Modulational Instability of Small Amplitude Wave Trains in the Novikov Equation

Brett Ehrman
Department of Mathematics
University of Kansas

Collaborators: Mathew A. Johnson and Stéphane Lafortune

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Overview

Consider the Novikov equation:

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx},$$

a 1-D dispersive model for nonlinear, shallow water waves with a Hamiltonian structure (like Camassa-Holm, Degasperis-Procesi, KdV, BBM, etc.)

We are concerned with stability of solutions of traveling waves of Novikov. Here is a road map for this talk:

- Mention Relevant Literature
- Show existence of and parameterize small-amplitude wave trains
- Examine the spectral problem using a Bloch decomposition
- Perform analysis to obtain a modulational instability (MI) result

Existing Literature on Stability

- First, Novikov is known to have smooth solitary waves, and they are orbitally stable: [BE, Johnson, Lafortune (2024)]
- Novikov also has peaked solitary waves, which are spectrally unstable on $W^{1,\infty}(\mathbb{R})$ but linearly and spectrally stable on $H^1(\mathbb{R})$: [Lafortune (2024)]

There are also smooth periodic traveling waves, for which essentially nothing is known regarding their stability.

Here, we construct family of smooth periodic traveling waves, study their spectral and "modulational stability."

Guess - there is a "critical wave number" k^* that tells us (for small amplitude) long waves are spectrally stable while short waves are not.

Wave Trains of Novikov

Again, here is the Novikov equation:

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx} \tag{1}$$

Definition (Wave Trains (or Traveling Wave Solutions))

We say $u(x, t)$ is a traveling wave solution or wave train for Novikov if there exists a solution of the form $u(x, t) = \varphi(\xi) := \varphi(x - ct)$, for some $c > 0$.

Setting $u(x, t) = \varphi(\xi)$ as above, u solves (1) iff φ solves

$$-c\varphi' + c\varphi''' + 4\varphi^2\varphi' = 3\varphi\varphi'\varphi'' + \varphi^2\varphi'''$$

which can be rearranged as:

$$\left(\varphi^2 - c\right)(\varphi - \varphi'')' + 3\varphi\varphi'(\varphi - \varphi'') = 0. \tag{2}$$

Existence of Wave Trains

After multiplying by the integrating factor $(\varphi - \varphi'')^{-\frac{1}{3}}$ and integrating, we arrive at

$$\varphi - \varphi'' = \frac{b}{(c - \varphi^2)^{\frac{3}{2}}}, \quad b \in \mathbb{R},$$

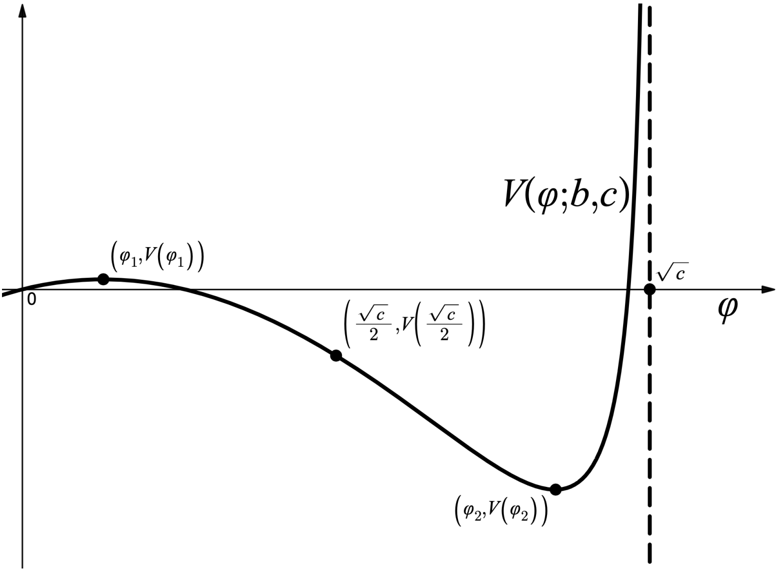
where along the way we chose $c > \varphi^2$ (see phase plane analysis). Reducing to quadrature gives

$$\frac{1}{2}\left(\varphi'\right)^2 = E - V(\varphi; b, c), \quad V(\varphi; b, c) := \frac{b\varphi}{c\sqrt{c - \varphi^2}} - \frac{\varphi^2}{2}, \quad E \in \mathbb{R}.$$

