SOLUTIONS OF LATTICE DIFFERENTIAL EQUATIONS OVER INHOMOGENEOUS MEDIA

By

MAILA BRUCAL - HALLARE

Submitted to the Department of Mathematics and the
Faculty of the Graduate School of the University of Kansas
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Committee members

PROF ERIK VAN VLECK, Chairperson

PROF WEIZHANG HUANG

PROF WEISHI LIU

PROF ATANAS STEFANOV

PROF SARAH KIEWEG

Date defended: 24 August 2012

The Dissertation Committee for MAILA BRUCAL - HALLARE certifies
that this is the approved version of the following dissertation:

SOLUTIONS OF LATTICE DIFFERENTIAL EQUATIONS OVER INHOMOGENEOUS MEDIA

Committee:

PROF ERIK VAN VLECK, Chairperson

PROF WEIZHANG HUANG

PROF WEISHI LIU

PROF ATANAS STEFANOV

PROF SARAH KIEWEG

Date approved: 24 August 2012

Abstract

We study one-dimensional spatially-discrete reaction-diffusion equations with a diffusion term that involves nearest-neighbor coupling and with a reaction-term that is a smooth-cubic nonlinearity. Specifically, we consider two nontrivial examples of lattice differential equations (LDEs) on $\mathbb{Z}$ that are related to the lattice Nagumo equation,

$$\frac{du_j}{dt} = d(u_{j-1} - 2u_j + d_{j+1}) - u_j(u_j - a)(u_j - 1), \qquad j \in \mathbb{Z}, t \in \mathbb{R},$$

where d is a positive number and $0 < a < 1$. The LDEs that we consider are used to model natural phenomena defined over an inhomogeneous medium, namely:

1. a lattice Nagumo equation with a negative diffusion coefficient. Such is still a well-posed problem in the LDE setting and has been shown to arise from a discrete model of phase transition for shape memory alloys [54]. This work shows that this antidiffusion lattice Nagumo equation has a period-2 traveling wavefront solution that is stable and unique. Utilizing the concrete expressions for the nonlinearities, we obtain criteria on the (d,a)-parameter plane that guarantee a display of bistable and monostable dynamics. Where there's bistable dynamics, we study the propagation failure phenomenon; where there's monostable dynamics, we compute a minimum wave speed for the traveling waves.

2. a lattice Nagumo equation that has a finite number of defects in the middle of $\mathbb{Z}$. This defect may occur due to deviations in the diffusive property of the medium. This work

shows that such an equation has an entire solution which behaves as two fronts coming from the both sides of $\mathbb{Z}$. A key idea for the existence proof is a characterization of the asymptotic behavior of the solutions as $t \to -\infty$ in terms of an appropriate super-solution, sub-solution pair.

Acknowledgements

I am indebted to many people in the process of doing this research work and completing my degree.

A single acknowledgement page is not sufficient to express my gratitude to the faculty and staff of the Department of Mathematics at the University of Kansas for all the support and guidance that I have received throughout these years. It has been a great privilege to spend my formative years as a mathematician in KU.

I had the wonderful opportunity to meet and/or have mathematics classes with my committee members (and I should say a short list of my favorite math teachers), Professors Liu, Huang, and Stefanov; my mentors in teaching, Professors Pasik-Duncan, Porter, Gavosto, and Sheu; my mentors in many other equally important things, Professors Stanislavova, Hukle, and Niknejad; and to my other fave math teachers with whom I wasn't able to have a formal mentor/mentee relationship, Professors Torres, Sheu, and Huneke. Additionally, I express my gratitude to my fellow math graduate students, Charles, Tim, Seungly, Branden, Xiaoming, Kabe, Xianping, and all others who I haven't yet thanked. I must also thank Gloria, Debbie, Kerrie, Sam, Erin, and Lori - for assisting me with many of my last-minute requests on various paperwork. To my students, too, thank you for listening while I sing the quadratic formula and for rolling your eyes when I deliver corny jokes that I heard somewhere else.

Most importantly, special mention should go to my advisor, Prof Erik Van Vleck for taking me in as a student; for leading me through many aspects of my mathematics career - from my

first big conference talk to my first publication; and for constantly encouraging me to finish my work - especially during those times that I just wanted to give up. Thank you, sir, thank you for all your help and patience and understanding. You have seen my best and my worst moments, but like a good parent, you have always believed that I can do it.

Outside the mathematics department, I received tremendous help from the teachers and staff of Educare at KU. Thank you for taking care of my kids during my final and most crucial year in graduate school. Each and everyone of you made an enormous impact to my kids and my kids love you dearly. You made me realize that there are people outside my small world who I can trust my kids with. Thanks to Ashley, Amy, Prof Claudia, Julie, Adam, Zach, Brian, and all the student-teachers.

Outside KU, I would like to thank the Rapada family for all their love and support to my family. Ate Jenet, you are always just a phone call away and I cannot remember a single instance when you did not accommodate my requests, whether they are big or small. You are more than a sister to me. Your family is our family here in Lawrence. Please visit us in Virginia! Thank you for everything.

And of course, I must record in this dissertation my unbounded gratitude and always increasing love for my mokong Ferdinand and my precious, beautiful creatures of God, Maxyn and Hally. I dedicate this research work, my small contribution to mathematics, to you. You are my inspiration and you bring meaning to my life. Thank you for always being there for Nanay. I am sorry for all those times that I couldn't play with you or that I shouted at you because I was too tired or I was too busy proving small lemmas in my head.

Surely, the path towards achieving my PhD dream was not without obstacles - another example of an inhomogeneous medium in nature that I studied in this work. Like the negative diffusion LDE problem, I had to re-structure my personal life in order to reach my goal; like the single-defect diffusion LDE problem, I had to manage my time and space wisely. In my dissertation, I used existing results to prove my claims; in the fulfillment of my dreams, I re-

ceived a lot of help in order to navigate my life that exists in various forms, as a PhD Student, Aspiring Mathematician, Teaching Assistant, Mommy, and Wife. Thank you, mathematics for the wonderful life that you paved for me.

Contents

Chapter 1

Introduction

Reaction-diffusion equations are used to model various problems that occur in nature, for example, in ecology, chemical kinetics, and materials science, etc. A one-dimensional reaction-diffusion equation is of the form

$$u_t = du_{xx} + f(u), \qquad x \in \mathbb{R}, t > 0, \tag{1.1}$$

where the parameter $d > 0$ is called the diffusion coefficient and the function f is the reaction-term. The unknown $u = u(x,t)$ may accordingly stand for population density of a certain biological species, concentration of a chemical reactant, or displacement of crystalline solids. Until recently, the object of investigations has been restricted mainly to the cases where the diffusion coefficient is constant. However, when we are dealing with systems which have an inhomogeneous spatial structure, we often encounter equations with a diffusion-coefficient that depends on space, like $u_t = d(x)u_{xx} + f(u)$, or $u_t = (d(x)u_x)_x + f(u)$, or in general, a reaction-diffusion-convection equation (with inhomogeneity at the reaction-term, too):

$$u_t = (d(x)u_x)_x + b(x)u_x + f(u,x), \qquad x \in \mathbb{R}, t > 0. \tag{1.2}$$

It is a fundamental problem to understand how the inhomogeneity of a medium affects the solutions of such equations. Results on the existence and non-existence of traveling waves for equations like (1.1) in periodic media, considered to be the simplest inhomogeneous media, can be found in [56] and various references therein.

Now, when the spatial medium of a natural phenomena to be modeled exhibits spatial-discreteness, lattice differential equations (LDEs) as models present an alternative path, and probably a more accurate tool of investigation. A one-dimensional LDE in $\vec{u}$ assumes the form:

$$\frac{du_j}{dt} = \sum_{k\in\mathbb{Z}} d_{j,k}u_{j+k} + f_j(u_j), \qquad j\in\mathbb{Z}, t\in\mathbb{R}, \tag{1.3}$$

where $\vec{u}(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$ for $t\in\mathbb{R}$ and each u_j is a bounded real-valued function of $t\in\mathbb{R}$. With some ellipticity condition like

$$d_{j,k} \geq 0,\ \ k\neq 0,\ \text{and}\ \ d_{j,0} = -\sum_{k\neq 0} d_{j,k} < 0,\ \ j\in\mathbb{Z}, \tag{1.4}$$

the sum $\sum_{k\in\mathbb{Z}} d_{j,k}u_{j+k}$ can be viewed as a diffusion-convection term and hence, LDEs can be considered as semi-discretizations of partial differential equations (PDEs) like (1.1) and take the form:

$$\frac{du_j}{dt} = \frac{1}{h^2}(d_{j-1/2}(u_{j-1}-u_j)+d_{j+1/2}(u_{j+1}-u_j)) + \frac{1}{2h}(b_j(u_{j+1}-u_{j-1})) + f_j(u_j), \qquad j\in\mathbb{Z}, t\in\mathbb{R},$$

where h is the mesh size, $d_{j+1/2} = d([j+1/2]h)$, $b_j = b(jh)$, and $f_j(s) = f(s, jh)$. Research on the existence of traveling wave solutions of LDEs treated as a semi-discretization system can be found in [19], [20].

If $d_{j,k}\neq 0$ for all k then the summation in (1.3) allows for infinite-range coupling. In the special case that $d_{j,k} = 0$ for $|k| > 1$, then we have the so-called nearest-neighbor coupling $d_{j,-1}u_{j-1} + d_{j,0}u_j + d_{j,1}u_{j+1}$; moreover, assuming ellipticity and homogeneity of the diffusion coefficients, such nearest-neighbor coupling gives rise to the one-dimensional discrete Laplacian, $d(u_{j-1} - 2u_j + u_{j+1})$.

Regarding the second term in (1.3), $f_j(u_j)$ is referred as the reaction or nonlinearity or source term. In the literature in both PDEs and LDEs, the following are some forms for f_j that are normally assumed (dropping the subscript j for brevity):

1. $f(u) = u(1-u)$, the monostable or KPP-Fisher nonlinearity;
2. $f(u) = u(u-a)(1-u)$, where $a \in (0,1)$, the bistable or Nagumo nonlinearity;
3. $f(u) = \begin{cases} -u, & 0 \le u \le a \\ 1-u, & a < u \le 1. \end{cases}$, where $a \in (0,1)$, the piecewise-linear or McKean caricature nonlinearity.

The McKean caricature nonlinearity is usually assumed as an approximation to the smooth cubic bistable nonlinearity, where the discontinuity at a, the detuning parameter, corresponds to the middle zero of the smooth cubic bistable nonlinearity.

Thus, with a one-dimensional discrete Laplacian and $f_j(u_j) = f(u_j) = u_j(u_j - a)(1-u_j)$ for all $j \in \mathbb{Z}$, the LDE

$$\frac{du_j}{dt} = d(u_{j-1} - 2u_j + u_{j+1}) + f(u_j), \qquad j \in \mathbb{Z}, t \in \mathbb{R} \tag{1.5}$$

is called the homogeneous lattice Nagumo equation; whereas an inhomogeneous lattice Nagumo equation has the form,

$$\frac{du_j}{dt} = d_j(u_{j-1} - u_j) + d_{j+1}(u_{j+1} - u_j) + f(u_j), \qquad j \in \mathbb{Z}, t \in \mathbb{R}, \tag{1.6}$$

where $d, d_j > 0$ for all j. The inhomogeneous lattice Fisher equation assumes a similar form, with its respective nonlinearity.

Both the homogeneous lattice Nagumo and lattice Fisher equations are examples of LDEs which have been used as direct models of natural phenomena. For example, the lattice Nagumo system has been used as a model for propagation of nerve impulses in myelinated nerve axons [39] while the lattice Fisher system has been used as a model in the spread of a population in patchy environments [35]. Many other examples of LDEs as models can be found in neurophysiology, fluid dynamics, and materials science (see [6], [21], [45], [54] and the references therein).

Traveling wavefront solutions are one of the simplest nontrivial solutions to spatially-continuous or spatially-discrete reaction-diffusion equations. The existence, uniqueness, and stability of traveling wavefronts to the lattice Nagumo problem has been solved by Zinner [57], [58] for positive diffusion coefficients d that are sufficiently large (see also [32], [35]), [31]; on the other hand, Keener [39] showed that only stationary waves exist if d is small. The existence, uniqueness, and stability of traveling wavefront solutions to the one-dimensional, lattice Fisher problem has been solved by Zinner and colleagues [32], [35] and by Chen, Fu, and Guo [14]. Note that, for the Fisher problem, it is essential that there exists a minimum wave speed c^* such that for $c \geq c^*$, there is a traveling wave solution with wave speed c. In both nonlinearities, as long as we assume a bounded initial condition, we have local existence and uniqueness results for (3.1) whether the diffusion coefficient d is positive or negative. Results on the existence and non-existence of traveling wavefront solutions in case the nonlinearity is piecewise-linear can be found in ([18], [19], [20], and the references therein).

Solutions to LDEs are sequences $\vec{u}(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$. Seeking traveling wave solutions requires solving for the wave profile $\phi : \mathbb{R} \to \mathbb{R}$ and the wave speed $c \in \mathbb{R}$ in the following ansatz:

$$u_j(t) = \phi(j - ct) \qquad \text{for every } j \in \mathbb{Z}.$$

Traveling wavefront solutions, in particular, are traveling wave solutions that have boundary conditions

$$\phi(-\infty) = \phi_-, \ \ \phi(+\infty) = \phi_+,$$

so that the traveling wave joins two equilibria $\phi_- < \phi_+$. Now, the traveling wave ansatz to the spatially discrete equation (1.5) yields a functional differential equation of mixed type (MFDE)

$$-c\phi'(\xi) = d(\phi(\xi-1) - 2\phi(\xi) + \phi(\xi+1)) + f(\phi(\xi)), \tag{1.7}$$

where $\xi = j - ct$, of which not much is known. It is called mixed-type because it has an advance $\xi + 1$ and a delayed $\xi - 1$ argument. Compared with the spatially-continuous equation like (1.1), a traveling wave ansatz of $u(x,t) = \phi(x - ct)$ yields a more manageable second-order ordinary differential equation,

$$-c\phi'(\xi) = d\phi''(\xi) + f(\phi(\xi)).$$

Theory for linear MFDEs started to develop in the late 1990s [43], [44], [45] inspired by the investigations of Rustichini [50] in the late 1980s; a general technique for proving existence of traveling waves for certain spatially discrete equations was developed in [44]; a center-manifold result was developed in [37]; and embedding approach was adapted in [51]. The first stumbling block in solving MFDEs like (1.7) is the fact that the associated initial-value problem is ill-posed [33]. Another issue is the fact that if $c \neq 0$, (1.7) is a differential-difference equation while if $c = 0$ (1.7) is a difference equation.

Observe that if one naïvely applies the traveling wave ansatz into the inhomogeneous lattice Nagumo (1.6), one obtains

$$-c\phi'(\xi) = d_j(\phi(\xi-1) - \phi(\xi)) + d_{j+1}(\phi(\xi+1) - \phi(\xi)) + f(\phi(\xi)),$$

or even if one assumes a different wave variable say, $\xi_j = j - c_j t$ due to the observation that the inhomogeneity of the medium will affect the wave speed, one obtains

$$-c_j\phi'(\xi_j) = d_j(\phi(\xi_{j-1}) - \phi(\xi_j)) + d_{j+1}(\phi(\xi_{j+1}) - \phi(\xi_j)) + f(\phi(\xi_j)), \qquad j \in \mathbb{Z}.$$

In either case, it is clear that analysis of such equations is difficult; this stems from the fact that in an inhomogeneous medium, there is no true traveling wave front solution. Due to these simple observations and the fact that LDEs also give rise to dynamics that cannot be directly observed in the spatially-continuous case (1.1), there is a need to develop theories for both homogeneous and inhomogeneous LDEs. An example of a dynamical behavior that was discovered in LDE (1.5) but is absent in (1.1) and observable in (1.2) with some periodicity assumptions, is the phenomenon on traveling waves called propagation failure. Propagation failure or pinning of the waves ([34] and references therein) happens when the wave speed becomes zero over a nontrivial interval of the detuning parameter a. From the applications point of view, with LDE used as a model of action potential propagation in the cardiac tissue, for example, one wants to obtain parameter values over which propagation failure happens.

This research is organized as follows. In Chapter 2, we discuss traveling wave solutions to LDEs (1.3) over a periodic medium and present recent results on the existence, uniqueness, and stability of such. In Chapter 3, we present some examples of LDEs over a periodic medium; in particular, we show that the antidiffusion lattice Nagumo equation (that is, equation (1.5) with $d < 0$) can be put in a period-2-medium framework. The results in Chapter 3 were published in a SIAM journal in 2011 [10]. In Chapter 4, we discuss time-global solutions to a an inhomogeneous lattice Nagumo equation (1.6) with that has a single diffusion-defect in middle and prove their existence by constructing super- and sub-solution pairs and using Comparison Principle. In Chapter 5, we present avenues for future work in this area and some immediate extensions of the results in this research.

Chapter 2

Existence, Uniqueness, and Stability of Traveling Fronts in Discrete Periodic Media

The classical results on the traveling wavefront problem to spatially continuous [2], [22], [23] or spatially discrete [39], [57], [58] reaction-diffusion equations are under the assumption that the medium over which the waves propagate is homogeneous. The simplest nonhomogeneous medium is a periodic medium. The theory presented by Chen, Guo, and Wu [16] deals with the spatially-discrete periodic media case. Unlike the homogeneous medium, the periodic case requires that one solves for N wave profiles, where N is the period of the wave.

Chen, Guo, and Wu introduced a general framework for the study of traveling wavefront solutions in spatially-discrete periodic media using comparison principles, spectrum analysis, and construction of super/sub-solutions. The existence question, in particular, was solved for both the bistable and monostable nonlinearities. If the nonlinearity is bistable, then [16] also proved the uniqueness and stability of traveling waves in periodic media; if the nonlinearity is monostable, then the uniqueness and stability of traveling waves in periodic media was studied in [27].

2.1 Assumptions

Here, we state the notation and assumptions for the spatially discrete periodic-media system. Consider a general system of spatially discrete reaction-diffusion equations for $\vec{u}(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$:

$$\dot{u}_j(t) = \sum_k d_{j,k} u_{j+k}(t) + f_j(u_j(t)), \qquad j \in \mathbb{Z}, \qquad t > 0, \tag{2.1}$$

where the coefficients $d_{j,k}$ are real numbers and have the following assumptions:

A1. *Periodic medium.* There exists positive integer N such that

$$d_{j+N,k} = d_{j,k} \quad \text{and} \quad f_{j+N}(\cdot) = f_j(\cdot) \ \text{ for all } j,k \in \mathbb{Z}.$$

A2. *Existence of ordered, periodic equilibria.* There exist $\vec{\Phi}^{\pm} = \{\phi_j^{\pm}\}_{j\in\mathbb{Z}}$ such that

$$\sum_k d_{j,k}\phi_{j+k}^{\pm} + f_j(\phi^{\pm}) = 0, \qquad \phi_{j+N}^{\pm} = \phi_j^{\pm}, \qquad \phi_j^- < \phi_j^+, \qquad j \in \mathbb{Z}.$$

After an appropriate change-of-variables, the equilibria take the form $\vec{\Phi}^- = \vec{0}$ and $\vec{\Phi}^+ = \vec{1}$.

A3. *Ellipticity.*

$$d_{j,k} > 0 \quad \text{for all } k \neq 0 \qquad \text{and} \qquad d_{j,0} = -\sum_{k\neq 0} d_{j,k} < 0 \quad j \in \mathbb{Z}.$$

A4. *Nondecoupledness.* For every integer pair $i \neq j$, there exist integers $i_0, i_1, \ldots, i_m$ such that $i_0 = i$ and $i_m = j$ with

$$\prod_{s=0}^{m-1} d_{i_s, i_{s+1}-i_s} > 0.$$

A5. *Finite-range interaction.* There exists a positive integer k_0 such that

$$d_{j,k} = 0 \text{ for } \quad |k| > k_0 \quad \text{for all } j \in \mathbb{Z}.$$

2.2 Statement of the Problem

The problem is to find traveling wavefront solutions $(c, \vec{u})$, where $\vec{u}(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$ with $t > 0$ to the system of equations (2.1) with the stated assumptions that connect two steady states $\vec{\Phi}^{\pm} = \{\phi_j^{\pm}\}_{j\in\mathbb{Z}}$ in the following sense:

$$\lim_{j\to\pm\infty}[u_j(t) - \phi_j^{\pm}] = 0 \qquad \text{for all } t > 0.$$

We also want to investigate the uniqueness and stability of such solutions. A traveling wave solution $\vec{w}(\xi) = \{w_j(\xi)\}_{j\in\mathbb{Z}}$ with $\xi \in \mathbb{R}$ is a solution which has the following ansatz

$$u_j(t) = w_j(\xi), \qquad \text{where } \xi = j - ct, \qquad \text{for all } j \in \mathbb{Z}, t \in \mathbb{R}. \tag{2.2}$$

The variable ξ is called the wave variable and the function $\vec{w}$ is called the wave profile. It can be shown that the wave profile $\vec{w}$ satisfies

$$w_{j+N}(\xi) = w_j(\xi) \qquad \text{for all } j \in \mathbb{Z}, \xi \in \mathbb{R},$$

and hence there are N functions to solve for, $\{w_1(\xi), ..., w_N(\xi)\}$, and we will assume that these N wave profiles have a common wave speed c. Indeed, for all $x \in \mathbb{R}$ and all $j \in \mathbb{Z}$,

$$w_{j+N}(\xi) = u_{j+N}(t)|_{ct=j+N-x} = u_{j+N}(\frac{1}{c}(j+N-x)) = u_j(\frac{1}{c}(j-x)) = w_j(\xi),$$

where we have used Theorem 2 (period of the profile).

The next section contains the results of the study of Chen, Guo, and Wu [16]. In particular, they proved the existence of traveling wavefronts in spatially discrete periodic media. Unlike the spatially continuous homogeneous case, finding a closed form for such solutions is not immediate (see [1] and [18] on how to construct explicit solutions in the spatially discrete homogeneous media case).

2.3 Results

In this section, we restate the theorems of Chen, Guo, and Wu [16] on the existence, uniqueness, asymptotics, and stability of a traveling wave solution in spatially discrete periodic media. (See Theorems 2, 3, 4, and 6 in [16]). The first theorem is an existence theorem that asserts that we can find a periodic traveling wavefront solution *without* requiring any stability condition on the boundary equilibria $\vec{0}$ and $\vec{1}$. This theorem will be applied to determine the bistable and monostable regions of the antidiffusion lattice Nagumo system.

The following existence theorem is theorem 6 in [16]

Theorem 1. *(see [16]) Assume that $\vec{0}, \vec{1}$ are steady-states and any other N-periodic state $\vec{\Phi} = \{\phi_j\}$ with $\phi_j \in (0,1)$, if it exists, is unstable. Then the problem (2.1) admits a solution $(c, \vec{w})$ satisfying*

$$\vec{w}(-\infty) = \vec{0} < \vec{w}(\xi) < \vec{1} = \vec{w}(+\infty) \qquad \textit{for all } \xi \in \mathbb{R}.$$

The above theorem is stated in a slightly different way; in particular, we included the fact that the other N-periodic state $\vec{\Phi} = \{\phi_j\}$ is such that $\vec{0} < \vec{\Phi} < \vec{1}$, where the ordering should be interpreted componentwise, that is, for all j, $0 < \phi_j < 1$. Of course, this change is consistent with the proofs and arguments in [16].

The next theorem (Theorems 2, 3, and 4 in [16]) is a result on the uniqueness, asymptotics, and stability of a traveling wave solution *when* the two steady-states $\vec{0}, \vec{1}$ are both stable.

Theorem 2. *(see [16]) Assume that $\vec{0}$ and $\vec{1}$ are stable steady-states. Suppose that $(c, \vec{u})$ with $\vec{u}(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$ is a traveling wave with $c \neq 0$ which connects $\vec{0}$ and $\vec{1}$.*

1. ***Exponential tail***. *There exist positive constants h^-, h^+ such that*

$$\lim_{j-ct\to-\infty} \frac{u_j(t)}{\psi_j^0 e^{(j-ct)\Lambda^0}} = h^-, \quad \lim_{j-ct\to+\infty} \frac{1-u_j(t)}{\psi_j^1 e^{(j-ct)\Lambda^1}} = h^+,$$

   where $(\{\psi_j^0\}_{j\in\mathbb{Z}}, \Lambda^0)$ and $(\{\psi_j^1\}_{j\in\mathbb{Z}}, \Lambda^1)$ with $\Lambda^1 < 0 < \Lambda^0$ are the eigenvector-eigenvalue pairs of a corresponding eigenvalue problem.

2. ***Uniqueness.*** *If $(\tilde{c}, \tilde{u})$ is another traveling wave solution, then $c = \tilde{c}$ and there exists $\tau > 0$ such that $\tilde{u}(t) = \vec{u}(t+\tau)$ for all $t \in \mathbb{R}$.*

3. ***Monotonicity in*** t. *For all $j \in \mathbb{Z}$,*

   (a) if $c > 0$, then $\frac{d}{dt}u_j < 0$; and

   (b) if $c < 0$, then $\frac{d}{dt}u_j > 0$.

4. ***Period of the profile.*** *The period of $\vec{u}$ is $\frac{N}{|c|}$, that is,*

$$u_j(t + \frac{N}{c}) = u_{j-N}(t) \quad \text{for all } t \in \mathbb{R}, j \in \mathbb{Z}.$$

5. ***Exponential stability.*** *$\vec{u}$ is globally exponentially stable; that is, if the left tail (near $j = -\infty$) of an initial datum $\vec{\tilde{u}}(0)$ is in the basin of attraction of $\vec{0}$ and the right tail (near $j = +\infty$) is in the basin of attraction of $\vec{1}$, then there are constants K, τ^* such that*

$$\|\vec{\tilde{u}}(t) - \vec{u}(t+\tau^*)\|_\infty \leq Ke^{-\nu t}, \qquad t \geq 0,$$

where ν *is a constant depending only on* $\{d_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{Z}}$ *and* $\{f_j\}_{j\in\mathbb{Z}}$.

Chapter 3

Antidiffusion Lattice Nagumo problem

3.1 A model for phase transition

We consider an antidiffusion lattice model that has been proposed as a simple model for shape memory alloys in which there are twinning microstructures, which arise from the phase transition of the material from the martensite phase to the austenite phase. In studying this problem, LDEs have been proposed because of the importance of the discreteness of the medium. We consider traveling wavefront solutions to an antidiffusion lattice Nagumo equation:

$$\dot{u}_n = d(u_{n+1} - 2u_n + u_{n-1}) - f_a(u_n), \;\; n \in \mathbb{Z}, \tag{3.1}$$

where $d < 0$, the nonlinearity is the smooth cubic function $f_a(u) = u(u-a)(u-1)$, and $a \in \mathbb{R}$. Although much is known about traveling wave solutions to both lattice Nagumo and lattice Fisher systems (with positive diffusion), comparatively little is known about the antidiffusion problem.

This chapter shows how to establish the existence, uniqueness, stability, and nonexistence of traveling wavefront solutions to (3.1). The method used is to transform the antidiffusion lattice Nagumo problem to a system of LDEs with periodic positive diffusion and periodic nonlinearity terms. Recent results on traveling waves for bistable and monostable LDEs in periodic media were used. In essence, the problem has been transformed to a system in which the connecting orbits are between spatially 2-periodic solutions of the original negative diffusion problem and uncover a rich structure of traveling wavefront solutions. Bistable and monostable dynamics can be observed from the antidiffusion lattice Nagumo problem by varying the values of the parameters (d,a). In the bistable region, we also study the phenomenon of propagation failure while in the monostable region, we study the minimum wave speed. The results in this chapter provide a concrete example of traveling wavefront solutions to a spatially discrete reaction-diffusion system in periodic media.

The antidiffusion lattice Nagumo model cannot arise from an approximation of PDEs; that is, it is not a semi-disretization of a certain PDE. It comes from a prototypical discrete model of phase transitions that was considered in [54]. This model consists of a chain of particles, each interacting with its nearest and next-to-nearest neighbors in which the long-range interaction between next-to-nearest neighbors is assumed to be harmonic, while the nearest-neighbor interactions are nonlinear and bistable. In the overdamped limit, after a suitable rescaling, a one-dimensional, spatially discrete reaction-diffusion equation with a negative diffusion coefficient and a bistable nonlinearity was obtained. So-called antidiffusion problems in two space dimensions were previously considered in [11], and the rewriting in terms of even and odd lattice sites that will be employed here was mentioned in [12] (see also [17]). Recently, Hupkes and Van Vleck studied the anti-diffusion problem in higher dimensions [38].

The antidiffusion lattice Nagumo problem can be stated as a periodic heterogeneous media system with periodic positive diffusion coefficients and periodic nonlinearities. The technique used is an appropriate choice of change-of-variables; see Section 3.2. It will be shown that unlike the positive diffusion lattice Nagumo traveling wavefront problem, co-existence of various bistable and monostable dynamical behavior can be observed.

It is not obvious that (3.1) can be converted into a 2-periodic system. Section 3.2 explains the appropriate choice of change-of-variables that is needed for this conversion. Hence, the traveling wavefront problem for the antidiffusion case turns out to be a search for a 2-periodic heteroclinic orbit. To introduce the notion of the period of a traveling wave, we need to look at a slightly different, but equivalent, way of defining a traveling wave solution. A traveling wave solution to a general spatially discrete equation with finite-range interaction

$$\begin{aligned} \dot{u}_n(t) &= \textstyle\sum_k a_{n,k} u_{n+k}(t) + f_n(u_n(t)), \;\; n \in \mathbb{Z}, \\ a_{n,k} &= 0, \;\; |k| > k_0, \qquad \text{for some integer } k_0 \geq 1 \end{aligned}$$

is a solution $\vec{u}(t) = \{u_n(t)\}_{n\in\mathbb{Z}}$ such that for some constants $c \in \mathbb{R}$ and $T > 0$ with $cT \in \mathbb{Z}$

$$u_n(T) = u_{n-cT}(0) \qquad \text{for all } n \in \mathbb{Z}.$$

For such a solution, the number c is called the wave speed and the smallest among the numbers T is called the period. In [16], it was proved that solving for the traveling waves of a spatially discrete N-periodic media problem requires solving for N wave profiles.

This chapter is organized as follows. In Section 3.2, we apply a change-of-variable to (3.1), analyze the equilibrium states, and rescale the system so that it is in the framework of [16]. It will be seen that unlike the positive diffusion problem, there may be more than one possible connecting orbit for the antidiffusion lattice Nagumo problem. In Section 3.3 we study the existence, uniqueness, and stability of the traveling wavefront solutions to the antidiffusion lattice Nagumo problem, present the different regions in the plane where there are bistable and monostable dynamics, and look into two special case studies. In Section 3.4, we look at some examples of traveling wavefront problems. With respect to its bistable region, we present some analysis on the phenomena of propagation failure. With respect to its monostable region, we present some preliminary computations for the minimum wave speed of wave solutions that is essential in establishing the existence of solutions. We also present an example showing that there are values for parameters d, a where bistable and monostable connections co-exist.

3.2 Preliminary Computations

In this section, we will show how to transform the antidiffusion problem (3.1) into a system of spatially discrete reaction-diffusion equations with periodic diffusion coefficients and nonlinearities. The first step (Section 3.2.1) is to relabel the even and odd nodes of (3.1) to obtain a system of two equations that will describe the behavior at the even nodes and the odd nodes, labeling the solution as a two-component vector $(\vec{x}, \vec{y}) = (\{x_j\}, \{y_j\})$ for $j \in \mathbb{Z}$; that is, each component of this 2-vector solution is a sequence of functions of t. Then in Section 3.2.2 we compute the equilibria of this system and observe that unlike the positive diffusion problem, the resulting system will have more than three equilibria (see Lemma 2). These equilibria are the candidates for the boundary conditions $(x_\pm, y_\pm)$ of the traveling wave solution; that is, the traveling wave solution is a heteroclinic orbit connecting (x_-, y_-) to (x_+, y_+), where $x_\pm = \lim_{j\to\pm\infty} x_j$ and $y_\pm = \lim_{j\to\pm\infty} y_j$.

The second step (see Section 3.2.3) is to rescale the system so that (x_-, y_-) is mapped to $\vec{0} = (0,0)$ and (x_+, y_+) is mapped to $\vec{1} = (1,1)$. The result is a system of two equations

$$\begin{cases} \dot{v}_j &= d_e(w_j - 2v_j + w_{j-1}) - f_e(v_j), \\ \dot{w}_j &= d_o(v_{j+1} - 2w_j + v_j) - f_o(w_j), \end{cases} \qquad j \in \mathbb{Z},$$

with solution $(\vec{v}, \vec{w}) = (\{v_j\}, \{w_j\})$ for $j \in \mathbb{Z}$, diffusion coefficient $\vec{d} = (d_e, d_o)$ and nonlinearity $\vec{f} = (f_e, f_o)$, where subscripts e, o stand for the even and odd nodes, respectively. The diffusion coefficients d_e, d_o are in terms of d, a, and we are interested in the case where d_e, d_o, this time, are both positive. An alternative to the change-of-variables outlined here is to apply the transformation $t \mapsto -t$ to the antidiffusion lattice Nagumo problem (3.1) so that the transformed problem has a positive discrete diffusion coefficient but the nonlinearity is no longer of bistable type (a bi-unstable nonlinearity).

3.2.1 Even and Odd Nodes

Let us look at the even and odd lattice nodes of the antidiffusion lattice Nagumo problem

$$\dot{u}_n(t) = d(u_{n+1} - 2u_n + u_{n-1}) - f_a(u_n), \qquad n \in \mathbb{Z}, t \in \mathbb{R} \tag{3.2}$$

with $d < 0$ and cubic nonlinearity $f_a(u) = u(u-a)(u-1)$ with $a \in \mathbb{R}$. For completeness, it is assumed that $a \in \mathbb{R}$ and as of this writing, the author is not aware of any physical rationale for considering $a \in \mathbb{R}$ (see Example 1, Section 3.3.2 for a different dynamical behavior when $a < 0$). The positive diffusion problem often assumes that $0 < a < 1$.

