

PROPERTIES OF H-SETS, KATĚTOV SPACES AND H-CLOSED EXTENSIONS
WITH COUNTABLE REMAINDER

By

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Abstract

In this work we obtain results related to H-sets, Katětov spaces, and H-closed extensions with countable remainder. As we shall see, these three areas are closely related but the results of each section carry their own definite flavor.

Our first results concern finding cardinality bounds of H-sets in Urysohn spaces. In particular, a Urysohn space X is constructed which has an H-set A with $|A| > 2^{\bar{\psi}(X)}$, where $\bar{\psi}(X)$ is the closed pseudocharacter of the space X . The space provides a counterexample to Fedeli's question in [16]. In addition, it is demonstrated that there is no θ -continuous map from a compact Hausdorff space to the space X with the H-set A as the image, giving a Urysohn counterexample to Vermeer's conjecture in [51]. Finally, it is shown that the cardinality of an H-set in a Urysohn space X is bounded by $2^{\chi(X_s)}$, where $\chi(X)$ is the character of X and X_s is the semiregularization of X . This refines Bella's result in [4] that the cardinality of such an H-set is bounded by $2^{\chi(X)}$.

The next section concerns the relationship of H-sets and Katětov spaces. We recall that a Katětov space can be embedded as an H-set in some space. Herrlich showed in [23] that the space of rational numbers, \mathbb{Q} , is not Katětov. Later Porter and Vermeer [41] refined this result with the fact that countable Katětov spaces are scattered. We obtain a similar refinement of Herrlich's result, and a generalization under an additional set-theoretic assumption. Our results include that a countable crowded space cannot be

embedded as an H-set and that, assuming the Continuum Hypothesis, neither can the minimal η_1 space.

Chapter 4 investigates necessary and sufficient conditions for a space to have an H-closed extension with countable remainder. For countable spaces we are able to give two characterizations of those spaces admitting an H-closed extension with countable remainder.

The general case appears more difficult, however, we arrive at a necessary condition – a generalization of Čech completeness, and several sufficient conditions for a space to have an H-closed extension with countable remainder. In particular, using the notation of Császár in [11], we show that a space X is a Čech g -space if and only if X is G_δ in σX or equivalently if EX is Čech complete. An example of a space which is a Čech f -space but not a Čech g -space is given answering a couple of questions of Császár. We show that if X is a Čech g -space and $R(EX)$, the residue of EX , is Lindelöf, then X has an H-closed extension with countable remainder. Finally, we investigate some natural extensions of the residue to the class of all Hausdorff spaces.

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Chapter 1

Introduction

In this work we consider two generalizations of compact Hausdorff spaces along a certain line. In particular, all spaces considered will be Hausdorff.

The class of compact Hausdorff spaces is perhaps the best known and useful class of topological spaces. As a few examples of how well-behaved the class of compact Hausdorff is recall the following facts.

- A closed subset of a compact Hausdorff space is compact Hausdorff.
- A continuous image of a compact Hausdorff space is compact Hausdorff.
- A product of compact Hausdorff spaces is compact Hausdorff.
- Every Tychonoff space can be densely embedded in a compact Hausdorff space.

Sometimes a salient feature of a property stands out more plainly in a well-chosen generalization of the property. Many different generalizations of compact Hausdorff have been considered in the literature, e.g. the Lindelöf property, paracompactness, metacompactness, and several others.

We consider here two similar generalizations of a compact space: an H-closed space and an H-set. We will see that many of the properties of compact spaces extend to H-

closed spaces and H-sets as well – with some subtle but important differences. As a preview, consider these properties parallel to the properties of compact spaces above.

- A regular closed subset of an H-set is H-closed.
- A θ -continuous image of an H-set is an H-set.
- A product of H-closed spaces is H-closed.
- Every Hausdorff space can be densely embedded in an H-closed space.

The first new results presented here concern finding cardinality bounds on H-sets in terms of well-known cardinal functions. These results extend a tradition of similar bounds. For example, in [35] Pol proved that if X is a compact space then $|X| \leq 2^{\chi(X)}$. Several years later, Dow and Porter [14] extended Pol's result to H-closed spaces, i.e. if X is an H-closed space then $|X| \leq 2^{\chi(X)}$. In fact, they showed $|X| \leq 2^{\bar{\psi}(X)}$ when X is H-closed. Finally, Bella [4] was able to extend Pol's result even to H-sets of Urysohn spaces: if A is an H-set in X and X is Urysohn then $|A| \leq 2^{\chi(X)}$. Our results here refine Bella's result by showing providing an example of an Urysohn space X with an H-set A with $|A| > 2^{\bar{\psi}(X)}$.

Recall that if τ is a Hausdorff topology on a set X and $\tau \subseteq \sigma$ then the topology on X generated by σ will also be Hausdorff. Also notice that if τ is a compact topology on X and $\sigma \subseteq \tau$ then the topology generated by σ is compact. If one considers the lattice of all topologies on an infinite set then the Hausdorff topologies occupy the upper part of the lattice, whereas the compact topologies occupy the lower part. These two major classes of topologies on a set meet in the family of compact Hausdorff topologies. Note that every infinite set must have some compact Hausdorff topology by the well-ordering lemma. It turns out that any two compact Hausdorff topologies are incomparable in this lattice, i.e. if one adds open sets the space is no longer compact whereas if one removes

open sets the space is no longer Hausdorff. The set of compact Hausdorff spaces then forms an anti-chain in the lattice of topologies. A reasonable question to ask then is whether this anti-chain is maximal. It turns out it is not. Indeed, there are spaces, which we will call minimal Hausdorff, which have no coarser Hausdorff topology, but are not compact as they are not Urysohn.

Examining the top part of the lattice of topologies, the Hausdorff topologies, in closer detail we find that some Hausdorff spaces don't even have a coarser minimal Hausdorff topology. The best known example of this is the space of rational numbers \mathbb{Q} , as shown by Herrlich [23]. For \mathbb{Q} , one may think of this property as meaning that every coarser Hausdorff topology on the space has sequences which do not converge. Topological spaces which *do* have a coarser minimal Hausdorff topology are called Katětov. One of our major results is the proof that a well known generalization of \mathbb{Q} , the minimal η_1 space, is not Katětov – if one assumes the Continuum Hypothesis.

The last part of this work concerns finding H-closed extension of Hausdorff spaces. We will see that the theory of H-closed extensions is in many aspects parallel to the theory of compactifications of a Tychonoff space. As Porter and Stephenson [38] say: “The starting point for the theory of H-closed [extensions] is a problem posed in 1924 by P. Alexandroff and P. Urysohn about whether a space can be densely embedded in some H-closed space.”

Just as the theory of compactifications is invaluable to the study of Tychonoff spaces, the theory of H-closed extensions is a major part of the study of general Hausdorff spaces. The most familiar, and perhaps oldest examples of compactifications, are the “small” compactifications of the real line, namely extended real line and the one point compactification, S^1 .

In 1930, Tychonoff [49] showed that every completely regular T_1 space can be densely embedded in a compact Hausdorff space – in fact, constructing the projec-

tive maximum of all compactifications of a space X : βX . However, it took until the late 1930s for Alexandroff and Urysohn's question to be answered positively [46, 29, 17, 1, 45]. As one might expect, each of these constructions was projectively large in the family of all H-closed extensions.

Returning to the “small” compactifications of the reals, we notice that S^1 is also a compactification of the rationals, \mathbb{Q} , and of the irrationals, \mathbb{P} . However, one might say that S^1 is a small compactification of \mathbb{P} since the difference, $S^1 \setminus \mathbb{P}$ is countable, while, on the other hand, it is a “large” compactification of \mathbb{Q} since $S^1 \setminus \mathbb{Q}$ is uncountable. In fact, \mathbb{Q} has *no* “small” compactification.

Here we will examine what conditions will guarantee that a space has a “small” extension – in particular, an H-closed extension with countable remainder. For the case of countable spaces, we obtain two characterizations of spaces with an H-closed extension with countable remainder – in particular, the fact that \mathbb{Q} has no H-closed extension with countable remainder will be a corollary. For the general case we consider a necessary condition and several sufficient conditions for a space to have an H-closed extension with countable remainder.

Chapter 2

Preliminaries

Several basic definitions will be required throughout the paper. We give some definitions, notation, and a few examples which the reader should keep in mind throughout. We will assume the reader is familiar with several basic notions of topology, in particular separation axioms, compactness, extensions, the Stone-Čech compactification, and filters on a family of sets. All spaces considered in this dissertation will be Hausdorff and examples will either easily be seen to be or will be explicitly shown to be Hausdorff.

2.1 Semiregular spaces

We will occasionally require our space to be “nicer” than the typical Hausdorff space. From the title of the section one might suspect we will require a stronger separation property, but this is a red herring – in fact, we will be coarsening the topology of the space.

Definition 2.1.1. Given a space X , a subset U is said to be regular open if $U = \text{int}_X \text{cl}_X U$. The family of regular open subsets of X is denoted $RO(X)$.

The collection of regular open sets has several nice properties we will make use of later. For now, we consider using them as a basis for a topology.

Definition 2.1.2. The semiregularization of a space (X, τ) , denoted X_s , is the set X with the topology, σ , generated by $RO(X)$. Note $RO(X)$ is in fact a basis and $\sigma \subseteq \tau$, i.e. σ is a coarser topology on X .

The semiregularization of X is very closely related to the original space, as we will see in more detail later. The following proposition is necessary before we can continue.

Proposition 2.1.3. [43] Given a Hausdorff space X , the space X_s is also Hausdorff.

A space Y is called *semiregular* if $RO(Y)$ forms a base for the topology on Y . Since $\text{int}_X \text{cl}_X U = \text{int}_{X_s} \text{cl}_{X_s} U$ for every open set U of a space X , it is clear that X_s is a semiregular space.

2.2 θ -continuity

Various possible properties of maps between topological spaces will be required throughout. The most common is a generalization of the usual concept of continuity commonly required in analysis.

Definition 2.2.1. A function $f : X \rightarrow Y$ is said to be θ -continuous at $x \in X$ if for each neighborhood, V , of $f(x)$ there is a neighborhood, U , of x such that $f[\text{cl}_X U] \subseteq \text{cl}_Y V$. A function is said to be θ -continuous if it is θ -continuous at each point in its domain.

Though the concept of θ -continuous is exactly what is required for our work, it is perhaps not as well behaved as we would hope. Some qualities of θ -continuous functions which are particularly nice or highlight differences from continuous functions are listed.

Fact 2.2.2. [43] Suppose $f : X \rightarrow Y$ is θ -continuous, then the following statements are true.

1. If $Y \subseteq Z$, then $f : X \rightarrow Z$ is θ -continuous.
2. If $A \subseteq X$, then $f|_A : A \rightarrow Y$ is θ -continuous.
3. If $f[X] \subseteq D \subseteq Y$ and D is dense in Y , then $f : X \rightarrow D$ is θ -continuous.
4. The identity map $\text{id} : X \rightarrow X_s$ is θ -continuous.
5. Compositions of θ -continuous functions are θ -continuous.

In particular, notice 4 above; this statement implies that X and X_s are θ -homeomorphic according to the following definition.

Notation 2.2.3. Given a function $f : X \rightarrow Y$, we use the notation “ f^{\leftarrow} ” for the set map $\mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ by $f[A] \mapsto A$. When f is a bijection we use the same notation for the inverse function.

Definition 2.2.4. Given two spaces X and Y and a function $f : X \rightarrow Y$. If f is bijective, θ -continuous and $f^{\leftarrow} : Y \rightarrow X$ is also θ -continuous, then X and Y are said to be θ -homeomorphic.

2.3 H-closed spaces and H-sets

Recall that a space X is called compact if every open cover of X has a finite subcover. In 1924, P. S. Alexandroff and P. S. Urysohn [2] proved that compact spaces are always closed.

Theorem 2.3.1. If X is a compact space and X is a subspace of Y , then X is closed in Y .

Extrapolating from the theorem above Alexandroff and Urysohn proposed a concept first called Hausdorff-closed, and now shortened to H-closed.

Definition 2.3.2. A Hausdorff space X is called H-closed if X is a closed subspace in every Hausdorff space in which it is embedded.

Alexandroff and Urysohn went on to demonstrate how close the concept of H-closed is to compactness, and provided an internal characterization with the following theorem.

Theorem 2.3.3. A Hausdorff space X is H-closed iff for every open cover \mathcal{U} of X there is a finite subfamily \mathcal{F} of \mathcal{U} for which the union of the closures of the members of \mathcal{F} cover X .

Recall that a compact Hausdorff is normal. The following example is an H-closed space which is not Urysohn – hence not compact. Ironically the space is commonly called Urysohn’s Example.

Example 2.3.4. Let

$$U' = \{(1/n, 0) : n \in \mathbb{N}\} \cup \{(1/n, 1/m) : n, m \in \mathbb{N}\} \cup \{(1/n, -1/m) : n, m \in \mathbb{N}\} \subset \mathbb{R}^2$$

and $\mathbb{U} = U' \cup \{p^-, p^+\}$ with topology as follows:

1. The points of the subset U' inherit basic open sets from the usual topology of \mathbb{R}^2 .
2. A basic open neighborhood of p^+ is of the form

$$\{p^+\} \cup \{(1/n, 1/m) : n > N, m \in \mathbb{N}\}.$$

Similarly a basic open neighborhood of p^- is of the form

$$\{p^-\} \cup \{(1/n, -1/m) : n > N, m \in \mathbb{N}\}.$$

In showing Urysohn's Example is Hausdorff we need only double check those points whose neighborhood base doesn't come directly from \mathbb{R}^2 , namely p^+ and p^- . First if $x \in \mathbb{U} \setminus \{p^+, p^-\}$, then $x = (1/N, r_0)$ for some $N \in \mathbb{N}$ and $r_0 \in \{0\} \cup \{1/n : n \in \mathbb{N}\}$. So the set $\{p^+, p^-\} \cup \{(1/n, 1/m) : n > N, m \in \mathbb{N}\}$ is an open neighborhood of p^+ and p^- whose closure misses x . Note that any basic open sets of p^+ and p^- of the form given in the construction of the space will serve to separate them. So Urysohn's Example is Hausdorff.

Next we show the space is not Urysohn. It is enough to exhibit two points which cannot be separated with closed neighborhoods. Note, in fact, that the closure of the basic open neighborhoods of p^- and p^+ given in the construction will always meet in a set of the form $\{(1/n, 0) : n > N\}$ for some $N \in \mathbb{N}$. Hence the space is not Urysohn.

