

INVARIANTS OF DETERMINANTS WITH BINARY
LINEAR ELEMENTS

By

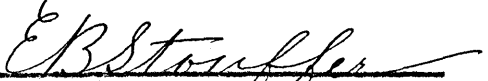
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TABLE OF CONTENTS

	Page
DEFINITIONS	1
THEORY OF INVARIANTS OF A GENERAL DETERMINANTAL FORM	5
THEORY OF COVARIANTS OF A GENERAL DETERMINANTAL FORM	12
FUNDAMENTAL SYSTEM OF COVARIANTS AND INVARIANTS	16
CONDITIONS FOR INVARIANTS AND COVARIANTS OF LESS GENERAL DETERMINANTAL FORMS	24

DEFINITIONS

A function of any number of variables x_1, x_2, \dots , which is rational, integral, and homogeneous in those variables is called a form or quantic in x_1, x_2, \dots . The coefficients in a form are constants so far as x_1, x_2, \dots are concerned. In this paper only binary forms, that is forms containing only two variables will be considered. The degree of the form in the variables is called the order of the form.

The usual way of writing a binary form of n^{th} order is

$$a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \frac{n(n-1)}{1 \cdot 2} a_2 x_1^{n-2} x_2^2 + \dots + n a_{n-1} x_1 x_2^{n-1} + a_n x_2^n.$$

Since every binary form can be expressed as a product of factors

$$a_0 (x_1 - \alpha_1 x_2) (x_1 - \alpha_2 x_2) \dots (x_1 - \alpha_n x_2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are solutions of the equation

$$a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \dots + n a_{n-1} x_1 x_2^{n-1} + a_n x_2^n = 0,$$

every binary form can be written as an n^{th} order determinant

$$\begin{vmatrix} a_0(x_1 - \alpha_1 x_2) & 0 & 0 & \dots & 0 \\ 0 & (x_1 - \alpha_2 x_2) & 0 & \dots & 0 \\ 0 & 0 & (x_1 - \alpha_3 x_2) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (x_1 - \alpha_n x_2) \end{vmatrix}$$

where all elements are zero except those in the principal diagonal.

On the other hand, the determinantal form

$$\begin{vmatrix} a_{11}x_1 + b_{11}x_2 & a_{12}x_1 + b_{12}x_2 & \dots & a_{1n}x_1 + b_{1n}x_2 \\ a_{21}x_1 + b_{21}x_2 & a_{22}x_1 + b_{22}x_2 & \dots & a_{2n}x_1 + b_{2n}x_2 \\ \dots & \dots & \dots & \dots \\ a_{n1}x_1 + b_{n1}x_2 & a_{n2}x_1 + b_{n2}x_2 & \dots & a_{nn}x_1 + b_{nn}x_2 \end{vmatrix} \quad (1)$$

when expanded gives a polynomial which is a binary form.

Its coefficients are determinants, or sums of determinants, whose elements are the coefficients a_{ij} and b_{ij} . This is the most general determinantal form with linear elements, which gives a binary form when expanded. It will be used as the basis of the discussion in the first part of this paper.

If in any binary form we substitute $\alpha X_1 + \beta X_2$ for x_1 , and $\gamma X_1 + \delta X_2$ for x_2 , we get a new form of the same order in the new variables X_1 and X_2 . This process of substituting a linear function of n new variables for each of the n original variables is called making a linear transformation. The most general linear transformation of a binary form is

$$\begin{cases} x_1 = \alpha X_1 + \beta X_2 \\ x_2 = \gamma X_1 + \delta X_2 \end{cases}$$

A linear transformation is determined by the coefficients $\alpha, \beta, \gamma, \delta$, and the determinant of these

arbitrary coefficients is called the modulus of the transformation and is represented by M ,

$$M = \begin{vmatrix} \alpha & \beta \\ r & \delta \end{vmatrix}.$$

If, by a linear transformation T , we change a function $\phi(x_1, x_2)$ into a corresponding function $\phi(X_1, X_2)$, in order to make the work general we should have a corresponding linear transformation T^{-1} which will change $\phi(X_1, X_2)$ back into $\phi(x_1, x_2)$. The necessary and sufficient condition that this be possible is that in the transformation T we be able to solve for X_1, X_2 in terms of x_1, x_2 . This can only be done when $\begin{vmatrix} \alpha & \beta \\ r & \delta \end{vmatrix}$ does not equal zero. Therefore the one condition imposed on a general linear transformation is that the determinant of the coefficients of the transformation shall not vanish.

If we apply a linear transformation

$$T: \begin{cases} x_1 = \alpha X_1 + \beta X_2 \\ x_2 = r X_1 + \delta X_2 \end{cases} \quad (2)$$

to any binary form

$$a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \dots + n a_{n-1} x_1 x_2^{n-1} + a_n x_2^n$$

there results a new form

$$A_0 X_1^n + n A_1 X_1^{n-1} X_2 + \dots + n A_{n-1} X_1 X_2^{n-1} + A_n X_2^n$$

in which the A 's are functions of the a 's and α, β, r, δ .

Any function of a_0, a_1, \dots is an invariant of the form if it is identically equal to the same function of A_0, A_1, \dots except for a factor which is an integral power

of the modulus. That is, if

$$I(A_0, A_1, \dots) \equiv M^\lambda I(a_0, a_1, \dots)$$

where λ is an integer, then $I(a_0, a_1, \dots)$ is an invariant.*

Exactly the same definition may be used for invariants of our determinantal form. Thus let the determinantal form (1) be changed by the linear transformation T (2) into the determinantal form

$$\begin{array}{cccc} A_{11} X_1 + B_{11} X_2 & A_{12} X_1 + B_{12} X_2 & \dots & A_{1n} X_1 + B_{1n} X_2 \\ A_{21} X_1 + B_{21} X_2 & A_{22} X_1 + B_{22} X_2 & \dots & A_{2n} X_1 + B_{2n} X_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} X_1 + B_{n1} X_2 & A_{n2} X_1 + B_{n2} X_2 & \dots & A_{nn} X_1 + B_{nn} X_2 \end{array}$$

where the coefficients A_{ij}, B_{ij} are functions of a_{ij}, b_{ij} , and $\alpha, \beta, \gamma, \delta$. Then $I(a, b)$ is an invariant of the determinantal form (1) if an identity holds of the form

$$I(A, B) \equiv M^\lambda I(a, b).$$

Any function of the coefficients and the variables of the form which is identically equal to the same function of the corresponding coefficients and variables in the transformed form, except for a factor which is an integral

* The proof that the factor involving the coefficient of the transformation must be an integral power of the modulus may be found in Grace and Young, "Algebra of Invariants" Chapter II, p 22.

power of the modulus, is a covariant of the form. Thus in the notation above, $K(a_1, b, x_1, x_2)$ is a covariant of the determinantal form (1) if

$$K(A, B, X_1, X_2) \equiv M^\lambda K(a, b, x_1, x_2).$$

In this paper only rational integral invariants and covariants will be considered.

