Fractional diffusion in Gaussian noisy environment

By

Guannan Hu

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Chairperson, Yaozhong Hu

Luke Huan

Committee members

David Nualart

Terry Soo

Xuemin Tu

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The Dissertation Committee for Guannan Hu certifies
that this is the approved version of the following dissertation :

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Yaozhong Hu, Chairperson

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# Abstract

Three types of stochastic partial differential equations are studied in this dissertation. We prove the existence and uniqueness of the solutions and obtain some properties of the solutions.

Chapter **??** studies the linear stochastic partial differential equation of fractional orders both in time and space variables

$$\left(\partial^{\beta} + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = au(t,x)\dot{W}(t,x).$$

Here $\dot{W}$ is a general Gaussian noise. $\partial^{\beta}$ is the Caputo fractional derivative of order $\alpha$ with respect to the time variable $t$.

$$\beta \in (1/2,2), \qquad \alpha \in (0,2],$$

and $a$ is some fixed real number.

For the case,

$$\begin{cases} \alpha \in (0,2], & \beta \in (1/2,1), & d \in \mathbb{N}, \\ \alpha \in (0,2], & \beta \in (1,\alpha), & d = 1, \\ \alpha = 2, & \beta \in (1,2), & d = 2,3. \end{cases}$$

We prove the existence and uniqueness of the solution and calculate the moment bounds of the solution when $\dot{W}$ has Reisz kernel as space covariance.

For the case when

$$\beta \in (1,2) \quad \text{and} \quad \alpha \in (0,2],$$

we prove the existence and uniqueness of the solution when $\dot{W}$ has Reisz kernel as space covariance. Along the way, we obtain some new properties of the fundamental solutions.

Chapter **??** studies the time-fractional diffusion in with fractional Gaussian noisy as described by the fractional order stochastic diffusion equations of the following form:

$$\left(\partial^{(\alpha)} - B\right) u(t,x) = u(t,x) \cdot \dot{W}^H(x),$$

where

$$\alpha \in (0,1),$$

$B$ is a second order elliptic operator with variable cooefficients and $\dot{W}^H$ is a time independent fractional Gaussian noise of Hurst parameter $H = (H_1, \cdots, H_d)$. We obtain conditions satisfied by $\alpha$ and $H$ so that the square integrable solution $u$ exists uniquely.

Chapter **??**, we prove the existence and uniqueness of mild solution for the stochastic partial differential equation

$$(\partial^\alpha - B) u(t,x) = u(t,x) \cdot \dot{W}(t,x),$$

where

$$\alpha \in (1/2,1) \cup (1,2);$$

$B$ is an uniform elliptic operator with variable coefficients and $\dot{W}$ is a Gaussian noise general in time with space covariance given by fractional, Riesz and Bessel kernel.

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# Chapter 1

# Introduction

In 1827, Robert Brown observed that minute particles suspended in liquid moved continuously in a jittery way. He found that the movement was not caused by the currents of the fluid. He also ruled out the explanation that this was the display of life in a microscopic form, because he observed the same phenomenon in non-living medium. He didn't put forward a theory explaining the mechanism underlying this motion, which is now referred to as Brownian motion.

In a 1905 paper, Einstein gave the explanation which is accepted among the science community nowadays. He suggested that the Brownian motion is caused by random buffeting from the numerous particles in the fluid. He showed that the mean squared displacement during a time interval of length $T$ is proportional to $T$.

Due to the irregular movement of the molecules, the substance in the area of higher concentration are more likely spread to area with lower concentration. This spreading process is called diffusion. For diffusion processes in more complex medium, the mean squared displacement of the particle during time T is proportional to $T^{\alpha}, \alpha \neq 1$. This kind of diffusion is called fractional diffusion.

The theory of fractional diffusion has wide applications in physics and biology. When $\alpha > 1$, the fractional diffusion is referred to as super-diffusion, which describes the diffusion in the case of turbulent plasmas, Levy flights, etc. When $\alpha < 1$, the fractional diffusion is referred to as sub-diffusion, which describes the diffusion in fractal, porous media, etc.

For tutorial introduction of fractional diffusion and its theoretical framework, we refer the reader to [**?**]. For more recent works on anomalous diffusions in the study of biophysics, we refer the reader to [**?**], [**?**], [**?**], [**?**].

In this dissertation, we consider two type of stochastic fractional differential equation.

The first type is the following equation which is fractional in time,

$$\partial^{\alpha} u(t,x) = Bu(t,x) + u(t,x)\dot{W}(t,x) \tag{1.0.1}$$

Here $t \geq 0$ ; $x \in \mathbb{R}^d$ ; $\alpha \in (1/2, 1)$ is a positive number; $B$ is an uniformly elliptic operator and $\partial^{\alpha}$ is the Caputo fractional derivative with respect to $t$ (see [**?**] for the study of various fractional derivatives). When $\alpha = 1, B = \Delta$, this equation has been studied extensively in for some examples [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

A. Kochubei [**?**] and W. Schneider el al [**?**] have considered the following.

$$\partial^{\alpha} u(t,x) = \Delta u(t,x) \tag{1.0.2}$$

Therein A. Kochubei [**?**] derived the explicit form of the solution of (**??**) in terms of the H function:

$$Z_0(t,x) = \pi^{-d/2}|x|^{-d} H^{2,0}_{1,2}\left( \frac{|x|^2}{4t^{\alpha}} \,\middle|\, \begin{matrix} (1,\alpha) \\ (d/2,1),(1,1) \end{matrix} \right).$$

With the asymptotic properties of H function, Eidelman et al [?] have constructed the following solution of the Cauchy problem of (??).

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x,\xi)u_0(\xi)d\xi + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, f(s,y)Y(t-s,x-y).$$

When $\alpha \in (1,2)$, A. V. Pskhu [?] considered the Cauchy problem of (??) and showed that when $B$ is $\Delta$, the Green's function $Y$ of (??) is the following:

$$Y(t,x) = C_d t^{\frac{\alpha}{2}(2-d)} f_{\frac{\alpha}{2}}(|x|t^{-\frac{\alpha}{2}}; d-1, \frac{\alpha}{2}(2-d)),$$

Based on the above results, in chapters **??** and **??** we extend some results of [?], [?] to

$$\partial^\alpha u(t,x) = Bu(t,x) + u(t,x)\dot{W}(t,x),$$

where $B$ is a second order differential operator and $\dot{W}$ is a Gaussian noise similar to [?] or [?].

The other type of stochastic fractional equation in this dissertation is of the following, which is referred as space-time fractional diffusion equation:

$$\left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = au(t,x)\dot{W}(t,x), \qquad t > 0,\, x \in \mathbb{R}^d.$$

The work by Chen and Dalang [?] deals with the case where $\beta = 1$, $\alpha \in (1,2]$.

When $\beta \in (0,1)$, $\alpha = 2$, $\Delta$ is replaced by a general elliptic operator, and $\dot{W}$ is a fractional noise, the equation was studied in [?].

When $\beta \in (0,1)$, $\alpha = 2$ and $\dot{W}$ is a fractional noise, the smoothed equation

$$\left(\partial^\beta - \frac{\nu}{2}\Delta\right) u(t,x) = I_t^{1-\beta}\left[u(t,x)\dot{W}(t,x)\right]$$

(see (**??**) for a generalization) was studied in [**?**]. In a series of papers [**?**, **?**, **?**], Nane and his coauthors studied the case $\alpha \in (0,2]$.

The case $\beta \in (0,1)$ corresponds to the slow diffusion (subdiffusion). For the fast diffusion case (super diffusion), i.e., $\beta \in (1,2)$, there have been only a few works. Le Chen has studied in [**?**] the smoothed equation with $\alpha = 2$, $d = 1$ and with space-time white noise. The corresponding non-smoothed equation is studied recently in [**?**]. Both papers [**?**, **?**] deal with the nonlinear equation, i.e., $\rho(u)\dot{W}$ with $\rho$ being a Lipschitz function.

Khoshnevisan and Foondun [**?**] and Song [**?**] has studied a similar equation with the $\alpha$-stable generator $(-\Delta)^{\alpha/2}$ replaced by a general Lévy generator.

In chapter **??** we consider the general case when both derivatives are fractional and $\dot{W}$ is the general multiplicative noise.

Chapter 3-5 are based on the following papers and draft.

- Le Chen, Guannan Hu, Yaozhong Hu and Jingyu Huang, Space-time fractional diffusions in Gaussian noisy environment *Preprint arXiv:1508.00252*, 2015.([**?**]).

- G. Hu and Y. Hu. Fractional diffusion in Gaussian noisy environment. *Mathematics*, 2015, 3(2), 131–152. ([**?**]).

- Stochastic time-fractional diffusion equations with variable coefficients, *draft*.

# Chapter 2

# Preliminaries

## 2.1 Fractional calculus

Here we give some definitions and basic properties of fractional calculus used in this dissertation. For detail account of Fractional calculus, we refer the reader to [**?**].

The following formula is well known.

$$\int_a^x \mathrm{d}x \int_a^x \mathrm{d}x \cdots \int_a^x \phi(x)\mathrm{d}x = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\phi(t)\mathrm{d}t$$

Therefore naturally we define the fractional integral the following way.

**Definition 2.1.1.** Let $\phi(x) \in L_1(a,b)$. The following integral is called the (left-handed) Riemann-Liouville fractional integral of order $\alpha$

$$(I^{\alpha}_{a+}\phi)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\phi(t)}{(x-t)^{1-\alpha}}\mathrm{d}t \qquad x > a$$

From the definition one can check that

$$I^{\alpha}_{a+}I^{\beta}_{a+}\phi = I^{\alpha+\beta}_{a+}\phi \qquad \alpha, \beta > 0.$$

**Definition 2.1.2.** $f:[a,b]\to\mathbb{R}$ and $\alpha\in(0,1)$. The following integral is called the (left-handed) Riemann-Liouville fractional derivative of order $\alpha$

$$(D^{\alpha}_{a+}f)(x):=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x\frac{f(t)}{(x-t)^{\alpha}}\mathrm{d}t$$

Denote by $AC([a,b])$ the absolutely continuous function on [a, b]. Denote $f(x)\in AC^n([a,b])$ if $f^{(n-1)}(x)\in AC([a,b])$.

**Theorem 2.1.3.** *Let $f(x)\in AC([a,b])$. Then $D^{\alpha}_{a+}$ exists a.e. for $\alpha\in(0,1)$. Furthermore*

$$(D^{\alpha}_{a+}f)(x)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(x-a)^{\alpha}}+\int_a^x\frac{f'(t)}{(x-t)^{\alpha}}\mathrm{d}t\right]$$

For any real number $\alpha$, denote $[\alpha]$ the integer part of $\alpha,\{\alpha\}$ the fractional part of $\alpha$. If $\alpha$ is not an integer, we define

$$(D^{\alpha}_{a+}f)(x):=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]}(D^{\{\alpha\}}_{a+}f)(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}(I^{1-\{\alpha\}}_{a+}f)(x) \tag{2.1.1}$$

Therefore we have

$$(D^{\alpha}_{a+}f)(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\int_a^x\frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t \qquad n=[\alpha]+1$$

The sufficient condition for the existence of above integral is $f(x)\in AC^{[\alpha]}([a,b])$

**Definition 2.1.4.** For $\alpha\in(0,1)$, the following is called Caputo derivative.

$$(\partial^{\alpha}f)(x):=\frac{1}{\Gamma(1-\alpha)}\left[\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x\frac{f(t)}{(x-t)^{\alpha}}\mathrm{d}t-(x-a)^{-\alpha}f(a)\right].$$

Let $f(x) \in AC^{[\alpha]}([a,b])$ for any non-integer real $\alpha$. By (**??**) and integration by parts we have

$$(\partial^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} \mathrm{d}t, \qquad n = [\alpha]+1.$$

## 2.2 H function

The H function generalizes many special functions including the Mittag-Leffler function, and the Wright function.

The Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta \in \mathbb{C}, \Re(\alpha), \Re(\beta) > 0.,$$

where $\Re(\alpha)$ is the real part of the complex number $\alpha$. It is as a natural generalization of $e^z$. As $e^z$ is involved with differential equation, Mittag-Leffler function is involved when using Fourier transform to solve fractional differential equation. The solutions of fractional differential equation involved in this dissertation are represented via H function. We rely heavily on the asymptote property and differential formula of H function to obtain the property of these solutions.

**Definition 2.2.1.** Let $m,n,p,q$ be integers such that $0 \le m \le q, 0 \le n \le p$. Let $a_i, b_i \in \mathbb{C}$ be complex numbers and let $\alpha_j, \beta_j$ be positive numbers, $i = 1,2,\cdots,p; j = 1,2,\cdots,q$. Let assume that the set of poles of the gamma functions $\Gamma(b_j+\beta_j s)$ doesn't intersect with that of the gamma functions $\Gamma(1-a_i-\alpha_i s)$, namely,

$$\left\{ b_{jl} = \frac{-b_j - l}{\beta_j}, l = 0,1,\cdots \right\} \bigcap \left\{ a_{ik} = \frac{1-a_i+k}{\alpha_i}, k = 0,1,\cdots \right\} = \emptyset$$

for all $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$. Denote

$$\mathscr{H}_{pq}^{mn}(s) := \frac{\prod_{j=1}^{m} \Gamma(b_j + \alpha_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_j + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \alpha_j s)}.$$

The *Fox H-function*

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n}\left[ z \middle| \begin{array}{ccc} (a_1, \alpha_1) & \cdots & (a_p, \alpha_p) \\ (b_1, \beta_1) & \cdots & (b_q, \beta_q) \end{array} \right]$$

is defined by the following integral

$$H_{pq}^{mn}(z) = \frac{1}{2\pi i} \int_L \mathscr{H}_{pq}^{mn}(s) z^{-s} ds\,, \quad z \in \mathbb{C}\,, \tag{2.2.1}$$

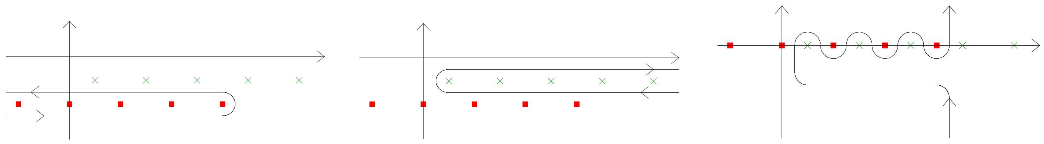
where an empty product in (**??**) means 1 and $L$ in (**??**) is the infinite contour which separates all the points $b_{jl}$ to the left and all the points $a_{ik}$ to the right of $L$. Specifically, $L$ is defined to be one of the following forms:

(1) $L = L_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$

(2) $L = L_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$

(3) $L = L_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$

See the corresponding figure as an example of the contour.



(1) (2) (3)

We will need the following notations in this section:

$$a^* := \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \geq 0; \tag{2.2.2}$$

$$\Delta := \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0; \tag{2.2.3}$$

$$\delta := \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{-\beta_j}; \tag{2.2.4}$$

$$\mu := \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2} \tag{2.2.5}$$

**Theorem 2.2.2.** *If the condition*

$$L = L_{-\infty}, \qquad \Delta > 0, \qquad z \neq 0; \tag{2.2.6}$$

$$L = L_{-\infty}, \qquad \Delta = 0, \qquad 0 < |z| < \delta; \tag{2.2.7}$$

$$L = L_{+\infty}, \qquad \Delta < 0, \qquad z \neq 0; \tag{2.2.8}$$

$$L = L_{+\infty}, \qquad \Delta = 0, \qquad |z| > \delta; \tag{2.2.9}$$

$$L = L_{i\gamma\infty}, \qquad a^* > 0, \qquad |argz| < \frac{a^*\pi}{2}, \qquad z \neq 0; \tag{2.2.10}$$

*then the integral* (**??**) *is well defined.*

Furthermore if any of the conditions in Theorem **??** are satisfied, then H-function **??** is a analytic function of $z$. Specifically we have

**Theorem 2.2.3.** *(1) If condition* (**??**) *or* (**??**) *are satisfied then*

$$H_{pq}^{mn}(z) = \sum_{j=1}^{m} \sum_{l=0}^{\infty} \operatorname*{Res}_{s=b_{jl}} [\mathscr{H}_{pq}^{mn}(s) z^{-s}],$$

*(2) If condition* (**??**) *or* (**??**) *are satisfied then*

$$H_{pq}^{mn}(z) = -\sum_{i=1}^{n} \sum_{k=0}^{\infty} \operatorname*{Res}_{s=a_{ik}} [\mathscr{H}_{pq}^{mn}(s) z^{-s}],$$

*(3) If condition* (**??**) *is satisfied then H-function* **??** *is a analytic function of $z$ on $|argz| < \frac{a^*\pi}{2}$.*

In this dissertation we mainly use the Theorem **??** to the following two special cases.

$$\Delta \geq 0 \quad L = L_{-\infty}, \quad \text{and} \qquad a^* \geq 0 \quad L = L_{i\gamma\infty}.$$

Based on Theorem **??**, we have the following asymptotic expansions of the H-functions at $\infty$ and 0.

Let's first consider the case where the poles of $\Gamma(b_j + \beta_j s)$ :

$$b_{jl} := -\frac{b_j + 1}{\beta_j}, \qquad j = 1, \cdots, m; \quad l = 0, 1, 2, \cdots$$

do not coincide, namely

$$\forall i, j, \qquad \beta_j(b_i + k) \neq \beta_i(b_j + l), \qquad i \neq j; \quad i, j = 1.2, \cdots m; \quad k, l = 0, 1, \cdots; \tag{2.2.11}$$

In this case, we apply the case 1 of Theorem **??** and the property of gamma function $\Gamma(z)$. We have

$$\operatorname*{Res}_{s=b_{jl}} [\mathscr{H}_{pq}^{mn}(s) z^{-s}] = h_{jl}^* z^{-b_{jl}},$$

where

$$h_{jl}^* := \frac{(-1)^l}{l!\beta_j} \frac{\prod_{i=1,i\neq j}^{m} \Gamma\left(b_i - [b_j + l]\frac{\beta_i}{\beta_j}\right) \prod_{i=1}^{n} \Gamma\left(1 - a_i + [b_j + l]\frac{\alpha_i}{\beta_j}\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i - [b_j + l]\frac{\alpha_i}{\beta_j}\right) \prod_{i=m+1}^{q} \Gamma\left(1 - b_i + [b_j + l]\frac{\beta_i}{\beta_j}\right)}. \tag{2.2.12}$$

Thus we have the asymptote expansion of $H_{pq}^{mn}(z)$ at 0.

Denote $A \sim B$, if $\lim_{z\to 0} \frac{A}{B} = C$, where $C$ is a constant.

**Theorem 2.2.4.** *Suppose $H_{pq}^{mn}(z)$ satisfies either $\Delta < 0, a^* > 0$ or $\Delta \geq 0$. When $z \to 0$, we have*

$$H_{pq}^{mn}(z) \sim \sum_{j=1}^{m} \sum_{l=0}^{\infty} h_{jl}^* z^{\frac{b_j+l}{\beta_j}}, \tag{2.2.13}$$

*if the poles of $\Gamma(b_j + \beta_j s)$ :*

$$b_{jl} := -\frac{b_j + 1}{\beta_j}, \qquad j = 1, \cdots, m; \quad l = 0, 1, 2, \cdots$$

*do not coincide. (See (**??**)).*

Similarly if the poles of $\Gamma(1 - a_i + \alpha_i s)$ do not coincide, namely

$$\forall i, j, \qquad \alpha_j(1 - a_i + k) \neq \alpha_i(1 - a_j + l), \qquad i \neq j; \quad i, j = 1.2, \cdots m; \quad k, l = 0, 1, \cdots. \tag{2.2.14}$$

Using the case 2 of Theorem **??**, we have the following asymptote expansion of $H_{pq}^{mn}(z)$ at $\infty$

**Theorem 2.2.5.** *Suppose $H_{pq}^{mn}(z)$ satisfies either $\Delta < 0, a^* > 0$ or $\Delta \geq 0$. When $z \to \infty$, we have*

$$H_{pq}^{mn}(z) \sim \sum_{i=1}^{n} \sum_{k=0}^{\infty} h_{ik} z^{\frac{a_i-1-k}{\alpha_i}}, \tag{2.2.15}$$

*if the poles of $\Gamma(1 - a_i + \alpha_i s)$ do not coincide.(See (**??**))*

Here

$$h_{ik} := \frac{(-1)^k}{k!\alpha_i} \frac{\prod_{j=1}^{m} \Gamma\left(b_j + [1-a_i+k]\frac{\beta_j}{\alpha_i}\right) \prod_{j=1, i\neq j}^{n} \Gamma\left(1 - a_j - [1-a_i+k]\frac{\alpha_j}{\alpha_i}\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_j + [1-a_i+k]\frac{\alpha_j}{\alpha_i}\right) \prod_{j=m+1}^{q} \Gamma\left(1 - b_j - [1-a_i+k]\frac{\beta_j}{\alpha_i}\right)}. \tag{2.2.16}$$

Now let's consider the case when poles of $\Gamma(b_j + \beta_j s)$ coincide. Suppose the order of the pole $b$ is $N^*$. Then

$$\operatorname*{Res}_{s=b}[\mathscr{H}_{pq}^{mn}(s)z^{-s}] = \frac{1}{(N^*-1)!} \lim_{s\to b} [(s-b)^{N^*} \mathscr{H}_{pq}^{mn}(s)z^{-s}]^{(N^*-1)}$$

Using the Leibniz rule to calculate above $N^*-1$ order derivative. We have

$$\begin{aligned} &[(s-b)^{N^*} \mathscr{H}_{pq}^{mn}(s)z^{-s}]^{(N^*-1)} \\ &\quad = z^{-s} \sum_{i=0}^{N^*-1} \left\{ \sum_{n=i}^{N^*-1} (-1)^i \binom{N^*-1}{n} \binom{n}{i} [\mathscr{H}_1^*(s)]^{(N^*-1-n)} [\mathscr{H}_2^*(s)]^{(n-i)} \right\} [\log z]^i, \end{aligned} \tag{2.2.17}$$

where

$$\mathscr{H}_1^*(s) = (s-b)^{N^*} \prod_{j=j_1}^{j_{N^*}} \Gamma(b_j + \beta_j s) \quad \text{and} \quad \mathscr{H}_2^*(s) = (s-b)^{N^*} \prod_{j=j_1}^{j_{N^*}} \Gamma(b_j + \beta_j s) \mathscr{H}_{pq}^{mn}(s). \tag{2.2.18}$$

Therefore we have

$$\operatorname*{Res}_{s=b}[\mathscr{H}_{pq}^{mn}(s)z^{-s}] = z^{\frac{b_j+l}{\beta_j}} \sum_{i=0}^{N_{jl}^*-1} H_{jli}^* [\log z]^i,$$

where

$$H^*_{jli} = \frac{1}{(N^*_{jl}-1)!}\sum_{n=i}^{N^*_{jl}-1}(-1)^i\binom{N^*_{jl}-1}{n}\binom{n}{i}[\mathscr{H}_1^*(b_{jl})]^{(N^*_{jl}-1-n)}[\mathscr{H}_2^*(b_{jl})]^{(n-i)}. \tag{2.2.19}$$

Thus we have

**Theorem 2.2.6.** *Suppose $H^{mn}_{pq}(z)$ satisfies either $\Delta < 0, a^* > 0$ or $\Delta \geq 0$. When $z \to 0$, we have*

$$H^{mn}_{pq}(z) \sim \sum\nolimits'_{j,l} h^*_{jl} z^{\frac{b_j+l}{\beta_j}} + \sum\nolimits''_{j,l}\sum_{i=0}^{N^*_{jl}-1} H^*_{jli} z^{\frac{b_j+l}{\beta_j}}[\log z]^i, \tag{2.2.20}$$

*if the poles of $\Gamma(b_j+\beta_j s)$ coincide. Here $\sum'_{j,l}$ is summation over $j,l$ such that the $b_{jl}$ do not coincides; $\sum''_{j,l}$ is the summation over $j,l$ such that $b_{jl}$ coincide with order $N^*_{jl}$;*

If the poles of $\Gamma(1-a_i+\alpha_i s)$ coincides, we have the a counterpart of Theorem **??** for the case when the poles of $\Gamma(1-a_i+\alpha_i s)$ coincides. We don't include it here because this case does not apply to the H function in this dissertation. For a detailed account of above theorems, we refer to [**?**].

# Chapter 3

# Space-time fractional diffusions in Gaussian noisy environment

## 3.1 Introduction

We consider the following linear stochastic partial differential equation of fractional orders both in time and space variables:

$$\begin{cases} \left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = au(t,x)\dot{W}(t,x), & t>0,\, x\in\mathbb{R}^d, \\ \frac{\partial^k}{\partial t^k} u(t,x)\Big|_{t=0} = u_k(x), & 0\le k\le \lceil\beta\rceil - 1,\ x\in\mathbb{R}^d, \end{cases} \tag{3.1.1}$$

with $\beta\in(1/2,2)$ and $\alpha\in(0,2]$, where $\lceil\beta\rceil$ is the smallest integer greater than or equal to $\beta$. We limit our consideration to the above parameter ranges of $\beta$ and $\alpha$ since we plan to use some particular properties of the corresponding *Fox H-functions* which will be proved only for these parameter ranges. Now let us give more detailed explanation on the terms appeared in the above equation. The fractional derivative in time $\partial^\beta = \dfrac{\partial^\beta}{\partial t^\beta}$

is understood in the *Caputo* sense:

$$\partial^\beta f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} \mathrm{d}\tau & \text{if } m-1<\beta<m\,, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } \beta = m\,, \end{cases}$$

where $t \ge 0$. $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian with respect to spatial variables and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian. $\dot{W}$ is a zero mean Gaussian noise with the following covariance structure

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \lambda(t-s)\Lambda(x-y),$$

where both (possibly generalized) functions $\gamma$ and $\Lambda$ are assumed to be nonnegative and nonnegative definite. We denote by $\mu$ the Fourier transformation measure of $\Lambda(x)$. Namely,

$$\Lambda(x-y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi(x-y)} \mu(\mathrm{d}\xi)\,.$$

This Fourier transform is understood in distributional sense (see Section 2). When $\lambda(t) = \delta_0(t)$ and $\Lambda(x) = \delta_0(x)$, this noise $\dot{W}$ reduces to the *space-time white noise*. $\nu > 0$ and $a$ are some real valued parameters. The given initial conditions $u_k(x)$ are assumed to be continuous and bounded functions. The product $u(t,x)\dot{W}(t,x)$ in the equation (**??**) is the Wick one (see e.g. [**?**]). So, the equation will be understood in the Skorohod sense. Let us point out that some of our results can also be extended to nonlinear equation (namely, replace $u(t,x)\dot{W}(t,x)$ in (**??**) by $\sigma(u(t,x))\dot{W}(t,x)$ for a Lipschitz nonlinear function $\sigma$). However, we limit ourselves to this linear case for two reasons: One is to simplify the presentation and to better explain the ideas and the other one is that we want to use chaos expansion method.

The deterministic counterparts of the equation (**??**) have received many attentions and are called anormalous diffusions. They appeared in biological physics and other fields. Equation (**??**) is an anormalous diffusion in a Gaussian noisy environment. More detailed motivations for the study of this type of equations are given in [**?**, **?**, **?**, **?**].

To study the equation (**??**) the important tools are the fundamental solutions corresponding to its deterministic counterpart. Let us briefly recall them. If $f$ is a continuous and bounded function on $\mathbb{R}_+ \times \mathbb{R}^d$, then there are two fundamental solutions

$$Z(t,x) := Z_{\alpha,\beta,d}(t,x) \quad \text{and} \quad Y(t,x) := Y_{\alpha,\beta,d}(t,x)$$

such that the solution $u(t,x)$ to the following deterministic equation (the deterministic counterpart of (**??**))

$$\begin{cases} \left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = f(t,x), & t > 0,\, x \in \mathbb{R}^d, \\ \frac{\partial^k}{\partial t^k} u(t,x)\Big|_{t=0} = u_k(x), & 0 \le k \le \lceil \beta \rceil - 1,\ x \in \mathbb{R}^d, \end{cases} \tag{3.1.2}$$

is represented by

$$u(t,x) = J_0(t,x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, f(s,y) Y(t-s, x-y), \tag{3.1.3}$$

where and throughout the chapter, we denote

$$J_0(t,x) := \sum_{k=0}^{\lceil \beta \rceil - 1} \int_{\mathbb{R}^d} u_{\lceil \beta \rceil - 1 - k}(y) \partial^k Z(t, x-y) \mathrm{d}y. \tag{3.1.4}$$

Here, we recall the notation $\partial^k = \frac{\partial^k}{\partial t^k}$, $k \in \mathbb{N}$.

This motivates us to study the mild solution to (**??**) (see e.g. Definition **??** below), namely, the solution to the following stochastic integral equation:

$$u(t,x) = J_0(t,x) + \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)u(s,y)W(ds,dy). \tag{3.1.5}$$

As in the classical case, the above equation can be studied by using the Itô-Wiener chaos expansion. To this end we need to understand well the two fundamental solutions $Z$ and $Y$. In particular, we need their nonnegativity and some heat kernel like estimates.

