we get

$$\begin{aligned} \mathbb{E} \left[\square_{(s,x)}^2 u(t,y) \right]^2 &= \int_0^{2s} [p_u(0) - p_u(x-y)] du + \int_0^{2t} [p_v(0) - p_v(x-y)] dv \\ &- 2 \int_{|s-t|}^{s+t} [p_w(0) - p_w(x-y)] dw \\ &= \left(\int_{s+t}^{2(s \vee t)} - \int_{2(s \wedge t)}^{s+t} + 2 \int_0^{|s-t|} \right) [p_r(0) - p_r(x-y)] dr. \end{aligned}$$

By Lemma 2.4.1, we see that

$$\left(\int_{s+t}^{2(s \vee t)} - \int_{2(s \wedge t)}^{s+t} \right) [p_r(0) - p_r(x-y)] dr \leq \frac{1}{\sqrt{2\pi}} \left(1 - e^{-\frac{(x-y)^2}{2(s+t)}} \right) (\sqrt{2s} + \sqrt{2t} - 2\sqrt{s+t}) \leq 0.$$

and

$$\int_0^{|s-t|} [p_r(0) - p_r(x-y)] dr \leq c_\alpha |x-y|^{2\alpha} |s-t|^{1/2-\alpha}$$

for every α in $[0, 1/2]$. Thus

$$\mathbb{E} \left[\square_{(s,x)}^2 u(t, y) \right]^2 \leq c_\alpha |x - y|^{2\alpha} |s - t|^{1/2-\alpha}.$$

An application of Theorem 2.3.1 immediately gives the desired result. \square

Remark 2.4.3. Using the method in Remark 2.3.3, the above result implies the following estimate

$$\lim_{\delta \rightarrow 0^+} \sup_{|t-s| \leq \delta, |x-y| \leq \delta} \frac{|u(s, x) - u(t, y)|}{|s - t|^{1/4} \sqrt{\log \frac{1}{|x||s-t|}} + |x - y|^{1/2} \sqrt{\log \frac{1}{|x-y||t|}}} \leq c \quad (2.38)$$

which is sharper than (2.35).

Remark 2.4.4. Compare to the current result of M. Meerschaert, W. Wang and Y. Xiao in [78], our result is less precise, due to the lack of lower bounds in the inequalities (2.28) and (2.38). However, lower bounds for (2.28) and (2.36) seem to be unknown. On the other hand, while in [78], the authors obtain the law of iterated logarithm of the type

$$\lim_{\delta \rightarrow 0^+} \sup_{|x-y| \leq \delta} \frac{W^H(x) - W^H(y)}{\sigma^H(x, y)} = \kappa \quad (2.39)$$

for some constant κ . It is still a challenging problem to determine the exact value of the constant κ .

Chapter 3

Nonlinear integrals

This chapter studies the nonlinear integrations of the form $\int W(ds, \varphi_s)$. This nonlinear integration appears in stochastic analysis as well as in the study of Feynman-Kac formulas. We introduce three type of nonlinear integrations:

1. Nonlinear Young integration treats the case when W is jointly Hölder continuous and φ is Hölder continuous (conditions **(W)** and **(ϕ)** below). This type of integration extends the classical Young integrals (cf. [98]) of the form $\int f(t)dg(t)$ where f and g are Hölder continuous functions such that the total of their Hölder exponents is greater than 1. We use the sewing lemma originated by Gubinelli in [44]. We will show that nonlinear Young integrals form compact mappings. An alternative representation of the nonlinear Young integrals is given using fractional calculus. This integration is described in Section 3.1.

2. The nonlinear Itô-Skorohod integration is described in Section 3.2. This type of integration assumes that W is a Gaussian process and relies on the covariance structure of W . It extends the classical Skorohod integral in stochastic analysis.

3. The nonlinear symmetric integration is described in Section 3.3. Here, we follow the work of Russo and Vallois in [88] where they introduce the symmetric integrals in stochastic analysis. We take another step forward and define symmetric integrals for nonlinear integration.

Finally, Section 3.4 connects the above three types of nonlinear integration by showing some identities between different types of nonlinear integration are valid in some situations.

Notations: We collect here some notations that we will use throughout the entire chapter. $A \lesssim B$ means there is a constant C such $A \leq CB$. We represent a vector x in \mathbb{R}^d as a matrix of dimension $d \times 1$, A^T represents the transpose of a matrix A . Sometimes we write x_\bullet for column vector x^T and x^\bullet for the row vector x . We use the Einstein convention on summation over repeated indices. For instance, $b_i c_i$ abbreviates for $\sum_{i=1}^d b_i c_i$

3.1 Nonlinear Young integral

Let W and φ be \mathbb{R}^d -valued functions defined on $\mathbb{R} \times \mathbb{R}^d$ and \mathbb{R}^d respectively. We define in the current section the nonlinear Young integration $\int W(ds, \varphi_s)$.

We make the following assumption on the regularity of W

(W) There are constants $\tau, \lambda \in (0, 1], \beta \geq 0$ such that for all $a < b$, the seminorm

$$\begin{aligned} & \|W\|_{\beta, \tau, \lambda; a, b} \\ := & \sup_{\substack{a \leq s < t \leq b \\ x, y \in \mathbb{R}^d; x \neq y}} \frac{|W(s, x) - W(t, x) - W(s, y) + W(t, y)|}{(1 + |x| + |y|)^\beta |t - s|^\tau |x - y|^\lambda} \\ & + \sup_{\substack{a \leq s < t \leq b \\ x \in \mathbb{R}^d}} \frac{|W(s, x) - W(t, x)|}{(1 + |x|)^{\beta + \lambda} |t - s|^\tau} + \sup_{\substack{a \leq t \leq b \\ x, y \in \mathbb{R}^d; x \neq y}} \frac{|W(t, y) - W(t, x)|}{(1 + |x| + |y|)^\beta |x - y|^\lambda}, \end{aligned} \tag{3.1}$$

is finite.

About the function φ , we assume

(ϕ) φ is locally Hölder continuous of order $\gamma \in (0, 1]$. That is the seminorm

$$\varphi_{\gamma; a, b} = \sup_{a \leq s < t \leq b} \frac{|\varphi(t) - \varphi(s)|}{|t - s|^\gamma},$$

is finite for every $a < b$.

Throughout the current section, we assume that $\tau + \lambda\gamma > 1$. Among three terms appearing in (3.1), we will pay special attention to the first term. Thus, we denote

$$[W]_{\beta,\tau,\lambda;a,b} = \sup_{\substack{a \leq s < t \leq b \\ x,y \in \mathbb{R}^d; x \neq y}} \frac{|W(s,x) - W(t,x) - W(s,y) + W(t,y)|}{(1 + |x| + |y|)^\beta |t - s|^\tau |x - y|^\lambda}.$$

When $\beta = 0$, then we denote $\|W\|_{\tau,\lambda;a,b} := \|W\|_{0,\tau,\lambda;a,b}$. If a, b are clear in the context, we frequently omit the dependence on a, b . For instance, $\|W\|_{\beta,\tau,\lambda}$ is an abbreviation for $\|W\|_{\beta,\tau,\lambda;a,b}$, $\|\varphi\|_\gamma$ is an abbreviation for $\|\varphi\|_{\gamma;a,b}$ and so on. We shall assume that a and b are finite. It is easy to see that for any $c \in [a, b]$

$$\sup_{a \leq t \leq b} |\varphi(t)| = \sup_{a \leq t \leq b} |\varphi(c) + \varphi(t) - \varphi(c)| \leq |\varphi(c)| + \|\varphi\|_\gamma |b - a|^\gamma < \infty.$$

Thus assumption (ϕ) also implies that

$$\|\varphi\|_{\infty;a,b} := \sup_{a \leq t \leq b} |\varphi(t)| < \infty.$$

For the results presented in this section, the condition (W) can be relaxed to

(W') There are constants $\tau, \lambda \in (0, 1]$, such that for all $a < b$ and compact set K in \mathbb{R}^d , the seminorm

$$\begin{aligned} & \sup_{\substack{a \leq s < t \leq b \\ x,y \in K; x \neq y}} \frac{|W(s,x) - W(t,x) - W(s,y) + W(t,y)|}{|t - s|^\tau |x - y|^\lambda} \\ & + \sup_{\substack{a \leq s < t \leq b \\ x \in K}} \frac{|W(s,x) - W(t,x)|}{|t - s|^\tau} + \sup_{\substack{a \leq t \leq b \\ x,y \in K; x \neq y}} \frac{|W(t,y) - W(t,x)|}{|x - y|^\lambda}, \end{aligned}$$

is finite.

However, the polynomial growth rate is needed in the following chapters to solve differential equations.

For later purpose, we denote $C_\beta^{(\tau, \lambda)}(\mathbb{R} \times \mathbb{R}^d)$ (respectively $C_{\text{loc}}^{(\tau, \lambda)}(\mathbb{R} \times \mathbb{R}^d)$) the collection of all functions W satisfying condition **(W)** (respectively **(W')**). κ denotes a universal generic constant depending only on λ, τ, α and independent of W, φ and a, b . The value of κ may vary from one occurrence to another.

3.1.1 Definition

We define the nonlinear integral $\int W(ds, \varphi_s)$ as follows.

Definition 3.1.1. Let a, b be two fixed real numbers, $a < b$. Let $\pi = \{a = t_0 < t_1 < \dots < t_m = b\}$ be a partition of $[a, b]$ with mesh size $|\pi| = \max_{0 \leq i \leq m-1} |t_{i+1} - t_i|$. The Riemann sum corresponding to π is

$$J_\pi = \sum_{i=1}^{m-1} W(t_{i+1}, \varphi_i) - W(t_i, \varphi_i). \quad (3.2)$$

If the sequence of Riemann sums J_π 's is convergent when $|\pi|$ shrinks to 0, we denote the limit as the nonlinear integral $\int_a^b W(ds, \varphi_s)$.

We observe that in the particular case when $W(t, x) = g(t)x$ for some functions $g : \mathbb{R} \rightarrow \mathbb{R}$, the nonlinear integral $\int_a^b W(ds, \varphi_s)$ defined above, if exists, coincides with the Riemann-Stieltjes integral $\int_a^b \varphi_s dg(s)$. It is well known that if φ and g are Hölder continuous with exponents α, β respectively and $\alpha + \beta > 1$, then the Riemann-Stieltjes integral $\int_a^b \varphi_s dg(s)$ exists and is called Young integral ([98]).

More generally, for each partition π of an interval $[a, b]$, one can consider the (abstract) Riemann sum

$$J_\pi(\mu) = \sum_{i=1}^{m-1} \mu(t_i, t_{i+1}) \quad (3.3)$$

where μ is a function defined on $[a, b]^2$ with values in a Banach space. A sufficient condition for convergence of the limit $\lim_{|\pi| \downarrow 0} J_\pi(\mu)$ is obtained by Gubinelli in [44] via the so-called sewing map. This point of view has important contributions to Lyons' theory of rough

paths ([73, 74]). Since we will apply Gubinelli's sewing lemma, we restate the result as follows.

Lemma 3.1.2 (Sewing lemma). *Let μ be a continuous function on $[a, b]^2$ with values in a Banach space $(B, \|\cdot\|)$ and let $\varepsilon > 0$. Suppose that μ satisfies*

$$\|\mu(s, t) - \mu(s, c) - \mu(c, t)\| \leq K|t - s|^{1+\varepsilon} \quad \forall a \leq s \leq c \leq t \leq b.$$

Then there exists a function $\mathcal{J}\mu(t)$ unique up to an additive constant such that

$$\|\mathcal{J}\mu(t) - \mathcal{J}\mu(s) - \mu(s, t)\| \leq K(1 - 2^{-\varepsilon})^{-1}|t - s|^{1+\varepsilon} \quad \forall a \leq s \leq t \leq b. \quad (3.4)$$

In addition, when $|\pi|$ shrinks to 0, the Riemann sums (3.3) converge to $\mathcal{J}\mu(b) - \mathcal{J}\mu(a)$.

In what follows, we adopt the notation $\mathcal{J}_a^b \mu = \mathcal{J}\mu(b) - \mathcal{J}\mu(a)$. The map $\mu \mapsto \mathcal{J}\mu$ is called the sewing map. The setting of Lemma 3.1.2 is adopted from [35]. In several occasions, one needs to prove a relation between two or more integrals. The following result provides a simple method for this problem.

Lemma 3.1.3. *Suppose μ_1 and μ_2 are two functions as in Lemma 3.1.2. In addition, assume that*

$$|\mu_1(s, t) - \mu_2(s, t)| \leq C|t - s|^{1+\varepsilon'} \quad \forall a \leq s \leq t \leq b$$

for some positive constant ε' . Then $\mathcal{J}\mu_1$ and $\mathcal{J}\mu_2$ are different by an absolute constant. That is $\mathcal{J}_s^t \mu_1 = \mathcal{J}_s^t \mu_2$ for all s, t .

Proof. From Lemma 3.1.2, $\mathcal{J}(\mu_1 - \mu_2) = \mathcal{J}\mu_1 - \mathcal{J}\mu_2$ and

$$|\mathcal{J}_s^t(\mu_1 - \mu_2)| \lesssim |\mu_1(s, t) - \mu_2(s, t)| + |t - s|^{1+\varepsilon} \lesssim |t - s|^{1+\varepsilon'} + |t - s|^{1+\varepsilon}$$

for all s, t . This implies $\mathcal{J}_s^t(\mu_1 - \mu_2) = 0$ for all s, t . □

Returning to our main objective of the current section, we consider

$$\mu(s, t) = W(t, \varphi_s) - W(s, \varphi_s).$$

Then the condition in Lemma 3.1.2 is guaranteed by **(W)**, and **(ϕ)**. Indeed, for every $s < c < t$,

$$\begin{aligned} & |\mu(s, t) - \mu(s, c) - \mu(c, t)| \\ &= |W(t, \varphi_s) - W(c, \varphi_s) - W(t, \varphi_c) + W(c, \varphi_c)| \\ &\leq [W]_{\beta, \tau, \lambda} (1 + \|\varphi\|_{\infty}^{\beta}) (t - s)^{\tau} |\varphi_s - \varphi_c|^{\lambda} \\ &\leq [W]_{\beta, \tau, \lambda} (1 + \|\varphi\|_{\infty}^{\beta}) \|\varphi\|_{\gamma}^{\lambda} (t - s)^{\tau + \lambda\gamma}. \end{aligned}$$

Hence, by combining the sewing lemma and the previous estimate, we obtain

Proposition 3.1.4. *Assuming the conditions **(W)**, **(ϕ)** with $\tau + \lambda\gamma > 1$, the sequence of Riemann sums (3.2) is convergent when $|\pi|$ goes to 0. In other words, the nonlinear integral $\int_a^b W(ds, \varphi_s)$ is well-defined.*

In addition, the following estimate holds

$$\left| \int_s^t W(dr, \varphi_r) - W(t, \varphi_c) + W(s, \varphi_c) \right| \leq \kappa \|W\|_{\tau, \lambda; a, b} (1 + \|\varphi\|_{\infty}^{\beta}) \|\varphi\|_{\gamma; a, b}^{\lambda} (t - s)^{\tau + \lambda\gamma} \quad (3.5)$$

for all $a \leq s \leq c \leq t \leq b$.

Remark 3.1.5. After the completion of this work, we are brought to the attention of the work [15] (and also [18, 19, 45]), where a similar nonlinear Young integral is studied. The objective of that paper is to define the averaging of the form $\int_0^t f(X_u) du$ for some process X_u and for some irregular function f . The sewing lemma that we follow is from [35], which is after the work of [44].

Remark 3.1.6. (i) In the particular case when $W(t, x) = g(t)x$, Proposition 3.1.4 reduces to

the existence of the Young integral $\int \varphi_s dg(s)$. Hence, from now on we refer the integral $\int W(ds, \varphi_s)$ as nonlinear Young integral.

(ii) In Proposition 3.1.4, we can also consider the Riemann sums with right-end points

$$J_{\pi}^+ = \sum_{i=0}^{m-1} [W(t_{i+1}, \varphi_{t_{i+1}}) - W(t_i, \varphi_{t_{i+1}})].$$

Then the corresponding limit exists and equals to $\int_a^b W(ds, \varphi_s)$. This is a straightforward consequence of Lemma 3.1.3.

It is evident that

$$\int_s^t W(dr, \varphi_r) = \int_s^c W(dr, \varphi_r) + \int_c^t W(dr, \varphi_r) \quad \forall s < c < t.$$

This together with (3.5) imply easily the following.

Proposition 3.1.7. *Assume that (W) and (ϕ) hold with $\lambda\gamma + \tau > 1$. As a function of t , the indefinite integral $\left\{ \int_a^t W(ds, \varphi_s), a \leq t \leq b \right\}$ is Hölder continuous of exponent τ .*

Fractional calculus is very useful in the study of (linear) Young integral. It leads to some detailed properties of the integral and solution of a differential equation (see [62], [63], and the references therein). It is interesting to extend this approach to nonlinear Young integral. In fact, the authors obtain in [60] the following presentation for the nonlinear Young integral by using fractional calculus. Since this method is not pursued in the current chapter, we refer the readers to [60] for further details.

Theorem 3.1.8. *Assume the conditions (W) and (ϕ) are satisfied. In addition, we suppose that*

$\lambda\gamma + \tau > 1$. Let $\alpha \in (1 - \tau, \lambda\tau)$. Then the following identity holds

$$\begin{aligned}
& \int_a^b W(dt, \varphi_t) \\
&= -\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left\{ \int_a^b \frac{W_{b-}(t, \varphi_t)}{(b-t)^{1-\alpha}(t-a)^\alpha} dt \right. \\
&+ \alpha \int_a^b \int_a^t \frac{W_{b-}(t, \varphi_t) - W_{b-}(t, \varphi_r)}{(b-t)^{1-\alpha}(t-r)^{\alpha+1}} dr dt \\
&+ (1-\alpha) \int_a^b \int_t^b \frac{W(t, \varphi_t) - W(s, \varphi_t)}{(s-t)^{2-\alpha}(t-a)^\alpha} ds dt \\
&\left. + \alpha(1-\alpha) \int_a^b \int_a^t \int_t^b \frac{W(t, \varphi_t) - W(s, \varphi_t) - W(t, \varphi_r) + W(s, \varphi_r)}{(s-t)^{2-\alpha}(t-r)^{\alpha+1}} ds dr dt \right\}, \tag{3.6}
\end{aligned}$$

where $W_{b-}(t, x) = W(t, x) - W(b, x)$.

3.1.2 Mapping properties

Let μ be a function as in Lemma 3.1.2. Let us define the quality

$$[\mu]_{1+\varepsilon; I} = \sup_{s, c, t \in I: s < c < t} \frac{|\mu(s, t) - \mu(s, c) - \mu(c, t)|}{|t - s|^{1+\varepsilon}}.$$

In several occasions, given two functions μ_1 and μ_2 such that $[\mu_1]_{1+\varepsilon}$ and $[\mu_2]_{1+\varepsilon}$ are finite, one would like to compare the integrals $\mathcal{J}\mu_1$ and $\mathcal{J}\mu_2$. The following result answers this question.

Lemma 3.1.9. *Let μ_1 and μ_2 be two continuous functions on $[a, b]^2$ such that $[\mu_1]_\alpha$ and $[\mu_2]_\alpha$ are finite for some $\alpha > 1$. Then for every $s, t \in [a, b]$*

$$|\mathcal{J}_s^t \mu_1 - \mathcal{J}_s^t \mu_2| \leq |\mu_1(s, t) - \mu_2(s, t)| + (1 - 2^{1-\alpha})^{-1} [\mu_1 - \mu_2]_{\alpha; [s, t]} |t - s|^\alpha$$

Proof. The proof is rather trivial thanks to the linearity nature of Lemma 3.1.2. Put $\mu = \mu_1 - \mu_2$. Notice that $[\mu]_\alpha \leq [\mu_1]_\alpha + [\mu_2]_\alpha < \infty$. Thus we can apply Lemma 3.1.2 to μ . The claim follows after observing that $\mathcal{J}\mu = \mathcal{J}\mu_1 - \mathcal{J}\mu_2$. \square

As an application, we study the dependence of the nonlinear Young integration $\int W(ds, \varphi_s)$ with respect to the medium W and the integrand φ .

Proposition 3.1.10. *Let W_1 and W_2 be real valued functions on $\mathbb{R} \times \mathbb{R}^d$ satisfying the condition (W). Let φ be a function in $C^\gamma(\mathbb{R}; \mathbb{R}^d)$ and let $\tau + \lambda\gamma > 1$. Then*

$$\begin{aligned} \left| \int_a^b W_1(ds, \varphi_s) - \int_a^b W_2(ds, \varphi_s) \right| &\leq |W_1(b, \varphi_a) - W_1(a, \varphi_a) - W_2(b, \varphi_a) + W_2(a, \varphi_a)| \\ &\quad + c(\|\varphi\|_\infty)[W_1 - W_2]_{\beta, \tau, \lambda} \|\varphi\|_\gamma |b - a|^{\tau + \lambda\gamma} \end{aligned}$$

Proof. Let $a < c < b$. Put

$$\begin{aligned} \mu_1(a, b) &= W_1(b, \varphi_a) - W_1(a, \varphi_a), \\ \mu_2(a, b) &= W_2(b, \varphi_a) - W_2(a, \varphi_a), \\ \mu &= \mu_1 - \mu_2. \end{aligned}$$

The argument before Proposition 3.1.4 shows that

$$[\mu]_{\tau + \lambda\gamma} \leq [W_1 - W_2]_{\beta, \tau, \lambda} (1 + \|\varphi\|_\infty^\beta) \|\varphi\|_\gamma.$$

The proposition follows from Lemma 3.1.9. □

Proposition 3.1.11. *Let W be a function on $\mathbb{R} \times \mathbb{R}^d$ satisfying the condition (W). Let φ^1 and φ^2 be two functions in $C^\gamma(\mathbb{R}; \mathbb{R}^d)$ and let $\tau + \lambda\gamma > 1$. Let $\theta \in (0, 1)$ such that $\tau + \theta\lambda\gamma > 1$. Then for any $u < v$*

$$\begin{aligned} \left| \int_u^v W(ds, \varphi_s^1) - \int_u^v W(ds, \varphi_s^2) \right| \\ \leq C_1[W]_{\beta, \tau, \lambda} \|\varphi^1 - \varphi^2\|_\infty^\lambda |v - u|^\tau + C_2[W]_{\beta, \tau, \lambda} \|\varphi^1 - \varphi^2\|_\infty^{\lambda(1-\theta)} |v - u|^{\tau + \theta\lambda\gamma}, \end{aligned}$$

where $C_1 = 1 + \|\varphi^1\|_\infty^\beta + \|\varphi^2\|_\infty^\beta$ and $C_2 = 2^{1-\theta} C_1 (\|\varphi^1\|_\gamma^\lambda + \|\varphi^2\|_\gamma^\lambda)^\theta$.

Proof. We put $\mu_1(a, b) = W(b, \varphi_a^1) - W(a, \varphi_a^1)$, $\mu_2(a, b) = W(b, \varphi_a^2) - W(a, \varphi_a^2)$ and $\mu = \mu_1 - \mu_2$. Applying Lemma 3.1.9, we obtain, for any $\theta \in (0, 1)$ such that $\tau + \theta\lambda\gamma > 1$

$$\begin{aligned} & \left| \int_u^v W(ds, \varphi_s^1) - \int_u^v W(ds, \varphi_s^2) \right| \\ & \leq |W(v, \varphi_u^1) - W(u, \varphi_u^1) - W(v, \varphi_u^2) + W(u, \varphi_u^2)| + [\mu]_{\tau+\theta\lambda\gamma} |v - u|^{\tau+\theta\lambda\gamma}. \end{aligned}$$

Notice that

$$|W(v, \varphi_u^1) - W(u, \varphi_u^1) - W(v, \varphi_u^2) + W(u, \varphi_u^2)| \leq C_1[W]_{\beta,\tau,\lambda} |u - v|^\tau \|\varphi^1 - \varphi^2\|_\infty^\lambda.$$

It remains to estimate $[\mu]_{\tau+\theta\lambda\gamma}$. It is obvious that for $i = 1, 2$

$$[\mu_i]_{\tau+\lambda\gamma} \leq [W]_{\beta,\tau,\lambda} (1 + \|\varphi^i\|_\infty^\beta) \|\varphi^i\|_\gamma^\lambda \leq C_1 [W]_{\beta,\tau,\lambda} \|\varphi^i\|_\gamma^\lambda$$

and hence

$$[\mu]_{\tau+\lambda\gamma} \leq [\mu_1]_{\tau+\lambda\gamma} + [\mu_2]_{\tau+\lambda\gamma} \leq C_1 [W]_{\beta,\tau,\lambda} \sum_{i=1}^2 \|\varphi^i\|_\gamma^\lambda.$$

On the other hand

$$\begin{aligned} & |\mu(a, b) - \mu(a, c) - \mu(c, b)| \\ & \leq |W(b, \varphi_a^1) - W(b, \varphi_a^2) - W(c, \varphi_a^1) + W(c, \varphi_a^2)| \\ & \quad + |W(b, \varphi_c^1) - W(b, \varphi_c^2) - W(c, \varphi_c^1) + W(c, \varphi_c^2)| \\ & \leq 2C_1 [W]_{\beta,\tau,\lambda} |b - c|^\tau \|\varphi^1 - \varphi^2\|_\infty^\lambda. \end{aligned}$$

Combining the two bounds for μ we get for any $\theta \in (0, 1)$ such that $\tau + \theta\lambda\gamma > 1$,

$$[\mu]_{\tau+\theta\lambda\gamma} \leq C_2 [W]_{\beta,\tau,\lambda} \|\varphi^1 - \varphi^2\|_\infty^{\lambda(1-\theta)}.$$

This completes the proof. □

Corollary 3.1.12. *Let I be a nonempty closed, bounded and connected interval. Let t_0 be in I . Assuming condition (W) with $\tau + \lambda\gamma > 1$. Then the map*

$$M : C^\gamma(I) \rightarrow C^\tau(I)$$

$$Mx(t) = \int_{t_0}^t W(ds, x_s)$$

is continuous and compact.

Proof. Continuity follows immediately from Proposition 3.1.11. For compactness, suppose B is a bounded subset of $C^\gamma(I)$. The estimate in Proposition 3.1.11 implies that $\{Mx\}_{x \in B}$ is bounded in $C^\tau(I)$. By the Arzelà-Ascoli theorem, the set $\{Mx\}_{x \in B}$ is relatively compact in $C^{\tau'}(I)$ for every $\tau' < \tau$. We show that $\{Mx\}_{x \in B}$ is indeed relatively compact in $C^\tau(I)$. More precisely, suppose $\{Mx^n\}$ is a convergent sequence in $M(B)$ in the norm of $C^{\tau'}(I)$, by taking further subsequence, we can assume that the sequence $\{x^n\}$ converges to x in $C^{\gamma'}(I)$, for some $\gamma' < \gamma$ (this is possible since B is bounded). It is sufficient to show that Mx^n converges to Mx in $C^\tau(I)$. To prove this, we choose $\theta \in (0, 1)$ and $\gamma' < \gamma$ such that $\tau + \theta\lambda\gamma' > 1$, and then we apply Proposition 3.1.11 to obtain

$$\|Mx - Mx^n\|_\tau \leq c \|W\|_{\beta, \tau, \lambda} (\|x - x^n\|_\infty^\lambda + \|x - x^n\|_\infty^{\lambda(1-\theta)}).$$

The constant c depends only on $\|x\|_\infty, \|x\|_{\gamma'}$ and $\|x^n\|_\infty, \|x^n\|_{\gamma'}$ which is uniformly bounded with respect to n . This shows Mx^n converges to Mx in $C^\tau(I)$ and completes the proof. □

3.2 Nonlinear Itô-Skorohod integral

Let $H \in (\frac{1}{2}, 1)$ and denote by $R_H(s, t) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H})$ the covariance function of a fractional Brownian motion of Hurst parameter H . Let $q(x, y)$ be a continuous and

positive definite function, namely, for any $x_i \in \mathbb{R}^d$, $i = 1, 2, \dots, m$ and complex numbers $\xi_i, i = 1, 2, \dots, m$, not all 0, we have

$$\sum_{i,j=1}^m q(x_i, x_j) \bar{\xi}_i \xi_j \geq 0,$$

where $\bar{\xi}_i$ is the conjugate number of ξ_i . For every $s, t \geq 0$ and $x, y \in \mathbb{R}^d$, we denote

$$Q(s, t, x, y) = \frac{\partial^2 R_H}{\partial s \partial t}(s, t) q(x, y) = \alpha_H |s - t|^{2H-2} q(x, y),$$

where $\alpha_H = H(2H - 1)$. Let \mathcal{S} be the set of all smooth functions $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $f(t, \cdot)$ has compact support for every $t \in [0, T]$. We introduce a scalar product on \mathcal{S} in the following way:

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_{[0, T]^2 \times \mathbb{R}^{2d}} \phi(s, x) \psi(t, y) Q(s, t, x, y) dx dy ds dt. \quad (3.7)$$

We denote by \mathcal{H} the Hilbert space of the closure of \mathcal{S} with respect to this inner product. Let T be a bijective Hilbert-Schmidt operator on \mathcal{H} . Define the Banach space (in fact, it is a Hilbert space) Ω as the completion of \mathcal{H} with respect to the norm $\|x\|_{\Omega} := \sqrt{\langle Tx, Tx \rangle_{\mathcal{H}}}$. Then, it follows from the Bochner-Minlos theorem (see [51], Theorem 3.1) that there is a probability measure P on (Ω, \mathcal{F}) such that $\langle h, \omega \rangle$ is a centered Gaussian random variable with covariance $\mathbb{E}[\langle h, \cdot \rangle \langle h', \cdot \rangle] = \langle h, h' \rangle_{\mathcal{H}}, \forall h, h' \in \Omega'$, where Ω' is the Banach space of all continuous linear functionals on Ω ; \mathcal{F} is the Borel σ -algebra generated by the open sets of Ω , and $\langle h, \omega \rangle$ the pairing between $h \in \Omega' \subset \mathcal{H}$ and Ω . We identify $\mathcal{H}' = \mathcal{H}$ so that the embeddings $\Omega' \subset \mathcal{H}' = \mathcal{H} \subset \Omega$ are continuous. We can define Gaussian random variable $\langle h, \omega \rangle$ for all $h \in \mathcal{H}$ by limiting argument.

First we give some specific elements in \mathcal{H} . For any $x \in \mathbb{R}^d$, we denote by δ_x the Dirac function on \mathbb{R}^d . Namely, δ_x is defined by $\int_{\mathbb{R}^d} \delta_x(y) f(y) dy = f(x)$ for any smooth function of compact support on \mathbb{R}^d .

Proposition 3.2.1. For any $s > 0$ and $x \in \mathbb{R}^d$, $I_{(0,s]}\delta_x$ is an element in \mathcal{H} and

$$\langle I_{(0,s]}\delta_x, I_{(0,t]}\delta_y \rangle_{\mathcal{H}} = R_H(s, t)q(x, y) \quad (3.8)$$

and

$$\|I_{(0,s]}\delta_x - I_{(0,t]}\delta_y\|_{\mathcal{H}}^2 = s^{2H}q(x, x) + t^{2H}q(y, y) - 2R_H(s, t)q(x, y). \quad (3.9)$$

Proof. For every $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we denote the elementary function

$$\delta_x^\varepsilon = (2\varepsilon)^{-d}I_{(x-\varepsilon, x+\varepsilon]}.$$

If ε tends to 0, the function δ_x^ε converges in \mathcal{H} to the generalized function δ_x . Indeed, fix (s, x) and (t, y) in $[0, T] \times \mathbb{R}^d$. For any positive numbers ε and ε' , we have

$$\langle I_{(0,s]}\delta_x^\varepsilon, I_{(0,t]}\delta_y^{\varepsilon'} \rangle_{\mathcal{H}} = R_H(s, t)(4\varepsilon\varepsilon')^{-d} \int_{y-\varepsilon'}^{y+\varepsilon'} \int_{x-\varepsilon}^{x+\varepsilon} q(x', y') dx' dy'.$$

Since $q(\cdot, \cdot)$ is continuous, the above right hand side converges to $q(x, y)$ as ε and ε' tend to 0. This shows easily that $I_{(0,s]}\delta_x^\varepsilon$ is a Cauchy sequence in \mathcal{H} when $\varepsilon \rightarrow 0$. The limit of $I_{(0,s]}\delta_x^\varepsilon$ in \mathcal{H} as $\varepsilon \rightarrow 0$ is $I_{(0,s]}\delta_x$. The equations (3.8) and (3.9) are immediate. \square

Since $I_{(0,s]}\delta_x \in \mathcal{H}$, we can define

$$W(s, x, \omega) = \langle I_{(0,s]}\delta_x, \omega \rangle, \quad \omega \in \Omega \quad (3.10)$$

Thus $\{W(s, x), t \geq 0, x \in \mathbb{R}^d\}$ is a multiparameter centered Gaussian process with the following covariance

$$\mathbb{E} [W(s, x)W(t, y)] = \langle I_{(0,s]}\delta_x, I_{(0,t]}\delta_y \rangle_{\mathcal{H}} = R_H(s, t)q(x, y).$$

We also denote

$$W(\phi) := \int_0^T \int_{\mathbb{R}^d} \phi(s, x) W(ds, x) dx := \langle \phi, \omega \rangle \quad \forall \phi \in \mathcal{H}.$$

The Itô integral is a fundamental concept in stochastic analysis. This integral can be defined under less condition than the Stratonovich one and has a completely different feature such as the famous Itô formula. From the modeling point of view, Itô type stochastic differential equations are more popular since all terms in the Itô equation $dx_t = b(x_t)dt + \sigma(x_t)\delta B_t$ (see also (5.6)) have clear meaning: $b(x_t)$ represents the mean rate of change and $\sigma(x_t)\delta B_t$ represents the fluctuation (it has zero mean contribution).

In this section, we will introduce nonlinear Itô-Skorohod integral. This integral is a probabilistic one and is defined for almost every sample path while nonlinear Young integral is defined for every sample path. The relation between these two integral is through the nonlinear symmetric (Stratonovich) integral.

We denote by \mathcal{P} the set of smooth and cylindrical random variables of the following form

$$F = f(W(\phi_1), \dots, W(\phi_n)), \quad (3.11)$$

$\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth). D denotes the Malliavin derivative. That is, if F is of the form (3.11), then DF is the \mathcal{H} -valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n)) \phi_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of \mathcal{P} under the norm

$$\|F\|_{1,2} = \sqrt{\mathbb{E}(F^2) + \mathbb{E}(\|DF\|_{\mathcal{H}}^2)}.$$

D can be extended uniquely to an operator from $\mathbb{D}^{1,2}$ into $L^2(\Omega; \mathcal{H})$. The *divergence operator*

δ is the adjoint of the Malliavin derivative operator D . We say that a random variable u in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator, denoted by $\text{Dom } \delta$, if there is a constant $c_u \in (0, \infty)$ such that

$$|\mathbb{E}(\langle DF, u \rangle_{\mathcal{H}})| \leq c_u \|F\|_{L^2(\Omega)} \quad \forall F \in \mathbb{D}^{1,2}.$$

In this case $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}(\delta(u)F) = \mathbb{E}(\langle DF, u \rangle_{\mathcal{H}}) \quad \forall F \in \mathbb{D}^{1,2}. \quad (3.12)$$

The following are two basic properties of the divergence operator δ .

(i) $\mathbb{D}^{1,2}(\mathcal{H}) \subset \text{Dom } \delta$ and for any $u \in \mathbb{D}^{1,2}(\mathcal{H})$

$$\mathbb{E}(\delta(u)^2) = \mathbb{E}(\|u\|_{\mathcal{H}}^2) + \mathbb{E}(\langle Du, (Du)^* \rangle_{\mathcal{H} \otimes \mathcal{H}}), \quad (3.13)$$

where $(Du)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

(ii) For any F in $\mathbb{D}^{1,2}(\mathcal{H})$ and any u in the domain of δ such that Fu and $F\delta(u) - \langle DF, u \rangle_{\mathcal{H}}$ are square integrable, then Fu is in the domain of δ and

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_{\mathcal{H}}. \quad (3.14)$$

The operator δ is also called the Skorohod integral because in the case of Brownian motion, it coincides with the generalization of the Itô stochastic integral to anticipating integrands introduced by Skorohod [93]. On the relation between δ and D , we have the identity

$$D\delta(u) = u + \delta(Du). \quad (3.15)$$

We refer to Nualart's book [80] for a detailed account of the Malliavin calculus with

respect to a Gaussian process. Using the specific definition of our \mathcal{H} , we also denote $\delta(u) = \int_0^T \int_{\mathbb{R}^d} u(t, x) W(\delta t, x) dx$. In addition, we can write the identity (3.13) as

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} u(t, x) W(\delta t, x) dx \right]^2 \\ &= \int_{[0, T]^2 \times \mathbb{R}^{2d}} \mathbb{E} [u(t, x) u(s, y)] Q(s, t, x, y) ds dt dx dy \\ &+ \int_{[0, T]^4 \times \mathbb{R}^{4d}} \mathbb{E} \left[D_{t_2, x_2} u(t_1, x_1) D_{s_2, y_2} u(s_1, y_1) \right] Q(t_1, s_2, x_2, y_1) Q(t_2, s_1, x_1, y_2) ds dt dx dy, \end{aligned} \tag{3.16}$$

where in the rest of the chapter we shall use $ds = ds_1 \cdots ds_k$, $dx = dx_1 \cdots dx_m$ and so on, the k and m being clear in the context.

Let $\{W(t, x), t \geq 0, x \in \mathbb{R}^d\}$ be the Gaussian field introduced in Section 3.2, whose mean is 0 and whose covariance is

$$\mathbb{E}(W(s, x)W(t, y)) = R_H(s, t)q(x, y).$$

Let $\varphi = \{\varphi_t, t \in [0, T]\}$ be a \mathbb{R}^d -valued stochastic process. Our aim in this section is to introduce and study the nonlinear stochastic integral $\int_0^T W(\delta t, \varphi_t)$.

This stochastic integral was studied earlier in order to establish the Feynman-Kac formula when φ_t is a Brownian motion, independent of W . The case $H > 1/2$ is discussed in [65] and the case $H < 1/2$ is discussed in [61]. When $\{W(t, x), t \geq 0\}$ is a semimartingale with respect to t (for fixed $x \in \mathbb{R}^d$), this type of stochastic integral has been studied extensively and generalized Itô formulas have been established. It has been applied to solve some stochastic partial differential equations. See for instance Kunita's book [71] and the references therein.

In this section, we will define the stochastic integral $\int W(\delta t, \varphi_t)$ based on the covariance structure of W . This method is closely tied to the nature of W as a Gaussian process. In particular, we introduce here two types of stochastic integrals, namely, the divergence

type and symmetric type. We also study their properties and relation. The divergence type integral turns out to have zero mean, thus one can think of it as a generalization of Itô-Skorohod integral. The symmetric integral does not have vanishing mean and differs from the divergence type integral by a correction term, related to the Malliavin derivative of some random variable. One can also view the symmetric integral as a generalization of Stratonovich integral.

We shall define the (nonlinear) Itô-Skorohod (divergence) type integral $\int_0^T W(\delta t, \varphi_t)$ by the (linear) multi-parameter integral $\int_0^T \int_{\mathbb{R}^d} \delta(\varphi_t - y)W(\delta t, y)dy$. Here and in the remaining part of the chapter, the symbol δ carries two meanings: the Itô-Skorohod integral and the Dirac delta function. Difference between the two meanings will be clear from the context.

Since $\delta(\varphi_t - y)$ is a distribution valued random process, to define its stochastic integral we need to approximate the Dirac delta function by smooth functions. Namely, we shall define $\int_0^T \int_{\mathbb{R}^d} \delta(\varphi_t - y)W(\delta t, y)dy$ as the limit $\lim_{\varepsilon \downarrow 0} \int_0^T \int_{\mathbb{R}^d} \eta_\varepsilon(\varphi_t - y)W(\delta t, y)dy$, where η_ε is an approximation of the Dirac delta function δ . To define such sequence η_ε , we denote by η the following bump function

$$\eta(x) = c_d \exp\{(|x|^2 - 1)^{-1}\}1_{\{|x| < 1\}}, x \in \mathbb{R}^d,$$

where $|x|$ is the Euclidean distance in \mathbb{R}^d and c_d is the positive constant so that

$$\int_{\mathbb{R}^d} \eta(x)dx = 1.$$

The function η is smooth and compactly supported. Its corresponding mollifier is

$$\eta_\varepsilon(x) = \varepsilon^{-d} \eta\left(\frac{x}{\varepsilon}\right). \quad (3.17)$$

Here is our definition.

Definition 3.2.2. Let $\varphi : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ be a measurable stochastic process. If $I_\varepsilon = \int_0^T \int_{\mathbb{R}^d} \eta_\varepsilon(\varphi_t - y) W(\delta t, y) dy$ is well-defined and it has a limit in $L^2(\Omega, \mathcal{F}, P)$ as $\varepsilon \rightarrow 0$, then we define $\int_0^T W(\delta t, \varphi_t)$ as the aforementioned limit.

Next, we shall give condition to ensure the existence of the stochastic integral $\int_0^T W(\delta t, \varphi_t)$, namely, to ensure the existence of the limit of I_ε in $L^2(\Omega, \mathcal{F}, P)$. To express the conditions in a more concise way we introduce the following notations.

$$q_\varphi(x, y) = \alpha \int_0^T \int_0^T \mathbb{E} q(x + \varphi_s, y + \varphi_t) |s - t|^{2H-2} ds dt$$

and

$$q_{D\varphi}^*(x, y) = \alpha_H^2 \int_{[0, T]^4 \times \mathbb{R}^{2d}} \mathbb{E} D_{s_1, x'} q(x + \varphi_{s_2}, y') D_{t_2, y'} q(x', y + \varphi_{t_1}) |s_1 - t_1|^{2H-2} |s_2 - t_2|^{2H-2} ds_1 ds_2 dt_1 dt_2 dx' dy'$$

whenever the integrals on the right hand side make sense. We make the following assumptions on the process φ_t .

(A1) φ_t belongs to $\mathbb{D}^{1,2}$ for all t , and for almost every $\omega \in \Omega$, the sample path φ_t is continuous in $t \in [0, T]$.

(A2) $|q|_\varphi$ is integrable on a neighborhood of $(0, 0)$, that is there exists an open set U in \mathbb{R}^{2d} containing $(0, 0)$ such that

$$\int_U \int_0^T \int_0^T \mathbb{E} |Q(s, t, x + \varphi_s, y + \varphi_t)| ds dt dx dy < \infty.$$

(A3) $q_\varphi(x, y)$ is well-define in neighborhood of $(0, 0)$ and it is continuous at $(0, 0)$.

(A4) There exists an open set U in \mathbb{R}^{2d} containing $(0, 0)$ such that

$$\int_U \int_{[0, T]^4 \times \mathbb{R}^{2d}} \mathbb{E} \left| D_{s_1, x'} q(x + \varphi_{s_2}, y') D_{t_2, y'} q(x', y + \varphi_{t_1}) \right| |s_1 - t_1|^{2H-2} |s_2 - t_2|^{2H-2} ds_1 ds_2 dt_1 dt_2 dx' dy' dx dy < \infty .$$

(A5) $q_{D\varphi}^*(x, y)$ is well-defined in neighborhood of $(0, 0)$ and it is continuous at $(0, 0)$.

Theorem 3.2.3. *We assume the conditions (A1)-(A5) are satisfied. Then $\int_0^T W(\delta t, \varphi_t)$ is well-defined and*

$$\begin{aligned} \mathbb{E} \left[\int_0^T W(\delta t, \varphi_t) \right]^2 &= q_{D\varphi}^*(0, 0) + q_\varphi(0, 0) \\ &= \int_{[0, T]^4} \int_{\mathbb{R}^{2d}} \mathbb{E} D_{s_1, x} Q(s_1, t_1, \varphi_{s_2}, y) D_{t_2, y} Q(s_2, t_2, x, \varphi_{t_1}) dx dy ds_1 ds_2 dt_1 dt_2 \\ &\quad + \int_0^T \int_0^T \mathbb{E} Q(s, t, \varphi_s, \varphi_t) ds dt . \end{aligned} \quad (3.18)$$

Before proceeding to the proof, let us make the following remark which we will use several times in the future.

Remark 3.2.4. Suppose that f and g are smooth functions, f has compact support, and φ is random variable in $\mathbb{D}^{1,2}$. Then the following integration by parts formula holds almost surely

$$\int_{\mathbb{R}^d} Df(x - \varphi) g(x) dx = - \int_{\mathbb{R}^d} f(x) Dg(x + \varphi) dx . \quad (3.19)$$

Indeed, the integration on the left hand side is

$$\int_{\mathbb{R}^d} \nabla f(x - \varphi) \cdot D\varphi g(x) dx .$$

Integrating by parts yields

$$- \int_{\mathbb{R}^d} f(x - \varphi) D\varphi \cdot \nabla g(x) dx.$$

With the change of the variable $x \mapsto x + \varphi$,

$$- \int_{\mathbb{R}^d} f(x) D\varphi \cdot \nabla g(x + \varphi) dx = - \int_{\mathbb{R}^d} f(x) Dg(x + \varphi) dx.$$

Proof of Theorem 3.2.3. For any $\varepsilon > 0$, the \mathcal{H} -valued random variable $\eta_\varepsilon(\cdot - \varphi_\cdot)$ belongs to $\mathbb{D}^{1,2}(\mathcal{H})$, hence belongs to $\text{Dom } \delta$. Thus, applying (3.13), for every positive numbers ε and ε' , we obtain

$$\begin{aligned} \mathbb{E}(\delta(\eta_\varepsilon(\cdot - \varphi_\cdot))\delta(\eta_{\varepsilon'}(\cdot - \varphi_\cdot))) &= \mathbb{E}\langle \eta_\varepsilon(\cdot - \varphi_\cdot), \eta_{\varepsilon'}(\cdot - \varphi_\cdot) \rangle_{\mathcal{H}} \\ &\quad + \mathbb{E}\langle D\eta_\varepsilon(\cdot - \varphi_\cdot), (D\eta_{\varepsilon'}(\cdot - \varphi_\cdot))^* \rangle_{\mathcal{H} \otimes \mathcal{H}} =: E_1 + E_2. \end{aligned} \quad (3.20)$$

Using a change of variable, we have

$$\begin{aligned} E_1 &= \alpha_H \mathbb{E} \int_0^T \int_0^T \int_{\mathbb{R}^{2d}} \eta_\varepsilon(x - \varphi_s) \eta_{\varepsilon'}(y - \varphi_t) q(x, y) |t - s|^{2H-2} dx dy ds dt \\ &= \alpha_H \mathbb{E} \int_0^T \int_0^T \int_{2\mathbb{R}^d} \eta_\varepsilon(x) \eta_{\varepsilon'}(y) q(x + \varphi_s, y + \varphi_t) |t - s|^{2H-2} dx dy ds dt \\ &= \alpha_H \int_0^T \int_0^T \int_{\mathbb{R}^{2d}} \eta_\varepsilon(x) \eta_{\varepsilon'}(y) \mathbb{E} q(x + \varphi_s, y + \varphi_t) |t - s|^{2H-2} dx dy ds dt. \end{aligned}$$

When ε and ε' tend to 0, using the conditions (A2), (A3), this quantity converges to

$$\alpha_H \int_0^T \int_0^T \mathbb{E} q(\varphi_s, \varphi_t) |t - s|^{2H-2} ds dt = q_\varphi(0, 0).$$

Hence, when ε tends to zero, $\eta_\varepsilon(\cdot - \varphi_\cdot)$ converges in $L^2(\Omega; \mathcal{H})$ to a \mathcal{H} -valued random variable, denoted by $\delta_\varphi = \delta(\varphi_t - y)$.

For the second expectation in (3.20), we use (3.16) to obtain

$$E_2 = \alpha_H^2 \mathbf{E} \int_{[0,T]^4 \times \mathbb{R}^{4d}} D_{s_1, x_1} \eta_\varepsilon(x_2 - \varphi_{s_2}) D_{t_2, y_2} \eta_{\varepsilon'}(y_1 - \varphi_{t_1}) q(x_1, y_1) q(x_2, y_2) |s_1 - t_1|^{2H-2} |s_2 - t_2|^{2H-2} ds dt dx dy .$$

An application of (3.19) yields

$$E_2 = \alpha_H^2 \mathbf{E} \int_{[0,T]^4 \times \mathbb{R}^{4d}} \eta_\varepsilon(x_2) D_{s_1, x_1} q(x_2 + \varphi_{s_2}, y_2) \eta_{\varepsilon'}(y_1) D_{t_2, y_2} q(x_1, y_1 + \varphi_{t_1}) |s_1 - t_1|^{2H-2} |s_2 - t_2|^{2H-2} ds dt dx dy .$$

When ε and ε' tend to 0, this converges to $q_{D\varphi}^*(0, 0)$ by using conditions (A4), (A5).

Therefore, $\delta(\eta_\varepsilon(\cdot - \varphi_\cdot))$ is a Cauchy sequence in $L^2(\Omega)$. Since δ is a closed operator and $\eta_\varepsilon(\cdot - \varphi_\cdot)$ converges to δ_φ , we obtain that δ_φ belongs to the domain of δ . As a consequence, $\delta(\eta_\varepsilon(\cdot - \varphi_\cdot))$ converges to $\delta(\delta_\varphi)$ when ε tends to zero. Thus the integration $\int_0^T W(\delta t, \phi_t)$ is well-defined. The equation (3.18) is immediate. \square

Remark 3.2.5. Under the hypothesis of the above theorem, the \mathcal{H} -valued random variable $\eta_\varepsilon(\cdot - \varphi_\cdot)$ converges in $L^2(\Omega; \mathcal{H})$ to $\delta_\varphi = \delta(\varphi_t - y)$ as ε tends to zero. Moreover, δ_φ also belongs to the domain of the divergence operator and the convergence also holds under the divergence δ . Hence, in this case, the stochastic integral in Definition 3.2.2 can be viewed as $\delta(\delta_\varphi)$, the divergence of δ_φ .

3.3 Nonlinear symmetric stochastic integral

We introduce and study symmetric type stochastic integral by using appropriate approximation. This stochastic integral will be different than the Itô-Skorohod type integral introduced in the previous section.

Recall that $W = \{W(s, x, \omega), \omega \in \Omega\}$ is the Gaussian random field (indexed by (s, x)) defined in the previous section. Throughout this subsection, we assume that W is almost surely continuous with respect to $s \geq 0$ and $x \in \mathbb{R}^d$. We define the composition of the random field W and a \mathbb{R}^d -valued process $\varphi = \{\varphi_s, s \in [0, T]\}$ by

$$\begin{aligned} W(s, \varphi_s) : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto W(s, \varphi_s(\omega), \omega). \end{aligned} \tag{3.21}$$

By convention, we will assume that all processes and functions vanish outside the interval $[0, T]$.

Definition 3.3.1. The nonlinear symmetric integral $\int_a^b W(d^{\text{sym}}s, \varphi_s)$ is defined as the limit as ε tends to zero of

$$(2\varepsilon)^{-1} \int_a^b (W(s + \varepsilon, \varphi_s) - W(s - \varepsilon, \varphi_s)) ds, \tag{3.22}$$

provided this limit exists in probability.

Example 3.3.2. In the particular case, when $W(s, x) = B_s f(x)$, where f is a nice deterministic function and $\{B_s, s \geq 0\}$ is a Brownian motion, the symmetric integral defined above coincides with Stratonovich integral. That is $\int_0^T W(d^{\text{sym}}s, \varphi_s) = \int_0^T f(\varphi_s) d^\circ B_s$.

In the following proposition we will see that for a suitable class of \mathbb{R}^d -valued processes $\{\varphi_t\}$, the symmetric stochastic integral $\int_0^T W(d^{\text{sym}}s, \varphi_s)$ exists almost surely. This result is an extension of [2, Proposition 3].

Proposition 3.3.3. *Let φ be a \mathbb{R}^d -valued process satisfying assumptions (A1)-(A5). In addition, suppose that φ satisfies*

$$\int_0^T \int_{|x|<1} [\mathbb{E}q(x + \varphi_s, x + \varphi_s)]^{1/2} dx ds < \infty, \tag{3.23}$$

$$\int_0^T \int_{|x|<1} \left[\mathbb{E} \left| \sum_{i,j=1}^d \langle D\varphi_s^i, D\varphi_s^j \rangle_{\mathcal{H}} \right| q(x + \varphi_s, x + \varphi_s) \right]^{1/2} dx ds < \infty \quad (3.24)$$

and the function

$$x \mapsto \int_0^T \int_0^T \int_{\mathbb{R}^d} |D_{t,y}q(x + \varphi_s, y)| |s - t|^{2H-2} dt ds dy \quad (3.25)$$

is a.s. well-defined and continuous on a neighborhood of 0. Assume also that the Gaussian field W has continuous sample path. Then the symmetric integral (3.22) exists and the following formula holds almost surely

$$\int_0^T W(\mathfrak{d}^{\text{sym}}_s, \varphi_s) = \int_0^T W(\delta s, \varphi_s) + \alpha_H \int_0^T \int_0^T \int_{\mathbb{R}^{2d}} D_{t,y}q(\varphi_s, y) |s - t|^{2H-2} dt ds dy. \quad (3.26)$$

Proof. We shall show the convergence in L^2 of (3.22). For every positive ε , since W has continuous sample path, we can write

$$\begin{aligned} W(s + \varepsilon, \varphi_s) - W(s - \varepsilon, \varphi_s) &= \lim_{\varepsilon' \rightarrow 0} \int_{\mathbb{R}^d} [W(s + \varepsilon, x) - W(s - \varepsilon, x)] \eta_{\varepsilon'}(x - \varphi_s) dx \\ &= \lim_{\varepsilon' \rightarrow 0} \int_{\mathbb{R}^d} \delta(I_{[s-\varepsilon, s+\varepsilon]} \delta_x) \eta_{\varepsilon'}(x - \varphi_s) dx, \end{aligned} \quad (3.27)$$

almost surely, where we have used (3.10) in the last equality. We notice $\eta_{\varepsilon'}(x - \varphi_s)$ belongs to $\mathbb{D}^{1,2}$ for every s and x . Using (3.14), we see that the integrand on the right hand side of (3.27) can be written as

$$\delta \left(I_{(s-\varepsilon, s+\varepsilon]} \delta_x \eta_{\varepsilon'}(x - \varphi_s) \right) + \left\langle D\eta_{\varepsilon'}(x - \varphi_s), I_{(s-\varepsilon, s+\varepsilon]} \delta_x \right\rangle_{\mathcal{H}}.$$

Taking integration with respect to x and s , we obtain

$$\begin{aligned}
& (2\varepsilon)^{-1} \int_0^T \int_{\mathbb{R}^d} [W(s + \varepsilon, x) - W(s - \varepsilon, x)] \eta_{\varepsilon'}(x - \varphi_s) dx ds \\
&= (2\varepsilon)^{-1} \int_0^T \int_{\mathbb{R}^d} \delta \left(I_{(s-\varepsilon, s+\varepsilon]} \delta_x \eta_{\varepsilon'}(x - \varphi_s) \right) dx ds \\
&\quad + (2\varepsilon)^{-1} \int_0^T \int_{\mathbb{R}^d} \left\langle D \eta_{\varepsilon'}(x - \varphi_s), I_{(s-\varepsilon, s+\varepsilon]} \delta_x \right\rangle_{\mathcal{H}} dx ds \\
&=: I_1 + I_2.
\end{aligned} \tag{3.28}$$

The proof is now decomposed into several steps.

Step 1. Let us show that the integration with respect to $dx ds$ in I_1 can be interchanged with the divergence operator to obtain

$$I_1 = \delta \left((2\varepsilon)^{-1} \int_0^T \int_{\mathbb{R}^d} I_{(s-\varepsilon, s+\varepsilon]} \delta_x \eta_{\varepsilon'}(x - \varphi_s) dx ds \right).$$

In fact, one can view the integral in I_1 in Bochner sense, that is integration with L^2 -valued integrand. In this setting, we have

$$\int_0^T \int_{\mathbb{R}^d} \delta(u(s, x)) dx ds = \delta \left(\int_0^T \int_{\mathbb{R}^d} u(s, x) dx ds \right)$$

provided that

$$\int_0^T \int_{\mathbb{R}^d} \|u(s, x)\|_{\mathbb{D}^{1,2}} dx ds < \infty \tag{3.29}$$

and δ is a bounded operator from $\mathbb{D}^{1,2}$ to L^2 . The later fact is automatically guaranteed by (3.13). It remains to check that $u(s, x) = I_{(s-\varepsilon, s+\varepsilon]} \delta_x \eta_{\varepsilon'}(x - \varphi_s)$ satisfies (3.29).

$$\begin{aligned}
\|u(s, x)\|_{\mathcal{H}}^2 &= \int_{s-\varepsilon}^{s+\varepsilon} \int_{s-\varepsilon}^{s+\varepsilon} \frac{\partial^2}{\partial s \partial t} R_H(s', t') ds' dt' q(x, x) \mathbb{E}[\eta_{\varepsilon'}^2(x - \varphi_s)] \\
&\leq R_H([0, T]^2) q(x, x) \mathbb{E}[\eta_{\varepsilon'}^2(x - \varphi_s)].
\end{aligned}$$

Thus by a change of variable, we obtain

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} \|u(s, x)\|_{\mathcal{H}} dx ds &\leq R_H^{1/2}([0, T]^2) \int_0^T \int_{\mathbb{R}^d} (\mathbb{E}q(x, x) \eta_{\varepsilon'}^2(x - \varphi_s))^{1/2} dx ds \\
&= R_H^{1/2}([0, T]^2) \int_0^T \int_{\mathbb{R}^d} (\mathbb{E}q(x + \varphi_s, x + \varphi_s) \eta_{\varepsilon'}^2(x))^{1/2} dx ds \\
&\leq c(\varepsilon', T) \int_0^T \int_{|x|<1} (\mathbb{E}q(x + \varphi_s, x + \varphi_s))^{1/2} dx ds.
\end{aligned}$$

The last integral is finite thanks to the condition (3.23). Similarly

$$\begin{aligned}
&\|Du(s, x)\|_{\mathcal{H}}^2 \\
&= \mathbb{E} \int_{s-\varepsilon}^{s+\varepsilon} \int_{s-\varepsilon}^{s+\varepsilon} \frac{\partial^2}{\partial s \partial t} R_H(s', t') ds' dt' q(x, x) \partial_i \eta_{\varepsilon'}(x - \varphi_s) \partial_j \eta_{\varepsilon'}(x - \varphi_s) \langle D\varphi_s^i, D\varphi_s^j \rangle_{\mathcal{H}} \\
&\leq R_H([0, T]^2) q(x, x) \mathbb{E} \left| \sum_{i,j} \partial_i \eta_{\varepsilon'}(x - \varphi_s) \partial_j \eta_{\varepsilon'}(x - \varphi_s) \langle D\varphi_s^i, D\varphi_s^j \rangle_{\mathcal{H}} \right|.
\end{aligned}$$

Thus, by a change of variable and by using the condition (3.24), we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \|Du(s, x)\|_{\mathcal{H} \otimes \mathcal{H}} dx ds \\
&\leq c(T) \int_0^T \int_{\mathbb{R}^d} [\mathbb{E}q(x, x)] \sum_{i,j} \partial_i \eta_{\varepsilon'}(x - \varphi_s) \partial_j \eta_{\varepsilon'}(x - \varphi_s) \langle D\varphi_s^i, D\varphi_s^j \rangle_{\mathcal{H}}]^{1/2} dx ds \\
&\leq c(\varepsilon', T) \int_0^T \int_{|x|<1} [\mathbb{E}q(x + \varphi_s, x + \varphi_s)] \sum_{i,j} \langle D\varphi_s^i, D\varphi_s^j \rangle_{\mathcal{H}}]^{1/2} dx ds < \infty.
\end{aligned}$$

Step 2. We show that

$$\delta \left((2\varepsilon)^{-1} \int_0^T \int_{\mathbb{R}^d} I_{(s-\varepsilon, s+\varepsilon]} \delta_x \eta_{\varepsilon'}(x - \varphi_s) dx ds \right) = \delta \left((2\varepsilon)^{-1} \int_0^T I_{(s-\varepsilon, s+\varepsilon]} \eta_{\varepsilon'}(\cdot - \varphi_s) ds \right).$$

It suffices to show for every smooth function ϕ with compact support

$$\phi = \int_{\mathbb{R}^d} \phi(y) \delta_y dy \tag{3.30}$$

is in \mathcal{H} , since with the choice $\phi = \eta_{\varepsilon'}$, (3.30) will yield the desired identity. Recall that \mathcal{S} is the space defined in Section 3.2 and is dense in \mathcal{H} . Thus to show (3.30), we verify

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \left\langle \int_{\mathbb{R}^d} \phi(y) \delta_y dy, \psi \right\rangle_{\mathcal{H}}$$

for every $\psi \in \mathcal{S}$. Indeed, we have

$$\begin{aligned} \left\langle \int_{\mathbb{R}^d} \phi(y) \delta_y dy, \psi \right\rangle_{\mathcal{H}} &= \int_{\mathbb{R}^d} \phi(y) \langle \delta_y, \psi \rangle_{\mathcal{H}} dy \\ &= \int_{\mathbb{R}^d} \int_0^T \int_0^T \int_{\mathbb{R}^d} \phi(y) \psi(x) Q(s, t, y, x) dx ds dt dy = \langle \phi, I_{(0,s]} \delta_x \rangle_{\mathcal{H}}, \end{aligned}$$

by Fubini's theorem.

Step 3. Combining the previous two steps, we obtain

$$I_1 = \delta \left((2\varepsilon)^{-1} \int_0^T I_{(s-\varepsilon, s+\varepsilon]} \eta_{\varepsilon'}(\cdot - \varphi_s) ds \right).$$

It is straightforward to check that when ε' and ε tend to zero, I_1 converges to $\int_0^T W(\delta s, \varphi_s)$ in L^2 .

Step 4. We now show the convergence of I_2 . A direct computation shows that

$$\begin{aligned} |I_2| &= (2\varepsilon)^{-1} \left| \int_0^T \int_{\mathbb{R}^d} \int_{-\varepsilon}^{\varepsilon} \int_0^T \int_{\mathbb{R}^d} D_{t,y} \eta_{\varepsilon'}(x - \varphi_s) q(x, y) |t - r - s|^{2H-2} dy dt dr dx ds \right| \\ &\leq d_H \int_0^T \int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} |D_{t,y} \eta_{\varepsilon'}(x - \varphi_s) q(x, y)| |t - s|^{2H-2} dy dt dx ds, \end{aligned}$$

where we have used the following inequality

$$(2\varepsilon)^{-1} \int_{-\varepsilon}^{\varepsilon} |t - r - s|^{2H-2} dr \leq d_H |t - s|^{2H-2}$$

for some constant d_H , independent $\varepsilon \in (0, 1)$ and $s, t \in \mathbb{R}$. By a change of variable

$x - \varphi_s \rightarrow x$, we obtain

$$I_2 \leq d_H \int_0^T \int_0^T \int_{\mathbb{R}^{2d}} \left| \eta_{\varepsilon'}(x) D_{t,y} q(x + \varphi_s, y) \right| |t - s|^{2H-2} dy dx ds dt.$$

Hence, by the dominated convergence theorem, when ε' and ε tend to zero, I_2 goes to $\int_0^T \int_0^T \int_{\mathbb{R}^d} D_{t,y} q(x + \varphi_s, y) |t - s|^{2H-2} dy ds dt$. Therefore, passing through the limits in (3.28), we obtain (3.26) □

3.4 Relationships of various nonlinear integrals

If the limit in Definition 3.3.1 exists for almost every sample path of W , then the symmetric integral can also be defined pathwise for a function $(W(t, x), t \geq 0, x \in \mathbb{R}^d)$. We also call such integral the symmetric integral and denoted by the same symbol $\int_0^T W(d^{\text{sym}}s, \varphi_s)$.

The following proposition establishes the relation between symmetric integral and nonlinear Young integral introduced in Section 3.1.

Proposition 3.4.1. *Assume the hypothesis of Proposition 3.1.4. Then the symmetric integral exists and the following relation holds*

$$\int_0^T W(d^{\text{sym}}s, \varphi_s) = \int_0^T W(ds, \varphi_s).$$

Proof. Fix $\varepsilon > 0$, we put

$$W_\varepsilon(s, x) = (2\varepsilon)^{-1} \int_{-\varepsilon}^{\varepsilon} W(s + \eta, x) d\eta.$$

We recall that $\int_0^T W(d^{\text{sym}}s, \varphi_s) = \lim_{\varepsilon \rightarrow 0} \int_0^T \partial_t W_\varepsilon(s, \varphi) ds$. We put

$$\mu_k(a, b) = W_{\varepsilon_k}(b, \varphi_a) - W_{\varepsilon_k}(a, \varphi_a),$$

$$\mu(a, b) = W(b, \varphi_a) - W(a, \varphi_a).$$

Since W_ϵ is continuously differentiable in time, the integral $\int W_\epsilon(ds, \varphi_s)$ is understood in classical sense and is equal to $\int \partial_t W_\epsilon(s, \varphi_s) ds$. Hence, applying Proposition 3.1.10 we obtain, for any $\theta \in (0, 1)$ such that $\theta\tau + \lambda\gamma > 1$

$$\begin{aligned} & \left| \int_0^T W(ds, \varphi_s) - \int_0^T \partial_t W_\epsilon(s, \varphi_s) ds \right| \\ & \leq |W(T, \varphi_0) - W(0, \varphi_0) - W_\epsilon(T, \varphi_0) + W_\epsilon(0, \varphi_0)| + c(\varphi)[W - W_\epsilon]_{\beta, \tau, \lambda} |b - a|^{\theta\tau + \lambda\gamma}. \end{aligned}$$

It remains to estimate the terms on the right side and show that they all converge to 0 when ϵ goes to 0. For the first term

$$\begin{aligned} & |W(T, \varphi_0) - W(0, \varphi_0) - W_\epsilon(T, \varphi_0) + W_\epsilon(0, \varphi_0)| \\ & \leq (2\epsilon)^{-1} \int_{-\epsilon}^\epsilon |W(T, \varphi_0) - W(0, \varphi_0) - W(T + \eta, \varphi_0) + W(\eta, \varphi_0)| d\eta \\ & \lesssim (2\epsilon)^{-1} \int_{-\epsilon}^\epsilon |\eta|^\tau d\eta \lesssim \epsilon^\tau. \end{aligned}$$

For the second term, we put $F = W - W_\epsilon$ and notice that

$$\begin{aligned} & |W_\epsilon(s, x) - W_\epsilon(s, y) - W_\epsilon(t, x) + W_\epsilon(t, y)| \\ & \leq (2\epsilon)^{-1} \int_{-\epsilon}^\epsilon |W(s + \eta, x) - W(s + \eta, y) - W(t + \eta, x) + W(t + \eta, y)| d\eta \\ & \leq [W](1 + |x|^\beta + |y|^\beta)(2\epsilon)^{-1} \int_{-\epsilon}^\epsilon |s - t|^\tau |x - y|^\lambda d\eta \\ & \leq [W](1 + |x|^\beta + |y|^\beta) |s - t|^\tau |x - y|^\lambda. \end{aligned}$$

Thus

$$|F(s, x) - F(s, y) - F(t, x) + F(t, y)| \leq 2[W](1 + |x|^\beta + |y|^\beta) |s - t|^\tau |x - y|^\lambda.$$

On the other hand,

$$\begin{aligned}
& |W_\epsilon(s, x) - W_\epsilon(s, y) - W_\epsilon(t, x) + W_\epsilon(t, y)| \\
& \leq (2\epsilon)^{-1} \int_{-\epsilon}^{\epsilon} d\eta |W(s + \eta, x) - W(s, x) - W(s + \eta, y) + W(s, y)| \\
& \quad + |W(t, x) - W(t + \eta, x) - W(t, y) + W(t + \eta, y)| \\
& \leq 2[W](1 + |x|^\beta + |y|^\beta) |x - y|^\lambda (2\epsilon)^{-1} \int_{-\epsilon}^{\epsilon} |\eta|^\tau d\eta \\
& \leq 2(1 + \tau)^{-1} [W](1 + |x|^\beta + |y|^\beta) |x - y|^\lambda \epsilon^\tau.
\end{aligned}$$

Hence, combining these two bounds, we get

$$|F(s, x) - F(s, y) - F(t, x) + F(t, y)| \lesssim [W](1 + |x|^\beta + |y|^\beta) |s - t|^{\theta\tau} |x - y|^\lambda \epsilon^{\tau(1-\theta)}.$$

Thus $[W - W_\epsilon]_{\beta, \theta\tau, \lambda} \lesssim \epsilon^{\tau(1-\theta)}$ which converges to 0 as $\epsilon \rightarrow 0$. □

Chapter 4

Differential equations associated with nonlinear Young integral operators

Let $W : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfy the condition **(W)** stated at the beginning of Section 3.1 with $\tau(1 + \lambda) > 1$. In this chapter we consider the following differential equation

$$\varphi_t = \varphi_{t_0} + \int_{t_0}^t W(ds, \varphi_s), \quad (4.1)$$

where $\int_{t_0}^t W(ds, \varphi_s)$ is a Young integral describe in Section 3.1. We are concerned with the existence, uniqueness, boundedness and the flow property of the solution. We shall also study the dependence of the solution on the initial conditions. Some related results on this direction are also obtained independently by Catellier and Gubinelli [15]. Applications of the results obtained are represented in Sections 4.3 and 4.4 where we consider a transport differential equation of the type

$$u(dt, x) = \nabla u(t, x)W(dt, x).$$

Literature on transport equations is vast and mostly focuses on irregularity of the spatial variables of the vector field (see for instance [29] for Sobolev vector fields, [3] for BV

vector fields and [7] for Besov vector fields). In the case W being a semi-martingale, the above equation is treated in [71]. It appears to be new in the context of nonlinear Young integration.