THEORY OF INVARIANTS OF A GENERAL DETERMINANTAL FORM

THEOREM I: All invariants of the determinantal form (1) must be homogeneous in a_{ij} and homogeneous in b_{ij} and of the same degree in a_{ij} and b_{ij} .

Apply to the determinant (1) the transformation

$$S_{rl} : \begin{cases} x_1 = rX_1 \\ x_2 = lX_2 \end{cases} \quad M = rl.$$

The coefficients A_{ij} and B_{ij} in the resulting determinant have the values

$$A_{ij} = ra_{ij},$$

and

$$B_{ij} = lb_{ij}.$$

By the definition of an invariant

$$I(A, B) \equiv (rl)^\lambda I(a, b). \quad (3)$$

After substituting for every A and B in the left member the equivalent in terms of $a, b, r,$ and $l,$ in order to make the equation (3) an identity, it is evident that $(rl)^\lambda$ must factor out of each term in the left member.

Therefore each term must contain λ factors of type A and

λ of type B and the invariant must be homogeneous in each and of the same degree in the a's as in the b's.

THEOREM II: Any invariant is unchanged if every a_{ij} is replaced by $-b_{ij}$ and every b_{ij} is replaced by a_{ij} .

Apply to the determinant (1) the transformation

$$V : \begin{cases} x_1 = -X_1 \\ x_2 = X_1 \end{cases} \quad M = 1$$

Under this transformation

$$A_{ij} = b_{ij}, \quad \text{and} \quad B_{ij} = -a_{ij}$$

and by our definition of an invariant

$$I(A, B) \equiv I(a, b).$$

If we substitute b_{ij} for A_{ij} and $-a_{ij}$ for B_{ij} , we get

$$I(A, B) \equiv I(b, -a).$$

Therefore

$$I(b, -a) \equiv I(a, b).$$

DEFINITION: A function ϕ is said to be annihilated by an operator Ω if when Ω operates on the function ϕ , it is reduced to identically zero.

THEOREM III: Every invariant is annihilated by the differential operator Ω

$$\Omega = \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial b_{ij}}.$$

If we apply the transformation T_k

$$T_k : \begin{cases} x_1 = X_1 + kX_2 \\ x_2 = X_2 \end{cases} \quad M = 1$$

to the determinantal form, the coefficients of the new determinant are

$$A_{ij} = a_{ij},$$

$$B_{ij} = ka_{ij} + b_{ij}.$$

Since by our definition of invariants

$$I(A, B) \equiv I(a, b)$$

under the particular transformation T_K , we have

$$\frac{\partial I(A, B)}{\partial k} = \frac{\partial I(a, b)}{\partial k}.$$

But $\frac{\partial I(a, b)}{\partial k}$ must necessarily equal zero since the coefficients a, b , do not involve k . Therefore

$$\frac{\partial I(A, B)}{\partial k} = \frac{\partial I}{\partial A_{11}} \cdot \frac{\partial A_{11}}{\partial k} + \frac{\partial I}{\partial A_{12}} \cdot \frac{\partial A_{12}}{\partial k} + \dots$$

$$\frac{\partial I}{\partial B_{11}} \cdot \frac{\partial B_{11}}{\partial k} + \dots + \frac{\partial I}{\partial B_{nn}} \cdot \frac{\partial B_{nn}}{\partial k} = 0, \quad \text{or}$$

$$\sum_{i,j=1}^n \frac{\partial I}{\partial A_{ij}} \cdot \frac{\partial A_{ij}}{\partial k} + \sum_{i,j=1}^n \frac{\partial I}{\partial B_{ij}} \cdot \frac{\partial B_{ij}}{\partial k} = 0. \quad (4)$$

Since $A_{ij} = a_{ij}$

and $B_{ij} = ka_{ij} + b_{ij}$,

we have $\frac{\partial A_{ij}}{\partial k} = 0$

and $\frac{\partial B_{ij}}{\partial k} = a_{ij} = A_{ij}$.

Making these substitutions in (4) we get

$$\sum_{i,j=1}^n 0 \cdot \frac{\partial I}{\partial A_{ij}} + \sum_{i,j=1}^n A_{ij} \frac{\partial I}{\partial B_{ij}} = 0.$$

That is

$$\left(\sum_{i,j=1}^n \frac{A_{ij} \partial}{\partial B_{ij}} \right) I(A, B) = 0, \quad \text{or}$$

$$\left(\sum_{i,j=1}^n \frac{a_{ij} \partial}{\partial b_{ij}} \right) I(a, b) = 0, \quad \text{or}$$

$$\Omega I = 0.$$

We shall now proceed to show that we have conditions sufficient for completely testing an invariant. Successive applications of the transformations

$$S_{\ell} : \begin{cases} x_1 = X_1 \\ x_2 = 1X_2 \end{cases} \quad V : \begin{cases} x_1 = -X_2 \\ x_2 = X_1 \end{cases}$$

$$T_{\kappa} : \begin{cases} x_1 = X_1 + \kappa X_2 \\ x_2 = X_2 \end{cases}$$

give the same result as does an application of the general transformation*. Consequently, a function which is an invariant under each of the transformations S_{ℓ} , V , T_{κ} is an invariant in the general sense. It was proved in Th. I that a function is an invariant under the transformation S_{ℓ} if and only if it is homogeneous in both a and b . In the same way it can be shown that a function is invariant under the transformation S_{ℓ} if and only if it is homogeneous in the b 's. We have proved that every function is invariant under the transformation V , if and only if it is unchanged if b_{ij} is put for a_{ij} and $-a_{ij}$ for b_{ij} , and that it is invariant under the transformation T , if and only if it is annihilated by Ω . Therefore the following theorem results:

THEOREM IV: If any rational integral function $I(a, b)$

* Proof may be found in Dickson, "Algebraic Invariants" p 34

is homogeneous in the b's, is annihilated by the operator Ω and is unaltered when every b_{ij} is replaced by $-a_{ij}$, and every a_{ij} is replaced by b_{ij} then $I(a, b)$ is an invariant.

