Certain Continuous Groups of Projective Transformations Treated Analytically

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Certain Continuous Groups of Projective Transformations Treated Analytically

It is the purpose of this paper to determine analytically certain continuous groups of plane collineations which have already been determined geometrically by Prof. H. B. Newson in an article printed in the American Journal of Mathematics, Vol. XXIV, No. 2.

The same author has shown elsewhere* that a plane collineation of the most general form leaving invariant a triangle can be written in the form:

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ A' & B' & 1 & KA' \\ A'' & B'' & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ A' & B' & 1 & KB' \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}}$$

where (AB), $(A'B')$ and $(A''B'')$ are the Cartesian

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* Kansas University Quarterly, Vol. VIII, No. 2, April 1899, Series A, page 48.

coordinates of the vertices of the invariant triangle and K and K' are the two independent cross-ratios along two of the sides of the invariant triangle. It will be noticed that every collineation of this type contains eight variable parameters of two distinct types, namely, the six coordinates of the three invariant points and the two characteristic cross-ratios.

The whole group of plane collineations, therefore, consists of ∞^8 different collineations and may be designated G_8.

In this paper is sought the determination of the most general sub-groups of G_8.

It is evident that there are ∞^6 plane collineations which leave a single point invariant, since the fixing of a point in the plane requires the changing of two parameters into constants. In order to test whether these ∞^6 collineations form a group let

$$T_1: \quad x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ A' & B' & 1 & KA' \\ A'' & B'' & 1 & KA'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ A' & B' & 1 & KB' \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}}$$

be a plane collineation of the most general type leaving a triangle invariant and transforming the point $(x\,y)$ into the point $(x_1 y_1)$, and let

$$T_2: \quad x_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A & B & 1 & A \\ A_1' & B_1' & 1 & KA_1' \\ A_1'' & B_1'' & 1 & K'A_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A & B & 1 & 1 \\ A_1' & B_1' & 1 & K_1 \\ A_1'' & B_1'' & 1 & K_1' \end{vmatrix}} \qquad y_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A & B & 1 & B \\ A_1' & B_1' & 1 & K_1B_1' \\ A_1'' & B_1'' & 1 & K_1'B_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A & B & 1 & 1 \\ A_1' & B_1' & 1 & K_1 \\ A_1'' & B_1'' & 1 & K_1' \end{vmatrix}}$$

be a second plane collineation transforming the point $(x_1\, y_1)$ into $(x_2\, y_2)$

and leaving invariant a different triangle except for the point (AB) which is to be left invariant by both T_1 and T_2.

The work may be simplified somewhat without affecting its generality by supposing that the point (AB) has been selected as origin in each collineation. Substituting, then, (00) for (AB) gives:

$$T_1: \quad x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & B' & 1 & KA' \\ A'' & B'' & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & B' & 1 & KB' \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}}, \qquad [1]$$

and an exactly similar expression for T_2 except that all of the letters in the determinants are subscipted one. Let this form of T_2 be numbered [2]. Hereafter in referring to T_1 and T_2 the forms [1] and [2] will be meant.

In order to get from these two transformations their resultant, which transforms (xy) directly into $(x_2 y_2)$, it is necessary to eliminate x_1 and y_1 from [1] and [2].

Expanding T_1:

$$[3]\quad \begin{cases} x_1 = \dfrac{x(KA'B''-K'A''B') + y(K'-K)A'A''}{x[B''(K-1)-B'(K'-1)] + y[A'(K'-1)-A''(K-1)] + A'B''-A''B'} \\[2ex] y_1 = \dfrac{x(K'-K)B'B'' + y(K'A'B''-KA''B')}{x[B''(K-1)-B'(K'-1)] + y[A'(K'-1)-A''(K-1)] + A'B''-A''B'} \end{cases}$$

Expanding T_2:

$$[4]\quad \begin{cases} x_2 = \dfrac{x_1(K_1A_1'B_1''-K_1'A_1''B_1') + y_1(K_1'-K_1)A_1'A_1''}{x_1[B_1''(K_1-1)-B_1'(K_1'-1)] + y_1[A_1'(K_1'-1)-A_1''(K_1-1)] + A_1'B_1''-A_1''B_1'} \\[2ex] y_2 = \dfrac{x_1(K_1'-K)B_1'B_1'' + y_1(K_1'A_1'B_1''-K_1A_1''B_1')}{x_1[B_1''(K_1-1)-B_1'(K_1'-1)] + y_1[A_1'(K_1'-1)-A_1''(K_1-1)] + A_1'B_1''-A_1''B_1'} \end{cases}$$

Let $KA'B''-K'A''B' = \alpha$, $\quad K'-K = \alpha'$,
$K'A'B''-KA''B' = \Delta$, $\quad B''(K-1)-B'(K'-1) = \lambda$,
$A'(K'-1)-A''(K-1) = \lambda'$, $\quad A'B''-A''B' = \gamma$,

and the same expressions in letters subscripted one equal the same Greek letters subscripted one.