4.1 Existence and uniqueness

Theorem 4.1.1 (Existence). *Suppose that W satisfies the assumption (W) with $\tau(1 + \lambda) > 1$ and $\beta + \lambda \leq 1$. Then the equation (4.1) has a solution in the space of Hölder continuous functions $C^\tau([t_0 - T, t_0 + T])$ for any $T > 0$. Moreover, if φ is a solution in $C^\tau([t_0 - T, t_0 + T])$, then*

$$\sup_{t \in [t_0 - T, t_0 + T]} |\varphi_t| + \sup_{t_0 - T \leq s < t \leq t_0 + T} \frac{|\varphi_t - \varphi_s|}{|t - s|^\tau} \leq C_{\tau, \lambda, T} e^{k_{\tau, \lambda, T} \|W\|_{\tau, \lambda}^{\frac{1 - \tau + \tau \lambda}{\tau \lambda}}} (1 \vee |\varphi_{t_0}|), \quad (4.2)$$

where the constant $k_{\tau, \lambda, T}$ and $C_{\tau, \lambda, T}$ depend only on λ , τ and T .

Proof. Fix $T > 0$, we denote $\|W\| = \|W\|_{\beta, \tau, \lambda; [t_0 - T, t_0 + T]}$. We define a mapping M acting on $C^\tau([t_0 - T, t_0 + T])$ as follows

$$Mx = x_0 + \int_{t_0}^{\cdot} W(ds, x_s), \quad \forall x \in C^\tau([t_0 - T, t_0 + T]).$$

We shall verify that M satisfies the hypothesis of Leray-Schauder theorem (see [41, Theorem 11.3]).

Step 1. M is well-defined, continuous and compact. This immediately follows from Corollary 3.1.12.

Step 2. Now we explain that the set $\{x \in C^\tau([t_0 - T, t_0 + T]) : x = \sigma Mx, 0 \leq \sigma \leq 1\}$ is bounded. Let x satisfy $x = \sigma Mx$ for some $\sigma \in [0, 1]$. By definition of M , we see $x = \sigma Mx$ can be written as

$$x_b - x_a = \sigma \int_a^b W(ds, x_s).$$

From (3.5), it follows that for any $a, b \in [t_0 - T, t_0 + T]$, we have

$$\begin{aligned} |x_b - x_a| &= \sigma \left| \int_a^b W(ds, x_s) \right| \\ &\leq \sigma \|W\| (1 + \|x\|_{\infty; a, b}^\beta) \|x\|_{\infty; a, b}^\lambda (b - a)^\tau + \sigma \kappa \|W\| (1 + \|x\|_{\infty; a, b}^\beta) \|x\|_{\tau; a, b}^\lambda |b - a|^{\tau + \lambda\tau}. \end{aligned}$$

Since $\sigma \leq 1$, this yields

$$\|x\|_{\tau; a, b} \leq \|W\| (1 + \|x\|_{\infty; a, b}^\beta) \|x\|_{\infty; a, b}^\lambda + \kappa \|W\| (1 + \|x\|_{\infty; a, b}^\beta) \|x\|_{\tau; a, b}^\lambda |b - a|^{\lambda\tau},$$

for every a, b in $[t_0, t_0 + T]$ with $a < b$. We emphasize that the constant κ appears in the previous inequality is independent of σ . An application of Young inequality gives

$$\|x\|_{\infty; a, b}^\beta \|x\|_{\tau; a, b}^\lambda \leq \|x\|_{\infty; a, b}^{\beta + \lambda} + \|x\|_{\tau; a, b}^{\beta + \lambda}.$$

Thus

$$\begin{aligned} \|x\|_{\tau; a, b} &\leq \|W\| (\|x\|_{\infty; a, b}^\lambda + \|x\|_{\infty; a, b}^{\beta + \lambda}) + \kappa \|W\| \|x\|_{\infty; a, b}^{\beta + \lambda} |b - a|^{\lambda\tau} \\ &\quad + \kappa \|W\| (\|x\|_{\tau; a, b}^\lambda + \|x\|_{\tau; a, b}^{\beta + \lambda}) |b - a|^{\lambda\tau}. \end{aligned}$$

Applying the inequality $z^\theta \leq 1 \vee z$ ($\theta \in [0, 1]$ and $z \geq 0$), we obtain

$$\|x\|_{\tau; a, b} \leq \|W\| (2 + \kappa |b - a|^{\lambda\tau}) (1 \vee \|x\|_{\infty; a, b}) + \kappa \|W\| (1 \vee \|x\|_{\tau; a, b}) |b - a|^{\lambda\tau}.$$

We further use

$$\|x\|_{\infty; a, b} \leq |x_a| + \|x\|_{\tau; a, b} |b - a|^\tau$$

to obtain

$$\|x\|_{\tau; a, b} \leq A \|W\| (1 \vee |x_a|) + A \|W\| (1 \vee \|x\|_{\tau; a, b}) |b - a|^{\lambda\tau}, \quad (4.3)$$

where A is a constant depending only on τ, λ and T . Let Δ be a positive number such that

$$A\|W\|\Delta^{\tau\lambda} = \frac{1}{2}. \quad (4.4)$$

If $|b - a| \leq \Delta$, then from (4.3)

$$\|x\|_{\tau;a,b} \leq 2A\|W\|(1 \vee |x_a|). \quad (4.5)$$

Hence, we obtain

$$(1 \vee \|x\|_{\infty;a,b}) \leq (2A\|W\|\Delta^{\tau} + 1)(1 \vee |x_a|). \quad (4.6)$$

Divide the interval $[t_0, t_0 + T]$ into $n = [T/\Delta] + 1$ subintervals of length less or equal than Δ . Applying the inequality (4.6) on the intervals $[t_0, t_0 + \Delta], [t_0 + \Delta, \dots], [t_0 + (n-1)\Delta, t_0 + n\Delta \wedge T]$, recursively, we obtain

$$(1 \vee \|x\|_{\infty,t_0,t_0+T}) \leq (2A\|W\|\Delta^{\tau} + 1)^n (1 \vee |x_{t_0}|). \quad (4.7)$$

We can also assume that $\Delta \leq T$. Thus $n \leq 2T/\Delta$. We use the bound $2A\|W\|\Delta^{\tau} + 1 \leq \exp(2A\|W\|\Delta^{\tau})$. Then (4.7) yields

$$(1 \vee \|x\|_{\infty,t_0,t_0+T}) \leq \exp(2A\|W\|\Delta^{\tau} \frac{2T}{\Delta}) (1 \vee |x_{t_0}|).$$

Using (4.4), namely,

$$\Delta = (2A\|W\|)^{-\frac{1}{\tau\lambda}},$$

we have

$$(1 \vee \|x\|_{\infty;t_0,t_0+T}) \leq e^{T(2A\|W\|)^{\frac{1-\tau+\tau\lambda}{\tau\lambda}}} (1 \vee |x_{t_0}|),$$

where $C_{\tau,\lambda}$ and $\kappa_{\tau,\lambda}$ are uniformly bounded in $\sigma \in [0, 1]$. The argument goes similarly on

the other interval $[t_0 - T, t_0]$. Thus

$$(1 \vee \|x\|_{\infty; t_0-T, t_0+T}) \leq e^{T(2A\|W\|)^{\frac{1-\tau+\tau\lambda}{\tau\lambda}}} (1 \vee |x_{t_0}|) . \quad (4.8)$$

Together with the estimate (4.5), this inequality (4.8) implies that the set

$$\{x \in C^\tau([t_0 - T, t_0 + T]) : x = \sigma Lx, 0 \leq \sigma \leq 1\}$$

is bounded in $C^\tau([t_0 - T, t_0 + T])$.

Step 3. Applying Leray-Schauder theorem, we see that the equation (4.1) has a solution $\{\varphi_t, t \in [t_0 - T, t_0 + T]\}$ in $C^\tau([t_0 - T, t_0 + T])$ for every T . The estimate (4.2) comes from (4.8) together with (4.5). \square

Next, we study some stability result. In particular, we want to know how the solution depends on the initial condition x_{t_0} .

Theorem 4.1.2. *Let the condition (W) be satisfied with $\tau + \tau\lambda > 1$. In addition, we assume that $W(t, x)$ is differentiable with respect to x for every t and the spatial gradient matrix of W is denoted by $\nabla W(t, x) = \left(\frac{\partial W_i(t, x)}{\partial x_j} \right)_{1 \leq i, j \leq d}$. Suppose*

$$\begin{aligned} \|\nabla W\|_{\tau, \lambda; [t_0-T, t_0+T] \times K} := & \sup_{\substack{t_0-T \leq s < t \leq t_0+T \\ x \in K}} \frac{|\nabla W(t, x) - \nabla W(s, x)|}{|t - s|^\tau} \\ & + \sup_{\substack{t_0-T \leq s < t \leq t_0+T \\ x, y \in K, x \neq y}} \frac{|\nabla W(t, x) - \nabla W(s, x) - \nabla W(t, y) + \nabla W(s, y)|}{|t - s|^\tau |x - y|^\lambda} \end{aligned}$$

is finite for all compact set K in \mathbb{R}^d . Let x_t and y_t be two solutions in $C^\tau([t_0 - T, t_0 + T])$ to the integral equation (4.1) with initial conditions x_0 and y_0 respectively. Then the following estimate holds

$$\sup_{t \in [t_0-T, t_0+T]} |x_t - y_t| \leq 2^{\kappa T A^{\frac{1}{\tau}}} |x_0 - y_0|, \quad (4.9)$$

where A is a constant depending on ∇W , x , y and T (precise formula is given in (4.10) below).

Proof. We put $R = \max\{\|x\|_{\infty;[t_0-T, t_0+T]}, \|y\|_{\infty;[t_0-T, t_0+T]}\}$, $K = \{x \in \mathbb{R}^d : |x| \leq R\}$ and $\|\nabla W\| = \|\nabla W\|_{\tau, \lambda; [t_0-T, t_0+T] \times K}$. We also denote $z_t = x_t - y_t$, $\rho_\tau = (\|x\|_\tau + \|y\|_\tau)^\lambda$ and $\eta_t = \eta x_t + (1 - \eta)y_t$ for each $\eta \in (0, 1)$. For every s, t and x , we use the notation $W([s, t], x) = W(t, x) - W(s, x)$.

We shall obtain estimate for z in $C([t_0 - T, t_0 + T])$. Fix $a < b$ in $[t_0 - T, t_0 + T]$. We then write

$$z_b - z_a = \int_a^b W(ds, x_s) - \int_a^b W(ds, y_s) = \mathcal{J}_a^b \mu,$$

where μ is the function

$$\mu(s, t) = W([s, t], x_s) - W([s, t], y_s) = \int_0^1 \nabla W([s, t], \eta_s) z_s d\eta.$$

For every $s \leq c \leq t$ in $[a, b]$, we can write

$$\begin{aligned} & \mu(s, t) - \mu(s, c) - \mu(c, t) \\ &= \int_0^1 ([\nabla W([c, t], \eta_s) - \nabla W([c, t], \eta_c)] z_s + \nabla W([c, t], \eta_c)(z_t - z_c)) d\eta. \end{aligned}$$

We note that $|\eta_t - \eta_s|^\lambda = |\eta(x_t - x_s) + (1 - \eta)(y_t - y_s)|^\lambda \leq \rho_\tau |u - v|^{\tau\lambda}$. It follows that

$$[\mu]_{\tau(1+\lambda); [a, b]} \leq \|\nabla W\| (\rho_\tau \|z\|_{\infty; a, b} + |b - a|^{\tau(1-\lambda)} \|z\|_{\tau; a, b}).$$

On the other hand, it is obvious that $|\mu(a, b)| \leq \|\nabla W\| |b - a|^\tau \|z\|_{\infty; a, b}$. Hence, the estimate (3.4) implies

$$|z_b - z_a| \leq \|\nabla W\| |b - a|^\tau \|z\|_{\infty; a, b} + \kappa \|\nabla W\| |b - a|^{\tau+\lambda\tau} (\rho_\tau \|z\|_{\infty; a, b} + |b - a|^{\tau(1-\lambda)} \|z\|_{\tau; a, b}).$$

In other words,

$$\|z\|_{\tau; a, b} \leq A \|z\|_{\infty; a, b} + A \|z\|_{\tau; a, b} (b - a)^\tau,$$

where

$$A = \kappa \|\nabla W\| [1 + \rho_\tau T^{\lambda\tau}]. \quad (4.10)$$

Therefore, using the bound $\|z\|_{\infty;a,b} \leq |z_a| + \|z\|_{\tau;a,b}$ one gets

$$\|z\|_{\tau;a,b} \leq A|z_a| + 2A\|z\|_{\tau;a,b}(b-a)^\tau. \quad (4.11)$$

Now we shall use the above inequality to show our theorem. Choose a, b such that

$$|b-a| \leq \Delta = \left(\frac{1}{4A}\right)^{\frac{1}{\tau}}.$$

Then inequality (4.11) implies $\|z\|_{\tau;a,b} \leq 2A|z_a|$ for all $a < b$. By the definition of the Hölder norm, we see that if $|b-a| \leq \Delta$, then

$$\begin{aligned} \|z\|_{\infty;a,b} &\leq |z_a| + \|z\|_{\tau;a,b}(b-a)^\tau \\ &\leq |z_a| + 2A|z_a|\Delta^\tau \\ &\leq 2|z_a|. \end{aligned}$$

Divide the interval $[t_0, t_0 + T]$ into $n = [T/\Delta] + 1$ subintervals of length less or equal than Δ . Applying the previous inequality on the intervals $[t_0, t_0 + \Delta], [t_0 + \Delta, t_0 + 2\Delta], \dots, [t_0 + (n-1)\Delta, t_0 + n\Delta \wedge T]$, recursively, we obtain

$$\|z\|_{\infty,t_0,t_0+T} \leq 2^n |z_{t_0}|.$$

We can assume $\Delta \leq T$. Thus

$$n = [T/\Delta] + 1 \leq \frac{2T}{\Delta} = 2T (4A)^{\frac{1}{\tau}}.$$

This implies

$$\|z\|_{\infty, t_0, t_0+T} \leq 2^{2^{1+2/\tau} T A^{\frac{1}{\tau}}} |z_{t_0}|.$$

which yields the bound (4.9) on the interval $[t_0, t_0 + T]$. Estimates on $[t_0 - T, t_0]$ are analogous. \square

An immediate consequence of the theorem is the following uniqueness result.

Corollary 4.1.3. *Under the hypothesis of Theorem 4.1.2 the equation (4.1) has a unique solution.*

4.2 Compositions

Given a function $G : \mathbb{R}^2 \rightarrow \mathbb{R}^d$, we may define the Riemann-Stieltjes integral $\int_a^b G(ds, s)$ as the limit of Riemann sums

$$\sum_i G(t_i, t_{i-1}) - G(t_{i-1}, t_{i-1}).$$

The sewing lemma (Lemma 3.1.2) gives a sufficient condition so that the aforementioned limit exists, namely G satisfies

$$|G(s, s) - G(s, t) - G(t, s) + G(t, t)| \lesssim |t - s|^{1+\varepsilon}$$

for some $\varepsilon > 0$. In such case, Lemma 3.1.3 also allows one to choose Riemann sums with right-end points. In other words, the Riemann sums with right-end points

$$\sum_i G(t_i, t_i) - G(t_{i-1}, t_i)$$

also converges to the Riemann-Stieltjes integral $\int_a^b G(ds, s)$. In what follows, all integrals are understood as Riemann-Stieltjes integration, except for a few occasions, which we will indicate. The following result can be regarded as Itô formula or chain rule for compositions

of functions in the context of nonlinear Young integration.

Theorem 4.2.1. *Let F be a function in $C_{\text{loc}}^{(\tau_F, \lambda_F)}(\mathbb{R} \times \mathbb{R}^d)$ (i.e. F satisfies the condition **(W')** with τ_F and λ_F), g and x be Hölder continuous functions with exponents τ_g and τ respectively. We suppose that $\tau_F + \lambda_F \tau > 1$ and $\tau_g + \tau_F > 1$. The following integration by parts formula holds*

$$\int_0^T g(t) dF(t, x_t) = \int_0^T g(t) F(dt, x_t) + \int_0^T g(t) F(t, dx_t). \quad (4.12)$$

*In particular, suppose that F belongs to $C_{\text{loc}}^{\tau_F}(\mathbb{R}; C_{\text{loc}}^{1+\lambda_F}(\mathbb{R}^d))$, x is of the form $x_t = \int_a^t W(ds, \phi_s)$, where W satisfy the condition **(W')** with τ and λ , ϕ satisfy **(\phi)** with γ , $\tau + \lambda\gamma > 1$ and $\tau\lambda_F + \tau > 1$. Then (4.12) becomes*

$$\int_0^T g(t) dF(t, x_t) = \int_0^T g(t) F(dt, x_t) + \int_0^T g(t) (\nabla F)(t, x_t) W(dt, \phi_t). \quad (4.13)$$

An important consequence of (4.13) is when g is a constant function

$$F(b, x_b) - F(a, x_a) = \int_a^b F(dt, x_t) + \int_a^b (\nabla F)(t, x_t) W(dt, \phi_t). \quad (4.14)$$

Proof. We choose a compact set K such that K contains $\{x_t, 0 \leq t \leq T\}$ and denote $\|F\| = \|F\|_{\tau_F, \lambda_F; [0, T] \times K}$. We put

$$\mu(a, b) = g(b)F(b, x_b) - g(b)F(a, x_b),$$

$$\nu(a, b) = g(a)F(a, x_b) - g(a)F(a, x_a),$$

$$\vartheta(a, b) = g(a)F(b, x_b) - g(a)F(a, x_a).$$

For every $a < c < b$, we have

$$\begin{aligned}
& |\mu(a, b) - \mu(a, c) - \mu(c, a)| \\
&= | -g(b)F(a, x_b) - g(c)F(c, x_c) + g(c)F(a, x_c) + g(b)F(c, x_b) | \\
&\leq |g(c)| |F(a, x_b) - F(c, x_c) + F(a, x_c) + F(c, x_b)| \\
&\quad + |g(c) - g(b)| |F(c, x_c) - F(a, x_c)| \\
&\leq \|g\|_\infty \|F\| \|x\|_\tau^{\lambda_F} |b - a|^{\tau_F + \lambda_F \tau} + \|g\|_{\tau_g} \|F\| |b - a|^{\tau_g + \tau_F},
\end{aligned}$$

and

$$\begin{aligned}
& |\nu(a, b) - \nu(a, c) - \nu(c, a)| \\
&= |g(a)F(a, x_b) - g(a)F(a, x_c) - g(c)F(c, x_b) + g(c)F(c, x_c)| \\
&\leq |g(c)| |F(a, x_b) - F(a, x_c) - F(c, x_b) + F(c, x_c)| \\
&\quad + |g(a) - g(c)| |F(a, x_b) - F(a, x_c)| \\
&\lesssim \|g\|_\infty \|F\| |b - a|^{\tau_F + \lambda_F \tau} + \|g\|_{\tau_g} \|F\| \|x\|_\tau^{\lambda_F} |b - a|^{\tau_g + \lambda_F \tau}.
\end{aligned}$$

Hence, from Lemmas 3.1.2 and 3.1.3, $\mathcal{J}_0^T \mu = \int_0^T g(t)F(dt, x_t)$ and $\mathcal{J}_0^T \nu = g(t)F(t, dx_t)$.

On the other hand,

$$\begin{aligned}
& |\vartheta(a, b) - \mu(a, b) - \nu(a, b)| \\
&= |[g(a) - g(b)][F(b, x_b) - F(a, x_b)]| \leq \|g\|_{\tau_g} \|F\| |b - a|^{\tau_g + \tau_F}.
\end{aligned}$$

This together with Lemma 3.1.3 implies (4.12).

To prove (4.13), it suffices to show

$$\int_0^T g(t)F(t, dx_t) = \int_0^T g(t)(\nabla F)(t, x_t)W(dt, \phi_t). \quad (4.15)$$

We put

$$\tilde{v}(a, b) = g(a) \nabla F(a, x_a) [W(b, \phi_a) - W(a, \phi_a)].$$

Then we write

$$\begin{aligned} v(a, b) &= g(a) \int_0^1 \nabla F(a, \eta x_a + (1 - \eta)x_b) d\eta (x_a - x_b) \\ &= g(a) \int_0^1 \nabla F(a, \eta x_a + (1 - \eta)x_b) d\eta \int_a^b W(ds, \phi_s). \end{aligned}$$

Using the estimate (3.5), we obtain

$$\begin{aligned} &|v(a, b) - \tilde{v}(a, b)| \\ &\leq |g(a) \int_0^1 [\nabla F(a, \eta x_a + (1 - \eta)x_b) - \nabla F(a, x_a)] d\eta \int_a^b W(ds, \phi_s)| \\ &\quad + |g(a) \nabla F(a, x_a) [\int_a^b W(ds, \phi_s) - W(b, \phi_b) + W(a, \phi_a)]| \\ &\lesssim |b - a|^{\lambda_F \tau + \tau} + |b - a|^{\tau + \lambda \gamma}. \end{aligned}$$

Identity (4.15) follows from Lemma 3.1.3 and the previous estimate. \square

4.3 Regularity of flow

Throughout the current section, we assume the hypothesis of Theorem 4.1.2. This assumption guarantees that $\varphi(t, x)$, the solution to

$$\varphi(t, x) = x + \int_0^t W(ds, \varphi(s, x))$$

is unique. Moreover, by the result in Section 4.1, for fixed t , $\varphi(t, \cdot)$ is an automorphism on \mathbb{R}^d , its inverse is $\varphi(t, \cdot)^{-1} = \varphi(-t, \cdot)$. Hence, the family $\{\varphi(t, \cdot) : t \in \mathbb{R}\}$ forms a flow of homeomorphism, i.e. it satisfies the following properties:

- $\varphi(t + s, \cdot) = \varphi(t, \varphi(s, \cdot))$ holds for all s, t ,
- $\varphi(0, \cdot)$ is the identity map,
- the map $\varphi(t, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a homeomorphism for all t .

Moreover, one can show that $\varphi(t, \cdot)$ is indeed a diffeomorphism.

Theorem 4.3.1. *Assume the hypothesis of Theorem 4.1.2. For any t in \mathbb{R} , the map $\varphi(t, \cdot)$ is a diffeomorphism. The following conclusions hold*

(i) *The gradient of φ_t at x , denoted by $\nabla\varphi(t, x) = \{\partial_j\varphi^i(t, x)\}_{i,j}$ satisfies the equation*

$$\partial_i\varphi^\bullet(t, x) = \delta_{\bullet i} + \int_0^t \partial_k W^\bullet(ds, \varphi(s, x)) \partial_k \varphi^i(s, x) \quad (4.16)$$

where δ_{ij} is the Kronecker symbol. Equation (4.16) can be written in short

$$\nabla\varphi(t, x) = I_d + \int_0^t \nabla W(ds, \varphi(s, x)) \nabla\varphi(s, x).$$

(ii) *For every t and x , the matrix $\nabla\varphi(t, x)$ is invertible and its inverse $M(t, x) = [\nabla\varphi(t, x)]^{-1}$ satisfies the equation*

$$M(t, x)^{j\bullet} = \delta_{j\bullet} - \int_0^t M(s, x)^{jk} \partial_\bullet W^k(ds, \varphi(s, x)) \quad (4.17)$$

or in short

$$M(t, x) = I_d - \int_0^t M(s, x) \nabla W(ds, \varphi(s, x)).$$

(iii) *φ is jointly Hölder continuous of order $(\tau, 1)$. That is*

$$|\varphi(s, x) - \varphi(s, y) - \varphi(t, x) + \varphi(t, y)| \lesssim |t - s|^\tau |x - y| \quad (4.18)$$

(iv) Let $J(t, x)$ denote the determinant of $\nabla\varphi(t, x)$. Then J satisfies the following scalar linear equation

$$J(t, x) = 1 + \int_0^t J(s, x) \operatorname{Div} (W(ds, \varphi(s, x))). \quad (4.19)$$

(v) The flow $\varphi(t, x)$ is a Lagrangian flow, namely there exists a constant L such that

$$\mathcal{L}^d(\varphi(t, \cdot)^{-1}(A)) \leq L \mathcal{L}^d(A) \quad \text{for every Borel set } A \subseteq \mathbb{R}^d \quad (4.20)$$

where \mathcal{L}^d is the Lebesgue measure on \mathbb{R}^d .

Proof. Let e be a unit vector in \mathbb{R}^d . For each h in \mathbb{R} , we denote

$$\eta_t^h = \frac{1}{h}(\varphi(t, x + he) - \varphi(t, x)).$$

To prove (i), it is sufficient to show that for every sequence h_n converging to 0, there is a subsequence h_{n_k} such that $\eta^{h_{n_k}}$ converges to the solution of the following equation

$$\eta_t = e + \int_0^t \nabla W(ds, \varphi(s, x)) \eta_s. \quad (4.21)$$

We remark that the equation (4.21) is linear and the existence and uniqueness of solution in $C^\tau(\mathbb{R})$ follows from our method discussed in Section 4.1. From Theorem 4.1.2 we see that

$$\|\eta^h\|_{\tau; K} \leq \kappa_K$$

uniformly in h for every compact interval K in \mathbb{R} . Hence, by the Arzelà-Ascoli theorem, there is a subsequence, still denoted by h_n such that η^{h_n} converges to η in $C^{\tau'}(K)$ for any arbitrary $\tau' < \tau$. On the other hand, we notice that η^h satisfies

$$\eta_t^h = e + \int_0^1 d\tau \int_0^t \nabla W(ds, \tau\varphi(s, x + he) - (1 - \tau)\varphi(s, x)) \eta_s^h. \quad (4.22)$$

Passing through the limit $h_n \rightarrow 0$, we see that η satisfies the equation (4.21) and then (i) follows. Assertion (iii) is a consequence of the estimate (4.9) in Theorem 4.1.2. In fact,

$$\begin{aligned} |\varphi(s, x) - \varphi(s, y) - \varphi(t, x) + \varphi(t, y)| &\leq \|\varphi(\cdot, x) - \varphi(\cdot, y)\|_{\tau; [s, t]} |t - s|^\tau \\ &\lesssim |t - s|^\tau |x - y|. \end{aligned}$$

Assertion (iv) follows from the Itô formula (4.14) applied to $J(t, x) = \det(\nabla\varphi(t, x))$ and the Jacobi's formula

$$d \det(M) = \det(M) \operatorname{tr}(M^{-1} dM).$$

To prove (v), we notice that the equation (4.19) can be solved explicitly thanks to (4.14)

$$J(t, x) = \exp \int_0^t \operatorname{Div} (W(dt, \varphi(t, x))). \quad (4.23)$$

Therefore, from (3.5), we obtain

$$|J(t, x)^{-1}| \leq e^{\kappa|t|^\tau}.$$

Together with the area formula

$$\mathcal{L}^d(\varphi(t, \cdot)^{-1}(A)) = \int_{\varphi(-t, A)} dx = \int_A |\det(\nabla\varphi)(-t, x)| dx$$

this estimate implies (4.20). □

4.4 Transport differential equation

As an application of the above Itô formula (4.13) and flow property (Theorem 4.3.1), we study the following transport differential equation in *Hölder media*. Specifically, let $W : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfy the conditions in Theorem 4.1.2. Consider the following first

order partial differential equations (transport equation in Hölder media W)

$$\frac{\partial}{\partial t} u(t, x) + \left(\frac{\partial}{\partial t} W(t, x) \right) \cdot \nabla u(t, x) = 0. \quad (4.24)$$

Here ∇ is the gradient operator (with respect to spatial variables). Since W is only Hölder continuous in time, the equation (4.24) is only formal. We can however define solutions in integral form. More precisely, a continuous function $u : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ is called a solution to (4.24) with the initial condition $u(0, x) = h(x)$ if it is differentiable with respect to $x \in \mathbb{R}^d$ and the following equation holds.

$$u(t, x) = h(x) - \int_0^t \nabla u(s, x) W(ds, x) \quad \forall t \geq 0, x \in \mathbb{R}^d. \quad (4.25)$$

Theorem 4.4.1. *Assuming W satisfies the conditions in Theorem 4.1.2. Let h be a function in $C_{\text{loc}}^{1+\lambda_0}(\mathbb{R}^d)$ where λ_0 satisfies $(1 + \lambda_0)\tau > 1$. Let $\varphi(t, x)$ be the unique solution to*

$$\varphi(t, x) = x + \int_0^t W(ds, \varphi(s, x)), \quad \forall t \geq 0.$$

Let $\psi(t, x)$ be the inverse of φ as a function $x \in \mathbb{R}^d$ to \mathbb{R}^d . Namely, $\varphi(t, \psi(t, x)) = x$ for all $t \geq 0, x \in \mathbb{R}^d$. Then the function u defined by

$$u(t, x) = h(\psi(t, x))$$

is a solution to the above transport equation.

Proof. From Theorem 4.3.1 such $\psi(t, x)$ exists and both $\varphi(t, x)$ and $\psi(t, x)$ are differentiable with respect to x . Differentiate $\varphi(t, \psi(t, x)) = x$ with respect to x and we see that

$$(\nabla \varphi)(t, \psi(t, x)) \nabla \psi(t, x) = I,$$

or

$$(\nabla\psi(t, x))^{-1} = (\nabla\varphi)(t, \psi(t, x)).$$

Let $\rho(r) = \varphi(r, \psi(r, x))$, $0 \leq r < \infty$. Thanks to Theorem 4.3.1(iii), Itô formula (4.13) is applicable. More precisely, for any C^τ -function $g(r)$, we have

$$\int_0^t g(r) d\rho(r) = \int_0^t g(r) \varphi(dr, \psi(r, x)) + \int_0^t g(r) (\nabla\varphi)(r, \psi(r, x)) \psi(dr, x).$$

Since $\rho(r) = x$, we have $d\rho(r) = 0$. Thus

$$\int_0^t g(r) (\nabla\varphi)(r, \psi(r, x)) \psi(dr, x) = - \int_0^t g(r) \varphi(dr, \psi(r, x)). \quad (4.26)$$

Now the Itô formula (4.14) applied to $h(\psi(t, x))$ yields

$$\begin{aligned} u(t, x) &= h(\psi(t, x)) = h(x) + \int_0^t (\nabla h)(\psi(r, x)) \psi(dr, x) \\ &= h(x) + \int_0^t \nabla [h(\psi(r, x))] (\nabla\psi(r, x))^{-1} \psi(dr, x) \\ &= h(x) + \int_0^t \nabla u(r, x) (\nabla\psi(r, x))^{-1} \psi(dr, x) \\ &= h(x) + \int_0^t \nabla u(r, x) (\nabla\varphi)(r, \psi(r, x)) \psi(dr, x). \end{aligned}$$

Using the equation (4.26) for $g(r) = \nabla u(r, x)$, we have

$$\begin{aligned} u(t, x) &= h(x) - \int_0^t \nabla u(r, x) \varphi(dr, \psi(r, x)) \\ &= h(x) - \int_0^t \nabla u(r, x) W(dr, \varphi(r, \psi(r, x))) \\ &= h(x) - \int_0^t \nabla u(r, x) W(dr, x). \end{aligned}$$

This completes the proof of the theorem. □

We also have the following uniqueness result.

Theorem 4.4.2. *Assuming W satisfies the conditions in Theorem 4.1.2. Let λ_0 be in $(0, 1]$ such that $(\lambda_0 + 1)\tau > 1$. Equation (4.25) has unique solution in the class $C_{\text{loc}}^{(\tau, \lambda_0)}(\mathbb{R} \times \mathbb{R}^d)$. More precisely, suppose u belongs to $C_{\text{loc}}^{(\tau, \lambda_0)}(\mathbb{R} \times \mathbb{R}^d)$ and satisfies (4.25), then u is uniquely defined by the relation $u(t, x) = h(\psi(t, x))$, where φ and ψ are the functions defined in Theorem 4.4.1.*

Proof. Let u be a solution to (4.25). Applying Itô formula (4.14) for the function $u(t, \varphi(t, x))$ we have

$$u(t, \varphi(t, x)) - h(x) = \int_0^t u(ds, \varphi(s, x)) + \int_0^t \nabla u(s, \varphi(s, x))W(ds, \varphi(s, x)).$$

It suffices to show the right hand side vanishes. In other words the following relation between the two nonlinear Young integrals holds

$$\int_0^t u(ds, \varphi(s, x)) = - \int_0^t \nabla u(s, \varphi(s, x))W(ds, \varphi(s, x)). \quad (4.27)$$

For clarity, we will omit x in the notations. We put

$$\begin{aligned} \mu_1(a, b) &= u(b, \varphi_a) - u(a, \varphi_a), \\ \mu_2(a, b) &= \nabla u(a, \varphi_a)[W(b, \varphi_a) - W(a, \varphi_a)]. \end{aligned}$$

Since u satisfies the equation (4.25), we can write

$$\mu_1(a, b) = - \int_a^b \nabla u(s, \varphi_a)W(ds, \varphi_a).$$

Thus

$$\mu_1(a, b) + \mu_2(a, b) = - \int_a^b \nabla u(s, \varphi_a)W(ds, \varphi_a) + \nabla u(a, \varphi_a)[W(b, \varphi_a) - W(a, \varphi_a)].$$

The estimate (3.4) (or (3.5)) implies

$$|\mu_1(a, b) + \mu_2(a, b)| \lesssim |b - a|^{2\tau}.$$

Since $2\tau > 1$, Lemma 3.1.3 yields $\mathcal{J}_0^t \mu_1 = -\mathcal{J}_0^t \mu_2$. This completes the proof after observing that the aforementioned identity is exactly the same as (4.27). \square

Remark 4.4.3. In the context of ordinary differential equation of the type

$$\frac{dX}{dt}(t, x) = b(t, X(t, x)),$$

with non-regular vector field b , existence and uniqueness and stability of regular Lagrangian flows were proved by R.J. DiPerna and P.-L. Lions ([29]) for Sobolev vector fields with bounded divergence. This result has been extended by L. Ambrosio ([3]) to BV coefficients with bounded divergence. In [20], it is shown that under slightly relaxed assumptions many of the ODE results of DiPerna-Lions theory can be recovered, from a priori estimates, similar to (4.20). The current chapter proposes another extension of this theory, where the vector field is distribution (rough) in time (derivative of a Hölder continuous function) and smooth in space. It is also interesting to extend the results presented here for vector fields which are rougher in time (see e.g. [63] for the linear case) or which are both rough in time and in space.

Chapter 5

Feynman-Kac formula, a pathwise approach

In this chapter we shall study the stochastic parabolic equation with Hölder continuous noise in a Hölder random media (see equation (5.4) below). A feature of this problem is that for the noise we don't assume any Hölder continuity in time variable. To make up for lack of regularity in time, we assume some regularity on spatial variables. In this case, the method presented in this chapter works for each sample path of the noise.

Throughout the current chapter, T is a fixed positive time. To describe the noise, we introduce the following space. Let β be a fixed non-negative number. We say that f is in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$ if it belongs to $C([0, T], C_{\text{loc}}^{1+\alpha}(\mathbb{R}^d))$ and satisfies the following condition

$$[\nabla f]_{\beta, \alpha} := \sup_{\substack{t \in [0, T]; \\ x, y \in \mathbb{R}^d; x \neq y}} \frac{|\nabla f(t, x) - \nabla f(t, y)|}{|x - y|^\alpha (1 + |x|^\beta + |y|^\beta)} < \infty. \quad (5.1)$$

We notice that the condition (5.1) implies the growth conditions on ∇f and f . More precisely, one has

$$[\nabla f]_{\alpha+\beta, \infty} := \sup_{t \in [0, T], x \in \mathbb{R}^d} \frac{|\nabla f(t, x)|}{1 + |x|^{\alpha+\beta}} < \infty, \quad (5.2)$$

and

$$[f]_{\alpha+\beta+1,\infty} := \sup_{t \in [0, T], x \in \mathbb{R}^d} \frac{|f(t, x)|}{1 + |x|^{\alpha+\beta+1}} < \infty. \quad (5.3)$$

It is easy to see that $\|f\|_{C_\beta^{0,1+\alpha}} := [f]_{\alpha+\beta+1,\infty} + [\nabla f]_{\alpha+\beta,\infty} + [\nabla f]_{\beta,\alpha}$ forms a norm on $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. In the rest of this chapter, we denote

$$C_\beta^{0,1+\alpha^-} = \bigcap_{0 < \alpha' < \alpha} C_\beta^{0,1+\alpha'}([0, T] \times \mathbb{R}^d).$$

Similar to the classical Hölder spaces, the space of smooth functions is not dense in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. However, we can still approximate a function in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$ by smooth functions with a little trade off in spatial regularity. More precisely, let η be function in $C_c^\infty(\mathbb{R}^{d+1})$ supported in $(-1, 1)^{d+1}$ and $\iint \eta(t, x) dt dx = 1$. For $\epsilon > 0$, we put $\eta_\epsilon(t, x) = \epsilon^{-d-1} \eta(\epsilon^{-1}(t, x))$. Let f be in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$, we define $f_\epsilon(t, x) = (f * \eta_\epsilon)(t, x)$. It is clear that f_ϵ belongs to $C_c^\infty(\mathbb{R}^{d+1})$. In addition, we have the following result.

Lemma 5.0.4. *For every $\alpha' < \alpha$, $[\nabla f_\epsilon - \nabla f]_{\beta,\infty}$ and $[\nabla f_\epsilon - \nabla f]_{\beta,\alpha'}$ converge to 0 as ϵ goes to 0.*

Proof. We have

$$\begin{aligned} |\nabla f_\epsilon(t, x) - \nabla f(t, x)| &\leq \iint |\nabla f(t, z) - \nabla f(t, x)| \eta_\epsilon(t, x - z) dt dz \\ &\leq [\nabla f]_{\beta,\alpha} \iint |x - z|^\alpha (1 + |x|^\beta + |z|^\beta) \eta_\epsilon(t, x - z) dt dz \\ &\lesssim [\nabla f]_{\beta,\alpha} \epsilon^\alpha (1 + |x|^\beta), \end{aligned}$$

which implied $[\nabla f_\epsilon - \nabla f]_{\beta,\infty} \rightarrow 0$. This also implies

$$|\nabla f_\epsilon(t, x) - \nabla f_\epsilon(t, y) - \nabla f(t, x) + \nabla f(t, y)| \lesssim [\nabla f]_{\beta,\alpha} \epsilon^\alpha (1 + |x|^\beta + |y|^\beta).$$

On the other hand

$$\begin{aligned}
|\nabla f_\epsilon(t, x) - \nabla f_\epsilon(t, y)| &\leq \iint |\nabla f(t, x - z) - \nabla f(t, y - z)| \eta_\epsilon(t, z) dt dz \\
&\leq [\nabla f]_{\beta, \alpha} |x - y|^\alpha \iint (1 + |x - z|^\beta + |y - z|^\beta) \eta_\epsilon(t, z) dt dz \\
&\lesssim [\nabla f]_{\beta, \alpha} |x - y|^\alpha (1 + |x|^\beta + |y|^\beta),
\end{aligned}$$

thus

$$|\nabla f_\epsilon(t, x) - \nabla f_\epsilon(t, y) - \nabla f(t, x) + \nabla f(t, y)| \lesssim [\nabla f]_{\beta, \alpha} |x - y|^\alpha (1 + |x|^\beta + |y|^\beta).$$

Interpolating these two bounds, we get

$$|\nabla f_\epsilon(t, x) - \nabla f_\epsilon(t, y) - \nabla f(t, x) + \nabla f(t, y)| \lesssim [\nabla f]_{\beta, \alpha} \epsilon^{\alpha - \alpha'} |x - y|^{\alpha'} (1 + |x|^\beta + |y|^\beta)$$

for every $\alpha' < \alpha$. This implies $[\nabla f_\epsilon - \nabla f]_{\beta, \alpha'} \rightarrow 0$. \square

In Chapter 6 we shall give conditions on the covariance of a Gaussian field $W(t, x)$ such that it is in $C_\beta^{0, 1+\alpha}([0, T] \times \mathbb{R}^d)$.

Assume that W belongs to the space $C_\beta^{0, 1+\alpha}([0, T] \times \mathbb{R}^d)$, throughout this chapter, we denote $W_n = W * \eta_{1/n}$. We consider the following parabolic equation with multiplicative noise:

$$\partial_t u + Lu + u \partial_t W = 0, \quad u(T, x) = u_T(x), \quad (5.4)$$

where the terminal function u_T is assumed to be measurable with polynomial growth and L is a second order differential operator of the form

$$L = \frac{1}{2} \sum_{i, j=1}^d a^{ij}(t, x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^d b^i(t, x) \partial_{x_i}. \quad (5.5)$$

Here the novelty is that we allow the coefficients $a^{ij}(t, x) = a^{ij}(t, x, W)$ and $b^i(t, x) =$

$b^i(t, x, W)$ depend on W . Since we are going to solve the equation and to establish a Feynman-Kac type formula for every sample paths of W , we omit the explicit dependence of a^{ij} and b^i on W . Notice that with a time reversal $t \rightarrow T - t$, we can solve the stochastic parabolic equation with initial condition:

$$\partial_t u = Lu - u \partial_t W, \quad u(0, x) = u_0(x).$$

The stochastic differential equations with random coefficients have been studied in a large amount of papers. For example, it has been used in the modeling of the pressure in an oil reservoir with a log normal random permeability in [53] (see in particular the references therein). Recently, there have been great amount of research work on uncertainty quantization from the numerical computation community. Many different types of stochastic partial differential equations with random coefficients have been studied. Let us only mention the books [43], [97], and the references therein. Since the classical Feynman-Kac formula has already experienced many applications including the so-called Monte-Carlos particle approximation (see [26, 27]), we expect that the Feynman-Kac formula we obtained will be a significant addition to this literature particularly in the use of Monte-Carlo method for the computations.

We assume the following conditions on the operator L appearing in the equation (5.4).

(L1) L is uniformly elliptic, that is there exist positive numbers λ and Λ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^d a^{ij}(t, x) \xi^i \xi^j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

(L2) For every t , the coefficients $a(t, \cdot)$ belong to $C_b^{2+\alpha}(\mathbb{R}^d)$ with bounded derivatives uniformly in t . That is

$$\sup_t \|a(t, \cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} \leq \Lambda.$$

(L3) b is Lipschitz continuous and has linear growth, that is, there exists a positive constant $\kappa(b)$ such that

$$\begin{aligned} \sup_t |b^i(t, x)| &\leq \kappa(b)(1 + |x|), \quad \forall x \in \mathbb{R}^d, \\ \sup_t |b^i(t, y) - b^i(t, x)| &\leq \kappa(b)|y - x|, \quad \forall x, y \in \mathbb{R}^d. \end{aligned}$$

Under our conditions on W , it turns out that we can define the Feynman-Kac solution to equation (5.4), namely,

$$u(r, x) = \mathbb{E}^B \left[u_T(X_T^{r,x}) \exp \left\{ \int_r^T W(ds, X_s^{r,x}) \right\} \right],$$

where $\{X_s^{r,x}, s \geq r\}$ is the diffusion process generated by L starting from x at time r . More precisely, for every $r \leq t \leq T$ and $x \in \mathbb{R}^d$, let $X_t^{r,x}$ be the diffusion process given by the stochastic differential equation

$$dX_t^{i,r,x} = \sigma^{ij}(t, X_t^{r,x}) \delta B_t^j + b^i(t, X_t^{r,x}) dt, \quad X_r^{r,x} = x, \quad (5.6)$$

where σ is the square root matrix of a , namely, $a^{ij} = \sum_{k=1}^d \sigma^{ik} \sigma^{jk}$ and δB_t denotes the Itô differential. We will occasionally omit the index r, x and write X_s for $X_s^{r,x}$. Under conditions **(L1)**-**(L3)**, it is well-known that the diffusion process $X_t^{r,x}$ exists and has finite moments of all orders.

Equation (5.4) with W replaced by W_n is classic and one can obtain a smooth solution u_n (see for instance [70] where a more general situation is studied). The main result of the current chapter is to show that u_n converges to the Feynman-Kac solution u defined above. There are three main tasks to be accomplished:

- (i) One needs to define the nonlinear integration $\int W(ds, X_s)$. Since here W is only continuous in time, this integration is different from the Young integration considered in Section 3.1.

- (ii) One needs to show exponential integrability of $\int W(ds, X_s)$. In particular, the function u defined by Feynman-Kac formula is well-defined.
- (iii) One needs to show that the exponential functional of this integration is stable under approximations by smooth functions.

The outline of this chapter is as follows. In Section 5.1, we define the nonlinear stochastic integration $\int W(ds, X_s)$ and show that it has finite moment of all orders. Exponential integrability is obtained if W has strictly sub-quadratic growth, namely, if α and β in (5.1)-(5.3) satisfy $\beta + \alpha < 1$. In Section 5.2, we show that the Feynman-Kac solution is indeed a solution in certain sense. When W has more regularity in time such as in the case of Brownian sheets or fractional Brownian sheets, one can use this regularity to reduce the regularity requirement in space. This case is considered in Section 5.3 when W satisfies the conditions in Section 3.1. Along the way, we will make use of some fundamental estimates for exponential moment of various norms of the diffusion X on finite intervals. These estimates are stated and proved in Section 5.4.

In what follows, \mathbb{E} denotes the expectation with respect to a Brownian motion B , $\|\cdot\|_p$ denotes the L^p norm corresponding to \mathbb{E} .

5.1 Nonlinear Stochastic integral with diffusive integrand

Let $X_t^{r,x}$ satisfy (5.6) and let W be in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. We shall define a new nonlinear integration $\int_r^T W(ds, X_s^{r,x})$. If W is differentiable in time, the natural definition for this type of integration is $\int_r^T \partial_t W(s, X_s^{r,x}) ds$. If W satisfies **(W)** then we can define it as in Section 3.1. However, in this chapter, Hölder continuity of W on t is not assumed. On the other hand, we shall use the crucial fact that $\{X_t^{r,x}, t \geq r\}$ is a semimartingale. We first give the following definition.

Definition 5.1.1. Let W_n be a sequence of smooth functions with compact support converging to W in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. We define

$$\int_r^T W(ds, X_s^{r,x}) = \lim_n \int_r^T \partial_s W_n(s, X_s^{r,x}) ds \quad (5.7)$$

if the above limit exists in probability.

Of course, at the first glance, there is no reason for the limit in (5.7) to converge. We will show, however, that the above definition is well-defined, thanks to smoothing effect of the diffusion process $X_s^{r,x}$. Our first task is to find an appropriate representation for the integration $\int_r^T \partial_t W_n(s, X_s^{r,x}) ds$. To accomplish this, we consider the partial differential equation

$$(\partial_t + L_0)v_n(r, x) = -\partial_t W_n(r, x), \quad v(T, x) = -W_n(T, x),$$

where we recall that L is defined by (5.5) and

$$L_0 = L - b\nabla = \frac{1}{2} \sum_{i,j=1}^d a^{ij}(t, x) \partial_{x_i} \partial_{x_j}.$$

We could have chosen $L_0 = L$ but the above choice of L_0 will allow us to show exponential integrability later. Since W_n is a smooth function, the solution v_n is a strong solution which is at least twice differentiable in space and once differentiable in time. We then apply Itô formula to obtain

$$\begin{aligned} v_n(s, X_s^{r,x}) &= (\partial_t + L)v_n(s, X_s^{r,x}) ds + \sigma^{ij}(s, X_s^{r,x}) \partial_{x_i} v_n(s, X_s^{r,x}) \delta B_s^j \\ &= -\partial_t W_n(s, X_s^{r,x}) ds - b(s, X_s^{r,x}) \cdot \nabla v_n(s, X_s^{r,x}) ds \\ &\quad + \sigma^{ij}(s, X_s^{r,x}) \partial_{x_i} v_n(s, X_s^{r,x}) \delta B_s^j. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \int_r^T \partial_t W_n(s, X_s^{r,x}) ds &= W_n(T, X_T^{r,x}) + v_n(r, x) - \int_r^T b(s, X_s^{r,x}) \cdot \nabla v_n(s, X_s^{r,x}) ds \\ &\quad + \int_r^T \sigma^{ij}(s, X_s^{r,x}) \partial_{x_i} v_n(s, X_s^{r,x}) \delta B_s^j. \end{aligned} \quad (5.8)$$

Notice that the time derivative in W_n is transferred to the spatial derivative in v_n . The next task is to show that v_n and its derivative ∇v_n converge. This is accomplished by some estimates which are in the same spirit of the well-known Schauder estimates for parabolic equations in Hölder spaces. More precisely, we have

Lemma 5.1.2. *Suppose that W belongs to $C_{\text{loc}}^2(\mathbb{R}^{d+1})$ and satisfies*

$$[W]_{\beta_1, \infty} := \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \frac{|\nabla W(t, x)|}{1 + |x|^{\beta_1}} < \infty$$

and

$$[W]_{\beta_2, \alpha} := \sup_{0 \leq t \leq T} \sup_{x \neq y} \frac{|\nabla W(t, x) - \nabla W(t, y)|}{|x - y|^\alpha (1 + |x|^{\beta_2} + |y|^{\beta_2})} < \infty$$

for some non-negative numbers β_1, β_2 . Let v be a strong solution with polynomial growth to the partial differential equation

$$(\partial_t + L_0)v = -\partial_t W, \quad v(T, x) = -W(T, x). \quad (5.9)$$

Let $t \mapsto \varphi_t$ be the diffusion process generated by L_0 , that is

$$\varphi_t^{r,x} = x + \int_r^t \sigma(s, \varphi_s^{r,x}) \delta B_s, \quad t \geq r. \quad (5.10)$$

Then v is uniquely defined and verifies

$$(v + W)(r, x) = -\mathbf{E} \int_r^T L_0 W(s, \varphi_s^{r,x}) ds. \quad (5.11)$$

In addition, the following estimates hold

$$\sup_{x \in \mathbb{R}^d} \frac{|(v + W)(r, x)|}{1 + |x|^{\beta_1}} \leq c(\beta_1, \lambda, \Lambda)[(T - r)^{1/2} + (T - r)][\nabla W]_{\beta_1, \infty}, \quad (5.12)$$

$$\sup_{x \in \mathbb{R}^d} \frac{|\nabla(v + W)(r, x)|}{1 + |x|^{\beta_2}} \leq c(\alpha, \beta_2, \lambda, \Lambda)[(T - r)^{\alpha/2} + (T - r)^{\alpha/2+1/2}][\nabla W]_{\beta_2, \alpha}, \quad (5.13)$$

and for every $\alpha' \in (0, \alpha)$,

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \frac{|\nabla(v + W)(r, x) - \nabla(v + W)(r, y)|}{(1 + |x|^{\beta_2} + |y|^{\beta_2})|x - y|^{\alpha'}} \\ \leq c(\alpha', \alpha, \beta_2, \lambda, \Lambda)[(T - r)^{(\alpha - \alpha')/2} + (T - r)^{(\alpha - \alpha')/2+1/2}][\nabla W]_{\beta_2, \alpha}. \end{aligned} \quad (5.14)$$

The proof of this result, even though lengthy, is straight forward and is provided in details in Appendix 5.5.

Proposition 5.1.3. *Suppose that W belongs $C_{\beta}^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. Then there exists a C^1 -generalized solution v to the parabolic partial differential equation*

$$(\partial_t + L_0)v = -\partial_t W, \quad v(T, x) = -W(T, x), \quad (5.15)$$

such that for every $0 < \alpha' < \alpha$, the following estimates hold

$$[v + W]_{\alpha+\beta+1, \infty} \leq c(\alpha, \beta, \lambda, \Lambda)[\nabla W]_{\alpha+\beta, \infty}, \quad (5.16)$$

$$[\nabla(v + W)]_{\beta, \infty} \leq c(\alpha, \beta, \lambda, \Lambda)[\nabla W]_{\beta, \alpha}, \quad (5.17)$$

$$[\nabla(v + W)]_{\beta, \alpha'} \leq c(\alpha, \alpha', \beta, \lambda, \Lambda)[\nabla W]_{\beta, \alpha}. \quad (5.18)$$

As a consequence, v belongs to the space $C_{\beta}^{0,1+\alpha^-}([0, T] \times \mathbb{R}^d)$.

Proof. We recall that η is the bump function defined at the beginning of this section and $W_n = W * \eta_{1/n}$. Lemma 5.0.4 yields $[W_n - W]_{\beta, \infty}$ and $[W_n - W]_{\beta, \alpha}$ converge to 0 as $n \rightarrow \infty$.

Thanks to linearity of the equation (5.15), $v_n - v_m$ is a strong solution to

$$(\partial_t + L_0)(v_n - v_m) = -\partial_t(W_n - W_m), \quad (v_n - v_m)(T, x) = (W_n - W_m)(T, x).$$

The results in Lemma 5.1.2 (with $\beta_1 = \beta_2 = \beta$) imply

$$[(v_n + W_n) - (v_m + W_m)]_{\beta, \infty} \lesssim [\nabla W_n - \nabla W_m]_{\beta, \infty},$$

$$[\nabla(v_n + W_n) - \nabla(v_m + W_m)]_{\beta, \infty} \lesssim [\nabla W_n - \nabla W_m]_{\beta, \alpha},$$

and for every $\alpha' \in (0, \alpha)$,

$$[\nabla(v_n + W_n) - \nabla(v_m + W_m)]_{\beta, \alpha'} \lesssim [\nabla W_n - \nabla W_m]_{\beta, \alpha}.$$

As a consequence, v_n is a Cauchy sequence in $C([0, T], C^1(K))$ for every compact set K in \mathbb{R}^d . Thus v_n converges to v in $C([0, T], C^1(K))$ for every compact set K . It is then straightforward to verify that v is a weak solution to (5.15). The estimates (5.16), (5.17) and (5.18) follow from a limiting argument. \square

Theorem 5.1.4. *Suppose that W belongs to $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. Let v be the $C_\beta^{0,1+\alpha'}$ -generalized solution to (5.15) constructed in Proposition 5.1.3. Then for every $t \in [r, T]$, the integration $\int_r^t W(ds, X_s^{r,x})$ is well-defined (in the sense of Definition 5.1.1). Moreover, it has moment of all positive orders and satisfies*

$$\begin{aligned} \int_r^t W(ds, X_s^{r,x}) &= v(r, x) - v(t, X_t^{r,x}) \\ &\quad - \int_r^t b(s, X_s^{r,x}) \cdot \nabla v(s, X_s^{r,x}) ds + \int_r^t \sigma^{ij}(s, X_s^{r,x}) \partial_{x_i} v(s, X_s^{r,x}) \delta B_s^j. \end{aligned} \quad (5.19)$$

Proof. We consider $W_n = W * \eta_{1/n}$ as in the proof of the previous proposition. It follows

from Itô formula that (see (5.8))

$$\begin{aligned} & \int_r^t \partial_t W_n(s, X_s^{r,x}) ds \\ &= v_n(r, x) - v_n(t, X_t^{r,x}) - \int_r^t b(s, X_s^{r,x}) \cdot \nabla v_n(s, X_s^{r,x}) ds + \int_r^t \sigma^{ij}(s, X_s^{r,x}) \partial_{x_i} v_n(s, X_s^{r,x}) \delta B_s^j. \end{aligned}$$

Lemma 5.1.2 and Proposition 5.1.3 say that v_n (and its derivatives) has polynomial growth and converges in $C([0, T]; C_{\text{loc}}^{1+\alpha'}(\mathbb{R}^d))$ to v for every $\alpha' < \alpha$. Hence, the right hand side of the above formula is convergent in $L^p(\Omega)$ for every $p > 1$. Passing through the limit in n yields the equation (5.19). \square

Remark 5.1.5. To define $\int_r^t W(ds, X_s^{r,x})$, usually one needs some regularity of W on the temporal variable t . The equation (5.19) states that the requirement of the regularity on t can be transformed to the one on spatial variable x of another function v (defined by (5.9)). The use of v appears in many situations. If L_0 is replaced by L in the definition of v (e.g. equation (5.9)) and the terminal condition is replaced $v(0, x) = \delta(x - y)$ for any fixed y , then v corresponds to the transition density of the process X_s . This transition density is a fundamental concept in Markov processes and some other fields. It has also been used to simplify the proofs of a number of inequalities (see e.g. [30], [55]). The reason to use L_0 instead of L is that we don't need to assume condition on b to define v and that $\partial_i v$ will appear in (5.19) even we use L . The removal of temporal regularity also appears in other context. For example, to study the equation $dX_t = b(X_t) + dB_t$, the transformation $Y_t = X_t - B_t$ will satisfy $\dot{Y}_t = b(Y_t + B_t)$. The map $(t, x) \mapsto \int_0^t b(x + B_s) ds$, averaging along the trajectories of a Brownian motion, then has better regularity than that of b . In the field of stochastic differential equations, this phenomena has been observed by A. M. Davie in [25] and is recently studied in more depth in [15].

As a direct consequence, we obtain

Corollary 5.1.6. *Let W be in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. Then for every $\alpha' < \alpha$, $p > 2$ and K compact*

subset of \mathbb{R}^d ,

$$\begin{aligned} & \left\| \int_r^T W(ds, X_s^{r,x}) - \int_r^T W(ds, X_s^{r,y}) - \int_r^T W_n(ds, X_s^{r,x}) + \int_r^T W_n(ds, X_s^{r,y}) \right\|_p \\ & \leq C(\alpha, \alpha', \beta, \lambda, \Lambda, K, T, p) ([\nabla(W - W_n)]_{\beta, \infty} + [\nabla(W - W_n)]_{\beta, \alpha}) |x - y|^{\alpha'} \end{aligned}$$

Proof. Fix $\alpha' < \alpha$, $p > 2$ and K compact subset of \mathbb{R}^d . We put $g(r, x) = \int_r^T W(ds, X_s^{r,x})$, $g_n(r, x) = \int_r^T W_n(ds, X_s^{r,x})$ and $h = v - v_n$. From (5.19),

$$\|g(r, x) - g(r, y) - g_n(r, x) + g_n(r, y)\|_p \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= |h(r, x) - h(r, y)| \\ I_2 &= \|h(T, X_T^{r,x}) - h(T, X_T^{r,y})\|_p \\ I_3 &= \int_r^T \|(b \cdot \nabla h)(s, X_s^{r,x}) - (b \cdot \nabla h)(s, X_s^{r,y})\|_p ds \\ I_4 &= \left\| \int_r^T (\sigma \nabla h)(s, X_s^{r,x}) - (\sigma \nabla h)(s, X_s^{r,y}) \cdot \delta B_s \right\|_p. \end{aligned}$$

Proposition 5.1.3 implies

$$|\nabla h(z)| \lesssim ([\nabla(W - W_n)]_{\beta, \infty} + [\nabla(W - W_n)]_{\beta, \alpha})(1 + |z|^\beta),$$

and

$$|\nabla h(x) - \nabla h(y)| \lesssim [\nabla(W - W_n)]_{\beta, \alpha} (1 + |x|^{\beta'} + |y|^{\beta'}) |x - y|^{\alpha'}$$

where $\beta' = \beta + \alpha - \alpha'$. Therefore we can estimate

$$I_1 = \left| \int_0^1 \nabla h(\tau x + (1 - \tau)y) d\tau (x - y) \right| \lesssim \|W - W_n\| |x - y|,$$

$$I_2 = \left\| \int_0^1 \nabla h(\tau X_T^{r,x} + (1 - \tau)X_T^{r,y}) d\tau (X_T^{r,x} - X_T^{r,y}) \right\|_p \lesssim \|W - W_n\| |x - y|,$$

$$\begin{aligned} I_3 &\leq \int_r^T \left\| [b(s, X_s^{r,x}) - b(s, X_s^{r,y})] \nabla h(s, X_s^{r,x}) \right\|_p ds \\ &\quad + \int_r^T \left\| b(s, X_s^{r,y}) [\nabla h(s, X_s^{r,x}) - \nabla h(s, X_s^{r,y})] \right\|_p ds \\ &\lesssim \|W - W_n\| |x - y|^{\alpha'}, \end{aligned}$$

where we have used Hölder inequality. Similarly, we can estimate I_4 using Burkholder-Davis-Gundy inequality to get $I_4 \lesssim |x - y|^{\alpha'}$. From these bounds, the result follows. \square

Proposition 5.1.7. *Suppose W belongs to $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$ with $\alpha + \beta < 1$. Then $\int_r^t W(ds, X_s^{r,x})$ is exponentially integrable uniformly over compact sets. More precisely, for every $\gamma > 0$, K compact subset of \mathbb{R}^d*

$$\sup_{x \in K} \mathbb{E} \exp \left\{ \gamma \int_r^t W(ds, X_s^{r,x}) \right\} < \infty \quad (5.20)$$

for all $\gamma > 0$.

Proof. From (5.19) it suffices to show for every $\gamma > 0$,

$$\sup_{x \in K} \mathbb{E} \exp \left\{ \gamma \int_r^t \sigma^{ij}(s, X_s^{r,x}) \partial_i v(s, X_s^{r,x}) dB_s^j \right\} < \infty, \quad (5.21)$$

$$\sup_{x \in K} \mathbb{E} \exp \left\{ \gamma \int_r^t b(s, X_s^{r,x}) \cdot \nabla v(s, X_s^{r,x}) ds \right\} < \infty, \quad (5.22)$$

$$\sup_{x \in K} \mathbb{E} \exp \left\{ \gamma |v(t, X_t^{r,x})| \right\} < \infty. \quad (5.23)$$

Let $0 < \theta < 2$. We claim that

$$\sup_{x \in K} \mathbb{E} \exp \left\{ \gamma \int_r^t |X_s^{r,x}|^\theta ds \right\} < \infty, \quad \forall \gamma > 0. \quad (5.24)$$

In fact, by Jensen inequality

$$\mathbb{E} \exp \left\{ \gamma \int_r^t |X_s^{r,x}|^\theta ds \right\} \leq (T-r)^{-1} \int_r^T \mathbb{E} e^{\gamma(T-r)|X_s^{r,x}|^\theta} ds.$$

The quality on the right hand side is finite thanks to (5.41).

For any martingale M_t with $\mathbb{E} e^{2\langle M \rangle_t} < \infty$ we have

$$\begin{aligned} \mathbb{E} e^{M_t} &= \mathbb{E} e^{M_t - \langle M \rangle_t} e^{\langle M \rangle_t} \\ &\leq \left\{ \mathbb{E} e^{2M_t - 2\langle M \rangle_t} \right\}^{1/2} \left\{ \mathbb{E} e^{2\langle M \rangle_t} \right\}^{1/2} = \left\{ \mathbb{E} e^{2\langle M \rangle_t} \right\}^{1/2}. \end{aligned}$$

Thus we have

$$\mathbb{E} \exp \left\{ \gamma \int_r^t \sigma^{ij}(s, X_s^{r,x}) \partial_i v(s, X_s^{r,x}) \delta B_s^j \right\} \leq \left\{ \mathbb{E} \exp \left[2\gamma^2 \int_r^t (a^{ij} \partial_i v \partial_j v)(s, X_s^{r,x}) ds \right] \right\}^{1/2}.$$

Taking into account the growth property of ∇v (see (5.17)) and a , we have

$$\sup_{x \in K} \mathbb{E} \exp \left[2\gamma^2 \int_r^t (a^{ij} \partial_i v \partial_j v)(s, X_s^{r,x}) ds \right] \lesssim \sup_{x \in K} \mathbb{E} \exp \left[c \int_r^t |X_s^{r,x}|^{2(\alpha+\beta)} ds \right],$$

which together with the previous claim shows (5.21) since $2(\alpha + \beta) < 2$. Similarly, since b has linear growth

$$\sup_{x \in K} \mathbb{E} \exp \left[\gamma \int_r^t b(s, X_s^{r,x}) \cdot \nabla v(s, X_s^{r,x}) ds \right] \lesssim \sup_{x \in K} \mathbb{E} \exp \left[c \int_r^t |X_s^{r,x}|^{1+\alpha+\beta} ds \right],$$

which shows (5.22) since $1 + \alpha + \beta < 2$.