If we apply the transformation

$$T'_k : \begin{cases} x_1 = X_1 \\ x_2 = kX_1 + X_2 \end{cases} \quad M = 1$$

to the determinantal form we get

$$A_{ij} = a_{ij} + kb_{ij}.$$

$$B_{ij} = b_{ij}.$$

Proceeding as we did to obtain the operator Ω , we get a new operator which we shall call O .

$$O = \sum_{i,j=1}^n b_{ij} \frac{\partial}{\partial a_{ij}}.$$

Therefore an invariant must also be annihilated by O , or

$$OI = 0.$$

This operator can also be obtained in another way.

We see that the determinantal form is unchanged if we interchange x_1 and x_2 , and a_{ij} and b_{ij} . Therefore an invariant must be annihilated by the operator formed by interchanging a_{ij} and b_{ij} . Making this change in Ω we get O .

THEOREM V: Any rational integral homogeneous function of the coefficients of the determinantal form, annihilated both by Ω and O is necessarily an invariant.

In order to prove this let us note that the

result of applying successively to a form the following transformations

$$S_{rl} : \begin{cases} x_1 = rX_1 \\ x_2 = lX_2 \end{cases}$$

$$T_K : \begin{cases} x_1 = X_1 + kX_2 \\ x_2 = X_2 \end{cases}$$

$$T'_s : \begin{cases} x_1 = X_1 \\ x_2 = sX_1 + X_2 \end{cases}$$

is the same as applying the transformation

$$T : \begin{cases} x_1 = (r + kls)X_1 + k l X_2 \\ x_2 = s l X_1 + l X_2 \end{cases}$$

This is a general transformation since the coefficients l , sl , kl and $r + kls$ are entirely arbitrary. That is, $S_{rl} \cdot T_K \cdot T'_s = T$, where $r + kls = \alpha$, $kl = \beta$, $sl = \gamma$ and $l = \delta$. We have shown that any function is an invariant under the transformation T_K if and only if it is annihilated by Ω ; that it is an invariant under the transformation T'_s if and only if it is annihilated by O ; that it is an invariant under the transformation S_{rl} if and only if it is homogeneous. Therefore any function is an invariant under all transformations if it satisfies all three conditions.

Consequently, all invariants must be simultaneous solutions of both partial differential equations,

$$\sum_{i,j=1}^n a_{ij} \frac{\partial I}{\partial b_{ij}} = 0 \quad (5)$$

and

$$\sum_{i,j=1}^n b_{ij} \frac{\partial I}{\partial a_{ij}} = 0. \quad (6)$$

DEFINITION: Any function S which is a solution of the partial differential equation $\sum_{i,j=1}^n a_{ij} \frac{\partial I}{\partial b_{ij}} = 0$ is a seminvariant. That is, all seminvariants are annihilated by Ω .

It is evident that expressions of the type a_{ij} are solutions of the differential equation (5) and that there are n^2 independent solutions of this type. It is also evident that expressions of the type $a_{ij} b_{kl} - a_{kl} b_{ij}$ are solutions of the same equation. There will be $n^2 - 1$ of these expressions of the type $a_{ij} b_{kl} - a_{kl} b_{ij}$, and since every new expression of this type contains a new term b_{kl} , these will evidently be independent solutions. Therefore there will be in all $2n^2 - 1$ independent seminvariants. There can be no more independent solutions since there are $2n^2$ independent variables in the linear partial differential equation.*

Furthermore it is evident that all expressions of the type $a_{ij} b_{kl} - a_{kl} b_{ij}$ are also solutions of (6). Since they are homogeneous in both a and b , they are invariants by the preceding theorem.

* Horn "Einführung in die Theorie der partiellen Differentialgleichungen" p 9.

THEORY OF COVARIANTS OF A GENERAL DETERMINANTAL FORM

THEOREM VI: Every covariant of the determinantal form (1) which is not homogeneous in the variables is a sum of covariants each of which is homogeneous in them.

If a_{ij} , b_{ij} are the coefficients of the determinantal form and K is a covariant, then by the definition

$$K(A, B, X_1, X_2) \equiv M^\lambda K(a, b, x_1, x_2).$$

When x_1, x_2 are replaced by their linear expressions in X_1, X_2 the terms of order ω in x_1, x_2 in the right member and only such terms give rise to terms of order ω in X_1, X_2 in the left member. Hence, if K_1 is the sum of all the terms of order ω of K ,

$$K_1(A, B, X_1, X_2) \equiv M^\lambda K(a, b, x_1, x_2)$$

and K_1 is a covariant. Thus our original covariant can be divided into separate covariants which are homogeneous in the variables and we have

$$K = K_1 + K_2 + \dots$$

Therefore we can limit our discussion to covariants which are homogeneous in the variables.

THEOREM VII: A covariant of the determinantal form which is of constant order ω in the variables is homogeneous in the coefficients.

Let us apply the transformation

$$S_\lambda : \begin{cases} x_1 = \lambda X_1 \\ x_2 = \lambda X_2 \end{cases} \quad M = \lambda^2$$

to the determinant. Then

$$A_{ij} = la_{ij},$$

$$B_{ij} = lb_{ij},$$

$$X_1 = l^{-1} x_1,$$

$$\text{and } X_2 = l^{-1} x_2.$$

By our definition of a covariant

$$K(A, B, X_1, X_2) \equiv l^{2\lambda} K(a, b, x_1, x_2).$$

If we substitute for the terms in the left member the corresponding values in terms of the original coefficients and variables of the transformation, we get

$$K(la, lb, l^{-1} x_1, l^{-1} x_2) \equiv l^{2\lambda} K(a, b, x_1, x_2).$$

Since K is homogeneous of order ω in the variables, from the variables in each term of the left member we obtain $l^{-\omega}$ as a factor. Therefore, since $l^{2\lambda}$ must be a factor of each term in the left member in order to make the equation an identity, the degree in a and b must be constant for each term and equal to $2\lambda + \omega$.

