ESSAYS IN MACROECONOMIC ANALYSIS

By

Victor J. Valcarcel

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(Chair) Prof. John W. Keating

Prof. Stephen Hillmer

Prof. Shigeru Iwata

Prof. Ted Juhl

Prof. Shu Wu

DATE 4/28/08

The dissertation committee for Victor J. Valcarcel certifies that this is the approved version of the following dissertation

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(Chair) Prof. John W. Keating

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ABSTRACT

This dissertation consists of four essays concerning the analysis of dynamic fluctuations in various U.S. macroeconomic aggregates with particular attention to the links between real private investment, real private consumption and different aspects of government spending. These essays constitute four separate approaches that are based on different assumptions and conduct different methodological avenues. While each chapter is meant to be "stand-alone" and independent of each other, all four center around one motivating factor: the recent evidence of differential growth rates between real consumption and real investment in the U.S.

CHAPTER 1: THE IMPACT OF FISCAL EXPANSIONS ON PRIVATE SPENDING: A NEW KEYNESIAN APPROACH.

This chapter promotes a structural model based on first principles of an economy populated by two types of consumer, with different saving behaviors, and two types of firms which produce either durable or non-durable goods in the presence of sticky prices. While the micro-foundations of this model, when fully specified, allow for price staggering and consumer myopia in the New Keynesian tradition, a Real Business Cycle (RBC) view is nested as well. Allowing for two different calibrations, of the state space representation of the model, yields different results. Based on New Keynesian assumptions, the model predicts positive and persistent increases in private consumption across sectors following a fiscal expansion. This evidence is consistent with the theoretical predictions of those models. Conversely, evidence from the RBC calibration shows that a positive government shock leads to a small decrease in non-durables (consistent with their theoretical predictions) but an increase in the consumption of durables which is consistent with the evidence and theoretical predictions of the New Keynesian models.

CHAPTER 2: THE IMPACT OF FISCAL EXPANSIONS ON PRIVATE SPENDING: A REAL BUSINESS CYCLE APPROACH.

While chapter 1 was essentially written from a New-Keynesian perspective, it was shown that under a particular calibration, an RBC model could be nested within the proposed system. Importantly, this model abstracted from technology shocks.

Incorporating technology shocks into the analysis, of the fiscal impact on private spending, is the main motivation for the second chapter. This chapter specifies a micro-founded structural dynamic stochastic general equilibrium (DSGE) model to estimate the impact of government shocks on private spending by applying a RBC model to post-war data that distinguishes between improvements in consumption versus investment-specific technologies.

Additionally, investment in either sector faces differential adjustment costs. In this context, the different technology shocks act as competitors to government and consumer preference shocks in eliciting different dynamic responses in the observable variables of interest. This chapter concludes that increases in government spending lead to higher levels of consumption while the response of investment is mixed. Consumption seems to be driven mainly by consumer preferences, consumption-specific technology shocks, and consumption-specific government shocks in that order. Conversely, investment seems to be driven mostly by investment-specific technology and government shocks. Hours worked are mainly driven by preference shocks. Finally, the results show that investment-specific technology seems to grow faster than consumption-specific productivity and this can inform the disparity in rates of growth of consumption and investment in the U.S.

CHAPTER 3: THE REAL INVESTMENT TO CONSUMPTION RATIO: INFERENCES FROM ALTERNATIVE PERMANENT-TRANSITORY DECOMPOSITIONS.

Recent empirical evidence seems to suggest that while consumption and investment, in *real* terms, do not grow at the same rate; these same aggregates seem to share a common trend in *nominal* terms. This contention seems to contradict the balanced growth prediction, and it has been termed the *nominal ratio balanced growth hypothesis*.

An intuitive explanation for this alternative concept of balanced growth might be that consumption and investment could be produced using different technologies, and, thus, two-sector models of the economy should be proposed. This is the main driving force that motivates the first two chapters of this dissertation.

The third chapter considers a number of reduced-form models based on permanent-transitory decompositions and structural break tests. A battery of models is used to derive some stylized facts about the U.S. investment-consumption ratio in real terms. In addition, an investigation on the relationship between the real investment-consumption ratio and prices is carried out. Standard unit root tests fail to reject the null of a unit root in the *real* investment-consumption ratio, but reject the null in the *nominal* investment-consumption ratio. Classical structural change tests in the tradition of Andrews (1993) show evidence of a structural break test in the real ratio consistent with the productivity miracle of the 90s. Within the context of permanent-transitory decompositions four unobserved component models are estimated. Results of these models substantiate the presence of a common trend between investment-consumption ratio and relative prices. A shared trend between these two series seems to be confirmed by the different specifications. In addition, lower volatility in both series seems to be prevalent in the latter part of the period. Finally, transitory fluctuations in relative prices remain more volatile but less persistent than fluctuations in the ratio of investment and consumption after the assumed break.

CHAPTER 4: A BAYESIAN APPROACH TO MODELING STOCHASTIC VOLATILITY IN THE U.S. INVESTMENT-CONSUMPTION RATIO

Recent studies in the Bayesian tradition have uncovered a substantial decrease in volatility in aggregate U.S. economic activity taking place in the mid-1980s consistent with findings of volatility reduction by "classical" econometric tests. However, no tests have been conducted to study such behavior in the great ratios thus far. Such investigation is important in light of the recent evidence of behavioral change in the level of the investment-consumption ratio described by other authors. This paper finds decisive evidence of a structural break in the variance of the ratio occurring in the third quarter of 1983. This conclusion is robust to a wide range of alternative priors and consistent with the early-to-mid 1980s volatility reduction observed in the major aggregates and sub-aggregates of U.S. economic activity. Such findings have a number of important implications for the behavior of real investment and consumption in the U.S. Additionally, they lead to the conclusion that any suggested specification that involves decomposing the investment-consumption ratio into a non-stationary process for the trend and a stationary process for the cycle should allow for a break in the covariance matrix.

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CHAPTER I

THE IMPACT OF FISCAL EXPANSIONS ON PRIVATE SPENDING: A NEW KEYNESIAN APPROACH

Abstract

How does fiscal policy, in particular government spending, affect aggregate economic activity? What are the mechanisms that affect the transmission of these effects? What are the implications for long-run economic growth? While the common wisdom in the field of macroeconomics anticipates that an increase in government spending will have an expansionary effect on output, there is no such agreement on the implied effect on consumption. Some economists—mostly advocates of the Real Business Cycle approach-- predict a rise in government expenditures precedes a decrease in consumption while others—New Keynesian followers-- argue for an increase in consumption after such a shock hits the economy. Most of these empirical studies base their theoretical foundation on a standard one-sector model of the economy and they differ solely on the dynamics of the consumer's behavior.

In other areas, particularly economic growth, some researchers have argued that a two-sector model of the economy that distinguishes between productions of durable goods from the rest of the economy provides a better picture of the long-run behavior of the U.S. economy. Given that the dynamics of private spending on durables differ from those of non-durables and services (as illustrated below), one might want to disaggregate the impact of a government spending shock to consumption into its effect on durables versus non-durables. Taking the philosophy of modeling the economy with two sectors to heart, this study represents an attempt to measure and allocate what effect an increase in government spending has on private spending while distinguishing between consumption of durables and consumption of non-durables and services. This line of research is, to my knowledge, as yet untried.

A benchmark model that allows for an even mix of rule-of-thumb and optimizer consumers along with the presence of sticky prices in a two-sector economy is considered. A second model (consistent with a RBC approach) features infinitely-lived optimizer consumers in a two-sector market with flexible prices.

Allowing for different calibrations of a state space dynamic model, my findings can be summarized as follows. First, a fiscal expansion leads to a permanent increase in government spending as well as the budget deficit. Second, the benchmark model shows that a positive increase in government

spending has a positive and persistent effect on consumption of durables as well as non-durables. In addition, the increase in public spending precedes a significant (but minor) increase in private spending on durables which is quickly reversed in the third quarter post shock. There is a negligibly positive contemporaneous effect on investment in non-durables, from the fiscal shock, which quickly turns negative (but not significant). Third, as expected, the RBC-consistent calibration *mostly* yields different results. The positive fiscal shock leads to a small decrease in consumption of non-durables and a persistently negative drop in private investment in both types of goods. Surprisingly, the result of an increase in durable-consumption following a fiscal expansion is robust to both specifications. This evidence runs counter to the RBC prediction of government spending crowding out private spending. It is worth mentioning that evidence from both calibrations agrees with the standard prediction of the theory (substantiated by other empirical findings) that an increase in government purchases raises hours. However, the increase in hours dedicated to the production of non-durables is much more muted than hours related to durables.

The paper is organized as follows. Section I provides motivation and describes the possible contribution of the study. Section II presents the two-sector, two-consumer model and establishes equilibrium conditions. Section III maps the theoretical model described in section II to a state space model by way of a *complex generalized Schur decomposition* and describes the calibration. Section IV describes the data and outlines a strategy to carry out estimation (in the future) based on a Kalman filtered maximum likelihood procedure. Section V presents results, and section VI concludes.

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I. Introduction

How does fiscal policy, in particular government expenditures, affect aggregate economic activity? What are the mechanisms that affect the transmission of these effects? What are the implications for long-run economic growth? While, *generally*, the common wisdom in the field of macroeconomics anticipates that an increase in government spending will have an expansionary effect on output, there is no such agreement on the implied effect on consumption. Some economists predict a rise in government expenditures precedes a decrease in consumption, while others argue for an increase in consumption after such a shock hits the economy. Most of these empirical studies base their theoretical foundation on a standard one-sector model of the economy and they differ solely on the dynamics of the consumer's behavior. In other areas, particularly economic growth, some researchers have argued that a two-sector model of the economy that distinguishes between productions of durable goods from the rest of the economy provides a better picture of the long-run behavior of the U.S. economy. Given that the dynamics of private spending on durables seem to differ from those of non-durables and services (as illustrated below), one might want to disaggregate the impact of a

government spending shock to consumption into its effect on durables versus non-durables. Taking the philosophy of modeling the economy with two sectors to heart, this study will attempt to measure and allocate what effect an increase in government spending has on consumption and investment while distinguishing between consumption of durables and consumption of non-durables and services. This line of research is, to my knowledge, as yet untried.

Since the 1950s, the canonical model of long-run macroeconomic fluctuations has been that of Solow (later modified to Solow-Ramsey). One of the well-known results of this model is that real aggregates for consumption, investment, output and the capital stock all grow together at an identical rate, referred to as the *balanced growth path*, which is determined by some exogenous technology process. Technology is assumed to affect aggregate output, which in standard growth models as well as real business cycles (RBC) and even New Keynesian (NK) models is derived from a single aggregate production function.

More recently, some researchers [Greenwood, Hercowitz and Krussell (1997), Whelan (2003) and Ireland and Schuh (2006), among others] have argued that the traditional one-sector model of economic

growth actually provides a poor description of the long-run behavior of the U.S. economy. As an alternative, Greenwood, Hercowitz, and Krussell (1997) (henceforth GHK) employed a two-sector model in which technology in the production of durable goods grows faster than in the rest of the economy. They argue that the official price deflators for durable equipment in the U.S. National Income and Product Accounts (NIPAs) understate the relative price in sectoral rates of technological progress. They, therefore, choose not to use NIPA data and opt instead for Gordon's equipment price index (1990). Whelan (2003) argues that the fact that most investment spending is allocated to durable goods while most consumption spending is on nondurable goods and services is enough to reject the balanced growth prediction.[1]

An even stronger balanced growth assumption implies that government expenditures, linked to aggregate output through the aggregate resource constraint, eventually grows at the same steady-state growth rate as other aggregates. RBC proponents typically use a one-sector neoclassical growth framework that includes a consumption-leisure choice as their canonical model. Preferences and

[1] This conclusion no doubt rests on the empirical evidence that private fixed investment has grown faster than real consumption in the 90s.

technology in these models are specified so that the underlying deterministic model is consistent with the "stylized fact" of balanced growth. These models typically assume a representative consumer who maximizes the present discounted value of her utility (from consumption and leisure), subject to some law of motion for the accumulation of the capital stock and some exogenous autoregressive process for aggregate technology. If this technology is well-characterized by an integrated process (unit root) with stationary zero-mean disturbances of the form $\log Z_t = \bar{Z} + \log Z_{t-1} + \varepsilon_t$, then it can easily be shown that C, I, K, G and Y all grow at the same average rate in the long run (King et al 1988). The implication is that while these aggregates may be well-characterized by a unit root process, the ratio of any of the variables will be stationary stochastic processes. The stationarity property of the "great ratios" is, therefore, a corollary of the balanced growth hypothesis. Holding both of these as "stylized facts" dates as far back as Kaldor (1957). More recently, King, Plosser, Stock and Watson (1991) (henceforth KPSW) presented evidence consistent with the balanced growth hypothesis and the stationarity of the great ratios using data up to 1988.

However, in recent years, evidence of the balanced growth seems less clear. Investment in real terms (relative to consumption) grew at an average 8.1% annually between 1991 and 2006, while consumption grew at an average 3.6% per year over the same period (see Fig. 1).[2]

Some factors such as the unusual length of the 1990s boom, the rapid increase in stock market capitalization or the federal budget surpluses might account for this substantive increase in private real investment in the last decade of the 20th century. If this were the case, we would expect the ratio of real investment to real consumption to reset down to its long-run average in the next downturn of the cycle. Tevlin and Whelan (2001) disaggregate the data for this period to show that this increase in real NIPA investment in the 1990s was entirely due to expenditures on producers' durable equipment, which, they report, grew 9.1% per year with real spending on structures growing only 2.2% annually between 1991 and 2000. As Fig. 2A shows, this is a long-term trend from 1960. Additionally, real consumer spending on durable goods has consistently grown faster than real consumption of non-durables and services. Fig. 2B shows that since the 1960s, total real

[2] These growth figures are similar to what Whelan (2003) reports for the 1991-2000 period.

output of durable goods has consistently grown faster than the total real business output (defined as GDP excluding the output of government and nonprofit institutions). Fig. 3 shows that, once expressed in nominal terms, the ratio of fixed investment to consumption has also been trendless throughout the postwar period: 13% of consumption expenditures go to durables, 47% of investment expenditures goes to durables. The implication is that relative price trends for durable goods have sustained a prolonged decline in the postwar period (see Fig. 4). One conclusion that can be drawn from this evidence is that nominal spending on investment and consumption have tended to grow at the same rate, but the higher share of available goods in investment and the declining relative price of these goods suggest that real investment tends to grow faster than real consumption. Similar conclusions can be drawn for government spending. Fig. 5 shows the lack of trend in the ratio of government's (federal+state+local) investment[3] to consumption during the postwar era, with government investment and consumption growing at an annual rate of 7.2% and 7.8%, respectively.

[3] The investment measure includes both state and local gross investment as well as federal non-defense investment. The ratio of defense investment (as well as the state measures) to government consumption seems to have had no trend only since 1971.

There has been a rich tradition in the research area to determine the effects of permanent and temporary fiscal-policy disturbances in macroeconomic aggregates [see Blanchard and Weil (2001) for one such recent study in the context of the neoclassical school of thought]. Given the evidence previously alluded to on the apparent different dynamics of durables vs. non-durables, one might then want to study what impact fiscal policy disturbances have on each sector of the economy. This paper attempts to measure the effects of changes in government purchases on private spending of durables and non-durables. Most macroeconomic models show that an increase in government expenditures tends to have an expansionary effect on output. However, there seems to be no such consensus on the implied fiscal effects on what constitutes the largest component of aggregate demand, namely consumption. This lack of agreement on the fiscal impact on private consumption presents a problem: without a clear answer on what the response of consumption to a change in government spending is, economists are ill-equipped to determine the size of the government spending multiplier.

Some economists propose that an increase in government purchases of goods and services (henceforth government spending)

precedes a decline in consumption, while others predict that consumption should rise, thereby amplifying the impact of increased government spending and aggregate economic activity (as measured by domestic output). One of the reasons for such a dispute arises from the behavior of the consumer. Proponents of the negative impact assume an infinitely lived Ricardian consumer who continually optimizes based on an inter-temporal budget constraint. An increase in government spending reduces the percent value of after-tax income for this consumer who, under the stress of this negative wealth effect, will most likely cut her consumption.

The mechanisms underlying those effects are described in detail in Aiyagari et al (1990), Baxter and King (1993), Christiano and Eichenbaum (1992), and Fatas and Mihov (2001), among others. According to these authors, an increase in government purchases, financed by lump-sum taxes, has a negative wealth effect, which is reflected in lower consumption. In addition, the quantity of labor supply increases at any given wage, which will lead to a lower real wage, higher employment, and higher output. If the resulting increase in employment is persistent enough, it will lead to higher expected return to capital, which could induce an increase in investment.

Proponents of the positive impact assume the consumer behaves in a non-Ricardian way, making her consumption choice based on her current disposable income and not her lifetime resources. For this consumer, the effect of an increase in government spending will depend crucially on how it is financed. According to Blanchard and Weil (2001), the total effect on output will depend on the extent of deficit financing and on the investment response. They explain that under the assumption of a constant money supply, the rise in consumption precedes an increase in interest rates, which will induce a decline in investment. Conversely, if the central bank holds the interest rate steady in the face of the increase in government spending, there is no resulting effect on investment. In fact, any response of the central bank that does not fully adjust for the higher money demand induced by the rise in output will ultimately lead to a decrease in investment.

In order to empirically estimate the effects mentioned above, a theoretical foundation that sufficiently describes the dynamics of government, consumption, investment and aggregate economic activity is needed.

II. A Two-Sector New Keynesian Model with Two Consumer Types

The economy consists of two types of consumers, a continuum of firms producing intermediate goods, a central bank carrying out monetary policy, a government authority in charge of fiscal policy and a set of perfectly competitive firms producing two types of final goods. In this two-sector economy, sector 1 produces durable goods and equipment that can be used by both consumers and producers, while sector 2 produces nondurable goods and services for consumption and structures for investment.

Details of the mechanisms driving this model economy are as follows. Sector 1 supplies C_1 units of durables to households, I_{12} units of capital goods for purchase by sector 2 and provides I_{11} units of capital for itself. Similarly, sector 2 supplies C_2 units of non-durables and services to households, I_{21} units of capital goods for purchase by sector 1, while I_{22} units of capital are kept in sector 2. The production technologies in the two sectors are assumed to be identical and, while future research could allow for different rates of technological progress to take place in each sector, for tractability the rates of technological

growth are each normalized to 1. Omitting time subscripts, the production technologies for each sector are as follows:

$$Y_1 = C_1 + I_1 + G_1 = K_{11}^{\beta_1} K_{21}^{\beta_2} N_1^{1-\beta_1-\beta_2} \tag{0.1}$$

$$Y_2 = C_2 + I_2 + G_2 = K_{12}^{\beta_1} K_{22}^{\beta_2} N_2^{1-\beta_1-\beta_2} \tag{0.2}$$

$$\text{where } I_1 = I_{11} + I_{12} \quad \textit{and} \quad I_2 = I_{21} + I_{22}$$

Equipment and structures are allowed to depreciate at different rates, so capital of both types used in the production of sector 1 accumulates according to:

$$\begin{aligned} K_{11,t+1} &= (1-\delta_1) K_{11,t} + I_{11,t} \\ K_{12,t+1} &= (1-\delta_1) K_{12,t} + I_{12,t} \end{aligned} \tag{0.3}$$

and similarly for sector 2:

$$\begin{aligned} K_{21,t+1} &= (1-\delta_2) K_{21,t} + I_{21,t} \\ K_{22,t+1} &= (1-\delta_2) K_{22,t} + I_{22,t} \end{aligned} \tag{0.4}$$

II.1. ***Households***

There is a continuum of infinitely lived households $i \in [0,1]$ with identical utility functions belonging to either of two types. The main difference between the two types of consumers is that one has access to capital markets, where a full set of contingent securities can be traded, and the other either has no access or chooses not to participate in the market for capital goods. I will refer to the first type as the *Ricardian* consumer. This Ricardian consumer, who continually optimizes based on a budget constraint, trades and accumulates physical capital that she then rents out to firms. The second type does not accumulate any assets nor does he incur any liabilities. He does not optimize his present value of lifetime resources, but fully consumes his current labor income. While this presents an extreme form of non-Ricardian behavior, it does serve to couch the limited participation aspect of this model. Borrowing Gregory Mankiw's (2000) terminology, I designate this consumer as the *rule-of-thumb consumer*. There are myriad reasons for the rule-of-thumb consumer's lack of participation in capital markets: lack of the precautionary saving motive, ignorance of inter-temporal trading opportunities, lack of access to the market, binding borrowing constraints to name a few.

1.1. ***The Ricardian Consumer***

Given that Ricardian consumers are inter-temporal optimizers, let C_t^o and N_t^o represent the consumption and labor for these Ricardian optimizing households. This household's objective is to maximize the present value of the expected utility $U(C_t^o, N_t^o)$ discounted by the factor $\frac{1}{1+\rho} \in (0,1)$. This consumer wants to maximize her leisure as well as her consumption of both durables and non-durables and services. Therefore, she keeps a stock of durables (DU) that evolves according to $DU_t = (1-\delta_U)DU_{t-1} + C_{1t}^{\ o}$.

This household type will, therefore, seek to maximize

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t U\left(C_t^o, N_t^o\right) \tag{1.1}$$

subject to the sequence of budget constraints

$$-T_t^o + R_t^k K_t^o + B_t^o + D_t^o + W_t N_t^o \geq \frac{1}{R_t} B^o{}_{t+1} + P_{1t}(C_{1t}^{\ o} + I_{1t}^{\ o}) + P_{2t}(C_{2t}^{\ o} + I_{2t}^{\ o}) \tag{1.2}$$

the accumulation equation for the stock of durables

$$DU_t = (1-\delta_U)DU_{t-1} + C_{1t}^{\ o} \tag{1.3}$$

and the capital accumulation equation for each sector given by (0.3) and (0.4).

At the beginning of the period (t) the consumer receives her labor income $W_t N_t^o$, where W_t represents the real wage and N_t^o the hours of work. B_t^o is her endowment of nominally risk-less one-period bonds carried over from last period, which pays one unit of the numeraire in the current period. R_t is the nominal return on bonds purchased in the current period. She receives dividends (D_t^o) from ownership of firms. In addition, she receives income from renting her stock of capital (K_t^o) to firms, which earns the real rental cost (R_t^k). Finally she pays taxes (or receives a transfer payment if negative) in the form of a lump-sum (T_t^o). Her expenditures go to consumption (C_t^o) and investment (I_t^o) goods in each sector (in real terms). P_{1t} and P_{2t} denote prices of final goods for each sector.

I assume the following utility function for the Ricardian households

$$U\left(C_t^o, N_t^o\right) \equiv \alpha_1 \log\left(DU_t\right) + \alpha_2 \log\left(C_{2t}^{\ o}\right) - \frac{N_{1t}^{o\ 1+\varphi}}{1+\varphi} - \frac{N_{2t}^{o\ 1+\varphi}}{1+\varphi}$$

where $\alpha_1, \alpha_2, \varphi \geq 0$.

The Euler equations that result from the optimizing consumer's problem in each sector can be written as:

$$1 = R_t \frac{1}{1+\rho} E_t \left[\frac{C_{1,t}^{\ o}}{C_{1,t+1}^{\ o}} \frac{1}{\pi_{t+1}} \right] \tag{1.4}$$

for sector 1 where $\pi_{t+1} = \frac{P_{1,t+1}}{P_{1,t}}$. Similarly for sector 2:

$$1 = R_t \frac{1}{1+\rho} E_t \left[\frac{C_{2,t}^{\ o}}{C_{2,t+1}^{\ o}} \frac{P_{2,t}}{P_{2,t+1}} \right] \tag{1.5}$$

In a competitive labor market, each household chooses the hours of labor they supply to the market for a given wage. Therefore, the following optimality conditions are derived for sector 1:

$$\alpha_1 \frac{W_t}{P_{1,t}} = C_{1,t}^{\ o} N^{o}_{\ 1,t}{}^{\varphi} \tag{1.6}$$

and similarly for sector 2:

$$\alpha_2 \frac{W_t}{P_{2,t}} = C_{2,t}^{\ o} N^{o}_{\ 2,t}{}^{\varphi} \tag{1.7}$$

where the α_s represent a preference shifter for each sector (durables and non-durables).

1.2. ***Linearized Conditions for the Optimizing Consumer***

The log-linearized Euler equation (1.4) for the optimizing household in sector 1 is given by

$$c_{1,t}{}^{o} = E_t c_{1,t+1}{}^{o} - \left(r_t - E_t \pi_{t+1}\right) + \ln\left(1+\rho\right) \tag{1.8}$$

where all lowercase variables represent the natural log of the aggregate variables in equation (1.4). Similarly a logarithmic transformation of Euler equation (1.5) is given by

$$c_{2,t}{}^{o} = E_t c_{2,t+1}{}^{o} - \left(r_t - E_t \pi_{t+1}\right) + \ln\left(1+\rho\right) \tag{1.9}$$

where the same above nomenclature rules apply.

Under perfectly competitive markets we can log-linearize the optimality condition (1.6) for the optimizing consumers in sector 1 to yield the following

$$(w_t - p_{1,t}) = c_{1,t}{}^{o} + \varphi n_{1,t}{}^{o}{}_{4} \tag{1.10}$$

and similarly for sector 2

$$(w_t - p_{2,t}) = c_{2,t}{}^{o} + \varphi n_{2,t}{}^{o} \tag{1.11}$$

The logarithm transformations of the Euler equations and the optimality conditions as given by equations (1.8)-(1.11) will be used

[4] Constants, such as α, drop in the steady state.

later in the development of the general equilibrium model for each sector.

1.3. *The Rule-of-Thumb Consumer*

As mentioned above, rule-of-thumb consumers behave in a (extreme form of) non-Ricardian fashion. These consumers, therefore, do not optimize inter-temporally. Let C_t^r and N_t^r represent the consumption and labor for these household types. I assume these rule-of-thumb consumers do not save, do not attempt to own firms and, therefore, earn no dividends. Additionally, they do not own any stock of capital and, therefore, do not rent out capital goods to firms, and, finally, do not lend out monetary assets and do not trade bonds. Given that they do not utilize any of the instruments typically used to smooth consumption, they make no attempts to stave off uncertainty in the fluctuation of their labor income, nor do they inter-temporally substitute to adjust to variation in the interest rate. I take no stand on the reasons for this behavior beyond enumerating the examples I mentioned earlier. This household type will, therefore, seek to maximize his period utility given by:

$$U(C_t^r, N_t^r) \tag{1.12}$$

by consuming all his disposable income net of taxes (which are allowed to differ from taxes paid by the optimizing consumer) as follows:

$$W_t(N_{1,t}{}^r + N_{2,t}{}^r) - T_t^r \geq P_{1,t}C_{1,t}{}^r + P_{2,t}C_{2,t}{}^r \tag{1.13}$$

As with the Ricardian consumer, the stock of durables evolves according to the following constraint.

$$DU_t = (1-\delta_U)DU_{t-1} + C_{1t}{}^r \tag{1.14}$$

Combining first order conditions derived from the rule-of-thumb consumer problem yields the familiar requirement associated with the competitive labor market assumption:

$$C_{1,t}{}^r(N_{1,t}{}^r)^{\varphi} = \frac{W_t}{P_{1,t}}\alpha_1 \tag{1.15}$$

Namely, that the marginal rate of technical substitution between leisure and consumption in sector 1 equals the relative real wage in that sector. Similarly, for sector 2:

$$C_{2,t}{}^r(N_{2,t}{}^r)^{\varphi} = \frac{W_t}{P_{2,t}}\alpha_2 \tag{1.16}$$

1.4. ***Linearized Equilibria for the Rule-of-Thumb Consumer***

Recall the general budget constraint facing the rule-of-thumb consumer given by equation (1.13). This general budget constraint affects the rule-of-thumb consumer regardless of what sector he purchases from or supplies to. The implication is that (1.13) reflects the combination of budgets under each sector and, therefore, a budget for either sector may be extrapolated. Thus, the budget constraint for the rule-of-thumb consumer in sector 1 is given by

$$P_{1,t}C_{1,t}^{\ r} = W_t P_{1,t} N_{1,t}^{\ r} - P_{1,t}T_t^r$$

or alternatively

$$C_{1,t}^{\ r} = W_t N_{1,t}^{\ r} - T_t^r \tag{1.17}$$

Given a particular steady state value for each of the aggregate variables present in the constraint (1.17) can be rewritten as follows:

$$C_{1,t}^{\ r}\left(\frac{C_1^r}{WN_1^r}\right) = \left(\frac{W_t}{W}\frac{N_{1,t}^{\ r}}{N_1^r}\right)C_1^r - \left(\frac{C_1^r}{WN_1^r}\right)T_t^r$$

where each variable missing a time subscript represents the steady state value of the corresponding aggregate (with the subscript). Dividing this equation through by the steady state personal

consumption in period 1 yields a first order approximation[5] of the rule-of-thumb household's level of consumption for sector 1 as given by

$$c_{1,t}{}^{r} = \frac{WN_1^{r}}{C_1^{r}}\left[w_t + n_{1,t}{}^{r} \right] - \left(\frac{Y}{C_1^{r}} \right) t_t^{r} \tag{1.18}$$

with consumption, labor and the real wage are reflected in log deviations from their respective steady state

$$c_{1,t} \equiv \log\left(\frac{C_{1,t}{}^{r}}{C_1^{r}} \right), \quad w_t \equiv \log\left(\frac{W_t}{W} \right), \quad n_{1,t}{}^{r} \equiv \log\left(\frac{N_{1,t}{}^{r}}{N_1^{r}} \right)$$

and the deviation of lump-sum taxes as a share of steady state output

$$t_t^{r} \equiv \left(\frac{T_t^{r} - T^{r}}{Y} \right).$$

By similar analysis (see Appendix I), a first order approximation of the rule-of-thumb consumer for sector 2 is given by

$$c_{2,t}{}^{r} = \frac{WN_2{}^{r}}{C_2^{r}}\left[w_t + n_{2,t}{}^{r} \right] - \left(\frac{Y}{C_2^{r}} \right) t_t^{r} \tag{1.19}$$

1.5. *Market Clearing and Aggregation*

As described later, the model includes a continuum of intermediate (j) firms supplying goods and services to final firms in either sector. Labor services hired by firm j ($N_{xt}(j)$ in either x sector) can be imputed to some measure of aggregate labor demand for each

[5] Around its steady state.

sector ($N_{x,t}$) in the usual way. Similarly, capital employed by each firm (j) can be aggregated to some measure of capital used to produce final goods in either sector.[6] Therefore, clearing of good and factor markets implies that the following conditions must hold for any period t in any given sector:

$$N_{1,t} = \int_0^1 N_{1,t}(j)dj$$

$$N_{2,t} = \int_0^1 N_{2,t}(j)dj$$

$$K_{1,t} = \int_0^1 K_{1,t}(j)dj \,^{7}$$

$$K_{2,t} = \int_0^1 K_{2,t}(j)dj \,^{8}$$

and

$$Y_{1,t} + Y_{2,t} = Y_t = C_t + I_t + G_t \tag{2.1}$$

where

$C_{1,t} + C_{2,t} = C_t$

$I_{11,t} + I_{12,t} + I_{21,t} + I_{22,t} = I_t$

$mG_t + (1-m)G_t = G_t$

with *m* representing the share of government spending on durables.

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[6] In the interest of clarity, the j subscripts are omitted without loss of generality throughout the rest of the paper.

[7] Where $K_{1,t}=K_{11,t}+K_{12,t}$

[8] Where $K_{2,t}=K_{21,t}+K_{22,t}$

Aggregate consumption, labor, investment and capital in each sector are given by a convex combination of the corresponding variables for each consumer type.

In sector 1:

$$C_{1,t} \equiv \lambda C_{1,t}^{\ r} + (1-\lambda) C_{1,t}^{\ o} \tag{2.2}$$

$$N_{1,t} \equiv \lambda N_{1,t}^{\ r} + (1-\lambda) N_{1,t}^{\ o} \tag{2.3}$$

and

$$I_{1,t} \equiv I_{1,t}^{\ o}$$
$$K_{1,t} \equiv K_{1,t}^{\ o}$$

Similarly for sector 2:

$$C_{2,t} \equiv \lambda C_{2,t}^{\ r} + (1-\lambda) C_{2,t}^{\ o} \tag{2.4}$$

$$N_{2,t} \equiv \lambda N_{2,t}^{\ r} + (1-\lambda) N_{2,t}^{\ o} \tag{2.5}$$

and

$$I_{2,t} \equiv I_{2,t}^{\ o}$$
$$K_{2,t} \equiv K_{2,t}^{\ o}$$

1.6. ***General Equilibrium Conditions for Households***

Under perfectly competitive markets, we can log-linearize the optimal condition for the rule-of-thumb consumer given by equation (1.15) for sector 1 and (1.16) for sector 2. A logarithmic transformation of (1.15) obtained from the first order conditions derived from the rule-of-thumb household problem shows

$$\left(w_t - p_{1,t}\right) = c_{1,t}{}^{r} + \varphi n_{1,t}{}^{r} \tag{3.1}$$

The marginal rate of technical substitution between leisure and consumption in sector 1 equals the relative real wage in that sector for that consumer type. Similarly, a log transformation of equation (1.16) for sector 2 yields the same relationship between the MRS and the real wage for the rule-of-thumb consumer in that sector, as follows:

$$\left(w_t - p_{2,t}\right) = c_{2,t}{}^{r} + \varphi n_{2,t}{}^{r} \tag{3.2}$$

Focusing first on sector 1, we can substitute the log linear conditions for the rule-of-thumb consumer given by (3.1) and the optimizing consumer given by (1.10) into our aggregation assumption (2.2) to yield the following:

$$c_{1,t} = \lambda\left[\left(w_t - p_{1,t}\right) - \varphi n_{1,t}{}^{r}\right] + \left(1-\lambda\right)\left[\left(w_t - p_{1,t}\right) - \varphi n_{1,t}{}^{o}\right]$$

or alternatively:

$$c_{1,t} = (\lambda + 1 - \lambda)\left[\left(w_t - p_{1,t}\right)\right] - \varphi\left[\lambda n_{1,t}{}^{r} + (1-\lambda) n_{1,t}{}^{o}\right].$$

Noting that the second term in brackets in the above equation is the aggregate labor as shown by equation (2.3), we arrive at the following equilibrium equation for sector 1:

$$\left(w_t - p_{1,t}\right) = c_{1,t} + \varphi n_{1,t} \tag{3.3}$$

The MRS between leisure and consumption in sector 1 is exactly offset by the relative real wage in that sector *regardless* of consumer type. A similar (but omitted) analysis shows that this condition

$$\left(w_t - p_{2,t}\right) = c_{2,t} + \varphi n_{2,t} \tag{3.4}$$

holds for sector 2 as well.

The goal of this subsection is to derive two theoretical Euler equations for consumption of sector 1 and sector 2 goods that might then have the capability to be cast into a state space model. To that end, I make use of the linearized equilibrium conditions and aggregation assumptions derived in section I.1 of this study.

I begin next with sector 1. Solving equation (3.3) for the real wage[9] and equation (3.1) for labor supply of the rule-of-thumb household in sector 1 and then combining them with the first order

[9] Allowing $w_t - p_{1,t} = w_t - p_{2,t} = w_t$ to be the real wage.

approximation of the rule-of-thumb household's level of steady state consumption for sector 1 as given by (1.18) yields:

$$c_{1,t}{}^{r} = \frac{WN_1{}^{r}}{C_1{}^{r}}\left[\left(\frac{1}{\varphi}+\frac{1}{\alpha_1}\right)c_{1,t}+\left(1+\frac{\varphi}{\alpha_1}\right)n_{1,t}-\frac{1}{\varphi}c_{1,t}{}^{r}\right]-\left(\frac{Y}{C_1{}^{r}}\right)t_t{}^{r} \qquad (3.5)$$

The focus of this study is on the response of aggregates to shocks and not on the difference in steady states across households. I, therefore, take the simplifying assumption that *steady state consumption* across household types is the same within each sector (i.e. $C_1{}^{r}=C_1{}^{o}=C_1$ and $C_2{}^{r}=C_2{}^{o}=C_2$). Consequently, this assumption for same steady state consumption across household types has a natural corollary (see Appendix I) in labor (i.e. $N_1{}^{r}=N_1{}^{o}=N_1$ and $N_2{}^{r}=N_2{}^{o}=N_2$). Letting $N_1{}^{r}\equiv N_1$, $C_1{}^{r}\equiv C_1$, and multiplying both sides of equation (3.5)by φ yields:

$$\varphi c_{1,t}{}^{r} = \left(\frac{WN_1}{Y_1}\frac{Y_1}{C_1}\right)\left[\left(1+\frac{\varphi}{\alpha_1}\right)c_{1,t}+\left(1+\frac{\varphi}{\alpha_1}\right)n_{1,t}-c_{1,t}{}^{r}\right]-\left(\frac{Y}{C_1{}^{r}}\right)\varphi t_t{}^{r}$$

After some manipulation it can be rewritten as:

$$\left(1+\hat{\mu}_p\gamma_c\varphi\right)c_{1,t}{}^{r} = \tilde{\alpha}_1 c_{1,t}+\tilde{\alpha}_1\varphi n_{1,t}-\hat{\mu}_p\varphi t_t{}^{r} \qquad (3.6)$$

where:

$$\mu_t{}^{p}\equiv\frac{Y_t}{W_tN_t} \quad or\ in\ ss \quad \hat{\mu}^{p}\equiv\frac{Y}{WN} \quad \tilde{\alpha}_1\equiv 1+\frac{\varphi}{\alpha_1}$$

and the consumption share of income in steady state is given by $\frac{C}{Y} \equiv \gamma_c$.

I will use equation (3.6) to ultimately derive an Euler equation for sector 1 that can be mapped into a dynamic state-space model (see Section II).

An analogous equation to (3.6) for sector 2 is given by

$$\left(1+\hat{\mu}_p \gamma_c \varphi\right) c_{2,t}{}^r = \tilde{\alpha}_2 c_{2,t} + \tilde{\alpha}_2 \varphi n_{1,t} - \hat{\mu}_p \varphi t_t{}^r \tag{3.7}$$

where $\tilde{\alpha}_2 \equiv 1 + \frac{\varphi}{\alpha_2}$ and the rest of the above nomenclature applies.

II.2. ***Producers***

As mentioned earlier, I assume an infinitely large subset of monopolistically competitive firms producing intermediate goods that are then used as inputs by firms producing final goods in either sector 1 (final durable goods) or sector 2 (final non-durables and services).

2.1. ***The Intermediate Goods Producer***

A given producer of intermediate goods will use labor as well as all types of capital available as inputs in order to supply to either sector but not both. Within the continuum of intermediate level goods and given the heterogeneity of raw materials, production processes, etc. that go into the assembly of a final good, it is as easy to envision intermediate firms that might supply to one sector as it is to imagine firms catering to both producers of durables and non-durables. Therefore, this constitutes a strong simplifying (but necessary) assumption for the analysis that follows. The production function for a typical firm producing intermediate good i to be used as an input by a final producer in, say, sector 1 is given by:

$$Y_{1,t}(i) = (K_{11,t}(i))^{\beta_1} (K_{21,t}(i))^{\beta_2} (N_{1,t}(i))^{1-\beta_1-\beta_2} \tag{4.1}$$

where $N_{1,t}(i)$ represents the labor services hired by firm i, while said firm employs $K_{11,t}(i)$ durable goods and $K_{21,t}(i)$ non-durables and services to supply $Y_{1,t}(i)$ intermediate goods to a final firm producing in sector 1.

Similarly, the technology supplied by a firm producing intermediate good j for sale to a final producer in sector 2 is given by:

$$Y_{2,t}(j) = (K_{12,t}(j))^{\beta_1} (K_{22,t}(j))^{\beta_2} (N_{2,t}(j))^{1-\beta_1-\beta_2} \tag{4.2}$$

where $N_{2,t}(j)$ represents the labor services hired by firm j, while said firm employs $K_{12,t}(i)$ durable goods and $K_{22,t}(i)$ non-durables and services to supply $Y_{2,t}(i)$ intermediate goods to a final firm producing in sector 2.

The model assumes no wage differential between the two sectors. Taking the real wage as given, I find the cost minimization condition for durables in sector 1 to be:

$$\frac{K_{11,t}}{N_{1,t}} = \left(\frac{\beta_1}{1-\beta_1-\beta_2} \right) W_t \tag{4.3}$$

Similarly the cost minimizing condition for non-durables in the same sector is:

$$\frac{K_{21,t}}{N_{1,t}} = \left(\frac{\beta_1}{1-\beta_1-\beta_2} \right) W_t \tag{4.4}$$

Given that identical technologies were assumed for both sectors, the cost minimization conditions for sector 2 analogous to (4.3) and (4.4) are:

$$\frac{K_{12,t}}{N_{2,t}} = \left(\frac{\beta_1}{1-\beta_1-\beta_2} \right) W_t \text{ and } \frac{K_{22,t}}{N_{2,t}} = \left(\frac{\beta_1}{1-\beta_1-\beta_2} \right) W_t \tag{4.5}$$

The marginal product of labor for firms catering to sector 1 is:

$$MPN_1 = (1-\beta_1-\beta_2) \left[\left(\frac{\beta_1}{1-\beta_1-\beta_2} \right)^{\beta_1} \left(\frac{\beta_2}{1-\beta_1-\beta_2} \right)^{\beta_2} W_t^{\beta_1+\beta_2} \right]$$

which, by construction, is identical to the marginal product of labor (MPN_2) of those firms supplying sector 2. This implies that the real marginal cost is common across *all* intermediate firms and given by the following:

$$MC_t = \Xi (W_t)^{1-\beta_1-\beta_2} \tag{4.6}$$

where $\frac{1}{\Xi} \equiv \beta_1^{\beta_1} \beta_2^{\beta_2} (1-\beta_1-\beta_2)^{1-\beta_1-\beta_2}$.

Intermediate good firms in both sectors are assumed to follow Calvo's (1983) staggered price setting model. According to this model, a fraction (η) of these firms leave their prices unchanged in the current

period. This fraction (η) of unchanged prices at period t is a subset of those prices that changed in period t-1, and they have the same relative frequency as in the previous period price distribution. For this last argument to hold, it is required that each price have an equal probability of being adjusted in a given period. Each firm that chooses a new price for its good in period t faces the same decision problem so the optimal price P_t^* is shared by all intermediate firms. The probability that any price will be adjusted in period t is, therefore, $(1-\eta)$ which is independent of the length of time the price has remained unchanged or what the current price of the good is. The aggregate price level P_t evolves according to a convex combination of the lagged price level (unchanged price level) P_{t-1} and the recently reset optimal price P_t^* selected by those firms that are able to change price at time t. The Calvo rule describing the evolution of the aggregate price level is as follows:

$$P_t = \eta P_{t-1} + (1-\eta) P_t^* \tag{4.7}$$

The intermediate goods firm resetting its price in period t will maximize

$$\underset{P_t^*}{Max}\, E_t \sum_{k=0}^{\infty} \eta^k E_t \left\{ \left(\frac{1}{1+\rho} \right)^k \left(\frac{C_t}{C_{t+k}} \right) Y_{t+k}(i) \left(\left(\frac{P_t^*}{P_{t+k}} \right) - MC_{t+k} \right) \right\} \tag{4.8}$$

subject to a sequence of demand constraints $Y_{t+k}(i) = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\theta_t} Y_{t+k}$ where P_t^* is the optimal price chosen by those firms that reset their prices at time t.

The first order condition associated with (4.8) is given by:

$$\sum_{k=0}^{\infty} \eta^k E_t \left\{ \left(\frac{1}{1+\rho} \right)^k \left(\frac{C_t}{C_{t+k}} \right) Y_{t+k}(i) \left[\left(\frac{P_t^*}{P_{t+k}} \right) - \mu^P MC_{t+k} \right] \right\} = 0$$

where $\mu^P \equiv \frac{\theta_t}{\theta_t - 1}$ is the gross price markup prevailing in a zero inflation steady state in the absence of frictions.

Combining the Dixit-Stiglitz (1977) price index

$$P_t^{1-\theta_t} = \int_0^1 P_t(i)^{1-\theta_t} di \tag{4.9}$$

with the Calvo price setting rule given by (4.7) yields the equation that describes the aggregate price level as:

$$P_t = \left(\eta P_{t-1}^{\,1-\theta_t} + (1-\eta)(P_t^*)^{1-\theta_t} \right)^{\frac{1}{1-\theta_t}} \tag{4.10}$$

This aggregate price level for the intermediate goods-producing firms is obtained regardless of choice of sector.

2.2. ***Linearized Conditions for the Intermediate Firm***

Given the aggregation assumptions of section I.1.3, the representative intermediate goods-producing firms in sector 1 behaves according to the following production technology

$$Y_{1,t} = K_{11,t}^{\beta_1} K_{21,t}^{\beta_2} N_{1,t}^{1-\beta_1-\beta_2} \tag{4.11}$$

Market clearance for sector 1 implies

$$Y_{1,t} = C_{1,t} + I_{11,t} + I_{12,t} + mG_t$$

where m indicates the share of total government expenditures allocated to durable goods. Substituting sector 1's cost minimization conditions (4.3) and (4.4) into the production technology given by (4.11) yields the following equation expressed in logs

$$y_{1,t} - n_{1,t} = (\beta_1 + \beta_1) w_t$$ [10]

Substituting this expression into the equilibrium condition for the sector 1 consumer given by (3.3) yields

$$y_{1,t} = \left(\frac{\beta_1 + \beta_2}{\alpha_1} \right) c_{1,t} + \left(1 + \frac{(\beta_1 + \beta_2)\varphi}{\alpha_1} \right) n_{1,t}$$

and combining this with the steady state resource constraint for that sector yields

10 Each variable is in logs of the corresponding aggregate; w_t is the log of the real wage, and the constant is omitted without loss of generality.

$$mg_t + \gamma_c c_{1,t} + \gamma_i i_{1,t} = \left(\frac{\beta_1 + \beta_2}{\alpha_1}\right) c_{1,t} + \left(1 + \frac{(\beta_1 + \beta_2)\varphi}{\alpha_1}\right) n_{1,t} \qquad (4.12)$$

where γ_c and γ_i are the consumption shares and investment shares of output respectively.

Recalling that capital accumulation in sector 1 is given by the following two laws of motion:

$$K_{11,t+1} = (1-\delta_1)K_{11,t} + I_{11,t}$$
$$K_{12,t+1} = (1-\delta_1)K_{12,t} + I_{12,t}$$

Taking a log transformation of these accumulation equations along with the assumption derived from the resource constraint --namely, that $I_{11,t} + I_{12,t} = I_{1,t}$ yields the following expression for the level of investment in sector 1:

$$i_{1,t} = \frac{1}{\delta_1}\left(k_{11,t+1} + k_{21,t+1}\right) - \left(\frac{1-\delta_1}{\delta_1}\right)\left(k_{11,t} + k_{21,t}\right) \qquad (4.13)$$

We can combine (4.12) and (4.13), given that the level of the capital stock for sector 1 is determined by the units of capital produced for purchase by sector 2 plus the units of capital sector 1 keeps for itself (i.e. $k_{11} + k_{21} = k_1$ in any given period), to yield an expression for the level of capital stock in sector 1

$$k_{1,t+1} = (1-\delta_1)k_{1,t} + \left(\frac{\alpha_1 + (\beta_1+\beta_2)\varphi}{\alpha_1}\right)\left(\frac{\delta_1}{\gamma_i}\right)n_{1,t} - \left(\gamma_c + \frac{(\beta_1+\beta_2)}{\alpha_1}\right)\left(\frac{\delta_1}{\gamma_i}\right)c_{1,t} - \left(\frac{\delta_1}{\gamma_i}\right)mg_t \tag{4.14}$$

By similar (but omitted) analysis, we arrive at a capital accumulation condition analogous to (4.14) for sector 2

$$k_{2,t+1} = (1-\delta_2)k_{2,t} + \left(\frac{\alpha_2 + (\beta_1+\beta_2)\varphi}{\alpha_2}\right)\left(\frac{\delta_2}{\gamma_i}\right)n_{2,t} - \left(\gamma_c + \frac{(\beta_1+\beta_2)}{\alpha_2}\right)\left(\frac{\delta_2}{\gamma_i}\right)c_{2,t} - \left(\frac{\delta_2}{\gamma_i}\right)(1-m)\,g_t \tag{4.15}$$

We now turn to the description of what the Phillips curve looks like in the context of this model. In order to arrive at a closed form of the Phillips curve, a measure of the marginal cost for each sector is needed along with the assumption that a generalized marginal cost can be derived from aggregating the marginal costs for both sectors.

Cost minimization in sector 1 based on a production technology of the form (4.11) implies that the marginal cost for sector 1 is given by (4.6). Combining a version of the consumer optimality condition given by (3.3) with equation (4.6) yields the following marginal cost function (in log-linear terms) for sector 1:

$$mc_t = \left(\frac{\beta_1}{1-\beta_1-\beta_2}\right)^{-\beta_1}\left(\frac{\beta_2}{1-\beta_1-\beta_2}\right)^{-\beta_2}\left[c_{1,t} + \varphi n_{1,t} - \log\alpha_1\right] \tag{4.16}$$

Analogously, the marginal cost for sector 2 is given by

$$mc_t = \left(\frac{\beta_1}{1-\beta_1-\beta_2}\right)^{-\beta_1}\left(\frac{\beta_2}{1-\beta_1-\beta_2}\right)^{-\beta_2}\left[c_{2,t}+\varphi n_{2,t}-\log\alpha_2\right] \qquad (4.17)$$

Summing up both marginal costs yields the following expression for the generalized marginal cost of the economy:

$$mc_t = \left(\frac{\beta_1}{1-\beta_1-\beta_2}\right)^{-\beta_1}\left(\frac{\beta_2}{1-\beta_1-\beta_2}\right)^{-\beta_2}\left[c_{1,t}+c_{2,t}+\varphi\left(n_{1,t}+n_{2,t}\right)-\log\left(\alpha_1+\alpha_2\right)\right]$$

or more succinctly by letting $\frac{1}{\tilde{\Xi}} \equiv \beta_1^{\beta_1}\beta_2^{\beta_2}\left(1-\beta_1-\beta_2\right)^{1-\beta_1-\beta_2}$

$$mc_t = \tilde{\Xi}c_{1,t}+\tilde{\Xi}c_{2,t}+\tilde{\Xi}\varphi n_{1,t}+\tilde{\Xi}\varphi n_{2,t} \qquad (4.18)$$

This form of the marginal cost will be used later in the development of a general equilibrium Phillips curve.

2.3. *The Final Goods Producer*

The final good is produced by two representative firms operating according to identical constant returns to scale (CRS) technologies under perfect competition. The production technology is given by the following constant elasticity of substitution (CES) aggregator function Y_t:

$$Y_t \leq \left[\int_0^1 \left(Y_t(i) \right)^{\frac{\theta_t - 1}{\theta_t}} di \right]^{\frac{\theta_t}{\theta_t - 1}} \tag{4.19}$$

where $Y_t(i)$ describes the amount of intermediate good i used as an input by the final goods firm in either sector. The parameter θ_t measures the time elasticity of demand for each intermediate good, and thus it constitutes a markup. Some researchers include it as a cost push shock in the model's Phillips curve relationship [Ireland (2002)].

The finished goods-producing firm maximizes its profits by choosing the following set of demand schedules:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\theta_t} Y_t \tag{4.20}$$

Given that the final goods firms operate under perfect competition, the final goods price P_t as well the prices for intermediate goods $P_t(i)$ are determined by the market and, therefore, taken as given. Competition drives the profits of the finished good-producing firms in each sector to zero. Combining (4.19) and (4.20) and rearranging yields the equilibrium zero-profit condition that determines the aggregate price level P_t to follow:

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t} \right]^{\frac{1}{1-\theta_t}} \tag{4.21}$$

the well-known Dixit-Stiglitz price index (1977).

As with the intermediate goods-producing firm, the final firms are assumed to be identical ex-ante across sectors with the exception of the differentiated good they produce, which again can be sold as a durable good (sector 1) or as a non-durable/service (sector 2). Each final firm faces a conventional constant price elasticity of demand for its product. Each final firm is a price taker within the context of a perfectly competitive market. Whatever the prevailing level of price stickiness is, therefore, inherited from the monopolistic competitive environment of the intermediate firms. The intermediate firm was assumed to follow the Calvo staggered price setting model (1983). For any given period t, a fraction (η) of final producers leave their prices unchanged from the previous period. This fraction (η) of unchanged prices (at period t) is a subset of those prices that were changed in period t-1 and have the same relative frequency as in the previous period (t-1) price distribution. Recall that each intermediate firm's price has an equal probability of being adjusted in a given period. Each intermediate firm that chooses a new price for its good in period t faces exactly the same decision problem, thus the optimal price (P_t^*) is the same for all intermediate, and thus, final firms.