Writing the even and odd nodes of (3.2) as $\vec{x} = \{x_j\}_{j\in\mathbb{Z}}$ and $\vec{y} = \{y_j\}_{j\in\mathbb{Z}}$, respectively, we get

$$\begin{cases} \dot{x}_j &= d(y_j - 2x_j + y_{j-1}) - f_a(x_j), \\ \dot{y}_j &= d(x_{j+1} - 2y_j + x_j) - f_a(y_j), \end{cases} \quad j \in \mathbb{Z}. \tag{3.3}$$

This simple change-of-variables allows us to look at a corresponding two-dimensional system; however, at this point, we cannot perform further analysis because the diffusion-coupling parameter d is negative. What we have done so far is to relabel $\vec{u}$ as

$$\begin{aligned} \vec{u}(t) &= \{\ldots, u_{-2}(t), u_{-1}(t), u_0(t), u_1(t), u_2(t), u_3(t), \ldots\} \\ &= \{\ldots, x_{-1}(t), y_{-1}(t), x_0(t), y_0(t), x_1(t), y_1(t), \ldots\}. \end{aligned}$$

At this point, notice that we have only one diffusion parameter d; in Section 3.2.3, after an appropriate change-of-variables, we will have two diffusion parameters that will be periodic and that we will require to both be positive.

3.2.2 Equilibrium Analysis

We need to compute the equilibria that will be used as boundary conditions of the traveling wavefront solutions to (3.3); that is, our goal in this sub-section is to compute $(x_\pm, y_\pm)$ defined by

$$\lim_{j\to-\infty}(x_j, y_j) = (x_-, y_-), \quad \lim_{j\to+\infty}(x_j, y_j) = (x_+, y_+).$$

Note that $(x_\pm, y_\pm)$ are in terms of the parameters d, a. The equilibria of (3.3) satisfy

$$d(y_j - 2x_j + y_{j-1}) - f_a(x_j) = 0 = d(x_{j+1} - 2y_j + x_j) - f_a(y_j), \qquad j \in \mathbb{Z},$$

and hence, as $j \to \pm\infty$, we obtain $y_\pm = x_\pm + \dfrac{f_a(x_\pm)}{2d}$ and $f_a(x_\pm) + f_a(y_\pm) = 0$. In the subsequent discussion, we drop the subscripts $\pm$. Hence, consider the two equations

$$y = x + \frac{f_a(x)}{2d}, \qquad f_a(x) + f_a(y) = 0, \tag{3.4}$$

the solutions of which are the candidates for $(x_\pm, y_\pm)$. From these two equations, define the ninth-degree polynomial

$$g(x) = f_a(x) + f_a\left(x + \frac{f_a(x)}{2d}\right). \tag{3.5}$$

The equilibria of (3.3) are zeroes of the polynomial g. Computing a closed-form expression of the zeroes of this ninth-degree polynomial in terms of the parameters d and a is not immediate. By applying some simple root analysis, we can characterize the zeroes of g, and hence the equilibria of (3.3), completely. These results are contained in Lemma 2.

Lemma 1. *Define the curves on the* (d,a)*-plane:*

$$L_0 = \{a + 4d = 0\}, \quad L_1 = \{1 - a + 4d = 0\}, \quad L_a = \{a(a-1) + 4d = 0\}.$$

Then for $a \neq 0, 1$*, the following hold:*

1. *If* $(d,a) \in L_0$*, then* 0 *is a root of* g *of multiplicity* 3.
2. *If* $(d,a) \in L_1$*, then* 1 *is a root of* g *of multiplicity* 3.
3. *If* $(d,a) \in L_a$*, then* a *is a root of* g *of multiplicity* 3.

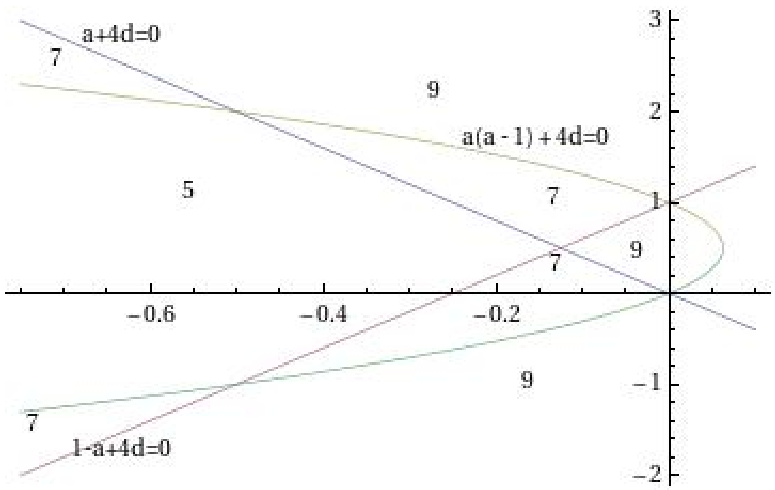
Proof. The lemma follows from a direct computation of $g'(x), g''(x), g'''(x)$ at $x = 0, a, 1$, whose values are tabulated below:

x	$2d\, g'(x)$	$2d^2\, g''(x)$	$4d^3\, g'''(x)$
0	$a(a+4d)$	$-(1+a)(a+2d)(a+4d)$	$n_0(a,d)$
1	$(1-a)(1-a+4d)$	$(2-a)(1-a+2d)(1-a+4d)$	$n_1(a,d)$
a	$a(a-1)(a(a-1)+4d)$	$(2a-1)(a(a-1)+2d)(a(a-1)+4d)$	$n_a(a,d)$

Here, n_0, n_1, n_a are polynomials in d and a that are never zero whenever $(d,a) \in L_0, L_1, L_a$, respectively. □

The next lemma is our main result on the number of distinct real zeroes of the ninth-degree polynomial g (see Figure 3.1). This lemma, in particular, claims that there are equilibria of (3.3) that cross 0 and 1 and in some parameter values, there are equilibria in between any two of the three zeroes of the cubic nonlinearity f_a. The proof uses a simple analysis of the zeroes of g as (d,a) varies on the left-half plane.

Figure 3.1: This shows how the number of distinct real zeroes changes in the (d,a)-plane.



The three curves L_0, L_1, L_a of Lemma 1 divide the left half of the (d,a)-plane into seven subregions (see Figure 3.1), namely

$$
\begin{aligned}
I &= \{1-a+4d>0,\ a+4d>0\},\\
II_{in} &= \{1-a+4d<0,\ a+4d>0,\ a(a-1)+4d<0\},\\
II_{out} &= \{1-a+4d<0,\ a+4d>0,\ a(a-1)+4d>0\},\\
III_{in} &= \{1-a+4d<0,\ a+4d<0,\ a(a-1)+4d<0\},\\
III_{out} &= \{1-a+4d<0,\ a+4d<0,\ a(a-1)+4d>0\},\\
IV_{in} &= \{1-a+4d>0,\ a+4d<0,\ a(a-1)+4d<0\},\\
IV_{out} &= \{1-a+4d>0,\ a+4d<0,\ a(a-1)+4d>0\},
\end{aligned}
$$

where the subscripts in, out serve to indicate that (d,a) is inside or outside the parabola L_a, respectively. The partitioning gives a picture that is symmetric with respect to the line $y = 1/2$. The number of zeroes of g will change as the curves L_0, L_a, L_1 are crossed. Figure 3.1 summarizes the results of the following lemma

Lemma 2. *For $a \neq 0,1$ and $d < 0$, g has at least five distinct real zeroes and there exist $\tau_1 > 1, \tau_2 < 0$ such that $g(\tau_i) = 0$. Specifically, the following hold:*

1. *if $(d,a) \in I \cup II_{out} \cup IV_{out}$ then g has nine distinct real zeroes;*

2. *if $(d,a) \in II_{in} \cup III_{out} \cup IV_{in}$ then g has seven distinct real zeroes; and*

3. *if $(d,a) \in III_{in}$ then g has five distinct real zeroes.*

Proof. From the definition of g (3.5), we can write $g(x) = f_a(x)H(x)$, where H is a sixth-degree polynomial with leading coefficient $1/d^3$. We note that

$$H(0) = \frac{a+4d}{2d}, \;\; H(1) = \frac{1-a+4d}{2d}, \;\; H(a) = \frac{a(a-1)+4d}{2d}, \;\; H(\pm\infty) < 0. \tag{3.6}$$

The main components of the proof of this lemma are shown for region I. To prove the result for the other regions, one applies similar arguments.

Region I. In this case, we have $0 < a < 1$ and $a(a-1)+4d < 0$. We claim that g has nine distinct real zeroes; that is, H has six distinct real zeroes, and we will show that two of these are less than 0, two are greater than 1, and the other two are located one each in the intervals $(0,a), (a,1)$.

From (3.6), since both $H(0)H(a)$ and $H(a)H(1)$ are negative, the intermediate value theorem says that there exist $0^+ \in (0,a)$, $1^- \in (a,1)$ that are zeroes of H. By construction of the polynomial g (see the first equation in (3.4)), 0^+ gives rise to another zero of H, $0^- := 0^+ + \frac{f_a(0^+)}{2d}$. Note that $0^- \notin \{0,a,1,0^+,1^-\}$. Moreover, using the second equation in (3.4), it follows that 0^- either is negative or is in the interval $(a,1)$ because $0 < a < 1$. Since $0^+ \in (0,a)$, it follows that $0^- < a$ and hence $0^- < 0$.

Similarly, 1^- defines another zero of H, $1^+ := 1^- + \dfrac{f_a(1^-)}{2d} \notin \{0,a,1,0^+,0^-,1^+\}$. Also, $1^+ > 1$. Now, we have obtained seven zeroes of g, ordered as follows: $0^- < 0 < 0^+ < a < 1^- < 1 < 1^+$. Note that each of $0^\pm, 1^\pm$ has multiplicity $n = 1$; otherwise, if 0^+ had multiplicity

$n > 1$, then $n = 2$ and 0^- would have multiplicity 2, too. Since $g'(0) < 0$ and $g'(a) < 0$, an even multiplicity for 0^+ would give rise to another zero for H in $(0,a)$. However, there is no room for another zero by the fundamental theorem of algebra. A similar argument shows that 1^+ has multiplicity 1. The fundamental theorem of algebra also guarantees that there can be no other zero within the intervals $(0^-,0)$ and $(1,1^+)$.

Since 0^- has multiplicity 1 and $g'(0) < 0 < g(-\infty)$, there exists a zero $\tau_2 < 0$ of H, distinct from the other four zeroes of H. By construction, a sixth zero is $\tau_1 := \tau_2 + \dfrac{f_a(\tau_2)}{2d} \notin \{0, a, 1, 0^\pm, 1^\pm, \tau_2\}$. Thus, we have found all nine zeroes of g whenever $(d,a) \in I$, namely,

$$\tau_2 < 0^- < 0 < 0^+ < a < 1^- < 1 < 1^+ < \tau_1. \tag{3.7}$$

The zeroes $0^-, 0^+$ coalesce when the boundary curve L_0 of region I is crossed, while the zeroes $1^-, 1^+$ coalesce when the boundary curve L_1 is crossed.

The following table is a summary of the subregions and the zeroes of g:

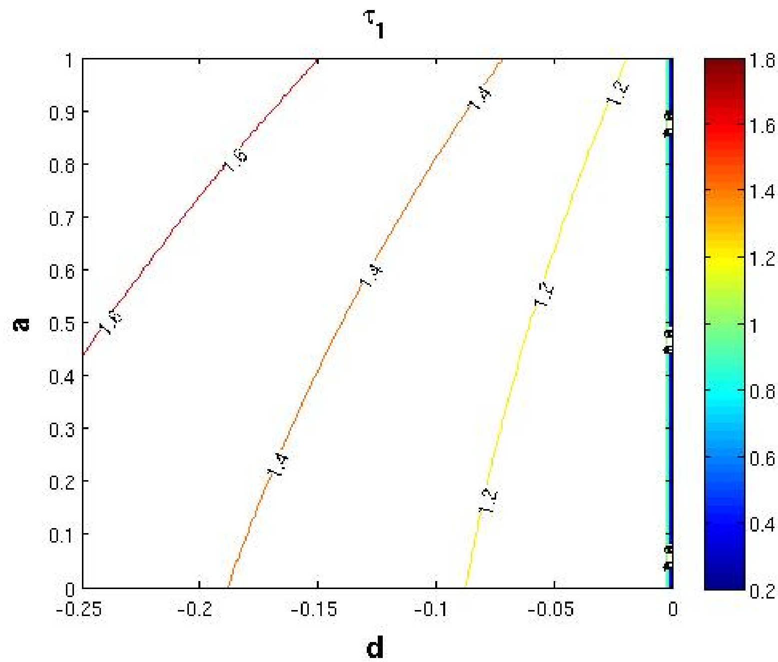
Region	Range of a	Zeroes of g
I	$0 < a < 1$	$\tau_2 < 0^- < 0 < 0^+ < a < 1^- < 1 < 1^+ < \tau_1$
II_{in}	$1/2 < a < 1$	$\tau_2 < 0^- < 0 < 0^+ < a < 1 < \tau_1$
	$a > 1$	$\tau_2 < 0^- < 0 < 0^+ < 1 < a < \tau_1$
II_{out}	$a > 1$	$\tau_2 < 0^- < 0 < 0^+ < 1 < a^- < a < a^+ < \tau_1$
III_{in}	$a < 0$	$\tau_2 < a < 0 < 1 < \tau_1$
	$0 < a < 1$	$\tau_2 < 0 < a < 1 < \tau_1$
	$a > 1$	$\tau_2 < 0 < 1 < a < \tau_1$
III_{out}	$a > 1$	$\tau_2 < 0 < 1 < a^- < a < a^+ < \tau_1$
	$a < 0$	$\tau_2 < a^- < a < a^+ < 0 < 1 < \tau_1$
IV_{in}	$0 < a < 1$	$\tau_2 < 0 < a < 1^- < 1 < 1^+ < \tau_1$
	$a < 0$	$\tau_2 < a < 0 < 1^- < 1 < 1^+ < \tau_1$
IV_{out}	$a < 0$	$\tau_2 < a^- < a < a^+ < 0 < 1^- < 1 < 1^+ < \tau_1$

□

Observe that we gain or lose two roots every time we cross one of the three bifurcation curves L_0, L_1, L_a. There is a saddle-node bifurcation on these three curves because zeroes are either created or annihilated when these curves are crossed. As proved in Lemma 2 and pictured in Figure 3.1, depending on a (with $d < 0$), there are three cases when g has nine distinct real zeroes. For example, if $0 < a < 1$, then g has nine distinct real zeroes whenever $(d,a) \in I$: $\tau_2 < 0^- < 0 < 0^+ < a < 1^- < 1 < 1^+ < \tau_1$. The superscripts $\pm$ indicate that the pairs $0^\pm, a^\pm, 1^\pm$ are created, respectively, from the saddle-node bifurcation at the curve L_0, L_a, L_1.

Unfortunately, we do not have closed-form expressions for the roots $\tau_i, 0^{\pm}, 1^{\pm}, a^{\pm}$ in terms of d and a. Contour plots of τ_1, τ_2 for $(d,a) \subseteq (-1/4,0) \times (0,1)$ are in Figures 3.2 and 3.3, respectively.

Figure 3.2: This is the contour plot of τ_1 over the region $(d,a) \subseteq (-1/4,0) \times (0,1)$. Observe that as d increases to 0, τ_1 decreases to 1.



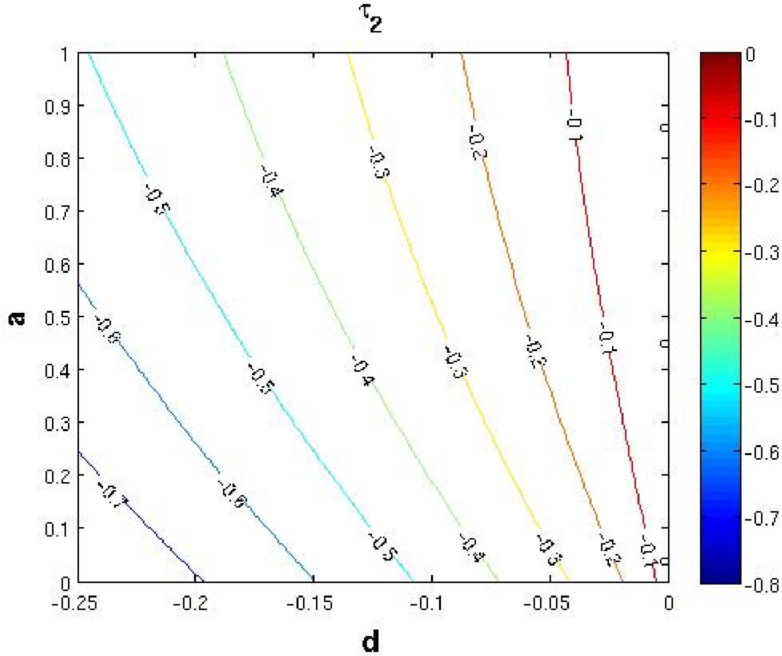
Finally, for subsequent reference, define the set

$$\hat{E} = \left\{ (x,y) \in \mathbb{R}^2 : y = x + \frac{f_a(x)}{2d}, f_a(x) + f_a(y) = 0 \right\}.$$

We shall call a point $(x,y) \in \hat{E}$ an equilibrium of (3.3). The points $(x_{\pm}, y_{\pm}) \in \hat{E}$ are the boundary conditions of the solution $(\vec{x}, \vec{y})$. By Lemma 2, the set $\hat{E}$ of equilibria has either five, seven, or nine ordered pairs, depending on the parameters (d,a) (see Figure 3.1), where $a \neq 0, 1$. In the special case that $\hat{E}$ has nine points, for example when $0 < a < 1$, the elements of $\hat{E}$ are

$$\hat{E} = \left\{ (0,0), (a,a), (1,1), (\tau_1, \tau_2), (\tau_2, \tau_1), (0^-, 0^+), (0^+, 0^-), (1^-, 1^+), (1^+, 1^-) \right\},$$

Figure 3.3: This is the contour plot of τ_2 over the region $(d,a) \subseteq (-1/4,0) \times (0,1)$. Observe that as d increases to 0, τ_2 increases to 0.



3.2.3 Rescaling the System

We wish to further transform the system (3.3) to a form in which $(x_-,y_-) \in \hat{E}$ is mapped to $(0,0)$ and $(x_+,y_+) \in \hat{E}$ is mapped to $(1,1)$. Since we are looking for heteroclinic connections, consider two distinct pairs (x_-,y_-) and (x_+,y_+), i.e., $x_- \neq x_+$ and $y_- \neq y_+$, taken from $\hat{E}$. We employ

$$v_j = \alpha^{-1}(x_j - x_-), \qquad w_j = \beta^{-1}(y_j - y_-) \tag{3.8}$$

for the even and odd lattice sites, respectively, where

$$\alpha := x_+ - x_-, \qquad \beta := y_+ - y_-. \tag{3.9}$$

Then upon substituting into (3.3) we obtain

$$\begin{cases} \dot{v}_j &= d_e(w_j - 2v_j + w_{j-1}) - f_e(v_j), \\ \dot{w}_j &= d_o(v_{j+1} - 2w_j + v_j) - f_o(w_j), \end{cases} \qquad j \in \mathbb{Z}, \tag{3.10}$$

where

$$d_e = \frac{d\beta}{\alpha}, \qquad d_o = \frac{d\alpha}{\beta} \tag{3.11}$$

and the reaction term-nonlinearities are

$$\begin{aligned} f_e(v_j) &:= \tfrac{1}{\alpha}\{f_a(\alpha v_j + x_-) + 2d[(\alpha-\beta)v_j + (x_- - y_-)]\}, \\ f_o(w_j) &:= \tfrac{1}{\beta}\{f_a(\beta w_j + y_-) - 2d[(\alpha-\beta)w_j - (x_- - y_-)]\}. \end{aligned} \tag{3.12}$$

In the above notation, the subscripts e, o of the diffusion parameter d and the nonlinearity f are used to denote even and odd nodes; that is, d_e, d_o are the diffusion parameters for the even, odd sites respectively, while f_e, f_o are the nonlinearity-reaction terms for the even, odd sites respectively. In fact, we can simplify these two nonlinearities to obtain

$$f_e(v_j) = \alpha^2 f_{a_e}(v_j), \qquad f_o(w_j) = \beta^2 f_{a_o}(w_j), \tag{3.13}$$

where

$$a_e = \frac{-f_a''(x_-)}{2\alpha} - 1, \qquad a_o = \frac{-f_a''(y_-)}{2\beta} - 1, \tag{3.14}$$

and we use the notation $f_a(u) = u(u-a)(u-1)$. This can be seen by applying the equilibria conditions (3.4) in the following way for the even nonlinearity:

$$
\begin{aligned}
f_e(v_j) &= \frac{1}{\alpha}\{f_a(\alpha v_j + x_-) + 2d(\alpha-\beta)v_j + 2d(x_- - y_-)\} \\
&= \frac{1}{\alpha}\{f_a(x_-) + f_a'(x_-)\alpha v_j + f_a''(x_-)\frac{(\alpha v_j)^2}{2} + (\alpha v_j)^3 + 2d(\alpha-\beta)v_j + 2d(x_- - y_-)\} \\
&= \frac{1}{\alpha}\{f_a'(x_-)\alpha v_j + f_a''(x_-)\frac{(\alpha v_j)^2}{2} + (\alpha v_j)^3 + 2d(\alpha-\beta)v_j\} \\
&= \alpha^2(v_j^3 + \frac{f_a''(x_-)}{2\alpha}v_j^2 + \frac{f_a'(x_-)\alpha + 2d(\alpha-\beta)}{\alpha^3}v_j) \\
&= \alpha^2 f_{a_e}(v_j),
\end{aligned}
$$

where

$$
a_e = \frac{-f_a''(x_-)}{2\alpha} - 1 = \frac{f_a'(x_-)\alpha + 2d(\alpha-\beta)}{\alpha^3},
$$

which follows from the following computation:

$$
\begin{aligned}
&\frac{-f_a''(x_-)}{2\alpha} - 1 - \frac{f_a'(x_-)\alpha + 2d(\alpha-\beta)}{\alpha^3} \\
&= \tfrac{1}{\alpha^3}\left(-f_a''(x_-)\tfrac{\alpha^2}{2} - \alpha^3 - f_a'(x_-)\alpha - 2d(\alpha-\beta)\right) \\
&= \tfrac{1}{\alpha^3}\left(-\left(f_a(x_-) + f_a'(x_-)\alpha + f_a''(x_-)\tfrac{\alpha^2}{2} + \alpha^3\right) + f_a(x_-) - 2d(\alpha-\beta)\right) \\
&= -\tfrac{1}{\alpha^3}(f_a(x_- + \alpha) + f_a(x_-) - 2d(\alpha-\beta)) \\
&= -\tfrac{1}{\alpha^3}(f_a(x_+) + f_a(x_-) - 2d(x_+ - y_+) + 2d(x_- - y_-)) \\
&= 0
\end{aligned}
$$

since $2d(x_\pm - y_\pm) = -f_a(x_\pm)$. The computations are similar for the odd-site nonlinearity $f_o(\cdot)$. We summarize the main result of Section 3.2:

Lemma 3. *By looking at the even and odd nodes, the one-dimensional antidiffusion lattice Nagumo system*

$$\dot{u}_n = d(u_{n+1} - 2u_n + u_{n-1}) - f_a(u_n), \ \ n \in \mathbb{Z},$$

where $d < 0$ and $f_a(u) = u(u-a)(u-1)$, $a \in \mathbb{R}$, can be converted into a two-periodic lattice system

$$\begin{cases} \dot{v}_j &= d_e(w_j - 2v_j + w_{j-1}) - f_e(v_j), \\ \dot{w}_j &= d_o(v_{j+1} - 2w_j + v_j) - f_o(w_j), \end{cases} \qquad j \in \mathbb{Z},$$

where d_e, d_o, f_e, f_o are given in (3.11), (3.13), (3.14), (3.9) and

$$(x_\pm, y_\pm) \in \{y = x + \frac{f_a(x)}{2d}, \ f_a(x) + f_a(y) = 0\}.$$

In this new system, a traveling wavefront, if it exists, connects $(v_-, w_-) = (0,0)$ to $(v_+, w_+) = (1,1)$. If $\alpha\beta < 0$, then the diffusion coefficients d_e, d_o are positive.

Thus, if we require $\alpha\beta < 0$ in (3.11), we have a system of spatially discrete ODEs with period 2, with periodic *positive* diffusion parameters d_e, d_o, and with periodic nonlinearities f_e, f_o.

3.3 Antidiffusion and Traveling Waves

In this section, we first show how to view the antidiffusion lattice Nagumo problem in the framework set forth by Chen, Guo, and Wu [16]. Then we will apply the existence theorem as stated in Chapter 2 in order to find conditions under which we have bistable or monostable dynamics.

3.3.1 Periodic Media

We have seen that computing for the traveling wave solutions $(c,\vec{u}(t))$ of the antidiffusion lattice Nagumo problem

$$\begin{cases} \dot{u}_n &= d(u_{n+1}-2u_n+u_{n-1})-f_a(u_n), \;\; n\in\mathbb{Z}, \\ f_a(u) &= u(u-a)(u-1), \\ d<0 \quad \text{and} & a\in\mathbb{R} \end{cases} \tag{3.15}$$

can be converted into a search for 2-periodic solutions $(\vec{v}(t),\vec{w}(t))$ of the 2-periodic system

$$\begin{cases} \dot{v}_j &= d_e(w_j-2v_j+w_{j-1})-f_e(v_j), \\ \dot{w}_j &= d_o(v_{j+1}-2w_j+v_j)-f_o(w_j), \end{cases} \qquad j\in\mathbb{Z}, \tag{3.16}$$

where the diffusion parameters are

$$d_e=\frac{d\beta}{\alpha}, \qquad d_o=\frac{d\alpha}{\beta} \tag{3.17}$$

and the reaction term-nonlinearities are

$$f_e(v)=\alpha^2 f_{a_e}(v), \;\; f_o(w)=\beta^2 f_{a_o}(w) \tag{3.18}$$

with

$$a_e=\frac{-f_a''(x_-)}{2\alpha}-1, \qquad a_o=\frac{-f_a''(y_-)}{2\beta}-1, \tag{3.19}$$

and $\alpha=x_+-x_-$, $\beta=y_+-y_-$ with equilibria $(x_\pm,y_\pm)\in\hat{E}$. Note that we have already rescaled the system so that $(v_-,w_-)=(0,0)$ and $(v_+,w_+)=(1,1)$, as required in assumption A2. We are now in a position to use the results obtained by [16]. From equations (2.1), we set

$$\begin{cases} a_{n,-1} &= d_e, \\ a_{n,0} &= -2d_e, \quad n = 2k, \\ a_{n,1} &= d_e, \end{cases} \qquad \begin{cases} a_{n,-1} &= d_o, \\ a_{n,0} &= -2d_o, \quad n = 2k+1, \\ a_{n,1} &= d_o, \end{cases} \qquad a_{n,k} = 0 \text{ for } |k| > 1m,\, n \in \mathbb{Z},$$

where the diffusion parameters are d_e, d_o as in (3.17); the nonlinearities are

$$\begin{cases} f_n(u_n(t)) &= f_e(u_n(t)), \quad n = 2k, \\ f_n(u_n(t)) &= f_o(u_n(t)), \quad n = 2k+1, \end{cases}$$

with f_e, f_o as in (3.18); and a_e, a_o as in (3.19). Thus, we have now fulfilled assumptions A1 and A5. Assumption A2 is satisfied with $\phi_n^- = 0, \phi_n^+ = 1$, for $n \in \mathbb{Z}$ - this is the rationale for the rescaling process that was performed in Section 2.3. In order to satisfy the ellipticity assumption (assumption A3), since we are solving the antidiffusion problem (that is, $d < 0$), we should require that $\alpha\beta < 0$, so that d_e, d_o are both positive. Finally, assumption A4 is clearly satisfied.

3.3.2 Existence of Solutions

The existence theorem (Theorem 1) in Chapter 2 does not require that the boundary conditions $\vec{0}$ and $\vec{1}$ are stable or unstable; however, this theorem requires that, in case there are other 2-periodic equilibria, such points must be unstable. In this section, we first study the existence of 2-periodic equilibria, other than $\vec{0}$ and $\vec{1}$, of the system (3.16) and then perform a linear stability analysis on these points.

Existence of Intermediate Points

Recall that we have rescaled the system (3.3) so that $(v_-, w_-) = (0,0) \equiv \vec{0}$ and $(v_+, w_+) = (1,1) \equiv \vec{1}$ using the change-of-variables

$$v := \frac{x - x_-}{\alpha}, \quad w := \frac{y - y_-}{\beta}, \tag{3.20}$$

where $(x_-, y_-) \in \hat{E}$. The set $\hat{E}$ is the set of equilibria of (3.3). Define the set

$$E = \left\{ (v,w) : v = \frac{x - x_-}{\alpha}, w = \frac{y - y_-}{\beta}, \ (x,y) \in \hat{E} \right\}.$$

We shall call a point $(v,w) \in E$ an equilibrium of (3.16). We are interested in those points $(v,w) \in E$ where $0 < v < 1$, $0 < w < 1$. By ellipticity, we have $\alpha\beta < 0$, so that from (3.20),

$$(v,w) \in E : \ 0 < v < 1, \ 0 < w < 1$$

is equivalent to

$$(x,y) \in E : \ x_- < x < x_+, \ y_+ < y < y_-, \qquad \text{or} \qquad x_- > x > x_+, \ y_+ > y > y_-. \tag{3.21}$$

If there exists a point $(x,y) \in \hat{E}$ such that (3.21) is satisfied, then we shall call the point (x,y) a point intermediate to the boundary conditions (x_-, y_-) and (x_+, y_+) in the system (3.3), or simply, an intermediate point. Correspondingly, the point $(v,w) \in E$ will be called a point intermediate to the boundary conditions $\vec{0}$ and $\vec{1}$ in the system (3.16), or simply, an intermediate point. Because of the change-of-variables (3.20), there is a one-to-one correspondence between the intermediate points of $\hat{E}$ and E. There may be equilibria $(x,y) \in \hat{E}, (v,w) \in E$ that are not intermediate points.

Example 1. *(See Section 3.4.1). Suppose* $(x_-, y_-) = (0,0)$ *and* $(x_+, y_+) = (\tau_1, \tau_2)$*. Then* $\alpha\beta = \tau_1\tau_2 < 0$*. In region* I*, the nine zeroes of* g *are* $\tau_2 < 0^- < 0 < 0^+ < a < 1^- < 1 < 1^+ < \tau_1$*. Applying the above criteria (3.21), there is exactly one intermediate point* $(x,y) = (0^+, 0^-)$*. In region* II_{out}*, the nine zeroes of* g *are* $\tau_2 < 0^- < 0 < 0^+ < 1 < a^- < a < a^+ < \tau_1$*, so that the only intermediate point is also* $(x,y) = (0^+, 0^-)$*. In region* IV_{out}*, where* a *is negative, the nine zeroes of* g *are* $\tau_2 < a^- < a < a^+ < 0 < 1^- < 1 < 1^+ < \tau_1$*, so that, in this case, there is no intermediate point.*

For $a > 0$*, the intermediate point* $(x,y) = (0^+, 0^-) \in \hat{E}$ *in* (v,w)*-coordinates corresponds to* $(v,w) = \left(\frac{0^+}{\tau_1}, \frac{0^-}{\tau_2}\right) \in E$*. This example shows that there is a different behavior for* $a < 0$ *and justifies the study of* $a \in \mathbb{R}$*, even though the picture in Figure 3.1 shows that there is symmetry about the line* $a = 1/2$*.*

Example 2. *(See Section 3.4.2). Suppose* $(x_-, y_-) = (\tau_1, \tau_2)$ *and* $(x_+, y_+) = (\tau_2, \tau_1)$*. Suppose* $0 < a < 1$*. In this case, the nine zeroes of g, if they exist, are*

$$\tau_2 < 0^- < 0 < 0^+ < a < 1^- < 1 < 1^+ < \tau_1.$$

Since the $x_\pm, y_\pm$ *are the largest and smallest numbers in this list, we see that the other seven pairs are intermediate points. The pairs* $(0^\pm, 0^\mp)$ *exist provided* $a + 4d > 0$ *while the pairs* $(1^\pm, 1^\mp)$ *exist provided* $1 - a + 4d > 0$*.*

Since the set of equilibria E to the system (3.16) has at most nine points and the boundary conditions $(x_-, y_-), (x_+, y_+) \in \hat{E}$ in a traveling wavefront solution should be distinct, the ellipticity assumption (assumption A3) restricts the number of possible traveling wavefront solutions for any pair of boundary conditions. As we have seen, assumption A3 is equivalent to $\alpha\beta < 0$. Hence, for example, the theory does not guarantee that there is a traveling wavefront solution to (3.3) that connects $(x_-, y_-) = (0,0)$ to $(x_+, y_+) = (1,1)$, since $\alpha = \beta = 1$.

In fact, with a fixed (d,a), for each $(x_-,y_-)\in\hat{E}$, there are either two, four, six, or eight possible equilibria $(x_+,y_+)\in\hat{E}$ such that $\alpha\beta<0$. In each region (d,a) where there are nine points $(x,y)\in\hat{E}$, there are 42 connections from (x_-,y_-) to (x_+,y_+) that satisfy $\alpha\beta<0$; that is, there are only 42 pairs (not $9\cdot 8=72$) of boundary conditions from a choice of nine equilibria. In each region region (d,a) where there are seven points $(x,y)\in\hat{E}$, there are 28 connections from (x_-,y_-) to (x_+,y_+) that satisfy $\alpha\beta<0$. In the region (d,a) where there are five points $(x,y)\in\hat{E}$, there are 14 connections from (x_-,y_-) to (x_+,y_+) that satisfy $\alpha\beta<0$. Adding them all up, we will have 252 connections from (x_-,y_-) to (x_+,y_+) that satisfy $\alpha\beta<0$ for any given d,a. Out of these 252 connections, 116 of them will have an intermediate point.

Next, by Theorem 1, we need to check that every intermediate point (v,w) is unstable. This stability analysis is accomplished in the succeeding discussion.

Stability of Equilibria.

