BIFURCATION ANALYSIS OF
OPEN ECONOMY NEW KEYNESIAN MODELS

By

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Submitted to the graduate degree program in Economics and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

In this study, we first review the bifurcation phenomena in dynamic economic systems and point out the importance of bifurcations together with a summary of the common types of bifurcations which have been encountered in economic research. Although bifurcation analysis is a relatively new subject with steadily growing interest in economic literature, previous research reveals the potential importance of further studies on bifurcation using different dynamic models.

Therefore, we continue exploring the bifurcation phenomena in an open economy New Keynesian model developed by Gali and Monacelli (2005). We find that open economy framework brings about more complex dynamics, a wider variety of qualitative behaviors and policy responses. Introducing parameters related to the open economy structure affects the values of bifurcation parameters and change the location of bifurcation boundaries. Thus, the stratification of the confidence region, as often seen in closed economy New Keynesian models, is still an important risk to be considered in the context of the open economy New Keynesian functional structures. Econometrics and optimal policy design become more complex with an open economy. Dynamical inferences need to be qualified by the increased risk of bifurcation boundaries crossing the confidence regions and policy design needs to take into consideration that a drastic change in monetary policy can produce an unanticipated bifurcation, if the econometrics research was not adequate.

Keywords: Stability; Bifurcation; Open Economy; New Keynesian; Determinacy; Macroeconomics; Dynamic Systems

ACKNOWLEDGEMENTS

It is my pleasure and duty to thank those people who helped me accomplish this work. First of all, I wish to thank and send my love to my parents and to my wife, Savili, who have always given me the moral strentgh and support, and to my children, Zeliha and Bahadir who always cheer me up with their existence.

I am wholeheartedly grateful to my supervisor, William A. Barnett, whose support, guidance and contribution gave me the courage and motivation to expand my knowledge on the subject. I wish to state my gratitude for the doctoral scholarship provided by the Economics Department at KU. During the course of my graduate studies at KU, I have benefited from the experience, knowledge and wisdom of many members of the Faculty of the Economics Department. I am much obliged, in particular, to the members of my doctoral committee; William Barnett, John Keating, Ted Juhl, Shu Wu, and Steve Hillmer. I sincerely value their expertise and the time they devoted to me for my dissertation progress.

I owe a special debt of gratitude to my close friends Talat Ulussever, Dogan Karaman and Yakup Asarkaya for their help and suggestions during my graduate studies.

Last but not least, I would like to express my regards to the people who helped me in any respect during my graduate studies.

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CHAPTER I:

BIFURCATION ANALYSIS IN DYNAMIC ECONOMIC SYSTEMS

1.1 Introduction:

Bifurcation analysis is a key tool for the analysis of dynamic systems in general and nonlinear systems in particular. When an exogenous (control) parameter of a dynamic system changes, qualitative behavior of the system may also change. For example, new fixed (equilibrium) points might emerge or current fixed points might disappear, or their stability properties (convergent vs. divergent, monotonic vs. damped, single periodic vs. multiperiodic, or chaotic) may change. These sorts of qualitative changes in a dynamic system are called bifurcations, and the values of the parameters at which these changes take place are called bifurcation points.

There exist an extensive literature on stability and bifurcation of systems in mathematics and engineering, despite the fact that it is a relatively new research area in economics. Nevertheless, interest in bifurcation analysis of dynamical economic systems has been increasing in order to understand the dynamic behavior of the systems and to find out possible dependence of the system's behavior on parameter values. Studying bifurcations provides information about the ocurrence and changes in stability of fixed points, limit cycles, and other solution paths, it helps model these changes and transitions from stable to unstable case or vice versa as some parameters change. Moreover, bifurcation analysis enables us to qualitatively estimate the behavior of trajectories without utilizing the solution of the underlying differential or difference equations. This is achieved by numerically approximating equilibrium solutions and their stability, even for problems that do not have analytic solutions.

1.2 Bifurcation Phenomena in Economics Literature:

It has been shown that dynamical economic systems might go through various types of bifurcations. As Grandmont (1985) and Barnett and Duzhak (2008) points out, even simple dynamic economic systems may exhibit various types of dynamic behaviors within the same functional structure and may be stratified into bifurcation regions. In nonlinear dynamic systems, bifurcations may even yield a transition to chaos as the evolution to chaos is preceded by infinite stages of bifurcation.

Most common types of bifurcations encountered in economic analysis include Saddle Node, Transcritical, Pitchfork, Flip and Hopf bifurcations. For instance, Pitchfork bifurcations in the tatonnement process by Scarf (1960) and Bala (1997) and in the Chamberlinian agglomeration model by Pflüger (2001), Transcritical and co-dimension two bifurcations in continuous time macroeconometric models by Barnett and He (1998, 1999a, 2002), Singularity bifurcations in Leeper and Sims' (1994) Euler equation macroeconometric model by Barnett and He (2006), Neimark-Sacker bifurcation in the Kaldor business cycle model by Dobrescu and Opris (2007) and in duopoly model by Agliari, Gardini, and Puu (2003), and Hopf bifurcations in growth and business cycle models by Benhabib and Nishimura (1979), Dockner and Feichtinger (1991), Nishimura and Shigoka (2006), Jia Xu et al. (2008), and Huang, Wang and Yi (2010) as well as in closed economy New Keynesian models by Barnett and Duzhak (2008, 2010) and in time-delayed model of asset prices by Qu and Wei (2010) are some of the bifurcation studies in economic literature. See also Puu (1991), Medio (1992), Lorenz (1993), Gandolfo (1996) and Zhang (2006) for a general treatment of nonlinear dynamics and bifurcation analysis in economics.

Bifurcation analysis has been commonly utilized to study the dynamic behavior of a economic models. Analyzing bifurcation boundaries helps understand the dynamic properties of a system especially when the true parameter values are unknown. Barnett and He (1999a, 2002, 2006) and Barnett and Duzhak (2008, 2010) point out that the existence of bifurcations in a dynamic system indicates the presence of different solutions corresponding to close parameter values in parameter space when these parameters are on different sides of the bifurcation boundary. Therefore, occurrence of bifurcation boundaries stratifies the parameter space. In this case, dynamic properties of the system can be quite different depending on the location of the parameter with respect to the boundary. As a result, robustness of inferences about dynamic properties of the system depends on the setting of such boundaries and the position of parameter values with respect to the boundaries. Bifurcation boundaries can be identified by investigating the eigenvalues of the Jacobian matrix and transversality conditions through numerical and analytical procedures, as we will do in this study.

Growth models and business cycles have been a popular source for bifurcation research in economics. Grandmont (1985) constructs a classical dynamical model based on an OLG model and established the conditions for coexistence of cycles over the periods and the Period Doubling bifurcations to occur in one-dimensional nonlinear dynamical systems. Medio and Negroni (1996), using a two-period overlapping generations (OLG) model, show that bifurcations causing cycles and chaos may happen depending on various parameters such as elasticities of utility functions, productivity coefficients and elasticity of substitutions between factors of production. Krawiec and Szydlowski (1999) apply the Hopf bifurcation theorem in the Kaldor-Kalecki business cycle model and they show that as the time-delay parameter is increased, the system bifurcates to limit cycle behavior, then to multiple periodic and aperiodic cycles, and eventually tends towards chaotic behavior. Cai (2005) uses the Hopf bifurcation theorem to investigate the occurrence of a limit cycle bifurcation based on a time delay parameter in an IS-LM business cycle model.

Barnett and He (1999a) investigate the stability of the Bergstrom, Nowman and Wymer (1992) continuous time macroeconometric model of the UK economy in cases of any change in the parameters of the model. They find both transcritical and Hopf bifurcations in the UK model in the parameter space of the model. Furthermore, they show that both types of bifurcations can coexist in the same subset of the parameter space. They also verify the presence of codimension-two bifurcations in this model. Woodford (1989) points out that the presence of complex dynamics might strengthen the effectiveness of economic policy if there is an imperfection in markets. Knowing the location of bifurcation points is crucial to investigating policy responses of economic models. In fact, as Barnett and He (2002) emphasized, stabilization policies can be regarded as certain policy tools to move the system from an unstable state to a stable state, assuming that the initial situation is unstable.

Barnett and He (2006) use the Leeper and Sims' (1994) Euler equations macroeconometric model of the U.S. economy to see how parameter changes influence the dynamic behavior of the system and the economy it represents. They find a Singularity bifurcation within a small neighborhood of estimated values of parameters in the parameter space. When parameter values get close to the boundary, one eigenvalue of the linearized model jumps to infinity while other eigenvalues stay bounded. This implies an almost instantaneous reaction of some variables to variations of other variables in the system.

Barnett and Duzhak (2008, 2010) analyze the bifurcation phenomenon using a closed economy New Keynesian model and they found both Hopf and Period Doubling bifurcations within the parameter space of the model.

As He and Barnett (2006) point out, Grandmont's (1985) findings indicate that the parameter space is stratified into bifurcation regions which involve different dynamical properties from monotonic stability to chaos and various forms of multiperiodic dynamics in between. Barnett and He (1998, 1999a, 2006), and Barnett and Duzhak (2008, 2010) confirm Grandmont's (1985) conclusions using macroeconomic models of the UK and the US economies, respectively. As Barnett and Duzhak (2008) indicate, Grandmont's views have important implications for macroeconometric models especially if bifurcation boundaries traverse the confidence regions for estimated values of parameters. The stratification of the confidence region into subsets separated by bifurcation boundaries affects the robustness of inferences about the dynamical system.

For the review of bifurcation analysis in macroeconomic models, the readers can also refer to Barnett and He (1998, 1999a,b, 2002, 2006) and Barnett and Duzhak (2008, 2010), which constitute the foundations upon which this study is built. Despite growing research interest in exploring the bifurcation phenomenon in economic systems, literature on this subject is still immature and needs an extensive study for a comprehensive understanding. In this study, we investigate the possibility of bifurcations in an open economy New Keynesian model developed by Gali and Monacelli (2005). It will help us extend the conclusions of Barnett and Duzhak (2008, 2010) and find out the differences between closed and open economy cases.

1.3 A Preliminary Review of Bifurcation Theory:

Consider a discrete-time dynamic system in the following difference equation form

$$y_{t+1} = f\left(y_t, \phi\right) \tag{1.3.1}$$

with f continuously differentiable at y^* where y^* is a solution to (1.3.1) given by $y^* = f\left(y^*, \phi\right)$, and ϕ is the vector of parameters.

An equilibrium $y^* = y_{t-1} = y_t$ of the system (1.3.1) is called (Lyapunov) stable if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\left\| y_0 - y^* \right\| \leq \delta$ implies $\left\| y_t - y^* \right\| \leq \varepsilon, \forall t > 0$. An equilibrium point of the system (1.3.1) would be asymptotically stable if it is both stable and $\exists \delta > 0$ such that $\forall y_0$ with $\left\| y_0 - y^* \right\| \leq \delta$, the solution $y_t \to y^*$ as $t \to \infty$.

If we write the system (1.3.1) in constant-coefficient (i.e. time-invariant) linear form as

$$y_{t+1} = C y_t, \tag{1.3.2}$$

where C is the Jacobian matrix, then the system (1.3.2) is asymptotically stable at the equilibrium point if and only if $\left|\lambda_i\right| < 1$ for all eigenvalues λ_i with $i = 1, 2, ..., n$.

The fundamental question in local bifurcation analysis is what happens to the behavior of the dynamic system when there are some variations in a parameter ϕ. As explained in Gandolfo (1996), the first step in the bifurcation analysis is to construct the Jacobian matrix of the system (1.3.1). The behavior of the solution of the system over time is governed by the sign and the absolute value of the roots. Thus, studying the stability properties of the system requires an investigation of the nature of the roots, even without having to compute them analytically.

In case of a hyperbolic equilibrium, the Jacobian matrix $C\left(y^*, \phi\right)$ has no eigenvalues on the unit circle. If an equilibrium point is hyperbolic, then small perturbations in parameter values do not cause any qualitative (structural) change in the system other than shifting the location of the equilibrium point. The qualitative structure of the system changes only if a variation in a parameter value causes an eigenvalue to reach the unit circle, i.e. $|\lambda_i| = 1$ for any i. In this case, the hyperbolicity disappears and a bifurcation emerges. In discrete dynamic systems, this instability may arise in three different forms of codimension-1 bifurcations:

1. A real eigenvalue i at $\lambda_i = 1$ leads to a steady-state bifurcation such as Transcritical, Saddle-Node or Pitchfork bifurcations. This is analogous to bifurcations which occur when $|\lambda_i| = 0$ in continuous-time systems.

2. A pair of complex-conjugate eigenvalues $\left(\lambda, \bar{\lambda}\right)$, crossing the unit circle with radius $R = 1$ (or crossing the imaginary axis of the complex plane in continuous case) gives rise to a Hopf (Neimark-Sacker) bifurcation.

3. A real eigenvalue i at $\lambda_i = -1$ leads to a Period Doubling (Flip) bifurcation. This form does not have an analog in continuous-time systems.

A *Saddle-Node (Fold) bifurcation* is known as the mechanism by which two fixed points of a dynamic system collide and then disappear. Lorenz (1993) presents an example of a Saddle-Node bifurcation in a labor market model assuming a parameterized labor demand function and a backward bending labor supply function. Bosi and Magris (2005) examine fluctuations and fiscal policy in a monetary model of growth. They show that the stationary rate of growth can be uncertain for a broad range of elasticities of intertemporal substitution in consumption. They note that this range of the elasticity is bounded from below by a value that encounters a Saddle-Node bifurcation and eventually both multiple stationary rates of growth and cycles may emerge in the model. Chian et al. (2006), using a forced oscillator model of business cycles, examine the behaviors of unstable periodic orbits and chaotic saddles which end up with a Saddle-Node bifurcation. Chiarella et al. (2004) analyze the financial market dynamics resulting from the actions of market players engaged in moving average rules and they establish the conditions for the occurrence of Saddle-Node bifurcations as well as for other types of bifurcations through changes in the reaction coefficient of some market players.

In case of a *Transcritical bifurcation*, the only change happens in the stability feature of the fixed point y^* as the parameter ϕ crosses over the bifurcation value $\phi = \phi_0$. Unlike the Saddle-Node bifurcation, the two fixed points do not vanish following the bifurcation event, but exchange their stability. Barnett and He (1998, 2002) find both Hopf and Transcritical bifurcations in Bergstom, Nowman and Wymer's (1992) continuous time macroeconometric model of the UK economy. Antinolfi et al. (2001) examine the dependence of an endogenous growth model on a parameter representing the degree of returns to scale and find that constant returns to reproducible factors of production is a Transcritical bifurcation point. Lorenz (1993) shows an example of a Transcritical bifurcation in a one-dimensional dynamical system based on neoclassical growth theory.