The probability that any price by the intermediate firm will be adjusted in period t is, therefore, $(1-\eta)$ which is independent of the length of time the price has remained unchanged or what the current price of the good is. The aggregate price level P_t evolves according to a convex combination of the lagged price level (unchanged price level) P_{t-1} and the recently reset optimal price P_t^* selected by those firms that are able to change price at time t. Therefore, the Calvo rule describing the evolution of the aggregate price level for the final firm is identical to that of the intermediate level firm and given by (4.7) since it is inherited. Combining the aggregate price level rule for the final firm given by (4.7) and the Dixit-Stiglitz price index given by (4.9) yields the same aggregate price level rule that was obtained for the intermediate goods firm given by (4.10).[11]

For an intermediate goods-producing firm that chooses a new price at time t to maximize expected discounted profits subject to the time-dependent pricing rules given by the Calvo staggered pricing formulation, the optimal reset price may be expressed as:

[11] This aggregate price level rule is identical for each of the final goods-producing firm operating in each sector.

$$P_t^* = \left(1 - \frac{1}{1+\rho}\eta\right)\sum_{k=0}^{\infty}\left(\frac{1}{1+\rho}\eta\right)^k E_t\left\{mc_{t+k}{}^n\right\} \tag{4.22}$$

where mc_t^n is the nominal marginal cost for the final goods-producing firm at time t[12] and $\frac{1}{1+\rho}$ denotes the subjective discount factor. This optimal reset price implicitly encompasses the case of perfect flexibility $(\eta_t = 0)$ where the firm adjusts its price proportionately to movements in its current marginal cost, as well as the case of price rigidities $(\eta_t > 0)$ where the future becomes relevant. By combining (4.7), (4.9) and (4.22), it is possible to derive an inflation equation of the following form:

$$\pi_t = \xi mc_t + \frac{1}{1+\rho}E_t\left\{\pi_{t+1}\right\} \tag{4.23}$$

Equation (4.22) is derived from the solutions to the Calvo price staggering maximization problem for which a first order condition is given by:

$$P_t^*\sum_{T=k}^{\infty}\left(\frac{1}{1+\rho}\eta\right)^{T-k} E_t\left\{\log P_t^* - \log P_t - mc_{t+k}{}^n\right\} = 0$$

which after some manipulation, yields[13]:

$$P_t^* = \left(\frac{1}{1+\rho}\eta\right)\sum_{T=k}^{\infty}\left(\frac{1}{1+\rho}\eta\right)^{T-k} E_t\left\{\sum_{s=k+1}^{T}\pi_S + mc_{t+k}{}^n\right\}$$

[12] In percentage deviation from steady state.

[13] Solving for $^{\log P_t}$, subtracting $^{\log P_t}$ from both sides and arranging terms.

or written another way:

$$\hat{P}_t^* = \sum_{T=k}^{\infty} \left(\frac{1}{1+\rho} \eta \right)^{T-k} E_t \left\{ \left(\frac{1}{1+\rho} \eta \right) \pi_{t+1} + \left(1 - \frac{1}{1+\rho} \eta \right) mc_{t+k}{}^n \right\} \qquad (4.24)$$

where $\hat{P}_t^* = \log\left(\frac{P_t^*}{P_t} \right)$ is the relative price chosen by those firms who revise their prices in period t. Woodford (2003) proves that (4.24) implies:

$$\hat{P}_t^* = \sum_{T=k}^{\infty} E_t \left\{ \left(\frac{1}{1+\rho} \eta \right) \pi_{t+1} + \left(1 - \frac{1}{1+\rho} \eta \right) mc_{t+k}{}^n \right\} + \left(\frac{1}{1+\rho} \eta \right) E_t \hat{P}_{t+1}{}^* \qquad (4.25)$$

A log transformation of the Calvo price setting rule given by (4.7) will yield:

$$\pi_t = \left(\frac{1-\eta}{\eta} \right) \hat{p}_t^* \qquad (4.26)$$

where $\hat{p}_t^* \equiv \log \frac{P_t^*}{P_t}$. Finally, substituting (4.26) into (4.25)[14] yields:

$$\pi_t = \frac{\left(1 - \frac{1}{1+\rho} \eta \right)(1-\eta)}{\eta} mc_t + \frac{1}{1+\rho} E_t \{ \pi_{t+1} \} \qquad (4.27)$$

a version of the standard New Keynesian Phillips curve given by (4.23)

[14] At K=0

where

$$\frac{(1-\eta)\left(1-\frac{1}{1+\rho}\eta\right)}{\eta} \equiv \xi$$

2.4. ***General Equilibrium Conditions for Producers***

Under perfectly competitive markets, we arrived at an expression for the marginal cost in a two-sector economy that looks like (4.18). In the previous section, we have derived a version of the New Keynesian Phillips curve that looks like equation (4.27). Combining this NK Phillips curve with the measure of marginal cost given by (4.18) yields the following general equilibrium condition:

$$\pi_t = \frac{1}{1+\rho} E_t\{\pi_{t+1}\} + \xi\left[\tilde{\Xi} c_{1,t} + \tilde{\Xi} c_{2,t} + \tilde{\Xi}\varphi n_{1,t} + \tilde{\Xi}\varphi n_{2,t}\right] \tag{5.1}$$

where

$$\xi \equiv \frac{(1-\eta)\left(1-\frac{1}{1+\rho}\eta\right)}{\eta}$$

I will map a version of equation (5.1) into a linearized expectational system of difference equations for each sector (see section III).

II.3. ***Central Bank***

Monetary policy is carried out by a central bank that is assumed to set nominal interest rates $1+r_t \equiv R_t$ every period according to the simplified Taylor rule:

$$r_t = r + \phi_\pi \pi_t \tag{5.2}$$

This linear interest rate rule is a simplified case of the Taylor rule with a zero coefficient on the output gap and a zero inflation target. If $\phi_\pi > 1$ in this linear interest rate rule (5.2), an increase in inflation will ultimately lead to a more than one-for-one increase in the nominal interest rate. This condition is necessary and sufficient to guarantee the uniqueness of equilibrium[15] in the absence of rule-of-thumb consumers.

[15] This is commonly referred to as the Taylor Principle. See Woodford (2003).

II.4. ***Central Government***

Government inflows come from the usual sources: tax revenues and loans to both consumers and firms. As mentioned in previous subsections, I allow for different levels of lump-sum taxation to occur across consumer types. Therefore, total tax revenue in any given period is given by a convex combination of taxes paid by Ricardian and non-Ricardian consumers so that $T_t \equiv \lambda T_t^o + (1-\lambda) T_t^r$. The second source of revenue comes from selling bonds in the open market.

Government spending come from spending as well as liquidating government debt. One of the main motivations of this study is to determine the effects fiscal policy has had on different types of consumption in the U.S. Therefore, I note that the definition of what constitutes government spending may have a direct bearing on the conclusions that I try to draw here. Therefore, I will consider both defense and non-defense spending as proxies for government spending.

Unless government seeks to run a deficit in some given period, expenditure must be no larger than tax revenues in the current period

and whatever debt was incurred last period. Consequently, the government budget constraint is

$$P_t T_t + R_t^{-1} B_{t+1} = B_t + P_t G_t \tag{5.3}$$

By normalizing the steady state deviations of government expenditures, taxes and bonds by steady state output

$$\left(g_t \equiv \frac{G_t - G}{Y}, t_t \equiv \frac{T_t - T}{Y}, and\ b_t \equiv \frac{\left(\frac{B_t}{P_{t-1}}\right) - \left(\frac{B}{P}\right)}{Y} \right)$$

equation (5.3) implies the following fiscal policy rule:

$$t_t = \phi_b b_t + \phi_g g_t \tag{5.4}$$

Government purchases fluctuate exogenously according to the following stationary autoregressive process:

$$g_t = \rho_g g_{t-1} + \varepsilon_t \tag{5.5}$$

where $0 < \rho_g < 1$ and the independently and identically distributed (*i.i.d*) government spending shock ε_t has a constant variance $\sigma_\varepsilon^{\ 2}$

III. General Equilibrium Dynamics of the Two-Sector, Two Consumer Economy

In this section, a complete mapping of all the linearized equilibrium conditions derived in previous sections is carried out. Once the mapping is complete and the model has been properly closed out, I proceed with a description of a possible benchmark calibration for the purpose of parameter estimations discussed in the following section.

III.1. *Solving the Model*

A system of stochastic difference equations that could describe the log-linearized conditions is given by the following form:

$$AE_t\left\{x_{t+1}^0\right\} = Bx_t^0 + C\upsilon_t \tag{5.6}$$

where $x_t \equiv \left[n_{1,t},\ n_{2,t},\ c_{1,t},\ c_{2,t},\ k_{1,t},\ k_{2,t},\ \pi_t,\ b_t\right]$. The elements of matrices A and B are constituted by the underlying structural parameters of the model that are derived in this section. A linear expectational system of difference equations of the form (5.6) can be derived from the equilibrium conditions for the consumer and the producer obtained in

the previous sections. In order to map these equilibrium conditions into (5.6), we must first develop the innovations by subtracting from each dependent variable the previous period expected value (E_{t-1}) of itself [see Keating (1990)]. The rest of this section will systematically develop the innovations for the relevant equilibrium conditions for mapping into the dynamic system given by (5.6).

Recall (3.7) describes the equilibrium equation for the rule-of-thumb consumer in sector 1. Applying an $\left(1-L^{-1}\right)$ operator to (3.7), where L stands for the lag operator, yields the following

$$\begin{aligned}\left(1+\hat{\mu}_p\gamma_c\varphi\right)\left[c_{1,t}{}^r - E_t c_{1,t+1}{}^r\right] = \\ = \tilde{\alpha}_1\left[c_{1,t} - E_t c_{1,t+1}\right] + \tilde{\alpha}_1\varphi\left[n_{1,t} - E_t n_{1,t+1}\right] - \hat{\mu}_p\varphi\left[t_t{}^r - E_t t_{t+1}{}^r\right]\end{aligned} \tag{5.7}$$

Similarly, applying the $\left(1-L^{-1}\right)$ operator to the aggregation condition (2.2) for consumers in sector 1 yields

$$\left[c_{1,t} - E_t c_{1,t+1}\right] = \lambda\left[c_{1,t}{}^r - E_t c_{1,t+1}{}^r\right] + \left(1-\lambda\right)\left[c_{1,t}{}^o - E_t c_{1,t+1}{}^o\right] \tag{5.8}$$

Finally, we can derive the log-linearized Euler equation for sector 1 by combining the optimizing household condition in sector 1 given by equation (1.9) with equations (5.7) and (5.8). After some tedious algebraic manipulation we arrive at the following form for the Euler equation in sector 1:

$$\frac{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi\right)}\left[c_{1,t}-E_t c_{1,t+1}\right]=-\left(\frac{\lambda\tilde{\alpha}_1\varphi}{1+\hat{\mu}_p\gamma_c\varphi}\right)\left[E_t n_{1,t+1}-n_{1,t}\right]$$
$$+\left(\frac{\lambda\hat{\mu}_p\varphi}{1+\hat{\mu}_p\gamma_c\varphi}\right)\left[E_t t_{t+1}{}^{r}-t_t^{r}\right]-(1-\lambda)\left[r_t-E_t\pi_{t+1}\right] \quad (5.9)$$

The Euler equation for sector 1 describes how the innovations in consumption (from both consumer types) are a function of innovations in labor hours as well as fiscal and monetary policies. The Euler equation can be written (more compactly) as follows:

$$c_{1,t}=E_t c_{1,t+1}-\frac{1}{\tilde{\sigma}}\left(r_t-E_t\pi_{t+1}\right)-\Lambda_n E_t\left\{\Delta n_{t+1}\right\}+\Lambda_t\left\{\Delta t_{t+1}{}^{r}\right\} \quad (5.10)$$

where

$$\frac{1}{\tilde{\sigma}}\equiv\frac{\left(1+\hat{\mu}_p\gamma_c\varphi\right)(1-\lambda)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)},\ \Lambda_n\equiv\frac{\left(\lambda\tilde{\alpha}_1\varphi\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)}\ \text{and}\ \Lambda_t\equiv\frac{\left(\lambda\hat{\mu}_p\varphi\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)}$$

One last step in mapping this Euler equation into the dynamic system given by equation (5.6) is to include information about the fiscal and monetary policy assumptions. To that end, recall the fiscal policy rule given by equation (5.4) along with the AR(1) process that government spending was assumed to follow as indicated by equation (5.5). Combining (5.4) and (5.5) and developing the innovation by

applying the $\left(1-L^{-1}\right)$ operator (in a similar fashion to what we used for the Euler equation) yields the following fiscal policy innovation:

$$E_t\left[t_{t+1}-t_t\right]=\phi_b E_t\left[b_{t+1}-b_t\right]+\phi_g\left(\rho_g-1\right)g_t$$

Finally, substituting this fiscal innovation along with the interest rate rule (5.2) into (5.10) yields the following Euler equation for consumption in sector 1:

$$\begin{aligned} &c_{1,t}-\Lambda_n n_{1,t}+\frac{\phi_\pi}{\tilde{\sigma}}\pi_t= \\ &\quad=E_t c_{1,t+1}+\frac{1}{\tilde{\sigma}}E_t\pi_{t+1}-\Lambda_n E_t n_{1,t+1}+\Lambda_t\phi_b E_t\left\{\Delta b_{t+1}\right\}+\Lambda_t\phi_b\left(\rho_g-1\right)g_t \end{aligned} \tag{5.11}$$

where

$$\frac{1}{\tilde{\sigma}}\equiv\frac{\left(1+\hat{\mu}_p\gamma_c\varphi\right)(1-\lambda)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)},\ \Lambda_n\equiv\frac{\left(\lambda\tilde{\alpha}_1\varphi\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)}\ \text{and}\ \Lambda_t\equiv\frac{\left(\lambda\hat{\mu}_p\varphi\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)}$$

Equation (5.11) is the expression describing the general equilibrium Euler rule for sector 1 to be inserted into the dynamic system (5.6).

An Euler equation, analogous to (5.11), can also be derived for sector 2. In the interest of brevity, I omit the analysis and simply state the Euler for sector 2 to look as follows:

$$\begin{aligned} &c_{2,t}-\Gamma_n n_{2,t}+\frac{\phi_\pi}{\tilde{\tilde{\sigma}}}\pi_t= \\ &\quad=E_t c_{2,t+1}+\frac{1}{\tilde{\tilde{\sigma}}}E_t\pi_{t+1}-\Gamma_n E_t n_{2,t+1}+\Gamma_t\phi_b E_t\left\{\Delta b_{t+1}\right\}+\Gamma_t\phi_b\left(\rho_g-1\right)g_t \end{aligned} \tag{5.12}$$

where

$$\frac{1}{\tilde{\tilde{\sigma}}} \equiv \frac{\left(1+\hat{\mu}_p\gamma_c\varphi\right)\left(1-\lambda\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_2\right)},\ \Gamma_n \equiv \frac{\left(\lambda\tilde{\alpha}_2\varphi\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_1\right)} \text{ and } \Gamma_t \equiv \frac{\left(\lambda\hat{\mu}_p\varphi\right)}{\left(1+\hat{\mu}_p\gamma_c\varphi-\lambda\tilde{\alpha}_2\right)}$$ [16]

Finally, it is useful to derive a relationship between investment, consumption, labor and inflation in a New Keynesian context that is sector-specific. To achieve this objective, one has but to relax the neoclassical assumption that the marginal product of labor *exactly* offsets the marginal product of labor by allowing a wedge to exist between the two and then, combine it with the log-linear equilibrium rules obtained from the consumer problems.

A separate inter-temporal first order condition that is derived from the optimizer consumer[17] problem is given by the following relationship (in logs):

$$r_t^k = -r_t + E_t\pi_{t+1} \tag{5.13}$$

where r_t^k describes the log real rate of return inflow to the household from renting out capital.

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[16] The wedge $\hat{\mu}_t^{\,p}$ between the MRS and MPN in sector 1 may not necessarily be the same as the wedge $\hat{\mu}_t^{\,p}$ for sector 2. Durable share of output γ_c (5.11) and nondurable shares of output γ_c (5.12) will likely be different and generally $\tilde{\alpha}_1 \neq \tilde{\alpha}_2$.

[17] We derive this equilibrium condition for the optimizing consumer only. There is no analogous first order condition to (5.13) for the non-Ricardian household given that we assumed the rule-of-thumb consumer does not invest and, thus, earns no returns on capital.

Now let there be a wedge μ_t^p between MRS and MPN (in logs). This wedge could be decomposed into the sum of the real wage plus some price markups. With the standard assumption that in equilibrium, the real wage is given by the MRS, the production technology assumed for sector 1 implies

$$\mu_t^p = y_{1,t} - n_{1,t} - w_t \tag{5.14}$$

Combining (5.14) with a version of the log-linear equilibrium condition for the consumer given by (3.3) yields the following condition for the wedge:

$$\mu_t^p = y_{1,t} - \frac{1}{\alpha_1} c_{1,t} - \left(1 + \frac{\varphi}{\alpha_1}\right) n_{1,t}$$

A corollary to equation (5.14) is given by $\mu_t^p = y_{1,t} - n_{1,t} - r_t^k$; therefore we have

$$r_t^k = \frac{1}{\alpha_1} c_{1,t} + k_{1,t} + \left(1 + \frac{\varphi}{\alpha_1}\right) n_{1,t} \tag{5.15}$$

Finally, substituting the interest rate rule given by (5.2) and (5.15) into the standard first order condition given by $r_t^k = E_t \pi_{t+1} - r_t$[18] yields the following condition

[18] Setting constants to 0 without loss of generality.

$$\frac{1}{\alpha_1} c_{1,t} + k_{1,t} + \left(1 + \frac{\varphi}{\alpha_1}\right) n_{1,t} = -\phi_\pi \pi_t + E_t \pi_{t+1} \tag{5.16}$$

This New Keynesian relationship between investment, consumption, labor and inflation will also hold sector 2. By similar analysis we can derive the counterpart to (5.16) for sector 2 to look like:

$$\frac{1}{\alpha_2} c_{2,t} + k_{2,t} + \left(1 + \frac{\varphi}{\alpha_2}\right) n_{2,t} = -\phi_\pi \pi_t + E_t \pi_{t+1} \tag{5.17}$$

The last piece of the puzzle would require us to obtain an equilibrium condition describing fiscal debt. Log-linearizing the government's constraint (5.3) around its steady state yields the following:

$$b_{t+1} = (1+\rho)\left[b_t + g_t - t_t\right] \tag{5.18}$$

The assumptions of zero debt in which the government initially runs a balanced budget (R_t=1) is implied by (5.18). Combining (5.18) with the fiscal policy rule given by (5.4) yields the following equilibrium condition for government debt

$$b_{t+1} = (1+\rho)(1-\phi_b) b_t + (1+\rho)\left(1-\phi_g\right) g_t \tag{5.19}$$[19]

[19] A sufficient condition for non-explosive dynamics in this debt equation is given by $\phi_b > \frac{\rho}{1+\rho}$

The equilibrium conditions describing the dynamics of the model are given by equations (5.5), (4.14), (4.15), (5.11), (5.12), (5.16), (5.17), (5.1) and (5.19).

Let $x_t^0 = \left[n_{1,t},\ n_{2,t},\ c_{1,t},\ c_{2,t},\ k_{1,t},\ k_{2,t},\ \pi_t,\ b_t \right]'$

and $\upsilon_t = \left[g_t \right]$

Then equations (4.14), (4.15), (5.11), (5.12), (5.16), (5.17), (5.1) and (5.19) can be written as:

$$AE_t\left\{x_{t+1}^0\right\} = Bx_t^0 + C\upsilon_t \qquad (5.20)$$

where A and B are square 8x8 matrices and C is an 8x1 vector. Specifically,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\Lambda_n & 0 & 1 & 0 & 0 & 0 & \frac{1}{\tilde{\sigma}}\Lambda_t\phi_b \\ 0 & -\Gamma_n & 0 & 1 & 1 & 0 & \frac{1}{\tilde{\sigma}}\Gamma_t\phi_b \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{8x8}$$

and equation (4.14) implies:

$$b_{11} = \frac{\alpha_1 + (\beta_1 + \beta_2)\varphi}{\alpha_1}\left(\frac{\delta_1}{\gamma_i}\right)$$

$$b_{13} = -\left(\gamma_c + \frac{(\beta_1 + \beta_2)}{\alpha_1}\right)\left(\frac{\delta_1}{\gamma_i}\right)$$

$$b_{15} = (1 - \delta_1)$$

$$c_{11} = -\left(\frac{\delta_1}{\gamma_i} m\right)$$

equation (4.15) implies:

$$b_{22} = \frac{\alpha_2 + (\beta_1 + \beta_2)\varphi}{\alpha_2}\left(\frac{\delta_2}{\gamma_i}\right)$$

$$b_{24} = -\left(\gamma_c + \frac{(\beta_1 + \beta_2)}{\alpha_2}\right)\left(\frac{\delta_2}{\gamma_i}\right)$$

$$b_{26} = (1 - \delta_2)$$

$$c_{21} = -\left(\frac{\delta_2}{\gamma_i}(1 - m)\right)$$

equation (5.11) implies:

$$b_{31} = -\Lambda_n$$

$$b_{33} = 1$$

$$b_{37} = \frac{\phi_\pi}{\tilde{\sigma}}$$

$$b_{38} = \Lambda_t \phi_b$$

$$c_{31} = -\Lambda_t \phi_g \left(\rho_g - 1 \right)$$

equation (5.12) implies:

$$b_{42} = -\Gamma_n$$

$$b_{44} = 1$$

$$b_{47} = \frac{\phi_\pi}{\tilde{\tilde{\sigma}}}$$

$$b_{48} = \Gamma_t \phi_b$$

$$c_{41} = -\Gamma_t \phi_g \left(\rho_g - 1 \right)$$

equation (5.16) implies:

$$b_{51} = \left(1 + \frac{\varphi}{\alpha_1} \right)$$

$$b_{53} = \left(\frac{1}{\alpha_1} \right)$$

$$b_{55} = 1$$

$$b_{57} = \phi_\pi$$

equation (5.17) implies:

$$b_{62} = \left(1 + \frac{\varphi}{\alpha_2} \right)$$

$$b_{64} = \left(\frac{1}{\alpha_2} \right)$$

$b_{66} = 1$

$b_{67} = \phi_{\pi}$

equation (5.1) implies:

$b_{71} = \lambda\tilde{\psi}\varphi$

$b_{72} = \lambda\tilde{\psi}\varphi$

$b_{73} = \lambda\tilde{\psi}$

$b_{74} = \lambda\tilde{\psi}$

$b_{77} = 1$

equation (5.19) implies:

$b_{88} = (1+\rho)(1-\phi_b)$

$c_{81} = (1+\rho)(1+\phi_g)$

Meanwhile equation (5.5) can be written as:

$$\upsilon_t = P\upsilon_{t-1} + \varepsilon_t \qquad (5.21)$$

where $P = \rho_g$ and $\varepsilon_t = \varepsilon_t^g$

Equation (5.6) describes a system of expectational difference equations driven by the exogenous government shock of (5.5). This system can be solved by uncoupling the unstable and stable

components and then solving the unstable components forward and the stable component backwards. One algorithm that accomplishes this is the one outlined by Klein (2000). Klein's method is based on the *complex generalized Schur decomposition*, which identifies unitary matrices Q and Z such that

$$QAZ = S$$

and

$$QBZ = T$$

Are both upper triangular. The eigenvalues of A and B can be recovered as the ratios of the diagonal elements of *T* and *S*:

$$\lambda(A,B) = \left\{ \frac{t_{ii}}{s_{ii}} \mid i = 1,2,...8 \right\}$$ [20]

Note that by the usual of assumptions associated with the capital stock, there are two predetermined variables and 6 non-predetermined in the vector x_t^0. Hence if two of the generalized eigenvalues in $\lambda(A,B)$ lie inside the unit circle and six are outside, the system has a unique solution. If more than six eigenvalues in $\lambda(A,B)$ lie outside the unit circle, then the system has no solution. Conversely, if less than six

[20] The matrices Q, Z, S, and T can always be re-arranged so that the generalized eigenvalues appear in ascending order

eigenvalues lie outside the unit circle then we have indeterminacy, see Blanchard and Kahn (1980).

Let's first define the vector x_t^1 of auxiliary variables as

$$x_t^1 = Z' x_t^0$$

where Z' is assumed to be a unitary matrix so that $Z'Z = I$ or $Z' = Z^{-1}$. Hence the equation above can be re-written as $x_t^0 = Z x_t^1$. Using this equation to rewrite (5.20) as

$$AZE_t\left\{x_{t+1}^1\right\} = BZx_t^1 + C\upsilon_t$$

pre-multiplying this equation by Q (and recalling the previous assumptions $QAZ = S$ and $QBZ = T$) we have the following equation

$$SE_t\left\{x_{t+1}^1\right\} = Tx_t^1 + QC\upsilon_t \tag{5.22}$$

From now on, assume that there are exactly six generalized eigenvalues that lie outside the unit circle. We can conformably partition the matrices Q, Z, S, and T so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

where Q_1 is 2x8 and Q_2 is 6x8, and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0_{6x2} & S_{22} \end{bmatrix}$$

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{6x2} & T_{22} \end{bmatrix}$$

where Z_{11}, $S_{11,}$ and T_{11} are 2x2, Z_{12}, $S_{12,}$ and T_{12} are 2x6, Z_{21} is 6x2, and Z_{22}, $S_{22,}$ and T_{22} are 6x6.

We now proceed to decouple our state vector (x_t^0 , which by the rotation expressed above is tantamount to decomposing) x_t^1 so that specifically we have

$$x_t^1 = \begin{bmatrix} x_{1t}^1 \\ x_{2t}^1 \end{bmatrix}$$

where

$$x_{1t}^1 = Z_{11}^{'} \begin{bmatrix} k_{1t} \\ k_{2t} \end{bmatrix} + Z_{21}^{'} \begin{bmatrix} n_{1t} \\ n_{2t} \\ c_{1t} \\ c_{2t} \\ \pi_t \\ b_t \end{bmatrix} \tag{5.23}$$

is a 2x1 vector and

$$x_{2t}^{1} = Z_{12}^{'}\begin{bmatrix} k_{1t} \\ k_{2t} \end{bmatrix} + Z_{22}^{'}\begin{bmatrix} n_{1t} \\ n_{2t} \\ c_{1t} \\ c_{2t} \\ \pi_{t} \\ b_{t} \end{bmatrix} \tag{5.24}$$

is 6x1.

In terms of these matrix partitions we can express (5.22) as

$$S_{11}E_{t}x_{1t+1}^{1} + S_{12}E_{t}x_{2t+1}^{1} = T_{11}x_{1t}^{1} + T_{12}x_{2t}^{1} + Q_{1}C\upsilon_{t} \tag{5.25}$$

and

$$S_{22}E_{t}x_{2t+1}^{1} = T_{22}x_{2t}^{1} + Q_{2}C\upsilon_{t} \tag{5.26}$$

Since the generalized eigenvalues associated with the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (5.26) can be solved forward to obtain

$$x_{2t}^{1} = -T_{22}R\upsilon_{t}$$

where the 6x1 R matrix is given by

$$R = \left[I_{(6x6)} - P \otimes \left(S_{22}T_{22}^{-1} \right) \right]^{-1} vec\left(Q_{2}C \right)$$ [21]

Use this result along with (5.24), to solve for

[21] See Klein (2000) for details.

$$\begin{bmatrix} n_{1t} \\ n_{2t} \\ c_{1t} \\ c_{2t} \\ \pi_t \\ b_t \end{bmatrix} = -\left(Z_{22}^{'}\right)^{-1} Z_{12}^{'} \begin{bmatrix} k_{1t} \\ k_{2t} \end{bmatrix} - \left(Z_{22}^{'}\right)^{-1} T_{22}^{-1} R v_t \tag{5.27}$$

As mentioned earlier, Z is unitary so that $Z^{'}Z = I$. This implies

$$\begin{pmatrix} Z_{11}^{'} & Z_{21}^{'} \\ Z_{12}^{'} & Z_{22}^{'} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} I_{2x2} & 0_{2x6} \\ 0_{6x2} & I_{6x6} \end{pmatrix}$$

Second row times first column implies

$$Z_{12}^{'} Z_{11} + Z_{22}^{'} Z_{21} = 0$$

or

$$-\left(Z_{22}^{'}\right)^{-1} Z_{12}^{'} = Z_{21} Z_{11}$$

Second row times second column implies

$$Z_{12}^{'} Z_{12} + Z_{22}^{'} Z_{22} = I$$

or

$$\left(Z_{22}^{'}\right)^{-1} = Z_{22} + \left(Z_{22}^{'}\right)^{-1} Z_{12}^{'} Z_{12} = Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}$$

Allowing (5.27) to be written more conveniently as

$$\begin{bmatrix} n_{1t} \\ n_{2t} \\ c_{1t} \\ c_{2t} \\ \pi_t \\ b_t \end{bmatrix} = N_1 \begin{bmatrix} k_{1t} \\ k_{2t} \end{bmatrix} + N_2 \upsilon_t \tag{5.28}$$

where $N_1 = Z_{21}Z_{11}^{-1}$ and $N_2 = -\left[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \right] T_{22}^{-1} R$.

Equation (5.23) now provides a solution for x_{1t}^1:

$$x_{1t}^1 = \left(Z_{11}^{'} + Z_{21}^{'} Z_{21} Z_{11}^{-1} \right) \begin{bmatrix} k_{1t} \\ k_{2t} \end{bmatrix} - Z_{21}^{'} \left[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \right] T_{22}^{-1} R \upsilon_t$$

Using first row times first column, we get

$$Z_{11}^{'} Z_{11} + Z_{21}^{'} Z_{21} = I$$

or

$$Z_{11}^{-1} = Z_{11}^{'} + Z_{21}^{'} Z_{21} Z_{11}^{-1}$$

and

$$Z_{21}^{-1} \left[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \right] = Z_{21}^{'} Z_{22} - Z_{21}^{'} Z_{21} Z_{11}^{-1} Z_{12} = -Z_{11}^{-1} Z_{12}$$

this lat result can more conveniently be written as

$$x_{1t}^1 = Z_{11}^{-1} \begin{bmatrix} k_{1t} \\ k_{2t} \end{bmatrix} + Z_{11}^{-1} Z_{12} T_{22}^{-1} R \upsilon_t$$

Finally, substitute these results into (5.25) to obtain the solution

$$E_t\begin{bmatrix} k_{1t+1} \\ k_{2t+1} \end{bmatrix} = N_3\begin{bmatrix} k_{1t} \\ k_{2t} \end{bmatrix} + N_4 \upsilon_t \tag{5.29}$$

where

$$N_3 = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}$$

and

$$N_4 = Z_{11}S_{11}^{-1}\left(T_{11}Z_{11}^{-1}Z_{12}T_{22}^{-1}R + Q_1C + S_{12}T_{22}^{-1}RP - T_{12}T_{22}^{-1}R\right) - T_{12}T_{22}^{-1}RP$$

Noting that the rows of N_1 and N_2 exactly reproduce two of the rows of N_3 and N_4, the solution to this model can be written efficiently by combining (5.21) and (5.29) as

$$x_{t+1} = \Pi x_t + W\varepsilon_{t+1}, \tag{5.30}$$

where

$$x_t = \left[n_{1,t},\ n_{2,t},\ c_{1,t},\ c_{2,t},\ \pi_t,\ b_t,\ \varepsilon_g^t\right]',$$

$$\varepsilon_{t+1} = \left[\varepsilon_{t+1}^g\right],$$

$$\Pi = \begin{pmatrix} N_3 & N_4 \\ 0_{6x1} & P \end{pmatrix},$$

$$W = \begin{pmatrix} 0_{6x1} \\ I_{1x1} \end{pmatrix}$$

In order to arrive at a just-identified system, I need to supplement the long-run restrictions implied by the model's equilibrium conditions

with an initial guess on some of the parameter values. To this end, I develop an initial calibration of the parameters of interest in the following sub-section.

III.2. ***Benchmark Calibration***

The following calibration is intended as an initial stand on the values of certain parameters as suggested by the current literature. The following values are fixed a priori. The discount factor is notoriously difficult to estimate based on aggregate data alone. Given that the frequency of data used in this study is quarterly, I set the discount factor $\left(\frac{1}{1+\rho}\right)$ equal to 0.99 so that each period in the model is interpreted to correspond to a one-quarter year in real time. The elasticity of capital with respect to output $(\beta_1 + \beta_2)$ is set to 0.31. This value is roughly consistent with what is empirically observed for the capital share of income and quite close to the standard assumption of one-third initially suggested by Solow himself and later popularized in the growth literature by Mankiw, Romer and Weil (1992) and others. The rate of depreciation of durable capital goods (δ_1) and stocks of nondurable goods (δ_2) are assumed to be 0.13 and 0.03, respectively.

These are the typical values for the rates of capital allowance used to construct NIPA capital stocks for durables, equipment and structures [Katz and Herman (1997)]. For the rate of return on capital $\left(R_t^k\right)$, I use the average value of 0.065 as suggested by King and Rebelo (2000). The preference shifters for nominal consumption of durables (α_1) and non-durables (α_2) are set to $\frac{1}{2}$. I follow Rotemberg and Woodford (1997-99) calibration of the elasticity of wages with respect to output of 0.3. My assumption, as mentioned above, of the elasticity of output with respect to capital of 0.31 implies that the elasticity of labor with respect to output is 0.69. I, therefore, set the benchmark value for elasticity of wages with respect to labor (φ) equal to 0.21. All these parameter values remain constant in the calibration that follows.

Next, I turn to those parameters that are allowed to vary in the calibration, to allow for the possibility of different specifications. As mentioned above, the wedge between the MRS and the MPN can be interpreted as a price markup. I assume the steady state price markup $\left(\mu^p\right)$ equal to 0.2, a value suggested by Gali, Lopez-Salido and Valles (2004). The baseline weight for Ricardian $(1-\lambda)$ and non-Ricardian (λ)

households is also set to $\frac{1}{2}$. This is within the range of estimates in the literature of the weight of rule-of-thumb behavior [Mankiw (2000)].

Next, I turn to the benchmark calibration for monetary and fiscal parameters of the model. The magnitude of the response of the central bank to inflation (ϕ_π) is set equal to 1.5, a standard value in empirical Taylor rules. I use a simple VAR to estimate the parameters of the fiscal policy rule (5.4) as well as government spending (5.5) (namely, $\phi_b, \phi_g, and\ \rho_g$). At first, assume the government shock is not explosive but close to non-stationary so $\rho_g = 0.9$. To obtain the value for ϕ_g, I take the difference between the estimated impulse response of government spending and deficit, respectively. Once we have these parameters $\phi_g\ and\ \rho_g$, I can calibrate the parameter ϕ_b so that the dynamics of government spending (5.5) and debt (5.4) are consistent with the horizon at which the deficit is back to zero in my estimates. For tractability, I assume government spending patterns on durables vs. non-durables are similar to private spending. Accordingly, I set (m) the share of government spending on durables to be 0.33.

IV. An Outline of the Empirical Model

The following outline describes the data I use in addition to the state-space methodology to carry out maximum likelihood-based estimation via the Kalman filter and finally expounds on the derivations needed to carry out innovation analysis.

IV.1. *Data*

Using U.S. quarterly data, I estimate the response of several macroeconomic variables to a government spending shock. The proposed model has implications for several observables including a measure of government spending, consumption of durables, consumption of non-durables and services, hours worked in production of both sectors, private nonresidential investment, private residential investment, inflation and the budget deficit.[22]

The budget deficit measure is constructed to be the difference between federal non-defense gross investment and gross government savings. In the interest of robustness, both total government purchases and military spending are used for the predetermined government variable. Both the budget deficit and government expenditures enter

[22] Unless otherwise stated, most variables (obtained from the FRED-II database) are expressed in logs and normalized by the civilian population over 16.

the model as a ratio to trend GDP[23]. Data for personal consumption on durables and personal consumption on non-durables and services from the FRED-II database is used for the consumption measures in both sectors. A measure of aggregate hours in each sector is arrived at by multiplying average weekly hours in manufacturing by the number of full-time equivalent employees in each sector obtained from the NIPA tables. I use investment on equipment and software as a measure for fixed private nonresidential investment and the commonly used fixed residential investment from FRED-II database. Finally, the consumer price index (CPI) is used for the Phillips curve embedded in the model (see section III).

IV.2. ***Estimation***

In accordance with the model setup of sections II and III, all encompassed by (5.30), and the data[24] described above residing in the following vector,

[23] Lagged potential output from the Congressional Budget Office is used as a proxy for trend GDP.

[24] All in log deviation form. The sample ranges from 1964:Q1 to 2006:Q1

$$d_t = \begin{bmatrix} n_{1,t} \\ n_{2,t} \\ c_{1,t} \\ c_{2,t} \\ \pi_t \\ b_t \\ g_t \end{bmatrix}$$

is used to estimate a seven-variable empirical model that can be described by

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{6.1}$$

and

$$d_t = Cs_t \tag{6.2}$$

where $A = \Pi$, $B = W$ and C is formed from the rows of Π as

$$C = \begin{pmatrix} \Pi_6 \\ \Pi_5 \\ \Pi_4 \\ \Pi_3 \\ \Pi_2 \\ \Pi_1 \end{pmatrix}$$

generally, the zero-mean, serially uncorrelated innovations would imply a diagonal covariance matrix of shocks. In the current example, however we consider a single stochastic process for government expenditures. Hence the covariance matrix for the model looks like

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}^{'} = \left[\sigma_g\right]$$

Thus, the model has been cast in state-space form where (6.1) acts as the transition equation and (6.2) as the measurement equation. Once a dynamic model has been written in state-space, basic filtering and smoothing algorithms such as the Kalman filter are available for inferences on the unobserved state vector s_t, conditional on the parameters of the model and the appropriate information set (Kim and Nelson 1999).

Hence, the likelihood function for the sample $\{d_t\}_{t=1}^{T}$ can be constructed as outlined by Hamilton (1994) or Kim and Nelson (1999).

The basic Kalman filter algorithm consists of two steps: A *prediction* and an *updating* step. The prediction step reflects our need to estimate an optimal predictor (at the beginning of period t), of our sample d_t, based on all available information up to time $t-1$; $d_{t|t-1}$. In order to do this, the state vector $s_{t|t-1}$ needs to be calculated as of period $t-1$. Once d_t is realized at the end of period t, the prediction error $\eta_{t|t-1}$ can be calculated. This prediction error contains new information about s_t beyond that contained in $s_{t|t-1}$. Therefore, once d_t is observed, a more accurate inference of s_t can be made by updating the information

set with the newest observation. Hence the state vector gets updated, since it is a linear combination of itself as of last period and the prediction error updated with new information. The prediction error is weighted by the *Kalman gain*, which determines how important is the new information about s_t in the newly updated prediction error.

For our estimation purposes in this model, let the estimate of the expectation (for $t = 1, 2, \ldots, T$ and $j = 0, 1$) of the state vector s_t conditional on information up to $t - j$ be

$$\hat{s}_{t|t-j} = E\left(s_t \mid d_{t-j}, d_{t-j-1}, \ldots d_1\right)$$

and the covariance matrix associated with the state vector, conditional on the same information set be

$$\Sigma_{t|t-j} = E\left(s_t - \hat{s}_{t-j}\right)\left(s_t - \hat{s}_{t-j}\right)'$$

and for forecasting purposes assume

$$\hat{d}_{t|t-j} = E\left(d_t \mid d_{t-j}, d_{t-j-1}, \ldots d_1\right)$$

Any recursion must begin with an initial value set. Let the initial values of the state vector $\hat{s}_{1|0}$, which describes the forecast of s_1 conditional on no observations of d_t, be the unconditional mean of s_1,

$$\hat{s}_{1|0} = E\left(s_1\right) = 0_{7\times 1} \tag{6.3}$$

and its associated initial value for its covariance matrix be

$$vec\left(\Sigma_{1|0}\right) = vec\left(Es_1 s_1^{'}\right) = \left[I_{(49\times 49)} - A \otimes A\right]^{-1} vec\left(BVB^{'}\right) \tag{6.4}$$

Let the prediction error be defined as

$$\eta_t = d_t - \hat{d}_{t|t-1}$$

and noting that (6.2) implies $\hat{d}_{t|t-1} = C\,\hat{s}_{t|t-1}$, we have that the covariance matrix of the prediction error looks like

$$E\eta_t \eta_t^{'} = C\Sigma_{t|t-1}C^{'}.$$

Using the formula for updating linear projections (see Hamilton 1994),

$$\begin{aligned}\hat{s}_{t|t} &= \hat{s}_{t|t-1} + \left[E\left(s_t - \hat{s}_{t|t-1}\right)\left(d_t - \hat{d}_{t|t-1}\right)^{'}\right]\left[E\left(d_t - \hat{d}_{t|t-1}\right)\left(d_t - \hat{d}_{t|t-1}\right)^{'}\right]\eta_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}\eta_t\end{aligned}$$

Thus from (6.1) we have,

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}\eta_t\,.$$

Subtracting this last result from (6.1) again

$$s_{t+1} - \hat{s}_{t+1|t} = A\left(s_t - \hat{s}_{t|t-1}\right) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}\eta_t$$

This last result implies

$$\Sigma_{t+1|t} = BVB^{'} + A\Sigma_{t|t-1}A^{'} - A\Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}C\Sigma_{t|t-1}A^{'}$$

These results can be summed up as follows, given a state vector

$$\hat{s}_t = \hat{s}_{t|t-1} = E\left(s_t \mid d_{t-1}, d_{t-2}, \ldots d_1\right)$$

and its covariance matrix

$$\Sigma_t = \Sigma_{t|t-1} = E\left(s_t - \hat{s}_{t|t-1}\right)\left(s_t - \hat{s}_{t|t-1}\right)'$$

then the state equation can be written as follows

$$\hat{s}_{t+1} = A\hat{s}_t + K_t\eta_t$$

where K_t denotes the Kalman gain. Likewise, the observation equation can be written as

$$d_t = C\hat{s}_t + \eta_t$$

where the prediction error and the covariance matrix of the prediction error are given, respectively, by

$$\eta_t = d_t - E\left(d_t \mid d_{t-1}, d_{t-2}, \ldots d_1\right) \text{ and } E\eta_t\eta_t' = C\Sigma_t C'.$$

Provided with the initial conditions for the state vector $\hat{s}_1$ and its covariance Σ_1 given by (6.3) and (6.4) respectively, we can generate recursively sequences for the Kalman gain K_t and the covariance matrix Σ_t using

$$K_t = A\Sigma_t C'\left(C\Sigma_t C'\right)^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'\left(C\Sigma_t C'\right)^{-1} C\Sigma_t A'$$

The forecast errors $\{\eta_t\}_{t=1}^{T}$ can then be used to construct the log likelihood function for the sample $\{d_t\}_{t=1}^{T}$

$$\ln L = -\left(\frac{3T}{2}\right)\ln(2\pi) - \frac{1}{2}\left[\sum_{t=1}^{T}\ln\left(C\Sigma_t C'\right) - \sum_{t=1}^{T}\ln\left(\eta_t'\left(C\Sigma_t C'\right)^{-1}\eta_t\right)\right]$$

V. Results

In the present section, I analyze the effect that an exogenous government spending shock has in the two-sector economy described above. As mentioned in section III, I calibrate a model with New Keynesian characteristics which allows for: a mix of "rule-of-thumb" and optimizing consumers, a wedge between the marginal rate of substitution and the marginal product of labor of the household, and the degree of annualized price adjustment. In what I dub the benchmark calibration, I allow for an even mix of consumer types by setting $\lambda = 0.5.$ Additionally, I set the share of firms that keep their prices unchanged $\eta = 0.75$, which corresponds to an average length of sticky prices of one year. Finally, I follow Gali et.al 2005 in setting the steady state markup μ^p equal 0.2. Figure 6 corresponds to the benchmark model responses to the major aggregates. First, as it is expected a fiscal expansion leads to a permanent increase in government spending as well as the budget deficit. Second, a positive shock to government spending seems to be expansionary in that it leads to increases in aggregate output and, to a lesser extent, a contemporaneous increase in labor hours. This is consistent with most theoretical predictions.

Figure 7 shows the responses by sector of an increase in government spending in the benchmark model. Both consumption of durables and non-durable goods see an increase following the positive shock to government spending. In addition, the increase in public spending precedes a significant (but minor) increase in private spending on durables which is quickly reversed in the third quarter post shock. There is a negligible (and not significant) contemporaneous effect on investment in non-durables, from the fiscal shock, which quickly turns negative.

Figure 8 corresponds to the responses of the durable and non-durable sectors to a positive government shock in the context of a RBC view. In other words, I consider the following: all consumers are optimizers so I set λ (the share of rule-of-thumb consumers in the economy) equal to zero. I allow the price staggering scheme to vanish by setting η the share of firms that keep their prices unchanged to zero. Finally, to allow for perfect market flexibility, I do away with the wedge $\mu^p = 0$. Hence, this specification assumes households that inter-temporally optimize subject to their resource constraints in a perfectly competitive market with no rigidities. The positive fiscal shock leads to a small decrease in consumption of non-durables and a persistently

negative drop in private investment in both types of goods. Surprisingly, consumption of durable goods rises following a positive fiscal shock. This evidence is in sharp contrast with the RBC prediction of government spending crowding out private spending and more in line with the New Keynesian theoretical prediction as well as the evidence from figure 7.

Remarkably, this result of an increase in durable-consumption following a fiscal expansion is robust to both specifications. Non-durable disbursements, in the U.S., are larger than the gross amounts of annual spending on durables; hence, it is reasonable to expect this increase in durable spending would overshadowed by a predicted non-durable spending drop in a traditional one-sector model.

Rule-of-thumb behavior by some households (who fully deplete their current labor income by consuming) and price rigidities, as those assumed recently in the New Keynesian literature, seem to engender this positive response of private consumption to public spending increases.[25]

While these fiscal expansions necessarily lead to increased taxation today or tomorrow, rule-of-thumb consumers may not internalize this

[25] This is first raised by Gali et.al (2005).

as a negative wealth-effect. Thus, this non-optimizing behavior "desensitizes" aggregate demand from the negative wealth effect. Additionally, in the presence of sticky prices, the price markup may correct downward enough to lead to increases in the real wage even when faced with a drop in the marginal product of labor. Importantly, the robustness of the positive effect on durable consumption between the widely different assumptions of the two views, argues for the necessity of modeling the economy with two sectors. It is worth mentioning that evidence from both calibrations agrees with the standard prediction of the theory (substantiated by other empirical findings) that an increase in government purchases raises hours. However, the increase in hours dedicated to the production of non-durables is much more muted than hours related to durables.

VI. Conclusion

The debate that centers on the impact of public spending on private spending rages on. Two schools of thought, which predict a diametrically opposite effect of public spending on private consumption, differ in household behavior and price flexibility assumptions but base their theoretical foundation on a single sector economy. Hinged on the recent empirical evidence of the differential growth rates of real consumption and real investment, this paper constructs an economy populated by two sectors: one engaged in the production of durables (which mostly go to investment spending) and one that produces non-durable goods (which mostly go to consumption). In addition, two types of consumers exist in this economy: one who inter-temporally optimizes her labor utility subject to her resource constraints and one who myopically consumes all of his current labor income. Finally, a dimension of price rigidity is considered. A benchmark model that allows for an even mix of rule-of-thumb and optimizer consumers along with the presence of sticky prices in a two-sector economy is considered. A second model (consistent with a RBC approach) features infinitely-lived optimizer consumers in a two-sector market with flexible prices. This framework

generates a positive relationship between consumption of durables and government spending which is robust to different specifications of household behavior and price stickiness. The effect on non-durable consumption, however, follows the theoretical predictions of the two views considered.

This paper conducts a parsimonious analysis in order to center on the key point of inquiry of this debate. For tractability it abstracts from a number of possible extensions. For instance, this theoretical analysis assumes this increase in public spending to be financed with lump-sum taxes. Allowing for distortionary taxes, will likely lead to different conclusions which will depend on the timing and composition of the taxation.

Opening the economy to trade would likely yield further insights in the composition of the response from either sector to the increase in government spending considered here.

Another extension that could prove beneficial would be to incorporate some form of rigidity in setting the nominal wage. This would likely have a significant effect on the marginal product of labor and, thus, labor income and consumption to a fiscal shock.

Finally, allowing for technology shocks to act as direct competitors with the fiscal shocks assumed here, in eliciting a different response from both sectors, would prove a more faithful representation of the RBC view. Chapter two of this thesis studies such an extension.

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VIII. Appendix A

Derivation of a first order approximation of the rule-of-thumb household's level of consumption for sector 2 can be accomplished in a similar fashion to that of sector 1.

The resource constraint for the rule-of-thumb consumer in sector 2 is given by:

$$P_{2,t}C_{2,t}{}^{r} = W_t P_{2,t} N_{2,t}{}^{r} - P_{2,t}T_t^{r} \qquad \text{(A1)}$$

or in real terms:

$$C_{2,t}{}^{r} = W_t N_{2,t}{}^{r} - T_t^{r} \qquad \text{(A2)}$$

Given a particular steady state value for each of the aggregate variables present in the constraint, equation (A2) can be rewritten as follows:

$$C_{2,t}{}^{r} = \left(\frac{C_2{}^{r}}{WN_2{}^{r}}\right)\left(\frac{WN_2{}^{r}}{C_2{}^{r}}\right) W_t N_{2,t}{}^{r} - T_t^{r} \qquad \text{(A3)}$$

Rearranging terms in (A3) yields:

$$C_{2,t}{}^{r} = \left(\frac{WN_2{}^{r}}{C_2{}^{r}}\right)\left[\frac{W_t}{W}\frac{N_{2,t}{}^{r}}{N_2{}^{r}}\right] C_2{}^{r} - T_t^{r} \qquad \text{(A4)}$$

Dividing both sides of (A4) by $C_2{}^{r}$ yields

$$\frac{C_{2,t}{}^{r}}{C_2{}^{r}} = \left(\frac{WN_2{}^{r}}{C_2{}^{r}}\right)\left[\frac{W_t}{W}\frac{N_{2,t}{}^{r}}{N_2{}^{r}}\right] - \frac{1}{C_2{}^{r}} T_t^{r} \qquad \text{(A5)}$$

and finally letting $c_2^r \equiv \log\left(\frac{C_{2t}^r}{C_2^r}\right)$, $w_t \equiv \log\left(\frac{W_t}{W}\right)$, $n_2^r \equiv \log\left(\frac{N_{2t}^r}{N_2^r}\right)$ and $t_t^r \equiv \frac{T_t^r}{Y}$ obtains equation (1.19) in the body of the paper.

For tractability, I follow the assumption proposed by Gali et al. (2005) in the analysis of both sectors: consumption in the *steady state* is the same across all household types. A suitable choice of T^r and T^o can ensure that assumption (namely that $C_1^r = C_1^o = C_1$ and $C_2^r = C_2^o = C_2$). Under perfectly competitive labor markets, all households share the same MRS. Thus, the assumption of equal *steady state* consumption levels implies that all households supply the same amount of labor *in the steady state* as well (i.e. $N_1^r = N_1^o = N_1$ and $N_2^r = N_2^o = N_2$).