By a method similar to that used in the case of invariants, we can obtain an annihilator for covariants.

If we apply the transformation

$$T_K : \begin{cases} x_1 = X_1 + kX_2 \\ x_2 = X_2 \end{cases} \quad M = 1 \quad (7)$$

to the determinantal form, there results

$$A_{ij} = a_{ij}$$

$$\text{and } B_{ij} = Ka_{ij} + b_{ij}.$$

The solution of (7) for X_1 and X_2 gives

$$X_1 = x_1 - kx_2$$

$$X_2 = x_2.$$

Therefore

$$\begin{aligned} \frac{\partial K(A, B, X_1, X_2)}{\partial k} &= \sum_{i,j=1}^n \frac{\partial K}{\partial A_{ij}} \cdot \frac{\partial A_{ij}}{\partial k} + \\ &\sum_{i,j=1}^n \frac{\partial K}{\partial B_{ij}} \cdot \frac{\partial B_{ij}}{\partial k} + \frac{\partial K}{\partial X_1} \cdot \frac{\partial X_1}{\partial k} + \frac{\partial K}{\partial X_2} \cdot \frac{\partial X_2}{\partial k} \\ &= \frac{\partial K(a, b, x_1, x_2)}{\partial k} = 0. \end{aligned}$$

But

$$\frac{\partial A_{ij}}{\partial k} = 0,$$

$$\frac{\partial B_{ij}}{\partial k} = a_{ij} = A_{ij},$$

$$\frac{\partial X_1}{\partial k} = -x_2 = -X_2,$$

and
$$\frac{\partial X_2}{\partial k} = 0.$$

Therefore

$$\sum_{i,j=1}^n A_{ij} \frac{\partial K}{\partial B_{ij}} - X_2 \frac{\partial K}{\partial X_1} = 0.$$

That is, the annihilator for covariats is

$$\sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial b_{ij}} - x_2 \frac{\partial}{\partial x_1}, \quad \text{or}$$

$$\Omega - x_2 \frac{\partial}{\partial x_1},$$

where
$$\Omega = \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial b_{ij}}.$$

Since the determinant is unchanged if we interchange

x_1 and x_2 and a_{ij} and b_{ij} , K is also annihilated by the

operator

$$0 - x_1 \frac{\partial}{\partial x_2},$$

where

$$0 = \sum_{i,j=1}^n b_{ij} \frac{\partial}{\partial a_{ij}}.$$

Let us denote a covariant of order ω of the determinantal form by

$$K = Sx_1^\omega + S_1x_1^{\omega-1}x_2 + \dots + S_{\omega-1}x_1x_2^{\omega-1} + S_\omega x_2^\omega.$$

By operating on K with $O = x_1 \frac{\partial}{\partial x_2}$, we obtain

$$(OS - S_1)x_1^\omega + (OS_1 - 2S_2)x_1^{\omega-1}x_2 + \dots \\ + (OS_{\omega-1} - \omega S_\omega)x_1x_2^{\omega-1} + OS_\omega x_2^\omega \equiv 0.$$

We can find expressions for all the coefficients S_i in terms of S and the operator O , by equating coefficients of like powers to zero. Then

$$K = Sx_1^\omega + OSx_1^{\omega-1}x_2 + \frac{1}{2!} O^2 Sx_1^{\omega-2}x_2^2 + \dots + \frac{1}{\omega!} O^\omega Sx_2^\omega.$$

Hence a covariant is uniquely determined by the coefficient S which is called the leader of the covariant.

Similarly K is annihilated by $\Omega = x_2 \frac{\partial}{\partial x_1}$, if and only if

$$\Omega S = 0,$$

$$\Omega S_1 = \omega S,$$

$$\Omega S_2 = (\omega - 1) S_1,$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$\Omega S_\omega = S_{\omega-1}.$$

Then if we know the coefficient of the x_2^ω term, we can obtain the other coefficients and finally S itself by means of the operator Ω . Since S is annihilated by Ω , it is a seminvariant. If it is also annihilated by O , and is homogeneous in a and b , it is an invariant. Therefore every solution of the equation $\Omega I = 0$ is either an invariant or a seminvariant which produces a covariant. Therefore the

leader of every covariant is a seminvariant and for every seminvariant there exists a covariant.

The solutions of $\Omega I = 0$ which are not invariants are of the form a_{ij} . Therefore for each a_{ij} as a leader there should be a covariant uniquely determined by the equation

$$K = Sx_1^\omega + OSx_1^{\omega-1}x_2 + \frac{1}{2!}O^2Sx_1^{\omega-2}x_2^2 + \dots + \frac{1}{\omega!}O^\omega Sx_2^\omega,$$

where $O^{\omega+1}S = 0.$ (8)

If $S = a_{ij},$ then
 $OS = b_{ij},$ and
 $O^2S = 0.$ (9)

Therefore from (8) and (9) $\omega + 1 = 2$ and $\omega,$ or the order of the covariant, is one. Therefore the covariant for every a_{ij} becomes $a_{ij}x_1 + b_{ij}x_2.$ That is, every element in the determinantal form is a covariant.

FUNDAMENTAL SYSTEM OF COVARIANTS AND INVARIANTS

The question now arises whether or not every covariant and invariant of the determinantal form can be expressed as a rational integral function of a finite number of covariants and invariants. We shall prove that this can be done.