[3] and [4] then become:

$$T_1: \left.\begin{aligned} x_1 &= \frac{\alpha x + \alpha' A' A'' y}{\lambda x + \lambda' y + \gamma} \\ y_1 &= \frac{\alpha' B' B'' x + \Delta y}{\lambda x + \lambda' y + \gamma} \end{aligned}\right\} \text{------} [5]$$

$$T_2: \left.\begin{aligned} x_2 &= \frac{\alpha_1 x_1 + \alpha_1' A_1' A_1'' y_1}{\lambda_1 x_1 + \lambda_1' y_1 + \gamma} \\ y_2 &= \frac{\alpha_1' B_1' B_1'' x_1 + \Delta_1 y_1}{\lambda_1 x_1 + \lambda_1' y_1 + \gamma} \end{aligned}\right\} \text{------} [6]$$

Eliminating x_1 and y_1 from [5] and [6] and collecting for x and y gives the resultant:

$$T_3: \left.\begin{aligned} x_2 &= \frac{(\alpha\alpha_1 + \alpha'\alpha_1' A_1' A_1'' B' B'')x + (\alpha'\alpha_1 A' A'' + \alpha_1' A_1' A_1'' \Delta) y}{(\alpha\lambda_1 + \alpha'\lambda_1' B' B'' + \lambda\gamma_1)x + (\alpha'\lambda_1 A' A'' + \lambda_1'\Delta + \lambda'\gamma_1)y + \gamma\gamma_1} \\ y_2 &= \frac{(\alpha\alpha_1' B_1' B_1'' + \alpha'\Delta_1 B' B'')x + (\alpha'\alpha_1' A' A'' B_1' B_1'' + \Delta\Delta_1) y}{(\alpha\lambda_1 + \alpha'\lambda_1' B' B'' + \lambda\gamma_1)x + (\alpha'\lambda_1 A' A'' + \lambda_1'\Delta + \lambda'\gamma_1)y + \gamma\gamma_1} \end{aligned}\right\} [7]$$

Writing a collineation (T_3') of the form of [1] which would transform $(x\,y)$ directly into $(x_2\,y_2)$ gives:

$$T_3': \quad x_2 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_2' & B_2' & 1 & K_2 A_2' \\ A_2'' & B_2'' & 1 & K_2' A_2'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_2' & B_2' & 1 & K_2 \\ A_2'' & B_2'' & 1 & K_2' \end{vmatrix}} \qquad y_2 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_2' & B_2' & 1 & K_2 B_2' \\ A_2'' & B_2'' & 1 & K_2' B_2'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_2' & B_2' & 1 & K_2 \\ A_2'' & B_2'' & 1 & K_2' \end{vmatrix}} \qquad [8]$$

Expanding T_3' gives:

$$T_3': \quad \left.\begin{aligned} x_2 &= \frac{\alpha_2 x + \alpha_2' A_2' A_2'' y}{\lambda_2 x + \lambda_2' y + \gamma_2} \\ y_2 &= \frac{\alpha_2' B_2' B_2'' x + \Delta_2 y}{\lambda_2 x + \lambda_2' y + \gamma_2} \end{aligned}\right\} \cdots\cdots [9]$$

in which $\alpha_2 = K_2 A_2' B_2'' - K_2' A_2'' B_2'$, $\alpha_2' = K_2' - K_2$,

$\lambda_2 = B_2'(K_2 - 1) - B_2'(K_2' - 1)$, $\lambda_2' = A_2'(K_2' - 1) - A_2''(K_2 - 1)$,

$\gamma_2 = A_2' B_2'' - A_2'' B_2'$, and $\Delta_2 = K_2' A_2' B_2'' - K_2 A_2'' B_2'$

If the collineations T_3 and T_3' are to be identical the corresponding coefficients of x and y must be equal. Dividing out the γ's and equating resulting ratios gives:

$$\left.\begin{aligned}
\frac{\alpha_2}{\gamma_2} &= \frac{\alpha\alpha_1 + \alpha'\alpha_1' A_1' A_1'' B' B''}{\gamma\gamma_1} \cdots\cdots (1) \\
\frac{\alpha_2' A_2' A_2''}{\gamma_2} &= \frac{\alpha'\alpha_1 A' A'' + \alpha_1' \Delta A_1' A_1''}{\gamma\gamma_1} \cdots\cdots (2) \\
\frac{\alpha_2' B_2' B_2''}{\gamma_2} &= \frac{\alpha\alpha_1' B_1' B_1'' + \alpha' \Delta_1 B' B''}{\gamma\gamma_1} \cdots\cdots (3) \\
\frac{\Delta_2}{\gamma_2} &= \frac{\alpha'\alpha_1' A' A'' B_1' B_1'' + \Delta\Delta_1}{\gamma\gamma_1} \cdots\cdots (4) \\
\frac{\lambda_2}{\gamma_2} &= \frac{\alpha\lambda_1 + \alpha'\lambda_1' B' B'' + \lambda\gamma_1}{\gamma\gamma_1} \cdots\cdots (5) \\
\frac{\lambda_2'}{\gamma_2} &= \frac{\alpha'\lambda_1 A' A'' + \lambda_1' \Delta + \lambda'\gamma_1}{\gamma\gamma_1} \cdots\cdots (6)
\end{aligned}\right\} [10]$$