Finally we show the space is H-closed. Consider an open cover of the space. We may refine the open cover to a cover with basic open sets, \mathcal{U} . So there exist sets $U^+, U^- \in \mathcal{U}$ such that $p^+ \in U^+$ and $p^- \in U^-$. Reserve U^+ and U^- and note that the subspace $\mathbb{U} \setminus (\text{cl } U^+ \cup \text{cl } U^-)$ is compact – and so has a finite subcover \mathcal{F} . Now the closures of the family of sets $\mathcal{F} \cup \{U^+, U^-\}$ cover \mathbb{U} .

Another example of a space which is H-closed but not compact, and demonstrates a common technique for generating H-closed spaces, is the following.

Example 2.3.5. Let \mathbb{J} be the set of real numbers $[0, 1]$ with basis $\{U \cup (V \cap \mathbb{Q}) : U, V \text{ are open in } [0, 1] \subseteq \mathbb{R}\}$. The space \mathbb{J} is H-closed, but not compact.

Several basic properties of H-closed spaces appear in [43]. We include some of these which will be frequently used and which highlight the differences from com-

compactness. In particular, 2.3.6.3 and 4 provide alternative internal characterizations of H-closed spaces which are invaluable.

Fact 2.3.6. [43]

1. A closed subset of an H-closed space need not be H-closed. (See 2.3.8.)
2. A regular closed subset of an H-closed space is H-closed.
3. A space X is H-closed iff every open filter has nonempty adherence.
4. A space X is H-closed iff every open ultrafilter converges.
5. If f is a θ -continuous surjection and X is H-closed, then Y is H-closed.

The concept of an H-set, a further generalization of the H-closed property, was introduced by Velicko [50] and Porter and Thomas [39].

Definition 2.3.7. Given a space X , we say that $A \subseteq X$ is an H-set of X if every family of open sets of X covering A has a finite subfamily for which the closure of the union contains A . More technically, given a family of open sets of X , \mathcal{U} , for which $A \subseteq \bigcup \mathcal{U}$, there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ for which $A \subseteq \bigcup \{\text{cl}_X V : V \in \mathcal{F}\}$.

Notice the H-set property is a property of a subspace of a parent space. A given space may be an H-set when embedded in one space, but not an H-set in another. Also notice a space X is H-closed iff it is an H-set in itself.

Urysohn's Example contains an example of an H-set which is infinite, discrete and not H-closed, and hence is not an H-set in every space which contains it.

Example 2.3.8. Consider the subset $A = \{(1/n, 0) : n \in \mathbb{N}\} \cup \{p^+\}$ in Urysohn's Example. In every cover of A with open sets of \mathbb{U} one of the sets must contain p^+ . But the closure of a basic open set of p^+ contains all but finitely many of the points of the form

$(1/n, 0)$. Hence A is an H -set of \mathbb{U} . However, A with the subspace topology from \mathbb{U} is discrete, i.e. it is homeomorphic to the countable discrete space ω , and therefore is not H -closed.

Porter and Woods [43] also examine H -sets in some detail, we list here some basic properties.

Fact 2.3.9. [43]

1. *If X is an H -set of a space Y , then X is closed in Y .*
2. *If X is an H -set of a space Y and Y is a subspace of Z , then X is an H -set of Z .*
3. *If X is an H -closed subspace of Y , then X is an H -set in Y .*
4. *If $f : X \rightarrow Y$ is θ -continuous and A is an H -set of X , then $f[A]$ is an H -set of Y .*
5. *If A is an H -set of X , B is a regular closed ($B = \text{cl}_X \text{int}_X B$) subset of X , and $B \subseteq A$, then B is H -closed.*
6. *An H -set of a regular space is compact.*

2.4 Minimal Hausdorff and Katětov spaces

For a given set X , one may consider the collection of Hausdorff topologies on the set. If the set X is finite, this collection has only one member, namely the discrete topology, but if X is infinite this collection is quite large, with cardinality $2^{2^{|X|}}$, and can be partially ordered by inclusion. Given this structure we may consider which members of the poset are maximal or minimal. In fact, there is only one maximal, hence maximum, member, namely the discrete topology, but, again assuming the set X is infinite, there are many minimal members. The minimal members of the partially ordered set of Hausdorff topologies on a set X are called minimal Hausdorff topologies.

Definition 2.4.1. A space (X, τ) is called minimal Hausdorff if there is no Hausdorff topology σ on X for which $\sigma \subsetneq \tau$.

In 1939, Parhomenko [34] showed that compact topologies are minimal Hausdorff. We will see, however, that the converse does not hold.

Still considering the poset of Hausdorff topologies on X , we may consider how the minimal topologies are related to the rest of the poset. To examine this question we make the following definition.

Definition 2.4.2. A space (X, τ) is called Katětov if there is a topology $\sigma \subseteq \tau$ for which (X, σ) is minimal Hausdorff.

A major result on this topic is that not all topologies on a countable space are Katětov. In particular, in 1965, Herrlich [23] showed the space of rational numbers, \mathbb{Q} , has no coarser minimal Hausdorff topology.

Noting that compact spaces are minimal Hausdorff, we may suspect that H-closed spaces and minimal Hausdorff spaces are closely related as well. This is indeed the case.

Now we can consider the relationship between H-closed spaces, minimal Hausdorff spaces, and Katětov spaces.

Theorem 2.4.3. [29] *A space is minimal Hausdorff iff it is H-closed and semiregular. In particular, if X is H-closed then X_s is minimal Hausdorff.*

Proof. First, suppose X is minimal Hausdorff, then X_s is also Hausdorff. Since X_s is a coarser topology, $X = X_s$, and X is semiregular. Let \mathcal{F} be a free open filter on X and $x \in X$. Note that $\{U \subseteq X : U \in \tau(X) \text{ and } x \notin U\} \cup \mathcal{F}$ is a strictly coarser Hausdorff topology on X ; this is a contradiction as X is minimal Hausdorff. Hence X is H-closed.

The other half relies heavily on 2.3.6 and 2.3.9. Now suppose X is H-closed and semiregular. Let X' be X with a coarser Hausdorff topology. Note $\text{id} : X \rightarrow X'$ is

continuous. If U is open in X , then $\text{cl}_X U$ is H-closed. Hence $\text{id}[\text{cl}_X U] = \text{cl}_X U$ is H-closed as a subspace of X' . Therefore $\text{cl}_X U$ is closed in X' . Since X is semiregular, $\{\text{cl}_X U : U \in \tau(X)\}$ is a closed base for X . Hence id^{\leftarrow} is continuous and a homeomorphism. Thus X is minimal Hausdorff. \square

The next corollary indicates the relationship between H-closed and Katětov topologies.

Corollary 2.4.4. *A space (X, τ) is Katětov iff there is an H-closed topology σ on X for which $\sigma \subseteq \tau$.*

2.5 The Katětov and Fomin Extensions

The Stone-Čech compactification of a Tychonoff space, defined to be the maximal Hausdorff compactification, is a well-known construction. Here we give the construction using ultrafilters due to Stone [46], rather than the product embedding construction due to Čech [9], as a warm-up for a generalization of the compactification to all Hausdorff space. First we recall the definition of a zero-set.

Definition 2.5.1. Given a space X , a set $Z \subseteq X$ is a zero-set (or z-set) if there exists a continuous function $f : X \rightarrow \mathbb{R}$ and $Z = f^{\leftarrow}(0)$.

We now construct a compactification equivalent to the Stone-Čech compactification of a Tychonoff space using z-ultrafilters.

Theorem 2.5.2. *Given a Tychonoff space X , the set $X \cup \{p : p \text{ is a free z-ultrafilter on } X\}$ with the closed sets for a topology generated by $\{Z \cup \mathcal{Z} : Z \text{ is a z-set of } X \text{ and } p \in \mathcal{Z} \text{ iff } Z \in p\}$ is equivalent to the the Stone-Čech compactification of X , βX .*

In their 1924 paper, Alexandroff and Urysohn [2] posed the question of whether every Hausdorff space can be densely embedded in an H-closed space. In the late 1930s and early 1940s several positive constructions were given [46, 29, 17, 1, 45]. One of these, the Katětov extension, was seen to be the maximal H-closed extension of a space.

Definition 2.5.3. [29] Given a Hausdorff space X , the Katětov extension, denoted κX , is the set $X \cup \{p : p \text{ is a free open ultrafilter on } X\}$, with the topology generated by sets of the form $\{p\} \cup U$ where $U \in p \in \kappa X \setminus X$.

Another of the constructions, by Fomin, is quite similar and will be useful in this dissertation.

Definition 2.5.4. [17] Given a Hausdorff space X , the Fomin extension, denoted σX , is the set $X \cup \{p : p \text{ is a free open ultrafilter on } X\}$, with the topology generated by sets of the form $oU = \{p \in \sigma X \setminus X : U \in p\} \cup U$.

Notice κX is the same set as σX , but with a finer topology in which the remainder of the extension, $\kappa X \setminus X$, is discrete. Since the sets are the same we will often use the notation oU even when considering κX . Concerning these sets the following lemma will prove useful.

Lemma 2.5.5. [43] *Given a Hausdorff space X , open sets U and V of X , and $Y = \kappa X$ or σX we have:*

1. $(oU) \cap X = U$.
2. $oU = \bigcup \{W : W \text{ is open in } Y \text{ and } W \cap X \subseteq U\}$.
3. $\text{cl}_Y U = (\text{cl}_X U) \cup oU$.
4. *There is a continuous θ -homeomorphism j from κX onto σX .*

5. If \mathcal{B} is an open neighborhood base of $x \in X$, then $\bigcap \{oB : B \in \mathcal{B}\} = \{x\}$.

2.6 The Iliadis Absolute

In analysis and topology we are used to the concepts of open and closed sets and how these families of sets may be used to define a topology on a space. The spaces we are most familiar with, \mathbb{R}^n , are rather nice in that they are connected. Equivalent to the statement “The space X is connected” is the statement “The space X contains no proper, non-trivial subsets which are both open and closed.” These sets which are both open and closed are called clopen and, though they destroy any hope of connectivity, lend some rather nice properties to some of the spaces we will consider.

Definition 2.6.1. Given a space X , a set $E \subseteq X$ is said to be clopen in X if E is both open and closed in the topology on X .

Definition 2.6.2. A space X is said to be zero-dimensional if the family of clopen sets of X forms a base for the topology on X .

Note that a zero-dimensional Hausdorff space X is Tychonoff. Given a point $p \in X$ and a closed set $A \subset X$ with $p \notin A$ there is a basic open (hence clopen) set U such that $p \in U$ and $A \cap \text{cl}_X U = \emptyset$. Let \mathbb{I} be the closed unit interval. Now the function $f : X \rightarrow \mathbb{I}$ defined by

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in X \setminus U \end{cases}$$

is a continuous real-valued function separating p and A .

Definition 2.6.3. A space X is said to be extremally disconnected if the closure of any open set of the space is also open. In other words, regular closed sets are clopen.

Besides θ -continuity, we will require a few more nice properties that maps between topological spaces may possess.

Definition 2.6.4. A map $f : X \rightarrow Y$ is said to be closed if $f[A] \subseteq Y$ is closed whenever $A \subseteq X$ is closed.

Definition 2.6.5. A map $f : X \rightarrow Y$ is called compact if $f^{-1}(y) \subseteq X$ is compact for each $y \in Y$.

Combining the previous two concepts gives us the following commonly used definition.

Definition 2.6.6. A map $f : X \rightarrow Y$ is said to be perfect if it is both closed and compact. Note we do not require that f be continuous.

Finally, we may require maps to have a property which forces images of closed sets to be small.

Definition 2.6.7. A map $f : X \rightarrow Y$ is called irreducible if for every closed set $A \subsetneq X$, $f[A] \neq Y$.

Since the properties above frequently appear together, functions which possess all of them are given a special designation.

Definition 2.6.8. A map $f : X \rightarrow Y$ is called a θ -cover if it is a θ -continuous, perfect and irreducible surjection.

Filters, and in particular ultrafilters, allow us to construct our most powerful tool: the Iliadis absolute of a space. There are several different constructions of EX given an initial Hausdorff space X ; one uses open ultrafilters, another regular open ultrafilters, and the last regular closed ultrafilters. It turns out that these different constructions are

equivalent (in a stronger sense than simply being homeomorphic) and we may occasionally conflate the open ultrafilter and regular open ultrafilter constructions.

One construction, by Gleason [20], of the absolute of a space X characterizes it as the dense subspace of the Stone space of the Boolean algebra of the regular open sets of X consisting of the regular open ultrafilters with non-empty adherence. A map is also constructed along with the absolute in which each ultrafilter is mapped to its (single) point of adherence in the original space.

Definition 2.6.9. The absolute of a space X , denoted (EX, k) , is the unique (up to equivalence) pair such that EX is a zero-dimensional and extremally disconnected Hausdorff space and $k : EX \rightarrow X$ is a θ -cover.

Equivalence in this case means if $(E'X, k')$ is another pair satisfying our requirements, then there exists a homeomorphism $h : EX \rightarrow E'X$ such that $k = k' \circ h$. In this case we will write $(EX, k) \equiv (E'X, k')$ or just $EX \equiv E'X$.

Corollary 2.6.10. For a space X , $EX \cong E(X_s)$.

Proof. Note $\text{id} : X \rightarrow X_s$ is a θ -homeomorphism, hence $\text{id} \circ k_X : EX \rightarrow X_s$ is a θ -cover. Therefore, by the uniqueness of the absolute, $EX \cong E(X_s)$. \square

When using the open filter construction, we can easily define a basis for EX .

Lemma 2.6.11. Given a space X the subsets of EX of the form $OU = \{p \in EX : U \in p\}$ form a basis for EX .

The following theorem lists some basic properties of the absolute.

Theorem 2.6.12. [43] Given a space X and $U, V \in \tau(X)$.

1. If X is H -closed then EX is compact.

2. If D is dense in X , then $ED \equiv k_X^{\leftarrow}[D]$ with the restriction map.
3. If hX is an H -closed extension of X then $E(hX)$ and $\beta(EX)$ are equivalent extensions of EX .
4. $\beta EX \setminus EX \cong \sigma X \setminus X$.

2.7 The θ -quotient topology

Given a Hausdorff space X and a quotient map $f : X \rightarrow Y$, we define a topology on Y , called the θ -quotient, or sometimes pseudo-quotient, or small image topology on Y . The name small image topology comes from the following characterization.

Definition 2.7.1. Let X and Y be sets and $f : X \rightarrow Y$ a function. For $A \subseteq X$ define $f^\#[A] = \{y \in Y : f^{\leftarrow}(y) \subseteq A\}$. The set $f^\#[A]$ is often called the small image of A .