IX. Appendix B

This subsection shows the necessary steps to formulate innovation analysis in this model along with the standard error calculations to draw inferences about the importance of a fiscal expansion, as given by a positive government shock, to the different sectors considered. Begin by rewriting the transition equation in the empirical state model given in (6.1) as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or using the backshift operator

$$(I - AL)s_t = B\varepsilon_t,$$

This state space model is an autoregressive process of the first order in the state vector. Thus a Wold moving average representation of (6.1) looks like the following

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}\ .$$

Given that this last equation holds for all periods, we can advance this expression by k periods to yield

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j+k}$$

so that

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t-j+k}$$

and thus, a forecast error will look like

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B\varepsilon_{t-j+k}$$

Therefore the covariance matrix of the state innovations will be given by

$$\begin{aligned}\Sigma_k^s &= E\left(s_{t+k} - E_t s_{t+k}\right)\left(s_{t+k} - E_t s_{t+k}\right)' \\ &= \left(BVB'\right) + A\left(BVB'\right)A' + A^2\left(BVB'\right)A^{2'} + \ldots + A^{k-1}\left(BVB'\right)A^{k-1'}\end{aligned}$$

Additionally, (6.1) implies that

$$\lim_{k\to\infty} \Sigma_k^s = \Sigma^s$$

where , as shown earlier, the covariance matrix of the state vector for this model is given by

$$vec\left(\Sigma^s\right) = \left[I_{(49\times 49)} - A \otimes A\right]^{-1} vec\left(BVB'\right)$$

We can now turn to the observation equation given in (6.2). The covariance matrix of the innovations of the sample are given by

$$\begin{aligned}\Sigma_k^d &= E\left(d_{t+k} - E_t d_{t+k}\right)\left(d_{t+k} - E_t d_{t+k}\right)' \\ &= C\Sigma_k^s C',\end{aligned}$$

and similarly to the state equation,

$$\lim_{k\to\infty} \Sigma_k^d = \Sigma^d$$

where the covariance matrix of the data vector is given by

$$\Sigma^d = C\Sigma^s C^{'}.$$

Let Θ denote the vector of estimated parameters and Ω the covariance matrix of these parameter estimates, so that asymptotically,

$$\Theta \sim N\left(\Theta^0, \Omega\right)$$

All of the elements in $\Sigma_k^s, \Sigma_k^d, \Sigma^s,$ and Σ^d can be expressed as nonlinear functions of Θ such as

$$\Sigma = g\left(\Theta\right),$$

so that the asymptotic standard errors for these elements can be found by calculating

$$\nabla g \Omega \nabla g^{'}.$$[26]

[26] Runkle (1987) shows a numerical calculation of the gradient ∇g

Figure 1
Ratio of Real Private Fixed Investment to Real Consumption
(1948:Q2-2006:Q2)

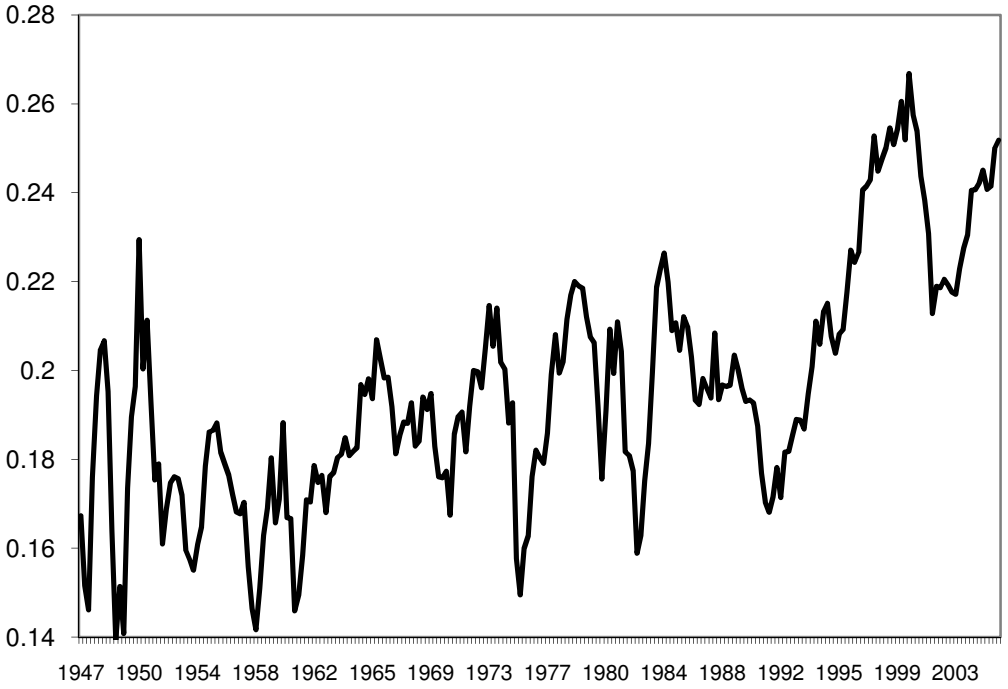


Figure 2a
Ratios of Major Components of Investment and Consumption

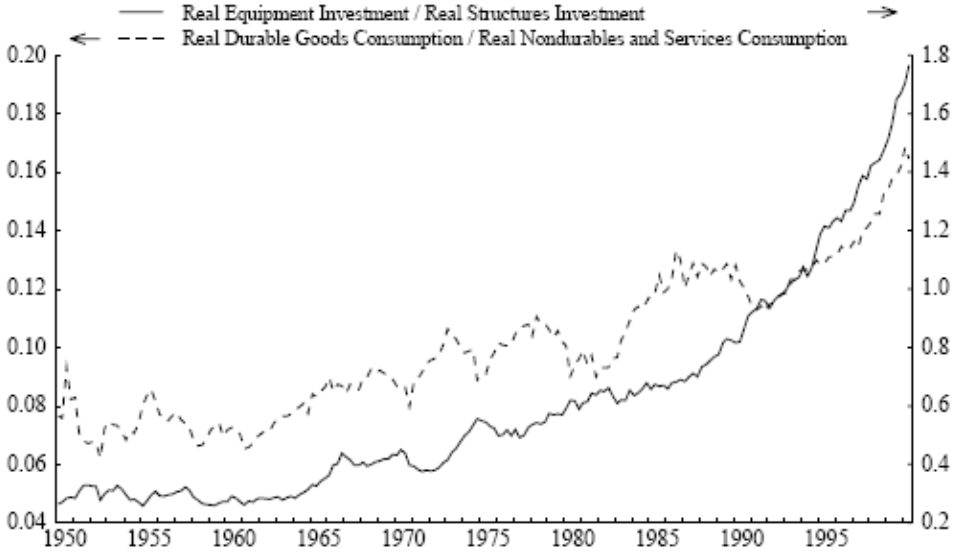


Figure 2b
Ratio of Total Real Durables Output to Real Business Output

Figure 3b
Ratio of Nominal Private Fixed Investment to Nominal Consumption

Figure 4
Relative Prices of Durables and Fixed Investment
(In Logs, All Prices Relative to Price of Consumer Nondurables and Services)

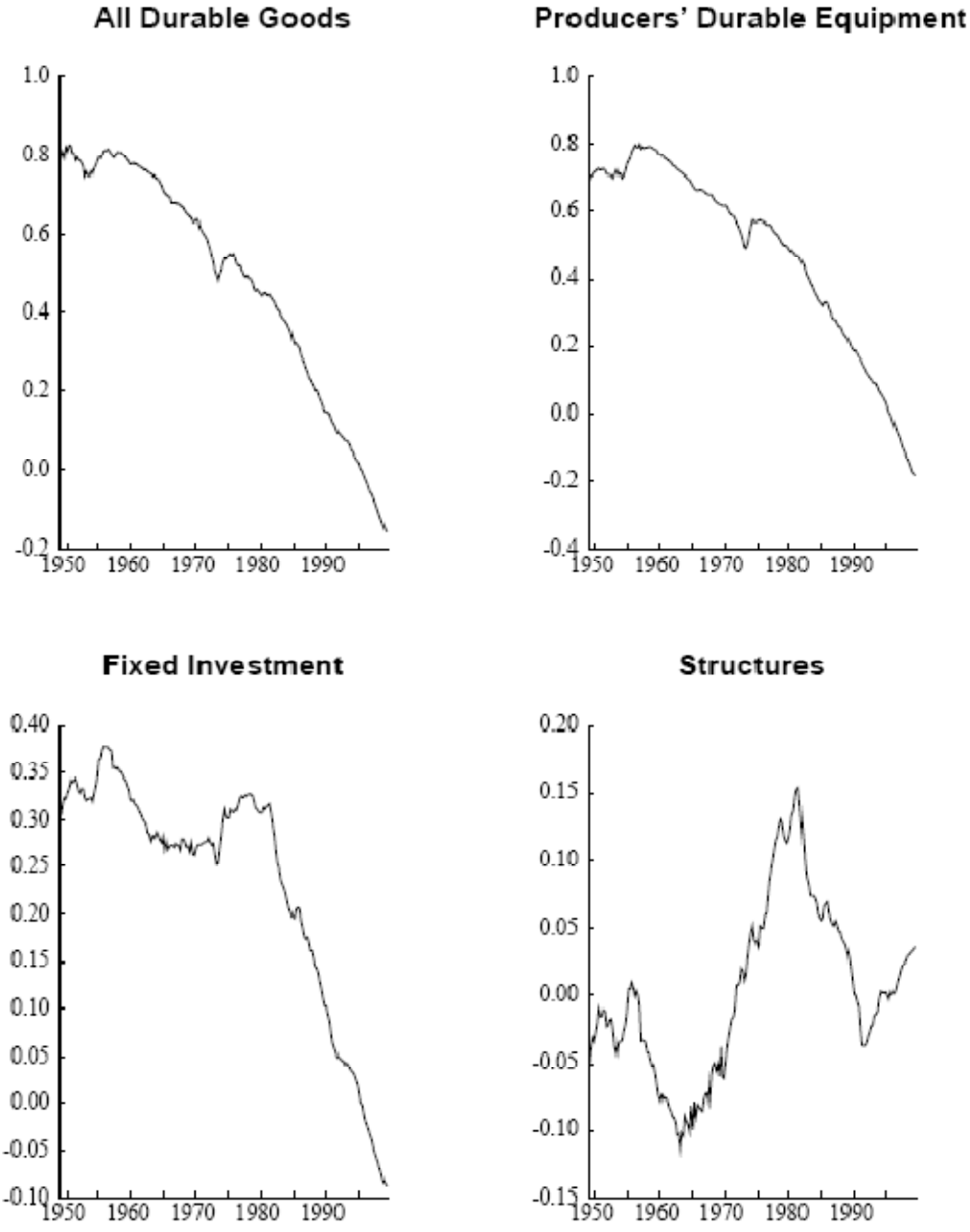
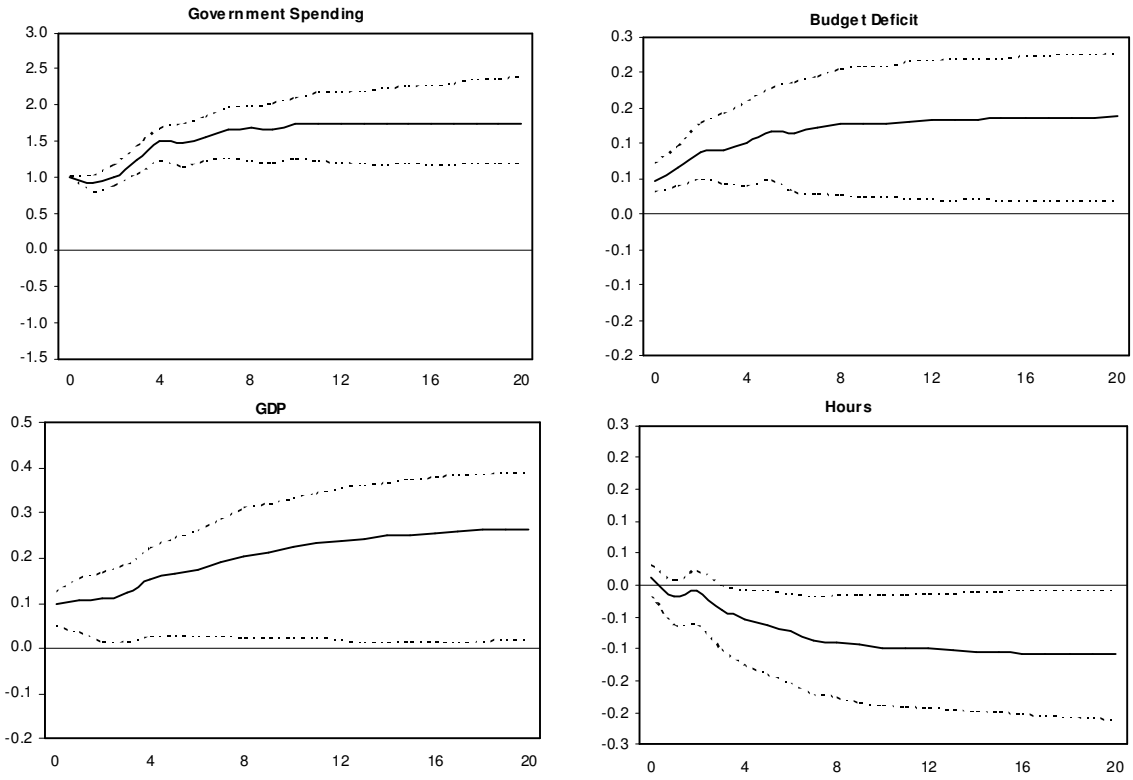


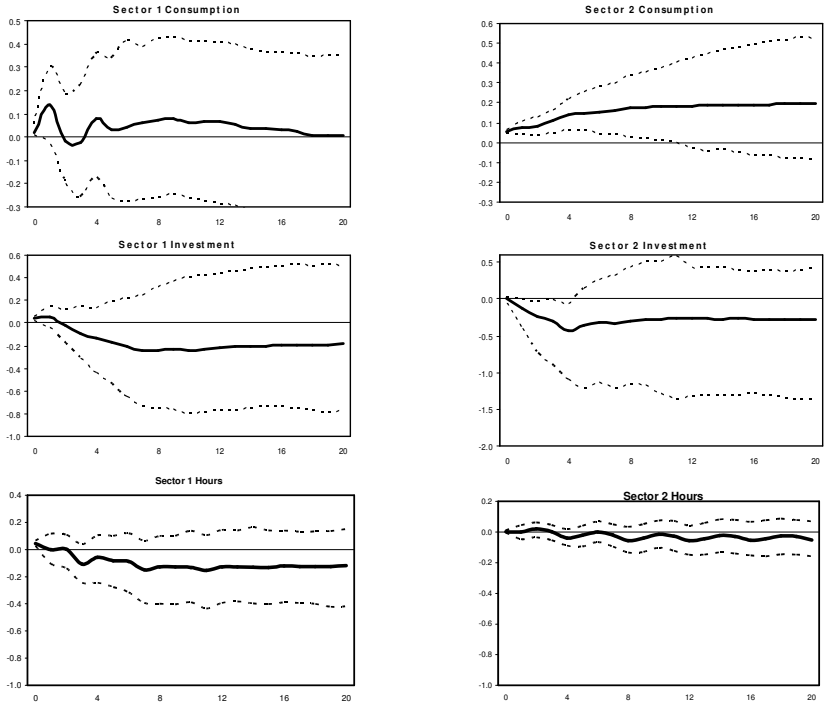
Figure 5.
Government Nondefense Investment to Consumption Ratio

Figure 6. Responses of the major aggregates to a positive government shock



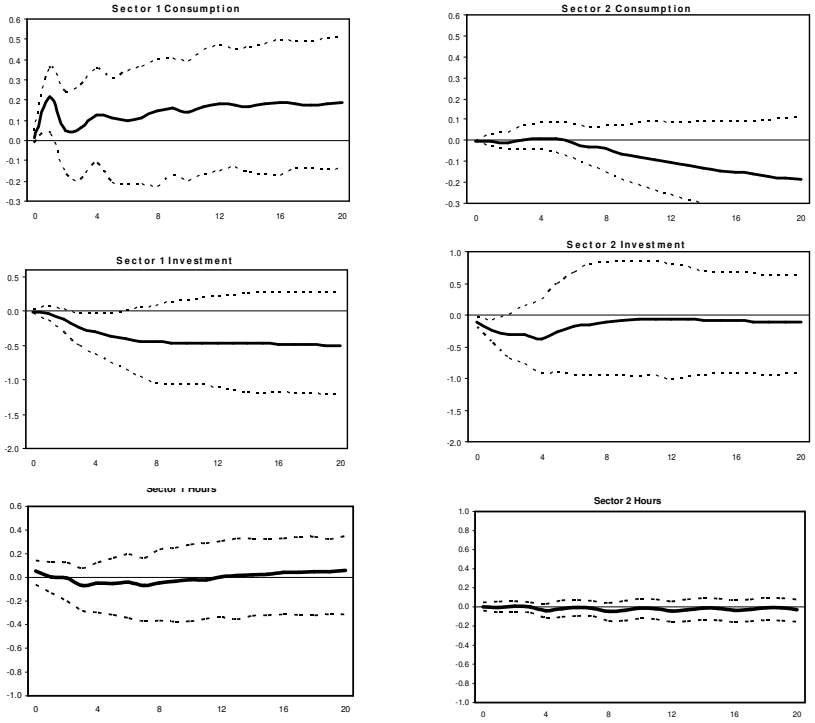
[27] Note: Estimated impulse responses to a government spending shock. Sample period 1964:Q1 – 2005Q2. The horizontal axis represents quarters after the shock. Confidence bounds are +/- 1 standard deviations of empirical distributions bootstrapped based on 1,000 replications.

Figure 7. Durable and non-durable sector responses to a positive government shock – Benchmark calibration



[28] Note: Estimated impulse responses to a government spending shock. Sample period 1964:Q1 – 2005Q2. The horizontal axis represents quarters after the shock. Confidence bounds are +/- 1 standard deviations of empirical distributions bootstrapped based on 1,000 replications.

Figure 8. Durable and non-durable sector responses to a positive government shock – RBC calibration



[29] Note: Estimated impulse responses to a government spending shock. Sample period 1964:Q1 – 2005Q2. The horizontal axis represents quarters after the shock. Confidence bounds are +/- 1 standard deviations of empirical distributions bootstrapped based on 1,000 replications.

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CHAPTER II

THE IMPACT OF FISCAL EXPANSIONS ON PRIVATE SPENDING: A REAL BUSINESS CYCLE APPROACH

Abstract

The present chapter of this thesis addresses the same question as what the previous chapter poses, namely: how does government spending affect private spending in the United States?

The framework adopted here, to answer that question, can be characterized as following a perspective most commonly associated with that of RBC models. Therefore, the goal of this paper is to consider differential growth rates between the two sectors considered, while abstracting from stickiness, in an attempt to answer the question from a different viewpoint.

Chapter 1 of this thesis shows how the impact of a government spending shock to private spending can be disaggregated into its effect on spending in two different sectors of the economy: one dedicated to the production of durables and the other to the production non-durable goods.

In this chapter, the analysis of two sectors is again motivated by the evident disparity in the growth rates of real consumption and investment (as shown in chapters one and three of this thesis) since the recent productivity revival of the nineties. Allowing for different consumer behaviors was key in addressing the distinction between the New-Keynesian and RBC models. While chapter 1 was essentially written from a New-Keynesian perspective, it was shown that under a particular calibration, an RBC model could be nested within the proposed system. Importantly, this model abstracted from technology shocks. The proper consideration of these shocks constitutes a necessary extension as it is well known that fluctuations in total factor productivity have played a key role, since the seminal Kydland and Prescott's (1982), in RBC models.

Incorporating technology shocks into the analysis, of the fiscal impact on private spending, is the main motivation for the present chapter. Therefore, chapter 2 represents an attempt to estimate the impact of government shocks on private spending by applying a RBC model to post-war data that distinguishes between improvements in consumption versus investment-specific technologies. Additionally, investment in either sector faces differential adjustment costs. In this context, the different technology shocks act as competitors to government and consumer preference shocks in eliciting different dynamic responses in the observable variables of interest.

This chapter finds evidence that increases in government spending lead to higher levels of consumption, and importantly, a positive investment

effect as well. While both adjustment costs seem to be important in describing the model, the degree of capital adjustment supersedes the labor adjustment costs both in terms of size and significance. According to this model, consumption seems to be driven mainly by consumer preferences, consumption-specific technology shocks, and consumption-specific government shocks in that order. Conversely, investment seems to be driven mostly by investment-specific technology and government shocks. Hours worked are mainly driven by preference shocks. All these features are highly intuitive, given the close relationship between preferences and the labor-leisure choice suggested by micro-founded general equilibrium models. Finally, the results show that investment-specific technology seems to grow faster than consumption-specific productivity and this can inform the disparity in rates of growth of consumption and investment in the U.S.

The paper is organized as follows. Section I provides motivation and describes the possible contribution of the study. Section II presents the two-sector model. Section III describes competitive equilibrium conditions. Section IV describes the stationarity conversion of the conditions described in section III. Sections V and VI, outline the linearization of equilibrium conditions around the steady state. Section VII casts the proposed model into state space by way of a *complex generalized Schur decomposition*, advances a strategy for solving such a model and outlines a procedure for estimating it. Section VIII presents results and section IX concludes.

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I. Introduction

How does government policy, specifically fiscal expansions, affect aggregate economic activity? What are the mechanisms that affect the transmission of these effects? While, *generally*, the common wisdom in the field of macroeconomics anticipates that an increase in government spending will have an expansionary effect on output, there is no such agreement on the implied effect on consumption. While New Keynesians (NK) disagree, Real Business Cycle (RBC) proponents predict a rise in government expenditures typically precedes a decrease in consumption. Some researchers [Greenwood et al. (1997, 2000), Whelan (2003), Kimball (1994) and Lindé (2004), among others] have argued that a two-sector model of the economy, that distinguishes investment-specific technological processes from the general "know-how" in the production of goods and services, provides a better picture of the long-run behavior of the U.S. economy. Sparked in part by this insight, Chapter 1 of this thesis showed how the impact of a government spending shock to private spending could be disaggregated into its effect on spending on durables versus non-durables. The analysis of two sectors is motivated by the evident disparity in the growth rates of real consumption and investment (as

shown in chapters one and three of this thesis) since the recent productivity revival of the nineties. Allowing for different consumer behaviors was key in addressing the distinction between the New-Keynesian and RBC models. While Chapter 1 was written from a New-Keynesian perspective, it was shown that under a particular calibration, an RBC model could be nested within the proposed system. Importantly, this model abstracted from technology shocks. Considering these shocks is a necessary extension as it is well-known that fluctuations in total factor productivity have played a key role, since the seminal work of Kydland and Prescott (1982), in RBC models. In more recent RBC research, Greenwood, Hercowitz and Krusell (1997, 2000); Fisher (2003); and Marquis and Trehan (2005) use data on relative prices to distinguish between technology shocks to consumption and investment-specific production sectors. Hobjin (2005) points out the unreliability of these price data for making inferences about the rate of investment-specific technology. In response to Hobjin's criticism, Basu, Fernald, Fisher and Kimball (2005) construct sector-specific technology measures from industry-level figures, rather than the problematic price data, to distinguish between consumption and investment-specific production processes.

Dejong, Ingram and Whiteman (2000) use aggregate data to estimate a model of neutral versus investment-specific technological change.

Incorporating technology shocks into the analysis of the fiscal impact on private spending is the main motivation for the present chapter. Therefore, Chapter 2 represents an attempt to estimate the impact of government shocks on private spending by applying a RBC model to post-war U.S. aggregate data that distinguishes between improvements in consumption versus investment-specific technologies. In this context, the different technology shocks act as competitors to government and consumer preference shocks in eliciting different dynamic responses in the observable variables of interest.

II. The Model

I consider an economy populated by an infinitely-lived representative agent who is endowed with one unit of time and draws additively separable utilities from private goods, public goods and leisure and produces consumer and investment goods each with different rates of technological progress. The second agent in this economy is a fiscal authority who purchases goods and services and balances its budget each period. This specification allows for government purchases to provide utility as a positive preference externality outside the household's control, and they enter the production functions as separable inputs that in a labor-augmenting way are combined with consumption and investment-specific technologies. This setup represents an extension of the model developed by Whelan (2003), where separate production functions for investment and consumption are combined with a logarithmic utility function for consumption to allow for a constant share of nominal investment and consumption expenditures in the steady state, while the real shares display differential growth rates driven by separate sector-specific technological processes. Chapter 3 of this thesis shows the stationarity of the U.S. investment-consumption ratio in nominal

terms while the real ratio is characterized by a trend. This evidence is in line with Whelan's (2003, 2004) assertion that constant nominal and trending real shares of consumption versus investment goods most accurately describes the behavior of U.S. postwar data. This model extends the utility function advocated by Whelan to include leisure as well as government. To enrich the dynamics of the model, it combines adjustment costs to the labor and capital accumulation with two sector-specific production functions. In addition to adjustment costs, the model combines preference shocks appearing in the utility function along with technology and government shocks appearing in the household's production function that are sector-specific. Following Pakko (2002, 2005), each of these shocks has a level and a growth component in order to derive inference on the persistence of the sector-specific shock. These (level and growth rate) shocks will all compete in explaining fluctuations in consumption, investment, hours worked and government spending.

II.1. ***The Representative Household***

The economy is populated by an infinitely-lived representative household that is endowed with one unit of time and has preferences described by the following expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln(C_t) + \ln(G_t) - \frac{1}{P_t}(N_{ct} + N_{it}) \right] \tag{1.1}$$

which is assumed to be additively separable among consumption (C_t), the labor supplied to produce consumption (N_{ct}) and investment (N_{it}) goods and public goods spending. G_t denotes *aggregate* government purchases of goods and services. The household enjoys a certain level of utility from available public goods, but public expenditures are not in the household's control.[30] The subjective discount factor β is assumed to lie between 0 and 1. The function $U(\bullet)$ is increasing and strictly concave in private consumption and also non-decreasing in public spending. Following Hansen's (1985) and Rogerson's (1988) formulations of indivisible labor, where the individual household is not allowed to work part time, is the reason for the assumption of

[30] The assumption of additive separable utility between private and public goods implies that the marginal utility of consumption is independent of government expenditures regardless of any given formulation of a balanced budget rule.

linearity of (1.1) in the labor supply assigned to the production of each good.

A generic solution to the household problem will yield a typical intra-temporal condition where the preference shock (P_t) affects the marginal rate of substitution between consumption and leisure. This specification relates an increase in P_t with an increase in equilibrium hours worked in each sector $(N_{ct} + N_{it})$. Francis and Ramey (2005) and Ireland and Schuh (2006) consider similar preference shocks in the context of RBC models, while Chang and Schorfheide (2003) and Gali, Gertler and López-Salido (2003), among others, show that preference shocks of this kind can be easily distinguished from technological disturbances as additional drivers of aggregate fluctuations at any horizon. Here, I extend Ireland and Schuh's by considering two other sources (besides P_t) of non-technology shocks: G_{ct} and G_{it} so that the estimated model is not forced to impute most of the post-war fluctuations in U.S. data to technology shocks alone.

The representative household supplies labor to two production sectors. Each sector takes capital, labor, technology and public goods as complementary inputs to produce its respective goods. This

specification represents a variation of the one-sector Cobb-Douglas production function advanced by Barro (1990)

$$Y_{xt} = K_{xt}^{\theta_x} \left(Z_{xt} N_{xt}\right)^{1-\theta_x} G_{xt}^{\varphi_x}, \qquad 0 < \theta_x < 1, \varphi_x \geq 0$$

where x represents either the consumption (c) or investment (i) goods.

This technology exhibits constant returns-to-scale with respect to private capital and labor-augmented technology; and φ captures the degree of positive external effect that public spending exerts on production. I take the simplifying assumption that sectoral government spending enters the production function in a labor-augmenting way, rather than as an exogenous externality. The assumption (as implied by $\varphi_x = 1 - \theta_x$) is that technological know-how as well as the availability of public goods increase labor productivity. This modification renders the following production function for consumer goods

$$Y_{ct} = K_{ct}^{\theta_c} \left(Z_{ct} N_{ct} G_{ct}\right)^{1-\theta_c}, \qquad 0 < \theta_c < 1 \tag{1.2}$$

along with the following production technology for investment goods

$$Y_{it} = K_{it}^{\theta_i} \left(Z_{it} N_{it} G_{it}\right)^{1-\theta_i}, \qquad 0 < \theta_i < 1 \tag{1.3}$$

where N_{ct} and N_{it} denote labor supply employed in producing consumption and investment goods, respectively. K_{ct} and K_{it} are the

stocks of capital allocated to each sector. G_{ct} and G_{it} are public purchases treated as additional inputs to the production of each sector. Finally, Z_{ct} and Z_{it} represent sector-specific technology shocks. Borrowing from Basu, Fernald and Shapiro (2001), the firm faces capital adjustment costs that deduct from production in each sector. These quadratic costs are given by

$$\frac{\phi_{kc}}{2}\left(\frac{I_{ct}}{K_{ct}}-\tilde{\kappa}_c\right)^2 Y_{ct} \qquad \text{and} \qquad \frac{\phi_{ki}}{2}\left(\frac{I_{it}}{K_{it}}-\tilde{\kappa}_i\right)^2 Y_{it} \tag{1.4}$$

for the consumption and investment good-producing sector, respectively. Similarly, the household, in its production of consumption and investment goods, also faces the following labor adjustment costs

$$\frac{\phi_{nc}}{2}\left(\frac{N_{ct}}{N_{ct-1}}-\tilde{\eta}_c\right)^2 Y_{ct} \qquad \text{and} \qquad \frac{\phi_{ni}}{2}\left(\frac{N_{it}}{N_{it-1}}-\tilde{\eta}_i\right)^2 Y_{it} \tag{1.5}$$

for the investment good-producing sector. These costs apply to all investment imputed to either the consumption (I_{ct}) or investment goods-producing sector (I_{it}). As Basu et al. point out, the implication of this specification is that the household incurs these costs whether it is incorporating new units of capital or merely reallocating existing units of capital across sectors. The parameters $\tilde{\kappa}_c$ and $\tilde{\kappa}_i$ represent the steady-

state investment-capital ratio in each sector, and $\tilde{\eta}_c$ and $\tilde{\eta}_i$ the growth rate of labor dedicated to each sector in the steady state. This particular designation of the parameters ensures that there are no adjustment costs in the steady state. Finally, the parameters $\phi_{kc}, \phi_{ki}, \phi_{nc}, \phi_{ni} \geq 0$ govern the extent of the capital and labor adjustment costs. Capital stock in each sector accumulates in the usual way

$$K_{ct+1} \leq (1-\delta_c)K_{ct} + I_{ct} \tag{1.6}$$

and

$$K_{it+1} \leq (1-\delta_i)K_{it} + I_{it} \tag{1.7}$$

for all t = 0, 1, 2,… The depreciation rates δ_c and δ_i, for the accumulation of capital dedicated to the production of consumption and investment goods, both lie between zero and one. In the aggregate, the budget constraint faced by the representative household is given by

$$K_{t+1} = (1-\tau_t)Y_t + (1-\delta)K_t - C_t \tag{1.8}$$

where τ_t is a proportional income tax. This tax rate is assumed to be nonnegative, to rule out the existence of income subsidies that could be financed by lump-sum taxation, but less than 1, to create an incentive for the representative household to produce consumption and investment goods.

II.2. ***The Fiscal Authority***

The government maximizes its purchases of intermediate (investment) goods and final (consumption) goods but never runs deficits. The government's instantaneous budget constraint therefore is given by

$$G_t = \tau_t Y_t \tag{1.9}$$

Therefore, combining (1.8) and (1.9) yields the aggregate resource constraint for the economy

$$Y_t = C_t + K_{t+1} - (1-\delta) K_t + G_t \tag{1.10}$$

III. Equilibrium Conditions

A competitive equilibrium can be defined in a standard way as a sequence of prices and allocations such that both the representative household and the firm optimize in a market clearing environment. However, since both welfare theorems hold in this economy, we can instead solve the equivalent and simpler social planner's problem that maximizes the utility of the representative household subject to the economy's resource constraints in each sector. These sector-specific resource constraints are specified from combining the production functions (1.2) and (1.3), their respective adjustment costs (1.4) and (1.5), and a sectoral version of the aggregate resource constraint (1.10). Thus, the aggregate resource constraint for consumption goods is given by

$$\left(1-\frac{\phi_{kc}}{2}\left(\frac{I_{ct}}{K_{ct}}-\tilde{\kappa}_c\right)^2\right)\left(1-\frac{\phi_{nc}}{2}\left(\frac{N_{ct}}{N_{ct-1}}-\tilde{\eta}_c\right)^2\right)K_{ct}^{\theta_c}\left(Z_{ct}G_{ct}N_{ct}\right)^{1-\theta_c}\geq C_t \quad (1.11)$$

and similarly for investment goods

$$\left(1-\frac{\phi_{ki}}{2}\left(\frac{I_{it}}{K_{it}}-\tilde{\kappa}_i\right)^2\right)\left(1-\frac{\phi_{ni}}{2}\left(\frac{N_{it}}{N_{it-1}}-\tilde{\eta}_i\right)^2\right)K_{it}^{\theta_i}\left(Z_{it}G_{it}N_{it}\right)^{1-\theta_i}\geq I_{ct}+I_{it} \quad (1.12)$$

where private investment can be allocated in order to produce other investment goods or consumption goods.

The representative household chooses the sequences of: $C_t, N_{ct}, N_{it}, I_{ct}, I_{it}, K_{ct+1}, K_{it+1}$ for all t = 0, 1, 2,… to maximize (1.1) subject to the laws of motion of capital (1.6) and (1.7) and the resource constraints (1.11) and (1.12) for each sector. The first order conditions for this problem are given by

$$\frac{1}{C_t} = \Lambda_{ct} \tag{1.13}$$

$$\begin{aligned}\frac{1}{P_t} = {} & \frac{(1-\theta_c)\Lambda_{ct}C_t}{N_{ct}} - \phi_{nc}\Lambda_{ct}\left(\frac{N_{ct}}{N_{ct-1}} - \tilde{\eta}_c\right)\left(\frac{1}{N_{ct-1}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{I_{ct}}{K_{ct-1}} - \tilde{k}_c\right)^2\right]K_{ct}^{\theta_c}\left(Z_{ct}G_{ct}N_{ct}\right)^{1-\theta_c} \\ & + \beta\phi_{nc}E_t\left\{\Lambda_{ct+1}\left(\frac{N_{ct+1}}{N_{ct}} - \tilde{\eta}_c\right)\left(\frac{N_{ct+1}}{N_{ct}}\right)\left(\frac{1}{N_{ct}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{I_{ct+1}}{K_{ct+1}} - \tilde{k}_c\right)^2\right]K_{ct+1}^{\theta_c}\left(Z_{ct+1}G_{ct+1}N_{ct+1}\right)^{1-\theta_c}\right\}\end{aligned} \tag{1.14}$$

$$\begin{aligned}\frac{1}{P_t} = {} & \frac{(1-\theta_i)\Lambda_{it}C_t}{N_{ct}} - \phi_{ni}\Lambda_{it}\left(\frac{N_{it}}{N_{it-1}} - \tilde{\eta}_i\right)\left(\frac{1}{N_{it-1}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{I_{it}}{K_{it-1}} - \tilde{k}_i\right)^2\right]K_{it}^{\theta_i}\left(Z_{it}G_{it}N_{it}\right)^{1-\theta_i} \\ & + \beta\phi_{ni}E_t\left\{\Lambda_{it+1}\left(\frac{N_{it+1}}{N_{it}} - \tilde{\eta}_i\right)\left(\frac{N_{it+1}}{N_{it}}\right)\left(\frac{1}{N_{it}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{I_{it+1}}{K_{it+1}} - \tilde{k}_i\right)^2\right]K_{it+1}^{\theta_i}\left(Z_{it+1}G_{it+1}N_{it+1}\right)^{1-\theta_i}\right\}\end{aligned} \tag{1.15}$$

$$\Xi_{ct} = \Lambda_{it} + \phi_{kc}\Lambda_{ct}\left[1 - \frac{\phi_{nc}}{2}\left(\frac{N_{ct}}{N_{ct-1}} - \tilde{\eta}_c\right)^2\right]\left(\frac{I_{ct}}{K_{ct}} - \tilde{k}_c\right)\left(\frac{1}{K_{ct}}\right)K_{ct}^{\theta_c}\left(Z_{ct}G_{ct}N_{ct}\right)^{1-\theta_c} \tag{1.16}$$

$$\Xi_{it} = \Lambda_{it}\left\{1 + \phi_{ki}\left[1 - \frac{\phi_{ni}}{2}\left(\frac{N_{it}}{N_{it-1}} - \tilde{\eta}_i\right)^2\right]\left(\frac{I_{it}}{K_{it}} - \tilde{k}_i\right)\left(\frac{1}{K_{it}}\right)K_{it}^{\theta_i}\left(Z_{it}G_{it}N_{it}\right)^{1-\theta_i}\right\} \tag{1.17}$$

$$\Xi_{ct} = \beta E_t\left[(1-\delta_c)\Xi_{ct+1}\right] + \beta\theta_c E_t\left[\frac{\Lambda_{ct+1}C_{t+1}}{K_{ct+1}}\right] + \\ + \beta\phi_{kc}E_t\left\{\Lambda_{ct+1}\left[1-\frac{\phi_{nc}}{2}\left(\frac{N_{ct+1}}{N_{ct}}-\tilde{\eta}_c\right)^2\right]\left(\frac{I_{ct+1}}{K_{ct+1}}-\tilde{\kappa}_c\right)\left(\frac{I_{ct+1}}{K_{ct+1}^2}\right)K_{ct+1}^{\theta_c}\left(Z_{ct+1}G_{ct+1}N_{ct+1}\right)^{1-\theta_c}\right\} \tag{1.18}$$

$$\Xi_{it} = \beta E_t\left[(1-\delta_i)\Xi_{it+1}\right] + \beta\theta_i E_t\left[\frac{\Lambda_{it+1}C_{t+1}}{K_{it+1}}\right] + \\ + \beta\phi_{ki}E_t\left\{\Lambda_{it+1}\left[1-\frac{\phi_{ni}}{2}\left(\frac{N_{it+1}}{N_{it}}-\tilde{\eta}_i\right)^2\right]\left(\frac{I_{it+1}}{K_{it+1}}-\tilde{\kappa}_i\right)\left(\frac{I_{it+1}}{K_{it+1}^2}\right)K_{it+1}^{\theta_i}\left(Z_{it+1}G_{it+1}N_{it+1}\right)^{1-\theta_i}\right\} \tag{1.19}$$

along with the four complementarity conditions given by (1.6), (1.7), (1.11)and (1.12) with equality in all periods. Aggregate labor supply is given by

$$N_t = N_{ct} + N_{it} \tag{1.20}$$

and aggregate investment is given by

$$I_t = I_{ct} + I_{it} \tag{1.21}$$

These equilibrium conditions include the nonnegative shadow prices Λ_{ct} and Λ_{it}, associated with the production possibility constraints (1.11) and (1.12), along with the nonnegative multipliers Ξ_{ct} and Ξ_{it} associated with the capital accumulation equations (1.6) and (1.7).

These equilibrium conditions can be interpreted in the usual way. Equation (1.13) shows how the Lagrange multiplier measures the nonnegative marginal utility of consumption. Equation (1.14) shows

the equivalence of the marginal rate of substitution, between consumption and leisure, and the marginal product of labor in the consumption goods-producing sector. Similarly equation (1.15) shows the equivalence between the marginal rate of substitution and the marginal product of labor for the investment goods-producing sector. Equations (1.16) and (1.17) show how capital adjustment costs create a wedge between the shadow price Λ_{it} of new investment goods and the shadow prices Ξ_{ct} and Ξ_{it} of installed capital in each sector. Lastly, the Euler equations (1.18) and (1.19) show how the shadow prices Ξ_{ct} and Ξ_{it} of newly installed capital in each sector equal the present discounted value of the additional output produced by a cost-adjusted additional unit of capital in that sector.

To close the model, we need to make some assumptions about the exogenous preference (P_t), the sector-specific technology (Z_{ct}, Z_{it}) and government (G_{ct}, G_{it}) shocks. To allow for different dynamics in the persistence characteristics of these driving processes, each shock is decomposed into two separate autoregressive components for its level and growth rate as follows

$$\ln(P_t) = \ln(p_t^l) + \ln(P_t^g) \tag{1.22}$$

$$\ln\left(p_t^l\right)=\rho_p^l\ln\left(p_{t-1}^l\right)+\varepsilon_{pt}^l \tag{1.23}$$

$$\ln\left(\frac{P_t^g}{P_{t-1}^g}\right)=\left(1-\rho_p^g\right)\ln\left(\overline{p}^g\right)+\rho_p^g\ln\left(\frac{P_{t-1}^g}{P_{t-2}^g}\right)+\varepsilon_{pt}^g \tag{1.24}$$

$$\ln\left(Z_{ct}\right)=\ln\left(z_{ct}^l\right)+\ln\left(Z_{ct}^g\right) \tag{1.25}$$

$$\ln\left(z_{ct}^l\right)=\rho_{zc}^l\ln\left(z_{ct-1}^l\right)+\varepsilon_{zct}^l \tag{1.26}$$

$$\ln\left(\frac{Z_{ct}^g}{Z_{ct-1}^g}\right)=\left(1-\rho_{zc}^g\right)\ln\left(\overline{z}_c^g\right)+\rho_{zc}^g\ln\left(\frac{Z_{ct-1}^g}{Z_{ct-2}^g}\right)+\varepsilon_{zct}^g \tag{1.27}$$

$$\ln\left(Z_{it}\right)=\ln\left(z_{it}^l\right)+\ln\left(Z_{it}^g\right) \tag{1.28}$$

$$\ln\left(z_{it}^l\right)=\rho_{zi}^l\ln\left(z_{it-1}^l\right)+\varepsilon_{zit}^l \tag{1.29}$$

$$\ln\left(\frac{Z_{it}^g}{Z_{it-1}^g}\right)=\left(1-\rho_{zi}^g\right)\ln\left(\overline{z}_i^g\right)+\rho_{zi}^g\ln\left(\frac{Z_{it-1}^g}{Z_{it-2}^g}\right)+\varepsilon_{zit}^g \tag{1.30}$$

$$\ln\left(G_{ct}\right)=\ln\left(g_{ct}^l\right)+\ln\left(G_{ct}^g\right) \tag{1.31}$$

$$\ln\left(g_{ct}^l\right)=\rho_{gc}^l\ln\left(g_{ct-1}^l\right)+\varepsilon_{gct}^l \tag{1.32}$$

$$\ln\left(\frac{G_{ct}^g}{G_{ct-1}^g}\right)=\left(1-\rho_{gc}^g\right)\ln\left(\overline{g}_c^g\right)+\rho_{gc}^g\ln\left(\frac{G_{ct-1}^g}{G_{ct-2}^g}\right)+\varepsilon_{gct}^g \tag{1.33}$$

$$\ln\left(G_{it}\right)=\ln\left(g_{it}^l\right)+\ln\left(G_{it}^g\right) \tag{1.34}$$

$$\ln\left(g_{it}^l\right)=\rho_{gi}^l\ln\left(g_{it-1}^l\right)+\varepsilon_{git}^l \tag{1.35}$$

$$\ln\left(\frac{G_{it}^g}{G_{it-1}^g}\right)=\left(1-\rho_{gi}^g\right)\ln\left(\overline{g}_i^g\right)+\rho_{gi}^g\ln\left(\frac{G_{it-1}^g}{G_{it-2}^g}\right)+\varepsilon_{git}^g \tag{1.36}$$

where the autoregressive parameters

$\rho_p^l,\rho_p^g,\rho_{zc}^l,\rho_{zc}^g,\rho_{zi}^l,\rho_{zi}^g,\rho_{gc}^l,\rho_{gc}^g,\rho_{gi}^l,\rho_{gi}^g$ all lie between zero and one.

The innovations $\varepsilon_p^l, \varepsilon_p^g, \varepsilon_{zc}^l, \varepsilon_{zc}^g, \varepsilon_{zi}^l, \varepsilon_{zi}^g, \varepsilon_{gc}^l, \varepsilon_{gc}^g, \varepsilon_{gi}^l, \varepsilon_{gi}^g$ are white noise processes with standard normal distributions and standard deviations $\sigma_p^l, \sigma_p^g, \sigma_{zc}^l, \sigma_{zc}^g, \sigma_{zi}^l, \sigma_{zi}^g, \sigma_{gc}^l, \sigma_{gc}^g, \sigma_{gi}^l, \sigma_{gi}^g$.

IV. The Stationary System

Equations (1.13)-(1.19), (1.6), (1.7), (1.11), (1.12), and (1.20)-(1.36) describe the behavior of the following 28 variables: $C_t, N_{ct}, N_{it}, N_t, I_{ct}, I_{it}, I_t, K_{ct}, K_{it}, \Lambda_{ct}, \Lambda_{it}, \Xi_{ct}, \Xi_{it}, P_t, p_t^l, P_t^g, Z_{ct}, z_{ct}^l, Z_{ct}^g, Z_{it}, z_{it}^l, Z_{it}^g,$

$G_{ct}, g_{ct}^l, G_{ct}^g, G_{it}, g_{it}^l$ and G_{it}^g. Thus, with this set up, I adopt Pakko's (2002, 2005) specification where in the long run, only the growth rate component can account for the nonstationary behavior of the variables of interest.[31]

At optimum, some of these variables inherit the nonstationarity characteristics from the presence of some of the exogenous shocks. To deal with this problem, I rescale the variables in the following way:

$$\text{Let } c_t = \frac{C_t}{P_{t-1}^g \left(Z_{it-1}^g G_{it-1}^g\right)^{\theta_c} \left(Z_{ct-1}^g G_{ct-1}^g\right)^{1-\theta_c}}, \quad i_{ct} = \frac{I_{ct}}{P_{t-1}^g Z_{it-1}^g G_{it-1}^g}, \quad i_{it} = \frac{I_{it}}{P_{t-1}^g Z_{it-1}^g G_{it-1}^g},$$

$$i_t = \frac{I_t}{P_{t-1}^g Z_{it-1}^g G_{it-1}^g}, \; k_{ct} = \frac{K_{ct}}{P_{t-1}^g Z_{it-1}^g G_{it-1}^g}, k_{it} = \frac{K_{it}}{P_{t-1}^g Z_{it-1}^g G_{it-1}^g}, k_t = \frac{K_t}{P_{t-1}^g Z_{it-1}^g G_{it-1}^g},$$

$$n_{ct} = \frac{N_{ct}}{P_{t-1}^g}, \; n_{it} = \frac{N_{it}}{P_{t-1}^g}, \; n_t = \frac{N_t}{P_{t-1}^g}, \; \lambda_{ct} = P_{t-1}^g \left(Z_{it-1}^g G_{it-1}^g\right)^{\theta_c} \left(Z_{ct-1}^g G_{ct-1}^g\right)^{1-\theta_c} \Lambda_{ct},$$

[31] This applies to the shock component that is stationary in growth rates. Of course, both the level-stationary and growth-stationary components may have an impact in the short run. Pakko adopts this set up for the effect of technology shocks to consumption and investment. I extend it to apply to the effects of technology, preferences as well as the government shocks to consumption, investment, hours worked and government spending in the U.S.

$\lambda_{it} = P^g_{t-1} Z^g_{it-1} G^g_{it-1} \Lambda_{it}$, $\xi_{ct} = P^g_{t-1} Z^g_{it-1} G^g_{it-1} \Xi_{ct}$, $\xi_{it} = P^g_{t-1} Z^g_{it-1} G^g_{it-1} \Xi_{it}$, $p_t = \dfrac{P_t}{P^g_{t-1}}$,

$p^g_t = \dfrac{P^g_t}{P^g_{t-1}}$, $z_{ct} = \dfrac{Z_{ct}}{Z^g_{ct-1}}$, $z^g_{ct} = \dfrac{Z^g_{ct}}{Z^g_{ct-1}}$, $z_{it} = \dfrac{Z_{it}}{Z^g_{it-1}}$, $z^g_{it} = \dfrac{Z^g_{it}}{Z^g_{it-1}}$, $g_{ct} = \dfrac{G_{ct}}{G^g_{ct-1}}$,

$g^g_{ct} = \dfrac{G^g_{ct}}{G^g_{ct-1}}$ $g_{it} = \dfrac{G_{it}}{G^g_{it-1}}$, and $g^g_{it} = \dfrac{G^g_{it}}{G^g_{it-1}}$.

All lower case variables have therefore been rendered stationary. Hence, we can write the first order conditions in the previous section in terms of these stationary variables as

$$1 = \lambda_{ct} c_t \tag{2.1}$$

$$\begin{aligned}\frac{1}{p_t} = {} & \frac{(1-\theta_c)\lambda_{ct} c_t}{n_{ct}} - \phi_{nc}\lambda_{ct}\left(\frac{p^g_{t-1} n_{ct}}{n_{ct-1}} - \tilde{\eta}_c\right)\left(\frac{p^g_{t-1}}{n_{ct-1}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{i_{ct}}{k_{ct-1}} - \tilde{k}_c\right)^2\right] k_{ct}^{\theta_c}\left(z_{ct} g_{ct} n_{ct}\right)^{1-\theta_c} \\ & + \beta\phi_{nc} E_t\left\{\lambda_{ct+1}\left(\frac{p^g_t n_{ct+1}}{n_{ct}} - \tilde{\eta}_c\right)\left(\frac{p^g_t n_{ct+1}}{n_{ct}}\right)\left(\frac{1}{n_{ct}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{i_{ct+1}}{k_{ct+1}} - \tilde{k}_c\right)^2\right] k_{ct+1}^{\theta_c}\left(z_{ct+1} g_{ct+1} n_{ct+1}\right)^{1-\theta_c}\right\}\end{aligned} \tag{2.2}$$

$$\begin{aligned}\frac{1}{p_t} = {} & \frac{(1-\theta_i)\lambda_{it} i_t}{n_{it}} - \phi_{ni}\lambda_{it}\left(\frac{p^g_{t-1} n_{it}}{n_{it-1}} - \tilde{\eta}_i\right)\left(\frac{p^g_{t-1}}{n_{it-1}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{i_{it}}{k_{it-1}} - \tilde{k}_i\right)^2\right] k_{it}^{\theta_i}\left(z_{it} g_{it} n_{it}\right)^{1-\theta_i} \\ & + \beta\phi_{ni} E_t\left\{\lambda_{it+1}\left(\frac{p^g_t n_{it+1}}{n_{it}} - \tilde{\eta}_i\right)\left(\frac{p^g_t n_{it+1}}{n_{it}}\right)\left(\frac{1}{n_{it}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{i_{it+1}}{k_{it+1}} - \tilde{k}_i\right)^2\right] k_{it+1}^{\theta_i}\left(z_{it+1} g_{it+1} n_{it+1}\right)^{1-\theta_i}\right\}\end{aligned} \tag{2.3}$$

$$\zeta_{ct} = \lambda_{it} + \phi_{kc}\lambda_{ct}\left[1 - \frac{\phi_{nc}}{2}\left(\frac{p^g_{t-1} n_{ct}}{n_{ct-1}} - \tilde{\eta}_c\right)^2\right]\left(\frac{i_{ct}}{k_{ct}} - \tilde{k}_c\right) k_{ct}^{\theta_c - 1}\left(z_{ct} g_{ct} n_{ct}\right)^{1-\theta_c} \tag{2.4}$$

$$\zeta_{it} = \lambda_{it}\left\{1 + \phi_{ki}\left[1 - \frac{\phi_{ni}}{2}\left(\frac{p^g_{t-1} n_{it}}{n_{it-1}} - \tilde{\eta}_i\right)^2\right]\left(\frac{i_{it}}{k_{it}} - \tilde{k}_i\right) k_{it}^{\theta_i - 1}\left(z_{it} g_{it} n_{it}\right)^{1-\theta_i}\right\} \tag{2.5}$$

$$p_t^g z_{it}^g g_{it}^g \zeta_{ct} = \beta(1-\delta_c) E_t[\zeta_{ct+1}] + \beta\theta_c E_t\left[\frac{\lambda_{ct+1} c_{t+1}}{k_{ct+1}}\right] + \\ + \beta\phi_{kc} E_t\left\{\lambda_{ct+1}\left[1 - \frac{\phi_{nc}}{2}\left(\frac{p_t^g n_{ct+1}}{n_{ct}} - \tilde{\eta}_c\right)^2\right]\left(\frac{i_{ct+1}}{k_{ct+1}} - \tilde{\kappa}_c\right)\left(\frac{i_{ct+1}}{k_{ct+1}}\right) k_{ct+1}^{\theta_c - 1}\left(z_{ct+1} g_{ct+1} n_{ct+1}\right)^{1-\theta_c}\right\} \tag{2.6}$$

$$p_t^g z_{it}^g g_{it}^g \zeta_{it} = \beta(1-\delta_i) E_t[\zeta_{it+1}] + \beta\theta_i E_t\left[\frac{\lambda_{it+1} i_{t+1}}{k_{it+1}}\right] + \\ + \beta\phi_{ki} E_t\left\{\lambda_{it+1}\left[1 - \frac{\phi_{ni}}{2}\left(\frac{p_t^g n_{it+1}}{n_{it}} - \tilde{\eta}_i\right)^2\right]\left(\frac{i_{it+1}}{k_{it+1}} - \tilde{\kappa}_i\right)\left(\frac{i_{it+1}}{k_{it+1}}\right) k_{it+1}^{\theta_i - 1}\left(z_{it+1} g_{it+1} n_{it+1}\right)^{1-\theta_i}\right\} \tag{2.7}$$

$$c_t = \left(1 - \frac{\phi_{kc}}{2}\left(\frac{i_{ct}}{k_{ct}} - \tilde{\kappa}_c\right)^2\right)\left(1 - \frac{\phi_{nc}}{2}\left(\frac{p_{t-1}^g n_{ct}}{n_{ct-1}} - \tilde{\eta}_c\right)^2\right) k_{ct}^{\theta_c}\left(z_{ct} g_{ct} n_{ct}\right)^{1-\theta_c} \tag{2.8}$$

$$i_t = \left(1 - \frac{\phi_{ki}}{2}\left(\frac{i_{it}}{k_{it}} - \tilde{\kappa}_i\right)^2\right)\left(1 - \frac{\phi_{ni}}{2}\left(\frac{p_{t-1}^g n_{it}}{n_{it-1}} - \tilde{\eta}_i\right)^2\right) k_{it}^{\theta_i}\left(z_{it} g_{it} n_{it}\right)^{1-\theta_i} \tag{2.9}$$

$$p_t^g z_{it}^g g_{it}^g k_{ct+1} = (1-\delta_c) k_{ct} + i_{ct} \tag{2.10}$$

$$p_t^g z_{it}^g g_{it}^g k_{it+1} = (1-\delta_i) k_{it} + i_{it} \tag{2.11}$$

$$i_t = i_{ct} + i_{it} \tag{2.12}$$

$$n_t = n_{ct} + n_{it} \tag{2.13}$$

$$\ln(p_t) = \ln(p_t^l) + \ln(p_t^g) \tag{2.14}$$

$$\ln(p_t^l) = \ln(p_{t-1}^l) + \varepsilon_{pt}^l \tag{2.15}$$

$$\ln(p_t^g) = (1-\rho_p^g)\ln(\overline{p}^g) + \rho_p^g \ln(p_{t-1}^g) + \varepsilon_{pt}^g \tag{2.16}$$

$$\ln(z_{ct}) = \ln(z_{ct}^l) + \ln(z_{ct}^g) \tag{2.17}$$

$$\ln(z_{ct}^l) = \rho_{zc}^l \ln(z_{ct-1}^l) + \varepsilon_{zct}^l \tag{2.18}$$

$$\ln(z_{ct}^g) = (1-\rho_{zc}^g)\ln(\overline{z}_c^g) + \rho_{zc}^g \ln(z_{ct-1}^g) + \varepsilon_{zct}^g \tag{2.19}$$

$$\ln\left(z_{it}\right)=\ln\left(z_{it}^{l}\right)+\ln\left(z_{it}^{g}\right) \tag{2.20}$$

$$\ln\left(z_{it}^{l}\right)=\rho_{zi}^{l}\ln\left(z_{it-1}^{l}\right)+\varepsilon_{zit}^{l} \tag{2.21}$$

$$\ln\left(z_{it}^{g}\right)=\left(1-\rho_{zi}^{g}\right)\ln\left(\overline{z}_{i}^{g}\right)+\rho_{zi}^{g}\ln\left(z_{it-1}^{g}\right)+\varepsilon_{zit}^{g} \tag{2.22}$$

$$\ln\left(g_{ct}\right)=\ln\left(g_{ct}^{l}\right)+\ln\left(g_{ct}^{g}\right) \tag{2.23}$$

$$\ln\left(g_{ct}^{l}\right)=\rho_{gc}^{l}\ln\left(g_{ct-1}^{l}\right)+\varepsilon_{gct}^{l} \tag{2.24}$$

$$\ln\left(g_{ct}^{g}\right)=\left(1-\rho_{gc}^{g}\right)\ln\left(\overline{g}_{c}^{g}\right)+\rho_{gc}^{g}\ln\left(g_{ct-1}^{g}\right)+\varepsilon_{gct}^{g} \tag{2.25}$$

$$\ln\left(g_{it}\right)=\ln\left(g_{it}^{l}\right)+\ln\left(g_{it}^{g}\right) \tag{2.26}$$

$$\ln\left(g_{it}^{l}\right)=\rho_{gi}^{l}\ln\left(g_{it-1}^{l}\right)+\varepsilon_{git}^{l} \tag{2.27}$$

$$\ln\left(g_{it}^{g}\right)=\left(1-\rho_{gi}^{g}\right)\ln\left(\overline{g}_{i}^{g}\right)+\rho_{gi}^{g}\ln\left(g_{it-1}^{g}\right)+\varepsilon_{git}^{g} \tag{2.28}$$

Finally, define $\gamma_t^c \equiv \frac{C_t}{C_{t-1}}$, $\gamma_t^i \equiv \frac{I_t}{I_{t-1}}$, $\gamma_t^n \equiv \frac{N_t}{N_{t-1}}$, $\gamma_t^{gc} \equiv \frac{G_{ct}}{G_{ct-1}}$, and $\gamma_t^{gi} \equiv \frac{G_{it}}{G_{it-1}}$.