Substituting in these equations [10] the values of the Greek letters in terms of the original parameters gives:

$$
\left.
\begin{aligned}
\frac{K_2 A_2' B_2'' - K_2' A_2'' B_2'}{A_2' B_2'' - A_2'' B_2'} &= \frac{(KA'B''-K'A''B')(K_1A_1'B_1''-K_1'A_1''B_1') + A_1'A_1''B'B''(K'-K)(K_1'-K_1)}{(A'B''-A''B')(A_1'B_1''-A_1''B_1')} \cdots (1) \\[2ex]
\frac{(K_2'-K_2)A_2'A_2''}{A_2'B_2''-A_2''B_2'} &= \frac{A'A''(K'-K)(K_1A_1'B_1''-K_1'A_1''B_1') + A_1'A_1''(K_1'-K_1)(K'A'B''-KA''B')}{(A'B''-A''B')(A_1'B_1''-A_1''B_1')} \cdots (2) \\[2ex]
\frac{(K_2'-K_2)B_2'B_2''}{A_2'B_2''-A_2''B_2'} &= \frac{B_1'B_1''(K_1'-K_1)(KA'B''-K'A''B') + B'B''(K'-K)(K_1'A_1'B_1''-K_1A_1''B_1')}{(A'B''-A''B')(A_1'B_1''-A_1''B_1')} \cdots (3) \\[2ex]
\frac{K_2'A_2'B_2''-K_2A_2''B_2'}{A_2'B_2''-A_2''B_2'} &= \frac{A'A''B_1'B_1''(K'-K)(K_1'-K_1) + (K'A'B''-KA''B')(K_1'A_1'B_1''-K_1A_1''B_1')}{(A'B''-A''B')(A_1'B_1''-A_1''B_1')} \cdots (4) \\[2ex]
\frac{B_2'(K_2-1)-B_2'(K_2'-1)}{A_2'B_2''-A_2''B_2'} &= \frac{(KA'B''-K'A''B')[B_1''(K_1-1)-B_1'(K_1'-1)] + B'B''(K'-K)[A_1'(K_1'-1)-A_1''(K_1-1)] + [B''(K-1)-B'(K'-1)](A_1'B_1''-A_1''B_1')}{(A'B''-A''B')(A_1'B_1''-A_1''B_1')} \cdots (5) \\[2ex]
\frac{A_2'(K_2'-1)-A_2''(K_2-1)}{A_2'B_2''-A_2''B_2'} &= \frac{A'A''(K'-K)[B_1''(K_1-1)-B'(K_1'-1)] + (K'A'B''-KA''B')[A_1'(K_1'-1)-A_1''(K_1-1)] + [A'(K'-1)-A''(K-1)](A_1'B_1''-A_1''B_1')}{(A'B''-A''B')(A_1'B_1''-A_1''B_1')} \cdots (6)
\end{aligned}
\right\} [11]
$$

These six equations fully determine the six parameters of T_3' in terms of the parameters of T_1 and T_2. Therefore the ∞^6 collineations which all leave the same point in the plane invariant possess the group property that the resultant of any two of them is a third collineation of the same kind.

In order to show conclusively that these ∞^6 collineations form a group it is necessary further to show that the inverse of each collineation is another of the same kind. This will be true if T_1, which may represent any one of the ∞^6 collineations leaving the point A, coordinates $(0\,0)$, invariant and transforming an arbitrary point $(x\,y)$ into another arbitrary point $(x_1\,y_1)$, can be solved for x and y in terms of x_1 and y_1 and a collineation of the same kind obtained which transforms $(x_1 y_1)$ back into $(x y)$.