Fact 2.7.2. [37] *Given the setup in the definition above, we have the following.*

1. $f^\#[X \setminus A] = Y \setminus f[A]$
2. $f^\#[A \cap B] = f^\#[A] \cap f^\#[B]$
3. If X is a space then the family $\{f^\#[U] : U \in \tau(X)\}$ is a base for a topology on Y , namely the small image topology.

If the map is compact we get a Hausdorff topology on the image set.

Fact 2.7.3. [37] *Let X be a space, Y a set and $f : X \rightarrow Y$ be a compact surjection. Let σ be the topology on Y generated by $\{f^\#[U] : U \in \tau(X_s)\}$, then (Y, σ) is a Hausdorff space.*

If we assume the map is irreducible as well we get the following.

Theorem 2.7.4. [44, 52] *Let X be a space, Y a set, and $f : X \rightarrow Y$ a compact and irreducible surjection, then the collection $\{f^\#[U] : U \in \tau(X)\}$ is a base for a Hausdorff topology σ on Y . Further, $f : X \rightarrow (Y, \tau)$ is θ -continuous and if X is compact then (Y, τ) is minimal Hausdorff.*

Chapter 3

H-sets and Katětov Spaces

3.1 Cardinality bounds of H-sets in Urysohn spaces

The bulk of this section can be found in [31]. Before we can start the bulk of this section we must continue with a few definitions. The following cardinal functions are well-known in topology.

Definition 3.1.1. The character at $x \in X$ is

$$\chi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local base for } x\},$$

while the character of X is $\chi(X) = \sup\{\chi(x, X) : x \in X\}$.

Definition 3.1.2. The pseudocharacter at $x \in X$ is

$$\psi(x, X) = \min\{|\mathcal{U}| : \bigcap \mathcal{U} = \{x\} \text{ and } \mathcal{U} \subseteq \tau(X)\},$$

while similarly the pseudocharacter of X is $\psi(X) = \sup\{\psi(x, X) : x \in X\}$.

When considering non-regular spaces the following definition is also interesting.

Definition 3.1.3. The closed pseudocharacter at $x \in X$ is

$$\bar{\psi}(x, X) = \min\{|\mathcal{V}| : \bigcap_{V \in \mathcal{V}} \text{cl}_X V = \{x\}$$

and $\mathcal{V} \subseteq \tau(X)\}$, and the closed pseudocharacter of X is $\bar{\psi}(X) = \sup\{\bar{\psi}(x, X) : x \in X\}$.

It is well known that for a Hausdorff space X , $\psi(X) \leq \bar{\psi}(X) \leq \chi(X)$ (see [24]), in fact, when we consider the semi-regularization, $\psi(X) \leq \bar{\psi}(X_s) = \bar{\psi}(X) \leq \chi(X_s) \leq \chi(X)$. In particular, it follows that $2^{\psi(X)} \leq 2^{\bar{\psi}(X_s)} = 2^{\bar{\psi}(X)} \leq 2^{\chi(X_s)} \leq 2^{\chi(X)}$.

Several of the basic properties of H-closed spaces and H-sets were given the previous chapter. Since the concepts are closely related, we would expect many of these properties to be similar. That is, if a statement P is true for all H-closed spaces a natural question to ask is whether P also holds for H-sets, or perhaps for H-sets embedded in a space with a particularly nice property. Here we consider the possibility of extending a particular cardinality bound for H-closed spaces. For example, Dow and Porter show in [14] that if X is an H-closed space then $|X| \leq 2^{\chi(X)}$ (in fact they show $|X| \leq 2^{\bar{\psi}(X)}$), and later, in [4], Bella shows that if X is Urysohn and A is an H-set of X then $|A| \leq 2^{\chi(X)}$. On the other hand, Bella and Yaschenko show in [6] that an H-set A of a Hausdorff space X may have cardinality larger than $2^{\chi(X)}$.

Similar results are those of Bella and Porter in [5] showing if X is H-closed then $|X|$ may be larger than $2^{\psi(X)}$, in fact one example of this has $X = \kappa\omega$, an H-closed Urysohn space. Here a space, X , is constructed demonstrating that the cardinality of an H-set of a Urysohn space is not always bounded by $2^{\bar{\psi}(X)}$. We also prove that the cardinality of an H-set of an Urysohn is bounded by $2^{\chi(X_s)}$. These two results constrain the maximum cardinality of an H-set of a Urysohn space as much as is possible with the inequalities listed above.

In [51], Vermeer conjectures that a subset A of a space X is an H-set if and only if there is a compact Hausdorff space K and a θ -continuous map $f : K \rightarrow X$ with $f[K] = A$. Bella and Yaschenko provide a Hausdorff counterexample with countable character with their construction. A counterexample to Vermeer’s conjecture, similar in construction to Bella and Yaschenko’s, but which is Urysohn and has countable closed pseudocharacter, is provided in this dissertation – by letting the character of the space grow we are able to achieve this surprising separation.

Bella and Yaschenko’s example also demonstrates that if the space X is not Urysohn we cannot be sure that the cardinality of an H-set is bounded by $2^{\chi(X)}$. Fedeli, recalling Dow and Porter’s tighter bound on H-closed spaces, then asked whether the bound could be improved if restricted to Urysohn spaces. In particular he asked the following question.

Question 3.1.4. [16] Let A be an H-set in the Urysohn space X . Is it true that $|A| \leq 2^{\psi(X)}$?

We will construct an example showing that the answer to Fedeli’s precise question is “no,” but will also provide a proof that the bound can be somewhat improved.

Bella and Yaschenko in [6] re-introduce under the name of *relatively H-closed* a concept first considered by Lambrinos in [30] under the name of *H-bounded* and investigated further by Mooney in [32]. Given a space X a subset A is called relatively H-closed if for every open cover \mathcal{U} of X there is a finite subfamily of \mathcal{U} whose closures cover A . Clearly an H-set is relatively H-closed, but the converse does not hold.

Example 3.1.5. Recalling Urysohn’s Example, we note that the subset $\{(1/n, 0) : n \in \mathbb{N}\}$ is relatively H-closed in \mathbb{U} , though it is not an H-set.

The following notation is introduced for the purpose of this work.

Notation 3.1.6. Let $X \subseteq Y \subseteq Z$ be spaces. We write $H(X;Y;Z)$ if every cover \mathcal{U} of Y with open sets of Z has a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ for which $X \subseteq \bigcup_{U \in \mathcal{F}} \text{cl}_Z U$.

Using this notation, X is H-closed iff $H(X;X;X)$, A is an H-set of X iff $H(A;A;X)$, and A is relatively H-closed in X iff $H(A;X;X)$.

Finally given a space X , we say an open filter $\mathcal{G} \subseteq \tau(X)$ has the *weak countable intersection property* if for every countable subset \mathcal{G}' of \mathcal{G} we have $\bigcap \{\text{cl}_X U : U \in \mathcal{G}'\} \neq \emptyset$. We then call a space X *weakly realcompact* if every open ultrafilter $\mathcal{U} \subseteq \tau(X)$ with the weak countable intersection property has nonempty adherence.

The first construction is a basic space from which the counterexample to Fedeli's question will be built. We modify a construction given by Bella and Yaschenko in [6].

Construction 3.1.7. Let X be a weakly realcompact space with countable closed pseudocharacter and κX Urysohn, e.g. $X = \omega$, and let $\widehat{X} = \kappa X \setminus X$. Define

$$Z_0 = X \cup (X \times \omega \times \widehat{X}) \cup \widehat{X}$$

with the following topology. If $U \in \tau(X)$ and $n \in \omega$ then

$$U(n) = U \cup (U \times [n, \omega) \times \widehat{X}) \in \tau(Z_0)$$

is a basic open neighborhood of $x \in X$, and if $p \in \widehat{X} = \kappa X \setminus X$ and $U \in p$ then a basic open neighborhood of p is

$$U(p) = (U \times \omega \times \{p\}) \cup \{p\} \in \tau(Z_0).$$

Finally, the points of $X \times \omega \times \widehat{X}$ are isolated.

Fact 3.1.8.

1. Z_0 is Urysohn.
2. X is relatively H -closed in Z_0 , in other words $H(X; Z_0; Z_0)$.
3. $H(X; X \cup \widehat{X}; Z_0)$.
4. Z_0 has countable closed pseudocharacter.

Proof.

1. If $x, y \in X$ then there exist U and V open neighborhoods, in X , of x and y respectively with $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. Hence $\text{cl}_{Z_0} U(0) \cap \text{cl}_{Z_0} V(0) = \emptyset$, and $U(0)$ and $V(0)$ are neighborhoods of x and y in Z_0 . If $p, q \in \widehat{X}$ then there exist $U \in p$ and $V \in q$ such that $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. Thus $p \in U(p)$, $q \in V(q)$, and $\text{cl}_{Z_0} U(p) \cap \text{cl}_{Z_0} V(q) = \emptyset$. If $x \in X$ and $p \in \widehat{X}$ then there exist U an open neighborhood of x in X and $V \in p$ with $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. Therefore $x \in U(0)$, $p \in V(p)$, and $\text{cl}_{Z_0} U(0) \cap \text{cl}_{Z_0} V(p) = \emptyset$. Finally, the points of $X \times \omega \times \widehat{X}$ are isolated.
2. Let \mathcal{C} be an open cover of Z_0 . Without loss of generality, for each $x \in X$ we can assume there exists an $U_x \in \tau(X)$ and an $n_x \in \omega$ with $x \in U_x(n_x) \in \mathcal{C}$; also we can assume that for each $p \in \widehat{X}$ there exists a $V_p \in p$ with $V_p(p) \in \mathcal{C}$. Now, $\{U_x : x \in X\} \cup \{V_p \cup \{p\} : p \in \widehat{X}\}$ is an open cover of κX . Since κX is H -closed, there exist finitely many x_1, x_2, \dots, x_n and p_1, \dots, p_m such that $\kappa X = \bigcup_{i=1}^n \text{cl}_{\kappa X} U_{x_i} \cup \bigcup_{i=1}^m \text{cl}_{\kappa X} V_{p_i}$. Hence

$$\begin{aligned} X &\subseteq \bigcup_{i=1}^n \text{cl}_X U_{x_i} \cup \bigcup_{i=1}^m \text{cl}_X V_{p_i} \\ &\subseteq \bigcup_{i=1}^n \text{cl}_{Z_0} U_{x_i}(n_{x_i}) \cup \bigcup_{i=1}^m \text{cl}_{Z_0} V_{p_i}(p_i). \end{aligned}$$

3. Take a cover \mathcal{U} of $X \cup \widehat{X}$ with open sets of Z_0 . Now extend \mathcal{U} to an open cover, \mathcal{U}' of all of Z_0 by adding in the isolated singletons not already covered. Then

there is a finite subfamily $\mathcal{V} \subset \mathcal{U}'$ with $X \subseteq \text{cl}_{Z_0} \cup \mathcal{V}$. However, it is clear that \mathcal{V} need not contain any of the isolated singletons added to extend \mathcal{U} . Hence we may take $\mathcal{V} \subset \mathcal{U}$.

4. We must show every point in Z_0 is the intersection of a countable collection of closed neighborhoods. This is certainly true for the isolated points of $X \times \omega \times \widehat{X}$. If, on the other hand, $x \in X \subset Z_0$ take $\{U_n : n \in \omega\} \subseteq \tau(X)$ with $\bigcap_{\omega} \text{cl}_X U_n = \{x\}$. Then $\bigcap_{\omega} \text{cl}_{Z_0} U_n(n) = \bigcap_{\omega} (\text{cl}_X U_n \cup (U_n \times [n, \omega) \times \widehat{X}) \cup o(U_n)) = \{x\}$. Finally, for points of $\widehat{X} \subseteq Z_0$ consider the following: let $p \in \widehat{X}$, then p is a free open ultrafilter on X and $\bigcap_{U \in p} \text{cl}_X U = \emptyset$. But X is also weakly real-compact, hence there is a countable family $\mathcal{C} \subseteq p$ with $\bigcap_{U \in \mathcal{C}} \text{cl}_X U = \emptyset$. Considering the family $\mathcal{C}' = \{U(p) : U \in \mathcal{C}\}$ we have $\bigcap_{\mathcal{C}'} \text{cl}_{Z_0} U(p) = \bigcap_{\mathcal{C}} \text{cl}_X U \cup (\bigcap_{\mathcal{C}} U \times \omega \times \{p\}) \cup \{p\} = \{p\}$. Hence $\{p\}$ is the intersection of a countable collection of closed neighborhoods and Z_0 has countable closed pseudocharacter. \square

Now we construct a space, again modifying a construction of Bella and Yaschenko in [6], which is Urysohn, has countable closed pseudocharacter and has a large H-set.

Theorem 3.1.9. *There is a space Z with the following properties:*

1. Z is Urysohn;
2. Z has countable closed pseudocharacter;
3. Z has an H-set H of cardinality greater than 2^ω .

Construction 3.1.10. *Let $\{X_n : n \in \omega\}$ be a sequence of spaces defined recursively as follows: $X_0 = \omega$ and $X_{n+1} = \widehat{X}_n = \kappa X_n \setminus X_n$. For each $n \in \omega$ let $Z_n = X_n \cup (X_n \times \omega \times \widehat{X}_n) \cup \widehat{X}_n$. Finally let Z_ω be the quotient space formed from $\bigcup_{\omega} Z_n$ by identifying \widehat{X}_{n-1} with X_n , and let $Z = Z_\omega \cup \{\infty\}$. A basic neighborhood of ∞ in Z will be $\{\infty\} \cup \bigcup_{i \in \omega \setminus n} \{X_i \times \omega \times \widehat{X}_i : n \in \omega\}$.*

Fact 3.1.11.

1. *The space Z is Urysohn.*
2. *Z has countable closed pseudocharacter.*
3. *The set $H = \bigcup_{\omega} X_n \cup \{\infty\}$ is an H -set of Z .*
4. *The cardinality of H is larger than $2^{\bar{\psi}(Z)} = \mathfrak{c}$.*

Proof.

1. That the points which are isolated can be separated from the other points of Z via closed neighborhoods is clear. Now for $x, y \in H$, if $x, y \in Z_n$ for some $n \in \omega$ then by Fact 3.1.8.1 the two points can be separated by closed neighborhoods. If, on the other hand, $x \in Z_n$ and $y \in Z_m$ where $n \neq m$, then may assume further that $x \in X_n$, $y \in X_m$ and $n < n + 2 \leq m$. Now let U_x be a basic open set of x and U_y a basic open set of y . Then

$$\text{cl}_Z U_x \subseteq X_{n-1} \cup (X_{n-1} \times \omega \times X_n) \cup X_n \cup (X_n \times \omega \times X_{n+1})$$

while

$$\text{cl}_Z U_y \subseteq X_{m-1} \cup (X_{m-1} \times \omega \times X_m) \cup X_m \cup (X_m \times \omega \times X_{m+1}).$$

Hence $\text{cl}_Z U_x \cap \text{cl}_Z U_y = \emptyset$. Finally, if $x = \infty$ and $y \in Z_n$ for some n , we may simply take the basic open neighborhood $\{\infty\} \cup \bigcup_{i=n+2}^{\infty} (X_i \times \omega \times \hat{X}_i)$ for $x = \infty$ and a typical neighborhood for y .