The nonnegativity of some $Z$'s is known. However, since $Y$ is the *Riemann-Liouville* fractional derivative of $Z$, its nonnegativity is a challenging problem. There have been only few results: As proved in Lemma 25 of [**?**], $Y_{2,\beta,d}$ with $\beta \in (1,2)$ is nonnegative if and only if $d \leq 3$. The one dimensional case is proved in [**?**], namely, $D_t Z_{\alpha,\beta,1}$, and hence $Y_{\alpha,\beta,1}$, is nonnegative either if $1 < \beta \leq \alpha \leq 2$, or if $\alpha \in (0,1]$ and $\beta \in (0,2)$. Here, we willshow the nonnegativity of $Y$ in the following three cases:

$$\begin{cases} \alpha \in (0,2], & \beta \in (1/2,1), & d \in \mathbb{N}, \\ \alpha \in (0,2], & \beta \in (1,\alpha), & d = 1, \\ \alpha = 2, & \beta \in (1,2), & d = 2,3. \end{cases} \tag{3.1.6}$$

This includes the above mentioned results as special cases. Let us also point out that for the smoothed SPDE, only the fundamental solution $Z$ is needed, which is usually more regular than the fundamental solution $Y$.

When $\beta = 1$ and $\alpha = 2$, to show the solution of (**??**) is square integrable, it is assumed in [**?**] and [**?**] that the covariance of noise satisfies the following conditions:

(i) $\gamma$ is locally integrable;

(ii) *Dalang's condition* $\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{1+|\xi|^2} < \infty$ is satisfied (see also [**?**, **?**]).

For the existence and uniqueness of the solution to the general equation (**??**), Dalang's condition will be replaced by the following condition:

$$\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{1+|\xi|^{2\alpha-\alpha/\beta}} < \infty. \tag{3.1.7}$$

It is obvious that if it is formally set $\alpha = 2$ and $\beta = 1$, then (**??**) is reduced to the usual Dalang's condition.

The remaining part of the chapter is organized as follows. We first specify the noise structure and present the definition of the solution in Section **??**. The main results are Theorem **??** on the existence and uniqueness of the mild solution and Theorem **??** on the moment bounds of the solution stated in Section **??**. The proof of these two theorems are based on some properties of the fundamental solutions represented in terms of the Fox H-functions. These results themselves are of particular interest and importance. We also list them as Theorem **??** and Theorem **??** in Section **??**. The properties of the fundamental solutions (Theorem **??**) are proved in Section **??** by using the Fox H-functions. In Section **??**, we obtain an expression of the density function for the $d$-dimensional spherically symmetric $\alpha$-stable distribution - an auxiliary result (Theorem **??**) which is used in the proof of Theorem **??**. The existence and uniqueness result (Theorem **??**) of the solution to (**??**) is proved in Section **??**. In Section **??**, we prove the explicit moment bounds when $\Lambda$ is the Riesz kernel.

Our main results (Theorem **??**) assume that the fundamental solutions are nonnegative. However, when $1 < \beta < 2$ and when the dimension is high, the nonnegativity of the fundamental solution $Y$ is not known yet. In this case, we shall show in Theorem **??** the existence and uniqueness of the solution of (**??**) for some specific Gaussian noise whose covariance function $\Lambda$ is the Riesz kernel.

## 3.2 Preliminaries

Let us start by introducing some basic notions on Fourier transforms. The space of real-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support is denoted by $\mathscr{D}(\mathbb{R}^d)$ or $\mathscr{D}$. The space of Schwartz functions is denoted by $\mathscr{S}(\mathbb{R}^d)$ or $\mathscr{S}$. Its dual, the space of tempered distributions, is denoted by $\mathscr{S}'(\mathbb{R}^d)$ or $\mathscr{S}'$. The Fourier transform is defined with the normalization

$$\mathscr{F}u(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u(x)\mathrm{d}x,$$

so that the inverse Fourier transform is given by $\mathscr{F}^{-1}u(\xi) = (2\pi)^{-d}\mathscr{F}u(-\xi)$.

Similarly to [**?**], on a complete probability space $(\Omega, \mathscr{F}, P)$ we consider a Gaussian noise $W$ encoded by a centered Gaussian family $\{W(\varphi);\, \varphi \in \mathscr{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$, whose covariance structure is given by

$$\mathbb{E}\left(W(\varphi)W(\psi)\right) = \int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}} \varphi(s,x)\psi(t,y)\lambda(s-t)\Lambda(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}s\mathrm{d}t, \tag{3.2.1}$$

where $\lambda : \mathbb{R} \to \mathbb{R}_+$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ are nonnegative definite functions and the Fourier transform $\mathscr{F}\Lambda = \mu$ such that $\mu(\mathrm{d}\xi)$ is a tempered measure, that is, there is an integer $m \geq 1$ such that $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-m}\mu(\mathrm{d}\xi) < \infty$. Throughout the paper, we assume that $\lambda$ is locally integrable and we denote

$$C_t := 2\int_0^t \lambda(s)\mathrm{d}s, \quad t > 0. \tag{3.2.2}$$

Let $\mathscr{H}$ be the completion of $\mathscr{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathscr{H}} \;=\; \int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}} \varphi(s,x)\psi(t,y)\lambda(s-t)\Lambda(x-y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}s\mathrm{d}t \tag{3.2.3}$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}_+^2 \times \mathbb{R}^d} \mathscr{F}\varphi(s,\xi)\overline{\mathscr{F}\psi(t,\xi)}\lambda(s-t)\mu(\mathrm{d}\xi)\,\mathrm{d}s\mathrm{d}t,$$

where $\mathscr{F}\varphi$ refers to the Fourier transform with respect to the space variable only. The mapping $\varphi \to W(\varphi)$ defined on $\mathscr{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ can be extended to a linear isometry between $\mathscr{H}$ and the Gaussian space spanned by $W$. We will denote this isometry by

$$W(\phi) = \int_0^\infty \int_{\mathbb{R}^d} \phi(t,x) W(\mathrm{d}t, \mathrm{d}x), \quad \text{for } \phi \in \mathscr{H}.$$

Notice that if $\phi$ and $\psi$ are in $\mathscr{H}$, then $\mathbb{E}\left(W(\phi)W(\psi)\right) = \langle \phi, \psi \rangle_{\mathscr{H}}$.

We will denote by $D$ the derivative operator in the sense of Malliavin calculus. That is, if $F$ is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \ldots, W(\phi_n)),$$

with $\phi_i \in \mathscr{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely $f$ and all its partial derivatives have polynomial growth), then $DF$ is the $\mathscr{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \ldots, W(\phi_n))\phi_j.$$

The operator $D$ is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathscr{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|F\|_{1,2} = \sqrt{\mathbb{E}[F^2] + \mathbb{E}[\|DF\|_{\mathscr{H}}^2]}.$$

We denote by $\delta$ the adjoint of the derivative operator given by the duality formula

$$\mathbb{E}\left(\delta(u)F\right) = \mathbb{E}\left(\langle DF, u \rangle_{\mathscr{H}}\right), \tag{3.2.4}$$

for all $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega;\mathscr{H})$ in the domain of $\delta$. The operator $\delta$ is also called the *Skorohod integral* because in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod. We refer to Nualart [**?**] for a detailed account of the Malliavin calculus with respect to a Gaussian process.

With the Skorohod integral introduced, the definition of the solution to equation (**??**) can be stated as follows.

**Definition 3.2.1.** Let $Z$ and $Y$ be the fundamental solutions defined by (**??**) and (**??**). An adapted random field $\{u = u(t,x) : t \geq 0, x \in \mathbb{R}^d\}$ such that $\mathbb{E}\left[u^2(t,x)\right] < +\infty$ for all $(t,x)$ is a *mild solution* to (**??**), if for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process

$$\left\{Y(t-s,x-y)u(s,y)1_{[0,t]}(s) : s \geq 0,\, y \in \mathbb{R}^d\right\}$$

is Skorohod integrable (see (**??**)), and $u$ satisfies

$$u(t,x) = J_0(t,x) + \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)u(s,y)W(\mathrm{d}s,\mathrm{d}y) \tag{3.2.5}$$

almost surely for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where $J_0(t,x)$ is defined by (**??**).

The main ingredient in proving the existence and uniqueness of the solution is the Wiener chaos expansion, to which we now turn.

Suppose that $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a square integrable solution to equation (**??**). Then for all fixed $(t,x)$ the random variable $u(t,x)$ admits the following Wiener chaos expansion

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,\cdot,t,x)), \tag{3.2.6}$$

where for each $(t,x)$, $f_n(\cdot,\cdot,t,x)$ is a symmetric element in $\mathscr{H}^{\otimes n}$. Then, as in [**?**, **?**, **?**], to show the existence and uniqueness of the solution it suffices to show that for all $(t,x)$

we have

$$\sum_{n=0}^{\infty} n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} < \infty\,. \tag{3.2.7}$$

For some technical reason, we will assume, throughout the paper, the following properties on $\Lambda$:

- $\Lambda(x) : \mathbb{R}^d \to [0,\infty]$ is a continuous function, where $[0,\infty]$ is the usual one-point compactification of $[0,\infty)$.
- $\Lambda(x) < \infty$ if and only if $x \neq 0$ or $\mathscr{F}(\Lambda)(\xi) \in L^\infty(\mathbb{R}^d)$ and $\Lambda(x) < \infty$ when $x \neq 0$.

With these two assumptions, according to Lemma 5.6 in [**?**], for any Borel probability measures $\mu_1(dx)$ and $\mu_2(dx)$, the following identity holds,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Lambda(x-y)\nu_1(\mathrm{d}x)\nu_2(\mathrm{d}y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathscr{F}\mu_1(\xi)\overline{\mathscr{F}\mu_2(\xi)}\mu(\mathrm{d}\xi)\,. \tag{3.2.8}$$

In particular, the above result can be applied to the case when $\mu_1(\mathrm{d}x) = f_1(x)\mathrm{d}x$ and $\mu_2(\mathrm{d}x) = f_2(x)\mathrm{d}x$ for two nonnegative functions $f_1$ and $f_2 \in L^1(\mathbb{R}^d)$.

## 3.3 Main results

### 3.3.1 Fundamental solutions: formulas and nonnegativity

Our first result is concerned with the fundamental solutions to (**??**) stated in the following theorem. We need the two parameter *Mittag-Leffler function* $E_{\alpha,\beta}(z)$:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}\,, \quad \Re(\alpha) > 0,\, \beta \in \mathbb{C},\, z \in \mathbb{C}\,, \tag{3.3.1}$$

where $\Re(\alpha)$ is the real part of the complex number $\alpha$. When $\beta = 1$, we also write $E_\alpha(z) := E_{\alpha,1}(z)$. The $H$-functions appearing in the following theorem and their properties are given in Chapter 2, section H function.

**Theorem 3.3.1.** *The fundamental solutions to* (**??**) *are given by*

$$Z(t,x) := Z_{\alpha,\beta,d}(t,x) = \pi^{-d/2} t^{\lceil\beta\rceil-1} |x|^{-d} H^{2,1}_{2,3}\left( \frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix} (1,1),(\lceil\beta\rceil,\beta) \\ (d/2,\alpha/2),(1,1),(1,\alpha/2) \end{matrix} \right) \tag{3.3.2}$$

*and*

$$Y(t,x) := Y_{\alpha,\beta,d}(t,x) = \pi^{-d/2} |x|^{-d} t^{\beta-1} H^{2,1}_{2,3}\left( \frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix} (1,1),(\beta,\beta) \\ (d/2,\alpha/2),(1,1),(1,\alpha/2) \end{matrix} \right). \tag{3.3.3}$$

*If $\beta \in (1,2)$, then*

$$Z^*(t,x) := Z^*_{\alpha,\beta,d}(t,x) = \frac{\mathrm{d}}{\mathrm{d}t} Z_{\alpha,\beta,d}(t,x) = \pi^{-d/2} |x|^{-d} H^{2,1}_{2,3}\left( \frac{|x|^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix} (1,1),(1,\beta) \\ (d/2,\alpha/2),(1,1),(1,\alpha/2) \end{matrix} \right). \tag{3.3.4}$$

*The Fourier transforms of the fundamental solutions are given by the following:*

$$\mathscr{F}Z(t,\cdot)(\xi) = t^{\lceil\beta\rceil-1} E_{\beta,\lceil\beta\rceil}(-2^{-1}\nu t^\beta |\xi|^\alpha), \tag{3.3.5}$$

$$\mathscr{F}Y(t,\cdot)(\xi) = t^{\beta-1} E_{\beta,\beta}(-2^{-1}\nu t^\beta |\xi|^\alpha), \tag{3.3.6}$$

$$\mathscr{F}Z^*(t,\cdot)(\xi) = E_\beta(-2^{-1}\nu t^\beta |\xi|^\alpha), \quad \textit{if } \beta \in (1,2); \tag{3.3.7}$$

*Moreover, we have the following results on the positivity of the fundamental solutions.*

*(a) If $\beta \in (0,1]$, $d \in \mathbb{N}$ and $\alpha \in (0,2]$, then both $Z(t,x)$ and $Y(t,x)$ are nonnegative;*

*(b)* *If* $\beta \in (1,2)$, $d \in \{2,3\}$, *and* $\alpha = 2$, *then both* $Z(t,x)$ *and* $Y(t,x)$ *are nonnegative;*

*(c)* *If* $\beta \in (1,2)$, $d = 1$ *and* $\alpha \in [\beta, 2]$, *then all* $Z(t,x)$, $Y(t,x)$ *and* $Z^*(t,x)$ *are nonnegative.*

The proof of this theorem is given in Section **??**.

**Remark 3.3.2.** Here are some known special cases:

(1) When $\alpha = 2$ and $\beta \in (0,1)$, it is proved in [**?**, **?**] and in [**?**], respectively, that

$$Z_0(t,x) = \pi^{-d/2}|x|^{-d} H^{2,0}_{1,2}\left(\frac{|x|^2}{2\nu t^\beta}\,\middle|\, \begin{matrix}(1,\beta)\\(d/2,1),(1,1)\end{matrix}\right), \tag{3.3.8}$$

and

$$Y_0(t,x) = \pi^{-d/2}|x|^{-d} t^{\beta-1} H^{2,0}_{1,2}\left(\frac{|x|^2}{2\nu t^\beta}\,\middle|\, \begin{matrix}(\beta,\beta)\\(d/2,1),(1,1)\end{matrix}\right), \tag{3.3.9}$$

which correspond to our $Z_{2,\beta,d}(t,x)$ and $Y_{2,\beta,d}(t,x)$, respectively. The equivalence is clear by applying Property 2.2 of [**?**]. For $Z_{2,\beta,d}$, see also [**?**, Chapter 6].

(2) When $\alpha = 2$ and $\beta \in (0,2)$, it is proved in [**?**] that

$$\Gamma_{\beta,d}(t,x) = \pi^{-d/2}|x|^{-d} t^{\beta-1} H^{2,0}_{1,2}\left(\frac{|x|^2}{4 t^\beta}\,\middle|\, \begin{matrix}(\beta,\beta)\\(d/2,1),(1,1)\end{matrix}\right), \tag{3.3.10}$$

which corresponds to our $Y_{2,\beta,d}$ with $\nu = 2$.

(3) In [**?**], the fundamental solution $Z^*_{\alpha,\beta,d}(t,x)$ has been studied for all $\alpha, \beta \in (0,2)$ and $d = 1$. From the Mellin-Barnes integral representation (6.6) of [**?**], we see that the reduced Green's function of [**?**] can be expressed by using the Fox H-function:

$$K^\theta_{\alpha,\beta}(x) = \frac{1}{|x|} H^{2,1}_{3,3}\left(|x|^\alpha\,\middle|\, \begin{matrix}(1,1),(1,\beta),(1,\frac{\alpha-\theta}{2})\\(1,1),(1,\alpha),(1,\frac{\alpha-\theta}{2})\end{matrix}\right), \quad x \in \mathbb{R}, \tag{3.3.11}$$

where $\alpha$ and $\beta$ have the same meaning as in this paper and $\theta$ is the skewness: $|\theta| \leq \min(\alpha, 2-\alpha)$. For the symmetric $\alpha$-stable case, i.e., $\theta = 0$, this expression can be simplified by using the definition of the Fox H-function and the fact that (see, e.g., [**?**, 5.5.5])

$$\frac{\Gamma(1+\alpha s)}{\Gamma(1+\alpha s/2)} = \frac{1}{\sqrt{\pi}} 2^{\alpha s} \Gamma(1/2+\alpha s/2). \tag{3.3.12}$$

Hence,

$$K^0_{\alpha,\beta}(x) = \frac{1}{\sqrt{\pi}|x|} H^{2,1}_{2,3}\left( (|x|/2)^\alpha \,\middle|\, \begin{matrix} (1,1),(1,\beta) \\ (1/2,\alpha/2),(1,1),(1,\alpha/2) \end{matrix} \right), \quad x \in \mathbb{R}. \tag{3.3.13}$$

This implies that the fundamental solution in [**?**, (1.3)]

$$G^0_{\alpha,\beta}(x,t) = t^{-\beta/\alpha} K^0_{\alpha,\beta}(t^{-\beta/\alpha} x) = \frac{1}{\sqrt{\pi}|x|} H^{2,1}_{2,3}\left( \frac{|x|^\alpha}{2^\alpha t^\beta} \,\middle|\, \begin{matrix} (1,1),(1,\beta) \\ (1/2,\alpha/2),(1,1),(1,\alpha/2) \end{matrix} \right)$$

corresponds to our $Z^*_{\alpha,\beta,1}(t,x)$ with $\nu = 2$.

The proof of the nonnegativity part in Theorem **??** requires a representation of the spherically symmetric $\alpha$-stable distribution from the Fox H-function, which is of interest by itself. The one-dimensional case can be found in [**?**]; see Remark **??** below.

**Theorem 3.3.3.** *Let $X$ be a centered, $d$-dimensional spherically symmetric $\alpha$-stable random variable with $\alpha \in (0,2]$. Then the characteristic function and the density of $X$ are, respectively,*

$$f_{\alpha,d}(\xi) = \exp\left(-|\xi|^\alpha\right), \quad \xi \in \mathbb{R}^d, \tag{3.3.14}$$

*and*

$$\rho_{\alpha,d}(x) = \pi^{-d/2}|x|^{-d} H_{1,2}^{1,1}\left((|x|/2)^{\alpha} \,\middle|\, \begin{matrix} (1,1) \\ (d/2,\alpha/2),(1,\alpha/2) \end{matrix}\right), \quad x \in \mathbb{R}^d. \tag{3.3.15}$$

The proof of this theorem is given in Section **??**.

**Remark 3.3.4.** When $d = 1$, the formula (**??**) yields a result in [**?**]. In particular, as proved in [**?**] (see (**??**)), when $d = 1$, we have

$$\rho_{\alpha,1}(x) = |x|^{-1} H_{2,2}^{1,1}\left(|x|^{\alpha} \,\middle|\, \begin{matrix} (1,1),(1,\alpha/2) \\ (1,\alpha),(1,\alpha/2) \end{matrix}\right) = \pi^{-1/2}|x|^{-1} H_{1,2}^{1,1}\left((|x|/2)^{\alpha} \,\middle|\, \begin{matrix} (1,1) \\ (1/2,\alpha/2),(1,\alpha/2) \end{matrix}\right),$$

where the second equality is due to (**??**) and the definition of the Fox H-function.

### 3.3.2 Existence and uniqueness of solutions to the SPDE

The following is one of the main theorem of the paper.

**Theorem 3.3.5.** *Assume the following conditions.*

*(1)* $Y_{\alpha,\beta,d}(t,x)$ *is nonnegative;*

*(2)* $\beta \in (1/2,2)$ *and* $\alpha \in (0,2]$*;*

*(3)* $\gamma$ *is locally integrable;*

*(4)* $\mu$ *satisfies Dalang's condition* (**??**)*;*

*(5) The initial conditions are such that for all* $t > 0$*,*

$$\widehat{C}_t := \sup_{y\in\mathbb{R}^d,\, s\in[0,t]} |J_0(s,y)| < +\infty. \tag{3.3.16}$$

*Then relation (**??**) holds for each $(t,x)$. Consequently, equation (**??**) admits a unique mild solution in the sense of Definition **??**.*

The proof of this theorem is given in Section **??**.

**Remark 3.3.6.** From Theorem **??**, it follows that the three cases in (**??**) satisfy the above assumptions (1) and (2). Moreover, if $u_k \in L^\infty(\mathbb{R}^d)$ for the first two cases in (**??**), or if $u_0 \in L^\infty(\mathbb{R}^d)$ and $u_1(x) \equiv u_1$ is a constant for the last case in (**??**), then by Lemma **??** below,

$$|J_0(t,x)| \le ||u_0||_{L^\infty(\mathbb{R}^d)} + t^{\beta-1} ||u_1||_{L^\infty(\mathbb{R}^d)} 1_{\{\beta>1\}}.$$

Hence the assumption (5) is also satisfied. The Dalang condition (**??**) imposes a further restriction on the possible values of $(\alpha,\beta)$ due to the spatial correlation $\Lambda(x)$.

**Remark 3.3.7** (Space-time white noise case)**.** When the noise $\dot{W}$ is a space-time white noise, i.e., $\lambda(t) = \delta_0(t)$ and $\Lambda(x) = \delta_0(x)$, then Dalang's condition (**??**) becomes

$$\frac{d}{\alpha} + \frac{1}{\beta} < 2. \tag{3.3.17}$$

This condition implies that $\beta > 1/2$. In particular, if $\alpha = 2$ and $d = 1$, then (**??**) reduces to

$$\beta > 2/3\,,$$

which recovers the condition in [**?**] and [**?**, Section 5.2]. If $\beta = 1$ and $d = 1$, then this condition becomes

$$\alpha > 1\,, \tag{3.3.18}$$

which recovers the condition in [**?**].

### 3.3.3 The smoothed equation

The methodology used in the proof of Theorem **??** can also be used to study the following equation

$$\left(\partial^{\beta}+\frac{\nu}{2}(-\Delta)^{\frac{\alpha}{2}}\right)u(t,x)=I_t^{\lceil\beta\rceil-\beta}\left[u(t,x)\dot{W}(t,x)\right], \tag{3.3.19}$$

with the same initial conditions as (**??**). Here $I_t^{\beta}$ is the *Riemann-Liouville fractional integral* of order $\beta$ (with an abuse of the notation $\beta$):

$$I_t^{\beta}f(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s)\mathrm{d}s,\quad \text{for } t>0 \text{ and } \beta>0\,.$$

Due to the fractional integral in equation (**??**) which plays a smoothing role, the mild formulation for the solution can be expressed by using $Z(t,x)$ only, namely,

$$u(t,x)=J_0(t,x)+\int_0^t\int_{\mathbb{R}^d}Z(t-s,x-y)u(s,y)W(\mathrm{d}s,\mathrm{d}y)\,. \tag{3.3.20}$$

Then, using the same procedure as in the proof of Theorem **??**, we have the following result.

**Theorem 3.3.8.** *Assume the conditions (3) and (5) in Theorem* **??** *and the other conditions are replaced by the following.*

*(1')* $Z_{\alpha,\beta,d}(t,x)$ *is nonnegative;*

*(2')* $\beta\in(1/2,1]\cup(3/2,2)$ *and* $\alpha\in(0,2]$*;*

*(4')* $\mu$ *satisfies*

$$\int_{\mathbb{R}^d}\frac{\mu(\mathrm{d}\xi)}{1+|\xi|^{\alpha(2\lceil\beta\rceil-1)/\beta}}<\infty\,. \tag{3.3.21}$$

*Then relation (**??**) holds for each $(t,x)$. Consequently, the smoothed equation (**??**) admits a unique mild solution in the sense of Definition **??** with $Y$ replaced by $Z$.*

**Remark 3.3.9.** The condition (1') is different and is usually easier to verify than the condition (1) in Theorem **??**. When $\beta \in (1/2,1]$, the condition (**??**) becomes $\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{1+|\xi|^{\alpha/\beta}} < \infty$ which is also weaker than (**??**) (since $\beta \le 1$). When $\beta \in (3/2,2)$, the condition (**??**) becomes $\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{1+|\xi|^{3\alpha/\beta}} < \infty$ which is also weaker than (**??**) (since $\beta < 2$).

The proof is essentially the same as that for Theorem **??**, the only change in the proof worthy to be pointed out is that instead of computing the integral

$$\int_0^\infty w^{2(\lceil\beta\rceil-1)} E^2_{\beta,\lceil\beta\rceil}(-2^{-1}\nu w^\beta |\xi|^\alpha)\mathrm{d}w,$$

we now need to compute the integral

$$\begin{aligned}\int_0^\infty w^{2(\lceil\beta\rceil-1)} E^2_{\beta,\lceil\beta\rceil}(-2^{-1}\nu w^\beta |\xi|^\alpha)\mathrm{d}w &= \frac{(2/\nu)^{(2\lceil\beta\rceil-1)/\beta}}{|\xi|^{\alpha(2\lceil\beta\rceil-1)/\beta}} \int_0^\infty \frac{1}{\beta} s^{\frac{1}{\beta}(2\lceil\beta\rceil-\beta-1)} E^2_{\beta,\lceil\beta\rceil}(-\nu s)\mathrm{d}s \\ &= \frac{C}{|\xi|^{\alpha(2\lceil\beta\rceil-1)/\beta}}. \qquad (3.3.22)\end{aligned}$$

The integrability condition of the above equation at zero and at infinity implies that $\beta > 0$ and $\beta \in (1/2,1] \cup (3/2,2]$ (which is equivalent to $\lceil\beta\rceil < \beta + 1/2$), respectively. Note that this condition on $\beta$ is more restrictive than the condition $\beta \in (0,2)$ in [**?**].

**Remark 3.3.10** (Space-time white noise case)**.** When the noise $\dot{W}$ is a space-time white (namely $\mu(d\xi) = d\xi$), then Dalang's condition (**??**) becomes

$$d < \alpha(2\lceil\beta\rceil - 1)/\beta \qquad \text{or} \qquad \frac{d}{\alpha} + \frac{1}{\beta} < \frac{2\lceil\beta\rceil}{\beta}. \qquad (3.3.23)$$

In particular, if $\alpha = 2$ and $d = 1$, then this condition reduces to $\beta < 2$. If $\beta = 1$ and $d = 1$, then this condition becomes (**??**), which recovers the condition in [**?**].

### 3.3.4 Moment bounds

In this subsection we give some upper bounds for the $p$-th moment and the lower bound of the second moment of the solution for some specific choice of the covariance kernel.

**Theorem 3.3.11.** *Assume the following conditions.*

*(1) The initial conditions satisfy condition (5) of Theorem* **??***;*

*(2)* $(\alpha, \beta, d)$ *satisfies one of the three conditions in* (**??**)*;*

*(3)* $\Lambda(x) = |x|^{-\kappa}$, $x \in \mathbb{R}^d$ *with*

$$0 < \kappa < \min(2\alpha - \alpha/\beta, d).$$

*Then the solution* $u(t,x)$ *to* (**??**) *satisfies that for all* $p \geq 1$,

$$\mathbb{E}\left[u(t,x)^p\right] \leq C^p \widehat{C}_t^p \exp\left( t(C_\kappa C_t \widetilde{C} C_* (2/\nu)^{\kappa/\alpha} (2\pi)^{-d})^{\frac{\alpha}{2\alpha\beta-\alpha-\beta\kappa}} p^{\frac{2\alpha\beta-\beta\kappa}{2\alpha\beta-\alpha-\beta\kappa}} \right), \quad (3.3.24)$$

*where* $C_t$ *and* $\widehat{C}_t$ *are defined in* (**??**) *and* (**??**)*, respectively,* $C = C(\alpha, \beta, \kappa) > 0$*, and*

$$C_* = \Gamma(2\beta - 1 - \beta\kappa/\alpha) \quad \text{and} \quad \widetilde{C} = \int_{\mathbb{R}^d} E_{\beta,\beta}^2(-|\xi|^\alpha)|\xi|^{\kappa-d} d\xi,$$

*and* $C_\kappa$ *appears in the Fourier transform of* $|x|^{-\kappa}$*, i.e.,* $\mu(d\xi) = C_\kappa |\xi|^{\kappa-d}$.

*In particular, if $\gamma$ is the Dirac delta function and if initial data $u_0(x) \equiv u_0 > 0$ is a constant and $u_1 \equiv 0$ when $\beta > 1$, then for some constant $c = c(\alpha, \beta, \kappa) > 0$,*

$$\mathbb{E}\left[|u(t,x)|^2\right] \geq c\, u_0^2 \exp\left(t\left(C_\kappa \widetilde{C}(4\pi)^{-d} C_*\,(2/\nu)^{\kappa/\alpha}\right)^{\frac{1}{2\beta-1-\beta\kappa/\alpha}}\right), \tag{3.3.25}$$

The proof of this theorem is given in Section **??**. The same method can be used to obtain the moment bound for the solution to the smoothed equation (**??**).