These definitions imply

$$\gamma_t^c = p_{t-1}^g\left(z_{it-1}^g g_{it-1}^g\right)^{\theta_c}\left(z_{ct-1}^g g_{ct-1}^g\right)^{1-\theta_c}\left(\frac{c_t}{c_{t-1}}\right) \tag{2.29}$$

$$\gamma_t^i = p_{t-1}^g z_{it-1}^g g_{it-1}^g\left(\frac{i_t}{i_{t-1}}\right) \tag{2.30}$$

$$\gamma_t^n = p_{t-1}^g\left(\frac{n_t}{n_{t-1}}\right) \tag{2.31}$$

$$\gamma_t^{gc} = \frac{g_{ct}}{g_{ct-1}}p_{t-1}^g \tag{2.32}$$

$$\gamma_t^{gi} = \frac{g_{it}}{g_{it-1}}p_{t-1}^g \tag{2.33}$$

V. The Steady State

Equations (2.1)-(2.33) imply that in the absence of shocks, the economy converges to a steady-state path along which all the stationary variables are constant. Letting every variable with a tilde denote a steady state for its corresponding time-varying stationary variable, we have :

$c_t = \tilde{c}, n_t = \tilde{n}, n_{ct} = \tilde{n}_c, n_{it} = \tilde{n}_i, i_t = \tilde{i}, i_{ct} = \tilde{i}_c, i_{it} = \tilde{i}_i, k_{ct} = \tilde{k}_c, k_{it} = \tilde{k}_i, \lambda_{ct} = \tilde{\lambda}_c, \lambda_{it} = \tilde{\lambda}_i,$
$\zeta_{ct} = \tilde{\zeta}_c, \zeta_{it} = \tilde{\zeta}_i, p_t = \tilde{p}, p_t^l = \tilde{p}^l, p_t^g = \tilde{p}^g, z_{ct} = \tilde{z}_c, z_{ct}^l = \tilde{z}_c^l, z_{ct}^g = \tilde{z}_c^g, z_{it} = \tilde{z}_i, z_{it}^l = \tilde{z}_i^l,$
$z_{it}^g = \tilde{z}_i^g, g_{ct} = \tilde{g}_c, g_{ct}^l = \tilde{g}_c^l, g_{ct}^g = \tilde{g}_c^g, g_{it} = \tilde{g}_i, g_{it}^l = \tilde{g}_i^l, g_{it}^g = \tilde{g}_i^g, \gamma_t^c = \tilde{\gamma}^c, \gamma_t^i = \tilde{\gamma}^i, \gamma_t^n = \tilde{\gamma}^n,$
$\gamma_t^{gc} = \tilde{\gamma}^{gc}$, *and* $\gamma_t^{gi} = \tilde{\gamma}^{gi}$ for all t = 0, 1, 2,…

Equations (2.16), (2.19), (2.22), (2.25), and (2.28) determine the steady-state values of $\tilde{p}^g, \tilde{z}_c^g, \tilde{z}_i^g, \tilde{g}_c^g$, and $\tilde{g}_i^g$, respectively. Equations (2.14), (2.15), (2.17), (2.18), (2.20), (2.21), (2.23), (2.24), (2.26), (2.27), (2.29)-(2.33) imply:

$$\tilde{p}^l = 1$$

$$\tilde{p} = \tilde{p}^g$$

$$\tilde{z}_c^l = 1$$

$$\tilde{z}_c = \tilde{z}_c^g$$

$$\tilde{z}_i^l = 1$$

$$\tilde{z}_i = \tilde{z}_i^g$$

$$\tilde{g}_c^l = 1$$

$$\tilde{g}_c = \tilde{g}_c^g$$

$$\tilde{g}_i^l = 1$$

$$\tilde{g}_i = \tilde{g}_i^g$$

$$\tilde{\gamma}^c = \tilde{p}^g \left(\tilde{z}_i^g \tilde{g}_i^g\right)^{\theta_c} \left(\tilde{z}_c^g \tilde{g}_c^g\right)^{1-\theta_c}$$

$$\tilde{\gamma}^i = \tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g$$

$$\tilde{\gamma}^n = \tilde{p}^g$$

$$\tilde{\gamma}^{gc} = \tilde{p}^g$$

In turn, equation (2.2)[32] determines $\tilde{n}_c$. Assuming solutions for $\tilde{k}_c$ and $\tilde{k}_i$ are available, we can determine the steady-state values of $\tilde{i}_c, \tilde{i}_i,$ and $\tilde{i}$ from (2.10), (2.11), and (2.12), respectively, as follows

$$\tilde{i}_c = \left[\tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - (1-\delta_c)\right]\tilde{k}_c$$

$$\tilde{i}_i = \left[\tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - (1-\delta_i)\right]\tilde{k}_i$$

$$\tilde{i} = \tilde{i}_c + \tilde{i}_i$$

Assuming zero adjustment costs in the steady state $\left(\frac{\tilde{i}_i}{\tilde{k}_i} - \tilde{\kappa}_i\right) = 0$, solving (2.9) for $\tilde{n}_i$ yields its steady-state value given by

$$\tilde{n}_i = \frac{1}{\tilde{z}_i^g \tilde{g}_i^g}\left(\frac{\tilde{i}}{\tilde{k}_i^{\theta_i}}\right)^{\frac{1}{1-\theta_i}}$$

[32] By combining it with (2.1).

This zero steady-state adjustment cost assumption reduces (2.8) to $\tilde{c} = \tilde{k}_c^{\theta_c} \left(\tilde{z}_c \tilde{n}_c \tilde{g}_c \right)^{1-\theta_c}$, the steady-state value for consumption.

Equation (2.13) determines the steady-state value of $\tilde{n}$.

$$\tilde{n} = \tilde{n}_c + \tilde{n}_i$$

Equation (2.1) determines the value of $\tilde{\lambda}_c$.

$$\tilde{\lambda}_c = \frac{1}{\tilde{c}}$$

Finally, substituting again the assumptions of $\left(\frac{\tilde{i}_i}{\tilde{k}_i} - \tilde{\kappa}_i \right) = 0$ and $\left(\frac{\tilde{i}_c}{\tilde{k}_c} - \tilde{\kappa}_c \right) = 0$ in (2.4) and (2.5) and combining those with (2.3)[33] yields the following steady-state values

$$\tilde{\zeta}_c = \tilde{\zeta}_i = \tilde{\lambda}_i = \frac{\tilde{n}_i}{(1-\theta_i)\tilde{p}^g \tilde{i}}$$

Solving (2.6) and (2.7) for $\tilde{k}_c$ and $\tilde{k}_i$, respectively, yields

$$\tilde{k}_c = \frac{\beta \theta_c}{\left(\tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - \beta(1-\delta_c) \right) \tilde{\zeta}_c}$$

and

[33] Along with the assumption $\left(\frac{\tilde{p}^g \tilde{n}_i}{\tilde{n}_i} - \tilde{\eta}_i \right) = 0$.

$$\tilde{k}_i = \left(\frac{\beta\theta_i \left[\tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - \beta(1-\delta_c) \right]}{\tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - \beta(1-\delta_i) - \beta\theta_i \left(\tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - 1 + \delta_i \right)} \right) \tilde{k}_c$$

Hence a systematic way to calculate all of these steady states is to start by calculating $\tilde{n}_c$, then calculate $\tilde{k}_c$ and $\tilde{k}_i$. Then go back and calculate the remaining steady-state values. These steady-state calculations imply that the parameters $\tilde{\kappa}_c$ and $\tilde{\kappa}_i$ must be given by $\tilde{\kappa}_c = \tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - (1-\delta_c)$ and $\tilde{\kappa}_i = \tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g - (1-\delta_i)$. Hence, in this context, capital allocated to both sectors ultimately grow at the same rate in the steady state.[34] Likewise, to have zero steady-state labor adjustment costs, parameters $\tilde{\eta}_c$ and $\tilde{\eta}_i$ must satisfy

$$\tilde{\eta}_c = \tilde{\eta}_i = \tilde{p}^g$$

[34] This is because by not specifying variable (deterministic) utilization rates in our specified technology, I have implicitly assumed a constant mix of usage between consumption and investment inputs in our production function. For a description of varying utilization rates in a similar context, see Basu et al. (2001) or Ireland and Schuh (2006).

VI. Linearization

In order to characterize how the variables in the model respond to the proposed shocks, we can log linearize the stationary dependent variables in (2.1)-(2.33) around their corresponding steady states. Let the circumflex denote log deviations from steady state as follows:

$$\hat{c}_t \equiv \ln\left(\frac{c_t}{\tilde{c}}\right), \hat{n}_t \equiv \ln\left(\frac{n_t}{\tilde{n}}\right), \hat{n}_{ct} \equiv \ln\left(\frac{n_{ct}}{\tilde{n}_c}\right), \hat{n}_{it} \equiv \ln\left(\frac{n_{it}}{\tilde{n}_i}\right), \hat{i}_t \equiv \ln\left(\frac{i_t}{\tilde{i}}\right), \hat{i}_{ct} \equiv \ln\left(\frac{i_{ct}}{\tilde{i}_c}\right),$$
$$\hat{i}_{it} \equiv \ln\left(\frac{i_{it}}{\tilde{i}_i}\right), \hat{k}_{ct} \equiv \ln\left(\frac{k_{ct}}{\tilde{k}_c}\right), \hat{k}_{it} \equiv \ln\left(\frac{k_{it}}{\tilde{k}_i}\right), \hat{\lambda}_{ct} \equiv \ln\left(\frac{\lambda_{ct}}{\tilde{\lambda}_c}\right), \hat{\lambda}_{it} \equiv \ln\left(\frac{\lambda_{it}}{\tilde{\lambda}_i}\right), \hat{\zeta}_{ct} \equiv \ln\left(\frac{\zeta_{ct}}{\tilde{\zeta}_c}\right),$$
$$\hat{\zeta}_{it} \equiv \ln\left(\frac{\zeta_{it}}{\tilde{\zeta}_i}\right), \hat{p}_t \equiv \ln\left(\frac{p_t}{\tilde{p}}\right), \hat{p}_t^l \equiv \ln\left(\frac{p_t^l}{\tilde{p}^l}\right), \hat{p}_t^g \equiv \ln\left(\frac{p_t^g}{\tilde{p}^l}\right), \hat{z}_{ct} \equiv \ln\left(\frac{z_{ct}}{\tilde{z}_c}\right), \hat{z}_{ct}^l \equiv \ln\left(\frac{z_{ct}^l}{\tilde{z}_c^l}\right),$$
$$\hat{z}_{ct}^g \equiv \ln\left(\frac{z_{ct}^g}{\tilde{z}_c^g}\right), \hat{z}_{it} \equiv \ln\left(\frac{z_{it}}{\tilde{z}_i}\right), \hat{z}_{it}^l \equiv \ln\left(\frac{z_{it}^l}{\tilde{z}_i^l}\right), \hat{z}_{it}^g \equiv \ln\left(\frac{z_{it}^g}{\tilde{z}_i^g}\right), \hat{g}_{ct} \equiv \ln\left(\frac{g_{ct}}{\tilde{g}_c}\right), \hat{g}_{ct}^l \equiv \ln\left(\frac{g_{ct}^l}{\tilde{g}_c^l}\right),$$
$$\hat{g}_{ct}^g \equiv \ln\left(\frac{g_{ct}^g}{\tilde{g}_c^g}\right), \hat{g}_{it} \equiv \ln\left(\frac{g_{it}}{\tilde{g}_i}\right), \hat{g}_{it}^l \equiv \ln\left(\frac{g_{it}^l}{\tilde{g}_i^l}\right), \hat{g}_{it}^g \equiv \ln\left(\frac{g_{it}^g}{\tilde{g}_i^g}\right), \hat{\gamma}_t^c \equiv \ln\left(\frac{\gamma_t^c}{\tilde{\gamma}^c}\right), \hat{\gamma}_t^i \equiv \ln\left(\frac{\gamma_t^i}{\tilde{\gamma}^i}\right),$$
$$\hat{\gamma}_t^n \equiv \ln\left(\frac{\gamma_t^n}{\tilde{\gamma}^n}\right), \hat{\gamma}_t^{gc} \equiv \ln\left(\frac{\gamma_t^{gc}}{\tilde{\gamma}^{gc}}\right), \text{and } \hat{\gamma}_t^{gi} \equiv \ln\left(\frac{\gamma_t^{gi}}{\tilde{\gamma}^{gi}}\right).$$

With these definitions in hand, the first order linear approximations to (2.1)-(2.33) are given by the following

$$\hat{c}_t + \hat{\lambda}_{ct} = 0 \tag{3.1}$$

$$\left((1-\theta_i)\left(\frac{1}{\tilde{p}^g}\right) + (1+\beta)\phi_{nc}\tilde{p}^g\right)\hat{n}_{ct} - (1-\theta_c)\left(\frac{1}{\tilde{p}^g}\right)\hat{p}_t = \\ = -\phi_{nc}\tilde{p}^g\left[\hat{p}_{t-1}^g\right] + \phi_{nc}\tilde{p}^g\left[\hat{n}_{ct-1}^g\right] + \beta\phi_{nc}\tilde{p}^g\left[\hat{p}_t^g\right] + \beta\phi_{nc}\tilde{p}^g E_t\left[\hat{n}_{ct+1}^g\right] \tag{3.2}$$

$$\left((1-\theta_i)\left(\frac{1}{\tilde{p}^g}\right)+(1+\beta)\phi_{ni}\tilde{p}^g\right)\hat{n}_{it}-(1-\theta_i)\left(\frac{1}{\tilde{p}^g}\right)\hat{p}_t=$$

$$=(1-\theta_i)\left(\frac{1}{\tilde{p}^g}\right)\left[\hat{\lambda}_{it}\right]+(1-\theta_i)\left(\frac{1}{\tilde{p}^g}\right)\left[\hat{i}_t\right]-\phi_{ni}\tilde{p}^g\left[\hat{p}^g_{t-1}\right]+\phi_{ni}\tilde{p}^g\left[\hat{n}_{it-1}\right]+ \quad (3.3)$$

$$+\beta\phi_{ni}\tilde{p}^g\left[\hat{p}^g_t\right]+\beta\phi_{ni}\tilde{p}^g E_t\left[\hat{n}_{it+1}\right]$$

$$\tilde{\zeta}_c\hat{\zeta}_{ct}=\tilde{\lambda}_i\hat{\lambda}_{it}+\phi_{kc}\left(\frac{\tilde{i}_c}{\tilde{k}_c}\right)\left(\frac{1}{\tilde{k}_c}\right)\left[\hat{i}_{ct}-\hat{k}_{ct}\right] \quad (3.4)$$

$$\hat{\zeta}_{it}=\hat{\lambda}_{it}+\phi_{ki}\left(\frac{\tilde{i}_i}{\tilde{k}_i}\right)\left(\frac{\tilde{i}}{\tilde{k}_i}\right)\left[\hat{i}_{it}-\hat{k}_{it}\right] \quad (3.5)$$

$$\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\tilde{\zeta}_c\left[\hat{p}^g_t\right]+\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\tilde{\zeta}_c\left[\hat{z}^g_{it}\right]+\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\tilde{\zeta}_c\left[\hat{g}^g_{it}\right]+\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\tilde{\zeta}_c\left[\hat{\zeta}_{ct}\right]=$$

$$=\beta(1-\delta_c)\tilde{\zeta}_c E_t\left[\hat{\zeta}_{ct+1}\right]+\beta\phi_{kc}\left(\frac{1}{\tilde{k}_c}\right)\left(\frac{\tilde{i}_c}{\tilde{k}_c}\right)^2 E_t\left[\hat{i}_{ct+1}\right]- \quad (3.6)$$

$$-\beta\left(\frac{1}{\tilde{k}_c}\right)\left[\theta_c+\phi_{kc}\left(\frac{\tilde{i}_c}{\tilde{k}_c}\right)^2\right]E_t\left[\hat{k}_{ct+1}\right]$$

$$\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\left[\hat{p}^g_t\right]+\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\left[\hat{z}^g_{it}\right]+\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\left[\hat{g}^g_{it}\right]+\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\left[\hat{\zeta}_{it}\right]=$$

$$=\beta(1-\delta_i)E_t\left[\hat{\zeta}_{it+1}\right]+\beta\theta_i\left(\frac{\tilde{i}}{\tilde{k}_i}\right)E_t\left[\hat{\lambda}_{it+1}\right]+\beta\theta_i\left(\frac{\tilde{i}}{\tilde{k}_i}\right)E_t\left[\hat{i}_{t+1}\right]+ \quad (3.7)$$

$$+\beta\phi_{ki}\left(\frac{\tilde{i}}{\tilde{k}_i}\right)\left(\frac{\tilde{i}_i}{\tilde{k}_i}\right)^2 E_t\left[\hat{i}_{it+1}\right]-\beta\left(\frac{\tilde{i}}{\tilde{k}_i}\right)\left[\theta_i+\phi_{ki}\left(\frac{\tilde{i}_i}{\tilde{k}_i}\right)^2\right]E_t\left[\hat{k}_{it+1}\right]$$

$$\hat{c}_t=\theta_c\hat{k}_{ct}+(1-\theta_c)\hat{z}_{ct}+(1-\theta_c)\hat{n}_{ct}+(1-\theta_c)\hat{g}_{ct} \quad (3.8)$$

$$\hat{i}_t=\theta_i\hat{k}_{it}+(1-\theta_i)\hat{z}_{it}+(1-\theta_i)\hat{n}_{it}+(1-\theta_i)\hat{g}_{it} \quad (3.9)$$

$$\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\tilde{k}_c\left[\hat{p}^g_t+\hat{z}^g_{it}+\hat{g}^g_{it}+\hat{k}_{ct+1}\right]=(1-\delta_c)\tilde{k}_c\left[\hat{k}_{ct}\right]+\tilde{i}_c\left[\hat{i}_{ct}\right] \quad (3.10)$$

$$\tilde{p}^g\tilde{z}^g_i\tilde{g}^g_i\tilde{k}_i\left[\hat{p}^g_t+\hat{z}^g_{it}+\hat{g}^g_{it}+\hat{k}_{it+1}\right]=(1-\delta_i)\tilde{k}_i\left[\hat{k}_{it}\right]+\tilde{i}_i\left[\hat{i}_{it}\right] \quad (3.11)$$

$$\tilde{i}\left[\hat{i}_t\right] = \tilde{i}_c\left[\hat{i}_{ct}\right] + \tilde{i}_i\left[\hat{i}_{it}\right] \tag{3.12}$$

$$\tilde{n}\left[\hat{n}_t\right] = \tilde{n}_c\left[\hat{n}_{ct}\right] + \tilde{n}_i\left[\hat{n}_{it}\right] \tag{3.13}$$

$$\hat{p}_t = \hat{p}_t^l + \hat{p}_t^g \tag{3.14}$$

$$\hat{p}_t^l = \rho_a^l \hat{p}_{t-1}^l + \varepsilon_{pt}^l \tag{3.15}$$

$$\hat{p}_t^g = \rho_p^g \hat{p}_{t-1}^g + \varepsilon_{pt}^g \tag{3.16}$$

$$\hat{z}_{ct} = \hat{z}_{ct}^l + \hat{z}_{ct}^g \tag{3.17}$$

$$\hat{z}_{ct}^l = \rho_{zc}^l \hat{z}_{ct-1}^l + \varepsilon_{zct}^l \tag{3.18}$$

$$\hat{z}_{ct}^g = \rho_{zc}^g \hat{z}_{ct-1}^g + \varepsilon_{zct}^g \tag{3.19}$$

$$\hat{z}_{it} = \hat{z}_{it}^l + \hat{z}_{it}^g \tag{3.20}$$

$$\hat{z}_{it}^l = \rho_{zi}^l \hat{z}_{it-1}^l + \varepsilon_{zit}^l \tag{3.21}$$

$$\hat{z}_{it}^g = \rho_{zi}^g \hat{z}_{it-1}^g + \varepsilon_{zit}^g \tag{3.22}$$

$$\hat{g}_{ct} = \hat{g}_{ct}^l + \hat{g}_{ct}^g \tag{3.23}$$

$$\hat{g}_{ct}^l = \rho_{gc}^l \hat{g}_{ct-1}^l + \varepsilon_{gct}^l \tag{3.24}$$

$$\hat{g}_{ct}^g = \rho_{gc}^g \hat{g}_{ct-1}^g + \varepsilon_{gct}^g \tag{3.25}$$

$$\hat{g}_{it} = \hat{g}_{it}^l + \hat{g}_{it}^g \tag{3.26}$$

$$\hat{g}_{it}^l = \rho_{gi}^l \hat{g}_{it-1}^l + \varepsilon_{git}^l \tag{3.27}$$

$$\hat{g}_{it}^g = \rho_{gi}^g \hat{g}_{it-1}^g + \varepsilon_{git}^g \tag{3.28}$$

$$\hat{\gamma}_t^c = \hat{p}_{t-1}^g + \theta_c\left[\hat{z}_{it-1}^g + \hat{g}_{it-1}^g\right] + \left(1-\theta_c\right)\left[\hat{z}_{ct-1}^g + \hat{g}_{ct-1}^g\right] + \hat{c}_t - \hat{c}_{t-1} \tag{3.29}$$

$$\hat{\gamma}_t^i = \hat{p}_{t-1}^g + \hat{z}_{it-1}^g + \hat{g}_{it-1}^g + \hat{i}_t - \hat{i}_{t-1} \tag{3.30}$$

$$\hat{\gamma}_t^n = \hat{p}_{t-1}^g + \hat{n}_t - \left(\frac{\tilde{n}_c}{\tilde{n}}\right)\hat{n}_{ct-1} - \left(\frac{\tilde{n}_i}{\tilde{n}}\right)\hat{n}_{it-1} \tag{3.31}$$

$$\hat{\gamma}_t^{gc} = p_{t-1}^g + \hat{g}_{ct}^g - \hat{g}_{ct-1}^g \tag{3.32}$$

$$\hat{\gamma}_t^{gi} = p_{t-1}^g + \hat{g}_{it}^g - \hat{g}_{it-1}^g \tag{3.33}$$

VII. Solving the Model

The log-linearized equilibrium conditions (3.1)-(3.33) form a system of linear expectational difference equations that can be solved by casting the system in state space and applying methods popularized by Blanchard and Kahn (1980), Klein (2000), or Sims (2001). The solution for this model will use information from five observable variables (consumption, investment, labor hours and two measures of government spending) to draw inferences on the dynamics of a set of unobservable variables that include ten shocks: $\hat{p}_t^l, \hat{p}_t^g, \hat{z}_{ct}^l, \hat{z}_{ct}^g, \hat{z}_{it}^l, \hat{z}_{it}^g, \hat{g}_{ct}^l, \hat{g}_{ct}^g, \hat{g}_{it}^l, and\ \hat{g}_{it}^g$. The model's structural parameters can, then, be estimated by maximum likelihood using Kalman filter algorithms outlined by Hamilton (1994) and Kim and Nelson (1999**).**

VII.1. ***State-Space Representation***

We can rewrite (3.1)-(3.33) in a slightly different way to make the mapping procedure as simple as possible.[35]

Let

$$f_t^0 = \left(\hat{n}_t\ \hat{i}_t\ \hat{i}_{ct}\ \hat{i}_{it}\ \hat{\lambda}_{it}\ \hat{p}_t\ \hat{z}_{ct}\ \hat{z}_{it}\ \hat{g}_{ct}\ \hat{g}_{it}\ \hat{\gamma}_t^c\ \hat{\gamma}_t^i\ \hat{\gamma}_t^n\ \hat{\gamma}_t^{gc}\ \hat{\gamma}_t^{gi}\right)'$$

$$s_t^0 = \left(\hat{k}_{ct}\ \hat{k}_{it}\ \hat{c}_{t-1}\ \hat{i}_{t-1}\ \hat{n}_{ct-1}\ \hat{n}_{it-1}\ \hat{p}_{t-1}^g\ \hat{z}_{ct-1}^g\ \hat{z}_{it-1}^g\ \hat{g}_{ct-1}^g\ \hat{g}_{it-1}^g\ \hat{c}_t\ \hat{n}_{ct}\ \hat{n}_{it}\ \hat{\lambda}_{ct}\ \hat{\xi}_{ct}\ \hat{\xi}_{it}\right)'$$

and $\upsilon_t = \left(\hat{p}_t^l\ \hat{p}_t^g\ \hat{z}_{ct}^l\ \hat{z}_{ct}^g\ \hat{z}_{it}^l\ \hat{z}_{it}^g\ \hat{g}_{ct}^l\ \hat{g}_{ct}^g\ \hat{g}_{it}^l\ \hat{g}_{it}^g\right)'$.

Then equations (3.4), (3.5), (3.9), (3.12)-(3.14), (3.17), (3.20), (3.23), (3.26), and (3.29)-(3.33) can be written as

$$Af_t^0 = Bs_t^0 + C\upsilon_t \tag{4.1}$$

where A is 15 x 15, B is 15 x 17 and C is 15 x 10.

Equation (3.4) implies

$$a_{13} = \phi_{kc}\left(\frac{\tilde{i}_c}{\tilde{k}_c}\right)\left(\frac{1}{\tilde{k}_c}\right)$$

$$a_{15} = \tilde{\lambda}_i$$

$$b_{11} = \phi_{kc}\left(\frac{\tilde{i}_c}{\tilde{k}_c}\right)\left(\frac{1}{\tilde{k}_c}\right)$$

$$b_{116} = \tilde{\zeta}_c$$

[35] The strategy followed here is to move the expectation terms and endogenous variables to the left-hand side of each equation and the predetermined variables and shocks to the right-hand side.

Equation (3.5) implies

$$a_{24} = \phi_{ki}\left(\frac{\tilde{i}_i}{\tilde{k}_i}\right)\left(\frac{\tilde{i}}{\tilde{k}_i}\right)$$

$$a_{25} = 1$$

$$b_{22} = \phi_{ki}\left(\frac{\tilde{i}_i}{\tilde{k}_i}\right)\left(\frac{\tilde{i}}{\tilde{k}_i}\right)$$

$$b_{217} = 1$$

Equation (3.9) implies

$$a_{32} = 1$$

$$a_{38} = \theta_i - 1$$

$$a_{310} = \theta_i - 1$$

$$b_{32} = \theta_i$$

$$b_{314} = 1 - \theta_i$$

Equation (3.12) implies

$$a_{42} = \tilde{i}$$

$$a_{43} = -\tilde{i}_c$$

$$a_{44} = -\tilde{i}_i$$

Equation (3.13) implies

$$a_{51} = \tilde{n}$$

$$a_{513} = \tilde{n}_c$$

$$b_{514} = \tilde{n}_i$$

Equation (3.14) implies

$$a_{66} = 1$$

$$c_{61} = 1$$

$$c_{62} = 1$$

Equation (3.17) implies

$a_{77} = 1$

$c_{73} = 1$

$c_{74} = 1$

Equation (3.20) implies

$a_{88} = 1$

$c_{85} = 1$

$c_{86} = 1$

Equation (3.23) implies

$a_{99} = 1$

$c_{97} = 1$

$c_{98} = 1$

Equation (3.26) implies

$a_{1010} = 1$

$c_{109} = 1$

$c_{1010} = 1$

Equation (3.29) implies

$a_{1111} = 1$

$b_{113} = -1$

$b_{117} = 1$

$b_{118} = 1 - \theta_c$

$b_{119} = \theta_c$

$b_{1110} = 1 - \theta_c$

$b_{1111} = \theta_c$

$b_{1112} = 1$

Equation (3.30) implies

$a_{122} = -1$

$a_{1212} = 1$

$b_{124} = -1$

$b_{127} = 1$

$b_{129} = 1$

$b_{1211} = 1$

Equation (3.31) implies

$a_{131} = -1$

$a_{1313} = 1$

$b_{135} = -\frac{\tilde{n}_c}{\tilde{n}}$

$b_{136} = -\frac{\tilde{n}_i}{\tilde{n}}$

$b_{137} = 1$

Equation (3.32) implies

$a_{149} = -1$

$a_{1414} = 1$

$b_{147} = 1$

$b_{1410} = 1$

Equation (3.33) implies

$a_{1510} = -1$

$a_{1515} = 1$

$b_{157} = 1$

$b_{1511} = 1$

Equations (3.1)-(3.3), (3.6)-(3.8), (3.10) and (3.11) can be rewritten as

$$DE_t s_{t+1}^0 + FE_t f_{t+1}^0 = Gs_t^0 + Hf_t^0 + J\upsilon_t \tag{4.2}$$

where D and G are 17 x 17, F and H are 17 x 15, and J is 17 x 10.

Equation (3.1) implies

$$d_{13} = 1$$

$$g_{115} = -1$$

Equation (3.2) implies

$$d_{27} = \beta\phi_{nc}\tilde{p}^{g}$$

$$d_{213} = \beta\phi_{nc}\tilde{p}^{g}$$

$$g_{25} = -\phi_{nc}\tilde{p}^{g}$$

$$g_{213} = (1-\theta_c)\left(\frac{1}{\tilde{p}^{g}}\right) + (1+\beta)\phi_{nc}\tilde{p}^{g}$$

$$h_{26} = (\theta_c - 1)\left(\frac{1}{\tilde{p}^{g}}\right)$$

Equation (3.3) implies

$$d_{37} = \beta\phi_{ni}\tilde{p}^{g}$$

$$d_{314} = \beta\phi_{ni}\tilde{p}^{g}$$

$$g_{36} = -\phi_{ni}\tilde{p}^{g}$$

$$g_{37} = \phi_{ni}\tilde{p}^{g}$$

$$g_{314} = (1-\theta_i)\left(\frac{1}{\tilde{p}^{g}}\right) + (1+\beta)\phi_{ni}\tilde{p}^{g}$$

$$h_{32} = (\theta_i - 1)\left(\frac{1}{\tilde{p}^{g}}\right)$$

$$h_{35} = (\theta_i - 1)\left(\frac{1}{\tilde{p}^{g}}\right)$$

$$h_{36} = (\theta_i - 1)\left(\frac{1}{\tilde{p}^{g}}\right)$$

Equation (3.6) implies

$$d_{41} = -\beta\left(\frac{1}{\tilde{k}_c}\right)\left[\theta_c + \phi_{kc}\left(\frac{\tilde{i}_c}{\tilde{k}_c}\right)^2\right]$$

$$d_{416} = \beta(1-\delta_c)\tilde{\zeta}_c$$

$$f_{43} = \beta\phi_{kc}\left(\frac{1}{\tilde{k}_c}\right)\left(\frac{\tilde{i}_c}{\tilde{k}_c}\right)^2$$

$$g_{416} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g\tilde{\zeta}_c$$

$$j_{42} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g\tilde{\zeta}_c$$

$$j_{46} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g\tilde{\zeta}_c$$

$$j_{410} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g\tilde{\zeta}_c$$

Equation (3.7) implies

$$d_{52} = -\beta\left(\frac{\tilde{i}}{\tilde{k}_i}\right)\left[\theta_i + \phi_{ki}\left(\frac{\tilde{i}_i}{\tilde{k}_i}\right)^2\right]$$

$$d_{517} = \beta(1-\delta_i)$$

$$f_{52} = \beta\theta_i\left(\frac{\tilde{i}}{\tilde{k}_i}\right)$$

$$f_{54} = \beta\phi_{ki}\left(\frac{\tilde{i}}{\tilde{k}_i}\right)\left(\frac{\tilde{i}_i}{\tilde{k}_i}\right)^2$$

$$f_{55} = \beta\theta_i\left(\frac{\tilde{i}}{\tilde{k}_i}\right)$$

$$g_{517} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g$$

$$j_{52} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g$$

$$j_{56} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g$$

$$j_{510} = \tilde{p}^g\tilde{z}_i^g\tilde{g}_i^g$$

Equation (3.8) implies

$$d_{63} = 1$$
$$d_{65} = \theta_c - 1$$
$$g_{61} = \theta_c$$
$$h_{67} = 1 - \theta_c$$
$$h_{69} = 1 - \theta_c$$

Equation (3.10) implies

$$d_{71} = \tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_c$$
$$g_{71} = (1 - \delta_c)\tilde{k}_c$$
$$h_{73} = \tilde{i}_c$$
$$j_{72} = -\tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_c$$
$$j_{76} = -\tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_c$$
$$j_{710} = -\tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_c$$

Equation (3.11) implies

$$d_{82} = \tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_i$$
$$g_{82} = (1 - \delta_i)\tilde{k}_i$$
$$h_{84} = \tilde{i}_i$$
$$j_{82} = -\tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_i$$
$$j_{86} = -\tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_c$$
$$j_{810} = -\tilde{p}\tilde{z}_i^g \tilde{g}_i^g \tilde{k}_c$$

Additionally, the presence of lagged values in the s_t^0 vector implies

$d_{93} = 1$
$g_{912} = 1$
$d_{104} = 1$
$h_{102} = 1$
$d_{115} = 1$
$g_{1113} = 1$
$d_{126} = 1$
$g_{1214} = 1$
$d_{137} = 1$
$j_{132} = 1$
$d_{148} = 1$
$j_{144} = 1$
$d_{159} = 1$
$j_{156} = 1$
$d_{1610} = 1$
$j_{168} = 1$
$d_{1711} = 1$
$j_{1710} = 1$

Finally, (3.15), (3.16), (3.18), (3.19), (3.21), (3.22), (3.24), (3.25), (3.27), and (3.28) can be written as

$$\upsilon_t = P\upsilon_{t-1} + \varepsilon_t \tag{4.3}$$

where P is 10 x 10, with $\rho_p^l\ \rho_p^g\ \rho_c^l\ \rho_c^g\ \rho_i^l\ \rho_i^g\ \rho_{gc}^l\ \rho_{gc}^g\ \rho_{gi}^l\ \rho_{gi}^g$ in its main diagonal and zero in the off-diagonal

and $\varepsilon_t = \left[\varepsilon_{pt}^l\ \ \varepsilon_{pt}^g\ \ \varepsilon_{zct}^l\ \ \varepsilon_{zct}^g\ \ \varepsilon_{zit}^l\ \ \varepsilon_{zit}^g\ \ \varepsilon_{gct}^l\ \ \varepsilon_{gct}^g\ \ \varepsilon_{git}^l\ \ \varepsilon_{git}^g\right]'$.

VII.2 ***Solving the System***

We can first solve (4.1) for f_t^0:

$$f_t^0 = A^{-1}Bs_t^0 + A^{-1}C\upsilon_t \tag{4.4}$$

Substituting this result into (4.2) yields the following system of linear expectational difference equations

$$KE_t s_{t+1}^0 = Ls_t^0 + M\upsilon_t \tag{4.5}$$

where $K = D + FA^{-1}B$, $L = G + HA^{-1}B$, and $M = J + HA^{-1}C - FA^{-1}CP$. This system, which is driven by the shocks in (4.3), can be solved by uncoupling the unstable and stable components and then solving the unstable component forward. Klein (2000) describes a way to achieve this uncoupling by applying the *complex generalized Schur decomposition*. When applied to this state-space representation, the Schur decomposition triangularizes (4.5) by identifying two unitary matrices Q and Z such that when pre and post-multiplied (respectively) to the matrices of coefficients in our transition equation (4.5) yields

$$QKZ = S$$

and

$$QLZ = T$$

two upper triangular matrices S and T. The generalized eigenvalues of L and K in (4.5) can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(L,K) = \{t_{ii} / s_{ii} \,|\, i = 1,2,\ldots,17\}$$

These generalized eigenvalues in matrices *Q*, *Z*, *S*, and *T* can be re-sorted to appear in ascending order in absolute value. All variables in s_t^0 are predetermined with the exception of the last six ($\hat{c}_t, \hat{n}_{ct}, \hat{n}_{it}, \hat{\lambda}_{ct}, \hat{\xi}_{ct}, and\ \hat{\xi}_{it}$). Therefore, if 11 of the generalized eigenvalues in $\lambda(L,K)$ lie inside the unit circle and six lie outside, then the system has a unique solution. However, if more than 11 eigenvalues lie inside, then the solution to the system is indeterminate, and if less than 11 lie inside the unit circle, then no solution exists [see Blanchard and Kahn (1980)].

Assuming that a single solution to the system exists,[36] we can conformably partition the matrices *Q*, *Z*, *S*, and *T* into a stable and an unstable component for each in the following way

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

[36] Assume there are exactly six eigenvalues that lie outside the unit circle.

where Q_1 is 11 x 17 and Q_2 is 6 x 17, and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$
$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0_{6\times 11} & S_{22} \end{bmatrix},$$
$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{6\times 11} & T_{22} \end{bmatrix},$$

where $Z_{11}, S_{11}, \text{and } T_{11}$ are 11 x 11, $Z_{12}, S_{12}, \text{and } T_{12}$ are 11 x 6, and $Z_{22}, S_{22}, \text{and } T_{22}$ are 6 x 6.

Next, we define the vector s_t^1 of auxiliary variables as

$$s_t^1 = Z' s_t^0$$

where we again partition, to isolate the unstable components from the stable ones, so that

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix}$$

where

$$s_{1t}^{1} = Z_{11}^{'}\begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{i}_{t-1} \\ \hat{n}_{ct-1} \\ \hat{n}_{it-1} \\ \hat{p}_{t-1}^{g} \\ \hat{z}_{ct-1}^{g} \\ \hat{z}_{it-1}^{g} \\ \hat{g}_{ct-1}^{g} \\ \hat{g}_{it-1}^{g} \end{bmatrix} + Z_{21}^{'}\begin{bmatrix} \hat{c}_{t} \\ \hat{n}_{ct} \\ \hat{n}_{it} \\ \hat{\lambda}_{ct} \\ \hat{\zeta}_{ct} \\ \hat{\zeta}_{it} \end{bmatrix} \tag{4.6}$$

is 11 x 1, and

$$s_{2t}^{1} = Z_{12}^{'}\begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{i}_{t-1} \\ \hat{n}_{ct-1} \\ \hat{n}_{it-1} \\ \hat{p}_{t-1}^{g} \\ \hat{z}_{ct-1}^{g} \\ \hat{z}_{it-1}^{g} \\ \hat{g}_{ct-1}^{g} \\ \hat{g}_{it-1}^{g} \end{bmatrix} + Z_{22}^{'}\begin{bmatrix} \hat{c}_{t} \\ \hat{n}_{ct} \\ \hat{n}_{it} \\ \hat{\lambda}_{ct} \\ \hat{\zeta}_{ct} \\ \hat{\zeta}_{it} \end{bmatrix} \tag{4.7}$$

is 6 x 1.

From the properties of unitary matrices, we have that $Z^{'}Z = I$ or $Z^{'} = Z^{-1}$, which implies that $s_t^0 = Zs_t^1$. We can then rewrite (4.5) as

$$KZE_t s_{t+1}^1 = LZs_t^1 + M\upsilon_t$$

Pre-multiplying this expression by Q and applying the definitions of the Schur decomposition stated earlier yields

$$SE_t s_{t+1}^1 = Ts_t^1 + QM\upsilon_t$$

where in terms of the partitions we have,

$$S_{11}E_t s_{1t+1}^1 + S_{12}E_t s_{2t+1}^1 = T_{11}s_{1t}^1 + T_{12}s_{2t}^1 + Q_1 M\upsilon_t \tag{4.8}$$

and

$$S_{22}E_t s_{2t+1}^1 = T_{22}s_{2t}^1 + Q_2 M\upsilon_t \tag{4.9}$$

Remembering that we partitioned S and T in such a way that the generalized eigenvalues corresponding to the elements of S_{22} and T_{22} all lie outside the unit circle, Klein's (2000) procedure allows us to solve the unstable component forward, so we can solve (4.9) forward to obtain

$$s_{2t}^1 = -T_{22}^{-1} R\upsilon_t$$

where the 6 x 10 matrix R is given by

$$vec(R) = vec\sum_{j=0}^{\infty}\left(S_{22}T_{22}^{-1}\right)^{j}Q_2MP^{j} = \sum_{j=0}^{\infty}vec\left[\left(S_{22}T_{22}^{-1}\right)^{j}Q_2MP^{j}\right] =$$

$$= \sum_{j=0}^{\infty}\left[P^{j}\otimes\left(S_{22}T_{22}^{-1}\right)^{j}\right]vec\left(Q_2M\right) =$$

$$= \sum_{j=0}^{\infty}\left[P\otimes\left(S_{22}T_{22}^{-1}\right)^{j}\right]vec\left(Q_2M\right) =$$

$$= \left[I_{60\times 60} - P\otimes\left(S_{22}T_{22}^{-1}\right)\right]^{-1}vec\left(Q_2M\right)$$

Using this result along with (4.7) to solve for

$$\begin{bmatrix}\hat{c}_t\\ \hat{n}_{ct}\\ \hat{n}_{it}\\ \hat{\lambda}_{ct}\\ \hat{\zeta}_{ct}\\ \hat{\zeta}_{it}\end{bmatrix} = -\left(Z_{22}^{'}\right)^{-1}Z_{12}^{'}\begin{bmatrix}\hat{k}_{ct}\\ \hat{k}_{it}\\ \hat{c}_{t-1}\\ \hat{i}_{t-1}\\ \hat{n}_{ct-1}\\ \hat{n}_{it-1}\\ \hat{p}_{t-1}^{g}\\ \hat{z}_{ct-1}^{g}\\ \hat{z}_{it-1}^{g}\\ \hat{g}_{ct-1}^{g}\\ \hat{g}_{it-1}^{g}\end{bmatrix} - Z_{22}^{'}T_{22}^{-1}R\upsilon_t$$

Since Z is unitary, $Z^{'}Z = I$, we have

$$\begin{bmatrix}Z_{11}^{'} & Z_{21}^{'}\\ Z_{12}^{'} & Z_{22}^{'}\end{bmatrix}\begin{bmatrix}Z_{11} & Z_{21}\\ Z_{12} & Z_{22}\end{bmatrix} = \begin{bmatrix}I_{11\times 11} & 0_{11\times 6}\\ 0_{6\times 11} & I_{6\times 6}\end{bmatrix}$$

Hence, multiplying the second row times the first column gives us

$$Z_{12}^{'}Z_{11} + Z_{22}^{'}Z_{21} = 0 \text{ or } Z_{21}Z_{11}^{-1} = -\left(Z_{22}^{'}\right)^{-1}Z_{12}^{'}$$

and multiplying the second row times the second column yields

$$Z_{12}^{'}Z_{12}+Z_{22}^{'}Z_{22}=I \text{ or} \left(Z_{22}^{'}\right)^{-1}=Z_{22}+\left(Z_{22}^{'}\right)^{-1}Z_{12}^{'}Z_{12}=Z_{22}-Z_{21}Z_{11}^{-1}Z_{12}$$

Using all these algebraic manipulations, we can rewrite this expression in a more succinct way as

$$\begin{bmatrix}\hat{c}_t\\ \hat{n}_{ct}\\ \hat{n}_{it}\\ \hat{\lambda}_{ct}\\ \hat{\zeta}_{ct}\\ \hat{\zeta}_{it}\end{bmatrix}=N_1\begin{bmatrix}\hat{k}_{ct}\\ \hat{k}_{it}\\ \hat{c}_{t-1}\\ \hat{i}_{t-1}\\ \hat{n}_{ct-1}\\ \hat{n}_{it-1}\\ \hat{p}^g_{t-1}\\ \hat{z}^g_{ct-1}\\ \hat{z}^g_{it-1}\\ \hat{g}^g_{ct-1}\\ \hat{g}^g_{it-1}\end{bmatrix}+N_2\upsilon_t \tag{4.10}$$

where

$$N_1=Z_{21}Z_{11}^{-1}$$

and

$$N_2=-\left[Z_{22}-Z_{21}Z_{11}^{-1}Z_{12}\right]T_{22}^{-1}R$$

We can now substitute (4.10) into (4.6) to derive a solution for s_{1t}^1 :

$$s_{1t}^{1} = \left(Z_{11}^{'} + Z_{21}^{'} Z_{21} Z_{11}^{-1}\right) \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{i}_{t-1} \\ \hat{n}_{ct-1} \\ \hat{n}_{it-1} \\ \hat{p}_{t-1}^{g} \\ \hat{z}_{ct-1}^{g} \\ \hat{z}_{it-1}^{g} \\ \hat{g}_{ct-1}^{g} \\ \hat{g}_{it-1}^{g} \end{bmatrix} - Z_{21}^{'} \left[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12} \right] T_{22}^{-1} R v_t$$

Again, we can multiply the first row times the first column to yield $Z_{11}^{'} Z_{11} + Z_{21}^{'} Z_{21} = I$, or $Z_{11}^{'} + Z_{21}^{'} Z_{21} Z_{11}^{-1} = Z_{11}^{-1}$ and from the second term in the expression above, we have

$$Z_{21}^{'} \left[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12} \right] = Z_{21}^{'} Z_{22} - Z_{21}^{'} Z_{21} Z_{11}^{-1} Z_{12} = -Z_{11}^{-1} Z_{12}.$$

Thus, this last result can be written more conveniently as

$$s_{1t}^{1} = Z_{11}^{-1} \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{i}_{t-1} \\ \hat{n}_{ct-1} \\ \hat{n}_{it-1} \\ \hat{p}_{t-1}^{g} \\ \hat{z}_{ct-1}^{g} \\ \hat{z}_{it-1}^{g} \\ \hat{g}_{ct-1}^{g} \\ \hat{g}_{it-1}^{g} \end{bmatrix} + Z_{11}^{-1} Z_{12} T_{22}^{-1} R \upsilon_t$$

We can substitute the results for s_{1t}^{1} and s_{2t}^{1} into (4.8) to obtain the solution

$$\begin{bmatrix} \hat{k}_{ct+1} \\ \hat{k}_{it+1} \\ \hat{c}_{t} \\ \hat{i}_{t} \\ \hat{n}_{ct} \\ \hat{n}_{it} \\ \hat{p}_{t}^{g} \\ \hat{z}_{ct}^{g} \\ \hat{z}_{it}^{g} \\ \hat{g}_{ct}^{g} \\ \hat{g}_{it}^{g} \end{bmatrix} = N_3 \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{i}_{t-1} \\ \hat{n}_{ct-1} \\ \hat{n}_{it-1} \\ \hat{p}_{t-1}^{g} \\ \hat{z}_{ct-1}^{g} \\ \hat{z}_{it-1}^{g} \\ \hat{g}_{ct-1}^{g} \\ \hat{g}_{it-1}^{g} \end{bmatrix} + N_4 \upsilon_t \qquad (4.11)$$

where

$$N_3 = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

and

$$N_4 = Z_{11} S_{11}^{-1} \left(T_{11} Z_{11}^{-1} Z_{12} T_{22}^{-1} R + Q_1 M + S_{12} T_{22}^{-1} RP - T_{12} T_{22}^{-1} R \right) - Z_{12} T_{22}^{-1} RP$$

Lastly, we can solve (4.4) for f_t^0:

$$f_t^0 = A^{-1} B \begin{bmatrix} I_{11\times 11} \\ N_1 \end{bmatrix} \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{i}_{t-1} \\ \hat{n}_{ct-1} \\ \hat{n}_{it-1} \\ \hat{p}_{t-1}^g \\ \hat{z}_{ct-1}^g \\ \hat{z}_{it-1}^g \\ \hat{g}_{ct-1}^g \\ \hat{g}_{it-1}^g \end{bmatrix} + A^{-1} B \begin{bmatrix} 0_{11\times 10} \\ N_2 \end{bmatrix} \upsilon_t + A^{-1} C \upsilon_t$$

which can be written more simply as

$$f_t^0 = N_5 \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{i}_{t-1} \\ \hat{n}_{ct-1} \\ \hat{n}_{it-1} \\ \hat{p}^g_{t-1} \\ \hat{z}^g_{ct-1} \\ \hat{z}^g_{it-1} \\ \hat{g}^g_{ct-1} \\ \hat{g}^g_{it-1} \end{bmatrix} + N_6 \upsilon_t , \tag{4.12}$$

where

$$N_5 = A^{-1} B \begin{bmatrix} I_{11\times 11} \\ N_1 \end{bmatrix}$$

and

$$N_6 = A^{-1} B \begin{bmatrix} 0_{11\times 10} \\ N_2 \end{bmatrix} + A^{-1} C$$

In closing, equations (4.3), (4.10), (4.11), and (4.12) can be used to solve the state-space model characterized by the following measurement and transition equations:

$$s_{t+1} = \Pi s_t + \Xi \varepsilon_{t+1} \tag{4.13}$$

and

$$s_t = U^{-1} f_t \tag{4.14}$$

where

$$f_t = \left(\hat{n}_t \ \hat{i}_t \ \hat{i}_{ct} \ \hat{i}_{it} \ \hat{\lambda}_{it} \ \hat{p}_t \ \hat{z}_{ct} \ \hat{z}_{it} \ \hat{g}_{ct} \ \hat{g}_{it} \ \hat{\gamma}_t^c \ \hat{\gamma}_t^i \ \hat{\gamma}_t^n \ \hat{\gamma}_t^{gc} \ \hat{\gamma}_t^{gi} \ \hat{c}_t \ \hat{n}_{ct} \ \hat{n}_{it} \ \hat{\lambda}_{ct} \ \hat{\xi}_{ct} \ \hat{\xi}_{it} \right)'_{21\times 1}$$

$$s_t = \left(\hat{k}_{ct} \ \hat{k}_{it} \ \hat{c}_{t-1} \ \hat{i}_{t-1} \ \hat{n}_{ct-1} \ \hat{n}_{it-1} \ \hat{p}_{t-1}^g \ \hat{z}_{ct-1}^g \ \hat{z}_{it-1}^g \ \hat{g}_{ct-1}^g \ \hat{g}_{it-1}^g \ \hat{p}_t^l \ \hat{p}_t^g \ \hat{z}_{ct}^l \ \hat{z}_{ct}^g \ \hat{z}_{it}^l \ \hat{z}_{it}^g \ \hat{g}_{ct}^l \ \hat{g}_{ct}^g \ \hat{g}_{it}^l \ \hat{g}_{it}^g \right)'_{21\times 1}$$

$$\varepsilon_t = \left[\varepsilon_{pt}^l \ \varepsilon_{pt}^g \ \varepsilon_{zct}^l \ \varepsilon_{zct}^g \ \varepsilon_{zit}^l \ \varepsilon_{zit}^g \ \varepsilon_{gct}^l \ \varepsilon_{gct}^g \ \varepsilon_{git}^l \ \varepsilon_{git}^g \right]'_{10\times 1}$$

$$\Pi = \begin{bmatrix} N_3 & N_4 \\ {\scriptstyle (11\times 11)} & {\scriptstyle (11\times 10)} \\ 0 & P \\ {\scriptstyle (10\times 11)} & {\scriptstyle (10\times 10)} \end{bmatrix}; \qquad \Xi = \begin{bmatrix} 0 \\ {\scriptstyle (11\times 10)} \\ I \\ {\scriptstyle (10\times 10)} \end{bmatrix};$$

&

$$U = \begin{bmatrix} N_5 & N_6 \\ {\scriptstyle (15\times 11)} & {\scriptstyle (15\times 10)} \\ N_1 & N_2 \\ {\scriptstyle (6\times 11)} & {\scriptstyle (6\times 10)} \end{bmatrix}$$

The linear method applied here (Klein 2000) -- as well as those of Blanchard and Kahn (1980) or Sims (2001) -- can be used to approximate a solution to a nonlinear Real Business Cycle model by way of casting the structural model into state space. In this case, the solution connects the behavior of the growth rates of U.S. hours worked, consumption, investment, and two measures of government spending (in all, five observable stationary variables) to a vector of unobserved state variables that includes ten autoregressive shocks $p_t^l, p_t^g, z_{ct}^l, z_{ct}^g, z_{it}^l, z_{it}^g, g_{ct}^l, g_{ct}^g, g_{it}^l,$ and g_{it}^g.