Using the growth property of v , i.e. the estimate (5.16),

$$\mathbb{E} \exp \left[\gamma |v(t, X_t^{r,x})| \right] \lesssim \mathbb{E} \exp \left[c |X_t^{r,x}|^{1+\alpha+\beta} \right],$$

which shows (5.23). □

Lemma 5.1.8. *Let W be in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$. Suppose $\alpha + \beta < 1$. For every $\gamma > 0$ and $r \in [0, T]$, we put $u(r, x) = \mathbb{E} \exp \left[\gamma \int_r^T W(ds, X_s^{r,x}) \right]$ and $u_n(r, x) = \mathbb{E} \exp \left[\gamma \int_r^T W_n(ds, X_s^{r,x}) \right]$. Then u_n converges to u in $C^{0,\alpha'}([0, T] \times K)$ for every $\alpha' < \alpha$ and K compact in \mathbb{R}^d .*

Proof. For a smooth function f , using fundamental theorem of calculus, we obtain

$$\begin{aligned} f(x) - f(a) - f(y) + f(b) &= \int_0^1 \int_0^1 f''(\xi) [\tau(x-y) + (1-\tau)(a-b)] d\eta d\tau(x-a) \\ &\quad + \int_0^1 f'(\theta) d\tau(x-a-y-b), \end{aligned}$$

where

$$\begin{aligned} \xi &= \tau\eta x + (1-\tau)\eta a + \tau(1-\eta)y + (1-\tau)(1-\eta)b, \\ \theta &= \tau y + (1-\tau)b. \end{aligned}$$

Thus, for every x, y in K , with $f(w) = \exp(\gamma w)$, we have

$$\begin{aligned} &u(r, x) - u_n(r, x) - u(r, y) + u_n(r, y) \\ &= \gamma^2 \mathbb{E} \int_0^1 \int_0^1 f(\xi) [\tau A(x, y) + (1-\tau)A_n(x, y)] d\eta d\tau B_n(x) + \gamma \mathbb{E} \int_0^1 f(\theta) d\tau C_n(x, y), \end{aligned} \tag{5.25}$$

where

$$\begin{aligned}
A(x, y) &= \int_r^T W(ds, X_s^{r,x}) - \int_r^T W(ds, X_s^{r,y}), \\
A_n(x, y) &= \int_r^T W_n(ds, X_s^{r,x}) - \int_r^T W_n(ds, X_s^{r,y}), \\
B_n(x) &= \int_r^T W(ds, X_s^{r,x}) - \int_r^T W_n(ds, X_s^{r,x}), \\
C_n(x, y) &= A(x, y) - A_n(x, y).
\end{aligned}$$

The random variables ξ and η are linear combinations of these terms. From Proposition 5.1.7, we know that moments of $f(\xi)$ and $f(\theta)$ are bounded uniformly in x and τ, η . On the other hand, from Corollary 5.1.6, for every $\alpha' < \alpha$ and $p > 2$

$$\begin{aligned}
\|A(x, y)\|_p &\lesssim |x - y|^{\alpha'}, \\
\sup_n \|A_n(x, y)\|_p &\lesssim |x - y|^{\alpha'}, \\
\limsup_{n \rightarrow 0} \sup_{x \in K} \|B_n(x)\| &= 0,
\end{aligned}$$

and

$$\|C_n(x, y)\|_p \lesssim ([\nabla(W - W_n)]_{\beta, \infty} + [\nabla(W - W_n)]_{\beta, \alpha}) |x - y|^{\alpha'}.$$

From (5.25), applying Hölder inequality and the above estimates for A, B, C we obtain

$$\begin{aligned}
&|u(r, x) - u_n(r, x) - u(r, y) + u_n(r, y)| \\
&\lesssim [\sup_{x \in K} \|B_n(x)\|_p + [\nabla(W - W_n)]_{\beta, \infty} + [\nabla(W - W_n)]_{\beta, \alpha}] |y - x|^{\alpha'}
\end{aligned}$$

for all x, y in K and $\alpha' < \alpha$. This completes the proof. \square

5.2 Feynman-Kac formula I

If W is a smooth function, then the classical Feynman-Kac formula asserts that

$$u(r, x) = \mathbb{E}^B \left[u_T(X_T^{r,x}) \exp \left(\int_r^T W(ds, X_s^{r,x}) \right) \right] \quad (5.26)$$

is the unique strong solution to (5.4). Indeed, suppose W is smooth and u is a strong solution to (5.4). Applying Itô formula to the process

$$t \mapsto u(t, X_t^{r,x}) \exp \left\{ \int_r^t \partial_t W(s, X_s^{r,x}) ds \right\}$$

we obtain

$$\begin{aligned} & \delta u(t, X_t^{r,x}) \exp \left\{ \int_r^t \partial_t W(s, X_s^{r,x}) ds \right\} \\ &= \exp \left\{ \int_r^t \partial_t W(s, X_s^{r,x}) ds \right\} (\partial_t + L + \partial_t W) u(t, X_t^{r,x}) dt \\ & \quad + \exp \left\{ \int_r^t \partial_t W(s, X_s^{r,x}) ds \right\} \sigma^{ij}(t, X_t^{r,x}) \partial_{x_i} u(t, X_t^{r,x}) \delta B_t^j \end{aligned}$$

Taking into account that $(\partial_t + L)u + \partial_t W u = 0$ and integrating over $[r, T]$, we have

$$\begin{aligned} & u_T(X_T^{r,x}) \exp \left\{ \int_r^T \partial_t W(s, X_s^{r,x}) ds \right\} - u(r, x) \\ &= \int_r^T \exp \left\{ \int_r^t \partial_t W(s, X_s^{r,x}) ds \right\} \sigma^{ij}(t, X_t^{r,x}) \partial_{x_i} u(t, X_t^{r,x}) \delta B_t^j \end{aligned}$$

Formula (5.26) is deduced by taking expectation on both sides.

Theorem 5.2.1. *Assume W belongs to $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$ with $\alpha + \beta < 1$. Let $W_n = W * \eta_{1/n}$. Let u_n be the solution to the parabolic equation*

$$\partial_t u_n + L u_n + u_n \partial_t W_n = 0, \quad u_n(T, x) = u_T(x).$$

Let u be the function defined in (5.26). Then u_n converges to u in $C^{0,\alpha'}([0, T] \times K)$ for every $\alpha' < \alpha$ and K compact set in \mathbb{R}^d . As a consequence, u belongs to $C_{\text{loc}}^{0,\alpha'}([0, T] \times \mathbb{R}^d)$ for all $\alpha' < \alpha$.

Proof. We notice that

$$u_n(r, x) - u(r, x) = \mathbb{E} \left\{ u_T(X_T^{r,x}) \left[\exp \left(\int_r^T W_n(ds, X_s^{r,x}) \right) - \exp \left(\int_r^T W(ds, X_s^{r,x}) \right) \right] \right\}.$$

This together with Lemma 5.1.8 yield the theorem. \square

We notice that if f and g are locally Hölder continuous functions on \mathbb{R}^d with exponents α and γ respectively. Suppose that f has compact support and $\alpha + \gamma > 1$. Then we can define the Young integral

$$\int_{\mathbb{R}^d} f(x)g(d^j x) = \int_{\mathbb{R}^d} f(x)g(x_1, \dots, x_{j-1}, dx_j, x_{j+1}, \dots, x_n)d\hat{x}_j$$

where $\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$.

We now show that if W is sufficiently regular in space, the Feynman-Kac solution u in (5.26) satisfies an equation derived from (5.4) by a change of variable. To better explain our procedure, let us first assume that W is smooth in space and time and u_T is also smooth. In such case, the equation (5.4) has unique smooth solution u such that

$$\partial_t u(t, x) + Lu(t, x) + u \partial_t W(t, x) = 0$$

for every $t \geq 0$ and $x \in \mathbb{R}^d$. We would like to obtain an equation of u such that the time derivative of W does not appear. To this end, we notice that

$$\partial_t u + u \partial_t W = e^{-W} \partial_t (u e^W).$$

Hence, multiply the equation with e^W and integrate in time, we obtain

$$u_t = e^{W_T - W_t} u_T + \int_t^T e^{W_s - W_t} L u_s ds. \quad (5.27)$$

In contrast with (5.4), the equation (5.27) does not contain the time derivative of W . One can also interpret (5.27) in weak sense. More precisely, the following result holds.

Theorem 5.2.2. *Assume W belongs to $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$ with $\alpha + \beta < 1$. Let u be the function defined in (5.26). Then there is a sequence of smooth functions W_n with compact supports convergent to W in $C_\beta^{0,1+\alpha}([0, T] \times \mathbb{R}^d)$ and a sequence of u_n such that u_n converges to u uniformly over all compact sets. Moreover, for every test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ the sequence*

$$\int_t^T \int_{\mathbb{R}^d} \partial_i [e^{W_n(s,x) - W_n(t,x)} \varphi(x)] a^{ij}(s, x) \partial_j u_n(s, x) dx ds$$

is convergent. If $\alpha > 1/2$, then we can identify the limit as

$$\int_t^T \int_{\mathbb{R}^d} \partial_i (e^{W(s,x) - W(t,x)} \varphi(x)) a^{ij}(s, x) u(s, x) dx ds.$$

In such case, u verifies the equation

$$\begin{aligned} \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}^d} e^{W(T,x) - W(t,x)} u_T(x) \varphi(x) dx \\ &+ \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} \partial_i (e^{W(s,x) - W(t,x)} \varphi(x)) a^{ij}(s, x) u(s, x) dx ds \\ &- \int_t^T \int_{\mathbb{R}^d} \partial_i \left(e^{W(s,x) - W(t,x)} \varphi(x) \left[b^i(s, x) - \frac{1}{2} \partial_j a^{ij}(s, x) \right] \right) u(s, x) dx ds. \end{aligned} \quad (5.28)$$

Proof. We recall that $W_n = W * \eta_{1/n}$ defined at the beginning of this section. Let u_n be the solution to the parabolic equation

$$\partial_t u_n + L u_n + u_n \partial_t W_n = 0, \quad u_n(T, x) = -W_n(T, x).$$

Then it is easily verified that

$$\begin{aligned} \int_{\mathbb{R}^d} u_n(t, x) \varphi(x) dx &= \int_{\mathbb{R}^d} e^{W_n(T, x) - W_n(t, x)} u_n(T, x) \varphi(x) dx \\ &+ \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} \partial_i [e^{W_n(s, x) - W_n(t, x)} \varphi(x)] a^{ij}(s, x) \partial_j u_n(s, x) dx ds \\ &+ \int_t^T \int_{\mathbb{R}^d} e^{W_n(s, x) - W_n(t, x)} \varphi(x) \left[b^i(s, x) - \frac{1}{2} \partial_j a^{ij}(s, x) \right] \partial_i u_n(s, x) dx ds. \end{aligned}$$

In other words,

$$\begin{aligned} \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} \partial_i [e^{W_n(s, x) - W_n(t, x)} \varphi(x)] a^{ij}(s, x) \partial_j u_n(s, x) dx ds \\ = \int_{\mathbb{R}^d} u_n(t, x) \varphi(x) dx - \int_{\mathbb{R}^d} e^{W_n(T, x) - W_n(t, x)} u_n(T, x) \varphi(x) dx \\ + \int_t^T \int_{\mathbb{R}^d} \partial_i \left(e^{W_n(s, x) - W_n(t, x)} \varphi(x) \left[b^i(s, x) - \frac{1}{2} \partial_j a^{ij}(s, x) \right] \right) u_n(s, x) dx ds. \end{aligned}$$

Since φ has compact support, it is clear that all the terms on the right hand side are convergent. This implies that

$$\int_t^T \int_{\mathbb{R}^d} \partial_i [e^{W_n(s, x) - W_n(t, x)} \varphi(x)] a^{ij}(s, x) \partial_j u_n(s, x) dx ds$$

is convergent. In case $\alpha > 1/2$, by Theorem 5.2.1, this limit is convergent in the context of Young integrations. Hence, taking the limit yields (5.28). \square

Remark 5.2.3. (i) The use of Itô formula in Section 5.1 is inspired from the work [37]. In that work, an Itô-Tanaka trick is applied to obtain some estimates to the commutator related to DiPerna-Lions' theory ([29]).

(ii) In the case W belongs to $C_{\text{loc}}^{0,2}([0, T] \times \mathbb{R}^d)$, the Itô-Tanaka formula (5.8) is negligible. In fact, using integration by part, one has

$$\int_r^T \partial_t W_n(s, X_s^{r,x}) ds = W_n(T, X_T^{r,x}) - W_n(r, x) - \int_r^T \nabla W_n(s, X_s^{r,x}) dX_s^{r,x}$$

where the last integral is in Stratonovich sense. By passing through the limit $n \rightarrow \infty$, we obtain

$$\int_r^T \partial_t W(s, X_s^{r,x}) ds = W(T, X_T^{r,x}) - W(r, x) - \int_r^T \nabla W(s, X_s^{r,x}) dX_s^{r,x}.$$

Assuming ∇W has linear growth in the spatial variable and $\nabla^2 W$ is globally bounded, one can also show exponential integrability

$$\mathbb{E}^B \exp \left[\int_r^T \partial_t W(s, X_s^{r,x}) ds \right] < \infty.$$

We consider u as in (5.26). Using the approximation as in the proof of Theorem 5.2.2, we can show that u verifies

$$\begin{aligned} \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}^d} e^{W(T,x)-W(t,x)} u(T, x) \varphi(x) dx \\ &\quad + \int_t^T \int_{\mathbb{R}^d} L^* [e^{W(s,x)-W(t,x)} \varphi(x)] u(s, x) dx ds \end{aligned}$$

for all test functions φ in $C_c^\infty(\mathbb{R}^d)$, where L^* is the adjoint of L .

5.3 Feynman-Kac formula II

In previous sections, to obtain the Feynman-Kac solution (5.26) (See Theorem 5.2.2) we assume that W is only continuous in time but satisfies (5.1)-(5.3) for $f = W$. This means that we suppose the the first spatial derivatives of W exist and are Hölder continuous in order to compensate the lack of regularity in time. For many other stochastic processes (such as Brownian sheet or fractional Brownian sheets), W is Hölder continuous in time. In this case, we may use this time regularity to relax the regularity requirement on space variable. In this section we obtain a Feynman-Kac formula for the solution to (5.4) when W satisfies the conditions of the type given in Section 3.1. For example, we do not require W to possess first derivatives. More precisely, we assume $W : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the

following condition.

(FK) There are constants $\tau, \lambda \in (0, 1]$ and $\beta > 0$ such that

$$\tau + \frac{1}{2}\lambda > 1, \quad \beta + \lambda < 2 \quad (5.29)$$

and such that the seminorm

$$\begin{aligned} & \|W\|_{\beta, \tau, \lambda} \\ := & \sup_{\substack{0 \leq s < t \leq T \\ x, y \in \mathbb{R}^d; x \neq y}} \frac{|W(s, x) - W(t, x) - W(s, y) + W(t, y)|}{(1 + |x| + |y|)^\beta |t - s|^\tau |x - y|^\lambda} \\ & + \sup_{\substack{0 \leq s < t \leq T \\ x \in \mathbb{R}^d}} \frac{|W(s, x) - W(t, x)|}{(1 + |x|)^{\beta + \lambda} |t - s|^\tau} + \sup_{\substack{0 \leq t \leq T \\ x, y \in \mathbb{R}^d; x \neq y}} \frac{|W(t, y) - W(t, x)|}{(1 + |x| + |y|)^\beta |x - y|^\lambda} \end{aligned} \quad (5.30)$$

is finite.

We continue to use the same notations introduced in previous sections. For example, $X_t = X_t^{r, x}$ denotes the solution to the equation (5.6). The objectives of this section are to show that the expression defined by (5.26) is well-defined under the above condition **(FK)** and is the solution to (5.4).

From $\tau + \frac{1}{2}\lambda > 1$, it follows that there is a $\gamma \in (0, 1/2)$ such that $\tau + \gamma\lambda > 1$. Since X_t is Hölder continuous of exponent γ , from Proposition 3.1.4, we know that $\int_r^T W(ds, X_s^{r, x})$ is well-defined and

$$\left| \int_r^T W(ds, X_s) \right| \leq C(1 + \|X\|_\infty^\beta)(1 + \|X\|_\gamma^\lambda). \quad (5.31)$$

Since $\beta + \lambda < 2$, Lemma 5.4.2 yields that

$$\mathbb{E} \exp \left\{ c \int_r^T W(ds, X_s) \right\} < \infty$$

for all $c \in \mathbb{R}$. Thus we have

Proposition 5.3.1. *Assume the conditions (L1)-(L3) are satisfied. Let (5.29)-(5.30) be satisfied. If there is an $\alpha_0 \in (0, 2)$ such that $|u_T(x)| \leq C_2 e^{C_1|x|^{\alpha_0}}$, then $u(r, x)$ defined by (5.26) is finite. Namely,*

$$u(r, x) = \mathbb{E}^B \left[u_T(X_T^{r,x}) \exp \left(\int_r^T W(ds, X_s^{r,x}) \right) \right] \quad (5.32)$$

is well-defined.

Now, let $W_n(t, x)$ be a sequence of functions in $C_0^\infty([0, T] \times \mathbb{R}^d)$ convergent to $W(t, x)$ under the norm $\|W\|_\infty + \|W\|_{\beta, \tau, \lambda}$. Denote $v_n(r, x) = \int_r^T W_n(ds, X_s^{r,x})$ and $v(r, x) = \int_r^T W(ds, X_s^{r,x})$ and $\tilde{v}_n(r, x) = v_n(r, x) - v(r, x)$. Thus, for any $0 \leq r < t \leq T$, we have

$$\begin{aligned} |\tilde{v}_n(t, x) - \tilde{v}_n(r, x)| &= \left| \int_t^T \tilde{W}_n(ds, X_s^{t,x}) - \int_r^T \tilde{W}_n(ds, X_s^{r,x}) \right| \\ &\leq \left| \int_r^t \tilde{W}_n(ds, X_s^{r,x}) \right| + \left| \int_t^T [\tilde{W}_n(ds, X_s^{t,x}) - \tilde{W}_n(ds, X_s^{r,x})] \right| \\ &=: I_1(r, t) + I_2(r, t). \end{aligned}$$

Applying the estimate in Proposition 3.1.4 to $\tilde{W}_n = W_n - W$, we obtain

$$\begin{aligned} \frac{I_1(r, t)}{(t-r)^\tau} &\leq \kappa \|\tilde{W}_n\|_{\tau, \lambda} (1 + \|X^{r,x}\|_\infty) \left[1 + \|X^{r,x}\|_\gamma (t-r)^{\lambda\gamma} \right] \\ &\leq C \|\tilde{W}_n\|_{\tau, \lambda} (1 + \|X^{r,x}\|_\infty) \left[1 + \|X^{r,x}\|_\gamma \right]. \end{aligned}$$

Lemma 5.4.2 states that $\|X^{r,x}\|_\infty$ is bounded in L^p for any $p \geq 1$. Thus

$$\lim_{n \rightarrow \infty} \mathbb{E}^B \left\{ \sup_{0 \leq r < t \leq T} \left| \frac{I_1(r, t)}{(t-r)^\tau} \right|^p \right\} = 0 \quad (5.33)$$

for any $p \geq 1$.

From Proposition 3.1.11 we have with $\tau + \theta\lambda\gamma > 1$,

$$\begin{aligned}
I_2(r, t) &\leq C \|\tilde{W}_n\|_{\beta, \tau, \lambda} \|X^{t,x} - X^{r,x}\|_\infty^\lambda (T-t)^\tau \\
&\quad + C \|\tilde{W}_n\|_{\beta, \tau, \lambda} \|X^{t,x} - X^{r,x}\|_\infty^{\lambda(1-\theta)} (T-t)^{\tau+\theta\lambda\gamma} \\
&\leq C \|\tilde{W}_n\|_{\beta, \tau, \lambda} \|X^{t,x} - X^{r,x}\|_\infty^{\lambda(1-\theta)} \left[1 + \|X^{t,x} - X^{r,x}\|_\infty^{\lambda\theta}\right]. \tag{5.34}
\end{aligned}$$

Notice that $X_s^{r,x} = X_s^{t, X_t^{r,x}}$. We have for any $p \geq 1$ and $\gamma' < 1$, by using the Markov property of the process $X_t^{r,x}$,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq r < t \leq T} \frac{\|X^{t,x} - X^{r,x}\|_\infty^p}{(t-r)^{\gamma'p/2}} &= \mathbb{E} \sup_{0 \leq r < t \leq T} \frac{\|X^{t,x} - X^{t, X_t^{r,x}}\|_\infty^p}{(t-r)^{\gamma'p/2}} \\
&\leq C \mathbb{E} \sup_{0 \leq r < t \leq T} \frac{\|X_t^{r,x} - x\|^p}{(t-r)^{\gamma'p/2}} \leq C,
\end{aligned}$$

where the last inequality follows from a similar argument as the proof of (5.42). Combining this with (5.34) implies

$$\mathbb{E} \sup_{0 \leq r < t \leq T} \left| \frac{I_2(r, t)}{(t-r)^{\gamma'\lambda(1-\theta)/2}} \right|^p \leq C.$$

Assume $\lambda/2 + \tau - 1 > 0$. For any $\tau' \in (0, \lambda/2 + \tau - 1)$ it is possible to find $\theta \in (0, 1)$ and $0 < \gamma < 1/2$ such that $\tau + \theta\lambda\gamma > 1$ and $\tau' < \gamma'\lambda(1-\theta)/2$. We see that $v(\cdot, x)$ is Hölder continuous of exponent τ' and

$$\lim_{n \rightarrow \infty} \|v_n(\cdot, x) - v(\cdot, x)\|_{\tau'} = 0$$

uniformly in compact set K of \mathbb{R}^d . From (5.32) it is easy to see that

$$\lim_{n \rightarrow \infty} \|u_n(\cdot, x) - u(\cdot, x)\|_{\tau'} = 0$$

uniformly in compact set K of \mathbb{R}^d . Thus we have

Proposition 5.3.2. *Let W_n be a sequence of smooth functions such that W_n converges to W in*

the norm $\|W\|_\infty + \|W\|_{\beta,\tau,\lambda}$ and u_n is the solution to (5.4) and u is given by (5.32). Then for any $\tau' < \lambda/2 + \tau - 1$, $u(t, x)$ is Hölder continuous of exponent τ' in time variable t and on any compact set K of \mathbb{R}^d ,

$$\lim_{n \rightarrow \infty} \|u_n(\cdot, x) - u(\cdot, x)\|_{\tau'} = 0 \quad (5.35)$$

uniformly on $x \in K$.

If $\tau + \tau' > 1$, then for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\int_0^t \int_{\mathbb{R}^d} u_n(s, x) (s, x) \varphi(x) \frac{\partial}{\partial s} W_n(s, x) ds dx$$

converges to the Young integral

$$\int_0^t \int_{\mathbb{R}^d} u(s, x) (s, x) \varphi(x) W(ds, x) dx .$$

It is obvious that the existence of $\tau' > 0$ such that $\tau + \tau' > 1$ and $\tau' < \lambda/2 + \tau - 1$ is equivalent to $\lambda + 4\tau > 4$. The above argument means that $u(t, x)$ is a weak solution to (5.4), in the sense of next theorem.

Theorem 5.3.3. Assume the conditions (L1)-(L3) are satisfied and assume there is an $\alpha_0 \in (0, 2)$ such that $|u_T(x)| \leq C_2 e^{C_1|x|^{\alpha_0}}$. Let $\|W\|_{\beta,\tau,\lambda}$ defined by (5.30) be finite, where the Hölder exponents λ and τ and the growth exponent β satisfy

$$\tau > 1/2, \quad \beta + \lambda < 2, \quad \lambda + 4\tau > 4. \quad (5.36)$$

Then u defined by (5.32) is a weak solution to (5.4) in the sense that u satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}^d} u_0(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}^d} u(s, x) L^* \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} u(s, x) \varphi(x) W(ds, x) dx, \end{aligned} \quad (5.37)$$

where φ is any smooth function with compact support and where the last integral is a Young integral.

Remark 5.3.4. Equation (5.37) is the definition of the weak solution used in [61], [63] and [65].

5.4 Estimates for diffusion process

In this section, we prove the exponential integrability of the Hölder norm and the supremum norm of a diffusion process which is needed in proving the existence of the Feynman-Kac solution in Chapter 5. The results obtained here are known in literature (see for instance [16], [36], [80]). However, it is difficult to find a single-source treatment that suits our purpose. Besides, our method is straightforward and unified. We present them here.

We recall that $X_t^{r,x}$ satisfies the equation (5.6). We denote

$$M_t^{r,x} = \sum_{j=1}^d \int_r^t \sigma^{ij}(s, X_s^{r,x}) \delta B_s^j. \quad (5.38)$$

Since σ is bounded (by condition **(L1)**), $(M_t^{r,x}; t \geq r)$ is a continuous L^2 martingale. In addition, we have the following properties.

Lemma 5.4.1. *Let α be a number in $(0, 1/2)$. There exist some positive constants γ_0 and γ_α such that*

$$\mathbb{E} \exp \left\{ \gamma_0 \sup_{r \leq t \leq T} |M_t^{r,x}|^2 \right\} \leq C(T - r, \Lambda) < \infty \quad (5.39)$$

and

$$\mathbb{E} \exp \left\{ \gamma_\alpha \left(\sup_{r \leq s, t \leq T} \frac{|M_t^{r,x} - M_s^{r,x}|}{|t - s|^\alpha} \right)^2 \right\} \leq C(T - r, \Lambda, \alpha) < \infty. \quad (5.40)$$

Proof. (5.39) is well-known and is a direct application of Doob's maximal inequality and Burkholder-Davis-Gundy inequality. (5.40) is proved in [10, Lemma 2]. However, for readers' convenience, we present a proof of (5.40) in the following. We will omit the

upper indices r, x . Applying the Garsia-Rodemich-Rumsey theorem (See [40] and [58], specifically [94, Theorem 2.1.3]) with $\Psi(x) = x^p$ and $p(x) = x^{\alpha+2/p}$, we have

$$|M_t - M_s| \leq 8\left(1 + \frac{2}{\alpha p}\right) 4^{1/p} |t - s|^\alpha \left\{ \int_r^T \int_r^T \left(\frac{|M_u - M_v|}{|u - v|^{\alpha+2/p}} \right)^p dudv \right\}^{1/p}.$$

Dividing both sides by $|t - s|^\alpha$ and taking the sup on $r \leq s < t \leq T$, we see that there is a constant $C = C(\alpha)$, independent of $p \geq 1$, such that

$$\mathbb{E} \left(\sup_{r \leq s < t \leq T} \frac{|M_t - M_s|}{|t - s|^\alpha} \right)^p \leq C^p \int_r^T \int_r^T \frac{\mathbb{E}|M_u - M_v|^p}{|u - v|^{\alpha p + 2}} dudv.$$

An application of the Burkholder-Davis-Gundy inequality gives

$$\|M_u - M_v\|_p \leq 2p^{1/2} \left\| \int_u^v a^{ii}(s, X_s^{r,x}) ds \right\|_{p/2}^{1/2} \leq 2\Lambda^{1/2} p^{1/2} (t - r)^{1/2}.$$

It follows that there is a constant C , which may be different than the above one, such that the p -moments of $\sup_{r \leq s < t \leq T} \frac{|M_t - M_s|}{|t - s|^\alpha}$ is at most $C^p p^{p/2} (T - r)^{(\frac{1}{2} - \alpha)p}$ for all $p > (\frac{1}{2} - \alpha)^{-1}$, which yields (5.40). \square

Lemma 5.4.2. Fix $\alpha \in (0, 1/2)$. There exist positive constants C_0, γ_0 and γ_α such that

$$\mathbb{E} \exp \left\{ \gamma_0 \sup_{r \leq t \leq T} |X_t^{r,x}|^2 \right\} \lesssim e^{C_0|x|^2} \quad (5.41)$$

and

$$\mathbb{E} \exp \left\{ \gamma_\alpha \left(\sup_{r \leq s, t \leq T} \frac{|X_t^{r,x} - X_s^{r,x}|}{|t - s|^\alpha} \right)^2 \right\} \lesssim e^{C_0|x|^2}. \quad (5.42)$$

Proof. We denote $X_t^* = \sup_{r \leq s \leq t} |X_s^{r,x}|$ and $M_t^* = \sup_{r \leq s \leq t} |M_s^{r,x}|$. We first prove (5.41). Since b has linear growth (by (L3)), from equation (5.6), we see that

$$|X_t| \leq |M_t| + |x| + \kappa(b) \int_r^t |X_s| ds.$$

An application of Gronwall's inequality yields

$$|X_t| \leq |M_t| + |x| + \kappa(b) \int_r^t (|M_s| + |x|) e^{\kappa(b)(t-s)} ds.$$

Hence, for all $p \geq 0$, applying Jensen's inequality,

$$\begin{aligned} \exp\{pX_T^*\} &\leq \exp\{p(M_T^* + |x|)\} \exp\{p\kappa(b) \int_r^T (|M_s| + |x|) e^{\kappa(b)(T-s)} ds\} \\ &\leq \frac{\exp\{p(M_T^* + |x|)\}}{e^{\kappa(b)(T-r)} - 1} \int_r^T \exp\{p(e^{\kappa(b)(t-r)} - 1)(|M_s| + |x|) e^{\kappa(b)(T-s)}\} ds \\ &\lesssim \exp\{p(M_T^* + |x|)\} \int_r^T \exp\{Cp(|M_s| + |x|)\} ds \end{aligned}$$

for some constant C depending on $T - r$ and $\kappa(b)$. We then apply Cauchy-Schwartz inequality

$$\mathbb{E}e^{pX_T^*} \lesssim \mathbb{E}e^{2p(M_T^* + |x|)} + \int_r^T \mathbb{E}e^{2Cp(|M_s| + |x|)} ds \lesssim \mathbb{E}e^{2Cp(M_T^* + |x|)},$$

where the constants (including the implied constant) are independent of p . Now we choose p according to the distribution $|N(0, a)|$ with a sufficient small, where $N(0, a)$ is a normal distribution independent of B . Using (5.39), the elementary estimate $\frac{1}{2}e^{\frac{a^2}{2}A^2} \leq \mathbb{E}^N e^{pA} \leq 2e^{\frac{a^2}{2}A^2}$ (with $A > 0$), and the previous estimate, we obtain (5.41).

From (5.6), we have

$$\frac{X_t - X_s}{(t-s)^\alpha} = \frac{\int_s^t b(u, X_u) du}{(t-s)^\alpha} + \frac{M_t - M_s}{(t-s)^\alpha} =: I_1 + I_2.$$

Since b has linear growth, $\sup_{r \leq s < t \leq T} |I_1| \leq c(\kappa(b), T, \alpha)(1 + X_T^*)$. (5.42) follows from (5.41) and (5.40). \square

5.5 Schauder estimates

We present the proof of Lemma 5.1.2. The estimates (5.12)-(5.14) are similar to Schauder estimates in the classical theory of parabolic equations. Beside the results obtained in Appendix 5.4, the method adopted here also makes use of Malliavin calculus. For this purpose, we need some preparations.

It is well-known (see e.g. [80]) that $X_t^{r,x}$ is differentiable (in Malliavin sense) with respect to the Brownian motion B_t . We denote the Malliavin derivative of X with respect to B^j by $D^j X$. It is shown in [80, Theorem 2.2.1] that $DX = (D^1 X, \dots, D^d X)^T$ has finite moments of all orders and satisfies

$$dD_\tau X_t^{i,r,x} = \sigma_k^{ij}(t) D_\tau X_t^{k,r,x} \delta B_t^j + b_k^i(t) D_\tau X_t^{k,r,x} dt, \quad D_\tau^j X_\tau^{r,x} = \sigma^{ij}(\tau, X_\tau^{r,x})$$

for $t \geq \tau \geq r$, $D_\tau X_t^{r,x} = 0$ if $t < \tau \leq T$. In the above equation, we have used the notations

$$\sigma_k^{ij}(t) = \partial_{x_k} \sigma^{ij}(t, X_t^{r,x}), \quad b_k^i(t) = \partial_{x_k} b^i(t, X_t^{r,x}).$$

The matrix DX is understood as $[DX]^{ij} = D^j X^i$. Following the proof of [80, Theorem 2.2.1], one can show that the map $x \mapsto X_t^{r,x}$ is differentiable. We denote $Y(t; r, x) = \frac{\partial}{\partial x} X_t^{r,x}$, the Jacobian of $x \mapsto X_t^{r,x}$. The matrix Y is understood as $[Y]^{ij} = Y_j^i = \partial_j X^i$. It follows that the $d \times d$ -matrix valued process $t \mapsto Y(t; r, x)$ satisfies

$$dY_\bullet^i(t; r, x) = \sigma_k^{ij}(t) Y_\bullet^k(t; r, x) \delta B_t^j + b_k^i(t) Y_\bullet^k(t; r, x) dt, \quad (5.43)$$

$$Y(r; r, x) = I_{d \times d}.$$

Let $Z(t)$ be the $d \times d$ matrix-valued process defined by

$$dZ_\bullet^i(t) = -Z_\bullet^\theta(t) \sigma_i^{\theta l} \delta B_t^l - Z_\bullet^\theta(t) \left[b_i^\theta(t) - \sigma_i^{\alpha l}(t) \sigma_\alpha^{\theta l}(t) \right] dt, \quad Z(r; r, x) = I_{d \times d}.$$

By means of Itô's formula, we have

$$\begin{aligned} d(Z_i^k Y_j^i) &= -Y_j^i Z_\theta^k \sigma_i^{\theta l} \delta B_t^l - Y_j^i Z_\theta^k b_i^\theta dt + Z_\theta^k Y_j^i \sigma_i^{\alpha l} \sigma_\alpha^{\theta l} dt \\ &\quad + Z_i^k Y_i^\theta \sigma_i^{\theta l} \delta B_t^l + Z_i^k Y_j^\theta b_i^\theta dt - Z_\theta^k Y_j^\alpha \sigma_\alpha^{ij} \sigma_i^{\theta l} dt = 0 \end{aligned}$$

and similarly for $Y_t Z_t$. Thus we obtain $Y_t Z_t = Y_t Z_t = I$. As a consequence, for every $t \geq r$, the matrix $Y(t; r, x)$ is invertible and its inverse is $Z(t; r, x)$. It is a standard fact that Y and Z have finite moments of all orders. More precisely, one has

$$\sup_{t \in [r, T], x \in \mathbb{R}^d} \mathbb{E} \left[|Y(t; r, x)|^p + |Y^{-1}(t; r, x)|^p \right] \leq c(p, T). \quad (5.44)$$

Since the coefficients of L are twice differentiable with bounded derivatives, DY exists and has finite moment of all orders and

$$\sup_{t \in [r, T], x \in \mathbb{R}^d} \mathbb{E} \sup_{\tau \in [r, T]} \left[|D_\tau Y(t; r, x)|^p + |D_\tau Y^{-1}(t; r, x)|^p \right] \leq c(p, T). \quad (5.45)$$

Moreover, it is well-known that the following representation holds (see, for instance [80, pg. 126])

$$D_\tau X_t^{r,x} = Y(t; r, x) Z(\tau; r, x) \sigma(\tau, X_\tau^{r,x}), \quad \forall \tau \in [r, t]. \quad (5.46)$$

As a consequence, if f is a smooth function, we have

$$D_\tau f(s, X_s^{r,x})^T = \nabla f(s, X_s^{r,x})^T Y(s; r, x) Y^{-1}(\tau; r, x) \sigma(\tau, X_\tau^{r,x}), \quad (5.47)$$

where and in what follows we denote $\nabla f(s, X_s^{r,x}) = (\nabla f)(s, X_s^{r,x})$. Later on, we occasionally make use of its variant

$$\nabla f(s, X_s^{r,x})^T Y(s; r, x) = D_\tau f(s, X_s^{r,x})^T \sigma^{-1}(\tau, X_\tau^{r,x}) Y(\tau; r, x), \quad \forall \tau \in [r, t]. \quad (5.48)$$

Lemma 5.5.1 (Bismut formula). *Suppose f belongs to $C^2(\mathbb{R}^{d+1})$ and suppose f and its derivatives have polynomial growth. Then*

$$\mathbb{E} [(\partial_i f)(s, X_s^{r,x})] = \frac{1}{s-r} \mathbb{E} \left[f(s, X_s^{r,x}) \int_r^s [\sigma^{-1}(\tau, X_\tau^{r,x}) Y(\tau; r, x) Y^{-1}(s; r, x)]^{ji} \delta B_\tau^j \right] \quad (5.49)$$

and

$$\partial_i \mathbb{E} f(s, X_s^{r,x}) = \frac{1}{s-r} \mathbb{E} \left[f(s, X_s^{r,x}) \int_r^s [\sigma^{-1}(\tau, X_\tau^{r,x}) Y(\tau; r, x)]^{ji} \delta B_\tau^j \right]. \quad (5.50)$$

Proof. Fix $\tau \in [r, s]$. The identity (5.47) yields

$$\nabla f(s, X_s)^T = [D_\tau f(s, X_s)]^T \sigma^{-1}(\tau) Y(\tau) Y^{-1}(s).$$

Integrating with respect to τ from r to s and taking the expectation give

$$\mathbb{E} \nabla f(s, X_s)^T = \frac{1}{s-r} \mathbb{E} \left[\int_r^s [D_\tau f(s, X_s)]^T \sigma^{-1}(\tau) Y(\tau) Y^{-1}(s) d\tau \right].$$

Formula (5.49) is then followed from the dual relationship (3.12) between the divergence operator δ and the Malliavin derivative D .

To show (5.50), we use (5.48). We integrate with respect to τ from r to s and then take the expectation to obtain

$$\nabla \mathbb{E} f(s, X_s) = \frac{1}{s-r} \mathbb{E} \left[\int_r^s [D_\tau f(s, X_s)]^T \sigma^{-1}(\tau) Y(\tau) d\tau \right].$$

Formula (5.50) follows from the dual relationship (3.12) between δ and D . □

Lemma 5.5.2. *Suppose that f is differentiable and satisfies*

$$\sup_{s \in [0, T], x \in \mathbb{R}^d} \frac{|f(s, x)|}{1 + |x|^\beta} \leq \kappa$$

for some nonnegative constants κ and β . Then we have

$$|\mathbb{E}(\nabla f)(s, X_s^{r,x})| \leq c(T, \Lambda, \lambda)\kappa(1 + |x|^\beta)[1 + (s - r)^{-1/2}] \quad (5.51)$$

and

$$|\nabla \mathbb{E}f(s, X_s^{r,x})| \leq c(T, \Lambda, \lambda)\kappa(1 + |x|^\beta)(s - r)^{-1/2}. \quad (5.52)$$

Proof. We only provide details for the proof of (5.51). The estimate (5.52) is proved similarly, perhaps in an easier manner. Motivated by the formula (5.49), we first estimate the moment of $\int_r^s [\sigma^{-1}(\tau)Y(\tau)Y^{-1}(s)]^{ji} \delta B_\tau^j$. From (3.14), we see that

$$\begin{aligned} \int_r^s [\sigma^{-1}(\tau)Y(\tau)Y^{-1}(s)]^{ji} \delta B_\tau^j &= \int_r^s [\sigma^{-1}(\tau)Y(\tau)]^{jk} \delta B_\tau^j [Y^{-1}(s)]^{ki} \\ &\quad - \int_r^s [\sigma^{-1}(\tau)Y(\tau)]^{jk} D_\tau^j [Y^{-1}(s)]^{ki} d\tau. \end{aligned}$$

From (5.44) and (5.45), it follows that

$$\sup_{s \in [r, T], x \in \mathbb{R}^d} \mathbb{E} \left| \int_r^s [\sigma^{-1}(\tau)Y(\tau)Y^{-1}(s)]^{ji} \delta B_\tau^j \right|^p \leq c(p, T)[(s - r)^{1/2} + (s - r)]^p.$$

Hence, applying Hölder inequality in (5.49),

$$|\mathbb{E} \nabla f(s, X_s^{r,x})| \lesssim (1 + (s - r)^{-1/2})[\mathbb{E}(1 + |X_s^{r,x}|^\beta)^2]^{1/2}.$$

Together with (5.41), this completes the proof of (5.51). \square

Proof of Lemma 5.1.2. Throughout the proof, we denote $\kappa_1 = [\nabla W]_{\beta_1, \infty}$, $\kappa_2 = [\nabla W]_{\beta_2, \alpha}$, $Y = \nabla \varphi$.

Uniqueness: Suppose v is a solution in $C^1([0, T]; C^2(\mathbb{R}^d))$. We apply Itô formula to the

process $s \mapsto (v + W)(s, \varphi_s^{r,x})$, taking into account the fact that L_0 is the generator of $\varphi_s^{r,x}$.

$$\begin{aligned} d(v + W)(s, \varphi_s^{r,x}) &= (\partial_t + L_0)(v + W)(s, \varphi_s^{r,x}) ds \\ &\quad + \sigma^{ij}(s, \varphi_s^{r,x}) \partial_{x_i}(v + W)(s, \varphi_s^{r,x}) \delta B_s^j. \end{aligned} \tag{5.53}$$

Since v is a strong solution, we see that $v + W$ satisfies

$$(\partial_t + L_0)(v + W) = L_0 W, \quad (v + W)(T, x) = 0.$$

Thus, integrating (5.53) from r to T yields

$$-(v + W)(r, x) = \int_r^T L_0 W(s, X_s^{r,x}) ds + \int_r^T \sigma^{ij}(s, X_s^{r,x}) \partial_{x_i}(v + W)(s, X_s^{r,x}) \delta B_s^j.$$

Taking expectation in the above identity, we obtain (5.11), which also shows the uniqueness of v .

C^0 -estimate: To prove the estimate (5.12), we write $L_0 W = \partial_i \left(\frac{1}{2} a^{ij} \partial_j W \right) + c^j \partial_j W$ where $c^j = -1/2 \partial_i a^{ij}$. Then

$$\mathbb{E} \int_r^T L_0 W(s, X_s^{r,x}) ds = I_1 + I_2$$

where

$$I_1 = \mathbb{E} \int_r^T \partial_i \left(\frac{1}{2} a^{ij}(s, X_s^{r,x}) \partial_j W(s, X_s^{r,x}) \right) ds$$

and

$$I_2 = \mathbb{E} \int_r^T c^j(s, X_s^{r,x}) \partial_j W(s, X_s^{r,x}) ds.$$

It follows from our conditions on L_0 and W that

$$\sup_{t \in [0, T], x \in \mathbb{R}^d} \frac{|a^{ij}(t, x) \partial_i W(t, x)|}{1 + |x|^{\beta_1}} \leq \Lambda \kappa_1 \quad \text{and} \quad \sup_{t \in [0, T], x \in \mathbb{R}^d} \frac{|c^j(t, x) \partial_j W(t, x)|}{1 + |x|^{\beta_1}} \leq \Lambda \kappa_1.$$

Applying Lemma 5.5.2, we obtain

$$I_1 \lesssim \kappa_1 \int_r^T ((s-r)^{-1/2} + 1) ds (1 + |x|^{\beta_1}) \lesssim \kappa_1 [(T-r)^{1/2} + (T-r)] (1 + |x|^{\beta_1}).$$

For the second term, we use (4.7)

$$I_2 \lesssim \kappa_1 \int_r^T \mathbb{E}(1 + |\varphi_s^{r,x}|^{\beta_1}) ds \lesssim \kappa_1 (T-r) (1 + |x|^{\beta_1}).$$

These inequalities altogether imply (5.12).

C¹-estimate: To show (5.13), we first apply (5.50)

$$\nabla \mathbb{E} L_0 W(s, \varphi_s^{r,x}) = (s-r)^{-1} \mathbb{E}[L_0 W(s, \varphi_s^{r,x}) H(s, x)]$$

where

$$H(s, x) = \int_r^s [\sigma^{-1}(\tau, \varphi_\tau^{r,x}) Y(\tau; r, x)]^T \delta B_\tau.$$

We denote

$$A(\tau, x) = \sigma^{-1}(\tau, X_\tau^{r,x}) Y(\tau; r, x).$$

From (5.47), we see that

$$\partial_{ij}^2 W(s, \varphi_s^{r,x}) = D_\tau^k [\partial_j W(s, \varphi_s^{r,x})] [A(\tau) Y^{-1}(s)]^{ki}, \quad \forall \tau \in [r, s].$$

Thus

$$\begin{aligned} L_0 W(s, \varphi_s^{r,x}) &= \frac{1}{2} a^{ij}(s, X_s^{r,x}) \partial_{ij}^2 W(s, \varphi_s^{r,x}) \\ &= \frac{1}{2} D_\tau^k [\partial_j W(s, \varphi_s^{r,x})] [A(\tau) Y^{-1}(s)]^{ki} a^{ij}(s, X_s^{r,x}) \\ &= \frac{1}{2} (s-r)^{-1} \int_r^s D_\tau^k [\partial_j W(s, \varphi_s^{r,x})] [A(\tau) Y^{-1}(s) a(s, X_s^{r,x})]^{kj} d\tau. \end{aligned}$$

Hence, applying (3.12),

$$\begin{aligned}
& \partial_l \mathbb{E} L_0 W(s, \varphi_s^{r,x}) \\
&= \frac{1}{2} (s-r)^{-2} \mathbb{E} \int_r^s D_\tau^k [\partial_j W(s, \varphi_s^{r,x})] [A(\tau) Y^{-1}(s) a(s, X_s^{r,x})]^{kj} H^l(s, x) d\tau \\
&= \frac{1}{2} (s-r)^{-2} \mathbb{E} \partial_j W(s, \varphi_s^{r,x}) \int_r^s [A(\tau) Y^{-1}(s) a(s, X_s^{r,x})]^{kj} H^l(s, x) \delta B_\tau^k.
\end{aligned}$$

Furthermore, since the random variable

$$G^{jl}(s; r, x) := \int_r^s [A(\tau) Y^{-1}(s) a(s, X_s^{r,x})]^{kj} H^l(s, x) \delta B_\tau^k$$

has mean zero, we can write

$$\partial_l \mathbb{E} L_0 W(s, \varphi_s^{r,x}) = \frac{1}{2} (s-r)^{-2} \mathbb{E} [\partial_j W(s, \varphi_s^{r,x}) - \partial_j W(s, x)] G^{jl}(s; r, x). \quad (5.54)$$

We now estimate the moment $G(s; r, x)$. Applying (3.14), we have

$$\begin{aligned}
G^{jl}(s; r, x) &= \int_r^s [A(\tau)]^{km} [Y^{-1}(s) a(s, X_s^{r,x})]^{mj} H^l(s, x) \delta B_\tau^k \\
&= [Y^{-1}(s) a(s, X_s^{r,x})]^{mj} H^l(s, x) \int_r^s [A(\tau)]^{km} \delta B_\tau^k \\
&\quad - \int_r^s D_\tau^k ([Y^{-1}(s) a(s, X_s^{r,x})]^{mj} H^l(s, x)) [A(\tau)]^{km} d\tau.
\end{aligned}$$

Using properties of Malliavin derivative, we have

$$\begin{aligned}
& D_\tau^k ([Y^{-1}(s) a(s, X_s^{r,x})]^{mj} H^l(s, x)) \\
&= D_\tau^k [Y^{-1}(s) a(s, X_s^{r,x})]^{mj} H^l(s, x) + [Y^{-1}(s) a(s, X_s^{r,x})]^{mj} D_\tau^k H^l(s, x).
\end{aligned}$$

Hence

$$\begin{aligned}
G^{jl}(s; r, x) &= [Y^{-1}(s)a(s, X_s^{r,x})]^{mj}H^l(s, x) \int_r^s [A(\tau)]^{km} \delta B_\tau^k \\
&\quad - \int_r^s D_\tau^k [Y^{-1}(s)a(s, X_s^{r,x})]^{mj}H^l(s, x)[A(\tau)]^{km} d\tau \\
&\quad - \int_r^s [Y^{-1}(s)a(s, X_s^{r,x})]^{mj}D_\tau^k H^l(s, x)[A(\tau)]^{km} d\tau.
\end{aligned} \tag{5.55}$$

Since a belongs to C_b^2 , estimate (5.45) is valid, the moments of $A(\tau)$ is also uniformly bounded (because a is strictly elliptic), and all the terms appear in G^{jl} has finite moments of all orders. In addition, observe that

$$D_\tau^i H^l(s, x) = 1_{\{r \leq \tau\}} A(\tau)^{il} + \int_r^s D_\tau^i A(u)^{kl} \delta B_u^k.$$

Thus, the L^p -norm of $H(s, x)$ and $DH(s, x)$ will contribute a factor $(r - s)^{1/2}$. Therefore, it follows from Burkholder-Davis-Gundy inequality and Hölder inequality that

$$\sup_{x \in \mathbb{R}^d} \|G^{jl}(s; r, x)\|_p \leq c(p, \lambda, \Lambda)[(s - r) + (s - r)^{3/2}], \forall p \geq 1. \tag{5.56}$$

Using the Hölder continuity of W , for every $p \geq 1$, we have

$$\|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x)\|_p \leq \kappa_2 \|(1 + |\varphi_s^{r,x}|^{\beta_2} + |x|^{\beta_2})|\varphi_s^{r,x} - x|^\alpha\|_p.$$

Taking into account the moment estimate (5.41) and Hölder inequality, this gives

$$\|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x)\|_p \leq c(\alpha, \beta_2, p, \Lambda) \kappa_2 (1 + |x|^{\beta_2}) (s - r)^{\alpha/2}. \tag{5.57}$$

Thus, applying Cauchy-Schwartz inequality in (5.54) yields

$$|\partial_l \mathbb{E} L_0 W(s, \varphi_s^{r,x})| \leq c(\lambda, \Lambda) (s - r)^{-2} \|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x)\|_2 \|G(s; r, x)\|_2.$$

Applying the moment estimate for G and (5.57), we obtain

$$|\partial_l \mathbb{E} L_0 W(s, \varphi_s^{r,x})| \leq c(\lambda, \Lambda)[(s-r)^{\alpha/2-1} + (s-r)^{\alpha/2-1/2}] \kappa_2(1 + |x|^{\beta_2}),$$

which together with (5.11) implies (5.13)

$C^{1,\alpha'}$ -**estimate:** This is the only place where we use the fact that the second derivatives of a are Hölder continuous. Each term appeared on the right hand side (5.55) is either differentiable or Hölder continuous in the x -variable. Thus, we obtain easily the estimate

$$\|G(s; r, x) - G(s; r, y)\|_p \leq c(p, \lambda, \Lambda)[(s-r) + (s-r)^{3/2}] |x - y|^\alpha. \quad (5.58)$$

From (5.57), we see that

$$\begin{aligned} & \|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x) - \nabla W(s, \varphi_s^{r,y}) + \nabla W(s, y)\|_p \\ & \leq \|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x)\|_p + \|\nabla W(s, \varphi_s^{r,y}) - \nabla W(s, y)\|_p \\ & \leq c(\alpha, p, \Lambda) \kappa_2(1 + |x|^{\beta_2} + |y|^{\beta_2})(s-r)^{\alpha/2}. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} & \|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x) - \nabla W(s, \varphi_s^{r,y}) + \nabla W(s, y)\|_p \\ & \leq \|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, \varphi_s^{r,y})\|_p + \|\nabla W(s, x) - \nabla W(s, y)\|_p \\ & \leq \kappa_2(W) \|(1 + |\varphi_s^{r,x}|^{\beta_2} + |\varphi_s^{r,y}|^{\beta_2}) |\varphi_s^{r,x} - \varphi_s^{r,y}|^\alpha\|_p \\ & \quad + \kappa_2(W)(1 + |x|^{\beta_2} + |y|^{\beta_2}) |x - y|^\alpha \\ & \leq c(\alpha, p, \Lambda) \kappa_2(1 + |x|^{\beta_2} + |y|^{\beta_2}) |x - y|^\alpha, \end{aligned}$$

where the last estimate comes from (5.41) and that fact that the derivative of the map

$x \mapsto \varphi_s^{r,x}$ has finite moments uniformly in x . Interpolating these two inequalities we obtain

$$\begin{aligned} & \|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x) - \nabla W(s, \varphi_s^{r,y}) + \nabla W(s, y)\|_p \\ & \leq c(\alpha, p, \Lambda) \kappa_2 (1 + |x|^{\beta_2} + |y|^{\beta_2}) |x - y|^\vartheta (s - r)^{(1-\vartheta)\alpha/2} \end{aligned} \quad (5.59)$$

for any $\vartheta \in [0, 1]$. Thus, from (5.54), applying Cauchy-Schwartz inequality we see that

$$\begin{aligned} & |\nabla \mathbb{E} L_0 W(s, \varphi_s^{r,x}) - \nabla \mathbb{E} L_0 W(s, \varphi_s^{r,y})| \\ & \leq (s - r)^{-2} \|\nabla W(s, \varphi_s^{r,x}) - \nabla W(s, x) - \nabla W(s, \varphi_s^{r,y}) + \nabla W(s, y)\|_2 \|G(s; r, x)\|_2 \\ & \quad + (s - r)^{-2} \|\nabla W(s, \varphi_s^{r,y}) - \nabla W(s, y)\|_2 \|G(s; r, x) - G(s; r, y)\|_2. \end{aligned}$$

Using (5.59), (5.57), (5.56) and (5.58), we obtain

$$\begin{aligned} & |\nabla \mathbb{E} L_0 W(s, \varphi_s^{r,x}) - \nabla \mathbb{E} L_0 W(s, \varphi_s^{r,y})| \\ & \leq c(\alpha, \lambda, \Lambda) \kappa_2 (1 + |x|^{\beta_2} + |y|^{\beta_2}) |x - y|^\vartheta [(s - r)^{(1-\vartheta)\alpha/2-1} + (s - r)^{(1-\vartheta)\alpha/2-1/2}] \\ & \quad + c(\alpha, \lambda, \Lambda) \kappa_2 (1 + |y|^{\beta_2}) |x - y|^\alpha [(s - r)^{\alpha/2-1} + (s - r)^{\alpha/2-1/2}]. \end{aligned}$$

Therefore, choosing $\vartheta < 1$, this estimate together with (5.11) implies (5.14). \square

Chapter 6

Asymptotic growth of Gaussian sample paths

In Chapters 3, 4 and 5, we assume the pathwise Hölder continuity and pathwise growth conditions on W in order to define and to solve (partial) differential equations related to the nonlinear integral $\int W(ds, \varphi_s)$. For instance, the conditions (3.1), (5.1), (5.2), (5.3) are essential in various parts of the dissertation. In probability theory, it is usually hard to obtain properties for (almost) every sample path of a stochastic process from its average properties (from its probability law). In this section, we investigate these pathwise Hölder continuity and pathwise growth problems for a stochastic process. We shall focus on Gaussian random fields. However, our method works well for other processes provided they satisfy some suitable normal concentration inequalities (for instance, see the assumptions in Theorem 6.2.2).

Let W be a stochastic process on $[0, T] \times \mathbb{R}^d$. An application of our results yields the asymptotic growth of the quantity

$$I(\delta, R) = \sup_{t \in [0, T]} \sup_{|x|, |y| \leq R; |x-y| \leq \delta} \frac{|W(t, \square[x, y])|}{|x_1 - y_1|^{\lambda_1} \cdots |x_d - y_d|^{\lambda_d}}$$

as $R \rightarrow \infty$, where $W(t, \square[x, y])$ denotes the d -increment of $W(t, \cdot)$ over the rectangle

$[x, y]$. More precise definition is given in Section 6.2. If R is fixed, the quality $I(\delta, R)$ is the objective in our previous work [58] via a multiparameter version of Garsia-Rodemich-Rumsey inequality.

Let us mention some historical remarks. (Pathwise) boundedness and continuity for stochastic processes have been studied thoroughly in literature. One of the central ideas is originated in an important early paper by Garsia, Rodemich and Rumsey (1970) [40]. This was developed further by Preston (1971,1972) [84, 85], Dudley (1973) [31] and Fernique (1975) [32]. In these considerations, the parameter space \mathbf{T} is bounded and treated as a "single-dimension" object. For instance, the well-known Dudley bound

$$\mathbb{E} \sup_{s,t \in \mathbf{T}} |W(t) - W(s)| \lesssim \int_0^{d_W(s,t)} \sqrt{\log N(\mathbf{T}, d_W, \varepsilon)} d\varepsilon$$

yields modulus of continuity in terms of the entropy number $N(T, d, \varepsilon)$. This is extended to a more precise bound in terms of majorizing measure

$$\mathbb{E} \sup_{\substack{s,t \in \mathbf{T} \\ d_W(s,t) \leq \delta}} |W(t) - W(s)| \lesssim \sup_{t \in \mathbf{T}} \int_0^\delta \log^{1/2} \frac{1}{\mu(B_{d_W}(t, u))} du .$$

The majorizing-measure bound turns out to be necessary for processes which satisfy normal concentration inequalities. This result by M. Talagrand is the milestone in theory of Gaussian processes. We refer the readers to [76, Chapter 6] and references therein for details and more historical facts. See also Talagrand's monograph [95] in which the role of majorizing measure is replaced by a variational quality called $\gamma_2(\mathbf{T}, d_W)$.

Estimates for the d -increment of W over a rectangle are quite different. Difficulties arise since $W(\square[s, t])$ does not behave nicely as increments. In particular, the corresponding entropic "metric"

$$(\mathbb{E} W(\square[s, t])^2)^{1/2}$$

does not satisfy the triangle inequality, but rather behaves like a volume metric. To elaborate

this point, let us consider the two dimensional case:

$$\begin{aligned} W(\square[s, t]) &= W(s_2, t_2) - W(s_2, t_1) - W(s_1, t_2) + W(s_1, t_1) \\ &= \Delta_{[t_2, t_1]}W(s_2) - \Delta_{[t_2, t_1]}W(s_1) = \Delta_{[s_2, s_1]}\Delta_{[t_2, t_1]}W, \end{aligned}$$

where $\tilde{W}(s) := \Delta_{[t_2, t_1]}W(s) = W(s; t_2) - W(s; t_1)$. This product-like property is essential in our current approach (see for instance inequality (6.5) below). Alternatively, to obtain a sharp bound for the difference, one can repeatedly apply the Garsia-Rodemich-Rumsey inequality first $\Delta_{[s_2, s_1]}\tilde{W}$ and then to $\Delta_{[s_2, s_1]}\Delta_{[t_2, t_1]}W$. Indeed, for bounded parameter domains equipped with Lebesgue measure, this direction was developed by the authors in [58]. This idea, while might be feasible, seems to be more complicated in our current setting with general (unbounded) parameter domains equipped with a general measure.

In Section 6.1, we will prove a deterministic inequality, which is more precise than the multiparameter Garsia-Rodemich-Rumsey inequality obtained in [58]. We then apply it to obtain a majorizing-measure bound on the d -increments of stochastic processes in Section 6.2. Our formulations are benefited from the treatment in [76]. We however did not consider the necessary conditions for these bounds (i.e. lower bounds). Results in these two subsections are applicable to general stochastic processes.

Given a well-developed toolbox to treat the case when \mathbf{T} is bounded (or for example, R is fixed in $I(\delta, R)$), the asymptotic growth for $I(\delta, R)$ as $R \rightarrow \infty$ can be obtained using concentration inequalities for Gaussian processes. More precise results are given for fractional Brownian fields. This is done in Section 6.3.

6.1 A deterministic inequality

Throughout the current subsection, we put $\Psi(u) = \exp(u^2) - 1$. Suppose μ is a nonnegative measure on \mathbf{T} and X is a measurable function on \mathbf{T} . We define

$$[X]_{\Psi,(\mathbf{T},\mu)} := \inf \left\{ \alpha > 0 : \int_{\mathbf{T}} \Psi \left(\frac{X(t)}{\alpha} \right) \mu(dt) \leq 1 \right\}.$$

When the parameter space \mathbf{T} and the measure μ are clear from the context, we often suppress them and write $[X]_{\Psi}$ instead. The following result, whose proof is given in [76, pg. 256-258], is an application of the Young inequality

$$ab \leq \int_0^a g(x)dx + \int_0^b g^{-1}(x)dx,$$

where g is a real-valued, continuous and strictly increasing function.

Lemma 6.1.1. *Let X and f be measurable functions on \mathbf{T} , μ be a nonnegative measure on \mathbf{T} . Assume that $[X]_{\Psi,(\mathbf{T},\mu)}$ is finite and $0 < \int |f|d\mu < \infty$. Then*

$$\int_{\mathbf{T}} |X(t)f(t)|\mu(dt) \leq 3[X]_{\Psi,(\mathbf{T},\mu)} \int_{\mathbf{T}} |f(t)| \log^{1/2} \left(1 + \frac{|f(t)|}{\int |f(s)|\mu(ds)} \right) \mu(dt).$$

We consider the case when \mathbf{T} has the form $\mathbf{T} = \mathbf{T}_1 \times \cdots \times \mathbf{T}_{\ell}$. A parameter t in \mathbf{T} has ℓ components, $t = (t_1, \dots, t_{\ell})$. For each $i = 1, \dots, \ell$, the space \mathbf{T}_i is equipped with a metric d_i . We also denote $d^*(s, t) = d_1(s_1, t_1) \dots d_{\ell}(s_{\ell}, t_{\ell})$ for every s, t in \mathbf{T} . Let X be a function on \mathbf{T} . We define the ℓ -increment of X over a "rectangle" $[s, t]$ as

$$X(\square[s, t]) = \prod_{j=1}^{\ell} (I - V_{j,s})X(t).$$

In the above expression, I is the identity operator, $V_{j,s}$ is the substitution operator which

substitutes the j -th component of a function on \mathbf{T} by s_j , more precisely,

$$V_{j,s}X(t) = X(t_1, \dots, t_{j-1}, s_j, t_{j+1}, \dots, t_\ell).$$

We refer to [58] for a more detailed description on this ℓ -increment.

For each i , $B^i(t_i, u)$ denotes the open ball with radius u in the metric space (\mathbf{T}_i, d_i) centered at t_i . For each t in \mathbf{T} , we denote $B(t, u) = B^1(t_1, u) \times \dots \times B^\ell(t_\ell, u)$. For each j , put $D_j = \sup_{s_j, t_j \in \mathbf{T}_j} d_j(s_j, t_j)$.

For each $i = 1, \dots, \ell$, let μ_i be a probability measure on \mathbf{T}_i . Let $k = (k_1, \dots, k_\ell)$ be a multi-index in \mathbb{N}^ℓ . We define

$$\begin{aligned} \mu_k^i(t_i) &= \mu^i(B^i(t_i, D_i 2^{-k_i})), & \rho_{k_i}(t_i, \cdot) &= \frac{1}{\mu_k^i(t_i)} 1_{B^i(t_i, D_i 2^{-k_i})}(\cdot) \\ \mu_k(t) &= \prod_{i=1}^{\ell} \mu_k^i(t_i), & \rho_k(t, \cdot) &= \prod_{i=1}^{\ell} \rho_{k_i}(t_i, \cdot) \end{aligned}$$

and

$$M_k(t) = \int_{\mathbf{T}} \rho_k(t, u) X(u) \mu(du). \quad (6.1)$$

We use the notations $k+1 = (k_1+1, \dots, k_\ell+1)$, $k+1_j = (k_1, \dots, k_{j-1}, k_j+1, k_{j+1}, \dots, k_\ell)$, $\hat{t}_i = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_\ell)$ and $\hat{\mathbf{T}}_i = \mathbf{T}_1 \times \dots \times \mathbf{T}_{i-1} \times \mathbf{T}_{i+1} \times \dots \times \mathbf{T}_\ell$.

Theorem 6.1.2. *Let $\{X(t), t \in \mathbf{T}\}$ be a measurable function on \mathbf{T} . We put $\mu = \mu^1 \times \dots \times \mu^\ell$ and*

$$Z = \inf \left\{ \alpha > 0 : \iint_{\mathbf{T} \times \mathbf{T}} \Psi \left(\frac{X(\square[u, v])}{\alpha d^*(u, v)} \right) \mu(du) \mu(dv) \leq 1 \right\}.$$

Assume that D_j , $j = 1, \dots, d$, and Z are finite. Then, for every s, t in \mathbf{T} such that the integral

$$\int_0^{d_1(s_1, t_1)} du_1 \cdots \int_0^{d_\ell(s_\ell, t_\ell)} du_\ell \left(\log^{1/2} \frac{1}{\mu(B(s, u))} + \log^{1/2} \frac{1}{\mu(B(t, u))} \right)$$

is finite, $M_k(\square[s, t])$ converges to a limit, denoted by $X'(\square[s, t])$, as k_1, \dots, k_ℓ go to infinity. In

addition, $X'(\square[s, t])$ satisfies

$$|X'(\square[s, t])| \leq C^\ell Z \int_0^{d_1(s_1, t_1)} du_1 \cdots \int_0^{d_\ell(s_\ell, t_\ell)} du_\ell \left(\log^{1/2} \frac{1}{\mu(B(s, u))} + \log^{1/2} \frac{1}{\mu(B(t, u))} \right). \quad (6.2)$$

Proof. Fix s, t in \mathbf{T} . We choose the multi-index n such that $D_j 2^{-n_j-1} \leq d_j(s_j, t_j) \leq D_j 2^{-n_j}$ for each $j = 1, \dots, \ell$. It suffices to show that the following series satisfies the bound in (6.2)

$$|M_n(\square[s, t])| + \sum_{k \geq n} |M_{k+1}(\square[s, t]) - M_k(\square[s, t])|. \quad (6.3)$$

We estimate the first term. Notice that we can write

$$M_n(\square[s, t]) = \iint_{\mathbf{T} \times \mathbf{T}} X(\square[u, v]) \rho_n(s, u) \rho_n(t, v) \mu(du) \mu(dv).$$

We consider the function $\{Y(u, v), (u, v) \in \mathbf{T} \times \mathbf{T}\}$ defined by

$$Y(u, v) = \begin{cases} \frac{X(\square[u, v])}{d^*(u, v)} & \text{when } d^*(u, v) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that

$$\begin{aligned} |M_n(\square[s, t])| &\leq \iint_{\mathbf{T} \times \mathbf{T}} |Y(u, v)| d^*(u, v) \rho_n(s, u) \rho_n(t, v) \mu(du) \mu(dv) \\ &\lesssim (D_1 2^{-n_1}) \cdots (D_\ell 2^{-n_\ell}) \iint_{\mathbf{T} \times \mathbf{T}} |Y(u, v)| \rho_n(s, u) \rho_n(t, v) \mu(du) \mu(dv), \end{aligned}$$

since the support of $\rho_n(s, \cdot) \rho_n(t, \cdot)$, $d^*(u, v) \lesssim (D_1 2^{-n_1}) \cdots (D_\ell 2^{-n_\ell})$. We now apply Lemma 6.1.1 to the functions Y and $\rho_n(s, \cdot) \otimes \rho_n(t, \cdot)$ on the product space $(\mathbf{T} \times \mathbf{T}, \mu \otimes \mu)$,

observing that $Z = [Y]_{\Psi}$, $\iint \rho_n(s, \cdot) \rho_n(t, \cdot) = 1$ and $\rho_n(s, u) \rho_n(t, v) \leq (\mu_n(s, u) \mu_n(t, v))^{-1}$,

$$\begin{aligned} & |M_n(\square[s, t])| \\ & \lesssim Z(D_1 2^{-n_1}) \cdots (D_\ell 2^{-n_\ell}) \iint_{\mathbf{T} \times \mathbf{T}} \rho_n(s, u) \rho_n(t, v) \log^{1/2} (1 + \rho_n(s, u) \rho_n(t, v)) \mu(du) \mu(dv) \\ & \lesssim Z(D_1 2^{-n_1}) \cdots (D_\ell 2^{-n_\ell}) \log^{1/2} \left(1 + \frac{1}{\mu_n(s) \mu_n(t)} \right). \end{aligned}$$

Since $d^*(s, t) \asymp (D_1 2^{-n_1}) \cdots (D_\ell 2^{-n_\ell})$, this shows

$$|M_n(\square[s, t])| \lesssim Z \int_0^{d_1(s_1, t_1)} du_1 \cdots \int_0^{d_\ell(s_\ell, t_\ell)} du_\ell \log^{1/2} \left(\frac{1}{\mu(B(s, u))} + \frac{1}{\mu(B(t, u))} \right). \quad (6.4)$$

We now estimate each term in the sum appear in (6.3). We denote $\tau_0 k = k$ and recursively $\tau_j k = \tau_{j-1} k + 1_j$ for each $j = 1, \dots, \ell$. For example, $\tau_1 k = (k_1 + 1, k_2, \dots, k_\ell)$ and $\tau_\ell k = k + 1$.

We then write

$$|M_{k+1}(\square[s, t]) - M_k(\square[s, t])| \leq \sum_{j=1}^{\ell} |M_{\tau_j k}(\square[s, t]) - M_{\tau_{j-1} k}(\square[s, t])|. \quad (6.5)$$

Note that the multi-indices $\tau_j k$ and $\tau_{j-1} k$ differs by exactly 1 unit at the j -th component.

Without loss of generality, we consider the case

$$|M_{\tilde{k}}(\square[s, t]) - M_k(\square[s, t])|,$$

where $\tilde{k} = k + 1_\ell = (k_1, \dots, k_{\ell-1}, k_\ell + 1)$. We adopt the notations $w = (w', w_\ell)$ for every w in \mathbf{T} ,

$$\rho'_k(s', u') = \rho_{k_1}(s_1, u_1) \cdots \rho_{k_{\ell-1}}(s_{\ell-1}, u_{\ell-1})$$

and similarly for $\rho'_k(t', v')$. We then write

$$M_k(\square[s, t]) = M_k(\square^{\ell-1}[s', t'], s_\ell) - M_k(\square^{\ell-1}[s', t'], t_\ell)$$

and similarly for $M_{\tilde{k}}(\square[s, t])$. Thus

$$\begin{aligned}
|M_{\tilde{k}}(\square[s, t]) - M_k(\square[s, t])| &\leq |M_{k+1_\ell}(\square^{\ell-1}[s', t'], s_\ell) - M_k(\square^{\ell-1}[s', t'], s_\ell)| \\
&\quad + |M_{k+1_\ell}(\square^{\ell-1}[s', t'], t_\ell) - M_k(\square^{\ell-1}[s', t'], t_\ell)| \quad (6.6) \\
&= I_1 + I_2.
\end{aligned}$$

We only need to estimate I_1 since I_2 is analogous. We have

$$\begin{aligned}
&M_k(\square^{\ell-1}[s', t'], s_\ell) \\
&= \iint_{\mathbf{T} \times \mathbf{T}} X(\square^{\ell-1}[u', v'], v_\ell) \rho'_k(s', u') \rho'_k(t', v') \rho_{\ell+1}(s_\ell, u_\ell) \rho_\ell(s_\ell, v_\ell) \mu(du) \mu(dv)
\end{aligned}$$

and similarly

$$\begin{aligned}
&M_{\tilde{k}}(\square^{\ell-1}[s', t'], s_\ell) \\
&= \iint_{\mathbf{T} \times \mathbf{T}} X(\square^{\ell-1}[u', v'], u_\ell) \rho'_k(s', u') \rho'_k(t', v') \rho_{\ell+1}(s_\ell, u_\ell) \rho_\ell(s_\ell, v_\ell) \mu(du) \mu(dv).
\end{aligned}$$

Note how the dummy variables v_ℓ and u_ℓ have been switched between the two formulas.