2. This is clear for each point in Z_n for some n . For ∞ we let \mathcal{U} be the neighborhood base $\{\{\infty\} \cup \bigcup_{i \in \omega \setminus n} (X_i \times \omega \times \widehat{X}_i) : n \in \omega\}$. Then $\bigcap_{U \in \mathcal{U}} \text{cl}_Z U = \{\infty\}$. $\mathcal{U} = \{U_n = \{\infty\} \cup \bigcup_{i \in \omega \setminus n} X_i \times \omega \times \widehat{X}_i : n \in \omega\}$, then $\{\infty\} = \bigcap_{\mathcal{U}} \text{cl}_Z U_n$.
3. Let \mathcal{U} be a cover of H with basic open sets of Z . There is some $U \in \mathcal{U}$ for which $\infty \in U$. Then for some $m \in \omega$, $\text{cl}_Z U$ contains X_i for all $i \geq m$. The cover \mathcal{U} contains, for each $n < m$, a subfamily, \mathcal{U}_n which covers $X_n \cup X_{n+1}$. Now as in 3.1.8.3 above, we have $H(X_n; X_n \cup X_{n+1}; Z_n)$. Hence we obtain a finite subfamily \mathcal{F}_n of \mathcal{U}_n , therefore of \mathcal{U} , for which $X_n \subseteq \bigcup_{U \in \mathcal{F}_n} \text{cl}_Z U$. The collection $\{U\} \cup \{\mathcal{F}_n : n < m\}$ is a finite family whose closure contains H .
4. This is clear since $|X_1| = |\kappa\omega| = 2^\epsilon > \mathfrak{c}$. □

We require a lemma before continuing.

Lemma 3.1.12. [3] *Let Y be a compact Hausdorff space and \mathcal{A} a partition of Y into closed G_κ subsets, then $|\mathcal{A}| \leq 2^\kappa$.*

Vermeer's [51] conjecture that an H-set of a space is the θ -continuous image of a compact space would impose a cardinality restriction on the H-set in the following way.

Theorem 3.1.13. *Let X be a space, $\kappa = \bar{\psi}(X)$, A an H-set of X , K a compact Hausdorff space, and $f : K \rightarrow X$ a θ -continuous function with $f[K] = A$, then $|A| \leq 2^\kappa$.*

Proof. The proof follows the outline of the comments after Theorem 5' of [6]. Note that point inverses under a θ -continuous map are closed. Let $x \in A$ and take a family of open neighborhoods of x , $\{U_\alpha : \alpha \leq \kappa\}$, such that $\bigcap_{\kappa} \text{cl}_X U_\alpha = \{x\}$. For every $p \in f^{-1}(x)$ fix an open neighborhood $W_{\alpha,p}$ satisfying $f[\text{cl}_K W_{\alpha,p}] \subseteq \text{cl}_X U_\alpha$ and let $W_\alpha = \bigcup \{W_{\alpha,p} : p \in f^{-1}(x)\}$. Then $f[W_\alpha] \subseteq \text{cl}_X U_\alpha$ and hence $f^{-1}(x) \subseteq \bigcap_{\kappa} W_\alpha \subseteq f^{-1}[\bigcap_{\kappa} \text{cl}_X U_\alpha] = f^{-1}(x)$. Now since $f^{-1}(x)$ is closed, and compact in this case, 3.1.12 implies the family $\{f^{-1}(x) : x \in A\}$ must have cardinality not more than 2^κ , and hence $|A| \leq 2^\kappa$. □

Corollary 3.1.14. *The H-set H of the space Z constructed above cannot be the θ -continuous image of a compact space.*

What is most notable about the above corollary is that the space Z is a *Urysohn* counterexample to Vermeer’s conjecture – rather than simply Hausdorff as Bella and Yaschenko’s example.

The following proposition pins down the cardinality of an H-set as much as may be possible with the cardinal functions we have considered.

Proposition 3.1.15. If A is an H-set of a Urysohn space X then $|A| \leq 2^{\mathcal{X}(X_s)}$.

Proof. Let A be an H-set in a Urysohn space X . In [4] Bella shows $|A| \leq 2^{\mathcal{X}(X)}$. Now $A \subset X_s$ is also an H-set of X_s , and X_s is also Urysohn. Hence $|A| \leq 2^{\mathcal{X}(X_s)}$. \square

3.2 Katětov spaces

In the preliminary chapter we considered a few of the relations between Katětov, H-closed, and minimal Hausdorff spaces. We now consider the relation between H-sets and Katětov spaces.

In 1941, Bourbaki [7] proved that \mathbb{Q} does not have a coarser compact Hausdorff topology, but it took until 1965 for Herrlich [23] to show \mathbb{Q} is not Katětov. A consequence of Herrlich’s proof is that a Katětov space is not the countable union of compact, nowhere dense subsets. In fact, had it been noticed, Herrlich’s result is an immediate corollary of this 1961 result of Bourbaki.

Theorem 3.2.1. [8] *The set of isolated points of a countable H-closed space is dense.*

Theorem 3.2.7 provides some characterizations of the Katětov property, one in terms of the remainder of an extension and the others as the image of a particular type

of function. What is lacking is an internal characterization of the Katětov property – in fact, this is one of the major unsolved problems of topology.

A few definitions and preliminary results will be useful before stating the theorem.

Definition 3.2.2. Given a space X and a set $A \subseteq X$, the θ -closure of A , denoted $\text{cl}_\theta A$, is the set $\{x \in X : \text{for all } U \in \mathcal{N}_x, \text{cl}_X U \cap A \neq \emptyset\}$. Further, a set A is called θ -closed if $A = \text{cl}_\theta A$. Note that $\text{cl}_\theta A$ is closed, $\text{cl}_X A \subseteq \text{cl}_\theta A$, but that, in general, the θ -closure is not a Kuratowski closure operator as it is not idempotent.

Fact 3.2.3. *Let X be a space and $A \subseteq X$.*

1. [43] X is H -closed iff EX is compact.
2. [12] A is θ -closed in X iff $k^\leftarrow[A]$ is closed in EX .

The next fact is two results of Dow and Porter providing half of one characterization of Katětov spaces. Vermeer [51] provided the other half as we will see in 3.2.7.

Fact 3.2.4. [14]

1. Let D be a space and hD an H -closed extension of D . Suppose X is a space such that there is a continuous bijection $f : X \rightarrow hD \setminus D$, then there is an H -closed extension $h'D$ of D such that $h'D \setminus D = X$.
2. If X is an infinite H -closed space with $|X| = \kappa$, then there is an H -closed extension $h\kappa$ of κ such that $h\kappa \setminus \kappa$ is homeomorphic to X .

Combining the two items above we get the following corollary.

Corollary 3.2.5. *If X is an infinite Katětov space then there is a discrete space D with an H -closed extension hD such that $hD \setminus D = X$.*

The following theorem provides several characterizations of the Katětov property – notice that, except for the definition itself, all other characterizations involve some other space. Before we state the proposition we make the following definition for the purposes of this dissertation.

Definition 3.2.6. A map $f : X \rightarrow Y$ is called *almost perfect* (or θ -perfect) if f is closed and $f^{-1}(y)$ is θ -closed for each $y \in Y$.

Theorem 3.2.7. [14, 41, 51] *For a space X , the following are equivalent:*

1. X is Katětov.
2. X is the remainder of an H -closed extension of a discrete space.
3. X is the perfect image of a compact space.
4. X is the perfect image of an H -closed space.
5. There is an H -closed space Y and an almost perfect surjection $f : Y \rightarrow X$.

Proof. The corollary above provides 1 implies 2. Vermeer [51] provides the following proof of 2 implies 1. Let D be a discrete space with H -closed extension hD and $hD \setminus D = X$. Notice $E(hD) \cong \beta D$ and $k[\beta D \setminus D] = hD \setminus D$. We note that $k|_{\beta D \setminus D} : \beta D \setminus D \rightarrow X$ is perfect surjection. Now there exists a compact subset $C \subseteq \beta D \setminus D$ such that $k|_C : C \rightarrow X$ is perfect, onto, and also irreducible. Theorem 2.7.4 shows that the closed base $\{k[A] : A \text{ is a closed subset of } C\}$ induces a minimal Hausdorff topology τ^* on X . But $\tau^* \subseteq \tau(X)$ since $k|_C : C \rightarrow X$ is closed.

The implications 3 implies 4 and 4 implies 5 are clear. To prove 1 implies 3, let σ be a coarser minimal Hausdorff on X . If $Y = E(X, \sigma)$ then Y is compact and since the function $k : Y \rightarrow (X, \sigma)$ is perfect and $\sigma \subseteq \tau(X)$, then $k : Y \rightarrow X$ is also perfect.

To prove 5 implies 1, let Y be an H-closed space and $f : Y \rightarrow X$ be a closed surjection such that $f^{\leftarrow}(x)$ is θ -closed for each $x \in X$. Now EY is compact and $k : EY \rightarrow Y$ is perfect. Since $k^{\leftarrow}[f^{\leftarrow}(x)]$ is compact for each $x \in X$, it follows that $f \circ k$ is perfect.

Now there is a closed set $D \subseteq EY$ such that $f \circ k|_D : D \rightarrow X$ is an irreducible surjection. Since $f \circ k|_D$ is compact, by 2.7.4 there is a minimal Hausdorff topology σ on X . Since $f \circ k|_D$ is closed, $\sigma \subseteq \tau(X)$. Hence X is a Katětov space. \square

Corollary 3.2.8. [41] *The perfect image of a Katětov space is Katětov.*

Corollary 3.2.9. [41] *If X has an H-closed extension hX and a closed discrete subspace A such that $|hX \setminus X| \leq |A|$, then X is a Katětov space.*

Proof. Let $f : hX \setminus X \rightarrow A$ be a one-to-one function. Define $g : hX \rightarrow X$ by $g|_X = \text{id}_X$ and $g|_{hX \setminus X} = f$. Since $g^{\leftarrow}(y)$ is finite for all $y \in X$, g is compact. Let B be a closed subset of hX . Since $g[B] = (B \cap X) \cup f[B \setminus X]$, $f[B \setminus X] \subseteq A$, and every subset of A is closed in X , it follows that $g[B]$ is closed. By 3.2.7, X is Katětov. \square

Corollary 3.2.10.

1. [40] *A space is θ -closed in some H-closed space iff it is Katětov.*
2. [14] *A Katětov space X is an H-set in some space.*

Proof. One direction of 1 is from 3.2.7.2 and the fact that the remainder of an H-closed extension of a discrete space is θ -closed in the extension. Conversely assume A is a θ -closed subspace of an H-closed space X . It follows from 3.2.3 that $k^{\leftarrow}[A]$ is a compact subspace of EX . Since $k|_{k^{\leftarrow}[A]} : k^{\leftarrow}[A] \rightarrow A$ is a perfect surjection (though not necessarily θ -continuous), it follows from 3.2.7 that A is Katětov.

For the proof of 2, consider these two statements: the remainder of an H-closed extension of a discrete space is θ -closed, and a θ -closed subspace of an H-closed space is an H-set. \square

Corollary 3.2.10 indicates the studies of Katětov spaces and H-sets are closely related – in fact, frequently to prove a space is not Katětov one proves instead that the space cannot be embedded as an H-set in a larger space.

Proposition 3.2.13 below indicates that the number of H-sets of an H-closed space can be quite large, and, assuming an image space can be found, provides a mechanism to partition an H-closed space into sets – which are both Katětov subspaces and H-sets. It is convenient to state a lemma first.

Lemma 3.2.11. *Given spaces X and Y , $A \subseteq X$ compact, and $f : X \rightarrow Y$ a θ -continuous function, then $f[A]$ is an H-set of Y .*

The next proposition demonstrates a nice property of θ -continuous functions.

Proposition 3.2.12. *Given spaces X and Y and a θ -continuous function $f : X \rightarrow Y$, $f^{\leftarrow}(y)$ is a θ -closed subset of X .*

Proof. Let $y \in f[X]$ and suppose $x \in X \setminus f^{\leftarrow}(y)$, then $f(x) \in Y$ and $f(x) \neq y$. Since Y is Hausdorff, there exist open sets of Y , V_1 and V_2 , such that $f(x) \in V_1$, $y \in V_2$, and $V_1 \cap V_2 = \emptyset$. Since f is θ -continuous there is U_1 an open neighborhood of x such that $f[\text{cl}_X U_1] \subseteq \text{cl}_Y V_1$. Now $\text{cl}_Y V_1 \cap V_2 = \emptyset$, so $f[\text{cl}_X U_1] \cap V_2 = \emptyset$. In particular $y \notin f[\text{cl}_X U_1]$ and $\text{cl}_X U_1 \cap f^{\leftarrow}(y) = \emptyset$. Hence $x \notin \text{cl}_\theta f^{\leftarrow}(y)$. \square

Proposition 3.2.13. *If X is an H-closed space, $f : X \rightarrow Y$ is θ -continuous and $p \in X$, then $f^{\leftarrow}(f(p))$ is an H-set of X .*

Proof. Note since X is H-closed then EX is compact and $k : EX \rightarrow X$ is θ -continuous. Hence $f \circ k : EX \rightarrow Y$ is θ -continuous. So $(f \circ k)^{\leftarrow}(y)$ is closed in EX , hence compact, for any point $y \in Y$. Now if $y = f(p)$ then $f^{\leftarrow}(f(p)) = k[(f \circ k)^{\leftarrow}(y)]$ is the image of a compact set under a θ -continuous map and is therefore an H-set. \square

We extend the definition of the closed pseudocharacter, $\bar{\psi}(X)$, for the next proposition.

Definition 3.2.14. Let X be a space and $A \subseteq X$, we define the closed pseudocharacter of A relative to X as $\bar{\psi}(A, X) = \min\{|\mathcal{U}| : \mathcal{U} \subset \tau(X) \text{ and } \bigcap_{\mathcal{U}} \text{cl}_X U = A\}$.

Definition 3.2.15. Let X be a space and \mathcal{K} be a family of subsets of X . We denote by $\bar{\psi}(\mathcal{K})$ the quantity $\sup_{\mathcal{K}} \bar{\psi}(K, X)$.

Proposition 3.2.16. If X is an H-closed space, $f : X \rightarrow Y$ is θ -continuous, and $\mathcal{K} = \{f^{\leftarrow}(f(p)) : p \in X\}$, then $|\mathcal{K}| = |f[X]| \leq 2^{\bar{\psi}(\mathcal{K})}$.