We want to investigate the (linear) stability of an equilibrium $(v,w)\in E$, intermediate or not. From (3.20), (v,w) is obtained from (x,y) by applying a linear change-of-variables. We observe that the linearization $A(v,w)$ of (3.16) about (v,w) is not symmetric in general because its off-diagonal elements, $2d_e, 2d_o$, may not be equal; however, the linearization $L(x,y)$ of (3.3) about (x,y) is symmetric. Hence, we study the linear stability of (v,w) by studying the matrix $L(x,y)$, given by

$$L(x,y)=\begin{pmatrix} -(2d+f_a'(x)) & 2d \\ 2d & -(2d+f_a'(y)) \end{pmatrix} \tag{3.22}$$

with trace and determinant

$$T(x,y)=-4d-(f_a'(x)+f_a'(y)),\quad D(x,y)=2d(f_a'(x)+f_a'(y))+f_a'(x)f_a'(y).$$

Since $L(x,y)$ is symmetric, its two eigenvalues are real so that the point (x,y) is unstable if and only if the larger eigenvalue of $L(x,y)$ is positive. The eigenvalues of $L(x,y)$ are given by $\frac{1}{2}(T \pm \sqrt{T^2-4D})$, where T,D are the trace and determinant, respectively.

For $(x,y) \neq (0,0),(a,a),(1,1)$, we use (3.4) to see that $D(x,y) \equiv 0$. Hence, the eigenvalues of $L(x,y)$ are either T or 0. The eigenvalue T is positive provided $2d + f_a'(x) < 0$; this comes from

$$T = -4d - f_a'(x) - f_a'(y) = -4d - f_a'(x) + \frac{2d\, f_a'(x)}{2d + f_a'(x)} = \frac{-[(2d + f_a'(x))^2 + 4d^2]}{2d + f_a'(x)}.$$

For $(x,y) = (0,0),(a,a),(1,1)$, we summarize the trace, determinant, and eigenvalues in the following table:

Equilibrium(x,y)	Trace	Determinant	Eigenvalues
$(0,0)$	$-2(a+2d)$	$a(a+4d)$	$-(a+4d), -a$
$(1,1)$	$-2(1-a+2d)$	$(1-a)(1-a+4d)$	$-(1-a+4d), -(1-a)$
(a,a)	$-2(a(a-1)+2d)$	$a(a-1)(a(a-1)+4d)$	$-(a(a-1)+4d), -a(a-1)$

Lemma 4. *Given a fixed $d < 0$ and $a \in \mathbb{R}$, suppose $\alpha\beta < 0$. The following are equivalent:*

1. *the equilibrium $(v,w) \in E$ is a 2-periodic unstable solution of (3.16);*

2. *the equilibrium $(x,y) \in \hat{E}$ is a 2-periodic unstable solution of (3.3);*

3. *$2d + f_a'(x) < 0$, where $x \notin \{0,a,1\}$;*

4. *$2d + f_a'(y) < 0$, where $y \notin \{0,a,1\}$;*

5. *$f_a'(x) f_a'(y) < 8d^2$, where $x,y \notin \{0,a,1\}$.*

In particular, if $f_a'(x) f_a'(y) < 0$, then (x,y) (and hence (v,w)) is unstable.

Testing analytically that the inequalities in this lemma are true is not immediate because we do not have closed form expressions for x and y as zeroes of the ninth-degree polynomial g. However, we can formulate some simple observations that will allow us to check $f_a'(x)f_a'(y) < 0$ for some pairs (x,y) (see Figure 3.4). Define the spinodal interval of f_a:

$$S = (s_-, s_+), \text{ where } s_\pm = \frac{1}{3}(a+1 \pm \sqrt{a^2 - a + 1}). \tag{3.23}$$

Note that for any $a \in \mathbb{R}$, $a^2 - a + 1 > 0$. The endpoints of the spinodal interval are the points where the derivative of f_a changes signs.

Lemma 5. *If $x \in S$ and $y \notin S$ or $y \in S$ and $x \notin S$, then $f_a'(x)f_a'(y) < 0$ so that (x,y) is unstable.*

Example 3. *See Figure 3.4, where the given (d,a)-values are in region I. The components of the equilibria*

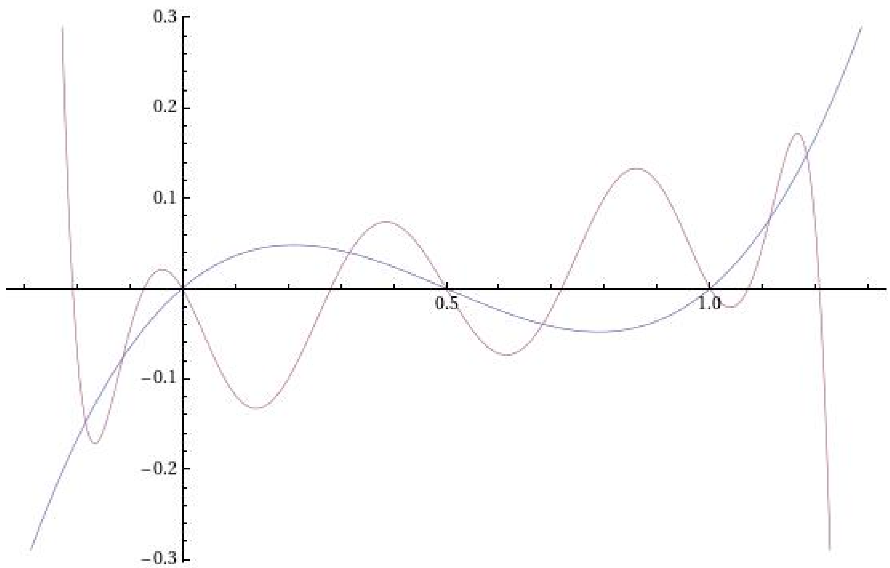
$$(0,0), (\tau_1, \tau_2), (\tau_2, \tau_1), (1,1)$$

are all outside S while the components of the point (a,a) are inside S. Hence, these five points are stable. The other four equilibria, $(0^\pm, 0^\mp)$ and $(1^\pm, 1^\mp)$, are unstable. From Example 1 and by the previous lemma, we see that $(0^+, 0^-)$ is the only intermediate unstable equilibria between the two stable equilibria $(x_-, y_-) = (0,0)$ and $(x_+, y_+) = (\tau_1, \tau_2)$. Similarly, $(1^+, 1^-)$ is the only intermediate unstable equilibrium between the two stable equilibrium $(x_-, y_-) = (1,1)$ and $(x_+, y_+) = (\tau_1, \tau_2)$.

Note that $2d + f_a'(x) = 3x^2 - 2(a+1)x + a + 2d$ and for $d < 0$, we have $Q(a,d) := a^2 - a + 1 - 6d > 0$. Hence if

$$\frac{a+1-\sqrt{Q(a,d)}}{3} < x < \frac{a+1+\sqrt{Q(a,d)}}{3},$$

Figure 3.4: This shows the graphs of f_a and g in the Cartesian plane when $a = 1/2$, $d = -1/16$. In this case, the spinodal interval is $(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$ so that $f_a'(0^-)f_a'(0^+) < 0$ and $f_a'(1^-)f_a'(1^+) < 0$.



then $2d + f_a'(x) < 0$ so that (x, y) is unstable. Otherwise, if

$$x < \frac{a+1-\sqrt{Q(a,d)}}{3} \quad \text{or} \quad x > \frac{a+1+\sqrt{Q(a,d)}}{3}$$

then $2d + f_a'(x) > 0$ so that (x, y) is stable.

Finally, our discussion of stability of equilibria agrees with the stability criteria in [16] (see Theorem 1, page 197), where they have proved that the characteristic equation of a general spatially discrete equation over periodic media has at most two real roots.

Bistable and Monostable Dynamics

In this final sub-section, we define what it means for the system (3.16) to exhibit bistable and monostable dynamics and then derive conditions under which (3.16) has such dynamics.

Bistable dynamics corresponds to $\vec{0}$ and $\vec{1}$ being stable 2-periodic equilibrium solutions to (3.16) and an unstable 2-periodic equilibrium solution $\vec{a} = (v, w)$ with $0 < v < 1$ and $0 < w < 1$ and no other stable 2-periodic equilibrium solutions (v, w) with values in $(0, 1) \times (0, 1)$.

Monostable dynamics corresponds to one of $\vec{0}$ or $\vec{1}$ being a stable 2-periodic equilibrium solution and the other an unstable 2-periodic equilibrium solution and no other stable 2-periodic equilibrium solutions with values in $(0, 1) \times (0, 1)$.

Theorem 3. *Given a fixed $d < 0$ and $a \in \mathbb{R}$, let $(x_\pm, y_\pm) \in \hat{E}$. Suppose $(x, y) \in \hat{E}$ is an intermediate point between (x_-, y_-) and (x_+, y_+):*

1. *If $f_a'(x_\pm) f_a'(y_\pm) \geq 8d^2 > f_a'(x) f_a'(y)$, then both (x_-, y_-) and (x_+, y_+) are stable and (x, y) is unstable. If there are no other stable 2-periodic equilibria solutions, then the antidiffusion lattice Nagumo system exhibits bistable dynamics.*

2. *If $f_a'(x_-) f_a'(y_-) \geq 8d^2 > f_a'(x_+) f_a'(y_+)$, then (x_-, y_-) is stable while (x_+, y_+) is unstable; or if $f_a'(x_+) f_a'(y_+) \geq 8d^2 > f_a'(x_-) f_a'(y_-)$, then (x_+, y_+) is stable while (x_-, y_-) is unstable. If there are no other stable 2-periodic solutions, then the antidiffusion lattice Nagumo system exhibits monostable dynamics.*

3.4 Examples

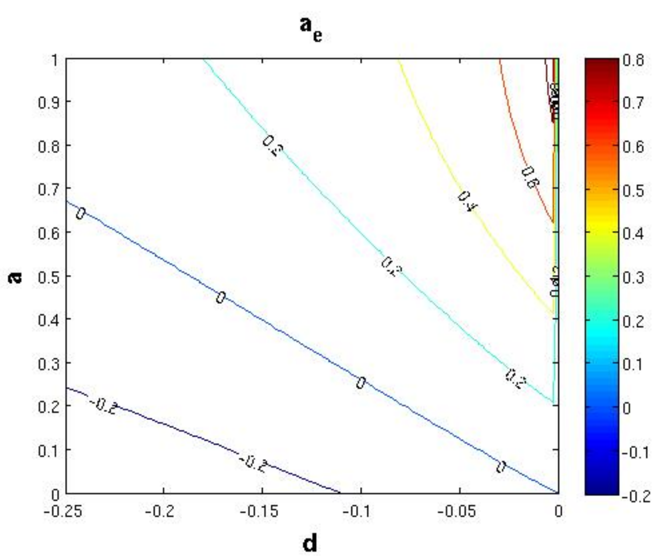
3.4.1 Case Study: $(0, 0)$ to (τ_1, τ_2) connection

Existence

Our equilibrium analysis has shown that for *any* value of the parameter (d, a) for $d < 0$, there always exist numbers outside the interval $(0, 1)$ that give rise to equilibria to the system (3.3); that is, there exist $\tau_1 > 1$ and $\tau_2 < 0$ with $g(\tau_i) = 0$ such that (τ_1, τ_2) or (τ_2, τ_1) is an equilibrium of (3.3). We want to know whether there exists a traveling wavefront solution to (3.3) that connects $(x_-, y_-) = (0,0)$, which we can call as a homogeneous equilibrium and $(x_+, y_+) = (\tau_1, \tau_2)$, which we can call as an inhomogeneous equilibrium. (In the positive diffusion problem, the traveling wavefront solution that is often studied is a solution that connects 0 to 1.)

In this case, $\alpha = \tau_1 > 1$, $\beta = \tau_2 < 0$. We set $v_j = \dfrac{x_j}{\tau_1}$, $w_j = \dfrac{y_j}{\tau_2}$. In system (3.16), the diffusion-coupling parameters are $d_e = \dfrac{d\tau_2}{\tau_1}$, $d_o = \dfrac{d\tau_1}{\tau_2}$, and the reaction-nonlinearity terms are $f_e(v) = \tau_1^2 f_{a_e}(v)$, $f_o(w) = \tau_2^2 f_{a_o}(w)$ with detuning parameters $a_e = \dfrac{a+1}{\tau_1} - 1, a_o = \dfrac{a+1}{\tau_2} - 1$, respectively. If $a > 0$, then the three detuning parameters are related by $a_o < 0 < a < a_e$. Figure 3.5 shows a contour plot of a_e for $-0.25 < d < 0$ and $0 < a < 1$ where we observe that when $a + 4d > 0$, the detuning parameter $a_e \in (0,1)$.

Figure 3.5: This is the contour plot of a_e over the region $(d, a) \subseteq (-1/4, 0) \times (0, 1)$. Note that as d increases to 0, a_e approaches $a \in (0,1)$.



Since $\alpha\beta < 0$, the diffusion-coupling parameters d_e, d_o are both positive (ellipticity condition). In (v, w)-coordinates, the set of equilibria of (3.16) are

$$E = \left\{ (v, w) : v = x/\tau_1, w = y/\tau_2, \text{ where } (x, y) \in \hat{E} \right\}.$$

For parameter values (d,a) where E has nine points, the elements of E are:

Case 1 ($a<0$).

$$\begin{aligned} E &= \{(0,0),(a/\tau_1,a/\tau_2),(1/\tau_1,1/\tau_2),(1,1),(\tau_2/\tau_1,\tau_1/\tau_2),(a^-/\tau_1,a^+/\tau_2), \\ &\quad (a^+/\tau_1,a^-/\tau_2),(1^-/\tau_1,1^+/\tau_2),(1^+/\tau_1,1^-/\tau_2)\}. \end{aligned}$$

Case 2 ($0<a<1$).

$$\begin{aligned} E &= \{(0,0),(a/\tau_1,a/\tau_2),(1/\tau_1,1/\tau_2),(1,1),(\tau_2/\tau_1,\tau_1/\tau_2),(0^-/\tau_1,0^+/\tau_2), \\ &\quad (0^+/\tau_1,0^-/\tau_2),(1^-/\tau_1,1^+/\tau_2),(1^+/\tau_1,1^-/\tau_2)\}. \end{aligned}$$

Case 3 ($a>1$).

$$\begin{aligned} E &= \{(0,0),(a/\tau_1,a/\tau_2),(1/\tau_1,1/\tau_2),(1,1),(\tau_2/\tau_1,\tau_1/\tau_2),(0^-/\tau_1,0^+/\tau_2), \\ &\quad (0^+/\tau_1,0^-/\tau_2),(a^-/\tau_1,a^+/\tau_2),(a^+/\tau_1,a^-/\tau_2)\}. \end{aligned}$$

Elements of E that are in the box $(0,1)\times(0,1)$ are called intermediate points of (3.16). For $a<0$, we do not have intermediate points. For $a>0$, the only intermediate point is $(0^+/\tau_1,0^-/\tau_2)$, which exists only when $a+4d>0$.

Next, let us investigate the stability of our equilibria. In (v,w)-coordinates, the boundary conditions are $\vec{0}\equiv\left(\dfrac{0}{\tau_1},\dfrac{0}{\tau_2}\right),\vec{1}\equiv\left(\dfrac{\tau_1}{\tau_1},\dfrac{\tau_2}{\tau_2}\right)$ while the intermediate point is $\vec{a}\equiv\left(\dfrac{0^+}{\tau_1},\dfrac{0^-}{\tau_2}\right)$. From the previous section, the stability of $(v,w)=\vec{0},\vec{1},\vec{a}$ can be inferred from the stability of $(x,y)=(0,0),(\tau_1,\tau_2),(0^+,0^-)$, respectively. In particular, using the results from the previous section, we have that (τ_1,τ_2) is stable if $2d+f_a'(\tau_1)>0$, which is satisfied when $1-a+2d>0$. Also, $(0,0)$ is stable (hence, $\vec{0}$ is stable) whenever $a+4d>0$. Finally, $(0^+,0^-)$ is unstable if $2d+f_a'(0^+)<0$ and exists only when $a+4d>0$.

Lemma 6. *1. If $a+4d>0$, $1-a+2d>0$, and $2d+f_a'(0^+)<0$, then the antidiffusion lattice Nagumo system exhibits bistable dynamics.*

2. *If $a+4d<0<1-a+2d$, then the antidiffusion lattice Nagumo system exhibits monostable dynamics.*

Finally, we apply the theorems of Chen, Guo, and Wu [16] as restated in Chapter 2. In particular, the existence theorem holds for all (d,a):

Lemma 7. *There exists a monotone traveling wavefront 2-periodic solution to the system (3.3) that connects $(0,0)$ to (τ_1,τ_2). If the conditions for bistable dynamics in Lemma 6 hold, then this traveling wavefront solution is unique, 2-periodic, monotonic, and globally exponentially stable.*

Propagation Failure

The first part of this section collects some results on propagation failure in the positive diffusion problem. The basic idea, as presented by Keener [39], is to consider a mapping Φ_K on the plane and to find points on the unit interval that will define some bounded areas in the unit square which will be mapped by Φ_K into themselves (see discussion in [39]). The existence of such points on the unit interval will guarantee the existence of propagation failure in the system.

Our goal is to investigate and derive conditions under which the traveling wavefronts from $(0,0)$ to (τ_1,τ_2) to the antidiffusion lattice Nagumo system with bistable dynamics fail to propagate. To this aim, we will, initially, look at the limiting equations of (3.16) to find an approximation $\hat{\Phi}$ to a map Φ that is analogous to Keener's mapping Φ_K. We use an approximation $\hat{\Phi}$ because the mapping Φ is of degree 9 and is complicated to analyze directly. This section ends with a result that contains sufficient conditions such that traveling waves connecting $(0,0)$ to (τ_1,τ_2) fail to propagate.

Positive Diffusion

In the reaction-diffusion PDE case, for example, $u_t = du_{xx} - f_a(u)$, where $x \in \mathbb{R}, t > 0$, and $d > 0$, the wave speed c of the traveling wave solution, if it exists, is a continuous and strictly monotonic function of a (that is, no propagation failure). In case of spatially periodic coefficients, for example, $\partial_t u = d\partial_x(b(x)\partial_x u) + f_a(u)$, where $x \in \mathbb{R}, t > 0$ and $d > 0$, with $b(x) = b(x+2\pi)$, stationary solutions $c = 0$ solve a time-periodic ODE. Heteroclinic orbits are now typically transverse, so that we expect pinned fronts for an interval of values of a.

In the positive diffusion lattice Nagumo equation LDE case,

$$\dot{u}_n(t) = d(u_{n+1} - 2u_n + u_{n-1}) - u_n(u_n - a)(u_n - 1), \qquad d > 0, 0 < a < 1, \tag{3.24}$$

the wave speed of the traveling wave solution may be zero for an open set of a. In particular, Keener [39] proved that for sufficiently small $d > 0$, there are numbers $a_- \neq a_+$ in the interval $(0,1)$ such that $c = c(a)$ satisfies

$$c \begin{cases} < 0, & 0 < a < a_-, \\ = 0, & a_- \leq a \leq a_+, \\ > 0, & a_+ < a < 1. \end{cases}$$

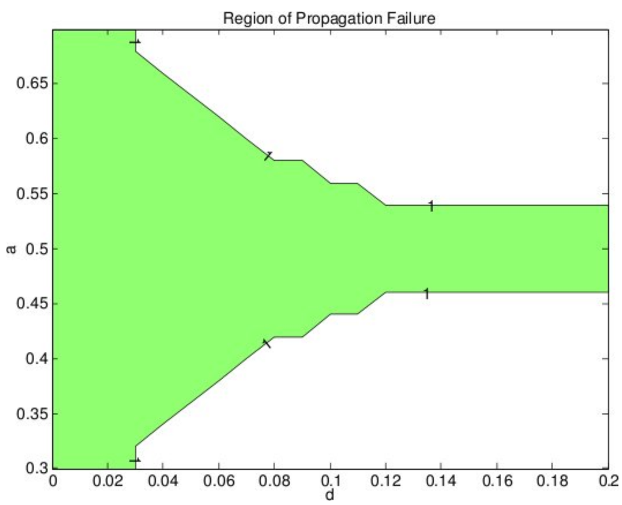
When the numbers $a_- \neq a_+$ exist, we say that pinning of the wave or propagation failure occurs; the wave is pinned and cannot propagate when a is in the nontrivial interval $[a_-, a_+]$. This interval is usually called the pinning interval, and the length of the interval gives a measure of the pinning of the waves.

A numerical approximation of the region of propagation failure for the positive diffusion problem is shown in Figure 3.6. This was obtained (as was Figure 3.7 in a similar way) by starting from a Heaviside initial condition, approximating the solution of the differential equations using the MATLAB code `ODE45` with tolerances of 10^{-10} to a final time of $T = 100$. If

$$u_{-1}(100) = w_{-1}(100) > 0.95 \qquad \text{or} \qquad u_0(100) = v_0(100) < 0.05, \tag{3.25}$$

the parameter value in the (d,a)-plane was deemed to not have propagation failure; otherwise it was labeled as having propagation failure. In Figure 3.7, we have observed numerically that the small rectangular strip close to $d = 0$ is just a numerical artefact and it approximately shrinks in width as $d \to 0-$.

Figure 3.6: The shaded area shows the region of propagation failure for the positive diffusion problem in the (d,a)-parameter plane.



To obtain a qualitative picture of the region of propagation failure, we analyze the range of existence of monotonic steady-state solutions, as argued by Keener in [39]. He proved the existence of a pinning interval by applying a result due to Moser. He showed that the mapping $\phi_{f_a,d}$ defined by

$$\phi_{f_a,d}(u,v) = \left(\frac{f_a(u)}{d} + 2u - v, u\right) \tag{3.26}$$

possesses symbolic dynamics (on two symbols). This mapping is obtained by setting $\dot{u}_n(t) = 0$ in (3.24) and writing the resulting difference equation

$$d(u_{n+1} - 2u_n + u_{n-1}) - f_a(u_n) = 0, \;\; d > 0,$$

as a map on the plane by taking

$$u_{n+1} = \frac{f_a(u_n)}{d} + 2u_n - v_n, \;\; v_{n+1} = u_n.$$

For the cubic-nonlinearity $f_a(u) = u(u-a)(u-1)$, where $0 < a < 1/2$, Keener derived a pinning interval

$$\sqrt{4d} < a < 1 - \sqrt{4d}, \;\; 0 \leq d \leq 1/16. \tag{3.27}$$

Negative Diffusion

We want to know if the propagation failure phenomenon exists in the antidiffusion lattice Nagumo traveling wavefront problem in a bistable region B for the connection $(0,0)$ to (τ_1, τ_2). From Lemma 6, B is a subset of $\{a + 4d > 0, 1 - a + 2d > 0, 2d + f_a'(0^+) < 0\}$, where $(0^+, 0^-)$ is the only intermediate point (for $a > 0$). In this part, we will assume that $0 < a < 1$ (similar arguments apply for the other case, $a > 1$). Thus, consider the bistable region

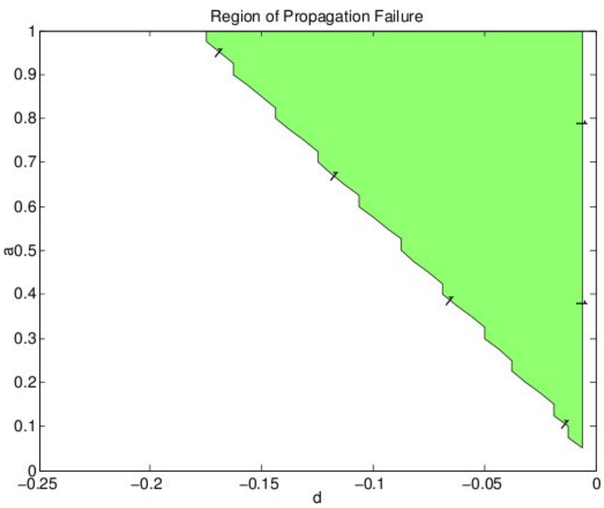
$$B = \{(d,a) : a + 4d > 0, 1 - a + 2d > 0, 2d + f_a'(0^+) < 0, 0 < a < 1\};$$

that is, if $(d,a) \in B$, then both $(v,w) = (0,0)$ and $(v,w) = (1,1)$ are stable equilibria to (3.16) and the only other intermediate point is $(v,w) = (\frac{0^+}{\tau_1}, \frac{0^-}{\tau_2})$, which is unstable.

Consider $\dot{u}_n = F_n(u_{n-1}, u_n, u_{n+1})$, where

$$F_n(u_{n-1}, u_n, u_{n+1}) = d_n(u_{n+1} - 2u_n + u_{n-1}) - f_n(u_n), \;\;\; n \in \mathbb{Z}, \tag{3.28}$$

Figure 3.7: The shaded area shows the region of propagation failure for the antidiffusion lattice Nagumo problem in a bistable region B.



where $(d_n, f_n) = (d_e, f_e)$ for n an even number and $(d_n, f_n) = (d_o, f_o)$ for n an odd number. Setting $\dot{u}_n = 0$ and then solving for u_{n+1}, we have

$$u_{n+1} = \frac{f_n(u_n)}{d_n} + 2u_n - v_n, \quad v_n = u_{n-1}.$$

Define the following maps:

$$\phi_e(u,v) = \left(\frac{f_e(u)}{d_e} + 2u - v, u\right), \quad \phi_o(u,v) = \left(\frac{f_o(u)}{d_o} + 2u - v, u\right) \tag{3.29}$$

$$\Phi = \phi_o \circ \phi_e. \tag{3.30}$$

The mapping Φ is of degree nine and our goal is to study the mapping Φ, which we can view as iterating the mapping (3.26) twice. The explicit expression for Φ is

$$\Phi(u,v) = \left(\frac{f_o(\frac{f_e(u)}{d_e} + 2u - v)}{d_o} + \frac{2f_e(u)}{d_e} + 3u - v, \frac{f_e(u)}{d_e} + 2u - v\right)$$

We want to derive sufficient conditions for d and a in region B that give rise to propagation failure. In B, the detuning parameter a_e for the even nodes satisfies $0 < a_e < 1$, and so we can view ϕ_e as the mapping ϕ_{f_e,d_e} in (3.26). However, the detuning parameter a_o for the odd nodes satisfies $a_o < -1$, and so we cannot apply an analysis similar to that in the even nodes. Instead, we will use an approximation of ϕ_o. To obtain such an approximation, let us study how the system (3.16) behaves as $d \to 0-$.

Lemma 8. *Consider the connection $(0,0)$ to (τ_1, τ_2) for a fixed (d,a) in a bistable region B. The equations (3.16) have limits*

$$\dot{v}_j = -f_a(v_j), \qquad \dot{w}_j = \frac{a}{2}(v_{j+1} - 2w_j + v_j) \tag{3.31}$$

as $d \to 0-$.

Proof. To see this, start from (3.16)

$$\begin{cases} \dot{v}_j &=& d_e(w_j - 2v_j + w_{j-1}) - f_e(v_j), \\ \dot{w}_j &=& d_o(v_{j+1} - 2w_j + v_j) - f_o(w_j), \end{cases}, j \in \mathbb{Z}.$$

Since $\tau_1 \to 1+$ and $\tau_2 \to 0-$ as $d \to 0-$, we clearly have that $d_e \to 0+$ and $f_e(u) \to f_a(u)$ (because $a_e \to a$) as $d \to 0-$. Furthermore,

$$f_o(u) = \tau_2^2 u(u-1)(u-a_o) = \tau_2 u(u-1)(\tau_2 u - (a+1-\tau_2))$$

so that $f_o(u) \to 0$ as $d \to 0-$.

Next, we wish to show that $d_o \to a/2$ as $d \to 0-$. To see this, we will consider for a fixed, $\tau_1 \equiv \tau_1(d)$ and $\tau_2 \equiv \tau_2(d)$. We obtain

$$\lim_{d\to 0-} d_o = \lim_{d\to 0-} d\frac{\tau_1(d)}{\tau_2(d)} = \lim_{d\to 0-} \frac{d\tau_1'(d) + \tau_1(d)}{\tau_2'(d)}$$

after applying L'Hôpital's rule. To evaluate $\tau_1'(d)$ and $\tau_2'(d)$ we differentiate the equations

$$f_a(\tau_1(d)) + f_a(\tau_2(d)) = 0, \qquad 2d(\tau_1(d) - \tau_2(d)) = f_a(\tau_2(d))$$

with respect to d to obtain the linear system

$$\begin{pmatrix} f_a'(\tau_1) & f_a'(\tau_2) \\ -2d & 2d + f_a'(\tau_2) \end{pmatrix} \begin{pmatrix} \tau_1' \\ \tau_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2(\tau_1 - \tau_2) \end{pmatrix}.$$

In the limit as $d \to 0-$ we have

$$\begin{pmatrix} 1-a & a \\ 0 & a \end{pmatrix} \begin{pmatrix} \tau_1'(0-) \\ \tau_2'(0-) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

which has solution $\tau_1'(0-) = -2/(1-a)$ and $\tau_2'(0-) = 2/a$. Thus,

$$\lim_{d \to 0-} d_o = \lim_{d \to 0-} \frac{d\tau_1'(d) + \tau_1(d)}{\tau_2'(d)} = \frac{1}{2/a} = a/2.$$

□

Setting $\dot{v}_j = \dot{w}_j = 0$, the limiting equations (3.31) yield $v_j = 0, a, 1$ ($a \in (0,1)$) and $w_j = \frac{1}{2}(v_j + v_{j+1})$. This implies that if, for example,

$$\vec{v} = \{..., 0, ..., 0, 0, a, ..., a, a, 1, 1...\}, \text{ then } \vec{w} = \{..., 0, ..., 0, \frac{a}{2}, a, ..., a, \frac{a+1}{2}, 1, ...\} \qquad (3.32)$$

and this pair $(\vec{v}, \vec{w})$ is a monotonic traveling wave solution (with $c = 0$) to the limiting equations that connect $(v_-, w_-) = (0,0)$ to $(v_+, w_+) = (1,1)$. Other monotonic solutions can be obtained by varying the location and the number of a's in the middle of $\vec{v}$. Our numerical criterion (3.25) says that there is propagation failure if $u_0(100) = v_0(100) > 0.05$ and $u_{-1}(100) = w_{-1}(100) < 0.95$. Let us study the particular solution (3.32) using this criterion. Since v_j is either $0, a, 1$ and w_j depends on the value of v_j and v_{j+1}, there are five possible cases for the values of (v_{-1}, v_0) and w_{-1}:

$$(v_{-1}, v_0) = (0,0), (0,a), (a,a), (a,1), (1,1), \text{ then } \; w_{-1} = 0, a/2, a, (a+1)/2, 1,$$

respectively. Thus, there is propagation failure if $0.05 < a < 0.95$ (see Figure 3.7).

Another possible monotonic traveling wave solution (with $c = 0$) from $(0,0)$ to $(1,1)$ to the limiting equations is

$$\vec{v} = \{\ldots, 0, \ldots, 0, 0, 1, 1 \ldots\}, \;\; \vec{w} = \{\ldots, 0, \ldots, 0, 1/2, 1, \ldots\}.$$

Using our numerical criterion (3.25) to this particular solution, if $(v_{-1}, v_0) = (0,1)$ then $w_{-1} = 1/2$; that is, we have $u_0 = 1 > 0.05$ and $u_{-1} = 1/2 < 0.95$ so that there is propagation failure to the limiting equations. These two examples prove that indeed, the small rectangular unshaded strip about $d = 0$ in Figure 3.7 is a discretization artefact.

Approximating the mapping Φ. Applying the analysis in Lemma 8 (in particular, the fact that $f_o \to 0$ as $d \to 0-$), we will initially approximate ϕ_o in (3.29) by the linear map

$$\phi_L(u, v) = (2u - v, u). \tag{3.33}$$

Hence, consider the map $\hat{\Phi}$ defined by

$$\hat{\Phi}(u, v) = \phi_L \circ \phi_e(u, v) = \left(\frac{2}{d_e} f_e(u) + 3u - 2v, \frac{1}{d_e} f_e(u) + 2u - v \right). \tag{3.34}$$

Note that the inverses of ϕ_e, ϕ_L, respectively, are

$$\phi_e^{-1}(u,v) = (v, \frac{1}{d_e} f_e(u) + 2v - u), \;\; \phi_L^{-1}(u,v) = (v, 2v - u)$$

so that

$$\hat{\Phi}^{-1}(u,v) = \phi_e^{-1} \circ \phi_L^{-1}(u,v) = (2v - u, \frac{1}{d_e} f_e(2v - u) + 3v - 2u).$$

The main idea of the proof of the next lemma lies on the fact that the mapping $\hat{\Phi}$ is a homeomorphism on the unit square $[0,1] \times [0,1]$ into the plane $\mathbb{R}^2$. This next lemma is analogous to Keener's Corollary 2.2 on page 560 of [39].

Lemma 9. *Suppose there are numbers $\hat{u}_j \in (0,1)$ and $\hat{u}_j^* \in (0,1)$ for $j \in \{0,1,2,...,5\}$ such that the following hold:*

1. $H_{-1}(\hat{u}_0) = H_{-2}(\hat{u}_1) = H_{-3}(\hat{u}_2) = H_0(\hat{u}_3) = H_{-1}(\hat{u}_4) = H_{-2}(\hat{u}_5) = 0$, *where*

$$H_{-k}(u) = \frac{2}{d_e} f_e(u) + 3u - k, \qquad \text{for } \; k \in \{0,1,2,3\}. \tag{3.35}$$

2.
$$\hat{u}_j^* = \begin{cases} \frac{1}{d_e} f_e(\hat{u}_j) + 2\hat{u}_j, & j = 0,3,4, \\ \frac{1}{d_e} f_e(\hat{u}_j) + 2\hat{u}_j - 1, & j = 1,2,5. \end{cases}$$

3.
$$\frac{2}{d_e} f_e'(u) + 3 > 0 \text{ for } \begin{cases} 0 \le u \le \hat{u}_j, & j = 0,1,2, \\ \hat{u}_j \le u \le 1, & j = 3,4,5. \end{cases}$$

Then $\hat{\Phi}$ has

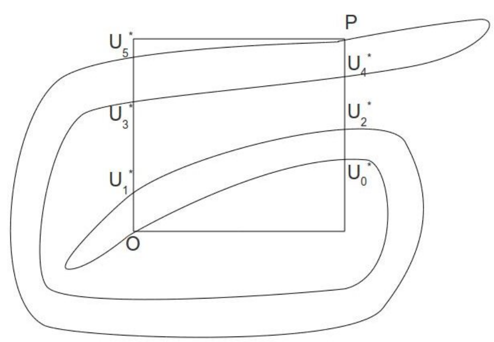
1. *a countable infinity of periodic orbits of arbitrarily high period,*

2. *an uncountable infinity of nonperiodic orbits, and*

3. *a dense orbit.*

Proof. To prove Lemma 9, we need to show that $\hat{\Phi}$ is a Smale-horseshoe type of mapping and as such, it will be topologically conjugate with the shift mapping σ on the space Σ of bi-infinite sequences on two symbols [55]. Geometrically, the mapping $\hat{\Phi}$ contracts the vertical direction, expands the horizontal direction, and folds the unit square around, laying it back on itself while fixing the points $(0,0)$ and $(1,1)$, in such a way that we can find disjoint regions that are mapped over themselves. A Smale-horseshoe mapping need not have the shape of a horseshoe (an inverted U); in fact, for our problem, the mapping $\hat{\Phi}$ yields a G-shaped horseshoe, as sketched in Figure 3.8.