VII.3. ***Estimation Procedure***

The model has implications for the behavior of five observables: consumption, investment, labor hours and two measures of government spending. The empirical model has 34 parameters: $\beta, \theta_c, \theta_i, \phi_{kc}, \phi_{ki}, \phi_{nc}, \phi_{ni}, \delta_c, \delta_i, p^g, \tilde{z}_c^g, \tilde{z}_i^g, \tilde{g}_c^g, \tilde{g}_i^g, \rho_p^l, \rho_p^g, \rho_c^l, \rho_c^g, \rho_i^l, \rho_i^g,$ $\rho_{gc}^l, \rho_{gc}^g, \rho_{gi}^l, \rho_{gi}^g, \sigma_p^l, \sigma_p^g, \sigma_c^l, \sigma_c^g, \sigma_i^l, \sigma_i^g, \sigma_{gc}^l, \sigma_{gc}^g, \sigma_{gi}^l,$ and σ_{gi}^g. The data considered is given by

$$d_t = \begin{pmatrix} \hat{\gamma}_t^c \\ \hat{\gamma}_t^i \\ \hat{\gamma}_t^n \\ \hat{\gamma}_t^{gc} \\ \hat{\gamma}_t^{gi} \end{pmatrix} = \begin{pmatrix} \ln(C_t) - \ln(C_{t-1}) - \ln\left[\tilde{p}^g \left(\tilde{z}_i^g \tilde{g}_i^g\right)^{\theta_c} \left(\tilde{z}_c^g \tilde{g}_c^g\right)^{1-\theta_c}\right] \\ \ln(I_t) - \ln(I_{t-1}) - \ln\left(\tilde{p}^g \tilde{z}_i^g \tilde{g}_i^g\right) \\ \ln(N_t) - \ln(N_{t-1}) - \ln\left(\tilde{p}^g\right) \\ \ln(G_{ct}) - \ln(G_{ct-1}) - \ln\left(\tilde{p}^g\right) \\ \ln(G_{it}) - \ln(G_{it-1}) - \ln\left(\tilde{p}^g\right) \end{pmatrix}$$

where $C_t, I_t, N_t, G_{ct}, G_{it}$, are levels of real per capita consumption, investment, labor hours and two measures of government expenditures. The system can be estimated with a structural model characterized with a state equation

$$S_{t+1} = AS_t + B\varepsilon_{t+1} \tag{4.15}$$

and an observation equation

$$d_t = CS_t \tag{4.16}$$

where $A = \Pi$, $B = \Xi$, and C is formed from the rows of U as

$$C = \begin{bmatrix} U_{11} \\ U_{12} \\ U_{13} \\ U_{14} \\ U_{15} \end{bmatrix}$$

and the vector of zero-mean, serially correlated innovations ε_{t+1} is normally distributed with a matrix that has $\left(\sigma_p^l\right)^2, \left(\sigma_p^g\right)^2, \left(\sigma_c^l\right)^2, \left(\sigma_c^g\right)^2, \left(\sigma_i^l\right)^2, \left(\sigma_i^g\right)^2, \left(\sigma_{gc}^l\right)^2, \left(\sigma_{gc}^g\right)^2, \left(\sigma_{gi}^l\right)^2$, and $\left(\sigma_{gi}^g\right)^2$ in its main diagonal and zeros in the off-diagonals.

The model defined by (4.15) and (4.16) is in state-space form with an easily estimable likelihood for the sample $\{d_t\}_{t=1}^T$ based on Kalman filtering techniques [Hamilton (1994), Kim and Nelson (1999)]. The basic Kalman filter algorithm consists of two steps: A *prediction* and an *updating* step. The prediction step reflects our need to estimate an optimal predictor (at the beginning of period t), of our sample d_t, based on all available information up to time $t-1$; $d_{t|t-1}$. In order to do this, the state vector $s_{t|t-1}$ needs to be calculated as of period $t-1$. Once d_t is realized at the end of period t, the prediction error $\eta_{t|t-1}$ can be calculated. This prediction error contains new information about s_t beyond that contained in $s_{t|t-1}$. Therefore, once d_t is observed, a more

accurate inference of s_t can be made by updating the information set with the newest observation. Hence the state vector gets updated, since it is a linear combination of itself as of last period and the prediction error updated with new information. The prediction error is weighted by the *Kalman gain*, which determines how important the new information about s_t is, in the newly updated prediction error.

For our estimation purposes in this model, let the estimate of the expectation (for $t = 1, 2, \ldots, T$ and $j = 0, 1$) of the state vector s_t conditional on information up to $t - j$ be

$$\hat{s}_{t|t-j} = E\left(s_t \mid d_{t-j}, d_{t-j-1}, \ldots d_1\right)$$

and the covariance matrix associated with the state vector, conditional on the same information set be

$$\Sigma_{t|t-j} = E\left(s_t - \hat{s}_{t-j}\right)\left(s_t - \hat{s}_{t-j}\right)'$$

and for forecasting purposes assume

$$\hat{d}_{t|t-j} = E\left(d_t \mid d_{t-j}, d_{t-j-1}, \ldots d_1\right)$$

Any recursion must begin with an initial value set. Let the initial values of the state vector $\hat{s}_{1|0}$, which describes the forecast of s_1 conditional on no observations of d_t, be the unconditional mean of s_1,

$$\hat{s}_{1|0} = E(s_1) = 0_{21\times 1} \tag{4.17}$$

and its associated initial value for its covariance matrix be

$$vec(\Sigma_{1|0}) = vec\left(Es_1 s_1^{'}\right) = \left[I - A \otimes A\right]^{-1} vec\left(BVB^{'}\right) \tag{4.18}$$

Let the prediction error be defined as

$$\eta_t = d_t - \hat{d}_{t|t-1}$$

and noting that (4.16) implies $\hat{d}_{t|t-1} = C\,\hat{s}_{t|t-1}$, we have that the covariance matrix of the prediction error looks like

$$E\left(\eta_t \eta_t^{'}\right) = C\Sigma_{t|t-1}C^{'}$$

Using the formula for updating linear projections (see Hamilton 1994),

$$\begin{aligned}\hat{s}_{t|t} &= \hat{s}_{t|t-1} + \left[E\left(s_t - \hat{s}_{t|t-1}\right)\left(d_t - \hat{d}_{t|t-1}\right)^{'}\right]\left[E\left(d_t - \hat{d}_{t|t-1}\right)\left(d_t - \hat{d}_{t|t-1}\right)^{'}\right]\eta_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}\eta_t\end{aligned}$$

Thus from (4.15) we have

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}\eta_t\,.$$

Subtracting this last result from (4.15) again we have

$$s_{t+1} - \hat{s}_{t+1|t} = A\left(s_t - \hat{s}_{t|t-1}\right) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}\eta_t$$

This last result implies

$$\Sigma_{t+1|t} = BVB^{'} + A\Sigma_{t|t-1}A^{'} - A\Sigma_{t|t-1}C^{'}\left(C\Sigma_{t|t-1}C^{'}\right)^{-1}C\Sigma_{t|t-1}A^{'}$$

These results can be summed up in the following way: given a state vector

$$\hat{s}_t = \hat{s}_{t|t-1} = E\left(s_t \mid d_{t-1}, d_{t-2}, \ldots d_1\right)$$

and its covariance matrix

$$\Sigma_t = \Sigma_{t|t-1} = E\left(s_t - \hat{s}_{t|t-1}\right)\left(s_t - \hat{s}_{t|t-1}\right)'$$

then the state equation can be written as follows

$$\hat{s}_{t+1} = A\hat{s}_t + K_t\eta_t$$

where K_t denotes the Kalman gain. Likewise, the observation equation can be written as

$$d_t = C\hat{s}_t + \eta_t$$

where the prediction error and the covariance matrix of the prediction error are given, respectively, by $\eta_t = d_t - E\left(d_t \mid d_{t-1}, d_{t-2}, \ldots d_1\right)$ and $E\eta_t\eta_t' = C\Sigma_t C'$.

Provided with the initial conditions for the state vector $\hat{s}_1$ and its covariance Σ_1 given by (6.3) and (6.4) respectively, we can recursively generate sequences for the Kalman gain K_t and the covariance matrix Σ_t using

$$K_t = A\Sigma_t C'\left(C\Sigma_t C'\right)^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C' \left(C\Sigma_t C'\right)^{-1} C\Sigma_t A'$$

The forecast errors $\{\eta_t\}_{t=1}^{T}$ can then be used to construct the log likelihood function for the sample $\{d_t\}_{t=1}^{T}$

$$\ln L = -\left(\frac{3T}{2}\right)\ln(2\pi) - \frac{1}{2}\left[\sum_{t=1}^{T}\ln\left(C\Sigma_t C'\right) - \sum_{t=1}^{T}\ln\left(\eta_t'\left(C\Sigma_t C'\right)^{-1}\eta_t\right)\right]$$

Hamilton (1994) shows how to generate a stream of smooth estimates $\left\{\hat{s}_{t|T}\right\}_{t=1}^{T}$ of the unobservable state variable where each point estimate equals the expected value of the state variable conditioned on all past sample information up to that point as follows

$$\hat{s}_{t|T} = E\left(s_t \mid d_T, d_{T-1}, \ldots, d_1\right)$$

Hence as mentioned earlier, for all t = 1, 2,…,T and j = 0, 1 the mean of the state equation is given by

$$\hat{s}_{t|t-j} = E\left(s_t \mid d_{t-j}, d_{t-j-1}, \ldots d_1\right)$$

and the variance is given by

$$\Sigma_{t|t-j} = E\left(s_t - \hat{s}_{t|t-j}\right)\left(s_t - \hat{s}_{t|t-j}\right)'$$

and for forecasting purposes we have

$$\hat{d}_{t|t-j} = E\left(d_t \mid d_{t-j}, d_{t-j-1}, \ldots d_1\right)$$

As before, let

$$\eta_t = d_t - \hat{d}_{t|t-1} = C\Sigma_{t|t-1}C'$$

so that the covariance matrix of forecast errors is given by

$$E\left(\eta_t\eta_t'\right) = C\Sigma_{t|t-1}C'$$

Based on the initializations of the mean and covariance given by (6.3) and (6.4) respectively, the sequences $\left\{\hat{s}_{t|t}\right\}_{t=1}^{T}$, $\left\{\hat{s}_{t|t-1}\right\}_{t=1}^{T}$, $\left\{\Sigma_{t|t}\right\}_{t=1}^{T}$, and $\left\{\Sigma_{t|t}\right\}_{t=1}^{T}$ can be generated recursively starting with estimates of the mean

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'\left(C\Sigma_{t|t-1}C'\right)^{-1}u_t,$$

and the variance

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}C'\left(C\Sigma_{t|t-1}C'\right)^{-1}C\Sigma_{t|t-1}$$

at time t (where $u_t = d_t - C\hat{s}_{t|t-1}$)

and then updating the estimate of the mean in the next period

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t}$$

with a covariance matrix given by

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t}A' \qquad \text{for all t = 1, 2,…,T}$$

Thus, following Hamilton (1994), we can begin the recursion by constructing a sequence $\{J_t\}_{t=1}^{T}$ that follows

$$J_t = \Sigma_{t|t} A' \Sigma_{t+1|t}^{-1} \text{ 37}$$

Then, noting that $\hat{s}_{T|T}$ corresponds to the last element of $\{\hat{s}_{t|t}\}_{t=1}^{T}$, we can use this terminal condition to recursively-backwards generate the rest of the sequence using

$$\hat{s}_{T-j|T} = \hat{s}_{T-j|T-j} + J_{T-j}\left(\hat{s}_{T-j+1|T} - \hat{s}_{T-j+1|T-j}\right) \text{ 38}$$

for j = 1, 2, …, T-1[39].

All data is drawn from either the Federal Reserve Bank of St. Louis FRED or the Bureau of Economic Analysis (NIPA tables) websites. The quarterly sample period encompasses 1947:Q1 through 2007:Q3. The measure of C_t comes from real personal consumption expenditures in chained 2000 dollars (PCECC96); I_t comes from real gross private domestic investment in chained 2000 dollars (GPDIC96); H_t comes from hours worked by all persons in the nonfarm business sector (HOANBS). Finally, the government measure comes from

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[37] Equation 13.6.11 pp. 395.

[38] Equation 13.6.16 pp. 396.

[39] Kohn and Ansley (1983) demonstrate that in cases where $\Sigma_{t+1|t}$ turns out not invertible, it can be replaced by the Moore-Penrose pseudoinverse in the expression for J_t.

government gross consumption and investment expenditures (GCE), which the BEA decomposes into the consumption and investment measure in 2000 chained dollars. All variables are seasonally adjusted and described in per capita terms by dividing by the civilian non-institutional population, ages 16 and over (CNP16OV). The logarithmic transformations of these variables are first-differenced prior to estimation. This is needed since the theoretical model admits nonstationary processes to drive the preference (P_t) and sector-specific technology (Z_{ct}, Z_{it}) and government (G_{ct}, G_{it}) shocks. Therefore, the aggregates inherit nonstationarity. Because there are more than one sector present, cointegrating relationships between the major aggregates[40] may not necessarily follow from this model. Thus, the growth rates of the observables are used for the estimation procedure.

While the U.S. National Income and Product Accounts provide, for example, the measures of investment (GPDIC96) I specify in my observables above, they do not decompose it into consumption and investment-goods-producing sectors as implied by this model in equation (1.21) so as to allow construction of sector-specific measures of the capital stock--through the sector-specific capital accumulation

[40] Unlike in the one sector growth model of King, Plosser, Stock and Watson (1991),where cointegration is found.

equations (1.6) and (1.7). Similarly, the index H_t is not decomposed in the data (into hours allocated to the production of consumer versus investment goods) as is implied by equation (1.20). Thus, the sector-specific inputs K_{ct}, K_{it}, H_{ct}, and H_{it} are treated as unobservable and a state-space model along with the observables described above are used to derive inferences about these unobservable components.

As mentioned above, the model has 34 deep parameters arising from a first principles micro-founded general equilibrium model. Prior to the estimation, I assume the following restrictions on some of the parameters. To ensure both capital and labor adjustment costs equal zero in the steady state, $\tilde{k}_c$, $\tilde{k}_i$, $\tilde{\eta}_c$, and $\tilde{\eta}_i$ are set to the model's implied steady-state growth rates of the capital-investment ratio and hours worked, respectively. The discount factor is infamously difficult to estimate accurately. I follow Hansen's (1985) suggestion to set $\beta = 0.99$ and Ireland's and Shuh's (2007) setting of $\delta_c = \delta_i = 0.025$. These values allow for the interpretation of the data of one-quarter year in real time. Also, the symmetry conditions $\phi_{kc} = \phi_{ki} = \phi_k$, $\phi_{hc} = \phi_{hi} = \phi_h$, and $\theta_c = \theta_i = \theta$,[41] are imposed. Once, the estimation is carried out, methods

[41] This symmetry condition is implicitly adopted by Greenwood, Hercowitz and Krussell's model (1997, 2000) and explicitly assumed by Whelan (2003).

to derive variance decompositions and impulse response functions are described below.

VII.4. *Innovation Analysis*

This subsection shows the necessary steps to formulate the variance decompositions of this model along with the standard error calculations to draw inferences about the importance of a fiscal expansion, as given by a positive government shock, to the different sectors considered. Begin by rewriting the transition equation in the empirical state model given in (4.15) as

$$s_t = As_{t-1} + B\varepsilon_t$$

or using the lag operator

$$(I - AL)s_t = B\varepsilon_t$$

This state space model is an autoregressive process of the first order in the state vector. Thus a Wold moving average representation of (4.15) looks like the following

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}$$

Given that this last equation holds for all periods, we can advance this expression by k periods to yield

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B \varepsilon_{t-j+k}$$

so that

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B \varepsilon_{t-j+k}$$

And, thus, a forecast error will look like

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B \varepsilon_{t-j+k}$$

Therefore, the covariance matrix of the state innovations will be given by

$$\begin{aligned}\Sigma_k^s &= E\left(s_{t+k} - E_t s_{t+k}\right)\left(s_{t+k} - E_t s_{t+k}\right)' \\ &= \left(BVB'\right) + A\left(BVB'\right)A' + A^2\left(BVB'\right)A^{2'} + \ldots + A^{k-1}\left(BVB'\right)A^{k-1'}\end{aligned}$$

Additionally, (4.15) implies that

$$\lim_{k\to\infty} \Sigma_k^s = \Sigma^s$$

where, as shown earlier, the covariance matrix of the state vector for this model is given by

$$vec\left(\Sigma^s\right) = \left[I - A \otimes A\right]^{-1} vec\left(BVB'\right)$$

We turn now to the covariance matrix of the innovations of f_t in equation (4.14), which is given by

$$\begin{aligned}\Sigma_k^d &= E\left(f_{t+k} - E_t f_{t+k}\right)\left(f_{t+k} - E_t f_{t+k}\right)' \\ &= \eta \Sigma_k^s \eta'\end{aligned}$$

which in the limit approaches the covariance matrix of the sample as

$$\lim_{k\to\infty} \Sigma_k^d = \Sigma^d$$

which is itself given by

$$\Sigma^d = C\Sigma^s C'$$

Hence, from equation (4.16) we have

$$\begin{pmatrix} \ln(C_t) - \ln(C_{t-1}) - \ln(\tilde{c}) \\ \ln(I_t) - \ln(I_{t-1}) - \ln(\tilde{i}) \\ \ln(N_t) - \ln(N_{t-1}) - \ln(\tilde{n}) \\ \ln(G_{ct}) - \ln(G_{ct-1}) - \ln(\tilde{g}_c) \\ \ln(G_{it}) - \ln(G_{it-1}) - \ln(\tilde{g}_i) \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{12} \\ U_{12} \\ U_{14} \\ U_{15} \end{pmatrix} s_t$$

This implies that, equation by equation, we have

$$\ln(C_{t+k}) = \ln(C_t) + k\ln(\tilde{c}) + U_{11}\sum_{j=1}^{k} s_{t+j}$$

$$\ln(I_{t+k}) = \ln(I_t) + k\ln(\tilde{i}) + U_{12}\sum_{j=1}^{k} s_{t+j}$$

$$\ln(N_{t+k}) = \ln(N_t) + k\ln(\tilde{n}) + U_{13}\sum_{j=1}^{k} s_{t+j}$$

$$\ln(G_{ct+k}) = \ln(G_{ct}) + k\ln(\tilde{g}_c)U_{14}\sum_{j=1}^{k} s_{t+j}$$

$$\ln(G_{it+k}) = \ln(G_{it}) + k\ln(\tilde{g}_i)U_{15}\sum_{j=1}^{k} s_{t+j}$$

Subtracting the expected values from the actuals, we have

$$\ln(C_{t+k}) - E_t(\ln(C_{t+k})) = U_{11}\sum_{j=1}^{k}(s_{t+j} - E_t s_{t+j})$$

$$\ln(I_{t+k}) - E_t(\ln(I_{t+k})) = U_{12}\sum_{j=1}^{k}(s_{t+j} - E_t s_{t+j})$$

$$\ln(H_{t+k}) - E_t(\ln(H_{t+k})) = U_{13}\sum_{j=1}^{k}(s_{t+j} - E_t s_{t+j})$$

$$\ln(G_{ct+k}) - E_t(\ln(G_{ct+k})) = U_{14}\sum_{j=1}^{k}(s_{t+j} - E_t s_{t+j})$$

$$\ln(G_{it+k}) - E_t(\ln(G_{it+k})) = U_{15}\sum_{j=1}^{k}(s_{t+j} - E_t s_{t+j})$$

From these, we can derive the following covariance matrices

$$\Sigma_k^C = E\left[\ln(C_{t+k}) - E_t(\ln(C_{t+k}))\right]\left[\ln(C_{t+k}) - E_t(\ln(C_{t+k}))\right]' = U_{11}\Sigma_k^s U_{11}'$$

$$\Sigma_k^I = E\left[\ln(I_{t+k}) - E_t(\ln(I_{t+k}))\right]\left[\ln(I_{t+k}) - E_t(\ln(I_{t+k}))\right]' = U_{12}\Sigma_k^s U_{12}'$$

$$\Sigma_k^H = E\left[\ln(H_{t+k}) - E_t(\ln(H_{t+k}))\right]\left[\ln(H_{t+k}) - E_t(\ln(H_{t+k}))\right]' = U_{13}\Sigma_k^s U_{13}'$$

$$\Sigma_k^{G_{gc}} = E\left[\ln(G_{ct+k}) - E_t(\ln(G_{ct+k}))\right]\left[\ln(G_{ct+k}) - E_t(\ln(G_{ct+k}))\right]' = U_{14}\Sigma_k^s U_{14}'$$

$$\Sigma_k^{G_{gi}} = E\left[\ln(G_{it+k}) - E_t(\ln(G_{it+k}))\right]\left[\ln(G_{it+k}) - E_t(\ln(G_{it+k}))\right]' = U_{15}\Sigma_k^s U_{15}'$$

are all a function of the covariance of the state equation Σ_k^s which is given by

$$\begin{aligned}\Sigma_k^s &= \left[\sum_{j=1}^{k}(s_{t+j} - E_t(s_{t+j}))\right]\left[\sum_{j=1}^{k}(s_{t+j} - E_t(s_{t+j}))\right]' = \\ &= \left[\sum_{j=1}^{k}\sum_{i=0}^{i-1}A^i B\varepsilon_{t+j-i}\right]\left[\sum_{j=1}^{k}\sum_{i=0}^{i-1}A^i B\varepsilon_{t+j-i}\right]' = \\ &= BVB' + (I+A)BVB'(I+A)' + \ldots + (I+A+\ldots+A^{k-1})BVB'(I+A+\ldots+A^{k-1})'\end{aligned} \quad [42]$$

[42] The necessary equation to derive variance decompositions.

VIII. Results

Estimates of the model's 28 unrestricted parameters are shown in Table 1. The standard errors associated with the parameter estimates were derived from a parametric bootstrapping procedure,[43] which generated 1,000 samples of artificial data for each of the observables studied in the model, each containing the same number of observations as the original sample of actual aggregate consumption, investment, hours worked and government spending in the U.S. The standard errors in Table 1 come from the standard deviations of the individual parameter estimates taken across these 1,000 replications.

The first result to highlight here is that adjustment costs enter the model significantly. The data seems to assign a higher importance to capital adjustments costs vs. labor adjustment costs as displayed by the larger estimate of $\hat{\phi}_{kc} = \hat{\phi}_{ki} = 66.68$ both in absolute terms and in relation to its own standard errors. However, labor adjustment costs $\hat{\phi}_{nc} = \hat{\phi}_{ni} = 49.98$ are also estimated to be important and statistically significant.

[43] The procedure is described in Ch 6 of Efron and Tibshirani's (1993) book.

The data, through equations (3.29)-(3.33), imply that the average quarterly growth rate of consumption is 1.0053 while the growth rate of investment is 1.0049.[44]

Parameters 9 through 28 of Table 1 summarize the persistence and volatility of the shocks in the model. The estimates $\rho_p^l = 0.0665$, $\rho_p^g = 0.9233$, $\sigma_p^l = 0.0126$, and $\sigma_p^g = 0.0012$ imply that the preference shock (P_t) contains a very persistent but small growth component and a larger but less persistent transitory component. In contrast, the consumption-specific technology shock (Z_{ct}) displays similar levels of persistence in its level and growth rate components. The estimates associated with the investment-specific technology shock (Z_{it}) given by $\rho_{zi}^l = 0.5773$, $\rho_{zi}^g = 0.4375$, $\sigma_{zi}^l = 0.3507$, and $\sigma_{zi}^g = 0.0410$ reveal that, while both components exhibit substantial persistence, the level component is both larger and more persistent than the growth component. Both the level and growth rate of the shock to government's consumption (g_{ct}) seem to display similar volatility, but the growth component exhibits twice the persistence of the level

[44] These rates are arrived at by combining the equations mentioned above with parameters 1 and 4-8 in Table 1.

component. Finally, the estimates $\rho_{gi}^{l} = 0.2384$, $\rho_{gi}^{g} = 0.1268$, $\sigma_{gi}^{l} = 0.0437$, and $\sigma_{gi}^{g} = 0.0083$ suggest that the shock to government's investment (g_{it}) has a larger level component, and it is all around more persistent than the shock to government's consumption (g_{ct}). The natural interpretation of these results is that the apparent differential rate in productivity between consumption and investment-producing sectors in the U.S. can be attributed to the high persistence of investment-specific technology and government shocks.

Table 2 shows the forecast error variances in consumption, investment, labor and government and decomposes them into attributed percentages to each of the ten shocks considered by the model. The first panel shows that most of the variance in consumption is driven by three growth shocks: the growths in preferences, consumption-specific technology and government consumption. Not surprisingly, as evidenced by panel 2, consumption-specific shocks (be they from technology or government) play no part in the volatility of investment. Instead, the levels of investment-specific technology and government shocks explain most of the variability in investment at the short horizons and then the growths of those same variables take over

in importance in the long run. Labor hours seem to be driven solely by preferences, with the latter's growth rate shock explaining the overwhelming majority of volatility in labor. Most volatility in government consumption is explained by the same three growth components that drive private consumption: namely, growth in preferences and consumption-specific technology and government shocks. Furthermore, as evidenced by the last panel of Table 2, most of the variance in government investment is explained by the level and growth of investment-specific government shocks and, to a lesser extent, shocks to the growth of preferences. Kimball (1994) shows that a two-sector Real Business Cycle model with logarithmic and time-separable utility over consumption has the property that consumption-specific technology shocks do not seem to impact investment or hours worked, but only consumption. These results are consistent with that finding where, as it is shown in Table 2, both the level and growth rate of consumption-specific technology shocks seem to only impact private consumption and to a lesser extent government consumption. It is worth noting that shocks to the growth of preferences seem to explain a large portion of the volatility in hours worked as well as consumption (both private and public) while having a much smaller

role in investment (both public and private). An attractive feature of these results is that estimates of the impact of technology shocks seem to be sector-specific, with consumption-specific technology affecting mostly consumption or government consumption while investment-specific shocks affecting mostly investment or government investment. Additionally, technology has no impact on preferences. This apparent orthogonality between technology and preferences is, again, another attractive feature of the empirical evidence yielded by this model. Finally, government shocks also seem to display a certain sectoral gravitation. While the level of consumption-specific government shock has no importance for any variable, the growth of consumption-specific government shocks seems to have most of their impact in consumption and government consumption. Likewise, investment-specific government shocks show up in the volatility in investment and government investment and not consumption (or hours, which seem to be solely explained by preferences).

Figures 1-4 show the estimated model's impulse responses of each of the observables to each of the unobservable shocks considered. These structural shocks are identified based on the dynamic effects that the Real Business Cycle model associates with each distinct type of

shock. Given that these structural disturbances come from a highly constrained dynamic stochastic general equilibrium model, it follows that they are not identified based on the usual timing assumptions typically associated with less-constrained vector auto-regressive time series models. Figure 1 displays the impulse responses of each observable variable to (both the level and growth of) preferences. Evidence from Figure 1 can be reconciled quite nicely with the variance decomposition results described earlier, where shocks to the growth of preferences explained a substantial portion of the volatility in consumption and hours worked, while assigning very little response to a shock in the level of preferences. Conversely, shocks to the growth rate of preferences seem to have the largest impact in consumption and labor hours. Responses of investment and government spending are more muted. Figure 2 shows the response to consumption-specific technology shocks. Here, the level of the consumption-specific technology shock has a temporary impact on consumption, while the growth of consumption-specific technology shocks lead to a permanently higher level of consumption,[45] along with a temporary increase in government. Figure 3 collects the response to investment-

[45] Both of these results are consistent with Kimball (1994) as mentioned above.

specific technology shocks. Not surprisingly, it shows evidence that investment-specific technology shocks have a marked impact on investment only with a shock in the level leading to a substantial temporary increase in investment, and a shock to the growth rate of investment-specific productivity shocks have a large and permanent effect on investment. Finally, Figure 4 shows the response to the level and growth rate of a government shock.[46] It shows that the growth of the government shock has a positive and permanent impact on consumption and investment. This constitutes one of the major highlights of this study since it stands diametrically opposed the Real Business Cycle prediction that government spending crowds out consumption. These results provide evidence that government spending is expansionary for private spending (both consumption and investment). This conclusion, while based on an entirely different theoretical model, stands in agreement with the evidence from Chapter 1 of this thesis.

Figures 5-7 graph how the estimates of the various shocks evolve over time within the construct of the model. All of these estimates reflect information contained in the full sample of the data.

[46] Only the more informative (from this model) of the two measures of the government shock is used here, namely government investment.

They are constructed by using Kalman smoothing algorithms as shown in section VIII.3. Figure 5 shows that the growth component of preferences explains most of the fluctuation in preference shocks where the contribution of the level component is negligible. The implication of this decomposition is that, according to this model and the available data, fluctuations in preferences tend to be permanent. Figure 6 shows that shocks to investment-specific productivity have grown faster than those that are consumption-specific. While consumption-specific productivity has been growing on par with its deterministic trend, the estimate of investment-specific productivity (Z_{it}) lies above its trend during most of the 1980s and 1990s and drops below trend in the early 2000s, only to recover back toward trend in the latter part of the sample. Although, we see higher volatility in the transitory component of productivity in investment (relative to consumption-specific productivity), a strong mean reversion is also evident. On the other hand, the estimate of consumption-specific productivity seems to experience a drop relative to trend in the last couple of years of the sample, which seems to be transitory. Taken together, all panels of Figure 6 imply that investment-specific technology grows at a faster pace than consumption-specific

productivity. This could be the main factor that explains the faster growth in private real investment (relative to consumption) we are seeing in the U.S. Furthermore, this pattern seems likely to continue in the future. Figure 7 seems more problematic in terms of drawing inferences. Both consumption and investment-specific government shocks seem to experience a structural break in growths that counterbalance each other, and the level components exhibit extreme persistence and little mean reversion. The connotation here is for the composition and timing of these shocks and how they affect the economy. If we were to assume a similar structure for government shocks as we see in technology and preferences, then Figure 7 seems to imply that the investment part seems to dominate the first part of the sample and the consumption-specific shock seems to take over importance (in terms of providing a positive rate of growth) in the latter part of the sample.

Finally, Figures 8-12 showcase the influence each shock has on the model's interpretation of the actual data by illustrating a counterfactual scenario. For example, the six charts in Figure 8 display the actual log of real per capita consumption (thin line) and an estimation of how this aggregate measure would have behaved in the

absence of one of the five shocks considered (thick lines), namely: preferences, consumption and investment-specific technologies and consumption and investment-specific government disturbances. Figure 8 reinforces the reasoning drawn from the variance decompositions in Table 2 where consumption-specific technology shocks as well as preferences seem to be the most important factors in allowing the model to capture the highly persistent movements in consumption. Conversely, the three charts on the second column show that consumption would have behaved in the same way in the absence of investment-specific productivity or government shocks. Figure 10 indicates how the persistent, but ultimately transitory, investment-specific productivity shocks allow the model to reproduce the volatility that is inherent in U.S. private investment. Figure 11 shows that, while preferences served to inhibit faster growth in government consumption since the 1970s, consumption-specific government shocks account for almost all growth in government consumption. Figure 12 shows that investment-specific government shocks account for some of the growth and most of the transitory movements in government investment for the full sample period. Finally, Figure 9 shows how preferences have served to soak up movements in labor hours,

especially in the latter part of the sample. Investment-specific technology shocks served to depress hours worked in the first part of the sample.

To sum up, Figures 5-12 show that a highly constrained micro-founded Real Business Cycle model based on first principles can provide a plausible interpretation of productivity and government fluctuations that is consistent with the postwar data in the U.S.

IX. *Conclusion*

The Real Business Cycle model studied here implies how different types of shocks--to the level and growth rate of productivity and government in distinct investment and consumption goods-producing sector have different effects on U.S. aggregate economic variables such as consumption, investment, labor hours worked and government spending. Of special interest is the response to a government shock, the focal point of this thesis.

The model is estimated via maximum likelihood and it yields a number of important insights for the postwar U.S. economy. First and foremost is the evidence that increases in government spending lead to higher levels of consumption, based on this model. This result is in line with the conclusion found in Chapter 1 with a simpler model that included no adjustment costs or productivity shocks to its structure. Furthermore, this model improves on Chapter 1 by finding evidence of a positive investment effect following an increase in government spending. The reader will recall that the results of Chapter 1 were mixed with regards to investment. Adding productivity shocks, as well adjustment costs to the structure of the model, was intended not only to better replicate the dynamics of a micro-founded economy but,

specifically, to get an answer to the investment response. To that effect, the model is more successful than that of the earlier chapter. Second, while both adjustment costs seem to be important in describing the model, the degree of capital adjustment supersedes the labor adjustment costs both in terms of size and significance. Third, according to this model, consumption seems to be driven mainly by consumer preferences, consumption-specific technology shocks, and consumption-specific government shocks in that order. Conversely, investment seems to be driven mostly by investment-specific technology and government shocks. Hours worked are mainly driven by preference shocks. All these features are highly intuitive, given the close relationship between preferences and the labor-leisure choice suggested by micro-founded general equilibrium models. Finally, the results show that investment-specific technology seems to grow faster than consumption-specific productivity and this can inform the disparity in rates of growth of consumption and investment in the U.S.

This model, though enriched with more structural dynamics than the one presented in Chapter 1 of this thesis, abstracts from a number of extensions that should be considered. First, the model developed here does not consider differential rates of utilization in the

production of consumption and investment-goods-producing firms. These, in combination with depreciation rate, seem like an important feature to include in the dynamic responses of shocks that might be sector-specific. Second, this model borrowed extensively from Pakko (2002, 2005) where the private agent can distinguish perfectly between shocks to the levels and growth rates of sector-specific technologies. Edge, Laubach and Williams (2004) argue that private agents in the U.S. took some time to recognize the persistent shifts in productivity growth of the 1970s and again in the 1990s. Their calibration shows that growth rate of shocks to consumption and investment-specific technologies can have different effects in the absence of full information. These results suggest that extending the analysis to include learning for households and firms is warranted. Finally, this model assumes a homogeneous capital stock that can be allocated across different sectors of the economy subject to adjustment costs. Unlike Chapter 1 of this thesis, and other works such as Tevlin and Whelan (2003), it does not consider different types of capital goods. Specifically, disaggregating capital good types into those associated with computing and software versus the rest of the economy would

contribute to the question of the role of information technology in the productivity revival of the 1990s.

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TABLE 1: ***Maximum Likelihood Estimates and Bootstrapped Standard Errors***

Parameter	*Estimate*	*Standard Error*
1. $\hat{\theta}_c$:	0.9141	(0.0000)
2. $\hat{\phi}_{kc}$:	66.6798	(0.8909)
3. $\hat{\phi}_{nc}$:	49.9864	(6.2100)
4. $\hat{p}^g$:	1.0013	(0.0002)
5. $\hat{z}_c^g$:	1.0009	(0.0002)
6. $\hat{z}_i^g$:	1.0004	(0.0001)
7. $\hat{g}_c^g$:	1.0075	(0.0001)
8. $\hat{g}_i^g$:	1.0032	(0.0001)
9. ρ_p^l :	0.0665	(0.0156)
10. ρ_p^g :	0.9233	(0.0014)
11. ρ_{zc}^l :	0.0549	(0.0004)
12. ρ_{zc}^g :	0.0681	(0.0003)
13. ρ_{zi}^l :	0.5773	(0.0005)
14. ρ_{zi}^g :	0.4375	(0.0001)
15. ρ_{gc}^l :	0.0274	(0.0000)
16. ρ_{gc}^g :	0.0498	(0.0024)
17. ρ_{gi}^l :	0.2384	(0.0005) [47]
18. ρ_{gi}^g :	0.1268	(0.0003)
19. σ_p^l :	0.0126	(0.0611)
20. σ_p^g :	0.0012	(0.0013)
21. σ_{zc}^l :	0.0228	(0.0011)
22. σ_{zc}^g :	0.0053	(0.0023)

[47] To ensure a global maximum was achieved, the parameter estimation procedure was replicated 1,500 times. The reported parameters are those associated with Ln (L) = -3,344, the highest maximum likelihood in absolute value yielded by the iterated procedure. The standard

TABLE 1: *(ctd).*

Parameter	*Estimate*	*Standard Error*
23. σ_{zi}^{l} :	0.3507	(0.0061)
24. σ_{zi}^{g} :	0.0410	(0.0130)
25. σ_{gc}^{l} :	0.0146	(0.0001)
26. σ_{gc}^{g} :	0.0177	(0.0020)
27. σ_{gi}^{l} :	0.0437	(0.0043)
28. σ_{gi}^{g} :	0.0083	(0.0060)

Prior to the estimation procedure, constraints on the following parameters $\beta = 0.99$, and $\delta_c = \delta_i = 0.025$ were imposed.

The symmetry conditions $\theta_c = \theta_i = \theta$, $\phi_{kc} = \phi_{ki} = \phi_k$, and $\phi_{nc} = \phi_{ni} = \phi_n$ were also assumed.

Finally, $\tilde{\eta}_c, \tilde{\eta}_i, \tilde{k}_c$, and $\tilde{k}_i$ are set to make steady-state capital and labor adjustments costs equal to zero.

errors were derived from a parametric bootstrap procedure which simulated the model to generate 1,000 samples of artificial data for per capita consumption, investment, labor hours and government spending in the U.S. Then the model is re-estimated 1,000 times using the simulated data sets. The reported standard errors come from the standard deviations of the individual parameter estimates taken across the 1,000 replications.

TABLE 2: ***Forecast Error Variance Decompositions***

Consumption

Quarters Ahead	1	2	8	12	20	40
ε_p^l	0.48	0.22	0.22	0.18	0.12	0.63
ε_p^g	53.60	69.11	65.45	62.46	58.50	54.59
ε_{zc}^l	15.28	1.38	0.78	0.56	0.35	0.18
ε_{zc}^g	30.62	16.92	15.48	15.23	14.01	14.40
ε_{zi}^l	0.00	0.15	0.22	0.22	0.17	0.10
ε_{zi}^g	0.00	0.27	1.16	2.27	4.37	7.61
ε_{gc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gc}^g	0.00	11.86	16.44	18.65	20.67	21.72
ε_{gi}^l	0.00	0.02	0.03	0.03	0.02	0.01
ε_{gi}^g	0.00	0.04	0.19	0.38	0.74	1.28

Investment

Quarters Ahead	1	2	8	12	20	40
ε_p^l	0.04	0.18	0.22	0.20	0.14	0.07
ε_p^g	0.10	0.96	3.06	5.40	9.46	15.40
ε_{zc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zc}^g	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zi}^l	71.51	50.24	33.93	24.71	15.05	6.69
ε_{zi}^g	15.38	35.36	49.50	56.53	62.56	65.73
ε_{gc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gc}^g	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gi}^l	10.35	7.26	4.90	3.57	2.17	0.96
ε_{gi}^g	2.60	5.98	8.37	9.56	10.58	11.12

Labor Hours

Quarters Ahead	1	2	8	12	20	40
ε_p^l	1.67	0.84	0.66	0.51	0.34	0.18
ε_p^g	98.24	99.04	99.13	99.22	99.37	99.61
ε_{zc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zc}^g	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zi}^l	0.00	0.00	0.01	0.02	0.02	0.02
ε_{zi}^g	0.06	0.09	0.16	0.19	0.20	0.14
ε_{gc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gc}^g	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gi}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gi}^g	0.01	0.01	0.02	0.03	0.03	0.02

Government (consumption sector)

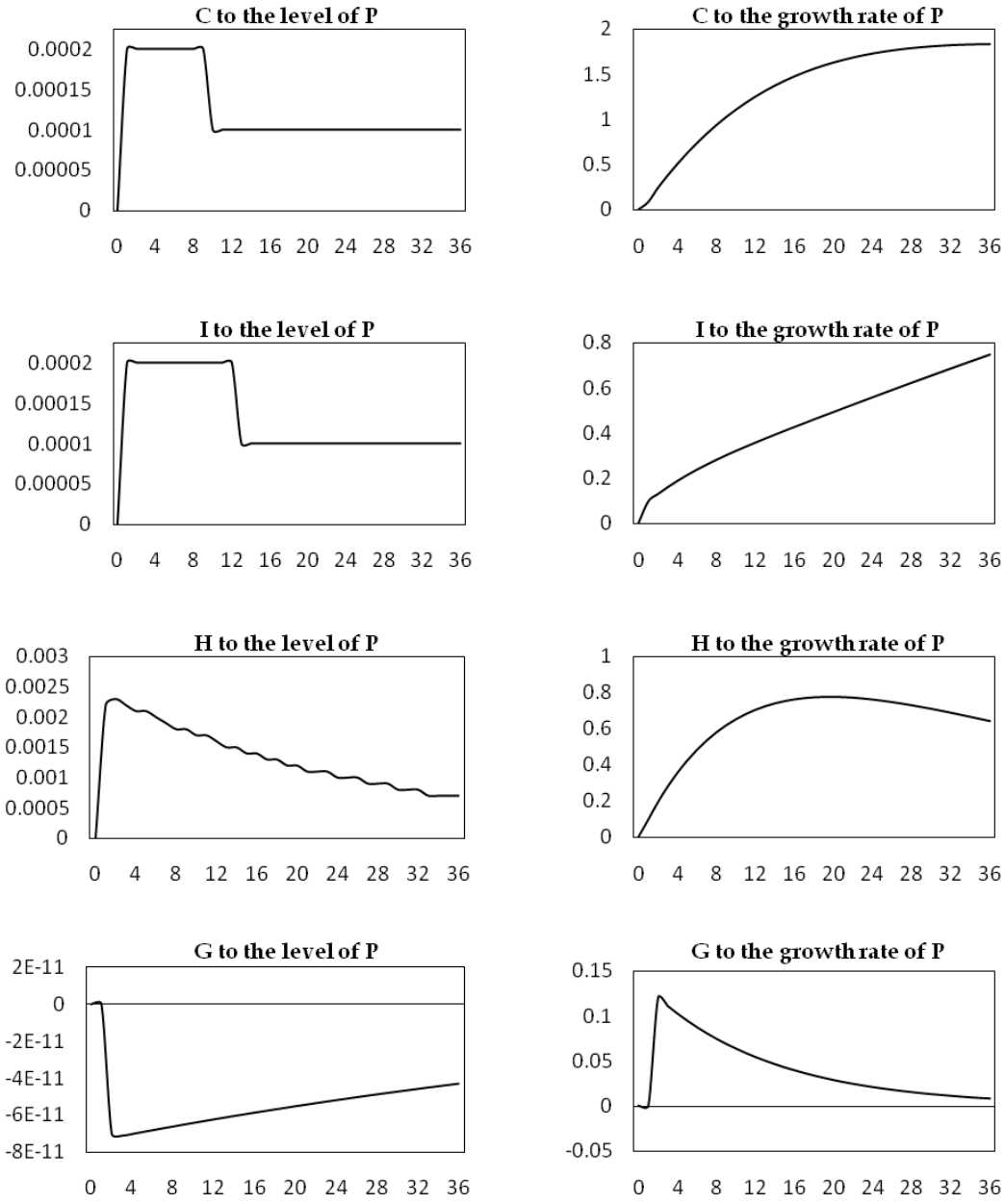
Quarters Ahead	1	2	8	12	20	40
ε_p^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_p^g	0.00	74.90	74.91	74.91	74.91	74.91
ε_{zc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zc}^g	0.00	9.44	9.44	9.44	9.44	9.44
ε_{zi}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zi}^g	0.00	0.00	0.10	0.00	0.00	0.00
ε_{gc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gc}^g	0.00	15.64	15.64	15.64	15.63	15.63
ε_{gi}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gi}^g	0.00	0.00	0.00	0.00	0.00	0.00

Government (investment sector)

Quarters Ahead	1	2	8	12	20	40
ε_p^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_p^g	0.00	12.66	12.38	12.25	12.12	12.04
ε_{zc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zc}^g	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zi}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{zi}^g	0.00	0.00	0.10	0.00	0.00	0.00
ε_{gc}^l	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gc}^g	0.00	0.00	0.00	0.00	0.00	0.00
ε_{gi}^l	78.76	61.61	63.87	64.68	65.28	65.68
ε_{gi}^g	21.23	25.72	23.73	23.07	22.59	22.27

[1]

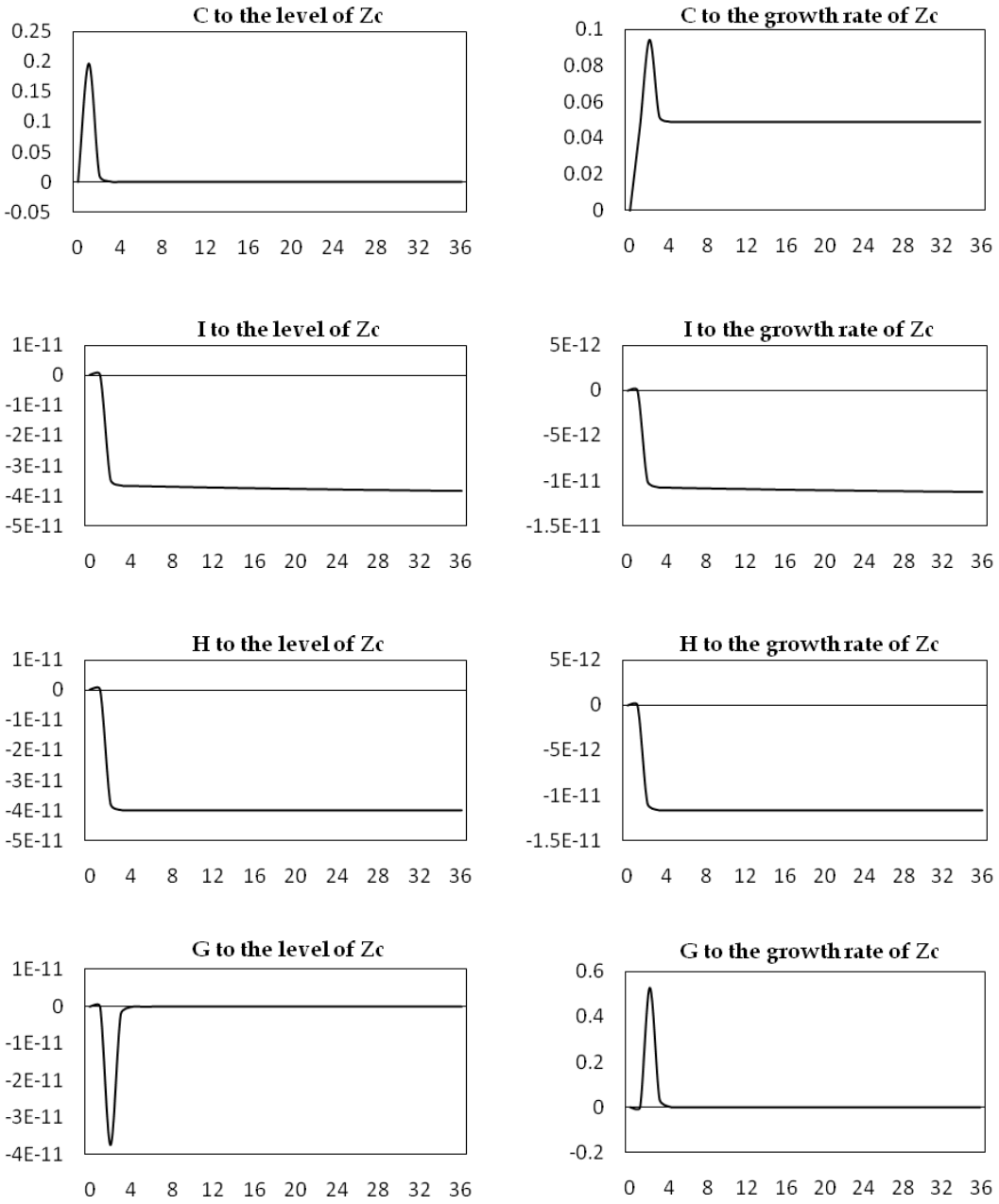
[1] Table 2 shows the decomposition of the forecast error variance for consumption, investment, labor hours and two measures of government spending at horizons ranging from one quarter to ten years. The figures shown represent the percentages due to each of the model's ten shocks.



[1]

[1] **Figure 1**. Impulse Responses to: Preference Shocks (A).