Proof. Note $f[X]$ is an H-set of Y and $|f[X]| \leq |X| \leq 2^{\bar{\psi}(X)}$.

Given spaces X and Y and a function f satisfying the hypotheses, we have $k^{\leftarrow}[\mathcal{K}]$ is a partition of the compact space EX into compact sets. By the proof of 3.1.12 $|\mathcal{K}| = |f[X]| \leq 2^{\bar{\psi}(\mathcal{K})}$. \square

Recall from above that \mathbb{Q} is not Katětov. This fact is an immediate corollary of the following proposition, the proof of which is a generalization of the method used in [23].

Theorem 3.2.17. [41] *A Katětov space is not the countable union of compact, nowhere dense sets.*

For countable spaces we can improve the result. Recall a space is called *crowded* if it contains no isolated points.

Theorem 3.2.18. *A countable crowded space is not an H-set of any space.*

Proof. Suppose A is a countable crowded space and A is a subspace of X . List A as $A = \{a_n : n \in \omega\}$. We construct by induction a chain of open sets which meets A , but the intersection of whose closures misses A . Let U_0 be an open set of X such that

$a_1 \in U_0$ and $a_0 \notin \text{cl}_X U_0$. Suppose we have $U_n \subseteq U_{n-1}$, $a_n \notin \text{cl}_X U_n$ and $U_n \cap A \neq \emptyset$ for all $n < m$. Since $U_{m-1} \cap A$ is open in A (and non-empty), there is some $a \in U_{m-1} \cap A$ such that $a \neq a_m$. Let U_m be an open set of X such that $a \in U_m$ but $a_m \notin \text{cl}_X U_m$ and $U_m \subseteq U_{m-1}$. Now the chain of open sets $\{U_n : n \in \omega\}$ meets A , but $\bigcap_{\omega} \text{cl}_X U_n \cap A = \emptyset$. Hence A is not an H-set of X . \square

The reader may also notice the above theorem applies to many spaces besides \mathbb{Q} . In particular, Bing's space, also known as the Sticky Foot space, is not an H-set of any space – hence not Katětov.

We quote the following proposition of Porter and Vermeer, which is in the same vein as the previous two results. First recall that, by definition, a space is *scattered* if every nonempty subspace has an isolated point.

Proposition 3.2.19. [41] A countable Katětov space is scattered.

Now \mathbb{Q} is an η_0 space – meaning that given two finite subsets $A, B \subseteq \mathbb{Q}$ with $A < B$ there exists an element $q \in \mathbb{Q}$ such that $A < q < B$. We consider here an η_1 space – which is a generalization of an η_0 -set in the following sense. A space X is an η_1 space if X is ordered and given two countable sets $A, B \subseteq X$ with $A < B$ there exists an element $r \in X$ such that $A < r < B$.

Gillman and Jerison [19] construct an η_1 space, Q , which is a generalization of \mathbb{Q} in the sense that Q is a minimal η_1 space, i.e. every η_1 space contains a copy of Q . Moreover, they show that the cardinality of Q is \mathfrak{c} .

The following theorem illustrates another commonality of Q and \mathbb{Q} , at least if we allow the Continuum Hypothesis as an axiom. Recall that the Continuum Hypothesis (CH) states that the cardinality of the real line, \mathfrak{c} , is equal to the cardinality of the power set of the natural numbers, i.e. $\mathfrak{c} = 2^{\omega}$. This statement is independent of the usual axioms of set theory, ZFC.

Theorem 3.2.20. *Assuming CH, the space Q is not an H-set of any space.*

Proof. The proof proceeds by induction, in many respects similar to the proof of 3.2.18. Assume $2^\omega = \omega_1$. Note then $|Q| = \mathfrak{c} = \omega_1$, so let $Q = \{q_\alpha : \alpha < \omega_1\}$. Suppose there exists a space X for which $Q \subseteq X$ is an H-set. That is for every cover of Q with open sets of X there is a finite subfamily whose closures (in X) contain Q . We will find a family of open sets, \mathcal{U} , of X which meets Q and for which $\bigcap_{\mathcal{U}} \text{cl}_X U \cap Q = \emptyset$. This will demonstrate Q is not an H-set of X . The proof proceeds by induction.

Base step: Choose U_0 open in X such that $U_0 \cap Q$ is an interval and $q_1 \in U_0$ but $q_0 \notin \text{cl}_X U_0$.

$\beta = \gamma + 1$: Assume we have $U_{\alpha+1} \subseteq U_\alpha$, $q_\alpha \notin \text{cl}_X U_\alpha$ and $U_\alpha \cap Q \neq \emptyset$ for all $\alpha < \gamma$. Since $U_\gamma \cap Q \neq \emptyset$, $U_\gamma \cap Q$ is infinite, so we may choose $q \in U_\gamma \cap Q$ with $q \neq q_{\gamma+1}$. Let $U_{\gamma+1}$ be an open subset of X such that $U_{\gamma+1} \cap Q$ is an interval, $q \in U_{\gamma+1} \subseteq U_\gamma$ and $q_{\gamma+1} \notin \text{cl}_X U_{\gamma+1}$.

β a limit ordinal: We note $\bigcap_{\alpha < \beta} U_\alpha \cap Q \neq \emptyset$. We can construct two sets in Q : $L = \{\inf U_\alpha \cap Q : \alpha < \beta\}$ and $R = \{\sup U_\alpha \cap Q : \alpha < \beta\}$. Note $L < R$ and since $\beta < |Q| = \omega_1$, the sets are countable. Now since Q is an η_1 set there is some $q \in Q$ such that $L < q < R$. Hence $\bigcap_{\alpha < \beta} U_\alpha \cap Q \neq \emptyset$, in fact the set is infinite (and contains the open set of Q : $(\sup L, \inf R)$). So we may choose some $q \in \bigcap_{\alpha < \beta} U_\alpha \cap Q$ with $q \neq q_\beta$ and find some U_β open in X such that $U_\beta \cap Q$ is an interval, $q \in U_\beta \subseteq \bigcap_{\alpha < \beta} U_\alpha$ and $q_\beta \notin \text{cl}_X U_\beta$.

By construction, the family $\{U_\alpha : \alpha < \omega_1\}$ satisfies our requirements, that is,

$$\bigcap_{\mathcal{U}} \text{cl}_X U \cap Q = \emptyset. \quad \square$$

Chapter 4

H-closed Extensions with Countable Remainder

4.1 Preliminaries

We begin with a definition necessary for the discussion of our topic.

Definition 4.1.1. Given a space X and Z , an extension of X , we say the space $Z \setminus X$ is the remainder of the extension.

In the theory of compactifications much research has been focused on finding compact extensions of a particular size or with remainders of a particular size – the most common examples being what would be called minimal compactifications, e.g. the one point compactification of the real line. Next to be thoroughly pinned down was the maximal compactification, i.e. the Stone-Čech compactification, βX , of a Tychonoff space X .

In this chapter we will concern ourselves with finding H-closed extensions with countable remainder, i.e. the smallest H-closed extensions. Our topic is a generalization of a question of Morita [33]: characterize those spaces which have compactifications with countable remainder – an area studied in depth by Henriksen [22], Hoshina [26, 27, 28], Terada [47] and Charalambous [10] but still not entirely resolved.

For subspaces of \mathbb{R} , we note \mathbb{P} clearly has a compactification with countable remainder, but \mathbb{Q} has none. In fact, for a Tychonoff space to have a compactification with countable remainder it must be Čech complete – a property we will investigate in the next section.

The question of which spaces allow H-closed extensions with countable remainder is an obvious generalization of the question of compactifications with countable remainder, and has been considered by Porter and Vermeer [41] and Tikoo [48].

The bulk of the results in this chapter are informed by the following facts.

Theorem 4.1.2. [36, 42, 43] *Let X be a Hausdorff space.*

1. *Then $\sigma X \setminus X$ is homeomorphic to $\beta EX \setminus EX$.*
2. *For each H-closed extension hX of X , there is a θ -continuous function $f_h : \sigma X \rightarrow hX$ such that $f_h = \text{id}_X$ and $\{f_h^{\leftarrow}(y) : y \in hX \setminus X\}$ is a partition of compact subsets of $\sigma X \setminus X$.*
3. *For each partition \mathcal{P} of nonempty compact sets of $\sigma X \setminus X$, there is an H-closed extension hX of X such that $\mathcal{P} = \{f_h^{\leftarrow}(y) : y \in hX \setminus X\}$.*
4. *Let η be a cardinal. There is an H-closed extension hX of X with $|hX \setminus X| = \eta$ iff $\sigma X \setminus X$ can be partitioned into η many compact sets.*

Corollary 4.1.3. *The space X has an H-closed extension with countable remainder iff $\sigma X \setminus X \cong \beta EX \setminus EX$ has a countable partition of compact sets.*

A few more facts about the Iliadis absolute will be useful in this chapter.

Fact 4.1.4. *Let X be a Hausdorff space and $k : EX \rightarrow X$ be the absolute map.*

1. [43] *If $U \in \tau(X)$, $OU = O(\text{int}_X \text{cl}_X U)$, $k[OU] = \text{cl}_X U$ and $\text{cl}_{EX} k^{\leftarrow}[U] = OU$.*

2. [43] For $x \in X$ and $U \in \tau(X)$, $k^{\leftarrow}(x) \subseteq OU$ iff $x \in \text{int}_X \text{cl}_X U$, in particular,
 $k^{\#}[OU] = \text{int}_X \text{cl}_X U$.
3. If T is clopen in EX then $T = O(k^{\#}[T])$.

Proof. Since T is clopen in EX , $T = OU$ for some $U \in \tau(X)$. By the above
 $k^{\#}[T] = \text{int}_X \text{cl}_X U$ and so $T = OU = O(\text{int}_X \text{cl}_X U) = O(k^{\#}[T])$. \square

4.2 Countable spaces

Our goal is to determine which spaces have H-closed extensions with a countable remainder. As a sub-goal we first consider which countable spaces have countable H-closed extensions. Recall the following definition.

Definition 4.2.1. A space X is said to be first countable if there is a countable neighborhood base for each point $x \in X$.

We know by 3.2.19 that a countable H-closed space is scattered. If Y is a countable H-closed extension of a space X , then Y is scattered and hence X is as well. Also Y_s is first countable and a minimal Hausdorff extension of X_s , therefore X_s is first countable.

Fact 4.2.2. A countable space X with a countable H-closed extension is Katětov.

Proof. By 3.2.9, it suffices to show X has an infinite closed discrete subspace. If X has no infinite closed discrete subspaces, then every infinite subset of X has a derived point. This means X is countably compact. As X is countable, it follows that X is compact – hence Katětov. \square

The other direction is to determine which countable spaces have a countable H-closed extension. We start with a countable, first countable, semiregular, Katětov space

X . We may also assume X is not countably compact; that is, X contains an infinite, closed discrete subspace A .

Theorem 4.2.3. *A countable Hausdorff space X has a countable H-closed extension iff X is Katětov and X_s is first countable.*

Proof. Suppose a countable space X is Katětov and X_s is first countable. We want to show X has an H-closed extension with countable remainder. By 4.1.2, it suffices to show $\beta EX \setminus EX$ has a countable partition of compact sets.

Let X' denote X with the coarser H-closed topology. So we have that the identity function $\text{id}_X : X \rightarrow X'$ is continuous.

1. By [13], there is a continuous function $f : EX \rightarrow EX'$ such that $k_{X'} \circ f = \text{id}_X \circ k_X$.

That is, the following diagram commutes:

$$\begin{array}{ccc} EX & \xrightarrow{f} & EX' \\ k_X \downarrow & & \downarrow k_{X'} \\ X & \xrightarrow{\text{id}_X} & X' \end{array}$$

As X' is H-closed, EX' is compact Hausdorff by 4.1.2. Also, there is a continuous extension $\beta f : \beta EX \rightarrow EX'$ and the following diagram commutes.

$$\begin{array}{ccc} & & \beta EX \\ & & \uparrow \\ & & \beta f \searrow \\ & & EX \xrightarrow{f} EX' \\ k_X \downarrow & & \downarrow k_{X'} \\ X & \xrightarrow{\text{id}_X} & X' \end{array}$$

Let $X = \{p_n : n \in \omega\}$ and $X' = \{p'_n : n \in \omega\}$ where $\text{id}_X(p_n) = p'_n$ for $n \in \omega$. Since k_X is perfect, we have that $\{k_X^{\leftarrow}(p_n) : n \in \omega\}$ is a partition of EX into compact

subsets, $\{k_{X'}^{\leftarrow}(p'_n) : n \in \omega\}$ is a partition of EX' into compact subsets, and $\{(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) : n \in \omega\}$ is a partition of βEX into compact subsets. By commutativity of the diagram, it follows that $k_X^{\leftarrow}(p_n) = (k_{X'} \circ f)^{\leftarrow}(p'_n) \subseteq (k_{X'} \circ \beta f)^{\leftarrow}(p'_n)$ and $(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \cap EX = k_X^{\leftarrow}(p_n)$ for $n \in \omega$.

2. As X_s is first countable, for each $x \in X$ there is a countable neighborhood base $\{U_n\}_\omega$ of regular open sets for $x \in X_s$. We now show $\{\text{cl}_{\beta EX} OU_n\}_\omega$ is a countable family of clopen sets for which if $k_X^{\leftarrow}(x) \subseteq T \in \tau(\beta EX)$ then there is some $m \in \omega$ such that $\text{cl}_{\beta EX} OU_m \subseteq T$. Let T be an open set in βEX such that $k_X^{\leftarrow}(x) \subseteq T$. As the clopen family $\{\text{cl}_{\beta EX} S : S \text{ is clopen in } EX\}$ is a base for βEX which is closed under finite unions and $k_X^{\leftarrow}(x)$ is compact, we can suppose $T = \text{cl}_{\beta EX} S$ for some clopen set S of EX . By 4.1.4, $S = OU$ for some $U \in \tau(X)$. As $k_X^{\leftarrow}(x) \subseteq OU$, it follows that $x \in \text{int}_X \text{cl}_X U$ and so for some $n \in \omega$, $x \in U_n \subseteq \text{int}_X \text{cl}_X U$. Hence we have $k_X^{\leftarrow}(x) \subseteq OU_n \subseteq O(\text{int}_X \text{cl}_X U) = OU = S$ and $k_X^{\leftarrow}(x) \subseteq \text{cl}_{\beta EX} OU_n \subseteq T$. Thus, $k_X^{\leftarrow}(x) = \bigcap_\omega \text{cl}_{\beta EX} OU_n$, and we can suppose

$$\text{cl}_{\beta EX} OU_{n+1} \subseteq \text{cl}_{\beta EX} OU_n$$

for $n \in \omega$.