**Remark 3.3.12.** When $\beta = 1$ and $\alpha = 2$, the equation (**??**) is reduced to the multiplicative stochastic heat equation (1.1) considered in [**?**]. In this case the exponent of $p$ in (**??**) becomes

$$\frac{2\alpha\beta - \beta\kappa}{2\alpha\beta - \alpha - \beta\kappa} = \frac{4-\kappa}{2-\kappa},$$

which is the same as in [**?**, Theorem 6.1, inequality (6.1)] (with $\kappa = a$). If we assume $\gamma(t) = t^{-\tilde{\beta}}$, then $C_t = Ct^{-\tilde{\beta}+1}$. The exponent of $t$ in (**??**) is

$$1 + (-\tilde{\beta}+1)\left(\frac{\alpha}{2\alpha\beta - \alpha - \beta\kappa}\right) = \frac{4 - 2\tilde{\beta} - \kappa}{2-\kappa}$$

which is the same exponent of $t$ as in [**?**], inequality (6.1). Hence, we conjecture that the bound (**??**) is sharp.

**Theorem 3.3.13.** *Under the conditions (1), (2) of Theorem* **??** *and*

*(3')* $\Lambda(x) = |x|^{-\kappa}$, $x \in \mathbb{R}^d$ *with*

$$0 < \kappa < \min(\alpha/\beta, d).$$

*Then the solution $u(t,x)$ to the smoothed equation (**??**) satisfies that for all $p \geq 1$,*

$$\mathbb{E}\left[u(t,x)^p\right] \leq C^p \widehat{C}_t^p \exp\left(t\,\left[C_\kappa C_t \bar{C} C_\#(2/\nu)^{\kappa/\alpha}\right]^{\frac{\alpha}{2\alpha\lceil\beta\rceil-\alpha-\beta\kappa}} p^{\frac{2\alpha\lceil\beta\rceil-\beta\kappa}{2\alpha\lceil\beta\rceil-\alpha-\beta\kappa}}\right), \qquad (3.3.26)$$

*where $C = C(\alpha,\beta,\kappa) > 0$, $\widehat{C}_t$ is defined in (**??**),*

$$C_\# = \Gamma(2\lceil\beta\rceil - 1 - \beta\kappa/\alpha) \quad \textit{and} \quad \bar{C} = \int_{\mathbb{R}^d} E_{\beta,\lceil\beta\rceil}(-|\eta|^\alpha)|\eta|^{\kappa-d} d\eta,$$

*and $C_\kappa$ is as defined in Theorem **??**. In particular, if $\gamma$ is the Dirac delta function and if initial data $u_0(x) \equiv u_0 > 0$ is a constant and $u_1 \equiv 0$ when $\beta > 1$, , then for some constant $c = c(\alpha,\beta,\kappa) > 0$,*

$$\mathbb{E}\left[|u(t,x)|^2\right] \geq c\, u_0^2 \exp\left(t\,\left[C_\kappa \bar{C}(4\pi)^{-d} C_\#(2/\nu)^{\kappa/\alpha}\right]^{\frac{1}{2\beta-1-\beta\kappa/\alpha}}\right). \qquad (3.3.27)$$

The proof of this theorem is a line-by-line change of the proof of Theorem **??**, and we leave it to the interested reader.

### 3.3.5 Case $1 < \beta < 2$ and $d \geq 2$

When $1 < \beta < 2$ and $\alpha \neq 2$, we could not show the nonnegativity of $Y(t,x)$ for high dimension ($d \geq 2$) (see Theorem **??** (b)). However, with a slightly different approach, it is possible to obtain similar results (to Theorem **??**) for Riesz kernels. Here is the main theorem of this subsection.

**Theorem 3.3.14.** *Assume the conditions (2), (3) and (5) of Theorem **??**, and assume*

*(4') $\Lambda(x) = |x|^{-\kappa}$, $x \in \mathbb{R}^d$ with*

$$0 < \kappa < \min(2\alpha - \alpha/\beta, d).$$

*Then relation (**??**) holds for each $(t,x)$. Consequently, equation (**??**) admits a unique mild solution in the sense of Definition **??**.*

This theorem is proved in Section **??**.

**Remark 3.3.15.** It is easy to see that the condition $\Lambda(x) = |x|^{-\kappa}$ with $0 < \kappa < 2\alpha - \alpha/\beta$ implies Dalang's condition (**??**). Condition $\kappa < d$ is to guarantee that $\Lambda$ is a locally integrable function.

## 3.4 Fox H-functions: Some proofs

### 3.4.1 Proof of Theorem ??

The proof of Theorem **??** will be based on following lemmas.

**Lemma 3.4.1.** *The function $Z_{\alpha,\beta,d}(t,x)$ has the Fourier transform given by* (**??**).

*Proof.* The proof needs the following relation between the Mittag-Leffler function and the Fox H-function (see [**?**, (2.9.27)]):

$$E_{\rho,\mu}(x) = H_{1,2}^{1,1}\left(-x \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\mu,\rho) \end{matrix}\right). \tag{3.4.1}$$

The case where $\beta \in (0,1]$, $\alpha = 2$ and $d \in \mathbb{N}$ can be found in [**?**, Section 4] or [**?**]. For $\beta \in (0,1]$ and for general $\alpha$, one can simply replace $|\xi|^2$ by $|\xi|^\alpha$ in the argument of [**?**, Section 4] and then use (**??**) to obtain (**??**). The case where $d = 1$, $\beta \in (0,2)$, and $\alpha \in (0,2)$ is proved by [**?**]. For the general case, denote $m = \lceil \beta \rceil - 1$. We know that $Z_{\alpha,\beta,d}$ solves

$$\left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = 0, \quad \frac{\partial^m}{\partial t^m} u(t,x)\bigg|_{t=0} = \delta_0(x).$$

Hence, the Fourier transform of $Z_{\alpha,\beta,d}$ satisfies

$$\partial^{\beta}\mathscr{F}Z(t,\cdot)(\xi)=-\frac{\nu}{2}|\xi|^{\alpha}\mathscr{F}Z(t,\cdot)(\xi),\quad \frac{\partial^m}{\partial t^m}\mathscr{F}Z(t,\cdot)(\xi)\bigg|_{t=0}=1.$$

This equation can be solved explicitly (see, e.g., [**?**, Theorem 7.2, on p. 135]) as

$$\mathscr{F}Z(t,\cdot)(\xi)=I_t^m E_{\beta}(-\nu|\xi|^{\alpha}t^{\beta}/2),$$

which gives immediately (**??**) when $m=0$. When $m=1$, the integral can be evaluated by [**?**, (1.99)] to give

$$\mathscr{F}Z(t,\cdot)(\xi)=tE_{\beta,2}(-\nu|\xi|^{2}t^{\beta}/2).$$

This completes the proof of Lemma **??**. □

**Lemma 3.4.2.** *The function $Z_{\alpha,\beta,d}(t,x)$ can be expressed in* (**??**).

*Proof.* Following Lemma **??**, we need to compute the inverse Fourier transform of (**??**). Instead of finding the inverse Fourier transform, it turns out that it is easier to verify that the Fourier transform of (**??**) is equal to the right hand side of (**??**). Let now $Z$ be defined by (**??**).

**Case I $d=1$.** Notice that $x\mapsto Z(t,x)$ is an even function. We have that

$$\mathscr{F}Z(t,\cdot)(\xi)=2\pi^{-1/2}t^{\lceil\beta\rceil-1}\int_0^{\infty}\mathrm{d}x\,x^{-1}H_{2,3}^{2,1}\left(\frac{x^{\alpha}}{2^{\alpha-1}\nu t^{\beta}}\,\middle|\,\begin{matrix}(1,1),(\lceil\beta\rceil,\beta)\\(1/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right)\cos(x\xi).$$

Write the $\cos(\cdot)$ function in the Fox H-function form by using (2.9.8) and Property 2.4 of [**?**]. We have

$$\mathscr{F}Z_{\alpha,\beta}(t,\cdot)(\xi)=t^{\lceil\beta\rceil-1}\int_0^{\infty}\mathrm{d}x\,x^{-1}H_{2,3}^{2,1}\left(\frac{x^{\alpha}}{2^{\alpha-1}\nu t^{\beta}}\,\middle|\,\begin{matrix}(1,1),(\lceil\beta\rceil,\beta)\\(1/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right)$$

$$\times H^{1,0}_{0,2}\left(x|\xi|/2 \,\middle|\, \overline{\phantom{(0,1/2),(1/2,1/2)}} \atop (0,1/2),\,(1/2,1/2)\right).$$

Now apply [**?**, Theorem 2.9 on p. 56]. We need to check the conditions there. The condition $a^* > 0$ (see (**??**) for the definition of $a^*$) imposes that $\beta < 2$. Note that since both $\nu t^\beta$ and $\xi$ are real numbers, $a_0^* = 0$ is allowed (see the paragraph before Theorem 2.10 of [**?**]). Hence,

$$\begin{aligned}\mathscr{F}Z(t,\cdot)(\xi) &= t^{\lceil\beta\rceil-1} H^{2,2}_{4,3}\left(2\left(\nu t^\beta|\xi|^\alpha\right)^{-1} \,\middle|\, \begin{matrix}(1,1),(1,\alpha/2),(1/2,\alpha/2),(\lceil\beta\rceil,\beta)\\(1/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right)\\ &= t^{\lceil\beta\rceil-1} H^{1,1}_{1,2}\left(2^{-1}\nu t^\beta|\xi|^\alpha \,\middle|\, \begin{matrix}(0,1)\\(0,1),(1-\lceil\beta\rceil,\beta)\end{matrix}\right),\end{aligned}$$

where the second equality follows from [**?**, Properties 2.2 and 2.4]. This proves the lemma when $d = 1$.

**Case II $d \geq 2$.** Because the function $x \mapsto Z_{\alpha,\beta,d}(t,x)$ is a radial function, by [**?**, Theorem 3.3 on p. 155],

$$\mathscr{F}Z(t,\cdot)(\xi) = 2^{d/2} t^{\lceil\beta\rceil-1}|\xi| \int_0^\infty \mathrm{d}x H^{2,1}_{2,3}\left(\frac{x^\alpha}{2^{\alpha-1}\nu t^\beta} \,\middle|\, \begin{matrix}(1,1),(\lceil\beta\rceil,\beta)\\(1/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right) J_{(d-2)/2}(x|\xi|)(|\xi|x)^{-d/2},$$

where $J_\nu(x)$ is the Bessel function of the first kind. Then we can apply Corollary 2.5.1 of [**?**]. Similar to the previous case, all conditions are satisfied with the condition $a^* > 0$ imposing that $\beta < 2$. Hence,

$$\begin{aligned}\mathscr{F}Z(t,\cdot)(\xi) &= t^{\lceil\beta\rceil-1} H^{2,2}_{4,3}\left(2\left(\nu t^\beta|\xi|^\alpha\right)^{-1} \,\middle|\, \begin{matrix}(1,1),(1,\alpha/2),(d/2,\alpha/2),(\lceil\beta\rceil,\beta)\\(d/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right)\\ &= t^{\lceil\beta\rceil-1} H^{1,1}_{1,2}\left(2^{-1}\nu t^\beta|\xi|^\alpha \,\middle|\, \begin{matrix}(0,1)\\(0,1),(1-\lceil\beta\rceil,\beta)\end{matrix}\right),\end{aligned}$$

where the second equality follows the same way as the previous case. This completes the proof of Lemma **??**. □

**Lemma 3.4.3.** *The fundamental solutions $Y_{\alpha,\beta,d}(t,x)$ and $Z^*_{\alpha,\beta,d}(t,x)$ are given by* (**??**) *and* (**??**)*, respectively.*

*Proof.* We first prove the expression for $Y_{\alpha,\beta,d}$. By Section 2 of [**?**], we know that $Y_{\alpha,\beta,d}(t,x)$ is the Riemann-Liouville fractional derivative in $t$ of $Z_{\alpha,\beta,d}(t,x)$ of order $\lceil\beta\rceil-\beta$. Notice that $Z_{\alpha,\beta,d}(0,x)=0$ for $|x|\neq 0$. Denote the Riemann-Liouville derivative of order $\beta$ by ${}_tD^{\beta}_{+}$. By [**?**, Property 2.3],

$$Z_{\alpha,\beta,d}(t,x)=\pi^{-d/2}t^{\lceil\beta\rceil-1}|x|^{-d}H^{1,2}_{3,2}\left(\frac{2^{\alpha-1}\nu t^{\beta}}{|x|^{\alpha}}\,\middle|\,\begin{matrix}(1-d/2,\alpha/2),(0,1),(0,\alpha/2)\\(0,1),(1-\lceil\beta\rceil,\beta)\end{matrix}\right).$$

Because $a^*=(2-\beta)+(2-\alpha)/2>0$, we can apply part (i) of [**?**, Theorem 2.8 on p. 55],

$${}_tD^{\lceil\beta\rceil-\beta}_{+}Z_{\alpha,\beta,d}(t,x)=\pi^{-d/2}|x|^{-d}t^{\beta-1}H^{1,3}_{4,3}\left(\frac{2^{\alpha-1}\nu t^{\beta}}{|x|^{\alpha}}\,\middle|\,\begin{matrix}(1-\lceil\beta\rceil,\beta),(1-d/2,\alpha/2),(0,1),(0,\alpha/2)\\(0,1),(1-\lceil\beta\rceil,\beta),(1-\beta,\beta)\end{matrix}\right).$$

Then we use Properties 2.2 and 2.4 of [**?**] to simplify the above expression to obtain (**??**). The expression for $Z^*_{\alpha,\beta,d}$ can be proved in a similar way. □

**Lemma 3.4.4.** *The Fourier transforms of $Y_{\alpha,\beta,d}(t,x)$ and $Z^*_{\alpha,\beta,d}(t,x)$ are given by* (**??**) *and* (**??**)*, respectively.*

*Proof.* We first consider $Y_{\alpha,\beta,d}$. From Lemma **??** and the proof of Lemma **??** it follows

$$\mathscr{F}Y_{\alpha,\beta,d}(t,\cdot)(\xi)={}_tD^{\lceil\beta\rceil-\beta}_{+}t^{\lceil\beta\rceil-1}H^{1,1}_{1,2}\left(2^{-1}\nu t^{\beta}|\xi|^{\alpha}\,\middle|\,\begin{matrix}(0,1)\\(0,1),(1-\lceil\beta\rceil,\beta)\end{matrix}\right).$$

Because $a^*=2-\beta>0$, we can apply part (i) of Theorem 2.8 of [**?**] to obtain

$$\mathscr{F}Y_{\alpha,\beta,d}(t,\cdot)(\xi)=t^{\beta-1}H^{1,2}_{2,3}\left(2^{-1}\nu t^{\beta}|\xi|^{\alpha}\,\middle|\,\begin{matrix}(1-\lceil\beta\rceil,\beta),(0,1)\\(0,1),(1-\lceil\beta\rceil,\beta),(1-\beta,\beta)\end{matrix}\right).$$

This is simplified to (**??**) by the properties 2.2 and 2.4 of [**?**]. The identity (**??**) can be obtained in a similar way. □

**Lemma 3.4.5.** *For all* $\mu > 0$ *and* $0 < \theta \le \min(1,\mu)$*, the following H-function is nonnegative:*

$$H_{1,1}^{1,0}\left(|x| \,\middle|\, \begin{matrix} (\mu,\theta) \\ (1,1) \end{matrix}\right) \ge 0, \quad \forall\, x \in \mathbb{R}. \tag{3.4.2}$$

*Proof.* We only need to prove that the following function is nonnegative

$$f(x) = |x|^{-1} H_{1,1}^{1,0}\left(|x| \,\middle|\, \begin{matrix} (\mu,\theta) \\ (1,1) \end{matrix}\right), \quad x \in \mathbb{R}.$$

By [**?**, Corollary 2.3.1] and the equation (**??**), the Laplace transform of $f$ is equal to

$$\int_0^\infty \mathrm{d}x\, e^{-xz} f(x) = E_{\theta,\mu}(-z).$$

By [**?**], we know that the above Mittag-Leffler function $E_{\alpha,\beta}(-z)$ is completely monotonic if and only if $0 < \alpha \le \min(\beta,1)$. Then the Bernstein theorem (see, e.g., [**?**, Theorem 12a]) implies that the function $f(x)$ is nonnegative. □

**Lemma 3.4.6.** *The nonnegative statements in Theorem* **??** *hold true.*

*Proof.* We first prove the case (a). In this case, $\beta \in (0,1]$. Because $\lim_{t\to 0} Z_{\alpha,\beta,d}(t,x) = 0$ for all $|x| \ne 0$ and from [**?**, Theorem 3.8] we see

$$Z_{\alpha,\beta,d}(t,x) = I_t^{1-\beta}\partial^{1-\beta} Z_{\alpha,\beta,d}(t,x) = I_t^{1-\beta} Y_{\alpha,\beta,d}(t,x).$$

Hence, it suffices to show the nonnegativity of $Y_{\alpha,\beta,d}(t,x)$. Applying Theorem 2.9 of [**?**] with $\eta = 0$, $\sigma = \beta$ and $z = |x|^\alpha/(2^\alpha \nu)$ to the expression of $Y(t,x)$ (it is easy to verify that all conditions are satisfied) yields

$$Y(t,x) = \beta\pi^{-d/2}t^{\beta-1}|x|^{-d}\int_0^\infty \mathrm{d}s\, s^{-1}H_{1,2}^{1,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}\nu}s^\beta \,\middle|\, \begin{matrix}(1,1)\\(d/2,\alpha/2),(1,\alpha/2)\end{matrix}\right) H_{1,1}^{1,0}\left((ts)^{-\beta}\,\middle|\,\begin{matrix}(\beta,\beta)\\(1,1)\end{matrix}\right).$$

By Lemma **??**, the second H-function in the above equation is nonnegative. On the other hand, Theorem **??** tells us that the first H-function is nonnegative. Thus, $Y(t,x)$ is nonnegative.

As for the case (b), it is known from [**?**] that $Y_{2,\beta,d}$ is nonnegative for $d \le 3$. By the same argument as in the proof of (a), $Z_{2,\beta,d}$ is also nonnegative.

Finally, for the case (c), it is proved in [**?**] that $Z^*_{\alpha,\beta,1}(t,x)$ is nonnegative. By the same reason as in the proof of (a), $Y_{\alpha,\beta,1}$ and $Z_{\alpha,\beta,1}$ are fractional integrals of $Z^*_{\alpha,\beta,1}$ of orders $1-\beta$ and 1, respectively. Therefore, both $Y_{\alpha,\beta,1}$ and $Z_{\alpha,\beta,1}$ are nonnegative as well. The proof of Lemma **??** is now complete. □

*Proof of Theorem* **??**. The Theorem **??** follows from the above lemmas. □

### 3.4.2 Proof of Theorem ??

*Proof of Theorem* **??**. The characteristic function (**??**) of $X$ is proved in [**?**, (7.5.3) on p. 211]. For the density $\rho_{\alpha,d}$, we need to compute the inverse Fourier transform. From [**?**, (7.5.5)] this inverse transform is

$$\rho_{\alpha,d}(r) = (2\pi)^{-d/2}r^{1-d/2}\int_0^\infty e^{-t^\alpha}J_{(d-2)/2}(rt)t^{d/2}\mathrm{d}t\,.$$

By (2.9.18) and (2,9.4) of [**?**], we have that

$$t^{(d+2)/2}J_{(d-2)/2}(rt) = (2/r)^{(d+2)/2}H_{0,2}^{1,0}\left(\frac{r^2t^2}{4}\,\middle|\,\begin{matrix}\overline{\phantom{(d/2,1),(1,1)}}\\(d/2,1),(1,1)\end{matrix}\right)$$

and

$$e^{-t^\alpha} = \frac{1}{\alpha} H_{0,1}^{1,0}\left(t \,\middle|\, \frac{\quad}{(0,1/\alpha)}\right).$$

Hence,

$$\rho_{\alpha,d}(r) = \pi^{-d/2} r^{-d} \int_0^\infty t^{-1} H_{0,2}^{1,0}\left(\left(\frac{rt}{2}\right)^\alpha \,\middle|\, \frac{\quad}{(d/2,\alpha/2),(1,\alpha/2)}\right) H_{0,1}^{1,0}\left(t \,\middle|\, \frac{\quad}{(0,1/\alpha)}\right) \mathrm{d}t.$$

Application of [**?**, Theorem 2.9] to evaluate the above integral yields the theorem. □

## 3.5 Proof of Theorem ??

*Proof of Theorem* **??**. Recall that $J_0(t,x)$ defined by (**??**) is the solution to the homogeneous equation. Using an iteration procedure as in [**?**], we have

$$f_n(s_1,x_1,\cdots,s_n,x_n,t,x) = g_n(s_1,x_1,\cdots,s_n,x_n,t,x) J_0(s_{\sigma(1)},x_{\sigma(1)})$$

where

$$g_n(s_1,x_1,\cdots,s_n,x_n,t,x) = \frac{1}{n!} Y(t-s_{\sigma(n)},x-x_{\sigma(n)})\cdots Y(s_{\sigma(2)}-s_{\sigma(1)},x_{\sigma(2)}-x_{\sigma(1)}),$$

and $\sigma$ denotes a permutation of $\{1,2,\cdots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Fix $t > 0$ and $x \in \mathbb{R}^d$, set $f_n(s,y,t,x) = f_n(s_1,y_1,\cdots,s_n,y_n,t,x)$. Then we have that

$$\begin{aligned} & n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \\ & = n! \int_{[0,t]^{2n}} \mathrm{d}s\mathrm{d}r \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, f_n(s,y,t,x) f_n(r,z,t,x) \prod_{i=1}^n \Lambda(y_i-z_i) \prod_{i=1}^n \lambda(s_i-r_i). \end{aligned} \quad (3.5.1)$$

where $\mathrm{d}y = \mathrm{d}y_1 \cdots \mathrm{d}y_n$, the differentials $\mathrm{d}z$, $\mathrm{d}s$ and $\mathrm{d}r$ are defined similarly. Set $\mu(\mathrm{d}\xi) := \prod_{i=1}^n \mu(\mathrm{d}\xi_i)$. Using the Fourier transform and Cauchy-Schwartz inequality together with (**??**), we obtain that

$$n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \leq \frac{\widehat{C}_t^2 n!}{(2\pi)^{nd}} \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} \mathscr{F} g_n(s,\cdot,t,x)(\xi) \overline{\mathscr{F} g_n(r,\cdot,t,x)(\xi)} \mu(\mathrm{d}\xi) \prod_{i=1}^n \lambda(s_i - r_i) \mathrm{d}s \mathrm{d}r \tag{3.5.2}$$

$$\leq \frac{\widehat{C}_t^2 n!}{(2\pi)^{nd}} \int_{[0,t]^{2n}} \left( \int_{\mathbb{R}^{nd}} \left( \mathscr{F} g_n(s,\cdot,t,x)(\xi) \right)^2 \mu(\mathrm{d}\xi) \right)^{1/2} \times \left( \int_{\mathbb{R}^{nd}} \left( \mathscr{F} g_n(r,\cdot,t,x)(\xi) \right)^2 \mu(\mathrm{d}\xi) \right)^{1/2} \prod_{i=1}^n \lambda(s_i - r_i) \mathrm{d}s \mathrm{d}r,$$

where the constant $\widehat{C}_t$ is defined in (**??**). Thus, thanks to the basic inequality $ab \leq 2^{-1}(a^2 + b^2)$ and the fact that $\lambda$ is locally integrable, we obtain

$$n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \leq \frac{\widehat{C}_t^2 n!}{(2\pi)^{nd}} \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} |\mathscr{F} g_n(s,\cdot,t,x)(\xi)|^2 \mu(\mathrm{d}\xi) \prod_{i=1}^n \lambda(s_i - r_i) \mathrm{d}s \mathrm{d}r$$
$$\leq \frac{\widehat{C}_t^2 C_t^n n!}{(2\pi)^{nd}} \int_{[0,t]^n} \mathrm{d}s \int_{\mathbb{R}^{nd}} |\mathscr{F} g_n(s,\cdot,t,x)(\xi)|^2 \mu(\mathrm{d}\xi),$$

where the constant $C_t$ is defined in (**??**). Furthermore, from the Fourier transform of $Y(t,\cdot)$ we can check that

$$|\mathscr{F} g_n(r,\cdot,t,x)(\xi)|^2 = \frac{1}{(n!)^2} \prod_{i=1}^n \left[ (s_{\sigma(i+1)} - s_{\sigma(i)})^{\beta-1} E_{\beta,\beta} \left( -2^{-1} \nu (s_{\sigma(i+1)} - s_{\sigma(i)})^{\beta} |\xi_{\sigma(i)} + \cdots + \xi_{\sigma(1)}|^{\alpha} \right) \right]^2,$$

where we have set $s_{\sigma(n+1)} = t$. As a consequence,

$$\int_{\mathbb{R}^{nd}} |\mathscr{F} g_n(s,\cdot,t,x)(\xi)|^2 \mu(\mathrm{d}\xi)$$

$$
\begin{aligned}
&\leq \frac{1}{(n!)^2}\prod_{i=1}^{n}\sup_{\eta}\left|\int_{\mathbb{R}^d}(Y(s_{\sigma(i+1)}-s_{\sigma(i)},\cdot)*Y(s_{\sigma(i+1)}-s_{\sigma(i)},\cdot))(x_{\sigma(i)})e^{i\eta\cdot x_{\sigma(i)}}\Lambda(x_{\sigma(i)})\mathrm{d}x_{\sigma(i)}\right| \\
&\leq \frac{1}{(n!)^2}\prod_{i=1}^{n}\left|\int_{\mathbb{R}^d}(Y(s_{\sigma(i+1)}-s_{\sigma(i)},\cdot)*Y(s_{\sigma(i+1)}-s_{\sigma(i)},\cdot))(x_{\sigma(i)})\Lambda(x_{\sigma(i)})\mathrm{d}x_{\sigma(i)}\right| \\
&\leq \frac{1}{(n!)^2}\prod_{i=1}^{n}\int_{\mathbb{R}^d}\left[(s_{\sigma(i+1)}-s_{\sigma(i)})^{\beta-1}E_{\beta,\beta}\left(-2^{-1}\nu(s_{\sigma(i+1)}-s_{\sigma(i)})^{\beta}|\xi_{\sigma(i)}|^{\alpha}\right)\right]^2\mu(\mathrm{d}\xi_{\sigma(i)}),
\end{aligned}
\tag{3.5.3}
$$

where we have used the fact that $|e^{ix_{\sigma(i)}\cdot\eta}|=1$ and that $Y$ and $\Lambda$ are nonnegative to get rid of the supremum in $\eta$. Therefore, using Fourier transform again we have

$$
\begin{aligned}
n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \leq \frac{\widehat{C}_t^2C_t^n}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}\mu(\mathrm{d}\xi)\int_{T_n(t)}\mathrm{d}s& \\
\times\prod_{i=1}^{n}\left(s_{i+1}-s_i\right)^{2\beta-2}E^2_{\beta,\beta}\left(-2^{-1}\nu(s_{i+1}-s_i)^{\beta}|\xi_i|^{\alpha}\right)&,
\end{aligned}
\tag{3.5.4}
$$

where $T_n(t)$ denotes the simplex

$$
T_n(t):=\{s=(s_1,\cdots,s_n):\,0<s_1<\cdots<s_n<t\}. \tag{3.5.5}
$$

By the change of variables $s_{i+1}-s_i=w_i$ for $1\leq i\leq n-1$ and $t-s_n=w_n$, we see that

$$
n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \leq \frac{\widehat{C}_t^2C_t^n}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}\int_{S_{t,n}}\prod_{i=1}^{n}w_i^{2\beta-2}E^2_{\beta,\beta}\left(-2^{-1}\nu w_i^{\beta}|\xi_i|^{\alpha}\right)\mathrm{d}w_i\mu(\mathrm{d}\xi_i),
$$

where

$$
S_{t,n}=\{(w_1,\cdots,w_n)\in[0,\infty)^n:\,w_1+\cdots+w_n\leq t\}.
$$

We take $N\geq 1$ which will be chosen later, and let

$$
C_N=\int_{|\xi|\geq N}\frac{\mu(d\xi)}{|\xi|^{2\alpha-\alpha/\beta}} \quad\text{and}\quad D_N=\mu\{\xi\in\mathbb{R}^d:|\xi|\leq N\}. \tag{3.5.6}
$$