Thus, (2) has a family of smooth periodic solutions when the potential V has a local min in the interval $\left(-\sqrt{c}, \sqrt{c}\right)$.
In [BE, Johnson, Lafortune (2024)], we found that to be true whenever

$$b > 0, \quad c > 4b^{\frac{1}{2}}3^{-\frac{3}{4}}.$$

Graph of Potential V



Understand the Constant Solution

Goal: analytically parameterize waves of small amplitude near the constant solution $w_0(b, c)$.

We start with the constant solution w_0 (with corresponding c_0). We rescale by $z = kx$ and seek 2π-periodic solutions of

$$\left(c - w^2\right)^{\frac{3}{2}} \left(w - k^2 w''\right) = b. \tag{3}$$

Note the constant solution w_0 with $c = c_0$ solves

$$\left(c_0 - w_0^2\right)^{\frac{3}{2}} w_0 = b.$$

If we linearize (3) about (c_0, w_0), have

$$\mathcal{Q} = 1 - 3\frac{w_0^2}{c_0 - w_0^2} - k^2 \partial_z^2$$

To have solutions possibly bifurcate from w_0, we require (by the IFT) $\mathcal{Q}$ to be singular. It turns out that...

Analytic Expansion of Small-Amplitude Waves

$$\mathcal{Q}e^{\pm iz} = 0 \Leftrightarrow c_0 = \left(\frac{k^2+4}{k^2+1}\right) w_0^2.$$

$$\text{This gives } w_0(b,k) = b^{\frac{1}{4}} \left(\frac{3}{k^2+1}\right)^{-\frac{3}{8}}$$

Lyapunov-Schmidt reduction now gives the following:

Theorem

Given $b, k > 0$, there exists a family of $2\pi/k$-per solutions of the form

$$u(x,t;a,b,k) = w\left(k\left(x - c(a,b,k)t\right); a, b, k\right),$$

where w and c are analytic in a and c is even in a. For $|a| \ll 1$, have

$$\begin{cases} w(z;a,b,k) = w_0(b,k) + a\cos(z) + a^2\left(d_1 + d_2\cos(2z)\right) + O\left(a^3\right), \\ c(a,b,k) = c_0(b,k) + c_2 a^2 + O\left(a^4\right). \end{cases}$$

Define our Spectral Problem

Fix $k > 0$ and let w be as in the previous theorem.
Linearizing Novikov about w in the traveling coordinate frame $(z, t) = \big(k(x - ct), t\big)$ gives

$$V_t = k\left(1 - k^2\partial_z^2\right)^{-1}\mathcal{L}[w]V, \text{ where}$$

$$\mathcal{L}[w] := c\partial_z - ck^2\partial_z^3 - 8ww_z - 4w^2\partial_z + 3k^2(w_z w_{zz} + ww_{zz}\partial_z + ww_z\partial_z^2) + k^2\left(w^2\partial_z^3 + 2ww_{zzz}\right)$$

and $V(z, t)$ is the governing perturbation.
Seek solutions of the form $V(z, t) = e^{\lambda t}v(z)$, with $v \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{C}$; this leads to the following spectral problem:

Refine Spectral Problem

$$\mathcal{A}[w]v = \lambda v, \quad \mathcal{A}[w] := k\left(1 - k^2\partial_z^2\right)^{-1}\mathcal{L}[w]$$

is closed and densely defined on $L^2(\mathbb{R})$.