3. Using the notation of 1, for each $n \in \omega$ we have $k_X^{\leftarrow}(p_n) \subseteq (k_{X'} \circ \beta f)^{\leftarrow}(p'_n)$ and $(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n) \subseteq \beta EX \setminus EX$ and finally

$$\bigcup_\omega ((k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n)) = \beta EX \setminus EX.$$

Note

$$[(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n)] \cap [\text{cl}_{\beta EX} OU_k \setminus \text{cl}_{\beta EX} OU_{k+1}] = K_{nk}$$

is a compact subset of $\beta EX \setminus EX$. Now, $\bigcup_{k \in \omega} K_{nk} = (k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n)$, $\beta EX \setminus EX = \bigcup_{n,k \in \omega} K_{nk}$ and $\{K_{nk} : n, k \in \omega\}$ is a partition of $\beta EX \setminus EX$. By 4.1.2, as $\beta EX \setminus EX$ has a countable partition of compact subsets, both EX and X have H-closed extensions with countable remainder.

Conversely, suppose the countable Hausdorff space X has a countable H-closed extension hX . By 4.1.2, $\sigma X \setminus X$ has a countable partition of compact sets. If X is not countably compact, X has a countably infinite closed discrete subspace. By 4.2.2, X is Katětov. If the countable space X is countably compact, then X is also compact and hence Katětov. As hX is countable and H-closed, hX_s is a countable minimal Hausdorff extension of X_s . But countable minimal Hausdorff spaces are first countable. Thus, X_s is first countable as well. \square

4.3 Generalizations of Čech completeness

We recall some basic definitions before considering the question of how generalizations of Čech completeness relate to finding H-closed extensions with countable remainder.

Definition 4.3.1. A Tychonoff space X is Čech complete if it is G_δ in every Hausdorff extension.

The following theorem is well-known and provides two important characterizations of Čech completeness. The first allows us a reduction in the number of compact Hausdorff extensions we must consider, and the second provides an internal characterization of the property.

Theorem 4.3.2. [18, 15] *The following are equivalent for a Tychonoff space X .*

1. *The space X is Čech complete.*

2. The space X is G_δ in βX .
3. There exists a sequence $(\mathcal{C}_n)_\omega$ of open covers of X such that every filter base of closed sets subordinate to $(\mathcal{C}_n)_\omega$ has non-empty intersection.

The following corollary is immediate.

Corollary 4.3.3. *If a space X has an H-closed extension with countable remainder then EX is Čech complete.*

Proof. Recall from 4.1.2 that a space X has an H-closed extension with countable remainder iff $\beta EX \setminus EX$ has a countable partition of compact sets. Of course, a prerequisite for $\beta EX \setminus EX$ to be the countable partition of compact sets is that it actually be the union of countably many compact sets. So if $\beta EX \setminus EX = \bigcup_\omega K_n$ where K_n is compact, then $G_n = \beta EX \setminus K_n$ is a family of open sets of βEX and $EX \subseteq G_n$ for all $n \in \omega$. Since $\bigcup_\omega K_n = \beta EX \setminus EX$, we have $\bigcap_\omega G_n = EX$. Hence EX is Čech complete. \square

Though Čech completeness of the absolute is a necessary condition for the existence of an H-closed extension with countable remainder, we will see that it is not sufficient – some additional property is required.

For metric space, restrictions related to the following definitions (along with Čech completeness) are sufficient to allow a compactification with countable remainder.

Notation 4.3.4. For a Tychonoff space X , let $R(X) = (\text{cl}_{\beta X} \beta X \setminus X) \cap X$. We call $R(X)$ the residue of X .

Definition 4.3.5. A space X called rim-compact (or semicompact) if X has a basis of open sets each of which has a compact boundary.

Definition 4.3.6. A space X is called Lindelöf if every open cover of X has a countable subfamily which covers.

The characterization of metric spaces allowing compactification with countable remainder is due to Hoshina.

Theorem 4.3.7. [26] *A metrizable space X has a compactification with countable remainder iff X is Čech complete, rim-compact and $R(X)$ is Lindelöf.*

For compactifications of Tychonoff spaces with countable remainder Hoshina also provides a sufficient condition.

Theorem 4.3.8. [26] *Let X be a Čech complete, rim-compact space. If $R(X)$ is separable metrizable then X has a compactification with countable remainder.*

We quote the following lemma of Hoshina [27], which is necessary for the next example.

Lemma 4.3.9. *If X has a countable compactification and \mathcal{U} is a collection of pairwise disjoint open sets of X with $U \cap R(X) \neq \emptyset$ for each $U \in \mathcal{U}$, then \mathcal{U} is countable.*

First we consider an example of Charalambous [10] showing that not only is Čech completeness is not enough to guarantee that a space has a compact extension with countable remainder but there exist two spaces X and X_1 with homeomorphic residues, $R(X) \cong R(X_1)$, one of which has a compactification with countable remainder – while the other does not.

Example 4.3.10. [10] The construction starts with the following setup due to Terada [47]. Note $X = \beta\mathbb{R} \setminus \mathbb{N}$ has a compactification with countable remainder, namely $\beta\mathbb{R}$, and $R(X) = \beta\mathbb{N} \setminus \mathbb{N}$.

Now let $Z = \mathbb{N} \cup \{\infty\}$, the one point compactification of \mathbb{N} , $Y = Z \times Z \times \beta\mathbb{N} \setminus \mathbb{N}$ and $X_1 = Y \setminus (\{\infty\} \times \mathbb{N} \times \beta\mathbb{N} \setminus \mathbb{N})$. Since Y is compact and $Y \setminus X_1$ is σ -compact and zero-dimensional, then X_1 is Čech complete and rim-compact. In addition, $R(X_1) =$

$\{\infty\} \times \{\infty\} \times \beta\mathbb{N} \setminus \mathbb{N}$ is homeomorphic with $R(X)$. But X_1 has no compactification with countable remainder. For let \mathcal{U} be an uncountable collection of pairwise disjoint nonempty open subsets of $\beta\mathbb{N} \setminus \mathbb{N}$. For each $U \in \mathcal{U}$ let $U' = Z \times Z \times U$, then $\{U' \cap X_1 : U \in \mathcal{U}\}$ is an uncountable collection of pairwise disjoint open sets of X_1 with $U' \cap X_1 \cap R(X_1) \neq \emptyset$ for each $U \in \mathcal{U}$. So by the lemma above, X_1 has no compactification with countable remainder.

We note here, however, that X_1 *does* have an H-closed extension with countable remainder, since $Y \setminus X_1 = \{\infty\} \times \mathbb{N} \times (\beta\mathbb{N} \setminus \mathbb{N})$ is zero-dimensional and the countable union of compact G_δ sets.

We now consider how it may be possible to partition the space $\beta EX \setminus EX$ into countably many compact sets – which would allow us to construct an H-closed extension of X with countable remainder. Since $\beta EX \setminus EX$ is zero-dimensional, the following proposition, communicated to Porter and Vermeer by F. Galvin, will be very useful.

Proposition 4.3.11. [41] A zero-dimensional space Y can be partitioned into a countable number of compact sets iff Y is the countable union of compact G_δ -sets.

Proof. If Y can be partitioned into a countable number of compact sets, then clearly each of the compact sets is G_δ . Conversely, suppose $Y = \bigcup_{\omega} A_n$ where each A_n is compact and G_δ in Y . Since the finite union of compact G_δ sets is a compact G_δ set, we will assume $A_n \subseteq A_{n+1}$ for all $n \in \omega$. From the fact that Y is zero-dimensional, it follows that for each $n \in \omega$, $A_n = \bigcap \{C_m^n : m \in \omega\}$, where C_m^n is clopen, $C_0^n = X$, and $C_{m+1}^n \subseteq C_m^n$ for all $m \in \omega$. Let $D_n^m = A_{n+1} \cap (C_m^n \setminus C_{m+1}^n)$ for $n, m \in \omega$, then $P = \{A_0\} \cup \{D_n^m : n, m \in \omega\} \setminus \{\emptyset\}$ is a countable partition of Y with compact sets. \square

Seeking to generalize Hoshina's characterization of metrizable spaces allowing compactifications with countable remainder, Porter and Vermeer found the following sufficient conditions for an H-closed with countable remainder.

Theorem 4.3.12. [41] *If cX is a zero-dimensional compactification of a Čech complete space X and $R(X)$ is Lindelöf, then $cX \setminus X$ has a countable partition of compact sets.*

Corollary 4.3.13. [41] *Let X be a space.*

1. *If X is not countably compact, EX is Čech complete, and $R(EX)$ is Lindelöf, then X has an H -closed extension with countable remainder and is Katětov.*
2. *If X is Tychonoff and Čech complete and $R(X)$ is Lindelöf, then X has an H -closed extension with a countable remainder*

Noting that Čech completeness of the absolute is necessary for a space to have an H -closed with countable remainder – we seek a generalization of Čech completeness to Hausdorff spaces which we may be able use directly. K. Császár in [11] modifies the internal characterization of a Čech complete space to obtain three different generalizations, two of which we will consider in depth.

Before we begin we will need the following definition also due to Császár:

Definition 4.3.14. A subset A of a topological space X is said to regularly embedded in X if whenever $x \in A \subseteq G$ and G is open, then there exists an open set V such that $x \in V \subseteq \text{cl}_X V \subseteq G$.

Proposition 4.3.15. [11] Suppose $A \subseteq X \subseteq Y$ are spaces. If A is regularly embedded in Y , then it is regularly embedded in X .

Theorem 4.3.16. [11] *If X is a Hausdorff space, then X is regularly embedded in σX .*

Proof. Let $G \in \tau(\sigma X)$ such that $X \subseteq G$ and let $x \in X$. There is a set $U \in \tau(X)$, such that $x \in oU \subseteq G$. We note that $\text{cl}_{\sigma X} oU \setminus X = oU \setminus X$ and hence $\text{cl}_{\sigma X} oU \subseteq G$. □

The following definitions generalize the internal characterization of Čech completeness for Tychonoff spaces to all Hausdorff spaces.

Definition 4.3.17. Let $(\mathcal{C}_n)_\omega$ be a sequence of families of sets of a set X and \mathcal{A} a family of sets. The family \mathcal{A} is subordinate to the sequence $(\mathcal{C}_n)_\omega$ if, for every $m \in \omega$, there is some set $A \in \mathcal{A}$ and also a set $C \in \mathcal{C}_m$ such that $A \subseteq C$.

Definition 4.3.18. Let X be a topological space. A Čech sequence (Čech f -sequence, Čech g -sequence) in X is a sequence $(\mathcal{C}_n)_\omega$ of open covers of X such that every filter base \mathcal{A} (of closed sets, of open sets) subordinate to $(\mathcal{C}_n)_\omega$ has an adherent point.

Definition 4.3.19. A Hausdorff space X is a Čech space (Čech g -space, Čech f -space) if there is a Čech sequence (Čech g -sequence, Čech f -sequence) in X .

Notice that for a Tychonoff space the concepts of Čech space, Čech g -space, Čech f -space, and Čech complete space coincide.

Theorem 4.3.20. [11] *A regularly embedded open subspace of a Čech g -space is a Čech g -space.*

Proof. Let Y be a Čech g -space, $X \subseteq Y$ regularly embedded and open, and $(\mathcal{C}_n)_\omega$ a Čech g -sequence on Y . Define \mathcal{B}_n to be the family of open subsets B of X for which $\text{cl}_Y B \subseteq X$ and $B \subseteq C$ for some $C \in \mathcal{C}_n$, then \mathcal{B}_n is a cover of X . In fact, if $x \in X$, there is a $C \in \mathcal{C}_n$ such that $x \in C$, and an open V such that $x \in V \subseteq \text{cl}_Y V \subseteq X$, then $x \in C \cap V \in \mathcal{B}_n$. Now we show $(\mathcal{B}_n)_\omega$ is a Čech g -sequence in X .

Suppose \mathcal{U} is a filter base of open subsets of X subordinate to $(\mathcal{B}_n)_\omega$, then it is also a filter base of open subsets of Y subordinate to $(\mathcal{C}_n)_\omega$. Hence there is some $x \in \bigcap \mathcal{U} \text{cl}_Y U$. Considering a particular $U \subseteq B \in \mathcal{B}_n$, we see $\text{cl}_Y U \subseteq X$ and so $x \in X$. \square

Theorem 4.3.21. [11] *A regularly embedded, dense G_δ subspace of a Čech g -space is a Čech g -space.*

Proof. Let $(\mathcal{C}_n)_\omega$ be a Čech g -sequence in Y , $X \subseteq Y$ dense, regularly embedded, and G_δ , so $X = \bigcap_\omega G_n$ where $G_n \in \tau(Y)$. Define \mathcal{B}_n to be the system of relatively open

subsets B of X for which $B = H \cap X$, where H is open in Y , $\text{cl}_Y H \subseteq G_n$, and $H \subset C_n$ for some $C_n \in \mathcal{C}_n$, then \mathcal{B}_n is an open cover of X . In fact, if $x \in X$, there is some $C_n \in \mathcal{C}_n$ such that $x \in C_n$ and an open set $V \in \tau(Y)$ such that $x \in V \subseteq \text{cl}_Y V \subseteq G_n$ and $x \in V \subseteq C_n \cap X \in \mathcal{B}_n$. We show $(\mathcal{B}_n)_\omega$ is a Čech g -sequence on X .

Let \mathcal{U} be a filter base of relatively open subsets of X which is subordinate to $(\mathcal{B}_n)_\omega$. Define \mathcal{U}' as the system of those open sets of Y , U' , for which $U' \cap X \in \mathcal{U}$, then \mathcal{U}' is an open filter base on Y subordinate to $(\mathcal{C}_n)_\omega$. In fact, for every $n \in \omega$, we may find a set $U_n \in \mathcal{U}$ such that $U_n \subseteq B_n \in \mathcal{B}_n$, hence sets D_n and H_n , open in Y , such that

$$U_n = D_n \cap X \subseteq B_n = H_n \cap X \subseteq C_n \in \mathcal{C}_n, \quad \text{cl}_Y H_n \subseteq G_n,$$

and

$$U_n = D_n \cap H_n \cap C_n \cap X, \quad D_n \cap H_n \cap C_n \in \mathcal{U}'.$$

Therefore there exists $x \in \{\text{cl}_Y U' : U' \in \mathcal{U}'\}$. Since, with the above notation, $x \in \text{cl}_Y (D_n \cap H_n \cap C_n) \subseteq \text{cl}_Y H_n \subseteq G_n$ for every n , we have $x \in X$. Finally if $U = D \cap X$, $D \in \mathcal{U}'$, then $x \in \text{cl}_Y D \subseteq \text{cl}_Y (D \cap X) = \text{cl}_Y U$ since D is open and X is dense in Y . Hence $x \in \text{cl}_Y U \cap X$ and x is an adherent point of \mathcal{U} in X . \square

Definition 4.3.22. A sequence of open covers (\mathcal{C}_n) is said to be monotone if \mathcal{C}_{n+1} refines \mathcal{C}_n .