Let $I$ be a subset of $\{1,2,\cdots,n\}$ and $I^c=\{1,2,\cdots,n\}\backslash I$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}}\int_{S_{t,n}}\prod_{i=1}^{n} w_i^{2\beta-2}E_{\beta,\beta}^2\big(-2^{-1}\nu w_i^{\beta}|\xi_i|^{\alpha}\big)\mathrm{d}w_i\mu(\mathrm{d}\xi_i)\\
&\qquad=\int_{\mathbb{R}^{nd}}\int_{S_{t,n}}\prod_{i=1}^{n} w_i^{2\beta-2}E_{\beta,\beta}^2\big(-2^{-1}\nu w_i^{\beta}|\xi_i|^{\alpha}\big)\big(1_{\{|\xi_i|\le N\}}+\mathbf{1}_{\{|\xi_i|>N\}}\big)\mathrm{d}w_i\mu(\mathrm{d}\xi_i)\\
&\qquad=\sum_{I\subset\{1,2,\cdots,n\}}\int_{\mathbb{R}^{nd}}\mathrm{d}w\int_{S_{t,n}}\mu(\mathrm{d}\xi)\prod_{i\in I}E_{\beta,\beta}^2\big(-2^{-1}\nu w_i^{\beta}|\xi_i|^{\alpha}\big)w_i^{2\alpha-2}1_{\{|\xi_i|\le N\}}\\
&\qquad\quad\times\prod_{j\in I^c}E_{\beta,\beta}^2\big(-2^{-1}\nu w_j^{\beta}|\xi_j|^{\alpha}\big)w_j^{2\beta-2}\mathbf{1}_{\{|\xi_j|>N\}}\,.
\end{aligned}$$

where $\mathrm{d}w=\mathrm{d}w_1\cdots\mathrm{d}w_n$. For the indices $i$ in the set $I$, for some constant $C_\beta\ge 1$ (one may choose $C_\beta=\Gamma(\beta)^{-2}$)

$$E_{\beta,\beta}^2\big(-2^{-1}\nu w_i^{\beta}|\xi_i|^{\alpha}\big)\le C_\beta. \tag{3.5.7}$$

Now using the inclusion $S_{t,n}\subset S_t^I\times S_t^{I^c}$ with

$$S_t^I=\left\{(w_i,i\in I):w_i\ge 0,\sum_{i\in I}w_i\le t\right\}\quad\text{and}\quad S_t^{I^c}=\left\{(w_i,i\in I^c):w_i\ge 0,\sum_{i\in I^c}w_i\le t\right\},$$

we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}}\int_{S_{t,n}}\prod_{i=1}^{n} w_i^{2\beta-2}E_{\beta,\beta}^2\big(-2^{-1}\nu w_i^{\beta}|\xi_i|^{\alpha}\big)\mathrm{d}w_i\mu(\mathrm{d}\xi_i)\\
&\quad\le C_\beta^{|I|}\sum_{I\subset\{1,2,\cdots,n\}}\int_{\mathbb{R}^{nd}}\mu(\mathrm{d}\xi)\int_{S_t^I\times S_t^{I^c}}\mathrm{d}w\\
&\qquad\quad\times\prod_{i\in I}w_i^{2\beta-2}1_{\{|\xi_i|\le N\}}\prod_{j\in I^c}w_j^{2\beta-2}1_{\{|\xi_j|>N\}}E_{\beta,\beta}^2\big(-2^{-1}\nu w_j^{\beta}|\xi_j|^{\alpha}\big)\,.
\end{aligned}$$

Furthermore, one can bound the integral over $S_t^{I^c}$ in the following way

$$\int_{S_t^{I^c}} \prod_{j\in I^c} w_j^{2\beta-2} E_{\beta,\beta}^2\big(-2^{-1}\nu w_j^{\beta}|\xi_j|^{\alpha}\big)\mathrm{d}w_j \le \int_{\mathbb{R}_+^{|I^c|}} \prod_{j\in I^c} w_j^{2\beta-2} E_{\beta,\beta}^2\big(-2^{-1}\nu w_j^{\beta}|\xi_j|^{\alpha}\big)\mathrm{d}w_j\,.$$

Then make the change of variables $w_j^{\beta}|\xi_j|^{\alpha} \to v_j$ to obtain

$$\begin{aligned}\int_{[0,\infty)^{|I^c|}} \prod_{j\in I^c} w_j^{2\beta-2} E_{\beta,\beta}^2\big(-2^{-1}\nu w_j^{\beta}|\xi_j|^{\alpha}\big)\mathrm{d}w_j &\le \prod_{j\in I^c} \frac{1}{|\xi_i|^{2\alpha-\alpha/\beta}} \int_0^{\infty} \frac{1}{\beta} v_j^{1-1/\beta} E_{\beta,\beta}^2(-2^{-1}\nu v_j)\mathrm{d}v_j \\ &\le C_{\nu,\beta}^{|I^c|} \prod_{j\in I^c} \frac{1}{|\xi_i|^{2\alpha-\alpha/\beta}}\,,\end{aligned}$$

where

$$C_{\nu,\beta} = \int_0^{\infty} \frac{1}{\beta} v^{1-1/\beta} E_{\beta,\beta}^2(-2^{-1}\nu v)\mathrm{d}v.$$

Note that the integrability of the above quantity at zero and at infinity implies that $\beta > 1/2$ and $\beta > 0$, respectively. Thus we have the following bound.

$$\begin{aligned}&\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^{n} w_i^{2\beta-2} E_{\beta,\beta}^2\big(-2^{-1}\nu w_i^{\beta}|\xi_i|^{\alpha}\big)\mathrm{d}w_i \mu(\mathrm{d}\xi_i) \\ &\quad\le \sum_{I\subset\{1,2,\cdots,n\}} C_{\beta}^{|I|} \int_{S_t^I} \prod_{i\in I} w_i^{2\beta-2}\mathrm{d}w_i \cdot \Big(\mu\{\xi\in\mathbb{R}^d : |\xi|\le N\}\Big)^{|I|} C_{\nu,\beta}^{|I^c|} \int_{|\xi_j|>N,\forall j\in I^c} \prod_{j\in I^c} \frac{\mu(\mathrm{d}\xi_j)}{|\xi_j|^{2\alpha-\alpha/\beta}} \\ &\quad\le \sum_{I\subset\{1,2,\cdots,n\}} \frac{C_{\beta}^{|I|} t^{(2\beta-1)|I|} C_{\nu,\beta}^{|I^c|}}{\Gamma((2\beta-1)|I|+1)} D_N^{|I|} C_N^{n-|I|} \\ &\quad\le C_*^n \sum_{k=0}^{n} \binom{n}{k} \frac{t^{(2\beta-1)k}}{\Gamma((2\beta-1)k+1)} D_N^k C_N^{n-k}\,.\end{aligned}$$

where $C_* = \max(C_\beta, C_{\nu,\beta})$, and $C_N$ and $D_N$ are defined in (**??**). Observing the trivial inequality $\binom{n}{k} \le 2^n$, we have

$$\sum_{n=0}^{\infty} n!\|f_n(\cdot,\cdot,t,x)\|_{\mathscr{H}^{\otimes n}}^2 \le \frac{\widehat{C}_t^2}{(2\pi)^{nd}} \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \binom{n}{k} (C_* C_t)^n \frac{t^{(2\beta-1)k}}{\Gamma((2\beta-1)k+1)} D_N^k C_N^{n-k}$$

$$\le \frac{\widehat{C}_t^2}{(2\pi)^{nd}} \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{t^{(2\beta-1)k}}{\Gamma((2\beta-1)k+1)} D_N^k C_N^{-k} (2C_* C_t C_N)^n \,.$$

Choosing $N$ sufficiently large so that $2C_* C_t C_N < 1$ yields

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot,\cdot,t,x)\|_{\mathscr{H}^{\otimes n}}^2 \le \frac{\widehat{C}_t^2}{(2\pi)^{nd}} \sum_{k=0}^{\infty} \frac{t^{(2\beta-1)k}}{\Gamma((2\beta-1)k+1)} D_N^k C_N^{-k} \frac{(2C_* C_t C_N)^k}{1-2C_* C_t C_N} < \infty \,.$$

This proves (**??**), and thus the existence and uniqueness of the solution. □

## 3.6 Proof of Theorem ??

**Lemma 3.6.1.** *Suppose that the initial conditions $u_k(x) \equiv u_k$ are constant. Then under the three cases of* (**??**)*, we have that*

$$J_0(t,x) = \begin{cases} u_0 & \text{if } \beta \in (0,1], \\ u_0 + t^{\beta-1} u_1 & \text{if } \beta \in (1,2). \end{cases} \tag{3.6.1}$$

*Proof.* By Theorem **??**, we know that under the first two cases of (**??**), the fundamental solutions are nonnegative and hence,

$$J_0(t,x) = \sum_{k=0}^{\lceil\beta\rceil-1} u_{\lceil\beta\rceil-1-k} \int_{\mathbb{R}^d} \partial^k Z(t,x-y)\mathrm{d}y = \sum_{k=0}^{\lceil\beta\rceil-1} u_{\lceil\beta\rceil-1-k} \mathscr{F}\left[\partial^k Z(t,\cdot)\right](0),$$

which is equal to the right hand side of (**??**). As for the last case in (**??**), because $Z$ is still nonnegative, the contribution by $u_0$ can be computed in the same way. However, we do not know whether $Z^*$ is nonnegative, and thus we cannot use the Fourier transform arguments to compute the contribution by $u_1$. Instead, we compute it directly:

$$\int_{\mathbb{R}^d} Z^*_{2,\beta,d}(t,x)\mathrm{d}x = S_{d-1}\pi^{-d/2} \int_0^\infty x^{-1} H^{2,0}_{1,2}\left(\frac{x^\alpha}{2\nu t^\beta} \,\middle|\, \begin{matrix}(1,\beta)\\(d/2,1),(1,1)\end{matrix}\right)\mathrm{d}x,$$

where

$$S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{3.6.2}$$

Then by [**?**, Corollary 2.3.1], we have the following Laplace transform:

$$g(z) := \int_0^\infty e^{-zx} x^{-1} H^{2,0}_{1,2}\left( \frac{x^\alpha}{2\nu t^\beta} \,\middle|\, \begin{matrix} (1,\beta) \\ (d/2,1),(1,1) \end{matrix} \right) \mathrm{d}x = H^{1,2}_{2,2}\left( 2\nu t^\beta z^2 \,\middle|\, \begin{matrix} (-1/3,1),(0,1) \\ (0,2),(0,\beta) \end{matrix} \right).$$

Then by [**?**, Theorem 1.3],

$$g(0) = h_{10} = \frac{\Gamma(3/2)}{2}.$$

Putting these identities together, we have that

$$\int_{\mathbb{R}^d} Z^*_{2,\beta,d}(t,x)\mathrm{d}x = 1.$$

This completes the proof of Lemma **??**. □

*Proof of Theorem* **??**. Since $\Lambda(x) = |x|^{-\kappa}$, we have $\mu(d\xi) = C_\kappa |\xi|^{\kappa-d}$, for some coefficient $C_\kappa$; see, e.g., [**?**]. We begin with the upper bound. By the hypercontractivity property of the $n$-th chaos, i.e.

$$\|I_n(f_n(\cdot,\cdot,t,x))\|_{L^p(\Omega)} \leq (p-1)^{\frac{n}{2}} \|I_n(f_n(\cdot,\cdot,t,x))\|_{L^2(\Omega)}. \tag{3.6.3}$$

On the other hand, from the proof of Theorem **??** (see (**??**)) it follows

$$\begin{aligned} \|I_n(f_n(\cdot,\cdot,t,x))\|^2_{L^2(\Omega)} &= n!\, ||f_n(\cdot,\cdot,t,x)||^2_{\mathscr{H}^{\otimes n}} \\ &\leq \frac{\widehat{C}_t^2 C_\kappa^n C_t^n}{(2\pi)^{nd}} \int_{T_n(t)} \int_{\mathbb{R}^{nd}} \prod_{i=1}^n (s_{i+1}-s_i)^{2\beta-2} E^2_{\beta,\beta}(-2^{-1}\nu(s_{i+1}-s_i)^\beta |\xi_i|^\alpha) |\xi_i|^{\kappa-d} \mathrm{d}\xi_i \mathrm{d}s_i \end{aligned}$$

$$= \frac{\widehat{C}_t^2 C_\kappa^n C_t^n (2/\nu)^{\kappa n/\alpha}}{(2\pi)^{nd}} \int_{T_n(t)} \int_{\mathbb{R}^{nd}} \prod_{i=1}^n (s_{i+1}-s_i)^{2\beta-2-\frac{\beta\kappa}{\alpha}} E_{\beta,\beta}^2(-|\eta_i|^\alpha)|\eta_i|^{\kappa-d} \mathrm{d}\eta_i \mathrm{d}s_i$$
$$= \frac{\widehat{C}_t^2 C_\kappa^n C_t^n (2/\nu)^{\kappa n/\alpha} \widetilde{C}^n}{(2\pi)^{nd}} \int_{T_n(t)} \prod_{i=1}^n (s_{i+1}-s_i)^{2\beta-2-\frac{\beta\kappa}{\alpha}} \mathrm{d}s_i,$$

where $\widehat{C}_t$ is defined in (**??**),

$$\widetilde{C} := \int_{\mathbb{R}^d} E_{\beta,\beta}^2(-|\eta|^\alpha)|\eta|^{\kappa-d}\mathrm{d}\eta = S_{d-1}\int_0^\infty E_{\beta,\beta}^2(-t^\alpha)t^{\kappa-1}\mathrm{d}t,$$

and $S_{d-1}$ is defined in (**??**). According to the property of the Mittag-Leffler function at zero and infinity, if $0 < \kappa < 2\alpha$, then the above constant $\widetilde{C}$ is finite. Then, under the condition that $\kappa < \alpha(2-1/\beta)$ (this condition implies $0 < \kappa < 2\alpha$), the integration over d$s$ can be evaluated explicitly; see [**?**, Lemma 4.5]. Hence,

$$\|I_n(f_n(\cdot,\cdot,t,x))\|_{L^2(\Omega)}^2 \le \frac{1}{(2\pi)^{nd}} (C_\kappa C_* C_t \widetilde{C})^n \frac{t^{(2\beta-1-\frac{\beta\kappa}{\alpha})n}(2/\nu)^{\kappa n/\alpha}}{\Gamma((2\beta-1-\frac{\beta\kappa}{\alpha})n+1)},$$

where $C_* := \Gamma(2\beta-1-\beta\kappa/\alpha)$. Denote

$$\Theta_t := \frac{1}{(2\pi)^d} C_\kappa C_* C_t \widetilde{C}(2/\nu)^{\kappa/\alpha}.$$

Thus we obtain

$$\|I_n(f_n(\cdot,\cdot,t,x))\|_{L^2(\Omega)} \le \widehat{C}_t \frac{\Theta_t^{n/2} t^{(\beta-\frac{1}{2}-\frac{\beta\kappa}{2\alpha})n}}{\Gamma((2\beta-1-\frac{\beta\kappa}{\alpha})n+1)^{\frac{1}{2}}}. \tag{3.6.4}$$

This bound together with the hypercontractivity implies that

$$\|I_n(f_n(\cdot,\cdot,t,x))\|_{L^p(\Omega)} \le \widehat{C}_t \frac{\Theta_t^{n/2} t^{(\beta-\frac{1}{2}-\frac{\beta\kappa}{2\alpha})n}(p-1)^{\frac{n}{2}}}{\Gamma((2\beta-1-\frac{\beta\kappa}{\alpha})n+1)^{\frac{1}{2}}}. \tag{3.6.5}$$

Therefore,

$$\|u(t,x)\|_{L^p(\Omega)} \le \sum_{n=0}^{\infty} \|I_n(f_n(\cdot,\cdot,t,x))\|_{L^p(\Omega)} \le \widehat{C}_t \sum_{n=0}^{\infty} \frac{\Theta_t^{n/2} t^{\theta n} p^{\frac{n}{2}}}{\Gamma(2\theta n+1)^{\frac{1}{2}}}.$$

where

$$\theta := \beta - 1/2 - \beta\kappa/(2\alpha). \tag{3.6.6}$$

Then by the fact that $\Gamma(1+2x) \ge \Gamma(1+x)^2$ for $x > -1$,

$$\begin{aligned}\|u(t,x)\|_{L^p(\Omega)} &\le \widehat{C}_t \sum_{n=0}^{\infty} \frac{\Theta_t^{n/2} t^{\theta n} p^{\frac{n}{2}}}{\Gamma(\theta n+1)} = \widehat{C}_t E_\theta\left(\Theta_t^{1/2} t^\theta p^{1/2}\right) \\ &\le C\widehat{C}_t \exp\left(t (C_\kappa C_t \widetilde{C} C_* (2/\nu)^{\kappa/\alpha} (2\pi)^{-d})^{\frac{\alpha}{2\alpha\beta-\alpha-\beta\kappa}} p^{\frac{\alpha}{2\alpha\beta-\alpha-\beta\kappa}}\right),\end{aligned}$$

for some positive constant $C = C(\alpha,\beta,\kappa)$, where in the last step, we have used the asymptotic property of the Mittag-Leffler function (see, e.g., [**?**, Theorem 1.3]).

Now we consider the special case when $\lambda$ is the Dirac delta function. By Lemma **??** and the assumptions on the initial conditions we have

$$J_0(t,x) = u_0 + t^{\beta-1} u_1 1_{\{\beta>1\}} = u_0.$$

From the proof of Theorem **??** (see (**??**) and (**??**)), we see that

$$\begin{aligned}\|I_n(f_n(\cdot,\cdot,t,x))\|^2_{L^2(\Omega)} =& \frac{1}{n!}\frac{u_0^2 C_\kappa^n}{(2\pi)^{nd}} \int_{[0,t]^n} \mathrm{d}s \int_{\mathbb{R}^{nd}} \mathrm{d}\xi \prod_{i=1}^n (s_{\sigma(i+1)} - s_{\sigma(i)})^{2\beta-2} \\ & \times E^2_{\beta,\beta}\left(-2^{-1}\nu (s_{\sigma(i+1)} - s_{\sigma(i)})^\beta |\xi_{\sigma(i)} + \cdots + \xi_{\sigma(1)}|^\alpha\right) |\xi_i|^{\kappa-d} \\ =& \frac{u_0^2 C_\kappa^n}{(2\pi)^{nd}} \int_{T_n(t)} \mathrm{d}s \int_{\mathbb{R}^{nd}} \mathrm{d}\xi \prod_{i=1}^n (s_{i+1} - s_i)^{2\beta-2}\end{aligned}$$

$$\times E^2_{\beta,\beta}\left(-2^{-1}\nu(s_{i+1}-s_i)^\beta|\xi_i+\cdots+\xi_1|^\alpha\right)|\xi_i|^{\kappa-d}.$$

Then by the change of variable $\xi_i+\cdots\xi_1=\eta_i$ and replacing $\mathbb{R}^{nd}$ by $\mathbb{R}^{nd}_+$, we obtain that

$$\begin{aligned}
\|I_n(f_n(\cdot,\cdot,t,x))\|^2_{L^2(\Omega)} &= \frac{u_0^2C_\kappa^n}{(2\pi)^{nd}}\int_{T_n(t)}\int_{\mathbb{R}^{nd}}\prod_{i=1}^n(s_{i+1}-s_i)^{2\beta-2}\\
&\qquad\times E^2_{\beta,\beta}\left(-2^{-1}\nu(s_{i+1}-s_i)^\beta|\eta_i|^\alpha\right)|\eta_i-\eta_{i-1}|^{\kappa-d}\mathrm{d}\xi_i\mathrm{d}s_i\\
&\geq \frac{u_0^2C_\kappa^n}{(2\pi)^{nd}}\int_{T_n(t)}\int_{\mathbb{R}^{nd}_+}\prod_{i=1}^n(s_{i+1}-s_i)^{2\beta-2}\\
&\qquad\times E^2_{\beta,\beta}\left(-2^{-1}\nu(s_{i+1}-s_i)^\beta|\eta_i|^\alpha\right)|\eta_i-\eta_{i-1}|^{\kappa-d}\mathrm{d}\xi_i\mathrm{d}s_i\\
&\geq \frac{u_0^2C_\kappa^n}{(2\pi)^{nd}}\int_{T_n(t)}\int_{\mathbb{R}^{nd}_+}\prod_{i=1}^n(s_{i+1}-s_i)^{2\beta-2}\\
&\qquad\times E^2_{\beta,\beta}\left(-2^{-1}\nu(s_{i+1}-s_i)^\beta|\eta_i|^\alpha\right)|\eta_i|^{\kappa-d}\mathrm{d}\xi_i\mathrm{d}s_i\,,
\end{aligned}$$

where $\eta_0=0$. Then with another change of variable $(\nu/2)^{1/\alpha}(s_{i+1}-s_i)^{\beta/\alpha}\eta_i\to\eta_i$, and by the same reasoning as before, we obtain that

$$\begin{aligned}
\|I_n(f_n(\cdot,\cdot,t,x))\|^2_{L^2(\Omega)} \geq&\frac{u_0^2C_\kappa^n}{(2\pi)^{nd}}\int_{T_n(t)}\int_{\mathbb{R}^{nd}_+}\prod_{i=1}^n(s_{i+1}-s_i)^{2\beta-2-\frac{\beta\kappa}{\alpha}}E^2_{\beta,\beta}\left(-|\eta_i|^\alpha\right)|\eta_i|^{\kappa-d}\mathrm{d}\xi_i\mathrm{d}s_i\\
=&\frac{u_0^2C_\kappa^n}{(2\pi)^{nd}}\left(\frac{\widetilde{C}}{2^d}\right)^n(2/\nu)^{\frac{\kappa n}{\alpha}}\int_{T_n(t)}\prod_{i=1}^n(s_{i+1}-s_i)^{2\beta-2-\frac{\beta\kappa}{\alpha}}\mathrm{d}s_i\\
=&\frac{t^{n(2\beta-1-\frac{\beta\kappa}{\alpha})}(2/\nu)^{\kappa n/\alpha}u_0^2C_\kappa^n\widetilde{C}^n(4\pi)^{-nd}C_*^n}{\Gamma(n(2\beta-1-\frac{\beta\kappa}{\alpha})+1)}\,.
\end{aligned}$$

Therefore, by the asymptotic property of the Mittag-Leffler function,

$$\mathbb{E}\left[u(t,x)^2\right]\geq\sum_{n=0}^\infty\frac{u_0^2\left(C_\kappa\widetilde{C}(4\pi)^{-d}C_*\right)^n t^{n(2\beta-1-\frac{\beta\kappa}{\alpha})}(2/\nu)^{\frac{\kappa n}{\alpha}}}{\Gamma(n(2\beta-1-\frac{\beta\kappa}{\alpha})+1)}$$

$$\geq c\, u_0^2 \exp\left( (C_\kappa \widetilde{C} (4\pi)^{-d} C_* (2/\nu)^{\kappa/\alpha})^{\frac{1}{2\beta-1-\beta\kappa/\alpha}} t \right),$$

for some positive constant $c = c(\alpha, \beta, \kappa)$. This completes the proof of Theorem **??**. □

## 3.7 Proof of Theorem ??

In this section, $C = C_{\alpha,\beta,\cdots}$ denotes a positive constant, possibly dependent on $\alpha, \beta, d, \nu, \cdots$.

**Lemma 3.7.1.** *Assume that $\beta \in (0,2)$, $\alpha > 0$ and $d \in \mathbb{N}$. Then there is a nonnegative constant $C_{\alpha,\beta,d}$ such that for all $0 < \zeta < \min(d/\alpha, 2)$,*

$$\left| H_{2,3}^{2,1} \left( z \,\middle|\, \begin{matrix} (1,1),(\beta,\beta) \\ (d/2,\alpha/2),(1,1),(1,\alpha/2) \end{matrix} \right) \right| \leq C_{\alpha,\beta,d} \frac{z^\zeta}{z^{\zeta+1}+1}, \qquad \text{for all } z \geq 0.$$

*Proof.* Notice $a^* = 2 - \beta > 0$ and condition (**??**) is sastidfied. We apply Theorem **??** to obtain

$$H_{2,3}^{2,1} \left( z \,\middle|\, \begin{matrix} (1,1),(\beta,\beta) \\ (d/2,\alpha/2),(1,1),(1,\alpha/2) \end{matrix} \right) \sim \sum_{i=1}^{1} \sum_{k=0}^{\infty} h_{ik} z^{\frac{a_i-1-k}{\alpha_i}}; \quad z \to \infty.$$

By the definition of $h_{ik}$ in (**??**), it's not difficult to see that all $h_{ik} \in \mathbb{R}$ for $\beta \in (0,2), \alpha > 0$. Especially, we have

$$h_{10} = \frac{\Gamma(\frac{d}{2})\Gamma(1)}{\Gamma(\beta)\Gamma(0)} = 0.$$

Therefore

$$\sum_{k=0}^{\infty} h_{ik} z^{\frac{a_i-1-k}{\alpha_i}} = h_{10} z^{\frac{a_1-1}{\alpha_1}} + h_{11} z^{\frac{a_1-1-1}{\alpha_1}} + o(z^{\frac{a_1-1-1}{\alpha_1}}) \leq C_{\alpha,\beta,d} \frac{1}{z}; \qquad z \to \infty.$$

When $(1+M)\alpha \neq d+2K$ for all $M,K=0,1,2,\ldots$, so condition (**??**) is satisfied. Theorem **??** case (1) implies that

$$H^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix}(1,1),(\beta,\beta)\\(d/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right) \sim \sum_{j=1}^{2}\sum_{\ell=0}^{\infty} h^*_{jl} z^{(b_j+\ell)/\beta_j}; \qquad z\to 0.$$

Our assumption $(1+M)\alpha \neq d+2K$ guarantees $\prod_{i=1,i\neq j}^{m}\Gamma\left(b_i-[b_j+l]\frac{\beta_i}{\beta_j}\right)\in\mathbb{R}$. With this observation and the definition of $h^*_{jl}$ in (**??**), it's not difficult to check that $h^*_{jl}\in\mathbb{R}$ for $\beta\in(0,2),\alpha>0$. Especially we have

$$h^*_{20}=\frac{\Gamma(\frac{d}{2}-\frac{d}{\alpha})\Gamma(1)}{\Gamma(0)\Gamma(\frac{\alpha}{2})}=0.$$

Therefore we have

$$\begin{aligned}\sum_{j=1}^{2}\sum_{\ell=0}^{\infty} h^*_{jl} z^{(b_j+\ell)/\beta_j} =& h^*_{10}z^{b_1/\beta_1}+o(z^{b_1/\beta_1})+h^*_{20}z^{b_2/\beta_2}+h^*_{21}z^{(b_2+1)/\beta_2}+o(z^{(b_2+1)/\beta_2})\\ \leq& C_{\alpha,\beta,d}z^{\frac{d}{\alpha}}+C_{\alpha,\beta,d}z^2; \qquad z\to 0.\end{aligned}$$

When $(1+M_J)\alpha = d+2K_J$ for $M_J,K_J\in\{0,1,2,\ldots\},\quad J\in\{1,2,\cdots\}$, we can apply Thereom **??**.

For the first summation We just proved

$$\sum\nolimits'_{j,l} h^*_{jl} z^{\frac{b_j+l}{\beta_j}} \leq C_{\alpha,\beta,d}z^{\frac{d}{\alpha}}+C_{\alpha,\beta,d}z^2; \qquad z\to 0.$$

For the second summation

$$\sum\nolimits''_{j,l}\sum_{i=0}^{N^*_{jl}-1} H^*_{jli} z^{\frac{b_j+l}{\beta_j}}[\log z]^i,$$

we first need to check $H^*_{jli} \in \mathbb{R}$ for $\beta \in (0,2), \alpha > 0$ but it's routine using by its definition (**??**).