- Say w is **spectrally unstable** if $\sigma\left(\mathcal{A}[w]\right)$ intersects the open right half-plane of $\mathbb{C}$.
- As expected from a PDE with a Hamiltonian structure, said spectrum is symmetric wrt real and imaginary axes.
- Thus, w is spectrally stable iff $\sigma\left(\mathcal{A}[w]\right) \subseteq i\mathbb{R}$.

Lemma (Bloch Decomposition)

Defining the family of Bloch operators:

$$A_\xi[w] := e^{-i\xi z}\mathcal{A}[w]e^{i\xi z}, \quad \xi \in \left[-1/2, 1/2\right) \text{ we have}$$

$$\sigma_{L^2(\mathbb{R})}\left(\mathcal{A}[w]\right) = \bigcup_{\xi\in\left[-1/2,1/2\right)} \sigma_{L^2_{per}(0,2\pi)}\left(\mathcal{A}_\xi[w]\right). \tag{4}$$

Bloch Decomposition Facts

We now have an eigenvalue-only spectral problem: good!

- The "Bloch operators" have compactly embedded domains in $L^2_{per}(0, 2\pi)$, and thus their spectra consist of **eigenvalues only.**
- Thus, spectral stability would follow from verifying that all such eigenvalues belong to $i\mathbb{R}$.
- To obtain spectral stability, the eigenvalues need to slide along the imaginary axis as we change ξ.
- It turns out that when $a = 0$ or $\xi = 0$, there is a triple eigenvalue at the origin, and these are typically the ones that cause instabilities when $a, \xi \neq 0$. If one of those leaves the imaginary axis, we call this a **"modulational instability."**

Modulational Stability/Instability

Definition (Modulational Stability)

A periodic traveling wave solution $u(x,t;a,b,k) = w\left(k\left(x - c(a,b,k)t\right); a, b, k\right)$ is **modulationally stable** if there exists an open neighborhood $B \subseteq \mathbb{C}$ of the origin and some $a_0, \xi_0 > 0$ s.t. for all $|\xi| < \xi_0$ and $|a| < a_0$,

$$\sigma_{L^2_{per}(0,2\pi)}\left(\mathcal{A}_\xi[w]\right) \cap B \subseteq i\mathbb{R}.$$

If we define $\mathcal{L}_\xi[w]$ with the same conjugation as $\mathcal{A}_\xi[w]$ in the Bloch decomposition lemma, have

$$\mathcal{A}_\xi[w]\varphi = \lambda\varphi \Leftrightarrow k\mathcal{L}_\xi[w]\varphi = \lambda\left(1 - k^2\left(\partial_z + i\xi\right)^2\right)\varphi.$$

This allows us to avoid the nonlocal term in the definition of $\mathcal{A}_\xi[w]$!

Spectrum when $a = 0$

First consider the case where $a = 0$, i.e., linearize about w_0. Fourier analysis shows that the eigenfunctions are e^{inz} and that the eigenvalues are

$$\lambda_{n,\xi} = \frac{i(n+\xi)\left((n+\xi)^2 - 1\right)k^3\left(c_0 - w_0^2\right)}{1 + k^2\left(n+\xi\right)^2}.$$

Note: w_0 is spectrally stable! Also, $\lambda = 0$ is an isolated e.v. of $\mathcal{A}_0[w_0]$ with mult. 3 and e-basis:

$$\ker\left(\mathcal{A}_0[w_0]\right) = \left\langle \left\{\cos(z), \sin(z), 1\right\} \right\rangle.$$

By spectral perturbation theory, for $|(a,\xi)| \ll 1$, there is a neighborhood of 0 that holds exactly 3 eigenvalues.

Spectrum when $\xi = 0$

Note: $\mathcal{A}_0[w]v = 0 \Leftrightarrow \mathcal{L}_0[w]v = 0$.