Proposition 4.3.23. [11] If there exists a Čech sequence (g -sequence, f -sequence) for a space X , then there exists a monotone Čech sequence (g -sequence, f -sequence).

The following proposition provides an external characterization of a Čech g -space comparable to that of a Čech complete space.

Proposition 4.3.24. [11] For a space X the following are equivalent.

1. X is G_δ in every Hausdorff extension.
2. X is G_δ in σX .
3. X is a Čech g -space.

Proof. That 1 implies 2 is obvious.

That 3 follows from 2 is a consequence of the facts that σX is a Čech g -space (since it is H-closed) and X is regularly embedded in σX . A more constructive proof is the following: Suppose X is G_δ in σX . Then we have $X = \bigcap_{\omega} U_k$ where U_k is open in σX . Note that $U_k = \bigcup oV_\alpha$ where V_α is open in X . Let

$$\mathcal{C}_k = \{V_\alpha : \bigcup oV_\alpha = U_k\}.$$

We show the sequence $(\mathcal{C}_n)_\omega$ is a Čech g -sequence. First notice that since $X \subseteq U_k$, \mathcal{C}_n is an open cover of X . Now suppose p is an open ultrafilter subordinate to $(\mathcal{C}_n)_\omega$, then for each $k \in \omega$ there is some $V \in \mathcal{C}_k$ such that $V \in p$. Now for each k , $p \in oV \subseteq U_k$, hence $p \in \bigcap_{\omega} U_k$, i.e. $p \in X$. That is to say p is a fixed open ultrafilter, and so $(\mathcal{C}_n)_\omega$ is a Čech g -sequence.

For 3 implies 1, suppose X is a Čech g -space so there exists a Čech g -sequence, $(\mathcal{C}_n)_\omega$, in X and let Y be an extension of X . Let

$$G_k = \bigcup \{V \subseteq Y : C \subseteq V \text{ and } C \in \mathcal{C}_k\}$$

for $k \in \omega$. Since \mathcal{C}_k is an open cover of X , $X \subseteq G_k \subseteq Y$. Clearly then $X \subseteq \bigcap_{\omega} G_k$, so to show X is G_δ we need only show $\bigcap_{\omega} G_k \subseteq X$. On the contrary suppose there is some $p \in \bigcap_{\omega} G_k \cap (Y \setminus X)$. Let \mathcal{B} denote the system of all open neighborhoods of p

and define

$$\mathcal{U} = \{B \cap X : B \in \mathcal{B}\}.$$

Then \mathcal{U} is a filter base of open subsets of X and is subordinate to $(\mathcal{C}_n)_\omega$ since, for $k \in \omega$, there is an open V such that $C = V \cap X \in \mathcal{C}_k$, $p \in V \subseteq G_k$ and then $V \in \mathcal{B}$, $C \in \mathcal{U} \cap \mathcal{C}_k$. Hence there is an adherent point of \mathcal{U} , $x \in X$. Now the points p and x have disjoint open neighborhoods W_p and W_x . But then $W_p \in \mathcal{B}$, $W_p \cap X \in \mathcal{U}$ so that $W_x \cap W_p \cap X = \emptyset$ contradicts the fact that x is an adherent point of \mathcal{U} . Hence $X = \bigcap_\omega G_n$. \square

With regard to finding countable H-closed extensions, the previous proposition indicates that Čech g -spaces may be the generalization of Čech complete spaces we should consider. The next theorem provides more support for this observation.

Proposition 4.3.25. A space X is a Čech g -space iff EX is Čech complete.

Proof. The space X is a Čech g -space iff X is G_δ in σX , i.e. $X = \bigcap_\omega U_n$ where $U_n \in \tau(\sigma X)$. Let $K_n = \sigma X \setminus U_n$, so $\sigma X \setminus X = \bigcup K_n$. Now recall $\sigma X \setminus X \cong \sigma EX \setminus EX$. Consider $K_n \subseteq \sigma EX \setminus EX$, and let $\hat{U}_n = \sigma EX \setminus K_n$. Note $EX \subseteq \hat{U}_n$, and since $\bigcup K_n = \sigma EX \setminus EX$, then $EX = \bigcap \hat{U}_n$ and EX is G_δ in σEX and hence Čech complete.

The argument can also be reversed. \square

Corollary 4.3.26. A space X is a Čech g -space iff X_s is a Čech g -space.

Proof. This follows from $EX = EX_s$. \square

The following proposition is another characterization of countable spaces admitting an H-closed extension with countable remainder. First we note that if X is countable then EX is Lindelöf.

Lemma 4.3.27. Let X be a countable space, then EX is Lindelöf.

Proof. Since $k : EX \rightarrow X$ is compact, $EX = \bigcup \{k^{\leftarrow}(x) : x \in X\}$ is the countable union of compact sets – hence Lindelöf. \square

Proposition 4.3.28. A countable space X admits an H-closed extension with countable remainder iff X is a Čech g -space.

Proof. Clearly if X admits an H-closed extension with countable remainder then it is a Čech g -space.

Now suppose X is countable and a Čech g -space, then EX is Tychonoff and Čech complete. Also note since X is countable that X is Lindelöf. Therefore EX is Lindelöf. Since $R(EX)$ is a closed subset of EX , it is Lindelöf as well. By 4.3.12, EX has an H-closed extension with countable remainder. Therefore X does as well. \square

Combining the above with 4.2.3 we have the following.

Theorem 4.3.29. For a countable space X the following are equivalent.

1. X has an H-closed extension with countable remainder.
2. X is Katětov and X_s is first countable.
3. X is a Čech g -space.

The following provides a characterization of all Hausdorff spaces having an H-closed extension with countable remainder in terms of a special class of Čech g -sequences.

Proposition 4.3.30. The space X has an H-closed extension with countable remainder iff X admits a Čech g -sequence $(\mathcal{C}_n)_\omega$ for which each free open ultrafilter p is not subordinate to \mathcal{C}_m only for $m = N$ for some $N_p \in \omega$.

Proof. Recall X has an H-closed extension with countable remainder iff $\sigma X \setminus X = \beta EX \setminus EX$ has a countable partition of compact sets $\{K_n\}$. Let $G_n = \sigma X \setminus K_n$, then

G_n is open in σX and so $G_n = \bigcup oU$ where $oU \subseteq G_n$ and $U \in \tau(X)$. Since $X \subseteq G_n$ and $oU \cap X = U$, $X = \bigcup \{U : oU \subseteq G_n\}$, i.e. $\{U : oU \subseteq G_n\}$ is an open cover of X . Note for each $p \in \sigma X \setminus X$, $p \in K_n$ implies $p \notin K_m$ for $m \neq n$, i.e. $p \notin \sigma X \setminus G_n$ implies $p \in \sigma X \setminus G_m$ for $m \neq n$. Finally we get $U \notin p$ for all U such that $oU \subseteq G_m$ implies $V \in p$ for all V such that $oV \subseteq G_m$ for $m \neq n$. Let $\mathcal{C}_n = \{U : oU \subseteq G_n\}$, then (\mathcal{C}_n) is a sequence of open covers of X . Also, for each $p \in \sigma X \setminus X$ there is an $N \in \omega$ such that $U \notin p$ for all $U \in \mathcal{C}_N$ (i.e. $p \in K_N$). In addition, for all $p \in \sigma X \setminus X$, p (as an open filter) is subordinate to all \mathcal{C}_n where $n \neq N$. Hence no free open ultrafilter on X is subordinate to (\mathcal{C}_n) and (\mathcal{C}_n) is a Čech g -sequence on X – one in which each open ultrafilter is excluded at exactly one level.

The argument above can be reversed. That is given an special Čech g -sequence $(\mathcal{C}_n)_\omega$, we simply notice that $\{K_n : K_n = \sigma X \setminus \bigcup \{oU : U \in \mathcal{C}_n\}\}$ is a countable compact partition of $\sigma X \setminus X$. □

Császár [11] gives an example showing not all Čech g -spaces are Čech f -spaces, a somewhat simpler example is provided by the following.

Example 4.3.31. Let X be the unit interval with the topology generated by open sets of the form $I \setminus M$ where I is an interval and M is countable. Then X is a Hausdorff Čech g -space which is not a Čech f -space.

Proof. Since X is H-closed, it is a Čech g -space.

To show X is not a Čech f -space, let (\mathcal{C}_n) be a sequence of open covers of X . Select $C_n \in \mathcal{C}_n$ such that $0 \in C_n$ and then I_n and M_n such that $0 \in I_n \setminus M_n \subseteq C_n$. Define

$$M_0 = \bigcup_1^\infty M_n \cup \{0\},$$

find some

$$x_k \in \left(\left(\bigcap_1^\infty I_n \right) \cap \left[0, \frac{1}{k} \right) \right) \setminus M_0,$$

and finally let

$$A_n = \{x_k : k \geq n\}.$$

After noting that A_n is closed by virtue of being countable, by $A_n \subseteq I_n \setminus M_0 \subseteq I_n \setminus M_n \subseteq C_n$ the system $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is a closed filter base subordinate to (\mathcal{C}_n) . So since $\bigcap A_n = \emptyset$, X is not a Čech f -space. \square

Császár goes on to ask whether every Čech f -space is also a Čech g -space. This is not the case.

Theorem 4.3.32. *There is a space which is a Čech f -space but not a Čech g -space.*

The following lemma is well known and can be found in Chapter 9 of [19].

Lemma 4.3.33. *If X is locally compact and realcompact, then every infinite closed subset of $\beta X \setminus X$ has cardinality at least $2^{\mathfrak{c}}$.*

We now construct a special subset of $\beta \omega \setminus \omega$.

Lemma 4.3.34. *There is a set $D \subseteq \beta \omega \setminus \omega = \omega^*$ for which D intersects every infinite compact subset of ω^* and $\omega^* \setminus D$ also intersects every infinite compact subset of ω^* .*

Proof. Note any infinite compact subset of ω^* has a countably infinite subset. We consider the family of sets $\mathcal{C} = \{C : C \text{ is a countably infinite subset of } \omega^*\}$. Note $|\mathcal{C}| = (2^{\mathfrak{c}})^{\omega} = 2^{\mathfrak{c}}$. Hence if $\mathcal{K} = \{K : K = \text{cl}_{\beta \omega} C \text{ for some } C \in \mathcal{C}\}$, then $|\mathcal{K}| \leq 2^{\mathfrak{c}}$. We construct D recursively; begin by well-ordering $\mathcal{K} = \{K_\beta : \beta < 2^{\mathfrak{c}}\}$. Let $p \in D_0$ and $q \in E_0$ where $p, q \in K_0$ and $p \neq q$.

For $\alpha + 1$ a successor ordinal, let $D_{\alpha+1} = D_\alpha \cup \{p\}$ and $E_{\alpha+1} = E_\alpha \cup \{q\}$ where $p, q \in K_{\alpha+1} \setminus (D_\alpha \cup E_\alpha)$ and $p \neq q$. Note $K_{\alpha+1} \setminus (D_\alpha \cup E_\alpha) \neq \emptyset$ since $|K_{\alpha+1}| = 2^{\mathfrak{c}}$ but $|D_\alpha \cup E_\alpha| < 2^{\mathfrak{c}}$.

For α a limit ordinal, let $D_\alpha = \bigcup_{\beta < \alpha} D_\beta \cup \{p\}$ and $E_\alpha = \bigcup_{\beta < \alpha} E_\beta \cup \{q\}$ where $p, q \in K_\alpha \setminus (\bigcup_{\beta < \alpha} D_\beta \cup \bigcup_{\beta < \alpha} E_\beta)$ and $p \neq q$. Note $K_\alpha \setminus (\bigcup_{\beta < \alpha} D_\beta \cup \bigcup_{\beta < \alpha} E_\beta) \neq \emptyset$ since $|K_\alpha| = 2^{\mathfrak{c}}$ but still $|\bigcup_{\beta < \alpha} D_\beta \cup \bigcup_{\beta < \alpha} E_\beta| < 2^{\mathfrak{c}}$.

Let $D = \bigcup_{2^{\mathfrak{c}}} D_\alpha$ and $E = \bigcup_{2^{\mathfrak{c}}} E_\alpha$. Note $D \cap E = \emptyset$ and for each infinite compact subset K of ω^* , $K \cap D \neq \emptyset$ and $K \cap E \neq \emptyset$. \square

The following lemma generalizes a theorem appearing in [42].

Lemma 4.3.35. *Let X be a space. If $A \subseteq \sigma X \setminus X$ and A is closed in $\sigma X \setminus X$, then $\text{cl}_{\sigma X} A$ is an H -set of σX .*

Proof. Let \mathcal{U} be an open cover of $\text{cl}_{\sigma X} A$. Extend, and possibly refine, \mathcal{U} to an open cover, \mathcal{C} , of all of σX with basic open sets of the form oU where $U \in \tau(X)$. Since σX is H -closed we can find a finite subfamily of \mathcal{C} with the closures covering σX , and since $\text{cl}_{\sigma X} oU = \text{cl}_X U \cup oU$ we get a finite subfamily covering A , hence finite subfamily whose closures cover $\text{cl}_{\sigma X} A$. \square

Corollary 4.3.36. [42] *Let X be a space. If $A \subseteq \sigma X \setminus X$ and A is closed in σX , then A is compact.*

Proof of 4.3.32. Consider the set D constructed above as a subset of $\kappa\omega$. Let $X = \kappa\omega \setminus D$, then X is a Čech f -space but not a Čech g -space.

To show X is a Čech f -space we must find a sequence of open covers $(\mathcal{C}_n)_\omega$ of X for which every subordinate closed filter base has nonempty adherence. The sequence $(\mathcal{C}_n)_\omega$ where $\mathcal{C}_n = \mathcal{C} = \{\{p\} \cup \omega : p \in X \setminus \omega\}$ suffices. For suppose \mathcal{F} is a subordinate closed filter base, then there is some $F \in \mathcal{F}$ and $U \in \mathcal{C}$ for which $F \subseteq U$. Now F

cannot contain an infinite subset V of ω because then $\omega V \subseteq F$, but $\omega V \not\subseteq U$. So $F \cap \omega$ is finite, and hence F is finite. Now \mathcal{F} contains a compact set and hence has nonempty adherence.