Furthermore

$$\begin{aligned}\sum\nolimits''_{j,l} \sum_{i=0}^{N^*_{jl}-1} H^*_{jli} z^{\frac{b_j+l}{\beta_j}} [\log z]^i =& \sum\nolimits''_{j=1,l=K_J} \sum_{i=0}^{N^*_{jl}-1} H^*_{jli} z^{\frac{b_j+l}{\beta_j}} [\log z]^i \\ =& H^*_{1K_1 0} z^{\frac{b_1+K_1}{\beta_1}} + H^*_{1K_1 1} z^{\frac{b_1+K_1}{\beta_1}} \log z \\ &+ H^*_{1K_2 0} z^{\frac{b_1+K_2}{\beta_1}} + H^*_{1K_2 1} z^{\frac{b_1+K_2}{\beta_1}} \log z + \cdots \\ \le& C_{\alpha,\beta,d} z^{\frac{d}{\alpha}} + C_{\alpha,\beta,d} z^{\frac{d}{\alpha}} |\log z|; \quad z \to 0.\end{aligned}$$

In sum, we have shown

$$H^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix}(1,1),(\beta,\beta) \\ (d/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right) \le \begin{cases} C_{\alpha,\beta,d}(z^{\frac{d}{\alpha}} + z^{\frac{d}{\alpha}} |\log z| + z^2); & z \to 0; \\ C_{\alpha,\beta,d} \frac{1}{z}; & z \to \infty. \end{cases}$$

Lastly, the conditions of Theorem **??** are satisfied, so $H^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix}(1,1),(\beta,\beta) \\ (d/2,\alpha/2),(1,1),(1,\alpha/2)\end{matrix}\right)$ is continuous when $z > 0$. This completes the proof of Lemma **??**. □

**Lemma 3.7.2.** *For all $\alpha \in (0,2]$, $d \in \mathbb{N}$ and $\kappa < \min\{2\alpha, d\}$, one can find $\zeta < \min(d/\alpha, 2)$ and a nonnegative constant $C$ (independent of $a$) such that*

$$\int_{\mathbb{R}^d} |x-a|^{-\kappa} \Theta(x) dx \le C < \infty \qquad \textit{for all } a \in \mathbb{R}^d,$$

*where*

$$\Theta(x) = \frac{1}{|x|^{\alpha+d} + |x|^{d-\zeta\alpha}}.$$

*Proof.* We divide the integral domain into $\{|x| \le 1\}$ and $\{|x| > 1\}$. Over the domain $\{|x| \le 1\}$, we have

$$\begin{aligned}
\int_{|x|\le 1} |x-a|^{-\kappa}\Theta(x)\mathrm{d}x &\le \int_{|x|\le 1} |x-a|^{-\kappa}\frac{1}{|x|^{d-\zeta\alpha}}\mathrm{d}x \\
&= \int_{|x|\le 1,\,|x|\le|x-a|} |x-a|^{-\kappa}\frac{1}{|x|^{d-\zeta\alpha}}\mathrm{d}x + \int_{|x-a|<|x|\le 1} |x-a|^{-\kappa}\frac{1}{|x|^{d-\zeta\alpha}}\mathrm{d}x \\
&\le \int_{|x|\le 1,\,|x|\le|x-a|} |x|^{-\kappa}\frac{1}{|x|^{d-\zeta\alpha}}\mathrm{d}x + \int_{|x-a|<|x|\le 1} |x-a|^{-\kappa}\frac{1}{|x-a|^{d-\zeta\alpha}}\mathrm{d}x \\
&\le 2\int_{|z|\le 1} \frac{1}{|z|^{\kappa+d-\zeta\alpha}}\mathrm{d}z \le C.
\end{aligned}$$

The last inequality is valid since we can choose $\zeta$ sufficiently close to $\min(d/\alpha, 2)$ so that $\kappa + d - \zeta\alpha < d$. On the other hand, over the domain $\{|x| > 1\}$, we have

$$\begin{aligned}
\int_{|x|>1} |x-a|^{-\kappa}\Theta(x)\mathrm{d}x &\le \int_{|x|>1} |x-a|^{-\kappa}\frac{1}{|x|^{\alpha+d}}\mathrm{d}x \\
&\le \int_{|x-a|\ge|x|>1} |x-a|^{-\kappa}\frac{1}{|x|^{\alpha+d}}\mathrm{d}x + \int_{|x|>|x-a|>1} |x-a|^{-\kappa}\frac{1}{|x|^{\alpha+d}}\mathrm{d}x \\
&\quad + \int_{|x|>1\ge|x-a|} |x-a|^{-\kappa}\frac{1}{|x|^{\alpha+d}}\mathrm{d}x \\
&\le 2\int_{|z|>1} \frac{1}{|z|^{\alpha+d}}\mathrm{d}z + \int_{|z|\le 1} |z|^{-\kappa}\mathrm{d}z \le C.
\end{aligned}$$

Note that the above constant $C$ does not depend on $a$. □

**Lemma 3.7.3.** *Assume $\kappa < \min\{2\alpha, d\}$. Then for all $s, r > 0$ and $x_2, y_2 \in \mathbb{R}^d$, we have that*

$$\int_{\mathbb{R}^{2d}} |Y(s, x_1 - x_2)Y(r, y_1 - y_2)|\, |x_1 - y_1|^{-\kappa}\mathrm{d}x_1\mathrm{d}y_1 \le C_{\alpha,\beta,d,\nu,\kappa}\,(s\,r)^{\theta},$$

*where $C$ does not depend on $x_2$ and $y_2 \in \mathbb{R}^d$, and*

$$\theta := \beta - 1 - \frac{\beta}{2\alpha}\kappa.$$

*Proof.* We use the notation $\Theta(x)$ in Lemma **??**. By Lemma **??** and the expression of $Y$ through Fox H-function (**??**), we see that for any $\zeta < \min(d/\alpha, 2)$, there is a constant $C_{\alpha,\beta,d,\nu,\zeta}$ such that

$$\begin{aligned} |Y(t,x)| &\leq C_{\alpha,\beta,d,\nu,\zeta}|x|^{-d}t^{\beta-1}\frac{|\frac{x}{t^{\beta/\alpha}}|^{\alpha\zeta}}{|\frac{x}{t^{\beta/\alpha}}|^{\alpha\zeta+\alpha}+1} \\ &= C_{\alpha,\beta,d,\nu,\zeta}t^{\beta-1-\frac{\beta d}{\alpha}}\Theta\left(\frac{x}{t^{\beta/\alpha}}\right). \end{aligned}$$

Therefore, by Lemma **??**, we have

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} |Y(s,x_1-x_2)Y(r,y_1-y_2)|\ |x_1-y_1|^{-\kappa}\mathrm{d}x_1\mathrm{d}y_1 \\ &\leq C_{\alpha,\beta,d,\nu,\zeta}(sr)^{\beta-1-\frac{\beta d}{\alpha}}\int_{\mathbb{R}^{2d}}\Theta\left(\frac{x_1-x_2}{s^{\beta/\alpha}}\right)\Theta\left(\frac{y_1-y_2}{r^{\beta/\alpha}}\right)|x_1-y_1|^{-\kappa}\mathrm{d}x_1\mathrm{d}y_1 \\ &\leq C_{\alpha,\beta,d,\nu,\zeta}r^{\beta-1-\frac{\beta d}{\alpha}}s^{\beta-1-\kappa\beta/\alpha}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|z_1-\frac{y_1-x_2}{s^{\beta/\alpha}}\right|^{-\kappa}\Theta(z_1)\mathrm{d}z_1\right)\Theta\left(\frac{y_1-y_2}{r^{\beta/\alpha}}\right)\mathrm{d}y_1 \\ &\leq C_{\alpha,\beta,d,\nu,\zeta}r^{\beta-1-\frac{\beta d}{\alpha}}s^{\beta-1-\kappa\beta/\alpha}\int_{\mathbb{R}^d}\Theta\left(\frac{y_1-y_2}{r^{\beta/\alpha}}\right)\mathrm{d}y_1 \\ &\leq C_{\alpha,\beta,d,\nu,\zeta}r^{\beta-1}s^{\beta-1-\kappa\beta/\alpha}. \end{aligned}$$

By symmetry, we also have

$$\iint_{\mathbb{R}^{2d}} |Y(s,x_1-x_2)Y(r,y_1-y_2)|\ |x_1-y_1|^{-\kappa}\mathrm{d}x_1\mathrm{d}y_1 \leq C_{\alpha,\beta,d,\nu,\zeta}s^{\beta-1}r^{\beta-1-\kappa\beta/\alpha}.$$

Now from the fact that $c \leq a$ and $c \leq b$ implies $c \leq \sqrt{ab}$, the lemma follows. □

The following lemma is from [**?**, Theorem 3.5].

**Lemma 3.7.4.** *Let $T_n$ be the simplex defined in* (**??**). *Then for all $h > -1$, it holds that*

$$\int_{T_n(t)} [(t-s_n)(s_n-s_{n-1})\dots(s_2-s_1)]^h \mathrm{d}s = \frac{\Gamma(1+h)^n}{\Gamma(n(1+h)+1)}t^{n(1+h)}.$$

*Proof of Theorem* **??**. Following the same notation and arguments as the proof of Theorem **??** until (**??**), we have

$$\begin{aligned} & n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \\ & \quad \leq C\frac{1}{n!}\int_{[0,t]^{2n}} \mathrm{d}s\mathrm{d}r \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(r,z,t,x) \prod_{i=1}^n \Lambda(y_i - z_i) \prod_{i=1}^n \lambda(s_i - r_i). \end{aligned}$$

Furthermore, by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(r,z,t,x) \prod_{i=1}^n \Lambda(y_i - z_i) \\ & \qquad \leq \quad \left\{ \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(s,z,t,x) \prod_{i=1}^n \Lambda(y_i - z_i) \right\}^{1/2} \\ & \qquad\quad \times \left\{ \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(r,y,t,x) g_n(r,z,t,x) \prod_{i=1}^n \Lambda(y_i - z_i) \right\}^{1/2} \end{aligned}$$

Applying Lemma **??** to the above two integrals, we have

$$\int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(r,z,t,x) \prod_{i=1}^n \Lambda(y_i - z_i) \leq C^n_{\alpha,\beta,d,\nu,\kappa} (\phi(s)\phi(r))^{\theta},$$

where

$$\phi(s) := \prod_{i=1}^n (s_{\sigma(i+1)} - s_{\sigma(i)}) \qquad \text{and} \qquad \phi(r) := \prod_{i=1}^n (r_{\rho(i+1)} - r_{\rho(i)}),$$

with

$$0 < s_{\sigma(1)} < s_{\sigma(2)} < \ldots < s_{\sigma(n)} \quad \text{and} \quad 0 < r_{\rho(1)} < r_{\rho(2)} < \ldots < r_{\rho(n)}.$$

Hence,

$$
\begin{aligned}
n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} &\leq \frac{C^n_{\alpha,\beta,d,\nu,\kappa}}{n!}\int_{[0,t]^{2n}}\prod_{i=1}^n \lambda(s_i-r_i)(\phi(s)\phi(r))^{\theta}\mathrm{d}s\mathrm{d}r \\
&\leq \frac{C^n_{\alpha,\beta,d,\nu,\kappa}}{n!}\frac{1}{2}\int_{[0,t]^{2n}}\prod_{i=1}^n \lambda(s_i-r_i)\left(\phi(s)^{2\theta}+\phi(r)^{2\theta}\right)\mathrm{d}s\mathrm{d}r \\
&= \frac{C^n_{\alpha,\beta,d,\nu,\kappa}}{n!}\int_{[0,t]^{2n}}\prod_{i=1}^n \lambda(s_i-r_i)\phi(s)^{2\theta}\mathrm{d}s\mathrm{d}r \\
&\leq \frac{C^n_{\alpha,\beta,d,\nu,\kappa}C^n_t}{n!}\int_{[0,t]^n}\phi(s)^{2\theta}\mathrm{d}s \\
&= C^n_{\alpha,\beta,d,\nu,\kappa}C^n_t\int_{T_n(t)}\phi(s)^{2\theta}\mathrm{d}s \\
&= \frac{C^n_{\alpha,\beta,d,\nu,\kappa}C^n_t\Gamma(2\theta+1)^n t^{(2\theta+1)n}}{\Gamma((2\theta+1)n+1)},
\end{aligned}
$$

where $C_t$ is defined in (**??**). Therefore,

$$
n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \leq \frac{C^n_{\alpha,\beta,d,\nu,\kappa}C^n_t}{\Gamma((2\theta+1)n+1)},
$$

and $\sum_{n\geq 0} n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}}$ converges if $\theta > -1/2$. Finally, the condition $\theta > -1/2$, which is equivalent to $\kappa < 2\alpha - \alpha/\beta$, guarantees both condition $\theta > -1$ in Lemma **??** and the assumption $\kappa < 2\alpha$ used in Lemma **??**. This completes the proof of Theorem **??**. □

# Chapter 4

# Stochastic time-fractional diffusion equations with variable coefficients and time independent noise

Here is the organization of the chapter, Section 2 will describe the operator $B$ and the noise $W^H$ and state the main result of the chapter. In our proof we need to use the properties of the two fundamental solutions (Green's functions) $Z(t,x,\xi)$ and $Y(t,x,\xi)$ associated with the equation $\partial^\alpha u(t,x) = Bu(t,x)$, which is represented by the Fox's $H$-function . we will recall some most relevant results on the $H$-function and the Green's function $Z(t,x,\xi)$ and $Y(t,x,\xi)$ in Section 3. A number of preparatory lemmas are needed to prove main results and they are presented in Section 4. Finally, the last section is devoted to the proof of our main theorem.

## 4.1 Main result

Let

$$B = \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

be a uniformly elliptic second-order differential operator with bounded continuous real-valued coefficients. Let $u_0$ be a given bounded continuous function (locally Hölder continuous if $d > 1$). Let $\{W^H(x), x \in \mathbb{R}^d\}$ be a time homogeneous (time-independent) fractional Brownian field on some probability space $(\Omega, \mathscr{F}, P)$ (Like elsewhere in probability theory, we omit the dependence of $W^H(x) = W^H(x, \omega)$ on $\omega \in \Omega$). Namely, the stochastic process $\{W^H(x), x \in \mathbb{R}^d\}$ is a (multi-parameter) Gaussian process with mean 0 and its covariance is given by

$$\mathbb{E}\left(W^H(x)W^H(y)\right) = \prod_{i=1}^{d} R_{H_i}(x_i, y_i), \tag{4.1.1}$$

where $H_1, \cdots, H_d$ are some real numbers in the interval $(0,1)$. Due to some technical difficulty, we assume that $H_i > 1/2$ for all $i = 1, 2, \cdots, d$; the symbol $\mathbb{E}$ denotes the expectation on $(\Omega, \mathscr{F}, P)$ and

$$R_{H_i}(x_i, y_i) = \frac{1}{2}\left(|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}\right), \quad \forall\, x_i, y_i \in \mathbb{R}$$

is the covariance function of a fractional Brownian motion of Hurst parameter $H_i$.

Throughout this chapter we fix an arbitrary parameter $\alpha \in (0,1)$ and a finite time horizon $T \in (0, \infty)$. We study the following stochastic partial differential equation of fractional order:

$$\begin{cases} \partial^\alpha u(t,x) = Bu(t,x) + u(t,x) \cdot \dot{W}^H(x), & t \in (0,T], \quad x \in \mathbb{R}^d; \\ u(0,x) = u_0(x), \end{cases} \tag{4.1.2}$$

where

$$\partial^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha} u(\tau,x)d\tau - t^{-\alpha}u(0,x)\right]$$

is the Caputo fractional derivative (see e.g. [**?**]) and $\dot{W}^H(x) = \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W^H(x)$ is the distributional derivative (generalized derivative) of $W^H$, called fractional Brownian noise.

Our objective is to obtain condition on $\alpha$ and $H$ such that the above equation has a unique solution. But since $W^H$ is not differentiable or since $\dot{W}^H(x)$ does not exist as an ordinary function, we have to describe under what sense a random field $\left\{u(t,x)\,, t \geq 0\,, x \in \mathbb{R}^d\right\}$ is a solution to the above equation (**??**).

To motivate our definition of the solution, let us consider the following (deterministic) partial differential equation of fractional order with the term $u(t,x) \cdot \dot{W}^H(x)$ in (**??**) replaced by $f(t,x)$:

$$\begin{cases} \partial^\alpha \tilde{u}(t,x) = B\tilde{u}(t,x) + f(t,x), \qquad t \in (0,T], \quad x \in \mathbb{R}^d\,; \\ \tilde{u}(0,x) = u_0(x)\,. \end{cases} \tag{4.1.3}$$

Here $u_0(x)$ is bounded continuous function(locally hölder continuous if $d > 1$). The function $f$ is bounded and jointly continuous in $(t,x)$ and locally Hölder continuous in x. In [**?**], it is proved that there are two Green's functions:

$$\left\{Z(t,x,\xi)\,,\ Y(t,x,\xi)\,, 0 < t \leq T\,, x, \xi \in \mathbb{R}^d\right\},$$

such that the solution to the Cauchy problem (**??**) is given by

$$\tilde{u}(t,x) = \int_{\mathbb{R}^d} Z(t,x,\xi) u_0(\xi) d\xi + \int_0^t ds \int_{\mathbb{R}^d} Y(t-s,x,y) f(s,y) dy. \tag{4.1.4}$$

In general, there is no explicit form for the two Green's functions $\{Z(t,x,\xi)\,,\ Y(t,x,\xi)\}$. However, their constructions and properties are known (see [**?**], [**?**], [**?**], and the references therein). we will recall some needed results in the next section.

From the classical solution expression (**??**), we expect that the solution $u(t,x)$ to (**??**) satisfies formally

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x,\xi)u_0(\xi)d\xi + \int_0^t ds \int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)\dot{W}^H(y)dy\,.$$

The above formal integral $\int_0^t ds \int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)\dot{W}^H(y)dy$ can be defined by Itô-Skorohod stochastic integral $\int_{\mathbb{R}^d}\left[\int_0^t Y(t-s,x,y)u(s,y)ds\right]W^H(dy)$ as given in [**?**].

Now, we can give the following definition.

**Definition 4.1.1.** A random field $\left\{u(t,x)\,, 0 \le t \le T\,, x \in \mathbb{R}^d\right\}$ is called a mild solution to the equation (**??**) if

(1) $u(t,x)$ is jointly measurable in $t \in [0,T]$ and $x \in \mathbb{R}^d$;

(2) $\forall (t,x) \in [0,T]\times\mathbb{R}^d$, $\int_0^t \int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)dsW^H(dy)$ is well defined in $\mathscr{L}^2$;

(3) The following holds in $\mathscr{L}^2$

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x,\xi)u_0(\xi)d\xi + \int_0^t \int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)W^H(dy)ds. \quad (4.1.5)$$

Let us return to the discussion of the two Green's functions $\{Z(t,x,\xi)\,,\, Y(t,x,\xi)\}$. If $\alpha = 1$, namely, if $\partial^\alpha$ in (**??**) is replaced by $\partial_t$ and $B = \Delta := \sum_{i=1}^d \partial_{x_i}^2$, then

$$Z(t,x,\xi) = Y(t,x,\xi) = (4\pi t)^{-d/2}\exp\left\{-\frac{|x-\xi|^2}{4t}\right\}. \quad (4.1.6)$$

In this case the stochastic partial differential equation of the form

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u\cdot\dot{W}^H(x), \qquad x \in \mathbb{R}^d, \quad (4.1.7)$$

was studied in [**?**]. The mild solution to the above equation (**??**) is proved to exist uniquely under conditions

$$H_i > 1/2\,, \quad i = 1, \cdots, d \qquad \text{and} \qquad \sum_{i=1}^{d} H_i > d - 1\,. \tag{4.1.8}$$

The main result of this chapter is to extend the above result in [**?**] to our equation (**??**).

**Theorem 4.1.2.** *Let the coefficients $a_{ij}(x)$, $b_i(x)\,, i, j = 1, \cdots, d\,,$ be bounded and continuous and let them be Hölder continuous with exponent $\gamma$. Let $a_{ij}(x)$ be uniformly elliptic. Namely, there is a constant $a_0 \in (0, \infty)$ such that*

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \forall\ \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d\,.$$

*Let $u_0$ be a bounded continuous (and locally Hölder continuous if $d > 1$). Assume*

$$H_i > \begin{cases} \frac{1}{2} & \text{if } d = 1, 2, 3, 4 \\ 1 - \frac{2}{d} - \frac{\gamma}{2d} & \text{if } d \geq 5 \end{cases} \tag{4.1.9}$$

*and*

$$\sum_{i=1}^{d} H_i > d - 2 + \frac{1}{\alpha}\,. \tag{4.1.10}$$

*Then, the mild solution to (**??**) exists uniquely in $L^2(\Omega, \mathscr{F}, P)$.*

**Remark 4.1.3.** (i) When $\alpha$ is formally set to 1, the condition (**??**) is the same as the condition (**??**) given in [**?**]. So, in some sense our condition (**??**) is optimal.

(ii) Since $H_i < 1$ for all $i = 1, 2, \cdots, d$ the condition is possible only when $\alpha > 1/2$.

## 4.2 Green's functions $Z$ and $Y$

**Example 4.2.1.** To compare with the classical case $\alpha = 1$, we consider the case $m = 2$, $n = 0$, $p = 1$, $q = 2$, $a_1 = \alpha_1 = b_2 = \beta_1 = \beta_2 = 1$ and $b_1 = \frac{d}{2}$. Let $L = L_{-\infty}$. Then, we have

$$\begin{aligned}
H_{12}^{20}\left[z\middle|\begin{matrix} (1,1) & \\ (\frac{d}{2},1), & (1,1)\end{matrix}\right] &= \frac{1}{2\pi i}\int_L \frac{\Gamma(\frac{d}{2}+s)\Gamma(1+s)}{\Gamma(1+s)} z^{-s} ds \\
&= \frac{1}{2\pi i}\int_L \Gamma(\frac{d}{2}+s) z^{-s} ds \\
&= \sum_{v=0}^{\infty} \lim_{s\to-(\frac{d}{2}+v)} (s+\frac{d}{2}+v)\Gamma(\frac{d}{2}+s) z^{-s} \\
&= \sum_{v=0}^{\infty} \lim_{s\to-(\frac{d}{2}+v)} \frac{\Gamma(v+\frac{d}{2}+s+1)}{(s+\frac{d}{2}+v-1)\cdots(s+\frac{d}{2})} z^{-s} \\
&= \sum_{v=0}^{\infty} z^{d/2}(-1)^v \frac{1}{v!} z^v \\
&= z^{d/2}\exp(-z). \qquad (4.2.1)
\end{aligned}$$

### 4.2.1 Green's functions $Z$ and $Y$ when $B$ has constant coefficients

In this subsection let us consider $Z$ and $Y$ when the operator $B$ in (**??**) has the following form

$$B = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where the matrix $A = (a_{ij})$ is positive definite. In this case, $Z$ and $Y$ (we call them $Z_0$ and $Y_0$ to distinguish with the general coefficient case) are given as follows.

$$Z_0(t,x) = \frac{\pi^{-d/2}}{(\det A)^{1/2}} \left[\sum_{i,j=1}^{d} A^{(ij)} x_i x_j\right]^{-d/2}$$

$$\times H_{12}^{20}\left[\frac{1}{4}t^{-\alpha}\sum_{i,j=1}^{d}A^{(ij)}x_ix_j\middle|\begin{array}{ll}(1,\alpha) & \\ (\frac{d}{2},1), & (1,1)\end{array}\right],$$

where $(A^{(ij)}) = A^{-1}$ and

$$Y_0(t,x) = \frac{\pi^{-d/2}}{(\det A)^{1/2}}\left[\sum_{i,j=1}^{d}A^{(ij)}x_ix_j\right]^{-d/2}t^{\alpha-1}$$

$$\times H_{12}^{20}\left[\frac{1}{4}t^{-\alpha}\sum_{i,j=1}^{d}A^{(ij)}x_ix_j\middle|\begin{array}{ll}(\alpha,\alpha) & \\ (\frac{d}{2},1), & (1,1)\end{array}\right].$$

It is easy to see that for the constant coefficients, both of the Green's functions are homogeneous in time and space. Namely,

$$Z_0(t,x,\xi) = Z_0(t,x-\xi)\,, \quad Y_0(t,x,\xi) = Y_0(t,x-\xi)\,.$$

In particular, when $\alpha = 1$, it is easy to see from the above expression and the explicit form (**??**) of $H_{12}^{20}(z)$ that

$$Z_0(t,x,\xi) = Y_0(t,x,\xi) = (4\pi)^{-d/2}\det(A)^{-1/2}\exp\left\{-\frac{\sum_{i,j=1}^{d}A^{(ij)}(x_i-\xi_i)(x_j-\xi_j)}{4t}\right\}.$$

which reduces to (**??**) when $A = I$ is the identity matrix.

With the above expression for $Z_0$ and $Y_0$ and the properties of the $H$-function, one can obtain the following estimates.

**Proposition 4.2.2.** *Denote*

$$p(t,x) = \exp\left(-\sigma t^{-\frac{\alpha}{2-\alpha}}|x|^{\frac{2}{2-\alpha}}\right), \quad t > 0\,,\ x \in \mathbb{R}^d\,, \qquad (4.2.2)$$

*where $\sigma \in (0,\infty)$ is generic positic constant whose exact value may vary at occurrence. Then, we have the following estimates:*

$$|Z_0(t,x)| \le \begin{cases} Ct^{-\frac{\alpha}{2}} p(t,x) & \textit{when} \quad d=1 \\ Ct^{-\alpha}[|\log \frac{|x|^2}{t^\alpha}|+1] p(t,x) & \textit{when} \quad d=2 \\ Ct^{-\alpha}|x|^{2-d} p(t,x) & \textit{when} \quad d \ge 3\,, \end{cases} \tag{4.2.3}$$

*where for instance, $|Z_0(t,x)| \le Ct^{-\frac{\alpha}{2}} p(t,x)$ means that there are positive constant $C$ and positive constant $\sigma$ such that the above inequality holds. In what follows the positive constants $C$ and $\sigma$ are generic, which may be different in different occurrences.*

*Proof.* Denote $R = |x|^2/t^\alpha$. From [**?**], Proposition 1, it follows that when $R \le 1$, we have

$$|Z_0(t,x)| \le \begin{cases} Ct^{-\frac{\alpha}{2}} & \text{when} \quad d=1 \\ Ct^{-\alpha}[|\log \frac{|x|^2}{t^\alpha}|+1] & \text{when} \quad d=2 \\ Ct^{-\alpha}|x|^{2-d} & \text{when} \quad d \ge 3\,, \end{cases}$$

Since when $R \le 1$, $p(t,x)$ is bounded from below. This proves the inequality (**??**) when $R \le 1$.

When $R > 1$, then by [**?**], Proposition 1 we have $|Z_0(t,x)| \le Ct^{-\frac{\alpha d}{2}} p(t,x)$. It is clear that this implies the inequality (**??**) when $d = 1$ and $d = 2$. Now, we assume that $d \ge 3$. We have

$$\begin{aligned} |Z_0(t,x)| &\le Ct^{-\frac{\alpha d}{2}} p(t,x) \le Ct^{-\alpha}|x|^{2-d} \left(\frac{|x|^2}{t^\alpha}\right)^{\frac{d}{2}-1} p(t,x) \\ &\le Ct^{-\alpha}|x|^{2-d} p(t,x)\,, \end{aligned}$$

where we used the fact that $\left(\frac{|x|^2}{t^\alpha}\right)^{\frac{d}{2}-1} p(t,x) \le p(t,x)$ for a different $\sigma$ in the later $p(t,x)$. $\square$

Similarly, we can use [**?**], Proposition 2 (for $d=1$ case) and [**?**], Section 4.2 (for $d \ge 2$ case) to obtain the following estimates for $Y_0(t,x)$.

**Proposition 4.2.3.** *We follow the same notation $p(t,x)$ as defined by (**??**). We have*

*(i) When $d=1$, we have the following estimates:*

$$|Y_0(t,x)| \le \begin{cases} Ct^{\frac{\alpha}{2}-1} p(t,x) & when \quad t^{-\alpha}|x|^2 \ge 1 \\ Ct^{\frac{\alpha}{2}-1} & when \quad t^{-\alpha}|x|^2 \le 1. \end{cases} \tag{4.2.4}$$

*(ii) When $d \ge 2$, we have the following estimates:*

$$|Y_0(t,x)| \le \begin{cases} Ct^{-1} p(t,x) & when \quad d=2 \\ Ct^{-\frac{\alpha}{2}-1} p(t,x) & when \quad d=3 \\ Ct^{-\alpha-1}[|\log \frac{|x|^2}{t^\alpha}|+1] p(t,x) & when \quad d=4 \\ Ct^{-\alpha-1}|x|^{4-d} p(t,x) & when \quad d \ge 5\,, \end{cases} \tag{4.2.5}$$

*where for instance, $|Y_0(t,x)| \le Ct^{-1}p(t,x)$ means that there are positive constant $C$ and positive constant $\sigma$ such that the above inequality holds. In what follows the positive constants $C$ and $\sigma$ are generic, which may be different in different occurrences. when*

When the constants $a_{ij}$ are dependent on $\xi$, we use

$$Z_0(t,x,\xi) = Z_0(t,x-\xi,\xi)\,, \quad Y_0(t,x,\xi) = Y_0(t,x-\xi,\xi)\,.$$

to denote the Green's functions. The above estimations are still valid and the constants in the estimations are independent of $\xi$.

### 4.2.2 Green's functions $Z$ and $Y$ in general coefficient case

If the coefficients of $B$ are not constant, then the Green's functions $Z$ and $Y$ are more complicated and may be obtained by a method similar to the Levi parametrix for the parabolic equations.

Denote

$$\begin{aligned}
M(t,x,\xi) &= \sum_{i,j=1}^{d}[a_{ij}(x)-a_{ij}(\xi)]\frac{\partial^2}{\partial x_i \partial x_j}Z_0(t,x-\xi,\xi) \\
&\quad + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}Z_0(t,x-\xi,\xi)+c(x)Z_0(t,x-\xi,\xi) \\
K(t,x,\xi) &= \sum_{i,j=1}^{d}[a_{ij}(x)-a_{ij}(\xi)]\frac{\partial^2}{\partial x_i \partial x_j}Y_0(t,x-\xi,\xi) \\
&\quad + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}Y_0(t,x-\xi,\xi)+c(x)Y_0(t,x-\xi,\xi)\,.
\end{aligned}$$

Let $Q(s,y,\xi)$ and $\Phi(s,y,\xi)$ be defined by

$$Q(t,x,\xi) = M(t,x,\xi) + \int_0^t ds \int_{\mathbb{R}^d} K(t-s,x,y)Q(s,y,\xi)dy,$$

$$\Phi(t,x,\xi) = K(t,x,\xi) + \int_0^t ds \int_{\mathbb{R}^d} K(t-s,x,y)\Phi(s,y,\xi)dy$$

The following proposition is prove in [**?**], see section 2.2, Theorem.