- $\mathcal{A}_0[w]w_z = 0$ by the translation invariance of Novikov.
- Differentiating the profile equation wrt a and again wrt b give $\mathcal{L}_0[w]\left(c_a w_b - c_b w_a\right) = 0$.
- Finally, $\mathcal{A}_0[w]w_b = -kc_b w_z$.

This means that for $\xi = 0, \lambda = 0$ is an e.v. of algebraic multiplicity 3 and geometric multiplicity 2, and that the (scaled to ensure continuity) generalized eigenspace for $\lambda = 0$ is spanned by

$$\begin{cases} \varphi_1 := \left(\frac{2b}{c_0}\right)\left(c_a w_b - c_b w_a\right) = \cos(z) + a\left[d_3 + 2d_2\cos(2z)\right] + O\left(a^2\right) \\ \varphi_2 := \left(\frac{-1}{a}\right) w_z = \sin(z) + a\left[2d_2\sin(2z)\right] + O\left(a^2\right) \\ \varphi_3 := \frac{w_b}{(w_0)_b} = 1 + O\left(a^2\right). \end{cases}$$

Now study $|(a, \xi)| \ll 1$

In fact, for small a and ξ, these span the three-dimensional generalized eigenspace for the continuation of the triple eigenvalue from $\lambda = 0$ up to higher-order.

The eigenvalues satisfy $\det\left(M_{a,\xi}(\lambda)\right) = 0$, where

$$M_{a,\xi}(\lambda) := \left[\frac{\left\langle \mathcal{A}_\xi[w]\varphi_j, \varphi_i\right\rangle}{\langle \varphi_i, \varphi_i\rangle} - \lambda \frac{\langle \varphi_j, \varphi_i\rangle}{\langle \varphi_i, \varphi_i\rangle}\right]_{i,j\in\{1,2,3\}}.$$

$$\mathcal{M}_{a,\xi}(\lambda) = \left[\frac{\left\langle k\mathcal{L}_\xi[w]\varphi_j, \left(1 - k^2(\partial_z + i\xi)^2\right)^{-1}\varphi_i\right\rangle}{\langle \varphi_i, \varphi_i\rangle} - \lambda \frac{\langle \varphi_j, \varphi_i\rangle}{\langle \varphi_i, \varphi_i\rangle}\right]_{i,j}$$

We obtain the following after very lengthy but straightforward calculations:

Taken together, it follows that

$$\begin{aligned}\mathcal{M}_{a,\xi}(\lambda) &= i\xi \begin{bmatrix} -2k\alpha m_1 & 0 & 0 \\ 0 & -2k\alpha m_1 & 0 \\ & 0 & 0 & k\alpha \end{bmatrix} + a \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & kw_0 m_1 \left(2k^2+8\right) \\ 0 & 0 & 0 \end{bmatrix} \\ &+ ai\xi \begin{bmatrix} 0 & 0 & \gamma_1 \\ 0 & 0 & 0 \\ \gamma_2 & 0 & 0 \end{bmatrix} + \xi^2 \begin{bmatrix} 0 & k\alpha\left(-3\xi^2 m_1 + 2\xi^2 y_1\right) & 0 \\ k\alpha\left(3\xi^2 m_1 - 2\xi^2 y_1\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ O\left(a^2 + a\xi^2 + \xi^3\right),\end{aligned} \tag{3.12}$$

where, for the sake of notational convenience, we have set

$$\alpha = c_0 - 4w_0^2, \quad y_1 = -\frac{2k^2}{\left(k^2+1\right)^2}, \quad m_1 = \frac{1}{k^2+1}$$

and

$$\gamma_1 = 2k\alpha d_3 + ky_1 w_0 \left(2k^2+8\right) - kw_0 m_1 \left(3k^2+8\right), \quad \gamma_2 = kd_3\alpha - \frac{kw_0\left(3k^2+8\right)}{2}.$$

Calculate the Determinant

We can write the determinant of the above as follows:

$$C_{a,\xi}(\lambda) := \left|M_{a,\xi}(\lambda)\right| = c_3\lambda^3 + ic_2\lambda^2 + c_1\lambda + ic_0, \text{ where}$$

- each c_i is real and analytic in both a and ξ
- each c_i is even in a
- c_3 and c_1 are even in ξ, whereas c_2 and c_0 are odd in ξ.