In case of a *Pitchfork bifurcation*, as the parameter ϕ traverses the bifurcation value $\phi = \phi_0$ at the fixed point y^*, two additional equilibria appear, which are unstable if x* is stable and stable if x* is unstable. In the first case, it is said that the system goes through a subcritical pitchfork bifurcation at x*. In the second case, it is called a supercritical bifurcation. There are various studies which find Pitchfork bifurcations in economic dynamics. Bala (1997), for instance, find a Pitchfork bifurcation in the tatonement process of a two-agent, two-commodity exchange economy. Azariadis and Guesnerie (1986) find a Pitchfork bifurcation in sunspots of order two. They also find that sunspots of higher order may have more complex bifurcations.

In a *Period Doubling (Flip) bifurcation*, the system switches to a different behavior with twice the period (the time it takes for the recurrence of solution path) of the original motion once the bifurcation parameter is altered. For example, a period-2 cycle bifurcates from the fixed point or the period of a current limit cycle doubles up. This is considered a transition to chaos in nonlinear dynamic systems as a series of Period Doubling bifurcations drive the system to chaotic motion. Period Doubling bifurcation has no counterpart in continuous time systems. To detect possible Period Doubling bifurcations in the open economy functional structure, we use the algorithm developed by Yuri A. Kuznetsov et al., which is known as CL_MatContM.

Assuming that certain conditions are satisfied, a *Hopf (Neimark-Sacker) bifurcation* emerges when a pair of complex conjugate eigenvalues goes over the unit circle (or the real part of the complex conjugate eigenvalues change sign by passing through zero in continuous time) upon a change in the control parameter. A limit cycle emerges from a fixed point while the fixed point changes stability. The bifurcation can be supercritical or subcritical depending on whether the emerging limit cycle is stable or unstable, respectively. This can be detected by performing a numerical experiment by increasing the value of the parameter after reaching the bifurcation value. If, for example, the limit cycle grows as we continue increasing the parameter value, it is called a supercritical Hop bifurcation. Hopf bifurcation, which causes a transition between stability and instability, is usually associated with oscillator behavor in the system. Hopf bifurcation can also be a trigger of the route to chaos. Since it was first introduced to economic literature by Torre (1977) for an IS-LM model, Hopf bifurcations have become a common subject for research and the Hopf bifurcation theorem has been widely used in the analysis of dynamic systems in economics literature. Foley (1989) examines the stability properties of closed orbits in a macroeconomic model that goes through endogenous financial-production cycles. Feichtinger (1992) examines a two-state advertising diffusion model and shows that, under certain assumptions, the model exhibits persistent fluctuations in sales and advertising over time and undergo a Hopf bifurcation. Economic interpretation of the Hopf bifurcation depends on the specific model and parameters which are under investigation. As Kind (1999) points out, the stable closed orbits in the supercritical Hopf bifurcations might be attributed to the stylized business or growth cycles, while the subcritical Hopf bifurcation can be interpreted within the notion of Leijonhufvud's (1973) corridor stability. Benhabib and Nishimura (1979) and Medio (1986) establish the conditions for the occurrence of closed orbits in optimal growth models. Using the Hopf bifurcation theorem, Semmler (1986), considering a Minskyan macroeconomic model, shows that introducing external financial perturbation terms has destabilizing effects on the real variables and change the characteristics of limit cycles. Diamond and Fudenberg (1989) examine the rational expectations equilibrium paths of Diamond's "search and barter model" and they show that the model can exhibit a Hopf bifurcation, so that cycles occur for some parameter values. For some other studies on Hopf bifurcation in economics, see also Dockner and Feichtinger (1991, 1993), Feichtinger, Novak and Wirl (1994), Feichtinger and Sorger (1986), Zhang (1988, 1990), Franke (1992), Krawiec and Szydlowski (1999), Guckenheimer, Myers and Sturmfels (1997), Asada and Yoshida (2001) and Cai (2005).

The readers who are seeking more detailed introduction on stability and bifurcation analysis can refer to Lorenz (1993), Strogatz (1994), Gandolfo (1996), Kuznetsov (1998), Elaydi (2005) and Zhang (2006), which were the main sources of this section.

CHAPTER II:

OPEN ECONOMY NEW KEYNESIAN MODELS

2.1 Introduction:

New Keynesian models have been used to analyze the macroeconomic consequences of monetary policy rules in a variety and to investigate various policy issues. These models have a dynamic stochastic general equilibrium structure, and are incorporated with nominal rigidities, usually through staggered price and/or wage settings, as well as imperfect competition and various forms of market frictions. In such models, monopolistically competitive firms produce differentiated goods whose prices are set under Calvo-type price stickiness constraint. Monetary policy has non-trivial effects on real variables, which appears as both a stabilization tool and another source of instability for the economy.

A standard New Keynesian model contains two behavioral equations, IS curve and New Keynesian (NK) Philips curve. The IS curve equation, which characterizes the demand side of the economy, formulates the the influences of the future output gap, and the real interest rate on the current output gap. The NK Philips curve equation describes the supply side of the economy and relates the current rate of inflation to the next period’s expected rate of inflation and the current output gap. Both equations are derived from the optimization problems of economic agents, based on microeconomic foundations. The IS curve is derived from the Euler equation for the representative household’s utility maximization problem and the NK Philips curve is derived from the pricing decision of a representative firm. Coefficients of these two equations are the functions of the deep parameters of the corresponding value functions. These two structural equations are accompanied by a monetary policy rule which formulates the setting of the nominal interest rate as a policy instrument whose value is determined based on the movements in the inflation rate and the output gap away from their pre-specified target levels. Most central banks today employ the short term interest rate as the main policy instrument for the monetary policy.

In recent years, there have been many studies extending the simple closed economy models to the open economy environment. Some contributions in the open economy literature include Ball (1998), Batini and Haldane (1999), Gali and Monacelli (1999), Svensson (2000), Obstfeld and Rogoff (2000), Corsetti and Pesenti (2000), McCallum and Nelson (2000), Clarida, Gali and Gertler (2001), Benigno and Benigno (2002), Batini, Harrison and Millard (2003), Laxton and Pesenti (2003), Smets and Wouters (2003), and Gali and Monacelli (2005). See also Lane (2001) and Walsh (2003) for a survey of open economy macroeconomic models. Regarding the review of and discussions about the New Keynesian models and the monetary policy rules, Walsh (2003) and Gali (2008) were the main sources being referred in this section.

2.2 Model:

In this study, we use Gali and Monacelli's (2005) model of a small open economy in New Keynesian tradition, which has been considered one of the mainstays of the New Keynesian literature in open economy environment.

Gali and Monacelli (1999) develop a two-country version of the open economy model, which is also used as a baseline by Clarida, Gali and Gertler (2001, 2002). Unlike Gali and Monacelli (1999), Gali and Monacelli (2005) define the small open economy as "one among a continuum of infinitesimally small economies making up the world economy", represented by the unit interval. Thus, domestic policy applications do not affect the other countries and the world economy. Each economy is assumed to have identical preferences as well well the same technology and market structure, although they might encounter different but imperfectly correlated productivity shocks. Our bifurcation and determinacy analyses are built upon Gali and Monacelli's (2005) model. In this section, we shortly summarize some of the basic characteristics of the model and consider the linearized version of the model referring to Gali and Monacelli (2005) for derivations and the other details. For comparison with the closed economy case, we will consider the results from Barnett and Duzhak (2008, 2010), which utilize a standard New Keynesian model of closed economy based on Walsh (2003).

Gali and Monacelli (2005) design a stylized model of the behaviors of consumers, firms and a policy maker. Both consumers and firms behave optimally so that consumers maximize the expected present value of their utility while firms maximize their profits.

A representative consumer supplies labor and purchases consumption goods in the small open economy and maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right)$$

where N_t denotes hours of labor, and C_t is a composite consumption index defined by

$C_t \equiv \left[(1-\alpha)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$. In the last expression, $C_{H,t} \equiv \left(\int_0^1 C_{H,t}^{\frac{\varepsilon-1}{\varepsilon}}(j)\,dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$ denotes an index of consumption of domestic goods with $j \in [0,1]$ representing the variety of goods and $C_{F,t} \equiv \left(\int_0^1 C_{i,t}^{\frac{\gamma-1}{\gamma}}\,di \right)^{\frac{\gamma}{\gamma-1}}$ is an index of imported goods with $C_{i,t} \equiv \left(\int_0^1 C_{i,t}^{\frac{\varepsilon-1}{\varepsilon}}(j)\,dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$ being an index of the quantity of goods imported from country i. The parameter $\varepsilon > 1$ denotes the elasticity of substitution between varieties produced within country i. The parameter $\alpha \in [0,1]$ is inversely related to the degree of home-bias in preferences, hence represents a natural index of openness. As Gali and Monacelli (2005) states, the parameter α corresponds to the share of imports in domestic consumption since price indices for domestic and imported goods are equal in steady state. Ried (2009) sets the boundaries of this parameter to 0 and 0.9, where the upper boundary implies a bias towards foreign goods. The larger α is, the smaller is the share of home-produced goods in domestic consumption. While $\alpha = 0.5$ implies no home bias, $\alpha > 0.5$ indicates a bias towards foreign goods. The parameter $\eta > 0$ denotes the elasticity of substitution between domestic and foreign goods, while γ measures the substitutability between goods produced in different foreign countries.

Maximization of the consumer's objective function is subject to the following sequence of budget constraints:

$$\int_0^1 P_{H,t}(j) C_{H,t}(j)\,dj + \int_0^1 \int_0^1 P_{i,t}(j) C_{i,t}(j)\,dj\,di + E_t\left\{Q_{t,t+1} D_{t+1}\right\} \le D_t + W_t N_t + T_t$$

for $t = 0, 1, 2, \ldots$, where $P_{i,t}(j)$ is the price of variety *j* imported from country *i*. D_{t+1} is the nominal pay-off in period $t+1$ of the portfolio held at the end of period *t* (which includes shares in firms), W_t is the nominal wage, and T_t denotes lump-sum transfers/taxes. $Q_{t,t+1}$ is the stochastic discount factor for one-period ahead nominal pay-offs received by domestic households. As described in detail by Gali and Monacelli (2005), the utility maximization problem yields the following dynamic (intertemporal) IS curve equation which is a log-linear approximation to the Euler equation:

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(r_t - E_t \pi_{t+1} - \overline{r}_t\right), \qquad (2.2.1)$$

where x_t is the gap between actual output and flexible-price equilibrium output,

$$\overline{r}_t = \rho - \sigma_\alpha\left(\frac{1+\varphi}{\sigma_\alpha + \varphi}\right)(1-\rho_a)\,a_t + \alpha\sigma_\alpha\left((\omega-1) - \frac{(\omega-1)\sigma_\alpha}{\sigma_\alpha+\varphi}\right)E_t\left\{\Delta y^*_{t+1}\right\}$$

is the small open economy's natural rate of interest, $\sigma_\alpha = \sigma(1-\alpha+\alpha\omega)^{-1}$ and $\omega = \sigma\gamma + (1-\alpha)(\sigma\eta - 1)$ are composite parameters. The lowercase letters denote the logs of the respective variables, $\rho = \beta^{-1} - 1$ denotes the time discount rate, and $a_t = \log A_t$ is the log of average product of labor. The negative coefficient on the interest rate in the IS relationship (2.2.1) reveals intertemporal substitution effects on consumption. As Woodford (2008) points out, the monetary policy decisions have impacts on the aggregate expenditure through the changes in the intertemporal substitution between expenditure in periods t and $t+1$. Note that there is no demand disturbance such as a taste shock in equation (2.2.1). Nevertheless, the absence of disturbance terms do not change the bifurcation analysis in our case since we only consider the Jacobian matrix of the endogenous variables, i.e. the output gap and the inflation rate.

Gali and Monacelli (2005) note that the fundamental difference between closed and open economy versions of the IS relationship is the fact that the elasticity of demand with respect to the real interest rate is no longer equal to the elasticity of intertemporal substitution, σ^{-1} in the open economy framework. Instead, it equals to $\sigma_\alpha^{-1} = (1-\alpha+\alpha\omega)\sigma^{-1}$, which depends on the openness of the economy through α as well as $\eta > 0$ and γ through the definition of ω.

Defining $r_t^{real} = r_t - E_t\pi_{t+1}$ as the one-period real interest rate and then recursively solving the equation (2.2.1) forward gives

$$x_t = -\frac{1}{\sigma_\alpha}\left(\sum_{i=0}^{\infty} E_t r_{t+i}^{real} + \sum_{i=0}^{\infty} E_t \overline{r}_{t+i}\right). \qquad (2.2.2)$$

This expression shows that the current and future values of the real interest rate and natural rate determine the current output level.

The representative firm hires labor and produces differentiated goods under monopolistic competition. Firms set prices in a staggered way so that each period only a fraction $1-\theta$ of randomly selected firms are permitted to change their prices. This price stickiness is due to Calvo (1983). θ is a measure of the degree of price rigidity. When $\theta = 0$, all firms are able to adjust their prices according to the conditions in every period. The larger the parameter θ, the fewer the firms are able to adjust their prices each period and the longer the time period between price adjustments for the representative firm. The problem of the representative firm j while resetting the price in period t is to maximize the expected present value of its dividend stream contingent on the new price, that is

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} \left[Y_{t+k} \left(\bar{P}_{H,t} - MC_{t+k}^n \right) \right] \right\}$$

subject to the following demand constraint that firm j faces

$$Y_{t+k}(j) \leq \left(\frac{\bar{P}_{H,t}}{P_{H,t+k}} \right)^{-\varepsilon} \left(C_{H,t+k} + \int_0^1 C_{H,t+k}^i \, di \right) \equiv Y_{t+k}^d \left(\bar{P}_{H,t} \right),$$

where $Y_t(j) = A_t N_t(j)$ is the linear production function with constant returns to scale and $\bar{P}_{H,t}$ denotes the newly set domestic prices. $MC_t^n = \frac{(1-\tau) W_t}{A_t}$ is the nominal marginal cost with τ being the constant employment subsidy which counterbalances the distortions due to the firms' market power and the monetary authority by influencing the terms of trade in favor of domestic consumers, in order to make the flexible price equilibrium allocation efficient.

This maximization problem of the representative firm yields, after some algebra, the aggregate supply curve, often called the New Keynesian (NK) Philips curve equation in log-linearized form:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t \qquad (2.2.3)$$

where $\mu = \frac{(1-\beta\theta)(1-\theta)}{\theta}$, and $\omega = \sigma\gamma + (1-\alpha)(\sigma\eta - 1)$.

By definition, the composite parameter $\kappa_\alpha = \mu(\sigma_\alpha + \varphi)$ where $\sigma_\alpha = \frac{\sigma}{(1-\alpha)+\alpha\omega}$, i.e. the impact of the output gap on the inflation rate, depends on the structural parameters $\alpha, \beta, \sigma, \theta$ and ω. A decrease in α, or ω results in a decrease in κ_α, hence inflation becomes less sensitive to the output gap.