Figure 3.8: The diagram shows a sketch of the image of the unit square under the mapping $\hat{\Phi}$.



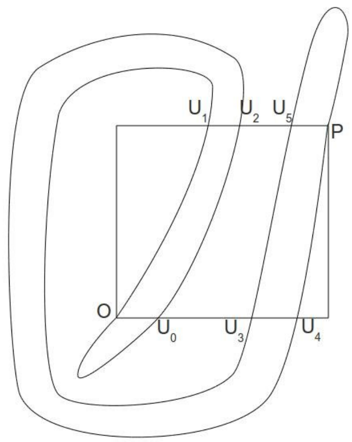
In Figures 3.8 and 3.9, we use the following notation to identify points:

$$U_j = (\hat{u}_j, 0), \ j = 0,3,4; \qquad U_j = (\hat{u}_j, 1), \ j = 1,2,5;$$
$$U_j^* = (0, \hat{u}_j^*), \ j = 1,3,5; \qquad U_j^* = (1, \hat{u}_j^*), \ j = 0,2,4. \tag{3.36}$$

To show that $\hat{\Phi}$ is a Smale-horseshoe type of mapping, we will apply the so-called Conley-Moser conditions [55] (see Theorem 25.1.5). For our problem, these conditions translate to finding 12 points U_j, U_j^* for $j = 0,1,2,3,4,5$ such that

1. $\hat{\Phi}(U_j) = U_j^*$ for $j = 0,1,2,3,4,5$.

Figure 3.9: The diagram shows a sketch of the image of the unit square under the inverse mapping $\hat{\Phi}^{-1}$.



2. The points O, U_0, U_2, U_1 define a vertical strip V_0. The points U_3, U_4, P, U_5 define a vertical strip V_1. The two strips V_0, V_1 are disjoint.

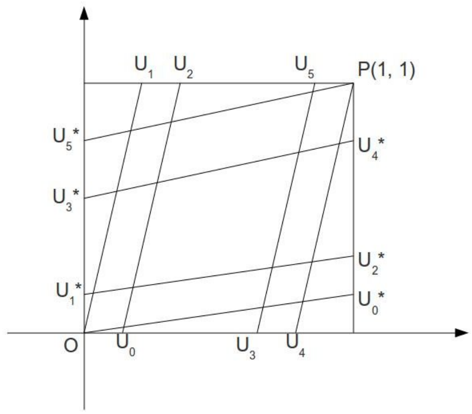
3. The points O, U_0^*, U_2^*, U_1^* define the horizontal strip H_0 while the points U_3^*, U_4^*, P, U_5^* define the horizontal strip H_1. The two strips H_0, H_1 are disjoint.

4. $\hat{\Phi}(V_i) = H_i$ for $i = 0, 1$.

5. The horizontal boundary curves of V_i map to the horizontal boundary curves of H_i and the vertical boundary curves of V_i map to the vertical boundary curves of H_i.

A horizontal strip is the set lying between two nonintersecting horizontal curves; a vertical strip is the set lying between two nonintersecting vertical curves. A horizontal curve $v = v(u)$ is the graph of a function $0 \leq v(u) \leq 1$ where $u \in [0, 1]$; a vertical curve $u = u(v)$ is the graph of a function $0 \leq v(u) \leq 1$ where $u \in [0, 1]$. Figure 10 illustrates the points U_j, U_j^* and the strips V_i, H_i in one unit square. Note that the boundary curves need not be straight lines.

Figure 3.10: The diagram shows a sketch of the 12 points that we are looking for and the vertical and horizontal strips.



The following table summarizes the mapping of the points and the conditions that the points $\hat{u}_j, \hat{u}_j^*$ must satisfy:

$\hat{\Phi}(U_0) = U_0^*$	$\frac{2}{d_e} f_e(\hat{u}_0) + 3\hat{u}_0 = 1$	$\hat{u}_0^* = \frac{1}{d_e} f_e(\hat{u}_0) + 2\hat{u}_0$
$\hat{\Phi}(U_1) = U_1^*$	$\frac{2}{d_e} f_e(\hat{u}_1) + 3\hat{u}_1 - 2 = 0$	$\hat{u}_1^* = \frac{1}{d_e} f_e(\hat{u}_1) + 2\hat{u}_1 - 1$
$\hat{\Phi}(U_2) = U_2^*$	$\frac{2}{d_e} f_e(\hat{u}_2) + 3\hat{u}_2 - 2 = 1$	$\hat{u}_2^* = \frac{1}{d_e} f_e(\hat{u}_2) + 2\hat{u}_2 - 1$
$\hat{\Phi}(U_3) = U_3^*$	$\frac{2}{d_e} f_e(\hat{u}_3) + 3\hat{u}_3 = 0$	$\hat{u}_3^* = \frac{1}{d_e} f_e(\hat{u}_3) + 2\hat{u}_3$
$\hat{\Phi}(U_4) = U_4^*$	$\frac{2}{d_e} f_e(\hat{u}_4) + 3\hat{u}_4 = 1$	$\hat{u}_4^* = \frac{1}{d_e} f_e(\hat{u}_4) + 2\hat{u}_4$
$\hat{\Phi}(U_5) = U_5^*$	$\frac{2}{d_e} f_e(\hat{u}_5) + 3\hat{u}_5 - 2 = 0$	$\hat{u}_5^* = \frac{1}{d_e} f_e(\hat{u}_5) + 2\hat{u}_5 - 1$

Thus, if we define

$$H_{-k}(u) = \frac{2}{d_e} f_e(u) + 3u - k \quad \text{for} \quad k \in \{0,1,2,3\}, \tag{3.37}$$

then the second column of the table gives the following conditions on the points $\hat{u}_j$:

$$H_{-1}(\hat{u}_0) = H_{-2}(\hat{u}_1) = H_{-3}(\hat{u}_2) = H_0(\hat{u}_3) = H_{-1}(\hat{u}_4) = H_{-2}(\hat{u}_5) = 0; \tag{3.38}$$

while the third column of the table gives the following conditions on the points u_j^*:

$$\hat{u}_j^* = \begin{cases} \dfrac{1}{d_e} f_e(\hat{u}_j) + 2\hat{u}_j, & j = 0,3,4, \\ \dfrac{1}{d_e} f_e(\hat{u}_j) + 2\hat{u}_j - 1, & j = 1,2,5. \end{cases}$$

For the two vertical strips V_i to be disjoint, we require that $\hat{u}_2 < \hat{u}_3$; for the two horizontal strips H_i to be disjoint, we require that $\hat{u}_3^* < \hat{u}_2^*$. Observe that there is a slight abuse of notation: H_0 may refer to the polynomial H_k for $k = 0$ or may refer to the horizontal strip H_0; the context should indicate which notion we are looking at. Finally, to prove that the boundary curves of the strips behave in a required manner, one applies arguments similar to Keener's proof [39]. □

There is a linear relationship between $\hat{u}_j$ and $\hat{u}_j^*$ for each j as follows:

$$\hat{u}_j^* = \frac{1}{2}\hat{u}_j, \;\; j \in \{1,3,5\}, \qquad \hat{u}_j^* = \frac{1}{2}(\hat{u}_j + 1), \;\; j \in \{0,2,4\}. \tag{3.39}$$

Remark 1. *The first two requirements on $\hat{u}_j, \hat{u}_j^*$ of Lemma 9 guarantee the existence of horizontal strips H_i and vertical strips V_i with $\hat{\Phi}(V_i) = H_i$ for $i = 0,1$ that satisfy the Conley-Moser conditions (in a weaker form). The third condition of Lemma 9 guarantees that the the boundary curves of these strips are monotone increasing.*

The next result contains sufficient conditions on the parameters (d,a) in a bistable region B that will imply the conditions of Lemma 9, and hence $\hat{\Phi}$ will be topologically conjugate to the shift mapping on two symbols $\{0,1\}$. Lemma 9 guarantees that for any sequence $\{s_j\}_{j\in\mathbb{N}}$ with $s_j \in \{0,1\}$, there is a sequence $\{(v_j, w_j)\}_{j\in\mathbb{N}}$ which is an equilibrium solution of

$$\begin{cases} \dot{v}_j &=& d_e(w_j - 2v_j + w_{j-1}) - f_e(v_j), \\ \dot{w}_j &=& d_o(v_{j+1} - 2w_j + v_j), \end{cases} \qquad j \in \mathbb{Z}, \tag{3.40}$$

with

$$\begin{cases} (v_j, w_j) \in [0, \hat{u}_2] \times [0, \hat{u}_2^*], & s_j = 0, \\ (v_j, w_j) \in [\hat{u}_3, 1] \times [\hat{u}_3^*, 1], & s_j = 1. \end{cases}$$

Equivalently, the next lemma will imply that the system (3.40) will have traveling wave solutions with zero speed, the propagation failure phenomenon

Lemma 10. *Consider the connection* $(0,0)$ *to* (τ_1, τ_2) *for a fixed* (d,a) *in a bistable region B. Suppose*

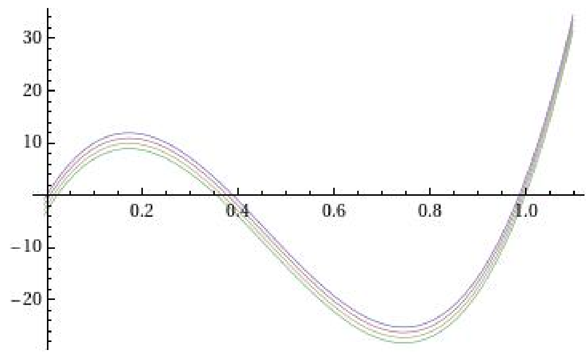
$$\frac{1}{\tau_1}(\tau_1^2 + \sqrt{6d\tau_2\tau_1} - \tau_1) \leq a \leq \frac{1}{\tau_1}(2\tau_1^2 - \sqrt{6d\tau_2\tau_1} - \tau_1).$$

Then $\hat{\Phi}$ *possesses the shift on sequences on two symbols as a subsystem.*

Proof. The previous lemma claims that the existence of $\{\hat{u}_j\}, \{\hat{u}_j^*\}$ for $j = 0, 1, ..., 5$ in the unit interval guarantees that there is propagation failure. Let us find a relationship between d_e and a_e that implies the hypotheses of this lemma. Observe that the equations satisfied by $\hat{u}_j$ in (3.38) are translates of each other. Conditions for the existence of roots of H_0 and H_{-3} can be computed because u is a factor of H_0 while $u-1$ is a factor of H_{-3}, and hence the two equations $H_0(u) = 0 = H_{-3}(u)$ are quadratic. Now, even without explicitly computing the other four roots $\hat{u}_k$ for $k = 0, 1, 4, 5$, which are roots of a cubic equation with positive discriminant, we can still ensure that these exist because these four roots are bounded by $0, \hat{u}_2, \hat{u}_3, 1$. In particular, the smallest root of H_0, which is 0, bounds the roots of the H_{-k}'s on the left while the largest root of H_{-3}, which is 1, bounds the roots of H_{-k}'s on the right. This is based on the observation that the H_{-k}'s for $k = 1, 2, 3, 4, 5$ are just vertical translations of H_0. Thus, the existence of three distinct real roots of H_0 and H_{-3} will imply the existence of the roots of H_{-1} and H_{-2}.

For example, if $a = 3/4$ and $d = -1/8$, then $(\tau_1, a_e) \approx (1.31383, 0.331979)$. The four curves $H_{-k}(u)$ for $k \in \{0, 1, 2, 3\}$ are illustrated in Figure 3.11.

Figure 3.11: Set $(d,a)=(-1/8,3/4)$. The translates $H_{-k}(u)$ are sketched, here the horizontal axis is u.



To guarantee the existence of the roots of H_0 and H_{-3}, since they are quadratic expressions, the discriminant of the quadratic equation $H_0 = 0 = H_{-3}$ must be nonnegative, that is,

$$(a_e+1)^2-4a_e-\frac{6d_e}{\tau_1^2}\geq 0,\;\; a_e^2-\frac{6d_e}{\tau_1^2}\geq 0$$

for H_0, H_{-3}, respectively. These inequalities are equivalent to

$$d_e \leq \frac{1}{6}\tau_1^2 a_e^2, \quad d_e \leq \frac{1}{6}\tau_1^2(a_e-1)^2, \tag{3.41}$$

respectively. In other words, the hypotheses of Lemma 9 will hold whenever

$$d_e = \min\{\frac{1}{6}\tau_1^2 a_e^2,\;\; \frac{1}{6}\tau_1^2(a_e-1)^2\} = \begin{cases} \frac{1}{6}\tau_1^2 a_e^2, & 0<a_e<1/2, \\ \frac{1}{6}\tau_1^2(a_e-1)^2, & 1/2<a_e<1. \end{cases}$$

Note that we do not allow $a_e = 1/2$ because in this case, $\hat{u}_2 = \hat{u}_3$, and so we would not have disjoint vertical strips V_0, V_1. Moreover, inequalities in (3.41) imply that

$$\frac{\sqrt{6d_e}}{\tau_1} \leq a_e \leq 1-\frac{\sqrt{6d_e}}{\tau_1}, \tag{3.42}$$

which can also be stated in terms of d and a:

$$\frac{1}{\tau_1}(\tau_1^2+\sqrt{6d\tau_2\tau_1}-\tau_1)\le a\le\frac{1}{\tau_1}(2\tau_1^2-\sqrt{6d\tau_2\tau_1}-\tau_1) \tag{3.43}$$

□

Compare the pinning interval in the positive diffusion case (3.27) with the above inequalities (3.42), (3.43). Observe that in (3.43), we have that $0<a<1$ as $d\to 0-$. These conditions on d and a can be further weakened by considering monotone functions, similar to what Keener did in the positive diffusion case [39].

Out of the two nonzero roots of H_0, we take the larger root as $\hat{u}_3$:

$$\hat{u}_3=(1/2)\left((a_e+1)+\sqrt{(a_e-1)^2-\frac{6d_e}{\tau_1^2}}\right); \tag{3.44}$$

while we take the smallest root of H_{-3} as $\hat{u}_2$:

$$\hat{u}_2=(1/2)\left(a_e-\sqrt{a_e^2-\frac{6d_e}{\tau_1^2}}\right). \tag{3.45}$$

We can, in fact, determine the precise ordering of all six numbers $\hat{u}_j$, $j\in\{0,1,...,5\}$. By monotonicity of the boundary curves of the horizontal and vertical strips, these numbers $\hat{u}_j$ in the unit interval must lie outside the spinodal interval $S=(s_-,s_+)$ of the polynomial H_{-k}. The roots $\hat{u}_0<\hat{u}_1<\hat{u}_2$ are the minimum roots of H_{-1},H_{-2},H_{-3}, respectively, while the roots $\hat{u}_3<\hat{u}_4<\hat{u}_5$ are the maximum roots of H_0,H_{-1},H_{-2}; that is, we have the ordering

$$0<\hat{u}_0<\hat{u}_1<\hat{u}_2<s_-<s_+<\hat{u}_3<\hat{u}_4<\hat{u}_5<1.$$

However, there is no such ordering for $\hat{u}_j^*$ because of the folding mechanism in $\hat{\Phi}$, (3.39):

$$0 < \hat{u}_1^* < \hat{u}_3^* < \min\{\hat{u}_0^*, \hat{u}_5^*\} < \max\{\hat{u}_0^*, \hat{u}_5^*\} < \hat{u}_2^* < \hat{u}_4^* < 1.$$

The mapping Φ. Gaining insight from the approximation mapping $\hat{\Phi}$, we now look at the full mapping $\Phi = \phi_o \circ \phi_e$. Note that $\Phi = \hat{\Phi} + E$, where E is the mapping on the plane defined by

$$E(u,v) = \left(\frac{1}{d_o} f_o(\frac{1}{d_e} f_e(u) + 2u - v),\ 0\right).$$

Certainly, other perturbations $\hat{\Phi}$ may be used to approximate Φ. We have chosen to approximate ϕ_o by the linear mapping ϕ_L because, as seen in the previous discussion, it gave rise to a simpler $\hat{\Phi}$. In particular, the definition of $\hat{\Phi}$ involves the even components d_e, f_e, a_e while the definition of E involves the odd components d_o, f_o, a_o. Separating the even from the odd contributions is helpful because the even detuning parameter a_e is in the unit interval $(0,1)$ while the odd detuning parameter a_o is negative.

A result similar to Lemma 9 is the following lemma

Lemma 11. *Suppose there are numbers $u_j \in (0,1)$, $u_j^* \in (0,1)$ for $j \in \{0,1,2,...,5\}$ such that*

1.

$$u_j = \begin{cases} \frac{1}{d_o} f_o(u_j^*) + 2u_j^*, & j = 1,3,5, \\ \frac{1}{d_e} f_o(u_j^*) + 2u_j^* - 1, & j = 0,2,4. \end{cases}$$

2.

$$u_j^* = \begin{cases} \frac{1}{d_e} f_e(u_j) + 2u_j, & j = 0,3,4, \\ \frac{1}{d_e} f_e(u_j) + 2u_j - 1, & j = 1,2,5. \end{cases}$$

3.

$$\frac{2}{d_e} f_e'(u) + 3 > 0 \text{ for } \begin{cases} 0 \le u \le u_j, & j = 0,1,2, \\ u_j \le u \le 1, & j = 3,4,5. \end{cases}$$

Then Φ *has*

1. *a countable infinity of periodic orbits of arbitrarily high period,*

2. *an uncountable infinity of non-periodic orbits, and*

3. *a dense orbit.*

Proof. The proof uses arguments similar to that of Lemma 9; that is, we need to find 12 points that satisfy the Conley-Moser conditions in this setup. The following table summarizes the mapping of the points and the conditions that the points u_j, u_j^* must satisfy:

$\Phi(U_0) = U_0^*$	$\frac{2}{d_e} f_e(u_0) + 3u_0 + \frac{1}{d_o} f_o(\frac{1}{d_e} f_e(u_0) + 2u_0) = 1$	$u_0^* = \frac{1}{d_e} f_e(u_0) + 2u_0$
$\Phi(U_1) = U_1^*$	$\frac{2}{d_e} f_e(u_1) + 3u_1 - 2 + \frac{1}{d_o} f_o(\frac{1}{d_e} f_e(u_1) + 2u_1 - 1) = 0$	$u_1^* = \frac{1}{d_e} f_e(u_1) + 2u_1 - 1$
$\Phi(U_2) = U_2^*$	$\frac{2}{d_e} f_e(u_2) + 3u_2 - 2 + \frac{1}{d_o} f_o(\frac{1}{d_e} f_e(u_2) + 2u_2 - 1) = 1$	$u_2^* = \frac{1}{d_e} f_e(u_2) + 2u_2 - 1$
$\Phi(U_3) = U_3^*$	$\frac{2}{d_e} f_e(u_3) + 3u_3 + \frac{1}{d_o} f_o(\frac{1}{d_e} f_e(u_3) + 2u_3) = 0$	$u_3^* = \frac{1}{d_e} f_e(u_3) + 2u_3$
$\Phi(U_4) = U_4^*$	$\frac{2}{d_e} f_e(u_4) + 3u_4 + \frac{1}{d_o} f_o(\frac{1}{d_e} f_e(u_4) + 2u_4) = 1$	$u_4^* = \frac{1}{d_e} f_e(u_4) + 2u_4$
$\Phi(U_5) = U_5^*$	$\frac{2}{d_e} f_e(u_5) + 3u_5 - 2 + \frac{1}{d_o} f_o(\frac{1}{d_e} f_e(u_5) + 2u_5 - 1) = 0$	$u_5^* = \frac{1}{d_e} f_e(u_5) + 2u_5 - 1$

In this table, we use a notation (see (3.36)) similar to the approximating case, where the hats in u_j, u_j^* have been dropped. Each of the equations in the second column in this table can be written in the form $u_j = F(u_j^*)$, where u_j^* is defined in the third column. For example, since $u_0^* = \frac{1}{d_e} f_e(u_0) + 2u_0$, the second column for u_0 gives $2u_0^* - u_0 + \frac{1}{d_o} f_o(u_0^*) = 1$, or $u_0 = \frac{1}{d_o} f_o(u_0^*) + 2u_0^* - 1$. □

Next, we want to find sufficient conditions that will guarantee the existence of the 12 points u_j, u_j^*. To this end, let us rewrite the equations for u_j, u_j^* in the following way. Define

$$\varepsilon_j = \frac{f_o(u_j^*)}{d_o}, \quad j \in \{0,1,2,3,4,5\}, \tag{3.46}$$

where

$$u_j^* = \begin{cases} \dfrac{1}{d_e} f_e(u_j) + 2u_j, & j = 0,3,4, \\ \dfrac{1}{d_e} f_e(u_j) + 2u_j - 1, & j = 1,2,5. \end{cases} \tag{3.47}$$

Since $a_o < 0$, we see that each perturbation ε_j is negative. For each j, observe that $\Phi(U_j) - \hat{\Phi}(U_j) = E(U_j) = (\varepsilon_j, 0)$.

Define the following family of functions (see (3.35)):

$$H_{-l}(u) = \frac{2}{d_e} f_e(u) + 3u - l, \;\; l \in \mathbb{R}. \tag{3.48}$$

Then the equations in the second column can be written as

$$H_{-(1-\varepsilon_0)}(u_0) = H_{-(2-\varepsilon_1)}(u_1) = H_{-(3-\varepsilon_2)}(u_2) = H_{\varepsilon_3}(u_3) = H_{-(1-\varepsilon_4)}(u_4) = H_{-(2-\varepsilon_5)}(u_5) = 0.$$

Thus, if we assume that $-1 < \varepsilon_j < 0$ for each j, then we have the following ordering of the family of functions H_{-l}, where $l \in \mathbb{R}$:

$$H_0 > H_{\varepsilon_3} > H_{-1} > H_{-(1+m_{0,4})} > H_{-(1+M_{0,4})} > H_{-(2+m_{1,5})} > H_{-(2+M_{1,5})} > H_{-3} > H_{-(3-\varepsilon_2)}, \tag{3.49}$$

where

$$m_{0,4} = \min\{-\varepsilon_0, -\varepsilon_4\}, \;\; M_{0,4} = \max\{-\varepsilon_0, -\varepsilon_4\},$$

$$m_{1,5} = \min\{-\varepsilon_1, -\varepsilon_5\}, \;\; M_{1,5} = \max\{-\varepsilon_1, -\varepsilon_5\}.$$

Because of this ordering of the translates, the numbers u_j exist if each of the top and bottom translates, H_0 and $H_{-(3-\varepsilon_2)}$, has three distinct real zeroes. The maximum real zero of H_0 is the number u_3 while the minimum real zero of $H_{-(3-\varepsilon_2)}$ is u_2. Sufficient conditions for the existence of three distinct real roots of H_0 and $H_{-(3-\varepsilon_2)}$ are $H_0(S_+) < 0 < H_{-(3-\varepsilon_2)}(S_-)$, which we can also write as

$$H_0(S_+) < 0 < H_0(S_-) - 3 + \varepsilon_2,$$

where (S_-, S_+) is the spinodal interval of H_{-l}, that is,

$$S_\pm = \frac{a_e + 1 \pm \sqrt{a_e^2 - a_e + 1 - \frac{9d_e}{2\tau_1^2}}}{3}. \tag{3.50}$$

Finally, a sufficient condition for $\varepsilon_j > -1$ is that $f_o(s_+) > -d_o$, where s_+ is the right endpoint of the spinodal interval of f_o, that is, $\min_{0\le u\le 1} f_o(u) = f_o(s_+)$:

$$s_+ = \frac{a_o + 1 + \sqrt{a_o^2 - a_o + 1}}{3}; \tag{3.51}$$

in particular, $\varepsilon_2 > \dfrac{f_o(s_+)}{d_o}$. Choose $\delta \in (-1,0)$ such that $\varepsilon_2 > \delta > \frac{f_o(s_+)}{d_o}$. Hence, if $H_0(S_-) - 3 + \delta > 0$, then $H_0(S_-) - 3 + \varepsilon_2 > 0$.

The next lemma collects all the requirements that a, d must satisfy in order to ensure that u_j, u_j^* exist; this is a result similar to Lemma 10. The first requirement is needed to guarantee that the translates are ordered as in (3.49) while the last two requirements guarantee that the top and bottom translates H_0 and $H_{-(3-\varepsilon_2)}$ have three distinct real zeroes. Finally, because the translates H_{-k} for $k = 0,1,2,3$ are squeezed in between H_0 and $H_{-(3-\varepsilon_2)}$, the proof of Lemma 10 will follow from the proof of the following lemma.

Lemma 12. *Consider the connection $(0,0)$ to (τ_1, τ_2) for a fixed (d,a) in a bistable region B. Assume that*

1. $f_o(s_+) + d_o > 0$,
2. $a_e^2 - a_e + 1 - \frac{9d_e}{2\tau_1^2} > 0$, *and*
3. $H_0(S_+) < 0 < H_0(S_-) - 3 + \delta$,

where $S_+, S-, s_-$ are defined by (3.50), (3.51), and $\delta \in (-1,0)$ satisfies $\delta d_o - f_o(s_+) > 0$. Then Φ possesses the shift on sequences on two symbols as a subsystem.

It is not easy to untangle the three conditions in Lemma 12 to get a pinning interval of the form similar to Lemma 10. To see that the above three conditions are viable assumptions, we apply a continuity argument on the parameters. Indeed, as $d \to 0^-$, we have that $f_o \to 0, d_o \to a/2$ so that the first requirement, asymptotically, is $a > 0$; as $d \to 0^-$, we have that $a_e \to a, d_e \to 0$ so that the second requirement is $1 > 0$; finally, as $d \to 0^-$, we have that $S_\pm \to T_\pm$, where (T_-, T_+) is the spinodal interval (3.23) of f_a so that the third inequality is $f_a(T_+) < 0 < f_a(T_-)$ (recall that $0 < a < 1$).

Minimum Wave Speed

The Fisher equation [24] is an example of a monostable scalar reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = du_{xx} + u(1-u), \qquad d > 0.$$

The monostable scalar spatially discrete reaction-diffusion LDE, that is, a homogeneous lattice Fisher equation is

$$\dot{u}_n(t) = d(u_{n-1} - 2u_n + u_{n+1}) - u_n(u_n - 1), \;\; d > 0, \; n \in \mathbb{Z}.$$

Let $f(u) = u(u-1)$, the Fisher nonlinearity. In both the PDE and the LDE, we have that $f(0) = f(1) = 0$ for $0 < u < 1$ and $f'(1) < 0 < f'(0)$; that is, the equilibrium point 0 is unstable while 1 is stable. The existence of a monotone traveling wavefront solution connecting 0 to 1 depends on the wave speed c - there is a number $c_* > 0$ such that a monotone traveling wavefront solution $u(x,t) = U(x - ct)$ (where $x \in \mathbb{R}$ for the PDE case while $x \in \mathbb{Z}$ for the LDE case) exists if and only if $c \geq c_*$. In other words, c_* is the smallest value of c for which there exists a monotone traveling wavefront and it is obtained by studying the linear stability of the equilibria of the system that is obtained by applying the traveling wave ansatz.

For the PDE case, a necessary condition for a monotone traveling wavefront solution to exist is

$$c \geq 2\sqrt{f'(0)} \tag{3.52}$$

while a sufficient condition for a monotone traveling wavefront to exist is

$$c \geq 2\sqrt{\beta}, \qquad \beta = \sup\left\{\frac{f(u)}{u} : 0 < u < 1\right\},$$

that is, $2\sqrt{f'(0)} \leq c_* \leq 2\sqrt{\beta}$. These are well-known results; for example, see [22]. For the Fisher nonlinearity, we have that $c_* = 2$ because $f'(0) = 1 = \beta$. The speed c_* is sometimes referred to as the linear minimum wave speed. In general, however, $\beta \neq f'(0)$ and for some choices of f, $c_* > 2$ (see [7]).

For the LDE case, the first results were obtained by Harris, Hudson, and Zinner [32] for the scalar case and by Hudson and Zinner [35] for the periodic case. In the scalar LDE case [32], if f, like the Fisher nonlinearity, satisfies the extra assumption $f'(0)x \geq f(x)$ for $x > 0$, then there is a traveling wavefront solution if and only if $d \leq \sup_{\lambda>0} \frac{\lambda c - f'(0)}{4\sinh^2(\lambda/2)}$, equivalently,

$$c \geq \inf_{\lambda>0} \frac{4d\sinh^2(\lambda/2) + f'(0)}{\lambda}. \tag{3.53}$$

In the periodic LDE case, Hudson and Zinner [35] obtained a sufficient condition for the existence of traveling wavefronts. A more recent study was conducted by Guo and Hamel [26], where they were able to obtain sufficient and necessary conditions for the existence of traveling wavefronts; however, their results are not directly applicable to the antidiffusion lattice Nagumo problem because [26] studied a slightly different system of periodic LDEs, a periodic LDE in divergence form:

$$\dot{u}_n(t) = d_{n+1}u_{n+1}(t) + d_n u_{n-1}(t) - (d_{n+1}+d_n)u_n(t) + f(u_n), \;\; t \in \mathbb{Z}.$$

They obtained that a sufficient and necessary condition for the existence of traveling wavefronts is that $c \geq c_*$, where $c_* = \min_{\lambda>0} \dfrac{M(\lambda)}{\lambda}$, where $M(\lambda)$ is the largest real eigenvalue of a certain matrix.

Let us now proceed to study the minimum wave speed for the antidiffusion lattice Nagumo problem in a region M where monostable dynamics occurs for the connection from $(0,0)$ to (τ_1, τ_2). Note that by our previous computations, M is a subset of $\{(d,a) : a+4d < 0, 0 < a < 1\}$.

Negative Diffusion

Consider the matrix $A(\lambda)$ defined for $\lambda > 0$ by

$$A(\lambda) = \begin{pmatrix} -(2d_e + f_e'(0)) & 2d_e \cosh\lambda \\ 2d_o \cosh\lambda & -(2d_o + f_o'(0)) \end{pmatrix}.$$

Note that $A(\lambda = 0)$ gives the linearization matrix at the point $(0,0)$. Let $M(\lambda)$ denote the largest positive eigenvalue of $A(\lambda)$. The eigenvalues of $A(\lambda)$ are given by $\dfrac{T \pm \sqrt{T^2 - 4D(\lambda)}}{2}$, where T is the trace given by $T = -2(d_e + d_o) - (f_e'(0) + f_o'(0))$ and $D(\lambda)$ is the determinant given by $D(\lambda) = (2d_e + f_e'(0))(2d_o + f_o'(0)) - 4d^2 \cosh^2 \lambda$. The discriminant is always nonnegative:

$$T^2 - 4D(\lambda) = ((2d_e + f_e'(0)) - (2d_o + f_o'(0)))^2 + 16d^2 \cosh^2 \lambda = 16d^2 \cosh^2 \lambda \geq 0$$

by using (3.12), and hence the eigenvalues of $A(\lambda)$ are both real.

The trace of $A(\lambda)$ is equal to

$$\begin{aligned} T &= -(2d_e + f_e'(0) + 2d_o + f_o'(0)) \\ &= -2d_e - (a - 2(d_e - d)) - 2d_o - (a - 2(d_o - d)) \\ &= -2(a + 2d). \end{aligned}$$

Denote the larger eigenvalue of $A(\lambda)$ by $M(\lambda)$. Then, noting that the minimum of $\cosh\lambda$ is 1, we have

$$M(\lambda) = (T + \sqrt{T^2 - 4D(\lambda)})/2 = -(a + 2d) - 2d\cosh\lambda \geq -(a + 4d) > 0.$$

In particular, $M(0) = -(a + 4d) > 0$. Since $M'(\lambda) = -2d\sinh\lambda$, we have $M'(0) = 0$. Hence, the minimum of $M(\lambda)/\lambda$ over all $\lambda > 0$ is achieved and is positive. We define c_*:

$$c_* = \min_{\lambda > 0} \frac{M(\lambda)}{\lambda}.$$

This value c_* is the so-called minimum wave speed for the traveling wavefront solution that connects $(0,0)$ to (τ_1, τ_2). Then, using an expansion of $\cosh\lambda$, we have

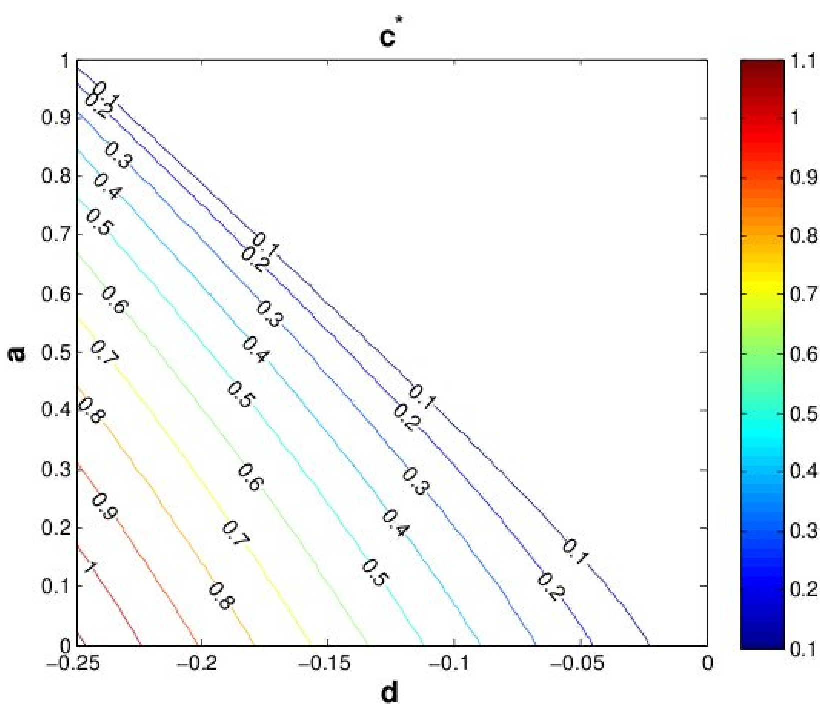
$$\begin{aligned} \frac{M(\lambda)}{\lambda} &= \frac{-a - 2d(1 + \cosh\lambda)}{\lambda} \\ &= \frac{-a}{\lambda} - 2d\left(\tfrac{2}{\lambda} + \tfrac{\lambda}{2!} + \tfrac{\lambda^3}{4!} + \tfrac{\lambda^5}{6!} + \ldots\right) \\ &> \frac{-a - 4d}{\lambda} - d\lambda \\ &= \frac{-a - 4d - d\lambda^2}{\lambda} := G(\lambda) \end{aligned}$$

for $\lambda > 0$. We see that $G'(\lambda) = 0 < G''(\lambda)$ when $\lambda = \sqrt{\dfrac{a+4d}{d}}$; that is, G is minimized at this λ. Thus, we have computed a (positive) lower bound for c^*:

$$c^* \geq G\left(\sqrt{\frac{a+4d}{d}}\right) = 2\sqrt{d(a+4d)}.$$

Compare this lower bound for c_* with the expression for the monostable scalar PDE case (3.52). In Figure 3.12 we show the computed values of the minimum wave speed c_* for the antidiffusion lattice Nagumo problem for $a+4d < 0$ and $0 < a < 1$.