As a result, if we set $q_j = \xi^{j-3}c_j$, then we have

$$q_{a,\xi}(\lambda) := C_{a,\xi}(-i\xi\lambda) = i\xi^3\left(q_3\lambda^3 - q^2\lambda^2 - q_1\lambda + q_0\right).$$

Then, if the roots of $C_{a,\xi}$ are imaginary iff each root of $q_{a,\xi}$ is real! The latter occurs when the following discriminant is positive:

$$\Delta(a,\xi;b,k) := 18q_3q_2q_1q_0 + q_2^2q_1^2 + 4q_2^3q_0 + 4q_3q_1^3 - 27q_3^2q_0^2. \quad (5)$$

Calculate the Discriminant

Since each q_j is even in ξ and even in a, all 5 terms in (5) are even in both a and ξ (and of course, analytic)! So, we may rewrite:

$$\Delta(a,\xi;b,k) = \xi^2 Y(b,k) + a^2 Z(b,k) + O\left(a^2\xi^2 + a^4 + \xi^4\right), \text{ where}$$

$$Y(b,k) \geq 0, \quad Z(b,k) = \left(3-k^2\right)\left(\frac{f(b,k)}{3^{\frac{3}{4}}\left(k^2+1\right)^{\frac{29}{4}}}\right),$$

where $f(b,k) > 0$ is a 26-degree polynomial in k, with:

- If $k^2 < 3, \Delta(a,\xi;b,k) > 0$!
- If $k^2 > 3, \Delta(a,\xi;b,k) < 0$, if we fix a and choose $|\xi| \ll 1$
- If $k^2 = 3$, then we can't calculate the sign of $\Delta(a,\xi;b,k) = O\left(a^2\xi^2 + a^4 + \xi^4\right)$ without much deeper analysis...

This leads to our final result!

Final Result

Theorem (Modulational Stability/Instability Result for Novikov)

The small amplitude $2\pi/k$-periodic traveling waves of Novikov considered here are modulationally unstable if $k > \sqrt{3}$ and modulationally stable if $0 < k < \sqrt{3}$.

Spectral Stability for $k < \sqrt{3}$!

Observe that when $\xi = 0$ we have

$$\lambda_{1,0} = \lambda_{-1,0} = \lambda_{0,0} = 0$$

and, noting that $n \mapsto \lambda_{n,0}$ is odd and strictly increasing in n for $n \geq 2$, we have that

$$\ldots < \lambda_{-3,0} < \lambda_{-2,0} < 0 < \lambda_{2,0} < \lambda_{3,0} < \ldots$$

Further, for $\xi \in (0, 1/2]$ one can readily verify that when $k^2 < 3$ we have

$$\ldots < \lambda_{-3,\xi} < \lambda_{-2,\xi} < \lambda_{0,\xi} < 0 < \lambda_{-1,\xi} < \lambda_{1,\xi} < \lambda_{2,\xi} < \lambda_{3,\xi} < \ldots.$$

so that, in particular, the Bloch eigenvalues associated to the constant state w_0 never collide away from $(\lambda, \xi) = (0, 0)$ when the condition $k^2 < 3$ is satisfied. As such, when $k^2 < 3$ the only spectral instability possible comes from the $(\lambda, \xi) = (0, 0)$ state.

Thank You! :)

References

- B. Ehrman, M. A. Johnson, and S. Lafortune, *Orbital Stability of Smooth Solitary Waves for the Novikov Equation*, preprint (2024).
- S. Lafortune, *Spectral and Linear Stability of Peakons in the Novikov Equation*, preprint (2024).