The New Keynesian Philips curve implies that the current inflation rate is a function of the output gap (or the real marginal cost) and the next period's expected inflation. As can be seen from equation (2.2.3), in open economy framework, the degree of openness, the substitutability between domestic and foreign goods, the terms of trade and world output are the factors that influence the rate of inflation.[1]

Solving equation (2.2.3) forward, we obtain,

$$\pi_t = \kappa_\alpha \sum_{i=0}^{\infty} \beta^i E_t x_{t+i} \qquad (2.2.4)$$

which says that the current inflation is the present discounted sum of future deviations in the output level from the flexible price level. According to (2.2.4), the real driving force of inflation dynamics appears to be the output gap. The output gap can be related to the domestic real marginal cost according to $mc_t = (\sigma_\alpha + \varphi)x_t$. As Gali and Monacelli (2005) states, in open

[1] Note that we do not consider any policy tradeoff between pursuing a stabilized inflation (which is reflected as a zero inflation in the steady state) and stabilized output (which is reflected as a zero output gap) in our study. That would require adding a disturbance term into to the NK Philips curve equation. For a detailed discussion of the subject, see, for example, Gali (2008) or Walsh (2003).

economy framework, its influences on employment (captured by φ), and the terms of trade (captured by σ_α) are the main channels that the domestic output affects the marginal cost. Notice that the composite parameter σ_α is also a function of the deep parameters, the degree of openness and the substitutability between domestic and foreign goods. The parameter φ denotes the elasticity of labor supply. The higher φ, the lower the elasticity. In a particular case, $\alpha = 0$ and/or $\sigma = \eta = \gamma = 1$ implies $\sigma_\alpha = \sigma$ and $\omega = 1$. In this case, the domestic real marginal cost is not affected by the movements in foreign output, and the economy reaches balanced trade (i.e. $nx_t = 0$). Then, the slope coefficient is given by $\lambda(\sigma + \varphi)$ as in the closed economy New Keynesian Philips curve.

As pointed out by Gali and Monacelli (2005), the degree of openness α has effects on the dynamic behaviors of domestic inflation through its impact on the the slope of the New Keynesian Phillips curve which measures the reaction of the inflation to any change in the output gap.

Although the structure of the open economy model, consisting of equations (2.2.1) and (2.2.3), is isomorphic to the closed economy version, both models differ in the functional construction of parameters. The parameters of the open economy model in Gali and Monacelli (2005) depend on factors which are exclusive to the open economy such as the degree of openness, the terms of trade, the substitutability among goods of different origin and the world output that is exogenously determined.

The model is closed by adding a simple (i.e. non-optimized) monetary policy rule conducted by the monetary authority:

$$r_t = \overline{r}_t + \phi_\pi \pi_t + \phi_x x_t \tag{2.2.5}$$

where the coefficients $\phi_x > 0$ and $\phi_\pi > 0$ measure the sensitivity of the nominal interest rate to changes in output gap and inflation rate, respectively. Bullard and Mitra (2002) suggest $0 < \phi_x < 4$ and $0 < \phi_\pi < 10$ for policy analysis. The third equation prevents the indeterminacy problem and allows for having a unique optimum equilibrium. In its particular form, the policy rule (2.2.5) is called the Taylor rule (Taylor 1993). Various variations of the Taylor rule are often employed to design the monetary policy in empirical DSGE models.

According to equation (2.2.5), the central bank utilizes the nominal interest rate as policy tool in order to implement the necessary monetary policy to encounter the variations in inflation and/or output gap with respect to their target levels.

Equations (2.2.1) and (2.2.2), in combination with a monetary policy rule such as (2.2.5) constitute a small open economy model in New Keynesian tradition. Like the closed economy counterpart, a standard open economy approach assumes long run neutrality, meaning that due to nominal rigidities monetary policy may have real effects on output, exchange rate and interest rate in the short run, while as the adjustment of wages and prices take place, the long run effect falls on the prices creating a nominal affect in the long run.

Complete financial markets (no frictions), constant, exogenous world interest rate, fully flexible exchange rates, complete exchange rate pass-through of nominal exchange rate changes to import prices, and law of one price are some of the assumptions made by Gali and Monacelli (2005) regarding the construction of the model. They also assume that in steady state purchasing power parity holds, real interest rate differential reverts to zero mean, trade is balanced, and nominal interest rate equals to the natural rate of interest.

Galí and Monacelli (2005) use Canadian data for their numerical analysis as it is considered an ideal “prototype small open economy" because of its relative size and proximity to the U.S. as well as its close economic ties with this country. Numerical values for the parameters in our study were chosen in line with those of the Gali and Monacelli (2005). Given that one period in the model corresponds to one quarter of a year, the (quarterly) discount factor is set at $\beta = 0.99$, which implies a steady state interest rate of 4 % per annum in the quarterly model. Empirically, the values of β between 0.90 and 1 are considered reasonable. The elasticity of substitution between domestic and foreign goods, η, typically takes values close to unity, as in Gali and Monacelli (2005) but some authors suggest values as high as 20, such as Obstfeld and Rogoff (2000*b*). Ried (2009) suggest η being between 1 and 12. $\eta = 1$ implies trade balance while an elasticity higher than unity means that home and foreign goods are highly substitutable. Engel (2000) argues that international substitutability must be lower than intranational substitutability. For the risk aversion parameter, σ, which is also the inverse of the intertemporal rate of substitution, Gali and Monacelli (2005) use $\sigma = 1$, although in the literature, it is allowed to be between 0 and 2 and as much as 10 as pointed out by Ried (2009). Gali and Monacelli (2005) choose the import share α equal to 0.4 which corresponds roughly to the Canadian import/GDP ratio. A degree of openness parameter close to zero is consistent with the small economy assumption. The Calvo sticky price parameter θ is assumed to be identical across countries and is set to 0.75 which implies a price duration of four quarters, $1/(1-\theta) = 4$, on average. The composite parameter κ_α is taken as 0.34 approximately by Gali and Monacelli (2005). The labor supply parameter, φ, equals 3 which implies a Frisch labor supply elasticity of $1/3$. Gali and Monacelli (2005) assume zero trade cost. In order to show the implications of the open economy environment through the deep parameters, we use the explicit expressions of the composite structural parameters of the model.

CHAPTER III:

STABILITY AND BIFURCATION ANALYSIS

As Gandolfo (1996) elaborately explains, the existence of a bifurcation around a fixed point y^* of a dynamic system such as (1.3.1) can be pinned down by checking the eigenvalues of the Jacobian matrix $C\left(y^*, \phi\right)$ computed at the fixed point. For a map, a bifurcation will occur when there are eigenvalues $C\left(y^*, \phi\right)$ on the unit circle. For a flow, it will occur when there are eigenvalues on the imaginary axis. The next problem is to identify the type of the bifurcation emerged in the system due to perturbations of the coefficients. This is done by checking the transversality conditions.

In order to determine if a Hopf bifurcation exists in the Gali and Monacelli Model, our methodology is that of Gandolfo (1996) and Barnett and Duzhak (2008, 2010). We first evaluate the Jacobian of the system at the equilibrium point $\pi_t = x_t = 0$ for all $t = 1, 2, \ldots$, and then check if the conditions of the Hopf Bifurcation Theorem are satisfied. For two dimensional systems, we apply the existence part of the Hopf Bifurcation Theorem given in Gandolfo (1996, page 492):

Theorem 1: Consider a two-dimesional map $y \to f\left(y, \phi\right)$, $y \in \mathbb{R}^2$, $\phi \in \mathbb{R}$. Assume that for each ϕ, there exist a local fixed point $y^* = y^*\left(\phi\right)$ in the relevant interval at which the eigenvalues of the Jacobian matrix evaluated at $\left(y^*\left(\phi\right), \phi\right)$ are complex conjugates $\lambda_{1,2} = a \pm ib$ and satisfy the following properties:

i) $\left|\lambda_1\right| = \left|\lambda_2\right| = +\sqrt{a^2 + b^2} = 1$, with $\lambda_i \neq 1$ for $i = 1, 2$. (3.1)

where $\left|\lambda_i\right|$ is the modulus of the eigenvalue λ_i.

ii) $\left.\frac{d\left|\lambda(\phi)\right|}{d\phi}\right|_{\phi=\phi_0} \neq 0$, (3.2)

Then, the system has a periodic solutions and a Hopf bifurcation occurs at $\phi = \phi_0$.[2]

The first condition says that the eigenvalue should actually step on the unit circle while crossing over it. The second condition (transversality condition) implies that the eigenvalue crosses the unit circle with nonzero speed with respect to the bifurcation parameter, that is it should be a smooth crossing over the unit circle with no jump or disconnection. Theorem 1 is valid only for two dimensional systems. It is not applicable for the dynamic systems in higher dimensions. In fact, presence of a Hopf bifurcation in a three dimensional system, for example, requires a pair of complex conjugate eigenvalues on the unit circle and one real-valued eigenvalue lying outside the unit circle. The following theorem from Wen, Xu and Han (2002) states the conditions for the existence of a Hopf bifurcation in a three dimensional dynamic system.

Theorem 2: Consider a 3x3 matrix *C* having a third order characteristic polynomial in the following form:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

The matrix *C* has a real negative root outside the unit circle and a pair of complex conjugate eigenvalues on the unit circle if and only if the transversality condition

$$\left.\frac{\partial\left|\lambda_i(\phi)\right|}{\partial\phi_j}\right|_{\phi_j=\phi_j^*} \neq 0 \quad (3.3)$$

holds and the following conditions are satisfied:

i) $|a_0| < 1$, (3.4)

[2] See Lorenz (1993) for an alternative but still straightforward version of the Hopf Bifurcation Theorem.

ii) $\left|a_0 + a_2\right| < 1 + a_2$, (3.5)

ii) $a_1 - a_0 a_2 = 1 - a_0^2$. (3.6)

The expressions (3.4)-(3.6) are the conditions for the pair of complex conjugate eigenvalues to lie on the unit circle at $\left(y^*, \phi^0\right)$, while (3.3) states that the pair of complex conjugate eigenvalues should cross the unit circle at a nonzero speed. We use Theorem 2 for analyzing the Gali and Monacelli Model in three dimension. The same theorems were employed by Barnett and Duzhak (2008, 2010) for the closed economy New Keynesian models. We follow Barnett and Duzhak (2008, 2010) to derive the conditions for the existence of Hopf bifurcation and to construct and interpret the bifurcation boundary diagrams.

For the numerical analysis, we follow the methodology developed by Govaerts et al. (2008). Consider the iteration $f^i : y \to f^i\left(y\right) = f\left(f\left(...f\left(y\right)\right)\right)$ that gives rise to a sequence of points $y = y_1, y_2, y_3, ..., y_{i+1}$, in which $y_{i+1} = f^i\left(y_1\right)$. Given the i$^{\text{th}}$ iterate of the equilibrium point, $f^i\left(y\right) - y = 0$, in order to identify bifurcations, we look at the eigenvalues of the Jacobian matrix $C\left(y, \phi\right) = f_y\left(y, \phi\right)$ along the equilibrium curve $f\left(y, \phi\right) = 0$ with $y \in \mathbb{R}^n$ and $\phi \in \mathbb{R}$ and through CL_MatcontM software, we check the following test functions according to Govaerts et al. (2008):

1. $g_1\left(y, \phi\right) = \det\left(C^{(i)} \odot C^{(i)} - I_m\right)$,

where $m = n\left(n-1\right)/2$, $C^{(i)}$ is the Jacobian matrix of iterated map f^i and $\odot$ is the bialternate matrix product.

2. $g_2\left(y, \phi\right) = \det\left(C^{(i)} + I_n\right)$.

3. $g_3\left(y, \phi\right) = v_{n+1}$.

where v is the tangent vector to the equilibrium curve in the (y, ϕ) -space.

4. $g_4(y, \phi) = \det \begin{pmatrix} F_Y \\ v^T \end{pmatrix}$,

where $F_Y = f^i(y, \phi) - y$.

Then, using the test functions, we can detect the codimension-1 bifurcations and branching points located as regular zeros of the above test functions, namely for Hopf, $g_3 = 0$; for Period Doubling, $g_2 = 0$; for Limit Point, $g_1 = 0$ and $g_4 \neq 0$; for branching point, $g_4 = 0$.

In this section, we consider varying the timing of the monetary policy rule and we run determinacy and bifurcation analyses for each case, following Barnett and Duzhak (2008, 2010). We will consider contemporaneous, forward and backward looking policy rules as well as their hybrid combinations. We derive analytical results and present numerical simulations for each case.

3.1 Gali and Monacelli Model Under Current Looking Policy Rules:

3.1.1 Under Current Looking Taylor Rule:

Following Barnett and Duzhak (2008, 2010), we start with a standard contemporaneous specification. Consider the following model in which the first two equations describe the economy while the third equation is the monetary policy rule followed by the central bank:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) x_t \qquad (3.1.1)$$

$$x_t = E_t x_{t+1} - \frac{1 + \alpha(\omega - 1)}{\sigma} \left(r_t - E_t \pi_{t+1} - \overline{r}_t \right) \qquad (3.1.2)$$

$$r_t = \overline{r}_t + \phi_\pi \pi_t + \phi_x x_t \qquad (3.1.3)$$

where x_t denotes the output gap, π_t is the inflation rate, and r_t is the nominal interest rate. E_t is the expectation operator. Equation (3.1.3) describes the policy rule as a current looking Taylor rule in which the interest rate is set according to the current inflation rate and the current output gap. The policy parameters ϕ_π and ϕ_x measure the central bank's response to changes in inflation and output gap, respectively. As mentioned before, there is no exogenous shock.

In order to avoid singularity, we substitute (3.1.3) into (3.1.2) for $r_t - \overline{r}_t$, and we obtain a first order stochastic system of two difference equations in terms of the domestic inflation and the output gap:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \tag{3.1.4}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(\phi_\pi \pi_t + \phi_x x_t - E_t \pi_{t+1}\right)$$

Note that the structure of system (3.1.4) is identical to the standard New Keynesian model of a closed economy. Clearly, $x_t = \pi_t = 0$ for all t constitutes an equilibrium solution to the system (3.1.4). Hence, at the equilibrium, inflation rate is zero since the price level is not changing, that is $\frac{P_t^*}{P_{t-1}} = 1$, and the output gap is zero since equilibrium output level will be at the flexible price level.