Taking the equations of T_1 in the form [5], page 6, clearing and collecting for x and y gives:

$$(\lambda x_1 - \alpha)x + (\lambda' x_1 - \alpha' A'A'')y + \gamma x_1 = 0$$

$$(\lambda y_1 - \alpha' B'B'')x + (\lambda' y_1 - \Delta)y + \gamma y_1 = 0$$

Solving these two equations for x and y gives:

$$T_1': \quad x = \frac{-\Delta\gamma x_1 + \alpha'\gamma A'A'' y_1}{(\lambda\Delta - \alpha'\lambda' B'B'')x_1 + (\alpha\lambda' - \lambda\alpha' A'A'')y_1 + (\alpha'^2 A'A''B'B'' - \alpha\Delta)}$$

$$y = \frac{\alpha'\gamma B'B'' x_1 - \alpha\gamma y_1}{(\lambda\Delta - \alpha'\lambda' B'B'')x_1 + (\alpha\lambda' - \lambda\alpha' A'A'')y_1 + (\alpha'^2 A'A''B'B'' - \alpha\Delta)}$$

These equations are of the same analytical form as the direct collineation [5]. The inverse of T_1, therefore exists and is a collineation of the same kind. If it were deemed necessary it is clearly possible to set up a collineation T_1''

of the form of T_1 which would transform $(x_1 y_1)$ into $(x y)$ and make it identical with T_1' by dividing out the absolute terms and equating the resulting ratios. Six equations of condition would be obtained which would fully determine the six parameters of T_1'' in terms of T_1.

We have shown, then, that the ∞^6 plane collineations each leaving a triangle invariant and all leaving the same point (A) invariant form a group. This group is, of course, a sub-group of G_8. It is a six-parameter-ed group, and, following Prof. Newson's notation, we shall designate it $G_6(A)$. There are ∞^2 such sub-groups $G_6(A)$ in G_8, one for each point in the plane.

Let us next seek to determine whether there is a five parametered sub-group of G_8. In order to fix a lineal element in the plane three parameters must be changed into constants. There are, therefore, ∞^5 plane collineations leaving the same lineal element invariant. In order to determine whether these ∞^5 collineations form a group we proceed exactly as in determining the six-parametered group. If the common point be taken as origin and the common line as the x-axis, the equations of two collineations each leaving a triangle invariant and both leaving the same lineal element invariant, may be obtained by substituting in T_1 and T_2 (equations [1] and [2]) $B' = B_1' = 0$. This gives for T_1:

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & KA' \\ A'' & B'' & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & 0 \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad [12]$$

and exactly similar equations for T_2 [13] except that the letters inside the determinants are subscripted one. As before we shall obtain the resultant of these two transformations by expanding them and eliminating x_1 and y_1.

Expanding T_1:

$$\left.\begin{aligned} x_1 &= \frac{KA'B''x + A'A''(K'-K)y}{B''(K-1)x + [A'(K'-1) - A''(K-1)]y + A'B''} \\ y_1 &= \frac{K'A'B''y}{B''(K-1)x + [A'(K'-1) - A''(K-1)]y + A'B''} \end{aligned}\right\} \quad [12a]$$

Expanding T_2:

$$\left.\begin{aligned} x_2 &= \frac{K_1A_1'B_1''x_1 + A_1'A_1''(K_1'-K_1)y}{B_1''(K_1-1)x_1 + [A_1'(K_1'-1) - A_1''(K_1-1)]y_1 + A_1'B_1''} \\ y_2 &= \frac{K_1'A_1'B_1''y_1}{B_1''(K_1-1)x_1 + [A_1'(K_1'-1) - A_1''(K_1-1)]y_1 + A_1'B_1''} \end{aligned}\right\} \quad [13a]$$

Eliminating x_1 and y_1 from these two collineations gives the resultant:

$$T_3:\quad x_2 = \frac{K K_1 A' A_1' B'' B_1'' x + \left[K_1 A_1' B_1'' A' A'' (K' - K) + K' A' B'' A_1' A_1'' (K_1' - K_1)\right]}{\left[K A' B'' B_1'' (K_1 - 1) + A_1' B'' B_1'' (K - 1)\right] x + \left[A' A'' B_1'' (K' - K)(K_1 - 1) + \left\{A_1'(K_1' - 1) - A_1''(K_1 - 1)\right\} K' A' B'' + A_1' B_1'' \left\{A'(K' - 1) - A''(K - 1)\right\}\right] + A' A_1' B'' B_1''}$$

$$y_2 = \frac{K' K_1' A' A_1' B'' B_1'' y}{\left[K A' B'' B_1'' (K_1 - 1) + A_1' B'' B_1'' (K - 1)\right] x + \left[A' A'' B_1'' (K' - K)(K_1 - 1) + \left\{A_1'(K_1' - 1) - A_1''(K_1 - 1)\right\} K' A' B'' + A_1' B_1'' \left\{A'(K' - 1) - A''(K - 1)\right\}\right] + A' A_1' B'' B_1''} \qquad [14]$$