Figure 3.12: This is a contour plot of c_* for $-1/4 < d < 0$ and $0 < a < 1$.



Finally, using a hyperbolic identity, we can further rewrite our result (compare with 3.53) in the following way:

$$c_* = \min_{\lambda>0} \frac{M(\lambda)}{\lambda} = \min_{\lambda>0} \frac{-a-2d(1+\cosh\lambda)}{\lambda} = \min_{\lambda>0} \frac{-f_a'(0) - 4d\cosh^2(\lambda/2)}{\lambda}.$$

3.4.2 Case Study: (τ_2, τ_1) to (τ_1, τ_2) connection

In this case, we want to know whether there exists a traveling wavefront solution to (3.3) that connects $(x_-, y_-) = (\tau_2, \tau_1)$ and $(x_+, y_+) = (\tau_1, \tau_2)$. Here, $\alpha = -\beta = \tau_1 - \tau_2 > 0$ and the diffusion-coupling parameters are $d_e = -d = d_o$. This case study is interesting because the boundary conditions are either both stable or both unstable (by Lemma 4), and hence monostable dynamics cannot occur. Furthermore, unlike the first case study, the intermediate point is not unique. In fact, in the regions where there are nine equilibria, there are seven intermediate points; where there are seven equilibria, there are five intermediate points; and where there are five equilibria, there are three intermediate points.

For example, consider the region

$$R = \{(d,a) : a + 4d < 0, 1 - a + 4d < 0, a(a-1) + 4d < 0\}.$$

This is the region in Figure 3.1 where there are only five distinct real zeroes to the ninth-degree polynomial g. The set R is quadrant III_{in} in the proof of Lemma 2. In R, for any value of a, the points $(0^\pm, 0^\mp), (1^\pm, 1^\mp), (a^\pm, a^\mp)$ do not exist and the points $(0,0), (a,a), (1,1)$ are unstable.

Thus, all three intermediate points between (τ_2, τ_1) and (τ_1, τ_2) are unstable. If $f_a'(\tau_1) + 2d > 0$ in R then the boundary conditions are stable so that the antidiffusion lattice Nagumo system has bistable dynamics. Since $\tau_1 > 1$, we have that $2d + f_a'(\tau_1) > 0$ if $1 - a + 2d > 0$. Therefore, the set of parameter values in the set

$$B = \{a + 4d < 0, 1 - a + 2d > 0 > 1 - a + 4d, a(a-1) + 4d < 0\}$$

will give rise to bistable dynamics to the antidiffusion lattice Nagumo system.

For completeness, let us look at the case where E has nine equilibria and determine which of the intermediate points from (τ_2, τ_1) to (τ_1, τ_2) will give rise to bistable dynamics, if any.

Consider the set of parameter values in the region

$$T = \{(d,a) : a + 4d > 0, 1 - a + 4d > 0\};$$

this is quadrant I in the proof of Lemma 2, where the equilibria $(0^{\pm}, 0^{\mp})$ and $(1^{\pm}, 1^{\mp})$ exist. Since $1 - a + 4d > 0$ implies that $1 - a + 2d > 0$, we have $2d + f_a'(\tau_1) > 0$; that is, the boundary conditions (τ_1, τ_2) and (τ_2, τ_1) are both stable. Bistable dynamics will not occur in this case because there are more than one stable intermediate points, namely $(0,0)$ and $(1,1)$. In fact, if $2d + f_a'(0^+) < 0$ and $2d + f_a'(1^+) < 0$, we see that in the chain from (x_-, y_-) to (x_+, y_+), there are four stable equilibria, underlined for emphasis:

$$\underline{(\tau_2, \tau_1)} \to (0^-, 0^+) \to \underline{(0,0)} \to (0^+, 0^-) \to (a,a) \to (1^-, 1^+) \to \underline{(1,1)} \to (1^+, 1^-) \to \underline{(\tau_1, \tau_2)}.$$

3.4.3 Co-Existence

In this final section of examples, we show that there are regions in the parameter space where bistable and monostable dynamics in the antidiffusion lattice Nagumo system co-exist. Define the parameter region

$$T = \{(d,a) : a + 4d > 0, 1 - a + 4d > 0\};$$

this is quadrant I in the proof of Lemma 2, where $0 < a < 1$. To show that bistable and monostable dynamics can both occur in T, we look at the stability of each of the nine equilibria.

Since $1 - a + 2d > 0$ in T, the equilibria (τ_1, τ_2) and (τ_2, τ_1) are stable. Also, for (d,a) in T, both $(0,0)$ and $(1,1)$ are stable because $a + 4d > 0$ and $1 - a + 4d > 0$, respectively; however, (a,a) is unstable because $a(a-1) + 4d < 0$. The equilibria $(0^{\pm}, 0^{\mp})$ are unstable if $2d + f_a'(0^+) < 0$ and the equilibria $(1^{\pm}, 1^{\mp})$ are unstable if $2d + f_a'(1^+) < 0$.

By ellipticity, we must require that $\alpha\beta < 0$. Hence, for example, even if $(0,0)$ and $(1,1)$ are both stable, the theory cannot be applied with these as boundary conditions because $\alpha = \beta = 1$. The following table summarizes the connections that exist between any two equilibria in $\hat{E}$ that satisfy $\alpha\beta < 0$ and whether there is bistable or monostable dynamics. In the table, $\gamma \in \{0,1\}$

and $\gamma^{\pm} \in \{0^{\pm}, 1^{\pm}\}$.

(x_-, y_-)	(x_+, y_+)	Intermediate points	Monostable or bistable
(τ_2, τ_1)	$(0^-, 0^+)$	none	monostable dynamics
	$(0, 0)$	$(0^-, 0^+)$	bistable dynamics if $2d + f_a'(0^+) < 0$
	(a, a)	none	monostable dynamics
	$(1^-, 1^+)$	none	monostable dynamics
	$(1, 1)$	$(1^-, 1^+)$	bistable dynamics if $2d + f_a'(1^+) < 0$
	(τ_1, τ_2)	other 7 equilibria	see Section 5.2
$(\gamma^{\mp}, \gamma^{\pm})$	(τ_2, τ_1)	none	monostable dynamics
	(γ, γ)	none	monostable dynamics
(γ, γ)	(τ_2, τ_1)	(γ^-, γ^+)	bistable dynamics if $2d + f_a'(\gamma^+) < 0$
	(τ_1, τ_2)	(γ^+, γ^-)	bistable dynamics if $2d + f_a'(\gamma^+) < 0$
	(γ^-, γ^+)	none	monostable dynamics
	(γ^+, γ^-)	none	monostable dynamics
(a, a)	(τ_2, τ_1)	none	monostable dynamics
	(τ_1, τ_2)	none	monostable dynamics
(τ_1, τ_2)	(τ_2, τ_1)	other 7 equilibria	see Section 5.2
	$(0, 0)$	$(0^+, 0^-)$	bistable dynamics if $2d + f_a'(0^+) < 0$
	$(0^+, 0^-)$	none	monostable dynamics
	(a, a)	none	monostable dynamics
	$(1, 1)$	$(1^+, 1^-)$	bistable dynamics if $2d + f_a'(1^+) < 0$
	$(1^+, 1^-)$	none	monostable dynamics

To determine whether there is monostable or bistable dynamics from (x_-, y_-) to (x_+, y_+), we determine the stability of $(x_{\pm}, y_{\pm})$ and the intermediate point, if such stability exists. From the table, we observe, for example, assuming $2d + f_a'(0^+) < 0$, that there is monostable dynamics from (τ_2, τ_1) to $(0^-, 0^+)$ and from $(0^-, 0^+)$ to $(0,0)$ while there is bistable dynamics from (τ_2, τ_1) to $(0,0)$; that is, like the positive diffusion problem, bistable dynamics may be viewed as a concatenation of two monostable dynamics that share an equilibrium.

Chapter 4

Lattice with a Single Defect

4.1 Introduction

We extend the classical notion of traveling wave solutions to one-dimensional lattice differential systems in case the medium over which the solution propagates is inhomogeneous. The inhomogeneity in the medium is given by a single defect in the coupling coefficients of the equations. We prove that there exists a time-global solution that behaves like a traveling wave front. The main technique is to construct appropriate sub- and super-solution pairs.

Consider the lattice differential system of equations

$$\dot{u}_j = d_{j+1}(u_{j+1} - u_j) + d_j(u_{j-1} - u_j) + f(u_j), \qquad j \in \mathbb{Z}, \tag{4.1}$$

where d_j are positive numbers and each u_j is a function of t. The dot notation represents derivative with respect to t. The coupling coefficients d_j determine how the j^{th}-component function of $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$ is related to its nearest neighbors $u_{j\pm1}(t)$. Here, we assume that f satisfies

$$f(0) = f(1) = 0, \qquad f'(0) < 0, \qquad f'(1) < 0. \tag{4.2}$$

In applications, it is considered that inhomogeneity is a characteristic of natural phenomenon as in the case of impulse propagation (see [49], [52], [53] and references therein) . In research, one usually starts with analyzing the homogeneous case and then extending the results to the inhomogeneous case. When $d_j = d$ for all j, we say that the system (4.1) is homogeneous over the one-dimensional lattice $\mathbb{Z}$, if, moreover, the bilinearity assumption (4.2) on f takes the cubic form $f(u) = u(u-a)(1-u)$ for some $0 < a < 1$, one obtains the classic lattice Nagumo system

$$\dot{u}_j = d(u_{j+1} - 2u_j + u_{j-1}) + f(u_j), \qquad \text{for } j \in \mathbb{Z},\ t \in \mathbb{R}. \tag{4.3}$$

For a sufficiently large diffusion coefficient d, Zinner ([57], [58]) proved the existence of a traveling wavefront solution $(\phi, c > 0)$ to the (homogeneous) lattice Nagumo system (4.3). This solution satisfies the traveling wave equation

$$c\phi'(\xi) = d(\phi(\xi+1) - 2\phi(\xi) + \phi(\xi-1)) + f(\phi(\xi)), \qquad \xi := j + ct \in \mathbb{R}, \tag{4.4}$$

which is obtained by setting $u_j(t) = \phi(\xi)$ in (4.3), with boundary conditions

$$\phi(-\infty) = 0, \qquad \phi(+\infty) = 1. \tag{4.5}$$

It is also known that $0 < \phi < 1$ and ϕ is monotonically increasing in ξ. On the other hand, Keener ([39]) proved that no such traveling wavefront solution exists if d is small.

When $d_j \neq d$ for at least one node j, we say that (4.1) is defined over an inhomogeneous medium. In this paper, we will investigate the lattice dynamical system (4.1) when we have a single defect over the lattice $\mathbb{Z}$ given by

$$d_j = \begin{cases} d, & j \neq 0, \\ d_0, & j = 0, \end{cases} \tag{4.6}$$

for some $d_0 \neq d$. In this case, we say that there is a local perturbation of the diffusion coefficients at the node $j = 0$. The system (4.1) then becomes

$$\dot{u}_j = \begin{cases} d(u_{j-1} - 2u_j + u_{j+1}) + f(u_j), & j \neq -1, 0 \\ d_{j+1}(u_{j+1} - u_j) + d_j(u_{j-1} - u_j) + f(u_j), & j = -1, 0 \end{cases} \tag{4.7}$$

In other words, a single diffusion defect gives rise to a defective homogeneous lattice Nagumo system so that the results in the un-defective (that is, homogeneous) system are not directly transferable to the defective (that is, inhomogeneous) system. In matrix form, the system (4.7) can be written as $\dot{u}(t) = Au + F(u)$ where A is the tri-diagonal infinite matrix of coupling coefficients

$$A = \begin{pmatrix} \ddots & & & & & & & \\ \ldots & d & -2d & d & & & & \ldots \\ \ldots & & d & -(d+d_0) & d_0 & & & \ldots \\ \ldots & & & d_0 & -(d_0+d) & d & & \ldots \\ \ldots & & & & d & -2d & d & \ldots \\ & & & & & \ddots & & \end{pmatrix} \tag{4.8}$$

and $F(u) = \{f_j(u_j)\}_{j\in\mathbb{Z}}$, where $f_j(u) = u(u-a)(1-u)$. Observe that the inhomogeneity of the system is captured in the middle portion of the symmetric matrix A.

The set $K := \{-1, 0\}$ will be called the obstacle set and define $\Omega := \mathbb{Z} \setminus K$. The inhomogeneity of the medium, in this case given by a nonempty set K, precludes the difficulty in proving the existence of traveling wavefront solutions (4.7) because solutions may not be translationally invariant anymore. However, since there is only one defect in the middle of the medium, one would guess that there would be a solution to the inhomogeneous equations (4.7) that is close to the traveling wavefront solution (ϕ, c) of the homogeneous equations (4.3). In this paper, we will show that there is a time-global solution $u_j(t)$ that behaves *like* a traveling wavefront solution, see Theorem A, which is the main result of this paper. In the next chapter, we will consider the case when there are many defects in the middle of $\mathbb{Z}$.

Theorem A - Main Result. *Suppose $|d-d_0|>0$ is a sufficiently small number. The inhomogeneous lattice differential equation (4.1) with a single diffusion-defect defined by $d_j = d$ for $j \neq 0$ and $d_j = d_0$ for $j = 0$ has a time-global solution $u_j(t)$ for $j \in \Omega$ such that*

1. *for all $t \in \mathbb{R}$ and $j \in \Omega$, we have $0 < u_j(t) < 1$, $\dot{u}_j(t) > 0$ and*

2. *$u_j(t) - \phi(j+ct) \to 0$ uniformly in both $j \in \Omega$ and $t \in \mathbb{R}$, that is,*

$$\sup_{j\in\Omega} |u_j(t) - \phi(j+ct)| \to 0, \qquad \text{as } |t| \to +\infty \tag{4.9}$$

and

$$\sup_{t\in\mathbb{R}} |u_j(t) - \phi(j+ct)| \to 0, \qquad \text{as } |j| \to +\infty \tag{4.10}$$

Since the obstacle set K is nonempty, one has to be careful in proving that the time-global solution $u_j(t)$ is close to the traveling wavefront solution (ϕ, c) in the sense described above. Notice that in the homogeneous case (see Theorem B), we send $j+ct$ to $\pm\infty$ while in the inhomogeneous case, we consider large space $|j|$ and large time $|t|$ separately. For comparison purposes, the analogous result in the (homogeneous) lattice Nagumo equation is:

Theorem B. *For a sufficiently large diffusion coefficient $d > 0$, the homogeneous lattice differential equation (4.3) has a traveling wave solution (ϕ, c) that satisfies (4.4) with*

1. *$0 < \phi < 1$, $\phi' > 0$ and*

2. *$\phi(j+ct) \to 0$ as $j+ct \to -\infty$ and $\phi(j+ct) \to 1$ as $j+ct \to +\infty$*

The homogeneous case supports an analysis via the wave equation (4.4), which is a mixed-type functional differential equation (MFDE). There are many existing results on how to analyze MFDEs, notably, research done by Chow, Hoffman, Hupkes, Lamb, Mallet-Paret, Shen, Van

Vleck, Wright, and many others (see references). The inhomogeneous case, however, does not support an equation analogous to (4.4) because there are no true traveling waves. Observe that if one substitutes $u_j(t) = \phi(j + c_j(t))$ into (4.1), then one obtains

$$\dot{c}_j \phi'(\xi_j) = d_{j+1}(\phi(\xi_{j+1}) - \phi(\xi_j)) + d_j(\phi(\xi_{j-1}) - \phi(\xi_j)) + f(\phi(\xi_j)),$$

where $\xi_j = j + c_j(t) \in \mathbb{R}$. The variable ξ_j takes into account that c_j may change as ϕ travels over $\mathbb{Z}$. We will not analyze (4.1) using such an ansatz. See [36] for an analysis that uses a slightly similar ansatz.

Our main result in this paper is that over an inhomogeneous medium with a single defect, there is a solution that can be considered as an extension of a traveling wave. Numerical simulations suggest that d_0 cannot be too different from d; indeed, if d_0 is too big, then a solution that starts out like a traveling wavefront at negative time may not survive the defect that exists in the middle of $\mathbb{Z}$. The requirements on d_0 are contained in the following:

Assumption on diffusion defect. *If $d - d_0 > 0$ then we assume that*

$$0 < \max\{1, R_1^+\} < \frac{d}{d_0} < R_2^+.$$

If $d - d_0 < 0$ then we assume that

$$0 < R_2^- < \frac{d}{d_0} < \min\{1, R_1^-\}.$$

The values of the constants $R_{1,2}^{\pm}$ can be found in the latter part of Section 4. These constants depend on f and on the particular wavefront ϕ (hence, on the detuning parameter a).

The proof of Theorem A - Main Result proceeds as follows. The convergence in (4.9) will imply the convergence in (4.10). The asymptotic behavior (4.9) is accomplished by first constructing, via the Maximum Principle, a time-global solution $u_j(t)$ to (4.7) that are asymptotic to ϕ for large negative time and uniformly over Ω (see Lemma A). Once the existence of such a solution has been proved (it is unique, see Lemma A), we then show that this solution is close to the traveling wavefront (ϕ, c) for large $|j|$ at some time and it converges to 1 for $j \in K$ as $t \to +\infty$ (see Lemma B). Using Lemma C, Lemma B implies the uniform convergence of $u_j(t)$ to $\phi(j+ct)$ for large positive time uniformly over Ω. The conclusion of Lemma C is the convergence in (4.9).

Theorem A is the spatially discrete analogue of Theorem 1.3 in a paper by Berestycki, Hamel, and Matano [8] (see also [9]), where they investigated the existence of solutions of a semilinear parabolic PDE $u_t = \Delta u + f(u)$ that is defined over $\mathbb{R}^N \setminus K$, where K is a compact obstacle. The main ideas in the present paper come from [8]. The results and techniques in [8], [9] have also been used in a defective scalar reaction-diffusion equation of the ignition type with a random, stationary and ergodic reaction rate (see [48]).

Lemma A - Negative Time. *The inhomogeneous lattice differential equation (4.1) with a single diffusion-defect defined by $d_j = d$ for $j \neq 0$ and $d_j = d_0$ for $j = 0$ has a time-global solution $u_j(t)$ for $j \neq -1, 0$ such that for all $t \in \mathbb{R}$ and $j \neq -1, 0$, we have $0 < u_j(t) < 1$, $\dot{u}_j(t) > 0$, and*

$$\sup_{j \neq -1,0} |u_j(t) - \phi(j+ct)| \to 0, \qquad \text{as } t \to -\infty. \tag{4.11}$$

Furthermore, the convergence in (4.11) determines a unique time-global solution $u_j(t)$ for $j \neq -1, 0$.

Lemma B. *The solution $u = \{u_j\}_{j \in \mathbb{Z}}$ from Lemma A satisfies the following: for any $\varepsilon > 0$, there exists a time $t_\varepsilon \geq 0$ and there exists an integer $n_\varepsilon > 1$ such that*

$$|u_j(t_\varepsilon) - \phi(j+ct_\varepsilon)| \leq \varepsilon, \ \ \text{for all} \ \ |j| \geq n_\varepsilon \tag{4.12}$$

and

$$u_j(t) \geq 1-\varepsilon, \;\; \textit{for all} \;\; t \geq t_\varepsilon; \;\; \textit{for} \;\; j=-1,0. \tag{4.13}$$

The next lemma, Lemma C, guarantees that the solution $u(t) = \{u_j(t)\}$ from Lemma A behaves like a wavefront as $t \to +\infty$ provided that we can find a positive time t_ε such that the solution $u(t_\varepsilon) = \{u_j(t_\varepsilon)\}_{j\in\mathbb{Z}}$ is arbitrarily close to $\phi(j+ct_\varepsilon)$ for almost all integers except finitely many consecutive ones in the middle of $\mathbb{Z}$. Recall that Lemma A guarantees the desired asymptotic behavior at negative time; Lemma C will prove the desired asymptotic behavior at positive time.

Lemma C - Positive Time. *Assume Lemma B is satisfied. Then*

$$\sup_{j\neq -1,0} |u_j(t) - \phi(j+ct)| \to 0, \qquad \textit{as } t \to \infty.$$

4.2 Behavior at Negative Time

In this section, we prove the existence of a time-global solution of (4.7) that is monotone increasing in t and converges to the traveling wave solution $\phi(j+ct)$ as $t \to -\infty$ uniformly in $j \in \Omega$. The main idea is to construct suitable super- and sub-solutions. We adapt the approach set by [8], [28], and [29], where they constructed a super-/sub-solution pair that at $-\infty$, sandwiches a desired entire solution to a continuous reaction-diffusion equation. See also [13], [15], [25], [30], [42], [46], [47]

Recall that in the homogeneous case (4.3), there exists a wave profile $\phi : \mathbb{R} \times \mathbb{R}$ and there exists a wave speed $c > 0$ that satisfy the wave equation (4.4) which is true for *all* wave variables $\xi \in \mathbb{R}$; this property clearly does not hold for the inhomogeneous case. Traveling wave solutions, when they exist, are translation invariant in a sense that any translate of ϕ is also a solution, that is, there exists a number θ such that $\phi(\xi) = \phi(\xi+\theta)$; and hence in order to work with a unique wave profile, let us fix a translate, say $\phi(0) = a$, where a is the middle zero of the cubic nonlinearity $f(u) = u(u-a)(1-u)$.

The next step is to record some known asymptotic bounds on ϕ, ϕ'; then, we construct candidate solutions and use the estimates to verify that such solutions form a super-/sub-solution pair.

4.2.1 Asymptotic Estimates

There are positive constants $k_{\pm}, K_{\pm}$ such that

$$k_{-}e^{\lambda z} \leq \phi(z) \leq K_{-}e^{\lambda z}, \qquad z \leq 0 \tag{4.14}$$

$$k_{+}e^{-\mu z} \leq 1-\phi(z) \leq K_{+}e^{-\mu z}, \qquad z \geq 0 \tag{4.15}$$

where $\mu, \lambda > 0$ satisfy the following characteristic equations

$$c\lambda = d(e^{\lambda} + e^{-\lambda} - 2) + f'(0) \tag{4.16}$$

$$-c\mu = d(e^{\mu} + e^{-\mu} - 2) + f'(1), \tag{4.17}$$

respectively. With our chosen translate, note that we have $0 < k_{-} \leq a \leq K_{-}$ and $k_{+} \leq 1-a \leq K_{+}$.

Observe that if $\lambda < \mu$ then $f'(1) < f'(0)$, or for the cubic nonlinearity, $0 < a < 1/2$. To see this, define the functions $G_0(x) = -cx + d(e^x + e^{-x} - 2) + f'(0)$ and $G_1(x) = cx + d(e^x + e^{-x} - 2) + f'(1)$ for $x > 0$. Since G_1 is increasing, for $0 < \lambda < \mu$ we have that $G_1(\lambda) < G_1(\mu) = 0 = G_0(\lambda)$. However, if it were true that $f'(1) - f'(0) > 0$ then we would have

$(G_1 - G_0)(x) = 2cx + f'(1) - f'(0) > 0$ for all $x > 0$. Thus, we have

$$\lambda < \mu \Rightarrow 0 < a < 1/2. \tag{4.18}$$

The derivative of ϕ also satisfies a similar bound: for some positive numbers $r_{\pm}, R_{\pm}$, we have

$$r_- e^{\lambda z} \le \phi'(z) \le R_- e^{\lambda z}, \qquad z \le 0 \tag{4.19}$$

$$r_+ e^{-\mu z} \le \phi'(z) \le R_+ e^{-\mu z}, \qquad z > 0 \tag{4.20}$$

With our chosen translate, we have $0 < r_{\pm} \le \phi'(0) \le R_{\pm}$. Finally, we also need the following uniform estimate for a function $f \in C^2$ with $f(0) = 0$: there exists $L > 0$ such that for $u, v \in [0,1]$, we have

$$|f(u+v) - f(u) - f(v)| \le Luv. \tag{4.21}$$

In fact, for the bistable nonlinearity $f(u) = u(u-a)(1-u)$, we have that

$$f(u+v) - f(u) - f(v) = uv(-3(u+v) + 2(a+1)). \tag{4.22}$$

For $(u,v) \in [0,1] \times [0,1]$, we have

$$L = \max|-3(u+v) + 2(a+1)| = \begin{cases} 4 - 2a, & 0 < a < 1/2 \\ 2(a+1), & 1/2 < a < 1 \end{cases}$$

4.2.2 Super-solution, Sub-solution

Define the following operators on a sequence of functions $w(t) = \{w_j(t)\}_{j\in\mathbb{Z}}$:

$$
\begin{aligned}
(Dw)_j(t) &= d(w_{j-1}(t) - 2w_j(t) + w_{j+1}(t)) \\
(\Delta w)_j(t) &= d_{j+1}(w_{j+1}(t) - w_j(t)) + d_j(w_{j-1}(t) - w_j(t))
\end{aligned}
\tag{4.23}
$$

In the single-defect problem with $d_j = d_0$ for $j = 0$ and $d_j = d$ elsewhere, we see that $(\Delta w)_j = (Dw)_j$ for $j \neq -1, 0$. Using these two operators, the system (4.7) can be written

$$
\dot{u}_j = \begin{cases} (Dw)_j(t) + f(u_j(t)), & j \neq -1, 0 \\ (\Delta w)_j(t) + f(u_j(t)), & j = -1, 0 \end{cases}
\tag{4.24}
$$

From (4.24), define the operator $\mathscr{L}$ on the sequence $w = \{w_j\}_{j\in\mathbb{Z}}$

$$
(\mathscr{L}w)_j(t) = \begin{cases} \dot{w}_j(t) - (Dw)_j(t) - f(w_j(t)), & j \neq -1, 0 \\ \dot{w}_j(t) - (\Delta w)_j(t) - f(w_j(t)), & j = -1, 0, \end{cases}
\tag{4.25}
$$

for all $t \in \mathbb{R}$.

Definition 1. *The functions $u = \{u_j\}_{j\in\mathbb{Z}}, v = \{v_j\}_{j\in\mathbb{Z}}$ for $(j,t) \in \mathbb{Z} \times (-\infty, -T)$ for some $T > 0$ are called a super-/sub-solution pair of (4.7) if $u_j(t) \leq v_j(t)$ and $(\mathscr{L}u)_j(t) \leq 0 \leq (\mathscr{L}v)_j(t)$ for all $j \in \mathbb{Z}$ and $t \leq -T$, where $\mathscr{L}$ is defined by (4.25).*

The super-/sub- solution pair that we will see in the succeeding sub-sections are defined using the traveling wave solution (ϕ, c) to the homogeneous lattice Nagumo system (4.3) and the function ξ whose t-derivative is

$$
\dot{\xi}(t) = Me^{\lambda(ct+\xi(t))}, \qquad t \leq -T,
\tag{4.26}
$$

where $M > 0$ is to be determined and λ is one of the characteristic values (see (4.16)). The value of T is given by

$$T := \frac{1}{\lambda c} \ln\left(\frac{c}{c+M}\right) > 0.$$

For $t \le -T$, we have that $0 < 1 - \frac{M}{c}e^{\lambda ct} < 1$ and so an explicit expression for ξ is

$$\xi(t) = \frac{1}{\lambda} \ln\left(\frac{1}{1 - \frac{M}{c}e^{\lambda ct}}\right) > 0.$$

Note that $ct + \xi(t) \le 0$ for $t \le -T$. Since $\dot{\xi}(-\infty) = 0$, the functions

$$p(t) := ct + \xi(t), \qquad q(t) := ct - \xi(t) \tag{4.27}$$

have derivatives that approach the wave speed c. Now we are ready to construct a super-/sub-solution pair of (4.7).

4.2.3 Super-solution

Consider the function $v^+(t) = \{v_j^+(t)\}_{j\in\mathbb{Z}}$ defined by

$$v_j^+(t) = \begin{cases} \phi(j + p(t)) + \phi(-j + p(t)), & j > 0, \\ 2\phi(p(t)), & j \le 0, \end{cases} \tag{4.28}$$

where $t \le -T$. We need to find conditions on M (see (4.26)) that will guarantee that $(\mathscr{L}v^+)_j(t) \ge 0$ for each integer. Recall that $p(t) = ct + \xi(t)$. For $j \le 0$, v_j^+ is independent of j so that both $(Dw)_j$ and $(\Delta w)_j$ are zero, and hence we have

$$(\mathscr{L}v^+)_j(t) = 2\dot{p}(t)\phi'(p(t)) - f(2\phi(p(t))),$$

so that $(\mathscr{L}v^+)_j(t) \geq 0$ if we require that $f(2\phi(p(t))) \leq 0$, that is, with our given bistable nonlinearity, we must choose a sufficiently negative time $-T_1$ such that

$$0 < \phi(p(t)) < \frac{a}{2}, \qquad \text{for } t \leq -T_1.$$

The number T_1 exists because the range of p is $(-\infty, p(0)]$ where $p(0) \leq 0$ and we have chosen $\phi(0) = a$. Now suppose that $j > 0$. We compute $(\mathscr{L}v^+)_j(t)$ as follows, where we drop the dependence on t for brevity:

$$\begin{aligned}(\mathscr{L}v^+)_j &= \dot{p}(\phi'(j+p)+\phi'(-j+p)) - d(\phi(j-1+p)-2\phi(j+p)+\phi(j+1+p)) \\ &\quad -d(\phi(-j-1+p)-2\phi(-j+p)+\phi(-j+1+p)) \\ &\quad -f(\phi(j+p)+\phi(-j+p)) \\ &= \dot{p}(\phi'(j+p)+\phi'(-j+p)) - c\phi'(j+p)+f(\phi(j+p)) \\ &\quad -c\phi'(-j+p)+f(\phi(-j+p))-f(\phi(j+p)+\phi(-j+p)) \\ &= \dot{\xi}(\phi'(j+p)+\phi'(-j+p))+G \end{aligned} \tag{4.29}$$

where

$$G = -f(\phi(j+p)+\phi(-j+p))+f(\phi(j+p))+f(\phi(-j+p)), \qquad j \geq 1. \tag{4.30}$$

Note that in some cases, $G > 0$ (see case when $\lambda < \mu$ below). Using (4.21), we have that

$$(\mathscr{L}v^+)_j \geq \dot{\xi}(\phi'(j+p)+\phi'(-j+p)) - L\phi(j+p)\phi(-j+p),$$

or using the definition of $\dot{\xi}$, we have

$$(\mathscr{L}v^+)_j \geq e^{\lambda p}(\phi'(j+p)+\phi'(-j+p))(M-\mathscr{G}), \tag{4.31}$$

where $\mathscr{G} = \dfrac{Le^{-\lambda p}\phi(j+p)\phi(-j+p)}{(\phi'(j+p)+\phi'(-j+p))}$.