Rearranging the terms, the system can be written in the form $E_t y_{t+1} = C y_t$,

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\frac{\mu}{\beta}+\left(1+\alpha(\omega-1)\right)\left(\frac{\beta\phi_x+\varphi\mu}{\beta\sigma}\right) & \frac{(\beta\phi_\pi-1)\left(1+\alpha(\omega-1)\right)}{\beta\sigma} \\ -\frac{\mu}{\beta}\left(\varphi+\frac{\sigma}{1+\alpha(\omega-1)}\right) & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \tag{3.1.5}$$

The system in (3.1.5) is in normal form in the sense that each equation has only one unknown variable evaluated at time $t+1$. Conditional upon the nonsingularity of the matrix *C*, the dynamic behavior of the system is governed by the eigenvalues of the coefficient matrix $C = A^{-1}B$ in (3.1.5). The characteristic equation of the coefficient matrix is given by:

$$p(\lambda) = \lambda^2 - a_1\lambda + a_0 = 0$$

where,

$$a_0 = \frac{\left(1+\alpha(\omega-1)\right)(\phi_x + \varphi\mu\phi_\pi)}{\beta\sigma} + \frac{\mu\phi_\pi + 1}{\beta} \text{ and, } a_1 = \frac{(\varphi\mu + \beta\phi_x)}{\beta\sigma}\left(1+\alpha(\omega-1)\right) + \frac{1+\beta+\mu}{\beta} \quad (3.1.6)$$

Eigenvalues of the Jacobian matrix of the system (3.1.5) are the roots of the characteristic polynomial, given as

$$\lambda_{1,2} = \left(\frac{\beta\phi_x + \kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right) \pm \sqrt{\left(\frac{\beta\phi_x + \kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right)^2 - 4\left(\frac{\phi_x + \kappa_\alpha\phi_\pi + \sigma_\alpha}{\beta\sigma_\alpha}\right)}$$

where $\kappa_\alpha = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)$, and $\sigma_\alpha = \frac{\sigma}{1+\alpha(\omega-1)}$.

In order to make sure that $x_t = \pi_t = 0$ is the only solution, we need to check the determinacy properties of the model (3.1.5). For the uniqueness and the stability of the equilibrium, both eigenvalues must be outside the unit circle. Note that there are two non-predetermined (endogeneous) variables: The inflation rate, π_t, and the output gap, x_t. There is no predetermined variables. Following Blanchard and Kahn (1980), the system has a unique, stationary equilibrium solution for inflation and output gap if and only if the number of eigenvalues of the 2x2 matrix *C* that lie outside the unit circle is equal to the number of forward looking (non-predetermined) variables which is two ($E_t x_{t+1}$ and $E_t\pi_{t+1}$). If, however, the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables, then the equilibrium is called locally indeterminate. Hence, determinacy requires that both eigenvalues of the coefficient matrix *C* are outside the unit circle (i.e. eigenvalues have modulus greater than one). Following Bullard and Mitra (2002), Proposition (3.1.1) establishes the necessary and sufficient conditions for the coefficcient matrix *C* to have both eigenvalues outside the unt circle, which implies a unique equilibrium solution.

Proposition 3.1.1. Given the monetary policy based on the current looking Taylor rule, the open economy New Keynesian model (3.1.5) has a unique stationary equilibrium if and only if

$$-\frac{\sigma(1-\beta)}{1+\alpha(\omega-1)} < \phi_x + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\phi_\pi \qquad (3.1.7)$$

and

$$(1-\beta)\phi_x + (\phi_\pi - 1)\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\mu > 0\,. \qquad (3.1.8)$$

Proof: Following the procedure suggested by J.P. Lasalle (1986, p.28) and Bullard and Mitra (2002), both eigenvalues would be outside the unit circle if and only if

$$|a_0| < 1 \text{ or } 1 - a_0 > 0 \qquad (3.1.9)$$

$$|a_1| < 1 + a_0 \text{ or } 1 - a_1 + a_0 > 0 \qquad (3.1.10)$$

We need to show that the inequalities (3.1.9) and (3.1.10) are satisfied. Condition (3.1.9) implies the inequality (3.1.7). Condition (3.1.10), on the other hand, implies the inequality (3.1.8). □

First note that if $\alpha = 0$, then the determinacy conditions of the model (3.1.5) reduces to the condition for the closed economy counterpart. Condition (3.1.7) is trivially satisfied since $\beta \in (0,1)$. Notice that $\phi_x > 0$ and $\phi_\pi > 1$ satisfy the condition (3.1.8) and is sufficient (although not necessary) for the system to have a unique equilibrium. So, the determinacy of the system depends upon the policy parameters ϕ_x and ϕ_π both. Since $\phi_x, \phi_\pi > 0$ by assumption , the determinacy condition (3.1.8) holds if $\phi_\pi > 1$, although the condition $\phi_\pi > 1$ can be relaxed a little if ϕ_x is large enough. Hence, a unique, stationary equilibrium can be achieved through an active interest rate policy satisfying the Taylor Principle so called by Woodford (2001, 2003b) and Bullard and Mitra (2002). Taylor Principle requires that the nominal interest rate must be raised more than the increase in inflation rate so that the real interest rate increases. As a result, open economy framework has no impact on determinacy condition under current looking Taylor rule. For any value of α and ω, following an active monetary policy is sufficient for the equilibrium determinacy.

The uniqueness of the equilibrium solution can be checked by computing the eigenvalues of the Jacobian matrix. For the baseline values of the parameters, the Jacobian matrix of the system (3.1.5) is

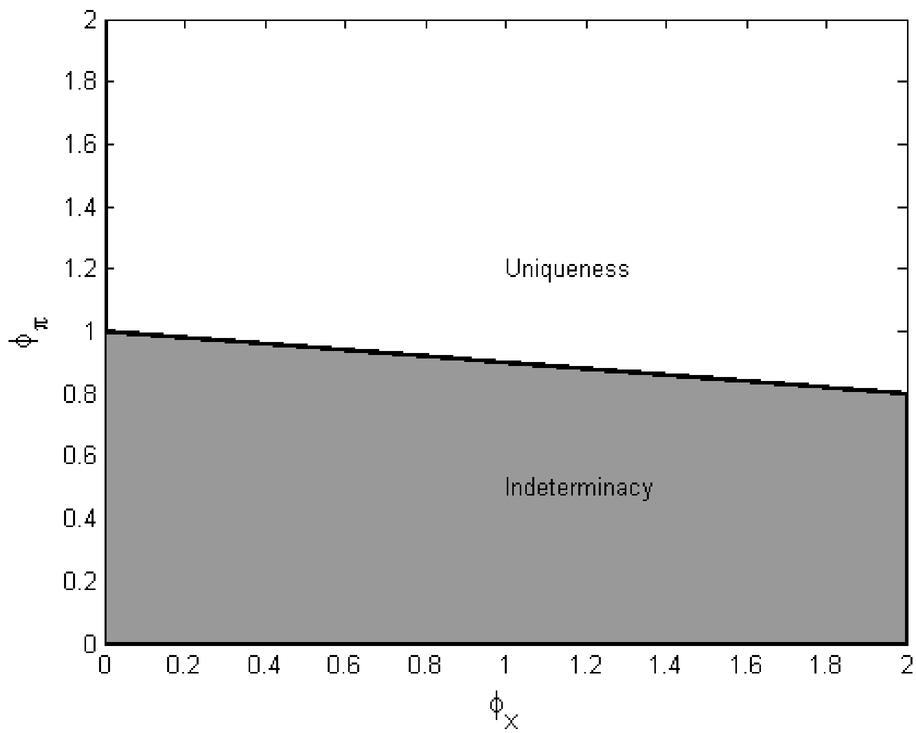
$$C = \begin{bmatrix} 1.4684 & 0.4899 \\ -0.3434 & 1.0101 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 1.2393 + 0.3402i$ and $\lambda_2 = 1.2393 - 0.3402i$, and with modulus $R = \sqrt{(1.2393)^2 + (0.3402)^2} = 1.2851$.[3] Note that the system has a pair of complex conjugate eigenvalues with modulus greater than one. Since the number of eigenvalues outside the unit circle is equal to the number of forward looking variables, there exists a unique solution of the system. Figure (3.1.1) illustrates the regions of the determinate and indeterminate equilibria in

[3] The equilibrium at the origin is a stable spiral point (focus), and is asymptotically stable. Since the complex eigenvalues are greater than one in modulus (R > 1), the amplitude of the sinusoids decays at the rate R^j as $j \to \infty$. Then, dynamic multiplier follows a pattern of decreasing oscillation whose frequency is defined by $\cos(\theta) = \frac{a}{R} = \frac{1.2393}{1.2851} = 0.9644$ or $\theta = \cos^{-1}(a/R) = \cos^{-1}(1.2393/1.2851) = 15.3426$. Hence, the cycles associated with the dynamic multiplier function have a period of $\frac{2\pi}{\theta} = \frac{2*3.14159}{15.3426} = 0.409524$, which are the peaks in the pattern appear about a half period apart.

$\left(x_t, \pi_t\right)$-space as implied by the condition (3.1.8). Geometrically, as shown in Figure (3.1.1), the determinacy region is characterized by an upper bound and a lower bound for ϕ_π as a function of ϕ_x as formulized by Proposition (3.1.1).

Figure 3.1.1: Determinacy region under the current looking Taylor rule.



At the point where $\left(1-\beta\right)\phi_x + \left(\phi_\pi - 1\right)\left(\dfrac{\sigma}{1+\alpha\left(\omega-1\right)} + \varphi\right)\mu = 0$, the system has a branching point that can be investigated by varying ϕ_π freely as shown by Barnett and Duzhak (2008). Using the calibration values of the parameters given in Gali and Monacelli (2005), we can solve the equation for ϕ_π which gives $\phi_\pi = 0.985437$. Thus, the system will have a branching point at the value $\phi_\pi = 0.985437$, which is just another way of saying that a bifurcation occurs here, but may not be one of the standard types of which we usually encounter in economic models. Some authors, however, distinguish both terms and prefer "bifurcation" for cases where branching is induced by varying external parameters.[4]

In order to examine the nature of the eigenvalues, we need to check the sign of the discriminant $\Delta \equiv a_1^2 - 4a_0$. If the discriminant of the quadratic equation is strictly negative, that is if

$$\begin{aligned}\Delta &\equiv a_1^2 - 4a_0 \\ &= \left(\frac{\beta\phi_x + \kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right)^2 - 4\left(\frac{\varphi_x + \kappa_\alpha\phi_\pi + \sigma_\alpha}{\beta\sigma_\alpha}\right) < 0\end{aligned} \tag{3.1.11}$$

where $\kappa_\alpha = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)$, and $\sigma_\alpha = \frac{\sigma}{1+\alpha(\omega-1)}$, then the roots are complex conjugate numbers in the form $\lambda_{1,2} = a \pm ib$, with $a, b \in \mathbb{R}$, $b \neq 0$ is the real part, while $i = +\sqrt{-1}$ is the imaginary unit.

Regarding the model (3.1.5), it is algebraically quite cumbursome to identify the sign of the modulus of the eigenvalues. Using the baseline values of the parameters, however, it is easy to verify that the discriminant Δ is strictly negative. Therefore, the eigenvalues of the system (3.1.5) are complex conjugate, $\lambda_{1,2} = a \pm ib$,

where

$$a = \frac{a_1}{2} = \frac{1}{2}\left(\frac{\beta\phi_x + \kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right) = \left(\frac{\beta\phi_x + \kappa_\alpha + \sigma_\alpha}{2\beta\sigma_\alpha} + \frac{1}{2}\right) \tag{3.1.12}$$

and

$$b = \frac{\sqrt{-\Delta}}{2} = \frac{1}{2}\sqrt{4\left(\frac{\phi_x + \kappa_\alpha\phi_\pi + \sigma_\alpha}{\beta\sigma_\alpha}\right) - \left(\frac{\beta\phi_x + \kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right)^2} \tag{3.1.13}$$

[4] See, for example, Buono, Lamb, and Roberts (2008).

with $\kappa_\alpha = \mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)$, and $\sigma_\alpha = \dfrac{\sigma}{1+\alpha(\omega-1)}$.

In order to verify the existence of a Hopf bifurcation, we need to apply Theorem 1. Using Theorem 1, the conditions for the existence of the Hopf bifurcation in the system (3.1.5) is presented in the following Proposition.

Proposition 3.1.2: The system (3.1.5) undergoes a Hopf bifurcation if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\sigma(\beta-1)}{1+\alpha(\omega-1)} - \mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\phi_\pi . \tag{3.1.14}$$

Proof: Suppose the system (3.1.5) goes through a Hopf bifurcation at (y^*, ϕ_x^*), where $y^* = (x^*, \pi^*)$. Then we need to show that $\Delta < 0$ and $\phi_x^* = \dfrac{\sigma(\beta-1)}{1+\alpha(\omega-1)} - \mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\phi_\pi$.

The existence of a Hopf bifurcation, by definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative, that is $\Delta < 0$.

For the second part, note that the existence of a Hopf bifurcation requires $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = +\sqrt{a^2+b^2} = 1$ by the first condition of Theorem 1. Rewriting the condition explicitly by substituting (3.1.12) and (3.1.13) into it, taking the square of both sides and then solving for ϕ_x, we obtain the critical value of the parameter as in (3.1.14). Therefore, the first condition of Theorem 1 holds only if $\phi_x = \dfrac{\sigma(\beta-1)}{1+\alpha(\omega-1)} - \mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\phi_\pi$.

From the other side, suppose $\Delta < 0$ and $\phi_x = \dfrac{\sigma(\beta-1)}{1+\alpha(\omega-1)} - \mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\phi_\pi$.

Substituting for ϕ_x^* into $\sqrt{a^2+b^2}$ yields $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = 1$, which is the first condition in

Theorem 1. In order to show that the critical value of the parameter ϕ_x is actually a Hopf bifurcation parameter, we check the second condition in Theorem 1, which gives

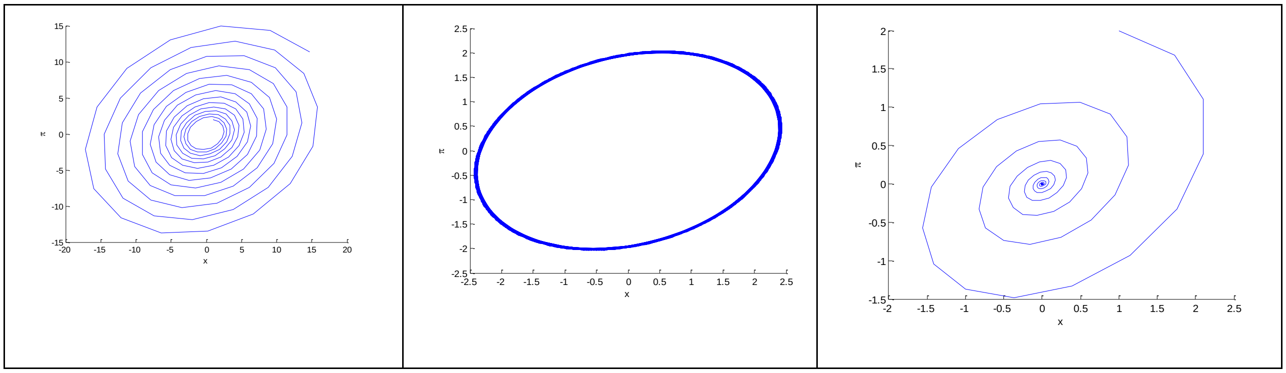
$$\left.\frac{d\left|\lambda_i\left(\phi_x\right)\right|}{d\phi_x}\right|_{\phi_x=\phi_x^*}=\left.\frac{d}{d\phi_x}\left(\sqrt{a^2+b^2}\right)\right|_{\phi_x=\phi_x^*}=\frac{1+\alpha\left(\omega-1\right)}{2\beta\sigma}\neq 0 \text{ for } i=1,2.$$

Thus, both conditions of Theorem 1 are satisfied and we have

$$\phi_x^*=\frac{\sigma\left(\beta-1\right)}{1+\alpha\left(\omega-1\right)}-\mu\left(\frac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right)\phi_\pi \quad . \qquad \square$$

Proposition (3.1.2) shows that taking the parameter ϕ_x free to vary and keeping the other parameters constant, the system (3.1.5) is likely to undergo a Hopf bifurcation at ϕ_x^*. In the closed economy case, the corresponding value of the bifurcation parameter is $\phi_x^*=\sigma\left(\beta-1\right)-\kappa\phi_\pi$ as given by Barnett and Duzhak (2008). Note that, for $\alpha=0$, Proposition (3.1.2) gives the same result as the closed economy counterpart. As pointed out in Gali and Monacelli (2005), while "the closed economy model is nested in the small open economy model as a limiting case", both versions differ basically in two aspects: First of all, some coefficients of the open economy model depends on the parameters that are exclusive to the open economy framework such as the degree of openness, terms of trade and substitutability between domestic and foreign goods. The degree of openness affects the inflation dynamics via its impact on the slope coefficient of the New Keynesian Philips curve equation which measures the inflation response to the changes in the output gap. For $\alpha=0$, the coefficient of the output gap in the Philips curve equation becomes $\kappa=\mu\left(\sigma+\varphi\right)$ as in the closed economy form of the Philips curve equation. Openness also affects the responsiveness of the output gap to the changes in interest rate through the dynamic IS equation. Secondly, the natural levels of output and interest rate depend upon both domestic and foreign disturbances in addition to openness and terms of trade.