Let $\alpha_x = a_{ij}x_1 + b_{ij}x_2$ represent an element of the determinantal form. Therefore α_x also represents a covariant which is the element itself. If we apply the transformation

$$T: \begin{cases} x_1 = \xi_1 X_1 + \eta_1 X_2 \\ x_2 = \xi_2 X_1 + \eta_2 X_2 \end{cases} \quad M = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} \neq 0$$

to the determinantal form, the new element becomes

$A_{ij} X_1 + B_{ij} X_2$, where $A_{ij} = a_{ij} \xi_1 + b_{ij} \xi_2$ which in the notation above we call α_ξ , and where $B_{ij} = a_{ij} \eta_1 + b_{ij} \eta_2$ or α_η . The determinant M we shall represent by $(\xi \eta)$. The invariant $a_{ij} b_{kl} - b_{il} a_{kj}$ can be expressed as a determinant and we shall represent it by (ab) . Since

$$\begin{vmatrix} A_{ij} & B_{ij} \\ A_{kl} & B_{kl} \end{vmatrix} = \begin{vmatrix} a_{ij} \xi_1 + b_{ij} \xi_2 & a_{ij} \eta_1 + b_{ij} \eta_2 \\ a_{kl} \xi_1 + b_{kl} \xi_2 & a_{kl} \eta_1 + b_{kl} \eta_2 \end{vmatrix} = \begin{vmatrix} a_{ij} & b_{ij} \\ a_{kl} & b_{kl} \end{vmatrix} \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix}$$

it is again evident that all forms of the type $a_{ij} b_{kl} - a_{kl} b_{ij}$ or (ab) are invariants.

LEMMA: If the operator V , where $V = \left(\frac{\partial^2}{\partial \xi_1 \partial \eta_2} - \frac{\partial^2}{\partial \xi_2 \partial \eta_1} \right)$, is applied r times to a product of k factors of the type α_ξ and l factors of the type α_η , there results a sum of terms each containing $k - r$ factors α_ξ , $l - r$ factors α_η , and r factors (ab) .

Let CD represent the product under consideration,

where $C = \alpha_{\xi_1}^{(1)} \cdot \alpha_{\xi_2}^{(2)} \cdot \dots \cdot \alpha_{\xi_k}^{(k)}$ and
 $D = \alpha_{\eta_1}^{(1)} \cdot \alpha_{\eta_2}^{(2)} \cdot \dots \cdot \alpha_{\eta_l}^{(l)}$.

Then $\frac{\partial CD}{\partial \xi_i} = \sum_{i=1}^k a^{(i)} \frac{CD}{\alpha_{\xi_i}^{(i)}}$ and

$$\frac{\partial^2 CD}{\partial \xi_i \partial \eta_j} = \sum_{j=1}^l \sum_{i=1}^k a^{(i)} b^{(j)} \frac{CD}{\alpha_{\xi_i}^{(i)} \alpha_{\eta_j}^{(j)}}.$$

Also $\frac{\partial^2 CD}{\partial \xi_2 \partial \eta_1} = \sum_{j=1}^l \sum_{i=1}^k a^{(i)} b^{(j)} \frac{CD}{\alpha_{\xi_i}^{(i)} \alpha_{\eta_j}^{(j)}}.$ Consequently

$$VCD = \sum_{j=1}^l \sum_{i=1}^k (ab) \frac{CD}{\alpha_{\xi_i}^{(i)} \alpha_{\eta_j}^{(j)}}.$$

This proves the lemma when $r = 1$. It is easily seen that if we operate again with V on each term in the summation of VCD, we add another factor of the type (ab) to each term in the result and decrease the number of factors of each type α_ξ and α_η by one. If this reasoning is continued the lemma is proved.

THEOREM VIII: Any covariant is a polynomial in terms of the expressions of the types α_x and (ab).

By the definition of covariants

$$K(A, B, X_1, X_2) \equiv (\xi\eta)^n K(a, b, x_1, x_2) \quad (10)$$

under the general transformation

$$T: \begin{cases} x_1 = \xi_1 X_1 + \eta_1 X_2 \\ x_2 = \xi_2 X_1 + \eta_2 X_2 \end{cases} \quad M = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} \neq 0 \quad (11)$$

Since K is homogeneous and of order ω in the variables we can write the left member as

$$\sum_{s=0}^{\omega} (\sum A, B) X_1^{\omega-s} X_2^s$$

where the inside summation represents a sum of various products of the coefficients A_{ij} and B_{kl} . That is, the inside summation represents a sum of terms each containing a certain number of factors of type α_ξ , and a certain number of factors of type α_η , since every A equals some α_ξ and every B equals some α_η .

Let us now make the substitutions $x_1 = y_2$ and $x_2 = -y_1$. Then by solving the equations (11)

$$x_1 = \frac{y_2 \eta_1 + y_1 \eta_2}{(\xi\eta)}$$

$$x_2 = -\frac{y_1 \xi_1 + y_2 \xi_2}{(\xi\eta)}.$$

Hence (10) becomes

$$\sum_{\rho=0}^{\omega} (\sum A, B) (y_1 \eta_1 + y_2 \eta_2)^{\omega-\rho} (-1)^{\rho} (y_1 \xi_1 + y_2 \xi_2)^{\rho} \\ \equiv (\xi \eta)^{\lambda+\omega} K(a, b, x, x).$$

Since the right member is of degree $\lambda+\omega$ in ξ and $\lambda+\omega$ in η , there must be in the left member $\lambda+\omega$ factors containing ξ and $\lambda+\omega$ factors containing η . If we operate with V $\lambda+\omega$ times on both members of the equation, the right member becomes cK , where $c \neq 0$.* The left member becomes a sum of products each of which contains $\lambda+\omega$ determinantal factors of which ω are of the type $(a_{ij} y_2 - b_{ij} y_1)$ and hence λ of the type (ab) . Substituting x_2 for y_1 and $-x_1$ for y_2 , the factors $a_{ij} y_2 - b_{ij} y_1$ become of the type $a_{ij} x_1 + b_{ij} x_2$ or α_x . Therefore every covariant is a polynomial in α_x and (ab) and the following theorem is proved.

THEOREM IX: Every covariant is a rational integral function of a finite number of invariants and covariants which are of the form $(a_{ij} b_{kl} - a_{kl} b_{ij})$ and $a_{ij} x_1 + b_{ij} x_2$, respectively.