Hence

$$\begin{aligned}
&|M_{\tilde{k}}(\square^{\ell-1}[s', t'], s_\ell) - M_k(\square^{\ell-1}[s', t'], s_\ell)| \\
&\leq \iint_{\mathbf{T} \times \mathbf{T}} |X(\square^\ell[u, v])| \rho'_k(s', u') \rho'_k(t', v') \rho_{\ell+1}(s_\ell, u_\ell) \rho_\ell(s_\ell, v_\ell) \mu(du) \mu(dv).
\end{aligned}$$

Similarly to the term $M_n(\square[s, t])$ one can obtain

$$\begin{aligned} & |M_{\tilde{k}}(\square^{\ell-1}[s', t'], s_\ell) - M_k(\square^{\ell-1}[s', t'], s_\ell)| \\ & \lesssim Z(D_1 2^{-k_1}) \cdots (D_\ell 2^{-k_\ell}) \log^{1/2} \left(1 + \frac{1}{\mu_{\tilde{k}}(s) \mu_k(s)} \right) \\ & \lesssim Z(D_1 2^{-k_1}) \cdots (D_\ell 2^{-k_\ell}) \log^{1/2} \left(1 + \frac{1}{\mu_k(s)} \right). \end{aligned}$$

Therefore, combining altogether (6.5), (6.6) and the previous estimate, we get

$$|M_{k+1}(\square[s, t]) - M_k(\square[s, t])| \lesssim Z \ell (D_1 2^{-k_1}) \cdots (D_\ell 2^{-k_\ell}) \log^{1/2} \left(1 + \frac{1}{\mu_k(s)} \right),$$

and hence,

$$\begin{aligned} & \sum_{k \geq n} |M_{k+1}(\square[s, t]) - M_k(\square[s, t])| \\ & \lesssim Z \ell (D_1 2^{-n_1}) \cdots (D_\ell 2^{-n_\ell}) \log^{1/2} \left(1 + \frac{1}{\mu_n(s)} \right) \\ & \lesssim Z \ell \int_0^{d_1(s_1, t_1)} du_1 \cdots \int_0^{d_\ell(s_\ell, t_\ell)} du_\ell \log^{1/2} \left(\frac{1}{\mu(B(s, u))} + \frac{1}{\mu(B(t, u))} \right). \end{aligned}$$

Together with the bound for $M_n(\square[s, t])$ (inequality (6.4) and (6.3), this completes the proof. \square

Remark 6.1.3. In Theorem 6.1.2, X' may not be defined as a function on \mathbf{T} , that is for each t in \mathbf{T} , there is no a priory reason for $X'(t)$ to be defined. However, in order to keep the representation compact, we have abused of notations and denote the limit as $X'(\square[s, t])$. This object is well-defined for every fixed s, t in \mathbf{T} .

6.2 Majorizing measure

We now suppose that X is a stochastic process with the probability space (Ω, \mathcal{F}, P) . We introduce the ℓ -fold volumetric

$$d^\ell(s, t) = \left(\mathbb{E}[X(\square[s, t])]^2 \right)^{1/2}.$$

Assume that $\sup_{s, t \in \mathbf{T}} d^\ell(s, t)$ is finite. In addition, for each i , there exists a metric d_i on \mathbf{T}_i such that

$$d^\ell(s, t) \leq d_1(s_1, t_1) \cdots d_\ell(s_\ell, t_\ell).$$

This is not a restriction since such collection of metrics always exists. For instance, one can choose

$$d_1(s_1, t_1) = \sup_{\hat{s}_1, \hat{t}_1 \in \hat{\mathbf{T}}_1} d^\ell(s, t)$$

and recursively

$$d_k(s_k, t_k) = \sup_{\hat{s}_k, \hat{t}_k \in \hat{\mathbf{T}}_k} \frac{d^\ell(s, t)}{\prod_{i=1}^{k-1} d_i(s_i, t_i)}$$

with the convention $0/0 = 0$.

We denote Z as in Theorem 6.1.2, that is

$$Z = \inf \left\{ \alpha > 0 : \iint_{\mathbf{T} \times \mathbf{T}} \Psi \left(\frac{X(\square[u, v])}{\alpha d^*(u, v)} \right) \mu(du) \mu(dv) \leq 1 \right\}. \quad (6.7)$$

We assume that Z is finite almost surely.

Example 6.2.1. Suppose X is a centered Gaussian process. Then Z has exponential tail. More precisely $P(Z > u) \leq (e \log 2)^{1/2} u 2^{-u^2/2}$ for all $u > (2 + 1/\log 2)^{1/2}$. This comes from a standard argument by Chebyshev inequality and Hölder inequality, see [76, pg. 256-258] for details.

As an application of Theorem 6.1.2, we have

Theorem 6.2.2. Let $\{X(t), t \in \mathbf{T}\}$ be a stochastic process such that Z , defined in (6.7), is finite a.s. Then X has a version X' such that for all $\omega \in \Omega$ and s, t in \mathbf{T}

$$|X'(\omega, \square[s, t])| \leq C^\ell Z(\omega) \int_0^{d_1(s_1, t_1)} du_1 \cdots \int_0^{d_\ell(s_\ell, t_\ell)} du_\ell \left(\log^{1/2} \frac{1}{\mu(B(s, u))} + \log^{1/2} \frac{1}{\mu(B(t, u))} \right).$$

In particular, if $\mathbb{E}Z$ is finite, then

$$\mathbb{E} \sup_{d_i(s_i, t_i) \leq \delta_i, 1 \leq i \leq \ell} |X(\square[s, t])| \leq C^\ell (\mathbb{E}Z) \sup_{s \in \mathbf{T}} \int_0^{\delta_1} du_1 \cdots \int_0^{\delta_\ell} du_\ell \log^{1/2} \frac{1}{\mu(B(s, u))}.$$

Proof. First note that for every t, v in \mathbf{T}

$$(\mathbb{E}|X(t) - X(v)|^2)^{1/2} \leq \sum_{i=1}^{\ell} d_i(t_i, v_i). \quad (6.8)$$

We recall the notation $M_k(t)$ in (6.1). We have

$$\begin{aligned} \mathbb{E}|X(t) - M_k(t)| &\leq \int_{\mathbf{T}} \mathbb{E}|X(t) - X(v)| \rho_k(t, v) \mu(dv) \\ &\leq \int_{\mathbf{T}} \sum_{i=1}^{\ell} d_i(t_i, v_i) \rho_k(t, v) \mu(dv) \leq \sum_{i=1}^{\ell} D_i 2^{-k_i}. \end{aligned}$$

Together with Borel-Cantelli lemma, this shows for all $t \in \mathbf{T}$, $M_k(t)$ converges to $X(t)$ almost surely. On the other hand, Theorem 6.1.2 shows for all $s, t \in \mathbf{T}$, $M_k(\square[s, t])$ converges to a limit, denoted by $X'(\square[s, t])$. This implies $X(\square[s, t]) = X'(\square[s, t])$ almost surely. The result is now followed from Theorem 6.1.2. \square

6.3 Asymptotic growth

Let $W(t, x)$ be a continuous Gaussian process on $[0, T] \times \mathbb{R}^d$ with mean 0. As in the previous subsection, we define the d -fold volumetric

$$d(x, y) = \sup_{t \in [0, T]} \left(\mathbb{E}[W(t, \square[x, y])]^2 \right)^{1/2}.$$

Without loss of generality, we assume there are metrics d_1, \dots, d_d on \mathbb{R} such that $d^*(x, y) = d_1(x_1, y_1) \dots d_d(x_d, y_d)$ satisfies $d(x, y) \leq d^*(x, y)$.

Let $\delta = (\delta_1, \dots, \delta_\ell)$ be in $(0, \infty)^\ell$. The notation $d^*(x, y) \leq \delta$ means $d_i(x_i, y_i) \leq \delta_i$ for all $i = 1, 2, \dots, \ell$. We denote $|x|^* = \max_{1 \leq i \leq d} d_i(0, x_i)$ for every $x \in \mathbb{R}^d$. We are interested in the asymptotic growth of the process

$$W^*(\delta, R) = \sup_{t \in [0, T]} \sup_{\substack{d^*(x, y) \leq \delta \\ |x|^*, |y|^* \leq R}} |W(t, \square[x, y])|$$

as R gets large and δ can range freely in a bounded neighborhood of 0. W^* also depends on T . However since T will always be fixed in our consideration, we suppress the dependence on T in our notations. We put

$$\mathcal{S}_R = \{x \in \mathbb{R}^d : |x|^* \leq R\},$$

$$m(\delta, R) = \mathbb{E}W^*(\delta, R),$$

and

$$\sigma(\delta, R) = \sup_{t \in [0, T]} \sup_{\substack{d^*(x, y) \leq \delta \\ x, y \in \mathcal{S}_R}} (E|W(t, \square[x, y])|^2)^{1/2}.$$

We first prove the following concentration inequality

Lemma 6.3.1. For any $r > 0$,

$$P\left(\frac{1}{\sigma(\delta, R)}|W^*(\delta, R) - m(\delta, R)| > r\right) \leq 2e^{-r^2/2}. \quad (6.9)$$

As a consequence,

$$\mathbb{E}\psi_\rho\left(\frac{|W^*(\delta, R) - m(\delta, R)|}{\sigma(\delta, R)}\right) \leq c_\rho < \infty \quad (6.10)$$

for every $\rho < 1/2$, where $\psi_\rho = \exp(\rho x^2)$.

Proof. It suffices to show (6.9). Let $\{X(u), u \in \mathbf{T}\}$ be a Gaussian process. Assume that \mathbf{T} is finite. The following concentration inequality is standard

$$P\left(\frac{1}{\sigma}\left|\sup_{u \in \mathbf{T}} |X(u)| - \mathbb{E}\left[\sup_{u \in \mathbf{T}} |X(u)|\right]\right| > r\right) \leq 2e^{-r^2/2}, \quad (6.11)$$

for every $\sigma \geq \sup_{u \in \mathbf{T}} (EX^2(u))^{1/2}$. We refer to [72] or [76, Theorem 5.4.3] for a proof of (6.11).

We now fix $(t_1, x_1), \dots, (t_m, x_m)$ in $[0, T] \times \mathbb{R}^d$ such that $d^*(x_j, x_k) \leq \delta$ and $|x_j|^*, |x_k|^* \leq R$ for all j, k . We denote $x_j \sqcup x_k$ the collection of points z in \mathbb{R}^d such that each component of z is the corresponding component of either x_j or x_k . We consider the centered Gaussian random process $X(t_i, x_j \sqcup x_k) := W(t_i, \square[x_j, x_k])$ indexed by the parameters $\{t_i\}_{1 \leq i \leq m}$ and $\{x_j \sqcup x_k\}_{1 \leq j, k \leq m}$. It is clear that

$$\mathbb{E}X^2(t_i, x_j \sqcup x_k) \leq \sigma^2(\delta, R).$$

Thus, the inequality (6.11) becomes

$$P\left(\frac{1}{\sigma(\delta, R)}\left|\sup_{i, j, k \leq m} |W(t_i, \square[x_j, x_k])| - \mathbb{E}\left[\sup_{i, j, k \leq m} |W(t_i, \square[x_j, x_k])|\right]\right| > r\right) \leq 2e^{-r^2/2}.$$

An approximation procedure yields (6.9). □

Theorem 6.3.2. *With probability one,*

$$\sup_{\delta \in (0,1]^\ell} \limsup_{R \rightarrow \infty} \frac{|W^*(\delta, R) - m(2\delta, 2R)|}{\sigma(2\delta, 2R) \sqrt{\log(\delta_1^{-1} \cdots \delta_\ell^{-1} \log R)}} \leq \sqrt{2} \quad (6.12)$$

Proof. We put $p(\delta, R) = \delta_1^{-1} \cdots \delta_\ell^{-1} (\log R)^2$ and consider the random variable

$$\Theta = \sup_{\delta \in (0,1]^\ell, R \geq 1} \frac{1}{p(\delta, R)} \psi_\rho \left(\frac{1}{\sigma(2\delta, 2R)} |W^*(\delta, R) - m(2\delta, 2R)| \right).$$

For each multi-index $j = (j_1, \dots, j_\ell)$ in \mathbb{N}^ℓ , we denote $2^{-j} = (2^{-j_1}, \dots, 2^{-j_\ell})$. The notation $\delta \leq 2^{-j}$ means $\delta_i \leq 2^{-j_i}$ for all $i = 1, 2, \dots, \ell$. Then using the monotonicity of p , ψ_ρ , W^* and σ , and (6.10) we have

$$\begin{aligned} \mathbb{E}\Theta &\leq \sum_{k \in \mathbb{N}, j \in \mathbb{N}^\ell} \mathbb{E} \sup_{\substack{2^{-j-1} \leq \delta \leq 2^{-j} \\ 2^{k-1} \leq R \leq 2^k}} \frac{1}{p(\delta, R)} \psi_\rho \left(\frac{1}{\sigma(2\delta, 2R)} |W^*(\delta, R) - m(2\delta, 2R)| \right) \\ &\leq \sum_{k \in \mathbb{N}, j \in \mathbb{N}^\ell} \frac{1}{p(2^{-j}, 2^{k-1})} \mathbb{E} \psi_\rho \left(\frac{1}{\sigma(2^{-j}, 2^k)} |W^*(2^{-j}, 2^k) - m(2^{-j}, 2^k)| \right) \\ &\leq c_\rho \sum_{k \in \mathbb{N}, j \in \mathbb{N}^\ell} \frac{1}{p(2^{-j}, 2^{k-1})} < \infty. \end{aligned}$$

Hence, with probability one, Θ is finite and

$$\psi_\rho \left(\frac{1}{\sigma(2\delta, 2R)} |W^*(\delta, R) - m(2\delta, 2R)| \right) \leq \Theta p(2\delta, R), \quad \forall \delta > 0, \forall R \geq 1.$$

In particular,

$$\frac{|W^*(\delta, R) - m(2\delta, 2R)|}{\sigma(2\delta, 2R)} \leq \sqrt{\frac{\log[\Theta p(2\delta, R)]}{\rho}}, \quad \forall \delta > 0, \forall R \geq 1.$$

We then use the trivial estimate

$$\sqrt{\log(\Theta p)} \leq \sqrt{|\log \Theta|} + \sqrt{|\log p|}$$

to get

$$\frac{|W^*(\delta, R) - m(2\delta, 2R)|}{\sigma(2\delta, 2R)\sqrt{\log(\delta_1^{-1} \cdots \delta_\ell^{-1} \log R)}} \leq \sqrt{\frac{|\log \Theta|}{\rho |\log \log R|}} + \sqrt{\frac{\log(\delta_1^{-1} \cdots \delta_\ell^{-1} (\log R)^2)}{\rho \log(\delta_1^{-1} \cdots \delta_\ell^{-1} \log R)'}}$$

for all $\delta > 0$ and $R \geq 1$. Since ρ can be chosen to be any constant less than $1/2$, we can choose a sequence ρ_n convergent to $1/2$. Since countable unions of events with probability zero still have probability zero, we can pass through the limit $n \rightarrow \infty$ to get, with probability one,

$$\frac{|W^*(\delta, R) - m(2\delta, 2R)|}{\sigma(2\delta, 2R)\sqrt{\log(\delta_1^{-1} \cdots \delta_\ell^{-1} \log R)}} \leq \sqrt{\frac{2|\log \Theta|}{|\log \log R|}} + \sqrt{\frac{2 \log(\delta_1^{-1} \cdots \delta_\ell^{-1} (\log R)^2)}{\log(\delta_1^{-1} \cdots \delta_\ell^{-1} \log R)'}}$$

for all $\delta > 0$ and $R \geq 1$. Finally, let $R \rightarrow \infty$ to complete the proof. \square

In general, it is hard to say anything about the growth of $m(\delta, R)$ as R gets large. In what follows, we restrict ourselves to a particular (but still sufficiently large) class of Gaussian random fields. To be more precise, for each $i = 1, \dots, \ell$, let ϕ_i be a majorant for d_i , that is, ϕ_i is strictly increasing with $\phi_i(0) = 0$ and

$$d_i(x_i, y_i) \leq \phi_i(|y_i - x_i|). \quad (6.13)$$

Define

$$\tilde{\omega}_i(\delta_i) = \delta_i \log^{1/2} \frac{1}{\phi_i^{-1}(\delta_i)} + \int_0^{\phi_i^{-1}(\delta_i)} \frac{\phi_i(u)}{u \log^{1/2}(1/u)} du.$$

We will always presume $\tilde{\omega}_i$'s are finite wherever they appear.

Proposition 6.3.3. Denote $\tilde{\delta}_i = \prod_{j \neq i} \delta_j$. Then we have

$$m(\delta, R) \lesssim \delta_1 \cdots \delta_\ell \log^{1/2} \left(\prod_{i=1}^{\ell} 2\phi_i^{-1}(R) \right) + \sum_{i=1}^{\ell} \tilde{\delta}_i \tilde{\omega}_i(\delta_i) \quad (6.14)$$

where the implied constant is independent of R and δ .

Proof. We take for the majorizing measure $\mu_i = \lambda / (2\phi_i^{-1}(R))$, where λ is the Lebesgue measure. By (6.13), the ball $B^i(x_i, u_i)$ contains the interval $(x_i - \phi_i^{-1}(u_i), x_i + \phi_i^{-1}(u_i)) \cap \{z_i : d_i(z_i, 0) \leq R\}$, thus,

$$\mu_i(B^i(x_i, u_i)) \geq \frac{\phi_i^{-1}(u_i)}{2\phi_i^{-1}(R)}.$$

Hence, for all x in \mathcal{S}_R ,

$$\log \frac{1}{\mu(B(x, u))} \leq \log \left(\prod_{i=1}^d \frac{2\phi_i^{-1}(R)}{\phi_i^{-1}(u_i)} \right).$$

Therefore, for δ sufficiently small,

$$\begin{aligned} & \int_0^{\delta_1} du_1 \cdots \int_0^{\delta_\ell} du_\ell \log^{1/2} \frac{1}{\mu(B(x, u))} \\ & \leq \int_0^{\delta_1} du_1 \cdots \int_0^{\delta_\ell} du_\ell \log^{1/2} \left(\prod_{i=1}^d \frac{2\phi_i^{-1}(R)}{\phi_i^{-1}(u_i)} \right) \\ & \leq \delta_1 \cdots \delta_\ell \log^{1/2} \left(\prod_{i=1}^{\ell} 2\phi_i^{-1}(R) \right) + \int_0^{\delta_1} du_1 \cdots \int_0^{\delta_\ell} du_\ell \log^{1/2} \left(\prod_{i=1}^d \frac{1}{\phi_i^{-1}(u_i)} \right). \end{aligned}$$

The last integral in the above formula can be estimated as following

$$\begin{aligned}
& \int_0^{\delta_1} du_1 \cdots \int_0^{\delta_\ell} du_\ell \log^{1/2} \left(\prod_{i=1}^d \frac{1}{\phi_i^{-1}(u_i)} \right) \\
&= \int_0^{\phi_1^{-1}(\delta_1)} d\phi_1(u_1) \cdots \int_0^{\phi_\ell^{-1}(\delta_\ell)} d\phi_\ell(u_\ell) \log^{1/2} \left(\prod_{i=1}^d \frac{1}{u_i} \right) \\
&\leq \sum_{i=1}^{\ell} \tilde{\delta}_i \int_0^{\phi_i^{-1}(\delta_i)} \log^{1/2}(1/u_i) d\phi_i(u_i).
\end{aligned}$$

Using integration by parts, $\int_0^{\phi_i^{-1}(\delta_i)} \log^{1/2}(1/u_i) d\phi_i(u_i) \leq \tilde{\omega}(\delta_i)$, which completes the proof. \square

Example 6.3.4. Let $W = (W(x), x \in \mathbb{R}^d)$ be a fractional Brownian sheet with Hurst parameter $H = (H_1, \dots, H_d) \in (0, 1)^d$. In particular, the covariance of W is given by

$$\mathbb{E}W(x)W(y) = \prod_{i=1}^d R_{H_i}(x_i, y_i)$$

where

$$R_{H_i}(s, t) = \frac{1}{2}(|s|^{2H_i} + |t|^{2H_i} - |s - t|^{2H_i}).$$

We see that

$$(\mathbb{E}|W(\square[x, y])|^2)^{1/2} = \prod_{i=1}^d |x_i - y_i|^{H_i},$$

thus $\phi_i(\delta_i) = |\delta_i|^{H_i}$ and $\sigma(\delta, R) = \delta_1 \cdots \delta_d$. We put

$$m(\delta, R) = \mathbb{E} \sup |W(\square[x, y])|.$$

where the supremum is taken over the domain $\{x, y : |x_i|^{H_i}, |y_i|^{H_i} \leq R \text{ and } |x_i - y_i|^{H_i} \leq \delta_i \forall 1 \leq i \leq d\}$. Note that

$$\tilde{\omega}_i(\delta_i) \lesssim \delta_i \log^{1/2} \frac{1}{\phi_i^{-1}(\delta_i)}$$

The bound (6.14) yields

$$m(\delta, R) \lesssim \delta_1 \cdots \delta_d \sqrt{\log(R\delta_1^{-1} \cdots \delta_d^{-1})}.$$

Theorem 6.2.2 yields

$$\sup_{|x_i|^{H_i}, |y_i|^{H_i} \leq R; d^*(x, y) \leq \delta} |W(\square[x, y])| \lesssim \delta_1 \cdots \delta_d \sqrt{\log(R\delta_1^{-1} \cdots \delta_d^{-1})}, \quad (6.15)$$

when R gets large. This implies the inequality of the form (3.1) for W .

Chapter 7

Linear stochastic convolution equation with rough dependence in space

Stochastic partial differential equation with multiplicative noise has become an attractive subject in recent years. In the current chapter, we consider the stochastic convolution equation of the type

$$u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) \sigma(u(s, y)) W(dy, ds) \quad (7.1)$$

where w is given a priori, σ is an affine function, i.e. $\sigma(z) = az + b$ for some constants a, b . G is referred to as Green function. W is a centered Gaussian process with covariance

$$\mathbb{E}[W(s, x)W(t, y)] = \frac{1}{2} \left(|x|^{2H} + |y|^{2H} - |x - y|^{2H} \right) (s \wedge t) \quad (7.2)$$

with $1/4 < H < 1/2$. That is, W is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter in space.

Many stochastic partial differential equations can be formulated into the form (7.1). In the following, we describe three main examples.

Stochastic heat equation (SHE): In differential form, it reads $(\partial_t - \frac{\nu}{2}\Delta)u = \sigma(u)\dot{W}$.

$\vartheta > 0$ is a fixed viscosity parameter. The Green function for the heat equation with viscosity constant ϑ is the Gaussian density,

$$G_t(x) = p_\vartheta(t, x) = \frac{1}{\sqrt{2\vartheta\pi t}} \exp\left(-\frac{x^2}{2\vartheta t}\right) \quad (7.3)$$

If an initial condition is specified at u_0 , the mild formulation of the stochastic heat equation is a special case of (7.1) with $w(t, x) = p_t * u_0(x)$

Stochastic wave equation (SWE): In differential form $(\partial_{tt}^2 - \frac{\vartheta^2}{2}\Delta)u = \sigma(u)\dot{W}$. The Green function of wave equation in dimension one is

$$G(t, x) = \frac{1}{2}I(|x| < \vartheta t).$$

If initial conditions are specified as $u(0, x) = u_0(x)$ and $\partial_t u(0, x) = u_1(x)$, then the stochastic wave equation with multiplicative is a special case of (7.1) with

$$w(t, x) = \frac{1}{2}[u_0(x + \vartheta t) + u_0(x - \vartheta t)] + \frac{1}{2} \int_{x-\vartheta t}^{x+\vartheta t} u_1(y) dy.$$

Stochastic fractional heat equation (SFHE): $(\partial_t - \frac{\vartheta}{2}\Delta^\alpha)u = \sigma(u)\dot{W}$. The Green function is specified by its Fourier transform $\hat{G}(t, \xi) = e^{-\frac{\vartheta}{2}|\xi|^\alpha t}$. The mild formulation of this equation is also a variation of (7.1) with $w(t, x) = G_t * u_0(x)$, where u_0 is the initial condition $u(0, x) = u_0(x)$.

Strictly speaking, the stochastic heat equation is a special case of the stochastic fractional heat equation, however, since this equation attracts much more attention than others, we consider it separately. Further, in Chapter 8, we will consider the stochastic heat equation with general coefficient σ .

Since the pioneering work by Peszat-Zabczyk [82] and Dalang (see [22]), there has been a lot of interest in stochastic partial differential equations driven by a Brownian motion in time with spatial homogeneous covariance. After more than a decade of investigations, the

standard assumptions on W under which existence and uniqueness hold take the following form:

- (i) $\beta[W(s, x)W(t, y)] = \Lambda(x - y) (s \wedge t)$, where Λ is a positive distribution of positive type.
- (ii) The Fourier transform of the spatial covariance Λ is a tempered measure μ that satisfies the integrability condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$.

In case of the covariance (7.2) under consideration, one can easily compute the measure μ , whose explicit expression is $\mu(d\xi) = c_{1,H}|\xi|^{1-2H}d\xi$, where $c_{1,H}$ is a constant depending on H (see expression (8.3) below). In addition, it is readily checked that μ fulfills the condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$ for all $H \in (0, 1)$. However, the corresponding covariance Λ is a distribution which fails to be positive when $H < \frac{1}{2}$, and the covariance of two stochastic integrals with respect to \dot{W} is expressed in terms of fractional derivatives. For this reason, the standard methodology used in the classical references [22, 24, 82] to handle homogeneous spatial covariances does not apply to our case of interest.

In a recent paper, Balan, Jolis and Quer-Sardanyons [8] proved the existence of a unique mild solution for equation (8.1) in the case $\sigma(u) = au + b$, using techniques of Fourier analysis. The method used in [8] cannot be extended to general nonlinear coefficients while the functional settings described here can, as we will see in Chapter 8.

Notations: Often the case, we will see a function of several variables. For this reason, we will write $\|f(s, x)\|_{T_s X_x}$ for $\| \|f(s, x)\|_{T_s} \|_{X_x}$.

7.1 Preliminaries

7.1.1 The noise

Let \mathcal{S} be the space of Schwartz functions on \mathbb{R} . The Fourier transform is defined with the normalization

$$\hat{\phi}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} \phi(x) dx,$$

so that the inverse Fourier transform is given by

$$\check{\phi}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \phi(\xi) d\xi.$$

We denote $\mathbb{R}_+ = [0, \infty)$ and let \mathcal{E} be the collection of functions on $\mathbb{R}_+ \times \mathbb{R}$ of the form $(t, x) \mapsto 1_A(t)\varphi(x)$, where A is Borel set in \mathbb{R}_+ and φ is in \mathcal{S} . On \mathcal{E} , we equip the following scalar inner product

$$\langle 1_A\varphi, 1_B\psi \rangle = c_H |A \cap B| \int_{\mathbb{R}} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} |\xi|^{1-2H} d\xi$$

where H is a fixed constant in $(0, 1/2)$, $c_H = \Gamma(2H+1) \sin(\pi H)/(2\pi)$ is a normalize constant, $|C|$ denotes the Lebesgue measure of a set C , $\hat{\phi}$ denotes the Fourier transform of ϕ and \bar{f} is the complex conjugate of f . Let \mathcal{H} be the Hilbert space obtained by completing \mathcal{E} with respect to the above inner product.

For $\beta \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^\beta(\mathbb{R})$ is the space of tempered distributions u over \mathbb{R} , the Fourier transform of which belongs to $L^1_{\text{loc}}(\mathbb{R})$ and satisfies

$$\|u\|_{\dot{H}^\beta} := \left(\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{u}(\xi)|^2 |\xi|^{2\beta} d\xi \right)^{\frac{1}{2}} < \infty.$$

If $\beta \in (0, 1)$ and f belongs to \dot{H}^β , then there exists a constant C_β such that

$$\|f\|_{\dot{H}^\beta}^2 = C_\beta \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+y) - f(x)|^2 |y|^{-1-2\beta} dx dy. \quad (7.4)$$

We refer to Proposition 1.37 in [7] for the proof of this identity. It is well known that \dot{H}^β is a Hilbert space if and only if $\beta < 1/2$ ([7, pg.26,27]). Moreover if $0 < \beta < 1/2$, then \mathcal{S} is dense in \dot{H}^β . It is not hard to check that in this case, \dot{H}^β contains indicator functions of finite intervals. With these notations, one realizes that the Hilbert space \mathcal{H} is indeed the space-time space $L^2(\mathbb{R}_+; \dot{H}^{1/2-H}(\mathbb{R}))$.

Let us now describe the noise and the stochastic integrations associated with it. There are several ways to construct it so that the identity (7.2) is satisfied. Here, we follow the approach from Nualart's book [80]. Let W be the isonormal Gaussian process with respect to the Hilbert space \mathcal{H} . That is, W is a centered Gaussian process indexed by \mathcal{H} with covariance structure

$$\mathbb{E}[W(h)W(k)] = \langle h, k \rangle_{\mathcal{H}}, \quad \forall h, k \in \mathcal{H}. \quad (7.5)$$

For every (t, x) in $\mathbb{R}_+ \times \mathbb{R}$, we denote $W(t, x) = W(I_{[0,t]} \times I_{[0,x]})$. It is elementary to check that the process $(W(t, x); t \in \mathbb{R}_+, x \in \mathbb{R})$ satisfies (7.2). For each $t \in \mathbb{R}$, let \mathcal{F}_t be the σ -algebra generated by the random variables $\{W(1_{[0,s]}\varphi), s \leq t, \varphi \in \mathcal{S}\}$. For a function h in \mathcal{H} , we define the stochastic integral

$$\int_0^\infty \int_{\mathbb{R}} h(t, x) W(dx, dt) := W(h).$$

We say that a random process $\{u(t, x), (t, x) \in \mathbb{R}^2\}$ is adapted if for every t , $u(t, x)$ is \mathcal{F}_t -measurable for every $x \in \mathbb{R}$. The process u is said to be adapted and elementary if u has the following form

$$u(t, x) = \sum_{i=1}^n F_i(x) I_{(t_i, t_{i+1}]}(t)$$

where F_i is \mathcal{F}_{t_i} -measurable and F_i belongs to \mathcal{S} almost surely. One can define stochastic integration of processes of the above form as following

$$\iint u(t, x) W(dx, dt) = \sum_{i=1}^n W(F_i(x) I_{(t_i, t_{i+1}]}(t)).$$

For two stochastic processes which are adapted and elementary, the following Itô isometry follows from (7.5)

$$\mathbb{E} \left(\iint u(t, x) W(dx, dt) \iint v(s, y) W(dy, ds) \right) = \mathbb{E} \langle u, v \rangle_{\mathcal{H}}. \quad (7.6)$$

By an approximation procedure (see [80]), the above formula holds true for all adapted stochastic processes u, v such that the right hand side is finite.

We conclude this subsection with the following moment estimate for stochastic integrals.

Proposition 7.1.1. *Let $p \geq 2$, f be an adapted random field. Then*

$$\left\| \int_0^t \int_{\mathbb{R}} f(s, y) W(dy, ds) \right\|_{L_{\omega}^p} \leq \sqrt{4p} \|f(s, y)\|_{\dot{H}_y^{1/2-H} L_{\omega}^p L_s^2} \quad (7.7)$$

Proof. Applying Burkholder inequality, we have

$$\left\| \int_0^t \int_{\mathbb{R}} f(s, y) W(dy, ds) \right\|_{L_{\omega}^p} \leq \sqrt{4p} \left\| \int_0^t \|f(s, y)\|_{\dot{H}_y^{1/2-H}}^2 ds \right\|_{L_{\omega}^{p/2}}^{1/2}.$$

The claim is then followed from Minkowski inequality. \square

7.1.2 Space-time function spaces

We introduce here the function spaces which form the underlying framework of our treatments.

Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. Let $\beta \in (0, 1)$ be a fixed number. For every function $f : \mathbb{R} \rightarrow B$, we introduce the function $V_f^{(\beta)} : \mathbb{R} \rightarrow [0, \infty]$

$$V_f^{(\beta)}(x) = \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}.$$

When the value of β is clear from the context, we will write V_f instead of $V_f^{(\beta)}$. As we will see later along the development of the chapter, $V_f(x)$ plays a role analogous to the modulus of continuity of f at x . It follows from Minkowski inequality that V satisfies

$$|V_f(x) - V_g(x)| \leq V_{f-g}(x) \quad (7.8)$$

for every functions f, g and every x in \mathbb{R} . Thus, V is a seminorm.

Suppose for instance that a bounded function f has modulus of continuity around x as $|h|^\beta \omega(h)$. Then $V_f^2(x)$ is majorized by the sum of $\int_0^1 \omega^2(h)h^{-1}dh$ and $\sup_x \|f(x)\|^2$. Thus, in order for $V_f(x)$ to be finite, it is sufficient that $\omega^2(h)h^{-1}$ is integrable near 0. Vice versa, if V_f is bounded over a domain, then f is necessary Hölder continuous as seen in the following result.

Proposition 7.1.2. *Let I be a non-empty open interval of \mathbb{R} . Let f be a function on \mathbb{R} such that $\sup_{x \in \bar{I}} V_f^{(\beta)}(x)$ is finite. Then*

$$\sup_{x \in I; |y| \leq \text{dist}(x, \partial I)} \frac{\|f(x+y) - f(y)\|}{|y|^\beta} \leq c(\beta) \sup_{x \in \bar{I}} V_f^{(\beta)}(x) \quad (7.9)$$

for some finite constant $c(\beta)$ depends only on β .

Proof. For every $x \in I$ and positive R , we denote $f_{x,R} = \frac{1}{2R} \int_{-R}^R f(y+x)dy$. We first estimate $\|f(x) - f_{x,R}\|$ as following

$$\begin{aligned} \|f(x) - f_{x,R}\| &\leq \frac{1}{2R} \int_{-R}^R \|f(x) - f(x+y)\| dy \\ &\leq \frac{1}{2R} \left(\int_{-R}^R \|f(x) - f(x+y)\|^2 |y|^{-1-2\beta} dy \right)^{1/2} \left(\int_{-R}^R |y|^{1+2\beta} dy \right)^{1/2} \\ &\leq \frac{1}{2\sqrt{1+\beta}} R^\beta \sup_{t \in \bar{I}} V_f(t). \end{aligned} \quad (7.10)$$

Let us now fix $x \in I$ and $y \in \mathbb{R}$ such that $|y| \leq \text{dist}(x, \partial I)$. We also choose $R = |y|$. It follows from triangle inequality that

$$\|f(x+y) - f(x)\| \leq \|f(x+y) - f_{x+y,R}\| + \|f_{x+y,R} - f_{x,R}\| + \|f(x) - f_{x,R}\|. \quad (7.11)$$

For the second term, we first use Minkowski inequality, then Cauchy-Schwartz inequality

$$\begin{aligned} \|f_{x+y,R} - f_{x,R}\| &\leq \frac{1}{4R^2} \int_{-R}^R \int_{-R}^R \|f(x+y+z) - f(x+w)\| dz dw \\ &\leq \frac{1}{4R^2} \int_{-R}^R \left(\int_{-R}^R \|f(x+y+z) - f(x+w)\|^2 |y+z-w|^{-2\beta-1} dz \right)^{\frac{1}{2}} \\ &\quad \left(\int_{-R}^R |y+z-w|^{2\beta+1} dz \right)^{\frac{1}{2}} dw. \end{aligned}$$

Because of the restrictions on the variables, $|y+z-w| \leq 3R$ and $x+w \in \bar{I}$. Hence

$$\|f_{x+y,R} - f_{x,R}\| \lesssim_{\beta} \sup_{t \in \bar{I}} V_f(t) R^{\beta}.$$

The first and third terms in the right hand side of (7.11) are estimated in (7.10). Combining these estimates with (7.11) yields (7.9). \square

Let θ be a nonnegative number. $\mathfrak{X}_{\theta}^{\beta}(B)$ is the space of all continuous functions $f : [0, \infty) \times \mathbb{R} \rightarrow B$ such that

(i) $(t, x) \mapsto V_f^{(\beta)}(t, x)$ is finite and bounded on $[0, \infty) \times \mathbb{R}$;

(ii) $\|f\|_{\mathfrak{X}_{\theta}^{\beta}(B)} := \sup_{t \geq 0; x \in \mathbb{R}} e^{-\theta t} \|f(t, x)\| + \sup_{t \geq 0; x \in \mathbb{R}} e^{-\theta t} V_f^{(\beta)}(t, x)$ is finite.

We equip $\mathfrak{X}_{\theta}^{\beta}(B)$ with the norm $\|\cdot\|_{\mathfrak{X}_{\theta}^{\beta}(B)}$ defined as above. Then $\mathfrak{X}_{\theta}^{\beta}(B)$ is a normed vector space.

$\mathfrak{X}_T^{\beta}(B)$ the space of all continuous functions $f : [0, T] \times \mathbb{R} \rightarrow B$ such that

(i) $(t, x) \mapsto V_f^{(\beta)}(t, x)$ is finite and bounded on $[0, T] \times \mathbb{R}$;

(ii) $\|f\|_{\mathfrak{X}_T^{\beta}(B)} := \sup_{t \leq T; x \in \mathbb{R}} \|f(t, x)\| + \sup_{t \geq 0; x \in \mathbb{R}} V_f^{(\beta)}(t, x)$ is finite.

We equip $\mathfrak{X}_T^{\beta}(B)$ with the norm $\|\cdot\|_{\mathfrak{X}_T^{\beta}(B)}$ defined as above. Then $\mathfrak{X}_T^{\beta}(B)$ is a normed vector space. In fact, these spaces are complete.

Proposition 7.1.3. $\mathfrak{X}_T^{\beta}(B)$ and $\mathfrak{X}_{\theta}^{\beta}(B)$ are Banach spaces.

Proof. We present the proof only for $\mathfrak{X}_T^\beta(B)$, one can show $\mathfrak{X}_\theta^\beta(B)$ is a Banach space with analogous arguments. Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{X}_T^\beta(B)$. Since the space $C_b([0, T] \times \mathbb{R}; B)$ is complete, there exists a bounded continuous function $f : [0, T] \times \mathbb{R} \rightarrow B$ such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]; x \in \mathbb{R}} \|f_n(t, x) - f(t, x)\| = 0.$$

Fix $\epsilon > 0$, there exists $n_0 > 0$ such that

$$\sup_{x \in \mathbb{R}} V_{f_n - f_m}(t, x) < \epsilon$$

for all $m, n > n_0$. It follows from Fatou's lemma that

$$V_{f_n - f}(t, x) \leq \liminf_{m \rightarrow \infty} V_{f_n - f_m}(t, x) \leq \epsilon$$

for every $t \in [0, T]$, $x \in \mathbb{R}$ and $n > n_0$. This implies $\sup_{t \leq T; x \in \mathbb{R}} |V_{f_n - f}(t, x)| \rightarrow 0$ as n goes to infinity, which means f_n converges to f in $\mathfrak{X}_T^\beta(B)$. \square

When $B = L^p(\Omega)$ with $p \in [1, \infty)$, we use the notations $\mathfrak{X}_T^{\beta, p} = \mathfrak{X}_T^\beta(L^p(\Omega))$ and $\mathfrak{X}_\theta^{\beta, p} = \mathfrak{X}_\theta^\beta(L^p(\Omega))$. A function f in $\mathfrak{X}_\theta^{\beta, p}$ can be considered as a stochastic process indexed by (t, x) in $[0, T] \times \mathbb{R}$ such that

$$\sup_{t, x} e^{-\theta t} \|f(t, x)\|_{L^p(\Omega)} + \sup_{t, x} e^{-\theta t} \left(\int_{\mathbb{R}} \|f(t, x + y) - f(t, x)\|_{L^p(\Omega)}^2 |y|^{-2\beta-1} dy \right)^{\frac{1}{2}} < \infty.$$

Likewise for $\mathfrak{X}_\theta^{\beta, p}$.

Finally, a useful feature of the operator $\sup_{t \geq 0} e^{-\theta t}$ in (7.13) which makes it appealing for white time noise is that it provides a good control on convolution of two functions. More precisely for every non-negative functions f and g ,

$$\sup_{t \geq 0} e^{-\theta t} \int_0^t f(t-s)g(s)ds \leq \left(\sup_{t \geq 0} e^{-\theta t} f(t) \right) \int_0^\infty e^{-\theta s} g(s)ds. \quad (7.12)$$

In the current article, in most of the cases we choose $\beta = 1/2 - H$. Thus, when $\beta = 1/2 - H$, we will write \mathfrak{X}_θ^p instead of $\mathfrak{X}_\theta^{1/2-H,p}$.

7.1.3 Mismatch of dimensions and equivalence of norms

Careful readers may notice the two qualities appearing in the norm of $\mathfrak{X}_\theta^{\beta,p}$ are not of the same dimension: the first term measures the amplitude while the second term measures oscillation in the spatial variable. More precisely, suppose that the amplitude of f has unit L , the spatial variable x has unit S , while the randomness ω is dimensionless. Then the first term in $\|\cdot\|_{\mathfrak{X}_\theta^{\beta,p}}$ has unit L while the second term has unit L/S^β . A fundamental principle in physics says that only qualities of the same dimension can be totaled. Hence, in order for the two terms to have the same dimension, we multiply the second term with a constant ϵ having unit of S^β . This leads to the following norm on $\mathfrak{X}_\theta^{\beta,p}$

$$\begin{aligned} \|u\|_{\mathfrak{X}_\theta^{\beta,p},\epsilon} := & \sup_{t \geq 0; x \in \mathbb{R}} e^{-\theta t} \|u(t, x)\|_{L_\omega^p} \\ & + \epsilon \sup_{t \geq 0; x \in \mathbb{R}} e^{-\theta t} \left(\int_{\mathbb{R}} \|u(t, x+y) - u(t, x)\|_{L_\omega^p}^2 |y|^{-2\beta-1} dy \right)^{\frac{1}{2}}. \end{aligned} \quad (7.13)$$

From mathematical perspective, the second term in the norm in (7.13) is not invariant by scaling while the first term is. Indeed, denote $u_\lambda(t, x) = u(t, \lambda x)$, then

$$\begin{aligned} \sup_{t \geq 0; x \in \mathbb{R}} e^{-\theta t} \left(\int_{\mathbb{R}} \|u_\lambda(t, x+h) - u_\lambda(t, x)\|_{L_\omega^p}^2 |h|^{2H-2} dh \right)^{1/2} \\ = \lambda^{1/2-H} \sup_{t \geq 0; x \in \mathbb{R}} e^{-\theta t} \left(\int_{\mathbb{R}} \|u(t, x+h) - u(t, x)\|_{L_\omega^p}^2 |h|^{2H-2} dh \right)^{1/2}. \end{aligned}$$

This is the very reason why various orders of $(t-s)$ appear in Lemma 7.2.1 below. Obviously, $\|\cdot\|_{\mathfrak{X}_\theta^{\beta,p},\epsilon}$ is equivalent to the usual norm of $\mathfrak{X}_\theta^{\beta,p}$ defined previously, thus does not alter the topology of the original space.

The same analysis can be carried out to $\mathfrak{X}_T^{\beta,p}$.

7.2 Linear stochastic convolution equation

7.2.1 An estimate for stochastic convolutions

The stochastic convolution of the type $\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u(s,y))W(dy, ds)$ appears on the right hand side of (7.1). Thus understanding this stochastic convolution is an important part in the study of (7.1). We first introduce some qualities that qualify the interplay between regularity of the Green kernel and the noise.

For every $\beta \in (0, 1)$, we define

$$\gamma_1(t) := \int_{\mathbb{R}} |G_t(y)|^2 dy, \quad (7.14)$$

$$\gamma_{2,\beta}(t) := \int_{\mathbb{R}} \int_{\mathbb{R}} |G_t(y+z) - G_t(y)|^2 |z|^{-1-2\beta} dz dy, \quad (7.15)$$

and

$$\gamma_{3,\beta}(t) := \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |G_t(y-z+h) - G_t(y+h) - G_t(y-z) + G_t(y)|^2 |z|^{-1-2\beta} |h|^{2H-2} dy dz dh. \quad (7.16)$$

When $\beta = 1/2 - H$ (so that $-1 - 2\beta = 2H - 2$), we omit the dependence of β in γ_2 and γ_3 . In frequency mode, γ_i 's take simpler forms. Indeed, we apply Fourier transform and Plancherel identity in the y variable, and make change of variables in z and h to get

$$\gamma_1(t) = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 d\xi,$$

$$\begin{aligned} \gamma_{2,\beta}(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 |e^{iz\xi} - 1|^2 |z|^{-1-2\beta} dz d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 |\xi|^{2\beta} d\xi \int_{\mathbb{R}} |e^{iz} - 1|^2 |z|^{-1-2\beta} dz, \end{aligned}$$

and

$$\begin{aligned}\gamma_{3,\beta}(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 |e^{ih\xi} - 1|^2 |e^{iz\xi} - 1|^2 |z|^{-1-2\beta} |h|^{2H-2} dz dh d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 |\xi|^{2(\frac{1}{2}-H+\beta)} d\xi \left(\int_{\mathbb{R}} |e^{ih} - 1|^2 |h|^{2H-2} dh \right) \left(\int_{\mathbb{R}} |e^{iz} - 1|^2 |z|^{-1-2\beta} dz \right).\end{aligned}$$

Since $H < 1/2$ and $\beta \in (0, 1)$, it follows that the integrals $\int_{\mathbb{R}} |e^{ih} - 1|^2 |h|^{2H-2} dh$ and $\int_{\mathbb{R}} |e^{iz} - 1|^2 |z|^{-1-2\beta} dz$ are finite. Therefore, modulo some finite constants, the γ_i 's can be replaced by $\hat{\gamma}_i$'s defined as following

$$\hat{\gamma}_1(t) = \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 d\xi, \quad (7.17)$$

$$\hat{\gamma}_{2,\beta}(t) = c_\beta \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 |\xi|^{2\beta} d\xi, \quad (7.18)$$

$$\hat{\gamma}_{3,\beta}(t) = c_{1/2-H} c_\beta \int_{\mathbb{R}} |\hat{G}_t(\xi)|^2 |\xi|^{2(\frac{1}{2}-H+\beta)} d\xi, \quad (7.19)$$

where $c_\beta = \int_{\mathbb{R}} |e^{iy} - 1|^2 |y|^{-1-2\beta} dy = 4(1+2\beta)\Gamma(-1-2\beta) \cos(\beta\pi)$ is finite provided $0 < \beta < 1$.

The following result gives a qualitative estimate for stochastic convolution of a random field against a deterministic Green kernel. It also qualifies the smoothing effect of the Green kernel.

Proposition 7.2.1. *Let $\beta \in (0, 1)$, $p \geq 2$, f be an adapted random field and $G_t(x)$ be a deterministic kernel. We denote*

$$A(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) f(s, y) W(dy, ds).$$

We assume that γ_1 , $\gamma_{2,\beta}$ and $\gamma_{3,\beta}$ are finite and integrable near 0. Then the following inequality hold

$$\|A\|_{\mathfrak{X}_\theta^{\beta,p}, \epsilon} \leq C_0 \sqrt{p} \|f\|_{\mathfrak{X}_\theta^{1/2-H,p}, \epsilon} \left(\int_0^\infty [\epsilon^{-2}\gamma_1(s) + \gamma_2(s) + \gamma_{2,\beta}(s) + \epsilon^2\gamma_{3,\beta}(s)] e^{-2\theta s} ds \right)^{\frac{1}{2}}, \quad (7.20)$$

where C_0 is a universal constant.

Proof. In what follows, we denote

$$N_{\beta,p,\epsilon}(f) = \sup_{x \in \mathbb{R}} \|f(x)\|_{L_\omega^p} + \epsilon \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_{L_\omega^p}^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}. \quad (7.21)$$

We observe that

$$\sup_{t \geq 0} e^{-\theta t} N_{\beta,p,\epsilon}(f(t, \cdot)) \leq \|f\|_{\mathfrak{X}_\theta^{\beta,p}, \epsilon}. \quad (7.22)$$

Applying inequality (7.7), we have

$$\|A(t, x)\|_{L_\omega^p} \leq \sqrt{4p} \|G_{t-s}(x-y)f(s, y)\|_{\dot{H}_y^{1/2-H} L_\omega^p L_s^2}. \quad (7.23)$$

Using (7.4), we have

$$\begin{aligned} & \|G_{t-s}(x-y)f(s, y)\|_{\dot{H}_y^{1/2-H}}^2 \\ & \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x-y-z)f(s, y+z) - G_{t-s}(x-y)f(s, y)|^2 |z|^{2H-2} dy dz \\ & \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x-y-z) - G_{t-s}(x-y)|^2 |f(s, y+z)|^2 |z|^{2H-2} dy dz \\ & \quad + \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x-y)|^2 |f(s, y+z) - f(s, y)|^2 |z|^{2H-2} dy dz. \end{aligned}$$

We then apply Minkowski inequality to obtain

$$\begin{aligned} \|G_{t-s}(x-y)f(s, y)\|_{\dot{H}_y^{1/2-H} L_\omega^p} &= \left\| \|G_{t-s}(x-y)f(s, y)\|_{\dot{H}_y^{1/2-H}} \right\|_{L_\omega^{p/2}}^{1/2} \\ &\lesssim (J_1)^{1/2} + (J_2)^{1/2} \end{aligned} \quad (7.24)$$

where

$$J_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x-y-z) - G_{t-s}(x-y)|^2 \|f(s, y+z)\|_{L_\omega^p}^2 |z|^{2H-2} dy dz$$

and

$$J_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x-y)|^2 \|f(s, y+z) - f(s, y)\|_{L_\omega^p}^2 |z|^{2H-2} dy dz$$

To estimate J_1 , we first bound $\|f(s, y+z)\|_{L_\omega^p} \leq N_{1/2-H, p, \epsilon}(f(s, \cdot))$, to get

$$J_1 \lesssim N_{1/2-H, p, \epsilon}^2(f(s, \cdot)) \gamma_2(t-s). \quad (7.25)$$

J_2 can be estimated as following. Obviously

$$\int_{\mathbb{R}} \|f(s, y+z) - f(s, y)\|_{L_\omega^p}^2 |z|^{2H-2} dz \leq \epsilon^{-2} N_{1/2-H, p, \epsilon}^2(f(s, \cdot)), \quad (7.26)$$

thus

$$J_2 \leq \epsilon^{-2} N_{1/2-H, p, \epsilon}^2(f(s, \cdot)) \gamma_1(t-s). \quad (7.27)$$

Hence, we combine altogether the estimates (7.23), (7.24), (7.25) and (7.27) to get

$$\sup_{x \in \mathbb{R}} \|A(t, x)\|_{L_\omega^p}^2 \lesssim p \int_0^t N_{1/2-H, p, \epsilon}^2(f(s, \cdot)) [\epsilon^{-2} \gamma_1(t-s) + \gamma_2(t-s)] ds. \quad (7.28)$$

Next, we estimate the second term in the norm $N_{\beta, p, \epsilon}(A(t, \cdot))$. For every $h \in \mathbb{R}$, we apply inequality (7.7) to get

$$\|A(t, x+h) - A(t, x)\|_{L_\omega^p} \leq \sqrt{4p} \| [G_{t-s}(x+h-y) - G_{t-s}(x-y)] f(s, y) \|_{\dot{H}_y^{1/2-H} L_\omega^p L_s^2} \quad (7.29)$$

The computations are carried out as before. From (7.4), we can write

$$\begin{aligned} & \| [G_{t-s}(x+h-y) - G_{t-s}(x-y)] f(s, y) \|_{\dot{H}_y^{1/2-H}} \\ & \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x+h-y-z) - G_{t-s}(x-y-z) \\ & \quad - G_{t-s}(x+h-y) + G_{t-s}(x-y)|^2 |f(s, y+z)|^2 |z|^{2H-2} dy dz \\ & \quad + \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 |f(s, y+z) - f(s, y)|^2 |z|^{2H-2} dy dz \end{aligned}$$

Using Minkowski inequality, we see that

$$\| [G_{t-s}(x+h-y) - G_{t-s}(x-y)]f(s, y) \|_{\dot{H}_y^{1/2-H} L_\omega^p} \lesssim (J'_1)^{1/2} + (J'_2)^{1/2} \quad (7.30)$$

where

$$J'_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x+h-y-z) - G_{t-s}(x-y-z) - G_{t-s}(x+h-y) + G_{t-s}(x-y)|^2 \|f(s, y+z)\|_{L_\omega^p}^2 |z|^{2H-2} dy dz,$$

and

$$J'_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 \|f(s, y+z) - f(s, y)\|_{L_\omega^p}^2 |z|^{2H-2} dy dz.$$

J'_1 is estimated similarly to J_1

$$J'_1 \leq N_{1/2-H,p,\epsilon}^2(f(s, \cdot)) \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x+h-y-z) - G_{t-s}(x-y-z) - G_{t-s}(x+h-y) + G_{t-s}(x-y)|^2 |z|^{2H-2} dy dz.$$

To estimate J'_2 , we use the estimate (7.26) to get

$$J'_2 \leq \epsilon^{-2} N_{1/2-H,p,\epsilon}^2(f(s, \cdot)) \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 dy.$$

Combining these estimates for J'_1 , J'_2 and (7.30), (7.29) altogether yields

$$\begin{aligned} & \epsilon^2 \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \|A(t, x+h) - A(t, x)\|_{L_\omega^p}^2 |h|^{-1-2\beta} dh \\ & \lesssim p \epsilon^2 \int_0^t \int_{\mathbb{R}} (J'_1 + J'_2) |h|^{-1-2\beta} dh ds \\ & \lesssim p \int_0^t N_{1/2-H,p,\epsilon}^2(f(s, \cdot)) \left[\epsilon^2 \gamma_{3,\beta}(t-s) + \gamma_{2,\beta}(t-s) \right] ds. \end{aligned} \quad (7.31)$$

Combining (7.28), (7.31), (7.22) and (7.12) altogether yields (7.20). \square

Remark 7.2.2. In terms of dimensional analysis (performed in Subsection 7.1.3), in Proposition 7.2.1, one should consider the quality $\|A\|_{\mathfrak{X}_\theta^{\beta,p},\epsilon'}$ with a different dimensional constant ϵ' if β is different from $1/2 - H$. This leads to the following estimate

$$\|A\|_{\mathfrak{X}_\theta^{\beta,p},\epsilon'} \leq C_0 \sqrt{p} \|f\|_{\mathfrak{X}_\theta^{1/2-H,p},\epsilon} \left(\int_0^\infty [\epsilon^{-2}\gamma_1(s) + \gamma_2(s) + (\epsilon'\epsilon^{-1})^2\gamma_{2,\beta}(s) + (\epsilon')^2\gamma_{3,\beta}(s)] e^{-2\theta s} ds \right)^{\frac{1}{2}}. \quad (7.32)$$

However, for our current purpose, the estimate (7.20) is sufficient.

7.2.2 Existence and uniqueness

We now assume that the kernel G (the *Green function*) satisfies the conditions in Proposition 7.2.1. In addition, we assume that for some values $\lambda > 0$ and $\tau \geq 0$, G can be written as

$$G(t, x) = t^{-\tau} K(x/t^\lambda) \quad (7.33)$$

where $K(x)$ is referred to as the *reduced Green function*. One can write the functions γ_i , ($i = 1, 2, 3$) according to the reduced Green function

$$\gamma_1(t) = \kappa_1 t^{-2\tau+\lambda}, \quad \gamma_{2,\beta}(t) = \kappa_{2,\beta} t^{-2\tau+2\lambda(1/2-\beta)}, \quad \gamma_{3,\beta}(t) = \kappa_{3,\beta} t^{-2\tau+2\lambda(H-\beta)}$$

where

$$\begin{aligned} \kappa_1 &= \int_{\mathbb{R}} |K(y)|^2 dy, \\ \kappa_{2,\beta} &= \int_{\mathbb{R}} \int_{\mathbb{R}} |K(y+z) - K(y)|^2 |z|^{-1-2\beta} dz dy, \end{aligned}$$

and

$$\kappa_{3,\beta} = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |K(y-z+h) - K(y+h) - K(y-z) + K(y)|^2 |z|^{-1-2\beta} |h|^{2H-2} dz dh dy .$$

Similarly,

$$\hat{\gamma}_1(t) = \hat{\kappa}_1 t^{-2\tau+\lambda} , \quad \hat{\gamma}_2(t) = c_\beta \hat{\kappa}_2 t^{-2\tau+2\lambda(1/2-\beta)} , \quad \hat{\gamma}_3(t) = c_{1/2-H} c_\beta \hat{\kappa}_3 t^{-2\tau+2\lambda(H-\beta)}$$

where

$$\hat{\kappa}_1 = \int_{\mathbb{R}} |\hat{K}(\xi)|^2 d\xi , \quad \hat{\kappa}_{2,\beta} = \int_{\mathbb{R}} |\hat{K}(\xi)|^2 |\xi|^{2\beta} d\xi , \quad \hat{\kappa}_{3,\beta} = \int_{\mathbb{R}} |\hat{K}(\xi)|^2 |\xi|^{2(\beta+\frac{1}{2}-H)} d\xi .$$

Theorem 7.2.3. *We denote*

$$\epsilon_0 = \left(\frac{\kappa_1}{\kappa_3} \right)^{\frac{1}{4}} \theta_0^{\lambda(H-1/2)} ; \quad \theta_0 = \left(2(\kappa_2^{\frac{1}{2}} + \kappa_1^{\frac{1}{4}} \kappa_3^{\frac{1}{4}}) C_1 \sqrt{p} \|\sigma\|_{\text{Lip}} \right)^{\frac{2}{2\lambda H+1-2\tau}} , \quad (7.34)$$

where C_1 is the constant in (7.37) below. Suppose that σ is affine and w belongs to the space $\mathfrak{X}_{\theta_0}^p$.

We assume that $\kappa_1, \kappa_2, \kappa_3$ are finite and

$$H > \frac{1}{4} + \frac{2\tau - 1}{4\lambda} . \quad (7.35)$$

There exists a unique solution u to the equation (7.1) in the space $\mathfrak{X}_{\theta_0}^p$. In particular, the following estimate holds

$$\begin{aligned} \sup_{x \in \mathbb{R}} \|u(t, x)\|_{L_\omega^p} + p^{\frac{\lambda(H-1/2)}{2\lambda H+1-2\tau}} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \|u(t, x+h) - u(t, x)\|_{L_\omega^p}^2 |h|^{2H-2} dh \right)^{1/2} \\ \lesssim \|w\|_{\mathfrak{X}_{\theta_0}^p, \epsilon_0} \exp(C_H ((\kappa_2^{\frac{1}{2}} + \kappa_1^{\frac{1}{4}} \kappa_3^{\frac{1}{4}}) \|\sigma\|_{\text{Lip}})^{\frac{2}{2\lambda H+1-2\tau}} p^{\frac{1}{2\lambda H+1-2\tau}} t) \end{aligned} \quad (7.36)$$

for some positive constant C_H .

Proof. Let u and v be in $\mathfrak{X}_{\theta_0}^p$. We define the operator

$$Lu(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) \sigma(u(s, y)) W(dy, ds).$$

Since σ has linear growth, it follows from (7.20) (with $\beta = 1/2 - H$) that L maps $\mathfrak{X}_{\theta_0}^p$ into itself. The estimate (7.20) thus implies

$$\begin{aligned} \|Lu - Lv\|_{\mathfrak{X}_{\theta_0}^p, \epsilon} &\leq C_0 \sqrt{p} \|\sigma\|_{\text{Lip}} \|u - v\|_{\mathfrak{X}_{\theta_0}^p, \epsilon} \\ &\quad \left(\int_0^\infty (\kappa_1 \epsilon^{-2} s^{-2\tau+\lambda} + \kappa_2 s^{-2\tau+2\lambda H} + \kappa_3 \epsilon^2 s^{-2\tau+\lambda(4H-1)}) e^{-2\theta s} ds \right)^{\frac{1}{2}} \end{aligned}$$

Condition (7.35) to ensure that the three integrals are finite, thus yields

$$\begin{aligned} \|Lu - Lv\|_{\mathfrak{X}_{\theta_0}^p, \epsilon} &\leq C_1 \sqrt{p} \|\sigma\|_{\text{Lip}} \|u - v\|_{\mathfrak{X}_{\theta_0}^p, \epsilon} \\ &\quad \left(\sqrt{\kappa_1} \epsilon^{-1} \theta^{\tau-\lambda/2-1/2} + \sqrt{\kappa_2} \theta^{\tau-\lambda H-1/2} + \sqrt{\kappa_3} \epsilon \theta^{\tau-\lambda(2H-1/2)-1/2} \right), \quad (7.37) \end{aligned}$$

We then choose $\epsilon = \epsilon_0$ and $\theta = \theta_0$. With this choice, the previous estimate becomes

$$\|Lu - Lv\|_{\mathfrak{X}_{\theta_0}^p, \epsilon_0} \leq \frac{1}{2} \|u - v\|_{\mathfrak{X}_{\theta_0}^p, \epsilon_0}.$$

Therefore the map L is a contraction map on $(\mathfrak{X}_{\theta_0}^p, \|\cdot\|_{\mathfrak{X}_{\theta_0}^p, \epsilon_0})$ with the choice of ϵ_0 and θ_0 in (7.34). By contraction principle, there exists a unique solution to the equation (7.1). The estimate (7.36) follows immediately from (7.20) and (7.34). \square

7.2.3 Examples

We revisit our three examples above and compute their corresponding coefficients.

Stochastic heat equation: The reduced Green function is $K(x) = \frac{1}{\sqrt{2s\pi}} e^{-\frac{x^2}{2s}}$, $\tau = \lambda = 1/2$.

The Fourier transform of K is $\hat{K}(\xi) = e^{-\frac{\vartheta}{2}\xi^2}$. Thus

$$\begin{aligned}\hat{\kappa}_1 &= \int_{\mathbb{R}} e^{-\vartheta\xi^2} d\xi = \sqrt{\frac{\pi}{\vartheta}}, \\ \hat{\kappa}_{2,\beta} &= \int_{\mathbb{R}} e^{-\vartheta\xi^2} |\xi|^{2\beta} d\xi = \vartheta^{-\beta-\frac{1}{2}} \Gamma(\beta + 1/2), \\ \hat{\kappa}_{3,\beta} &= \int_{\mathbb{R}} e^{-\vartheta\xi^2} |\xi|^{2(\beta+1/2-H)} d\xi = \vartheta^{-1+H-\beta} \Gamma(\beta - H + 1)\end{aligned}$$

are finite numbers. In addition, $\gamma_1(t) = c_1 t^{-1/2}$, $\gamma_{2,\beta}(t) = c_2 t^{-1/2-\beta}$, $\gamma_{3,\beta}(t) = c_3 t^{H-\beta-1}$. Thus, from Theorem 7.2.3, equation (7.1) with σ affine has a unique solution if and only if the condition (7.35) is satisfied. This yields the restriction $H > 1/4$. θ_0 is given by

$$\theta_0 = \text{const} \|\sigma\|_{\text{Lip}}^{\frac{2}{H}} \vartheta^{1-\frac{1}{H}} p^{\frac{1}{H}}.$$

Stochastic wave equation: $G(t, x) = \frac{1}{2}I(|x| < \vartheta t)$ is the fundamental solution to the wave equation. The reduced Green function is $\frac{1}{2}I(|x| < \vartheta)$, whose Fourier transform is $\frac{\sin(\vartheta\xi)}{\xi}$. The scaling factors are $\tau = 0$ and $\lambda = 1$. It follows that $\kappa_1 = \vartheta/2$ and

$$\begin{aligned}\hat{\kappa}_{2,\beta} &= \int_{\mathbb{R}} \frac{\sin^2(\vartheta\xi)}{|\xi|^2} |\xi|^{2\beta} d\xi = \vartheta^{1-2\beta} \int_{\mathbb{R}} \frac{\sin^2(\xi)}{|\xi|^2} |\xi|^{2\beta} d\xi \\ \hat{\kappa}_{3,\beta} &= \int_{\mathbb{R}} \frac{\sin^2(\vartheta\xi)}{|\xi|^2} |\xi|^{2(\beta+\frac{1}{2}-H)} d\xi = \vartheta^{2(H-\beta)} \int_{\mathbb{R}} \frac{\sin^2(\xi)}{|\xi|^2} |\xi|^{2(\beta+\frac{1}{2}-H)} d\xi\end{aligned}$$

are finite if $0 \leq \beta < H$. Thus, κ_2, κ_3 (with $b = 1/2 - H$) are finite if and only if $H > 1/4$. The condition (7.35) becomes the trivial inequality $H > 0$. The exponential growth rate of p -moments is

$$\theta_0 = \text{const} \|\sigma\|_{\text{Lip}}^{\frac{2}{2H+1}} \vartheta^{\frac{2H}{2H+1}} p^{\frac{1}{2H+1}}.$$

Stochastic fractional heat equation: The Fourier transform of the Green function is $e^{-\frac{\vartheta}{2}t|\xi|^\alpha}$. Thus the Fourier transform of the reduced Green function is $\hat{K}(\xi) = e^{-\frac{\vartheta}{2}|\xi|^\alpha}$. In

addition, $\tau = \lambda = \frac{1}{\alpha}$. From here we see that

$$\begin{aligned}\hat{\kappa}_1 &= \int_{\mathbb{R}} e^{-\vartheta|\xi|^\alpha} d\xi = \vartheta^{-\frac{1}{\alpha}} \frac{2}{\alpha} \Gamma\left(\frac{1}{\alpha}\right), \\ \hat{\kappa}_{2,\beta} &= \int_{\mathbb{R}} e^{-\vartheta|\xi|^\alpha} |\xi|^{2\beta} d\xi = \vartheta^{-\frac{1+2\beta}{\alpha}} \frac{2}{\alpha} \Gamma\left(\frac{1+2\beta}{\alpha}\right), \\ \hat{\kappa}_{3,\beta} &= \int_{\mathbb{R}} e^{-\vartheta|\xi|^\alpha} |\xi|^{2(\beta+1/2-H)} d\xi = \vartheta^{-\frac{2-2(H-\beta)}{\alpha}} \frac{2}{\alpha} \Gamma\left(\frac{2-2(H-\beta)}{\alpha}\right)\end{aligned}$$

are finite numbers if $\beta > 0$. The condition (7.35) becomes

$$H > \frac{3-\alpha}{4}.$$

Since H is at most $1/2$, this inequality implicitly restricts α to the region $\alpha \in (1, 2]$. The exponential growth rate of p -moments is

$$\theta_0 = \text{const} \|\sigma\|_{\text{Lip}}^{\frac{2\alpha}{2H+\alpha-2}} \vartheta^{-\frac{2(1-H)}{2H+\alpha-2}} p^{\frac{\alpha}{2H+\alpha-2}}$$

7.3 The Anderson model with more general initial data

In the special case $\sigma(u) = \lambda u$, equation (7.1) takes form

$$u(t, x) = w(t, x) + \lambda \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) u(s, y) W(dy, ds). \quad (7.38)$$

Equation (7.38) is a generalized version of the Anderson model. In the case of heat equation, the previous equation is the well-known parabolic Anderson model, and is related to challenging systems in random environment like KPZ equation [11, 49] or polymers [1, 12]. The localization and intermittency properties of the parabolic Anderson model have thus been thoroughly studied for equations driven by a Brownian motion (see [68] for a nice survey), while a recent trend consists in extending this kind of result to

equations driven by very general Gaussian noises [17,57,64,65].