Writing a collineation (T_3') of the form of [12] which would transform $(x\,y)$ directly into $(x_2\,y_2)$ we have:

$$T_3':\quad x_2 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_2' & 0 & 1 & K_2 A_2' \\ A_2'' & B_2'' & 1 & K_2' A_2'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_2' & 0 & 1 & K_2 \\ A_2'' & B_2'' & 1 & K_2' \end{vmatrix}} \qquad y_2 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_2' & 0 & 1 & 0 \\ A_2'' & B_2'' & 1 & K_2' B_2'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_2' & 0 & 1 & K_2 \\ A_2'' & B_2'' & 1 & K_2' \end{vmatrix}} \qquad [15]$$

Expanding T_3' gives:

$$T_3':\quad \left\{\begin{aligned} x_2 &= \frac{K_2 A_2' B_2'' x + A_2' A_2'' (K_2' - K_2)\, y}{B_2''(K_2 - 1)x + \left[A_2'(K_2' - 1) - A_2''(K_2 - 1)\right] y + A_2' B_2''} \\ y_2 &= \frac{K_2' A_2' B_2'' y}{B_2''(K_2 - 1)x + \left[A_2'(K_2' - 1) - A_2''(K_2 - 1)\right] y + A_2' B_2''} \end{aligned}\right\} \quad [16]$$

If the collineations T_3 [14] and T_3' [16] are to be identical the corresponding coefficients of x and y must be equal.

Dividing the numerator and denominator of the right hand member of each equation in [14] and [16] by the absolute term of the denominator and equating the resulting ratios gives:

$$\left.\begin{array}{ll}
K_2 = K K_1 & \cdots\cdots (1) \\[6pt]
\dfrac{A_2''(K_2'-K_2)}{B_2''} = \dfrac{K_1 A''(K'-K)}{B''} + \dfrac{K' A_1''(K_1'-K_1)}{B_1''} & \cdots\cdots (2) \\[6pt]
K_2' = K' K_1' & \cdots\cdots (3) \\[6pt]
\dfrac{K_2-1}{A_2'} = \dfrac{K(K_1-1)}{A_1'} + \dfrac{K-1}{A'} & \cdots\cdots (4) \\[6pt]
\dfrac{K_2'-1}{B_2''} - \dfrac{A_2''(K_2-1)}{A_2' B_2''} = \dfrac{(K_1-1)(K'-K)A''}{A_1' B''} + \dfrac{K'(K_1'-1)}{B_1''} & \\[6pt]
\qquad - \dfrac{K'(K_1-1)A_1''}{A_1' B_1''} + \dfrac{(K'-1)}{B''} - \dfrac{A''(K-1)}{A' B''} & \cdots\cdots (5)
\end{array}\right\} [17]$$

As was to be expected this gives five equations of condition which fully determine the five parameters of T_3'' [15] in terms of T_1 [12]. The resultant, therefore of two plane collineations each leaving a triangle invariant and both leaving the same lineal element invariant is another collineation of the same kind. In attempting to determine whether the resultant of any two of the ∞^5 collineations leaving the same lineal element invariant is another collineation of the same kind, the work has been done independently of the work of determining the groups $G_6(A)$ in order that the one work may check the other. If the work has been done in each case correctly the substitution of the condition $B' = B_1' = B_2' = 0$ in equations [11], page 8, (the six equations of condition which determined the six parameters of the resultant in $G_6(A)$,) should reduce these to five equations of conditions, and, moreover, the same five equations of condition that we have in [17] on the preceding page. Upon making the substitution we find this to be the case. Of equations [11] no. (3) becomes $0=0$ and drops out leaving (1), (2), (4), (5) and (6) identical respectively with (1), (2), (3), (4) and (5) of [17]

In order to show that the ∞^5 collineations having a lineal element in common possess the second group property, namely, that the inverse collineation to each one exists and is another collineation of the same kind, let us take the equations of T_1 in the expanded form [12a], page 13, and attempt to solve them for x and y. Clearing, transposing and collecting for x and y, we have:

$$\left.\begin{aligned} &[B''(K-1)x_1 - KA'B'']x + \left[\{A'(K'-1) - A''(K-1)\}x_1 - A'A''(K'-K)\right]y + A'B''x_1 = 0 \\ &B''(K-1)y_1x + \left[\{A'(K'-1) - A''(K-1)\}y_1 - K'A'B''\right]y + A'B''y_1 = 0 \end{aligned}\right\} [18]$$

Solving these two equations for x and y gives:

$$\left.\begin{aligned} x &= \frac{K'A'B''x_1 - (K'-K)A'A''y_1}{K'B''(1-K)x_1 + \left[A''(K-1)(K'-K) - K\{A'(K'-1) - A''(K-1)\}\right]y_1 - KK'A'B''} \\ y &= \frac{KA'B''y_1}{K'B''(1-K)x_1 + \left[A''(K-1)(K'-K) - K\{A'(K'-1) - A''(K-1)\}\right]y_1 - KK'A'B''} \end{aligned}\right\} [19]$$

These equations give us the inverse

collineation since they are of the same analytical form as the direct collineation [12a]. The inverse of T_1, [12a] therefore exists and is a collineation of the same kind. To verify the correctness of the solution of equations [18] for equations [19] it is only necessary to substitute in the equations obtained for the inverse of T_1' [1] on page 10 the original expressions for the Greek letters and the condition $B' = B_1' = 0$ and reduce. Performing the substitution and reduction readily shows them to be identical with [19].