We have now proved that for a determinantal form of any order there is a finite fundamental system of rational integral covariants K_1, K_2, \dots, K_r such that any rational integral covariant of the determinantal form is a polynomial in K_1, K_2, \dots, K_r with numerical coefficients.

It is not difficult to determine the number of invariants and covariants in the fundamental system for a

*Dickson, "Algebraic Invariants", 43, p. 68

determinantal form of n^{th} order. The invariants in the fundamental system are of the form $a_{ij}b_{\kappa\epsilon} - a_{\kappa\epsilon}b_{ij}$. For an n^{th} order determinant, there are n^2 choices for the term a_{ij} . With each of these a_{ij} 's, we can combine any $b_{\kappa\epsilon}$ except b_{ij} . Therefore we have a choice of $n^2 - 1$ terms for $b_{\kappa\epsilon}$. Since the last term is determined by the first we see that there may be $n^2(n^2 - 1)$ or $n^4 - n^2$ different invariants of the form $a_{ij}b_{\kappa\epsilon} - a_{\kappa\epsilon}b_{ij}$. However since the invariant $a_{ij}b_{\kappa\epsilon} - a_{\kappa\epsilon}b_{ij}$ is identical to $a_{\kappa\epsilon}b_{ij} - a_{ij}b_{\kappa\epsilon}$ except for a numerical factor -1 , we have $\frac{n^4 - n^2}{2}$ different invariants in the fundamental system.

The number of covariants is the number of elements in the determinant, which is n^2 .

THEOREM X: A fundamental system of invariants and covariants of a determinantal form of n^{th} order consists of no more than $\frac{n^4 - n^2}{2}$ invariants and n^2 covariants.

We shall now illustrate this last theorem with an example. That is, we shall take a particular invariant of the binary determinantal form and show that it is a polynomial in the invariants of the fundamental set. Any invariant or covariant of the ordinary binary form can be expressed by direct substitution in terms of the coefficients and variables of the determinantal form and when so expressed becomes of course an invariant or covariant of the determinantal form. Consequently, we can obtain an invariant of our determinantal form by starting with the

discriminant of the ordinary binary quadratic form. The determinant

$$\begin{vmatrix} a_{11}x_1 + b_{11}x_2 & a_{12}x_1 + b_{12}x_2 \\ a_{21}x_1 + b_{21}x_2 & a_{22}x_1 + b_{22}x_2 \end{vmatrix} \quad (12)$$

when expanded becomes

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} x_1^2 + \left(\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} \right) x_1 x_2 + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} x_2^2,$$

a binary quadratic form of the type

$$a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2 \quad \text{where} \quad (13)$$

$$a_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$a_1 = \frac{1}{2} \left(\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} \right) \quad \text{and}$$

$$a_2 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}.$$

The discriminant of (13) is

$$a_1^2 - a_2 a_0.$$

Therefore

$$\left(\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} \right)^2 - 4 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

must be an invariant of the determinant (12). This when

expanded gives

$$\begin{aligned} & a_{11}^2 b_{22}^2 + a_{21}^2 b_{12}^2 - 2a_{11} a_{21} b_{12} b_{22} + a_{22}^2 b_{11}^2 - 2a_{12} a_{22} b_{11} b_{21} \\ & + a_{12}^2 b_{21}^2 + 2a_{11} a_{22} b_{11} b_{22} - 2a_{11} a_{12} b_{21} b_{22} - 2a_{21} a_{22} b_{12} b_{11} \\ & + 2a_{12} a_{21} b_{12} b_{21} - 4a_{11} a_{21} b_{11} b_{12} - 4a_{11} a_{22} b_{11} b_{22} - 4a_{12} a_{21} b_{12} b_{21} \\ & - 4a_{12} a_{22} b_{21} b_{22} + 4a_{11} a_{21} b_{11} b_{12} + 4a_{11} a_{22} b_{11} b_{22} + 4a_{12} a_{21} b_{11} b_{22} \\ & + 4a_{12} a_{22} b_{21} b_{22}. \end{aligned}$$

After like terms are collected this can be expressed as

$$\begin{aligned} & (a_{11}b_{22} - a_{22}b_{11})^2 + (a_{12}b_{21} - a_{21}b_{12})^2 \\ & - 2(a_{11}b_{21} - a_{21}b_{11})(a_{12}b_{22} - a_{22}b_{12}) \\ & - 2(a_{11}b_{12} - a_{12}b_{11})(a_{21}b_{22} - a_{22}b_{21}), \end{aligned} \quad (14)$$

which involves each of the 6 or $\frac{n^4 - n^2}{2}$ invariants of the fundamental set. However the same invariant can be expressed in terms of fewer than 6 of these fundamental invariants. It can be easily verified that

$$\begin{aligned} & (a_{11}b_{22} - a_{22}b_{11})(a_{12}b_{21} - a_{21}b_{12}) + \\ & (a_{11}b_{12} - a_{12}b_{11})(a_{21}b_{22} - a_{22}b_{21}) \equiv \\ & (a_{11}b_{21} - a_{21}b_{11})(a_{12}b_{22} - a_{22}b_{12}). \end{aligned} \quad (15)$$

Therefore the substitution of the left member of equation (15) for the third term in the invariant (14) makes it possible to reduce the invariant (14) which is the discriminant of the ordinary binary quadratic form, to

$$\begin{aligned} & [(a_{11}b_{22} - a_{22}b_{11}) - (a_{12}b_{21} - a_{21}b_{12})]^2 \\ & - 4(a_{11}b_{12} - a_{12}b_{11})(a_{21}b_{22} - a_{22}b_{21}). \end{aligned}$$

The syzygy (15) may be changed into either of the two following relations by transposing and changing signs.

$$\begin{aligned} & (a_{11}b_{22} - a_{22}b_{11})(a_{12}b_{21} - a_{21}b_{12}) \equiv \\ & (a_{11}b_{21} - a_{21}b_{11})(a_{12}b_{22} - a_{22}b_{12}) + \\ & (a_{12}b_{11} - a_{11}b_{12})(a_{21}b_{22} - a_{22}b_{21}), \\ \text{or} \quad & (a_{11}b_{12} - a_{12}b_{11})(a_{21}b_{22} - a_{22}b_{21}) \equiv \\ & (a_{11}b_{21} - a_{21}b_{11})(a_{12}b_{22} - a_{22}b_{12}) + \\ & (a_{22}b_{11} - a_{11}b_{22})(a_{12}b_{21} - a_{21}b_{12}). \end{aligned}$$

It will be noticed in this class of syzygies that in any product every subscript represented in the coefficients of the determinant appears just twice, once with the "a" term and once with the "b" term, and that every invariant of the fundamental set appears once and only once in the syzygy.