In certain situations, $w(t, x)$ may not satisfy the conditions in Theorem 7.2.3. For instance, in the case of heat equation, the initial datum u_0 can be a Dirac mass at x_0 , in which $w(t, x) = p_t(x - x_0)$ is singular at $t = 0$. To treat this class of initial conditions, we introduce another family of space-time function spaces as following.

Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. Let $\beta \in (0, 1)$ be a fixed number. $\mathfrak{Y}_T^\beta(B)$ is the space of all (strongly) measurable functions $f : [0, T] \times \mathbb{R} \rightarrow B$ such that

$$\|f\|_{\mathfrak{Y}_T^\beta(B)}^2 := \int_0^T \int_{\mathbb{R}} \|f(t, x)\|^2 dx dt + \int_0^T \iint_{\mathbb{R}^2} \|f(t, x+h) - f(t, x)\|^2 |h|^{-2\beta-1} dh dx dt$$

is finite. We equip $\mathfrak{Y}_T^\beta(B)$ with the norm $\|\cdot\|_{\mathfrak{Y}_T^\beta(B)}$ defined as above.

Similarly, for every $\theta > 0$, $\mathfrak{Y}_\theta^\beta(B)$ is the space of all measurable functions $f : [0, T] \times \mathbb{R} \rightarrow B$ such that

$$\|f\|_{\mathfrak{Y}_\theta^\beta(B)}^2 := \int_0^\infty \int_{\mathbb{R}} e^{-2\theta t} \|f(t, x)\|^2 dx dt + \int_0^\infty e^{-2\theta t} \iint_{\mathbb{R}^2} \|f(t, x+h) - f(t, x)\|^2 |h|^{-2\beta-1} dh dx dt$$

is finite. We equip $\mathfrak{Y}_\theta^\beta(B)$ with the norm $\|\cdot\|_{\mathfrak{Y}_\theta^\beta(B)}$ defined as above.

Proposition 7.3.1. $\mathfrak{Y}_T^\beta(B)$ is a Banach space.

Proof. The proof is similar to that of Proposition 7.1.3 and therefore is skipped to the sack of conciseness. \square

When $B = L^p(\Omega)$ with $p \in [1, \infty)$, we use the notations $\mathfrak{Y}_T^{\beta,p} = \mathfrak{Y}_T^\beta(L^p(\Omega))$.

Proposition 7.3.2. Let $\beta \in (0, 1)$, $p \geq 2$, f be an adapted random field and $G_t(x)$ be a deterministic kernel. We denote

$$A(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y) f(s, y) W(dy, ds).$$

We assume that γ_1 , $\gamma_{2,\beta}$ and $\gamma_{3,\beta}$ are finite and integrable near 0. Then the following inequality hold

$$\|A\|_{\mathfrak{Y}_{\theta}^{\beta,p},\epsilon} \leq C_0 \sqrt{p} \|f\|_{\mathfrak{Y}_{\theta}^{\frac{1}{2}-H,p},\epsilon} \left(\int_0^{\infty} [\epsilon^{-2}\gamma_1(s) + \gamma_2(s) + \gamma_{2,\beta}(s) + \epsilon^2\gamma_{3,\beta}(s)] e^{-2\theta s} ds \right)^{\frac{1}{2}}, \quad (7.39)$$

where C_0 is the universal constant in (7.20).

Proof. The argument is very similar to the proof of Proposition 7.2.1. We illustrate the similarity by showing the following estimate and leave further details for interested readers,

$$\int_0^{\infty} \int_{\mathbb{R}} e^{-2\theta t} \|A(t, x)\|_{L_{\omega}^p}^2 dx dt \lesssim p \int_0^{\infty} e^{-2\theta t} (\epsilon^{-2}\gamma_1(t) + \gamma_2(t)) dt \|f\|_{\mathfrak{Y}_{\theta}^p,\epsilon}^2. \quad (7.40)$$

In fact, following the estimates there, we have

$$\|A(t, x)\|_{L_{\omega}^p}^2 \lesssim p \int_0^t (J_1(s) + J_2(s)) ds$$

where

$$J_1(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x - y - z) - G_{t-s}(x - y)|^2 \|f(s, y + z)\|_{L_{\omega}^p}^2 |z|^{2H-2} dy dz$$

and

$$J_2(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{t-s}(x - y)|^2 \|f(s, y + z) - f(s, y)\|_{L_{\omega}^p}^2 |z|^{2H-2} dy dz.$$

It follows that

$$\begin{aligned} \int_0^{\infty} e^{-2\theta t} \int_{\mathbb{R}} \|A(t, x)\|_{L_{\omega}^p}^2 dx dt &\lesssim p \int_0^{\infty} \int_0^t e^{-2\theta t} \gamma_2(t-s) \|f(s, y)\|_{L_{\omega}^p L_y^2}^2 ds dt \\ &\quad + \int_0^{\infty} \int_0^t e^{-2\theta t} \gamma_1(t-s) \|f(s, y)\|_{L_{\omega}^p \dot{H}_y^{\frac{1}{2}-H}}^2 ds dt. \end{aligned}$$

By interchanging the order of integration, we obtain (7.40). \square

The following existence and uniqueness result is a direct application of the previous

proposition. The proof is similar to that of Theorem 7.2.3 and hence will be omitted.

Theorem 7.3.3. *Let θ_0 be the constant in (7.34). Suppose w belongs to the space $\mathfrak{Y}_{\theta_0}^p$. We assume that $\kappa_1, \kappa_2, \kappa_3$ are finite and condition (7.35) is satisfied. There exists a unique solution u to the equation (7.38) in the space $\mathfrak{Y}_{\theta_0}^p$.*

We conclude the chapter with the following remark.

Remark 7.3.4. With a little more work, one can show that the solution found in the above theorem is indeed a random-field solution. In other words, the second moment $\mathbb{E}u(t, x)^2$ is finite for every fixed t and x . However, we will not pursue this direction here.

Chapter 8

Nonlinear stochastic heat equation

In this chapter we are interested in the one-dimensional stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2} \frac{\partial^2 u}{\partial x^2} + \sigma(u) \dot{W}, \quad t \geq 0, \quad x \in \mathbb{R}, \quad (8.1)$$

where W is a centered Gaussian process with covariance given by (7.2) with $\frac{1}{4} < H < \frac{1}{2}$. That is, W is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter H in the space variable. For this stochastic heat equation with a rough noise in space, understood in the Itô sense, our aim is twofold: on one hand, for a differentiable coefficient σ with a Lipschitz derivative and satisfying $\sigma(0) = 0$, we will obtain the existence and uniqueness of the solution. On the other hand, we shall further investigate the special relevant case $\sigma(u) = u$. We now detail those two main points.

Indeed, the isometry property of stochastic integrals with respect to W involves the semi norm

$$\mathcal{N}_{\frac{1}{2}-H,2} u(t, x) = \left(\int_{\mathbb{R}} \mathbb{E} |u(t, x+h) - u(t, x)|^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}.$$

Then, if u and v are two solutions, $\mathcal{N}_{\frac{1}{2}-H,2}(\sigma(u) - \sigma(v))$ cannot be bounded in terms of $\mathcal{N}_{\frac{1}{2}-H,2}(u - v)$, due to the presence of a double increment of the form $\sigma(u(s, z+h)) - \sigma(v(s, z+h)) - \sigma(u(s, z)) + \sigma(v(s, z))$. To overcome this difficulty we have used a truncation

argument to show the uniqueness of mild solutions, inspired by the work [48] on the stochastic Burgers equation on the whole real line driven by a space-time white noise. The main ingredient is a uniform estimate of the $L^p(\Omega)$ -norm of a stochastic convolution (see Lemma 8.3.1). Due to this argument, the uniqueness is obtained in the space \mathcal{Z}_T^p which requires integrability in the space variable.

The existence of a solution is much more involved. The methodology, inspired by the work of Gyöngy and Krylov [47] on stochastic differential equations and Gyöngy [46] on semi linear stochastic partial differential equations, consists of taking approximations obtained by regularizing the noise and using a compactness argument on a suitable space of trajectories, together with the strong uniqueness result.

We also establish the Hölder continuity of the solution u in both space and time variables. We also derive exponential upper bounds for the moments using a sharp Burkholder's inequality, and the matching lower bounds for second moments using Sobolev embedding argument. A more detail description can be found in [56].

The chapter is organized as follows. Section 8.1 contains some preliminaries on stochastic integration with respect to the noise W and elements of Malliavin calculus. Section 8.2 deals with basic moment estimates and Hölder continuity properties of stochastic convolutions. We establish the uniqueness of a solution in Section 8.3. To do this, first we derive moment estimates for the supremum norm in space and time for stochastic convolution. In order to show the existence we need to introduce several spaces of functions in Section 8.3.2 and derive compactness criteria.

8.1 Preliminaries

8.1.1 Noise structure and stochastic integration

Our noise W can be seen as a Brownian motion with values in an infinite dimensional Hilbert space. One might thus think that the stochastic integration theory with respect to

W can be handled by classical theories (see e.g [14,21,22]). However, the spatial covariance function of W , which is formally equal to $H(2H - 1)|x - y|^{2H-2}$, is not locally integrable when $H < 1/2$ (in other words, the Fourier transform of $|\xi|^{1-2H}$ is not a function), and W thus lies outside the scope of application of these classical references. Due to this fact, we provide some details about the construction of a stochastic integral with respect to our noise.

Let us start by introducing our basic notation on Fourier transforms of functions. The space of Schwartz functions is denoted by \mathcal{S} . Its dual, the space of tempered distributions, is \mathcal{S}' . The Fourier transform of a function $u \in \mathcal{S}$ is defined with the normalization

$$\mathcal{F} u(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u(x) dx,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1} u(\xi) = (2\pi)^{-1} \mathcal{F} u(-\xi)$.

Let $\mathcal{D}((0, \infty) \times \mathbb{R})$ denote the space of real-valued infinitely differentiable functions with compact support on $(0, \infty) \times \mathbb{R}$. Taking into account the spectral representation of the covariance function of the fractional Brownian motion in the case $H < \frac{1}{2}$ proved in [83, Theorem 3.1], we represent our noise W by a zero-mean Gaussian family $\{W(\varphi); \varphi \in \mathcal{D}((0, \infty) \times \mathbb{R})\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose covariance structure is given by

$$\mathbb{E} [W(\varphi) W(\psi)] = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F} \varphi(s, \xi) \overline{\mathcal{F} \psi(s, \xi)} |\xi|^{1-2H} ds d\xi, \quad (8.2)$$

where the Fourier transforms $\mathcal{F} \varphi, \mathcal{F} \psi$ are understood as Fourier transforms in space only and

$$c_{1,H} = \frac{1}{2\pi} \Gamma(2H + 1) \sin(\pi H). \quad (8.3)$$

The inner product appearing in (8.2) can be expressed in terms of fractional derivatives. Let $\beta \in (0, 1)$. Define (see [89]) the Marchaud fractional derivative with respect to the space

variable D_-^β of order β of a function $\varphi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$D_-^\beta \varphi(s, x) = \lim_{\varepsilon \rightarrow 0} D_{-, \varepsilon}^\beta \varphi(s, x), \quad (8.4)$$

where

$$D_{-, \varepsilon}^\beta \varphi(s, x) = \frac{\beta}{\Gamma(1-\beta)} \int_\varepsilon^\infty \frac{\varphi(s, x) - \varphi(s, x+y)}{y^{1+\beta}} dy,$$

and define the fractional integral of order β of a function $\psi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$I_-^\beta \psi(s, x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \psi(s, u) (x-u)^{\beta-1} du.$$

Note that here the fractional differentiation and integration are only with respect to space variables. Observe that if $\varphi = I_-^\beta \psi$ for some $\psi \in L^2(\mathbb{R}_+ \times \mathbb{R})$, then by Theorem 6.1 in [89] we have

$$D_-^\beta \varphi = D_-^\beta (I_-^\beta \psi) = \psi$$

and, hence,

$$\int_{\mathbb{R}_+ \times \mathbb{R}} \left[D_-^\beta \varphi(s, x) \right]^2 ds dx = \int_{\mathbb{R}_+ \times \mathbb{R}} \psi^2(s, x) ds dx < \infty.$$

Then for our noise it is known (cf. [83] for further details) that

$$\mathbb{E} [W(\varphi) W(\psi)] = c_{2,H} \int_{\mathbb{R}_+ \times \mathbb{R}} D_-^{\frac{1}{2}-H} \varphi(s, x) D_-^{\frac{1}{2}-H} \psi(s, x) ds dx, \quad (8.5)$$

where

$$c_{2,H} = \left[\Gamma \left(H + \frac{1}{2} \right) \right]^2 \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right)^{-1}.$$

for any $\varphi, \psi \in \mathcal{D}((0, \infty) \times \mathbb{R})$.

Based on the previous observation and relation (8.5), we introduce a new set of function spaces. Indeed, let \mathfrak{S} be the class of functions $\varphi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that there exists $\psi \in L^2(\mathbb{R}_+ \times \mathbb{R})$ satisfying $\varphi(s, x) = I_-^{\frac{1}{2}-H} \psi(s, x)$. The relation between \mathfrak{S} and our noise W

is given in the following proposition.

Proposition 8.1.1. *The class of functions \mathfrak{S} is a Hilbert space with the inner product*

$$\langle \varphi, \psi \rangle_{\mathfrak{S}} := c_{2,H} \int_{\mathbb{R}_+ \times \mathbb{R}} D_{\pm}^{\frac{1}{2}-H} \varphi(s, x) D_{\pm}^{\frac{1}{2}-H} \psi(s, x) ds dx, \quad (8.6)$$

and $\mathcal{D}((0, \infty) \times \mathbb{R})$ is dense in \mathfrak{S} . Moreover if \mathfrak{S}_0 denotes the class of functions $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})$ such that $\int_{\mathbb{R}_+ \times \mathbb{R}} |\mathcal{F} \varphi(s, \xi)|^2 |\xi|^{1-2H} d\xi ds < \infty$, then \mathfrak{S}_0 is not complete and the inclusion $\mathfrak{S}_0 \subset \mathfrak{S}$ is strict. Also for any $\varphi, \psi \in \mathfrak{S}_0$,

$$\langle \varphi, \psi \rangle_{\mathfrak{S}} = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F} \varphi(s, \xi) \overline{\mathcal{F} \psi(s, \xi)} |\xi|^{1-2H} d\xi ds. \quad (8.7)$$

For the proof of this proposition, we refer to [83]. Note that in [83], the functions considered there are from \mathbb{R} to \mathbb{R} , but by scrutinizing the proofs we see that the results of this paper can be easily extended to our case, i.e. the functions from $\mathbb{R}_+ \times \mathbb{R}$ to \mathbb{R} . We omit the details.

For any $\beta \in (0, 1)$, the homogeneous Sobolev space \dot{H}^β is the completion of the space of infinitely differentiable functions with compact support with respect to the norm (see Proposition 1.37 in [7])

$$\|f\|_{\dot{H}^\beta}^2 = \int_{\mathbb{R}} |D_{\pm}^\beta f(x)|^2 dx = c_{3,\beta}^2 \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+y) - f(x)|^2 |y|^{-1-2\beta} dx dy, \quad (8.8)$$

where $c_{3,\beta}^2 = \frac{(\frac{1}{2}-\beta)\beta}{c_{2,\frac{1}{2}-\beta}}$. As a consequence, our Hilbert space \mathfrak{S} can be identified with the homogenous Sobolev space of order $\beta = \frac{1}{2} - H$ of functions with values in $L^2(\mathbb{R}_+)$, that is $\mathfrak{S} = \dot{H}^{\frac{1}{2}-H}(L^2(\mathbb{R}_+))$ and for any $f \in \mathfrak{S}$,

$$\|f\|_{\mathfrak{S}}^2 = c_{3,\frac{1}{2}-H}^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(s, x+y) - f(s, x)|^2 |y|^{2H-2} dx dy ds.$$

From Proposition 8.1.1, we see that the Gaussian family W can be extended as an isonormal

Gaussian process $W = \{W(\phi), \phi \in \mathfrak{H}\}$ indexed by the Hilbert space \mathfrak{H} .

Let us now turn to the stochastic integration with respect to W . Since we are handling a Brownian motion in time, one can start by integrating elementary processes.

Definition 8.1.2. Let \mathcal{F}_t be the σ -algebra generated by W up to time t . An elementary process g is given by

$$g(s, x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \mathbf{1}_{(a_i, b_i]}(s) \mathbf{1}_{(h_j, l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \dots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are \mathcal{F}_{a_i} -measurable random variables for $i = 1, \dots, n$. The integral of such a process with respect to W is defined as

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s, x) W(ds, dx) &= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} W(\mathbf{1}_{(a_i, b_i]} \otimes \mathbf{1}_{(h_j, l_j]}) \\ &= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} [W(b_i, l_j) - W(a_i, l_j) - W(b_i, h_j) + W(a_i, h_j)]. \end{aligned} \quad (8.9)$$

We can now extend the notion of integral with respect to W to a broad class of adapted processes.

Proposition 8.1.3. Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathfrak{H}$ and $\mathbb{E}[\|g\|_{\mathfrak{H}}^2] < \infty$. Then, we have:

- (i) The space of elementary processes defined in Definition 8.1.2 is dense in Λ_H .
- (ii) For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s, x) W(ds, dx)$ is defined as an $L^2(\Omega)$ -limit of Riemann sums along elementary processes approximating g , and we have:

$$\mathbb{E} \left[\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s, x) W(ds, dx) \right)^2 \right] = \mathbb{E} [\|g\|_{\mathfrak{H}}^2]. \quad (8.10)$$

Proof. Let us prove item (i). To this aim, consider $g \in \Lambda_H$ and set $\varphi(t, x) = D^{\frac{1}{2}-H} g(t, x)$.

According to the definition of Λ_H , we have: $\mathbb{E}[\int_{\mathbb{R}_+} \int_{\mathbb{R}} |\varphi(s, x)|^2 dx ds] < \infty$. Then we will show that $g(t, x)$ can be approximated by elementary processes in $L^2(\Omega; \mathfrak{H})$ in three steps.
Step 1. Recall that we have set $\dot{H}^{\frac{1}{2}-H}$ for the class of functions f , such that there exists $h \in L^2(\mathbb{R})$ satisfying $f = I_{\mathbb{Z}}^{\frac{1}{2}-H} h$. We show that the process g can be approximated in $L^2(\Omega; \mathfrak{H})$ by functions of the form

$$\psi_m(s, x; \omega) = \sum_{i=1}^m \mathbf{1}_{(a_i, b_i]}(s) \phi_i(x; \omega), \quad (8.11)$$

where for each i , $\phi_i(x; \omega)$ is an \mathcal{F}_{a_i} -measurable $L^2(\Omega; \dot{H}^{\frac{1}{2}-H})$ -valued random field. To see this, we just set

$$\psi_m(s, x; \omega) = \sum_{k=1}^{m2^m} \mathbf{1}_{((k-1)2^{-m}, k2^{-m}]}(s) 2^m \int_{(k-1)2^{-m}}^{k2^{-m}} g(r, x; \omega) dr,$$

and we easily get that $D_{\mathbb{Z}}^{\frac{1}{2}-H} \psi_m(s, x; \omega) \rightarrow D_{\mathbb{Z}}^{\frac{1}{2}-H} g(s, x; \omega)$ in $L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ as m tends to infinity. In this way we get the desired approximation.

Step 2. We show that each $\psi_m(s, x; \omega)$ of the form (8.11) can be approximated by a linear combination of elements of the form $X \mathbf{1}_{(a, b]}(s) h(x)$, in $L^2(\Omega; \mathfrak{H})$. Indeed, for each $\phi_i(x; \omega)$, we notice that since

$$\mathbb{E} \int_{\mathbb{R}} |D_{\mathbb{Z}}^{\frac{1}{2}-H} \phi_i(x; \omega)|^2 dx < \infty,$$

$D_{\mathbb{Z}}^{\frac{1}{2}-H} \phi_i(x; \omega)$ can be approximated by functions with the form

$$\sum_{j=1}^N X_j h_j(x)$$

in $L^2(\Omega; L^2(\mathbb{R}))$, where each X_j is an \mathcal{F}_{a_i} -measurable random variable and each h_j is an element in $L^2(\mathbb{R})$. Thus, it is easily seen that $\phi_i(x; \omega)$ can be approximated a sequence of

functions of the form

$$\sum_{j=1}^N X_j I_{\frac{1}{2}}^{\frac{1}{2}-H} h_j(x).$$

So we conclude that $\psi_m(s, x; \omega)$ can be approximated by

$$\sum_{i=1}^m \mathbf{1}_{(a_i, b_i]}(s) \sum_{j=1}^N X_{i,j} I_{\frac{1}{2}}^{\frac{1}{2}-H} h_{i,j}(x)$$

in $L^2(\Omega; \mathcal{H})$, where $X_{i,j}$ are \mathcal{F}_{a_i} -measurable random variables and $h_{i,j} \in L^2(\mathbb{R})$.

Step 3. Owing to Theorem 3.3 in [83] we know that

$$\text{Span} \left\{ D_{\frac{1}{2}}^{\frac{1}{2}-H} \mathbf{1}_{(h,l]}; h < l \right\}$$

is dense in $\Lambda_0 := \{D_{\frac{1}{2}}^{\frac{1}{2}-H} f : f \in \dot{H}^\beta\}$, in $L^2(\mathbb{R})$ norm. This observation and the results in Step 2 immediately shows that $\psi_m(s, x; \omega)$ can be approximated by elementary processes in $L^2(\Omega; \mathfrak{H})$. This completes the proof. \square

With this stochastic integral defined, we are ready to state the definition of the solution to equation (8.1).

Definition 8.1.4. Let $u = \{u(t, x), 0 \leq t \leq T, x \in \mathbb{R}\}$ be a real-valued predictable stochastic process such that for all $t \in [0, T]$ and $x \in \mathbb{R}$ the process $\{p_{t-s}(x-y)\sigma(u(s, y))\mathbf{1}_{[0,t]}(s), 0 \leq s \leq t, y \in \mathbb{R}\}$ is an element of Λ_H , where $p_t(x)$ is the heat kernel on the real line related to $\frac{\kappa}{2}\Delta$. We say that u is a mild solution of (8.1) if for all $t \in [0, T]$ and $x \in \mathbb{R}$ we have:

$$u(t, x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s, y))W(ds, dy) \quad a.s., \quad (8.12)$$

where the stochastic integral is understood in the sense of Proposition 8.1.3.

8.1.2 Elements of Malliavin calculus

We recall that the Gaussian family W can be extended to \mathfrak{H} and this produces an isonormal Gaussian process, where \mathfrak{H} is the Hilbert space introduced in Proposition 8.1.1. We refer to [80] for a detailed account of the Malliavin calculus with respect to a Gaussian process. On our Gaussian space, the smooth and cylindrical random variables F are of the form

$$F = f(W(\phi_1), \dots, W(\phi_n)),$$

with $\phi_i \in \mathfrak{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely f and all its partial derivatives have polynomial growth). For this kind of random variable, the derivative operator D in the sense of Malliavin calculus is the \mathfrak{H} -valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n)) \phi_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathfrak{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{\mathbb{E}[F^2] + \mathbb{E}[\|DF\|_{\mathfrak{H}}^2]}.$$

We denote by δ the adjoint of the derivative operator (or divergence) given by the duality formula

$$\mathbb{E}[\delta(u)F] = \mathbb{E}[\langle DF, u \rangle_{\mathfrak{H}}], \quad (8.13)$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathfrak{H})$ in the domain of δ .

For any integer $n \geq 0$ we denote by \mathbf{H}_n the n th Wiener chaos of W . We recall that \mathbf{H}_0 is simply \mathbb{R} and for $n \geq 1$, \mathbf{H}_n is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(\phi)); \phi \in \mathfrak{H}, \|\phi\|_{\mathfrak{H}} = 1\}$, where H_n is the n th Hermite polynomial. For any $n \geq 1$, we denote by $\mathfrak{H}^{\otimes n}$ (resp. $\mathfrak{H}^{\odot n}$) the n th tensor product (resp. the n th symmetric

tensor product) of \mathfrak{H} . Then, the mapping $I_n(\phi^{\otimes n}) = H_n(W(\phi))$ can be extended to a linear isometry between $\mathfrak{H}^{\odot n}$ (equipped with the modified norm $\sqrt{n!} \|\cdot\|_{\mathfrak{H}^{\otimes n}}$) and \mathbf{H}_n .

Consider now a random variable $F \in L^2(\Omega)$ which is measurable with respect to the σ -field \mathcal{F} generated by W . This random variable can be expressed as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \quad (8.14)$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \mathfrak{H}^{\odot n}$, $n \geq 1$, are determined by F . This identity is called the Wiener-chaos expansion of F .

The Skorohod integral (or divergence) of a random field u can be computed by using the Wiener chaos expansion. More precisely, suppose that $u = \{u(t, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a random field such that for each (t, x) , $u(t, x)$ is an \mathcal{F}_t -measurable and square integrable random variable. Then, for each (t, x) we have a Wiener chaos expansion of the form

$$u(t, x) = \mathbb{E}[u(t, x)] + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)). \quad (8.15)$$

Suppose that $\mathbb{E}[\|u\|_{\mathfrak{H}}^2]$ is finite. Then, we can interpret u as a square integrable random function with values in \mathfrak{H} and the kernels f_n in the expansion (8.15) are functions in $\mathfrak{H}^{\otimes(n+1)}$ which are symmetric in the first n variables. In this situation, u belongs to the domain of the divergence operator (that is, u is Skorohod integrable with respect to W) if and only if the following series converges in $L^2(\Omega)$

$$\delta(u) = \int_0^{\infty} \int_{\mathbb{R}^d} u(t, x) \delta W(t, x) = W(\mathbb{E}[u]) + \sum_{n=1}^{\infty} I_{n+1}(\tilde{f}_n(\cdot, t, x)), \quad (8.16)$$

where \tilde{f}_n denotes the symmetrization of f_n in all its $n + 1$ variables. We note that whenever $u \in \Lambda_H$ the integral $\delta(u)$ coincides with the Itô integral.

Along the chapter we denote by C a generic constant that may vary from line to line.

8.2 Moment estimates and Hölder continuity of stochastic convolutions

8.2.1 Moment bound of the solution

First we introduce some notation, which makes some of our formulae easier to read, and which will prevail until the end of the article. Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. Let $\beta \in (0, 1)$ be a fixed number. For every function $f : \mathbb{R} \rightarrow B$, we introduce the function $\mathcal{N}_\beta f : \mathbb{R} \rightarrow [0, \infty]$ defined by

$$\mathcal{N}_\beta f(x) = \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}. \quad (8.17)$$

With this notation, the norm of the homogeneous Sobolev space \dot{H}^β can be written as $c_{3,\beta} \|\mathcal{N}_\beta f\|_{L^2(\mathbb{R})}$. The following technical lemma will be used along the chapter.

Lemma 8.2.1. *For any $\beta \in (0, 1)$,*

$$\int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 dx \leq C_\beta (\kappa s)^{-\frac{1}{2}-\beta},$$

where in the definition of \mathcal{N}_β we take $B = \mathbb{R}$.

Proof. Recalling that $\mathcal{F} p_s(\xi) = e^{-\frac{\kappa}{2}s\xi^2}$ and invoking Plancherel's identity we can write

$$\begin{aligned} \int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} |p_s(x+h) - p_s(x)|^2 |h|^{-1-2\beta} dh dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\kappa s|\xi|^2} |e^{-i\xi z} - 1|^2 |z|^{-1-2\beta} dz d\xi \\ &= C_{1,\beta} \int_{\mathbb{R}} e^{-\kappa s\xi^2} |\xi|^{2\beta} d\xi, \end{aligned}$$

where the second relation is obtained by a scaling $v \equiv \xi z$ in the integral in z , and

$C_{1,\beta} = \frac{1}{2\pi} \int_{\mathbb{R}} |e^{iv} - 1|^2 |v|^{-1-2\beta} dv$. Setting now $\eta = (\kappa s)^{1/2} \xi$ in the integral in ξ , we get

$$\int_{\mathbb{R}} [\mathcal{N}_{\beta} p_s(x)]^2 dx \leq C_{\beta} (\kappa s)^{-\frac{1}{2}-\beta},$$

where $C_{\beta} = C_{1,\beta} \int_{\mathbb{R}} e^{-\eta^2} |\eta|^{2\beta} d\eta$. □

The transformation \mathcal{N}_{β} can also be defined for functions f defined on $\mathbb{R}_+ \times \mathbb{R}$ acting on the spacial variable, and in this case, $\mathcal{N}_{\beta} f : \mathbb{R}_+ \times \mathbb{R} \rightarrow [0, \infty]$.

Fix $p \geq 2$. Suppose now that $f = \{f(t, x), t \geq 0, x \in \mathbb{R}\}$ is a random field such that $\mathbb{E}|f(t, x)|^p < \infty$ for all (t, x) . Then we can consider f as an $L^p(\Omega)$ -valued function and we will denote by $\mathcal{N}_{\beta,p} f$ the transformation introduced in (8.17) for $B = L^p(\Omega)$, that is,

$$\mathcal{N}_{\beta,p} f(t, x) = \left(\int_{\mathbb{R}} \|f(t, x+h) - f(t, x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}. \quad (8.18)$$

With the above notation in mind, the following inequality is essential in our approach. Let W be the Gaussian noise defined by the covariance (8.2).

Proposition 8.2.2. *Consider a predictable random field $f \in \Lambda_H$. Then, for any $p \geq 2$ we have*

$$\left\| \int_0^t \int_{\mathbb{R}} f(s, y) W(ds, dy) \right\|_{L^p(\Omega)} \leq \sqrt{4p} c_{3, \frac{1}{2}-H} \left(\int_0^t \int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,p} f(s, y)]^2 dy ds \right)^{\frac{1}{2}}. \quad (8.19)$$

Proof. Applying Burkholder's inequality, we have

$$\left\| \int_0^t \int_{\mathbb{R}} f(s, y) W(ds, dy) \right\|_{L^p(\Omega)} \leq \sqrt{4p} \left\| \int_0^t \|f(s, \cdot)\|_{\dot{H}^{\frac{1}{2}-H}}^2 ds \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{1}{2}}. \quad (8.20)$$

Using (8.8), we can write

$$\|f(s, \cdot)\|_{\dot{H}^{\frac{1}{2}-H}}^2 = c_{3, \frac{1}{2}-H}^2 \int_{\mathbb{R}^2} |f(s, y+h) - f(s, y)|^2 |h|^{2H-2} dh dy. \quad (8.21)$$

We now invoke Minkowski's inequality, under the form

$$\left\| \int_S U(\xi) \mu(d\xi) \right\|_{L^q(\Omega)} \leq \int_S \|U(\xi)\|_{L^q(\Omega)} \mu(d\xi),$$

for a measure μ on the state space S . Together with (8.21), this yields

$$\begin{aligned} \left\| \int_0^t \|f(s, \cdot)\|_{\dot{H}^{\frac{1}{2}-H}}^2 ds \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{1}{2}} &\leq c_{3, \frac{1}{2}-H} \int_{\mathbb{R}^2} \left\| (f(s, y+h) - f(s, y))^2 \right\|_{L^{\frac{p}{2}}(\Omega)} |h|^{2H-2} dh dy \\ &= c_{3, \frac{1}{2}-H} \int_{\mathbb{R}^2} \|f(s, y+h) - f(s, y)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy, \end{aligned}$$

from which identity (8.19) is easily deduced. \square

Next, for $\theta > 0$, $\varepsilon > 0$ and $\beta \in (0, 1)$, we consider the space $\mathfrak{X}_\theta^{\beta, p}$ which consists of all random fields such that the following norm is finite

$$\|u\|_{\mathfrak{X}_{\theta, \varepsilon}^{\beta, p}} := \sup_{t \geq 0, x \in \mathbb{R}} e^{-\theta t} \|u(t, x)\|_{L^p(\Omega)} + \varepsilon \sup_{t \geq 0, x \in \mathbb{R}} e^{-\theta t} \mathcal{N}_{\beta, p} u(t, x). \quad (8.22)$$

Remark 8.2.3. (a) In the case $\varepsilon = 1$, we simply write $\|\cdot\|_{\mathfrak{X}_\theta^{\beta, p}}$.

(b) The second term in the norm in (8.22) is not invariant by scaling while the first term is.

Indeed, denote $f_\lambda(t, x) = f(t, \lambda x)$, then

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \|f_\lambda(t, x+h) - f_\lambda(t, x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh \right)^{1/2} \\ = \lambda^\beta \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \|f(t, x+h) - f(t, x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh \right)^{1/2}. \end{aligned}$$

This is the very reason why various orders of $(t-s)$ appear in the proof of Proposition 8.2.4 below. We bypass this technical difficulty by the introduction of an additional scaling factor ε in (8.22).

(c) Another way to see the role of ε is via dimensional analysis. Suppose that the amplitude of f has unit L , the spatial variable x has unit S , while the randomness ω is dimensionless.

Then the first term in (8.22) has unit L while the second term has unit L/S^β . Hence, in order for the two terms to have the same dimension, we multiply the second term with a constant ε having unit of S^β .

Proposition 8.2.4. *Consider a predictable random field $f \in \Lambda_H$ and define process $\{\Phi(t, x), t \geq 0, x \in \mathbb{R}\}$ by*

$$\Phi(t, x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f(s, y) W(ds, dy). \quad (8.23)$$

Then, for any $\beta < H$ and $p \geq 2$, the following inequality holds:

$$\begin{aligned} \|\Phi\|_{\mathfrak{X}_{\theta, \varepsilon}^{\beta, p}} &\leq C_0 \sqrt{p} \|f\|_{\mathfrak{X}_{\theta, \varepsilon}^{\frac{1}{2}-H, p}} \\ &\times \left(\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \kappa^{-\frac{1}{4}-\frac{\beta}{2}} \theta^{\frac{\beta}{2}-\frac{1}{4}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}} + \varepsilon \kappa^{\frac{H}{2}-\frac{\beta}{2}-\frac{1}{2}} \theta^{\frac{\beta}{2}-\frac{H}{2}} \right), \end{aligned} \quad (8.24)$$

where C_0 is a constant depending only on H and β .

Proof. According to our definition (8.22), we have $\|\Phi\|_{\mathfrak{X}_{\theta, \varepsilon}^{\beta, p}} = \mathcal{A}_1 + \varepsilon \mathcal{A}_2$, with

$$\mathcal{A}_1 = \sup_{t \geq 0, x \in \mathbb{R}} e^{-\theta t} \|\Phi(t, x)\|_{L^p(\Omega)}, \quad \text{and} \quad \mathcal{A}_2 = \sup_{t \geq 0, x \in \mathbb{R}} e^{-\theta t} \mathcal{N}_{\beta, p} u(t, x).$$

We now estimate those terms separately. Along the proof C will denote a generic constant depending only on H and β .

Step 1: Upper bound for \mathcal{A}_1 . The term $\Phi(t, x)$ is of the form

$$\int_0^t \int_{\mathbb{R}} g_{t,x}(s, y) W(ds, dy), \quad \text{with} \quad g_{t,x}(s, y) = p_{t-s}(x-y) f(s, y).$$

Applying inequality (8.19), we thus have

$$\|\Phi(t, x)\|_{L^p(\Omega)} \leq C \sqrt{p} \left(\int_0^t \int_{\mathbb{R}^2} \|g_{t,x}(s, y+h) - g_{t,x}(s, y)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds \right)^{\frac{1}{2}}.$$

A simple decomposition of the increment $g_{t,x}(s, y + h) - g_{t,x}(s, y)$ then yields

$$\|\Phi(t, x)\|_{L^p(\Omega)} \leq C\sqrt{p} \left[\left(\int_0^t J_1(s) ds \right)^{\frac{1}{2}} + \left(\int_0^t J_2(s) ds \right)^{\frac{1}{2}} \right],$$

where

$$J_1(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x - y - z) - p_{t-s}(x - y)|^2 \|f(s, y + z)\|_{L^p(\Omega)}^2 |z|^{2H-2} dy dz$$

and

$$J_2(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} p_{t-s}^2(x - y) \|f(s, y + z) - f(s, y)\|_{L^p(\Omega)}^2 |z|^{2H-2} dy dz.$$

To estimate $J_1(s)$, we write

$$J_1(s) \leq \sup_{x \in \mathbb{R}} \|f(s, x)\|_{L^p(\Omega)}^2 \int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(y)]^2 dy.$$

Applying Lemma 8.2.1 with $\beta = \frac{1}{2} - H$, we obtain

$$J_1(s) \leq C \sup_{x \in \mathbb{R}} \|f(s, x)\|_{L^p(\Omega)}^2 [\kappa(t - s)]^{H-1}.$$

Let us now turn to estimate $J_2(s)$. Recalling our notation (8.18), we have

$$\begin{aligned} J_2(s) &= \int_{\mathbb{R}} p_{t-s}^2(x - y) [\mathcal{N}_{\frac{1}{2}-H, p} f(s, y)]^2 dy \leq \sup_{x \in \mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H, p} f(s, x)]^2 \int_{\mathbb{R}} p_{t-s}^2(x - y) dy \\ &\leq [2\pi\kappa(t - s)]^{-\frac{1}{2}} \sup_{x \in \mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H, p} f(s, x)]^2. \end{aligned} \quad (8.25)$$

Hence, putting together our bounds on J_1 and J_2 , we get

$$e^{-\theta t} \sup_{x \in \mathbb{R}} \|\Phi(t, x)\|_{L^p(\Omega)} \leq C\sqrt{p} \sup_{\substack{s \geq 0 \\ x \in \mathbb{R}}} e^{-\theta s} \|f(s, x)\|_{L^p(\Omega)} \left(\int_0^t e^{-2\theta(t-s)} [\kappa(t-s)]^{H-1} ds \right)^{\frac{1}{2}} \\ + C\sqrt{p} \varepsilon \sup_{\substack{s \geq 0 \\ x \in \mathbb{R}}} e^{-\theta s} \sup_{x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H, p} f(s, x) \frac{\left(\int_0^t e^{-2\theta(t-s)} [\kappa(t-s)]^{-\frac{1}{2}} ds \right)^{\frac{1}{2}}}{\varepsilon},$$

and some elementary computations for the integrals above yield

$$\mathcal{A}_1 = \sup_{t \geq 0, x \in \mathbb{R}} e^{-\theta t} \|\Phi(t, x)\|_{L^p(\Omega)} \leq C\sqrt{p} \|f\|_{\mathfrak{X}_{\theta, \varepsilon}^{\frac{1}{2}-H, p}} (\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}}).$$

Step 2: Upper bound for \mathcal{A}_2 . For every $h \in \mathbb{R}$, we apply inequality (8.19) to get

$$\|\Phi(t, x+h) - \Phi(t, x)\|_{L^p(\Omega)} \leq C\sqrt{p} \\ \times \left(\int_0^t \int_{\mathbb{R}} \left[\mathcal{N}_{\frac{1}{2}-H, p} \{[(p_{t-s}(x+h-\cdot) - p_{t-s}(x-\cdot))]f(s, \cdot)\} \right]^2 (y) dy ds \right)^{\frac{1}{2}}. \quad (8.26)$$

Furthermore, along the same lines as for Step 1 above, we can write

$$\left(\int_0^t \int_{\mathbb{R}} \left[\mathcal{N}_{\frac{1}{2}-H, p} \{[(p_{t-s}(x+h-\cdot) - p_{t-s}(x-\cdot))]f(s, \cdot)\} \right]^2 (y) dy ds \right)^{\frac{1}{2}} \\ \leq \int_0^t (J'_1(s, h))^{1/2} ds + \int_0^t (J'_2(s, h))^{1/2} ds, \quad (8.27)$$

where

$$J'_1(s, h) = \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x+h-y-z) - p_{t-s}(x-y-z) - p_{t-s}(x+h-y) + p_{t-s}(x-y)|^2 \\ \times \|f(s, y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2} dy dz,$$

and

$$J'_2(s, h) = \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x+h-y) - p_{t-s}(x-y)|^2 \|f(s, y+z) - f(s, y)\|_{L^p(\Omega)}^2 |z|^{2H-2} dy dz .$$

Using the same arguments as in the proof of Lemma 8.2.1, we can show that

$$\int_{\mathbb{R}} J'_1(s, h) |h|^{-1-2\beta} dh \leq C[\kappa(t-s)]^{H-\beta-1} \sup_{x \in \mathbb{R}} \|f(s, x)\|_{L^p(\Omega)}^2 .$$

On the other hand, applying Lemma 8.2.1 leads to

$$\int_{\mathbb{R}} J'_2(s, h) |h|^{-1-2\beta} dh \leq C[\kappa(t-s)]^{-\frac{1}{2}-\beta} \sup_{x \in \mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H, p} f(s, x)]^2 .$$

Combining these estimates for J'_1, J'_2 and resorting to (8.27), (8.26), similarly as the estimate for $e^{-\theta t} \|\Phi(t, x)\|_{L^p(\Omega)}$, we obtain

$$\mathcal{A}_2 \leq C\sqrt{p} \left(\|f\|_{\mathfrak{X}_{\theta, \varepsilon}^{\frac{1}{2}-H, p}} \kappa^{\frac{H}{2}-\frac{\beta}{2}-\frac{1}{2}} \theta^{\frac{\beta}{2}-\frac{H}{2}} + \varepsilon^{-1} \|f\|_{\mathfrak{X}_{\theta, \varepsilon}^{\frac{1}{2}-H, p}} \kappa^{-\frac{1}{4}-\frac{\beta}{2}} \theta^{\frac{\beta}{2}-\frac{1}{4}} \right) .$$

Putting together Step 1 and Step 2, our claim (8.24) is now easily checked. \square

In the particular case $\beta = \frac{1}{2} - H$, and using the simplified notation $\|\cdot\|_{\mathfrak{X}_{\theta, \varepsilon}^{\frac{1}{2}-H, p}} = \|\cdot\|_{\mathfrak{X}_{\theta, \varepsilon}^p}$, the estimate (8.24) can be written as

$$\|\Phi\|_{\mathfrak{X}_{\theta, \varepsilon}^p} \leq C_0 \sqrt{p} \|f\|_{\mathfrak{X}_{\theta, \varepsilon}^p} \left(\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}} + \varepsilon \kappa^{H-\frac{3}{4}} \theta^{\frac{1}{4}-H} \right) , \quad (8.28)$$

8.2.2 Hölder continuity estimates

A natural question arising from the definition (8.23) of the process Φ is the derivation of Hölder type exponents in both time and space. Some estimates in this direction are provided in the next proposition. We recall that $\mathfrak{X}_{\theta_0}^p = \mathfrak{X}_{\theta_0}^{\frac{1}{2}-H, p}$, and the norm $\|\cdot\|_{\mathfrak{X}_{\theta_0}^p}$ is given by (8.22) with $\varepsilon = 1$ and $\beta = \frac{1}{2} - H$.

Proposition 8.2.5. *Recall that the noise W is given by the covariance (8.2). Consider $p \geq 2$ and a predictable random field $f \in \mathfrak{X}_{\theta_0}^p$, where θ_0 is any positive number. We define the random field Φ as in (8.23), and fix a finite time horizon T . Then for every $x, h \in \mathbb{R}$, $t_1, t_2 \in [0, T]$ and every $\gamma \in [0, H]$ we have*

$$\|\Phi([t_1, t_2], x + h) - \Phi([t_1, t_2], x)\|_{L^p(\Omega)} \leq C \|f\|_{\mathfrak{X}_{\theta_0}^p} e^{\theta_0 T} |t_2 - t_1|^{\frac{H-\gamma}{2}} |h|^\gamma. \quad (8.29)$$

In the above, the constant C depends on T and we are using the notation

$$\Phi([t_1, t_2], x) = \Phi(t_2, x) - \Phi(t_1, x).$$

In particular, if we let $t_1 = 0$, we get the Hölder estimate of the space variable. For the Hölder estimate of the time variable, we have

$$\|\Phi(t_2, x) - \Phi(t_1, x)\|_{L^p(\Omega)} \leq C \|f\|_{\mathfrak{X}_{\theta_0}^p} e^{\theta_0 T} |t_2 - t_1|^{\frac{H}{2}}. \quad (8.30)$$

Proof. First we prove (8.29). Without loss of generality, we assume $t_1 < t_2$ and denote $\Delta t = t_2 - t_1$. We denote

$$V_1(f) = \sup_{t \leq T} \sup_{x \in \mathbb{R}} \|f(t, x)\|_{L^p(\Omega)}, \quad V_2(f) = \sup_{t \leq T} \sup_{x \in \mathbb{R}} \mathcal{N}_{H,p} f(t, x), \quad (8.31)$$

and $V(f) = V_1(f) + V_2(f)$. Observe that according to (8.22), we have $V(f) \leq \exp(\theta_0 T) \|f\|_{\mathfrak{X}_{\theta_0}^p}$.

As in the proof of Proposition 8.2.4, we first write $\Phi([t_1, t_2], x + h) - \Phi([t_1, t_2], x) = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1 = \int_0^{t_1} \int_{\mathbb{R}} [p_{[t_1-s, t_2-s]}(x + h - y) - p_{[t_1-s, t_2-s]}(x - y)] f(s, y) W(ds, dy),$$

and

$$\mathcal{A}_2 = \int_{t_1}^{t_2} \int_{\mathbb{R}} [p_{t_2-s}(x + h - y) - p_{t_2-s}(x - y)] f(s, y) W(ds, dy).$$

We now treat those two terms separately.

Step 1: Upper bound for \mathcal{A}_1 . The computations are carried out analogously to the proof of Proposition 8.2.4, and we have

$$\|\mathcal{A}_1\|_{L^p(\Omega)}^2 \leq C \int_0^{t_1} (A_{11}(s) + A_{12}(s)) ds,$$

where A_{11} and A_{12} are analogous to J_1, J_2 in the proof of Proposition 8.2.4, and are respectively defined by

$$\begin{aligned} A_{11}(s) = & \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{[t_1-s, t_2-s]}(x+h-y-z) - p_{[t_1-s, t_2-s]}(x-y-z) \\ & - p_{[t_1-s, t_2-s]}(x+h-y) + p_{[t_1-s, t_2-s]}(x-y)|^2 \|f(s, y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2} dy dz, \end{aligned}$$

and

$$\begin{aligned} A_{12}(s) = & \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{[t_1-s, t_2-s]}(x+h-y) - p_{[t_1-s, t_2-s]}(x-y)|^2 \\ & \times \|f(s, y+z) - f(s, y)\|_{L^p(\Omega)}^2 |z|^{2H-2} dy dz. \end{aligned}$$

Let us now bound A_{11} . Invoking Plancherel's identity with respect to y and the explicit formula for $\mathcal{F} p_t$, we have

$$\begin{aligned} A_{11}(s) & \leq CV_1^2(f) \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{[t_1-s, t_2-s]}(h+y-z) - p_{[t_1-s, t_2-s]}(y-z) \\ & \quad - p_{[t_1-s, t_2-s]}(h+y) + p_{[t_1-s, t_2-s]}(y)|^2 |z|^{2H-2} dy dz \\ & \leq CV_1^2(f) \int_{\mathbb{R}} \int_{\mathbb{R}} |e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2 |e^{-i\xi z} - 1|^2 |e^{i\xi h} - 1|^2 |z|^{2H-2} d\xi dz \\ & \leq CV_1^2(f) \int_{\mathbb{R}} |e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2 |e^{i\xi h} - 1|^2 |\xi|^{1-2H} d\xi. \end{aligned}$$

Moreover, owing to the identity

$$\int_0^{t_1} |e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2 ds \leq \frac{|e^{-\frac{\Delta t \kappa}{2}|\xi|^2} - 1|^2}{\kappa|\xi|^2}, \quad (8.32)$$

we obtain

$$\begin{aligned} \int_0^{t_1} A_{11}(s) ds &\leq C\kappa^{-1}V_1^2(f) \int_{\mathbb{R}} |e^{-\frac{\Delta t \kappa}{2}|\xi|^2} - 1|^2 |e^{i\xi h} - 1|^2 |\xi|^{-1-2H} d\xi \\ &\leq C\kappa^{-1}V_1^2(f) I, \end{aligned} \quad (8.33)$$

where

$$I := \int_{\mathbb{R}} |1 - e^{-\frac{\Delta t \kappa}{2}|\xi|^2}|^2 \sin^2(\xi h/2) |\xi|^{-1-2H} d\xi. \quad (8.34)$$

Our next step is to bound I in two elementary and different ways.

(i) The change of variable $h\xi := \xi$ yields

$$I = |h|^{2H} \int_{\mathbb{R}} \left(1 - e^{-\frac{\kappa \Delta t}{2|h|^2}|\xi|^2}\right)^2 \sin^2(\xi/2) |\xi|^{-1-2H} d\xi,$$

and we then bound $1 - e^{-\frac{\kappa \Delta t}{2|h|^2}|\xi|^2}$ by 1 to obtain $I \leq C|h|^{2H}$.

(ii) On the other hand, the change of variable $(\kappa \Delta t)^{1/2} \xi := \xi$ in (8.34) leads to

$$I = (\kappa \Delta t)^H \int_{\mathbb{R}} \left(1 - e^{-\xi^2/2}\right)^2 \sin^2\left(\frac{h\xi}{2(\kappa \Delta t)^{1/2}}\right) |\xi|^{-1-2H} d\xi,$$

and we bound the trigonometric function \sin^2 by 1 to obtain $I \leq C(\kappa \Delta t)^H$.

Interpolating the two estimates we have obtained for I , with a coefficient $\delta = \frac{\gamma}{2H} \in [0, 1]$, we see that

$$I \leq C|h|^{2H\delta} (\kappa \Delta t)^{H(1-\delta)} \leq C(\kappa \Delta t)^{\frac{2H-\gamma}{2}} |h|^\gamma. \quad (8.35)$$

Plugging this identity back into (8.33), we have shown

$$\int_0^{t_1} A_{11}(s) ds \leq C \kappa^{-1} (\kappa \Delta t)^{\frac{2H-\gamma}{2}} |h|^\gamma V_1^2(f),$$

for all $\gamma \in [0, 2H]$. Let us now turn to the estimate for A_{12} . Similarly to what has been done for A_{11} we get

$$\begin{aligned} \int_0^{t_1} A_{12}(s) ds &\leq C V_2^2(f) \int_0^{t_1} \int_{\mathbb{R}} |p_{[t_1-s, t_2-s]}(h+y) - p_{[t_1-s, t_2-s]}(y)|^2 dy ds \\ &\leq C V_2^2(f) \int_{\mathbb{R}} \int_0^{t_1} |e^{-\frac{t_2-s}{2} \kappa |\xi|^2} - e^{-\frac{t_1-s}{2} \kappa |\xi|^2}|^2 ds |e^{i\xi h} - 1|^2 d\xi. \end{aligned}$$

Thanks to (8.32), we thus end up with

$$\int_0^{t_1} A_{12}(s) ds \leq C \kappa^{-1} V_2^2(f) \int_{\mathbb{R}} |1 - e^{-\frac{\Delta t \kappa}{2} |\xi|^2}|^2 \sin^2(h\xi/2) |\xi|^{-2} d\xi.$$

In addition, the integral on the right hand side can be estimated as I above, and we get

$$\int_0^{t_1} A_{12}(s) ds \leq C V_2^2(f) (\kappa \Delta t)^{\frac{1-\gamma'}{2}} |h|^{\gamma'}$$

for all $\gamma' \in [0, 1]$. Since $1 > 2H$, we may choose $\gamma' = \gamma$ to obtain

$$\int_0^{t_1} A_{12}(s) ds \leq C \kappa^{-1} (\kappa \Delta t)^{\frac{2H-\gamma}{2}} |h|^\gamma V_2^2(f),$$

for all $\gamma \in [0, 2H]$. Hence, the bounds on A_{11} and A_{12} yield

$$\|\mathcal{A}_1\|_{L^p(\Omega)}^2 \leq C V^2(f) (\Delta t)^{\frac{2H-\gamma}{2}} h^\gamma,$$

for all $\gamma \in [0, 2H]$.

Step 2: Upper bound for \mathcal{A}_2 . The term $\|\mathcal{A}_2\|_{L^p(\Omega)}^2$ can be estimated analogously to \mathcal{A}_1 . Indeed,

the reader can check that, owing to inequality (8.19) and Plancherel's identity, we have

$$\|\mathcal{A}_2\|_{L^p(\Omega)}^2 \leq CV_1^2(f) \int_0^{\Delta t} \int_{\mathbb{R}} e^{-s\kappa|\xi|^2} \sin^2(h\xi/2)(|\xi|^{1-2H} + 1) d\xi ds ,$$

where we recall that V_1 is defined by (8.31). Taking integration in ds first, we see that

$$\|\mathcal{A}_2\|_{L^p(\Omega)}^2 \leq C\kappa^{-1}V_1^2(f) \int_{\mathbb{R}} (1 - e^{-\Delta t\kappa|\xi|^2}) \sin^2(h\xi/2)(|\xi|^{-1-2H} + |\xi|^{-2}) d\xi .$$

These two integrals can be estimated as the term I in (8.35), and we get

$$\|\mathcal{A}_2\|_{L^p(\Omega)}^2 \leq CV_1^2(f)(\Delta t)^{\frac{2H-\gamma}{2}} |h|^\gamma ,$$

for all $\gamma \in [0, 2H]$. Let us remark that the constants in all previous estimates depend only on T, p and κ^{-1} . In addition, as functions of (p, κ^{-1}) , these constants have at most polynomial growth. Hence, gathering the estimates for $\|\mathcal{A}_1\|_{L^p(\Omega)}^2$ and $\|\mathcal{A}_2\|_{L^p(\Omega)}^2$ the proof of our claim (8.29) is finished.

Step 3: Proof of (8.30). Again, we assume that $t_1 < t_2$, and we proceed as in the previous steps and the proof of Proposition 8.2.4. Indeed, we begin by writing

$$\|\Phi(t_2, x) - \Phi(t_1, x)\|_{L^p(\Omega)} \leq B_1 + B_2,$$

where

$$B_1 = \left\| \int_0^{t_1} \int_{\mathbb{R}} p_{[t_1-s, t_2-s]}(x-y) f(s, y) W(ds, dy) \right\|_{L^p(\Omega)}$$

and

$$B_2 = \left\| \int_{t_1}^{t_2} \int_{\mathbb{R}} p_{t_2-s}(x-y) f(s, y) W(ds, dy) \right\|_{L^p(\Omega)} .$$

Once again we handle those two terms separately.

For the term B_1 , we resort to inequality (8.19) in our usual way. We get

$$\begin{aligned}
B_1 \leq & C \left(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} p_{[t_1-s, t_2-s]}^2(x-y) \|f(s, y) - f(s, y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2} dz dy ds \right)^{\frac{1}{2}} \\
& + C \left(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{[t_1-s, t_2-s]}(x-y) - p_{[t_1-s, t_2-s]}(x-y-z)|^2 \right. \\
& \quad \left. \times \|f(s, y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2} dz dy ds \right)^{\frac{1}{2}}.
\end{aligned}$$

With the definition (8.31) in mind, it is now readily checked that

$$B_1 \leq C (B_{11} V_2(f) + B_{12} V_1(f)), \quad (8.36)$$

with

$$B_{11} = \left(\int_0^{t_1} \int_{\mathbb{R}} |p_{[t_1-s, t_2-s]}(x-y)|^2 dy ds \right)^{\frac{1}{2}}$$

and

$$B_{12} = \left(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{[t_1-s, t_2-s]}(x-y) - p_{[t_1-s, t_2-s]}(x-y-z)|^2 |z|^{2H-2} dz dy ds \right)^{\frac{1}{2}}.$$

We now appeal to Plancherel's identity to get

$$B_{11} = C \left(\int_0^{t_1} \int_{\mathbb{R}} \left| e^{-\frac{t_2-s}{2} \kappa |\xi|^2} - e^{-\frac{t_1-s}{2} \kappa |\xi|^2} \right|^2 d\xi ds \right)^{\frac{1}{2}} = C(t_2 - t_1)^{\frac{1}{4}},$$

and

$$\begin{aligned}
B_{12} &= C \left(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| e^{-\frac{t_2-s}{2} \kappa |\xi|^2} - e^{-\frac{t_1-s}{2} \kappa |\xi|^2} \right|^2 |e^{-i\xi z} - 1|^2 |z|^{2H-2} dz d\xi ds \right)^{\frac{1}{2}} \\
&= C \left(\int_0^{t_1} \int_{\mathbb{R}} \left| e^{-\frac{t_2-s}{2} \kappa |\xi|^2} - e^{-\frac{t_1-s}{2} \kappa |\xi|^2} \right|^2 |\xi|^{1-2H} d\xi ds \right)^{\frac{1}{2}} \\
&= C(t_2 - t_1)^{\frac{H}{2}}.
\end{aligned}$$

Reporting these estimates in (8.36) and observing that $H < \frac{1}{2}$, we end up with

$$B_1 \leq C(t_2 - t_1)^{\frac{H}{2}} [V_1(f) + V_2(f)] \leq C(t_2 - t_1)^{\frac{H}{2}} \|f\|_{\mathbb{X}_{\theta_0}^p} e^{\theta_0 T}.$$

The patient reader might check that the same kind of upper bound is valid for B_2 , and gathering the estimates for B_1 and B_2 yields inequality (8.30). \square

8.3 Existence and uniqueness of the solution

In this section we will establish a result regarding the uniqueness of the solution. Then we will describe the structure of some new spaces of stochastic processes which will be used to show the existence of the solution.

8.3.1 Uniqueness of the solution

In this subsection we give some results about the uniqueness of the solution assuming that the solution has enough regularity. To this end, we first introduce a norm $\|\cdot\|_{\mathcal{Z}_T^p}$ for a random field $v(t, x)$ as follows

$$\|v\|_{\mathcal{Z}_T^p} = \sup_{t \leq T} \|v(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} + \sup_{t \leq T} \mathcal{N}_{\frac{1}{2}-H, p}^* v(t), \quad (8.37)$$

where $p \geq 2$ and

$$\mathcal{N}_{\frac{1}{2}-H, p}^* v(t) = \left(\int_{\mathbb{R}} \|v(t, \cdot) - v(t, \cdot + h)\|_{L^p(\Omega \times \mathbb{R})}^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}. \quad (8.38)$$

Then the space \mathcal{Z}_T^p will consist all the random fields such that the above quantity is finite.

The proof of the uniqueness theorem requires a localization argument, based on uniform estimates (in space and time) of stochastic convolutions, provided by the following lemma.

Lemma 8.3.1. *Suppose that $p > \frac{6}{4H-1}$. Let v be a process in the space \mathcal{Z}_T^p . Define*

$$\Phi(t, x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)v(s, y)W(ds, dy). \quad (8.39)$$

Then, there exists a constant C depending on T, p and H , such that

$$\left\| \sup_{t \in [0, T], x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \Phi(t, x) \right\|_{L^p(\Omega)} \leq C \|v\|_{\mathcal{Z}_T^p}. \quad (8.40)$$

Notice that in the above inequality, the operator $\mathcal{N}_{\frac{1}{2}-H}$, defined in (8.18), acts on the trajectories of the random field $\Phi(t, x)$, and, as a consequence, $\mathcal{N}_{\frac{1}{2}-H} \Phi(t, x)$ is a random variable.

Proof. We shall apply the factorization method to handle the stochastic convolution (see, for instance, [21]). Namely, an application of a stochastic version of Fubini's theorem enables to write

$$\Phi(t, x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} p_{t-r}(x-z)Y(r, z)dzdr,$$

with

$$Y(r, z) = \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} p_{r-s}(z-y)v(s, y)W(ds, dy),$$

and where $\alpha \in (0, 1)$ is a parameter whose value will be chosen later. The proof will be done in two steps.

Step 1: Uniform estimate of $\mathcal{N}_{\frac{1}{2}-H} \Phi(t, x)$. In order to estimate $\mathcal{N}_{\frac{1}{2}-H} \Phi(t, x)$, we bound the difference $\Phi(t, x) - \Phi(t, x+h)$ as follows

$$\begin{aligned} & |\Phi(t, x) - \Phi(t, x+h)| \\ &= \frac{\sin(\alpha\pi)}{\pi} \left| \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} (p_{t-r}(x-z) - p_{t-r}(x+h-z)) Y(r, z) dz dr \right| \\ &\leq \frac{\sin(\alpha\pi)}{\pi} \int_0^t (t-r)^{\alpha-1} \|p_{t-r}(\cdot) - p_{t-r}(\cdot+h)\|_{L^q(\mathbb{R})} \|Y(r, \cdot)\|_{L^p(\mathbb{R})} dr, \end{aligned}$$

where q satisfies $p^{-1} + q^{-1} = 1$. So using Minkowski's integral inequality, we get

$$\begin{aligned}
& \int_{\mathbb{R}} |\Phi(t, x) - \Phi(t, x + h)|^2 |h|^{2H-2} dh \\
& \leq C \int_{\mathbb{R}} \left(\int_0^t (t-r)^{\alpha-1} \|p_{t-r}(x-\cdot) - p_{t-r}(x+h-\cdot)\|_{L^q(\mathbb{R})} \|Y(r, \cdot)\|_{L^p(\mathbb{R})} dr \right)^2 |h|^{2H-2} dh \\
& \leq C \left(\int_0^t (t-r)^{\alpha-1} \|Y(r, \cdot)\|_{L^p(\mathbb{R})} [K_t(r)]^{1/2} dr \right)^2, \tag{8.41}
\end{aligned}$$

where we have set

$$K_t(r) \equiv \int_{\mathbb{R}} \|p_{t-r}(x-z) - p_{t-r}(x+h-z)\|_{L^q(\mathbb{R}, dz)}^2 |h|^{2H-2} dh.$$

Now the kernel K_t can be bounded by elementary methods: with the change of variable $z \rightarrow \sqrt{t-r}z$ and $h \rightarrow \sqrt{t-r}h$, we obtain:

$$\begin{aligned}
& \int_{\mathbb{R}} \|p_{t-r}(x-z) - p_{t-r}(x+h-z)\|_{L^q(\mathbb{R}, dz)}^2 |h|^{2H-2} dh \\
& = C (t-r)^{-\frac{3}{2} + \frac{1}{q} + H} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| e^{-\frac{z^2}{2}} - e^{-\frac{(z+h)^2}{2}} \right|^q dz \right)^{\frac{2}{q}} |h|^{2H-2} dh = C (t-r)^{-\frac{1}{2} - \frac{1}{p} + H},
\end{aligned}$$

where we have used the fact that $q^{-1} = 1 - p^{-1}$. Going back to (8.41), the following holds true

$$\begin{aligned}
\int_{\mathbb{R}} |\Phi(t, x) - \Phi(t, x + h)|^2 |h|^{2H-2} dh & \leq C \left(\int_0^t (t-r)^{\alpha-1 + \frac{1}{2}(H - \frac{1}{p} - \frac{1}{2})} \|Y(r, \cdot)\|_{L^p(\mathbb{R})} dr \right)^2 \\
& \leq C \left(\int_0^t (t-r)^{q[\alpha-1 + \frac{1}{2}(H - \frac{1}{2} - \frac{1}{p})]} dr \right)^{\frac{2}{q}} \left(\int_0^t \|Y(r, \cdot)\|_{L^p(\mathbb{R})}^p dr \right)^{\frac{2}{p}}.
\end{aligned}$$

We can now start to tune our parameters. It is easily checked that the first integral in the right hand side above is finite (uniformly in $0 < t \leq T$) if and only if:

$$\alpha > \frac{3}{2p} + \frac{1}{4} - \frac{H}{2}. \tag{8.42}$$

With this choice of α , we get

$$\int_{\mathbb{R}} |\Phi(t, x) - \Phi(t, x + h)|^2 |h|^{2H-2} dh \leq C \left(\int_0^t \|Y(r, \cdot)\|_{L^p(\mathbb{R})}^p dr \right)^{\frac{2}{p}},$$

and since this bound is uniform in x , this yields

$$\sup_{0 \leq t \leq T, x \in \mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H} \Phi(t, x)]^2 \leq C \left(\int_0^T \|Y(r, \cdot)\|_{L^p(\mathbb{R})}^p dr \right)^{\frac{2}{p}}. \quad (8.43)$$

Then, to prove (8.40) it suffices to show that

$$\mathbb{E} \int_{\mathbb{R}} |Y(r, z)|^p dz \leq C \|v\|_{\mathcal{Z}_T^p}^p. \quad (8.44)$$

Step 2: Proof of (8.44). Set $g_{r,z}(s, y) = (r - s)^{-\alpha} p_{r-s}(z - y)v(s, y)$, so that

$$Y(r, z) = \int_0^r \int_{\mathbb{R}} g_{r,z}(s, y) W(ds, dy).$$

Then apply the Burkholder type inequality (8.19), an elementary decomposition of the increments of $g_{r,z}$ in order to get

$$\mathbb{E} \int_{\mathbb{R}} |Y(r, z)|^p dz \leq C [D_1(r) + D_2(r)],$$

where

$$D_1(r) = \int_{\mathbb{R}} \left(\int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} |p_{r-s}(y) - p_{r-s}(y + h)|^2 \right. \\ \left. \times \|v(s, y + z + h)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds \right)^{\frac{p}{2}} dz$$

and

$$D_2(r) = \int_{\mathbb{R}} \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |p_{r-s}(y)|^2 \times \|v(s, y+z+h) - v(s, y+z)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds \right)^{\frac{p}{2}} dz.$$

Let us now bound the term D_1 . Invoking Minkowski's integral inequality, it is easily seen that

$$D_1(r) \leq \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |p_{r-s}(y) - p_{r-s}(y+h)|^2 \|v(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 |h|^{2H-2} dh dy ds \right)^{\frac{p}{2}}.$$

Integrating this identity in h and y , we end up with

$$D_1(r) \leq C \left(\int_0^r (r-s)^{-2\alpha+H-1} \|v(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 ds \right)^{\frac{p}{2}}.$$

Similarly we get the following estimate for $D_2(r)$

$$\begin{aligned} D_2(r) &\leq C \left(\int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha-\frac{1}{2}} \|v(s, \cdot+h) - v(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 |h|^{2H-2} dh ds \right)^{\frac{p}{2}} \\ &= C \left(\int_0^r (r-s)^{-2\alpha-\frac{1}{2}} \left[\mathcal{N}_{\frac{1}{2}-H, p}^* v(s) \right]^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

Combining the estimates for $D_1(r)$ and $D_2(r)$ we obtain

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}} |Y(r, z)|^p dz &\leq C \left(\int_0^r \left[(r-s)^{-2\alpha+H-1} \|v(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 \right. \right. \\ &\quad \left. \left. + (r-s)^{-2\alpha-\frac{1}{2}} \left[\mathcal{N}_{\frac{1}{2}-H, p}^* v(s) \right]^2 \right] ds \right)^{\frac{p}{2}}. \end{aligned} \quad (8.45)$$

Let us go back now to the values of our parameters α, p . One can check that the two singularities in the integrals on the right hand side above are non divergent whenever

$\alpha < \frac{H}{2}$. Combining this condition with the restriction (8.42), we end up with the relation:

$$\frac{3}{2p} + \frac{1}{4} - \frac{H}{2} < \alpha < \frac{H}{2}. \quad (8.46)$$

Those two conditions can be jointly met if and only if $H > \frac{1}{4}$ and $p > \frac{6}{4H-1}$. This completes the proof of the lemma. \square

Notice that the previous lemma implies that for any process $v \in \mathcal{Z}_T^p$, the random variable $\sup_{t \leq T} \sup_{x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \Phi(t, x)$ is finite almost surely, if Φ is given by (8.39).

The uniqueness result for equation (8.1) is the following.

Theorem 8.3.2. *Assume the following conditions hold true:*

1. For $p > \frac{6}{4H-1}$, the initial condition u_0 is in $L^p(\mathbb{R})$ and

$$\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|_{L^p(\mathbb{R})}^2 |h|^{2H-2} dh < \infty. \quad (8.47)$$

2. σ is differentiable, its derivative is Lipschitz and $\sigma(0) = 0$.