Figure 3.1.2: Phase diagrams indicating a Hopf bifurcation



Furthermore, we numerically examine the Jacobian matrix of the system (3.1.5), keeping the structural parameters and the policy parameter ϕ_π constant at their baseline values while varying ϕ_x over a certain range. Although the parameter ϕ_x is assumed to be non-negative, it is worth mentioning what happens for negative values of it. For $-0.52 < \phi_x \leq 0$, the Jacobian matrix has complex eigenvalues with radius greater than one, hence the equilibrium is a stable spiral. At $\phi_x = -0.52$, complex conjugate eigenvalues has radius 1 implying a limit cycle that occurs at the bifurcation point. When an orbit gets inside the limit cycle it then settles to the stable equilibrium point. For $-1.154 \leq \phi_x < -0.52$, the Jacobian matrix has complex conjugate eigenvalues with radius smaller than unity, which implies that the equilibrium is an unstable spiral. So, $\phi_x = -0.52$ is a Hopf bifurcation parameter, which is not in the feasible subset of parameter space of ϕ_x. Hence, the limit cycle and periodic behaviors can only be encountered through parameter values that are not feasible. Figure (3.1.2) illustrates different phase plots for the system (3.1.5). In the first plot, the solution path is an unstable spiral which happens for ϕ_x = -0.50 and higher values. The system is still unstable at ϕ_x =0.2 and diverges very quickly from the initial point for ϕ_x = 0.2. There exists a divergent path for larger values of ϕ_x up to ϕ_x =12. The second diagram illustrates a limit cycle at ϕ_x = -0.52, while the third one shows a stable spiral at ϕ_x = -0.70.

Our numerical analysis also indicates the existence of a Period Doubling bifurcation at $\phi_x = -2.427$, given the benchmark values of the parameters as given in Gali and Monacelli (2005). For $-2.4271 < \phi_x < -1.154$, both eigenvalues are real and inside the unit circle. At $\phi_x = -2.4271$, we have $\lambda_1 = -1$ and $0 < \lambda_2 < 1$. Thus, $\phi_x = -2.4271$ is a Period Doubling bifurcation parameter value. For $\phi_x < -2.4271$, we have $\lambda_1 < -1$ and $0 < \lambda_2 < 1$. This implies multiple equilibria as the number of of eigenvalues outside the unit circle is less than the number of non-predetermined variables.[5]

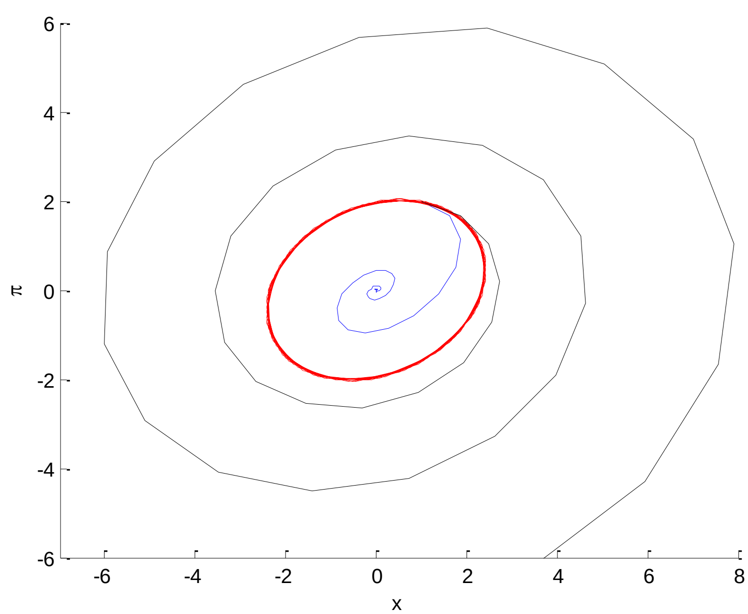
As a result, we numerically find a Period Doubling bifurcation at $\phi_x = -2.43$ and a Hopf bifurcation at $\phi_x = -0.52$. Decreasing the value of ω results in a higher value of bifurcation parameter value in absolute value, except when $\alpha = 0$ at which changes in ω does not make any difference. On the other hand, decreasing the value of α results in a lower value of bifurcation parameter value in absolute value, except when $\omega = 1$ at which changes in α does not make any difference.

Numerical computations indicate that the monetary policy rule (3.1.3) should have $\phi_x^* < 0$ for a Hopf or Period Doubling bifurcation to occur. This means that the feasible subset of the parameter space does not have a risk of bifurcation. The negative coefficient for *(y-y*)* indicates a procyclical monetary policy, i.e., rising interest rates when the output gap is negative

[5] See Blanchard and Kahn (1980). Assuming that λ_1, λ_2 lie in the complex plane, one eigenvalue is inside and the other one is outside the unit circle if and only if $|\lambda_1 + \lambda_2| > |1 + \lambda_1\lambda_2|$. For the calibrated values of the parameters this condition is met.

or vice versa. Schettkat and Sun (2009) state that various situations such as exchange rate stabilization or an underestimation of the potential output level may explain such a result, but otherwise it is difficult to rationalize a negative policy parameter. There is a large volume of literature trying to explain procyclicality in monetary policy. Demirel (2010), for example, shows that the existence of country spread may help explain how optimal fiscal and monetary policies can be procyclical. Schettkat and Sun (2009) claim that the Bundesbank responds to the changes in output gap asymmetrically in different economic situations. In some cases, the Bank did not pursue active policies against recessions. Leith, Moldovan and Rossi (2009) argue that in case of superficial habits and under the benchmark value of θ = 0.65 of the degree of habit formation, the optimal simple rule implies a negative response to the output gap and a perverse policy response to output gap and inflation induce an instability in the model.

Figure 3.1.3: Phase diagram indicating a Hopf bifurcation under the current looking Taylor Rule[6]



[6] Inner trajectory for ϕ_x = -0.70, limit cycle for ϕ_x = -0.52 , outer trajectory ϕ_x = -0.45

Phase diagram in Figure (3.1.3) illustrate a Hopf bifurcation under the current looking Taylor Rule. Notice that there is only one periodic solution and other solutions diverge from the periodic solution as t→∞. In this case, the periodic solution is called an unstable limit cycle.

Based on our findings, under current looking Taylor rule, we can say that procyclical monetary policy gives rise to bifurcations which stratify the confidence region in the parameter space of ϕ_x. Counter-cyclical monetary policy would be bifurcation-free and would yield a more robust confidence region for the paramater space.

On the other hand, we may obtain similar analytic results for the parameter ϕ_π being a potential source of bifurcation as formulized by the following Proposition.

Proposition 3.1.3: The system (3.1.5) undergoes a Hopf bifurcation if and only if $\Delta < 0$ and

$$\phi_\pi^* = \frac{\sigma(\beta - 1) - \phi_x\left(1 + \alpha(\omega - 1)\right)}{\mu\sigma + \mu\varphi\left(1 + \alpha(\omega - 1)\right)}. \tag{3.1.15}$$

Proof: Suppose the system (3.1.5) goes through a Hopf bifurcation at $\left(y^*, \phi_\pi^*\right)$, where $y^* = (x^*, \pi^*)$. Then we need to show that $\Delta < 0$ and $\phi_\pi^* = \frac{\sigma(\beta - 1) - \phi_x\left(1 + \alpha(\omega - 1)\right)}{\mu\sigma + \mu\varphi\left(1 + \alpha(\omega - 1)\right)}$. The existence of a Hopf bifurcation, by definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative, that is $\Delta < 0$.

For the second part, note that the existence of a Hopf bifurcation requires $\text{mod}(\lambda_1) = \text{mod}(\lambda_2) = +\sqrt{a^2 + b^2} = 1$ by the first condition of Theorem 1. Rewriting the condition explicitly by substituting (3.1.12) and (3.1.13) into it, taking the square of both sides and solving for ϕ_π, we have the critical value of the parameter ϕ_π as in (3.1.15). Therefore, the first condition of the Theorem 1 holds only if $\phi_\pi = \dfrac{\sigma(\beta-1)-\phi_x(1+\alpha(\omega-1))}{\mu\sigma+\mu\varphi(1+\alpha(\omega-1))}$.

From the other side, suppose $\Delta < 0$ and $\phi_\pi = \dfrac{\sigma(\beta-1)-\phi_x(1+\alpha(\omega-1))}{\mu\sigma+\mu\varphi(1+\alpha(\omega-1))}$. Substituting for ϕ_π^* into $\sqrt{a^2+b^2}$ yields $\operatorname{mod}(\lambda_1)=\operatorname{mod}(\lambda_2)=1$, which is the first condition in Theorem 1. In order to show that the critical value of the parameter ϕ_π is actually a Hopf bifurcation parameter, we check the second condition in Theorem 1, which yields,

$$\left.\frac{d\left|\lambda_i(\phi_\pi)\right|}{d\phi_\pi}\right|_{\phi_\pi=\phi_\pi^*} = \left.\frac{d}{d\phi_\pi}\left(\sqrt{a^2+b^2}\right)\right|_{\phi_\pi=\phi_\pi^*} = \mu\left(\frac{\sigma+\varphi(1+\alpha(\omega-1))}{2\beta\sigma}\right) \neq 0 \text{ for } i=1,2.$$

Thus, both conditions of Theorem 1 are satisfied and we have

$$\phi_\pi^* = \frac{\sigma(\beta-1)-\phi_x(1+\alpha(\omega-1))}{\mu\sigma+\mu\varphi(1+\alpha(\omega-1))}. \qquad \square$$

Proposition (3.1.3) shows that taking the parameter ϕ_π free to vary and keeping the other parameters constant, the system (3.1.5) is likely to undergo a Hopf bifurcation at ϕ_π^*. We also numerically find a Period Doubling bifurcation at $\phi_\pi = -13.3$, which is not in the feasible subset of parameter space for ϕ_π^*. Decreasing the value of ω causes a higher value of the bifurcation parameter in absolute value, except when $\alpha = 0$ at which changes in ω does not make any difference. On the other hand, decreasing the value of α results in a lower value of the bifurcation parameter in absolute value, except when $\omega = 1$ at which changes in α does not make any difference.

In conclusion, when we consider ϕ_π as the bifurcation parameter, we numerically find that a bifurcation can be possible only for negative values of ϕ_π. This implies that assuming $\phi_x > 0$ and $\phi_\pi > 0$, Gali and Monacelli Model under current looking Taylor rule does not nest a risk of a bifurcation for the feasible parameter space even though it is theoretically possible within the functional structure of the system (3.1.5).

As Rotemberg and Woodford (1999) and Leith, Moldovan and Rossi (2009) argue, failure to satisfy the Taylor rule with $\phi_\pi < 1$ implies that inflation can be driven by self-fulfilling expectations which are validated by passive or even perverse monetary policies. If, in addition, the output gap response is negative, both destabilizing elements in the policy rule may compel the system to a saddle path from which any deviation will result in an explosive path for inflation.

3.1.2 Under Pure Current Looking Inflation Targeting Rule:

Consider the following model in which the first two equations describe the economy while the third equation represents the monetary policy rule followed by the central bank:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t \qquad (3.1.1)$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(r_t - E_t \pi_{t+1} - \overline{r}_t\right) \qquad (3.1.2)$$

$$r_t = \overline{r}_t + \phi_\pi \pi_t \qquad (3.1.16)$$

The last equation describes the policy rule based on pure current looking inflation targeting where ϕ_π is the policy parameter that represents the sensitivity of the central bank's policy rate to current inflation. Equation (3.1.16) states that the nominal interest rate is determined according to the changes in current inflation rate. Monetary policy rule (3.1.16) does not include an interest rate response to the output gap unlike the standard Taylor rule. Corsetti, Meier and Muller (2010) use this version of the policy rule in a way that the nominal interest rate, r_t, responds to current deviations of the inflation rate, π_t, from target level by a factor of 1.5, that is $r_t = \bar{r}_t + 1.5(\pi_t - \bar{\pi}_t)$.[7] On the other hand, Ball (1998) argues that pure inflation targeting brings some risks in an open economy environment since it gives rise to large fluctuations in exchange rate and output, and he suggests following long-run inflation targeting to avoid such problems.

Substituting (3.1.16) into (3.1.2) for $r_t - \bar{r}_t$, we have a two-equation first order stochastic difference equation system in terms of domestic inflation and output gap:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \tag{3.1.17}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(\phi_\pi \pi_t - E_t \pi_{t+1}\right)$$

Rearranging the terms, the system can be written in the form $E_t y_{t+1} = C y_t$,

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \frac{1}{\beta}\begin{bmatrix} \left(\mu + \beta + \dfrac{1+\alpha(\omega-1)}{\sigma}\mu\varphi\right) & -\dfrac{1+\alpha(\omega-1)}{\sigma}(1-\beta\phi_\pi) \\ -\mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) & 1 \end{bmatrix}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \tag{3.1.18}$$

The eigenvalues of the coefficient matrix *C*, λ_1 and λ_2, are computed by setting $\det(C - \lambda I) = 0$ which gives a second-order characteristic polynomial in λ:

$$p(\lambda) = \lambda^2 - a_1\lambda + a_0 = 0$$

[7] Here, a bar over a variable refers to its steady-state value.

where,

$$a_1 = \frac{\varphi\mu\left(1+\alpha\left(\omega-1\right)\right)}{\beta\sigma} + \frac{1+\beta+\mu}{\beta} \text{ and } a_0 = \frac{1}{\beta}\left(1+\left(\mu+\frac{1+\alpha\left(\omega-1\right)}{\sigma}\mu\varphi\right)\phi_\pi\right).$$

The solutions to the characteristic polynomial are given by the following eigenvalues:

$$\lambda_{1,2} = \left(\frac{\kappa_\alpha+\sigma_\alpha}{\beta\sigma_\alpha}+1\right) \pm \sqrt{\left(\frac{\kappa_\alpha+\sigma_\alpha}{\beta\sigma_\alpha}+1\right)^2 - 4\left(\frac{\kappa_\alpha\phi_\pi+\sigma_\alpha}{\beta\sigma_\alpha}\right)}$$

where $\kappa_\alpha = \mu\left(\frac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right)$, and $\sigma_\alpha = \frac{\sigma}{1+\alpha\left(\omega-1\right)}$.