**Proposition 4.2.4.** *Let the coefficients $a_{ij}(x)$ and $b_i(x)$ satisfy the conditions in Theorem* **??**. *Recall that $\gamma$ is the Hölder exponent of the coefficients with respect to the spatial* *variable x. Then, the Green's functions* $\{Z(t,x,\xi),Y(t,x,\xi)\}$ *have the following form:*

$$Z(t,x,\xi) = Z_0(t,x-\xi,\xi) + V_Z(t,x,\xi);$$

$$Y(t,x,\xi) = Y_0(t,x-\xi,\xi) + V_Y(t,x,\xi), \tag{4.2.6}$$

*where*

$$V_Z(t,x,\xi) = \int_0^t ds \int_{\mathbb{R}^d} Y_0(t-s,x,y)Q(s,y,\xi)dy;$$
$$V_Y(t,x,\xi) = \int_0^t ds \int_{\mathbb{R}^d} Y_0(t-s,x,y)\Phi(s,y,\xi)dy.$$

*Moreover, the function* $V_Z(t,x,\xi), V_Y(t,x,\xi)$ *satisfy the following estimates.*

$$|V_Z(t,x,\xi)| \le \begin{cases} Ct^{(\gamma-1)\frac{\alpha}{2}} p(t,x-\xi), & when \;\; d=1; \\ Ct^{\frac{\gamma\alpha}{2}-\alpha} p(t,x-\xi), & when \;\; d=2; \\ Ct^{\frac{\gamma_0\alpha}{2}-\alpha}|x-\xi|^{2-d+\gamma-\gamma_0} p(t,x-\xi), & when \;\; d=3 \text{ or } d\ge 5; \\ Ct^{(\gamma-\gamma_0)\frac{\alpha}{2}-\alpha}|x-\xi|^{-2+\gamma-2\gamma_0} p(t,x-\xi), & when \;\; d=4 \end{cases} \tag{4.2.7}$$

*and*

$$|V_Y(t,x,\xi)| \le \begin{cases} Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}} p(t,x-\xi), & when \;\; d=1; \\ Ct^{\frac{\gamma\alpha}{2}-1} p(t,x-\xi), & when \;\; d=2; \\ Ct^{(\gamma_0+\gamma)\frac{\alpha}{4}-1}|x-\xi|^{2-d+(\gamma-\gamma_0)/2} p(t,x-\xi), & when \;\; d=3 \text{ or } d\ge 5; \\ Ct^{(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{-2+\gamma-2\gamma_0} p(t,x-\xi), & when \;\; d=4 \end{cases} \tag{4.2.8}$$

*Here* $\gamma_0$ *is any number such that* $0<\gamma_0<\gamma$ *and in the case* $d\ge 3$*, the constant* $C$ *depends on* $\gamma_0$.

## 4.3 Auxiliary lemmas

To prove our main theorem, we need to dominate certain multiple integral involving $Y(t,x,\xi)$ and $Z(t,x,\xi)$. Since both $Y(t,x,\xi)$ and $Z(t,x,\xi)$ are complicated, we will first bounded them by $p(t,x-\xi)$ from the estimations of $|Y_0(t,x,\xi)|$ and $|V_Y(t,x,\xi)|$. More precisely, we have the following bounds for $Y(t,x,\xi)$.

**Lemma 4.3.1.** *Let* $x\in\mathbb{R}^d, t\in(0,T]$. *Then*

$$|Y(t,x,\xi)|\le\begin{cases}Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi), & d=1;\\ Ct^{-1}p(t,x-\xi), & d=2;\\ Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1}|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi), & d=4;\\ Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{2-d+(\gamma-\gamma_0)/2}p(t,x-\xi), & d=3\ \text{ or } d\ge 5.\end{cases} \tag{4.3.1}$$

*Proof.* we will prove the lemma case by case. First, when $d=1$, by Proposition **??**, we have

$$|Y_0(t,x-\xi,\xi)| \le \begin{cases}Ct^{\frac{\alpha}{2}-1}p(t,x-\xi), & t^{-\alpha}|x-\xi|^2\ge 1;\\ Ct^{\frac{\alpha}{2}-1}, & t^{-\alpha}|x-\xi|^2\le 1.\end{cases}$$

If $t^{-\alpha}|x-\xi|^2\le 1$, then

$$|Y_0(t,x-\xi,\xi)|\le Ct^{-1+\frac{\alpha}{2}}\cdot\frac{p(x,t)}{e^{-\sigma}}\le Ct^{\frac{\alpha}{2}-1}p(t,x-\xi).$$

Therefore

$$\begin{aligned}|Y(t,x,\xi)| &\le |Y_0(t,x-\xi,\xi)|+|V_Y(t,x,\xi)|\\ &\le Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}}p(t,x-\xi)+Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi)\end{aligned}$$

$$\leq \; Ct^{-1+\frac{\alpha}{2}} p(t, x-\xi)\,.$$

Now we consider the case $d = 2$. From the following inequalities:

$$\begin{aligned} |V_Y(t,x,\xi)| &\leq Ct^{\gamma\frac{\alpha}{2}-1} p(t,x-\xi)\,; \\ |Y_0(t,x-\xi,\xi)| &\leq Ct^{-1} p(t,x-\xi) \end{aligned}$$

we have easily

$$|Y(t,x,\xi)| \leq |Y_0(t,x-\xi,\xi)| + |V_Y(t,x,\xi)| \leq Ct^{-1} p(t,x-\xi)\,.$$

We re going to prove the lemma when $d = 3$. From Proposition **??** we have

$$\begin{aligned} |Y_0(t,x-\xi,\xi)| &\leq Ct^{-\frac{\alpha}{2}-1} p(t,x-\xi) \\ &= Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1} |x-\xi|^{-1+(\gamma-\gamma_0)/2} \left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{1-(\gamma-\gamma_0)/2} p(t,x-\xi) \\ &\leq Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1} |x-\xi|^{-1+(\gamma-\gamma_0)/2} p(t,x-\xi)\,. \end{aligned}$$

Combining this inequality with Proposition **??** we obtain

$$|Y(t,x,\xi)| \leq Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1} |x-\xi|^{-1+(\gamma-\gamma_0)/2} p(t,x-\xi)\,.$$

We turn to consider the case $d = 4$. Proposition **??** yields that for any $\theta > 0$ the following holds true:

$$\begin{aligned} |Y_0(t,x-\xi,\xi)| &\leq Ct^{-\alpha-1}\left[\left(\frac{|x-\xi|^2}{t^\alpha}\right)^\theta + \left(\frac{t^\alpha}{|x-\xi|^2}\right)^\theta\right] p(t,x-\xi)\,; \\ &= Ct^{-\alpha-1}\left(\frac{t^\alpha}{|x-\xi|^2}\right)^\theta \left[\left(\frac{|x-\xi|^2}{t^\alpha}\right)^{2\theta} + 1\right] p(t,x-\xi)\,. \end{aligned}$$

If $\frac{|x-\xi|^2}{t^\alpha} > 1$, then

$$\left[\left(\frac{|x-\xi|^2}{t^\alpha}\right)^{2\theta}+1\right]p(t,x-\xi) \;\leq\; 2\left(\frac{|x-\xi|^2}{t^\alpha}\right)^{2\theta}p(t,x-\xi) \leq Cp(t,x-\xi)\,.$$

As a consequence, we have

$$|Y_0(t,x-\xi,\xi)| \leq Ct^{-\alpha-1}\left(\frac{t^\alpha}{|x-\xi|^2}\right)^{\theta}p(t,x-\xi)\,.$$

If $\frac{|x-\xi|^2}{t^\alpha} \leq 1$, then the above inequality is obviously true. Now, we can choose $\theta > 0$, such that $-2\theta \geq (-2+\gamma-2\gamma_0)$. Thus, we have

$$\begin{aligned}|Y_0(t,x-\xi,\xi)| &= Ct^{-\alpha-1+\alpha\theta+(-2\theta-(-2+\gamma-2\gamma_0))\frac{\alpha}{2}}|x-\xi|^{-2+\gamma-2\gamma_0} \\ &\quad \cdot\left(\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)^{-2\theta-(-2+\gamma-2\gamma_0)}p(t,x-\xi) \\ &\leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1}\cdot|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi)\,.\end{aligned}$$

Combining the above inequality with Proposition **??** we have

$$\begin{aligned}|Y(t,x,\xi)| &\leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1}|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi) \\ &\quad +Ct^{(\gamma_0+\gamma)\frac{\alpha}{4}-1}|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi) \\ &\leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1}|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi)\end{aligned}$$

since $-(\gamma-2\gamma_0)\frac{\alpha}{2}-1 \leq (\gamma_0+\gamma)\frac{\alpha}{4}-1$.

Finally we consider the case $d \geq 5$. From the estimates $|Y_0(t,x-\xi,\xi)| \leq Ct^{-\alpha-1}|x-\xi|^{4-d}p(t,x-\xi)$ we obtain

$$|Y_0(t,x-\xi,\xi)| \;\leq\; Ct^{-(\gamma_0+\gamma)\frac{\alpha}{4}-1}|x-\xi|^{2-d+(\gamma-\gamma_0)/2}\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{2-(\gamma-\gamma_0)/2}p(t,x-\xi)$$

$$\leq \quad t^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{2-d+(\gamma-\gamma_0)/2}p(t,x-\xi)\,.$$

Therefore, we have

$$|Y(t,x,\xi)| \leq Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{2-d+(\gamma-\gamma_0)/2}p(t,x-\xi)\,.$$

The proposition is then proved. □

The bound (**??**) will greatly help to simplify our estimation of the multiple integrals that we are going to encounter. However, when the dimension $d$ is greater than or equal to 2, the multiple integrals are still complicated to estimate and our main technique is to reduce the computation to one dimensional. This means we will further bound the right hand side of the inequality (**??**) by product of functions of one variable. Before doing so, we denote the exponents of $t$ and $|x-\xi|$ in (**??**) by $\zeta_d$ and $\kappa_d$. Namely, we denote

$$\zeta_d := \begin{cases} -1+\frac{\alpha}{2}, & d=1; \\ -1, & d=2; \\ -(\gamma-2\gamma_0)\frac{\alpha}{2}-1, & d=4; \\ -(\gamma-\gamma_0)\frac{\alpha}{4}-1, & d=3 \text{ or } d\geq 5\,. \end{cases} \tag{4.3.2}$$

and

$$\kappa_d := \begin{cases} 0, & d=1,2; \\ -2+\gamma-2\gamma_0, & d=4; \\ 2-d+(\gamma-\gamma_0)/2, & d=3 \text{ or } d\geq 5\,. \end{cases} \tag{4.3.3}$$

From now on we will exclusively use $p(t,x) = \exp\big(-\sigma t^{-\frac{\alpha}{2-\alpha}}|x|^{\frac{2}{2-\alpha}}\big)$ to denote a function of one variable. However, the constant $\sigma$ may be different in different appearances of $p(t,x)$ (for notational simplicity, we omit the explicit dependence on $\sigma$ of $p(t,x)$).

With these notation Lemma **??** yields

**Lemma 4.3.2.** *The following bound holds true for the Green's function $Y$:*

$$|Y(t,x,\xi)| \le C\prod_{i=1}^{d} t^{\zeta_d/d}|x_i-\xi_i|^{\kappa_d/d}p(t,x_i-\xi_i)\,. \tag{4.3.4}$$

*Proof.* It is easy to see that

$$|x| = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2} \ge \max_{1\le i\le d}|x_i| \ge \prod_{i=1}^{d}|x_i|^{\frac{1}{d}}\,.$$

Thus for any positive number $\alpha > 0$, $|x|^{-\alpha} \le \prod_{i=1}^{d}|x_i|^{-\frac{\alpha}{d}}$.

On the other hand,

$$\begin{aligned} |x|^{\frac{2}{2-\alpha}} &= \left[\sum_{i=1}^{d}|x_i|^2\right]^{\frac{1}{2-\alpha}} \ge \left[\max_{1\le i\le d}|x_i|^2\right]^{\frac{1}{2-\alpha}} \\ &= \max_{1\le i\le d}|x_i|^{\frac{2}{2-\alpha}} \ge \frac{1}{d}\sum_{i=1}^{d}|x_i|^{\frac{2}{2-\alpha}}\,. \end{aligned}$$

Combining the above with (**??**) yields (**??**) since the exponents in $|x-\xi|$ in (**??**) are negative. □

**Lemma 4.3.3.** *Let $-1 < \beta \le 0, x \in \mathbb{R}$. Then, there is a constant $C$, dependent on $\sigma$, $\alpha$ and $\beta$, but independent of $\xi$ and $s$ such that*

$$\sup_{\xi\in\mathbb{R}}\int_{\mathbb{R}}|x|^{\beta}p(s,x-\xi)dx \le Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}\,.$$

*Proof.* Making the substitution $x = ys^{\frac{\alpha}{2}}$ we obtain

$$\begin{aligned}\int_{\mathbb{R}} |x|^{\beta} p(s,x-\xi)dx &= s^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}} \int_{\mathbb{R}} |y|^{\beta} \cdot \exp\left(-\sigma\left|y-\frac{\xi}{s^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right)dy \\ &\leq s^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}\left(\int_{|y|\leq 1} |y|^{\beta}dy + \int_{\mathbb{R}} \exp\left(-\sigma\left|y-\frac{\xi}{s^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right)dy\right) \\ &\leq Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}\end{aligned}$$

since the two integrals inside the parenthesis are finite (and independent of $s$ and $\xi$). □

The following is a slight extension of the above lemma.

**Lemma 4.3.4.** *There is a constant $C$, dependent on $\sigma$, $\alpha$ and $\beta$, but independent of $\xi$ and $s$ such that*

$$\sup_{\xi\in\mathbb{R}} \int_{\mathbb{R}} |x|^{\beta} |\log|x|| \, p(s,x-\xi)dx \leq Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}} [1+|\log s|] \, .$$

*Proof.* we will follow the same idea as in the proof of Lemma **??**. Making the substitution $x = ys^{\frac{\alpha}{2}}$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}} |x|^{\beta} |\log|x|| \, p(s,x-\xi)dx \\ &\quad\leq Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}} \int_{\mathbb{R}} |y|^{\beta} [|\log|y|| + |\log s|] \cdot \exp\left(-\sigma\left|y-\frac{\xi}{s^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right)dy \\ &\quad\leq Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}(1+|\log s|)\left(\int_{|y|\leq e} |y|^{\beta}|\log|y||dy + \int_{\mathbb{R}} \exp\left(-\sigma\left|y-\frac{\xi}{s^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right)dy\right) \\ &\quad\leq Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}(1+|\log s|) \, .\end{aligned}$$

This proves the lemma. □

**Lemma 4.3.5.** *Let $\theta_1$ and $\theta_2$ satisfy $-1 < \theta_1 < 0, -1 < \theta_2 \leq 0$. Then for any $\rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2$,* *(i) If* $\theta_1+\theta_2=-1$*, then*

$$\int_{\mathbb{R}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \le C+C|\log(\rho_2-\tau_1)|\,.$$

*(ii) If* $\theta_1+\theta_2<-1$, *then*

$$\int_{\mathbb{R}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \le C|\rho_2-\tau_1|^{1+\theta_1+\theta_2}.$$

*Proof.* Without loss of generality, suppose $\tau_1\le\rho_2$. We divide the integral domain into four intervals.

$$\begin{aligned}
&\int_{\mathbb{R}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1\\
=\;&\int_{-\infty}^{\frac{3\tau_1-\rho_2}{2}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1\\
&+\int_{\frac{3\tau_1-\rho_2}{2}}^{\frac{\tau_1+\rho_2}{2}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1\\
&+\int_{\frac{\tau_1+\rho_2}{2}}^{\frac{3\rho_2-\tau_1}{2}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1\\
&+\int_{\frac{3\rho_2-\tau_1}{2}}^{\infty}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1\\
=:\;&I_1+I_2+I_3+I_4\,.
\end{aligned}$$

Let us consider $I_2$ first. When $\rho_1\in\left[\frac{3\tau_1-\rho_2}{2},\frac{\tau_1+\rho_2}{2}\right]$, we have $|\rho_2-\rho_1|\ge\frac{\rho_2-\tau_1}{2}$. Noticing $p(s_2-s_1,\rho_2-\rho_1)\le 1$, we have the following estimate for $I_2$:

$$\begin{aligned}
I_2 &\le \left(\frac{\rho_2-\tau_1}{2}\right)^{\theta_2}\int_{\frac{3\tau_1-\rho_2}{2}}^{\frac{\tau_1+\rho_2}{2}}|\rho_1-\tau_1|^{\theta_1}d\rho_1\\
&\le \left(\frac{\rho_2-\tau_1}{2}\right)^{\theta_2}\left[\int_{\tau_1}^{\frac{\tau_1+\rho_2}{2}}(\rho_1-\tau_1)^{\theta_1}d\rho_1+\int_{\frac{3\tau_1-\rho_2}{2}}^{\tau_1}(\tau_1-\rho_1)^{\theta_1}d\rho_1\right]
\end{aligned}$$

$$= \; C\Big(\rho_2 - \tau_1\Big)^{1+\theta_1+\theta_2}.$$

With the same argument, we have

$$I_3 \le C\Big(\rho_2 - \tau_1\Big)^{1+\theta_1+\theta_2}.$$

Now, we study $I_1$. The term $I_4$ can be analyzed in a similar way. Since $\rho_1 < \frac{3\tau_1-\rho_2}{2} < \tau_1 < \rho_2$, we have

$$I_1 \le \int_{-\infty}^{\frac{3\tau_1-\rho_2}{2}} (\tau_1 - \rho_1)^{\theta_1+\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 .$$

To estimate the above integral, we divide our estimation into three cases.

*Case i):* $\theta_1 + \theta_2 < -1$.

In this case, we bound $p(s_2 - s_1, \rho_2 - \rho_1)$ by 1. Thus, we have

$$I_1 \le \int_{-\infty}^{\frac{3\tau_1-\rho_2}{2}} (\tau_1 - \rho_1)^{\theta_1+\theta_2} d\rho_1 = \frac{1}{1+\theta_1+\theta_2}\left(\frac{\rho_2 - \tau_1}{2}\right)^{1+\theta_1+\theta_2}.$$

*Case ii):* $\theta_1 + \theta_2 = -1, \frac{\rho_2-\tau_1}{2} \ge 1$.

In this case, we have $\frac{3\tau_1-\rho_2}{2} \le \tau_1 - 1$. Thus, we have

$$\begin{aligned} I_1 &\le \int_{-\infty}^{\tau_1-1} (\tau_1 - \rho_1)^{-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\ &\le \int_{-\infty}^{\tau_1-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\ &\le \int_{-\infty}^{\infty} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \end{aligned}$$

which is bounded when $s_1$ and $s_2$ are in a bounded domain.

*Case iii):* $\theta_1 + \theta_2 = -1, \frac{\rho_2-\tau_1}{2} < 1$.

In this case, we divide the integral into two intervals as follows.

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\frac{3\tau_1-\rho_2}{2}} (\tau_1-\rho_1)^{\theta_1+\theta_2} p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \\
&\le \int_{-\infty}^{\tau_1-1} (\tau_1-\rho_1)^{-1} p(s_2-s_1,\rho_2-\rho_1)d\rho_1 + \int_{\tau_1-1}^{\frac{3\tau_1-\rho_2}{2}} (\tau_1-\rho_1)^{-1} p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \\
&\le C + \int_{\tau_1-1}^{\frac{3\tau_1-\rho_2}{2}} (\tau_1-\rho_1)^{-1} d\rho_1 \\
&\le C + C|\ln(\rho_2-\tau_1)|.
\end{aligned}$$

Similar argument works for $I_4$. Combining the estimates for $I_k$,$k=1,2,3,4$ yields the lemma. □

**Lemma 4.3.6.** *Let $\theta_1$ and $\theta_2$ satisfy $-1<\theta_1<0, -1<\theta_2\le 0$ and $\theta_1+2\theta_2>-2$. Let $0\le r_1<r_2\le T$ and $0\le s_1<s_2\le T$. Then for any $\rho_1,\tau_2\in\mathbb{R},\rho_1\neq\tau_2$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}|\tau_2-\tau_1|^{\theta_2} p(s_2-s_1,\rho_2-\rho_1)p(r_2-r_1,\tau_2-\tau_1)d\rho_1 d\tau_1 \\
&\le \begin{cases} C(s_2-s_1)^{\frac{\alpha(\theta_1+\theta_2+1)}{2}}(r_2-r_1)^{\frac{\alpha(\theta_2+1)}{2}}, & \theta_1+\theta_2>-1; \\ C(r_2-r_1)^{\frac{\alpha(\theta_1+2\theta_2+2)}{2}}, & \theta_1+\theta_2<-1; \\ C(r_2-r_1)^{\frac{\alpha(\theta_2+1)}{2}}\left[1+|\log(r_2-r_1)|\right], & \theta_1+\theta_2=-1. \end{cases}
\end{aligned} \tag{4.3.5}$$

*Proof.* First, we write

$$\begin{aligned}
I &:= \int_{\mathbb{R}^2} |\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}|\tau_2-\tau_1|^{\theta_2} p(s_2-s_1,\rho_2-\rho_1)p(r_2-r_1,\tau_2-\tau_1)d\rho_1 d\tau_1 \\
&= \int_{\mathbb{R}} f(\tau_1,\rho_2,s_1,s_2,\theta_1,\theta_2)|\tau_2-\tau_1|^{\theta_2} p(r_2-r_1,\tau_2-\tau_1)d\tau_1\,,
\end{aligned} \tag{4.3.6}$$

where

$$f(\tau_1,\rho_2,s_1,s_2,\theta_1,\theta_2) = \int_{\mathbb{R}} |\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2} p(s_2-s_1,\rho_2-\rho_1)d\rho_1\,.$$

We divide the situation into three cases.

*Case i):* $\theta_1+\theta_2>-1$.

In this case we apply the Hölder's inequality to obtain

$$\begin{aligned} f(\tau_1,\rho_2,s_1,s_2,\theta_1,\theta_2) &\le \left\{\int_{\mathbb{R}}|\rho_1-\tau_1|^{\theta_1+\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1\right\}^{\frac{\theta_1}{\theta_1+\theta_2}} \\ &\quad \cdot\left\{\int_{\mathbb{R}}|\rho_2-\rho_1|^{\theta_1+\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1\right\}^{\frac{\theta_2}{\theta_1+\theta_2}} \\ &\le C(s_2-s_1)^{\frac{\alpha(\theta_1+\theta_2)}{2}+\frac{\alpha}{2}}, \qquad (4.3.7) \end{aligned}$$

where the last inequality follows from Lemma **??**. Substituting the above estimate (**??**) into (**??**), we have

$$\begin{aligned} I &= \int_{\mathbb{R}} f(\tau_1,\rho_2,s_1,s_2,\theta_1,\theta_2)|\tau_2-\tau_1|^{\theta_2}p(r_2-r_1,\tau_2-\tau_1)d\tau_1 \\ &\le C(s_2-s_1)^{\frac{\alpha(\theta_1+\theta_2)}{2}+\frac{\alpha}{2}}\int_{\mathbb{R}}|\tau_2-\tau_1|^{\theta_2}p(r_2-r_1,\tau_2-\tau_1)d\tau_1\,. \end{aligned}$$

Using Lemma **??** again we have,

$$I\le C(s_2-s_1)^{\frac{\alpha(\theta_1+\theta_2)}{2}+\frac{\alpha}{2}}(r_2-r_1)^{\frac{\alpha\theta_2}{2}+\frac{\alpha}{2}}\,.$$

*Case ii):* $\theta_1+\theta_2<-1$.

In this case, from Lemma **??**, part (ii) it follows

$$f(\tau_1,\rho_2,s_1,s_2,\theta_1,\theta_2)\le C|\rho_2-\tau_1|^{\theta_1+\theta_2+1}\,.$$

Hence, we have

$$I \le C\int_{\mathbb{R}}|\rho_2-\tau_1|^{\theta_1+\theta_2+1}|\tau_2-\tau_1|^{\theta_2}p(r_2-r_1,\tau_2-\tau_1)d\tau_1\,.$$

Now, since from the condition of the lemma, $\theta_1 + 2\theta_2 + 1 > -1$, we can use Hölder's inequality such as in the inequality (**??**) in the case (i), to obtain

$$\begin{aligned} I &\leq C\int_{\mathbb{R}} |\rho_2 - \tau_1|^{\theta_1+\theta_2+1} |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \\ &\leq C(r_2 - r_1)^{\frac{\alpha(\theta_1+2\theta_2)}{2}+\alpha}. \end{aligned}$$

*Case iii):* $\theta_1 + \theta_2 = -1$.

In this case, we first use Lemma **??**, part (i) to obtain

$$f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) \leq C\left[1 + |\log|\rho_2 - \tau_1||\right].$$

Thus, using Lemma **??**, we have

$$\begin{aligned} I &\leq C\int_{\mathbb{R}} \{1 + |\log|\rho_2 - \tau_1||\} |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \\ &\leq C(r_2 - r_1)^{\frac{\alpha(\theta_2+1)}{2}} \left[1 + |\log|r_2 - r_1||\right]. \end{aligned}$$

The lemma is then proved. □

**Corollary 4.3.7.** *Let* $\theta_1$ *and* $\theta_2$ *satisfy* $-1 < \theta_1 < 0, -1 < \theta_2 \leq 0$ *and* $\theta_1 + 2\theta_2 > -2$. *Let* $0 \leq r_1 < r_2 \leq T$ *and* $0 \leq s_1 < s_2 \leq T$. *Then for any* $\rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2$, *we have*

$$\begin{aligned} &\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} |\tau_2 - \tau_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1 \\ &\leq \begin{cases} C(s_2 - s_1)^{\frac{\alpha(\theta_1+2\theta_2+2)}{4}} (r_2 - r_1)^{\frac{\alpha(\theta_1+2\theta_2+2)}{4}}; & \theta_1 + \theta_2 \neq -1 \\ C(s_2 - s_1)^{\frac{\alpha(\theta_2+1)}{4}} (r_2 - r_1)^{\frac{\alpha(\theta_2+1)}{4}} \left[1 + |\log(r_2 - r_1)| + |\log(s_2 - s_1)|\right]; & \theta_1 + \theta_2 = -1. \end{cases} \end{aligned} \tag{4.3.8}$$

*Proof.* Consider first the case $\theta_1+\theta_2<-1$. Denote the integral on the left hand side of (**??**) by $I$. Then the inequality (**??**) implies

$$I \le C(r_2-r_1)^{\frac{\alpha(\theta_1+2\theta_2)}{2}+\alpha}.$$

In the same way we have

$$I \le C(s_2-s_1)^{\frac{\alpha(\theta_1+2\theta_2)}{2}+\alpha}.$$

Now we use the fact that if three numbers satisfying $a\le b$ and $a\le c$, then $a=a^{1/2}a^{1/2}\le b^{1/2}c^{1/2}$.

$$I \le C(r_2-r_1)^{\frac{\alpha(\theta_1+2\theta_2)}{4}+\alpha/2}(s_2-s_1)^{\frac{\alpha(\theta_1+2\theta_2)}{4}+\alpha/2}$$

which simplifies to (**??**). The exactly the same argument applied to the case $\theta_1+\theta_2=-1$ and the case $\theta_1+\theta_2>-1$. Thus, the inequality (**??**) implies (**??**). $\square$

**Lemma 4.3.8.** *Let $p_1,\cdots,p_n>0$. Then for any $T>0$,*

$$\int_{0\le s_1<\cdots<s_n\le T}(s_n-s_{n-1})^{p_n-1}\cdots(s_2-s_1)^{p_2-1}s_1^{p_1-1}ds=\frac{T^n\prod_{k=1}^n\Gamma(p_k)}{\Gamma(p_1+\cdots+p_n+1)}. \tag{4.3.9}$$

**Lemma 4.3.9.** *Assume that $u_0$ is bounded. Then*

$$\sup_{x\in\mathbb{R}}\int_{\mathbb{R}^d}Z(t,x,\xi)u_0(\xi)d\xi\le C.$$

*Proof.* We use $Z(t,x,\xi)=Z_0(t,x-\xi,\xi)+V_Z(t,x,\xi)$. Since $u_0$ is bounded,

$$\left|\int_{\mathbb{R}^d}Z_0(t,x,\xi)u_0(\xi)d\xi\right| \le C\int_{\mathbb{R}^d}|Z_0(t,x,\xi)|d\xi$$

which is bounded by the estimates in (**??**) and a substitution $\xi = x + t^{\frac{\alpha}{2}} y$. In fact, we have, for example, when $d \geq 3$,

$$\int_{\mathbb{R}^d} |Z_0(t, x-\xi)| d\xi \quad \leq \quad C \int_{\mathbb{R}^d} t^{-\alpha} t^{\frac{(2-d)\alpha}{2}} t^{\frac{d\alpha}{2}} |y|^{2-d} \exp\{-\sigma |y|^{\frac{2}{2-\alpha}}\} dy \leq C t^{1-\alpha} \leq C.$$

Similarly, using the estimation for $V_Z(t,x,\xi)$ given in Proposition **??** we can bound $\int_{\mathbb{R}^d} |V_Z(t,x,\xi)| d\xi$ by a constant. In fact, for example, when $d = 3$, we have

$$\int_{\mathbb{R}^d} |V_Z(t,x,\xi)| d\xi \quad \leq \quad C t^{\frac{\gamma_0 \alpha}{2} - \alpha} \int_{\mathbb{R}^d} t^{\frac{3\alpha}{2}} t^{\frac{(\gamma-\gamma_0-1)\alpha}{2}} |y|^{\gamma-\gamma_0-1} \exp\{-\sigma |y|^{\frac{2}{2-\alpha}}\} dy \leq C t^{\frac{\gamma\alpha}{2}} \leq C.$$

The other dimension case can be dealt with the same way. □

## 4.4 Proof of the main theorem ??

Change $t$ to $s$ and $x$ to $y$ and the equation (**??**) for mild solution becomes

$$u(s,y) = \int_{\mathbb{R}^d} Z(s,y,\xi) u_0(\xi) d\xi + \int_0^s \int_{\mathbb{R}^d} Y(s-r,y,z) u(r,z) W^H(dz) dr.$$

Substituting the above into (**??**), we have

$$\begin{aligned} u(t,x) \quad = \quad & \int_{\mathbb{R}^d} Z(t,x,\xi) u_0(\xi) d\xi + \int_0^t \int_{\mathbb{R}^{2d}} Y(t-s,x,y) Z(s,y,\xi) u_0(\xi) d\xi W^H(dy) ds \\ & + \int_0^t \int_0^s \int_{\mathbb{R}^{2d}} Y(t-s,x,y) Y(s-r,y,z) u(r,z) W^H(dz) dr W^H(dy) ds. \end{aligned}$$

We continue to iterate this procedure to obtain

$$u(t,x) = \sum_{n=0}^{\infty} \Psi_n(t,x), \tag{4.4.1}$$

where $\Psi_n$ satisfies the following recursive relation:

$$\begin{aligned}\Psi_0(t,x) &= \int_{\mathbb{R}^d} Z(t,x,\xi)u_0(\xi)d\xi \\ \Psi_{n+1}(t,x) &= \int_0^t \int_{\mathbb{R}^d} Y(t-s,x,y)\Psi_n(s,y)W^H(dy)ds\,, \quad n=0,1,2,\cdots\end{aligned}$$

To write down the explicit expression for the expansion (**??**), we denote

$$f_n(t,x;x_1,\cdots,x_n) = \int_{T_n}\int_{\mathbb{R}^d} Y(t-s_n,x,x_n)\cdots Y(s_2-s_1,x_2,x_1)Z(s_1,x_1,\xi)u_0(\xi)d\xi d\mathbf{s}\,, \tag{4.4.2}$$

where

$$T_n = 0 \le s_1 < s_1 < \cdots < s_1 \le t \quad \text{and} \quad d\mathbf{s} = ds_1 ds_2 \cdots ds_n\,.$$

With these notations, we see from the above iteration procedure that

$$\begin{aligned}\Psi_n(t,x) &= I_n(\tilde{f}_n(t,x)) \\ &= \int_{\mathbb{R}^{nd}} f_n(t,x;x_1,\cdots,x_n)W^H(dx_1)W^H(dx_2)\cdots W^H(dx_n) \\ &= \int_{\mathbb{R}^{nd}} \tilde{f}_n(t,x;x_1,\cdots,x_n)W^H(dx_1)W^H(dx_2)\cdots W^H(dx_n)\,. \end{aligned} \tag{4.4.3}$$

where $I_n$ denotes the multiple Itô type integral with respect to $W(x)$ (see [**?**]) and $\tilde{f}_n(t,x;x_1,\cdots,x_n)$ is the symmetrization of $f_n(t,x;x_1,\cdots,x_n)$ with respect to $x_1,\cdots,x_n$:

$$\tilde{f}_n(t,x;x_1,\cdots,x_n) = \frac{1}{n!}\sum_{i_1,\cdots,i_n\in\sigma(n)} f_n(t,x;x_{i_1},\cdots,x_{i_n})\,,$$

where $\sigma(n)$ denotes the set of permutations of $(1,2,\cdots,n)$.