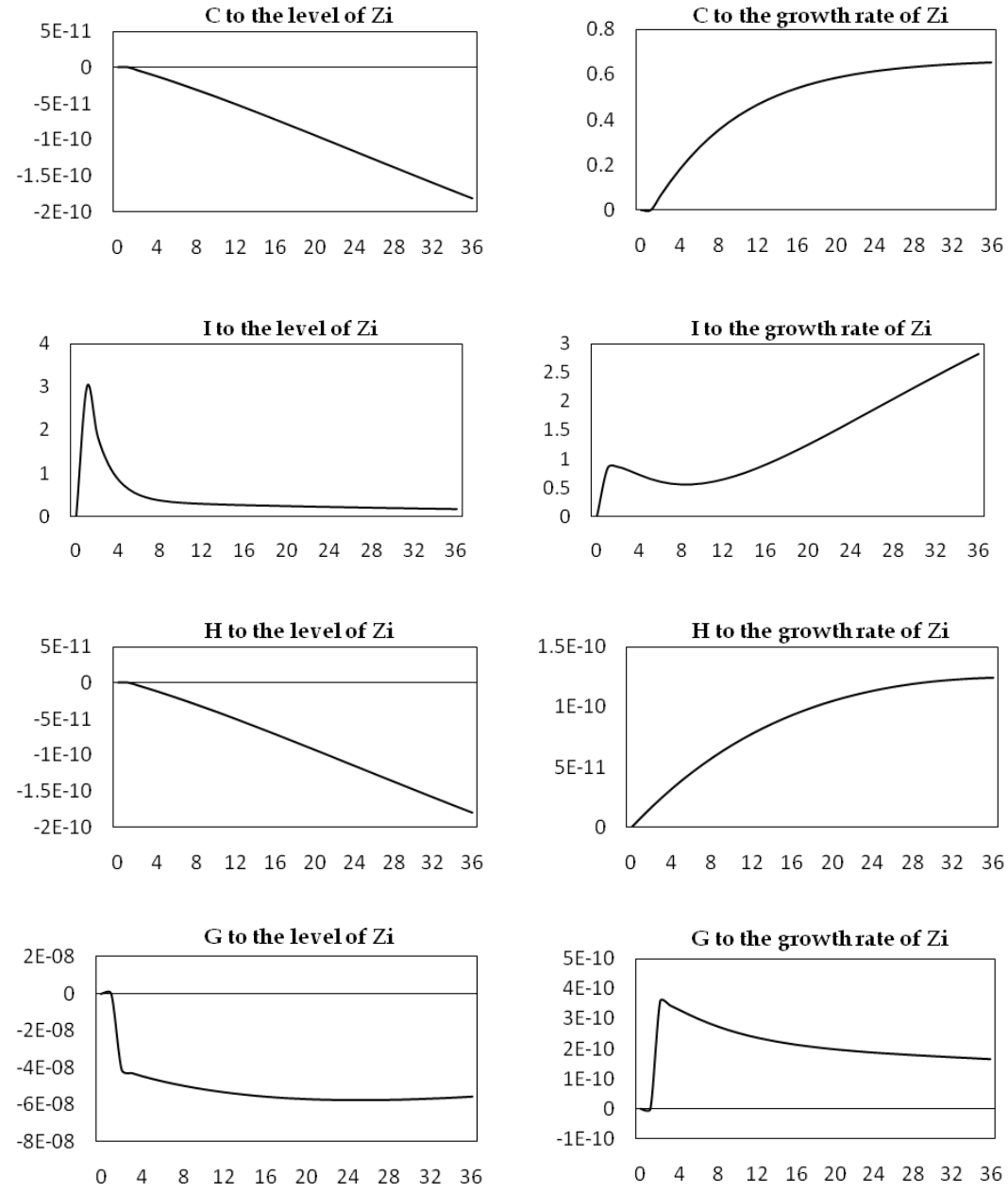
[2] Each chart shows the response of consumption (C), investment (I), hours worked (H) or government spending (G) to a one-standard-deviation increase to the level or growth rate of preferences (A).



[1]

[1] **Figure 2**. Impulse Responses to: Consumption-specific Tech Shock (Z_c).

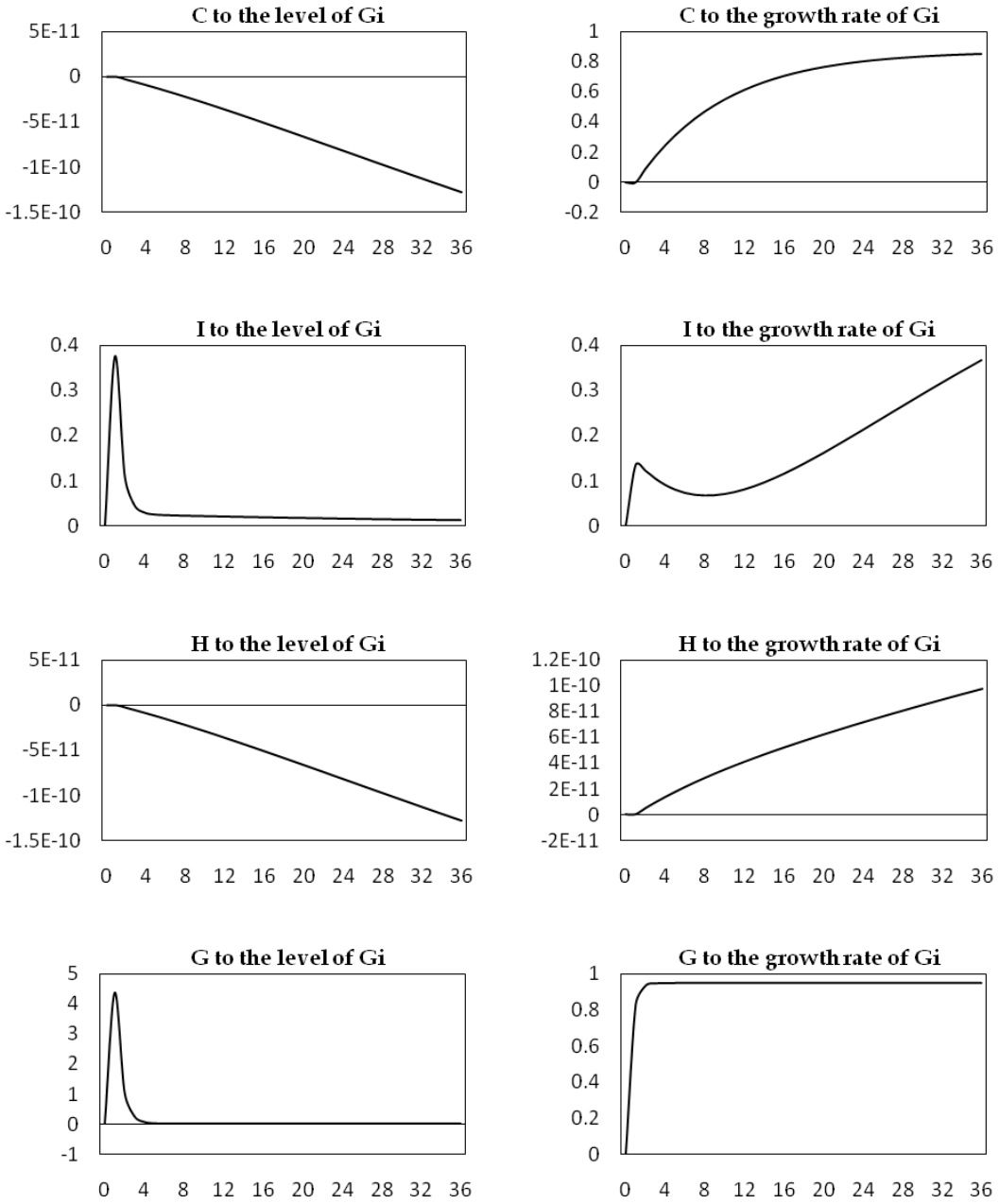
[2] Each chart shows the response of consumption (C), investment (I), hours worked (H) or government spending (G) to a one-standard-deviation increase to the level or growth rate of consumption-specific technology (Z_c).



[1]

[1] **Figure 3**. Impulse Responses to: Investment-specific Tech Shock (Zi).

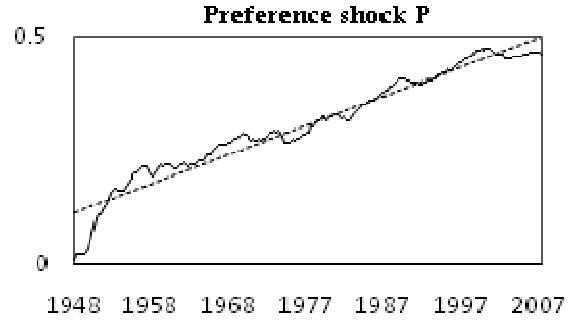
[2] Each chart shows the response of consumption (C), investment (I), hours worked (H) or government spending (G) to a one-standard-deviation increase to the level or growth rate of investment-specific technology (Zi).

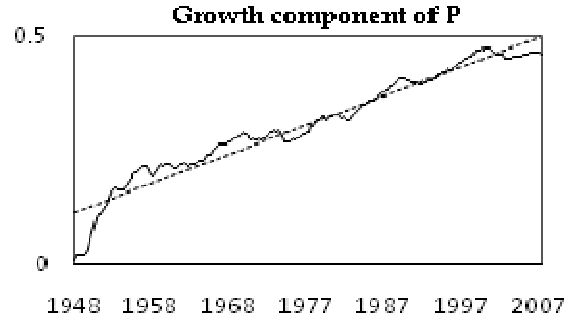


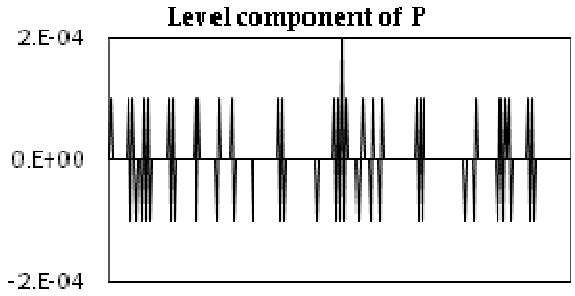
[1]

[1] **Figure 4**. Impulse Responses to: Governmental Investment-specific Shock (Gi).

[2] Each chart shows the response of (C), (I), (H) or government spending (G) to a one-standard-deviation increase to the level or growth rate of governmental investment-specific shock (Gi).







[48]

[48] **Figure 5.** Smoothed estimate of the preference shock (in logs), decomposed into level and growth rate components.

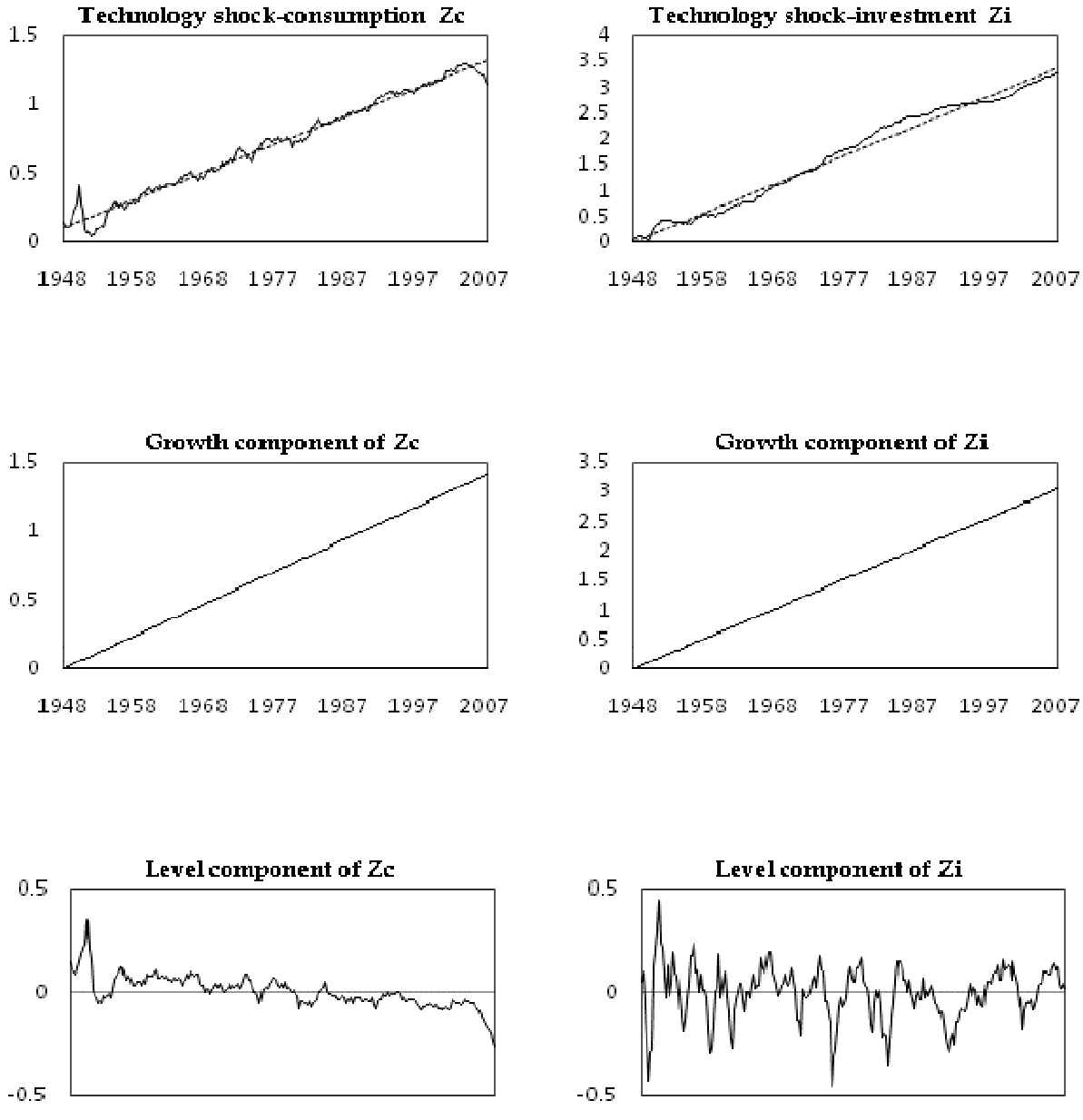


Figure 6. Smooth Estimates of Sector-specific Technology Shocks (in logs).

Each estimate is decomposed into level and growth components. The dotted line represents a deterministic trend.

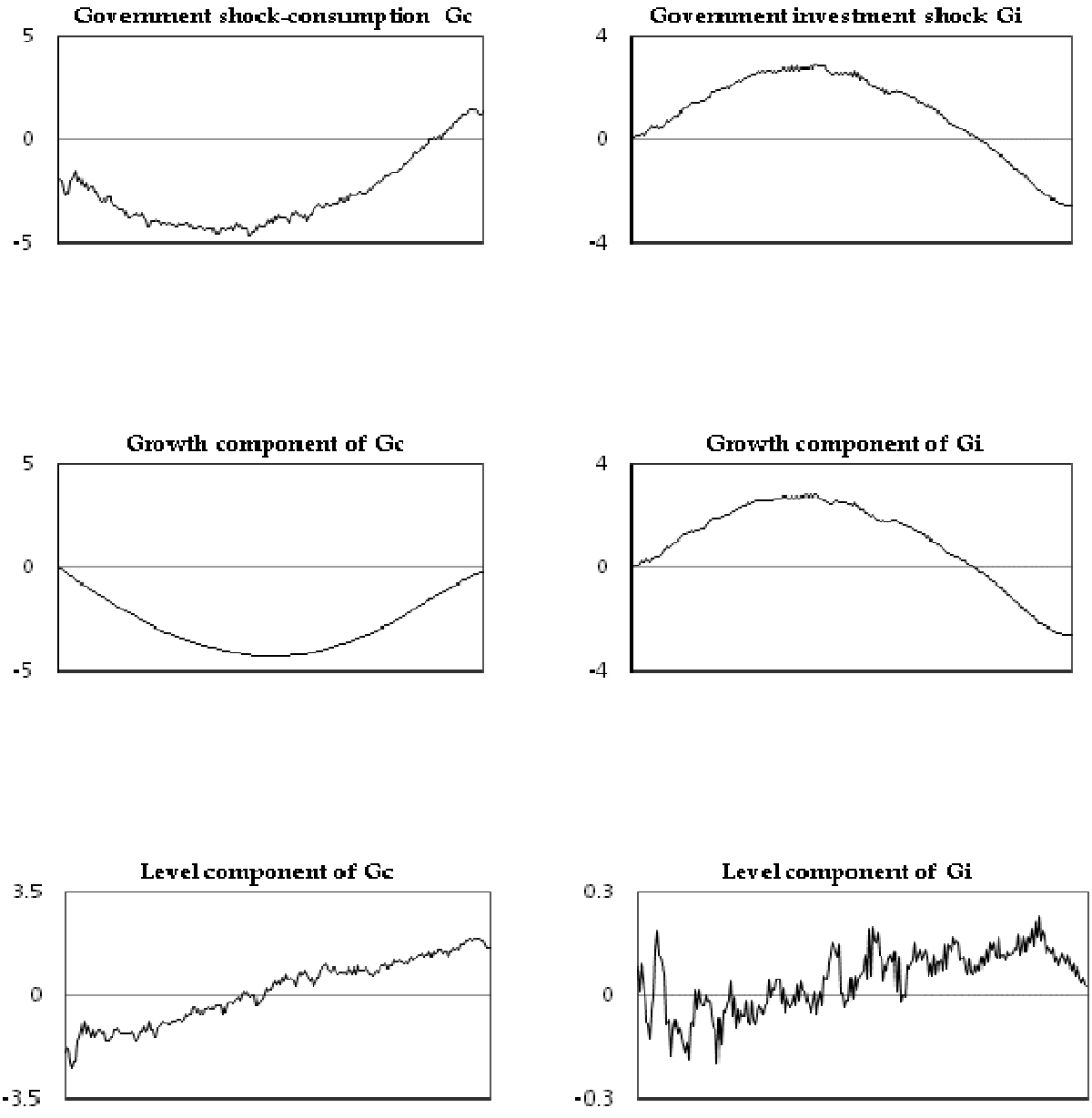


Figure 7. Smooth Estimates of Sector-specific Government Shocks (in logs).

Each estimate is decomposed into level and growth components. The dotted line represents a deterministic trend.

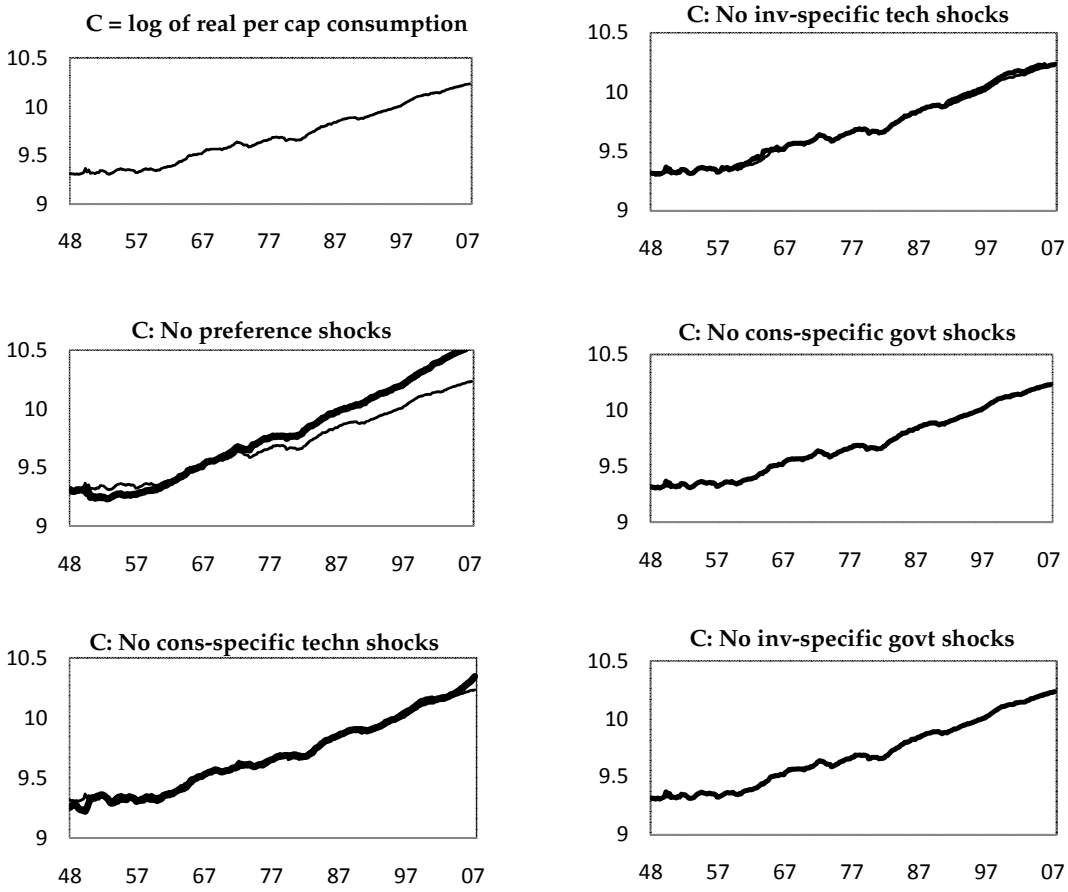


Figure 8. The top left chart shows the log of actual U.S consumption expressed in chained 2000 dollars.

The thick line in the remaining graphs shows the model's estimate of how the actual series (thin line) would have behaved in the absence of each of the shocks considered.

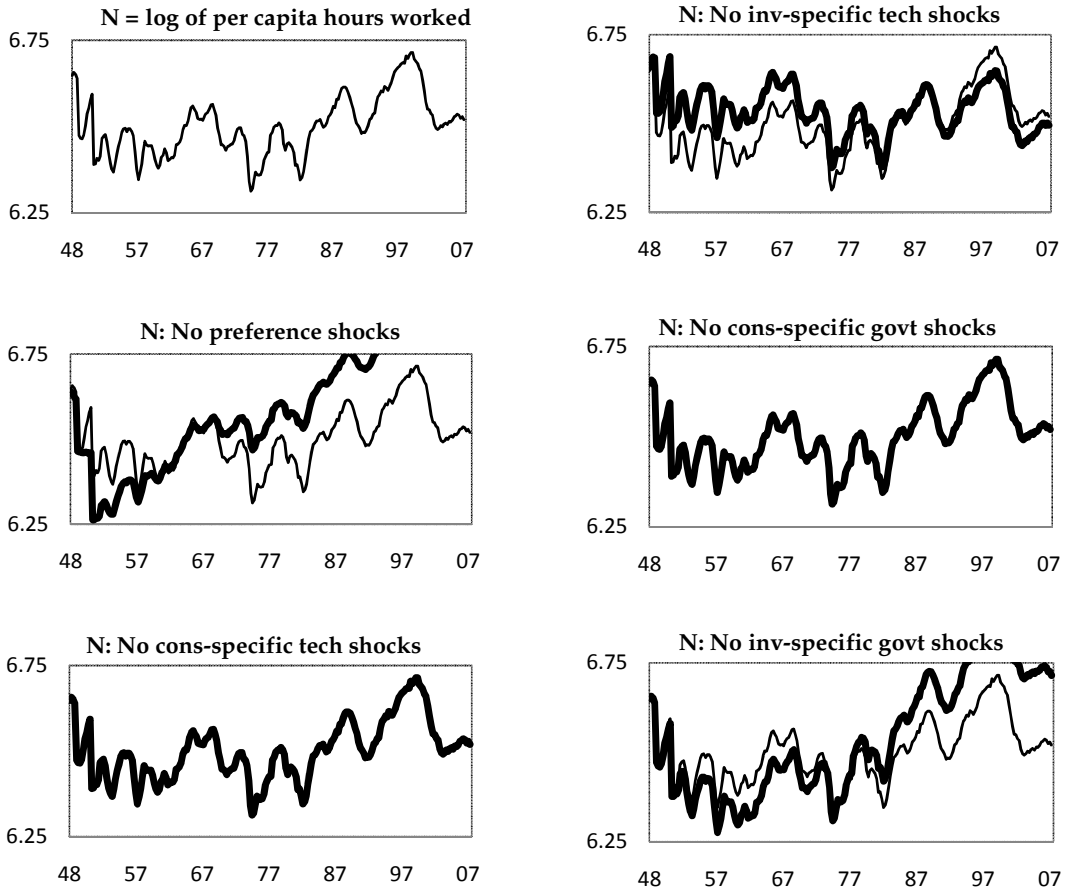


Figure 9. The top left chart shows the log of actual U.S hours worked (indexed 1992=100).

The thick line in the remaining graphs shows the model's estimate of how the actual series (thin line) would have behaved in the absence of each of the shocks considered.

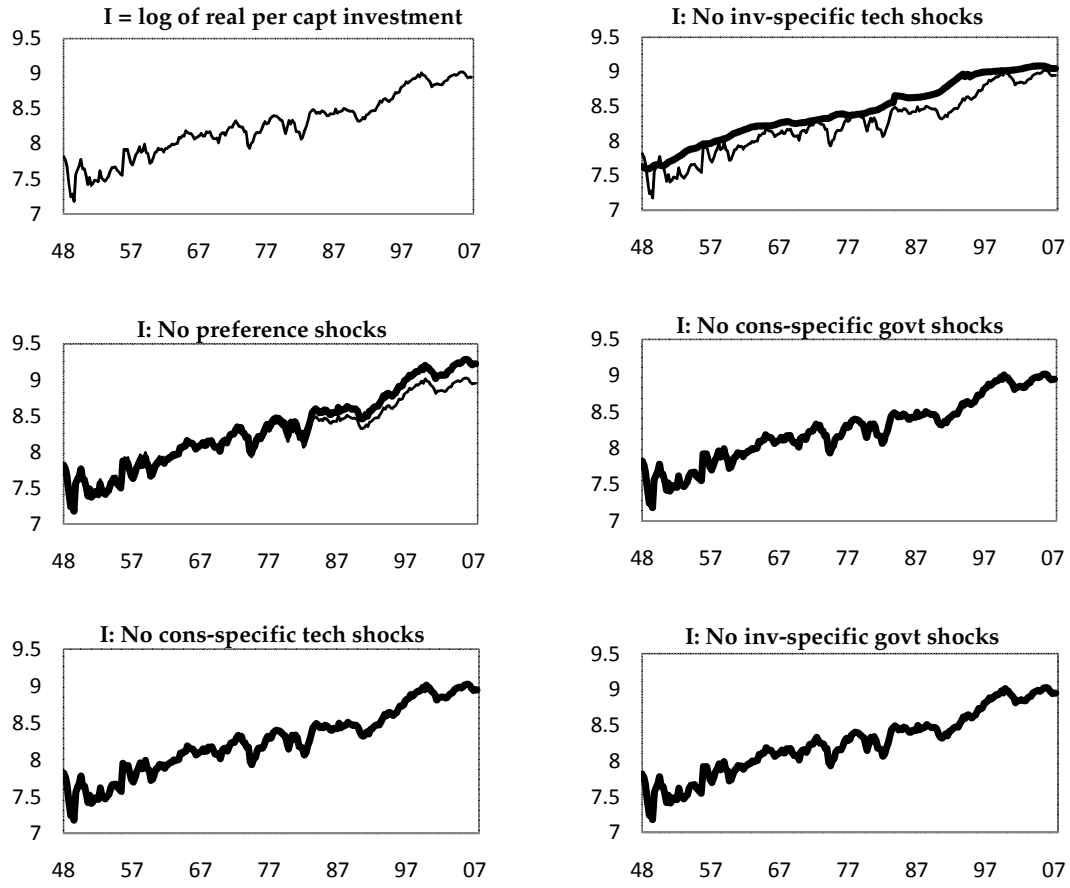


Figure 10. The top left chart shows the log of actual U.S investment expressed in chained 2000 dollars.

The thick line in the remaining graphs shows the model's estimate of how the actual series (thin line) would have behaved in the absence of each of the shocks considered.

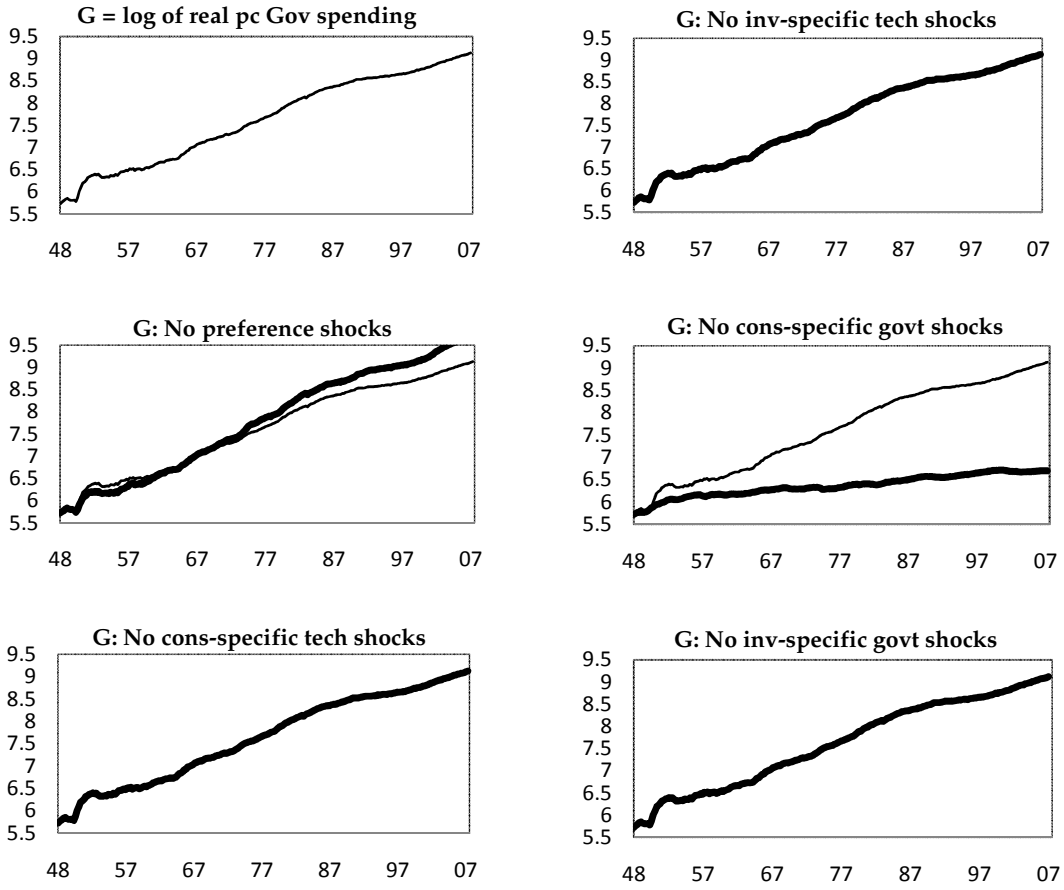


Figure 11. The top left chart shows the log of actual U.S government consumption expressed in chained 2000 dollars.

The thick line in the remaining graphs shows the model's estimate of how the actual series (thin line) would have behaved in the absence of each of the shocks considered.

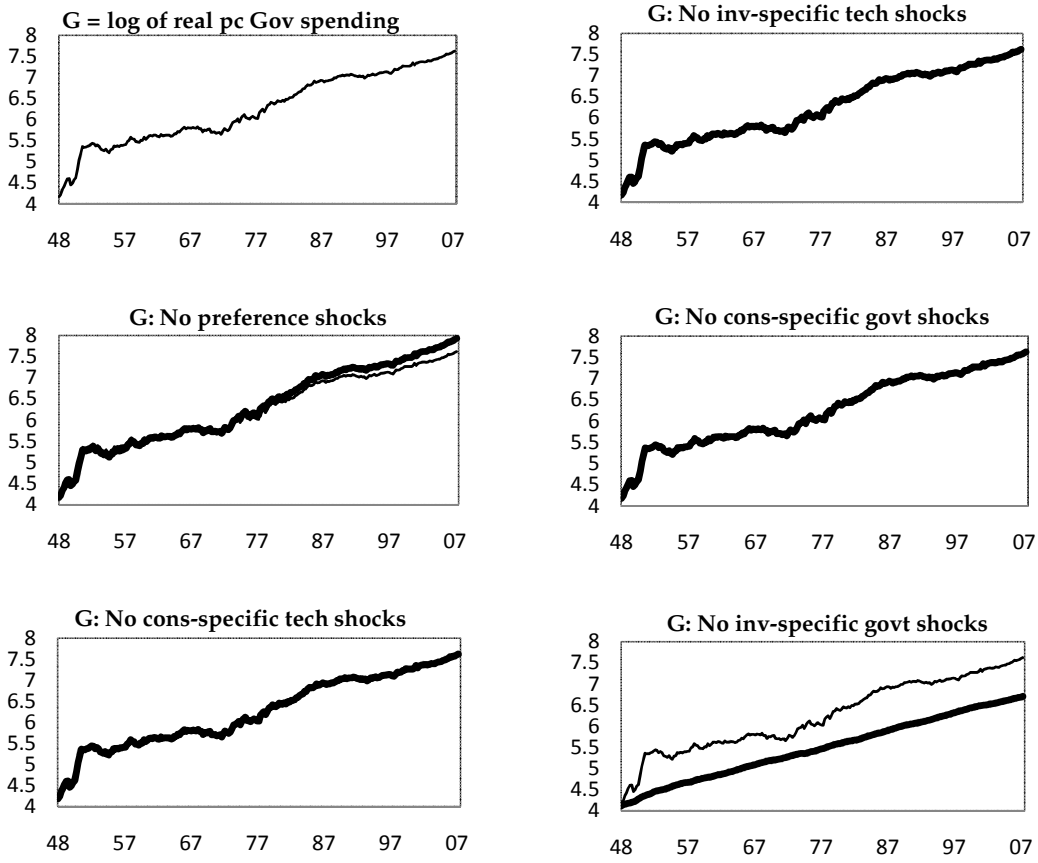


Figure 12. The top left chart shows the log of actual U.S government investment expressed in chained 2000 dollars.

The thick line in the remaining graphs shows the model's estimate of how the actual series (thin line) would have behaved in the absence of each of the shocks considered.

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CHAPTER III

THE REAL INVESTMENT TO CONSUMPTION RATIO: INFERENCES FROM ALTERNATIVE PERMANENT-TRANSITORY DECOMPOSITIONS.

Abstract

One of the well-known predictions of the balanced growth hypothesis is that real aggregates for consumption, investment, output and the capital stock all grow together at an identical rate, referred to as the *balanced growth path*, which is determined by the exogenous technology process. Any correlated deviation from this model's long-run balanced growth path is assumed to be part of the business cycle.

Recent empirical evidence seems to suggest that while consumption and investment, in *real* terms, seem to not grow together at the same rate; the same aggregates seem to share a common trend in *nominal* terms. This contention seems to contradict the balanced growth prediction, and it has been termed the *nominal ratio balanced growth hypothesis*.

This paper shows that a two-sector model is not a necessary condition for having different price dynamics in consumption and investment. An alternative balanced growth hypothesis-- namely, that the ratio of nominal investment to nominal consumption is stationary-- is proposed. An empirical test for the possibility of this alternative balanced growth path within the context of a one-sector model is carried out using a new time-series technique as yet untried within the context of this new balanced growth hypothesis.

Evidence is presented of a shift in the dynamics of the real investment-to-consumption ratio in the United States in the early 1990s. This seems to clearly contradict the widely accepted stylized fact of stability of the "great ratios". Within the context of permanent-transitory decompositions four unobserved component models are estimated. Results of these models substantiate the presence of a common trend between investment-consumption ratio and relative prices. A shared trend between these two series seems to be confirmed by the different specifications. In addition, lower volatility in both series seems to be prevalent in the latter part of the period. Finally, transitory fluctuations in relative prices remain more volatile but less persistent than fluctuations in the ratio of investment and consumption after the assumed break.

The rest of this paper is organized as follows. Section II briefly outlines the balanced growth literature, motivates the need for an alternative hypothesis and develops a simple model with common technology across the aggregates mentioned that displays the stationarity of the investment-

consumption ratio in nominal terms. Section III (very) briefly summarizes the different philosophies in the realm of time series to deal with non-stationarity and motivates the need for the permanent-transitory decomposition that is used in this paper. Section IV describes the evolution of the Unobserved Components in the literature. Section V describes and tests the time series properties of the variables in question. Section VI considers four different permanent-transitory decompositions of the series. Sections VII and VIII discuss results and conclude.

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I. Introduction

The Solow-Ramsey growth model has been the canonical model of long-run macroeconomic fluctuations and, to date, remains the principal axiom underlying a large part of macroeconomic research. This canonical model is based on a one-sector economy in which all goods are produced using the same technology and a representative consumer maximizes the discounted present value of his utility from consuming subject to the law of motion for capital and a process for aggregate technology. One of the well-known predictions of this model is that real aggregates for consumption, investment, output and the capital stock all grow together at an identical rate, referred to as the *balanced growth path*, which is determined by the exogenous technology process. Any correlated deviation from this model's long-run balanced growth path is assumed to be part of the business cycle.

Recently, a new avenue of research critical of the one-sector model has taken root. This criticism of the one-sector model has developed some traction based on the latest empirical evidence, which seems to contradict the balanced growth prediction. This empirical evidence seems to suggest that while consumption and investment, in *real* terms, seem to not grow together at the same rate; the same

aggregates seem to share a common trend in *nominal* terms. This has been termed the *nominal ratio balanced growth hypothesis*. An intuitive explanation for this alternative concept of balanced growth might be that consumption and investment could be produced using different technologies, and, thus, two-sector models of the economy are proposed. Once we allow for the idea of an economy with two independent technology processes for consumption and investment, then any decentralized market equilibrium will feature consumption and investment prices that do not *generally* move together.

This paper shows that a two-sector model is not a necessary condition for having different price dynamics in consumption and investment. An alternative balanced growth hypothesis-- namely, that the ratio of nominal investment to nominal consumption is stationary-- is proposed. An empirical test for the possibility of this alternative balanced growth path within the context of a one-sector model is carried out using a new time-series technique as yet untried within this the context of this new balanced growth hypothesis.

II. Balanced Growth

Empirically, since the canonical paper of Nelson and Plosser (1982), there has been a pervasive agreement in the field about the representation of output, consumption and investment as unit root processes. While these aggregates may be well-characterized by a unit-root process, standard macroeconomic models suggest that the "great ratios" of consumption-to-output and investment-to-output should be stationary. Since the 1950s, the one-sector Solow-Ramsey growth model has been the canonical model of long-run macroeconomic fluctuations. One of the well-known predictions of this model is that real aggregates for consumption, investment, output and the capital stock all grow together at an identical rate, referred to as the *balanced growth path*, which is determined by some exogenous technology process. Technology is assumed to affect aggregate output, which in standard growth models as well as real business cycles (RBC) and even New Keynesian (NK) models, is derived from a single aggregate production function. The implication is that the stationarity property of the "great ratios" is therefore a corollary of the balanced growth hypothesis. Including both of these "stylized facts" in macroeconomic modeling dates as far back as Kaldor (1957). The balanced growth

prediction, that the logs of the level of macroeconomic aggregates share a common stochastic trend, has played an important role in modern macroeconomics. Sample averages of these great ratios are often used to calibrate the long-run properties of many theoretical macroeconomic models. Furthermore, the common trend prediction has been used as the foundation for the empirical characterization of macroeconomic trends and cycles.

There has been a vast amount of research on empirical macroeconomic analysis based on the balanced growth prediction. Some notable examples are those of King, Plosser, Stock and Watson (1991) (henceforth KPSW); Cochrane (1994); and Rotenberg and Woodford (1996). KPSW (1991) developed long-run restrictions in a structural vector auto-regression (VAR) framework to identify a time series for this common trend. Based on earlier theoretical work of King, Plosser and Rebelo (KPR 1988), they presented evidence consistent with the balanced growth hypothesis and the stationarity of the great ratios using data up to 1988. Cochrane (1994), relying on the balanced growth assumption, uses a vector error correction model (VECM) to address cointegration issues of real consumption and real output and identifies real consumption as the permanent component

of real output. Rotemberg and Woodford (1996) rely on the stationarity of the ratio of real consumption to real output to provide econometric estimates of the transitory components of consumption, investment and output. As mentioned previously, most of this empirical literature of balanced growth has relied either overtly or implicitly on the one-sector Solow-Ramsey model with a common productivity process across every sector of the economy.

More recently, a number of researchers have pointed out that new empirical evidence seems to reject the prediction of the balanced growth hypothesis. This rejection stems from the recent evidence that the ratio of real investment to real consumption has been moving substantially upward over time and mean-reversion does not look apparent. Three such dissenters of the one-sector common technology implication of the balanced growth hypothesis are Greenwood, Hercowitz and Krussell (henceforth GHK). GHK (1997) describe a model with investment-specific technological change and argue that real equipment investment[49] rises relative to real GDP. They conclude that investment-specific technological change accounts for most of the U.S. long-run growth. Another skeptic of the balanced growth

[49] The real equipment series is defined using Robert Gordon's (1990) price index.

prediction is Whelan (2003, 2005). Whelan (2005) estimates a traditional one-sector model with the balanced growth assumption and argues that the ensuing estimated VECM produces estimates of the transitory components of consumption, investment and output (all in real terms) that, contrary to all intuition, trend upward.[50] He then describes a two-sector model in which technology shocks are consumption-specific and investment-specific to match the apparent stationarity of the *nominal* ratios (see Fig. 2). His argument, in a nutshell, is that the reason for a trend in the real ratio, while the nominal ratio seems stationary, is that there must be different technology processes driving consumption and investment.

This paper attempts to model the relationship between nominal consumption and investment in a way that is consistent with the U.S. evidence on the time evolution of both series.

[50] This result is no doubt related to the upward trend in the real investment-consumption ratio described by Fig. 1. Whelan, and others, suspect that the KPR (1988) and KPSW (1991) models, which advocate the balanced growth prediction, are in no small part motivated by the apparent mean reversion of this ratio up to 1988. Evidence suggests, however, that this ratio becomes upward trending (some argue irretrievably) in the 1990s. See Whelan (2003) for a discussion of possible catalysts for this fundamental change in the relationship of real investment and consumption.

II.1. ***Reconciling Balanced Growth with Recent Economic Activity***

The prediction of the balanced growth hypothesis that was supported by U.S. data up to 1988 seems to have lost some gravitas based on the upsurge of private investment relative to consumption in the 1990s. Investment in real terms (relative to consumption) grew at an average 8.1% annually between 1991 and 2006 while consumption grew at an average 3.6% per year over the same period (see Figure. 1).[51]

Some factors such as the unusual length of the 1990s boom, the rapid increase in stock market capitalization or the federal budget surpluses, might account for this substantive increase in private real investment in the last decade of the twentieth century. If this were the case, we would expect the ratio of real investment to real consumption to reset down to its long-run average in the next downturn of the cycle. Figure 2 shows that, once expressed in nominal terms, the ratio of fixed investment-to-consumption has also been trendless throughout the postwar period. Whelan (2003, 2005) argues that the apparent difference between the two ratios (nominal and real) of investment relative to consumption reflect different price developments for capital

[51] These growth figures are similar to what Whelan (2003) reports for the 1991-2000 period.

versus consumer goods. Tevlin and Whelan (2003) showed that relative price trends for durable goods (which in large part affect private investment) have sustained a prolonged decline in the postwar period. One conclusion that can be drawn from this evidence is that nominal spending on investment and consumption have tended to grow at the same rate, but the higher share of available goods in investment and the declining relative price of these goods suggest that real investment tends to grow faster than real consumption. This evidence seems to contradict the prediction of the balanced growth hypothesis. Supporters of this evidence suggest that different prices in each sector imply different technologies driving each process. However, as I show in the following subsection, the existence of different technology processes for consumption and investment is not a necessary condition to show trend-less nominal ratios. This begs the following question: Can a "common technology" economy be used to describe the relationship between nominal consumption and investment in a way that informs the recent evidence and reconciles it with the balanced growth hypothesis?

While subsection 2 will illustrate the possibility with a simple model of a two-sector economy with common technology shock, empirical models are left for later sections.

II.2. *A Simple Illustration of a Model of Balanced Growth in Nominal Terms with Common Technology*

Consider an economy inhabited by a representative agent who maximizes his expected lifetime utility as given by

$$E\left[\sum_{t=0}^{\infty}\beta^{t}U\left(C_{t},N_{t}\right)\right] \tag{1.1}$$

where C and N represent consumption and labor, respectively. The production functions for consumption and investment in this model take the form (omitting time subscripts whenever there is no risk of ambiguity)

$$C\equiv Zf\left(K_{c},N_{c}\right)=ZK_{c}^{\beta_{c}}N_{c}^{1-\beta_{c}} \tag{1.2}$$

$$I\equiv Zf\left(K_{I},N_{I}\right)=ZK_{I}^{\beta_{I}}N_{I}^{1-\beta_{I}} \tag{1.3}$$

where K_C, K_I, N_C, N_I are the capital and labor inputs of the consumption and investment-producing sectors.

To date, an unobserved components (henceforth, UC) model of the private domestic aggregates of the economy has not been

proposed. The goal of this paper is therefore to fill that gap by proposing alternative UC models of consumption and investment. Models similar to this one can be found in GHK (1997) and Whelan (2003, 2005). What distinguishes this model from the other ones is that the same measure (Z) of total factor productivity is common to both the production of capital goods as well consumer goods. This, as it is shown later, renders the permanent component to both nominal consumption and nominal investment its usual interpretation (of an I(1) technology process).

Firms in both sectors take the price that the market bears and maximize the following problem:

$$\underset{\{K_c,N_c\}}{Max}\ \pi_C = P_C Zf(K_C, N_C) - WN_C - P_I\left(i + \delta - \frac{\Delta P_I}{P_I}\right)K_C \tag{1.4}$$

and

$$\underset{\{K_N,N_N\}}{Max}\ \pi_N = P_I Zf(K_I, N_I) - WN_I - P_I\left(i + \delta - \frac{\Delta P_I}{P_I}\right)K_I \tag{1.5}$$

where a Jorgensonian user cost of capital formulation specifies the rental rate of capital with i as the required nominal return on investment in capital goods and δ the depreciation rate of capital. Due to the constant returns to scale (CRS) assumption, the firm nets zero

profits each period. The aggregate resource constraint holds each period, so in equilibrium we have

$$C + I = Zf\left(K_C, K_I, N_C, N_I\right) \tag{1.6}$$

If capital in some sector (j) grows at a constant rate (γ_j), then, clearly, at a steady-state growth path in which all real variables grow at constant rates, investment in that type of capital is given by

$$I_j = \left(\delta + \gamma_j\right) K_j \tag{1.7}$$

Solving the first-order conditions of firm's problem in sector 1 specified by (1.4) yields the following demand equations:

$$K_C = \left(\frac{P_C}{P_I}\right) \frac{\beta_C C}{i + \delta - \dfrac{\Delta P_I}{P_I}} \tag{1.8}$$

$$N_C = \frac{\left(1 - \beta_C\right) P_C C}{W} \tag{1.9}$$

Similarly, solving the firm's problem in sector 2 as given by (1.5) yields

$$K_I = \frac{\beta_I I}{i + \delta - \dfrac{\Delta P_I}{P_I}} \tag{1.10}$$

$$N_I = \frac{\left(1 - \beta_I\right) P_I I}{W} \tag{1.11}$$

Conditions (1.8)-(1.11) imply that for either factor, the ratios of the amount of each factor used in the two sectors are proportional to the ratio of nominal outputs, i.e.

$$\frac{K_I}{K_C} = \left(\frac{P_I}{P_C}\right)\left(\frac{\beta_I}{\beta_C}\right)\frac{I}{C} \tag{1.12}$$

$$\frac{N_I}{N_C} = \left(\frac{P_I}{P_C}\right)\left(\frac{1-\beta_I}{1-\beta_C}\right)\frac{I}{C} \tag{1.13}$$

Our investment assumption in (1.7) can be rewritten for the two sectors here as $I = (\delta + \gamma_C) K_C + (\delta + \gamma_I) K_I$. Therefore, the composition of output in the investment sector along a balanced growth path can be written as

$$I = \left(\frac{P_C}{P_I}\right)\frac{\beta_C (\delta + \gamma_C) C}{i + \delta - \frac{\Delta P_I}{P_I}} + \frac{\beta_I (\delta + \gamma_I) I}{i + \delta - \frac{\Delta P_I}{P_I}} \tag{1.14}$$

Rearranging (1.14) yields the following equation for the nominal ratio

$$\frac{P_I I}{P_C C} = \frac{\beta_C (\delta + \gamma_C)}{i + \delta - \frac{\Delta P_I}{P_I} - \beta_I (\delta + \gamma_I)} \tag{1.15}$$

The fact that the right-hand side of (1.15) can be assumed to be constant along a steady-state path renders the ratio of nominal aggregates to be constant as well.[52]

In a model that assumes a common technology process in the production of both consumer and capital goods, a technology shock for either sector evolves stochastically according to some measure of trend (μ_z) and transitory fluctuations (ω) so that

$$Z_t = Z_{t-1} + \mu_Z + \omega_{Z,t} \tag{1.16}$$

In this case, the economy's dynamic stochastic general equilibrium will feature the growth rates of *nominal* consumption and investment equaling their non-stochastic steady-state values plus some stationary deviations. [53]

Most empirical studies modeling the balanced growth hypothesis take place within the context of VAR/VECM analysis. This paper investigates the relationship between private consumption and investment using an alternative approach to the estimation of cointegrated systems, in the context of state space, which builds on

[52] It's important to note that this result holds regardless of whether the model describes a common measure of TFP, as in this model, or technology processes that are sector-specific, as those described by GHK (1997) or Whelan (2003, 2005).

[53] As Whelan (2005) points out, the assumption of unit-elastic factor demands implied by a Cobb-Douglas technology is important for this result of a stable long-run nominal ratio of consumption to investment to hold. While the assumption may seem restrictive, Jones (2005) shows Cobb-Douglas-shaped production functions can be derived from idea-based models under very general assumptions.

Stock and Watson's (1988) common trends representation. In order to motivate the use of this method, a brief outline of time series methodologies is useful.

III. A Brief General Outline of Time Series Methodologies

Macroeconomic time series fluctuate. It has been a long-running goal of macroeconomists to describe the characteristics of these fluctuations. In particular, finding some permanence (or lack thereof) in the movements of these series may yield interesting insights into the underlying mechanisms that govern them. In turn, these insights may also provide invaluable information about the underlying relationships between macroeconomic variables. For example, some fluctuations seem to dissipate (rather quickly) over time while other movements exhibit some permanence over the long run. Knowing which of the two behaviors is the dominating force describing a given series will help in understanding how it relates to other series and economic activity in general.

An old concept, which is still very much at the center of attention in the study of time series, is that of *stationarity*. Throughout this discussion, I will be using the term *stationarity* to describe a *covariance-stationary* or *weakly stationary* process. Simply stated, a process is weakly stationary if the first two moment conditions are not time-dependent. Put another way, the influence of historical events on the long-term forecast of a stationary process will continue to diminish

so that the variance of the forecast error will converge to some bound. The implication is that a shock to a stationary process will have some temporary effect that diminishes and eventually vanishes for some long horizon. We should, therefore, expect no permanence in a stationary process. The converse applies for a *non-stationary* series in which we would expect some permanence since the variance of its error is not bounded as it accumulates.

As a preamble, I will discuss three (not-altogether unrelated) methodologies that macroeconomists have used to extract information from the persistence and non-persistence of macroeconomic fluctuations-- namely, differencing, filtering, and decomposition.

III.1. *Differencing*

In a seminal paper, Nelson and Plosser (1982) set about trying to answer the following question: Can macroeconomic time series be characterized by a stationary process around a deterministic trend, or is the accumulation of stationary and non-stationary behaviors in each series over time better explained by a non-stationary process? They consider a specific case of non-stationarity: a *random walk with drift*, the prototypical example of what is known as a *unit root* process. One

would like to be able to tell whether a macroeconomic series displays unit root behavior because of the well-known Granger and Newbold's (1974) problem of spurious relationships between unit root processes. Nelson and Plosser make two important contributions.

First, by relying heavily on the Dickey-Fuller (1981) test for unit root, they conclude that most macroeconomic variables are integrated of order one. This first contribution has been widely accepted. Most macroeconomists acquiesce to the notion of the prevalence of difference stationarity in macroeconomic variables, even while maintaining that Dickey-Fuller as well as many other tests for unit root has inherent power problems. These tests do not distinguish well between a series with a difference stationary stochastic trend and a level stationary process with characteristics roots close to unity.

Their second contribution has been as well-accepted as the first. Namely, that the stationarity in economic activity can be removed more appropriately by first differencing rather than linear detrending, making the trend a random walk with drift rather than a straight line. After applying the first difference filter to annual U.S. output, they find the stationary component of output becomes high-frequency noise, making the strongly autocorrelated patterns of the business cycle

disappear. This conclusion is contingent on the strong assumption that the first difference of output zeroes out *exactly* after the first lag.

In summary— and to borrow Granger's famous notation— most macroeconomic series are I(1) processes. In other words, they seem to be *difference-stationary*, displaying unit root behavior that when differenced become integrated of order zero I(0) or stationary. Differencing non-stationary series has, therefore, become common practice before any multivariate analysis is to be attempted and any inference on the comovement of (or relationship between) macroeconomic variables are to be concluded.