Every syzygy between the invariants of a determinant of order k will be a syzygy for any determinant of order greater than k , since every invariant of a determinant of order k is an invariant of a determinant of order greater than k . That is, the syzygy between the invariants of the determinant of second order will be a syzygy for the third order form. Another syzygy of the third order form which involves more than the invariants of a binary form is

$$\begin{aligned}
 & (a_{11} b_{21} - a_{21} b_{11})(a_{12} b_{23} - a_{23} b_{12})(a_{13} b_{32} - a_{32} b_{13}) + \\
 & (a_{12} b_{11} - a_{11} b_{12})(a_{13} b_{21} - a_{21} b_{13})(a_{32} b_{23} - a_{23} b_{32}) + \\
 & (a_{11} b_{13} - a_{13} b_{11})(a_{12} b_{32} - a_{32} b_{12})(a_{21} b_{23} - a_{23} b_{21}) + \\
 & (a_{11} b_{23} - a_{23} b_{11})(a_{21} b_{32} - a_{32} b_{21})(a_{12} b_{13} - a_{13} b_{12}) + \\
 & (a_{11} b_{32} - a_{32} b_{11})(a_{12} b_{21} - a_{21} b_{12})(a_{23} b_{13} - a_{13} b_{23}) = \\
 & 2(a_{21} b_{32} - a_{32} b_{21})(a_{13} b_{11} - a_{11} b_{13})(a_{23} b_{12} - a_{12} b_{23}).
 \end{aligned}$$

Other syzygies similar to this involving still other invariants are

$$\begin{aligned}
 & (a_{22}b_{21} - a_{21}b_{22})(a_{31}b_{32} - a_{32}b_{31})(a_{33}b_{23} - a_{23}b_{33}) + \\
 & (a_{22}b_{31} - a_{31}b_{22})(a_{21}b_{33} - a_{32}b_{21})(a_{23}b_{32} - a_{32}b_{23}) + \\
 & (a_{22}b_{32} - a_{32}b_{22})(a_{21}b_{23} - a_{23}b_{21})(a_{31}b_{33} - a_{33}b_{31}) + \\
 & (a_{33}b_{22} - a_{22}b_{33})(a_{21}b_{32} - a_{32}b_{21})(a_{31}b_{23} - a_{23}b_{31}) + \\
 & (a_{22}b_{23} - a_{23}b_{22})(a_{21}b_{31} - a_{31}b_{21})(a_{32}b_{33} - a_{33}b_{32}) = \\
 & 2(a_{22}b_{32} - a_{32}b_{22})(a_{31}b_{21} - a_{21}b_{31})(a_{33}b_{23} - a_{23}b_{33})
 \end{aligned}$$

and

$$\begin{aligned}
 & (a_{22}b_{11} - a_{11}b_{22})(a_{31}b_{33} - a_{33}b_{31})(a_{12}b_{13} - a_{13}b_{12}) + \\
 & (a_{22}b_{31} - a_{31}b_{22})(a_{11}b_{12} - a_{12}b_{11})(a_{13}b_{33} - a_{33}b_{13}) + \\
 & (a_{22}b_{33} - a_{33}b_{22})(a_{11}b_{13} - a_{13}b_{11})(a_{31}b_{12} - a_{12}b_{31}) + \\
 & (a_{22}b_{12} - a_{12}b_{22})(a_{11}b_{33} - a_{33}b_{11})(a_{31}b_{13} - a_{13}b_{31}) + \\
 & (a_{22}b_{13} - a_{13}b_{22})(a_{11}b_{31} - a_{31}b_{11})(a_{12}b_{33} - a_{33}b_{12}) = \\
 & 2(a_{12}b_{22} - a_{22}b_{12})(a_{13}b_{11} - a_{11}b_{13})(a_{31}b_{33} - a_{33}b_{31}).
 \end{aligned}$$

CONDITIONS FOR INVARIANTS AND COVARIANTS OF
LESS GENERAL DETERMINANTAL FORMS

A determinantal form whose elements are of the form $a_{ij}x_1 + b_{ij}x_2$ can be changed into a new determinant,

$$\begin{vmatrix}
 a_{11}x_1 + b_{11}x_2 & & & b_{12}x_2 & \dots & & & b_{1n}x_2 \\
 & b_{21}x_2 & a_{22}x_1 + b_{22}x_2 & & & & & b_{2n}x_2 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & b_{n1}x_2 & & & & a_{nn}x_1 + b_{nn}x_2
 \end{vmatrix} \quad (16)$$

by adding or subtracting rows and columns. The determinant

$$\begin{vmatrix} a_{11}x_1 + b_{11}x_2 & a_{12}x_1 + b_{12}x_2 \\ a_{21}x_1 + b_{21}x_2 & a_{22}x_1 + b_{22}x_2 \end{vmatrix}$$

for example, can be changed by adding the first row multiplied by $-\frac{a_{21}}{a_{11}}$ to the second row into

$$\begin{vmatrix} a_{11}x_1 + b_{11}x_2 & a_{12}x_1 + b_{12}x_2 \\ \frac{b_{21}a_{11} - a_{21}b_{11}}{a_{11}}x_2 & \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}}x_1 + \frac{b_{22}a_{11} - a_{21}b_{12}}{a_{11}}x_2 \end{vmatrix}$$

If we add the first column multiplied by $-\frac{a_{12}}{a_{11}}$ to the second column the determinant becomes

$$\begin{vmatrix} a_{11}x_1 + b_{11}x_2 & \frac{a_{11}b_{12} - a_{12}b_{11}}{a_{11}}x_2 \\ \frac{b_{21}a_{11} - a_{21}b_{11}}{a_{11}}x_2 & \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}}x_1 + \frac{a_{11}b_{22} - a_{12}a_{21}b_{12} - a_{11}a_{12}b_{21} + a_{12}a_{21}b_{11}}{a_{11}^2}x_2 \end{vmatrix}$$

which is of the desired form.