Notice that there are two endogeneous variables: Inflation rate, π_t, and output gap, x_t. There is no predetermined variable in the model. Following Blanchard and Kahn (1980), the system (3.1.18) has a unique equilibrium solution for the inflation rate and the output gap if and only if the number of eigenvalues of the 2x2 matrix *C* that are outside the unit circle is equal to the number of forward looking (non-predetermined) variables, which is two ($E_t x_{t+1}$ and $E_t \pi_{t+1}$) in this case. Then, following Bullard and Mitra (2002), Proposition (3.1.4) characterizes the necessary and sufficient conditions for the determinacy.

Proposition 3.1.4: Given the monetary policy based on pure current looking inflation targeting, the open economy New Keynesian model (3.1.18) has a unique stationary equilibrium if and only if

$$1 < \phi_\pi < \frac{\left(\beta-1\right)\sigma}{\mu\sigma+\mu\varphi\left(1+\alpha\left(\omega-1\right)\right)}. \qquad (3.1.19)$$

Proof: In order to check if the system (3.1.18) has a unique stable equilibrium, we follow the procedure suggested by J.P. Lasalle (1986, p.28) and Bullard and Mitra (2002). Then, both eigenvalues would be outside the unit circle if and only if

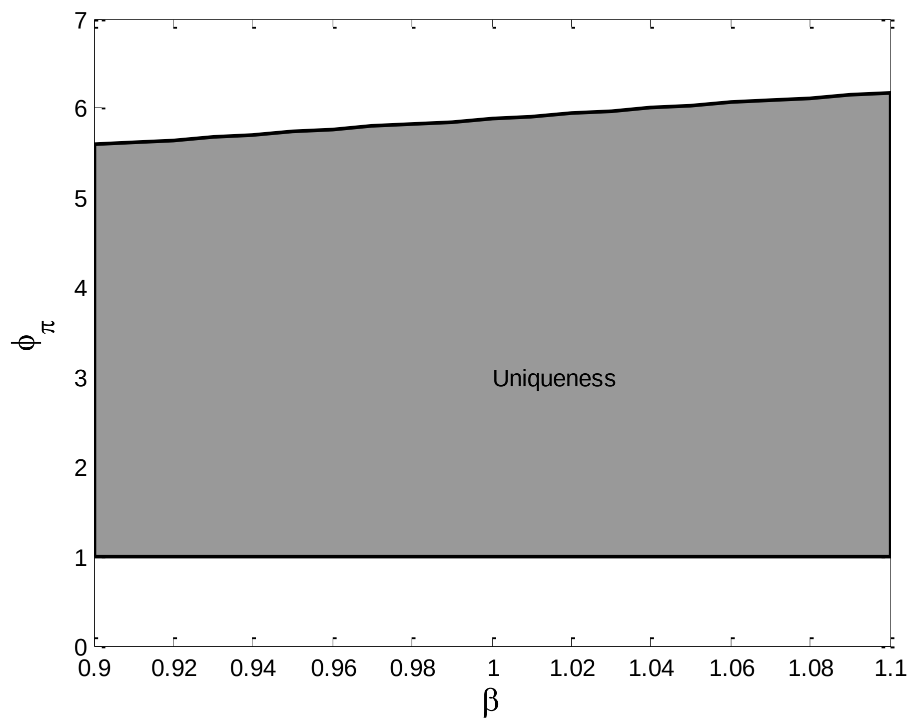
$|a_0| < 1$

$|a_1| < 1 + |a_0|$

From the first condition we obtain the upper boundary, we obtain

$\phi_\pi < \dfrac{(\beta - 1)\sigma}{\mu\sigma + \mu\varphi\left(1 + \alpha(\omega - 1)\right)}$. From the second condition we obtain the lower boundary, we obtain $\phi_\pi > 1$. Combining them, we have the determinacy condition (3.1.19).[8] □

Figure 3.1.4: Determinacy region under pure current looking inflation targeting



Regarding determinacy, the condition $\phi_\pi > 1$ requires an active monetary policy so that the central bank adjusts nominal interest rates more than one-for-one in response to a deviation in inflation rate from its target level. On the other hand, Minford and Srinivasan (2010) argue that with $\phi_\pi > 1$, explosive solutions are also possible within the model, just as multiple solutions

[8] Gali (2008, ch.4) argue that for uniqueness it is necessary and sufficient to have $1 < \phi_\pi$ without any upper bound.

are possible with $\phi_\pi < 1$. But the upper boundary implies that the policy should not be too reactionary which would also lead to indeterminacy. Thus, the upper boundary prevents the overreaction of the monetary authority to changes in inflation which might result in explosive solutions. However, as the degree of openness (captured by α) increases the upper bound gets lower and the determinacy region shrinks.

As illustrated in Figure (3.1.4), the pure current looking inflation targeting based monetary policy yields a unique equilibrium for a feasible set of parameter values. Given the baseline values of the parameters, the Jacobian matrix of the system is

$$C = \begin{bmatrix} 1.3434 & 0.5051 \\ -0.3434 & 1.0101 \end{bmatrix}$$

with complex conjugate eigenvalues $\lambda_1 = 1.1768 + 0.3817\text{i}$ and $\lambda_2 = 1.1768 - 0.3817\text{i}$, having modulus $R = \sqrt{1.1768^2 + 0.3817^2} = 1.2372$. Having a radius greater than unity, both eigenvalues are outside the unit circle.[9] Since the number of eigenvalues outside the unit circle and the number of forward looking variables are equal, the system (3.1.18) has a unique, stationary equilibrium solution.

As shown in Gandolfo (1996), in order to examine the nature of the eigenvalues we need to check the sign of the discriminant $\Delta \equiv a_1^2 - 4a_0$. If the discriminant of the quadratic equation is strictly negative, that is if

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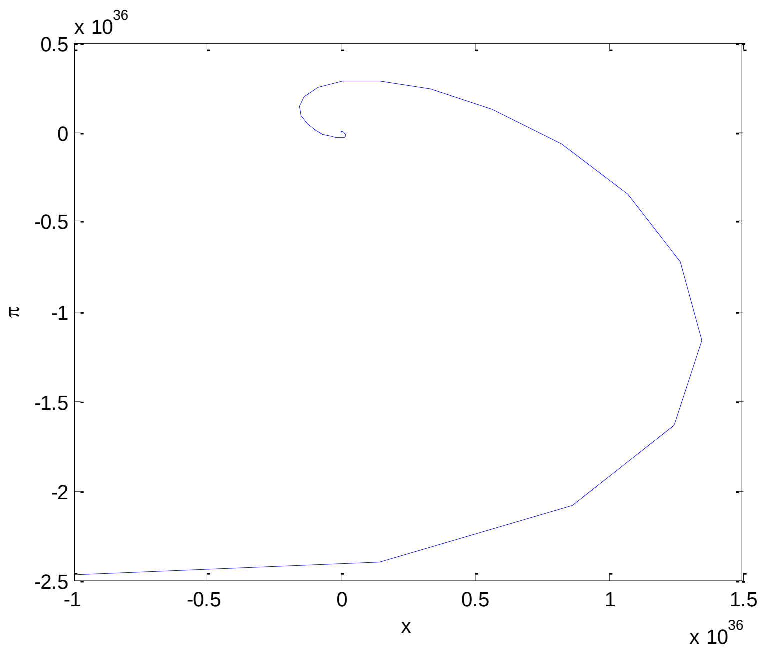
[9] Given the parameter values, the equilibrium is a stable spiral. Since the complex eigenvalues are greater than one in modulus (R > 1), the amplitude of the sinusoids decay at the rate R^j as $j \to \infty$. Then, dynamic multiplier follows a pattern of decreasing oscillation whose frequency is defined by $\cos(\theta) = \frac{a}{R} = \frac{1.1768}{1.2372} = 0.9512$ or $\theta = \cos^{-1}(a/R) = \cos^{-1}(1.1768/1.2372) = 17.9771$ in degrees. Hence, the cycles associated with the dynamic multiplier function have a period of $\frac{2\pi}{\theta} = \frac{2*3.14159}{17.9771} = 0.3495$, which means the peaks in the pattern appear about 0.35 periods apart.

$$\Delta \equiv a_1^2 - 4a_0 = \left(\frac{\beta\phi_x + \kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right)^2 - 4\left(\frac{\varphi_x + \kappa_\alpha\phi_\pi + \sigma_\alpha}{\beta\sigma_\alpha}\right) < 0$$

where $\kappa_\alpha = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)$, and $\sigma_\alpha = \frac{\sigma}{1+\alpha(\omega-1)}$, then the roots are complex conjugate numbers in the form $\lambda_{1,2} = a \pm ib$, with $a,b \in \mathbb{R}$, $b \neq 0$ as the real part, while $i = +\sqrt{-1}$ is the imaginary unit. Figure (3.1.5) shows a phase diagram of the model (3.1.2).

Figure 3.1.5: Phase diagram for Model (3.1.2) with $\phi_\pi = 1.5$



Although it is algebraically quite cumbursome to determine the sign of the modulus of the eigenvalues, using the baseline values of the parameters, we can verify that the inside of the square roots in both eigenvalues is negative. Hence the discriminant Δ is strictly negative. Therefore, the eigenvalues of the system (3.1.18) are complex conjugate, $\lambda_{1,2} = a \pm ib$, where

$$a = \frac{a_1}{2} = \frac{1}{2}\left(\frac{\kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right) = \left(\frac{\kappa_\alpha + \sigma_\alpha}{2\beta\sigma_\alpha} + \frac{1}{2}\right) \tag{3.1.20}$$

and

$$b = \frac{\sqrt{-\Delta}}{2} = \frac{1}{2}\sqrt{4\left(\frac{\kappa_\alpha \phi_\pi + \sigma_\alpha}{\beta\sigma_\alpha}\right) - \left(\frac{\kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1\right)^2} \tag{3.1.21}$$

with $\kappa_\alpha = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)$, and $\sigma_\alpha = \frac{\sigma}{1+\alpha(\omega-1)}$.

With the presence of complex conjugate eigenvalues, we may expect to see a Hopf bifurcation if certain conditions are satisfied. The only possible source for a bifurcation is the policy parameter ϕ_π. Using Theorem 1, the conditions for the existence of a Hopf bifurcation is stated in the following Proposition.

Proposition 3.1.5: The system (3.1.18) undergoes a Hopf bifurcation if and only if $\Delta < 0$ and

$$\phi_\pi^* = \frac{(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}. \tag{3.1.22}$$

Proof: Suppose the system (3.1.18) goes through a Hopf bifurcation at (y^*, ϕ_π^*), where $y^* = (x^*, \pi^*)$. Then, we need to show that $\Delta < 0$ and $\phi_\pi^* = \frac{(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}$. The existence of a Hopf bifurcation, by definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative (i.e. $\Delta < 0$).

For the second part, note that the existence of a Hopf bifurcation requires $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = +\sqrt{a^2 + b^2} = 1$ by the first condition of Theorem 1. Rewriting the condition explicitly by substituting (3.1.20) and (3.1.21) into it, taking the square of both sides, and then solving for ϕ_π, we obtain the critical value of the parameter as in (3.1.22). Therefore, the first condition of Theorem 1 holds only if $\phi_\pi = \frac{(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}$.

Conversely, suppose $\Delta < 0$ and $\phi_\pi = \frac{(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}$. Substituting for ϕ_π^* into $\sqrt{a^2+b^2}$ yields $\operatorname{mod}(\lambda_1) = \operatorname{mod}(\lambda_2) = 1$, which is the first condition in Theorem 1. In order to show that the critical value of the parameter ϕ_π is actually a Hopf bifurcation parameter, we check the second condition of Theorem 1, which gives

$$\left.\frac{d\left|\lambda_i(\phi_\pi)\right|}{d\phi_\pi}\right|_{\phi_\pi=\phi_\pi^*} = \left.\frac{d}{d\phi_\pi}\left(\sqrt{a^2+b^2}\right)\right|_{\phi_\pi=\phi_\pi^*} = \frac{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}{2\beta\sigma} \neq 0 \text{ for } i = 1, 2.$$

Thus, both conditions of Theorem 1 are satisfied and we have $\phi_\pi^* = \frac{(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}$. □

Proposition (3.1.5) shows formally that taking the parameter ϕ_π free to vary and keeping the other parameters constant, the system (3.1.18) is likely to undergo a Hopf bifurcation at ϕ_π^*. Numerical analysis also indicates a Period Doubling bifurcation at ϕ_π = -12.59 and a Limit Point (Fold) bifurcation (which is also a branching point) at $\phi_\pi = 1$. Increasing the value of ω reduces the bifurcation parameter ϕ_π in absolute value, except when $\alpha = 0$ at which changes in ω does not make any difference. On the other hand, decreasing the value of α causes a lower value of the bifurcation parameter ϕ_π in absolute value, except when $\omega = 1$ at which changes in α does not make any difference.

3.1.3 With Credibility Gap under the Current Looking Inflation Targeting:

We now modify the policy rule to evaluate the effects of a credibility gap which shows to what extend agents discount the central bank’s decisions as described in Galí (2008). As before, we assume that the economy is described by equations (3.1.1) and (3.1.2) while the central bank follows the pure current looking inflation targeting rule (3.1.23).

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \tag{3.1.1}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(r_t - E_t\pi_{t+1} - \overline{r}_t\right) \tag{3.1.2}$$

$$r_t = \overline{r}_t + \phi_\pi \pi_t \tag{3.1.23}$$

Suppose that the public, on the other hand, believe that the monetary policy rule is given by

$$r_t = \overline{r}_t + \phi_\pi (1-\delta)\pi_t \tag{3.1.24}$$

where $\phi_\pi > 1$ and $\delta \in \mathbb{R}$ measures the credibility gap. Hence, we consider the system consisting of the equations (3.1.1), (3.1.2) and (3.1.24).