We have shown, then, that the ∞^5 collineations each leaving a triangle invariant and all leaving the same lineal element invariant form a group. It is a five-parametered sub-group of G_8 and will be referred to as $G_5(Al)$. Since there are ∞^3 lineal elements in the plane, it follows that there are ∞^3 sub-groups $G_5(Al)$ in G_8.

We shall next investigate whether there are four-parametered sub-groups in G_8. Four parameters become constants in the fixing of two points. We, therefore, let T_1 and T_2 be two plane collineations each leaving a triangle invariant and both leaving the same two points invariant – say the points (AB) and $(A'B')$. If the points A and B, [coordinates (AB) and $(A'B')$] are common the line joining them will be common to T_1 and T_2. Let A be taken for origin as in the work for $G_6(A)$ and $G_5(Al)$ and let the line AB be chosen for the x-axis. We then have for T_1:

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & KA' \\ A'' & B'' & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & 0 \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}}, \quad [20]$$

and for T_2:

$$x_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & K_1A' \\ A_1'' & B_1'' & 1 & K_1'A_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K_1 \\ A_1'' & B_1'' & 1 & K_1' \end{vmatrix}} \qquad y_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & K_1A' \\ A_1'' & B_1'' & 1 & K_1'A_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K_1 \\ A_1'' & B_1'' & 1 & K_1' \end{vmatrix}}. \qquad [21]$$

We proceed as before to obtain the resultant of T_1 [20] and T_2 [21].

Expanding [20] and [21] gives:

$$T_1: \begin{cases} x_1 = \dfrac{KA'B''x + A'A''(K'-K)y}{B''(K-1)x + [A'(K'-1) - A''(K-1)]y + A'B''} \\[2ex] y_1 = \dfrac{K'A'B''y}{B''(K-1)x + [A'(K'-1) - A''(K-1)]y + A'B''} \end{cases} \qquad [22]$$

$$T_2: \begin{cases} x_2 = \dfrac{K_1A'B_1''x_1 + A'A_1''(K_1'-K_1)y_1}{B_1''(K_1-1)x_1 + [A'(K_1'-1) - A_1''(K_1-1)]y_1 + A'B_1''} \\[2ex] y_2 = \dfrac{K_1'A'B_1''y_1}{B_1''(K_1-1)x_1 + [A'(K_1'-1) - A_1''(K_1-1)]y_1 + A'B_1''} \end{cases} \qquad [23].$$

Eliminating x_1 and y_1 from [22] and [23] gives the resultant:

$$T_3: \begin{cases} x_2 = \dfrac{KK_1A'B''B_1''x + [K_1A'A''B_1''(K'-K) + K'A'A_1''B''(K_1'-K_1)]y}{\begin{gathered}[KB''B_1''(K_1-1) + B''B_1''(K-1)]x + [A''B_1''(K'-K)(K_1-1) + K'B''\{A'(K_1'-1) - A_1''(K_1-1)\} \\ + B_1''\{A'(K'-1) - A''(K-1)\}]y + A'B''B_1''\end{gathered}} \\[4ex] y_2 = \dfrac{K'K_1'A'B''B_1''y}{[\text{Same denominator as for } x_2]} \end{cases} \qquad [24]$$

Writing a collineation (T_3') of the form of T_1 and T_2 which would leave the same two points A and B invariant and transform the point (xy) directly into $(x_2 y_2)$ and expanding T_3' gives:

$$T_3': \begin{cases} x_2 = \dfrac{K_2 A' B_2'' x + A' A_2'' (K_2' - K_2) y}{B_2''(K_2-1)x + \left[A'(K_2'-1) - A_2''(K_2'-1)\right] y + A' B_2''} \\[2ex] y_2 = \dfrac{K_2' A' B_2'' y}{B_2''(K_2-1)x + \left[A'(K_2'-1) - A_2''(K_2-1)\right] y + A' B_2''} \end{cases} \qquad [25]$$