The expansion (**??**) with the explicit expression (**??**) for $\Psi_n$ is called the *chaos expansion of the solution*.

If the equation (**??**) has a square integrable solution, then it has a chaos expansion according to a general theorem of Itô. From the above iteration procedure, it is easy to see that this chaos expansion of the solution is given uniquely by (**??**)-(**??**). This is the uniqueness.

If we can show that the series (**??**) is convergent in $L^2(\Omega,\mathscr{F},P)$, then it is easy to verify that $u(t,x)$ defined by (**??**)-(**??**) satisfies the equation (**??**). Thus, the existence of the solution to (**??**) is solved and the explicit form of the solution is also given (by (**??**)-(**??**)). We refer to [**?**] for more detail.

Thus, our remaining task is to prove that the series defined by (**??**) is convergent in $L^2(\Omega,\mathscr{F},P)$. To this end, we need to use the lemmas that we just proved.

Let now $u(t,x)$ be defined by (**??**)-(**??**). Then we have

$$\begin{aligned} \mathbb{E}[u(t,x)^2] &= \sum_{n=0}^{\infty} \mathbb{E}\left[I_n(\tilde{f}_n(t,x))\right]^2 \\ &= \sum_{n=0}^{\infty} n!\langle \tilde{f}_n, \tilde{f}_n\rangle_H \\ &\leq \sum_{n=0}^{\infty} n!\langle f_n, f_n\rangle_H\,, \end{aligned} \tag{4.4.4}$$

where

$$\langle f,g\rangle_H = \int_{\mathbb{R}^{2nd}} \prod_{i=1}^{n} \varphi_H(u_i,v_i) f(u_1,\cdots,u_n) g(v_1,\cdots,v_n) du_1 dv_1 du_2 dv_2 \cdots du_n dv_n \tag{4.4.5}$$

and the last inequality follows from Hölder inequality. Here and in the remaining part of the chapter, we use the following notations:

$$\begin{aligned} u_i &= (u_{i1},\cdots,u_{id})\,, \quad du_i = du_{i1}\cdots du_{id}\,, \quad i=1,2,\cdots,n; \\ \varphi_H(u_i,v_i) &= \prod_{j=1}^{d} \varphi_{H_j}(u_{ij},v_{ij}) = \prod_{j=1}^{d} H_j(2H_j-1)|u_{ij}-v_{ij}|^{2H_j-2}\,. \end{aligned}$$

We use the idea in [**?**] to estimate each term $\Theta_n(t,x) = n!\langle f_n, f_n\rangle_H$ in the series (**??**). By the defining formula (**??**) for $f_n$ we have

$$\begin{aligned}\Theta_n(t,x) &= n!\int_{T_n^2}\int_{\mathbb{R}^{2nd+2}}\prod_{i=1}^n \varphi_H(\xi_i-\eta_i)\, Y(t-s_n,x,\xi_n)\cdots Y(s_2-s_1,\xi_2,\xi_1)\\ &\quad \cdot\int_{\mathbb{R}^d} Z(s_1,\xi_1,\xi_0)u_0(\xi_0)d\xi_0\cdot Y(t-r_n,x,\eta_n)\cdots Y(r_2-r_1,\eta_2,\eta_1)\\ &\quad \cdot\int_{\mathbb{R}^d} Z(r_1,\eta_1,\eta_0)u_0(\eta_0)d\eta_0 d\xi d\eta d\mathbf{s}d\mathbf{r}.\end{aligned}$$

Application of lemma **??** to the above integral yields

$$\begin{aligned}\Theta_n(t,x) &\le Cn!\int_{T_n^2}\int_{\mathbb{R}^{2nd}}\prod_{i=1}^n \varphi_H(\xi_i-\eta_i) Y(t-s_n,x,\xi_n)\cdots Y(s_2-s_1,\xi_2,\xi_1)\\ &\quad \cdot Y(t-r_n,x,\eta_n)\cdots Y(r_2-r_1,\eta_2,\eta_1)d\xi d\eta d\mathbf{s}d\mathbf{r}.\end{aligned}$$

Using lemma **??** to the above integral, we have

$$\Theta_n(t,x) \le C^n n!\int_{T_n^2}\prod_{i=1}^d \Theta_{i,n}(t,x_i,\mathbf{s},\mathbf{r})d\mathbf{s}d\mathbf{r}, \tag{4.4.6}$$

where

$$\begin{aligned}\Theta_{i,n}(t,x_i,\mathbf{s},\mathbf{r}) &= \int_{\mathbb{R}^{2n}}\left\{\prod_{k=1}^n \varphi_{H_i}(\rho_k-\tau_k)\right\}|t-s_n|^{\frac{\zeta_d}{d}}|x_i-\rho_n|^{\frac{\kappa_d}{d}}p(t-s_n,x_i-\rho_n)\\ &\quad \cdots|s_2-s_1|^{\frac{\zeta_d}{d}}|\rho_2-\rho_1|^{\frac{\kappa_d}{d}}p(s_2-s_1,\rho_2-\rho_1)\\ &\quad \cdot|t-r_n|^{\frac{\zeta_d}{d}}|x_i-\tau_n|^{\frac{\kappa_d}{d}}p(t-r_n,x_i-\tau_n)\cdots|r_2-r_1|^{\frac{\zeta_d}{d}}\\ &\quad \cdot|\tau_2-\tau_1|^{\frac{\kappa_d}{d}}p(r_2-r_1,\tau_2-\tau_1)d\rho d\tau.\end{aligned}$$

Here we use the notation $\rho_k = \xi_{ki}$ and $\tau_k = \eta_{ki}$, $k = 1, \cdots, n$. The quantity $\Theta_{i,n}$ can be written as

$$\begin{aligned}\Theta_{i,n}(t,x_i,\mathbf{s},\mathbf{r}) &= |t-s_n|^{\frac{\zeta_d}{d}}|t-r_n|^{\frac{\zeta_d}{d}}\cdots|s_2-s_1|^{\frac{\zeta_d}{d}}|r_2-r_1|^{\frac{\zeta_d}{d}} \\ &\quad \cdot\int_{\mathbb{R}^{2n}}\left\{\prod_{k=1}^{n}\varphi_{H_i}(\rho_k-\tau_k)\right\}|x_i-\rho_n|^{\frac{\kappa_d}{d}}p(t-s_n,x_i-\rho_n) \\ &\quad \cdot|x_i-\tau_n|^{\frac{\kappa_d}{d}}p(t-r_n,x_i-\tau_n)\cdots|\rho_2-\rho_1|^{\frac{\kappa_d}{d}}p(s_2-s_1,\rho_2-\rho_1) \\ &\quad \cdots|\tau_2-\tau_1|^{\frac{\kappa_d}{d}}p(r_2-r_1,\tau_2-\tau_1)d\rho d\tau\,. \end{aligned} \tag{4.4.7}$$

From the definition (**??**) of $\kappa_d$ we see easily $\frac{\kappa_d}{d} > -1$. We assume

$$2H_i + \frac{2\kappa_d}{d} > 0\,. \tag{4.4.8}$$

Under the above condition we can apply the Corollary **??** with $\theta_1 = 2H_i - 2 > -1$, $\theta_2 = \frac{\kappa_d}{d} > -1$ to the integration $d\rho_1 d\tau_1$ in the expression (**??**) (Condition (**??**) implies that $\theta_1 + 2\theta_2 > -2$). Then, when $\theta_1 + \theta_2 \neq -1$, we have

$$\begin{aligned}\Theta_{i,n}(t,x_i,\mathbf{s},\mathbf{r}) &\le C|t-s_n|^{\frac{\zeta_d}{d}}|t-r_n|^{\frac{\zeta_d}{d}}\cdots|s_3-s_2|^{\frac{\zeta_d}{d}}|r_3-r_2|^{\frac{\zeta_d}{d}} \\ &\quad \cdot|s_2-s_1|^{\frac{\zeta_d}{d}+\frac{H_id+\kappa_d}{2d}\alpha}|r_2-r_1|^{\frac{\zeta_d}{d}+\frac{H_id+\kappa_d}{2d}\alpha} \\ &\quad \cdot\int_{\mathbb{R}^{2n-2}}\left\{\prod_{k=2}^{n}\varphi_{H_i}(\rho_k-\tau_k)\right\}|x_i-\rho_n|^{\frac{\kappa_d}{d}}p(t-s_n,x_i-\rho_n) \\ &\quad \cdot|x_i-\tau_n|^{\frac{\kappa_d}{d}}p(t-r_n,x_i-\tau_n)\cdots|\rho_3-\rho_2|^{\frac{\kappa_d}{d}}p(s_3-s_2,\rho_3-\rho_2) \\ &\quad \cdots|\tau_3-\tau_3|^{\frac{\kappa_d}{d}}p(r_3-r_2,\tau_3-\tau_2)d\rho_n\cdots d\rho_2 d\tau_n\cdots d\tau_2\,. \end{aligned}$$

Repeatedly applying this argument, we obtain

$$\Theta_{i,n}(t,x_i,\mathbf{s},\mathbf{r}) \le C^n\prod_{k=1}^{n}|t_{k+1}-t_k|^{\ell_i}|s_{k+1}-s_k|^{\ell_i}\,, \tag{4.4.9}$$

where we recall the convention that $t_{n+1}=t$ and $s_{n+1}=s$ and where

$$\ell_i=\frac{\zeta_d}{d}+\frac{H_id+\kappa_d}{2d}\alpha\,.$$

Substituting the above estimate of $\Theta_{i,n}$ into the expression for $\Theta_n$, we have

$$\begin{aligned}\Theta_n(t,x) &\le C^n\int_{T_n^2}\prod_{k=1}^n(s_{k+1}-s_k)^{\ell}(r_{k+1}-r_k)^{\ell}d\mathbf{s}d\mathbf{r}\\ &= C^n\left[\int_{T_n}\prod_{k=1}^n(s_{k+1}-s_k)^{\ell}d\mathbf{s}\right]^2,\end{aligned}$$

where

$$\ell=\sum_{i=1}^d\ell_i=\zeta_d+\frac{|H|\alpha}{2}+\frac{\kappa_d\alpha}{2}\quad\text{with}\quad|H|=\sum_{i=1}^dH_i\,.$$

Now, we apply Lemma **??** to obtain

$$\begin{aligned}\Theta_n(t,x) &\le C^n\left[\frac{\Gamma(\ell+1)}{\Gamma(n(\ell+1))}\right]^2\\ &\le \frac{C^n}{\Gamma(2n(\ell+1))}\,.\end{aligned}$$

This estimate combined with (**??**) proves that if

$$2(\ell+1)>1\,, \tag{4.4.10}$$

then $\sum_{n=0}^{\infty}\Theta_n(t,x)$ is bounded which implies that the series (**??**) is convergent in $L^2(\Omega,\mathscr{F},P)$.

Using the explicit expressions of $\zeta_d$ and $\kappa_d$, we analyze the condition (**??**) for the cases $d=1$, $d=2$, $d=4$ and $d=3$ or $d\ge5$ separately, then we see that the condition (**??**) is equivalent to

$$\sum_{i=1}^dH_i>d-2+\frac{1}{\alpha}\,. \tag{4.4.11}$$

For instance, when $d=1$,

$$2(\ell+1)>1 \Leftrightarrow |H|>\frac{1}{\alpha}-\kappa_d-\frac{2}{\alpha}\delta_d \Leftrightarrow |H|>\frac{1}{\alpha}-\frac{2}{\alpha}(-1+\frac{\alpha}{2})=\frac{1}{\alpha}-1.$$

When $\theta_1+\theta_2=-1$, Corollary **??** implies that for any $\varepsilon<0$,

$$\begin{aligned} &\int_{\mathbb{R}^2}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}|\tau_2-\tau_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)p(r_2-r_1,\tau_2-\tau_1)d\rho_1 d\tau_1 \\ \le\;& C(s_2-s_1)^{\frac{\alpha(\theta_2+1+\varepsilon)}{4}}(r_2-r_1)^{\frac{\alpha(\theta_2+1++\varepsilon)}{4}}. \end{aligned}$$

Now we can follow the above same argument to obtain that if

$$2(\ell+1)>1\,, \tag{4.4.12}$$

where $\ell=\tau_d+\frac{\tau_d}{2}\frac{d\varepsilon+\kappa_d+d}{4}\alpha$, then $\Theta_n(t,x)$ is bounded. In the same way as in the case $\theta_1+\theta_2\neq-1$, we can show that the condition (**??**) implies (**??**).

Now we consider the condition (**??**). From the definition (**??**) of $\kappa_d$, we see that when $d=1,2,3,4$, $H_i>1/2$ implies (**??**). When $d\geq 5$, then the condition (**??**) is implied by the following

$$H_i>1-\frac{2}{d}-\frac{\gamma}{2d}$$

by choosing $\gamma_0$ sufficiently small.

Theorem 2 is then proved. $\Box$.

# Chapter 5

# Stochastic time-fractional diffusion equations with variable coefficients and time dependent noise

## 5.1 Introduction

In this article we prove the existence and uniqueness of the mild solution of the equation

$$\begin{cases} (\partial^\alpha - B)\, u(t,x) = u(t,x)\dot{W}(t,x), & t \in (0,T],\, x \in \mathbb{R}^d, \\ \dfrac{\partial^k}{\partial t^k} u(t,x)\Big|_{t=0} = u_k(x), & 0 \le k \le \lceil \alpha \rceil - 1,\ x \in \mathbb{R}^d, \end{cases} \tag{5.1.1}$$

with any fixed $T \in \mathbb{R}^+$, $\alpha \in (1/2,1) \cup (1,2)$, where $\lceil \alpha \rceil$ is the smallest integer not less than $\alpha$. Here we assume

- $u_0(x)$ is bounded continuously differentiable. Its first order derivative bounded and Hölder continuous. The Hölder exponent $\gamma > \frac{2-\alpha}{\alpha}$
- $u_1(x)$ is bounded continuous function(locally hölder continuous if $d > 1$)

In this equation, $\dot{W}$ is a zero mean Gaussian noise with the following covariance structure

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \lambda(t-s)\Lambda(x-y),$$

where $\lambda(\cdot)$ is nonnegative definite and locally intergrable and $\Lambda(\cdot)$ is one of the following situations:

(i) Fractional kernel. $\Lambda(x) := \prod_{i=1}^{d} 2H_i(2H_i-1)|x_i|^{2H_i-1}$, $x \in \mathbb{R}^d$ and $1/2 < H_i < 1$.

(ii) Reisz kernel. $\Lambda(x) := C_{\alpha,d}|x|^{-\kappa}$, $x \in \mathbb{R}^d$ and $0 < \kappa < d$ and $C_{\alpha,d} = \Gamma(\frac{\kappa}{2})2^{-\alpha}\pi^{-d/2}/\Gamma(\frac{\alpha}{2})$.

(iii) Bessel kernel. $\Lambda(x) := C_\alpha \int_0^\infty \omega^{-\frac{\kappa}{2}-1} e^{-\omega} e^{\frac{-|x|^2}{4\omega}} d\omega$, $x \in \mathbb{R}^d$, $0 < \kappa < d$, and $C_\alpha = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$.

$$B := \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

is uniformly elliptic. Namely it satisfies the following conditions:

(i) $a_{ij}(x), b_j(x)$ and $c(x)$ are bounded Hölder continuous functions on $\mathbb{R}^d$

(ii) $\exists a_0 > 0$, such that $\forall x, \xi \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geq a_0|\xi|^2.$$

The fractional derivative in time $\partial^\alpha$ is understood in *Caputo* sense:

$$\partial^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \mathrm{d}\tau \, \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} & \text{if } m-1 < \alpha < m \,, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{if } \alpha = m \,. \end{cases}$$

Throughout this chapter, the initial conditions $u_k(x)$ are bounded continuous(Hölder continuous, if $d > 1$) functions. The study of the mild solution relies on the asymptote property of the Green's function $Z, Y$ of the following deterministic equation.

$$\begin{cases} (\partial^{\alpha} - B)\, u(t,x) = f(t,x), & t > 0, x \in \mathbb{R}^d, \\ \dfrac{\partial^k}{\partial t^k} u(t,x)\Big|_{t=0} = u_k(x), & 0 \le k \le \lceil \alpha \rceil - 1,\ x \in \mathbb{R}^d, \end{cases} \tag{5.1.2}$$

In chapter **??** we cover the case $\alpha \in (1/2, 1)$. When $\alpha \in (1,2)$, [**?**] showed that when $B$ is $\Delta$, Green's function $Y$ of (**??**) the following:

$$Y(t,x) = C_d t^{\frac{\alpha}{2}(2-d)} f_{\frac{\alpha}{2}}(|x| t^{-\frac{\alpha}{2}}; d-1, \frac{\alpha}{2}(2-d)),$$

where

$$f_{\frac{\alpha}{2}}(z; \mu, \delta) = \begin{cases} \dfrac{2}{\Gamma(\frac{\mu}{2})} \displaystyle\int_1^{\infty} \phi(-\frac{\alpha}{2}, \delta; -zt)(t^2-1)^{\frac{\mu}{2}-1} \mathrm{d}t, & \mu > 0, \\ \phi(-\dfrac{\alpha}{2}, \delta; -z), & \mu = 0; \end{cases}$$

$C_d = 2^{-n} \pi^{\frac{1-d}{2}}$ and the wright's function

$$\phi(-\frac{\alpha}{2}, \delta; -z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\delta - \frac{\alpha}{2} n)}$$

The solution of (**??**) has the following form:

$$u(t,x) = J_0(t,x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, f(s,y) Y(t-s, x-y), \tag{5.1.3}$$

where and throughout the chapter, we denote

$$J_0(t,x) := \sum_{k=0}^{\lceil\alpha\rceil-1} \int_{\mathbb{R}^d} u_k(y) Z_{k+1}(t,x-y)\mathrm{d}y\,. \tag{5.1.4}$$

For case of $\alpha \in (1/2,1)$, we use $Z$ in place of $Z_1$. We have the following facts about $Z_1(t,x), Z_2(t,x)$ and $Y(t,x)$.

$$Z_1(t,x) = D^{\alpha-1}Y(t,x); \qquad Z_1(t,x) = \frac{\partial}{\partial t} Z_2(t,x)$$

As in the chapter **??**, We first get the estimation of $Y$, then use Wiener chaos expansion to obtain relation between the parameter $\alpha, d, H_i$ and $\kappa$ such that the mild solution exist.

The rest of the article is organized as follows. Section 2 gives more details about the solution of (**??**), estimation of $Y$ for $\alpha \in (1/2,1)$ and some preliminaries about Wiener spaces. Section 3 gives the estimation of $Y$ for $\alpha \in (1,2)$ and further estimations before proving the existence of the mild solution.

**Notation:** Throughout this chapter we denote

$$p(t,x) := \exp\left\{ -\sigma \left| \frac{x}{t^{\frac{\alpha}{2}}} \right|^{\frac{2}{2-\alpha}} \right\},$$

where $\sigma > 0$ is a generic positive constant whose values may vary at different occurrence, so is C.

## 5.2 Preliminary

We consider a Gaussian noise $W$ on a complete probability space $(\Omega, \mathscr{F}, P)$ encoded by a centered Gaussian family $\{W(\varphi); \varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)\}$, whose covariance structure $\lambda(s-t)$ is given by

$$\mathbb{E}\left(W(\varphi)W(\psi)\right) = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(s,x)\psi(t,y)\lambda(s-t)\Lambda(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}s\mathrm{d}t, \tag{5.2.1}$$

where $\lambda : \mathbb{R} \to \mathbb{R}_+$ is nonnegative definite and locally intergrable. Throughout the chapter, we denote

$$C_t := 2\int_0^t \lambda(s)\mathrm{d}s, \quad t > 0. \tag{5.2.2}$$

$\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ is a fractional, Reisz or Bessel kernel.

**Definition 5.2.1.** Let $Z$ and $Y$ be the fundamental solutions defined by (**??**) and (**??**). An adapted random field $\{u = u(t,x) : t \geq 0, x \in \mathbb{R}^d\}$ such that $\mathbb{E}\left[u^2(t,x)\right] < +\infty$ for all $(t,x)$ is a *mild solution* to (**??**), if for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process

$$\left\{Y(t-s,x-y)u(s,y)1_{[0,t]}(s) : s \geq 0, y \in \mathbb{R}^d\right\}$$

is Skorodhod integrable (see (**??**)), and $u$ satisfies

$$u(t,x) = J_0(t,x) + \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)u(s,y)W(\mathrm{d}s,\mathrm{d}y) \tag{5.2.3}$$

almost surely for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where $J_0(t,x)$ is defined by (**??**).

We use a similar chaos expansion to the one used in chapter 3. To prove the existence and uniqueness of the solution we show that for all $(t,x)$,

$$\sum_{n=0}^{\infty} n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} < \infty. \tag{5.2.4}$$

## 5.3 Estimations of the Green's functions

The fundamental solution of (**??**) is constructed by Levi's parametrix method. We refer the reader to [**?**] for detail of this method. In this section $x := (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d, \xi, \eta$ are defined the same way; $t \in (0,T]$. We use $\gamma$ to denote the Hölder exponents with respect to spatial variables. We can assume they are equal. For $\alpha \in (1,2)$, we assume

$$\gamma > 2 - \frac{2}{\alpha}.$$

For $\alpha \in (\frac{1}{2}, 1)$, Chapter 4 gives the estimations the $Z$ and $Y$. For $\alpha \in (1,2)$, we need some lemmas before we can estimate $Z_1, Z_2$ and $Y$.

From [**?**] we have

$$\begin{aligned}
Z_j(t,x-\xi) =& Z_j^0(t,x-\xi,\xi) + V_{Z_j}(t,x;\xi), \qquad j = 1,2. \\
Y(t,x-\xi) =& Y_0(t,x-\xi,\xi) + V_Y(t,x;\xi).
\end{aligned}$$

We refer the reader to [**?**] for the definitions of $Z_k^0(t,x-\xi,\xi), Y_0(t,x-\xi,\xi)$ and $V_Y(t,x;\xi)$. Here we list their estimations which we use to get the estimations of $Z_k$ and $Y$ in section 3. These estimations are given in section 2.2 of [**?**] or Lemma 15 in [**?**].

**Lemma 5.3.1.**

$$|Z_1^0(t,x-\xi,\eta)| \leq Ct^{-\frac{\alpha d}{2}}\mu_d(t^{-\frac{\alpha}{2}}|x-\xi|)p(t,x-\xi),$$

$$|Z_2^0(t,x-\xi,\eta)| \leq Ct^{-\frac{\alpha d}{2}+1}\mu_d(t^{-\frac{\alpha}{2}}|x-\xi|)p(t,x-\xi),$$

*where*

$$\mu_d(z) := \begin{cases} 1, & d=1; \\ 1+|\log z|, & d=2; \\ z^{2-d}, & d \geq 3. \end{cases} \tag{5.3.1}$$

**Lemma 5.3.2.**

$$|Y_0(t,x-\xi,\eta)| \leq Ct^{\alpha-\frac{\alpha d}{2}-1}\mu_d(t^{-\frac{\alpha}{2}}|x-\xi|)p(t,x-\xi),$$

*where*

$$\mu_d(z) := \begin{cases} 1, & d \leq 3; \\ 1+|\log z|, & d=4; \\ z^{4-d}, & d \geq 5. \end{cases} \tag{5.3.2}$$

The following estimations of $V_{Z_1}, V_{Z_2}$ and $V_Y$ are from Theorem 1 of [**?**], where $\nu_1 \in (0,1)$, such that $\gamma > \nu_1 > 2 - \frac{2}{\alpha}$ and $\nu_0 = \nu_1 - 2 + \frac{2}{\alpha}$.