We first substitute (3.1.24) for $r_t - \overline{r}_t$ into equation (3.1.2). Then, we obtain the reduced form of the model in terms of inflation and output gap as follows:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(\phi_\pi (1-\delta)\pi_t - E_t\pi_{t+1}\right)$$

Then, rearranging the terms, the system can be written in the form $E_t y_{t+1} = C y_t$ as follows

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \frac{1}{\beta}\begin{bmatrix} \left(\mu + \beta + \frac{1+\alpha(\omega-1)}{\sigma}\mu\varphi\right) & -\left(\beta\phi_\pi(1-\delta)+1\right)\frac{1+\alpha(\omega-1)}{\sigma} \\ -\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) & 1 \end{bmatrix}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix}. \tag{3.1.25}$$

The eigenvalues of the Jacobian matrix *C*, i.e. λ_1 and λ_2, are computed by setting $\det(C - \lambda I) = 0$ which gives a second-order characteristic polynomial,

$$p(\lambda) = \lambda^2 - a_1\lambda + a_0 = 0$$

where

$$a_1 = \frac{\varphi\mu(1+\alpha(\omega-1))}{\beta\sigma} + \frac{1+\beta+\mu}{\beta} \text{ and } a_0 = \frac{\varphi\mu(1+\alpha(\omega-1))\phi_\pi(1-\delta)}{\beta\sigma} + \frac{\mu\phi_\pi(1-\delta)+1}{\beta}.$$

The solution to the characteristic polynomial is given by the following eigenvalues

$$\lambda_{1,2} = \left(\frac{\kappa_\alpha+\sigma_\alpha}{\beta\sigma_\alpha}+1\right) \pm \sqrt{\left(\frac{\kappa_\alpha+\sigma_\alpha}{\beta\sigma_\alpha}+1\right)^2 - 4\left(\frac{\kappa_\alpha\phi_\pi(1-\delta)+\sigma_\alpha}{\beta\sigma_\alpha}\right)}$$

with $\kappa_\alpha = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)$, and $\sigma_\alpha = \frac{\sigma}{1+\alpha(\omega-1)}$.

The proposition below characterizes the necessary and sufficient conditions for the determinacy of the system (3.1.25), following Bullard and Mitra (2002).

Proposition 3.1.6: Given the monetary policy based on current looking inflation targeting with credibility gap, the open economy New Keynesian model (3.1.25) has a unique stationary equilibrium if and only if

$$1 < \phi_\pi(1-\delta) < \frac{\sigma(\beta-1)}{\mu\sigma+\mu\varphi(1+\alpha(\omega-1))}. \tag{3.1.26}$$

Proof: Determinacy requires that both eigenvalues are outside the unit circle which happens if and only if

$|a_0| < 1$ and

$|a_1| < 1 + |a_0|$.

From the first condition we obtain the upper boundary,

$$\frac{\sigma(\beta-1)}{\mu\sigma+\mu\varphi(1+\alpha(\omega-1))} > \phi_\pi(1-\delta).$$

From the second condition we obtain the lower boundary

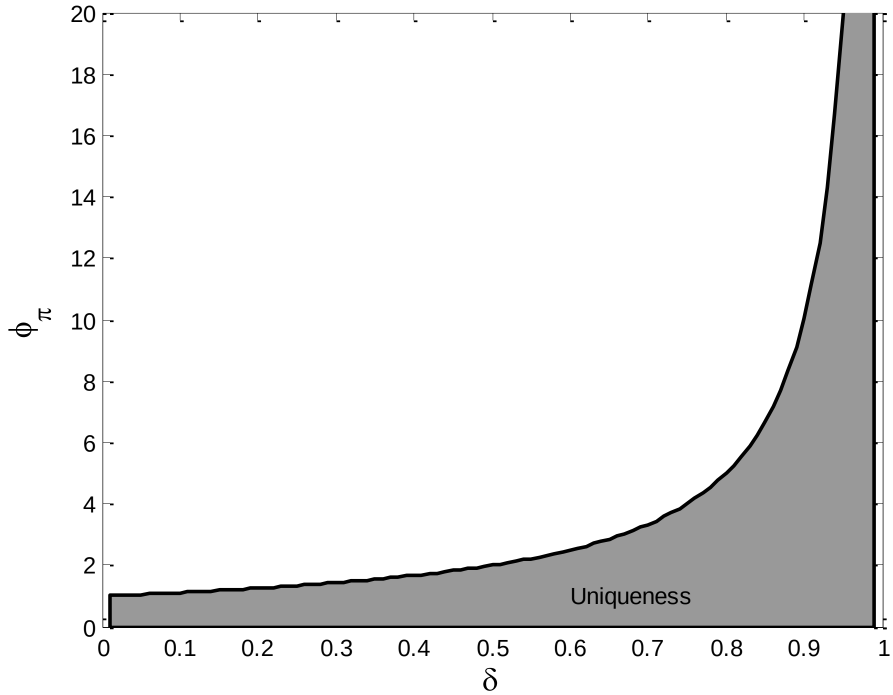
$\phi_\pi (1-\delta) > 1$.

Combining them, we have the determinacy condition (3.1.26). Thus, following Bullard and Mitra (2002), and assuming $\phi_\pi > 0$ and $\delta \in (0,1)$, both eigenvalues will be outside the unit circle if and only if

$$1 < \phi_\pi (1-\delta) < \frac{\sigma(\beta-1)}{\mu\sigma + \mu\varphi(1+\alpha(\omega-1))}. \qquad \square$$

Note that, since $\phi_\pi > 0$ by assumption, the lower boundary of the determinacy condition (3.1.26) equivalently holds if $\phi_\pi > \frac{1}{1-\delta}$. This resembles the Taylor rule, with the exception of the credibility gap. As the credibility gap rises the determinacy region shrinks which gives rise to a smaller room for the existence of a unique solution. When $\delta = 0$, it collapses to the model (3.1.2). Therefore, in case of credibility gap, the monetary policy authority has to compensate the lack of credibility by pursuing a more aggressive policy which involves raising the nominal interest rate much more than the deviation of the inflation rate from its target level and this results in a larger increase in the real interest rate. Figure (3.1.6) illustrates the determinacy region as a function of the credibility parameter δ. As we can see clearly on the figure, credibility gap dramatically narrows the region of unique equilibrium.

Figure 3.1.6: Determinacy diagram under current looking inflation targeting with credibility gap



At the point where $\left(\phi_\pi(1-\delta)-1\right)\left(\frac{\mu\sigma}{1+\alpha(\omega-1)}+\mu\varphi\right)=0$, the system (3.1.25) has a branching point. Using the baseline values of the parameters given by Gali and Monacelli (2005), and by Gali (2008) and solving the equation for ϕ_π we obtain $\phi_\pi = 2$. Thus, the branching point occurs at $\phi_\pi = 2$. This implies that in the open economy framework, the occurrence of a branching point requires that the monetary policy instrument responds to changes in inflation rate, π_t, twice as much in the presence of a credibility gap.

As shown in Gandolfo (1996), in order to examine the nature of the eigenvalues we need to check the sign of the discriminant $\Delta \equiv a_1^2 - 4a_0$. If the discriminant of the characteristic equation is strictly negative, that is if

$$\Delta \equiv a_1^2 - 4a_0$$
$$= \left(\frac{\kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1 \right)^2 - 4\left(\frac{\kappa_\alpha \phi_\pi (1-\delta) + \sigma_\alpha}{\beta\sigma_\alpha} \right) < 0,$$

then the roots of the 2x2 matrix C will be complex conjugate numbers in the form $\lambda_{1,2} = a \pm ib$, $a, b \in \mathbb{R}$, $b \neq 0$ as the real part, while $i = +\sqrt{-1}$ is the imaginary unit.

It is algebraically quite cumbursome to identify the sign of the modulus of the eigenvalues. So, we assume that the eigenvalues of the system (3.1.25) are complex conjugate, $\lambda_{1,2} = a \pm ib$, where

$$a = \frac{a_1}{2} = \frac{1}{2}\left(\frac{\kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1 \right) = \left(\frac{\kappa_\alpha + \sigma_\alpha}{2\beta\sigma_\alpha} + \frac{1}{2} \right), \tag{3.1.27}$$

and

$$b = \frac{\sqrt{-\Delta}}{2} = \frac{1}{2}\sqrt{4\left(\frac{\kappa_\alpha \phi_\pi (1-\delta) + \sigma_\alpha}{\beta\sigma_\alpha} \right) - \left(\frac{\kappa_\alpha + \sigma_\alpha}{\beta\sigma_\alpha} + 1 \right)^2}, \tag{3.1.28}$$

with $\kappa_\alpha = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right)$, and $\sigma_\alpha = \frac{\sigma}{1+\alpha(\omega-1)}$.

With the presence of a pair of complex conjugate eigenvalues, we may expect to see a Hopf bifurcation if certain conditions are satisfied. The only possible source for a bifurcation is the monetary policy parameter ϕ_π. Applying Theorem 1 with respect to the parameter ϕ_π, the conditions for the existence of a Hopf bifurcation is derived in the following Proposition.

Proposition 3.1.7: The system (3.1.25) undergoes a Hopf bifurcation if and only if $\Delta < 0$ and

$$\phi_\pi^* = \frac{(1-\delta)^{-1}(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}. \tag{3.1.29}$$

Proof: Suppose the system (3.1.25) goes through a Hopf bifurcation at $\left(y^*, \phi_\pi^*\right)$, where

$y^* = \left(x^*, \pi^*\right)$. Then we need to show that $\Delta < 0$ and $\phi_\pi^* = \dfrac{(1-\delta)^{-1}(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}$. The existence of a Hopf bifurcation, by definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative, that is $\Delta < 0$.

For the second part, notice that the existence of a Hopf bifurcation requires $\operatorname{mod}(\lambda_1) = \operatorname{mod}(\lambda_2) = +\sqrt{a^2+b^2} = 1$ from the first condition of Theorem 1. Rewriting the condition explicitly by substituting (3.1.27) and (3.1.28) into it, taking the square of both sides, and then solving for ϕ_π, we obtain the critical value of the parameter ϕ_π as in (3.1.22). Therefore, the first condition of Theorem 1 holds only if $\phi_\pi = \dfrac{\sigma(\beta-1)}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}\dfrac{1}{1-\delta}$.

Conversely, suppose $\Delta < 0$ and $\phi_\pi = \dfrac{\sigma(\beta-1)}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}\dfrac{1}{1-\delta}$. Substituting for ϕ_π^* into $\sqrt{a^2+b^2}$ yields $\operatorname{mod}(\lambda_1) = \operatorname{mod}(\lambda_2) = 1$, which is the first condition in Theorem 1. In order to show that the critical value of the parameter ϕ_π is actually a Hopf bifurcation parameter, we check the second condition of Theorem 1, which yields

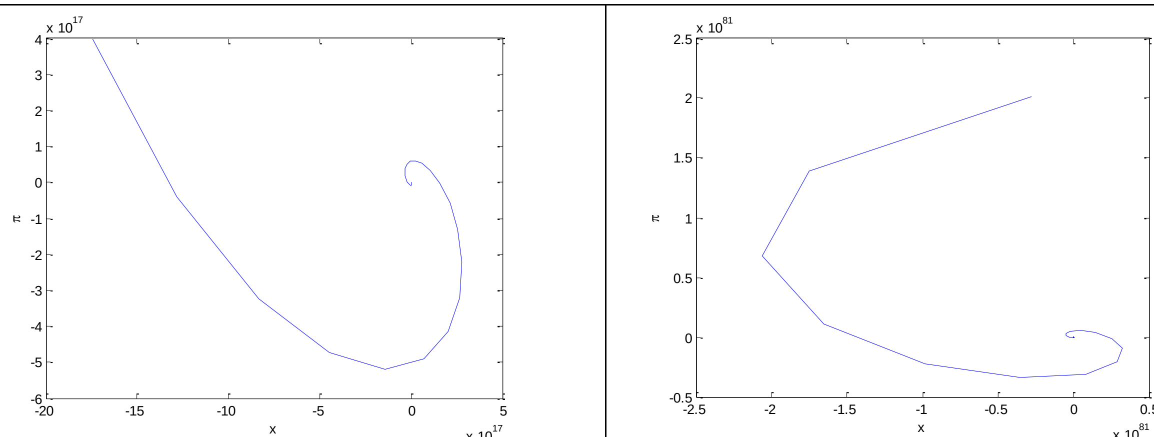
$$\left.\frac{d\left|\lambda_i(\phi_\pi)\right|}{d\phi_\pi}\right|_{\phi_\pi=\phi_\pi^*} = \left.\frac{d}{d\phi_\pi}\left(\sqrt{a^2+b^2}\right)\right|_{\phi_\pi=\phi_\pi^*} = \frac{\mu\sigma + \varphi\mu\left(1+\alpha(\omega-1)\right)}{2\beta\sigma}(1-\delta) \neq 0 \text{ for } i = 1,2.$$

Thus, both conditions of Theorem 1 are satisfied and we have $\phi_\pi^* = \dfrac{(1-\delta)^{-1}(\beta-1)\sigma}{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}$. □

Proposition (3.1.7) shows formally that taking the parameter ϕ_π free to vary and keeping the other parameters constant at their baseline values, the system (3.1.25) is likely to undergo a Hopf bifurcation at ϕ_π^*.

Futhermore, we numerically determine a Period Doubling bifurcation at $\phi_\pi = -25.4$, which is not a feasible value and is far from the bifurcation point occured in the case without a credibility gap. We also numerically find a Limit Point bifurcation and a branching point at $\phi_\pi = 2$. The equilibrium is a stable node for $2 < \phi_\pi$ with both eigenvalues outside the unit circle. The equilibrium point is a saddle point for $\phi_\pi < 2$. The saddle point is unstable as there are trajectories diverging from $\left(\pi^*, x^*\right) = (0,0)$ even though they begin arbitrarily close to $\left(\pi^*, x^*\right) = (0,0)$. Figure (3.1.7) illustrates two phase plots constructed for different values of the parameter ϕ_π.

Figure 3.1.7: Phase diagram for various values of ϕ_π for Model (3.1.3)



$\phi_\pi = 3$ $\phi_\pi = 5$

The Period Doubling bifurcation dissappears for lower values of ω. On the other hand, decreasing the value of α reduces the bifurcation parameter ϕ_π in absolute value, except when $\omega = 1$ at which changes in α does not make any difference.

3.1.4 Under Current Looking Taylor Rule with Interest Rate Smoothing:

It has been empirically shown that the lagged interest rate usually receives a statistically significant coefficient estimate when interest rate is regressed on inflation and output gap. Some authors claim that it reflects the inertial behavior while others argue that it is due to the gradual adjustment policy conducted by the monetary authority. Parameter uncertainty, imperfect information, pursuing financial stability are considered as some of the motivations which leads the policy maker to pursue such a precautinary policy. See, for example, Sack (2000), Rudebusch (2005) and Walsh (2003) for further discussion of the subject.

Consider the following model in which the first two equations describe the economy while the third equation is the interest rate rule followed by the central bank in order to conduct the monetary policy:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \tag{3.1.1}$$

$$x_t = E_t x_{t+1} - \frac{1-\alpha+\alpha\omega}{\sigma}\left(r_t - E_t\pi_{t+1} - \overline{r}_t\right) \tag{3.1.2}$$

$$r_t = \overline{r}_t + \phi_\pi \pi_t + \phi_x x_t + \phi_r r_{t-1} \tag{3.1.30}$$

where ϕ_r is the degree of interest rate smoothing, ϕ_π and ϕ_x are the central bank's relative policy weights assigned to the inflation rate and the output gap, respectively. Equation (3.1.30) states that the nominal interest rate is assigned according to the current values of the inflation rate and the output gap as well as the policy rate in the previous period.