III.2. *Filtering*

A different school of thought is based on filtering. While differencing can be thought of as a yet another filtering technique (as alluded to previously), it relies on the strong assumption that absolutely no permanence in the variable remains after the first lag. Filtering techniques relax this assumption in order to extract the underlying dimension of permanence from the series in question to yield a measure of non-permanent fluctuations that may or may not be idiosyncratic. Filtering has become very popular in business cycle

research, where these non-permanent fluctuations are interpreted as the cycle that can be thought of as a residual that obtains from detrending the series in question. The seminal, and perhaps most popular, filtering technique is that of Hodrick and Prescott (1997)—henceforth referred to as the *HP filter*. They propose decomposing a series as the sum of a smoothly varying trend and a cyclical component. They establish a measure of smoothness of the trend as the sum of squares of its second differences. The cyclical component is a (zero frequency or mean zero) deviation from this trend over long periods. The HP filter involves finding a measure of the trend that will minimize the sum of the squared cycles and the weighted sum of the second differences of this trend.[54] This weight, or smoothing factor, commonly referred to as λ is the squared of the ratio of variances of the cycle and trend. They select $\lambda = 1{,}600$ and concede results to be somewhat sensitive to the selection of this smoothing factor.

Other filtering techniques are those proposed by Baxter and King (1999), band-pass filter, and Christiano and Fitzgerald (2003).

[54] This minimization problem was borrowed from actuarial science.

III.3. *Decomposition*

The last methodology I review here is decomposition-- more specifically, the permanent-transitory decomposition. The idea is that any macroeconomic series can be decomposed into two components: one exhibiting long-run permanence, and one depicting short-run fluctuation. Its application is far from exclusive to the business cycle research, but I use it in that context here. Filtering techniques, as those previously stated, can be thought of as permanent-transitory decompositions. Hodrick and Prescott build their HP filter around the idea that any series can be decomposed into a smooth trend (by definition, a permanent component) and some dispersion from this trend (a transitory component). As it will be mentioned below, Beveridge and Nelson (1982) make use of first-differencing to arrive at a permanent transitory decomposition for output. Therefore, permanent-transitory decompositions can be thought of as encompassing both differencing and filtering and, thus, can be viewed as the most general of the three techniques described here.

As mentioned above, perhaps the most famous permanent transitory-decomposition is that of Beveridge and Nelson (1982). Their result is that any homogeneous non-stationary I(1) process can be

decomposed into a random walk (by definition, a permanent component) process, which is I(1), and an I(0) stationary process (the transitory component). The insight of the B-N decomposition comes from the relationship of current values of the series in question to the forecast profile for future values of the series. They make use of first-differencing and the Wold[55] (1938) representation theorem to stand for some deterministic trend that is then used as the benchmark for measuring the cyclical component (Beveridge and Nelson 1982). The B-N decomposition is tailored to the stochastic structure of any given process. The permanent component is always assumed to be a random walk with the same rate of drift as the original data and a proportional innovation to the original process. The transitory component is a stationary process that represents the forecastable momentum present at each time period, which dissipates as the series trends to its permanent levels. B-N decomposition of any I(1) process into a trend and cycle is arrived at with traditional time series methods and is itself inherent to the process.

A different kind of permanent-transitory decomposition involves decomposing a process into permanent and transitory

55 Any linear covariance-stationary stochastic process (LCSSP) process can be represented as an infinite order moving average.

components that are taken to be unobserved. Dynamic systems that involve these UC models depart from traditional time series representations and make use of state-space modeling originally developed by control engineers (Kalman 1960).

This paper considers a number of UC models to shed some light into the relationship between nominal consumption and investment in a way consistent with the theoretical model outlined in section II.

IV. A Primer on Unobserved Components Models

As described in the previous section, a popular method of decomposing a series into permanent and transitory components is the UC model. The UC model introduced by Harvey (1985) and popularized by Clark (1987) specifies a given observable process into the sum of two (or potentially more) components that are unobserved. After taking a stand on the structure of these components, the model is cast into state-space. Once a dynamic time series is written in state-space form, maximum likelihood estimation can be carried out by recursive Kalman filtering. Under Gaussian conditions, the Kalman filter (Kalman 1960) yields the minimum mean square estimator of the state vector based on past information. Therefore, basic Kalman filtering is readily available for inferences on the unobserved state vector conditional on the parameters of the model and the appropriate information set.

Clark (1987) describes the canonical UC model of output. He attributes unforecastable surprises from past economic activity to be purely cyclical and stationary in nature since the trend component is smooth and predictable. Clark's model decomposes aggregate output into trend (permanent) and cycle (transitory) components that are not

observed but follow a random walk with stochastic drift and the stationary ARIMA (2,0,0) process, respectively. His UC model's key assumption is that innovations in the trend and cycle are independent. This simplifying assumption that permanent and transitory components are not correlated has indeed become the standard restriction on UC models. Clark contends his assumption to not be very realistic but more desirable than the alternative (perfect correlation between the permanent and transitory components). He claims this restriction is not too restrictive and allows him to make inferences about innovations to the trend and cycle of U.S. GDP by merely inspecting the size of the variance of each component. For example, if the drift is deterministic and the variance of the transitory component is small, most of the short-run variation in output would be attributed to the trend. However, if a stochastic drift with small variance is present and the variance of the cyclical component is nonzero, the trend becomes much smoother, and more of the short-run variation in output is assigned to the stationary transitory component. His results allocated more than 50% of the quarter-to-quarter innovations in real output for the United States to come from the transitory component and, hence, be cyclical.

Watson (1986) also employs a canonical UC model for aggregate output in the same fashion as Clark but assumes a deterministic drift (rather than Clark's stochastic one) in the permanent component. Additionally, and for comparison purposes, he also models using traditional ARIMA processes. He concludes that both UC and ARIMA models display equivalent short-run characteristics but different long-run behavior. The short-run forecasting abilities of the models were essentially identical, but UC models provided uniformly better long-run forecasts (Watson 1986).

Campbell and Mankiw (1987) used three methods to estimate persistence in output: a fully specified ARMA (2,2) model, Cochrane's (1986) nonparametric approach, and a UC model (a la Clark 1987). The identification restriction of their UC model is the polar opposite from Clark's UC and on par with the B-N decomposition— namely, that the permanent and transitory components are *perfectly* correlated. Incidentally, they conclude fluctuations in aggregate economic activity seem to be persistent, which agrees with modern common wisdom in the field but is tangential to the objectives of my thesis.

Permanent-transitory decompositions, particularly UC models, have been typically used to characterize aggregate economic activity,

more specifically, fluctuations in output (see Clark 1987, Campbell and Mankiw 1987, Watson 1988, and more recently Kim Nelson 1999 among others). Other measures, such as industrial production (Clark 1987), durables consumption (Watson 1986) or labor supply (Clark 1989), have been studied within the context of UC modeling and aggregate economic activity, yet have not received as much attention.

V. Properties of the Investment-Consumption Ratio: Evidence of a Structural Break in Levels.

As mentioned previously, some researchers (GHK 1997, Whelan 2003) have pointed out that real consumption and real investment seem to have been growing at different rates (specifically since the early 1990s). This suggestion is consistent with the evidence presented in Figure 1 in which the ratio of real investment to consumption seems to display a trend starting in the 1990s with no sign of mean reversion to date. Conversely, the ratio of nominal investment to consumption seems to display strong mean reversion with an apparent lack of a trend (see Fig. 2). I, therefore, set about studying the stationarity properties of both ratios by employing four unit root tests: the augmented Dickey-Fuller test (ADF), the Phillips-Perron test, the Elliott, Rotemberg and Stock (ERS)[56] test, and the Banerjee, Lumsdaine and Stock (BLS) test. As Table 1 indicates, all four tests of the investment-to-consumption *real* ratio fail to reject the null hypothesis of a unit root. The robustness of this conclusion with regard to lag length is somewhat mixed.[57] However, all but the BLS test reject the

[56] BLS tests the null of a unit root in the presence of a structural break. I use the sequential rather than the recursive technique for these series.

[57] The selection of the lag length is important. Adding too few (k) lags may not get rid of all the serial correlation, causing the regression residuals in the test to not behave like white-noise

null of a unit root in the nominal ratio. Thus, there is strong evidence (which is quite robust to the choice of lag length) that the nominal investment-to-consumption ratio is in fact stationary (see Table 2).

Even a cursory inspection of Figure 1 denotes a shift in the structure of the real consumption-to-investment ratio in the 1990s in which a trend seems to be established. On the other hand, there is no obvious long-run trend in the nominal ratio in which mean reversion is apparent throughout the sample period. Having analyzed the stationarity of both ratios, I now test for the presence of a structural break. To that end, I use Andrews (1993) test for structural change with unknown change point. I test an autoregressive model of the form:

$$y_t = \beta_t y_{t-1} + \varepsilon_t$$

where y_t is the *real* or the *nominal* investment-to-consumption ratio and β_t is the OLS coefficient of the AR(1) model, which, being a function of time, could change. My null hypothesis of no structural change is as follows:

processes so that some dependence in the errors may remain. Adding too many lags may reduce the power of the test since additional unnecessary parameters need to be estimated, causing a loss of degrees of freedom. There are a number of criteria to select the lag length. I chose the modified Akaike Information Criterion (MAIC).

$$H_0 : \ \beta_t = \beta_0 \quad \forall t$$
$$H_A : \ \beta_t = \begin{cases} \beta_1(\pi) & t = 1,...,t\pi \\ \beta_2(\pi) & t = t\pi,...,T \end{cases}$$

and the alternative is that a break exists where π is the break function. Following Andrews' *Partial Generalized Method of Moments* methodology (1993), I run a Wald test at every possible break π.[58] This Wald statistic for testing H_0 against H_A is given by

$$\omega_t(\pi) = T(\hat{\beta}_1(\pi) - \hat{\beta}_2(\pi))^T \left(\frac{\widehat{V}_1(\pi)}{\pi} + \frac{\widehat{V}_2(\pi)}{1-\pi}\right)^{-1} (\hat{\beta}_1(\pi) - \hat{\beta}_2(\pi))$$

where I estimate the model by OLS before ($\hat{\beta}_1(\pi)$) and after the break ($\hat{\beta}_2(\pi)$). We might suspect a break to exist wherever the difference between the estimate before and after break $(\hat{\beta}_1(\pi) - \hat{\beta}_2(\pi))$ is the largest.[59] Like any Wald statistic, however, we don't only need the restriction but also a form of the variance. In this case, as suggested by Andrews, I use his result of the optimal partial sample GMM using the optimal weighing matrix γ, namely:

[58] Andrews recommends checking for breaks in the $\pi = [.15,.85]$ interval.

[59] I am looking for the largest Wald statistic yielded by (7).

$$\sqrt{T}(\hat{\beta}_1(\pi) - \hat{\beta}_2(\pi)) \xrightarrow{d} N(0, \frac{V_1}{\pi} + \frac{V_2}{1-\pi})$$

where

$$V_1 = \left(M_1(\pi)^T S_1^{-1}(\pi) M_1(\pi)\right)^{-1}$$

$$V_2 = \left(M_2(\pi)^T S_2^{-1}(\pi) M_2(\pi)\right)^{-1}$$

with S_1 and M_1 [60] being the long-run variance and moment condition, respectively, for the first part of the sample, and S_2 and M_2 being the counterparts for the second part of the sample (after the supposed break).

Hence, I follow Andrews' methodology and run a Wald test (3.0) at every point between the 15th and 85th percentile of the series and look for the largest Wald statistic within that interval. The sup of that statistic denotes where the break occurs. Therefore, I test the sup of (3.0), namely:

$$\underset{\pi \in \Pi}{Sup}\, \omega_t(\pi) = T(\hat{\beta}_1(\pi) - \hat{\beta}_2(\pi))^T \left(\frac{\hat{V}_1(\pi)}{\pi} + \frac{\hat{V}_2(\pi)}{1-\pi}\right)^{-1} (\hat{\beta}_1(\pi) - \hat{\beta}_2(\pi))$$

As Table 3 shows, this yields an Andrews test for the real investment-to-consumption ratio of 25.47 and Andrews statistic of 5.77 for the nominal ratio. Andrews' critical values for two restrictions (p = 2) and π =.15 are 10.01 at 10%, 11.79 at 5% and 15.51 at 1%.[61] Hence we

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[60] In my case $\hat{M}_1(\pi) = -\frac{1}{T\pi}\sum_{t=1}^{T\pi} y_{t-1} y_{t-1}{}^T$ and $\hat{M}_2(\pi) = \frac{1}{T - T\pi}\sum_{t=T\pi+1}^{T} y_{t-1} y_{t-1}{}^T$

[61] Andrews provides a table of asymptotic critical values for his test in his 1993 paper.

reject the null hypothesis of *no structural breaks* at the 1% level of significance for the *real* ratio, whereas we fail to reject the null of *no structural breaks* for the *nominal* ratio.

VI. Measuring Trends and Cycles.

This chapter contends that by treating technology as a common unobserved component of both the major aggregate series, a number of permanent-transitory decompositions or UC models may yield useful insights with direct applicability for this "nominal-ratio balanced growth" emerging literature. As will be shown later, what motivates this chapter, and indeed the whole dissertation, is the evidence (shown below) that investment and consumption have not been growing at the same rate in the U.S. Each chapter of the dissertation considers a potentially different answer to this evidence. Chapter 1 allows for the production of differential types of goods: durables (which mostly go to investment) versus non-durables (which mostly go to consumption) as a possible explanation for the disparity in the growth rates of real per capita consumption and investment in the U.S. Chapter 2 advocates that the difference in the growth rates could be attributed to the possibility that there may be more than one type of productivity shocks, where different (investment versus consumption-specific) technology processes drive the economy. The present chapter considers that fluctuation in relative prices (between consumption and investment type goods) may help account for the

disparity in the growth rates of these two aggregates. To that end, in the following sections, I consider different specifications of a bivariate unrestricted UC model of the investment-consumption ratio and relative prices.

As a first step, I must take a stand on the structural form of the permanent component of each series. Most permanent-transitory models (Beveridge and Nelson 1982, Clark 1987, Watson 1986, Campbell and Mankiw 1987, and more recently Kim and Nelson 1999, Morley Nelson and Zivot 2003) build their decompositions on the assumption that the permanent component is a random walk with serially uncorrelated increments. Others (Shapiro and Watson 1988, Blanchard and Quah 1989) consider models with permanent components that have more complex dynamics than a simple random walk. With these less stringent specifications, the variances of the permanent components maybe under-identified with just the information of the second moments of the original sequence (Quah 1992). Since my proposal rests on the treatment of the UC model, I will make the stronger assumption of a random walk in the permanent component.

The next simplifying assumption deals with the relationship between the two components of each series. I could assume perfect (negative) correlation of the innovations between the permanent and transitory components as in Beveridge and Nelson (1982) or Campbell and Mankiw (1987). Alternatively, I could consider independence between the trend and cycle components as in Clark's (1987) standard UC model and Quah (1992). Watson (1986) and Cochrane (1988) consider the middle ground, with the weaker assumption that the increments in the permanent component are correlated (but not perfectly) with the temporary component. Recent findings by Morley, Nelson and Zivot (2003) show that given model dynamics that are sufficiently rich, correlations between movements in unobserved components can be easily identified. I will, therefore, consider some UC models in which I allow for perfect correlation between components, no correlation between components, and, finally, the presence of correlation. I will remain agnostic to a priori assumptions and will attempt to estimate these correlations instead. An UC approach can be an alternative way to directly estimate these correlations-- without making explicit long-run restrictions of that sort-- and derive inferences on innovations from technology and non-

technology shocks. Additionally, a UC model of the investment-consumption ratio and relative prices will filter the series for the cycle and trend, which, for empirical robustness, could then be used as input variables in alternative time series methods such as error correction models (ECM) for permanent components or a bivariate VAR model for the transitory components with further insights to be derived there.

VI.1. *UC Models of the Investment-Consumption Ratio and Relative Prices*

As mentioned earlier, a corollary of the balanced growth hypothesis is the stationarity of the great ratios. However, empirical evidence seems to suggest that the investment-consumption ratio (i/c) ratio has in fact become an integrated process of the first order since the late 1980s/early 1990s. I aim to show that this ratio has become I(1) since the late 1980s/early 1990s due in part to large permanent movements in relative prices. Specifically, prices of durables (which account for a large portion of private investment) have decreased steadily since the late 1980s, some have argued that this decline, while more pronounced in the last 15 yrs, has taken place since the postwar [GHK (1997, 2000) and Whelan (2005).] If the

(i/c) ratio seems to be cointegrated with relative prices, with cointegrating vector (1 -1), then a common-trend bivariate model of the investment-consumption ratio and relative prices is warranted.

Initially, I consider a stationary bivariate model of the i/c and ppi/cpi ratios. The model looks like the following

$$\begin{aligned} \left(\frac{I}{C}\right)_t &= \mu_1 + \omega_{(i/c)t} \\ RP_t &= \mu_2 + \omega_{(RP)t} \\ \\ \left(1 - \phi_{ci1}(L) - \phi_{ci2}(L^2)\right)\omega_{(i/c)t} &= \varepsilon_{(i/c)t} \\ \left(1 - \phi_{RP1}(L) - \phi_{RP2}(L^2)\right)\omega_{(RP)t} &= \varepsilon_{(RP)t} \end{aligned} \tag{2.1}$$

Where both ratios are characterized by two drifts and separate stationary AR(2) processes. There is no permanent component specified here. A priori, this model should not stand well to the data, given that both series seem to be trending as evidence by Figures 1 and 3, respectively.

To account for a trend, I consider a second model where both series share the same trend component. This is inherently a *common trends* representation in the spirit of Stock and Watson (1988) and, more recently, Morley (2005). The benefit behind the specification is that it allows us to relax the standard UC model assumption of independence between the permanent and transitory components.

Instead, this model allows for the estimation of correlations between the components.[62] Any time a correlation is estimated, there raises the question of structural interpretation. In an economic model that is fully identified, the common assumption is that there is no correlation left in the underlying structural shocks. Therefore, the presence of correlation between variables or components indicates that the model has not been fully specified and, perhaps, there is some form of causality between those variables or components that has not been properly accounted for by the model. Typically, it is possible to remain agnostic about the source of correlation between variables or components and still draw conclusions about their dynamic behavior. On the other hand, economic theory may provide a structural interpretation of the correlated unobserved components. I use Model 2 as my benchmark to be used as a means of comparison with the evidence for other models. This benchmark model looks like the following:

[62] Typically, when studying long-run relationships between aggregates with a VAR/VECM analysis, the researcher must a priori assume zero-correlation between the technology shock and shocks to the transitory component. A UC model needs no such restriction to be identified and can instead estimate the correlation between the permanent and transitory components. This is particularly appealing to advocates of learning by doing models or simply to anyone who might buy into the idea of different rates of diffusion of technology that might affect both the cycle as well as the long-term trend.

$$\left(\frac{I}{C}\right)_t = z_t + \omega_{(i/c)t}$$
$$RP_t = \overline{rp} + z_t + \omega_{(RP)t}$$

$$z_t = \overline{z} + z_{t-1} + \nu_t$$

$$\begin{aligned}\left(1-\phi_{ci1}(L)-\phi_{ci2}(L^2)\right)\omega_{(i/c)t} &= \varepsilon_{(i/c)t} \\ \left(1-\phi_{RP1}(L)-\phi_{RP2}(L^2)\right)\omega_{(RP)t} &= \varepsilon_{(RP)t}\end{aligned} \tag{2.2}$$

Cov matrix

$$Q = \begin{pmatrix} \sigma_{ci}^2 & & \\ \rho_{cirp} & \sigma_{rp}^2 & \\ \rho_{civ} & \rho_{rpv} & \sigma_{v}^2 \end{pmatrix}$$

Where technology, which is now a common permanent component of both series, is again assumed to follow an unobservable random walk with a deterministic drift where $\nu_t \sim i.i.d.N(0,\sigma_\nu{}^2)$. Each transitory components of (i/c) and relative prices follow an *invertible* finite order *autoregressive* (AR) processes where $\varepsilon_{(i/c)t} \sim i.i.d.N\left(0,\sigma_{\varepsilon_{(i/c)}}{}^2\right)$, and $\varepsilon_{(RP)t} \sim i.i.d.N\left(0,\sigma_{\varepsilon_{RP}}{}^2\right)$, respectively. This model is set up to estimate the AR coefficients as well as the relationships between the transitory and permanent components that inhabit the covariance matrix. For example, ρ_{civ} represents the correlation between transitory and permanent (i/c) ratio, ρ_{cirp} represents the correlation between

transitory (i/c) ratio and relative prices whereas σ_{ci}^2 and σ_v^2 represent the variance of the transitory (i/c) ratio and the shared trend, respectively.

Chapter 4 finds evidence of a break in volatility of the investment-consumption ratio to take place in the third quarter of 1983. Taking this evidence along with a close inspection of Figure 1, I specify a model where I allow for the possibility that the balanced growth hypothesis used to hold until 1983 and then permanent shocks start to become more prevalent afterward. In essence, a decrease in temporary shocks and an increase in permanent shocks hitting the system after this break could account for the trending and lower volatility behavior of the investment-consumption ratio. Thus, this (third) model allows for a break in volatility to happen in the third quarter of 1983 while imposing a zero-restriction on the variance of the permanent component prior to the break. The model looks as follows:

$$\left(\frac{I}{C}\right)_t = z_t + \omega_{(i/c)t}$$
$$RP_t = \overline{rp} + z_t + \omega_{(RP)t}$$

$$z_t = \overline{z} + z_{t-1} + \nu_t$$

$$\begin{aligned}\left(1-\phi_{ci1}(L)-\phi_{ci2}(L^2)\right)\omega_{(i/c)t} &= \varepsilon_{(i/c)t} \\ \left(1-\phi_{RP1}(L)-\phi_{RP2}(L^2)\right)\omega_{(RP)t} &= \varepsilon_{(RP)t}\end{aligned} \tag{2.3}$$

$$Cov\ matrix\ pre-brake \qquad\qquad Cov\ matrix\ post-brake$$
$$Q = \begin{pmatrix} \sigma_{ci}^2 & & \\ \rho_{cirp} & \sigma_{rp}^2 & \\ - & - & 0 \end{pmatrix} \qquad \tilde{Q} = \begin{pmatrix} \tilde{\sigma}_{ci}^2 & & \\ \tilde{\rho}_{cirp} & \tilde{\sigma}_{rp}^2 & \\ \tilde{\rho}_{civ} & \tilde{\rho}_{rpv} & \tilde{\sigma}_{v}^2 \end{pmatrix}$$

Finally, and, again motivated by the evidence found in Chapter 4, I specify a model where I impose a break in 1983:Q3 to happen in *most* of the covariance matrix (Q) in the bivariate UC setup. This specification is a restriction of the benchmark model expressed earlier where the variance of –and correlation with-- the permanent shock remains fixed across the break. This final model looks as follows:

$$\left(\frac{I}{C}\right)_t = z_t + \omega_{(i/c)t}$$
$$RP_t = \overline{rp} + z_t + \omega_{(RP)t}$$

$$z_t = \overline{z} + z_{t-1} + \nu_t$$

$$\begin{aligned}\left(1-\phi_{ci1}(L)-\phi_{ci2}(L^2)\right)\omega_{(i/c)t} &= \varepsilon_{(i/c)t} \\ \left(1-\phi_{RP1}(L)-\phi_{RP2}(L^2)\right)\omega_{(RP)t} &= \varepsilon_{(RP)t}\end{aligned} \tag{2.4}$$

$$Cov\ matrix\ pre-brake \qquad Cov\ matrix\ post-brake$$
$$Q = \begin{pmatrix} \sigma_{ci}^2 & & \\ \rho_{cirp} & \sigma_{rp}^2 & \\ \rho_{civ} & \rho_{rpv} & \sigma_{\nu}^2 \end{pmatrix} \qquad \tilde{Q} = \begin{pmatrix} \tilde{\sigma}_{ci}^2 & & \\ \tilde{\rho}_{cirp} & \tilde{\sigma}_{rp}^2 & \\ \rho_{civ} & \rho_{rpv} & \sigma_{\nu}^2 \end{pmatrix}$$

These model specifications would also provide insights as to the speed of adjustment of each of the series when subjected to a shock of their transitory component. VAR modeling would also be capable of estimating these speeds of adjustment, which it would allow to differ for different shocks. However, without further restrictions to justify an orthogonalization of the structural shocks driving the cycles, VAR modeling poses no advantage over this UC specification. Such analysis of the speed of adjustment is also useful as an informal measure of the length of the cycle in each of the series as they are disturbed by technology or non-technology shocks.

In conclusion, I propose multiple specifications of a bivariate UC model of the relationship between the investment-consumption ratio and relative prices. This paper empirically investigates this relationship to shed some light on the balanced growth hypothesis and what has been coined by Whelan as the *nominal ratio balanced growth hypothesis*. It is within the realm of permanent-transitory decompositions that this paper attempts to make its contribution. For estimation, I cast the correlated UC models given by (2.1)-(2.4) and apply the Kalman filter and maximum likelihood based on forecast error decomposition (Harvey, 1993).

The data used are 100 times the natural logarithms of the ratios of U.S. per capita real consumption and fixed investment. Consumer and producer price indices are used to compute the relative price measure. Future extensions of this model will consider other component prices for analysis. The demographic measure used is civilians over 16 years old. I consider two sample periods: The main one encompasses data from 1948:Q2-2006:Q2.[63]

[63] These data are seasonally adjusted and available from the St. Louis Federal Reserve publicly available FRED-II database.

VII. Results

This section presents results derived from the analysis of the bivariate UC model of the investment-consumption ratio and relative prices in its various specifications.

Table 3 reports parameter estimates for the correlated UC model of the investment-consumption ratio and relative prices where both series are forced to be stationary. This is the first model I consider where each series is characterized by separate autoregressive transitory component and drift. While there is no permanent component specified here, being a stationary model, the sum of the autoregressive coefficients of transitory (i/c) ratio and relative prices are estimated to be 0.98 and 0.99, respectively. This constitutes very persistent transitory components and thus a likely misspecification where the transitory components may not be stationary. Nelson (1988) cautions that a decomposition that yields highly persistent stationary components, may be spurious. This result is not surprising given the strong evidence for nonstationarity of both series as shown by Tables 1 and 2. Thus, we can move on to consider a different specification where both series are allowed to be nonstationary.

Table 4 shows results for what I dub the benchmark model. Parameter estimates of a bivariate UC decomposition of the series of interest into a common random walk process for the trend and two separate AR(2) processes describing the transitory component. The transitory component of relative prices seems less persistent that that of the investment-consumption ratio as evidenced by a lower sum of the autoregressive coefficients of the former. Variances of the transitory prices and the ratio are very similar in magnitude. This result suggests that the I/c ratio and relative prices may have always been influenced by permanent shocks. Two scenarios arise: If, say, the variance of the transitory shocks is smaller for RP than the IC ratio, then the implication is that permanent shocks are more evident in relative prices which might lead us to conclude that balanced growth may have never held in the sense that i/c ratio could have always undergone permanent shocks related to relative prices. Conversely, if the variance of the transitory shocks is smaller for IC than the RP, then the implication is that permanent shocks are more evident in IC which might lead us to conclude that prices may not be the only factor affecting the ratio.[64]

64 This may or may not refute the balanced growth hypothesis. It could be that balanced growth held up to the break (productivity miracle of the 1990s) so that the IC ratio changes from I(0) to I(1) due to prices and/or other factors.

Motivated by the findings in Chapter 4 of this thesis, that a break in volatility of the investment-consumption ratio took take place in the third quarter of 1983, I specify a model where I allow for the possibility that the balanced growth hypothesis used to hold until 1983 and then permanent fluctuations might have become more prevalent afterward. In essence, a decrease in temporary shocks and an increase in permanent shocks hitting the system after this break could account for the trending and lower volatility behavior of the investment-consumption ratio. Thus, this (third) model allows for a break in volatility to happen in the third quarter of 1983 while imposing a zero-restriction on the variance of the permanent component prior to the break. Table 5 shows results for this specification. The variance of the permanent component σ_v is restricted to zero pre-break to correspond with the idea of I(0) before 1983. Consequently the correlations involving σ_v were left blank. It is worth noting that the variances of transitory components reduce post break with variance of the transitory RP being smaller than that of the IC ratio. This evidence is consistent with evidence found in Chapter 4 of this thesis. However, the sum of the autoregressive for RP is again too close to 1 which again could be due to misspecification. To that effect this model, which has yields a smaller likelihood estimate in absolute value than

the previous model, does not improve on the benchmark presented earlier. These results provide little support for the hypothesis that the ratio may have been best characterized by an I(0) process in the first part of the sample only to become an I(1) process later.

Finally, Table 6 reports results of a model that is somewhat less restricted than the previous one, though more restricted than the benchmark. This is a bivariate UC model of the (i/c) ratio and relative prices that share a common random walk component and different transitory components. I impose a break in 1983:Q3 to happen in *most* of the covariance matrix (Q) in the setup. As I mentioned earlier, this specification is a restriction of the benchmark model expressed earlier where the variance of –and correlation with-- the permanent shock remains fixed across the break but the variances and correlations of the transitory components are allowed to vary across the break. First, the variances of both transitory components reduce post break, though the reduction in transitory prices is more modest. This, again, is consistent with the evidence presented in Chapter 4 of this thesis, in addition to the earlier model. Second, while the estimates for transitory prices are less persistent than the IC ratio, both are well within the stationarity condition (sum of the $\phi_{i/c}$ parameters add up to 0.96, and the sum of

ϕ_{RP} estimates add to 0.75). Third, the variance of transitory prices is smaller than that of the (i/c) ratio pre-break but the variance of transitory prices is larger post-break. However, something that we can note here is that while variances of transitory components reduce post break, such reduction seems much more modest for prices than for the ratio of aggregates. The evidence of negative correlations between the common trend and transitory components seems plausible (and consistent with the benchmark model) along with a stronger correlation between transitory fluctuation in prices and the (i/c) ratio after the assumed the break. These results, though not reported, were fairly robust to different break dates assumed to take place throughout the 1980s. Thus, this last model seems to be an improvement over the benchmark and yields interesting insights discussed in the conclusion.

VIII. Conclusion

This paper presents evidence of a shift in the dynamics of the real investment-to-consumption ratio in the United States occurring in the early 1990s. As it is evident in Figure 1, the real investment-to-consumption ratio (contrary to the theoretical prediction) seems to exhibit a fundamental break in behavior in the early 1990s, shifting from a stationary series pre-1990s to a trending process thereafter. Using data from 1948:Q2 to 2006:Q2, the Andrews (1993) structural break test detects a shift in the real ratio around 1992:Q2. This is an important finding because the stability of the "great ratios" is a natural corollary to the balanced growth hypothesis and a widely accepted stylized fact in macroeconomic theory. Thus, this finding implies that real investment seems to have a different long-run growth rate from real consumption in the United States.

Next, I consider Whelan's (2005) alternative formulation of balanced growth, which he terms *Nominal Ratio Balanced Growth Hypothesis*, which predicts the stationarity of the investment-to-consumption ratio in *nominal* terms. Unit root tests as well as Andrews (1993) break test find evidence of stability in the nominal ratio. In the spirit of GHK (1987) and Whelan (2003, 2005), a simple two-sector

model is proposed. As a point of departure from those authors, the two sectors are allowed to share a common technology. I show that the assumption of investment specific technologies, as advocated by those authors, while intuitively appealing is not necessary to establish the stationarity of the nominal ratio. Analysis based on unobserved components modeling is implemented to explore the viability of the traditional balanced growth with the inclusion of relative prices in the analysis and the implications for the macroeconomic impact of technology shocks.

Four different models described by equations (2.1)-(2.4) for the real investment-consumption ratio and relative prices for the period 1948:Q2 to 2006:Q2 are studied. These four different specifications of bivariate correlated unobserved components models yield interesting insights. The first model, which forces stationary behavior on two nonstationary variables by omitting a trend, probably suffers from misspecification error. The third model which imposes stationarity in the earlier part of the sample and nonstationarity in the latter probably suffers from similar misspecification problems. Model 2 allows for a common trend to be shared between the two series. The transitory components of both series had very similar volatility estimates, with

transitory prices exhibiting much less persistence than the investment-consumption ratio. Finally the fourth model, allowed for a shared trend between the series as well as a break in transitory behavior of both variables. Evidence is supportive of a common trend between the investment-consumption ratio and relative prices. This model's results suggest that the investment-consumption ratio seems to have undergone a fundamental change in behavior becoming less volatile and much less mean-reverting. Permanent fluctuations seem to have more importance than transitory ones in driving the ratio in the latter part of the sample as evidenced by $\tilde{\sigma}_{ci} < \sigma_{v}$. On the other hand, there is more volatility in transitory fluctuations than permanent ones in relative prices throughout the sample, as evidenced by $\sigma_{v} < \tilde{\sigma}_{RP} < \sigma_{RP}$.

IX. References

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Appendix I

As mentioned in previous sections, I will be considering a number of UC models. One of those models will specify a common trends representation between both series; the shared trend is assumed to follow an unobserved random walk with drift process. Alternatively, a *vector error correction model* (VECM) can be used to describe a system of series that share common trends. There are some advantages and disadvantages to each representation (UC vs. VECM) of a bivariate system characterized by a common trend.

X.1. *Advantages of VECM under Cointegration*

A VECM (or conversely, a VAR in levels) allows for multivariate dynamics and permits the user to analyze the effects of both supply and demand shocks (with corresponding restrictions and assumptions provided).

VAR/VECM offers an alternative to permanent-transitory decompositions by forcing the long-run response to a specified shock to zero out with straightforward interpretation (Blanchard and Quah 1989).

Typically, VAR/VECM models allow the analyst to remain more general in the inferences she might derive with respect to shocks.

A priori, I expect more robustness with respect to lag specification in the VAR/VECM framework. The UC model might be more susceptible to changes in lag specification--especially in the permanent component.

X.2. *Advantages of UC Models under Cointegration*

A correlated bivariate UC model allows variables that share a trend to have different speeds of adjustment, or half-life, and a VECM model generally will not.[65] Therefore, a UC model can determine which component adjusts quicker after a given shock hits the system.

A bivariate UC model will allow the analyst to remain agnostic with respect to long-run restrictions of shocks. Generally, a bivariate VAR requires long-run restrictions for estimation.

UC models do not require relying on low power tests for unit root before the model is set up for estimation.[66]

UC, within a two variable model, will allow for the possibility of more than two shocks to the economy. The bivariate VAR must

[65] Unless one of the error correction coefficients is exactly zero.
[66] UC models do not require an estimate of the spectral density at zero frequency.

impose an economic model in which there are exactly two true shocks (one with no permanent effects) and the model's responses are linear and invertible. If there is one underlying technology shock, the VAR will yield two shocks, which may be interpreted as the permanent and transitory components to that single shock. However, if there are many underlying shocks in the economy, the VAR will still only find two shocks.

As mentioned in the last paragraph, the UC model allows the user to estimate the variance of the unobserved trend, as well as its effect on innovation of the permanent and transitory components of each series. This establishes more than two shocks. Conversely, a VAR/VECM poses no specification of the structure of the trend but may *overstate* the effect that common permanent component has on each series. To illustrate this point, let's assume the UC model specified by (2.1)-(2.5) is an adequate representation of the economy.

X.3. ***A Simple Illustration***

Assume a version of the bivariate UC under cointegration model of (2.1)-(2.5). Omitting drifts and shifters without loss of generality:

$$y_t = A_t + \omega_{yt} \tag{2.5}$$

$$n_t = A_t + \omega_{nt} \tag{2.6}$$

$$A_t = A_{t-1} + \nu_t \tag{2.7}$$

$$\phi_y(L)\omega_{yt} = \varepsilon_{yt} \tag{2.8}$$

$$\phi_n(L)\omega_{nt} = \varepsilon_{nt} \tag{2.9}$$

From (3.1) and (3.2), we have:

$$y_t = n_t - \omega_{nt} + \omega_{yt} \qquad \text{(a)}$$

and from (3.2) and (3.3), we have:

$$n_t = n_{t-1} + \nu_t + \omega_{nt} - \omega_{nt-1} \qquad \text{(b)}$$

Taking first differences of (a) we have:

$$\Delta y_t = \Delta n_t - \Delta\omega_{nt} + \Delta\omega_{yt} \qquad \text{(c)}$$

and substituting into (b) into (c) yields:

$$\Delta y_t = \nu_t + \Delta\omega_{yt} \qquad \text{(d)}$$

The right-hand side of (d) is a sum of white noise processes with lag 1 as highest order, therefore (d) has an MA(1) representation like the following:

$$\Delta y_t = \eta_t + \theta \eta_{t-1} \qquad \text{(e)}$$

with $\eta_t \sim w.n$ and $\theta \neq 1$ since the coefficient on

$\omega_{yt} \neq 0$ & $E(\omega_{yt}{}^2) > 0$

so $y_t \sim I(1), n_t \sim I(1), and\ (y_t - n_t) \sim I(0)$

so we can infer that $Y_t = (y_t\ n_t)'$ are cointegrated with the following cointegrated vector a = (1,-1).

Solving (a) for lagged ω_{yt}

$$\omega_{yt-1} = y_{t-1} - n_{t-1} - \omega_{nt-1}$$

and substituting into (d) yields

$$\Delta y_t = \nu_t + \omega_{yt} - (y_{t-1} - n_{t-1} - \omega_{nt-1})$$

which simplifies to

$$\Delta y_t = \nu_t + \omega_{nt-1} + \omega_{yt} - (y_{t-1} - n_{t-1}) \qquad \text{(f)}$$

and from (b), we have

$$\Delta n_t = \nu_t + \omega_{nt} - \omega_{nt-1} \quad \text{(b')}$$

stacking (f) and (b') we get:

$$\begin{pmatrix}\Delta y_t\\ \Delta n_t\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}y_{t-1}\\ n_{t-1}\end{pmatrix}+\begin{pmatrix}\nu_t+\omega_{nt-1}+\omega_{yt}\\ \nu_t+\omega_{nt}-\omega_{nt-1}\end{pmatrix} \quad \text{(g)}$$

The VECM form of (g) is

$$\begin{pmatrix}\Delta y_t\\ \Delta n_t\end{pmatrix}=\begin{pmatrix}-1\\ 0\end{pmatrix}(y_t-n_t)+\begin{pmatrix}\varepsilon_{yt}\\ \varepsilon_{nt}\end{pmatrix} \qquad \text{(h)}$$

when letting $\begin{aligned}\varepsilon_{yt}&\equiv\nu_t+\omega_{nt}+\omega_{yt}\\ \varepsilon_{nt}&\equiv\nu_t+\omega_{nt}-\omega_{nt-1}\end{aligned}$

Notice that under cointegration, a typical bivariate VAR of the form

$$y_t=n_t+\omega_{yt} \qquad \text{(a′)}$$

$$n_t=n_{t-1}+\omega_{nt} \qquad \text{(b′)}$$

might ultimately be represented as

$$\begin{pmatrix}\Delta y_t\\ \Delta n_t\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}y_{t-1}\\ n_{t-1}\end{pmatrix}+\begin{pmatrix}\omega_{yt}+\omega_{nt}\\ \omega_{nt}\end{pmatrix} \qquad \text{(g′)}$$

and the VECM form of (g′) would be

$$\begin{pmatrix}\Delta y_t\\ \Delta n_t\end{pmatrix}=\begin{pmatrix}-1\\ 0\end{pmatrix}(y_t-n_t)+\begin{pmatrix}\varepsilon_{yt}\\ \varepsilon_{nt}\end{pmatrix} \qquad \text{(h′)}$$

when letting $\begin{matrix} \varepsilon_{yt} \equiv \omega_{yt} + \omega_{nt} \\ \varepsilon_{nt} \equiv \omega_{nt} \end{matrix}$ From the estimates of $\varepsilon_{yt}, \varepsilon_{nt}$ from the VECM of the VAR system given by (a′) and (b′), ω_{yt}, ω_{nt} can be easily recovered.

However, when letting $\begin{matrix} \varepsilon_{yt} \equiv \nu_t + \omega_{nt-1} + \omega_{yt} \\ \varepsilon_{nt} \equiv \nu_t + \omega_{nt} - \omega_{nt-1} \end{matrix}$ the estimates of $\varepsilon_{yt}, \varepsilon_{nt}$ from the UC model system given by (a) and (b) would be overstated by the presence of the error term component ν_t. Therefore, ω_{yt}, ω_{nt} would *not* be recoverable.

X.4. ***A Simple VECM Illustration***

Recall one of the advantages of correlated bivariate UC models as stated above was that UC models would allow variables that share a trend to have different speeds of adjustment, or half-life, and A VECM model generally would not. The VECM model would typically assume constant speeds of adjustment (though non-constant magnitudes of the adjustment). This can be shown with a simple example.

Assume y_t and n_t admit a finite order VAR representation. The VECM representation (Hamilton 1995) is given by:

$$\Delta y_t = \pi_y z_{t-1} + \sum_{j=1}^{p-1} \xi_{yy,j} \Delta y_{t-j} + \sum_{j=1}^{p-1} \xi_{yn,j} \Delta n_{t-j} + \varepsilon_{yt} \quad (2.10)$$

$$\Delta n_t = \pi_n z_{t-1} + \sum_{j=1}^{p-1} \xi_{ny,j} \Delta y_{t-j} + \sum_{j=1}^{p-1} \xi_{nn,j} \Delta n_{t-j} + \varepsilon_{nt} \quad (2.11)$$

where π_y and π_n are the error correction coefficients that measure the expected adjustment of y_t and n_t each quarter to their long-run equilibrium

and where $z_{t-1} = y_{t-1} - \alpha - \beta n_{t-1}$

One could estimate the following equation by *ordinary least squares* (OLS)

$$y_t = \hat{\alpha} + \hat{\beta} n_t + z_t$$

If $y_t \sim I(1)$ & $n_t \sim I(1)$ but $z_t \sim I(0)$, then y_t and n_t are said to be cointegrated.

The ECM connects the speed of adjustment of both y_t and n_t to the cointegrated error term, which according to Cochrane (1994) follows an implied AR(1) process

$$z_t = (1 + \pi_y - \beta \pi_n) z_{t-1} + \varepsilon_{yt} - \beta \varepsilon_{nt} \quad (2.12)$$

While $\hat{\pi}_y$ and $\hat{\pi}_n$ imply the magnitude of the adjustment, $(1 + \hat{\pi}_y - \beta \hat{\pi}_n)$ implies the speed of adjustment and, generally, it is always constant in the VECM unless either $\hat{\pi}_y$ or $\hat{\pi}_n$ is exactly zero. If $\hat{\pi}_n = 0$, then n is said

to be weakly exogenous[67] with respect to the cointegrating relationship. This would imply an instantaneous adjustment to its long-run equilibrium level.

[67] Weak exogeneity tests tend to have low power against alternatives.

TABLE 1: Unit Root Test for the Real Ratio $\frac{i}{c}$

Test	Statistic	5% / 10% Critical Values
ADF t-statistic	-1.53	-2.88 / -2.57
Elliot-Rothenberg-Stock DF^{GLS}	-1.15	-1.95 / -1.62
Phillips-Perron	-1.68	-2.88 / -2.57
Banerjee-Lumsdaine-Stock	-2.59	-4.76 / -4.48

Lag Selection calculated by MAIC

TABLE 2: Unit Root Test for the Nominal Ratio $\frac{i^*}{c^*}$

Test	Statistic	5% / 10% Critical Values
ADF t-statistic	-3.08	-2.88 / -2.57
Elliot-Rothenberg-Stock DF^{GLS}	-2.08	-1.95 / -1.62
Phillips-Perron	-3.29	-2.88 / -2.57
Banerjee-Lumsdaine-Stock	-3.18	-4.76 / -4.48

Lag Selection calculated by MAIC

TABLE 3: Stationary (I/C) ratio and Relative Prices
Correlated UC Model Parameter Estimates

Permanent Component	*Estimate*	*Standard Error*
μ_1	-161.89	(5.8295)
μ_2	-35.810	(588.37)
Transitory Ratio	***Estimate***	***Standard Error***
ϕ_{ci1}	1.43542	(0.0583)
ϕ_{ci2}	-0.4575	(0.0588)
σ_{ci}	1.9224	(0.0893)
Transitory Prices	***Estimate***	***Standard Error***
ϕ_{RP1}	1.6087	(0.0527)
ϕ_{RP2}	-0.6096	(0.0523)
σ_{RP}	0.5552	(0.0258)
Correlations	***Estimate***	***Standard Error***
ρ_{cirp}	-0.0569	(0.0675)

Parameters smoothed by Kalman filter and estimated by MLE where estimated likelihood equals -608.04. Data encompasses 1948:Q2-2006:Q2

TABLE 4: (I/C) ratio and Relative Prices Correlated UC Model Parameter Estimates Benchmark Model (1948:Q2-2006:Q2)

Permanent Component	*Estimate*	*Standard Error*
$\overline{rp}$	165.03	(2.0158)
$\overline{z}$	0.0566	(0.0856)
γ	0.9889	(0.0273)
σ_v	1.5081	(0.1957)
Transitory Ratio	*Estimate*	*Standard Error*
ϕ_{ci1}	1.5036	(0.0737)
ϕ_{ci2}	-0.5546	(0.0720)
σ_{ci}	1.7381	(0.2237)
Transitory Prices	*Estimate*	*Standard Error*
ϕ_{RP1}	0.8808	(0.0684)
ϕ_{RP2}	-0.0875	(0.0268)
σ_{RP}	1.7753	(0.2042)
Correlations	*Estimate*	*Standard Error*
$\rho_{ci,rp}$	0.4977	(0.1051)
ρ_{civ}	-0.3857	(0.1334)
ρ_{RPv}	-0.9921	(0.0048)

Parameters smoothed by Kalman filter and estimated by MLE where estimated likelihood equals -673.19. Data encompasses 1948:Q2-2006:Q2[1]

[1] To begin the Kalman smoothing process and estimate these parameters, we set z_{010} = -165.67, the value of the (real) investment to consumption ratio in 1948:Q2 1st observation o the ratio. Given the arbitrariness of this initial value for the trend, we allow a relatively large diffuse prior by setting Var(z_{010}) = 100.

TABLE 5: (I/C) ratio and Relative Prices Correlated UC Model Parameter Estimates with Volatility Break in 1983:Q3 and zero-variance of the permanent component pre-brake

Permanent Component	*Estimate*	*Standard Error*
$\overline{rp}$	187.90	(2.0169)
$\overline{z}$	0.0220	(0.0009)
σ_v [1]	Zero-restriction	
$\tilde{\sigma}_v$	0.7818	(0.00)
Transitory Ratio	***Estimate***	***Standard Error***
ϕ_{ci1}	1.3663	(0.0001)
ϕ_{ci2}	-0.4800	(0.0002)
σ_{ci}	2.1341	(0.00)
$\tilde{\sigma}_{ci}$	1.4426	(0.0001)
Transitory Prices	***Estimate***	***Standard Error***
ϕ_{RP1}	1.5222	(0.0001)
ϕ_{RP2}	-0.5325	(0.0001)
σ_{RP}	0.6905	(0.0001)
$\tilde{\sigma}_{RP}$	0.5627	(0.0008)

[1] Parameters with tildes represent estimates of the covariance matrix post-break.

Correlations	*Estimate*	*Standard Error*
ρ_{cirp}	-0.0443	(0.0000)
$\tilde{\rho}_{cirp}$	0.2698	(0.0006)
ρ_{civ}	-	-
$\tilde{\rho}_{civ}$	-0.5094	(0.0001)
ρ_{RPv}	-	-
$\tilde{\rho}_{RPv}$	-0.9661	(0.0001)

Parameters smoothed by Kalman filter and estimated by MLE where estimated likelihood equals -664.34. Data encompasses 1948:Q2-2006:Q2[1]

[1] To begin the Kalman smoothing process and estimate these parameters, we set $z_{0|0} = -165.67$, the value of the (real) investment to consumption ratio in 1948:Q2 1st observation o the ratio. Given the arbitrariness of this initial value for the trend, we allow a relatively large diffuse prior by setting $\text{Var}(z_{0|0}) = 100$.

TABLE 6: (I/C) ratio and Relative Prices Correlated UC Model Parameter Estimates with Break (in Q) in 1983:Q3 except the variance of the permanent and its correlations

Permanent Component	*Estimate*	*Standard Error*
$\overline{rp}$	167.70	(2.710)
$\overline{z}$	0.0689	(0.0852)
σ_v [1]	1.4131	(0.1707)
Transitory Ratio	***Estimate***	***Standard Error***
ϕ_{ci1}	1.5263	(0.0855)
ϕ_{ci2}	-0.5640	(0.0833)
σ_{ci}	1.9093	(0.2128)
$\tilde{\sigma}_{ci}$	1.2423	(0.2606)
Transitory Prices	***Estimate***	***Standard Error***
ϕ_{RP1}	0.8174	(0.0783)
ϕ_{RP2}	-0.0650	(0.0313)
σ_{RP}	1.6732	(0.1841)
$\tilde{\sigma}_{RP}$	1.6412	(0.1866)
Correlations	***Estimate***	***Standard Error***
ρ_{cirp}	0.4512	(0.0917)
$\tilde{\rho}_{cirp}$	0.7232	(0.1155)
ρ_{civ}	-0.3246	(0.1171)

[1] Parameters with tildes represent post-break estimates.

ρ_{RPv}	-0.9905	(0.0063)

Parameters smoothed by Kalman filter and estimated by MLE where estimated likelihood equals -689.03. Data encompasses 1948:Q2-2006:Q2[1]

[1] To begin the Kalman smoothing process and estimate these parameters, we set $z_{0|0} = -165.67$, the value of the (real) investment to consumption ratio in 1948:Q2 1st observation o the ratio. Given the arbitrariness of this initial value for the trend, we allow a relatively large diffuse prior by setting $Var(z_{0|0}) = 100$.