If T_3 [24] and T_3' [25] are to be identical the corresponding coefficients of x and y must be equal. Dividing out the absolute terms and equating resulting ratios we have:

$$\left.\begin{aligned} & K_2 = K K_1 \text{ .... (1)} \\ & \frac{A_2''}{B_2''}(K_2' - K_2) = \frac{K_1 A''}{B''}(K' - K) + \frac{K' A_1''}{B_1''}(K_1' - K_1) \text{ .... (2)} \\ & K_2' = K' K_1' \text{ .... (3)} \\ & \frac{K_2' - 1}{B_2''} - \frac{A_2''(K_2 - 1)}{A' B_2''} = \frac{A''}{A' B''}(K' - K)(K_1 - 1) + \frac{K'(K_1' - 1)}{B_1''} + \frac{K' - 1}{B''} \\ & \qquad - \frac{K'(A_1'')(K_1 - 1)}{A' B_1''} - \frac{A''(K - 1)}{A' B''} \text{ .... (4)} \end{aligned}\right\} [26]$$

It will be noticed that the equation obtained by equating the coefficients of x in the denominator and the equations obtained by equating the coefficients of y in the numerator of the right member of the equation for y_2 both reduce to the same equation (3) above.

This gives four independent equations of condition which completely determine the four parameters of T_3' [25] in terms of the original collineations T_1 and T_2.

The ∞^4 collineations each leaving a triangle invariant and all leaving the same two points invariant, therefore, possess the fundamental group property that the resultant of any two of them is another of the same kind.

We may check the correctness of the work in obtaining equations [26] by substituting in equations [17] page 15, the condition $A_2' = A_1' = A'$. We find upon making the substitution that equation (4) of [17] becomes identical with (3) and equations [17] reduce to equations [26].

In order to show that the ∞^4 collineations under discussion possess the other necessary group property we would take equations of T_1 in the [22], page 20, and solve for x and y in terms of x_1 and y_1. But we find that this work has already been done on page 17, since equations [22] and [12a] are exactly the same.

The ∞^4 plane collineations each of which leaves invariant a triangle and all of which leave the same two points invariant, possess the group properties and therefore form a sub-group of G_8. Since there are ∞^4 pairs of points in the plane it follows that there are ∞^4 such sub-groups in G_8. We shall refer to this four-parametered sub-group as $G_4(AB)$.

A set of ∞^4 plane collineations may also be obtained in a different way. To fix a straight line and a point not on the line in the plane would require the change of four parameters into constants. Hence there are ∞^4 plane collineations each leaving a triangle invariant and all leaving the same straight line invariant and the same point not on the common line. Let the common invariant point be chosen for origin and ~~the~~ a line through the origin parallel to the common invariant line be chosen for the x-axis. Then any two plane collineations (T_1 and T_2) which would each leave a triangle invariant and both leave the same line and the same point not on the common line invariant would take the forms:

$$T_1: \quad x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & B' & 1 & KA' \\ A'' & B' & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & B' & 1 & KB' \\ A'' & B' & 1 & K'B' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad [27]$$

$$T_2: \quad x_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_1' & B' & 1 & K_1 A_1' \\ A_1'' & B' & 1 & K_1' A_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_1' & B' & 1 & K_1 \\ A_1'' & B' & 1 & K_1' \end{vmatrix}} \qquad y_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_1' & B' & 1 & K_1 B' \\ A_1'' & B' & 1 & K_1' B' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_1' & B' & 1 & K_1 \\ A_1'' & B' & 1 & K_1' \end{vmatrix}} \qquad [28]$$

It will be noticed. that the only variations of these equations [27] and [28] from [1] and [2] of page 4 is that $B' = B'' = B_1' = B_1''$ also a transformation of the same form which would transform $(x\,y)$ directly into $(x_2 y_2)$ would have $B_2' = B_2'' = B'$. If the resultant of T_1 [27] and T_2 [28] is another of the same kind the substitution of the above conditions in equations [11] page 8, should reduce the six to only four independent equations. Making the substitutions and reducing we find the first four of [11] reduce to the following forms [29] given on the following page and that equation (5) of [11] reduces to (4) of [29] and equation (6) of [11] reduces to (3) of [29].

Equations [11], therefore, reduce to the following four independent equations:

$$
\left.
\begin{aligned}
\frac{K_2A_2'-K_2'A_2''}{A_2'-A_2''} &= \frac{(KA'-K'A'')(K_1A_1'-K_1'A_1'')+A_1'A_1''(K_1'-K_1)(K'-K)}{(A'-A'')(A_1'-A_1'')} \quad \ldots(1)\\[2ex]
\frac{A_2'A_2''(K_2'-K_2)}{A_2'-A_2''} &= \frac{A'A''(K'-K)(K_1A_1'-K_1'A_1'')+A_1'A_1''(K_1'-K_1)(K'A'-KA'')}{(A'-A'')(A_1'-A_1'')} \quad \ldots(2)\\[2ex]
\frac{K_2'-K_2}{A_2'-A_2''} &= \frac{(K_1'-K_1)(KA'-K'A'')+(K'-K)(K_1'A_1'-K_1A_1'')}{(A'-A'')(A_1'-A_1'')} \quad \ldots(3)\\[2ex]
\frac{K_2'A_2'-K_2A_2''}{A_2'-A_2''} &= \frac{A'A''(K'-K)(K_1'-K_1)+(K'A'-KA'')(K_1'A_1'-K_1A_1'')}{(A'-A'')(A_1'-A_1'')} \quad \ldots(4)
\end{aligned}
\right\}[29]
$$