In a similar manner, any determinant (1) can be changed into one in which the x_1 appears only in elements of the main diagonal.

THEOREM XI: Any invariant of the binary determinantal form (16) is homogeneous in the a 's and homogeneous in the b 's and of the same degree in each.

If we apply the transformation

$$S_{re} : \begin{cases} x_1 = rX_1 \\ x_2 = lX_2 \end{cases}$$

to the determinant (16), we obtain a new determinant in

the variables X_1, X_2 whose coefficients are

$$\begin{aligned} A_{ij} &= ra_{ij} \\ B_{ij} &= lb_{ij}. \end{aligned} \tag{17}$$

By our definition of an invariant

$$I(A, B) \equiv (rl)^\lambda I(ab).$$

This becomes, after substituting the values from (17) into the left member of the identity above,

$$I(ra, lb) \equiv (rl)^\lambda I(a, b).$$

Since we must be able to factor r^λ and l^λ from each term in the left member in order to make the above equation an identity, the invariant must be homogeneous in a and b and of the same degree in each.

THEOREM XII: Every covariant of the determinant (16) is annihilated by the operator

$$\sum_{i=1}^n a_{ii} \frac{\partial}{\partial b_{ii}}.$$

If we apply the transformation

$$T_k: \begin{cases} x_1 = X_1 + kX_2 \\ x_2 = X_2 \end{cases} \quad M = 1.$$

to the determinant (16), the elements of the principal diagonal are $a_{ij}X_1 + (ka_{ii} + b_{ii})X_2$, while the other elements are $b_{ij}X_2$. That is

$$\begin{aligned} A_{ij} &= a_{ij}, \\ B_{ij} &= b_{ij}, \quad (\text{when } i \neq j) \end{aligned} \tag{18}$$

and

$$B_{ii} = ka_{ii} + b_{ii}.$$

Since by our definition of an invariant

$$I(A, B) \equiv I(a, b),$$

we have

$$\frac{\partial I(A, B)}{\partial k} \equiv \frac{\partial I(a, b)}{\partial k}.$$

But

$$\frac{\partial I(a, b)}{\partial k} \equiv 0$$

since $I(a, b)$ does not involve k . Therefore

$$\begin{aligned} \frac{\partial I(A, B)}{\partial k} &= \sum_{i=1}^{\infty} \frac{\partial I}{\partial A_{ii}} \cdot \frac{\partial A_{ii}}{\partial k} + \sum_{i=1}^{\infty} \frac{\partial I}{\partial B_{ii}} \cdot \frac{\partial B_{ii}}{\partial k} \\ &+ \sum_{\substack{i, j=1 \\ (i \neq j)}}^{\infty} \frac{\partial I}{\partial B_{ij}} \cdot \frac{\partial B_{ij}}{\partial k} = 0. \end{aligned} \quad (19)$$

From (18)

$$\frac{\partial A_{ii}}{\partial k} = 0,$$

$$\frac{\partial B_{ij}}{\partial k} = 0, \quad (i \neq j)$$

$$\text{and} \quad \frac{\partial B_{ii}}{\partial k} = a_{ii} = A_{ii}.$$

The equation (19) will then simplify into

$$\sum_{i=1}^{\infty} A_{ii} \frac{\partial I}{\partial B_{ii}} = 0 \quad \text{or} \quad \sum_{i=1}^{\infty} a_{ii} \frac{\partial I}{\partial b_{ii}} = 0,$$

if we make the above substitutions.

Since the determinantal form (16) is changed if a_{ij} and b_{ij} , and x_1 and x_2 , are interchanged, we therefore cannot find a corresponding annihilator as we did in the most general form. This means that we cannot find invariants and covariants of the form (16)

by the method used for the determinant (1). One advantage of changing the form (1) into the form (16) is that the annihilator is simpler. The solutions of the differential equation

$$\sum_{i=1}^n a_{ii} \frac{\partial I}{\partial b_{ii}} = 0$$

are of the type $a_{ii} b_{kk} - a_{kk} b_{ii}$ and these are seminvariants of the determinant (16).

The determinant (16) can be simplified still more by dividing each column by the coefficient of the x_1 term in that column. This gives us the determinant form

$$\begin{vmatrix} x_1 + \frac{b_{11}}{a_{11}} x_2 & \frac{b_{12}}{a_{22}} x_2 & \dots & \frac{b_{1n}}{a_{nn}} x_2 \\ \frac{b_{21}}{a_{11}} x_2 & x_1 + \frac{b_{22}}{a_{22}} x_2 & \dots & \frac{b_{2n}}{a_{nn}} x_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{b_{n1}}{a_{11}} x_2 & \frac{b_{n2}}{a_{22}} x_2 & \dots & x_1 + \frac{b_{nn}}{a_{nn}} x_2 \end{vmatrix}$$

or

$$\begin{vmatrix} x_1 + c_{11} x_2 & c_{12} x_2 & \dots & c_{1n} x_2 \\ c_{21} x_2 & x_1 + c_{22} x_2 & \dots & c_{2n} x_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} x_2 & c_{n2} x_2 & \dots & x_1 + c_{nn} x_2 \end{vmatrix} \quad (20)$$

This last determinant can be treated as (16) was treated and it can be proved that an invariant of (20) is homogeneous in the coefficients and annihilated by the operator

$$\sum_{i=1}^n \frac{\delta}{\delta c_{ii}}$$

One advantage of using the determinant (20) is that the annihilator for invariants of this form is simpler than either of the other annihilators and the seminvariants, solutions of

$$\sum_{i=1}^n \frac{\delta I}{\delta c_{ii}} = 0$$

are of the form $c_{ii} - c_{kk}$.

The methods used for finding invariants and covariants of the most general determinantal form will not apply to the form (20) since when a general transformation is made the new determinant is not of the same type as the original determinant. Therefore we are unable to find invariants and covariants by the methods used in the first part of this paper.

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