3. u and v are two solutions of (8.1) and $u, v \in \mathcal{Z}_T^p$.

Then for every $t \in [0, T]$ and $x \in \mathbb{R}$, $u(t, x) = v(t, x)$, a.s.

Proof. Assume that u solves (8.1) and $u \in \mathcal{Z}_T^p$. From the mild formulation of the solution we have

$$u(t, x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u(s, y)) W(ds, dy). \quad (8.48)$$

We claim that

$$\sup_{t \leq T} \sup_{x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} u(t, x) < \infty, \quad \text{a.s.} \quad (8.49)$$

This follows from the decomposition (8.48). Indeed, on one hand, (8.47) implies that, if $g(t, x) = p_t u_0(x)$, then

$$\sup_{t \leq T} \sup_{x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} g(t, x) < \infty.$$

On the other hand, from the properties of σ , it follows that $\sigma(u)$ also belongs to \mathcal{Z}_T^p , and by Lemma 8.3.1

$$\sup_{t \leq T} \sup_{x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \sigma(u)(t, x) < \infty, \quad \text{a.s.}$$

Notice that to estimate the first term of (8.37) for $\sigma(u)$, we need to assume $\sigma(0) = 0$. If v is another solution of equation (8.1) belonging also to \mathcal{Z}_T^p , then (8.49) also holds for v . In this way, we can define the stopping times

$$\begin{aligned} T_k &= \inf \left\{ 0 \leq t \leq T : \sup_{0 \leq s \leq t, x \in \mathbb{R}} \int_{\mathbb{R}} |u(s, x) - u(s, x+h)|^2 |h|^{2H-2} dh \geq k \right. \\ &\quad \left. \text{or} \quad \sup_{0 \leq s \leq t, x \in \mathbb{R}} \int_{\mathbb{R}} |v(s, x) - v(s, x+h)|^2 |h|^{2H-2} dh \geq k \right\}, \end{aligned}$$

and $T_k \uparrow T$, almost surely, as k tends to infinity. Our strategy will be to control the two following quantities:

$$I_1(t, x) = \mathbb{E} \left[\mathbf{1}_{\{t < T_k\}} |u(t, x) - v(t, x)|^2 \right]$$

and

$$I_2(t, x) = \mathbb{E} \left[\int_{\mathbb{R}} \mathbf{1}_{\{t < T_k\}} |u(t, x) - v(t, x) - u(t, x+h) + v(t, x+h)|^2 |h|^{2H-2} dh \right].$$

We also set $\bar{I}_j(t) = \sup_{x \in \mathbb{R}} I_j(t, x)$ for $j = 1, 2$.

In order to bound I_1 , let us first use elementary properties of Itô's integral, which entail

$$\begin{aligned} \mathbf{1}_{\{t < T_k\}} (u(t, x) - v(t, x)) &= \int_0^{t \wedge T_k} \int_{\mathbb{R}} p_{t-s}(x-y) [\sigma(u(s, y)) - \sigma(v(s, y))] W(ds, dy) \\ &= \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \mathbf{1}_{\{s < T_k\}} [\sigma(u(s, y)) - \sigma(v(s, y))] W(ds, dy). \end{aligned}$$

We thus get $I_1(t, x) \leq C(I_{11}(t, x) + I_{12}(t, x))$, where

$$I_{11}(t, x) = \mathbb{E} \int_0^t \int_{\mathbb{R}^2} |p_{t-s}(x-y) - p_{t-s}(x-y-h)|^2 \\ \times \mathbf{1}_{\{s < T_k\}} |\sigma(u(s, y+h)) - \sigma(v(s, y+h))|^2 |h|^{2H-2} dh dy ds,$$

and

$$I_{12}(t, x) = \mathbb{E} \int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(x-y) \mathbf{1}_{\{s < T_k\}} |\sigma(u(s, y)) - \sigma(v(s, y)) \\ - \sigma(u(s, y+h)) + \sigma(v(s, y+h))|^2 |h|^{2H-2} dh dy ds.$$

Next we bound the term $I_{11}(t, x)$ as follows

$$I_{11}(t, x) \leq C \mathbb{E} \int_0^t \int_{\mathbb{R}^2} |p_{t-s}(x-y) - p_{t-s}(x-y-h)|^2 \\ \times \mathbf{1}_{\{s < T_k\}} |u(s, y+h) - v(s, y+h)|^2 |h|^{2H-2} dh dy ds \leq C \int_0^t (t-s)^{H-1} \mathcal{I}_1(s) ds,$$

where we recall that $\mathcal{I}_1(t) = \sup_{x \in \mathbb{R}} I_1(t, x)$. Let us now invoke the following elementary bound on the rectangular increments of σ , valid whenever σ' is Lipschitz

$$|\sigma(a) - \sigma(b) - \sigma(c) + \sigma(d)| \leq C|a - b - c + d| + C|a - b|(|a - c| + |b - d|),$$

With this additional ingredient, and along the same lines as for $I_{11}(t, x)$, we get

$$I_{12}(t, x) \leq Ck \int_0^t (t-s)^{-\frac{1}{2}} [\mathcal{I}_1(s) + \mathcal{I}_2(s)] ds.$$

Finally, gathering our estimates on I_{11} and I_{12} we end up with:

$$\mathcal{I}_1(t) \leq Ck \int_0^t (t-s)^{H-1} [\mathcal{I}_1(s) + \mathcal{I}_2(s)] ds.$$

The term $I_2(t, x)$ above is dealt with exactly the same way, and we leave to the reader the task of showing that:

$$\mathcal{I}_2(t) \leq Ck \int_0^t (t-s)^{2H-\frac{3}{2}} [\mathcal{I}_1(s) + \mathcal{I}_2(s)] ds .$$

As a consequence,

$$\mathcal{I}_1(t) + \mathcal{I}_2(t) \leq Ck \int_0^t (t-s)^{2H-\frac{3}{2}} [\mathcal{I}_1(s) + \mathcal{I}_2(s)] ds ,$$

which implies $\mathcal{I}_1(t) + \mathcal{I}_2(t) = 0$ for all $t \in [0, T]$. In particular,

$$\mathbb{E} \left[\mathbf{1}_{\{t < T_k\}} |u(t, x) - v(t, x)|^2 \right] = 0 ,$$

which implies $u(t, x) = v(t, x)$ a.s. on $\{t < T_k\}$ for all $k \geq 1$ and $t \in [0, T]$. Therefore, taking into account that $T_k \uparrow \infty$ a.s. as k tends to infinity, we conclude that $u(t, x) = v(t, x)$ a.s. for all $(t, x) \in [0, T] \times \mathbb{R}$. This proves the uniqueness. \square

8.3.2 Space-time function spaces

We introduce here the function spaces which form the underlying spaces of our treatment for the existence of the solution. Since these spaces do not belong to standard classes of function spaces, we describe them in detail.

We denote by $C_{uc}([0, T] \times \mathbb{R})$ the space of all real-valued continuous functions on $[0, T] \times \mathbb{R}$ equipped with the topology of convergence uniformly over compact sets.

Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. Let $\beta \in (0, 1)$ be a fixed number. For every $\delta \in (0, \infty]$ and every function $f : \mathbb{R} \rightarrow B$, we introduce the function

$\mathcal{N}_\beta^{(\delta)} f : \mathbb{R} \rightarrow [0, \infty]$ defined by

$$\mathcal{N}_\beta^{(\delta)} f(x) = \left(\int_{|h| \leq \delta} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}. \quad (8.50)$$

Notice that for $\delta = \infty$, this coincides with the function $\mathcal{N}_\beta^{(\infty)} f = \mathcal{N}_\beta f$ introduced in (8.17).

As we will see later along the development of the chapter, $\mathcal{N}_\beta^{(\delta)} f$ plays a role analogous to the modulus of continuity of f near x . It follows from the triangular inequality, that \mathcal{N} satisfies

$$|\mathcal{N}_\beta^{(\delta)} f(x) - \mathcal{N}_\beta^{(\delta)} g(x)| \leq \mathcal{N}_\beta^{(\delta)} (f - g)(x) \quad (8.51)$$

for all $\delta \in (0, \infty]$, functions f, g and x in \mathbb{R} . Thus, \mathcal{N} is a seminorm.

Suppose, for instance, that a function f has modulus of continuity $|h|^\beta \omega(h)$ at x , for any $|h| \leq \delta$. Then $[\mathcal{N}_\beta^{(\delta)} f(x)]^2$ is majorized by $2 \int_0^\delta \omega^2(h) h^{-1} dh$. Thus, for $\mathcal{N}_\beta^{(\delta)} f(x)$ to be finite, it is sufficient that $\omega^2(h) h^{-1}$ is integrable near 0. On the other hand, if $\mathcal{N}_\beta^{(\delta)} f$ is bounded over a domain, the following proposition asserts that f is necessarily Hölder continuous.

Proposition 8.3.3. *Let I be a non-empty open interval of \mathbb{R} and $\delta \in (0, \infty]$. Let f be a function on \mathbb{R} such that $\sup_{x \in \bar{I}} \mathcal{N}_\beta^{(\delta)} f(x)$ is finite. Then*

$$\sup_{x \in I, |y| \leq \frac{\delta}{3} \wedge \text{dist}(x, \partial I)} \frac{\|f(x+y) - f(x)\|}{|y|^\beta} \leq c(\beta) \sup_{x \in \bar{I}} \mathcal{N}_\beta^{(\delta)} f(x) \quad (8.52)$$

for some finite constant $c(\beta)$ which depends only on β .

Proof. For every $x \in I$ and positive $R, R \leq \delta$, we denote $f_{x,R} = \frac{1}{2R} \int_{-R}^R f(y+x) dy$. We first

estimate $\|f(x) - f_{x,R}\|$ as follows

$$\begin{aligned}
\|f(x) - f_{x,R}\| &\leq \frac{1}{2R} \int_{-R}^R \|f(x) - f(x+y)\| dy \\
&\leq \frac{1}{2R} \left(\int_{-R}^R \|f(x) - f(x+y)\|^2 |y|^{-1-2\beta} dy \right)^{\frac{1}{2}} \left(\int_{-R}^R |y|^{1+2\beta} dy \right)^{\frac{1}{2}} \\
&\leq \frac{R^\beta}{2\sqrt{(1+\beta)}} \sup_{x \in \bar{I}} \mathcal{N}_\beta^{(\delta)} f(x). \tag{8.53}
\end{aligned}$$

Let us now fix $x \in I$ and $y \in \mathbb{R}$ such that $|y| \leq \delta/3 \wedge \text{dist}(x, \partial I)$. We also choose $R = |y|$. It follows from triangle inequality that

$$\|f(x+y) - f(x)\| \leq \|f(x+y) - f_{x+y,R}\| + \|f_{x+y,R} - f_{x,R}\| + \|f(x) - f_{x,R}\|. \tag{8.54}$$

For the second term, we apply Minkowski's and Cauchy-Schwarz' inequalities to get

$$\|f_{x+y,R} - f_{x,R}\| \leq \frac{1}{4R^2} \int_{-R}^R \int_{-R}^R \|f(x+y+z) - f(x+w)\| dz dw,$$

and this yields

$$\begin{aligned}
\|f_{x+y,R} - f_{x,R}\| &\leq \frac{1}{4R^2} \int_{-R}^R \left(\int_{-R}^R \|f(x+y+z) - f(x+w)\|^2 |y+z-w|^{-2\beta-1} dz \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{-R}^R |y+z-w|^{2\beta+1} dz \right)^{\frac{1}{2}} dw.
\end{aligned}$$

Notice that because of the restrictions on the variables, the domain of integration above satisfies $|y+z-w| \leq 3R \leq \delta$ and $x+w \in \bar{I}$. Hence

$$\|f_{x+y,R} - f_{x,R}\| \leq C_\beta \sup_{y \in \bar{I}} \mathcal{N}_\beta^{(\delta)} f(y) R^\beta.$$

The first and third terms on the right hand side of (8.54) are estimated in (8.53). Combining these estimates with (8.54) yields (8.52). \square

The function $\mathcal{N}_\beta^{(\delta)} f$ can be defined for functions defined on $\mathbb{R}_+ \times \mathbb{R}$, and in this case, $\mathcal{N}_\beta^{(\delta)} f : \mathbb{R}_+ \times \mathbb{R} \rightarrow [0, \infty]$. We introduce here a new space which will be used for the existence part of the solution to (8.1).

Definition 8.3.4. Let $\mathfrak{X}_T^\beta(B)$ be the space of all continuous functions $f : [0, T] \times \mathbb{R} \rightarrow B$ such that

$$\|f\|_{\mathfrak{X}_T^\beta(B)} := \sup_{t \in [0, T], x \in \mathbb{R}} \|f(t, x)\| + \sup_{t \in [0, T], x \in \mathbb{R}} \mathcal{N}_\beta f(t, x) < \infty.$$

We equip $\mathfrak{X}_T^\beta(B)$ with the norm $\|\cdot\|_{\mathfrak{X}_T^\beta(B)}$ defined as above. Then $\mathfrak{X}_T^\beta(B)$ is a normed vector space. In fact, the following proposition states that $\mathfrak{X}_T^\beta(B)$ is complete.

Proposition 8.3.5. $\mathfrak{X}_T^\beta(B)$ is a Banach space.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{X}_T^\beta(B)$. Since the space $C_b([0, T] \times \mathbb{R}; B)$ of bounded continuous functions from $[0, T] \times \mathbb{R}$ to B is complete, there exists a bounded continuous function $f : [0, T] \times \mathbb{R} \rightarrow B$ such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T], x \in \mathbb{R}} \|f_n(t, x) - f(t, x)\| = 0.$$

For any $\varepsilon > 0$ there exists $n_0 > 0$ such that

$$\sup_{x \in \mathbb{R}} \mathcal{N}_\beta(f_n - f_m)(t, x) < \varepsilon$$

for all $m, n \geq n_0$. It follows from Fatou's lemma that

$$\mathcal{N}_\beta(f_n - f)(t, x) \leq \liminf_{m \rightarrow \infty} \mathcal{N}_\beta(f_n - f_m)(t, x) \leq \varepsilon$$

for every $t \in [0, T]$, $x \in \mathbb{R}$ and $n \geq n_0$. This implies that $\lim_{n \rightarrow \infty} \sup_{t \leq T, x \in \mathbb{R}} \mathcal{N}_\beta(f_n - f)(t, x) = 0$ which means f_n converges to f in $\mathfrak{X}_T^\beta(B)$. \square

When $B = L^p(\Omega)$ with $p \in [1, \infty)$, we use the notation $\mathfrak{X}_T^{\beta, p} = \mathfrak{X}_T^\beta(L^p(\Omega))$. A function f

in $\mathfrak{X}_T^{\beta,p}$ can be considered as a stochastic process indexed by (t, x) in $[0, T] \times \mathbb{R}$ such that

$$\sup_{t \in [0, T], x \in \mathbb{R}} \|f(t, x)\|_{L^p(\Omega)} + \sup_{t \in [0, T], x \in \mathbb{R}} \left(\int_{\mathbb{R}} \|f(t, x+y) - f(t, x)\|_{L^p(\Omega)}^2 |y|^{-2\beta-1} dy \right)^{\frac{1}{2}} < \infty.$$

Notice that the restriction of an element f in the space $\mathfrak{X}_\theta^{\beta,p}$, introduced in (8.22), to the interval $[0, T]$ belongs to $\mathfrak{X}_T^{\beta,p}$ and $\|f\|_{\mathfrak{X}_T^{\beta,p}} \leq e^{\theta T} \|f\|_{\mathfrak{X}_\theta^{\beta,p}}$.

Whenever σ is an affine function (i.e. $\sigma(u) = au + b$ for some constants a, b), these spaces are sufficient to show existence and uniqueness for equation (8.1). On the other hand, the case of general Lipschitz function σ leads to the consideration of additional spaces, that we are going to study now.

For every $h \in \mathbb{R}$, let τ_h be the translation map in the spatial variable, that is $\tau_h f(t, x) = f(t, x - h)$.

Definition 8.3.6. Let X_T^β be the space of all real-valued continuous functions f on $[0, T] \times \mathbb{R}$ such that

(i) $(t, x) \mapsto \mathcal{N}_\beta^{(1)} f(t, x)$ is finite and continuous on $[0, T] \times \mathbb{R}$.

(ii) $\lim_{h \downarrow 0} \sup_{t \in [0, T], x \in [-R, R]} \mathcal{N}_\beta^{(1)} (\tau_h f - f)(t, x) = 0$ for every positive R .

We equip X_T^β with the following topology. A sequence $\{f_n\}$ in X_T^β converges to f in X_T^β if for all $R > 0$, the sequences $\{f_n\}$ and $\{\mathcal{N}_\beta^{(1)}(f_n - f)\}$ converge uniformly on $[0, T] \times [-R, R]$ to f and 0 respectively. We define a metric on X_T^β as follows

$$d_\beta(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_{n,\beta}}{1 + \|f - g\|_{n,\beta}}, \quad (8.55)$$

where $\|\cdot\|_{n,\beta}$ is the seminorm

$$\|f\|_{n,\beta} := \sup_{t \in [0, T], x \in [-n, n]} |f(t, x)| + \sup_{t \in [0, T], x \in [-n, n]} \mathcal{N}_\beta^{(1)} f(t, x).$$

Since functions in X_T^β are locally bounded, the topology of X_T^β is not altered if in the previous definition $\mathcal{N}_\beta^{(1)} f$ is replaced by $\mathcal{N}_\beta^{(\delta)} f$ for some *finite* and positive δ . We emphasize that replacing δ by ∞ would create a strictly smaller space.

Remark 8.3.7. The space which satisfies only condition (i) in Definition 8.3.6 would be too big and fail to be separable. Analogous situations occur frequently in analysis. In the study of Morrey spaces, this fact was first observed by Zorko in [99]. Continuity spatial translations with respect to a norm is therefore sometimes called Zorko condition.

Proposition 8.3.8. X_T^β is a complete metric space.

Proof. Let $\{f_n\}$ be a Cauchy sequence in X_T^β . Since the space $C_{uc}([0, T] \times \mathbb{R})$ is complete, there exists continuous function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that for all compact intervals I ,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T], x \in I} |f_n(t, x) - f(t, x)| = 0.$$

Let us fix a compact interval $I = [-N, N]$, and $\varepsilon > 0$. There exists $n_0 > 0$ such that

$$\sup_{t \in [0, T], x \in I} \mathcal{N}_\beta^{(1)}(f_n - f_m)(t, x) < \varepsilon$$

for all $m, n \geq n_0$. It follows from Fatou's lemma that

$$\mathcal{N}_\beta^{(1)}(f_n - f)(t, x) \leq \liminf_{m \rightarrow \infty} \mathcal{N}_\beta^{(1)}(f_n - f_m)(t, x) \leq \varepsilon,$$

for every $t \in [0, T]$, $x \in I$ and $n \geq n_0$. This implies that $\mathcal{N}_\beta^{(1)}(f_n - f)$ converges to 0 uniformly on $[0, T] \times I$. In addition, from (8.51), it follows that $\mathcal{N}_\beta^{(1)} f_n$ converges to $\mathcal{N}_\beta^{(1)} f$ uniformly on $[0, T] \times I$, thus the continuity of $\mathcal{N}_\beta^{(1)} f_n$ implies that of $\mathcal{N}_\beta^{(1)} f$.

It remains to check that f satisfies the condition (ii) of Definition 8.3.6. For every $\varepsilon > 0$ and $|h| \leq 1$, choose n sufficiently large so that $\sup_{t \in [0, T], x \in [N-1, N+1]} \mathcal{N}_\beta^{(1)}(f_n - f)(t, x) < \varepsilon$.

Applying Minkowski's inequality, for every $(t, x) \in [0, T] \times [-N, N]$, we have

$$\begin{aligned} \mathcal{N}_\beta^{(1)}(\tau_h f - f)(t, x) &\leq \mathcal{N}_\beta^{(1)}(\tau_h f - \tau_h f_n)(t, x) + \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t, x) + \mathcal{N}_\beta^{(1)}(f_n - f)(t, x) \\ &\leq 2\varepsilon + \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t, x). \end{aligned}$$

Since f_n belongs to X_T^β , $\lim_{h \rightarrow 0} \sup_{t \in [0, T], x \in [-N, N]} \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t, x) = 0$ which implies f belongs to X_T^β . \square

The next results give some characterizations of the space X_T^β .

Lemma 8.3.9. *Let $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $t \mapsto \mathcal{N}_\beta^{(1)} f(t, x)$ is continuous for every fixed x . Suppose in addition that for every $R > 0$,*

$$\lim_{\delta \downarrow 0} \sup_{t \in [0, T], x \in [-R, R]} \int_{-\delta}^{\delta} |f(t, x + y) - f(t, x)|^2 |y|^{-2\beta-1} dy = 0.$$

Then $\mathcal{N}_\beta^{(1)} f$ is continuous and f belongs to X_T^β .

Proof. Fix $R, \varepsilon > 0$, and choose δ such that

$$\sup_{t \in [0, T], x \in [-R-1, R+1]} \int_{-\delta}^{\delta} |f(t, x + y) - f(t, x)|^2 |y|^{-2\beta-1} dy < \varepsilon.$$

Then for every $t \in [0, T]$, $x \in [-R, R]$ and $|h| \leq 1$

$$[\mathcal{N}_\beta^{(1)}(\tau_h f - f)(t, x)]^2 \leq 2\varepsilon + \sup_{t \in [0, T], x \in [-R-1, R+1]} 2|\tau_h f(t, x) - f(t, x)|^2 \int_{|y| > \delta} |y|^{-2\beta-1} dy.$$

Since f is continuous, $\lim_{h \rightarrow 0} \sup_{t \in [0, T], x \in [-R-1, R+1]} |\tau_h f(t, x) - f(t, x)| = 0$. Together with the previous estimate, this yields $\lim_{h \rightarrow 0} \sup_{t \in [0, T], x \in [-R, R]} \mathcal{N}_\beta^{(1)}(\tau_h f - f)(t, x) = 0$ which on one hand, together with (8.51) implies the continuity of $\mathcal{N}_\beta^{(1)} f$. On the other hand, it obviously implies $f \in X_T^\beta$. \square

Proposition 8.3.10. Let $\phi \in C^\infty(\mathbb{R})$ be supported in $[-1, 1]$, such that $\int_{\mathbb{R}} \phi(x)dx = 1$ and $0 \leq \phi \leq 1$. Set $\phi_n(x) = n\phi(nx)$. Then

1. If $f \in X_T^\beta$, then $f * \phi_n \rightarrow f$ in X_T^β as $n \rightarrow \infty$, where $*$ denotes the convolution with respect to the space variable.
2. $C^{0,1}([0, T] \times \mathbb{R})$ i.e., the space of functions which are continuous in time and continuously differentiable in space, is dense in X_T^β .
3. Suppose that f is a continuous function on $[0, T] \times \mathbb{R}$ such that $t \mapsto \mathcal{N}_{\beta,1} f(t, x)$ is finite and continuous in time for every fixed $x \in \mathbb{R}$. Then f belongs to X_T^β if and only if for every $R > 0$

$$\lim_{\delta \downarrow 0} \sup_{t \in [0, T], x \in [-R, R]} \int_{-\delta}^{\delta} |f(t, x+y) - f(t, x)|^2 |y|^{-2\beta-1} dy = 0. \quad (8.56)$$

Proof. We denote $f_n = f * \phi_n$. To show (1), we observe that

$$\begin{aligned} & f_n(t, x+y) - f_n(t, x) - f(t, x+y) + f(t, x) \\ &= \int_{\mathbb{R}} [\tau_h f(t, x+y) - \tau_h f(t, x) - f(t, x+y) + f(t, x)] \phi_n(h) dh \end{aligned}$$

and hence, for every $x \in [-R, R]$, applying Jensen's inequality, we get

$$\begin{aligned} & \int_{-1}^1 |f_n(t, x+y) - f_n(t, x) - f(t, x+y) + f(t, x)|^2 |y|^{-2\beta-1} dy \\ & \leq \int_{\mathbb{R}} \int_{-1}^1 |\tau_h f(t, x+y) - \tau_h f(t, x) - f(t, x+y) + f(t, x)|^2 |y|^{-2\beta-1} \phi_n(h) dh dy \\ & \leq \sup_{r \in [0, T], z \in [-R-1, R+1]} \sup_{|h| \leq \frac{1}{n}} [\mathcal{N}_\beta^{(1)}(\tau_h f - f)(r, z)]^2. \end{aligned}$$

By assumption f belongs to X_T^β . Therefore, owing to condition (ii) in Definition 8.3.6, this integral converges to 0 when $n \rightarrow \infty$. This proves item (1).

To show (2), we first prove that X_T^β contains $C^{0,1}([0, T] \times \mathbb{R})$. Indeed, if g is a function in $C^{0,1}([0, T] \times \mathbb{R})$, by dominated convergence theorem, it is easy to show that $\mathcal{N}_\beta^{(1)} g(t, x)$

is finite and continuous in time for every fixed x . Moreover, for every $R > 0$, we have

$$\sup_{t \in [0, T], x \in [-R, R]} \int_{-\delta}^{\delta} |g(t, x + y) - g(t, x)|^2 |y|^{-2\beta-1} dy \leq \sup_{x \in [-R, R]} \|\partial_x g\|_{\infty} \int_{|y| \leq \delta} |y|^{1-2\beta} dy. \quad (8.57)$$

Since $\lim_{\delta \rightarrow 0} \int_{|y| \leq \delta} |y|^{1-2\beta} dy = 0$, Lemma 8.3.9 implies that g belongs to X_T^{β} . We have thus proved that $C^{0,1} \subset X_T^{\beta}$. Together with (1), this yields (2).

The sufficiency of (3) is in fact the content of Lemma 8.3.9. We focus on the necessity of (8.56). Assume that f belongs to X_T^{β} . Fix $R > 0$, $\varepsilon > 0$ and choose g in $C^{0,1}$ so that

$$\sup_{t \in [0, T], x \in [-R, R]} \mathcal{N}_{\beta}^{(1)}(f - g)(t, x) < \varepsilon.$$

Then for every $\delta > 0$ we have

$$\begin{aligned} \sup_{t \leq T, |x| \leq R} \int_{-\delta}^{\delta} |f(t, x + y) - f(t, x)|^2 |y|^{-2\beta-1} dy \\ \leq 2\varepsilon^2 + 2 \sup_{t \leq T, |x| \leq R} \int_{-\delta}^{\delta} |g(t, x + y) - g(t, x)|^2 |y|^{-2\beta-1} dy. \end{aligned} \quad (8.58)$$

Since g is $C^{0,1}$, the last term converges to 0 when $\delta \downarrow 0$ (see relation (8.57)). Due to the fact that ε can be chosen arbitrarily small, this implies that f satisfies the condition (8.56). \square

Corollary 8.3.11. X_T^{β} is a Polish (complete and separable) space.

Proof. Completeness comes from Proposition 8.3.8. For separability, we invoke Proposition 8.3.10(2) and the fact that the functions in $C^{0,1}([0, T] \times \mathbb{R})$ can be approximated by polynomials with rational coefficients, using a truncation argument. \square

Proposition 8.3.12. The inclusion $X_T^{\beta} \subset X_T^{\alpha}$ holds continuously for $\beta > \alpha$.

Proof. Suppose f belongs to X_T^{β} . Fix $n \geq 1$. By Proposition 8.3.3, we see that

$$\sup_{t \in [0, T], |x| \leq n} |f(t, x + y) - f(t, x)| \leq C \sup_{t \in [0, T], |x| \leq n+1} \mathcal{N}_{\beta}^{(3)} f(t, x) |y|^{\beta}$$

for every $|y| \leq 1$. Hence for every $t \leq T, |x| \leq n$

$$\int_{|y| \leq 1} |f(t, x + y) - f(t, x)|^2 |y|^{-2\alpha-1} dy \leq C \sup_{t \in [0, T], |x| \leq n+1} \mathcal{N}_\beta^{(3)} f(t, x).$$

is finite. The continuity of $(t, x) \mapsto \int_{|y| \leq 1} |f(t, x + y) - f(t, x)|^2 |y|^{-2\alpha-1} dy$ follows at once from dominated convergence theorem. \square

Next we derive a compactness criterion for X_T^β . We first recall some well-known facts. An ε -cover of a metric space is a cover of the space consisting of sets of diameter at most ε . A metric space is called *totally bounded* if it admits a finite ε -cover for every $\varepsilon > 0$. It is well known that a metric space is compact if and only if it is complete and totally bounded. The following lemma is the key ingredient for many compactness results

Lemma 8.3.13. *Let X be a metric space. Assume that, for every $\varepsilon > 0$, there exists a $\delta > 0$, a metric space W , and a mapping $\Phi : X \rightarrow W$ such that $\Phi(X)$ is totally bounded, and for all $x, y \in X$ with $d(\Phi(x), \Phi(y)) < \delta$, we have $d(x, y) < \varepsilon$. Then X is totally bounded.*

The proof of this lemma is elementary, we refer readers to Lemma 1 in [50] for details. The following result provides sufficient conditions for relative compactness in X_T^β .

Proposition 8.3.14. *A set \mathfrak{F} in X_T^β is relatively compact if*

[A1] $\sup_{f \in \mathfrak{F}} |f(0, 0)|$ is finite;

[A2] For every fixed $x \in \mathbb{R}$, $\{f(\cdot, x) : f \in \mathfrak{F}\}$ is equicontinuous in time;

[A3] For every $R > 0$, $\limsup_{\delta \downarrow 0} \sup_{f \in \mathfrak{F}} \sup_{t \in [0, T]} \sup_{x \in [-R, R]} \int_{-\delta}^{\delta} \frac{|f(t, x + y) - f(t, x)|^2}{|y|^{1+2\beta}} dy = 0$.

Proof. Suppose that \mathfrak{F} satisfies the three conditions. We first observe that condition [A3] together with (8.52) implies the following equicontinuity property. For every $R > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t \in [0, T]} |f(t, x) - f(t, y)| < \varepsilon$$

whenever $f \in \mathfrak{F}$ and $x, y \in [-R, R]$ satisfy $|x - y| < \eta$. Together with [A2], this implies equicontinuity for \mathfrak{F} in $(t, x) \in [0, T] \times [-R, R]$. Indeed, take N to be a sufficiently large integer, and set $x_i = -R + \frac{j}{N}R$, $j = 0, 1, \dots, 2N$. According to [A2], $\{f(\cdot, x_i) : f \in \mathfrak{F}\}$ is equicontinuous in time, uniformly for $j = 0, 1, \dots, 2N$. By writing

$$|f(t, x) - f(s, x)| \leq |f(t, x) - f(t, x_i)| + |f(t, x_i) - f(s, x_i)| + |f(s, x_i) - f(s, x)|,$$

where x_i is chosen in such a way that $|x - x_i| < \eta$, this shows the uniformity in x .

Fix now $R > 0$ and $\varepsilon > 0$. From [A3], we can choose a positive number $\delta_1 = \delta_1(\varepsilon)$, such that $\delta_1 < 1$ and

$$2 \sup_{f \in \mathfrak{F}} \sup_{t \in [0, T], x \in [-R, R]} \int_{-\delta_1}^{\delta_1} \frac{|f(t, x + y) - f(t, x)|^2}{|y|^{1+2\beta}} dy < \varepsilon^2.$$

We now choose $\delta_2 \leq \varepsilon$ satisfying

$$2(3\delta_2)^2 \int_{|y| > \delta_1} \frac{dy}{|y|^{1+2\beta}} < \varepsilon^2.$$

By the equicontinuity, we can also choose a positive number $\eta = \eta(\varepsilon)$, $\eta < 1$, such that

$$\|f(t, x) - f(s, y)\| < \delta_2, \tag{8.59}$$

whenever $f \in \mathfrak{F}$ and $(t, x), (s, y) \in [0, T] \times [-R - 2, R + 2]$ satisfy $|t - s| + |x - y| < \eta$. Since $[0, T] \times [-R - 2, R + 2]$ is compact, we can find a finite set of points $\{(t_a, x_i) : 1 \leq a, i \leq n\}$ in $[0, T] \times [-R - 2, R + 2]$ such that for every $(t, x) \in [0, T] \times [-R - 1, R + 1]$, there is some (t_a, x_j) so that $|t - t_a| + |x - x_j| < \eta$ and $[x_j - 1, x_j + 1] \subset [-R - 2, R + 2]$.

Define $\Phi : \mathfrak{F} \rightarrow \mathbb{R}^{n^2}$ by

$$\Phi(f) = (f(t_a, x_i) : 1 \leq a, i \leq n).$$

Condition [A1] and equicontinuity imply that the image $\Phi(\mathfrak{F})$ is bounded and thus totally bounded in \mathbb{R}^{n^2} . Furthermore, consider $f, g \in \mathfrak{F}$ with $\|\Phi(f) - \Phi(g)\|_\infty < \delta_2$. Resorting to the fact that for any $(t, x) \in [0, T] \times [-R - 1, R + 1]$ there are some a, j so that $|t - t_a| + |x - x_j| < \eta$, we can write

$$|f(t, x) - g(t, x)| \leq |f(t, x) - f(t_a, x_j)| + |f(t_a, x_j) - g(t_a, x_j)| + |g(t_a, x_j) - g(t, x)| \leq 3\delta_2,$$

where we bounded the first and third term on the right hand side thanks to (8.59), and the second one according to the fact that $\|\Phi(f) - \Phi(g)\|_\infty < \delta_2$. We end up with

$$\sup_{t \in [0, T], x \in [-R-1, R+1]} |f(t, x) - g(t, x)| \leq 3\delta_2 \leq 3\varepsilon.$$

In addition, for every $(t, x) \in [0, T] \times [-R, R]$ we have

$$\begin{aligned} [\mathcal{N}_\beta(f - g)(t, x)]^2 &\leq 2 \sup_{h \in \{f, g\}} \int_{|y| \leq \delta_1} |h(t, x + y) - h(t, x)|^2 \frac{dy}{|y|^{1+2\beta}} \\ &\quad + 2 \sup_{r \in [0, T], z \in [-R-1, R+1]} |f(r, z) - g(r, z)|^2 \int_{|y| > \delta_1} \frac{dy}{|y|^{1+2\beta}} \leq 2\varepsilon^2. \end{aligned}$$

Therefore, by the definition of the metric on X_T^β (see (8.55)) and Lemma 8.3.13, the set \mathfrak{F} is totally bounded in X_T^β . \square

A useful consequence of the previous proposition is the following corollary.

Corollary 8.3.15. *Suppose $\alpha > \beta$. Let \mathfrak{F} be a subset of X_T^α such that \mathfrak{F} is equicontinuous in time for every fixed x , $\sup_{f \in \mathfrak{F}} |f(0, 0)| < \infty$ and $\sup_{f \in \mathfrak{F}} \sup_{t \in [0, T], |x| \leq R} \mathcal{N}_\alpha^{(1)} f(t, x) < \infty$ for every positive R . Then \mathfrak{F} is relatively compact in X_T^β .*

Proof. It suffices to check that \mathfrak{F} satisfies condition [A3]. Applying (8.52), for δ small

enough, the assumption on \mathfrak{F} implies

$$\sup_{f \in \mathcal{F}} \sup_{t \in [0, T], |x| \leq R} |f(t, x + y) - f(t, x)| \leq C|y|^\alpha,$$

for all $|y| \leq \delta$. Hence,

$$\sup_{f \in \mathcal{F}} \sup_{t \in [0, T], |x| \leq R} \int_{|y| \leq \delta} |f(t, x + y) - f(t, x)|^2 |y|^{-2\beta-1} dy \leq C \int_{|y| \leq \delta} |y|^{2(\alpha-\beta)-1} dy,$$

which clearly implies [\[A3\]](#) since $\alpha > \beta$. □

The following result provides sufficient conditions for relative compactness in $\mathfrak{X}_T^\beta(B)$. Its proof is completely analogous to that of [Proposition 8.3.14](#) and is omitted for the sake of conciseness.

Proposition 8.3.16. *Suppose that a set \mathfrak{F} in $\mathfrak{X}_T^\beta(B)$ satisfies the following properties.*

1. *For every $t \in [0, T]$ and $x \in \mathbb{R}$, $\mathfrak{F}(t, x) := \{f(t, x) : f \in \mathfrak{F}\}$ is relatively compact in the Banach space B .*
2. *For every fixed $x \in \mathbb{R}$, $\{f(\cdot, x) : f \in \mathfrak{F}\}$ is equicontinuous in time.*
3. *For every $R > 0$, we have*

$$\limsup_{\delta \downarrow 0} \sup_{f \in \mathfrak{F}} \sup_{t \in [0, T], x \in [-R, R]} \int_{-\delta}^{\delta} \frac{\|f(t, x + y) - f(t, x)\|^2}{|y|^{1+2\beta}} dy = 0.$$

Then \mathfrak{F} is relatively compact in $\mathfrak{X}_T^\beta(B)$.

In order to handle the nonlinearity in [equation \(8.1\)](#), the following composition rule is crucial.

Proposition 8.3.17 (Left composition). *Let σ be a Lipschitz function on \mathbb{R} and let f be a function in X_T^β . Suppose that for every fixed x , the map $t \mapsto \mathcal{N}_\beta^{(1)} \sigma(f)(t, x)$ is continuous. Then $\sigma(f)$*

belongs to X_T^β . Furthermore, if f_n is a sequence converging to f in X_T^β , then for every positive R and for any $\delta > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T], |x| \leq R} \mathcal{N}_\beta^{(\delta)}(\sigma(f_n) - \sigma(f))(t, x) = 0.$$

Proof. We first show that $\sigma(f)$ belongs to X_T^β . For any $\delta > 0$ we have

$$\int_{|y| \leq \delta} |\sigma(f(t, x + y)) - \sigma(f(t, x))|^2 |y|^{-2\beta-1} dy \leq \|\sigma\|_{Lip}^2 [\mathcal{N}_\beta^{(\delta)} f(t, x)]^2$$

which together with the criterion (3) in Proposition 8.3.10 implies that $\sigma(f)$ belongs to X_T^β .

For the second assertion, for every positive R and any $\varepsilon > 0$, we can choose $\delta_0 > 0$ and $n_0 > 0$, so that, for any $n \geq n_0$,

$$\sup_{t \in [0, T], |x| \leq R} \mathcal{N}_\beta^{(\delta_0)}(\sigma(f_n) - \sigma(f))(t, x) \leq \varepsilon. \quad (8.60)$$

Indeed, it is easily seen that

$$\begin{aligned} \mathcal{N}_\beta^{(\delta_0)}(\sigma(f_n) - \sigma(f))(t, x) &\leq \mathcal{N}_\beta^{(\delta_0)}\sigma(f_n)(t, x) + \mathcal{N}_\beta^{(\delta_0)}\sigma(f)(t, x) \\ &\leq \|\sigma\|_{Lip} \left(\mathcal{N}_\beta^{(\delta_0)} f_n(t, x) + \mathcal{N}_\beta^{(\delta_0)} f(t, x) \right) \\ &\leq \|\sigma\|_{Lip} \left(\mathcal{N}_\beta^{(\delta_0)}(f_n - f)(t, x) + 2\mathcal{N}_\beta^{(\delta_0)} f(t, x) \right), \end{aligned}$$

and the last term is readily bounded by ε if δ_0 is chosen small enough. Now with (8.60) in hand we obtain, for any $\delta > 0$,

$$\begin{aligned} &\sup_{t \in [0, T], |x| \leq R} \mathcal{N}_\beta^{(\delta)}(\sigma(f_n) - \sigma(f))(t, x) \\ &\leq C \varepsilon + C \|\sigma\|_{Lip} \sup_{t \in [0, T], |x| \leq R+1} |f_n(t, x) - f(t, x)| \left(\int_{|y| > \delta_0} |y|^{-2\beta-1} dy \right)^{\frac{1}{2}}. \end{aligned}$$

We conclude the proof by taking the limit as n tends to infinity. \square

The next lemma asserts that if a process is in $\mathfrak{X}_T^{\alpha,p}$, then its paths almost surely lie in the space X_T^β for a certain value of β .

Lemma 8.3.18. *Let f be a stochastic process in $\mathfrak{X}_T^{\alpha,p}$ with $p\alpha > 1$. Assume that for any $R > 0$,*

$$\sup_{s,t \in [0,T]} \sup_{|x| \leq R} \|f(t,x) - f(s,x)\|_{L^p(\Omega)} \leq C_R |t-s|^\lambda, \quad (8.61)$$

where $\lambda > 0$ satisfies $p\lambda > 1$. Then f has a version \tilde{f} such that with probability one, \tilde{f} belongs to X_T^β for every $\beta < \alpha - \frac{1}{p}$.

Proof. Since f belongs to $\mathfrak{X}_T^{\alpha,p}$, inequality (8.52) implies

$$\sup_{t \in [0,T]} \sup_{x,y \in \mathbb{R}} \frac{\|f(t,x+y) - f(t,x)\|_{L^p(\Omega)}}{|y|^\alpha} \leq C \sup_{t \in [0,T], x \in \mathbb{R}} \int_{\mathbb{R}} \|f(t,x+y) - f(t,x)\|_{L^p(\Omega)}^2 |y|^{-2\alpha-1} dy.$$

Then by Kolmogorov continuity criterion, f has a version \tilde{f} such that with probability one, \tilde{f} satisfies

$$\sup_{s,t \in [0,T], |x| \leq R} |\tilde{f}(t,x+y) - \tilde{f}(s,x)| \leq C |y|^{\beta'} |t-s|^{\lambda'}$$

for every R and $|y| \leq 1$, where β' and λ' are fixed and such that $\beta < \beta' < \alpha - \frac{1}{p}$ and $\lambda < \lambda' < \lambda - \frac{1}{p}$. This implies that a.s. $\mathcal{N}_\beta^{(1)} f(t,x)$ is finite and $\mathcal{N}_\beta^{(\delta)}$ satisfies condition (8.56). The continuity of $\mathcal{N}_\beta^{(1)} f$ follows from dominated convergence theorem. These facts imply that \tilde{f} belongs to X_T^β almost surely. \square

8.3.3 Probability measures on X_T^β

To show the existence of solution to equation (8.1) we need some tightness arguments for some probability measures defined on X_T^β . We have the following result towards this aim.

Theorem 8.3.19. *Let $\{\mathbb{P}_n, n \geq 1\}$ be a sequence of probability measures on X_T^β . This sequence is tight if the following three conditions hold:*

1. For each positive η , there exist a and n_0 such that for all $n \geq n_0$

$$\mathbb{P}_n(f \in X_T^\beta : |f(0,0)| \geq a) \leq \eta. \quad (8.62)$$

2. For every $x \in \mathbb{R}$, and every positive ε and η , there exist δ satisfying $0 < \delta < 1$, and n_0 such that for all $n \geq n_0$

$$\mathbb{P}_n \left(f \in X_T^\beta : \sup_{s,t \leq T, |t-s| < \delta} |f(t,x) - f(s,x)| \geq \varepsilon \right) \leq \eta. \quad (8.63)$$

3. For every $R > 0$, for each positive ε and η , there exist $\delta \in (0,1)$ and n_0 such that for all $n \geq n_0$

$$\mathbb{P}_n \left(f \in X_T^\beta : \sup_{t \in [0,T], |x| \leq R} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy \geq \varepsilon \right) \leq \eta. \quad (8.64)$$

Proof. Without loss of generality we assume $n_0 = 1$. For a given $\eta > 0$, we choose a so that $\mathbb{P}_n(B^c) \leq \eta$ for all $n \geq 1$, where

$$B = \{f \in X_T^\beta : |f(0,0)| < a\}.$$

According to condition (3), for any integer k, N , we also choose and fix $\delta_{k,N}$ such that $\mathbb{P}_n(A_{k,N}^c) \leq \eta 2^{-k-N}$ for all $n \geq 1$, where

$$A_{k,N} = \left\{ f \in X_T^\beta : \sup_{t \in [0,T], |x| \leq N} \int_{-\delta_{k,N}}^{\delta_{k,N}} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy \leq \frac{1}{k^2} \right\}.$$

Then for each $\tilde{x} \in [-N, N] \cap \frac{\delta_{k,N}}{3}\mathbb{Z}$, where \mathbb{Z} is the set of integers (note that the number of such \tilde{x} has order $\frac{N}{\delta_{k,N}}$), we choose $\delta'_{k,N}(\tilde{x})$ according to condition (2) such that $\mathbb{P}_n(B_{k,N}^c(\tilde{x})) \leq$

$\delta_{k,N}\eta 2^{-k-N}$, where

$$B_{k,N}(\tilde{x}) = \left\{ f \in X_T^\beta : \sup_{t,s \leq T, |t-s| \leq \delta'_{k,N}(\tilde{x})} |f(t, \tilde{x}) - f(s, \tilde{x})| \leq \frac{1}{k^2} \right\}.$$

Consider now $B_{k,N} = \bigcap_{\tilde{x} \in [-N,N] \cap \frac{\delta_{k,N}}{3} \mathbf{Z}} B_{k,N}(\tilde{x})$. It is easy to see that

$$\mathbb{P}_n(B_{k,N}^c) \leq \sum_{\tilde{x} \in [-N,N] \cap \frac{\delta_{k,N}}{3} \mathbf{Z}} \mathbb{P}_n(B_{k,N}^c(\tilde{x})) \leq C \frac{N}{\delta_{k,N}} \eta \delta_{k,N} 2^{-k-N} = C \eta 2^{-k-N} N.$$

We thus set $A = \bigcap_{k,N} (A_{k,N} \cap B_{k,N}) \cap B$. Then according to Proposition 8.3.14 we see that the closure of A is compact in X_T^β , and $\mathbb{P}_n(A) \geq 1 - C\eta$. This shows the tightness of \mathbb{P}_n . \square

The following proposition states that under some conditions, a sequence of processes u_n can be regarded as a tight sequence of probability measures on the space X_T^β .

Proposition 8.3.20. *Assume that $\alpha, \lambda \in (0, 1)$ and $p \geq 1$ satisfy $p\alpha > 1$, $p\lambda > 1$ and $\beta < \alpha - \frac{1}{p}$.*

Let $\{u_n, n \geq 1\}$ be a sequence of stochastic processes such that

1. $\lim_{\delta \rightarrow \infty} \limsup_n \mathbb{P}(|u_n(0, 0)| > \delta) = 0$,
2. For every $R > 0$, $\sup_n \sup_{s, t \in [0, T], |x| \leq R} \|u_n(t, x) - u_n(s, x)\|_{L^p(\Omega)} \leq C_R |t - s|^\lambda$,
3. $\sup_n \|u_n\|_{\mathbf{x}_T^{\alpha, p}}$ is finite.

From Lemma 8.3.18, the law of u_n can be considered as a probability measure on X_T^β . In addition, as probability measures on X_T^β , the sequence $\{u_n, n \geq 1\}$ is tight.

Proof. This proposition can be easily proved using the same ideas as in the proof of Lemma 8.3.18 and Theorem 8.3.19, we omit the details. \square

8.3.4 Existence of the solution

The main result of this subsection is the existence of a solution for equation (8.1). The methodology, inspired by the work of Gyöngy [46] on semilinear stochastic partial

differential equations, consists in proving tightness of a sequence of solutions obtained by regularizing the noise, and then using the uniqueness result. The space \mathcal{Z}_T^p , where we proved our uniqueness result, consists of $L^p(\mathbb{R})$ -valued processes, and it is not clear how to characterize compactness of probability laws on the space of trajectories of these processes. For this reason, we prove the existence of a solution with paths in the space $X_T^{\frac{1}{2}-H}$ introduced in Definition 8.3.6, equipped with the metric (8.55).

Theorem 8.3.21. *Assume that for equation (8.1) the following conditions hold:*

1. For some $p > \frac{6}{4H-1}$, the initial condition u_0 is in $L^p(\mathbb{R})$ and

$$\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|_{L^p(\mathbb{R})}^2 |h|^{2H-2} dh < \infty. \quad (8.65)$$

2. σ is differentiable and the derivative of σ is Lipschitz and $\sigma(0) = 0$.

Then there exists a solution u to (8.1) with paths in the space $X_T^{\frac{1}{2}-H}$.

Proof. We follow the methodology developed in [46]. We consider a regularization of the noise in space. Indeed, for $\varepsilon > 0$ and $\varphi \in \mathfrak{S}$, we define

$$W_\varepsilon(\varphi) = \int_0^t \int_{\mathbb{R}} [\rho_\varepsilon * \varphi](s, x) W(ds, dy) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s, x) \rho_\varepsilon(x - y) W(ds, dy) dx, \quad (8.66)$$

where $\rho_t(x) = (2\pi t)^{-\frac{1}{2}} e^{-x^2/2t}$. Notice that relation (8.66) can be also read (either in Fourier or direct coordinates) as

$$\begin{aligned} \mathbb{E} [W_\varepsilon(\varphi) W_\varepsilon(\psi)] &= c_{1,H} \int_0^t \int_{\mathbb{R}} \mathcal{F} \varphi(s, \xi) \overline{\mathcal{F} \psi(s, \xi)} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} d\xi ds \\ &= c_{1,H} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s, x) f_\varepsilon(x - y) \psi(s, y) dx dy ds, \end{aligned} \quad (8.67)$$

where f_ε is given by $f_\varepsilon(x) = \mathcal{F}^{-1}(e^{-\varepsilon|\xi|^2} |\xi|^{1-2H})$. In other words, our noise is still a white noise in time but its space covariance is now given by f_ε . Note that f_ε is a real positive

definite function, but is not necessarily positive. As assessed by (8.67), we however have

$$\mathbb{E} \left[|W_\varepsilon(\varphi)|^2 \right] \leq \mathbb{E} \left[|W(\varphi)|^2 \right], \quad (8.68)$$

for all φ in \mathfrak{S} .

For every fixed $\varepsilon > 0$, the noise W_ε induces an approximation to equation (8.12), namely

$$u_\varepsilon(t, x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u_\varepsilon(s, y)) W_\varepsilon(ds, dy), \quad (8.69)$$

where the integral is understood in the Itô sense. Since $|\xi|^{1-2H} e^{-\varepsilon|\xi|^2}$ is in $L^1(\mathbb{R})$, $|f_\varepsilon|$ is bounded. Thus, using Picard iteration, it is easy to see that (8.69) has a unique random field solution, and by estimating the p th moment of $|u_\varepsilon(t, x) - u_\varepsilon(t, x')|$, we see that each solution $u_\varepsilon(t, x)$ is Hölder continuous in space with order β for all $\beta \in (0, 1)$. Therefore we conclude that u_ε is in $\mathfrak{X}_T^{\beta, p}$ for all $\beta \in (0, 1)$. We remark that $\|u_\varepsilon\|_{\mathfrak{X}_T^{\beta, p}}$ may not be bounded uniformly in ε . However, using (8.68), (8.28) and Gronwall's lemma, we can obtain the following uniform bound

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{\mathfrak{X}_T^{\beta, p}} < \infty,$$

for all $\beta < H$ and $p \geq 2$. In particular we can choose β and p such that $\frac{1}{2} - H < \beta - \frac{1}{p}$. In addition, we can show that u_ε is Hölder continuous in time on $[0, T] \times \mathbb{R}$, uniformly on ε . With these properties, we can check that the three conditions in Proposition 8.3.20 are satisfied. Hence the laws of the processes u_ε , considered as probability measures on the space $X_T^{\frac{1}{2}-H}$, are tight and hence weakly relatively compact.

We now base our final considerations on the forthcoming Lemmas 8.3.22 - 8.3.25. Fix a sequence ε_n converging to zero and set $u_n = u_{\varepsilon_n}$. We shall hinge on Lemma 8.3.23 in order to prove that the sequence u_n actually converges in probability. To apply this lemma, we consider now two sequences $u_{m(n)}$ and $u_{l(n)}$, where $\{m(n), n \geq 1\}$ and $\{l(n), n \geq 1\}$ are

strictly increasing sequences of positive integers. For each $n \geq 1$, the triplet $(u_{m(n)}, u_{l(n)}, W)$ defines probability measure on the space

$$\mathcal{B} := X_T^{\frac{1}{2}-H} \times X_T^{\frac{1}{2}-H} \times C_{uc}([0, T] \times \mathbb{R}).$$

Since the family $\{u_\varepsilon, \varepsilon > 0\}$ is weakly relatively compact, there exists a subsequence of the form $\{(u_{m(n_k)}, u_{l(n_k)}, W), k \geq 1\}$ which converges in distribution as k tends to infinity. Thus, by Skorokhod embedding theorem, there is a probability space $(\Omega', \mathcal{F}', P')$ and a sequence of random elements $z_k = (u'_{m(n_k)}, u'_{l(n_k)}, W')$ with values on \mathcal{B} such that z_k has the same distribution as $(u_{m(n_k)}, u_{l(n_k)}, W)$ and z_k converges almost surely (in the topology of \mathcal{B}) to (u', v', W') . By Lemma 8.3.25 we see that both u' and v' are solutions to equation (8.12), with W replaced by W' . Then by Lemma 8.3.24 and the uniqueness result Theorem 8.3.2 we thus get that $u' = v'$ in $X_T^{\frac{1}{2}-H}$. We can now apply Lemma 8.3.23 in order to assert that u_n converges to some random field u in $X_T^{\frac{1}{2}-H}$, in probability. Moreover, taking a subsequence if necessary, we see that u_n converges to u in $X_T^{\frac{1}{2}-H}$ a.s. Hence, thanks to another application of Lemma 8.3.25 we see that u satisfies equation (8.12). This proves the existence of the solution. \square

We now state the lemmata on which the proof of Theorem 8.3.21 relies. The first lemma is a version of Gronwall's lemma, borrowed from [22, Lemma 15], and the correction to this paper [23].

Lemma 8.3.22. *Let $g \in L^1([0, T]; \mathbb{R}_+)$ and consider a sequence of functions $\{f_n; n \geq 0\}$ with $f_n : [0, T] \rightarrow \mathbb{R}_+$, such that f_0 is bounded and for all $n \geq 1$*

$$f_n(t) \leq c_1 + c_2 \int_0^t g(t-s) f_{n-1}(s) ds, \quad (8.70)$$

for two positive constants c_1, c_2 . Then $\sup_{n \geq 1} f_n$ is bounded. If we assume moreover that $c_1 = 0$ in inequality (8.70), we obtain that $\sum_{n \geq 0} f_n^{1/p}$ converges uniformly in $[0, T]$, for all $1 \leq p < \infty$.

The second lemma is a general result on convergence of random variables borrowed from [46].

Lemma 8.3.23. *Let \mathbb{E} be a Polish space equipped with the Borel σ -algebra. A sequence of \mathbb{E} -valued random elements z_n converges in probability if and only if for every pair of subsequences $z_{l(n)}$, $z_{m(n)}$ there exists a subsequence $w_k := (z_{l(n_k)}, z_{m(n_k)})$ converging weakly to a random element w supported on the diagonal $\{(x, y) \in \mathbb{E} \times \mathbb{E} : x = y\}$.*

The next result asserts that the approximate solution to the stochastic heat equation is uniformly bounded in the space \mathcal{Z}_T^p defined by (8.38).

Lemma 8.3.24. *The approximate solutions u_ε satisfy the condition*

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{\mathcal{Z}_T^p} < \infty. \quad (8.71)$$

Furthermore, if $u_\varepsilon \rightarrow u$ in $X_T^{\frac{1}{2}-H}$ a.s., as ε tends to zero, then u is also in \mathcal{Z}_T^p .

Proof. We will use Picard iteration to show that for each ε , $u_\varepsilon \in \mathcal{Z}_T^p$. Then we will use Gronwall's lemma to show that the processes u_ε are uniformly (in ε) bounded in \mathcal{Z}_T^p . To this end, we first define

$$u_\varepsilon^0(t, x) = u_0(x),$$

and recursively

$$u_\varepsilon^{n+1}(t, x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u_\varepsilon^n(s, y)) W_\varepsilon(ds, dy).$$

We wish to bound $\|u_\varepsilon^n\|_{\mathcal{Z}_T^p}$ uniformly in n . First recall that

$$\|u_\varepsilon^n\|_{\mathcal{Z}_T^p} = \sup_{t \in [0, T]} \|u_\varepsilon^n(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} + \sup_{t \in [0, T]} \mathcal{N}_{\frac{1}{2}-H, p}^* u_\varepsilon^n(t),$$

where $\mathcal{N}_{\frac{1}{2}-H, p}^*$ is defined in (8.38). Let us now bound the terms $\|u_\varepsilon^n(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}$ and $\mathcal{N}_{\frac{1}{2}-H, p}^* u_\varepsilon^n(t)$.

Step 1. We shall bound $\|u_\varepsilon^n(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}$ uniformly in n by considering the differences of Picard's iterations. Indeed, by Burkholder's inequality we have

$$\begin{aligned} & \mathbb{E}|u_\varepsilon^{n+1}(t, x) - u_\varepsilon^n(t, x)|^p \\ &= \mathbb{E} \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) [\sigma(u_\varepsilon^n(s, y)) - \sigma(u_\varepsilon^{n-1}(s, y))] W_\varepsilon(ds, dy) \right|^p \\ &\leq C_p \mathbb{E} \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) p_{t-s}(x-z) [\sigma(u_\varepsilon^n(s, y)) - \sigma(u_\varepsilon^{n-1}(s, y))] \right. \\ &\quad \left. \times [\sigma(u_\varepsilon^n(s, z)) - \sigma(u_\varepsilon^{n-1}(s, z))] f_\varepsilon(y-z) dy dz ds \right|^{\frac{p}{2}}. \end{aligned}$$

Thus, since $\|f_\varepsilon\|_\infty \leq C_\varepsilon$ and owing to the fact that σ is a Lipschitz function, we have

$$\begin{aligned} & \mathbb{E}|u_\varepsilon^{n+1}(t, x) - u_\varepsilon^n(t, x)|^p \\ &\leq C_\varepsilon \mathbb{E} \left| \int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y) |u_\varepsilon^n(s, x+y) - u_\varepsilon^{n-1}(s, x+y)| dy \right)^2 ds \right|^{\frac{p}{2}}, \end{aligned}$$

where C_ε denotes a generic constant depending on ε and p . We now integrate with respect to the space variable and invoke Minkowski's inequality. In this way we obtain

$$\begin{aligned} & \mathbb{E} \left\| u_\varepsilon^{n+1}(t, \cdot) - u_\varepsilon^n(t, \cdot) \right\|_{L^p(\mathbb{R})}^p \\ &\leq C_\varepsilon \mathbb{E} \left\| \int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y) |u_\varepsilon^n(s, y + \cdot) - u_\varepsilon^{n-1}(s, y + \cdot)| dy \right)^2 ds \right\|_{L^{\frac{p}{2}}(\mathbb{R})}^{\frac{p}{2}} \\ &\leq C_\varepsilon \mathbb{E} \left(\int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y) \left\| u_\varepsilon^n(s, \cdot) - u_\varepsilon^{n-1}(s, \cdot) \right\|_{L^p(\mathbb{R})} dy \right)^2 ds \right)^{\frac{p}{2}} \\ &\leq C_\varepsilon \left(\int_0^t \left\| u_\varepsilon^n(s, \cdot) - u_\varepsilon^{n-1}(s, \cdot) \right\|_{L^p(\Omega \times \mathbb{R})}^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

This relation easily entails

$$\left\| u_\varepsilon^{n+1}(t, \cdot) - u_\varepsilon^n(t, \cdot) \right\|_{L^p(\Omega \times \mathbb{R})}^2 \leq C_\varepsilon \int_0^t \left\| u_\varepsilon^n(s, \cdot) - u_\varepsilon^{n-1}(s, \cdot) \right\|_{L^p(\Omega \times \mathbb{R})}^2 ds,$$

and a direct application of Gronwall's lemma as stated in Lemma 8.3.22 yields that the quantity $\sup_n \sup_{t \in [0, T]} \|u_\varepsilon^n(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}$ is finite for each fixed $\varepsilon > 0$. This implies that $\sup_{t \in [0, T]} \|u_\varepsilon(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} < \infty$ for each fixed $\varepsilon > 0$.

Step 2. Next we estimate $\mathcal{N}_{\frac{1}{2}-H, p}^* u_\varepsilon(t)$, and observe that we are able to handle this term directly (namely without invoking Picard's iterations). We can write

$$\begin{aligned} \int_{\mathbb{R}} \mathbb{E} |u_\varepsilon(t, x) - u_\varepsilon(t, x+h)|^p dx &\leq C \int_{\mathbb{R}} |p_t u_0(x) - p_t u_0(x+h)|^p dx \\ &\quad + C_\varepsilon \int_{\mathbb{R}} \mathbb{E} \left| \int_0^t \left(\int_{\mathbb{R}} |p_{t-s}(y) - p_{t-s}(y+h)| |u_\varepsilon(s, y+x)| dy \right)^2 ds \right|^{\frac{p}{2}} dx \\ &\leq C \int_{\mathbb{R}} |p_t u_0(x) - p_t u_0(x+h)|^p dx \\ &\quad + C_\varepsilon \left(\int_0^t \left(\int_{\mathbb{R}} |p_{t-s}(y) - p_{t-s}(y+h)| dy \right)^2 \|u_\varepsilon(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

We thus end up with

$$\begin{aligned} \mathcal{N}_{\frac{1}{2}-H, p}^* u_\varepsilon(t) &= \int_{\mathbb{R}} \frac{\|u_\varepsilon(t, \cdot) - u_\varepsilon(t, \cdot+h)\|_{L^p(\Omega \times \mathbb{R})}^2}{|h|^{2-2H}} dh \\ &\leq C \int_{\mathbb{R}} \frac{\|p_t u_0(\cdot) - p_t u_0(\cdot+h)\|_{L^p(\mathbb{R})}^2}{|h|^{2-2H}} dh + C_\varepsilon \sup_{s \in [0, T]} \|u_\varepsilon(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 \\ &\quad \times \int_0^t \int_{\mathbb{R}} \frac{\left(\int_{\mathbb{R}} |p_{t-s}(y) - p_{t-s}(y+h)| dy \right)^2}{|h|^{2-2H}} dh ds, \end{aligned}$$

and the right-hand side in the above inequality is easily seen to be finite. Putting together the last two steps, we can conclude that for each fixed ε , $u_\varepsilon \in \mathcal{Z}_T^p$.

Step 3: Uniform bounds in ε . To prove the norms of u_ε in \mathcal{Z}_T^p are uniformly bounded in ε , we note that u_ε satisfies the equation

$$u_\varepsilon(t, x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} [(p_{t-s}(x - \cdot) \sigma(u_\varepsilon(s, \cdot))) * \rho_\varepsilon](y) W(ds, dy).$$

Hence we have

$$\begin{aligned} \mathbb{E}|u_\varepsilon(t, x)|^p &\leq C|p_t u_0(x)|^p + C\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} |\mathcal{F}(p_{t-s}(x - \cdot)\sigma(u_\varepsilon(s, \cdot)))(\xi)|^2 e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} d\xi ds\right)^{\frac{p}{2}} \\ &\leq C|p_t u_0(x)|^p + C\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} |\mathcal{F}(p_{t-s}(x - \cdot)\sigma(u_\varepsilon(s, \cdot)))(\xi)|^2 |\xi|^{1-2H} d\xi ds\right)^{\frac{p}{2}}. \end{aligned} \quad (8.72)$$

Going back from Fourier to direct coordinates, one can check that

$$\mathbb{E}|u_\varepsilon(t, x)|^p \leq C|p_t u_0(x)|^p + \mathcal{D}_1(t) + \mathcal{D}_2(t),$$

with

$$\mathcal{D}_1(t) = \left(\int_0^t \int_{\mathbb{R}^2} |p_{t-s}(y) - p_{t-s}(y+h)|^2 \|u_\varepsilon(s, y+x+h)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds\right)^{\frac{p}{2}}$$

and

$$\mathcal{D}_2(t) = \left(\int_0^t \int_{\mathbb{R}^2} |p_{t-s}(y)|^2 \|u_\varepsilon(s, y+x+h) - u_\varepsilon(s, y+x)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds\right)^{\frac{p}{2}}.$$

These terms are treated exactly as the terms D_1, D_2 in the proof of Lemma 8.3.1, except for the fact that $\alpha = 0$ in the current situation. We obtain

$$\begin{aligned} \|u_\varepsilon(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 &\leq C \|u_0\|_{L^p(\mathbb{R})}^2 + C \int_0^t (t-s)^{H-1} \|u_\varepsilon(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 ds \\ &\quad + C \int_0^t (t-s)^{-\frac{1}{2}} \int_{\mathbb{R}} \|u_\varepsilon(s, \cdot) - u_\varepsilon(s, \cdot+h)\|_{L^p(\Omega \times \mathbb{R})}^2 |h|^{2H-2} dh ds. \end{aligned} \quad (8.73)$$

Similarly we get (see also the bounds for the terms $\mathcal{I}_1, \mathcal{I}_2$ in the proof of Theorem 8.3.2)

$$\begin{aligned}
[\mathcal{N}_{\frac{1}{2}-H,p}^* u_\varepsilon(t)]^2 &\leq C \int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|_{L^p(\mathbb{R})}^2 |h|^{2H-2} dh \\
&+ C \int_0^t (t-s)^{2H-\frac{3}{2}} \|u_\varepsilon(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 ds \\
&+ C \int_0^t \int_{\mathbb{R}} (t-s)^{H-1} \|u_\varepsilon(s, \cdot) - u_\varepsilon(s, \cdot + l)\|_{L^p(\Omega \times \mathbb{R})}^2 |l|^{2H-2} dl ds. \quad (8.74)
\end{aligned}$$

Set

$$\Psi(t) = \|u_\varepsilon(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 + [\mathcal{N}_{\frac{1}{2}-H,p}^* u_\varepsilon(t)]^2.$$

Thus combining the estimates (8.73) and (8.74) yields

$$\Psi(t) \leq C \|u_0\|_{L^p(\mathbb{R})}^2 + C \int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|_{L^p(\mathbb{R})}^2 |h|^{2H-2} dh + C \int_0^t (t-s)^{2H-\frac{3}{2}} \Psi(s) ds.$$

Since we have shown that for each fixed ε , $\|u_\varepsilon\|_{\mathcal{Z}_T^p} < \infty$, we can apply the Gronwall type Lemma 8.3.22 to the above inequality to show that

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{\mathcal{Z}_T^p} < \infty.$$

Step 4: u is an element of \mathcal{Z}_T^p . Recall once again that we have decomposed $\|u\|_{\mathcal{Z}_T^p}$ according to relation (8.37). We now bound $\|u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2$ and $\mathcal{N}_{\frac{1}{2}-H,p}^* u(t)$ in this decomposition.

Since u_ε converges to u in $X_T^{\frac{1}{2}-H}$ a.s., we have $u_\varepsilon(t, x) \rightarrow u(t, x)$ a.s. for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Thus by Fatou's lemma,

$$\|u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} = \left(\mathbb{E} \int_{\mathbb{R}} \lim_{\varepsilon \rightarrow 0} |u_\varepsilon(t, x)|^p dx \right)^{\frac{1}{p}} \leq \lim_{\varepsilon \rightarrow 0} \left(\mathbb{E} \int_{\mathbb{R}} |u_\varepsilon(t, x)|^p dx \right)^{\frac{1}{p}} \leq C.$$

Therefore we conclude that $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}$ is finite. On the other hand, for each x and h we have $|u_\varepsilon(t, x+h) - u_\varepsilon(t, x)|^2 \rightarrow |u(t, x+h) - u(t, x)|^2$ a.s., so by Fatou's lemma

again we obtain

$$\begin{aligned} \int_{|h| \leq 1} \frac{\|u(t, \cdot + h) - u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2}{|h|^{2-2H}} dh &\leq \int_{|h| \leq 1} \frac{\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(t, \cdot + h) - u_\varepsilon(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2}{|h|^{2-2H}} dh \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_{|h| \leq 1} \frac{\|u_\varepsilon(t, \cdot + h) - u_\varepsilon(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2}{|h|^{2-2H}} dh. \end{aligned}$$

The desired bound on $\mathcal{N}_{\frac{1}{2}-H, p}^* u(t)$ is obtained from the inequality above, by handling the integral on the domains $|h| \leq 1$ and $|h| > 1$. In the latter case, we simply bound $\|u(t, \cdot + h) - u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2$ by $2\|u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2$. By doing so, we conclude that

$$\sup_{t \in [0, T]} \mathcal{N}_{\frac{1}{2}-H, p}^* u(t) = \sup_{t \in [0, T]} \int_{\mathbb{R}} \frac{\|u(t, \cdot + h) - u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2}{|h|^{2-2H}} dh < \infty.$$

Together with the previous estimate on $\|u(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}$, we conclude that $u \in \mathcal{Z}_T^p$. \square

We now state a convergence result for stochastic integrals, with respect to the approximating noise W_ε .