Hence the ∞^4 plane collineations under discussion possess the fundamental group property that the resultant of any two of them is another of the same kind. In order to show that the inverse to each transformation exists let the condition $B'=B''$ ~~$A_1=A$~~ be substituted in the equation T_1' - the inverse of [1] - as given on page 10. Inspection shows that such substitution cannot alter its analytical form. Therefore the inverse to T_1 [27] exists and is another collineation of the same kind. We have shown, then, that the ∞^4 collineations under discussion possess the group properties and therefore form a sub-group of G_8. It is a four-parametered group and we shall refer to it as $G_4(A, l'')$. Since there are ∞^4 combinations of point and line in the plane where the point is not on the line, it follows that there are ∞^4 sub-groups $G_4(A, l'')$ in G_8.

Let us next seek to determine whether there is a three-parametered sub-group of G_8. Let T_1 and T_2 be two plane collineations having two points $[(AB) = (A_1 B_1)]$ and $[(A'B') = (A_1' B_1')]$ in common and the line through (AB) and $(A''B'')$ also common. We may choose this last line for the y-axis and then $A'' = A_1'' = A_2'' = 0$.

T_1 then takes the form:

$$T_1: \quad x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & KA' \\ 0 & B'' & 1 & 0 \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ 0 & B'' & 1 & K' \end{vmatrix}}, \quad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & 0 \\ 0 & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ 0 & B'' & 1 & K' \end{vmatrix}} \qquad [30]$$

and T_2 becomes:

$$T_2: \quad x_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_1' & 0 & 1 & KA_1' \\ 0 & B_1'' & 1 & 0 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_1' & 0 & 1 & K \\ 0 & B_1'' & 1 & K' \end{vmatrix}} \qquad y_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A_1' & 0 & 1 & 0 \\ 0 & B_1'' & 1 & K'B_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A_1' & 0 & 1 & K \\ 0 & B_1'' & 1 & K' \end{vmatrix}} \qquad [31]$$

If the ∞^5 collineations of the form of T_1 [30] and T_2 [31] possess the group property that the resultant of any two of them is another of the same kind the substitution of the condition $A'' = A_1'' = A_2'' = 0$ should reduce equations [26] to only three independent equations: Making the substitution and reducing gives:

$$\left.\begin{aligned} K_2 &= K K_1 \text{ .... } (1) \\ K_2' &= K' K_1' \text{ .... } (2) \\ \frac{K_2' - 1}{B_2''} &= \frac{K'(K_1' - 1)}{B_1''} + \frac{K' - 1}{B''} \text{ ..... } (3) \end{aligned}\right\} [32]$$

In the reduction equation (2) of [26] reduces to $0 = 0$ and drops out leaving only three independent equations. The ∞^5 collineations under discussion, therefore possess the fundamental group property.

If we expand T_1 [30] and solve for x and y we obtain for its inverse:

$$
\left.
\begin{aligned}
x &= \frac{K'A'B''x_1}{K'B''(1-K)x_1 - KA'(K'-1)y_1 - KK'A'B''} \\
y &= \frac{KA'B''y_1}{K'B''(1-K)x_1 - KA'(K'-1)y_1 - KK'A'B''}
\end{aligned}
\right\} \quad [33]
$$

which is of the same analytic form as [30] is when expanded. The ∞^5 collineations under discussion, therefore form a group. It is a three-parametered sub-group of G_8 and is designated $G_3(ABl')$. Since ∞^5 combinations can be made of two points and a line through one of them and not the other it follows that there are ∞^5 sub-groups $G_3(ABl')$ in G_8.

If we let the collineations T_1 and T_2 have three points in common invariant we may test by substituting $B'' = B_1'' = B_2''$ in [32]. The equations reduce to only two independent equations; $K_2 = KK_1$ and $K_2' = K'K_1'$. Substitution of the same condition in $[33]_2$ shows that the inverse exists. The ∞^2 of such collineations therefore form a sub-group of G_8. Since ∞^6 combinations can be made of 3 points and 3 lines it follows that there are ∞^6 such sub groups in G_8. Such a sub-group may be designated $G_6(ABC)$.

From the relative positions of points in the invariant figure common to any two collineations of a system we have determined six sub-groups of G_8, namely $G_6(A)$, $G_5(Al)$, $G_4(AB)$, $G_4(A, l'')$, $G_3(ABl')$ and $G_2(ABC)$.