To show X is not a Čech g -space we consider the following diagram:

$$\begin{array}{ccccc} \omega = E\omega & \hookrightarrow & EX = X_s & \hookrightarrow & E(\kappa\omega) = \beta\omega \\ \downarrow & & \downarrow & & \downarrow \\ \omega & \hookrightarrow & X & \hookrightarrow & \kappa\omega \end{array}$$

In this case if X is a Čech g -space then $EX = X_s$ is Čech complete. But then EX is G_δ in every Hausdorff extension, in particular $\beta\omega$ – contradicting the construction of D . □

From the above a space must be a Čech g -space if it is to have an H-closed extension with countable remainder. By 4.3.12, if we also have that the residue of EX , $R(EX)$, is Lindelöf, then this is sufficient to guarantee an H-closed extension of the space with countable remainder. Hence we have the following corollary.

Corollary 4.3.37. *If a space X is a Čech g -space and $R(EX)$ is Lindelöf, then X has an H-closed extension with countable remainder.*

It seems that the next step would be to generalize the condition on $R(EX)$ to a condition on the original space X . What follows are several theorems and examples obtained while trying to find conditions both necessary and sufficient for a space to have an H-closed extension with countable remainder.

Lemma 4.3.38. *The countable intersection of σ -compact subspaces in a regular space is Lindelöf.*

Proof. Let X be a regular space, $B_n \subseteq X$ where B_n is σ -compact for $n \in \omega$, and $A = \bigcap_{\omega} B_n$. Note $\prod_{\omega} B_n$ is Lindelöf. The function $e : A \rightarrow \prod_{\omega} B_n$ defined by $e(x)(n) = x$ is an embedding and $e[A]$ is closed in the product. Therefore A is Lindelöf. \square

Proposition 4.3.39. [41] Let X be a Tychonoff, nowhere locally compact space. If X has an H-closed extension with countable remainder, then X has a dense Lindelöf subspace.

Proof. From 4.1.2 and since $k : EX \rightarrow X$ is irreducible, it follows that EX is nowhere locally compact and has an H-closed extension with countable remainder. By the continuity of k (when X is regular), it suffices to show EX has a dense Lindelöf subspace. Now $\beta EX \setminus EX$ has a countable partition $\{A_n : n \in \omega\}$ of compact sets. Fix $n \in \omega$, then for each $m \in \omega$ there is an open set U_m in βEX such that $A_n \subseteq U_m$ and $(\text{cl}_{\beta EX} U_m) \cap A_k = \emptyset$ for $0 \leq k < n$ and $n < k \leq n + m + 1$, and $\text{cl}_{\beta EX} U_m \subseteq U_{m-1}$ if $m \geq 1$. Hence $B_n = \bigcap \{\text{cl}_{\beta EX} U_m : m \in \omega\}$ is compact, $\beta EX \setminus B_n$ is σ -compact, and $B_n \setminus EX = A_n$. If V is an open set in βEX and $V \subseteq B_n$, then $\text{cl}_{\beta EX} V = \text{cl}_{\beta EX}(V \setminus EX)$ as $\beta EX \setminus EX$ is dense in βEX . Since $V \setminus EX \subseteq B_n \setminus EX = A_n$, $\text{cl}_{\beta EX} V \subseteq \text{cl}_{\beta EX} A_n = A_n$. So $V \subseteq A_n \subseteq \beta EX \setminus EX$ implying $V = \emptyset$. Hence, $\beta EX \setminus B_n$ is open and dense. By Baire's Theorem and 4.3.38, $\bigcap \{\beta EX \setminus B_n : n \in \omega\}$ is dense in EX and Lindelöf. \square

Fact 4.3.40. [41] A complete metric space is Katětov.

Example 4.3.41. [41] Let D be the discrete space of cardinality \aleph_1 , and \mathbb{P} be the irrationals. Note both D and \mathbb{P} have compact extensions with countable remainder. Also, the space $D \times \mathbb{P}$ is locally Lindelöf and a complete metric space – hence Čech complete, first countable and Katětov. Recall \mathbb{P} has a coarser compact Hausdorff topology. In particular, $\mathbb{P} \cong \prod_{\omega} \omega$, and there is a continuous bijection $f : \prod_{\omega} \omega \rightarrow \prod_{\omega} (\omega \cup \{\infty\})$. Let \mathbb{P}' denote \mathbb{P} with this coarser compact Hausdorff topology, then $D \times \mathbb{P}'$ is locally

compact and Hausdorff. Thus, $D \times \mathbb{P}$ has a coarser compact Hausdorff topology. However, since the space is nowhere locally compact and has no dense Lindelöf subspace, $D \times \mathbb{P}$ has no H-closed extension with countable remainder.

The converse of 4.3.39 is false, for consider the space \mathbb{Q} . Also consider the following example, which has a dense subspace admitting an H-closed extension with countable remainder, but has none itself.

Example 4.3.42. Again let D be the discrete space of cardinality \aleph_1 and let D^* be the one point compactification of D . Let \mathbb{R} denote the real numbers with the usual topology and let \mathbb{R}^+ denote the two point compactification of \mathbb{R} . Let $X = \mathbb{P} \times D^* \times \mathbb{R}^+$ and note that $cX = \mathbb{R}^+ \times D^* \times \mathbb{R}^+$ is a compactification of X where $cX \setminus X = \mathbb{Q} \times D^* \times \mathbb{R}^+$ has a countable partition into compact sets. So X has an H-closed extension with countable remainder. Let $Y = X \cup (\mathbb{Q} \times D \times \mathbb{P})$, then cX is also a compactification of Y . However $cX \setminus Y = \mathbb{Q} \times [(D^* \times \mathbb{R}^+) \setminus (D \times \mathbb{P})]$ does not have a countable partition of compact sets, so Y has no H-closed extension with countable remainder. This is despite the fact Y is nowhere locally compact, X is a dense Lindelöf subspace of Y , and X itself has an H-closed extension with countable remainder.

Example 4.3.43. The space $X = \mathbb{P} \times 2$ with the lexicographic order has an H-closed extension with countable remainder, namely $Y = \mathbb{R}^+ \times 2$ with the lexicographic order, since X is both a Čech g -space and Lindelöf. The space X^2 also has an H-closed extension with countable remainder, though X^2 is not Lindelöf. In particular, notice Y^2 is a zero-dimensional compactification of X^2 , which has a remainder that can be expressed as the countable union of compact G_δ sets. Namely,

$$Y^2 \setminus X^2 = \bigcup_{q \in \mathbb{R}^+ \setminus \mathbb{P}} [(\{q\} \times 2) \times (\mathbb{R} \times 2)] \cup \bigcup_{q' \in \mathbb{R}^+ \setminus \mathbb{P}} [(\mathbb{R} \times 2) \times (\{q'\} \times 2)].$$

Consider the following fact.

Fact 4.3.44. *Let a Tychonoff space X have an H -closed extension hX with a countable remainder. If \mathcal{U} is a family of pairwise disjoint open sets in X , then $\{U \in \mathcal{U} : U \cap R(X) \neq \emptyset\}$ is countable.*

Proof. If U is an open set of X we denote by $o_h U$ the largest open set in hX such that $o_h U \cap X = U$. By the denseness of X in hX , $\{o_h U : U \in \mathcal{U}\}$ is a family of pairwise disjoint open sets in hX . If $U \cap R(X) \neq \emptyset$, then $o_h U \setminus X \neq \emptyset$. As $hX \setminus X$ is countable, $\{U \in \mathcal{U} : U \cap R(X) \neq \emptyset\}$ is countable. \square

We define the *relative cellularity* of a space X relative to a subspace A as follows:
 $c(A, X) = \sup\{\mathcal{U} : \mathcal{U} \text{ is a family of pairwise disjoint nonempty open subsets of } X \text{ such that } U \cap A \neq \emptyset \text{ for all } U \in \mathcal{U}\}$.

Thus by the fact above, if X is a Tychonoff space with an H -closed extension with countable remainder, then $c(R(X), X) = \omega$.

Corollary 4.3.45. *If X is Tychonoff, nowhere locally compact and has an H -closed extension with a countable remainder then $c(X) = \omega$.*

Remark 4.3.46. As the space $D \times \mathbb{P}$ described in 4.3.41 is nowhere locally compact and $c(X) = \omega_1$, it follows from the above that X has no H -closed extension with a countable remainder.

The next result extends a result of Hoshina [27] which states that if a paracompact space X has a compactification with a countable remainder then $R(X)$ is Lindelöf, and answers a question of Porter and Vermeer [41].

Definition 4.3.47. A family of sets which is composed of countably many pairwise disjoint families is called σ -discrete.

Proposition 4.3.48. Let X be a paracompact Tychonoff space which has an H-closed extension hX with a countable remainder, then $R(X)$ is Lindelöf.

Proof. Let \mathcal{C} be an open cover of $R(X)$. Extend each $C \in \mathcal{C}$ to an open set C' of X such that $C' \cap R(X) = C$. Now $\{C' : C \in \mathcal{C}\} \cup \{X \setminus R(X)\}$ is an open cover of X and has a σ -discrete open refinement $\{\mathcal{U}_n\}_\omega$. For each $n \in \omega$, \mathcal{U}_n is a family of pairwise disjoint open sets in X . Also, $\{U \cap R(X) : U \in \mathcal{U}_n, n \in \omega, U \cap R(X) \neq \emptyset\}$ is a refinement of \mathcal{C} . By 4.3.44, for each $n \in \omega$, $\{U \cap R(X) : U \in \mathcal{U}_n, U \cap R(X) \neq \emptyset\}$ is also countable. Hence \mathcal{C} has a countable subcover. \square

Considering the importance $R(X)$ seems to play in finding extension with countable remainder for Tychonoff spaces, we seek to generalize it all Hausdorff spaces. There are a few possibilities to consider. To begin we make the following notational definitions.

Definition 4.3.49. Given a space X set $R_\sigma(X) = X \cap \text{cl}_{\sigma X}(\sigma X \setminus X)$.

Notice that $x \in R_\sigma(X)$ iff for every open neighborhood U of x there is some $p \in \sigma X \setminus X$ such that $U \in p$.

Definition 4.3.50. Given a space X set $R_{EX}(X) = k[R(EX)]$.

Another characterization of $R_{EX}(X)$ is $x \in R_{EX}(X)$ iff for each $U \in \tau(X)$ with $x \in \text{cl}_X U$ there is some $p \in \sigma X \setminus X$ such that $U \in p$.

Definition 4.3.51. Given a space X let

$$R_H(X) = \{x \in X : x \text{ has no H-closed neighborhood}\}.$$

Note that if $U \in \tau(X)$, A is an H-set of X and $U \subseteq A$ then $\text{cl}_X U$ is H-closed, so replacing “H-closed” with “H-set” in the previous definition does not obtain a larger set.

Proposition 4.3.52. For a space X , $R_{EX}(X) \subseteq R_\sigma(X) = R_H(X)$.

Proof. Suppose $x \in R_{EX}(X)$, then there is some $p \in R(EX)$ such that $k(p) = x$. Now $p \in R(EX)$ iff for each $U \in p$ there is some $q \in \sigma EX \setminus EX$ such that $U \in q$. Since $k(p) = x$ then $\mathcal{N}_p \subseteq p$. So for every open neighborhood U of there is some $q \in \sigma X \setminus X$ such that $U \in q$.

Now suppose $x \notin R_H(X)$, then there is some $U \in \mathcal{N}_x$ such that $\text{cl}_X U$ is H-closed. Now if p is an open ultrafilter on X then $\text{ad}(p) = \bigcap_p \text{cl}_X V = \bigcap_p \text{cl}_X (U \cap V) \neq \emptyset$. So every open ultrafilter containing U is fixed and $x \notin R_\sigma(X)$. Therefore $R_\sigma(X) \subseteq R_H(X)$.

Finally suppose $x \notin X \cap R_\sigma(X)$, then there is some $U \in \mathcal{N}_x$ for which if p is a open ultrafilter and $U \in p$, then $\text{ad}(p) \neq \emptyset$. This means every open filter on $\text{cl}_X U$ has nonempty adherence and hence $\text{cl}_X U$ is H-closed. \square

The next example shows that the containment in the previous proposition can be strict.

Example 4.3.53. Let $X = [0, 1] \cup ([1, 2] \cap \mathbb{Q})$ with the usual topology as a subspace of \mathbb{R} . Let $x = 1$, then x has no H-closed neighborhood so $1 \notin R_\sigma(X)$. But $1 \in \text{cl}_X(0, 1)$ so $1 \in R_{EX}(X)$.

Chapter 5

Concluding Remarks

We mention here some of the author's continuing research and questions raised by the results presented.

With regard to Katětov spaces, the major problem of finding an internal characterization remains unsolved. However, in the countable case we may be close to an answer. Our dual characterization of countable spaces with countable H-closed extensions indicates that a countable space is a Čech g -space exactly when its semiregularization is first countable and it is Katětov.

In the general case, we note that since many examples of spaces which are not Katětov rely heavily on the topology of \mathbb{Q} (a countable space), our proof that, under the Continuum Hypothesis, the minimal η_1 space, Q , is not Katětov provides a somewhat different direction for this study. A natural question is whether or not Q is Katětov, or can even be embedded as an H-set, under $\neg\text{CH}$.

Question 5.0.54. Is the minimal η_1 space Q Katětov under $\neg\text{CH}$?

Concerning H-closed extensions with countable remainder, though we have given two internal characterizations for countable spaces as well as an internal characterization for an arbitrary Hausdorff spaces, the author feels that the relationship among the

concepts of Čech g -spaces, Katětov spaces, and spaces admitting H-closed extensions with countable remainder has not been fully explored.

We note, for example, that the concept of a Katětov refers explicitly to a coarsening of the topology on a space. In addition, in analogy to the family of open sets of \mathbb{P} , $\mathcal{C}_n = \{(\frac{n}{m}, \frac{n+1}{m}) \cap \mathbb{P} : n \in \mathbb{Z}, m \in \mathbb{N}\}$ – demonstrating that \mathbb{P} is Čech-complete – we note that the Čech property can also be thought of as a type of coarsening of the topology. For, though in this in example $\bigcup \mathcal{C}_n$ forms a basis for the topology, that need not always be the case.

On the other hand, in the Tychonoff setting additional assumptions on $R(X)$ – in particular $R(X)$ being Lindelöf (or having a countable network) – have been useful in showing a space has an H-closed (or compact) extension with countable remainder. However, none of these conditions has been proven both necessary and sufficient, except in the metric case. Generalizing these conditions to Hausdorff spaces also poses a problem. As earlier, one can easily pull the problem back to a condition on $R(EX)$ – but understanding how conditions on $R(EX)$ relate to the original space has proven difficult.

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