TABLE 7: Andrews Structural Break Test

Test	Statistic	Break
Real Ratio $\frac{i}{c}$	25.47	Break detected in 1992:Q2
Nominal Ratio $\frac{i^*}{c^*}$	5.77	No break detected

The critical values for the 1%, 5% and 10% level of significance are 15.51, 11.79 and 10.01 respectively

Figure 1.
Ratio of Real Private Fixed Investment to Real Consumption
(1948:Q2-2006:Q2)

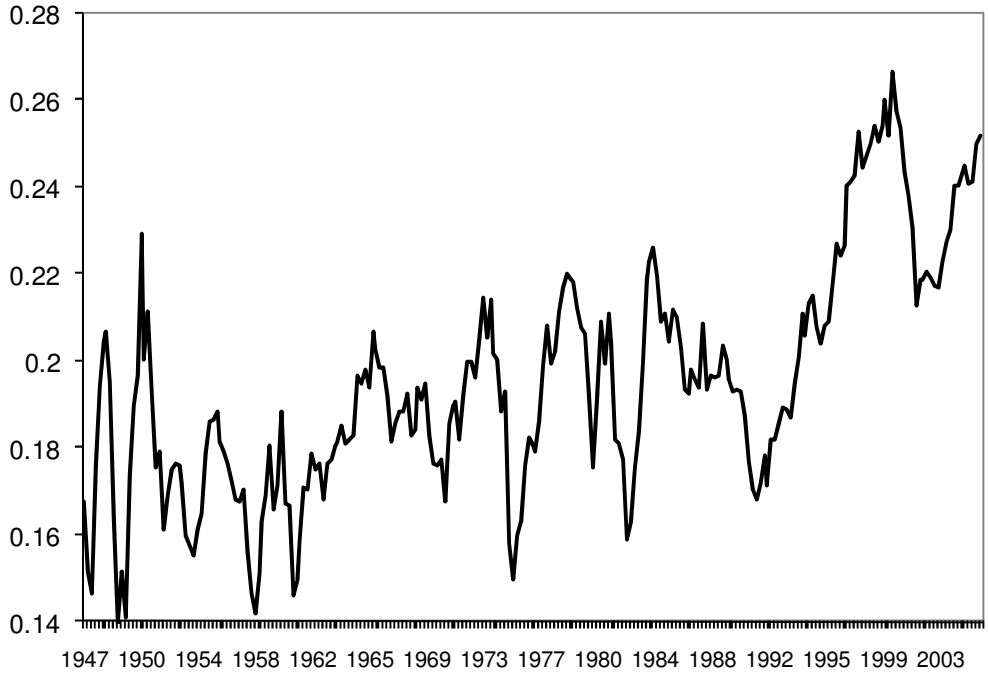


Figure 2
Ratio of Nominal Private Fixed Investment to Nominal Consumption
(1948:Q2-2006:Q2)

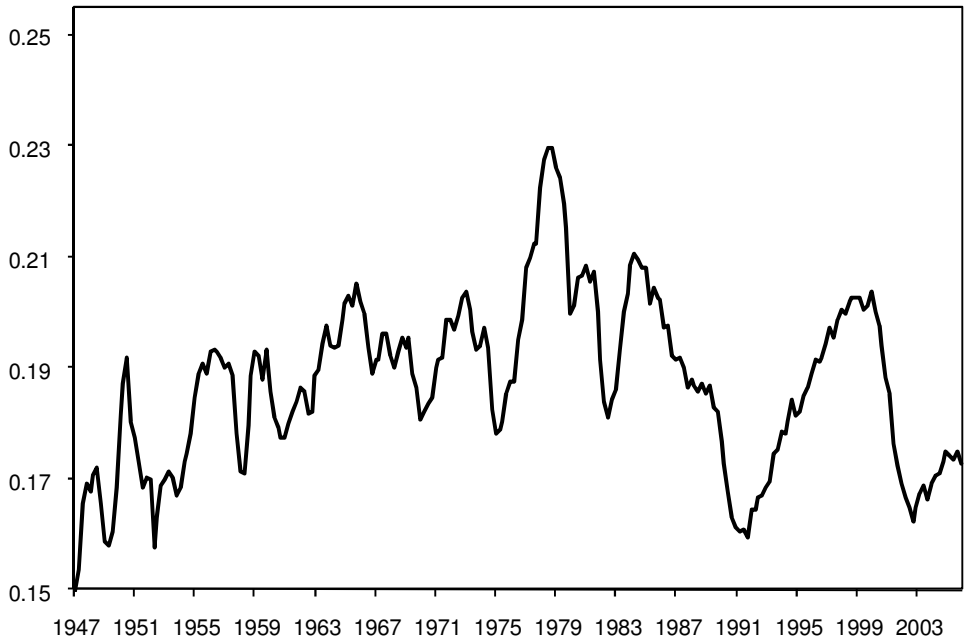
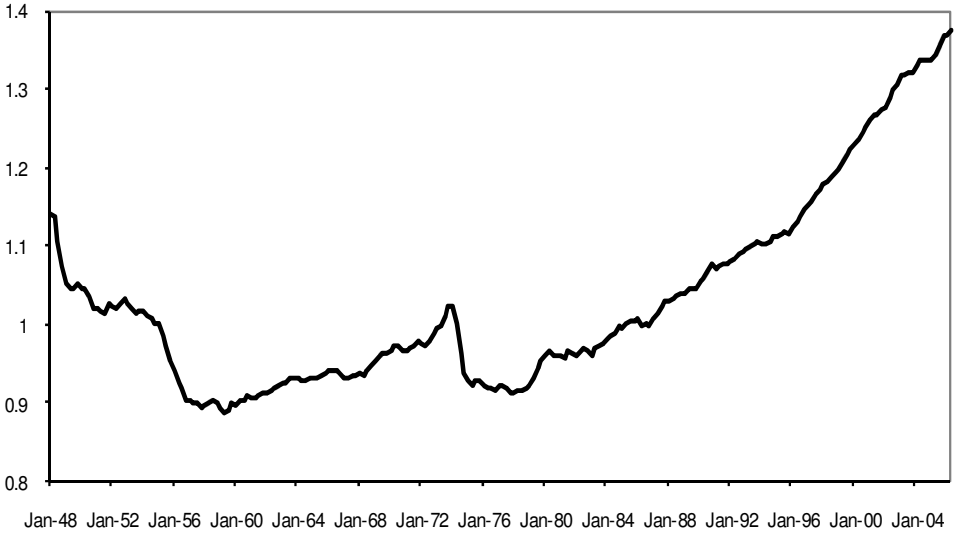


Figure 3
Ratio of CPI to PPI (1948:Q2-2006:Q2)



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CHAPTER IV

A BAYESIAN APPROACH TO MODELING STOCHASTIC VOLATILITY IN THE U.S. INVESTMENT-CONSUMPTION RATIO

Abstract

Recent studies in the Bayesian tradition have uncovered a substantial decrease in volatility in aggregate U.S. economic activity taking place in the mid-1980s consistent with findings of volatility reduction by "classical" econometric tests. However, no tests have been conducted to study such behavior in the great ratios thus far. Such investigation is important in light of the recent evidence of behavioral change in the level of the investment-consumption ratio described by other authors. This paper finds decisive evidence of a structural break in the variance of the ratio occurring in the third quarter of 1983. This conclusion is robust to a wide range of alternative priors and consistent with the early-to-mid 1980s volatility reduction observed in the major aggregates and sub-aggregates of U.S. economic activity. Such findings have a number of important implications for the behavior of real investment and consumption in the U.S. Additionally, they lead to the conclusion that any suggested specification that involves decomposing the investment-consumption ratio into a non-stationary process for the trend and a stationary process for the cycle should allow for a break in the covariance matrix.

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I. Introduction

Post-war economic activity in the U.S. appears to have stabilized substantially since the early-to-mid 1980s. For example, the standard deviation of quarterly growth rates of real GDP for 1950-1984 was more than twice as large as that for 1985-2006. There has been an upsurge of literature engaged in statistically testing this volatility reduction and substantiating various stylized facts about the nature of the stabilization (a non-exhaustive set of tests and stylized facts are advanced by Niemira and Klein 1994; Kim and Nelson 1999; McConnell and Perez-Quiros 2000; Warnock and Warnock 2000; Blanchard and Simon 2001; Chauvet and Potter 2001; Stock and Watson 2002; Kim, Nelson and Piger 2004).

For example, using classical methods, McConnell and Perez-Quiros (henceforth MPQ) find evidence of volatility reduction in GDP taking place in the mid-1980s, while Warnock and Warnock (henceforth WW) find a volatility reduction in aggregate employment. From the Bayesian approach, Chauvet and Potter (henceforth CP) find evidence of stabilization in output, consumption, personal income and employment, and Kim, Nelson and Piger (henceforth KNP) find

evidence of reduced volatility in GDP and its cyclical component and disaggregates.

One point of contention between MPQ and KNP, as it pertains to the source of the phenomenon, is worth mentioning. MPQ explored structural breaks using classical tests based on work of Andrews (1993) and Andrews and Ploberger (1994) and concluded that the source of the volatility reduction in economic activity was the durable goods sector. They went on to show that measures of final sales have not become more stable and suggest the reduction can be attributed to inventory behavior. Conversely, KNP find (using Bayesian methods) a break in volatility in final sales of durables as well as non-durables and conclude that the source of the reduction may not be due to inventories and in fact may not be easily attributed to any one factor: "…evidence from broad production sectors of real GDP is not sufficiently sharp to help invalidate potential candidates for the source of the volatility reduction" (KNP 2004).

The goal in this section is to test for a break in volatility of the great ratio of real investment and consumption. The motivation for the test is two-pronged:

First, earlier in this thesis, I show recent evidence that the real investment-to-consumption ratio is not trend-less (implying that they have not been growing at the same rate since the late 1980s / early 1990s). A possible explanation (which I promote in that discussion) could be that the growth of durables (the larger portion of which is associated with private investment) has outpaced the growth of non-durables (which mostly goes to consumption) since the late 1980s. Could the discrepancy in the growth rates of both series be associated with a discrepancy in the volatility of those growths? In other words, consumption and investment seem to grow together, but their ratio does not reflect this stylized fact. Some measures of consumption and investment seem to display a reduction in volatility since the mid-80s. Will the ratio fail to show that as well?

The second motivation is much more pragmatic. If I find no evidence of structural break in the volatility of the ratio, I do not have a rationale for imposing a break in the covariance matrix of the UC models I advance in the present chapter.

This subchapter is organized as follows: Section 2 specifies the model I use to investigate the structural change in the volatility of the investment-to-consumption ratio. Section 3 describes the Bayesian

methodology, the mechanics and strategy for implementing the Gibbs sampler as well as consideration of priors. Section 4 describes the strategy for model comparison and testing. Section 5 presents evidence of volatility reduction in the ratio of real investment-to-consumption. Section 6 examines the robustness of the results to three alternative considerations of prior distributions for the break. Section 7 concludes.

II. Model Specification

To investigate a possible volatility reduction in the growth rate of the investment-to-consumption ratio, I use the following empirical model:

$$y_t = \sum_{j=1}^{k} \phi_j y_{t-j} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma_{S_t}^2) \qquad (1.1)$$

where y_t is the demeaned growth rate of the series under consideration. The variance of the process is allowed to hold a two-state Markov-switching process of the following form:

$$\sigma_{S_t}^2 = \sigma_0^2(1 - S_t) + \sigma_1^2(S_t) \qquad (1.2)$$

$$S_t = \begin{cases} 0, & 1 \le t \le \tau \\ 1, & \tau \le t \le T-1 \end{cases} \qquad (1.3)$$

where S_t is a discrete latent variable that determines the date of the structural break τ and evolves according to a two-state first-order Markov-switching process with the following transition probabilities[68]:

$$\begin{aligned} &\Pr(S_{t+1} = 0 \mid S_t = 0) = q \\ &\Pr(S_{t+1} = 1 \mid S_t = 1) = 1 \\ &0 < q < 1 \end{aligned} \qquad (1.4)$$

[68] This allows for the possibility of an endogenous but permanent structural break in the conditional variance (Chib 1998).

Along with S_t , the model's unknown parameters $\{[\phi_1 \ldots \phi_k], \sigma_0^2, \sigma_1^2, q, \tau\}$ are treated as random variables. In this setting, conditional on no break in the current period (S_t=0), there is always a 1-q probability that a break will occur next period (S_{t+1}=1). Hence the expected duration of a regime before the onset of a break (length of time S_t remains zero) is given by $E(\tau) = \frac{1}{1-q}$.

Once a structural break occurs at t = τ , every S_t after that becomes a one ($S_{\tau+j} = 1$) for all j > 0.

III. A Brief Description of the Bayesian Methodology

To draw Bayesian inferences about these T+5[69] variates, we need to derive a joint posterior density of the parameters by approximating it with the sample distribution[70] given the parameters and a prior distribution.

$$\begin{aligned} g\left(\tilde{S}_T, \phi_1, \phi_2, \sigma_0^2, \sigma_1^2, q \mid \tilde{y}_T\right) &= g\left(\phi_1, \phi_2, \sigma_0^2, \sigma_1^2, q \mid \tilde{y}_T, \tilde{S}_T\right) g\left(\tilde{S}_T \mid \tilde{y}_T\right) \\ &= g\left(\phi_1, \phi_2, \sigma_0^2, \sigma_1^2 \mid \tilde{y}_T, \tilde{S}_T\right) g\left(q \mid \tilde{y}_T, \tilde{S}_T\right) g\left(\tilde{S}_T \mid \tilde{y}_T\right) \quad (1.5) \\ &= g\left(\phi_1, \phi_2, \sigma_0^2, \sigma_1^2 \mid \tilde{y}_T, \tilde{S}_T\right) g\left(q \mid \tilde{S}_T\right) g\left(\tilde{S}_T \mid \tilde{y}_T\right) \end{aligned}$$

where $\tilde{y}_T = [y_1 \quad y_2 \quad \dots \quad y_T]$ and $\tilde{S}_T = [S_1 \quad S_2 \quad \dots \quad S_T]$.

Equation (1.5) assumes that, conditional on the states $\tilde{S}_T$, the transition probability q is independent of the other parameters of the model as well as the data $\tilde{y}_T$. This conditioning assumption allows us to estimate the joint and marginal distributions from information of the conditional distributions.

---

[69] Assuming an AR(2) process for equation (1.1).

[70] There is a functional equivalence between the sample distribution of the parameters and the likelihood function.

III.1 Illustration of the Gibbs Sampler General Basic Mechanism

The Markov-chain Monte Carlo (MCMC) simulation method used to approximate joint and marginal distributions by sampling from conditional distributions is known as the *Gibbs-sampling*[71] tool for Bayesian inference. While the method is quite powerful, the basic idea is simple.

Suppose we are given a model (m) and a collection of available data ($\tilde{y}$) and we want to estimate k parameters[72] $(\theta_1,\ldots,\theta_k)$. The model's joint density (or likelihood function) is not available but we are given the complete set of conditional distributions of each single parameter given the others, i.e. the following k conditional distributions are known:

$$\begin{aligned} &f_1(\theta_1 \mid \theta_2,\theta_3,\ldots\theta_k,\tilde{y},m);\\ &f_2(\theta_2 \mid \theta_1,\theta_3,\ldots\theta_k,\tilde{y},m);\\ &\quad\vdots\\ &\quad\vdots\\ &f_k(\theta_k \mid \theta_1,\theta_2,\theta_3,\ldots\theta_{k-1},\tilde{y},m) \end{aligned} \qquad (1.6)$$

[71] See Geman and Geman (1984), Gelfand and Smith (1990) and Casella and George (1993).
[72] I.E. the goal is to obtain characteristics of the marginal density $f(\theta)$ of the parameters such as the mean or the variance.

where $f_i(\theta_i \mid \theta_{j\neq i}, \tilde{y}, m)$ denotes the conditional distribution of the parameters θ_i given the data, the model and the other k parameters. The strength of the method comes from the little number of prerequisites. The Gibbs sampler requires no knowledge of either the joint density or the marginal densities. We do not need to know the exact forms of the conditional distributions. All that is needed is the ability to draw a random number from each of the k conditional distributions in (1.6) based on an arbitrary set of prior information.

Assume we have an arbitrary set of initial values $(\theta_2^0, \ldots, \theta_k^0)$ of the parameters $(\theta_2, \ldots \theta_k)$. The Gibbs sampler consists of k steps:

1. Draw a random sample from $f_1(\theta_1 \mid \theta_2^0, \ldots \theta_k^0, \tilde{y}, m)$. Denote the random draw by θ_1^1. Next, we incorporate this posterior θ_1^1 into the information set of the 2nd conditional distribution and draw a random sample again.
2. Then draw a random sample θ_2^1 from $f_2(\theta_2 \mid \theta_1^1, \theta_3^0, \ldots \theta_k^0, \tilde{y}, m)$.
3. Then draw a random sample θ_3^1 from $f_3(\theta_3 \mid \theta_1^1, \theta_2^1, \theta_4^0, \ldots \theta_k^0, \tilde{y}, m)$.

   ⋮

k. Finally, draw a random sample θ_k^1 from $f_k(\theta_k \mid \theta_1^1, \ldots, \theta_{k-1}^1, \tilde{y}, m)$.

This completes one iteration and the parameters become $(\theta_1^1, \theta_2^1, \ldots, \theta_k^1)$. Next, using these new parameters as starting values and repeating the prior iteration of random draws, another Gibbs iteration is completed to obtain the updated parameters $(\theta_1^2, \theta_2^2, \ldots, \theta_k^2)$.

Steps 1 through k can be repeated *j* times to obtain a sequence of random draws

$$(\theta_1^1, \theta_2^1, \ldots, \theta_k^1), \ldots, (\theta_1^j, \theta_2^j, \ldots, \theta_k^j) \qquad (1.7)$$

Geman and Geman (1984) showed that the joint and marginal distributions of $(\theta_1^j, \theta_2^j, \ldots, \theta_k^j)$ converge at an exponential rate to the joint and marginal distributions of $(\theta_1, \theta_2 \ldots \theta_k)$ as $j \to \infty$. The regularity conditions under which $(\theta_1^j, \theta_2^j, \ldots, \theta_k^j)$ are approximately equivalent to a random draw from the joint distribution $f(\theta_1 \mid \theta_2, \theta_3, \ldots, \theta_k, \tilde{y}, m)$ are fairly weak. The requirement is simply that for an arbitrary starting value $(\theta_2^0, \ldots, \theta_k^0)$ the prior Gibbs iterations have a chance to visit the full parameter space (Tierney 1994).

In practice, a sufficiently large number of iterations (L) is run to ensure the Gibbs sampler has converged [i.e. equation (1.7) is indeed close enough to a random sample from the joint distribution

$f(\theta_1 \mid \theta_2, \theta_3, \ldots, \theta_k, \tilde{y}, m)$]. Therefore, the joint and marginal distributions of $(\theta_1, \ldots \theta_k)$ can be simulated by the empirical distributions of M simulated values of (1.7), with j = L+1, L+M, with L iterations being discarded (commonly referred to as the *burn-in* sample). M can be chosen to give sufficient precision to the empirical distribution of interest. For example, a point estimate and variance of θ_i may be approximated by

$$\hat{\theta}_i = \frac{1}{M}\sum_{j=1}^{M} \theta_i^j, \qquad \hat{\sigma}_i^2 = \frac{1}{M-1}\sum_{j=1}^{M} \left(\theta_i^j - \hat{\theta}_i\right)^2 \tag{1.8}$$

III.2 Gibbs Strategy

To derive the distributions of the blocks of each of the T+5 variates in (1.5) conditional on all other blocks of variates, I iterate the following three steps until convergence is achieved:

Step 1 Generate the block of $\tilde{S}_T$ from $g\left(\tilde{S}_T \mid \phi_1, \phi_2, \sigma_0^2, \sigma_1^2, q, \tilde{y}_T\right)$

Step 2 Generate the transition probability q from $g\left(q \mid \tilde{S}_T\right)$

Step 3 Generate $\phi_1, \phi_2, \sigma_0^2, \sigma_1^2$ from $g\left(\phi_1, \phi_2, \sigma_0^2, \sigma_1^2, q \mid \tilde{y}_T, \tilde{S}_T\right)$[73]

[73] Kim and Nelson (99) submit a comprehensive treatment on the generation of posteriors, delineate specific details about these steps and offer illustrative examples.

III.3 Prior Distributions

Let $\theta = [\beta_i, \sigma_i^2]'$ be a vector of unknown *i* parameters for some given model and $\tilde{y}$ be the data. Classical analysis assumes the parameters to be discrete point estimates and uses the sample $\tilde{y}$ to make inferences about the population. Bayesian analysis treats the parameters as random variables having probability distributions. These distributions constitute knowledge (which may be subjective and reflect different prior beliefs) of the parameters the modeler has before observing the sample ($\tilde{y}$). For this reason, they are typically referred to as *prior distributions*. A prior distribution $P(\theta)$ (information about the parameter) is combined with the likelihood function $f(\tilde{y} \mid \theta)$ of the data (information contained in the sample). Then by the definition of conditional probability we have,

$$f(\theta \mid \tilde{y}) = \frac{f(\theta, \tilde{y})}{f(\tilde{y})} = \frac{f(\tilde{y} \mid \theta) P(\theta)}{f(\tilde{y})} \tag{1.9}$$

where the marginal distribution $f(\tilde{y})$ can be obtained by

$$f(\tilde{y}) = \int f(\tilde{y}, \theta) d\theta = \int f(\tilde{y} \mid \theta) P(\theta) d\theta$$

The distribution $f(\theta \mid \tilde{y})$ in equation (1.9) is called the *posterior distribution* of θ. Using Bayes rule[74] we obtain,

$$f(\theta \mid \tilde{y}) \propto f(\tilde{y} \mid \theta) P(\theta) \tag{1.10}$$

where $P(\theta)$ is the prior distribution and $f(\tilde{y} \mid \theta)$ is the sample distribution given parameter θ or likelihood function.

In general, obtaining a posterior distribution as in (1.10) is far from trivial because it relies on the choice of a prior distribution, which can be difficult. In theory, a prior distribution of any functional form can be combined with the likelihood function to yield a posterior distribution. However, not every distribution is adequate. For example, a normal density cannot be used as a prior distribution for σ^2 because variances cannot take negative values. Therefore, there are "families" of densities that can be more easily combined with the likelihood function than others. A *natural conjugate prior* is an example of these. When a conjugate prior is combined with a likelihood function, the posterior distribution has the same form as the prior. For MCMC methods, use of a conjugate prior means that there exists a closed-form solution for the conditional posterior distributions. I use a

[74] Because $f(X)$ has no operational significance.

conjugate prior specification for the model specified by (1.1) through (1.4):

Prior A:
$$[\phi_1,\ldots,\phi_k]' \sim N(0_k, I_k); \qquad q \sim beta(8,.1);$$
$$\frac{1}{\sigma_0^2} \sim gamma(1,1); \qquad \frac{1}{\sigma_1^2} \sim gamma(1,1);$$

This prior places the mean value of q at a relatively high value, thus placing most of the probability mass of a break date late in the sample. For robustness, section 6 considers three additional prior distributions of q which place probability mass in different parts of the sample.

IV. Model Testing

I estimate two versions of the model implied by equations (1.1) through (1.4). The first version—model 1—freely estimates the model's parameters. The second version—model 0—constrains the model's parameters to have no structural break, and the priors used for this model are based on those of $[\phi_1,\ldots,\phi_k]'$ and σ_0^2 in priors A and B.

The null hypothesis is described by the "no break in volatility" constraint of model 0, while the alternative is given by model 1 (which allows for the structural break). Hence, to test for evidence of a structural break at an unknown date, I compare models 0 and 1 using the following Bayes' factor:

$$BF_{10} = \frac{g\left(\tilde{y}_t \mid \text{model } 1\right)}{g\left(\tilde{y}_t \mid \text{model } 0\right)} \tag{1.11}$$

where $\tilde{y}_T = [y_1,\ldots y_T]'$ are the data and $g\left(\tilde{y}_T \mid \bullet\right)$ the marginal likelihood conditional on the specified model. To calculate the Bayes factor, I follow Chib's (1995) procedure by calculating the marginal likelihood from the output that is yielded by the Gibbs sampler procedure.[75]

[75] Kass and Raftery (1995) provide an overview of the Bayes factor for model comparison.

I use the well-known Jeffrey's scale (1961), which is often alluded to in the Bayesian statistics literature, for a descriptive interpretation of the Bayes factor.

Jeffreys' Bayes Factor scale

ln(BF) < 0:	Evidence supporting the null hypothesis
0 < ln(BF) < 1.15:	Very slight evidence against the null hypothesis
1.15 < ln(BF) < 2.23:	Slight evidence against the null hypothesis
2.23 < ln(BF) < 4.6:	Very strong evidence against the null hypothesis
ln(BF) > 4.6:	Decisive evidence against the null hypothesis

See H. Jeffreys (1961)

V. Evidence of a Break in Volatility of the Investment-to-Consumption Ratio

In this section I investigate evidence of a structural change in the growth rate of the investment-to-consumption ratio. All data were obtained from the FRED II database maintained by the Federal Reserve Bank of St. Louis. Data on personal consumption and private investment are seasonally adjusted and normalized[76] and cover the sample period 1955:q2 to 2006:q2. The modified Akaike Information Criterion (MAIC) was used, and it chose a lag order (k) of 2 for model 0[77]. Evaluation is based on 5,000 Gibbs simulations with a *burn-in* period of 1,000 to reduce the initial distorting effects and ensure convergence of the sampling distribution. As part of the simulation, and in order to generate $[\phi_1, \phi_2]'$, I employ rejection sampling to ensure the roots of $(1 - \phi_1 L - \phi_2 L^2) = 0$ lie outside the unit circle. For example, if the generated values for ϕ_1, ϕ_2 do not satisfy the stationarity condition, they are discarded and re-simulated until the condition is satisfied.

Conditional on the set of priors (A), the log of the Bayes factor for the investment-to-consumption ratio is 6.2. According to the Jeffrey's scale displayed in section 4, this is decisive evidence against

[76] The ratio is expressed in demeaned and standardized quarterly growth rates.
[77] The largest lag length I allowed for the MAIC was k=5.

the model with no structural change in volatility. The posterior mean of σ_1^2 is about 42% that of σ_0^2. This implies that the volatility reduction in this ratio is relatively large. The posterior mean date of the unknown structural break in the I-C ratio is 1983:3 based on the prior A for q[78]. This date is comparable with the break dates in GDP (and sub-aggregates) reported by MPQ (using classical methods) and KNP (with the Bayesian approach). Figure 2(b) shows the posterior distribution is fairly clustered around the break date but with a bit of skewness in its diffusion to the right.

[78] See figure 1(a).

VI. Robustness to Various Prior Specifications

In this section I investigate the sensitivity of these results to alternative priors. The priors of interest are those that pertain to the Markov transition probability (q). This key parameter of the model controls the placement of the unknown structural change date. Nevertheless, prior distributions of (q) are subjective and, therefore, it is interesting to determine the robustness of the reduced volatility evidence when alternative priors for q are considered. As mentioned in section 5, the value of the log Bayes factor of 6.2 in favor of the alternative hypothesis was obtained using prior A, which includes $q \sim beta(8, 0.1)$. Figure 1(a) shows the histogram of this prior based on 5,000 simulated draws from the distribution. This distribution has a mean of roughly 0.988 and a standard deviation of 0.037. From the 5,000 draws of q plotted in figure 1(a), about 80% corresponded to $E(\tau) = \frac{1}{1-q} > 101$ quarters, which is the midpoint of the sample size considered. Hence, roughly 20% of the draws from the distribution of q yielded expected break dates less than 100 quarters into the sample.

To investigate the robustness of the findings for the investment-to-consumption ratio, I use the same set of priors (prior A) for all

parameters except the prior of q. I consider different specifications of the prior distribution of q and retest the sample for a change in volatility. Table 5 shows the Bayes factor statistic prior A as well as three alternative specifications for the prior distribution of q I discuss below.

First, I consider a relatively flat prior $q \sim beta(1,1)$, the histogram of which is displayed in figure 1(b). This prior has a mean near one-half and roughly equal probability placed on all realizations of q. Hence, this prior is quite uninformative about the initial guess of where the break might occur. Table 2 shows parameter estimates for model 1 under this prior. Posterior parameters under this prior are quite close to those yielded by prior A. The ratio of posterior means of the variance of e_t before and after the structural break was also fairly close to that of prior A at $\frac{\sigma_1^2}{\sigma_0^2} = 0.43$. Figure 3(a) shows the posterior probability of q under this prior to be slightly flatter than the one yielded by prior A. The posterior distribution of q in Figure 3(b) shows a lower peak than prior A, though slightly less diffuse. Taken together, Figure 3 and Table 2 suggest evidence of a break in the volatility ratio with a lower probability than that of prior A. This is consistent with

evidence from the second row of table 5, where the log Bayes factor for this prior specification is 2.9. The statistic is lower than that obtained from prior A, but there is still strong evidence in favor of the alternative hypothesis.

The second alternative prior I consider, displayed in figure 1(c), looks symmetric to prior A. It places most probability mass of the break date early in the sample, with the mean of q at a low value. This prior could be considered quite incongruous with the sample I consider. More than 99% of the draws of q would yield expected break dates in the first 15 quarters of the sample. Despite the seeming inadequacy of this prior, the log Bayes factor still predicts strong evidence in favor of a structural break, with posterior estimates close to those of the $q \sim beta(1,1)$.[79]

Finally, I increase the scatter three-fold and consider the last prior $q \sim beta(.1,3)$. This is the prior distribution graphed in figure 1(d), which places almost all probability mass on a brake very early in the sample. None of the 5,000 draws of q corresponded to $E(\tau) = \frac{1}{1-q} > 50$ quarters. Here the ratio of posterior means of the variance of e_t before and after the structural break is the farthest away from the other priors

[79] See Figure 4 and Table 3.

at $\frac{\sigma_1^2}{\sigma_0^2} = 0.36$. Figure 5(a) shows the probability of a structural change to be very flat, relative to the other priors. It finds a 30% probability of a break in volatility to occur as early as 1964 up to 1984. Figure 5(b) shows the posterior of q to be bimodal and fairly diffuse. The last row in Table 5 suggest that, under this model, there is very evidence (albeit slight) in favor of the alternative hypothesis. It takes an extreme prior, which places a probability mass implausibly early in the sample with a large prior variance, in order not to get strong evidence of a structural change.

In conclusion, it appears that the evidence for a structural break in the investment-to-consumption ratio is robust to a large family of prior distributions.

VII. Conclusion

In this subchapter, I used Bayesian tests to find evidence of a volatility reduction in the investment-to-consumption ratio. I find decisive evidence of a structural break in the variance of the ratio occurring in the third quarter of 1983. This finding is consistent with the early-to-mid 1980s volatility reduction observed in the major aggregates and sub-aggregates of U.S. economic activity. This finding has a number of implications: First, while the investment-to-consumption ratio has been growing since the late 80s, the variance of this growth seems to have decreased. This leads one to the conclusion that while, this faster growth in real investment vs. consumption cannot be permanent, it may continue to persist if fueled by stronger expenditures on durables (which could be due to lower durable prices, technological advances and an increase in global competition). Second, MPQ (00) concluded that volatility in expenditures on durable goods lowered and expenditures on non-durables did not experience such a break. While I don't promote a formal test for each series here, the strong and robust evidence in favor of a break in the ratio suggests that both series should have experienced a volatility reduction. A constant (or even increasing) volatility in non-durables (coming from the

consumption side) would surely "drown out" most evidence of a break in durables (coming from the investment side). Evidence of a break in the volatility of both expenditures on durables and non-durables is actually confirmed by KNP (04). Finally, while the ratio of real investment-to-consumption seems to be a trending series since the late 1980s, its growth rate has become less volatile since the mid 1980s. While, empirically these two phenomena may not be related and their sources are yet to be determined, any suggested specification that involves decomposing this series into a non-stationary process for the trend and a stationary process for the cycle should allow for a change in its covariance matrix.

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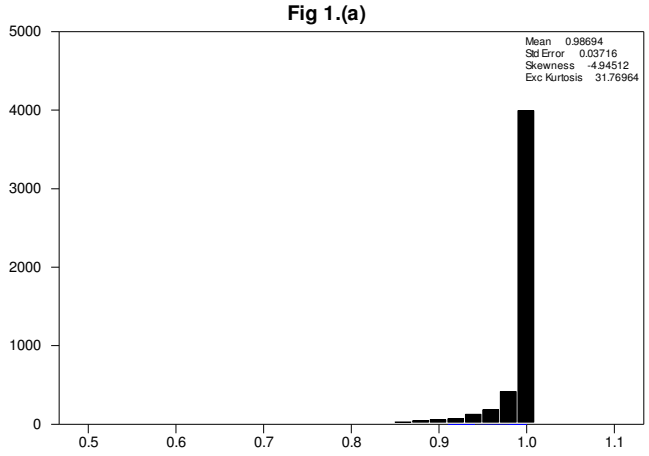
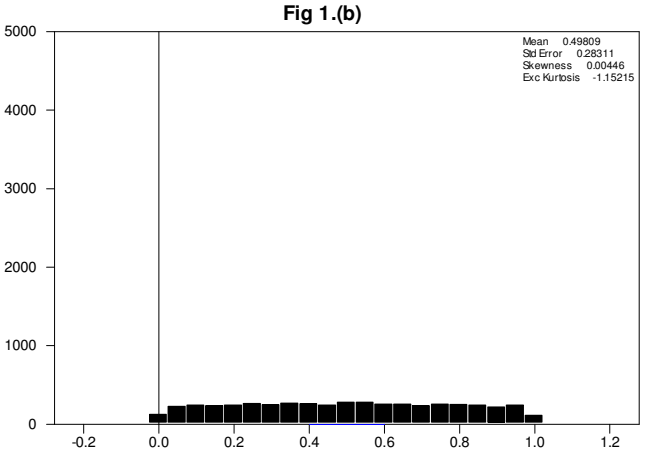
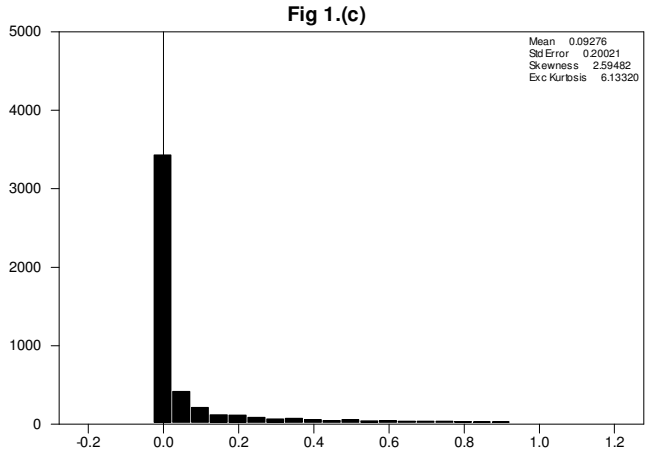






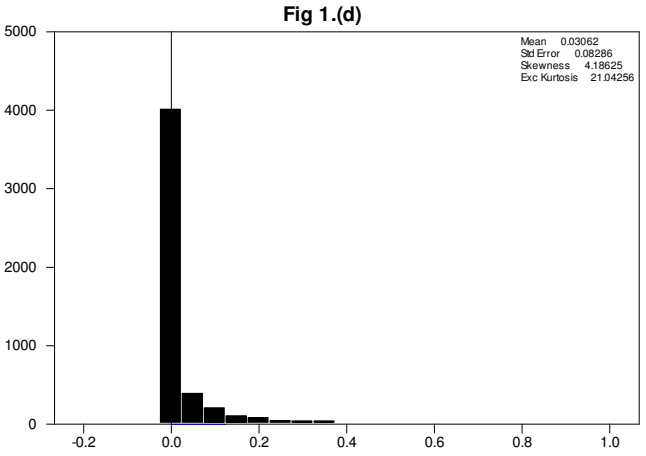


Figure 1. *Histogram for 5,000 draws from: (a)* $q \sim beta(8,0.1)$ *(b)* $q \sim beta(1,1)$ *(c)* $q \sim beta(0.1,1)$ *(d)* $q \sim beta(0.1,3)$.

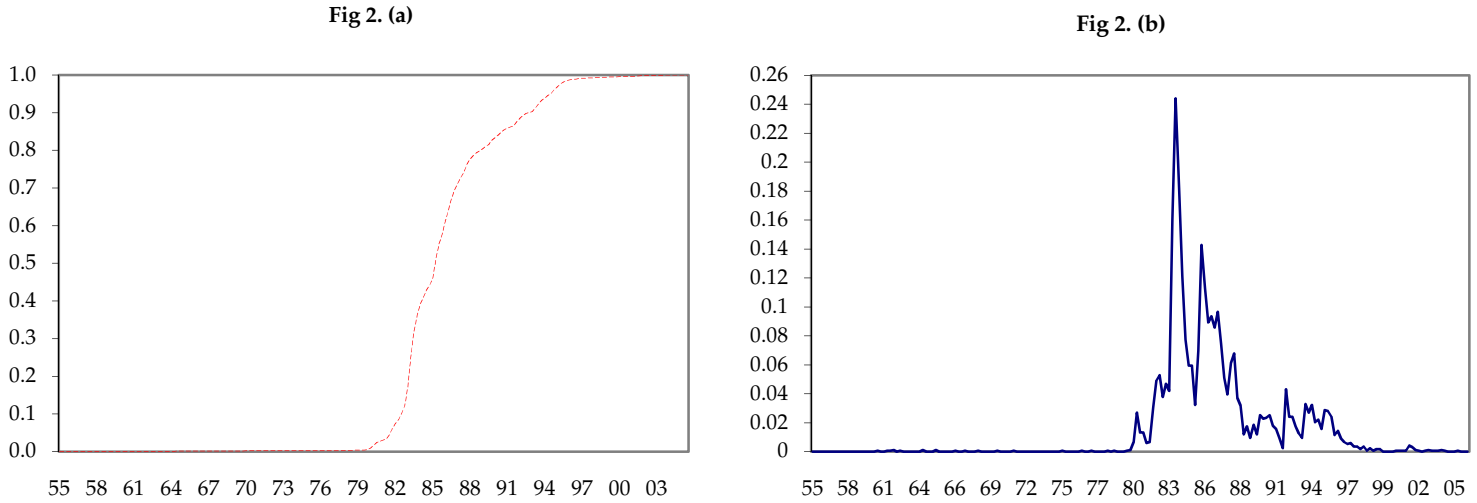


Figure 2. *Growth Rate of Investment-to-Consumption Ratio:*

(a) Probability of Structural Change (q) *(b) Posterior Distribution of Structural Break Date (τ)*

Table 1

U.S. I/C Ratio: Model 1 Posterior Estimates (1955:Q2 – 2006:Q2)			
Prmtrs.	Mean	Std. Dev	Median
ϕ_1	0.3817	0.0740	0.3812
ϕ_2	0.1388	0.0760	0.1373
σ_0^2	1.0379	0.1485	1.0229
σ_1^2	0.4386	0.0836	0.4318
Based on prior: $q \sim beta(8, 0.1)$			

Fig 3. (a)

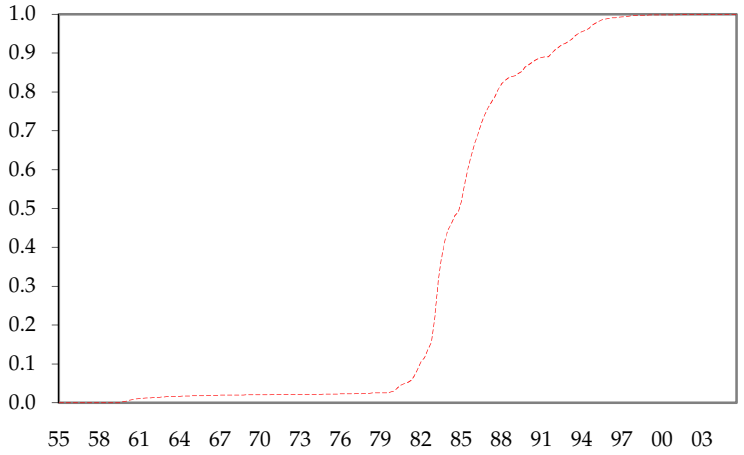


Fig 3. (b)

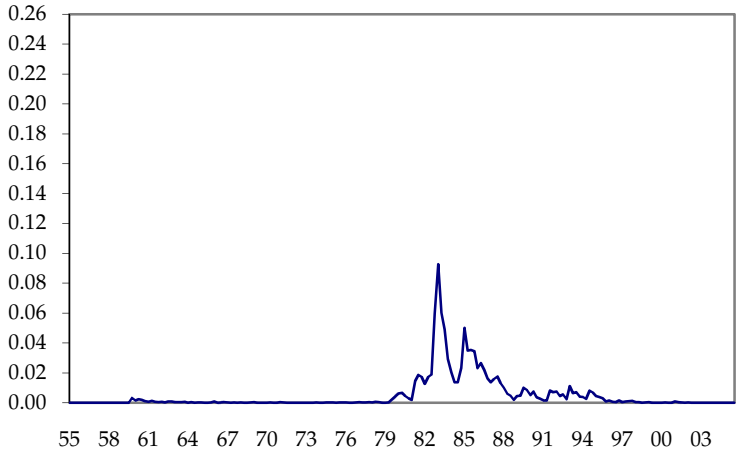


Figure 3. *Growth Rate of Investment-to-Consumption Ratio:*

(a) Probability of Structural Change (q) *(b) Posterior Distribution of Structural Break Date (τ)*

Table 2

U.S. I/C Ratio: Model 1 Posterior Estimates (1955:Q2 – 2006:Q2)			
Prmtrs.	Mean	Std. Dev	Median
ϕ_1	0.3825	0.0738	0.3806
ϕ_2	0.1378	0.0769	0.1375
σ_0^2	1.0463	0.1528	1.0320
σ_1^2	0.4520	0.0912	0.9858
Based on prior: $q \sim beta(1,1)$			

Fig 4. (a)

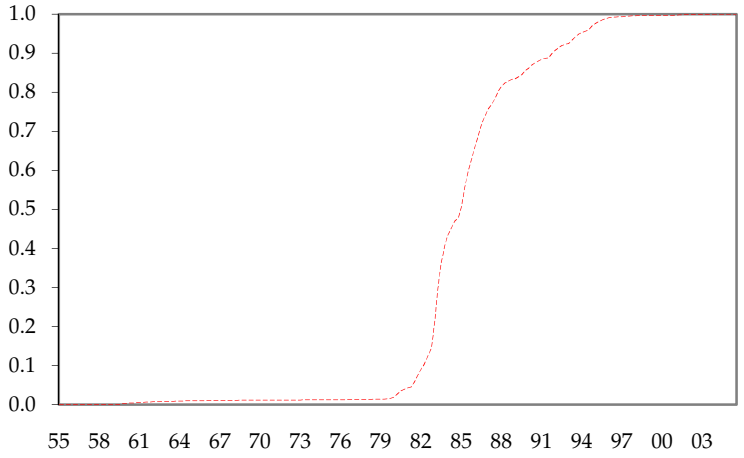


Fig 4. (b)

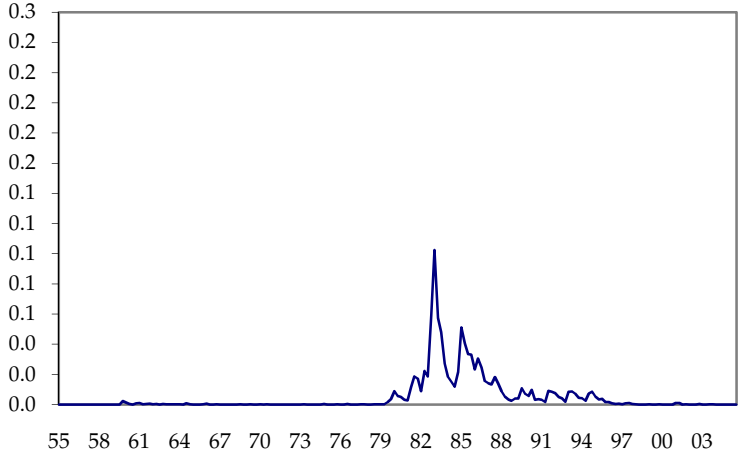


Figure 4. *Growth Rate of Investment-to-Consumption Ratio:*

(a) Probability of Structural Change (q) *(b) Posterior Distribution of Structural Break Date (τ)*

Table 3

U.S. I/C Ratio: Model 1 Posterior Estimates (1955:Q2 – 2006:Q2)			
Prmtrs.	Mean	Std. Dev	Median
ϕ_1	0.3783	0.0740	0.3744
ϕ_2	0.1385	0.0716	0.1392
σ_0^2	1.0507	0.1543	1.0332
σ_1^2	0.4491	0.0896	0.4391
Based on prior: $q \sim beta(0.1,1)$			

Fig 5. (a)

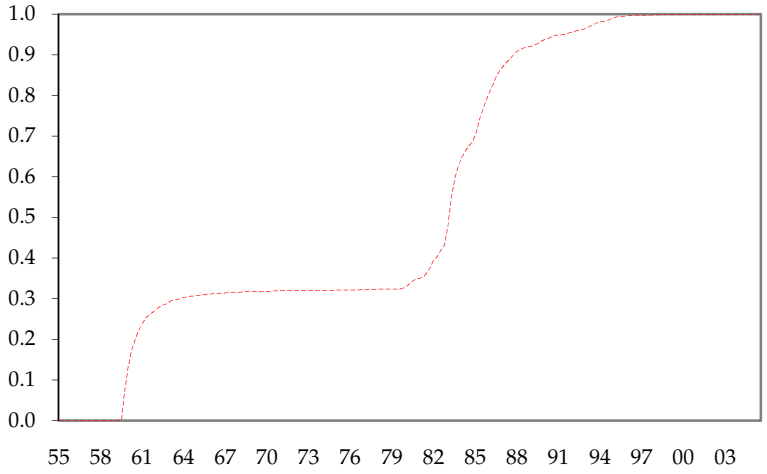


Fig 5. (b)

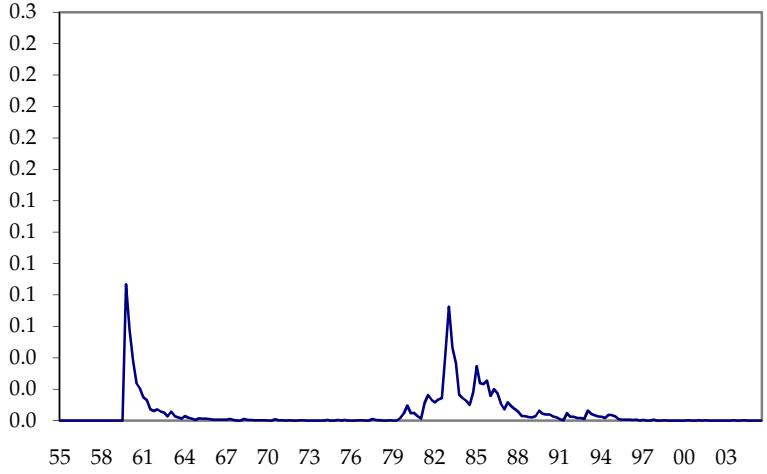


Figure 5. *Growth Rate of Investment-to-Consumption Ratio:*

(a) Probability of Structural Change (q) *(b) Posterior Distribution of Structural Break Date (τ)*

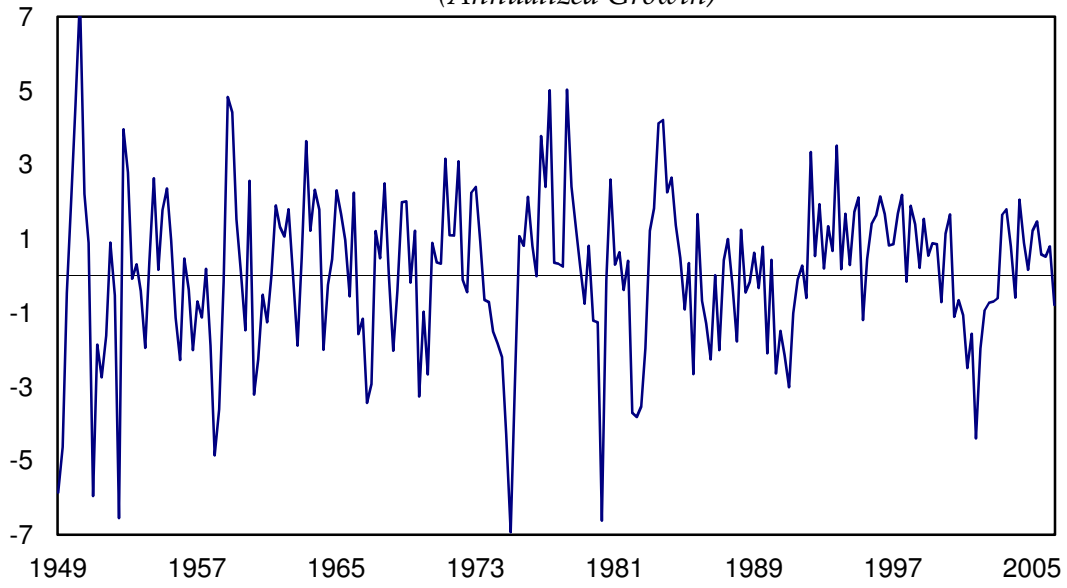
Table 4

U.S. I/C Ratio: Model 1 Posterior Estimates (1955:Q2 – 2006:Q2)			
Prmtrs.	Mean	Std. Dev	Median
ϕ_1	0.3894	0.0721	0.3909
ϕ_2	0.1260	0.0769	0.1301
σ_0^2	1.1502	0.3231	1.0741
σ_1^2	0.5425	0.1567	0.4940
Based on prior: $q \sim beta(0.1,3)$			

Table 5 Structural Break Tests
for Alternative Priors of q

Prior Distribution	Bayes Factor
q~beta(8,0.1)	6.2
q~beta(1,1)	2.9
q~beta(0.1,1)	2.4
q~beta(0.1,3)	1.0

Fig 6. U.S. Investment-Consumption Ratio
(Annualized Growth)



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DISSERTATION CONCLUSION

This dissertation consists of four essays concerning the analysis of dynamic fluctuations in various U.S. macroeconomic aggregates with particular attention to the links among real private investment, real private consumption and different aspects of government spending. These essays constitute four separate approaches that are based on different assumptions and draw inferences based on different methodological avenues. While each chapter is meant to be "stand-alone" and independent, all four center around one motivating factor: the recent evidence of differential growth rates between real consumption and real investment in the U.S.

The first two chapters of this thesis consider two *structural* models of the U.S. economy motivated by evidence of faster growth in real private investment (relative to consumption) that has prevailed since the early 1990s in the U.S. Conversely, the last two chapters explore *reduced form* models aimed at developing some stylized facts to better understand possible causes for this apparent growth rate disparity between consumption and investment.

Each chapter explores a separate alternative explanation for the difference in growths between consumption and investment. Chapter I, titled *"THE IMPACT OF FISCAL EXPANSIONS ON PRIVATE*

SPENDING: A NEW KEYNESIAN APPROACH," models a two-sector economy that produces differentiated goods that are inherent to either consumption or investment. Thus, a possible reason why investment has grown at a different rate than consumption is *different production processes* for each. Chapter II, titled "*THE IMPACT OF FISCAL EXPANSIONS ON PRIVATE SPENDING: A REAL BUSINESS CYCLE APPROACH,*" posits that different technology factor productivity shocks that separately affect consumption or investment can help explain the difference in growth rates of these aggregates. Thus, Chapter II advocates *sector-specific technology shocks* as a possible reason why investment has grown at a different rate than consumption. Chapter III, titled "*THE REAL INVESTMENT TO CONSUMPTION RATIO: INFERENCES FROM ALTERNATIVE PERMANENT-TRANSITORY DECOMPOSITIONS,*" considers fluctuations in *relative prices* as the driving cause for the upsurge in real investment growth relative to consumption. Chapter IV, titled "*A BAYESIAN APPROACH TO MODELING TOCHASTIC VOLATILITY IN THE U.S. INVESTMENT-CONSUMPTION RATIO,*" is a reduced form model aimed at complementing those reduced form models considered in Chapter III.

Chapters III and IV focus on fluctuations in the investment-consumption ratio. If investment grows at the same rate as consumption in the long run, then the ratio of the two variables should be stationary (in

the absence of permanent shocks). Results from the reduced form models in Chapters III and IV can be summarized as follows. Chapter IV finds evidence of a break in volatility in the ratio of private investment to private consumption occurring in the third quarter of 1983 consistent with what has been dubbed the great moderation in U.S. aggregate economic activity of the mid 1980s. Chapter III finds that modeling both the investment-consumption ratio and relative prices as stationary variables likely suffers from misspecification, whereas a model that considers a common trend between the two variables does better. The estimated coefficient on the trend of relative prices is 0.98 and statistically indistinguishable from 1. Parameter estimates of that model do not vary much from a similar model where this coefficient is assumed to be 1. Finally, allowing for a break in the covariance matrix of the UC model (as suggested by the findings in Chapter IV) seems to be warranted by the results. The variances of transitory components of the investment-consumption ratio as well as relative prices have reduced after the imposed break. The highlights of Chapter III and IV are: first, the investment-consumption ratio has become less volatile and less mean-reverting in recent years. Second, permanent fluctuations seem to have become more important, relative to transitory ones, in driving the investment-consumption ratio post break. Finally, there is evidence of a

stronger correlation between transitory components of prices and the investment-consumption ratio in recent years.

Chapter I establishes a micro-founded dynamic stochastic general equilibrium (DSGE) model that allows for two sectors that produce durable goods (which mostly go to investment) and nondurables and services (which mostly go to consumption) as a possible explanation for the differential growth rate in real consumption and investment in the U.S. Within this context, this model examines the impact of an increase in government spending on consumption and investment in the short-run. The model finds an increase in consumption (both of durables and nondurables) follows a positive government shock in a New Keynesian environment, i.e. if we restrict a portion of households to behave in a non-Ricardian way in the presence of sticky prices. If, however, we allow for a RBC economy, the response of durable spending implied by the model remains positive. This is contradictory to the theoretical RBC prediction. Thus, this model suggests that the response of durable spending (roughly 39% of this durable spending goes to computing equipment and software today) is fairly insensitive to the assumed behavior of the household, which is the main point of contention between the two views. A shortcoming of this model is that productivity shocks are normalized away from the theoretical construct. Additionally, results on the investment response were mixed at best.

Finally, Chapter II builds a micro-founded DSGE model that allows for sector-specific productivity shocks, adjustment cost and labor-augmenting government shocks. The model predicts a positive effect on consumption following a positive government spending increase. It also improves over the results of Chapter I by predicting a similar increase in investment after such a shock hits the economy. The unobservable driving forces seem to be well-delineated across sectors, so that consumption is estimated to be driven mostly by preferences as well as consumption-specific technology and government shocks, whereas investment-specific technology and government shocks explain most of the variance in investment. Results suggest that investment-specific technology grows faster than consumption-specific technology, which could explain the growth disparity in the aggregates. This model is not without its shortcomings: first, while technology shocks to investment grow faster than consumption, the quarterly investment is not estimated to grow faster than consumption in this model. Second, permanent-transitory decompositions of both government shocks are not very intuitive with a growth component that is hump-shaped, and fairly symmetric between both government shocks, and level components that trend slightly. Finally, this model fails to incorporate learning for households. For example, Edge, Laubach and Williams (2004) show that the growth rate of

consumption and investment-specific government shocks can have different effects in the absence of full information.

In closing, this thesis studies alternative potential causes that might account for the apparent differential rates of growth in private fixed investment and consumption since the 1990s in the U.S. While this is an unlikely permanent event, whether it is a persistent level shift or a temporary break in trends, it has lasted for roughly 17 years to date. Even in the dubious case, this faster growth in investment (relative to consumption) would reverse itself in the immediate future, any attempt at modeling the postwar U.S. economy henceforth should take account of this phenomenon.