Lemma 8.3.25. *Let $u_n(t, x)$ be a solution to the equation*

$$u_n(t, x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u_n(s, y)) W_n(ds, dy),$$

where we have set $W_n = W_{\varepsilon_n}$ (recall that W_ε is defined by (8.66)) for a sequence $\{\varepsilon_n, n \geq 1\}$ satisfying $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. We assume the following conditions:

(i) with probability one, u_n converges to u in $X_T^{\frac{1}{2}-H}$,

(ii) $\sup_n \|u_n\|_{\mathfrak{X}_T^{\beta, p}} < \infty$, with $\beta > \frac{1}{2} - H$.

Then the process u belongs to $\mathfrak{X}_T^{\frac{1}{2}-H, 2}$. Furthermore, for any fixed $t \leq T$ and $x \in \mathbb{R}$, the random variable $\Phi^n(t, x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u_n(s, y)) W_n(ds, dy)$ converges a.s. to $\Phi(t, x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u(s, y)) W(ds, dy)$, as $n \rightarrow \infty$.

Proof. We decompose the difference $\Phi(t, x) - \Phi^n(t, x)$ into $(\Phi(t, x) - \Phi^{n,1}(t, x)) + (\Phi^{n,1}(t, x) - \Phi^n(t, x))$, where

$$\Phi^{n,1}(t, x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u(s, y)) W^n(ds, dy).$$

Now we note that $\Phi(t, x) - \Phi^{n,1}(t, x)$ can be expressed as

$$\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u(s, y)) W(ds, dy) - \int_0^t \int_{\mathbb{R}} [(p_{t-s}(x-\cdot) \sigma(u(s, \cdot))) * \rho_{\varepsilon_n}](y) W(ds, dy),$$

and thus

$$\begin{aligned} & \mathbb{E} \left| \Phi(t, x) - \Phi^{n,1}(t, x) \right|^2 \\ &= C \mathbb{E} \int_0^t \int_{\mathbb{R}} \left| e^{-\frac{\varepsilon_n |\xi|^2}{2}} - 1 \right|^2 |\mathcal{F}(p_{t-s}(x-\cdot) \sigma(u(s, \cdot))) (\xi)|^2 |\xi|^{1-2H} d\xi ds. \end{aligned}$$

The latter quantity obviously converges to 0 as ε_n goes to 0 because of the finiteness of

$$\mathbb{E} \int_0^t \int_{\mathbb{R}} |\mathcal{F}(p_{t-s}(x-\cdot) \sigma(u(s, \cdot))) (\xi)|^2 |\xi|^{1-2H} d\xi ds,$$

which can be seen by an application of Fatou's lemma (as in Step 4 of the proof of Lemma 8.3.24).

It remains to show that $\lim_{n \rightarrow \infty} \mathbb{E} |\Phi^{n,1}(t, x) - \Phi^n(t, x)|^2 = 0$. However, similarly to (8.72), we have

$$\mathbb{E} \left[|\Phi^{n,1}(t, x) - \Phi^n(t, x)|^2 \right] \leq \mathbb{E} \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f_n(s, y) W(ds, dy) \right|^2,$$

where we have set $f_n = \sigma(u_n) - \sigma(u)$. Furthermore, appealing to Proposition 8.3.17, we see that f_n converges to 0 in $X_T^{\frac{1}{2}-H}$. Then an application of Lemma 8.3.26 completes the proof. Indeed, it is not difficult to check that the sequence f_n satisfies conditions (C1)-(C3)

of Lemma 8.3.26 below. □

Lemma 8.3.26. *Suppose that $\{f_n, n \geq 1\}$ is a sequence of stochastic processes belonging to $\mathfrak{X}_T^{\beta,p} \cap \mathfrak{X}_T^{\frac{1}{2}-H,2}$ with $\frac{1}{2} - H < \beta < H$ and $p > 2$. Assume that the following conditions hold:*

- (C1) *With probability one, f_n converges uniformly to 0 over compact sets of $[0, T] \times \mathbb{R}$.*
- (C2) *For every $R > 0$, $\sup_n \sup_{s,t \in [0,T], |x| \leq R} \mathbb{E}|f_n(t, x) - f_n(s, x)|^2 \leq C|t - s|^\lambda$ for some $\lambda > \frac{1}{2}$.*
- (C3) *$\sup_n \|f_n\|_{\mathfrak{X}_T^{\beta,p}} \leq M$, where M is a finite number.*

Then for every $t \leq T$ and $x \in \mathbb{R}$ the random variable $Y_n(t, x)$ defined by:

$$Y_n(t, x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y) f_n(s, y) W(ds, dy)$$

converges to 0 in $L^2(\Omega)$.

Proof. We first show that $\{f_n, n \geq 1\}$ is relatively compact and converges to 0 in $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. For this purpose, we verify the three conditions (1)-(3) of Proposition 8.3.16. Condition (2) in Proposition 8.3.16 is evident from (C2). Condition (3) in Proposition 8.3.16 follows from the following inequality, where $\delta \leq 1$

$$\int_{|y| \leq \delta} \frac{\|f(t, x + y) - f(t, x)\|_{L^2(\Omega)}^2}{|y|^{2-2H}} dy \leq \sup_{|y| \leq 1} \frac{\|f(t, x + y) - f(t, x)\|_{L^2(\Omega)}^2}{|y|^{2\beta}} \int_{|y| \leq \delta} |y|^{2\beta+2H-2} dy.$$

In fact, the first factor is uniformly bounded in $(t, x) \in [0, T] \times \mathbb{R}$ because of inequality (8.52) and the fact that f_n is bounded in $\mathfrak{X}_T^{\beta,2}$ by condition (C3). Taking into account that $\beta > \frac{1}{2} - H$, the second factor converges to zero as δ tends to zero. To verify condition (1) in Proposition 8.3.16, we fix t, x and note that (C1) implies $f_n(t, x)$ converges almost surely to 0. On the other hand, $\mathbb{E}|f_n(t, x)|^p$ is uniformly bounded, where $p > 2$. These two facts imply $\{f_n(t, x)\}$ converges to 0 in $L^2(\Omega)$, thus condition (1) in Proposition 8.3.16 is verified. Furthermore, condition (C1) ensures that 0 is the only possible limit point of $\{f_n\}$ in $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. We conclude that f_n converges to 0 in $\mathfrak{X}_T^{\frac{1}{2}-H,2}$.

Let us now prove that $Y_n(t, x)$ converges to 0 in $L^2(\Omega)$. Applying (8.19) we get $\mathbb{E}|Y_n(t, x)|^2 \leq C (J_1(t) + J_2(t))$ with

$$J_1(t) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x-y-z) - p_{t-s}(x-y)|^2 \mathbb{E} f_n^2(s, y+z) |z|^{2H-2} dy dz ds$$

and

$$J_2(t) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x-y)|^2 \mathbb{E} |f_n(s, y+z) - f_n(s, y)|^2 |z|^{2H-2} dy dz ds .$$

Now for every fixed $\varepsilon > 0$ and $R > 0$, we choose n so that

$$\sup_{s \in [0, T], |y| \leq R} \mathbb{E} f_n^2(s, y) + \sup_{s \in [0, T], |y| \leq R} \int_{\mathbb{R}} \mathbb{E} |f_n(s, y+z) - f_n(s, y)|^2 |y|^{2H-2} dy < \varepsilon .$$

By making a shift in y , we end up with

$$\begin{aligned} J_1(t) &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p_{t-s}(x-y) - p_{t-s}(x-y+z)|^2 \mathbb{E} f_n^2(s, y) |z|^{2H-2} dy dz ds \\ &\leq \int_0^t \sup_{|y| \leq R} \mathbb{E} f_n^2(s, y) \int_{|y-x| \leq R} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2 dy ds \\ &\quad + \sup_{r \in [0, T], w \in \mathbb{R}} \mathbb{E} f_n^2(r, w) \int_0^t \int_{|y-x| > R} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2 dy ds \\ &\leq C\varepsilon + CM \int_0^t \int_{|y| > R} [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2 dy ds . \end{aligned}$$

Similarly,

$$J_2(t) \leq C\varepsilon + CM \int_0^t \int_{|y| > R} |p_{t-s}(y)|^2 dy ds .$$

We now choose R sufficiently large so that

$$\int_0^t \int_{|y| > R} [|p_{t-s}(y)|^2 + [\mathcal{N}_{\frac{1}{2}-H} p_{t-s}(x-y)]^2] dy ds < \varepsilon .$$

Then $\mathbb{E}|Y_n(t, x)|^2 \leq C\varepsilon$ for n sufficiently large. This implies the result. \square

Finally, the techniques we have designed to get existence and uniqueness for equation (8.1) also allow to obtain the following moment bound for the solution.

Theorem 8.3.27. *Assume the conditions in Theorem 8.3.21, then for the solution we have the following moment bound*

$$\sup_{x \in \mathbb{R}} \|u(t, x)\|_{L^p(\Omega)} \leq C \exp\{Ctp^{\frac{1}{H}} \kappa^{1-\frac{1}{H}} \|\sigma\|_{\text{Lip}}^{\frac{2}{H}}\},$$

and

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \|u(t, x+y) - u(t, x)\|_{L^p(\Omega)}^2 |y|^{2H-2} dy \right)^{\frac{1}{2}} &\leq C \kappa^{\frac{1}{2}-\frac{1}{4H}} p^{\frac{1}{4H}-\frac{1}{2}} \\ &\times \exp\{Ctp^{\frac{1}{H}} \kappa^{1-\frac{1}{H}} \|\sigma\|_{\text{Lip}}^{\frac{2}{H}}\}. \end{aligned}$$

If, in addition, we assume that the initial condition u_0 is Hölder continuous with order γ , then by Proposition 8.2.4 we have

$$\|u(t, x) - u(s, y)\|_{L^p(\Omega)} \leq C(|t-s|^{\frac{H}{2} \wedge \frac{\gamma}{2}} + |x-y|^{H \wedge \gamma}) \quad (8.75)$$

for all $s, t \in [0, T]$ and $x, y \in \mathbb{R}$.

Proof. We will hinge our considerations on the spaces $\mathfrak{X}_\theta^p = \mathfrak{X}_\theta^{\frac{1}{2}-H, p}$ defined by (8.22). Along the same lines as in the proof of Lemma 8.3.24 we can show that $u \in \mathfrak{X}_\theta^p$. Now apply Proposition 8.2.4 by taking f to be the solution u to equation (8.1), and combine it with the mild formulation of the solution. We get the following bound

$$\|u\|_{\mathfrak{X}_{\theta, \varepsilon}^p} \leq C \|u_0\|_{L^\infty} + C \|\sigma\|_{\text{Lip}} \sqrt{p} \|u\|_{\mathfrak{X}_{\theta, \varepsilon}^p} \left(\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}} + \varepsilon \kappa^{H-\frac{3}{4}} \theta^{\frac{1}{4}-H} \right).$$

We optimize the formula above by choosing $\varepsilon = \kappa^{\frac{1}{4}-\frac{H}{2}} \theta^{-\frac{1}{4}+\frac{H}{2}}$, in order to obtain

$$\|u\|_{\mathfrak{X}_{\theta, \varepsilon}^p} \leq C + C \|\sigma\|_{\text{Lip}} \sqrt{p} \|u\|_{\mathfrak{X}_{\theta, \varepsilon}^p} \kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}},$$

then choose θ such that $C\|\sigma\|_{\text{Lip}}\sqrt{p}\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}} = \frac{1}{2}$, that is

$$\theta = Cp^{\frac{1}{H}}\kappa^{1-\frac{1}{H}}\|\sigma\|_{\text{Lip}}^{\frac{2}{H}}, \quad \text{and} \quad \varepsilon = C\kappa^{\frac{1}{4H}-\frac{1}{2}}p^{\frac{1}{2}-\frac{1}{4H}}\|\sigma\|_{\text{Lip}}^{1-\frac{1}{2H}}.$$

Plugging this choice into the above inequality gives the bound

$$\|u\|_{\mathfrak{X}_{\theta,\varepsilon}^p} \leq C.$$

from which our claims are easily deduced. □

We now show the matching lower bound for the second moment.

Proposition 8.3.28. *Let u be a solution to the equation*

$$u(t, x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s, y))W(ds, dy). \quad (8.76)$$

Suppose that u_0 is a bounded nontrivial function and there is a positive constant σ_ such that $|\sigma(z)| \geq \sigma_*|z|$ for all $z \in \mathbb{R}$. Then there exist some universal constants C and L such that*

$$\mathbb{E}|u(t, x)|^2 \geq C \frac{|p_t u_0(x)|^3}{\|u_0\|_{L^\infty}} \exp\{L\sigma_*^{\frac{2}{H}}\kappa^{1-\frac{1}{H}}t\}. \quad (8.77)$$

Proof. From the equation of u , applying Itô isometry, we see that to get

$$\mathbb{E}|u(t, x)|^2 = |p_t u_0(x)|^2 + c_1(H)\mathbb{E} \int_0^t \|p_{t-s}(x-y)\sigma(u(s, y))\|_{\dot{H}^{\frac{1}{2}-H}}^2 ds. \quad (8.78)$$

Let us recall the well-known Sobolev embedding inequality

$$\|g\|_{\dot{H}^{\frac{1}{2}-H}} \geq c\|g\|_{L^{\frac{1}{H}}}, \quad \forall g \in \dot{H}^{\frac{1}{2}-H}(\mathbb{R}).$$

Hence, together with our assumption on σ , it follows that there exists some positive

constant b such that

$$\mathbb{E}|u(t, x)|^2 \geq |p_t u_0(x)|^2 + b\sigma_*^2 \mathbb{E} \int_0^t \|p_{t-s}(x - \cdot)u(s, \cdot)\|_{L^{\frac{1}{H}}(\mathbb{R})}^2 ds.$$

Since $2H < 1$, applying Jensen inequality we see that

$$\begin{aligned} \|p_{t-s}(x - \cdot)u(s, \cdot)\|_{L^{\frac{1}{H}}(\mathbb{R})}^2 &= \left(\int_{\mathbb{R}} |p_{t-s}(x - y)|^{\frac{1}{H}-1} |u(s, y)|^{\frac{1}{H}} p_{t-s}(x - y) dy \right)^{2H} \\ &\geq \int_{\mathbb{R}} |p_{t-s}(x - y)|^{3-2H} |u(s, y)|^2 dy. \end{aligned}$$

It follows that

$$\mathbb{E}|u(t, x)|^2 \geq |p_t u_0(x)|^2 + b\sigma_*^2 \int_0^t \int_{\mathbb{R}} |p_{t-s}(x - y)|^{3-2H} \mathbb{E}|u(s, y)|^2 dy ds.$$

Iterating the previous inequality yields

$$\mathbb{E}|u(t, x)|^2 \geq |p_t u_0(x)|^2 + \sum_{n=1}^{\infty} (b\sigma_*^2)^n I_n(t, x). \quad (8.79)$$

In the above, we have adopted the notation

$$I_n(t, x) = \int_{D_n(t)} \int_{\mathbb{R}^n} p_{t-s_n}^{3-2H}(x - y_n) \cdots p_{s_2-s_1}^{3-2H}(y_2 - y_1) |p_{s_1} u_0(y_1)|^2 d\bar{y} d\bar{s}$$

where $D_n(t) = \{(s_1, \dots, s_n) \in [0, t]^n : 0 < s_1 < \dots < s_n < t\}$ and $d\bar{y} = dy_1 \cdots dy_n$, $d\bar{s} = ds_1 \cdots ds_n$. Note that for every $x, z \in \mathbb{R}$ and $a, b > 0$, the following identity holds

$$\int_{\mathbb{R}} p_a^{3-2H}(x - y) p_b^{3-2H}(y - z) dy = (3 - 2H)^{-\frac{1}{2}} \left(\frac{2\pi\kappa ab}{a + b} \right)^{H-1} p_{a+b}^{3-2H}(x - z).$$

We thus can compute $I_n(t, x)$ by integrating y_j 's in descending order starting from y_n . This

procedure yields

$$I_n(t, x) = (3 - 2H)^{-\frac{n-1}{2}} \times \int_{D_n(t)} \left(\frac{t - s_n}{t - s_1} \prod_{j=2}^n 2\pi\kappa(s_j - s_{j-1}) \right)^{H-1} \int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x - y_1) |p_{s_1} u_0(y_1)|^2 dy_1 d\bar{s}. \quad (8.80)$$

On the other hand, for every fixed $R > 0$, applying Jensen inequality, we see that

$$\begin{aligned} \int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x - y_1) |p_{s_1} u_0(y_1)|^2 dy_1 &\geq p_{t-s_1}^{1-2H}(R) \int_{|x-y_1|<R} p_{t-s_1}^2(x - y_1) |p_{s_1} u_0(y_1)|^2 dy_1 \\ &\geq p_{t-s_1}^{1-2H}(R) R^{-1} \left(\int_{|x-y_1|<R} p_{t-s_1}(x - y_1) p_{s_1} * u_0(y_1) dy_1 \right)^2. \end{aligned} \quad (8.81)$$

The integral on the right side can be rewritten as

$$p_t u_0(x) - \int_{|x-y_1|\geq R} p_{t-s_1}(x - y_1) p_{s_1} * u_0(y_1) dy_1.$$

Since u_0 is bounded, we see that $|p_{s_1} * u_0(y_1)| \leq \|u_0\|_{L^\infty}$ and hence

$$\begin{aligned} \left| \int_{|x-y_1|\geq R} p_{t-s_1}(x - y_1) p_{s_1} * u_0(y_1) dy_1 \right| &\leq \|u_0\|_{L^\infty} \int_{|y|>R} p_{t-s_1}(y) dy \\ &= \|u_0\|_{L^\infty} \pi^{-\frac{1}{2}} \int_{|z|>\frac{R}{\sqrt{2\kappa(t-s_1)}}} e^{-z^2} dz. \end{aligned}$$

For every fixed ϵ in $(0, 1)$, we now choose $R = M\sqrt{2\kappa(t-s_1)}$ where M is such that $e^{-M^2} M^{-1} = \epsilon$. It follows that

$$p_{t-s_1}^{1-2H}(R) R^{-1} = \pi^{H-\frac{1}{2}} (2\kappa(t-s_1))^{H-1} e^{-M^2} M^{-1}$$

and

$$\left| \int_{|x-y_1|<R} p_{t-s_1}(x-y_1) p_{s_1} * u_0(y_1) dy_1 \right| \geq |p_t u_0(x)| - \|u\|_\infty e^{-M^2} M^{-1}.$$

Together with (8.81), we see that

$$\int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x-y_1) |p_{s_1} u_0(y_1)|^2 dy_1 \geq c e^{-M^2} M^{-1} (\kappa(t-s_1))^{H-1} \left(|p_t u_0(x)| - e^{-M^2} M^{-1} \|u_0\|_{L^\infty} \right)^2$$

for some universal constant c . Hence, upon combining the previous estimate and (8.80), we arrive at

$$I_n(t, x) \geq \epsilon c^n \kappa^{(H-1)n} \int_{D_n(t)} \prod_{j=2}^{n+1} (s_j - s_{j-1})^{H-1} d\bar{s} \left(|p_t u_0(x)| - \epsilon \|u_0\|_{L^\infty} \right)^2$$

where $s_{n+1} = t$ and c is some universal constant. It is elementary to compute

$$\int_{D_n(t)} \prod_{j=2}^{n+1} (s_j - s_{j-1})^{H-1} d\bar{s} = \frac{\Gamma(H)^n t^{nH}}{\Gamma(nH+1)}.$$

Therefore, together with (8.79), we obtain

$$\mathbb{E}|u(t, x)|^2 \geq \epsilon \left(|p_t u_0(x)| - \epsilon \|u_0\|_{L^\infty} \right)^2 \sum_{n=0}^{\infty} (cb\Gamma(H))^n \frac{(\sigma_*^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} t)^{nH}}{\Gamma(nH+1)}$$

which yields the following estimate

$$\mathbb{E}|u(t, x)|^2 \geq C\epsilon \left(|p_t u_0(x)| - \epsilon \|u_0\|_{L^\infty} \right)^2 e^{L\sigma_*^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} t}. \quad (8.82)$$

By choosing $\epsilon = \frac{|p_t u_0(x)|}{3\|u_0\|_{L^\infty}}$, we conclude the proof. \square

Chapter 9

Stochastic differential equation for Brox diffusion

Ever since the work of 2014 Abel medalist, Yakov Sinai, in [92] on the random walk in random medium there has been a great amount of work on random processes in a random environment. One of the continuous time and continuous space analogues of Sinai's random walk is the Brownian motion in a white noise medium, namely, the Brox diffusion, which can be described briefly as follows. Let $(\mathcal{B}(t), t \geq 0)$ be a one dimensional standard Brownian motion and let $(W(x), x \in \mathbb{R})$ be a two sided one dimensional Brownian motion, independent of \mathcal{B} . Its derivative $\dot{W}(x)$ with respect to x in the sense of Schwartz distribution is called the white noise (see [52]). The Brox diffusion is a diffusion process $X(t)$ determined formally by the following stochastic differential equation

$$X(t) = -\frac{1}{2} \int_0^t \dot{W}(X(s)) ds + \mathcal{B}(t). \quad (9.1)$$

Throughout the chapter, we assume the initial condition $X(0) = 0$ for simplicity. Since \dot{W} is a distribution (generalized function), the conventional theory of stochastic differential equations does not apply to the above equation (9.1).

In the case W is nice (for example, $\dot{W}(x)$ is deterministic and globally Lipschitz

continuous), then the solution $X(t)$ to (9.1) exists uniquely and it is a Markov process with generator

$$A = \frac{1}{2} e^{W(x)} \frac{d}{dx} \left(e^{-W(x)} \frac{d}{dx} \right). \quad (9.2)$$

In [13], the process $X(t)$ defined (formally) by (9.1) is identified as a Feller diffusion with the above generator A . The Itô-McKean's construction of this Feller diffusion from a Brownian motion via scale-transformation and time change is particularly used there. Let us briefly recall this construction. Let B be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, independent of $(W(x), x \in \mathbb{R})$ (Note that, if it is not stated otherwise, we assume throughout the chapter that $(W(x), x \in \mathbb{R})$ is a two sided Brownian motion).

Set

$$S_W(x) = \int_0^x e^{W(z)} dz, \quad (9.3)$$

and

$$T_{W,B}(t) = \int_0^t e^{-2W \circ S_W^{-1}(B(s))} ds. \quad (9.4)$$

Then, the Feller diffusion $(X(t), t \geq 0)$ associated with (9.1) is represented as

$$X(t) = S_W^{-1} \circ B \circ T_{W,B}^{-1}(t), \quad 0 \leq t < \infty. \quad (9.5)$$

We shall call (9.5) the Itô-McKean representation of the Feller diffusion. With this representation Th. Brox (in [13]) studied the limit of the scaled process $\alpha^{-2}X(e^\alpha)$ (and the limit of the form $\alpha^{-2}X(e^{\alpha h(\alpha)})$, where $h(\alpha) \rightarrow 1$) as $\alpha \rightarrow \infty$.

After this work of Brox ([13]) there have been a number of papers devoted to the study of the process $X(t)$ defined by (9.5). Let us only mention the papers [4, 28, 91] where the local time of $X(t)$ is studied. Some ideas in these papers will be used later. Let us also mention that about the same time as [13] the process $X(t)$ was also studied in the paper [90].

It may be interesting to note that if W were continuously differentiable, it could be

easily checked by Itô's calculus that such an X defined by (9.5) is a weak solution to (9.1) (see Remark 9.2.3 in Section 9.2).

By definition a diffusion is a Markov process with continuous sample paths. Probabilists are interested in more detailed properties of the sample paths. By fixing an almost sure realization of two-sided Brownian motion W , the equation (9.1) can be considered as a stochastic differential equations with singular drift in the form

$$X_t = X_0 + \int_0^t \sigma(X_s) d\mathcal{B}(s) + \int_0^t b'(X_s) ds, \quad (9.6)$$

where \mathcal{B} is a Brownian motion, and σ and b are continuous function. In fact, there have been already a number of work on such (one dimensional) equations (see e.g. [9], [38], [39], [87], and the references therein). In some cases strong existence and uniqueness has been proved for such equations. In the case $\sigma \equiv 1$ (which, in fact, is the situation in (9.1)) if b is Hölder continuous of order α for some $\alpha > 1/2$, then the existence and uniqueness of the strong solution to (9.6) were derived in [9]. Under similar conditions, these results have been also proved in [87]. We would like to mention that existence and uniqueness of the strong solution to (9.6) has been also obtained in [87] under some technical assumption $\mathcal{A}(\nu_0)$ (see [87, pg. 2229]). It is not clear whether this technical assumption can be verified for the equation (9.1) which corresponds to (9.6) with $\sigma = 1$ and $b' = -\frac{1}{2}\dot{W}$.

The current chapter offers the following contributions: First, we show that for any Brownian motion B , independent of W , the Itô-McKean representation (9.5) is a weak solution of the equation (9.1); second, for any given Brownian motion \mathcal{B} we construct a particular Brownian motion B , independent of W , such that the Itô-McKean representation (9.5) is a strong solution of the equation (9.1); third, we show both the strong uniqueness of the solution; and finally, we develop an Itô calculus for the solution. Note that the regularity of the generalized drift $b' = -\frac{1}{2}\dot{W}$ (where W is Hölder continuous with exponent α , for any α less than $1/2$) is at the border of what the papers mentioned above handled

to show that X is a solution of the stochastic differential equation with generalized drift. While proving our result, we will be able to represent the “drift” term

$$-\frac{1}{2} \int_0^t \dot{W}(X(s)) ds$$

via a precise Stratonovich integral with respect to W . Our approach is probabilistic and we will crucially use the fact that W is a Brownian motion. In comparison with the results obtained in the aforementioned papers, the other results can be applied to (almost) every sample path of W , but need to assume that W has a Hölder continuity higher than $1/2$, which cannot be verified by a Brownian motion. Our result can be applied to Brownian motion but is not for every sample path.

Notations: Throughout the chapter we will use a number of different filtrations and σ -fields. Set $\mathcal{F}^B = \{\mathcal{F}_t^B\}_{t \geq 0}$ be the filtration generated by the Brownian motion B . We will also need the extended filtration $\mathcal{F}^{B,W} = \{\mathcal{F}_t^{B,W}\}_{t \geq 0}$ given by

$$\mathcal{F}_t^{B,W} = \mathcal{F}_t^B \vee \sigma(W(x), x \in \mathbb{R}), t \geq 0.$$

$C_b(\mathbb{R})$ denotes the space of all bounded continuous functions on \mathbb{R} . For $\lambda \in (0, 1)$, and $a < b$, let $\|\cdot\|_{\lambda,[a,b]}$ the λ -Hölder norm for functions on $[a, b]$, that is,

$$\|f\|_{\lambda,[a,b]} \equiv \|f\|_{\infty,[a,b]} + \sup_{x,y \in [a,b]} \frac{|f(x) - f(y)|}{|x - y|^\lambda} \quad (9.7)$$

where $\|\cdot\|_{\infty,[a,b]}$ is the supremum norm. Similarly $\|\cdot\|_\lambda$ will denote the λ -Hölder norm for functions on \mathbb{R} . Let $C^\lambda([a, b])$ (resp. C^λ) be the space of Hölder continuous functions f on $[a, b]$ (resp. on \mathbb{R}) with $\|f\|_{\lambda,[a,b]} < \infty$ (resp. $\|f\|_{\lambda,\mathbb{R}} < \infty$). The notation $A \lesssim B$ means $A \leq CB$ for some non-negative constant C .

9.1 Main results

It is evident that to understand equation (9.1), one should first properly define the drift term $\int_0^t \dot{W}(X(s))ds$. For a two-sided Brownian motion W , \dot{W} is not a function but a distribution (generalized functions), this integral has no canonical meaning. However, if the process X admits the Itô-McKean presentation (9.5) for some Brownian motion B independent of W , we can define this integral in such a way that the map $W \mapsto \int \dot{W}(X(s))ds$ is an extension of the integration on smooth functions, i.e $\int \dot{f}(X(s))ds$ for a regular function f .

Let us now describe our method in more details by the following heuristic argument. We first fix W and B , and adopt the following strategy. Let $L_X(t, x)$ be the local time of the process X which is defined as the unique process such that

$$\int_0^t f(X(s))ds = \int_{\mathbb{R}} L_X(t, x)f(x)dx, \quad \forall t \geq 0 \quad \text{and} \quad \forall f \in C_b(\mathbb{R}). \quad (9.8)$$

From the representation (9.5), we see that

$$L_X(t, x) = e^{-W(x)}L_B(T_{W,B}^{-1}(t), S_W(x)), \quad (9.9)$$

where $L_B(t, x)$ is the local time for Brownian motion B , S_W and $T_{W,B}$ are defined by (9.3) and (9.4). Using the definition (9.8) of the local time, we formally write

$$\int_0^t \dot{W}(X(s))ds = \int_{\mathbb{R}} L_X(t, x)\dot{W}(x)dx = \int_{\mathbb{R}} L_X(t, x)W(d^0x). \quad (9.10)$$

A fundamental problem arises: in what sense should one interpret $W(d^0x)$, the above stochastic integral with respect to W ? Note that for fixed t , the process $x \mapsto L(t, x)$ is not necessarily adapted, which is one of the difficulties. If W were a smooth function the above integral would be the usual (pathwise) integral. Hence the last integral in (9.10) should be defined as the (anticipative) Stratonovich stochastic integral so that the integrations in (9.10) are extensions of the classical setting of smooth functions. It turns out that with this

interpretation, the process X given by (9.5) will indeed solve (9.1) (weakly). This can also be seen from our approximation argument (see Section 9.2).

Let us explain how the Stratonovich integral $\int_{\mathbb{R}} L_X(t, x)W(d^0x)$ can be defined rigorously. Presumably, one may use the anticipative stochastic calculus ([79]) (with the help of Malliavin calculus) to define this integral. However, we immediately encountered a difficulty to show the square integrability of $L_X(t, x)$. Instead, we use (9.9) and (9.10) to formally write

$$\begin{aligned} \int_0^t \dot{W}(X(s))ds &= \int_{\mathbb{R}} L_X(t, x)W(d^0x) = \int_{\mathbb{R}} e^{-W(x)} L_B(T_{W,B}^{-1}(t), S_W(x))W(d^0x) \\ &= \int_{\mathbb{R}} e^{-W(x)} L_B(\xi, S_W(x))W(d^0x) \Big|_{\xi=T_{W,B}^{-1}(t)}. \end{aligned} \quad (9.11)$$

The expression on the right hand side of (9.11) enables us to give a meaning to $\int_0^t \dot{W}(X(s))ds$.

Furthermore, throughout the chapter, we can consider a more general situation, namely the integral of the type

$$\int_0^t g(X(s), W(X(s)))\dot{W}(X(s))ds. \quad (9.12)$$

This generalization will later allow us to develop Itô calculus on equation (9.1) and obtain strong uniqueness result. Concerning the function g , we assume that $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a *deterministic* continuous function such that

- For every $x \in \mathbb{R}$, the function $u \mapsto g(x, u)$ is continuously differentiable,
- For every $u \in \mathbb{R}$, the functions $x \mapsto g(x, u)$ and $x \mapsto \partial_u g(x, u)$ are Hölder continuous of order λ with $\lambda > 1/2$.

In addition, we assume that g satisfies the analytic bounds

$$\sup_{x \in K} |g(x, u)| \leq c_1(K)e^{\theta|u|} \quad (9.13)$$

and

$$\sup_{x,y \in K} \frac{|g(x,u) - g(y,u)|}{|x-y|^\lambda} + \sup_{x,y \in K} \frac{|\partial_u g(x,u) - \partial_u g(y,u)|}{|x-y|^\lambda} \leq c_2(K)e^{\theta|u|} \quad (9.14)$$

for every $u \in \mathbb{R}$ and compact interval K , where $\theta, c_1(K)$ and $c_2(K)$ are some positive constants.

Note that for any fixed $\xi \geq 0$, the mapping $x \mapsto g(x, W(x))L_B(\xi, S_W(x)), x \in \mathbb{R}_+$ is *adapted* with respect to the filtration generated by $\{W(z), z \in [0, x]\}_{x \geq 0}$. Similarly the mapping $x \mapsto g(x, W(x))L_B(\xi, S_W(x)), x \in \mathbb{R}_-$ is *adapted* with respect to the filtration generated by $\{W(z), z \in [x, 0]\}_{x \leq 0}$. To elaborate the notation we define

$$\tilde{W}(x) = W(-x), \quad x \geq 0.$$

Let $W(dx)$ and $\tilde{W}(dx)$ denote Itô differentials. Then for any $a \leq b$, and continuous function g on \mathbb{R}^2 , we define the Itô integral

$$\int_a^b g(x, W(x))L_B(\xi, S_W(x))W(dx) = \begin{cases} \int_a^b g(x, W(x))L_B(\xi, S_W(x))W(dx), & \text{if } 0 \leq a \leq b \\ \int_0^{|a|} g(x, W(-x))L_B(\xi, S_W(-x))\tilde{W}(dx) \\ \quad + \int_0^b g(x, W(x))L_B(\xi, S_W(x))W(dx), & \text{if } a \leq 0 \leq b, \\ \int_{|b|}^{|a|} g(x, W(-x))L_B(\xi, S_W(-x))\tilde{W}(dx), & \text{if } a \leq b \leq 0. \end{cases} \quad (9.15)$$

Now for any $a \leq b$, $\xi > 0$, and any continuous function g satisfying (9.13) and (9.14), we

define

$$\int_a^b g(x, W(x))L_B(\xi, S_W(x))W(d^0x) := \int_a^b g(x, W(x))L_B(\xi, S_W(x))W(dx) - \frac{1}{2} \int_a^b \partial_u g(x, W(x))L_B(\xi, S_W(x))dx, \quad (9.16)$$

where $\int_a^b g(x, W(x))L_B(\xi, S_W(x))W(dx)$ is the Itô stochastic integral defined in (9.15). While the right hand side of (9.16) is valid for a bigger class of functions, we restricted ourselves to conditions (9.13) and (9.14) because it is this specific class in which most of the limiting results of the current work hold. The following result, whose proof can be found in Subsection 9.5.1, confirms that the integration defined in (9.16) is indeed of Stratonovich type.

Proposition 9.1.1. *Assume that g satisfies the conditions (9.13) and (9.14) with some $\lambda > 1/2$. In addition, we assume that $u \mapsto \partial_u g(x, u)$ is continuously differentiable. Fix arbitrary $a < b$. Let $\pi : a = x_0 < x_1 < \dots < x_n = b$ be a partition of the interval $[a, b]$ and let $|\pi| = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i)$. Let*

$$W_\pi(x) = W(x_i) + (W(x_{i+1}) - W(x_i)) \frac{x - x_i}{x_{i+1} - x_i}, \quad x_i \leq x < x_{i+1}, \quad (9.17)$$

be the linear interpolation of W associated with the partition π . Then

$$\int_a^b g(x, W(x))L_B(\xi, S_W(x))W(d^0x) = \lim_{|\pi| \rightarrow 0} \int_a^b g(x, W(x))L_B(\xi, S_W(x))\dot{W}_\pi(x)dx, \quad (9.18)$$

where the limit in (9.18) is in L^2 .

The regularity of this integration is described in the following result, whose proof is provided in Subsection 9.5.3.

Lemma 9.1.2. *Let g be a continuous function satisfying (9.13) and (9.14). Then there exists a*

version of the process

$$(\xi, a) \mapsto \int_{-a}^a g(x, W(x))L_B(\xi, S_W(x))W(d^0x)$$

which is jointly continuous in $(\xi, a) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. From (9.16), it is sufficient to show the process

$$H(\xi, y) = \int_0^y g(x, W(x))L_B(\xi, S_W(x))dW(x)$$

has a jointly continuous version. Fix $y_1 < y_2 < N$, $\xi_1 < \xi_2$, using martingale moment inequality and (9.13), we obtain

$$\begin{aligned} & \mathbb{E}|H(\xi_1, y_1) - H(\xi_1, y_2) - H(\xi_2, y_1) + H(\xi_2, y_2)|^4 \\ &= \mathbb{E} \left| \int_{y_1}^{y_2} g(x, W(x))L_B([\xi_1, \xi_2], S_W(x))dW(x) \right|^4 \\ &\lesssim |y_2 - y_1| \int_{y_1}^{y_2} \mathbb{E} e^{4\theta|W(x)|} |L_B([\xi_1, \xi_2], S_W(x))|^4 dx. \end{aligned}$$

It is straightforward to verify that (see also the identity (9.59) below)

$$\mathbb{E}^B |L_B([\xi_1, \xi_2], S_W(x))|^4 \leq C|\xi_2 - \xi_1|^2.$$

Hence,

$$\begin{aligned} & \mathbb{E}|H(\xi_1, y_1) - H(\xi_1, y_2) - H(\xi_2, y_1) + H(\xi_2, y_2)|^4 \\ &\leq C|y_2 - y_1||\xi_2 - \xi_1|^2 \int_{y_1}^{y_2} e^{8\theta^2|x|} dx \leq C_N|y_2 - y_1|^2|\xi_2 - \xi_1|^2. \end{aligned}$$

The result then follows from two-parameter Kolmogorov theorem. \square

As an immediate consequence, we have

Lemma 9.1.3. *Let g be a continuous function satisfying (9.13) and (9.14). Then for any fixed $\xi \geq 0$, the limit*

$$\lim_{a \rightarrow \infty} \int_{-a}^a g(x, W(x)) L_B(\xi, S_W(x)) W(d^0 x)$$

exists almost surely. We will denote the limiting process as

$$\int_{-\infty}^{\infty} g(x, W(x)) L_B(\xi, S_W(x)) W(d^0 x).$$

Furthermore, for any fixed $\xi \geq 0$, we define

$$\tau_{W,B}(\xi) = \inf\{x > 0 : S_W(x) > |\max_{s \in [0, \xi]} B_s|\}. \quad (9.19)$$

Then,

$$\tau_{W,B}(\xi) < \infty, \text{ a.s.}, \quad (9.20)$$

and for all $\xi \geq 0$,

$$\int_{-\infty}^{\infty} g(x, W(x)) L_B(\xi, S_W(x)) W(d^0 x) = \int_{-\tau_{W,B}(\xi)}^{\tau_{W,B}(\xi)} g(x, W(x)) L_B(\xi, S_W(x)) W(d^0 x). \quad (9.21)$$

As a consequence, the process $\xi \mapsto \int_{-\infty}^{\infty} g(x, W(x)) L_B(\xi, S_W(x)) W(d^0 x)$ has a continuous version.

Proof. We denote $M_B(\xi) = |\max_{s \in [0, \xi]} B_s|$. A result of Matsumoto and Yor in [77, identity (4.5)] shows that

$$\lim_{K \rightarrow \infty} \sqrt{2\pi K} \mathbb{E}[S_W(K)^{-1}] = 1. \quad (9.22)$$

On the other hand, for each $K > 0$ (recall also that B and W are independent)

$$\begin{aligned} P(\tau_{W,B}(\xi) > K) &= P(S_W(K)^{-1} \geq M(\xi)^{-1}) \\ &\leq \mathbb{E}[M_B(\xi)] \mathbb{E}[S_W(K)^{-1}] \lesssim \mathbb{E}[S_W(K)^{-1}]. \end{aligned}$$

Together with (9.22), it follows that $\lim_{K \rightarrow \infty} P(\tau_{W,B}(\xi) > K) = 0$. From here, we deduce (9.20).

Since $S_W(\cdot)$ is strictly increasing, if y is such that $y > \tau_{W,B}(\xi)$, then $S_W(y) > |\max_{s \in [0, \xi]} B_s|$, and hence $L_B(\xi, S_W(y))$ vanishes. As a consequence, with probability one, the map $x \mapsto g(x, W(x))L_B(\xi, S_W(x))$ is supported in the interval $[-\tau_{W,B}(\xi), \tau_{W,B}(\xi)]$. Therefore, the limit of $\int_{-a}^a g(x, W(x))L_B(\xi, S_W(x))W(d^0x)$ as a goes to ∞ exists almost surely. From here, we also obtain (9.21). By Lemma 9.1.2, the map $(\xi, a) \mapsto \int_{-a}^a g(x, W(x))L_B(\xi, S_W(x))W(d^0x)$ is continuous. This together with continuity of $\xi \mapsto \tau_{W,B}(\xi)$ implies that the process

$$\xi \mapsto \int_{-\infty}^{\infty} g(x, W(x))L_B(\xi, S_W(x))W(d^0x)$$

has a continuous version. □

With the help of Lemmas 9.1.2, 9.1.3 we can now define the integral of the type (9.12) for sufficiently regular functions g and X as in (9.5).

Definition 9.1.4. Let X be the process in (9.5). Suppose that g is a function satisfying conditions (9.13) and (9.14). Then for every $t \geq 0$, we define

$$\begin{aligned} \int_0^t g(X(s), W(X(s)))\dot{W}(X(s))ds \\ := \int_{-\infty}^{\infty} g(x, W(x))e^{-W(x)}L_B(\xi, S_W(x))W(d^0x)|_{\xi=T_{W,B}^{-1}(t)}, \end{aligned} \quad (9.23)$$

where $T_{W,B}$ is defined by (9.4) and $T_{W,B}^{-1}$ is the inverse of $T_{W,B}$. In particular, for $g \equiv 1$ we have

$$\int_0^t \dot{W}(X(s))ds = \int_{-\infty}^{\infty} e^{-W(x)}L_B(\xi, S_W(x))W(d^0x)|_{\xi=T_{W,B}^{-1}(t)} \quad (9.24)$$

for all $t \geq 0$.

From Lemma 9.1.3, the process $\xi \mapsto \int_{-\infty}^{\infty} g(x, W(x))e^{-W(x)}L_B(\xi, S_W(x))W(d^0x)$ has a continuous version. In addition, since the map $t \mapsto T_{W,B}^{-1}(t)$ is also continuous, we see that

the process

$$t \mapsto \int_0^t g(X(s), W(X(s))) \dot{W}(X(s)) ds$$

also has a continuous version. From now on, we will only consider this continuous version whenever we write either $\int_0^t g(X(s), W(X(s))) \dot{W}(X(s)) ds$ or alternatively its two other equivalent presentations

$$\int_{-\infty}^{\infty} g(x, W(x)) e^{-W(x)} L_B(T_{W,B}^{-1}(t), S_W(x)) W(d^0 x) = \int_{-\infty}^{\infty} g(x, W(x)) L_X(t, x) W(d^0 x).$$

In the above, the equality can be seen from (9.9).

Now with a rigorous definition of $\int_0^t \dot{W}(X(s)) ds$ at hand we can now precisely describe the notions of strong and weak solutions to (9.1).

Definition 9.1.5 (Strong solution). Let $(W(x), x \in \mathbb{R})$ be a two-sided Brownian motion, and $(\mathcal{B}(t), t \geq 0)$ be a Brownian motion with respect to a usual filtration $(\mathcal{F}^{\mathcal{B}})_{t \geq 0}$, independent of W . Let $\mathcal{F}^{\mathcal{B}, W} = (\mathcal{F}_t^{\mathcal{B}, W})_{t \geq 0}$ be the extended filtration given by

$$\mathcal{F}_t^{\mathcal{B}, W} = \mathcal{F}_t^{\mathcal{B}} \vee \sigma(W(x), x \in \mathbb{R}), \forall t \geq 0.$$

We assume that $\mathcal{F}^{\mathcal{B}, W}$ also satisfies the usual conditions. A continuous process $(X(t), t \geq 0)$ is a strong solution to (9.1) if it satisfies the following conditions:

- (i) X is adapted to the extended filtration $\mathcal{F}^{\mathcal{B}, W}$.
- (ii) There exists a Brownian motion $(B(t), t \geq 0)$ independent of W such that $X(t)$ admits the Itô-McKean representation (9.5).
- (iii) For every t , the integral $\int_0^t \dot{W}(X(s)) ds$ is well defined as in Definition 9.1.4.
- (iv) For every $t \geq 0$, the equation

$$X(t) = \mathcal{B}(t) - \frac{1}{2} \int_0^t \dot{W}(X(s)) ds$$

holds almost surely.

Definition 9.1.6 (Weak solution). Let $(W(x), x \in \mathbb{R})$ be a two-sided Brownian motion. A pair (X, \mathcal{B}) in which X is a continuous process, \mathcal{B} is a Brownian motion independent of W , is a weak solution to (9.1) if X is a strong solution to (9.1). More precisely, let $\mathcal{F}^{\mathcal{B}, W}$ be the filtration defined as in Definition 9.1.5. X satisfies the following conditions:

- (i) X is adapted to the filtration $\mathcal{F}^{\mathcal{B}, W}$.
- (ii) There exists a Brownian motion $(B(t), t \geq 0)$ independent of W such that $X(t)$ admits the Itô-McKean representation (9.5).
- (iii) For every $t \geq 0$, the integral $\int_0^t \dot{W}(X(s))ds$ is well defined as in Definition 9.1.4.
- (iv) For every $t \geq 0$, the equation

$$X(t) = \mathcal{B}(t) - \frac{1}{2} \int_0^t \dot{W}(X(s))ds$$

holds almost surely.

The major contribution of the current chapter is the strong existence and uniqueness result for the Brox equation (9.1).

Theorem 9.1.7 (Existence and uniqueness of strong solution). *Let W be a two-sided Brownian motion and \mathcal{B} be a Brownian motion independent of W . Then there exists a unique strong solution X to (9.1).*

In proving Theorem 9.1.7, we are able to obtain existence of a pair (X, \mathcal{B}) satisfying (9.1). The precise statement is following.

Proposition 9.1.8 (Existence of a weak solution). *Let $(W(x), x \in \mathbb{R})$ be a two-sided Brownian motion and let $(B(t), t \geq 0)$ be a Brownian motion, independent of W . Let $X(t)$ be the Itô-McKean*

representation given by the equation (9.5) and let $\int_0^t W(X(s))ds$ be defined by (9.24). Then, there is a Brownian motion \mathcal{B} determined by

$$\mathcal{B}(t) = \int_0^t e^{-W \circ S_W^{-1} \circ B \circ T_{W,B}^{-1}(s)} dB \circ T_{W,B}^{-1}(s), \quad (9.25)$$

which is independent of W , such (X, \mathcal{B}) is a weak solution to equation (9.1).

In fact, Theorem 9.1.8 claims a bit more than just weak existence. It states that any Brox diffusion given by the Itô-McKean representation (9.5) is a weak solution to the equation (9.1). In addition, the Brownian motion \mathcal{B} appeared in the equation is given explicitly by the equation (9.25).

As an application of our method, we can easily obtain the following Itô formula whose proof is provided in Section 9.3.

Theorem 9.1.9 (Itô formula). *Let (X, \mathcal{B}) be a weak solution to (9.1). Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a deterministic continuous function such that*

- *For every x , the map $u \mapsto f(x, u)$ is continuously differentiable*
- *f and $\partial_u f$ satisfy the conditions (9.13) and (9.14).*

We define the function $F(x) = \int_0^x f(y, W(y))dy + F(0)$, where $F(0)$ is some constant. Then, with probability one,

$$F(X(t)) = F(0) + \int_0^t f(X(s), W(X(s)))d\mathcal{B}(s) + \frac{1}{2} \int_0^t \partial_x f(X(s), W(X(s)))ds \\ - \frac{1}{2} \int_{-\infty}^{\infty} f(x, W(x))L_X(t, x)W(d^0 x) + \frac{1}{2} \int_{-\infty}^{\infty} \partial_u f(x, W(x))L_X(t, x)W(d^0 x).$$

An immediate consequence is the following

Corollary 9.1.10 (Itô formula). *Let (X, \mathcal{B}) be a weak solution to (9.1). Let $F : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable deterministic function which is continuously differentiable in t and twice continuously*

differentiable in x . Then, with probability one,

$$\begin{aligned} F(t, X(t)) &= F(0, 0) + \int_0^t \partial_s F(s, X(s)) ds + \int_0^t \partial_x F(s, X(s)) d\mathcal{B}(s) \\ &\quad + \frac{1}{2} \int_0^t \partial_{xx} F(s, X(s)) ds - \frac{1}{2} \int_{-\infty}^{\infty} \partial_x F(s, x) L_X(t, x) W(d^0 x). \end{aligned}$$

The rest of this chapter is organized as follows. In the next section, we provide some preliminaries and show how Proposition 9.1.8 can be derived. Theorem 9.1.9 is proved in Section 9.3. The proof of Theorem 9.1.7 is given in Section 9.4. The proof of Proposition 9.1.1 is provided in Subsection 9.5.1. Proofs of some further technical results (described in Section 9.2) are provided in Section 9.5.

9.2 Preliminary and proof of Proposition 9.1.8

We present in the current section some necessary results which serve as the backbone of our approach. Since Proposition 9.1.8 follows directly from these results, we provide its proof at the end of the section.

Let $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_t\}_{t \geq 0}$ be a filtration under which B is a Brownian motion. We assume that the filtration $\tilde{\mathcal{F}}$ satisfies the usual conditions for a filtration; namely, it is right-continuous and $\tilde{\mathcal{F}}_0$ contains all the null sets. In what follows, $\tilde{\mathcal{F}}$ is usually chosen to be $\mathcal{F}^{B, W}$.

An $\{\tilde{\mathcal{F}}_t\}$ -time-change is a càdlàg, increasing family of $\{\tilde{\mathcal{F}}_t\}$ -stopping times. It is said to be *finite* if each stopping time is finite almost surely, and *continuous* if it is almost surely continuous with respect to time. Let $T = \{T(t) : t \geq 0\}$ be a finite $\{\tilde{\mathcal{F}}_t\}$ -time change and consider the time-changed filtration $\{\tilde{\mathcal{F}}_{T_t}\}_{t \geq 0}$. The right-continuity of $\{\tilde{\mathcal{F}}_t\}$ and $\{T_t\}$ imply that $\{\tilde{\mathcal{F}}_{T_t}\}_{t \geq 0}$ satisfies the usual conditions. Moreover, the time-changed process $\{B \circ T(t)\}$ is an $\{\tilde{\mathcal{F}}_{T_t}\}$ -semimartingale (see [67, Corollary 10.12]). As a consequence, one can define the Itô integral of the form $\int_0^t g(B \circ T(s)) dB \circ T(s)$. In the following proposition we gather some useful facts.

Proposition 9.2.1. *Let f be a function in $C^2(\mathbb{R})$, the set of continuous functions with continuous derivatives up to second order. Let $T = (T(t), t \geq 0)$ be a continuous finite time change. Then, with probability one, for all $t \geq 0$, the following identities hold*

$$f(B \circ T(t)) = f(B \circ T(0)) + \int_{T(0)}^{T(t)} f'(B(s))dB(s) + \frac{1}{2} \int_{T(0)}^{T(t)} f''(B(s))ds, \quad (9.26)$$

$$\int_{T(0)}^{T(t)} f'(B(u))dB(u) = \int_0^t f'(B \circ T(s))dB \circ T(s), \quad (9.27)$$

$$\int_{T(0)}^{T(t)} f''(B(u))du = \int_0^t f''(B \circ T(s))dT(s). \quad (9.28)$$

Finally, the process $t \mapsto \int_0^t f'(B \circ T(s))dB \circ T(s)$ is a semimartingale with respect to the filtration $\{\tilde{\mathcal{F}}_{T_t}\}_{t \geq 0}$, and its quadratic variation is given by

$$\langle \int_0^t f'(B \circ T(s))dB \circ T(s) \rangle = \int_0^t |f'(B \circ T(s))|^2 dT(s). \quad (9.29)$$

In fact, in [69], the author has obtained time-changed Itô formula (such as (9.26)) for semimartingales possibly with jumps. However, we do not need such general result in the current chapter. We refer the reader to [69, Theorem 3.3] for a justification of (9.26) and (9.29). Identities (9.27) and (9.28) follow from [67, Proposition 10.21], see also in [69].

Throughout the chapter, we will approximate $W(x)$ by its polygonal approximations. Since $W(x)$ is defined for all $x \in \mathbb{R}$, we now partition the whole line \mathbb{R} . Let π be any partition with nodes $\{x_i \in \mathbb{R} : x_i < x_{i+1} \ \forall i \in \mathbb{Z}\}$. Then the polygonal approximation of W associated with this partition, denoted by W_π , is the piecewise function such that for every $i \in \mathbb{Z}$

$$W_\pi(x) = W(x_i) + \frac{W(x_{i+1}) - W(x_i)}{x_{i+1} - x_i} (x - x_i), \quad x_i \leq x < x_{i+1}. \quad (9.30)$$

Fix arbitrary Brownian motion B independent of W . Then, for any polygonal approximation W_π of W , we can define X_π via an analogue to the Itô-McKean representation

(9.5):

$$X_\pi(t) = S_{W_\pi}^{-1} \circ B \circ T_{W_\pi, B}^{-1}(t), \quad 0 \leq t < \infty, \quad (9.31)$$

where

$$S_{W_\pi}(x) = \int_0^x e^{W_\pi(z)} dz, \quad 0 \leq t < \infty, \quad (9.32)$$

and

$$T_{W_\pi, B}(t) = \int_0^t e^{-2W_\pi \circ S_{W_\pi}^{-1}(B(s))} ds, \quad 0 \leq t < \infty. \quad (9.33)$$

We also denote

$$\mathcal{B}_\pi(t) = \int_0^t e^{-W_\pi(X_\pi(s))} dB \circ T_{W_\pi, B}^{-1}(s), \quad (9.34)$$

Since W_π is piecewise differentiable it follows from Proposition 9.2.1 that

Lemma 9.2.2. *Let $X_\pi(t)$ be defined by (9.31)-(9.33) and $\mathcal{B}_\pi(t)$ be defined in (9.34). Then \mathcal{B}_π is a Brownian motion with respect to the time-changed filtration $\{\mathcal{F}_{T_{W_\pi, B}^{-1}(t)}^{B, W}\}_{t \geq 0}$. In addition, \mathcal{B}_π is independent of W and X_π satisfies*

$$X_\pi(t) = -\frac{1}{2} \int_0^t \dot{W}_\pi(X_\pi(s)) ds + \mathcal{B}_\pi(t). \quad (9.35)$$

Proof. The Itô formulas (9.26)-(9.28) from Proposition 9.2.1 give

$$\begin{aligned} X_\pi(t) = & \int_0^t (S_{W_\pi}^{-1})' \circ B \circ T_{W_\pi, B}^{-1}(s) dB \circ T_{W_\pi, B}^{-1}(s) \\ & + \frac{1}{2} \int_0^t (S_{W_\pi}^{-1})'' \circ B \circ T_{W_\pi, B}^{-1}(s) \frac{d}{ds} T_{W_\pi, B}^{-1}(s) ds. \end{aligned} \quad (9.36)$$

Note that we apply Proposition 9.2.1 by, first, fixing a realization of W ; we also use the fact that B is a Brownian motion with respect to $\mathcal{F}^{B, W}$. From the definition of $S_{W_\pi}(x)$, we have

$$\frac{d}{dx} S_{W_\pi}^{-1}(x) = e^{-W_\pi(S_{W_\pi}^{-1}(x))},$$

$$\frac{d^2}{dx^2} S_{W_\pi}^{-1}(x) = -e^{-2W_\pi(S_{W_\pi}^{-1}(x))} \dot{W}_\pi(S_{W_\pi}^{-1}(x)).$$

Thus

$$\begin{aligned} \left[\frac{d}{dx} S_{W_\pi}^{-1} \right] \circ B \circ T_{W_\pi, B}^{-1}(s) &= e^{-W_\pi(X_\pi(s))}, \\ \left[\frac{d^2}{dx^2} S_{W_\pi}^{-1} \right] \circ B \circ T_{W_\pi, B}^{-1}(s) &= -e^{-2W_\pi(X_\pi(s))} \dot{W}_\pi(X_\pi(s)). \end{aligned}$$

Similarly, we have

$$\frac{d}{dt} T_{W_\pi, B}^{-1}(s) = e^{2W_\pi(S_{W_\pi}^{-1} \circ B \circ T_{W_\pi, B}^{-1}(s))} = e^{2W_\pi(X_\pi(s))}.$$

Thus (9.36) can be written as

$$\begin{aligned} X_\pi(t) &= \int_0^t e^{-W_\pi(X_\pi(s))} dB \circ T_{W_\pi, B}^{-1}(s) + \frac{1}{2} \int_0^t -e^{-2W_\pi(X_\pi(s))} \dot{W}_\pi(X_\pi(s)) e^{2W_\pi(X_\pi(s))} ds \\ &= \mathcal{B}_\pi(t) - \frac{1}{2} \int_0^t \dot{W}_\pi(X_\pi(s)) ds. \end{aligned} \quad (9.37)$$

From Doob's optional stopping (sampling) theorem it is easy to see that $(\mathcal{B}_\pi(t), t \geq 0)$ is a local martingale with respect to $\{\mathcal{F}_{T_{W_\pi, B}^{-1}(t)}^{B, W}\}_{t \geq 0}$. Moreover, its quadratic variation is

$$\int_0^t e^{-2W_\pi(X_\pi(s))} \frac{d}{ds} T_{W_\pi, B}^{-1}(s) ds = \int_0^t e^{-2W_\pi(X_\pi(s))} e^{2W_\pi(X_\pi(s))} ds = t. \quad (9.38)$$

Thus by Lévy's characterization theorem $\mathcal{B}_\pi(t)$ is a Brownian motion with respect to $\{\mathcal{F}_{T_{W_\pi, B}^{-1}(t)}^{B, W}\}_{t \geq 0}$.

To complete the proof of Lemma 9.2.2, it remains to show that \mathcal{B}_π and W are independent processes. Since both of them are Gaussian, it suffices to show that they are uncorrelated. Indeed, using (9.27) and (9.31), we can write

$$\mathcal{B}_\pi(t) = \int_0^{T_{W_\pi, B}^{-1}(t)} e^{-W_\pi \circ S_{W_\pi}^{-1} \circ B(u)} dB(u).$$

Hence, for every $t \geq 0$ and $x \in \mathbb{R}$, we use the fact that \mathcal{B}_π is $\{\mathcal{F}_{T_{W_\pi, B}^{-1}(t)}^{B, W}\}_{t \geq 0}$ -Brownian motion, and the fact that W is measurable with respect to $\mathcal{F}_{T_{W_\pi, B}^{-1}(0)}^{B, W}$ to get

$$\mathbb{E}[\mathcal{B}_\pi(t)W(x)] = \mathbb{E}\left[\mathbb{E}\left[\mathcal{B}_\pi(t) \middle| \mathcal{F}_{T_{W_\pi, B}^{-1}(0)}^{B, W}\right] W(x)\right] = \mathbb{E}[\mathcal{B}_\pi(0)W(x)] = 0.$$

Hence, we complete the proof of Lemma 9.2.2. \square

Remark 9.2.3. (i) Lemma 9.2.2 implies that $(X_\pi(t), t \geq 0)$ is the weak solution of the equation:

$$dX_\pi(t) = -\frac{1}{2}\dot{W}_\pi(X_\pi(t))dt + d\tilde{B}(t),$$

where \tilde{B} is a Brownian motion independent of W_π .

(ii) The result of Lemma 9.2.2 holds true when $W_\pi(x)$ is replaced by any continuously differentiable function.

Now Proposition 9.1.8 follows from (9.35) by shrinking the mesh size $|\pi|$ to 0. This step is verified through the following propositions.

Proposition 9.2.4. *For every $T \geq 0$, $\lim_{|\pi| \rightarrow 0} \mathbb{E} \sup_{t \leq T} |\mathcal{B}_\pi(t) - \mathcal{B}(t)|^2 = 0$.*

Proposition 9.2.5. *Then for every $\delta > 0$, there exists a partition $\pi(\delta)$ of \mathbb{R} such that for any $T > 0$,*

$$\limsup_{\delta \rightarrow 0} \left| \int_0^t g(X_{\pi(\delta)}(s), W_{\pi(\delta)} \circ X_{\pi(\delta)}(s)) \dot{W}_{\pi(\delta)}(X_{\pi(\delta)}(s)) ds - \int_{-\infty}^{\infty} g(x, W(x)) L_X(t, x) W(d^0 x) \right| = 0,$$

with probability one.

The proofs of these propositions are provided in Subsection 9.5.2 and Subsection 9.5.4 respectively. Proposition 9.2.5 in turn is relied on the following moment estimates for local time of Brownian motion, which are of independent interest.

Proposition 9.2.6. (i) Let $x, y \in \mathbb{R}$. For every $\beta \in [0, 1/2]$, the following estimates holds

$$\left| \mathbb{E} (L_B([\xi, \eta], y) - L_B([\xi, \eta], x))^{2n} \right| \leq C_{\beta, n} |\eta - \xi|^{n(1-\beta)} |x - y|^{2\beta n}. \quad (9.39)$$

(ii) For every $x_1, y_1, \dots, x_k, y_k$ satisfying

$$x_1 < y_1 \leq x_2 < y_2 \leq \dots \leq x_{2n} < y_{2n}. \quad (9.40)$$

and every $\alpha \in [0, 1]$ we have

$$\left| \mathbb{E} \prod_{k=1}^{2n} (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \right| \leq C_{\alpha, n} |\eta - \xi|^{n\alpha} \prod_{k=1}^{2n} |y_k - x_k|^{1-\alpha}. \quad (9.41)$$

The proof of the previous proposition is given in Subsection 9.5.3.

Remark 9.2.7. (i) The former inequality (9.39) is well known. The above second estimate (9.41) is new and quite interesting itself. Since in our proof of (9.41) we shall obtain some results which can be used to prove (9.39) easily, we shall also present a straightforward proof of (9.39).

(ii) From [81], it is known that $L(\xi, x)$ is a semimartingale on x . A consequence is that $\mathbb{E} (L_B(\xi, y) - L_B(\xi, x))^{2n} \leq C_{\beta, n} |x - y|^n$. (9.39) is an extension of this equality.

We will also need the following analytic result.

Lemma 9.2.8. Let f and f_n , ($n = 1, 2, \dots$) be bijective functions on \mathbb{R} which are continuous and strictly increasing. Suppose that $f_n(x)$ converges to $f(x)$ for every x in \mathbb{R} . Then for any compact $A \subset \mathbb{R}$, $\lim_{n \rightarrow \infty} \sup_{y \in A} |f_n^{-1}(y) - f^{-1}(y)| = 0$.

Proof. The proof follows by contradiction. Suppose there exists ϵ_0 and a subsequences $\{f_{n_k}\}$ and $\{y_{n_k}\}$ such that

$$y_{n_k} \rightarrow y, \text{ as } n_k \rightarrow \infty,$$

$$|f_{n_k}^{-1}(y_{n_k}) - f^{-1}(y)| > \epsilon_0, \quad \forall n_k.$$

Thus, for infinitely many n_k 's, either $f_{n_k}^{-1}(y_{n_k}) > f^{-1}(y) + \epsilon_0$ or $f_{n_k}^{-1}(y_{n_k}) < f^{-1}(y) - \epsilon_0$. Without loss of generality, we consider only the former case in which $y_{n_k} > f_{n_k}(f^{-1}(y) + \epsilon_0)$ for infinitely many n_k 's. Upon passing the limit $n_k \rightarrow \infty$, we obtain $y \geq f(f^{-1}(y) + \epsilon_0) > f(f^{-1}(y))$, which is a contradiction. \square

Let us see how Proposition 9.1.8 follows from these propositions.

Proof of Proposition 9.1.8. By Proposition 9.2.5 (with $g \equiv 1$) we see that $\int_0^t W_\pi(X_\pi(s))ds$ converges almost surely to $\int_0^t \dot{W}(X(s))ds$ uniformly on the compacts of \mathbb{R}_+ . It is also obvious from the definitions of $S_{W_\pi}(x)$, $T_{W_\pi, B}(t)$, $X_\pi(t)$, and application of Lemma 9.2.8 that $X_\pi(t)$ converges almost surely to $X(t)$ uniformly on compact intervals of \mathbb{R}_+ . From Proposition 9.2.4, it follows that $\mathcal{B}_\pi(t)$ converges almost surely to the process \mathcal{B} defined in (9.25) uniformly on compact intervals of \mathbb{R}_+ . By passing through the limit $|\pi| \rightarrow 0$ in (9.35), we see that X satisfies (9.1). In addition, by Lemma 9.2.2, for every π , \mathcal{B}_π is the Brownian motion independent of W , hence, it is trivial to see that the limiting process \mathcal{B} is also a Brownian motion independent of W . To conclude the proof, it remains to show that X is adapted to the filtration $\mathcal{F}^{\mathcal{B}, W}$. Indeed, because the integrand in (9.25) is non-vanishing, it follows that the filtration $\mathcal{F}^{\mathcal{B}, W}$ coincides with the time-changed filtration $\mathcal{F}_{T_{W, B}^{-1}}^{\mathcal{B}, W}$. In addition, it is evident that the process X defined by Itô-McKean representation (9.5) is adapted to the later filtration. These two facts complete the proof. \square

9.3 Itô formula - Proof of Theorem 9.1.9

Proof of Theorem 9.1.9. Let π be any partition of \mathbb{R} . Let W_π be the linear interpolation of W defined by (9.30). Denote $F_\pi(x) = \int_0^x f(y, W_\pi(y))dy + F(0)$ and $X_\pi(t) = S_{W_\pi}^{-1} \circ B \circ T_{W_\pi, B}^{-1}(t)$. We apply the time-changed Itô formula (9.26) for $F_\pi(X_\pi(t)) = F_\pi \circ S_{W_\pi}^{-1} \circ B \circ T_{W_\pi, B}^{-1}(t)$, recall that in order to apply Itô formula we first fix a realization of W and we also use the

fact that B is a Brownian motion with respect to $\mathcal{F}^{B,W}$.

$$\begin{aligned} F_\pi(X_\pi(t)) &= F(0) + \int_0^t (F_\pi \circ S_{W_\pi}^{-1})' \circ B \circ T_{W_\pi, B}^{-1}(s) dB \circ T_{W_\pi, B}^{-1}(s) \\ &\quad + \frac{1}{2} \int_0^t (F_\pi \circ S_{W_\pi}^{-1})'' \circ B \circ T_{W_\pi, B}^{-1}(s) dT_{W_\pi, B}^{-1}(s). \end{aligned}$$

It is now easy to see that

$$(F_\pi \circ S_{W_\pi}^{-1})' \circ B \circ T_{W_\pi, B}^{-1}(s) = f(X_\pi(s), W_\pi(X_\pi(s)))e^{-W_\pi(X_\pi(s))},$$

$$\begin{aligned} (F_\pi \circ S_{W_\pi}^{-1})'' \circ B \circ T_{W_\pi, B}^{-1}(s) &= \partial_x f(X_\pi(s), W_\pi(X_\pi(s)))e^{-2W_\pi(X_\pi(s))} \\ &\quad + \partial_u f(X_\pi(s), W_\pi(X_\pi(s)))e^{-2W_\pi(X_\pi(s))}\dot{W}_\pi(X_\pi(s)) \\ &\quad - f(X_\pi(s), W_\pi(X_\pi(s)))e^{-2W_\pi(X_\pi(s))}\dot{W}_\pi(X_\pi(s)), \end{aligned}$$

and

$$dT_{W_\pi, B}^{-1}(s) = e^{2W_\pi(X_\pi(s))} ds.$$

Upon combining the above four identities, we obtain

$$\begin{aligned} F_\pi(X_\pi(t)) &= F(0) + \int_0^t f(X_\pi(s), W_\pi(X_\pi(s))) d\mathcal{B}_\pi(s) \\ &\quad + \frac{1}{2} \int_0^t \partial_x f(X_\pi(s), W_\pi(X_\pi(s))) ds \\ &\quad - \frac{1}{2} \int_0^t f(X_\pi(s), W_\pi(X_\pi(s))) \dot{W}_\pi(X_\pi(s)) ds \\ &\quad + \frac{1}{2} \int_0^t f'(X_\pi(s), W_\pi(X_\pi(s))) \dot{W}_\pi(X_\pi(s)) ds, \end{aligned} \tag{9.42}$$

where $\mathcal{B}_\pi(t) = \int_0^t e^{-W_\pi(X_\pi(s))} dB \circ T_{W_\pi, B}^{-1}(s)$ is a Brownian motion, as seen from Lemma 9.2.2.