Observe that the arguments of both ϕ and ϕ' lead us to consider two cases: $-j+p<j+p<0$ and $-j+p<0<j+p$ (recall that $j>0$). Assume the first case. Then using (4.14) and (4.19),

$$\begin{aligned}\mathscr{G} &\leq \frac{Le^{-\lambda p}\phi(-j+p)\phi(j+p)}{\phi'(j+p)} \\ &\leq \frac{LK_-^2 e^{-\lambda p}e^{\lambda(-j+p)}e^{\lambda(j+p)}}{r_- e^{\lambda(j+p)}} \\ &< \frac{LK_-^2}{r_-}e^{-\lambda j}e^{\lambda p} \\ &< \frac{LK_-^2}{r_-}e^{\lambda p},\end{aligned}$$

Thus, (4.31) becomes

$$(\mathscr{L}v^+)_j \geq e^{\lambda p}(\phi'(j+p)+\phi'(-j+p))(M-\frac{LK_-^2}{r_-})$$

or, we should require that

$$M > \frac{LK_-^2}{r_-}. \tag{4.32}$$

Now assume that $-j+p<0<j+p$. Then using (4.14) and (4.20),

$$\begin{aligned}\mathscr{G} &\leq \frac{Le^{-\lambda p}\phi(-j+p)}{\phi'(j+p)} \\ &\leq \frac{LK_- e^{-\lambda p}e^{\lambda(-j+p)}}{r_+ e^{-\mu(j+p)}} \\ &= \frac{LK_-}{r_+}e^{-\lambda p}e^{\lambda(-j+p)}e^{\mu(j+p)}\end{aligned} \tag{4.33}$$

Consider two sub-cases: if $\lambda>\mu$ then $e^{-(\lambda-\mu)x}<1$ so that continuing from the above inequalities,

$$\mathscr{G} \leq \frac{LK_-}{r_+}e^{-(\lambda-\mu)p}e^{-(\lambda-\mu)j}e^{\lambda p} < \frac{LK_-}{r_+}e^{\lambda p}.$$

Thus, if $\lambda > \mu$, (4.31) becomes

$$(\mathscr{L}v^+)_j \geq e^{\lambda p}(\phi'(j+p)+\phi'(-j+p))(M-\frac{LK_-}{r_+}) \tag{4.34}$$

or, we should require that

$$M > \frac{LK_-}{r_+}. \tag{4.35}$$

However, if $\lambda < \mu$ then by (4.18), $0 < a < 1/2$ which guarantees that $\frac{2}{3}(a+1) < 1$. Now, with $u = \phi(j+p), v = \phi(-j+p)$, G in (4.30) becomes $G = -uv(-3(u+v)+2(a+1))$ using (4.22) which is always nonnegative provided

$$u+v > \frac{2}{3}(a+1). \tag{4.36}$$

Since $\phi(+\infty) = 1$, we can always choose $L_1 > 0$ to be a sufficiently large number so that if $j+p > L_1$ then $1 > \phi(j+p) > \frac{2}{3}(a+1)$. Thus, for sufficiently large $j+p$, (4.36) is satisfied so that from the last line of (4.29), $(\mathscr{L}v)_j > G \geq 0$. In case $0 < j+p < L_1$, we have that $e^{-\mu(j+p)} > e^{-\mu L_1}$ then continuing from the inequality (4.33) into (4.31),

$$Me^{-\mu(j+p)} - \frac{LK_-}{r_+} > Me^{-\mu L_1} - \frac{LK_-}{r_+}$$

so that $(\mathscr{L}v^+)_j \geq 0$ if we require that

$$M > \frac{LK_-}{r_+}e^{\mu L_1}. \tag{4.37}$$

Lemma 13. *If M is chosen such that it satisfies (4.32), (4.35), (4.37) then $v^+(t)$ defined by (4.28) satisfies $(\mathscr{L}v^+)_j(t) \geq 0$ for $j \in \mathbb{Z}$ and $t \leq T_1 < 0$, for some sufficiently large $-T_1$.*

4.2.4 Sub-solution

Consider the function $v^-(t) = \{v_j^-(t)\}_{j\in\mathbb{Z}}$ defined by

$$v_j^-(t) = \begin{cases} \phi(j+q(t)) - \phi(-j+q(t)), & j > 0 \\ 0, & j \le 0 \end{cases} \tag{4.38}$$

where $t \le -T$. We need to find conditions on M (see (4.26)) that will guarantee that $(\mathscr{L}v^-)_j(t) \le 0$ for $j \in \mathbb{Z}$. Recall that $q(t) := ct - \xi(t) < ct + \xi(t) \le 0$. Note also that $v_j^-(t) \le v_j^+(t)$.

For negative integers j, it is clear that $(\mathscr{L}v^-)_j(t) \le 0$. Suppose $j > 0$. We then compute (again dropping dependence on t for brevity):

$$\begin{aligned} (\mathscr{L}v^-)_j &= (\phi'(j+q) - \phi'(-j+q))(c - \dot{\xi}) - d(\phi(j-1+q) - 2\phi(j+q) + \phi(j+1+q)) \\ &\quad + d(\phi(-j-1+q) - 2\phi(-j+q) + \phi(-j+1+q)) \\ &\quad - f(\phi(j+q) - \phi(-j+q)) \\ &= (\phi'(j+q) + \phi'(-j+q))(c - \dot{\xi}) - c\phi'(j+q) + f(\phi(j+q)) \\ &\quad + c\phi'(-j+q) - f(\phi(-j+q)) - f(\phi(j+q) - \phi(-j+q)) \\ &= -\dot{\xi}(\phi'(j+q) - \phi'(-j+q)) + H, \end{aligned} \tag{4.39}$$

where

$$H = f(\phi(j+q)) - f(\phi(-j+q)) - f(\phi(j+q) - \phi(-j+q)).$$

Using (4.21), we have

$$H \le L\phi(-j+q)(\phi(j+q) - \phi(-j+q)),$$

so that

$$(\mathscr{L}v^-)_j \le -\dot{\xi}(\phi'(j+q) - \phi'(-j+q)) + L\phi(-j+q)(\phi(j+q) - \phi(-j+q)).$$

As before, we consider two cases: $-j+q<j+q<0$ and $-j+q\leq 0<j+q$.

Suppose that $-j+q<j+q<0$. It can be shown that $\phi'(j+q)-\phi'(-j+q)\geq K(\phi(j+q)-\phi(-j+q))$ for some positive constant K. Then

$$\begin{aligned}(\mathscr{L}v^-)_j &\leq -\dot{\xi}K(\phi(j+q)-\phi(-j+q))+L\phi(-j+q)(\phi(j+q)-\phi(-j+q))\\ &= (\phi(j+q)-\phi(-j+q))[-KMe^{\lambda(ct+\xi(t))}+L\phi(-j+q)]\\ &\leq (\phi(j+q)-\phi(-j+q))[-KMe^{\lambda(ct+\xi)}+LK_-e^{\lambda(-j+q)}]\\ &\leq e^{\lambda ct}(\phi(j+q)-\phi(-j+q))[-KMe^{\lambda\xi}+LK_-e^{-\lambda(j+\xi)}]\end{aligned}$$

Thus, if

$$M>\frac{LK_-}{K} \tag{4.40}$$

then we have $-KMe^{\lambda\xi}+LK_-e^{-\lambda(j+\xi)}<0$ so that in the first case, $(\mathscr{L}v^-)_j\leq 0$.

Consider the case $-j+q<0<j+q$. As before, we consider two sub-cases, $\lambda\geq\mu$ and $\lambda<\mu$. Suppose $\lambda\geq\mu$. Then using the estimates (4.19) and (4.14), we have

$$\begin{aligned}(\mathscr{L}v^-)_j &\leq -Me^{\lambda(ct+\xi)}(r_-e^{\lambda(j+q)}-R_-e^{\lambda(-j+q)})+LK_-e^{\lambda(-j+q)}\\ &= Me^{\lambda(ct+\xi)}e^{-\lambda j}[-r_-e^{(\lambda-\mu)j}e^{-\mu q}+R_-e^{\lambda q}+\tfrac{LK_-}{M}e^{-2\lambda\xi}]\end{aligned} \tag{4.41}$$

Since $\lambda\geq\mu$, the first term in the brackets is less than $-r_-e^{-\mu q}$. Since $\xi>0$, the third term in the brackets is less than $\frac{LK_-}{M}$. Thus, we need

$$-r_-e^{-\mu q}+R_-e^{\lambda q}+\frac{LK_-}{M}<0. \tag{4.42}$$

Since we imposed that $p(t)=ct+\xi(t)\leq 0$ for $t\leq -T$, it is also true that $q(t)<0$ for $t\leq -T$. Hence, we should choose a sufficiently negative time $-T_2$ such that if $t\leq -T_2\leq -T$ then (4.42) is true, that is,

$$M>\frac{LK_-}{r_-e^{-\mu q}-R_-e^{\lambda q}},\qquad \text{for } t\leq -T_2 \tag{4.43}$$

Now assume that $\lambda < \mu$ so that $0 < a < 1/2$. Then

$$(\mathscr{L}v^-)_j \leq Me^{\lambda(ct+\xi)}\phi'(-j+q) + H \leq MR_- e^{\lambda(-j+2ct)} + H,$$

by using (4.19). The trick is again to make H sufficiently negative (by choosing time to be sufficiently negative) so that the entire left-hand side is non-negative. Hence, we choose $L_2 > 0$ such that if $j+q > L_2$ then $\phi(j+q) > \frac{2}{3}(a+1)$ so that $H < 0$. Consider the case when $0 < j+q < L_2$. Then $e^{-\mu(j+q)} > e^{-\mu L_2}$ so that from the second line of (4.41), the first term in the brackets is

$$\begin{aligned} -r_- e^{(\lambda-\mu)j} e^{-\mu q} &\leq -r_1 e^{(\lambda-\mu)j} e^{-\mu q} e^{(\lambda-\mu)q} \\ &= -r_- e^{(\lambda-\mu)(j+q)} e^{-\mu q} \\ &\leq -r_- e^{(\lambda-\mu)L_2} e^{-\lambda q} \end{aligned}$$

Thus, we must choose a sufficiently negative time $-T_3$ such that

$$-r_- e^{(\lambda-\mu)L_2} e^{-\lambda q} + R_- e^{\lambda q} + \frac{LK_-}{M} < 0, \qquad \text{for } t \leq -T_3$$

equivalently,

$$M > \frac{LK_-}{r_- e^{(\lambda-\mu)L_2} e^{-\lambda q} - R_- e^{\lambda q}}. \tag{4.44}$$

Lemma 14. *If M is chosen such that it satisfies (4.40), (4.43), (4.44) then the function defined by (4.38) satisfies $(\mathscr{L}v^-)_j(t) \leq 0$ for $j \in \mathbb{Z}$ and $t \leq T_2$, for some sufficiently large $-T_2$.*

4.2.5 Construction of the time-global wave-like solution

Now that we have established the existence of a super-/sub-solution pair to (4.7), we will now construct a time-global solution that behaves in the required manner at negative time. This section finishes the proof of Lemma A.

Let $u_n(t) = \{u_{j,n}(t)\}_{j\in\mathbb{Z}}$ be the solution to (4.7) for $t \geq -n$ with initial condition

$$u_n(-n) = v^-(-n) = \{v_j^-(-n)\}_{j\in\mathbb{Z}}.$$

Since $u_{j,n}(-n) = v_j^-(-n) \leq v_j^+(-n)$, Comparison Principle guarantees that

$$v_j^-(t) \leq u_{j,n}(t) \leq v_j^+(t)$$

for $-n \leq t \leq T$ and $j \in \Omega$. Setting $t = -(n-1)$, we have

$$u_{j,n}(-n+1) \geq v_j^-(-n+1) = u_{j,n-1}(-n+1).$$

Hence, by Comparison Principle,

$$u_{j,n}(t) \geq u_{j,n-1}(t), \qquad -n+1 \leq t \leq T, \;\; j \in \Omega, \tag{4.45}$$

that is, $u_n(t)$ is a monotone increasing sequence in n. Letting $n \to +\infty$ and by using parabolic estimates, the sequence u_n converges to an entire solution defined for $t \in \mathbb{R}$ and $j \in \Omega$. Denote this solution by $\tilde{u}(t) = \{\tilde{u}_j(t)\}_{j\in\mathbb{Z}}$. Then letting $n \to +\infty$ in (4.45),

$$v_j^-(t) \leq \tilde{u}_j(t) \leq v_j^+(t), \qquad -\infty < t \leq T, \;\; j \in \Omega.$$

Since $\phi' > 0$ and $\dot{p} \to 0$ as $t \to -\infty$ (so that $c - \dot{p} > 0$ for sufficiently negative time), we have

$$\dot{v}_j^-(t) = (\phi'(j+q) - \phi'(-j+q))(c - \dot{p}) > 0.$$

Thus,

$$\dot{\tilde{u}}_{j,n}(-n) > 0, \qquad \text{for sufficiently large} \;\; n.$$

By Comparison Principle,

$$\dot{\tilde{u}}_{j,n}(t) > 0, \qquad t > -n, \;\; j \in \Omega.$$

Letting $n \to +\infty$, $\dot{\tilde{u}}_j(t) \geq 0$ for $t \in \mathbb{R}$ and $j \in \Omega$. Since $\dot{\tilde{u}}_j \neq 0$, we have that $\dot{\tilde{u}}_j > 0$ for $t \in \mathbb{R}$ and $j \in \Omega$.

Finally, we prove that condition (4.11) guarantees the uniqueness of the time-global monotone increasing function that we established above (from now on, we drop the tilde sign in u). The key again is the Comparison Principle. Suppose there is another time-global solution $v = \{v_j\}_{j\in\mathbb{Z}}$ that satisfies (4.11). The main idea is to squeeze v_j in between two time-translates of u_j. These two time-translates form a super-/sub-solution pair for $j \neq -1, 0$.

Choose $\eta > 0$ sufficiently small such that $f'(s) \leq -\beta$ for $s \in [-2\eta, 2\eta] \cup [1-2\eta, 1+2\eta]$ for some $\beta > 0$. Then for any $\varepsilon \in (0, \eta)$, we can find $t_\varepsilon \in \mathbb{R}$ such that

$$||v - u||_\infty < \varepsilon, \qquad -\infty < t < t_\varepsilon, \tag{4.46}$$

by triangle inequality. For each $t_0 \in (-\infty, T-\varepsilon)$ and $j \neq -1, 0$, define $W^{\pm}(t) = \{W_j^{\pm}(t)\}$ by

$$W_j^+(t) = u_j(t_0 + t + \varepsilon(1 - e^{-\beta t})) + \varepsilon e^{-\beta t}$$

and

$$W_j^-(t) = u_j(t_0 + t + \varepsilon(1 - e^{-\beta t})) + \varepsilon e^{-\beta t}.$$

In particular, $W_j^{\pm}(0) = u_j(t_0) \pm \varepsilon$. Then by (4.46), we have

$$W_j^-(0) \leq v_j(t_0) \leq W_j^+(0), \qquad j \neq -1, 0.$$

Let us now verify that W^+ is a super-solution for $t \in [0, T - t_0 - \varepsilon]$ and $j \neq -1, 0$. Let $\xi = t_0 + t + \varepsilon(1 - e^{-\beta t})$. Then for $j \neq -1, 0$:

$$\begin{aligned}
(\mathscr{L}W^+)_j &= \dot{W}_j^+ - (DW^+)_j - f(W_j^+) \\
&= u_j'(\xi)(1 + \varepsilon\beta e^{-\beta t}) - \varepsilon\beta e^{-\beta t} - (Du)_j(\xi) - f(u_j(\xi) + \varepsilon e^{-\beta t}) \\
&= u_j'(\xi)\varepsilon\beta e^{-\beta t} - \varepsilon\beta e^{-\beta t} + f(u_j(\xi)) - f(u_j(\xi) + \varepsilon e^{-\beta t}).
\end{aligned}$$

By the Mean Value Theorem, for $\theta \in (0,1)$ we have

$$\begin{aligned}
(\mathscr{L}W^+)_j &= \varepsilon e^{-\beta t}(\beta u_j'(\xi) - \beta - f'(u_j(\xi) + \theta\varepsilon e^{-\beta t})) \\
&\geq \varepsilon e^{-t}(-\beta - \max_{[0,1]} f'(s)) \\
&\geq \varepsilon e^{-t}(-\beta + \beta) = 0.
\end{aligned}$$

Similarly, it can be shown that $(\mathscr{L}W^+)_j \leq 0$ for $j \neq -1, 0$ and $t \in [0, T - t_0 - \varepsilon]$. Thus, by triangle inequality, we have

$$W_j^-(t) \leq v_j(t_0 + t) \leq W_j^+(t), \qquad t \in [0, T - t_0 - \varepsilon],\ j \neq -1, 0.$$

Rewriting $t_0 + t$ by t, this inequality is

$$u_j(t - \varepsilon(1 - e^{-\beta(t - t_0)})) - \varepsilon e^{-\beta(t - t_0)} \leq v_j(t) \leq u_j(t + \varepsilon(1 - e^{-\beta(t - t_0)})) + \varepsilon e^{-\beta(t - t_0)}$$

for $t \in [t_0, T - \varepsilon]$ and $t_0 \in (-\infty, T - \varepsilon)$. Letting $t_0 \to -\infty$, we have

$$u_j(t - \varepsilon) \leq v_j(t) \leq u_j(t + \varepsilon), \qquad t \in (-\infty, T - \varepsilon),\ j \neq -1, 0.$$

By Comparison Principle, this inequality also holds for $t \in \mathbb{R}$ ($j \neq -1, 0$). Letting $\varepsilon \to 0$, we get $v \equiv u$. This finishes the proof of Lemma A.

4.3 Behavior at Positive Time

4.3.1 Proof of Lemma B

Lemma B. *The solution $u_j(t)$ from Lemma A satisfies the following: for any $\varepsilon > 0$, there exists a time $t_\varepsilon \geq 0$ and there exists an integer $n_\varepsilon > 1$ such that*

$$|u_j(t_\varepsilon) - \phi(j + ct_\varepsilon)| \leq \varepsilon, \ \textit{for all} \ \ |j| \geq n_\varepsilon \tag{4.47}$$

and

$$u_j(t) \geq 1 - \varepsilon, \ \textit{for all} \ \ t \geq t_\varepsilon; \ \textit{for} \ \ j = -1, 0. \tag{4.48}$$

Proof. From Lemma A, there exists a unique time-global solution u such that $\dot{u}_j > 0$ for all $(t, j) \in \mathbb{R} \times \Omega$ and that satisfies (4.11) uniformly in $j \in \Omega$, where ϕ is the unique solution of (4.4) with $\phi(0) = a$ and boundary conditions (4.5). The strong maximum principle implies that $0 < u_j(t) < 1$ for all $(t, j) \in \mathbb{R} \times \Omega$. Note that $u_j(t) \to 1$ as $t \to +\infty$ for $j = -1, 0$.

Let $\varepsilon > 0$ be an arbitrary positive real number. Since $u_j(t) \to 1$ as $t \to +\infty$ for $j = -1, 0$, there is a time $t_\varepsilon \geq 0$ such that

$$u_j(t) \geq 1 - \varepsilon, \qquad \text{for all} \ \ t \geq t_\varepsilon, j = -1, 0.$$

From (4.11), there exists a time $\mathscr{T}_1 \leq 0$ such that $|u_j(\mathscr{T}_1) - \phi(j + c\mathscr{T}_1)| \leq \frac{\varepsilon}{2}$ for all $j \in \Omega$. Since $\phi(+\infty) = 1$, there exists $\xi^+ > 0$ such that if $j \geq \xi^+$ then $1 - \varepsilon \leq u_j(\mathscr{T}_1) \leq 1$. Since $t_\varepsilon \geq 0 \geq \mathscr{T}_1$ and u_j increases with t, we have

$$1 - \varepsilon \leq u_j(t_\varepsilon) \leq 1, \qquad \text{for all } j \geq \xi^+.$$

As a consequence, for $j \geq \xi^+$ we have

$$|u_j(t_\varepsilon) - \phi(j + ct_\varepsilon)| \leq \varepsilon.$$

Since $\phi(-\infty) = 0$, there exists $\xi^- < -1$ such that if $j \leq \xi^-$ then

$$|u_j(t_\varepsilon) - \phi(j + ct_\varepsilon)| \leq \varepsilon.$$

Let $n_\varepsilon = \max\{\xi^+, |\xi^-|\}$. Then for all $|j| > n_\varepsilon$

$$|u_j(t_\varepsilon) - \phi(j + ct_\varepsilon)| \leq \varepsilon.$$

□

4.3.2 Proof of the Main Result

Assuming Lemma C is true (see next section), we can now prove Theorem A (analogue of Theorem 1.3 in [8]).

Proof. Lemma A guarantees that there exists a time-global solution $u = \{u_j\}_{j \in \mathbb{Z}}$ of (4.1) such that $\dot{u}_j > 0$ for all $(j,t) \in \Omega \times \mathbb{R}$ and that satisfies (4.11). By Lemma C, we have that Lemma B is true so that (4.9) is proved. To see (4.10), we need to show that $u_j(t) - \phi(j + ct) \to 0$ as $|j| \to +\infty$ uniformly in $t \in \mathbb{R}$. Let $\varepsilon > 0$ be an arbitrary positive real number. Since $u_j(t) \to \phi(j + ct)$ as $|t| \to +\infty$ uniformly in Ω, there exists $\tau > 0$ such that $|u_j(t) - \phi(j + ct)| \leq \varepsilon$ for all $|t| \geq \tau$ and for all $j \in \Omega$. On the other hand, the same arguments as above work locally in time, that is, there exists an integer $n_\varepsilon > 1$ such that

$$|u_j(t) - \phi(j + ct)| \leq \varepsilon, \qquad \text{for all } |t| \leq \tau,\ |j| > n_\varepsilon.$$

Thus,

$$u_j(t) - \phi(j+ct) \to 0, \qquad \text{as} \quad |j| \to +\infty$$

uniformly in $t \in \mathbb{R}$. This completes the proof of Theorem A. □

4.4 A Stability Result.

The result that remains to be proved is Lemma C. In this section, first we state a result, Lemma E, that has the same conclusion as Lemma C but with an initial condition that is localized and a small perturbation of ϕ. Lemma C can be viewed as a generalization of Lemma E. Due to its assumption (a bounded initial condition), Lemma E can be viewed as a stability result. The proof of Lemma C involves similar arguments to the proof of Lemma E.

Lemma D - Positive Time with Bounded Initial Conditions. *Suppose that we have sequences $s = \{s_j\}_{j\in\mathbb{Z}}$ and $S = \{S_j\}_{j\in\mathbb{Z}}$ that satisfy Assumption s and S. Assume that Lemma B is satisfied and suppose, further, we have an initial condition $u_j(0)$ such that*

$$w_j^-(0) := \phi(j+P(0)) - s_j v(0) \le u_j(0) \le \phi(j+Q(0)) + S_j v(0) =: w_j^+(0).$$

where P and Q are defined in (4.55), (4.60) and $w_j^{\pm}$ are defined in (4.54), (4.59). Then

$$\sup_{j\in\Omega} |u_j(t) - \phi(j+ct)| \to 0, \text{ as } t \to \infty.$$

To prove Lemma D, we again invoke the Comparison Principle by constructing an appropriate super-/sub-solution pair, $w^{\pm} = \{w_j^{\pm}\}_{j\in\mathbb{Z}}$ that sandwiches the time-global solution $u = \{u_j\}_{j\in\mathbb{Z}}$ from Lemma A. Note that this super-/sub-solution pair $w^{\pm}$ is vastly different from the super-/sub-solution pair $v^{\pm} = \{v_j^{\pm}\}$ for negative time (see Section 4.2).

Assumption on sequences s and S. *The sequences $s = \{s_j\}_{j\in\mathbb{Z}}$ and $S = \{S_j\}_{j\in\mathbb{Z}}$ are bounded sequences of positive numbers that satisfy the following:*

$$s_j = S_j = 1, \qquad j \le -3, \; or \; j \ge 2,$$

$$(Ds)_{-2} = (Ds)_1 = (DS)_{-2} = (DS)_1 = 0, \tag{4.49}$$

and if $d - d_0 > 0$,

$$(\Delta s)_{-1} + (d-d_0)C_1 = (\Delta s)_0 = (\Delta S)_{-1} = (\Delta S)_0 + (d-d_0)C_1 = 0, \tag{4.50}$$

while if $d - d_0 < 0$,

$$(\Delta s)_{-1} = (\Delta s)_0 - (d-d_0)C_1 = (\Delta S)_{-1} - (d-d_0)C_1 = (\Delta S)_0 = 0. \tag{4.51}$$

for some positive number C_1.

4.4.1 Some Constants

Extend the function f outside the interval $[0,1]$ in the following way:

$$\begin{cases} f(s) = f'(0)s, & s \le 0, \\ f(s) = f'(1)(s-1), & s \ge 1. \end{cases}$$

Set $\omega = \min\{|f'(0)|/2, |f'(1)|/2, ||f'||_\infty\} > 0$. Let $\rho > 0$ be such that

$$\begin{cases} |f'(s) - f'(0)| \le \omega, & s \le \rho, \\ |f'(s) - f'(1)| \le \omega, & s \ge 1-\rho, \end{cases} \tag{4.52}$$

and let $A > 0$ be such that

$$\begin{cases} \phi(z) \geq 1-\rho/2, & z \geq A, \\ \phi(z) \leq \rho, & z \leq -A. \end{cases} \tag{4.53}$$

Let $\underline{\delta} := \min_{z\in[-A,A]} \phi'(z) > 0$.

4.4.2 Sub-solution

Consider the function $w^-(t) = \{w_j^-(t)\}_{j\in\mathbb{Z}}$ defined by

$$w_j^-(t) = \phi(j+P(t)) - s_j\underline{v}(t) \tag{4.54}$$

where

$$P(t) = ct - e^{-\omega t} + \underline{V}(t) - \underline{V}(0), \qquad t \geq 0. \tag{4.55}$$

We assume that there exists $C_1' > 0$ such that $\phi(j+1+P(t)) - \phi(j+P(t)) \leq C_1'\underline{v}(t)$. Set

$$C_2 = \frac{\underline{\delta}\omega}{4||f'||_\infty||s||_\infty||\phi'||_\infty}.$$

Let

$$0 < \underline{\eta} \leq \rho/C_2, \qquad \underline{\mu} = \min\{\rho/(2||s||_\infty), C_2\underline{\eta}\}. \tag{4.56}$$

Hence, in particular,

$$||s||_\infty \leq \frac{\rho}{2\underline{\mu}}. \tag{4.57}$$

Define the functions

$$\underline{v}(t) = \underline{\mu}e^{-\omega t}, \;\; \underline{V}(t) = \frac{4||f'||_\infty||s||_\infty}{\underline{\delta}\omega}\underline{\mu}e^{-\omega t}$$

where we have $\underline{V}(+\infty) = 0$ and $\underline{V}(0) = \dfrac{4||f'||_\infty||s||_\infty}{\underline{\delta}\omega}\underline{\mu}$ so that $||\phi'||_\infty\underline{V}(0) = \dfrac{\underline{\mu}}{C_2} \leq \underline{\eta}$. Also, $\dot{\underline{V}} \leq 0$. Since $\dot{\underline{V}}(+\infty) = 0$, we have that $\dot{P}(+\infty) = c$ (compare with (4.27)).

Next consider $(\mathscr{L}w^-)_j$ where $\mathscr{L}$ is defined in (4.25) . Our goal is to show that $(\mathscr{L}w^-)_j \le 0$ for $j \in \mathbb{Z}$ and $t \ge 0$. As before, we drop the dependence on t for brevity.

For $j \ge 2$ and $j \le -3$, we assume $s_j = 1$ so that

$$(\mathscr{L}w^-)_j = \phi'(j+P)[\omega e^{-\omega t} + \underline{\dot{V}}] - \underline{\dot{v}} + f(\phi(j+P)) - f(\phi(j+P) - \underline{v})$$

For $j = -1$ we have

$$\begin{aligned}(\mathscr{L}w^-)_j \quad &= \phi'(j+P)[\omega e^{-\omega t} + \underline{\dot{V}}] - s_j\underline{\dot{v}} + [d_0 s_{j+1} - (d+d_0)s_j + d s_{j-1}]\underline{v} \\ &+(d-d_0)[\phi(j+1+P) - \phi(j+P)] + f(\phi(j+P)) - f(\phi(j+P) - s_j\underline{v})\end{aligned}$$

For $j = 0$ we have

$$\begin{aligned}(\mathscr{L}w^-)_j \quad &= \phi'(j+P)[\omega e^{-\omega t} + \underline{\dot{V}}] - s_j\underline{\dot{v}} + [d s_{j+1} - (d+d_0)s_j + d_0 s_{j-1}]\underline{v} \\ &+(d-d_0)[\phi(j-1+P) - \phi(j+P)] + f(\phi(j+P)) - f(\phi(j+P) - s_j\underline{v})\end{aligned}$$

For $j = -2$ and $j = 1$ we have

$$\begin{aligned}(\mathscr{L}w^-)_j \quad &= \phi'(j+P)[\omega e^{-\omega t} + \underline{\dot{V}}] - s_j\underline{\dot{v}} + d[s_{j+1} - 2s_j + s_{j-1}]\underline{v} \\ &+f(\phi(j+P)) - f(\phi(j+P) - s_j\underline{v})\end{aligned}$$

Remark 2. *What is important to note here is that at the obstacle K, the diffusion term Δs explicitly appears; at the nearest nodes $j = -2, 1$, the diffusion term Ds explicitly appears; while at the nodes far from the obstacle ($j \ge 2, j \le -3$), both Δs and Ds do not appear. This observation is useful when one considers the problem of multiple defects in the middle of $\mathbb{Z}$.*

Write

$$(\mathscr{L}w^-)_j = L_1 + L_2 + L_3$$

where

$$L_1 = f(\phi(j+P)) - f(\phi(j+P) - s_j\underline{v}),$$

$$L_3 = \phi'(j+P)[\omega e^{-\omega t} + \frac{1}{2}\dot{\underline{V}}],$$

and L_2 is the remaining term depending on the value of j.

To bound the L_3 term we write $\dot{\underline{V}}(t) = -\underline{C}_3 e^{-\omega t} = \frac{-\underline{C}_3}{\underline{\mu}} v < 0$ where $\underline{C}_3 = \frac{4||f'||_\infty ||s||_\infty \underline{\mu}}{\underline{\delta}}$ so we need $\omega - \underline{C}_3/2 \le 0$, or

$$||s||_\infty \ge \frac{\underline{\delta}\omega}{2\underline{\mu}||f'||_\infty}. \tag{4.58}$$

4.4.3 Bounding L_1 and L_2

Case 1: $j + P(t) \le -A$

For these values of j and t, we have $\phi(j+P) - s_j\underline{v} \le \rho$ since $\underline{v} \ge 0$. By definition of ρ and ω and the Mean Value theorem, we have

$$L_1 = f(\phi(j+P)) - f(\phi(j+P) - s_j\underline{v}) \le (f'(0) + \omega)s_j\underline{v}.$$

We have $\phi' \ge 0$ and $\dot{\underline{V}} \le 0$ so that for $j \ge 2$ and $j \le -3$,

$$(\mathscr{L}w^-)_j \le (f'(0)+\omega)\underline{v} - \dot{\underline{v}} \le (f'(0)+2\omega)\underline{v} \le 0,$$

using that $-\dot{\underline{v}} = \omega\underline{v}$.

For $j = -2$ and $j = 1$, we have

$$\begin{aligned}(\mathscr{L}w^-)_j &\le (f'(0)+\omega)s_j\underline{v} - s_j\dot{\underline{v}} + d[s_{j+1} - 2s_j + s_{j-1}]\underline{v} \\ &\le [(f'(0)+2\omega)s_j + (Ds)_j]\underline{v} \le 0,\end{aligned}$$

provided $(Ds)_{-2} = (Ds)_1 = 0$.

For $j=-1$, we have for $d-d_0>0$,

$$\begin{aligned}(\mathscr{L}w^-)_j &\leq (f'(0)+\omega)s_j\underline{v}-s_j\dot{\underline{v}}+(\Delta s)_j\underline{v}+(d-d_0)[\phi(j+1+P)-\phi(j+P)]\\ &\leq [(f'(0)+2\omega)s_j+(\Delta s)_j+C_1'(d-d_0)]\underline{v}\leq 0,\end{aligned}$$

provided $(\Delta s)_{-1}+C_1'(d-d_0)\leq 0$. The case $d-d_0<0$ is similar.

For $j=0$, we have for $d-d_0>0$ (since $\phi(j-1+P)-\phi(j+P)<0$),

$$\begin{aligned}(\mathscr{L}w^-)_j &\leq (f'(0)+\omega)s_j\underline{v}-s_j\dot{\underline{v}}+(\Delta s)_j\underline{v}+(d-d_0)[\phi(j-1+P)-\phi(j+P)]\\ &\leq [(f'(0)+2\omega)s_j+(\Delta s)_j]\underline{v}\leq 0,\end{aligned}$$

provided $(\Delta s)_0=0$. The case $d-d_0<0$ is similar.

Case 2: $j+P(t)\geq A$

. For these values of j and t, we have

$$\phi(j+P)\geq\phi(j+P)-s_j\underline{v}\geq 1-\frac{\rho}{2}-\frac{\rho}{2}=1-\rho$$

using $s_j\underline{v}\leq\frac{\rho}{2}$, by definition of $\underline{\mu}$ and $\underline{v}$. Now continue as in the first case. By definition of ρ and ω and the Mean Value theorem, we have

$$L_1=f(\phi(j+P))-f(\phi(j+P)-s_j\underline{v})\leq(f'(1)+\omega)s_j\underline{v}.$$

We have $\phi'\geq 0$ and $\dot{\underline{V}}\leq 0$ so that for $j\geq 2$ and $j\leq -3$,

$$(\mathscr{L}w^-)_j\leq(f'(1)+\omega)\underline{v}-\dot{\underline{v}}\leq(f'(1)+2\omega)\underline{v}\leq 0,$$

using that $-\dot{\underline{v}}=\omega\underline{v}$.

For $j=-2$ and $j=1$, we have

$$\begin{aligned}(\mathscr{L}w^-)_j &\le (f'(1)+\omega)s_j\underline{v} - s_j\dot{\underline{v}} + d[s_{j+1}-2s_j+s_{j-1}]\underline{v}\\ &\le [(f'(1)+2\omega)s_j + (Ds)_j]\underline{v} \le 0,\end{aligned}$$

provided $(Ds)_{-2} = (Ds)_1 = 0$.

For $j=-1$, we have for $d-d_0>0$,

$$\begin{aligned}(\mathscr{L}w^-)_j &\le (f'(1)+\omega)s_j\underline{v} - s_j\dot{\underline{v}} + (\Delta s)_j\underline{v} + (d-d_0)[\phi(j+1+P)-\phi(j+P)]\\ &\le [(f'(1)+2\omega)s_j + (\Delta s)_j + C_1'(d-d_0)]\underline{v} \le 0,\end{aligned}$$

provided $(\Delta s)_{-1} + C_1'(d-d_0) \le 0$. The case $d-d_0<0$ is similar.

For $j=0$, we have for $d-d_0>0$ (since $\phi(j-1+P)-\phi(j+P)<0$),

$$\begin{aligned}(\mathscr{L}w^-)_j &\le (f'(1)+\omega)s_j\underline{v} - s_j\dot{\underline{v}} + (\Delta s)_j\underline{v} + (d-d_0)[\phi(j-1+P)-\phi(j+P)]\\ &\le [(f'(1)+2\omega)s_j + (\Delta s)_j]\underline{v} \le 0,\end{aligned}$$

provided $(\Delta s)_0 = 0$. The case $d-d_0<0$ is similar.