**Lemma 5.3.3.**

$$|V_{Z_1}(t,x;\xi)| \leq \begin{cases} Ct^{(\gamma-1)\frac{\alpha}{2}}p(t,x-\xi), & d=1; \\ Ct^{\nu_0\alpha-1}|x-\xi|^{-d+\gamma-\nu_1+2-\nu_0}p(t,x-\xi), & d \geq 2 \end{cases} \tag{5.3.3}$$

**Lemma 5.3.4.**

$$|V_{Z_2}(t,x;\xi)| \le \begin{cases} Ct^{(\gamma-1)\frac{\alpha}{2}+1}p(t,x-\xi)\,, & d=1\,; \\ Ct^{\frac{v_0\alpha}{2}+1-\alpha}|x-\xi|^{-d+\gamma-v_1+2-v_0}p(t,x-\xi)\,, & d\ge 2 \end{cases} \tag{5.3.4}$$

**Lemma 5.3.5.**

$$|V_Y(t,x;\xi)| \le \begin{cases} Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}}p(t,x-\xi)\,, & d=1\,; \\ Ct^{v_0\alpha-1}|x-\xi|^{-d+\gamma-v_1+2-v_0}p(t,x-\xi)\,, & d\ge 2 \end{cases} \tag{5.3.5}$$

Based on the above three lemmas we have

**Lemma 5.3.6.** *Let* $x\in\mathbb{R}^d, t\in(0,T]$. *Then*

$$|Y(t,x-\xi)| \le \begin{cases} Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi), & d=1; \\ Ct^{\alpha-\frac{\alpha}{2}\gamma+v_0\alpha-2}|x-\xi|^{-d+\gamma-2v_0+\frac{2}{\alpha}}p(t,x-\xi), & d\ge 2. \end{cases} \tag{5.3.6}$$

*Proof.* We "add" together the estimation of $Y_0$ in Lemma **??** and $V_y$ in Lemma **??** to get the estimation of $Y$. We use the following inequality throughout the proof.

$$a,b,\sigma>0, \quad \text{then} \quad \exists\sigma,C>0, \quad s.t. \quad x^a e^{-\sigma x^b} < Ce^{-\sigma' x^b},$$

First when $d=1$,

$$\begin{aligned} |Y(t,x-\xi)| &\le |Y_0(t,x-\xi,\xi)|+|V_Y(t,x,\xi)| \\ &\le Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}}p(t,x-\xi)+Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi) \\ &\le Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi)\,. \end{aligned}$$

When $d \geq 5$, by the fact

$$\nu_0 = \nu_1 - 2 + 2/\alpha \qquad \text{and} \qquad \gamma > \nu_1 > 2 - \frac{2}{\alpha},$$

we have

$$4 - \gamma + 2\nu_0 - \frac{2}{\alpha} = -\gamma + 2\nu_1 + \frac{2}{\alpha} \geq 0.$$

Therefore

$$\begin{aligned}
|Y_0(t, x-\xi, \xi)| &\leq Ct^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x-\xi}{t^{\frac{\alpha}{2}}} \right|^{4-d} p(t, x-\xi) \\
&= Ct^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x-\xi}{t^{\frac{\alpha}{2}}} \right|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} \left| \frac{x-\xi}{t^{\frac{\alpha}{2}}} \right|^{4-\gamma+2\nu_0-\frac{2}{\alpha}} p(t, x-\xi) \\
&\leq Ct^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x-\xi).
\end{aligned}$$

Furthermore because of the assumption

$$\gamma > 2 - \frac{2}{\alpha},$$

we have

$$\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2 < \nu_0\alpha - 1.$$

Therefore

$$\begin{aligned}
|Y(t, x-\xi)| &\leq |Y_0(t, x-\xi, \xi)| + |V_y(t, x, \xi)| \\
&\leq Ct^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x-\xi) \\
&+ Ct^{\nu_0\alpha - 1} |x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x-\xi) \\
&\leq Ct^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x-\xi).
\end{aligned}$$

When $d=2$ and $d=3$, as in the previous cases, we first have

$$\begin{aligned}|Y_0(t,x-\xi,\xi)| &\leq Ct^{\alpha-\frac{\alpha n}{2}-1}p(t,x-\xi)\\ &\leq Ct^{\alpha-\frac{\alpha}{2}\gamma+\nu_0\alpha-2}|x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}}p(t,x-\xi).\end{aligned}$$

Then as the last step in the case of $d\geq 5$, we have

$$|Y(t,x-\xi)| \leq Ct^{\alpha-\frac{\alpha}{2}\gamma+\nu_0\alpha-2}|x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}}p(t,x-\xi).$$

When $d=4$, let's first transform the estimation of $Y_0$ into the following form:

$$t^{\zeta_d}|x-\xi|^{\kappa_d}p(t,x-\xi).$$

We have

$$\begin{aligned}|Y_0(t,x-\xi,\xi)| &\leq Ct^{\alpha-\frac{\alpha d}{2}-1}\left\{\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{\theta}+\left|\frac{t^{\frac{\alpha}{2}}}{x-\xi}\right|^{\theta}\right\}p(t,x-\xi)\\ &\leq Ct^{\alpha-\frac{\alpha d}{2}-1}\left|\frac{t^{\frac{\alpha}{2}}}{x-\xi}\right|^{\theta}\left\{\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{2\theta}+1\right\}p(t,x-\xi),\end{aligned}$$

for $\forall\,\theta>0$.

If $|\frac{x-\xi}{t^{\frac{\alpha}{2}}}|\leq 1$, then

$$\left\{\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{2\theta}+1\right\}p(t,x-\xi) \leq 2p(t,x-\xi);$$

if $|\frac{x-\xi}{t^{\frac{\alpha}{2}}}| > 1$, then

$$\begin{aligned} \left\{\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{2\theta}+1\right\} p(t,x-\xi) &\leq 2\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{2\theta} p(t,x-\xi) \\ &\leq Cp(t,x-\xi). \end{aligned}$$

Therefore if we choose $\theta > 0$ such that

$$-\theta > -d+\gamma-2\nu_0+\frac{2}{\alpha},$$

we have

$$\begin{aligned} |Y_0(t,x-\xi,\xi)| &\leq Ct^{\alpha-\frac{\alpha d}{2}-1}\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{-\theta} p(t,x-\xi) \\ &\leq Ct^{\alpha-\frac{\alpha d}{2}-1}\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t,x-\xi) \\ &\leq Ct^{\alpha-\frac{\alpha}{2}\gamma+\nu_0\alpha-2}|x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t,x-\xi). \end{aligned}$$

As in previous two cases, we end up with

$$|Y(t,x-\xi)| \leq Ct^{\alpha-\frac{\alpha}{2}\gamma+\nu_0\alpha-2}|x-\xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t,x-\xi).$$

□

Let's denote the the estimation function of $Y$ by $t^{\zeta_d}|x-\xi|^{\kappa_d}p(t,x-\xi)$. For the estimation of integral (**??**) involving $Y$ and fractional kernel it more convenient to represent the estimation of $Y$ as the the product of one dimensional functions. To this purpose, as in the case of $0<\alpha<1$, the estimation of $Y$ is represented as the product of one dimensional functions, which is shown in the following lemma.

**Lemma 5.3.7.** *Let $x_i, \xi_i \in \mathbb{R}, t \in (0,T]$*

$$|Y(t,x-\xi)| \le C \prod_{i=1}^{d} t^{\zeta_d/d} |x_i - \xi_i|^{\kappa_d/d} p(t, x_i - \xi_i)\,, \tag{5.3.7}$$

*where $\zeta_d$ and $\kappa_d$ are the powers of $t$ and $x-\xi$ in the estimation of $Y$, i.e.,*

$$\zeta_d = \begin{cases} -1 + \frac{\alpha}{2}, & d = 1; \\ \alpha - \frac{\alpha}{2}\gamma + \nu_0 \alpha - 2, & d \ge 2\,. \end{cases} \tag{5.3.8}$$

*and*

$$\kappa_d = \begin{cases} 0, & d = 1; \\ -d + \gamma - 2\nu_0 + \frac{2}{\alpha}, & d \ge 2\,. \end{cases} \tag{5.3.9}$$

**Lemma 5.3.8.**

$$\sup_{t,x} \left| \int_{\mathbb{R}^d} Z_{k+1}(t,x-\xi) u_k(t,\xi) d\xi \right| \le C \qquad k = 0, 1.$$

*Proof.* First recall that $u_k(x)$ are bounded. Thanks to the following fact from [**?**]

$$\int_{\mathbb{R}^d} Z_1^0(t,x,\xi) d\xi = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} Z_2^0(t,x,\xi) d\xi = t,$$

we only need to show

$$\sup_{t,x} \int_{\mathbb{R}^d} V_{Z_j}(t,x,\xi) \mathrm{d}\xi \le C,$$

since $u_k$ are bounded.

Let's consider the case $d \geq 3$ and $d = 2$ for $V_{z_1}$ as examples. When $d \geq 3$, by the estimation of $V_{Z_1}$ in Lemma **??**, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |V_{Z_1}(t,x,\xi)| d\xi &\leq \int_{\mathbb{R}^d} Ct^{-\frac{\alpha d}{2}} \mu_d(t^{-\frac{\alpha}{2}}|x-\xi|) p(t,x-\xi) dy \\
&\leq \int_{\mathbb{R}^d} Ct^{-\frac{\alpha d}{2}+d} \mu_d(z) p(t,x-\xi) dz \\
&\leq Ct^{-\frac{\alpha d}{2}+d} \\
&\leq C,
\end{aligned}$$

due to the fact $t \in (0,T]$.

For the case $d = 2, Z_1$, notice that

$$\forall \theta > 0, \exists C > 0 \quad s.t. \quad (\log|z| + 1) < c|z|^{\theta},$$

as shown in the case of d=4 in the proof of **??**. Then the above argument ends proof. The proof for the rest of the cases is almost the same, so we omit it.

□

## 5.4 Miscellaneous estimations

For fractional kernel, we need the following estimation, which is immediate from Corollary 15, [**?**].

**Lemma 5.4.1.** *Let* $0 < r, s \leq T$ *and*

$$2H_i + \frac{2\kappa_d}{d} > 0. \tag{5.4.1}$$

*Then for any* $\rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2$*, we have*

$$\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{2H_i - 2} |\rho_2 - \rho_1|^{\frac{\kappa_d}{d}} |\tau_2 - \tau_1|^{\frac{\kappa_d}{d}} p(s, \rho_2 - \rho_1) p(r, \tau_2 - \tau_1) \mathrm{d}\rho_1 \mathrm{d}\tau_1 \leq C(s\,r)^{\theta_i},$$

*where*

$$\theta_i = \begin{cases} C(s\,r)^{\frac{H_i d + \kappa_d}{2d}\alpha}, & 2H_i - 2 + \kappa_d/d \neq -1; \\ C(s\,r)^{\frac{d\varepsilon + \kappa_d + d}{4d}\alpha}, & 2H_i - 2 + \kappa_d/d = -1\,. \end{cases}$$

*Proof.* In Corollary 15, [**?**], let $\theta_1 = 2H_i - 2, \theta_2 = \kappa_d/d$. Then notice that for $0 < r \leq T$

$$\forall \varepsilon < 0, \exists C > 0, \quad s.t. \quad \log r < Cr^{\varepsilon}.$$

□

The next lemma can be proved as in Lemma 11, [**?**].

**Lemma 5.4.2.** *Let* $-1 < \beta \leq 0, x \in \mathbb{R}^d$*. Then, there is a constant C, dependent on* $\sigma$*,* $\alpha$ *and* $\beta$*, but independent of* $\xi$ *and s such that*

$$\int_{\mathbb{R}^d} |x|^{\beta} p(s, x - \xi) dx \leq Cs^{\frac{\alpha\beta}{2} + \frac{\alpha}{2}d}.$$

For Bessel kernel, we need the following lemma.

**Lemma 5.4.3.** *Assume* $0 < s, r \leq T$ *and* $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$*, we have that*

$$\int_{\mathbb{R}^{2d}} |Y(r, y_1 - y_2) Y(s, z_1 - z_2)| \int_0^{\infty} \omega^{-\frac{\kappa}{2} - 1} e^{-\omega} e^{\frac{-|y_1 - z_1|^2}{4\omega}} d\omega \mathrm{d}y_1 \mathrm{d}z_1 \leq C \cdot (r\,s)^{\ell},$$

*where*

$$\ell := \zeta_d - \frac{\alpha}{4}\kappa + \frac{\alpha}{2}\kappa_d + \frac{\alpha}{2}d$$

*Proof.* Recall that the estimation of $Y(t,x)$ in (**??**) and (**??**) has the following form:

$$Cs^{\zeta_d}|x|^{\kappa_d}p(t,x).$$

By substituting $Y$, we have

$$\int_{\mathbb{R}^{2d}} |Y(r,y_1-y_2)Y(s,z_1-z_2)| \int_0^\infty \omega^{-\frac{\kappa}{2}-1}e^{-\omega}e^{\frac{-|y_1-z_1|^2}{4\omega}}\,d\omega \mathrm{d}y_1\mathrm{d}z_1$$

$$\leq C\int_{\mathbb{R}^d} s^{\zeta_d}|z_2-z_1^{\kappa_d}|p(s,z_2-z_1)r^{\zeta_d}\int_0^\infty I\cdot\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega\mathrm{d}z_1,$$

where

$$I := \int_{\mathbb{R}_d} |y_2-y_1|^{\kappa_d}\exp\left\{-\sigma\left|\frac{y_2-y_1}{r^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right\}\exp\left\{-\frac{|y_1-z_1|^2}{4\omega}\right\}\mathrm{d}y_1.$$

For $I$, we have two estimations:

$$\begin{aligned} I &\leq \int_{\mathbb{R}_d} |y_2-y_1|^{\kappa_d}\exp\left\{-\sigma\left|\frac{y_2-y_1}{r^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right\}\mathrm{d}y_1 \\ &\leq Cr^{\frac{\alpha}{2}\kappa_d+\frac{\alpha}{2}d}, \end{aligned}$$

and

$$\begin{aligned} I &\leq \int_{\mathbb{R}_d} |y_2-y_1|^{\kappa_d}\exp\left\{-\frac{|y_1-z_1|^2}{4\omega}\right\}\mathrm{d}y_1 \\ &\leq C\omega^{\frac{\kappa_d}{2}+\frac{d}{2}}, \end{aligned}$$

by Lemma **??**.

With the estimations of $I$, we have

$$\begin{aligned}\int_0^\infty I\cdot\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega &= \int_0^{r^\alpha} I\cdot\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega + \int_{r^\alpha}^\infty I\cdot\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega \\ &\leq r^{\frac{\alpha}{2}\kappa_d+\frac{\alpha}{2}d-\frac{\alpha}{2}\kappa} + \int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2}+\frac{d}{2}}\,\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega.\end{aligned}$$

For $\int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2}+\frac{d}{2}}\,\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega$,

if $\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2}<0$

$$\begin{aligned}\int_{r^\alpha}^\infty \omega^{\frac{\kappa}{2}+\frac{d}{2}}\,\omega^{-\frac{\kappa_d}{2}-1}e^{-\omega}\mathrm{d}\omega &\leq \int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2}-1}\mathrm{d}\omega \\ &= Cr^{\alpha(\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2})};\end{aligned}$$

if $\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2}\geq 0$

$$\begin{aligned}\int_{r^\alpha}^\infty \omega^{\frac{\kappa}{2}+\frac{d}{2}}\,\omega^{-\frac{\kappa_d}{2}-1}e^{-\omega}\mathrm{d}\omega &= \int_{r^\alpha}^\infty \omega^{\frac{\kappa}{2}+\frac{d}{2}-\frac{\kappa_d}{2}-1}e^{-\omega}\mathrm{d}\omega \\ &= C \\ &\leq Cr^{\alpha(\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2})}.\end{aligned}$$

Therefore we end up with

$$\int_0^\infty I\cdot\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega \leq Cr^{\alpha(\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2})}.$$

The estimation of integration with respect to $z_1$ is straightforward thank to fact that C is independent of $z_1$.

We have

$$\int_{\mathbb{R}^d} s^{\zeta_d}|z_2-z_1|p(s,z_2-z_1)r^{\zeta_d}\int_0^\infty I\cdot\omega^{-\frac{\kappa}{2}-1}e^{-\omega}\mathrm{d}\omega\mathrm{d}z_1$$

$$\leq Cr^{\alpha(\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2})}\cdot r^{\zeta_d}\cdot s^{\alpha(\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2})}s^{\zeta_d},$$

by Lemma **??**.

By symmetry, we have

$$\int_{\mathbb{R}^{2d}}|Y(r,y_1-y_2)Y(s,z_1-z_2)|\int_0^\infty \omega^{-\frac{\kappa}{2}-1}e^{-\omega}e^{\frac{-|y_1-z_1|^2}{4\omega}}d\omega\mathrm{d}y_1\mathrm{d}z_1$$

$$\leq Cs^{\alpha(\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2})}\cdot s^{\zeta_d}\cdot r^{\alpha(\frac{\kappa_d}{2}+\frac{d}{2}-\frac{\kappa}{2})}r^{\zeta_d}.$$

Combining the two estimations we get the estimation in the lemma. □

The following lemma is Theorem 3.5 from [**?**].

**Lemma 5.4.4.** *Let $T_n(t)=\{s=(s_1,\dots,s_n):0<s_1<s_2<\dots<s_n<t\}$. Then*

$$\int_{T_n(t)}[(t-s_n)(s_n-s_{n-1})\dots(s_2-s_1)]^h\mathrm{d}s=\frac{\Gamma(1+h)^n}{\Gamma(n(1+h)+1)}t^{n(1+h)},$$

*if and only if $1+h>0$.*

## 5.5 Existence and uniqueness of the solution

**Theorem 5.5.1.** *Assume the following conditions:*

*(1) $\lambda(t)$ is a nonnegative definite locally integrable function ;*

*(2) $\alpha\in(1/2,1)\cup(1,2)$.*

*Then relation (**??**) holds for each $(t,x)$, if any of the following is true. Consequently, equation (**??**) admits a unique mild solution in the sense of Definition **??**.*

*(i)* $\Lambda(x)$ *is fractional kernel with condition:*

$$H_i > \begin{cases} \frac{1}{2}, & d = 1,2,3,4 \\ 1 - \frac{2}{d} - \frac{\gamma}{2d}, & d \geq 5, \alpha \in (0,1) \\ 1 - \frac{2}{d}, & d \geq 5, \alpha \in (1,2) \end{cases}$$

*and*

$$\sum_{i=1}^{d} H_i > d - 2 + \frac{1}{\alpha}.$$

*(ii)* $\Lambda(x)$ *is the Reisz or Bessel kernel with condition:*

$$\kappa < 4 - 2/\alpha;$$

*Proof.* Fix $t > 0$ and $x \in \mathbb{R}^d$.

Let

$$(s,y,t,x) := (s_1, y_1, \cdots, s_n, y_n, t, x);$$

$$g_n(s,y,t,x) := \frac{1}{n!} Y(t - s_{\sigma(n)}, x - y_{\sigma(n)}) \cdots Y(s_{\sigma(2)} - s_{\sigma(1)}, y_{\sigma(2)} - y_{\sigma(1)});$$

$$f_n(s,y,t,x) := g_n(s,y,t,x) J_0(s_{\sigma(1)}, x_{\sigma(1)}),$$

where $\sigma$ denotes a permutation of $\{1,2,\cdots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$.

By iteration of $u(t,x)$, we have

$$n! \| f_n(\cdot,\cdot,t,x) \|^2_{\mathscr{H}^{\otimes n}}$$

$$= n!\int_{[0,t]^{2n}} \mathrm{d}s\mathrm{d}r \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, f_n(s,y,t,x) f_n(r,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \prod_{i=1}^{n} \lambda(s_i - r_i), \quad (5.5.1)$$

where $\mathrm{d}y := \mathrm{d}y_1 \cdots \mathrm{d}y_n$, the differentials $\mathrm{d}z$, $\mathrm{d}s$ and $\mathrm{d}r$ are defined similarly. Set $\mu(\mathrm{d}\xi) := \prod_{i=1}^{n} \mu(\mathrm{d}\xi_i)$.

Recall that $J_0$ is bounded, so we have

$$\begin{aligned} & n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \\ & \quad \le C\frac{1}{n!}\int_{[0,t]^{2n}} \mathrm{d}s\mathrm{d}r \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(r,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \prod_{i=1}^{n} \lambda(s_i - r_i). \end{aligned}$$

Furthermore by Cauchy-Schwarz inequality,

$$\begin{aligned} & \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(r,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \\ & \quad \le \left\{ \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(s,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \right\}^{1/2} \\ & \qquad \cdot \left\{ \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(r,y,t,x) g_n(r,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \right\}^{1/2} \end{aligned}$$

(i) Let $\Lambda(\cdot) = \varphi_H(\cdot)$ and use the estimation of $Y$ in Lemma **??**, we have

$$\begin{aligned} & \int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(s,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \\ & \quad \le C\prod_{i=1}^{d} \int_{\mathbb{R}^{2n}} \prod_{k=1}^{n} \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i \quad (5.5.2) \end{aligned}$$

where

$$\Theta_n(t,y_{ik},s) := |s_{\sigma(k+1)} - s_{\sigma(k)}|^{\frac{\zeta_d}{d}} |y_{i\sigma(k+1)} - y_{i\sigma(k)}|^{\frac{\kappa_d}{d}} p(s_{\sigma(k+1)} - s_{\sigma(k)}, y_{i\sigma(k+1)} - y_{i\sigma(k)});$$

$$y_i = (y_{i1}, y_{i2}, \cdots, y_{ik}, \cdots, y_{in}), \qquad z_i = (z_{i1}, z_{i2}, \cdots, z_{ik}, \cdots, z_{in});$$

$$dy_i := \prod_{k=1}^{n} dy_{ik} \qquad dz_i := \prod_{k=1}^{n} dz_{ik};$$

and

$$y_{\sigma(k+1)} = z_{\sigma(k+1)} := x_i; \qquad s_{\sigma(n+1)} = r_{\sigma(n+1)} := t.$$

Let's first consider the case $2H_i - 2 + \kappa_d/d \neq -1$. Applying Lemma **??** to

$$\int_{\mathbb{R}^{2n}} \prod_{k=1}^{n} \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t,y_{ik},s) \Theta_n(t,z_{ik},s) dy_i dz_i \tag{5.5.3}$$

for $dy_{i\sigma(1)} dz_{i\sigma(1)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \prod_{k=1}^{n} \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t,y_{ik},s) \Theta_n(t,z_{ik},s) dy_i dz_i \\ \leq C(s_{i\sigma(2)} - s_{i\sigma(1)})^{2\ell_i} \int_{\mathbb{R}^{2n}} \prod_{k=2}^{n} \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t,y_{ik},s) \Theta_n(t,z_{ik},s) dy_i dz_i\end{aligned}$$

where

$$\ell_i = \frac{\zeta_d}{d} + \theta_i.$$

Applying Lemma **??** to (**??**) for $dy_{i\sigma(k)}dz_{i\sigma(k)}, k=2,\cdots,n$, we have

$$\prod_{i=1}^{d}\int_{\mathbb{R}^{2n}}\prod_{k=1}^{n}\varphi_{H_i}(y_{ik}-z_{ik})\Theta_n(t,y_{ik},s)\Theta_n(t,z_{ik},s)dy_idz_i \le \prod_{k=1}^{n}C^n(s_{\sigma(k+1)}-s_{\sigma(k)})^{2\ell}$$

where

$$\ell=\sum_{i=1}^{d}\ell_i=\zeta_d+\frac{|H|\alpha}{2}+\frac{\kappa_d\alpha}{2} \quad \text{with} \quad |H|=\sum_{i=1}^{d}H_i\,. \tag{5.5.4}$$

Due to the same argument, we have

$$\prod_{i=1}^{d}\int_{\mathbb{R}^{2n}}\prod_{k=1}^{n}\varphi_{H_i}(y_{ik}-z_{ik})\Theta_n(t,y_{ik},r)\Theta_n(t,z_{ik},r)dy_idz_i \le \prod_{k=1}^{n}C^n(r_{\rho(k+1)}-r_{\rho(k)})^{2\ell}$$

Therefore

$$\int_{\mathbb{R}^{2nd}}\mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x)g_n(r,z,t,x)\prod_{i=1}^{n}\Lambda(y_i-z_i)\le C^n\,(\phi(s)\phi(r))^{\ell},$$

where

$$\phi(s):=\prod_{i=1}^{n}(s_{\sigma(i+1)}-s_{\sigma(i)}), \qquad \phi(r):=\prod_{i=1}^{n}(r_{\rho(i+1)}-r_{\rho(i)}),$$

with

$$0<s_{\sigma(1)}<s_{\sigma(2)}<\ldots<s_{\sigma(n)} \quad \text{and} \quad 0<r_{\rho(1)}<r_{\rho(2)}<\ldots<r_{\rho(n)}.$$

Hence,

$$\begin{aligned} n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} &\le \frac{C^n}{n!}\int_{[0,t]^{2n}}\prod_{i=1}^{n}\lambda(s_i-r_i)(\phi(s)\phi(r))^{\ell}\mathrm{d}s\mathrm{d}r \\ &\le \frac{C^n}{n!}\frac{1}{2}\int_{[0,t]^{2n}}\prod_{i=1}^{n}\lambda(s_i-r_i)\left(\phi(s)^{2\ell}+\phi(r)^{2\ell}\right)\mathrm{d}s\mathrm{d}r \\ &= \frac{C^n}{n!}\int_{[0,t]^{2n}}\prod_{i=1}^{n}\lambda(s_i-r_i)\phi(s)^{2\ell}\mathrm{d}s\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C^n C_t^n}{n!} \int_{[0,t]^n} \phi(s)^{2\ell} \mathrm{d}s \\
&= C^n C_t^n \int_{T_n(t)} \phi(s)^{2\ell} \mathrm{d}s \\
&= \frac{C^n C_t^n \Gamma(2\ell+1)^n t^{(2\ell+1)n}}{\Gamma((2\ell+1)n+1)},
\end{aligned}
$$

where $C_t = 2\int_0^t \lambda(r)dr$. The last step is by Lemma **??**.

Therefore,

$$
n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \leq \frac{C^n C_t^n}{\Gamma((2\ell+1)n+1)},
$$

and $\sum_{n\geq 0} n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}}$ converges if $\ell > -1/2$.

Next we need to show

$$
\ell > -1/2 \iff |H| > d - 2 + \frac{1}{\alpha}.
$$

Firstly by definition of $\ell$ (**??**)

$$
\ell > -1/2 \iff |H| > -\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha}\zeta_d.
$$

Then using the definition of $\zeta_d$ and $\kappa_d$ in (**??**), (**??**), (**??**), (**??**) we have:

when $1/2 < \alpha < 1$,

$$
\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha}\zeta_d = \begin{cases}
-1 + \frac{1}{\alpha}, & d = 1; \\
\frac{1}{\alpha}, & d = 2; \\
\frac{1}{\alpha} + 2, & d = 4; \\
\frac{1}{\alpha} - 2 + d, & d = 3 \text{ or } d \geq 5;
\end{cases}
$$

when $1 < \alpha < 2$,

$$\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha}\zeta_d = \begin{cases} -1 + \frac{1}{\alpha}, & d = 1; \\ d - 2 + \frac{1}{\alpha}, & d \geq 2; \end{cases}$$

For case $2H_i - 2 + \kappa_d/d = -1$, applying Lemma **??** to (**??**), we have

$$\prod_{i=1}^{d} \int_{\mathbb{R}^{2n}} \prod_{k=1}^{n} \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i \leq \prod_{k=1}^{n} C^n (s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell'},$$

where

$$\ell' = \zeta_d + \frac{d\varepsilon + \kappa_d + d}{4}\alpha \quad \text{with} \quad |H| = \sum_{i=1}^{d} H_i\,.$$

Using the relation $2H_i - 2 + \kappa_d/d = -1$, we have

$$\ell' = \ell + \frac{d\alpha}{4}\varepsilon.$$

Since

$$|H| > d - 2 + \frac{1}{\alpha} \implies \ell > -1/2\,,$$

we can choose $\varepsilon$ big enough such such

$$|H| > d - 2 + \frac{1}{\alpha} \implies \ell' > -1/2\,.$$

Lastly, when $\alpha \in (1/2, 1)$, for $d \leq 4, H_i > 1/2$ implies condition (**??**); for $d > 4$, condition (**??**) is implied by

$$H_i > 1 - \frac{2}{d} - \frac{\gamma}{2d}$$

with $\gamma_0$ sufficiently small; when $\alpha \in (1,2)$ for $d=1, H_i > 1/2$ implies (**??**); for $d \geq 2$, (**??**) is implied by

$$H_i > 1 - \frac{2}{d}$$

with $\nu_0$ sufficiently small. This completes the proof of Theorem **??** for case of $\Lambda(\cdot) = \varphi_H(\cdot)$

(ii) Let $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$. For Reisz kernel, notice that

$$|x|^{-\kappa} \leq C \prod_{i=1}^{d} |x_i|^{\frac{\kappa}{d}},$$

so this case is reduced to case (i) with $H_i = (-\frac{\kappa}{d} + 2)\frac{1}{2}$, $i = 1, 2, \cdots, d$.

Correspondingly

$$|H| > d - 2 + \frac{1}{\alpha}$$

is

$$\kappa < 4 - 2/\alpha,$$

which also guarantees condition (**??**) .

For Bessel kernel, applying Lemma **??** for $\mathrm{d}y_{\sigma(i)}\mathrm{d}z_{\sigma(i)}$ in the order of $i = 1, 2, \cdots, n$ to

$$\int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(s,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i)$$

yields

$$\int_{\mathbb{R}^{2nd}} \mathrm{d}y\mathrm{d}z\, g_n(s,y,t,x) g_n(s,z,t,x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \leq \prod_{k=1}^{n} C^n (s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell},$$

where

$$\ell := \zeta_d - \frac{\alpha}{4}\kappa + \frac{\alpha}{2}\kappa_d + \frac{\alpha}{2}d$$

As in case (i), $\sum_{n\geq 0} n!\|f_n(\cdot,\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}}$ converges if $\ell > -1/2$. Then using the definition of $\zeta_d$ and $\kappa_d$ in (**??**), (**??**), (**??**), (**??**), we have

$$\ell > -1/2 \iff \kappa < 4 - 2/\alpha.$$

This finishes the the proof of the theorem. □

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