For every $\delta > 0$, from Proposition 9.2.5, we can choose a partition $\pi = \pi(\delta)$ such that

$$\int_0^t f(X_{\pi(\delta)}(s), W_{\pi(\delta)}(X_{\pi(\delta)}(s))) \dot{W}(X_{\pi(\delta)}(s)) ds$$

and

$$\int_0^t f'(X_{\pi(\delta)}(s), W_{\pi(\delta)}(X_{\pi(\delta)}(s))) \dot{W}(X_{\pi(\delta)}(s)) ds$$

converge to

$$\int_{\mathbb{R}} f(x, W(x)) e^{-W(x)} L_B(T_{W, B}^{-1}(t), S_W(x)) W(d^0 x)$$

and

$$\int_{\mathbb{R}} f'(x, W(x)) e^{-W(x)} L_B(T_{W, B}^{-1}(t), S_W(x)) W(d^0 x)$$

respectively as $\delta \downarrow 0$. In addition, since X_π and W_π converge to X and W , respectively, uniformly over compact intervals, with probability one, the integral $\int_0^t \partial_x f(X_\pi(s), W_\pi(X_\pi(s))) ds$ converges to $\int_0^t \partial_x f(X(s), W(X(s))) ds$. Hence, by passing through the limit $\delta \downarrow 0$ in (9.42), it remains to show that the stochastic integral

$$\int_0^t f(X_\pi(s), W_\pi(X_\pi(s))) d\mathcal{B}_\pi(s)$$

converges to $\int_0^t f(X(s), W(X(s))) d\mathcal{B}(s)$ in probability as the mesh size of π shrinks to 0. For this purpose, we fix a continuous sample path of W and further denote $\tilde{f}(x) = f(x, W(x))$ and $\tilde{f}_\pi(x) = f(x, W_\pi(x))$. Since for fixed $t > 0$, X_π converges uniformly to X on $[0, t]$, for each $M > 0$ we can find a stopping time T_M such that

$$\sup_{s \leq t} \sup_{\pi} |X_\pi(s \wedge T_M)| \leq M.$$

Since X has finite range, we can also require $\lim_{M \rightarrow \infty} T_M = \infty$. Thus, it suffices to show the following limit in L^2

$$\lim_{|\pi| \downarrow 0} \int_0^{t \wedge T_M} \tilde{f}_\pi(X_\pi(s)) d\mathcal{B}_\pi(s) = \int_0^{t \wedge T_M} \tilde{f}(X(s)) d\mathcal{B}(s).$$

Similarly to the proof of Proposition 9.2.4, it is equivalent to show

$$\begin{aligned} \lim_{|\pi| \rightarrow 0} \mathbb{E}^B \left[\int_0^{t \wedge T_M} \tilde{f}_\pi(X_\pi(s)) d\mathcal{B}_\pi(s) \int_0^{t \wedge T_M} \tilde{f}(X(s)) d\mathcal{B}(s) \right] \\ = \mathbb{E}^B \int_0^{t \wedge T_M} |\tilde{f}(X(s))|^2 ds. \end{aligned} \quad (9.43)$$

Indeed, by the Itô isometry, the expectation on the left side equals to

$$\mathbb{E}^B \left[\int_0^{T_{W_\pi, B}^{-1}(t \wedge T_M) \wedge T_{W, B}^{-1}(t \wedge T_M)} (\tilde{f}_\pi \circ S_{W_\pi}^{-1})' \circ B(u) \cdot (\tilde{f} \circ S_W^{-1})' \circ B(u) du \right].$$

It follows from Lemma 9.2.8 that with probability one

$$\begin{aligned} \lim_{|\pi| \rightarrow 0} \int_0^{T_{W_\pi, B}^{-1}(t \wedge T_M) \wedge T_{W, B}^{-1}(t \wedge T_M)} (\tilde{f}_\pi \circ S_{W_\pi}^{-1})' \circ B(u) \cdot (\tilde{f} \circ S_W^{-1})' \circ B(u) du \\ = \int_0^{T_{W, B}^{-1}(t \wedge T_M)} |(\tilde{f} \circ S_W^{-1})' \circ B(u)|^2 du = \int_0^{t \wedge T_M} |\tilde{f}(X(s))|^2 ds. \end{aligned}$$

As in the proof of Proposition 9.2.4 we can use the Cauchy-Schwarz inequality and some

changes of variables to see that

$$\begin{aligned}
& \left(\int_0^{T_{W\pi, B}^{-1}(t \wedge T_M) \wedge T_{W, B}^{-1}(t \wedge T_M)} (\tilde{f}_\pi \circ S_{W\pi}^{-1})' \circ B(u) \cdot (\tilde{f} \circ S_W^{-1})' \circ B(u) du \right)^2 \\
& \leq \int_0^{T_{W\pi, B}^{-1}(t \wedge T_M)} |(\tilde{f}_\pi \circ S_{W\pi}^{-1})' \circ B(u)|^2 du \int_0^{T_{W, B}^{-1}(t \wedge T_M)} |(\tilde{f} \circ S_W^{-1})' \circ B(u)|^2 du \\
& = \int_0^{t \wedge T_M} |\tilde{f}_\pi(X_\pi(s))|^2 ds \int_0^{t \wedge T_M} |\tilde{f}(X(s))|^2 ds \\
& \leq t^2 \sup_{|x| \leq M} |\tilde{f}(x)|^4.
\end{aligned}$$

We may use uniform integrability to get (9.43) and then to conclude the proof. \square

9.4 Strong solution - Proof of Theorem 9.1.7

9.4.1 Existence part of Theorem 9.1.7

Because the methods proving existence and uniqueness are quiet different, we consider them separately. In this subsection, we focus on showing existence of a strong solution to equation (9.1). Throughout the current section, W is a (given) two-sided Brownian motion and \mathcal{B} is a (given) Brownian motion independent of W . We first seek for a Brownian motion B such that relation (9.25) is verified. For this purpose, we first prove the following result.

Lemma 9.4.1. *Let \mathcal{B} be a Brownian motion and let W be two-sided Brownian motion independent of \mathcal{B} . Then, for P -a.s. W , the equation*

$$M(t) = \int_0^t e^{W \circ S_W^{-1} \circ M(u)} d\mathcal{B}(u), \quad t \geq 0 \quad (9.44)$$

has unique strong solution $(M(t), t \geq 0)$ which has continuous sample paths.

Proof. First, we show the existence of the weak solution to (9.44). In fact, let \tilde{B} be a Brownian

independent from W . We define

$$\tilde{\mathcal{B}}(t) = \int_0^t e^{-W \circ S_W^{-1} \circ \tilde{B} \circ T_{W, \tilde{B}}^{-1}(s)} d\tilde{B} \circ T_{W, \tilde{B}}^{-1}(s).$$

Then, it follows from Proposition 9.1.8 that $\tilde{\mathcal{B}}(t)$ is a Brownian motion, independent of W . Denote $\tilde{M} = \tilde{B} \circ T_{W, \tilde{B}}^{-1}$. Then $d\tilde{\mathcal{B}}(t) = e^{-W \circ S_W^{-1} \circ M(t)} dM(t)$ or $dM(t) = e^{W \circ S_W^{-1} \circ M(t)} d\tilde{\mathcal{B}}(t)$. This means that (\tilde{M}, \tilde{B}) is a weak solution to equation (9.44).

Let us prove the pathwise uniqueness for equation(9.44). Note that by the classical Lévy theorem W satisfies the following modulus of continuity condition: for each $n \geq 1$,

$$|W(x, \omega) - W(x', \omega)| \leq c_n(\omega) \log(|x - x'|) \sqrt{|x - x'|} \quad \forall x, x' \in [-n, n],$$

for some $c_n(\omega) \geq 0$, for P - a.s. ω . (9.45)

Thus we can find a set $A \subset \Omega$ with $P(A) = 1$, such that, for all $\omega \in A$, the following holds: for any $n \geq 1$, there exists $c_n(\omega) \geq 0$, such that

$$|W(x, \omega) - W(x', \omega)| \leq \rho_n(x, x'), \quad \forall x, x' \in [-n, n],$$

where $\rho_n(x, x') := c_n(\omega) \log(|x - x'|) \sqrt{|x - x'|}$. Fix arbitrary $\omega \in A$. For any $k \geq 1$, we define

$$\phi_k(z) = \phi_k(z, \omega) = e^{W(S_W^{-1}(-kV(z \wedge k)), \omega)} \tag{9.46}$$

and consider the following stochastic differential equation

$$M_k(t) = \int_0^t \phi_k(M_k(u)) d\mathcal{B}(u). \tag{9.47}$$

Note that

$$\begin{aligned} \int_{0+}^1 (\sqrt{|\log(u)u|})^{-2} du &= - \int_{0+}^1 (\log(u))^{-1} d(\log u) \\ &= \int_1^{\infty} \frac{1}{v} dv = \infty. \end{aligned} \tag{9.48}$$

We now take

$$n(k, w) = \lfloor |S_W^{-1}(k)| + |S_W^{-1}(-k)| + 1 \rfloor,$$

where $\lfloor a \rfloor$ denotes the integer part of a . Then

$$|W(x, \omega) - W(x', \omega)| \leq c_n(\omega) \log(|x - x'|) \sqrt{|x - x'|}$$

for all x, x' in the interval $[-n(k, w), n(k, w)]$. Define

$$S^*(\omega) = \sup_{|x| \leq |S_W^{-1}(-k)| + |S_W^{-1}(k)|} (e^{W(x, \omega)} + e^{-W(x, \omega)}).$$

Then

$$\begin{aligned} |\phi_k(z) - \phi_k(z')| &\leq S^* |W(S_W^{-1}(-k \vee (z \wedge k))) - W(S_W^{-1}(-k \vee (z' \wedge k)))| \\ &\leq S^* \rho(|S_W^{-1}(-k \vee (z \wedge k)) - S_W^{-1}(-k \vee (z' \wedge k))|). \end{aligned}$$

Note that S_W^{-1} is Lipschitz function and we can easily derive:

$$|S_W^{-1}(-k \vee (z \wedge k)) - S_W^{-1}(-k \vee (z' \wedge k))| \leq S^* |z - z'|,$$

and hence

$$|\phi_k(z) - \phi_k(z')| \leq S^* \rho(S^* |z - z'|).$$

This together with (9.48) implies the pathwise uniqueness of the equation (9.47) by standard

Yamada-Watanabe criterion (see [66], Chapter IV, Theorem 3.2).

Now, let M^1 and M^2 be two continuous solutions to (9.44). Define the following stopping times:

$$\begin{aligned} T_k^{M_1, W} &= \inf\{t \geq 0 : M^1(t) = S_W(k) \text{ or } M^1(t) = S_W(-k)\}, \\ T_k^{M_2, W} &= \inf\{t \geq 0 : M^2(t) = S_W(k) \text{ or } M^2(t) = S_W(-k)\}, \\ \tilde{T}_k^W &= \min(T_k^{M_1, W}, T_k^{M_2, W}). \end{aligned}$$

Since the processes $(M^1(t), t \geq 0)$ and $(M^2(t), t \geq 0)$ have continuous sample paths, we see $\tilde{T}_k^W \uparrow \infty$ a.s. when $k \rightarrow \infty$. When $t \leq \tilde{T}_k^W$, both $(M^1(t), t \geq 0)$ and $(M^2(t), t \geq 0)$ satisfy (9.47). Thus $M^1(t) = M^2(t)$ when $t \leq \tilde{T}_k^W$. Passing through the limit $k \rightarrow \infty$ yields the strong uniqueness of the equation (9.44).

Finally, because weak existence and strong uniqueness together imply strong existence, we see that the equation (9.44) has a unique strong solution. \square

We are now ready to prove the existence part of Theorem 9.1.7.

Proof of existence part of Theorem 9.1.7. Let M be the unique strong solution to equation (9.44). Define a stopping $\tau(t)$ so that

$$\int_0^{\tau(t)} e^{2W \circ S_W^{-1} \circ M(s)} ds = t. \quad (9.49)$$

We note that if M and W are provided, τ is uniquely determined by (9.49) because the map $u \mapsto \int_0^u e^{2W \circ S_W^{-1} \circ M(s)} ds$ is strictly increasing on \mathbb{R}_+ . We define $B = M \circ \tau$. It follows from (9.44) that

$$\langle B \rangle_t = \int_0^{\tau(t)} e^{2W \circ S_W^{-1} \circ M(s)} ds = t.$$

Thus, from Lévy's characterization theorem, B is a Brownian motion. In addition, the

relation (9.49) is equivalent to

$$\tau(t) = \int_0^t e^{-2W \circ S_W^{-1} \circ M \circ \tau(s)} ds .$$

Hence, taking into account the relation $M \circ \tau = B$, we have $\tau(t) = \int_0^t e^{-2W \circ S_W^{-1} \circ B(s)} ds = T_{W,B}(t)$. From here and the equation (9.44) it follows that B and \mathcal{B} satisfy the relation (9.25). In addition, similar to the proof of Proposition 9.1.8 it is clear that B is independent of W .

We now define $X = S_W^{-1} \circ B \circ T_{W,B}^{-1}$. Proposition 9.1.8 then shows that X is adapted to the extended filtration $\mathcal{F}^{\mathcal{B},W}$ and satisfies the equation (9.1) almost surely for the given Brownian motion \mathcal{B} . This proves that X is a strong solution to the equation (9.1). \square

9.4.2 Uniqueness part of Theorem 9.1.7

To show uniqueness for strong solutions of (9.1), we rely on Itô formula, Theorem 9.1.9.

Proof of uniqueness part of Theorem 9.1.7. Let \mathcal{B} be a Brownian motion independent of W . We would like to show that X constructed in the proof of the existence part of Theorem 9.1.7 is indeed the unique strong solution to the equation (9.1). Let \tilde{X} be another strong solution, and let \tilde{B} the corresponding Brownian motion in the Itô-McKean representation, that is

$$\tilde{X} = S_W^{-1} \circ \tilde{B} \circ T_{W,\tilde{B}}^{-1}. \quad (9.50)$$

Here, as usual,

$$T_{W,\tilde{B}}(t) = \int_0^t e^{-2W \circ S_W^{-1}(\tilde{B}(s))} ds , \quad (9.51)$$

or alternatively $T_{W,\tilde{B}}(t)$ satisfies

$$\int_0^{T_{W,\tilde{B}}(t)} e^{2W \circ \tilde{X}(s)} ds = t . \quad (9.52)$$

The advantage of the later definition is that $T_{W,\tilde{B}}(t)$ is given only via \tilde{X} . By a simple

transformation one can see that \tilde{B} can be expressed via \tilde{X} as

$$\tilde{B}(t) = S_W \circ \tilde{X} \circ T_{W, \tilde{B}}. \quad (9.53)$$

Now we would like to express \tilde{B} as a solution to certain stochastic equation driven by \mathcal{B} . To this end we apply Itô formula from Theorem 9.1.9 to the function $S_W(x) = \int_0^x e^{W(y)} dy$. However, we cannot do it directly, since $x \mapsto e^x$ does not have bounded derivatives. Therefore an approximation is needed. Let R be a fixed positive number. Let f_R be a C^3 -function with bounded derivatives such that $f_R(x) = e^x$ for every $x \in [-R, R]$ and $f_R = 0$ outside $[-R-1, R+1]$. We then apply Itô formula from Theorem 9.1.9 to the function $F_R(x) = \int_0^x f_R(W(y)) dy$ to get

$$\begin{aligned} F_R(\tilde{X}(t)) &= \int_0^t f_R(W(\tilde{X}(s))) d\mathcal{B}(s) - \frac{1}{2} \int_{-\infty}^{\infty} f_R(W(x)) L_{\tilde{X}}(t, x) W(d^0 x) \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} f_R'(W(x)) L_{\tilde{X}}(t, x) W(d^0 x). \end{aligned}$$

Since \tilde{X} has continuous sample paths, $L_{\tilde{X}}(t, \cdot)$ vanishes outside of a compact interval (independent from R). We can pass easily to the limit, as $R \rightarrow \infty$, and obtain

$$S_W(\tilde{X}(t)) = \int_0^t e^{W(\tilde{X}(s))} d\mathcal{B}(s). \quad (9.54)$$

The previous equation, (9.53) and (9.50) imply

$$\begin{aligned} \tilde{B}(t) &= \int_0^{T_{W, \tilde{B}}(t)} e^{W(\tilde{X}(s))} d\mathcal{B}(s) \\ &= \int_0^{T_{W, \tilde{B}}(t)} e^{W \circ S_W^{-1} \circ \tilde{B} \circ T_{W, \tilde{B}}^{-1}(s)} d\mathcal{B}(s) \end{aligned} \quad (9.55)$$

Then we immediately obtain

$$\tilde{B}(T_{W, \tilde{B}}^{-1}(t)) = \int_0^t e^{W \circ S_W^{-1} \circ \tilde{B} \circ T_{W, \tilde{B}}^{-1}(s)} d\mathcal{B}(s).$$

Thus $(\tilde{B} \circ T_{W, \tilde{B}}^{-1}(t), t \geq 0)$ satisfies (9.44). However, Lemma 9.4.1 states that the equation (9.44) has the unique strong solution. That is, if $M(t) = \tilde{B} \circ T_{W, \tilde{B}}^{-1}(t)$ then M is uniquely determined from the equation (9.44). In addition, upon comparing (9.52) with (9.49), we see that $T_{W, \tilde{B}}(t) = \tau(t)$ where $\tau(t)$ is uniquely defined by (9.49). Note that both M and τ are solutions of equations ((9.44)) and (9.49) respectively) which do not depend on *particular* solution \tilde{X} for (9.1). Then we have

$$\begin{aligned}\tilde{B} &= M \circ \tau \\ &= B, \quad \text{a.s.}\end{aligned}$$

where B is the Brownian motion constructed in the proof of Theorem 9.1.7. This and (9.50) imply that

$$\tilde{X} = X, \quad \text{a.s.}$$

and uniqueness follows. □

9.5 Proofs

9.5.1 Proof of Proposition 9.1.1

We have the following decomposition

$$\int_a^b g(x, W(x)) L_B(\xi, S_W(x)) \dot{W}_\pi(x) dx = I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned}
I_1 &= \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} [g(x, W(x)) - g(x_k, W(x))] L_B(\xi, S_W(x)) \frac{W(x_{k+1}) - W(x_k)}{x_{k+1} - x_k} dx, \\
I_2 &= \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} g(x_k, W(x)) [L_B(\xi, S_W(x)) - L_B(\xi, S_W(x_k))] \frac{W(x_{k+1}) - W(x_k)}{x_{k+1} - x_k} dx, \\
I_3 &= \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} [g(x_k, W(x)) - g(x_k, W(x_k))] L_B(\xi, S_W(x_k)) \frac{W(x_{k+1}) - W(x_k)}{x_{k+1} - x_k} dx, \\
I_4 &= \sum_{k=0}^{n-1} g(x_k, W(x_k)) L_B(\xi, S_W(x_k)) [W(x_{k+1}) - W(x_k)].
\end{aligned}$$

From the Cauchy-Schwarz inequality we see that I_1^2 is at most

$$(b-a) \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |g(x, W(x)) - g(x_k, W(x))|^2 |L_B(\xi, S_W(x))|^2 \frac{[W(x_{k+1}) - W(x_k)]^2}{(x_{k+1} - x_k)^2} dx.$$

Taking expectation and applying the Hölder inequality and (9.13) we obtain

$$\mathbb{E}I_1^2 \lesssim \sum_{k=0}^{n-1} |x_{k+1} - x_k|^{2\lambda} \lesssim |\pi|^{2\lambda-1}$$

which implies $\mathbb{E}I_1^2$ goes to 0 since $\lambda > 1/2$.

Denote each term in the expression of I_2 by I_{2k} . Then

$$\mathbb{E}(I_2^2) = \sum_{k=0}^{n-1} \mathbb{E}(I_{2k}^2) + \sum_{k \neq j} \mathbb{E}(I_{2k} I_{2j}) =: I_{2,1} + I_{2,2}.$$

From the Cauchy-Schwarz inequality we see that $I_{2,1}$ is at most

$$\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} \left\{ |g(x_k, W(x))|^2 [L_B(\xi, S_W(x)) - L_B(\xi, S_W(x_k))]^2 \frac{[W(x_{k+1}) - W(x_k)]^2}{x_{k+1} - x_k} \right\} dx$$

By conditioning on the σ -algebra generated by W (namely taking the expectation with

respect to the Brownian motion B first) and applying (9.39) with $\beta = 1/2$, we see that

$$I_{2,1} \lesssim \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} |g(x_k, W(x))|^2 \left[|S_W(x) - S_W(x_k)| \frac{(W(x_{k+1}) - W(x_k))^2}{x_{k+1} - x_k} \right] dx.$$

which is majorized by a constant multiple of $|\pi|$. It follows that $\lim_{|\pi| \rightarrow 0} I_{2,1} = 0$. If $k \neq j$ and if $x \in [x_j, x_{j+1})$ and $z \in [x_k, x_{k+1})$, then the intervals $[S_W(x_j), S_W(x))$ and $[S_W(x_k), S_W(z))$ are disjoint. Then we have from (9.41) with $\alpha = 0$,

$$\begin{aligned} & \mathbb{E}[L_B(\xi, S_W(x)) - L_B(\xi, S_W(x_j))][L_B(\xi, S_W(z)) - L_B(\xi, S_W(x_k))] \\ & \leq \mathbb{E}|S_W(x) - S_W(x_j)| |S_W(z) - S_W(x_k)|. \end{aligned}$$

Therefore, together with (9.13), we have

$$\begin{aligned} I_{2,2} \lesssim & \sum_{j < k} \int_{x_j}^{x_{j+1}} \int_{x_k}^{x_{k+1}} \mathbb{E} \left[e^{\theta|W(x)| + \theta|W(z)|} |S_W(x) - S_W(x_k)| |S_W(z) - S_W(x_k)| \right. \\ & \left. \left| \frac{W(x_{j+1}) - W(x_j)}{x_{j+1} - x_j} \frac{W(x_{k+1}) - W(x_k)}{x_{k+1} - x_k} \right| \right] dx dz. \end{aligned}$$

It is now easy to check that $I_{2,2}$ converges to 0, hence so does I_2 .

Using the Taylor expansion, we have

$$g(x_k, W(x)) - g(x_k, W(x_k)) = \partial_u g(x_k, W(x))(W(x) - W(x_k)) + R_k(x)$$

with $\sup_{0 \leq x \leq y} \mathbb{E}|R_k(x)|^p \leq C_p |x_{k+1} - x_k|^p$. Hence, we can decompose $I_3 = I_{3,1} + I_{3,3} + I_{3,3}$, where

$$\begin{aligned} I_{3,1} = & \sum_{k=0}^{n-1} \left[\int_{x_k}^{x_{k+1}} (W(x) - W(x_k)) dx \frac{W(x_{k+1}) - W(x_k)}{x_{k+1} - x_k} - \frac{1}{2}(x_{k+1} - x_k) \right] \\ & \times \partial_u g(x_k, W(x_k)) L_B(\xi, S_W(x_k)), \end{aligned}$$

$$I_{3,2} = \frac{1}{2} \sum_{k=0}^{n-1} \partial_u g(x_k, W(x_k)) L_B(\xi, S_W(x_k)) (x_{k+1} - x_k),$$

and

$$I_{3,3} = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} R_k(x) dx L_B(\xi, S_W(x_k)) \frac{W(x_{k+1}) - W(x_k)}{x_{k+1} - x_k}.$$

$I_{3,1}$ is a sum of martingale difference. It is easy to see that

$$\begin{aligned} \mathbb{E}(I_{3,1})^2 &\leq \sum_{k=0}^{n-1} \mathbb{E} \left[|\partial_u g(x_k, W(x_k))|^2 L_B^2(\xi, S_W(x_k)) \right] \\ &\quad \times \left[\int_{x_k}^{x_{k+1}} (W(x) - W(x_k)) dx \frac{W(x_{k+1}) - W(x_k)}{x_{k+1} - x_k} - \frac{1}{2} (x_{k+1} - x_k) \right]^2 \\ &\leq C \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 \rightarrow 0. \end{aligned}$$

$I_{3,2}$ is the Riemann sum of the integral $\frac{1}{2} \int_a^b \partial_u g(x, W(x)) L_B(\xi, S_W(x)) dx$. A straightforward estimation yields that $I_{3,3}$ converges to 0 in L^2 . Hence, we have I_3 converges to the integral

$$\frac{1}{2} \int_a^b \partial_u g(x, W(x)) L_B(\xi, S_W(x)) dx$$

in L^2 . By standard Itô calculus, we see that I_4 converges in L^2 to the Itô integral

$$\int_a^b g(x, W(x)) L_B(\xi, S_W(x)) W(dx).$$

This completes the proof. □

9.5.2 Proof of Proposition 9.2.4

From Doob's maximal inequality, it suffices to show

$$\lim_{|\pi| \rightarrow 0} \mathbb{E} |\mathcal{B}_\pi(t) - \mathcal{B}(t)|^2 = 0,$$

for every fixed $t > 0$. We write

$$\begin{aligned}\mathcal{B}_\pi(t) &= \int_0^t e^{-W_\pi(X_\pi(s))} dB \circ T_{W_\pi, B}^{-1}(s) \\ &= \int_0^{T_{W_\pi, B}^{-1}(t)} e^{-W_\pi(X_\pi \circ T_{W_\pi, B}^{-1}(u))} dB(u) \\ &= \int_0^{T_{W_\pi, B}^{-1}(t)} e^{-W_\pi(S_{W_\pi, B}^{-1} \circ B(u))} dB(u)\end{aligned}$$

and similarly

$$\mathcal{B}(t) = \int_0^{T_{W, B}^{-1}(t)} e^{-W(S_{W, B}^{-1} \circ B(u))} dB(u).$$

By a change of variable (similar to the one used in the proof of Lemma 9.2.2), we can immediately get that the quadratic variation of \mathcal{B} is given by

$$\int_0^{T_{W, B}^{-1}(t)} e^{-2W(S_{W, B}^{-1} \circ B(u))} du = \int_0^t e^{-2W(S_{W, B}^{-1} \circ B \circ T_{W, B}^{-1}(s))} dT_{W, B}^{-1}(s) = t, \quad (9.56)$$

and hence \mathcal{B} is a Brownian motion with respect to $\{\mathcal{F}_{T_{W, B}^{-1}(t)}^{B, W}\}_{t \geq 0}$. In addition, Lemma 9.2.2 asserts that \mathcal{B}_π is a Brownian motion with respect to $\{\mathcal{F}_{T_{W_\pi, B}^{-1}(t)}^{B, W}\}_{t \geq 0}$. Since \mathcal{B}_π and \mathcal{B} are square integrable martingales we get

$$\begin{aligned}\mathbb{E}(\mathcal{B}_\pi(t)\mathcal{B}(t)) &= \mathbb{E}\left[\int_0^{T_{W_\pi, B}^{-1}(t)} e^{-W_\pi(S_{W_\pi, B}^{-1} \circ B(u))} dB(u) \int_0^{T_{W, B}^{-1}(t)} e^{-W(S_{W, B}^{-1} \circ B(u))} dB(u)\right] \\ &= \mathbb{E}\left[\int_0^{T_{W_\pi, B}^{-1}(t) \wedge T_{W, B}^{-1}(t)} e^{-W_\pi(S_{W_\pi, B}^{-1} \circ B(u)) - W(S_{W, B}^{-1} \circ B(u))} du\right].\end{aligned} \quad (9.57)$$

From Lemma 9.2.8, $T_{W_\pi, B}^{-1}$ and $S_{W_\pi}^{-1}$ converge uniformly over finite intervals, almost surely, to $T_{W, B}^{-1}$ and S_W^{-1} respectively. Hence, for each $t \geq 0$,

$$\int_0^{T_{W_\pi, B}^{-1}(t) \wedge T_{W, B}^{-1}(t)} e^{-W_\pi(S_{W_\pi, B}^{-1} \circ B(u)) - W(S_{W, B}^{-1} \circ B(u))} du \rightarrow \int_0^{T_{W, B}^{-1}(t)} e^{-2W(S_{W, B}^{-1} \circ B(u))} du = t, \quad (9.58)$$

with probability one, as $|\pi| \rightarrow 0$. The last equality follows from (9.56).

Now, by first applying Cauchy-Schwarz inequality, and then equalities (9.38) and (9.56) we get

$$\begin{aligned} & \left(\int_0^{T_{W\pi, B}^{-1}(t) \wedge T_{W, B}^{-1}(t)} e^{-W_\pi(S_{W\pi, B}^{-1} \circ B(u)) - W(S_{W, B}^{-1} \circ B(u))} du \right)^2 \\ & \leq \int_0^{T_{W\pi, B}^{-1}(t)} e^{-2W_\pi(S_{W\pi, B}^{-1} \circ B(u))} du \int_0^{T_{W, B}^{-1}(t)} e^{-2W(S_{W, B}^{-1} \circ B(u))} du = t^2. \end{aligned}$$

The above bound implies uniform integrability of random variables

$$\int_0^{T_{W\pi, B}^{-1}(t) \wedge T_{W, B}^{-1}(t)} e^{-W_\pi(S_{W\pi, B}^{-1} \circ B(u)) - W(S_{W, B}^{-1} \circ B(u))} du,$$

and hence by (9.58) we get that the right hand side of (9.57) converges to t , and this immediately implies that $\lim_{|\pi| \rightarrow 0} \mathbb{E} \mathcal{B}_\pi(t) \mathcal{B}(t) = t$. Therefore,

$$\begin{aligned} \mathbb{E}(\mathcal{B}_\pi(t) - \mathcal{B}(t))^2 &= \mathbb{E}(\mathcal{B}_\pi(t)^2) + \mathbb{E}(\mathcal{B}(t)^2) - 2\mathbb{E}(\mathcal{B}_\pi(t)\mathcal{B}(t)) \\ &= 2t - 2\mathbb{E}(\mathcal{B}_\pi(t)\mathcal{B}(t)) \end{aligned}$$

converges to 0 as $|\pi| \rightarrow 0$. □

9.5.3 Proof of Proposition 9.2.6

Let $p(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ be the heat kernel and \mathfrak{S}_m denote the symmetric group of permutations of $\{1, 2, \dots, m\}$. It is easy to verify that for generic points u_1, \dots, u_m in \mathbb{R} , we have

$$\mathbb{E} \prod_{j=1}^m L_B([\xi, \eta], u_j) = \sum_{\sigma \in \mathfrak{S}_m} \int_{D_m} \prod_{j=1}^m p(s_j - s_{j-1}, u_{\sigma_j} - u_{\sigma_{j-1}}) d\bar{s}, \quad (9.59)$$

where D_m is the domain $\{\bar{s} \in [\xi, \eta]^m : \xi < s_1 < \dots < s_m < \eta\}$, $d\bar{s} = ds_1 \cdots ds_m$, and $u_0 = 0$ by convention. (9.59) is in fact the so-called Kac moment formula (see Marcus-Rosen's

book [76]).

To use (9.59) to compute the two moments in (9.39) and (9.41), we need to introduce some notations. As introduced in [58], for $k = 1, \dots, n$ and $x \in \mathbb{R}$, $V_k(x)$ denotes the substitution operator, i.e. for a generic function $f = f(u_1, \dots, u_n)$, $V_k(x)f(u) = f(u_1, \dots, u_{k-1}, x, u_{k+1}, \dots, u_m)$. It is clear that if $f(u)$ is a random process, then

$$\mathbb{E}V_k(x)f(u) = \mathbb{E}f(u_1, \dots, u_{k-1}, x, u_{k+1}, \dots, u_m) = V_k(x)\mathbb{E}f(u).$$

Thus the operator V_k commutes with the expectation operator.

For any points x_1, \dots, x_m and y_1, \dots, y_m in \mathbb{R} , we denote $\bar{x} = (x_1, \dots, x_m)$ and $\bar{y} = (y_1, \dots, y_m)$. The notation $[\bar{x}, \bar{y}]$ denotes the rectangle $[x_1, y_1] \times \dots \times [x_m, y_m]$ in \mathbb{R}^m . The operator $\square^m([\bar{x}, \bar{y}])$ is defined as $\square^m([\bar{x}, \bar{y}]) := \prod_{k=1}^m [V_k(y_k) - V_k(x_k)]$. When applied to an m -multivariate function, $\square^m([\bar{x}, \bar{y}])$ is the rectangular increment of the function over the rectangle $[\bar{x}, \bar{y}]$. In particular, when $f(x) = f(x_1) \cdots f(x_m)$, then $\square^m([\bar{x}, \bar{y}])f = \prod_{k=1}^m [f(y_k) - f(x_k)]$. Moreover, for sufficiently smooth function f , the rectangular increment of f can be computed as follows

$$\square^m([\bar{x}, \bar{y}])f = \int_{[x, y]} \frac{\partial^m}{\partial z_1 \partial z_2 \cdots \partial z_m} f(\bar{z}) d\bar{z}. \quad (9.60)$$

With these notations, we can write $\prod_{k=1}^m (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k))$ as follows

$$\prod_{k=1}^m (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) = \square^m([\bar{x}, \bar{y}]) \prod_{j=1}^m L_B([\xi, \eta], u_j).$$

Notice that the operator \square also commutes with the expectation operator. In particular,

when combined with (9.59), we obtain the formula

$$\begin{aligned} \mathbb{E} \prod_{k=1}^m (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \\ = \sum_{\sigma \in \mathfrak{S}_m} \int_{D_m} d\bar{s} \square^m([\bar{x}, \bar{y}]) \prod_{j=1}^m p(s_j - s_{j-1}, u_{\sigma_j} - u_{\sigma_{j-1}}). \end{aligned} \quad (9.61)$$

First, let us assume $x_1 = \dots = x_m = x$ and $y_1 = \dots = y_m = y$. Denote $\bar{x}_{\hat{m}} = (x_{\sigma_1}, \dots, x_{\sigma_{m-1}})$ and $\bar{y}_{\hat{m}, \widehat{m-1}} = (x_{\sigma_1}, \dots, x_{\sigma_{m-2}})$ etc. From (9.61) it follows

$$\begin{aligned} \mathbb{E} \prod_{k=1}^m (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \\ = \sum_{\sigma \in \mathfrak{S}_m} \int_{D_m} d\bar{s} \square^{m-1}([\bar{x}_{\hat{m}}, \bar{y}_{\hat{m}}]) \prod_{j=1}^{m-1} p(s_j - s_{j-1}, u_{\sigma_j} - u_{\sigma_{j-1}}) \\ \times [p(s_m - s_{m-1}, y - u_{\sigma_{m-1}}) - p(s_m - s_{m-1}, x - u_{\sigma_{m-1}})] \end{aligned}$$

By unpacking the $(m-1)^{\text{th}}$ difference operator, the above sum is equal to

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_m} \int_{D_m} d\bar{s} \square^{m-2}([\bar{x}_{\hat{m}, \widehat{m-1}}, \bar{y}_{\hat{m}, \widehat{m-1}}]) \prod_{j=1}^{m-2} p(s_j - s_{j-1}, u_{\sigma_j} - u_{\sigma_{j-1}}) \times \\ [p(s_{m-1} - s_{m-2}, y - u_{\sigma_{m-2}}) + p(s_{m-1} - s_{m-2}, x - u_{\sigma_{m-2}})] \\ [p(s_m - s_{m-1}, 0) - p(s_m - s_{m-1}, x - y)]. \end{aligned}$$

If we continue to apply the operator V this way, we shall obtain

$$\begin{aligned} \mathbb{E} [L([\xi, \eta], x) - L([\xi, \eta], y)]^{2n} \\ = (2n)! \int_{D_{2n}} \prod_{k=2}^{2n} [p(s_k - s_{k-1}, 0) + (-1)^{k+1} p(s_k - s_{k-1}, x - y)] [p(s_1, x) + p(s_1, y)] d\bar{s}. \end{aligned} \quad (9.62)$$

The estimate (9.39) follows from (9.62) and the following inequality

$$\int_a^b \int_s^b (t-s)^\gamma [p(t-s, 0) - p(t-s, x-y)][p(s-a, 0) + p(s-a, x-y)] dt ds \leq c_{\beta, \gamma} |b-a|^{\gamma+1-\beta} |x-y|^{2\beta},$$

is valid for all $\beta \in [0, 1/2]$ and $\gamma \geq 0$.

Now we assume a condition which is slightly more restricted than (9.40):

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{2n} < y_{2n}. \quad (9.63)$$

The functions $p(t, x)$ and all its partial derivatives are continuously differentiable on the any interval $(t, x) \in [0, \infty) \times (-\infty, -a] \cup [a, \infty)$ for any positive a . Thus the function on the right hand side of (9.59) is continuously differentiable on the $[\bar{x}, \bar{y}]$ satisfying (9.63). Using the equations (9.60), (9.61) and interchanging order of integrations, we have

$$\begin{aligned} \mathbb{E} \prod_{k=1}^m (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \\ = \sum_{\sigma \in \mathfrak{S}_m} \int_{D_m} d\bar{s} \int_{[\bar{x}, \bar{y}]} d\bar{z} \frac{\partial^m}{\partial z_1 \cdots \partial z_m} \prod_{j=1}^m p(s_j - s_{j-1}, z_{\sigma_j} - z_{\sigma_{j-1}}). \end{aligned} \quad (9.64)$$

Notice that each partial derivative $\partial/\partial z_{\sigma_j}$ contributes one derivative to either $p(s_j - s_{j-1}, z_{\sigma_j} - z_{\sigma_{j-1}})$ or $p(s_{j+1} - s_j, z_{\sigma_{j+1}} - z_{\sigma_j})$. We record the results by a binary index e_j , $e_j = 1$ represents the former case, $e_j = 0$ represents the later case. Moreover, if the later case happens, it also contributes a factor -1 . Since z_m only appears in the last term $p(s_m - s_{m-1}, z_{\sigma_m} - z_{\sigma_{m-1}})$, we

must have the restriction $e_m = 1$. Thus, we can write (9.64) as

$$\begin{aligned} & \mathbb{E} \prod_{k=1}^m (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \\ &= \sum_{\sigma \in \mathfrak{S}_m} \sum_{\bar{e} \in \mathfrak{C}} \text{sign}(\bar{e}) \int_{D_m} d\bar{s} \int_{[\bar{x}, \bar{y}]} d\bar{z} \prod_{j=1}^m p^{(e_j+1-e_{j-1})}(s_j - s_{j-1}, z_{\sigma_j} - z_{\sigma_{j-1}}), \end{aligned} \quad (9.65)$$

where \mathfrak{C} denotes all the m -tuple $\bar{e} = (e_1, \dots, e_m) \in \{0, 1\}^m$ such that $e_m = 1$ and $\text{sign}(\bar{e})$ is the sign of \bar{e} , defined by $\text{sign} \bar{e} := (-1)^{\sum_{j=1}^m (1-e_j)}$ and $e_0 = 1$ by convention.

For instance, in the case $m = 4$, when σ is the identity map in \mathfrak{S}_4 , the integrand in (9.65) is

$$\begin{aligned} & - p''_{s_4-s_3}(z_4 - z_3) p'_{s_3-s_2}(z_3 - z_2) p'_{s_2-s_1}(z_2 - z_1) p_{s_1}(z_1) \\ & + p'_{s_4-s_3}(z_4 - z_3) p''_{s_3-s_2}(z_3 - z_2) p'_{s_2-s_1}(z_2 - z_1) p_{s_1}(z_1) \\ & + p''_{s_4-s_3}(z_4 - z_3) p_{s_3-s_2}(z_3 - z_2) p''_{s_2-s_1}(z_2 - z_1) p_{s_1}(z_1) \\ & - p'_{s_4-s_3}(z_4 - z_3) p'_{s_3-s_2}(z_3 - z_2) p''_{s_2-s_1}(z_2 - z_1) p_{s_1}(z_1) \\ & + p''_{s_4-s_3}(z_4 - z_3) p'_{s_3-s_2}(z_3 - z_2) p_{s_2-s_1}(z_2 - z_1) p'_{s_1}(z_1) \\ & - p'_{s_4-s_3}(z_4 - z_3) p''_{s_3-s_2}(z_3 - z_2) p_{s_2-s_1}(z_2 - z_1) p'_{s_1}(z_1) \\ & - p''_{s_4-s_3}(z_4 - z_3) p_{s_3-s_2}(z_3 - z_2) p'_{s_2-s_1}(z_2 - z_1) p'_{s_1}(z_1) \\ & + p'_{s_4-s_3}(z_4 - z_3) p'_{s_3-s_2}(z_3 - z_2) p'_{s_2-s_1}(z_2 - z_1) p'_{s_1}(z_1). \end{aligned} \quad (9.66)$$

Combining the estimate in Lemma 9.5.1 (below) with (9.65), we see that there exists a constant c_n depending only on n such that

$$\left| \mathbb{E} \prod_{k=1}^{2n} (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \right| \leq c_n \prod_{j=1}^{2n} |x_j - y_j|. \quad (9.67)$$

An application (9.39) with $\beta = 0$ yields

$$\left| \mathbb{E} \prod_{k=1}^{2n} (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \right| \leq c_n |\eta - \xi|^n. \quad (9.68)$$

Now given $\alpha \in [0, 1]$, an interpolating between (9.67) and (9.68) yields

$$\left| \mathbb{E} \prod_{k=1}^{2n} (L_B([\xi, \eta], y_k) - L_B([\xi, \eta], x_k)) \right| \leq c_{\alpha, n} |\eta - \xi|^{n\alpha} \prod_{j=1}^{2n} |x_j - y_j|^{1-\alpha}.$$

This is (9.41) under the condition (9.63). The estimate (9.41) under the general condition (9.40) follows by a limiting argument since both sides of (9.41) are continuous function of x_k, y_k 's. This finishes the proof of Proposition 9.2.6 modulo the proof of the following lemma which was used in the above proof.

Lemma 9.5.1. *Let $\bar{e} = (e_1, \dots, e_m)$ ($m \geq 2$) be an m -tuple in $\{0, 1\}^m$ such that $e_m = 1$ and we take $e_0 = 1$ by convention. Let u_k ($k = 1, 2, \dots, m$) be non-zero real numbers and let D_m be the domain $\{\bar{s} \in [\xi, \eta]^m : \xi < s_1 < \dots < s_m < \eta\}$. Then the following estimate holds*

$$\left| \int_{D_m} \prod_{j=1}^m p^{(e_{j+1}-e_{j-1})}(s_j - s_{j-1}, u_j) d\bar{s} \right| \leq 1. \quad (9.69)$$

Proof. We denote \mathcal{L} the Laplace transform with respect to the t variable and put

$$J = \int_{D_m} \prod_{j=1}^m p^{(e_j - e_{j-1} + 1)}(s_j - s_{j-1}, u_j) d\bar{s}. \quad (9.70)$$

Let $*$ denote the convolution operator, i.e. for two functions f and g , $f * g(t) = \int_0^t f(s)g(t-s)ds$. Then we can rewrite J into the form

$$J = \int_{\xi}^{\eta} p^{(e_1)}(s_1, z_{\sigma_1}) f(\eta - s_1) ds_1,$$

where f is the function defined by

$$f(t) = \int_0^t [p^{(e_2-e_1+1)}(\cdot, u_2) * \dots * p^{(e_m-e_{m-1}+1)}(\cdot, u_m)](s) ds .$$

It is well known (see for example, [42], Formula 3.471 (9) and Formula 8.469 (3)) that

$$\mathcal{L}[p(\cdot, x)](s) = \frac{1}{\sqrt{2s}} e^{-|x|\sqrt{2s}} . \quad (9.71)$$

By taking derivative under the integral sign (noticing that we assume $x \neq 0$), we obtain

$$\mathcal{L}[p'(\cdot, x)](s) = -\text{sign}(x) e^{-|x|\sqrt{2s}} .$$

We further notice that $p'' = 2\partial_t p$, thus

$$\mathcal{L}[p''(\cdot, x)](s) = \sqrt{2s} e^{-|x|\sqrt{2s}} .$$

Writing all three formulas in one, for $k = 0, 1, 2$, we have

$$\mathcal{L}[p^{(k)}(\cdot, x)](s) = (\sqrt{2s})^{k-1} [-\text{sign}(x)]^k e^{-|x|\sqrt{2s}} . \quad (9.72)$$

Since convolution becomes product under Laplace transform, the Laplace transform of f is

$$\begin{aligned} \mathcal{L}[f](s) &= s^{-1} \prod_{j=2}^m (\sqrt{2s})^{e_j-e_{j-1}} [-\text{sign}(u_j)]^{e_j-e_{j-1}+1} \exp \left\{ -|u_j|\sqrt{2s} \right\} \\ &= \sqrt{2} (\sqrt{2s})^{-1-e_1} \exp \left\{ -\sqrt{2s} \sum_{j=2}^m |u_j| \right\} \prod_{j=2}^m [-\text{sign}(u_j)]^{e_j-e_{j-1}+1} , \end{aligned}$$

where the factor s^{-1} comes from the fact that the Laplace transform of $\int_0^t f(r) dr$ is $s^{-1} \mathcal{L}f(s)$.

To simplify notations, we will denote $|u| = \sum_{j=2}^m |u_j|$. We consider now two cases. Case 1:

$e_1 = 0$. Inverting the Laplace transform, using (9.71), we see that

$$f(t) = \sqrt{2} \prod_{j=2}^m [-\text{sign}(u_j)]^{e_j - e_{j-1} + 1} p(t, |u|). \quad (9.73)$$

Thus

$$|J| \leq \sqrt{2} \int_{\xi}^{\eta} p(s_1, u_1) p(\eta - s_1, |u|) ds_1 \leq 1/\sqrt{2}.$$

Case 2: $e_1 = 1$. We notice that

$$\mathcal{L} \left[\text{erfc} \left(\frac{|x|}{\sqrt{2t}} \right) \right] (s) = \frac{1}{s} e^{-|x|\sqrt{2s}},$$

where $\text{erfc}(z)$ is the complementary error function $\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-y^2} dy$. Inverting the Laplace transform as in the former case, we obtain

$$f(t) = \prod_{j=2}^m [-\text{sign}(u_j)]^{e_j - e_{j-1} + 1} \text{erfc} \left(\frac{|u|}{\sqrt{2t}} \right). \quad (9.74)$$

Thus if we use the fact that $0 \leq \text{erfc}(z) \leq 1$, we have

$$\begin{aligned} |J| &\leq \int_{\xi}^{\eta} |p'(s_1, u_1)| \text{erfc} \left(\frac{|u|}{\sqrt{2(\eta - s_1)}} \right) ds_1 \\ &\leq \int_{\xi}^{\eta} \frac{|u_1|}{\sqrt{2\pi s_1}} e^{-\frac{|u_1|^2}{2s_1}} \frac{ds_1}{s_1}. \end{aligned}$$

By the change of variable $t = \frac{|u_1|}{\sqrt{2s_1}}$, we see that $|J| \leq \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt \leq 1$. □

9.5.4 Proof of Proposition 9.2.5

To outline the strategy proving Proposition 9.2.5, let us first observe that using the representation $X_\pi = S_{W_\pi}^{-1} \circ B \circ T_{W_\pi, B}^{-1}$ we can write

$$\begin{aligned} \int_0^t g(X_\pi(s), W_\pi \circ (X_\pi(s))) \dot{W}_\pi(X_\pi(s)) ds \\ = \int_{-\infty}^{\infty} g(x, W_\pi(x)) e^{-W_\pi(x)} L_B(T_{W_\pi, B}^{-1}(t), S_{W_\pi}(x)) \dot{W}_\pi(x) dx . \end{aligned}$$

We observe that from Lemma 9.2.8, with probability one, $T_{W_\pi, B}^{-1}(\cdot)$ converges to $T_{W, B}^{-1}(\cdot)$ uniformly over compacts of \mathbb{R}_+ . In addition, the function e^{-u} can be combined with $g(x, u)$. Therefore, to prove Proposition 9.2.5, it suffices to show

- For every function g satisfying conditions (9.13) and (9.14), with probability one, the process $\xi \mapsto \int_{-\infty}^{\infty} g(x, W_\pi(x)) L_B(\xi, S_{W_\pi}(x)) \dot{W}_\pi(x) dx$ converges to $\int_{-\infty}^{\infty} g(x, W(x)) L_B(\xi, S_W(x)) \dot{W}(x) dx$ uniformly over compact sets.

The remaining of this subsection is devoted to verify the previous statement. In what follows, $\{\ell_\pi(g, \xi), \xi \geq 0\}$ denote the process

$$\ell_\pi(g, \xi) = \int_{\mathbb{R}} g(x, W_\pi(x)) L_B(\xi, S_{W_\pi}(x)) \dot{W}_\pi(x) dx , \quad (9.75)$$

which is well-defined for all continuous sample paths of W . For every compact set K , we denote

$$c_3(K) = c_1(K) + c_2(K)$$

where c_1 and c_2 are the constant in (9.13) and (9.14).

In the former part of the current subsection, we will truncate the processes ℓ_π and show the corresponding truncated processes converges uniformly. In the later part, the claim is verified completely via a gluing argument.

Let us remark that for all the results in this subsection holds, we employ the two estimates (9.39) and (9.41) for the local time of Brownian motion B, L_B .

Convergence over bounded interval: We consider an interval $[a, b]$ with length $L = b - a$.

Let $\pi = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a partition of $[a, b]$ with mesh size

$$\Delta = \max_{k=0, \dots, n-1} |x_{k+1} - x_k|.$$

We denote

$$\ell_\pi^{[a,b]}(g, \xi) = \int_a^b g(x, W_\pi(x)) L_B(\xi, S_{W_\pi}(x)) \dot{W}_\pi(x) dx, \quad (9.76)$$

where as usual W_π is the linear interpolation of W associated with π .

We first decompose $\ell_\pi^{[a,b]}(g, \xi)$ as follows

$$\begin{aligned} \ell_\pi^{[a,b]}(g, \xi) &= \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} g(x, W_\pi(x)) [L_B(\xi, S_{W_\pi}(x)) - L_B(\xi, S_{W_\pi}(x_k))] \dot{W}_\pi(x) dx \\ &\quad + \sum_{k=0}^{n-1} L_B(\xi, S_{W_\pi}(x_k)) \int_{x_k}^{x_{k+1}} [g(x, W_\pi(x)) - g(x_k, W_\pi(x))] \dot{W}_\pi(x) dx \\ &\quad + \sum_{k=0}^{n-1} L_B(\xi, S_{W_\pi}(x_k)) \int_{x_k}^{x_{k+1}} g(x_k, W_\pi(x)) \dot{W}_\pi(x) dx \end{aligned}$$

Let G be a function such that $\partial_u G(x, u) = g(x, u)$. The integral inside the last summand can be computed as follows

$$\begin{aligned} \int_{x_k}^{x_{k+1}} g(x_k, W_\pi(x)) \dot{W}_\pi(x) dx &= G(x_k, W(x_{k+1})) - G(x_k, W(x_k)) \\ &= \int_{x_k}^{x_{k+1}} g(x_k, W(x)) W(dx) + \frac{1}{2} \int_{x_k}^{x_{k+1}} \partial_u g(x_k, W(x)) dx, \end{aligned}$$

where the last line follows from the classical Itô formula. Therefore, we can further decompose $\ell_\pi^{[a,b]}(g, \xi)$ as

$$\ell_\pi^{[a,b]}(g, \xi) = I_1(\xi) + I_2(\xi) + I_3(\xi) + I_4(\xi)$$

where

$$\begin{aligned}
I_1(\xi) &= \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} g(x, W_\pi(x)) [L_B(\xi, S_{W_\pi}(x)) - L_B(\xi, S_{W_\pi}(x_k))] \dot{W}_\pi(x) dx, \\
I_2(\xi) &= \sum_{k=0}^{n-1} L_B(\xi, S_{W_\pi}(x_k)) \int_{x_k}^{x_{k+1}} [g(x, W_\pi(x)) - g(x_k, W_\pi(x))] \dot{W}_\pi(x) dx, \\
I_3(\xi) &= \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} L_B(\xi, S_{W_\pi}(x_k)) g(x_k, W(x)) W(dx), \\
I_4(\xi) &= \frac{1}{2} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \partial_u g(x_k, W(x)) L_B(\xi, S_{W_\pi}(x_k)) dx.
\end{aligned}$$

To simplify notation, we omit dependence of I_i 's on g . For a generic function f on \mathbb{R} , we will denote

$$f([\xi, \eta]) \equiv f(\eta) - f(\xi), \quad \forall \eta, \xi \in \mathbb{R}.$$

Lemma 9.5.2. *Suppose g satisfies the conditions in Proposition 9.2.5. There exist positive constants ϵ, γ, κ which does not depend on (a, b) such that the following estimates holds: for all $\eta, \xi \in \mathbb{R}_+$,*

$$\mathbb{E}|I_1([\xi, \eta])|^6 \lesssim c_1^6 (b-a) e^{\kappa(|a| \vee |b|)} |\eta - \xi|^{1+\epsilon} \Delta^\gamma, \quad (9.77)$$

$$\mathbb{E}|I_2([\xi, \eta])|^6 \lesssim c_2^6 (b-a) e^{\kappa(|a| \vee |b|)} |\eta - \xi|^{1+\epsilon} \Delta^\gamma, \quad (9.78)$$

$$\begin{aligned}
\mathbb{E}|I_3([\xi, \eta]) - \int_a^b g(x, W(x)) L_B([\xi, \eta], S_W(x)) W(dx)|^6 \\
\lesssim c_3^6 (b-a) e^{\kappa(|a| \vee |b|)} |\eta - \xi|^{1+\epsilon} \Delta^\gamma, \quad (9.79)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}|I_4([\xi, \eta]) - \frac{1}{2} \int_a^b \partial_u g(x, W(x)) L_B([\xi, \eta], S_W(x)) dx|^6 \\
\lesssim c_3^6 (b-a) e^{\kappa(|a| \vee |b|)} |\eta - \xi|^{1+\epsilon} \Delta^\gamma, \quad (9.80)
\end{aligned}$$

where the implied constants depend only on $b - a$. As a consequence, for all $\eta, \xi \in \mathbb{R}_+$,

$$\mathbb{E}|\ell_\pi^{[a,b]}(g, [\xi, \eta]) - \int_a^b g(x, W(x))L_B([\xi, \eta], S_W(x))W(d^0x)|^6 \lesssim c_3^6(b-a)e^{\kappa(|a| \vee |b|)}|\eta - \xi|^{1+\epsilon}\Delta^\gamma. \quad (9.81)$$

Proof. To deal with I_1 , we denote

$$a_k = \int_{x_k}^{x_{k+1}} g(x, W_\pi(x)) [L_B([\xi, \eta], S_{W_\pi}(x)) - L_B([\xi, \eta], S_{W_\pi}(x_k))] \dot{W}_\pi(x) dx.$$

Then

$$\begin{aligned} & \mathbb{E}I_1([\xi, \eta])^6 \\ &= \sum_{k_1=0}^{n-1} \mathbb{E}a_{k_1}^6 + 6 \sum_{k_1 \neq k_2} \mathbb{E}a_{k_1}^5 a_{k_2} + 15 \sum_{k_1, k_2} \mathbb{E}a_{k_1}^4 a_{k_2}^2 + 30 \sum_{k_1, k_2, k_3} \mathbb{E}a_{k_1}^4 a_{k_2} a_{k_3} \\ &+ 20 \sum_{k_1, k_2} \mathbb{E}a_{k_1}^3 a_{k_2}^3 + 60 \sum_{k_1, k_2, k_3} \mathbb{E}a_{k_1}^3 a_{k_2}^2 a_{k_3} + 120 \sum_{k_1, k_2, k_3, k_4} \mathbb{E}a_{k_1}^3 a_{k_2} a_{k_3} a_{k_4} \\ &+ 90 \sum_{k_1, k_2, k_3} \mathbb{E}a_{k_1}^2 a_{k_2}^2 a_{k_3}^2 + 180 \sum_{k_1, k_2, k_3, k_4} \mathbb{E}a_{k_1}^2 a_{k_2}^2 a_{k_3} a_{k_4} \\ &+ 360 \sum_{k_1, k_2, k_3, k_4, k_5} \mathbb{E}a_{k_1}^2 a_{k_2} a_{k_3} a_{k_4} a_{k_5} + 6! \sum_{k_1, k_2, k_3, k_4, k_5, k_6} \mathbb{E}a_{k_1} a_{k_2} a_{k_3} a_{k_4} a_{k_5} a_{k_6} \end{aligned}$$

where the indices k_1, \dots, k_6 are pairwise disjoint if they appear under the same summation notation. Among these sums, the most difficult term to estimate is the last one. All other sums can be handled by mean of the Hölder inequality and (9.39) (similar to the method of estimating A below). To illustrate our method while maintain a decent length of the chapter, we will give detailed estimates for the two sums

$$\begin{aligned} A &= \sum_{k_1, k_2, k_3, k_4, k_5} \mathbb{E}a_{k_1}^2 a_{k_2} a_{k_3} a_{k_4} a_{k_5}, \\ \tilde{A} &= \sum_{k_1, k_2, k_3, k_4, k_5, k_6} \mathbb{E}a_{k_1} a_{k_2} a_{k_3} a_{k_4} a_{k_5} a_{k_6}. \end{aligned}$$

To avoid lengthy formula, we denote $\Delta_k W = W(x_{k+1}) - W(x_k)$, $\Delta_k = x_{k+1} - x_k$. We also omit the indices under the sigma notation. By the Cauchy-Schwarz inequality and (9.13)

$$a_k^2 \leq c_1^2(b-a) \frac{|\Delta_k W|^2}{\Delta_k} \int_{x_k}^{x_{k+1}} e^{2\theta|W_\pi(z)|} [L_B([\xi, \eta], S_{W_\pi}(z)) - L_B([\xi, \eta], S_{W_\pi}(x_k))]^2 dz.$$

Hence, A is bounded from the above by

$$c_1^6(b-a) \sum \mathbb{E} \int_{x_{k_1}}^{x_{k_1+1}} e^{2\theta|W_\pi(z)|} [L_B([\xi, \eta], S_{W_\pi}(z)) - L_B([\xi, \eta], S_{W_\pi}(x_{k_1}))]^2 \frac{|\Delta_{k_1} W|^2}{\Delta_{k_1}} dz_1 a_{k_2} a_{k_3} a_{k_4} a_{k_5}.$$

Taking the expectation with respect to the Brownian motion B first and applying (9.39) with $\beta = 1/2$ we see that A is bounded from the above by

$$c_1^6(b-a)|\eta - \xi|^{3/2} \sum \mathbb{E} \int_{[x_k, x_{k+1}]} e^{2\theta|W_\pi(z_1)| + \theta|W(z_2)| + \dots + \theta|W(z_5)|} |S_{W_\pi}(z_1) - S_{W_\pi}(x_{k_1})| \prod_{j=2}^5 |S_{W_\pi}(z_j) - S_{W_\pi}(x_{k_j})|^{1/2} \frac{|\Delta_{k_1} W|^2}{\Delta_{k_1}} \prod_{j=2}^5 \frac{|\Delta_{k_j} W|}{\Delta_{k_j}} d\bar{z},$$

where $\int_{[x_k, x_{k+1}]} d\bar{z}$ denotes $\prod_{j=1}^5 \int_{x_{k_j}}^{x_{k_j+1}} dz_j$. We further apply the Hölder inequality and the simple estimate $\mathbb{E} e^{\theta|W_\pi(z)|} \leq e^{\theta^2|z|/2}$. The above quality is bounded by a constant multiple of

$$c_1^6(b-a)|\eta - \xi|^{3/2} \sum \int_{[x_k, x_{k+1}]} \left\{ \mathbb{E} |S_{W_\pi}(z_1) - S_{W_\pi}(x_{k_1})|^2 \prod_{j=2}^5 |S_{W_\pi}(z_j) - S_{W_\pi}(x_{k_j})| \right\}^{1/2} \times e^{\kappa(|z_1| + \dots + |z_5|)} |\Delta_{k_j}|^{-1/2} d\bar{z}.$$

Applying the Hölder inequality again, we obtain

$$A \lesssim c_1^6(b-a)|\eta - \xi|^{3/2} \Delta \left(\int_a^b e^{\kappa|x|} dx \right)^5.$$

To estimate \tilde{A} , we first take the expectation with respect to the Brownian motion B . Using (9.41) with $\alpha \in [0, 1]$ we have

$$\tilde{A} \lesssim c_1^6 (b-a) |\eta - \xi|^{3\alpha} \sum_{j=1}^6 \prod_{j=1}^6 \int_{x_{k_j}}^{x_{k_{j+1}}} \mathbb{E} e^{\theta W_\pi(z_j)} |S_{W_\pi}(z_j) - S_{W_\pi}(x_{k_j})|^{1-\alpha} \frac{\Delta_{k_j} W}{\Delta_{k_j}} dz_j.$$

Applying the Hölder inequality yields

$$\tilde{A} \lesssim c_1^6 (b-a) |\eta - \xi|^{3\alpha} \Delta^{6(\frac{1}{2}-\alpha)} \left(\int_a^b e^{\kappa|x|} dx \right)^6. \quad (9.82)$$

Choosing α between 1/3 and 1/2 yields (9.77).

Proof of (9.78): From the Hölder inequality we have

$$\begin{aligned} \mathbb{E} |I_2([\xi, \eta])|^6 &\leq (b-a)^5 \times \\ &\mathbb{E} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |g(x, W_\pi(x)) - g(x_k, W_\pi(x))|^6 |L_B([\xi, \eta], S_{W_\pi}(x_k))|^6 |W_\pi(x)|^6 dx. \end{aligned}$$

An further application of the Hölder inequality, condition (9.14) and the estimate (9.39) with $\beta = 0$ yields

$$\mathbb{E} |I_1([\xi, \eta])|^6 \lesssim (b-a)^5 c_2^6 (b-a) e^{\kappa(|a| \vee |b|)} |\eta - \xi|^3 \sum_{k=0}^{n-1} |x_{k+1} - x_k|^{6\lambda-2},$$

which implies (9.78).

Proof of (9.79): Applying the moment inequality for martingales, we see that the expression on its left hand side is at most a constant times

$$\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} [g(x, W(x)) L_B([\xi, \eta], S_W(x)) - g(x_k, W(x)) L_B([\xi, \eta], S_{W_\pi}(x_k))]^6 dx,$$

which is again bounded by the sum of a certain constant multiple of

$$D := c_1^6(b-a) \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} [L_B([\xi, \eta], S_W(x)) - L_B([\xi, \eta], S_{W_\pi}(x_k))]^6 e^{6\theta|W(x)|} dx,$$

and

$$\tilde{D} := \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} [L_B([\xi, \eta], S_{W_\pi}(x_k))]^6 [g(x, W(x)) - g(x_k, W(x))]^6 dx,$$

Similar to the estimation for I_2 , it is easy to see that \tilde{D} satisfies

$$\tilde{D} \lesssim c_2^6(b-a)e^{\kappa(|a| \vee |b|)} |\eta - \xi|^3 \sum_{k=0}^{n-1} |x_{k+1} - x_k|^{6\lambda-2}$$

which in turn satisfies the bound (9.79).

By mean of inequality (9.39) with $\beta \in (0, 1/2]$, D is bounded by a constant times

$$c_1^6(b-a) |\eta - \xi|^{3(1-\beta)} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} |S_W(x) - S_{W_\pi}(x_k)|^{6\beta} e^{6\theta|W(x)|} dx.$$

By the Hölder inequality, we see that above expression is at most a constant times

$$c_1^6(b-a) |\eta - \xi|^{3(1-\beta)} |\Delta|^{3\beta} \int_a^b e^{\kappa|x|} dx,$$

which also yields (9.79).

Proof of (9.80): By the Hölder inequality, the quality on the left hand side of (9.80) is at most a constant times

$$\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \mathbb{E} [\partial_u g(x, W(x)) L_B([\xi, \eta], S_W(x)) - \partial_u g(x_k, W(x)) L_B([\xi, \eta], S_{W_\pi}(x_k))]^6 dx.$$

From here, (9.80) follows similarly. □

Convergence over \mathbb{R} : Let γ and κ be the constants in Lemma 9.5.2. Let π be a partition of \mathbb{R} . For every $N \in \mathbb{Z}$, let π_N be the partition on $[N - 1, N]$ induced by π and $|\pi_N|$ denote the mesh size of π_N . For every $\delta > 0$, we now choose a partition $\pi(\delta)$ such that

$$\sum_N c_3([N - 1, N]) (e^{\kappa|N|} |\pi_N|^\gamma)^{\frac{1}{6}} \leq \delta. \quad (9.83)$$

With the notations in the previous subsection, the process $\ell_\pi(g, \cdot)$ (defined in (9.75)) can be written as

$$\ell_\pi(g, \xi) = \sum_{N \in \mathbb{Z}} \ell_{\pi_N}^{[N-1, N]}(g, \xi), \quad \xi \geq 0, \quad (9.84)$$

where $\ell_{\pi_N}^{[N-1, N]}(g, \cdot)$ is the process defined in (9.76). Finiteness of the process $\ell_\pi(g, \cdot)$ will become clear at the end of this subsection. For a random variable Y , we denote the L^6 -norm $\|Y\|_6 := (\mathbb{E}Y^6)^{1/6}$. To simplify notations, we further denote

$$\ell(g, [\xi, \eta]) = \int_{-\infty}^{\infty} g(x, W(x)) L_B(\xi, S_W(x)) W(d^0 x)$$

and

$$\ell^{[N-1, N]}(g, [\xi, \eta]) = \int_{N-1}^N g(x, W(x)) L_B(\xi, S_W(x)) W(d^0 x).$$

From the estimate (9.81), we obtain

$$\begin{aligned} \|\ell_\pi(g, [\xi, \eta]) - \ell(g, [\xi, \eta])\|_6 &\leq \sum_{N \in \mathbb{Z}} \left\| \ell_{g, \pi_N}^{[N-1, N]}(g, [\xi, \eta]) - \ell^{[N-1, N]}(g, [\xi, \eta]) \right\|_6 \\ &\lesssim |\eta - \xi|^{(1+\epsilon)/6} \sum_{N \in \mathbb{Z}} c_3([N - 1, N]) \left(|\pi_N|^\gamma e^{\kappa|N|} \right)^{1/6}. \end{aligned}$$

We now choose $\pi = \pi(\delta)$ and use the condition (9.83) to obtain

$$\|\ell_{\pi(\delta)}(g, [\xi, \eta]) - \ell(g, [\xi, \eta])\|_6 \lesssim |\eta - \xi|^{(1+\epsilon)/6} \delta. \quad (9.85)$$

Let K be any positive number. Applying the Garsia-Rodemich-Rumsey inequality (see [40]), we see that there exists a continuous version of the process $\ell_{\pi}(g, \cdot) - \ell(g, \cdot)$ which satisfies the following estimate almost surely

$$\sup_{0 < \xi < \eta < K} \frac{|\ell_{\pi(\delta)}(g, [\xi, \eta]) - \ell(g, [\xi, \eta])|}{|\eta - \xi|^{\epsilon/8}} \leq C_K \delta. \quad (9.86)$$

Since $\ell(g, \cdot)$ has a continuous version and is finite almost surely, this implies the same properties holds for $\ell_{\pi(\delta)}(g, \cdot)$. Moreover, we have also proved the uniform convergence

$$\lim_{\delta \rightarrow 0} \sup_{0 < \xi < \eta < K} \frac{|\ell_{\pi(\delta)}(g, [\xi, \eta]) - \ell(g, [\xi, \eta])|}{|\eta - \xi|^{\epsilon/8}} = 0. \quad (9.87)$$

which holds almost surely. This finishes the proof of Step 2, and hence of Proposition 9.2.5.

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