Case 3: $-A < j+P(t) < A$**.**

For these values of j and t, we have $\phi'(j+P) \ge \underline{\delta} > 0$. Since $\dot{\underline{V}} \le 0$, we have

$$\begin{aligned}(\mathscr{L}w^-)_j &\le ||f'||_\infty s_j\underline{v} - s_j\dot{\underline{v}} + \tfrac{\underline{\delta}}{2}\dot{\underline{V}} + (\Delta s)_j\underline{v}\\ &+\delta_{-1}(j)(d-d_0)[\phi(j+1+P)-\phi(j+P)] + \delta_0(j)(d-d_0)[\phi(j-1+P)-\phi(j+P)]\end{aligned}$$

where $\delta_n(j) = 0$ for $j \neq n$ and $\delta_n(j) = 1$ for $j = n$. Using $-\dot{\underline{v}} = \omega \underline{v}$ and $\omega \leq ||f'||_\infty$ to obtain for $d - d_0 > 0$,

$$\begin{aligned}(\mathscr{L}w^-)_j \quad &\leq 2||f'||_\infty s_j \underline{v} - \tfrac{\delta}{2}\underline{C}_3 \underline{v} + (\Delta s)_j \underline{v} \\ &+\delta_{-1}(j)(d-d_0)C_1' \underline{v} - \delta_0(j)(d-d_0)C_1' \underline{v} \leq 0,\end{aligned}$$

by definition of $\underline{C}_3$. The case $d - d_0 < 0$ is similar.

Lemma 15. *Suppose the sequence of positive numbers $s = \{s_j\}_{j\in\mathbb{Z}}$ is chosen such that it is bounded according to (4.57), (4.58) and that $s_j = 1$ for $j \leq -3$ and $j \geq 2$, with $(Ds)_{-2} = (Ds)_1 = 0$. Moreover, if $d - d_0 > 0$,*

$$(\Delta s) = 0, \;\; (\Delta s)_{-1} + (d-d_0)C_1' \leq 0,$$

while if $d - d_0 < 0$,

$$(\Delta s)_{-1} = 0, \;\; (\Delta s)_0 - (d-d_0)\tilde{C}_1' \leq 0.$$

Then the function $w^-(t) = \{w_j^-\}_{j\in\mathbb{Z}}$ defined by (4.54) satisfies $(\mathscr{L}w^-)_j(t) \leq 0$ for $j \in \mathbb{Z}$ and $t \geq 0$.

Remark 3. *The proof tells us that we can find a slightly less restrictive assumption on the sequence s by assuming that $0 < s_j < \frac{\rho}{2\underline{\mu}}$ for $j \leq -3$ and $j \geq 2$ and the following:*

$$\textit{Case 1.1: } f'(1) < f'(0) \textit{ and } d - d_0 > 0. \; \begin{cases} (f'(0)+2\omega)s_{-2} + (\Delta s)_{-2} = 0 \\ (f'(0)+2\omega)s_{-1} + (\Delta s)_{-1} + (d-d_0)C_1 = 0 \\ (f'(0)+2\omega)s_0 + (\Delta s)_0 = 0 \\ (f'(0)+2\omega)s_1 + (\Delta s)_1 = 0 \end{cases}$$

Case 1.2: $f'(1) < f'(0)$ *and* $d - d_0 < 0$. $\begin{cases} (f'(0)+2\omega)s_{-2} + (\Delta s)_{-2} = 0 \\ (f'(0)+2\omega)s_{-1} + (\Delta s)_{-1} = 0 \\ (f'(0)+2\omega)s_0 + (\Delta s)_0 - (d-d_0)C_1' = 0 \\ (f'(0)+2\omega)s_1 + (\Delta s)_1 = 0 \end{cases}$

Case 2.1: $f'(1) \geq f'(0)$ *and* $d - d_0 > 0$. $\begin{cases} (f'(1)+2\omega)s_{-2} + (\Delta s)_{-2} = 0 \\ (f'(1)+2\omega)s_{-1} + (\Delta s)_{-1} + (d-d_0)C_1' = 0 \\ (f'(1)+2\omega)s_0 + (\Delta s)_0 = 0 \\ (f'(1)+2\omega)s_1 + (\Delta s)_1 = 0 \end{cases}$

Case 2.2: $f'(1) \geq f'(0)$ *and* $d - d_0 < 0$. $\begin{cases} (f'(1)+2\omega)s_{-2} + (\Delta s)_{-2} = 0 \\ (f'(1)+2\omega)s_{-1} + (\Delta s)_{-1} = 0 \\ (f'(1)+2\omega)s_0 + (\Delta s)_0 - (d-d_0)C_1' = 0 \\ (f'(1)+2\omega)s_1 + (\Delta s)_1 = 0 \end{cases}$

4.4.4 Super-solution

Consider the function $w_j^+(t) = \{w_j^+(t)\}_{j\in\mathbb{Z}}$ defined by

$$w_j(t) = \phi(j + Q(t)) + S_j\bar{v}(t) \tag{4.59}$$

where

$$Q(t) = ct + e^{-\omega t} - \overline{V}(t) + \overline{V}(0), \tag{4.60}$$

We assume that there exists $\tilde{C}_1 > 0$ such that $\phi(j+1+Q(t)) - \phi(j+Q(t)) \leq \tilde{C}_1\bar{v}(t)$. Let ω and ρ be as in the construction of the subsolution and let $A' > 0$ be such that

$$\begin{cases} \phi(z) \geq 1-\rho, & z \geq A', \\ \phi(z) \leq \rho/2, & z \leq -A'. \end{cases}$$

Define $\overline{\delta} = \min_{z\in[-A',A']} \phi'(z)$.

Set

$$C_2' = \frac{\overline{\delta}\omega}{(4||f'||_\infty||\phi'||_\infty||S||_\infty)}.$$

Let

$$0 < \overline{\eta} \le \rho/C_2' \qquad \overline{\mu} = \min\{\rho/(2||S||_\infty), C_2'\overline{\eta}\}, \tag{4.61}$$

in particular,

$$||S||_\infty \le \frac{\rho}{2\overline{\mu}}. \tag{4.62}$$

Define the functions

$$\overline{v}(t) = \overline{\mu}e^{-\omega t},\ \ \overline{V}(t) = \frac{4||f'||_\infty||S||_\infty}{\overline{\delta}\omega}\overline{\mu}e^{-\omega t}$$

so that $\overline{V}(+\infty) = 0$ and $\overline{V}(0) = \dfrac{4||f'||_\infty||S||_\infty}{\overline{\delta}\omega}\overline{\mu}$ so that $||\phi'||_\infty\overline{V}(0) = \frac{\overline{\mu}}{C_2'} \le \overline{\eta}$. Also, $\dot{\overline{V}} \le 0$. Since $\dot{V}(+\infty) = 0$, we have that $\dot{Q}(+\infty) = c$ (compare with (4.27)).

Next consider $(\mathscr{L}w^+)_j$ where $\mathscr{L}$ is defined in (4.25) . Our goal is to show that $(\mathscr{L}w^+)_j \ge 0$ for $j \in \mathbb{Z}$ and $t \ge 0$. As before, we drop the dependence on t for brevity.

For $j \ge 2$ and $j \le -3$, we assume $S_j = 1$ so that

$$(\mathscr{L}w^+)_j = -\phi'(j+Q)[\omega e^{-\omega t} + \dot{\overline{V}}] + \dot{\overline{v}} + f(j+Q) - f(\phi(j+Q) - \overline{v})$$

For $j = -1$, we have

$$\begin{aligned}(\mathscr{L}w^+)_j \ &= -\phi'(j+Q)[\omega e^{-\omega t} + \dot{\overline{V}}] + S_j\dot{\overline{v}} - [d_0S_{j+1} - (d+d_0)S_j + dS_{j-1}]\overline{v} \\ &\quad +(d-d_0)[\phi(j+1+Q) - \phi(j+Q)] + f(\phi(j+Q)) - f(\phi(j+Q) - S_j\overline{v}).\end{aligned}$$

For $j=0$, we have

$$\begin{aligned}(\mathscr{L}w^+)_j \;&= -\phi'(\xi_j)[\omega e^{-\omega t}+\dot{\overline{V}}]+S_j\dot{\overline{v}}-[dS_{j+1}-(d+d_0)S_j+d_0S_{j-1}]\overline{v}\\ &\quad +(d-d_0)[\phi(j-1+Q)-\phi(j+Q)]+f(\phi(j+Q))-f(\phi(j+Q)-S_j\overline{v}).\end{aligned}$$

For $j=-2$ and $j=1$, we have

$$\begin{aligned}(\mathscr{L}w^+)_j \;&= -\phi'(j+Q)[\omega e^{-\omega t}+\dot{\overline{V}}]+S_j\dot{\overline{v}}-d[S_{j+1}-2S_j+S_{j-1}]\overline{v}\\ &\quad +f(\phi(j+Q))-f(\phi(j+Q)-S_j\overline{v}).\end{aligned}$$

Write

$$(\mathscr{L}w^+)_j = L_1+L_2+L_3$$

where

$$L_1 = f(\phi(j+Q))-f(\phi(j+Q)-S_j\overline{v}),$$

$$L_3 = -\phi'(j+Q)[\omega e^{-\omega t}+\frac{1}{2}\dot{\overline{V}}],$$

and L_2 is the remaining term depending on the value of j.

To bound the L_3 term, we write $\dot{\overline{V}}(t) = -\overline{C}_3e^{-\omega t} = \frac{-\overline{C}_3}{\overline{\mu}}\overline{v}$ where $\overline{C}_3 = 4||f'||_\infty||S||_\infty\overline{\mu}\,\overline{\delta}^{-1}$ so we need $-[\omega-\overline{C}_3/2]\geq 0$, or

$$||S||_\infty \geq \frac{\overline{\delta}\omega}{2\overline{\mu}||f'||_\infty}. \tag{4.63}$$

4.4.5 Bounding L_1 and L_2

Case 1: $j+Q(t) \le -A'$**.**

For these values of j and t, we have $\phi(j+Q) \le \phi(j+Q)+S_j\overline{v} \le \rho/2+\rho/2=\rho$ since $S_jv \le \rho/2$, by definition of $\overline{\mu}$ and $\overline{v}$. By definition of ρ and ω and the Mean Value theorem, we have

$$L_1 = f(\phi(j+Q)) - f(\phi(j+Q)+S_j\overline{v}) \ge -(f'(0)+\omega)S_j\overline{v}.$$

We have $\phi' \ge 0$ and $\dot{\overline{v}} \le 0$ so that for $j \ge 2$ and $j \le -3$,

$$(\mathscr{L}w^+)_j \ge -(f'(0)+\omega)S_j\overline{v} + S_j\dot{\overline{v}} \ge -(f'(0)+2\omega)S_j\overline{v} \ge 0$$

using that $\dot{\overline{v}} = -\omega\overline{v}$.

For $j=-2$ and $j=1$, we have

$$\begin{aligned}(\mathscr{L}w^+)_j &\ge -(f'(0)+\omega)S_j\overline{v} + S_j\dot{\overline{v}} - d[S_{j+1}-2S_j+S_{j-1}]\overline{v}\\ &\ge -[(f'(0)+2\omega)S_j + (DS)_j]\overline{v} \ge 0,\end{aligned}$$

provided $(DS)_{-2} = (DS)_1 = 0$.

For $j=-1$, we have for $d-d_0>0$ (since $\phi(j+1+Q)-\phi(j+Q)>0$),

$$\begin{aligned}(\mathscr{L}w^+)_j &\ge -(f'(0)+\omega)S_j\overline{v} + S_j\dot{\overline{v}} - (\Delta S)_j\overline{v} + (d-d_0)[\phi(j+1+Q)-\phi(j+Q)]\\ &\ge -[(f'(0)+2\omega)S_j + (\Delta S)_j]\overline{v} \ge 0,\end{aligned}$$

provided $(\Delta S)_{-1} = 0$. The case $d-d_0<0$ is similar.

For $j=0$, we have for $d-d_0>0$,

$$\begin{aligned}(\mathscr{L}w^+)_j &\geq -(f'(0)+\omega)S_j\overline{v}+S_j\dot{\overline{v}}-(\Delta S)_j\overline{v}+(d-d_0)[\phi(j-1+Q)-\phi(j+Q)]\\ &\geq -[(f'(0)+2\omega)S_j+(\Delta S)_j+\tilde{C}_1(d-d_0)]\overline{v}\geq 0,\end{aligned}$$

provided $(\Delta S)_0+\tilde{C}_1(d-d_0)\leq 0$. The case $d-d_0<0$ is similar.

Case 2: $j+Q(t)\geq A'$

For these values of j and t, we have $\phi(j+Q)+S_j\overline{v}\geq\phi(j+Q)\geq 1-\rho$. By definition of ρ and ω and the Mean Value theorem, we have

$$L_1=f(\phi(j+Q))-f(\phi(j+Q)-S_j\overline{v})\geq -(f'(1)+\omega)S_j\overline{v}.$$

Since $\phi'\geq 0$ and $\dot{\overline{v}}\leq 0$, for $j\geq 2$ and $j\leq -3$,

$$(\mathscr{L}w^+)_j\geq -(f'(1)+\omega)\overline{v}+\dot{\overline{v}}\geq -(f'(1)+2\omega)\overline{v}\geq 0$$

using that $\dot{\overline{v}}=-\omega\overline{v}$.

For $j=-2$ and $j=1$, we have

$$\begin{aligned}(\mathscr{L}w^+)_j &\geq -(f'(1)+\omega)S_j\overline{v}+S_j\dot{\overline{v}}-d[S_{j+1}-2S_j+S_{j-1}]\overline{v}\\ &\geq -[(f'(1)+2\omega)S_j+(DS)_j]\overline{v}\geq 0,\end{aligned}$$

provided $(DS)_{-2}=(DS)_1=0$.

For $j=-1$, we have for $d-d_0>0$ (since $\phi(j+1+Q)-\phi(j+Q)>0$),

$$\begin{aligned}(\mathscr{L}w^+)_j &\geq -(f'(1)+\omega)S_j\overline{v}+S_j\dot{\overline{v}}-(\Delta S)_j\overline{v}+(d-d_0)[\phi(j+1+Q)-\phi(j+Q)]\\ &\geq -[(f'(1)+2\omega)S_j+(\Delta S)_j]\overline{v}\geq 0,\end{aligned}$$

provided $(\Delta S)_{-1} = 0$. The case $d - d_0 < 0$ is similar.

For $j = 0$, we have for $d - d_0 > 0$,

$$\begin{aligned}(\mathscr{L}w^+)_j &\geq -(f'(1)+\omega)S_j\overline{v} + S_j\dot{\overline{v}} - \overline{v}(\Delta S)_j + (d-d_0)[\phi(j-1+Q) - \phi(j+Q)] \\ &\geq -[(f'(1)+2\omega)S_j + (\Delta S)_j + \tilde{C}_1(d-d_0)]\overline{v} \leq 0,\end{aligned}$$

provided $(\Delta S)_0 + \tilde{C}_1(d-d_0) \leq 0$. The case $d - d_0 < 0$ is similar.

Case 3: $-A' \leq j + Q(t) \leq A'$**.**

For these values of j and t, we have $\phi'(j+Q) \geq \overline{\delta} > 0$. Since $\dot{\overline{v}} \leq 0$, we have

$$\begin{aligned}(\mathscr{L}w^+)_j &\geq -||f'||_\infty S_j\overline{v} + S_j\dot{\overline{v}} - \tfrac{\overline{\delta}}{2}\dot{\overline{V}} - (\Delta S)_j\overline{v} \\ &+\delta_{-1}(j)(d-d_0)[\phi(j+1+Q) - \phi(j+Q)] + \delta_0(j)(d-d_0)[\phi(j-1+Q) - \phi(j+Q)]\end{aligned}$$

Use $\dot{\overline{v}} = -\omega\overline{v}$ and $-\omega \geq ||f'||_\infty$ to obtain for $d - d_0 > 0$,

$$\begin{aligned}(\mathscr{L}w^+)_j &\geq -2||f'||_\infty S_j\overline{v} + \tfrac{\overline{\delta}}{2\mu}\overline{C}_3\overline{v} - (\Delta S)_j\overline{v} \\ &+\delta_{-1}(j)(d-d_0)\tilde{C}_1\overline{v} - \delta_0(j)(d-d_0)\tilde{C}_1\overline{v} \geq 0,\end{aligned}$$

by definition of $\overline{C}_3$. The case $d - d_0 < 0$ is similar.

Lemma 16. *Suppose the sequence of positive numbers $S = \{S_j\}_{j\in\mathbb{Z}}$ is chosen such that it is bounded according to (4.62), (4.63) and that $S_j = 1$ for $j \leq -3$ and $j \geq 2$ with $(DS)_{-2} = (DS)_1 = 0$. Moreover, if $d - d_0 > 0$,*

$$(\Delta S)_{-1} = 0, \;\; (\Delta S)_0 + (d-d_0)\tilde{C}_1 \leq 0,$$

while if $d - d_0 < 0$,

$$(\Delta S)_0 = 0, \;\; (\Delta S)_{-1} - (d-d_0)\tilde{C}_1 \leq 0.$$

Then the function $w^+(t) = \{w_j^+\}_{j\in\mathbb{Z}}$ *defined by (4.59) satisfies* $(\mathscr{L}w^+)_j(t) \geq 0$ *for* $j \in \mathbb{Z}$ *and* $t \geq 0$.

Remark 4. *The proof tells us that we can find a slightlyless restrictive assumption on the sequence* S *by assuming that* $S_j \leq \frac{\rho}{2\mu}$ *for* $j \leq -3$ *and* $j \geq 2$ *and the following:*

$$\text{Case 1.1: } f'(1) < f'(0) \text{ and } d - d_0 > 0. \begin{cases} (f'(0)+2\omega)S_{-2} + (\Delta S)_{-2} = 0 \\ (f'(0)+2\omega)S_{-1} + (\Delta S)_{-1} = 0 \\ (f'(0)+2\omega)S_0 + (\Delta S)_0 + (d-d_0)\tilde{C}_1 = 0 \\ (f'(0)+2\omega)S_1 + (\Delta S)_1 = 0 \end{cases}$$

$$\text{Case 1.2: } f'(1) < f'(0) \text{ and } d - d_0 < 0. \begin{cases} (f'(0)+2\omega)S_{-2} + (\Delta S)_{-2} = 0 \\ (f'(0)+2\omega)S_{-1} + (\Delta S)_{-1} - (d-d_0)\tilde{C}_1 = 0 \\ (f'(0)+2\omega)S_0 + (\Delta S)_0 = 0 \\ (f'(0)+2\omega)S_1 + (\Delta S)_1 = 0 \end{cases}$$

$$\text{Case 2.1: } f'(1) \geq f'(0) \text{ and } d - d_0 > 0. \begin{cases} (f'(1)+2\omega)S_{-2} + (\Delta S)_{-2} = 0 \\ (f'(1)+2\omega)S_{-1} + (\Delta S)_{-1} = 0 \\ (f'(1)+2\omega)S_0 + (\Delta S)_0 + (d-d_0)\tilde{C}_1 = 0 \\ (f'(1)+2\omega)S_1 + (\Delta S)_1 = 0 \end{cases}$$

$$\text{Case 2.2: } f'(1) \geq f'(0) \text{ and } d - d_0 < 0. \begin{cases} (f'(1)+2\omega)S_{-2} + (\Delta S)_{-2} = 0 \\ (f'(1)+2\omega)S_{-1} + (\Delta S)_{-1} - (d-d_0)\tilde{C}_1 = 0 \\ (f'(1)+2\omega)S_0 + (\Delta S)_0 = 0 \\ (f'(1)+2\omega)S_1 + (\Delta S)_1 = 0 \end{cases}$$

4.4.6 Proof of Lemma D

Now we are ready to prove Lemma D, our preparation lemma for Lemma C:

Lemma D - Positive Time with Bounded Initial Conditions. *Suppose that we have sequences* $s = \{s_j\}_{j\in\mathbb{Z}}$ *and* $S = \{S_j\}_{j\in\mathbb{Z}}$ *that satisfy Assumption s and S. Assume that Lemma B is satisfied* *and suppose, further, we have an initial condition $u_j(0)$ such that*

$$w_j^-(0) := \phi(j+P(0)) - s_j v(0) \leq u_j(0) \leq \phi(j+Q(0)) + S_j v(0) =: w_j^+(0). \tag{4.64}$$

where P and Q are defined in (4.55), (4.60) and $w_j^\pm$ are defined in (4.54), (4.59). Then

$$\sup_{j\in\Omega} |u_j(t) - \phi(j+ct)| \to 0, \text{ as } t \to \infty.$$

Proof. Levermore and Xin [41] proved that if the initial condition $u_j(0)$ is a small perturbation of the traveling wave solution ϕ, then the solution of the initial value problem converges to the traveling wave solutions as $t \to +\infty$. Using the assumption and the Comparison Principle (via Lemmas 15 and 16), we have that

$$w_j^-(t) \leq u_j(t) \leq w_j^+(t), \qquad t \geq 0. \tag{4.65}$$

Use $\overline{\eta} > 0$ and $\underline{\eta} > 0$ arbitrary (see (4.56), (4.61)). By (4.65), we have

$$\begin{aligned} \inf_{j\in\Omega}[u_j(t) - \phi(j+ct)] &\geq \inf_{j\in\Omega}[\phi(j+P(t)) - s_j\underline{v}(t) - \phi(j+ct)] \\ &\geq -[e^{-\omega t} + \underline{V}(0) - \underline{V}(t)]||\phi'||_\infty - \inf_{j\in\Omega} s_j\underline{v}(t) \\ &\geq -[e^{-\omega t} + \underline{V}(0) - \underline{V}(t)]||\phi'||_\infty - ||s||_\infty\underline{v}(t) \end{aligned}$$

using the Mean Value Theorem and the definition of $P(t)$. The right-hand side converges to $-\underline{V}(0)||\phi'||_\infty \geq -\underline{\eta}$ using the definition of $\underline{\eta}$ (4.56) as $t \to +\infty$. Since $\underline{\eta} > 0$ is arbitrary,

$$\liminf_{t\to\infty}\{\inf_j u_j(t) - \phi(j+ct)\} \geq 0.$$

Similarly, from (4.65), we have

$$\begin{aligned}
\sup_{j\in\Omega}[u_j(t)-\phi(j+ct)] &\leq \sup_{j\in\Omega}[\phi(j+Q(t))+S_j\overline{v}(t)-\phi(j+ct)] \\
&\leq [e^{-\omega t}+\overline{V}(0)-\overline{V}(t)]||\phi'||_\infty+\sup_{j\in\Omega}S_j\overline{v}(t) \\
&\leq [e^{-\omega t}+\overline{V}(0)-\overline{V}(t)]||\phi'||_\infty+||S||_\infty\overline{v}(t)
\end{aligned}$$

using the Mean Value Theorem and the definition of $Q(t)$. The right-hand side converges to $\overline{V}(0)||\phi'||_\infty \leq \overline{\eta}$ using the definition of $\overline{\eta}$ (4.61) as $t \to +\infty$. Since $\overline{\eta} > 0$ is arbitrary,

$$\limsup_{t\to\infty}\{\sup_{j\in\Omega} u_j(t)-\phi(j+ct)\} \leq 0.$$

Thus,

$$\sup_{j\in\Omega}|u_j(t)-\phi(j+ct)| \to 0, \quad \text{as} \quad t\to\infty.$$

□

4.4.7 Determining s and S

The existence of the super-/sub-solution pair was the key in the proof of Lemma D, whose existence, in turn, depend on the sequences s and S (see Lemmas 15 and 16). In this section, we explicitly compute the components of these two sequences.

Let us extract the middle portion of the tri-diagonal infinite matrix A in (4.8) and set

$$\hat{A} = \begin{pmatrix} -2d & d & 0 & 0 \\ d & -(d+d_0) & d_0 & 0 \\ 0 & d_0 & -(d+d_0) & d \\ 0 & 0 & d & -2d \end{pmatrix}$$

Let $C_1 = \min\{C_1', \tilde{C}_1\}$.

Define $\hat{s} = [s_{-2}, s_{-1}, s_0, s_1]$ and $\hat{\underline{b}} = [-d, (d_0 - d)C_1, 0, -d]$ if $d - d_0 > 0$ and $\hat{\underline{b}} = [-d, 0, (d - d_0)C_1, -d]$ if $d - d_0 < 0$. The linear system that determines $\hat{s}$ is $\hat{A}\hat{s} = \hat{\underline{b}}$. By Cramer's Rule, as long as $d + 4d_0 \neq 0$, then in case $d - d_0 > 0$,

$$\hat{s} = \frac{1}{d(d+4d_0)} \begin{pmatrix} (1+C_1)d^2 + (4+C_1)dd_0 - 2C_1 d_0^2 \\ (1+2C_1)d^2 + 2(2+C_1)dd_0 - 4C_1 d_0^2 \\ d^2 + 4(1+C_1)dd_0 - 4C_1 d_0^2 \\ d^2 + 2(2+C_1)dd_0 - 2C_1 d_0^2 \end{pmatrix}$$

It can be directly shown that

$$s_{-1} > \max\{s_{-2}, s_0\} > \min\{s_{-2}, s_0\} > s_1 > 0$$

and $s_{-1} \geq 1$. Hence, $||s||_\infty = s_{-1}$.

In case $d - d_0 < 0$,

$$\hat{s} = \frac{1}{d(d+4d_0)} \begin{pmatrix} d^2 + 2(2-C_1)dd_0 + 2C_1 d_0^2 \\ d^2 + 4(1-C_1)dd_0 + 4C_1 d_0^2 \\ (1-2C_1)d^2 + 2(2-C_1)dd_0 + 4C_1 d_0^2 \\ (-1+C_1)d^2 + (-4+C_1)dd_0 - 2C_1 d_0^2 \end{pmatrix}$$

It can be directly shown that

$$s_0 > \max\{s_1, s_{-1}\} > \min\{s_1, s_{-1}\} > s_{-2} > 0$$

and $s_0 \geq 1$. Hence, $||s||_\infty = s_0$.

Define $\hat{S} = [S_{-2}, S_{-1}, S_0, S_1]$ and $\hat{\bar{b}} = [-d, 0, (d_0 - d)C_1, -d]$ if $d - d_0 > 0$ and $\hat{\bar{b}} = [-d, (d - d_0)C_1, 0, -d]$ if $d - d_0 < 0$. The linear system that determines $\hat{S}$ is $\hat{A}\hat{S} = \hat{\bar{b}}$. By Cramer's Rule, as long as $d+4d_0 \neq 0$, then in case $d-d_0>0$,

$$\hat{S} = \frac{1}{d(d+4d_0)} \begin{pmatrix} d^2+2(2+C_1)dd_0-2C_1d_0^2 \\ d^2+4(1+C_1)dd_0-4C_1d_0^2 \\ (1+2C_1)d^2+2(2+C_1)dd_0-4C_1d_0^2 \\ (1+C_1)d^2+(4+C_1)dd_0-2C_1d_0^2 \end{pmatrix}$$

It can be directly shown that

$$S_{-1} > \max\{S_{-2}, S_0\} > \min\{S_{-2}, S_0\} > S_1 > 0$$

and $S_0 \geq 1$. Hence, $||S||_\infty = S_{-1}$.

In case $d-d_0<0$,

$$\hat{S} = \frac{1}{d(d+4d_0)} \begin{pmatrix} (1-C_1)d^2+(4-C_1)dd_0+2C_1d_0^2 \\ (1-2C_1)d^2+2(2-C_1)dd_0+4C_1d_0^2 \\ d^2+4(1-C_1)dd_0+4C_1d_0^2 \\ d^2+2(2-C_1)dd_0+2C_1d_0^2 \end{pmatrix}$$

It can be directly shown that

$$S_0 > \max\{S_1, S_{-1}\} > \min\{S_1, S_{-1}\} > S_2 > 0$$

and $S_0 \geq 1$. Hence, $||S||_\infty = S_0$.

We summarize our computations in a lemma:

Lemma 17. *If* $d-d_0>0$ *then*

$$||s||_\infty = \frac{(1+2C_1)d^2+2(2+C_1)dd_0-4C_1d_0^2}{d(d+4d_0)} = ||S||_\infty$$

and if $d - d_0 < 0$ *then*

$$||s||_\infty = \frac{(1-2C_1)d^2 + 2(2-C_1)dd_0 + 4C_1 d_0^2}{d(d+4d_0)} = ||S||_\infty$$

From the bounds on the sequence s in (4.58), (4.57) and the bounds on the sequence S in (4.63), (4.62), we have

$$\frac{\omega\,\underline{\delta}}{2\underline{\mu}||f'||_\infty} \le ||s||_\infty \le \frac{\rho}{2\underline{\mu}}$$

and

$$\frac{\omega\,\overline{\delta}}{2\overline{\mu}||f'||_\infty} \le ||S||_\infty \le \frac{\rho}{2\overline{\mu}}.$$

The constants ω, ρ depend on f; the constants $\underline{\delta},, \overline{\delta}$, depend on ϕ, while $\underline{\mu}, \overline{\mu}$ depend on both f and ϕ.

Let

$$C^- = \max\{\frac{\omega\,\underline{\delta}}{2\underline{\mu}||f'||_\infty}, \frac{\omega\,\overline{\delta}}{2\overline{\mu}||f'||_\infty}\} = \frac{\omega}{2||f'||_\infty}\max\{\frac{\underline{\delta}}{\underline{\mu}}, \frac{\overline{\delta}}{\overline{\mu}}\}$$

and

$$C^+ = \min\{\frac{\rho}{2\underline{\mu}}, \frac{\rho}{2\overline{\mu}}\} = \frac{\rho}{2}\min\{\frac{1}{\underline{\mu}}, \frac{1}{\overline{\mu}}\}.$$

We can now justify the Assumption on the diffusion defect. Define

$$R_1^+ = \frac{2 - 2C^- + C_1 - \sqrt{16C_1(1 - C^- + 2C_1) + (4 - 4C^- + 2C_1)^2}}{-(1 - C^- + 2C_1)},$$

$$R_2^+ = \frac{2 - 2C^+ + C_1 - \sqrt{16C_1(1 - C^+ + 2C_1) + (4 - 4C^+ + 2C_1)^2}}{-(1 - C^+ + 2C_1)}.$$

If $d - d_0 > 0$ then we choose d_0 such that

$$0 < \max\{1, R_1^+\} < \frac{d}{d_0} < R_2^+. \tag{4.66}$$

Define

$$R_1^- = \frac{C_1 + 2C^- - 2 - \sqrt{-4C_1(1-2C_1-C^-)+(2-C_1-2C^-)^2}}{(1-2C_1-C^-)}$$

and

$$R_2^- = \frac{C_1 + 2C^+ - 2 - \sqrt{-4C_1(1-2C_1-C^+)+(2-C_1-2\overline{C}^+)^2}}{(1-2C_1-C^+)}.$$

If $d - d_0 < 0$ then we choose d_0 such that

$$0 < R_2^- < \frac{d}{d_0} < \min\{1, R_1^-\}. \tag{4.67}$$

4.5 Proof of Lemma C

Recall that the result left to prove is Lemma C, which we re-state:

Lemma C - Positive Time. *Let $u_j(t)$ be a classical solution of*

$$\dot{u}_j(t) = \begin{cases} d(u_{j+1} - 2u_j + u_{j-1}) + f(u_j) & j \neq -1, 0, \\ d_{j+1}(u_{j+1} - u_j) + d_j(u_{j-1} - u_j) + f(u_j) & j = -1, 0, \end{cases}$$

for $t \geq 0$ with $0 \leq u_j(t) \leq 1$ for $j \neq -1, 0$. If for any $\varepsilon > 0$, there exists a time $t_\varepsilon \geq 0$ and there exists an integer $n_\varepsilon > 1$ such that

$$|u_j(t_\varepsilon) - \phi(j + ct_\varepsilon)| \leq \varepsilon, \ \textit{for all} \ \ |j| \geq n_\varepsilon \tag{4.68}$$

and

$$u_j(t) \geq 1 - \varepsilon, \ \textit{for all} \ \ t \geq t_\varepsilon; \ \textit{for} \ \ j = -1, 0. \tag{4.69}$$

then

$$\sup_{j\neq -1,0} |u_j(t)-\phi(j+ct)| \to 0, \qquad \text{as } t\to\infty.$$

Again, the main difference between Lemma D and Lemma C is that the former requires a bounded initial condition while the latter has a more general assumption on u_j. As mentioned earlier, the proof of Lemma C follows similar arguments as the proof of Lemma D, with some modifications because of $t_\varepsilon \geq 0$.

Proof. Similar to the proof of Lemma D, we need to show that for an arbitrary $\eta > 0$, we have

$$\inf_{j\in\Omega}(u_j(t)-\phi(j+ct)) \geq -\eta \tag{4.70}$$

and

$$\sup_{j\in\Omega}(u_j(t)-\phi(j+ct)) \leq \eta \tag{4.71}$$

because these two will imply that

$$\limsup_{t\to+\infty}\sup_{j\in\Omega}(u_j(t)-\phi(j+ct)) \leq 0 \leq \liminf_{t\to+\infty}\inf_{j\in\Omega}(u_j(t)-\phi(j+ct)) \tag{4.72}$$

and hence,

$$\sup_{j\in\Omega}|u_j(t)-\phi(j+ct)| \to 0 \qquad \text{as} \quad t\to+\infty. \tag{4.73}$$

To show (4.70), we will construct a sub-solution that resembles (4.54) ; to show (4.71), we will construct a super-solution that resembles (4.59). Some modifications are needed because the initial conditions that are required in Lemma D may not hold in the more general assumptions of Lemma C. In particular, we need to define a super-/sub-solution pair that behaves nicely at $t_\varepsilon \geq 0$, that generalizes the bounded initial condition assumption in Lemma D.

For the sub-solution, the gist of the proof is: for all $t \geq t_\varepsilon$, we have

$$\begin{aligned} \inf_{j\in\Omega}(u_j(t)-\phi(j+ct)) &= \inf_{j\in\Omega}(\tilde{u}_j(t_\varepsilon+t)-\phi(j+ct)) \\ &\geq \inf_{j\in\Omega}(\phi(j+c(t_\varepsilon+t)-F(t_\varepsilon+t))-s_jv(t_\varepsilon+t)-\phi(j+ct)), \end{aligned}$$

where we want

$$\tilde{u}_j(t) \geq \phi(j+c(t_\varepsilon+t)-F(t_\varepsilon+t))-s_jv(t_\varepsilon+t), \tag{4.74}$$

for some functions F and $v > 0$, and positive sequence s to be determined. Observe that $u_j(-t_\varepsilon) = \tilde{u}_j(0)$.

By the Mean Value Theorem,

$$\inf_{j\in\Omega}(u_j(t)-\phi(j+ct)) \quad \geq \quad -||\phi'||_\infty(ct_\varepsilon-F(t_\varepsilon+t))||\phi'||_\infty-||s||_\infty v(t_\varepsilon+t),$$

which we want to converge to $(ct_\varepsilon - F(+\infty))||\phi'||_\infty$, that is, we want $ct_\varepsilon - F(+\infty)$ and $v(+\infty) = 0$. Other characteristics of F and v will appear in the computation to show that we $\phi(j+ct-F(t)_{-}s_jv(t)$ is a sub-solution; these are similar to the proof of Lemma D. If $(ct_\varepsilon - F(+\infty))||\phi'||_\infty \geq -\eta$, then

$$\liminf_{t\to+\infty}(\inf_{j\in\Omega}(u_j(t)-\phi(j+ct))) \geq 0.$$

Finally, we need initial conditions at t_ε that are analogous to the assumptions of Lemma D and that uses the assumption of Lemma C. By assumption, there exists $n_\varepsilon > 1$ such that if $|j| \geq n_\varepsilon$ then

$$|u_j(t_\varepsilon)-\phi(j+ct_\varepsilon)| < \varepsilon,$$

that is, $\phi(j+ct_\varepsilon)-\varepsilon < u_j(t_\varepsilon)$; while if $|j| < n_\varepsilon$ then there exists $\beta > 0$ such that

$$\phi(j+ct_\varepsilon-\beta) \leq u_j(t_\varepsilon),$$

because $\phi(-\infty) = 0$.

We apply similar arguments for the super-solution. $\square$

Chapter 5

Extensions and Future Work

5.1 Next-to-nearest neighbor coupling

Consider the following LDE that involves nearest-neighbor and next-to-nearest neighbor interactions:

$$\dot{u}_j = \alpha(u_{j+1} - 2u_j + u_{j-1}) + \beta(u_{j-2} - u_j + u_{j-2}) - f_a(u_j), \;\; j \in \mathbb{Z}, \tag{5.1}$$

where f_a is the smooth cubic bistable nonlinearity $f_a(u) = u(u-a)(u-1)$ with $0 < a < 1$. It is expected that the analysis for the traveling wavefront problem to (5.1) will require looking at a higher-degree equilibrium equation because it has three parameters, α, β, a. There are four cases to consider

$$\alpha, \beta > 0; \qquad \alpha, \beta < 0; \qquad \beta < 0 < \alpha; \qquad \alpha < 0 < \beta$$

If $\beta < 0 < \alpha$ then the states u_j have a propensity to be similar to their nearest-neighbors while having a propensity to be different from their next-to-nearest neighbors. If $\alpha < 0 < \beta$ then we have the reverse propensity behavior.

The traveling wavefront to (5.1), with the usual traveling wave ansatz, gives rise to the following MFDE,

$$-c\phi'(\xi) = \alpha(\phi(\xi+1) - 2\phi(\xi) + \phi(\xi-1)) + \beta(\phi(\xi+2) - 2\phi(\xi) + \phi(\xi-2)) - f_a(\phi(\xi)). \tag{5.2}$$

If $\alpha, \beta > 0$ then some results by Mallet-Paret [43], [44], [45] can be applied . For the other cases, where either coefficient can be negative, one can apply the method and some results in [4], [5].

In case $\alpha < 0 < \beta$, we adapt the method presented in [10] and in Chapter 3 to show that (5.1) can be viewed in a period-2 framework and period-4 framework. In both cases, the method starts with an assumption that, respectively, there is a stable period-2 equilibria (τ_0, τ_1) and there is a stable period-4 equilibria $(\tau_0, \tau_1, \tau_2, \tau_3)$ such that they alternate signs, that is, $sign(\tau_i) = (-1)^i$. The existence of such equilibria depends on the interaction of the parameters α, β, a. These equilibria will define the boundary conditions of our traveling wavefront solutions, that is, we want to find a heteroclinic solution ϕ that connects the stable equilibria $(0,0)$ and (τ_0, τ_1) in the period-2 case and similarly for the period-4 scenario. Then we will apply the existence, stability, and uniqueness results for traveling waves in discrete periodic media that were stated in Chapter 2 (see [16]).

5.1.1 Existence of Period-2 Traveling Fronts

With $\alpha < 0 < \beta$, consider the even and odd nodes of u_j in (5.1) and represent them by $u_{2j} = x_j$ and $u_{2j+1} = y_j$ for $j \in \mathbb{Z}$, to get

$$\begin{cases} \dot{x}_j &= \alpha(y_j - 2x_j + y_{j-1}) + \beta(x_{j-1} - 2x_j + x_{j+1}) - f_a(x_j) \\ \dot{y}_j &= \alpha(x_{j+1} - 2y_j + x_j) + \beta(y_{j-1} - 2y_j + y_{j+1}) - f_a(y_j) \end{cases}, \quad j \in \mathbb{Z}. \tag{5.3}$$

In this form, notice that if $\alpha = 0$, then we get 2 copies of the positive-diffusion ($\beta > 0$) lattice Nagumo equation. Obviously, $(x,y) = (0,0)$ is a stable equilibrium of (5.3); let us assume, in the meantime, that there is another stable equilibrium (τ_0, τ_1) of (5.3). We want a heteroclinic connection from $(x_-, y_-) = (0,0)$ to $(x_+, y_+) = (\tau_0, \tau_1)$, where $(x_\pm, y_\pm) = (\lim_{j\to\pm\infty})(x_j, y_j)$.

Re-scale (5.3) using the following linear change-of-variables: $v_j = \dfrac{x_j}{\tau_0}$ and $w_j = \dfrac{y_j}{\tau_1}$. The purpose of the re-scaling is to insure that we are working in the unit interval, that is, we want $(v,w) \in (0,1)\times(0,1)$. The re-scaled system is

$$\begin{cases} \dot{v}_j &= \frac{\alpha\tau_1}{\tau_0}(w_{j-1} - 2v_j + w_j) + \beta(v_{j-1} - 2v_j + v_{j+1}) - \frac{1}{\tau_0} f_a(\tau_0 v_j) - 2\alpha(1 - \frac{\tau_1}{\tau_0})v_j \\ \dot{w}_j &= \frac{\alpha\tau_0}{\tau_1}(v_j - 2w_j + v_{j+1}) + \beta(w_{j-1} - 2w_j + w_{j+1}) - \frac{1}{\tau_1} f_a(\tau_1 w_j) - 2\alpha(1 - \frac{\tau_0}{\tau_1})w_j. \end{cases} \tag{5.4}$$

Since $(x,y) = (\tau_0, \tau_1)$ satisfies $y = x + \frac{f_a(x)}{2\alpha}$, it can be shown that

$$\begin{cases} \frac{1}{\tau_0} f_a(\tau_0 v) - 2\alpha(1 - \frac{\tau_1}{\tau_0})v &= \tau_0^2 f_{a_0}(v) \\ \frac{1}{\tau_1} f_a(\tau_1 w) - 2\alpha(1 - \frac{\tau_0}{\tau_1})w &= \tau_1^2 f_{a_1}(w) \end{cases}$$

with the same notation $f_a(u) = u(u-a)(u-1)$ and where the detuning parameters (a_0, a_1) satisfy $a_0 = \dfrac{a+1}{\tau_0} - 1$ and $a_1 = \dfrac{a+1}{\tau_1} - 1$. Thus,

Lemma 18. *By looking at the even and nodes of u_j, the system (5.1) with $\alpha < 0 < \beta$ exhibits the same dynamics as the anti-diffusion lattice Nagumo system that was considered in Chapter 3.*

5.1.2 Existence of Period-4 Traveling Fronts

With $\alpha < 0 < \beta$, it can be shown that (5.1) has a 2^2-periodic traveling wavefront solution. The procedure starts with re-casting the equations (5.1) into

$$\begin{cases} \dot{\hat{w}}_j &= \alpha(\hat{z}_{j-1} - 2\hat{w}_j + \hat{x}_j) + \beta(\hat{y}_{j-1} - 2\hat{w}_j + \hat{y}_j) - f(\hat{w}_j) \\ \dot{\hat{x}}_j &= \alpha(\hat{w}_j - 2\hat{x}_j + \hat{y}_j) + \beta(\hat{z}_{j-1} - 2\hat{x}_j + \hat{z}_j) - f(\hat{x}_j) \\ \dot{\hat{y}}_j &= \alpha(\hat{x}_j - 2\hat{y}_j + \hat{z}_j) + \beta(\hat{w}_j - 2\hat{y}_j + \hat{w}_{j+1}) - f(\hat{y}_j) \\ \dot{\hat{z}}_j &= \alpha(\hat{y}_j - 2\hat{z}_j + \hat{w}_{j+1}) + \beta(\hat{x}_j - 2\hat{z}_j + \hat{x}_{j+1}) - f(\hat{z}_j) \end{cases} \tag{5.5}$$

via the change-of-variables $u_{4j} = \hat{w}_j$, $u_{4j+1} = \hat{x}_j$, $u_{4j+2} = \hat{y}_j$, and $u_{4j+3} = \hat{z}_j$. We want to look for traveling wavefront solutions that connect the equilibria $(\hat{w}_-, \hat{x}_-, \hat{y}_-, \hat{z}_-) = (0,0,0,0)$ to $(\hat{w}_+, \hat{x}_+, \hat{y}_+, \hat{z}_+) = (\tau_0, \tau_1, \tau_2, \tau_3)$ where $\tau_0 > 0, \tau_1 < 0, \tau_2 > 0, \tau_3 < 0$.

The equilibrium $(w,x,y,z) = (\tau_0, \tau_1, \tau_2, \tau_3)$ satisfies four nonlinear equations:

$$\begin{cases} \alpha(z - 2w + x) + 2\beta(y - w) - f(w) = 0 \\ \alpha(w - 2x + y) + 2\beta(z - x) - f(x) = 0 \\ \alpha(x - 2y + z) + 2\beta(w - y) - f(y) = 0 \\ \alpha(y - 2z + w) + 2\beta(x - z) - f(z) = 0 \end{cases}$$

In fact, we have

$$\tau_1 < \tau_3 < 0 < s_- < s_+ < 1 < \tau_2 < \tau_0 \tag{5.6}$$

where s_-, s_+ are the critical points of f. To prove that there exist such τ_i with (5.6), we work out the equilibrium equations to obtain the following equivalent system:

$$\begin{cases} f(w)+f(x)+f(y)+f(z) & = & 0 \\ 2\alpha+4\beta+\frac{f(y)-f(w)}{y-w} & = & 0 \\ 2\alpha+4\beta+\frac{f(z)-f(x)}{z-x} & = & 0 \\ \alpha(x-2y+z)+2\beta(w-y)-f(y) & = & 0 \end{cases} \tag{5.7}$$

The system (5.5) can be viewed in the periodic framework (see Chapter 2), where the boundary conditions are satisfied by setting $\hat{w}_j = \tau_0 w_j$, $\hat{x}_j = \tau_1 x_j$, $\hat{y}_j = \tau_2 y_j$, $\hat{z}_j = \tau_3 z_j$ and the ellipticity conditions are satisfied by setting

$$\begin{cases} (d_0, \hat{d}_0, e_0) & = & (\frac{\alpha\tau_3}{\tau_0}, \frac{\alpha\tau_1}{\tau_0}, \frac{\beta\tau_2}{\tau_0}) \\ (d_1, \hat{d}_1, e_1) & = & (\frac{\alpha\tau_0}{\tau_1}, \frac{\alpha\tau_2}{\tau_1}, \frac{\beta\tau_3}{\tau_1}) \\ (d_2, \hat{d}_2, e_2) & = & (\frac{\alpha\tau_1}{\tau_2}, \frac{\alpha\tau_3}{\tau_2}, \frac{\beta\tau_0}{\tau_2}) \\ (d_3, \hat{d}_3, e_3) & = & (\frac{\alpha\tau_2}{\tau_3}, \frac{\alpha\tau_0}{\tau_3}, \frac{\beta\tau_1}{\tau_3}). \end{cases}$$

Note that $d_j, \hat{d}_j, e_j$ are all positive. With respect to (w,x,y,z), the boundary conditions are the equilibria $\vec{0} = (0,0,0,0)$ and $\vec{1} = (1,1,1,1)$. If there are no other stable 2^2-periodic equilibria (with $w_j, x_j, y_j, z_j \in (0,1)$), then there exists a traveling wavefront solution to (5.5). If $\vec{0}, \vec{1}$ are both stable (under some conditions on α, β, a) then the theory says that the 2^2-periodic traveling wavefront solution is unique (up to a time translation) and globally exponentially stable. Recently, Van Vleck and Zhang [53] worked on the nearest and next-to-nearest coupling problem where they used results by [5].

5.1.3 Questions

Thus, based on these preliminary computations, in case of nearest and next-to-nearest neighbor coupling (5.1) with $\alpha < 0 < \beta$, an investigation of the existence of period-2 and period-4 traveling wavefront solutions requires answers to...

1. Existence of τ_i (that is, a careful investigation of the equilibrium equations and a bifurcation analysis of the parameters)? The equilibrium equations exhibit some structure and symmetry that can be used.

2. Stability of τ_i (the theory for periodic media requires the knowledge of stability of equilibria at the boundary and instability of equilibria in the interior)?

3. Propagation failure phenomenon (Conley-Moser conditions will give rise to an analysis of the Keener map (see Chapter 3.4.1) that will be iterated four times)

5.2 Period Power of 2

For some integer $L \geq 1$, consider the lattice dynamical system having L-range interaction of the form

$$\dot{u}_j = \sum_{k=1}^{L} \alpha_k (u_{j+k} - 2u_j + u_{j-k}) - f(u_j), \quad j \in \mathbb{Z} \tag{5.8}$$

where $f(u) = u(u-a)(u-1)$, $0 < a < 1$ and $\alpha_k = (-1)^k \beta_k$ for $\beta_k \in \mathbb{R}_{\geq 0}$.

For $L = 1$, (5.8) is the spatially discrete anti-diffusion ($\alpha_1 < 0$) Nagumo equation, to which it has been established that there is a traveling 2-periodic wavefront solution that connects $(0,0)$ to (τ_0, τ_1) with $\tau_0 > 0 > \tau_1$ where τ_j satisfies a degree-9 equation.

For $L = 2$, (5.8) takes the form

$$\dot{u}_j = \alpha_1 (u_{j+1} - 2u_j + u_{j-1}) + \alpha_2 (u_{j+2} - 2u_j + u_{j-2}) - f(u_j), \quad j \in \mathbb{Z}, \tag{5.9}$$

where $\alpha_1 < 0 < \alpha_2$, which is what we have considered in the previous section.

Applying a similar process and performing a careful analysis of the equilibrium equation of degree 9^L, it can be shown that (5.8) has a 2^L-periodic traveling wavefront solution that connects $\vec{0}$ to $\vec{1}$ (with L components) provided that there are no other stable equilibria between $\vec{0}$ and $\vec{1}$. Hence, we have constructed a lattice dynamical system that has a traveling wavefront solution with period 2^L, for any integer $L \geq 1$. Nontrivial preliminary questions in this case are similar to Section 5.1.3.

5.3 Period infinity

What happens if we send L to infinity? Consider the lattice dynamical system

$$\dot{u}_j = \sum_{k=1}^{\infty} \alpha_k (u_{j+k} - 2u_j + u_{j-k}) - f(u_j), \quad j \in \mathbb{Z} \tag{5.10}$$

where $f(u) = u(u-a)(u-1)$, $0 < a < 1$ and ellipticity conditions

$$\begin{array}{ll} \alpha_k = (-1)^k \beta_k, & \text{where } \beta_k \in \mathbb{R}_{\geq 0} \\ \sum_{k=1}^{\infty} \alpha_k < \infty & \text{finite diffusion coefficients} \\ \sum_{k=1}^{\infty} k|\alpha_k| < \infty & \text{finite first moment} \\ \sum_{k=1}^{\infty} k^2|\alpha_k| < \infty & \text{finite second moment.} \end{array}$$

We require finite diffusion coefficients as can be seen from:

$$\begin{aligned} \dot{u}_j &= \sum_{k=1}^{\infty} \alpha_k (u_{j+k} - 2u_j + u_{j-k}) - f(u_j) \\ &= \sum_{k=1}^{\infty} \alpha_k (u_{j+k} + u_{j-k}) - 2u_j \left(\sum_{k=1}^{\infty} \alpha_k\right) - f(u_j) \end{aligned}$$

We require finite second moment so that the first term in (5.10) approximates the second derivative in wave variable ξ, as follows: with the traveling wave ansatz $u_j(t) = u(x-ct) =$

$u(\xi)$, we have

$$cu'(\xi) = \sum_{k=1}^{\infty} \alpha_k k^2 \left(\frac{u(\xi+k) - 2u(\xi) + u(\xi-k)}{k^2} \right) - f(u(\xi))$$

so that

$$\begin{aligned} u(\xi+k) &= u(\xi) + u'(\xi)k + u''(\xi)\tfrac{k^2}{2} + O(k^3) \\ u(\xi-k) &= u(\xi) - u'(\xi)k + u''(\xi)\tfrac{k^2}{2} - O(k^3) \end{aligned}$$

implies

$$\frac{u(\xi+k) - 2u(\xi) + u(\xi-k)}{k^2} = u''(\xi) + O(k^2)$$

and hence

$$\begin{aligned} \textstyle\sum_{k=1}^{\infty} k^2 \alpha_k \left(\frac{u(\xi+k)-2u(\xi)+u(\xi-k)}{k^2} \right) - u''(\xi) &= \textstyle\sum_{k=1}^{\infty} k^2 \alpha_k \left(u''(\xi) + O(k^2) \right) - u''(\xi) \\ &= u''(\xi) \left(\textstyle\sum_{k=1}^{\infty} k^2 \alpha_k - 1 \right) + \textstyle\sum_{k=1}^{\infty} k^2 \alpha_k O(k^2) \end{aligned}$$

In this case, a similar analysis shown in this work can be applied which requires a careful analysis of the equilibrium equations; or noting that there is infinite-range coupling in this case, one can apply the method done in [4] to establish existence of the wavefronts.

5.4 Finite number of multiple defects in the middle of $\mathbb{Z}$

The analysis and computations in Chapter 4 can be applied in case when there are finitely many defects located in the middle of $\mathbb{Z}$. For example, assume an inhomogeneity given by

$$d_j = \begin{cases} d_0, & 0 \le j \le n \\ d, & \text{otherwise} \end{cases} \tag{5.11}$$

where n is a positive integer and $d \neq d_0$. Then the inhomogeneous system

$$\dot{u}_j = d_{j+1}(u_{j+1} - u_j) + d_j(u_{j-1} - u_j) + f(u_j), \qquad j \in \mathbb{Z},$$

becomes

$$\dot{u}_j = \begin{cases} d(u_{j-1} - 2u_j + u_{j+1}) + f(u_j), & j < -1, \ j > n \\ d_0(u_{j+1} - u_j) + d(u_{j-1} - u_j) + f(u_j), & j = -1 \\ d_0(u_{j-1} - 2u_j + u_{j+1}) + f(u_j), & 0 \leq j \leq n-1 \\ d(u_{j+1} - u_j) + d_0(u_{j-1} - u_j) + f(u_j), & j = n, \end{cases}$$

in other words, the set of obstacles is $K = [-1, n] \cap \mathbb{Z}$. From the analysis in Chapter 4 (see Theorem A), we need to first construct a super-/sub-solution pair $v^{\pm}(t) = \{v_j^{\pm}(t)\}_{j \in \mathbb{Z}}$ that will squeeze in a time-global solution $u_j(t)$ for $j \in \Omega$ that behaves in the desired manner at negative time. This super-/sub-solution pair will resemble (4.28) and (4.38) in Section 4.2; in particular, one may take a super-solution of the form

$$v_j^{+}(t) = \begin{cases} \phi(j + p(t)) + \phi(-j + p(t)), & j > 0 \\ 2\phi(j + p(t)), & j \leq 0 \end{cases}$$

and a sub-solution of the form

$$v_j^{-}(t) = \begin{cases} \phi(j + q(t)) - \phi(-j + q(t)), & j > 0 \\ 0, & j \leq 0. \end{cases}$$

Then, one constructs another super-/sub-solution pair $w^{\pm}(t) = \{w_j^{\pm}(t)\}_{j \in \mathbb{Z}}$ in positive time, where one has to choose appropriate bounded positive sequences $s = \{s_j\}_{j \in \mathbb{Z}}$ and $S = \{S_j\}_{j \in \mathbb{Z}}$. Recall that these two sequences guarantee the existence of the super-/sub-solution pair $w^{\pm}$ (see

Lemmas 15, 16). A slight generalization of this particular diffusion defect is when $d_j \neq d_0$ for all $0 \leq j \leq n$.

A more challenging case is when

$$d_j = \begin{cases} d_0, & -m \leq j \leq n \\ d, & \text{otherwise} \end{cases} \tag{5.12}$$

for some positive integers m,n. In this case, the super-/sub-solution pair at negative time will not be as simple as the above ansatz.

5.5 Finite number of scattered defects

How does one handle a defect that is scattered in $\mathbb{Z}$ where j does not fall in an interval $[-m,n]$ for positive integers m,n? Or, even if the d_j's fall in an interval but the intervals are located far away from the middle? Perhaps, one can start with an inhomogeneous media whose diffusion coefficients are defined by

$$d_j = \begin{cases} d, & j < -n_1 \\ d_1, & -n_1 \leq j < -n_2 \\ d, & -n_2 \leq j < n_3 \\ d_1, & n_3 \leq j < n_4 \\ d, & j > n_4, \end{cases}$$

where $d_1 \neq d$ and n_j are positive integers.

5.6 Propagation failure in inhomogeneous media

Wave blocking in inhomogeneous media in the spatially-continuous case was investigated in [3] and [40]. An analysis of the propagation failure phenomenon for LDEs over an inhomogeneous medium, whether in the periodic media case or having defects in the middle (or scattered) are open problems. The periodic-N media case may be handled using the analysis set forth by Keener (see Chapter 3.4.1 and [39]) which will require an N-iterate analysis of a Henon-like mapping ϕ. For example, in the anti-diffusion system in Chapter 3, we iterated ϕ twice (see Section 3.4.1).

In the single-defect inhomogeneous system, we observe that we cannot obtain a similar homeomorphism ϕ as defined in [39] due to the inhomogeneity. Instead, one can proceed with a direct computation of the invariant sets of the system. Consider the set K consisting of all functions $U = \{U_j\}_{j\in\mathbb{Z}}$ such that

$$\begin{aligned}
0 \le U_j \le r_1, &\quad j \le -2\\
r_2 \le U_j \le r_3, &\quad j = -1\\
r_4 \le U_j \le r_5, &\quad j = 0\\
r_6 \le U_j \le 1, &\quad j > 0.,
\end{aligned}$$

where $0 < r_i < 1$ for all $i = 1, \ldots, 6$ are parameters. Then one investigates the behavior of the vector field at the boundaries of K.

Bibliography

[1] K. ABELL, ET AL, *Computation of mixed type functional differential boundary value problems*, SIAM J. Appl. Dyn. Sys., 4(3), 755 - 781, 2005. Cited on 10

[2] D. G. ARONSON AND H. WEINBERGER, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in Partial Differential Equations and Related Topics, Lecture Notes in Math 446, Springer-Verlag, Berlin, 5 - 49, 1975. Cited on 7

[3] D. G. ARONSON, N. V. MANTZARIS, AND H. G. OTHMER, *Wave propagation and blocking in inhomogeneous media*, Disc. Cont. Dyn. Sys., 13(4), 843 - 876, 2005. Cited on 125

[4] P. BATES AND A. CHMAJ, *A discrete convolution model for phase transitions*, Arch. Rational Mech. Anal., 150, 281 - 305, 1999. Cited on 117, 123

[5] P. BATES, X. CHEN, AND A. CHMAJ, *Traveling waves of bistable dynamics on a lattice*, SIAM J. Math. Anal., 35 (2), 520 -546, 2003. Cited on 117, 120

[6] J. BELL AND C. COSNER, *Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons*, Quart. Appl. Math. 42(1), 1-14, 1984. Cited on 4

[7] R. D. BENGURIA AND M. C. DEPASSIER, *Validity of the linear speed selection mechanism for fronts of the nonlinear diffusion equation*, Phys. Rev. Lett., 73 (16), 2272 - 2274, 1994. Cited on 61

[8] H. BERESTYCKI, F. HAMEL, AND H. MATANO, *Bistable traveling waves around an obstacle*, Comm. Pure App. Math., 62(6), 729 - 788, 2009. Cited on 75, 76, 91

[9] H. BERESTYCKI AND F. HAMEL,*Generalized travelling waves for reaction-diffusion equations*, Perspectives in Nonl. Partial Diff. Eqns. - In honor off H.Brezis, Contemp. Math., Amer. Math. Soc., 2007. Cited on 75

[10] M. BRUCAL-HALLARE, E.S. VAN VLECK, *Traveling Wavefronts in an antidiffusion lattice Nagumo model*, SIAM J. Appl. Dyn. Sys., 10 (3), 921 - 959, 2011. Cited on 6, 117

[11] J. CAHN, S.N. CHOW, AND E.S. VAN VLECK, *Spatially discrete nonlinear diffusion equations*, Rocky Mntn. J. Math. 25(1), 87 118, 1995. Cited on 14

[12] J. CAHN, J. MALLET-PARET, AND E.S. VAN VLECK, *Traveling wave solutions for systems of ODEs on a two-dimensional spatial lattice*, SIAM J. Appl. Math, 59(2), 455 - 493, 1998. Cited on 14

[13] X. CHEN AND J-S. GUO,*Existence and uniqueness of entire solutions for a reaction-diffusion equation*, J. Diff. Eqn., 212, 62-84, 2005. Cited on 76

[14] X. CHEN, S-C. FU, AND J-S. GUO, *Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices*, SIAM J. Math. Anal., 38 (1), 233 - 258, 2006. Cited on 4

[15] X. CHEN AND J-S. GUO, *Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics*, Math. Ann., 326, 123 - 146, 2003. Cited on 76

[16] X. CHEN, J-S. GUO, AND C-C. WU, *Traveling waves in discrete periodic media for bistable dynamics*, Arch. Ratl. Mech. Anal., 189, 189-236, 2008. Cited on 7, 10, 11, 15, 28, 30, 36, 40, 117

[17] S-N. CHOW, J.MALLET-PARET, AND E.S. VAN VLECK, *Dynamics of Lattice Differential Equations*, Intl. J. Bif. Chaos, 6(9), 1605–1621, 1996. Cited on 14

[18] C. ELMER, *Finding stationary fronts for a discrete Nagumo and wave equation; construction*, Physica D, 218, 11-23, 2006. Cited on 4, 10

[19] C. ELMER AND E. VAN VLECK, *Anisotropy, propagation failure, and wave speedup in traveling waves of discretizations of a Nagumo PDE*, J. Comp. Physics, 185 (2), 562 582, 2003. Cited on 2, 4

[20] C. ELMER AND E. VAN VLECK, *Dynamics of monotone traveling fronts for discretizations of Nagumo PDE's*, Nonlinearity, 18, 1605 - 1628, 2005. Cited on 2, 4

[21] T. ERNEUX AND G. NICHOLS, *Propagating waves in discrete bistable reaction-diffusion systems*, Physica D, 67 (1-3), 237-244, 1993. Cited on 4

[22] P.C.FIFE, *Mathematical Aspects of Reacting and Diffusing Systems*, Springer-Verlag, New York, Berlin, 1979. Cited on 7, 61

[23] P.C.FIFE AND J.B. MCLEOD, *The approach of solutions of nonlinear diffusion equations to traveling front solutions*, Arch. Ratl. Mech. Anal, 65, 335-361, 1977. Cited on 7

[24] R. FISHER, *The wave advance of advantageous genes*, Ann. Eugenics 7, 355- 369, 1937. Cited on 60

[25] Y. FUKAO, Y. MORITA, AND H. NINOMIYA, *Some entire solutions of the Allen-Cahn equation*, Taiwanese J. Math., 8(1), 15 - 32, 2004. Cited on 76

[26] J-S. GUO AND F. HAMEL, *Front propagation for discrete periodic monostable equations*, Math. Ann., 335, 489-525, 2006. Cited on 61

[27] J-S. GUO AND C-C. WU, *Uniqueness and stability of traveling waves for periodic monostable lattice dynamical system*, J. Diff. Eqns. 246, 3818-3833, 2009. Cited on 7

[28] J-S. GUO AND Y. MORITA, *Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations*, Disc. Cont. Dynam. Sys., 12(2), 193 - 212, 2005. Cited on 76

[29] Y-J LIN GUO, *Entire solutions for a discrete diffusive equation*, J. Math. Anal. Appl., 347, 450 - 458, 2008. Cited on 76

[30] F. HAMEL AND N. NADIRASHVILI, *Entire solutions of the KPP equation*, Comm. Pure Appl. Math., 52, 1255 - 1276, 1999. Cited on 76

[31] D. HANKERSON AND B. ZINNER, *Wavefronts for a cooperative tridiagonal system of differential equations*, J. Dynam. Diff. Eqn., 5, 359-373, 1993. Cited on 4

[32] G. HARRIS, W. HUDSON, AND B. ZINNER, *Traveling wavefronts for the discrete Fisher's equation*, J. Diff. Eqn., 105, 46-62, 1993. Cited on 4, 61

[33] J. HARTERICH, B. SANDSTEDE AND A. SCHEEL, *Exponential dichotomies for linear non-autonomous functional differential equations of mixed type*, Indiana Univ. Math. J., 51, 1081-1109, 2002. Cited on 5

[34] A. HOFFMAN AND J. MALLET-PARET, *Universality of crystallographic pinning*, J. Dyn. Diff. Eqn., 22, 79-119, 2010. Cited on 6

[35] W. HUDSON AND B. ZINNER, *Existence of traveling waves for a generalized discrete Fisher's equation*, Comm. App. Nonlinear Analysis, 1, No 3, 23-46, 1994. Cited on 4, 61

[36] A. R. HUMPHRIES, B. E. MOORE, AND E. S. VAN VLECK, *Front solutions for bistable differential-difference equations with inhomogeneous diffusion*, SIAM J. Appl. Math., 71(4), 1374 - 1400, 2011. Cited on 74

[37] H. J. HUPKES AND S.M. VERDUYN-LUNEL, *Center manifold theory for functional differential equations of mixed type*, J. Dynam. Diff. Eqn. 19, 497-560, 2007. Cited on 5

[38] H. J. HUPKES AND E. VAN VLECK, Negative diffusion and travelling waves in high dimensional lattice systems, Submitted 2012. Cited on 14

[39] J. P. KEENER, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math, 47(3), 556-572, 1987. Cited on 4, 7, 40, 41, 42, 48, 52, 55, 71, 126

[40] T. LEWIS AND J. P. KEENER, *Wave-block in excitable media due to regions of depressed excitability*, SIAM J. Appl. Math., 61 (1), 293 - 316, 2000. Cited on 125

[41] C. D. LEVERMORE AND J. X. XIN,*Multidimensional stability of traveling waves in a bistable reactiondiffusion equation, II*, Comm. Partial Diff. Eqns., 17 (11-12),1901-1924, 1992. Cited on 107

[42] N-W. LIU AND W-T. LI, *Entire solutions in reaction-advection-diffusion equations with bistable nonlinearities in heterogeneous media*, Sci. China Math., 53(7), 1775 - 1786, 2010. Cited on 76

[43] J. MALLET-PARET, *The Fredholm alternative for functional differential equations of mixed type*, J. Dynam. Diff. Eqn., 11, 1-48, 1999. Cited on 5, 117

[44] J. MALLET-PARET *The global structure of traveling waves in spatially discrete dynamical systems*, J. Dynam. Diff. Eqn., 11, 49-127, 1999. Cited on 5, 117

[45] J. MALLET-PARET, *Traveling waves in spatially discrete dynamical systems of diffusive type*, Lecture Notes in Math, 1822, 231-298, 2003. Cited on 4, 5, 117

[46] T. MIYAZAWA AND T. IZUYAMA, *Diffusion in one-dimensional inhomogeneous media*, Phys. Rev. A., 36 (12), 5791 - 5811, 1987. Cited on 76

[47] Y. MORITA AND H. NINOMIYA,*Entire solutions with merging fronts to reaction-diffusion equations*, J. Dynam. Diff. Eqn., 18(4), 841 - 861, 2006. Cited on 76

[48] J. NOLEN AND L. RYZHIK, *Traveling waves in a one-dimensional heterogeneous medium*, preprint. Cited on 75

[49] J. P. PAUWELUSSEN, *Nerve impulse propagation in a branching nerve system: a simple model*, Phs. D., 4(1), 67 - 88, 1981. Cited on 71

[50] A. RUSTICHINI, *Functional differential equations of mixed type: the linear autonomous case*, J. Dynam. Fiff. Eqn., 1(2), 121-143, 1989. Cited on 5

[51] A. SCHEEL AND E.S. VAN VLECK, *Lattice differential equations embedded into reaction-diffusion systems*, Proc. Royal Soc. Edinburgh, 139, 193 - 207, 2009. Cited on 5

[52] J. SNEYD AND J. SHERRATT, *On the propagation of calcium waves in an inhomogeneous medium*, SIAM J. Appl. Math., 57(1), 73 - 94, 1997. Cited on 71

[53] E. VAN VLECK AND A. ZHANG, *Competing interactions and traveling wave solutions in lattice differential equations*, Submitted, 2012. Cited on 71, 120

[54] A. Vainchtein and E. Van Vleck, *Nucleation and propagation of phase mixtures in a bistable chain*, Phys. Rev. B, Vol 79, No 144123, 2009. Cited on iii, 4, 14

[55] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Second Edition, Springer-Verlag, New York, 2000. Cited on 49

[56] J. Xin, *Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media*, J. Stat. Phys, 73(5/6), 893-926, 1993. Cited on 2

[57] B. Zinner, *Stability of traveling wavefronts for the discrete Nagumo equation*, SIAM J. Math. Anal., 22, 1016-1020, 1991. Cited on 4, 7, 71

[58] B. Zinner, *Existence of traveling wavefront solutions for the discrete Nagumo equation*, J. Diff. Eqn, 96, 1-27, 1992. Cited on 4, 7, 71