Woodford (2003a) finds the interest rate inertia coefficient ϕ_r equal to 0.46 which implies that interest rates should be adjusted roughly half of the way toward the target level within a quarter.

Moving the equation (3.1.30) one period forward and adding expectations, then rearranging the terms and defining $y_t = \left[x_t, \pi_t, r_t\right]'$, we can write the system of equations in the form $E_t y_{t+1} = Cy_t + D$:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ E_t r_{t+1} \end{bmatrix} = C \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix} + \begin{bmatrix} -\frac{1-\alpha+\alpha\omega}{\sigma}\overline{r}_t \\ 0 \\ E_t\overline{r}_{t+1} - \phi_x \overline{r}_t \frac{1-\alpha+\alpha\omega}{\sigma} \end{bmatrix} \tag{3.1.31}$$

where

$$C = \begin{bmatrix} \frac{\mu}{\beta}\left(1+\varphi\frac{1-\alpha+\alpha\omega}{\sigma}\right)+1 & -\frac{1-\alpha+\alpha\omega}{\beta\sigma} & \frac{1-\alpha+\alpha\omega}{\sigma} \\ -\frac{\mu}{\beta}\left(\frac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right) & \frac{1}{\beta} & 0 \\ \phi_x + \frac{\mu}{\beta}\left(1+\varphi\frac{1-\alpha+\alpha\omega}{\sigma}\right)\left(\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}-\phi_\pi\right) & -\frac{1}{\beta}\left(\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}-\phi_\pi\right) & \phi_r + \phi_x\frac{1-\alpha+\alpha\omega}{\sigma} \end{bmatrix}.$$

It is the coefficient matrix *C* that is relevant for determinacy and bifurcation analysis. Its characteristic polynomial is given by

$$\begin{aligned} p(\lambda) &= \det\left(C-\lambda I_3\right) \\ &= \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0 = 0 \end{aligned}$$

where

$$a_2 = -\left(\phi_r + \frac{1+\alpha\left(\omega-1\right)}{\sigma}\left(\phi_x + \frac{\varphi\mu}{\beta}\right) + \frac{1}{\beta}\left(1+\mu\right)+1\right), \tag{3.1.32}$$

$$a_1 = -\left(\phi_r\left(\frac{1}{\beta}+\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+1\right)+\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}+\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+\frac{1}{\beta}\right), \tag{3.1.33}$$

and $a_0 = -\frac{\phi_r}{\beta}$. (3.1.34)

Note that there are three endogeneous variables; the rate of inflation, π_t, the output gap, x_t, and the nominal interest rate, r_t. Then, following Blanchard and Kahn (1980), the system (3.1.31) has a unique, stationary equilibrium solution if and only if the number of eigenvalues of the 3x3 coefficient matrix *C* outside the unit circle is equal to the number of forward looking (non-predetermined) variables which is three ($E_t x_{t+1}$, $E_t \pi_{t+1}$ and $E_t r_{t+1}$). Consequently, we should have all the eigenvalues to be outside the unit circle for uniqueness.

Proposition 3.1.8: Given the monetary policy based on current looking Taylor rule with interest rate smoothing, the open economy New Keynesian model (3.1.31) leads to indeterminacy.

Proof: Following the methodology employed by Bullard and Mitra (2001), the number of the roots with negative and/or positive signs can be determined by Descartes' rule of signs theorem as follows. Looking at the polynomial in the positive case without changing the signs of λ, we count the number of sign changes, which is one (from positive to negative after the first term) in this case. Hence the maximum number of positive roots for the characteristic polynomial is one:

$$p(\lambda) = +\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$$

In the negative case by having changed the sign on λ, we look at $p(-\lambda)$:

$$p(-\lambda) = -\lambda^3 - a_2\lambda^2 + a_1\lambda - a_0$$

There are two sign changes in this "negative" case, so there are at most two negative roots. Therefore, by Descartes' rule of signs, the characteristic polynomial has two negative roots as well as having one positive root.

We can also evaluate $p(x)$ at $x=1$ for the roots with positive sign and at $x=-1$ for the roots with negative sign to see whether the negative and positive roots lie outside (or inside) the unit circle. Note that for $\lambda=0$, we have $p(0)=-\frac{\phi_r}{\beta}<0$.

For $\lambda=-1$, $p(-1)<0$ yields

$$\phi_x(1-\beta)+\left(\frac{\sigma\mu}{1-\alpha+\alpha\omega}+\varphi\mu\right)(\phi_r+\phi_\pi-1)<\frac{2\beta\sigma}{1-\alpha+\alpha\omega}$$

for the given values of the parameters in Gali and Monacelli (2005) and for $\phi_r=0.5$.

For $\lambda=1$,

$$p(1)=1-\left(\phi_r+\frac{1+\alpha(\omega-1)}{\sigma}\left(\phi_x+\frac{\varphi\mu}{\beta}\right)+\frac{1}{\beta}(1+\mu)+1\right)$$
$$-\left(\phi_r\left(\frac{1}{\beta}+\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+1\right)+\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}+\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+\frac{1}{\beta}\right)-\frac{\phi_r}{\beta}<0$$

for the given values of the parameters in Gali and Monacelli (2005). □

We can numerically verify whether Proposition (3.1.8) holds for the given values of the parameters in Gali and Monacelli (2005) whose Jacobian matrix is

$$C=\begin{bmatrix}1.3434 & -1.0101 & 1\\ -0.3434 & 1.0101 & 0\\ -0.3472 & 1.3889 & 0.6250\end{bmatrix}$$

with eigenvalues $\lambda_1=1.3743+0.5546\mathrm{i}$, $\lambda_2=1.3743-0.5546\mathrm{i}$, $\lambda_3=0.23$. Note that one solution is real, positive and inside the unique circle while the other two solutions are complex conjugate with radius greater than one. Since the number of eigenvalues outside the unit circle (which is two) is less than the number of forward looking variables (which is three), there is no unique solution to the system. The indeterminacy result suggests that there are other stationary equilibrium solutions to the model (3.1.31) under the current looking Taylor rule with interest rate smoothing. Figure (3.1.8) shows a phase plot of the system (3.1.31).

Having a pair of complex conjugate eigenvalues and a real-valued eigenvalue outside the unit circle, the following Proposition states the conditions for the system (3.1.31) to undergo a Hopf bifurcation:

Proposition 3.1.9: The system (3.1.31) undergoes a Hopf bifurcation if and only if the following transversality condition

$\left.\frac{\partial\left|\lambda_i(\phi)\right|}{\partial\phi_j}\right|_{\phi_j=\phi_j^*} \neq 0$ holds and

i) $\phi_r - \beta < 0$, (3.1.35)

ii) $\phi_r\left(\frac{\sigma(2+\mu+2\beta)}{1-\alpha+\alpha\omega}+\varphi\mu\right)+\phi_x(\beta+1)+\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(\phi_\pi+1)+\frac{2\sigma}{1-\alpha+\alpha\omega}<0$, (3.1.36)

iii) $\phi_r^2\xi_4+\phi_r\xi_3+(\phi_x\phi_r+\phi_x)\xi_2+\phi_\pi\xi_1+\xi_0=-1$. (3.1.37)

Proof: For the necessary part of the proof, we use Theorem 2. From the condition (*i*) in Theorem 2, we have $|a_0|=\left|-\frac{\phi_r}{\beta}\right|=\left|\frac{\phi_r}{\beta}\right|<1$, which implies $\frac{\phi_r}{\beta}<1$.

From the condition (*ii*) in Theorem 2, we have $|a_0+a_2|<1+a_1$, which implies

$$\phi_r\left(\frac{\sigma(2+\mu+2\beta)}{1-\alpha+\alpha\omega}\right)+\phi_x(\beta+1)+\phi_\pi\left(\frac{\sigma\mu}{1+\alpha(\omega-1)}+\varphi\mu\right)+\frac{\sigma(2+\mu)}{1-\alpha+\alpha\omega}+\varphi\mu<0.$$

But this condition requires negative values for the parameters thus, it cannot be satisfied for the feasible set of values of the parameters.

Finally, from the condition (*iii*) in Theorem 2, we obtain

$a_1-a_0a_2=1-a_0^2$,

which implies

$$\phi_r^2\xi_4+\phi_r\xi_3+\left(\phi_x\phi_r+\phi_x\right)\xi_2+\phi_\pi\xi_1+\xi_0=-1$$

where

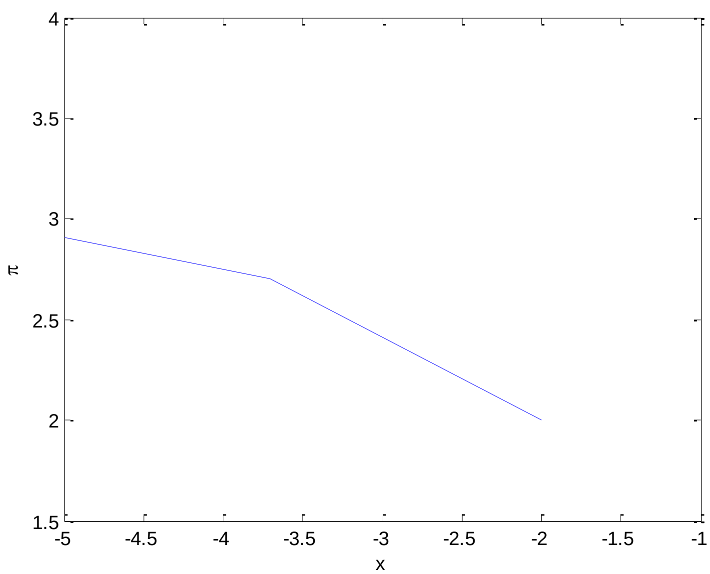
$$\xi_0=\frac{1}{\beta},\ \xi_1=\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma},\ \xi_2=\frac{1-\alpha+\alpha\omega}{\beta\sigma},\ \xi_3=\left(1+\frac{1}{\beta}\right)\left(\frac{1}{\beta}+\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right),$$

and $\xi_4=\frac{1}{\beta}\left(1+\frac{1}{\beta}\right)$.

For the sufficient part, assume the conditions (*i*)-(*iii*) hold true. Then, we need to show that the transversality condition holds which can easily be verified numerically. □

Since the condition (3.1.36) in Proposition (3.1.9) does not hold, it is not theoretically possible for a Hopf bifurcation to occur in the Gali and Monacelli Model under the current looking Taylor rule with interest rate smoothing setting.

Figure 3.1.8: A phase diagram of Model (3.1.4) given the baseline values



Furthermore, we analyze the system (3.1.31) for the existence of a Period Doubling bifurcation. Keeping the structural parameters and policy parameters ϕ_π , ϕ_r constant at their baseline values while varying the policy parameter ϕ_x over a certain range, we numerically find Period Doubling bifurcation at $\phi_x = 0.83$. Within the closed economy framework under the same policy rule, Barnett and Duzhak (2010) report a Period Doubling bifurcation close to $\phi_x = 2$. Comparing the results from Barnett and Duzhak (2010) with the open economy case, however, does not give us a clear statement about whether or not the openness makes the New Keynesian model more sensitive to bifurcations since Gali and Monacelli Model incorporates a wider set of parameters including some deep parameters relevant to the open economy. Furthermore, both studies use different set of benchmark values for the parameters. Hence, a direct comparison becomes even harder.

Airaudo and Zanna (2005), using a money-in-utility function and assuming non-separability, show that cyclical and chaotic dynamics are more likely the more open the economy and the higher the exchange rate pass-through into import prices. They also show that the existence of cyclical and chaotic dynamics depend on some open economy features and is in general robust to different timings in the policy rule. Analyzing the effects of a change in the parameters α and ω in the Gali and Monacelli Model, on the other hand, does not seem to be indicative for more sensitivity in open economy environment. Lowering ω and raising α increase the value of the bifurcation parameter. There is no bifurcation in any type at $(\omega, \alpha) = (0, 1)$.

Figure 3.1.9: Period Doubling bifurcation boundary at $\phi_x = 0.827$ for Model (3.1.4)

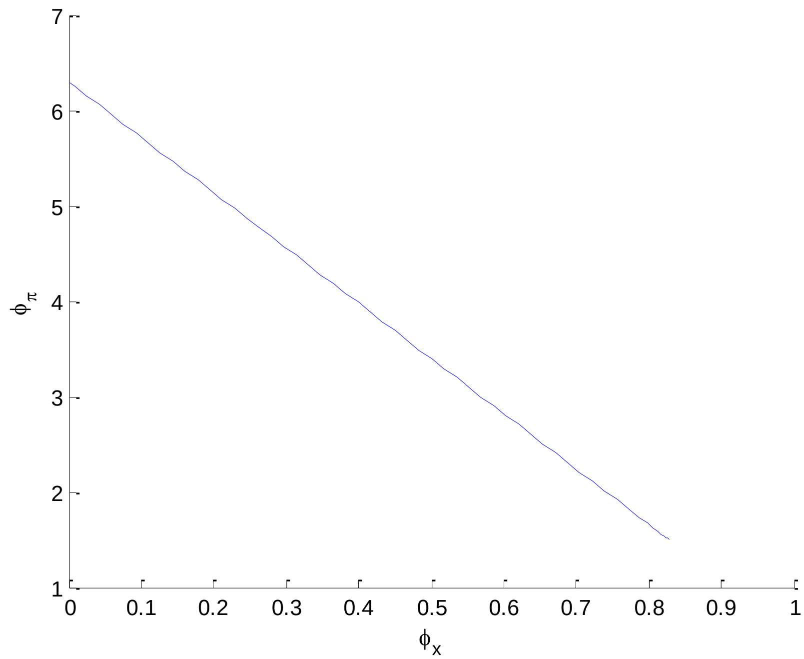


Figure (3.1.9) illustrates the Period Doubling bifurcation boundary for the parameter ϕ_x. Note that along the bifurcation boundary which is the set of bifurcation points of the same type, the values of the bifurcation parameter ϕ_x lie between 0 and 0.83. As the magnitude of the reaction of central bank to inflation, that is ϕ_π, increases, even the smaller values of parameter ϕ_x would be sufficient to cause a Period Doubling bifurcation. Crossing the boundary causes the bifurcation.

When we consider ϕ_π as the bifurcation parameter, we numerically find a Period Doubling bifurcation at $\phi_\pi = 5.57$ and a branching point at $\phi_\pi = -1.5$. For ϕ_π being the bifurcation parameter, lowering ω and raising α increase the value of the bifurcation parameter. Furthermore, there is no bifurcation in any type at $(\omega, \alpha) = (0, 1)$.

Figure 3.1.10: Period Doubling bifurcation boundary at $\phi_\pi = 5.57$ for Model (3.1.4)

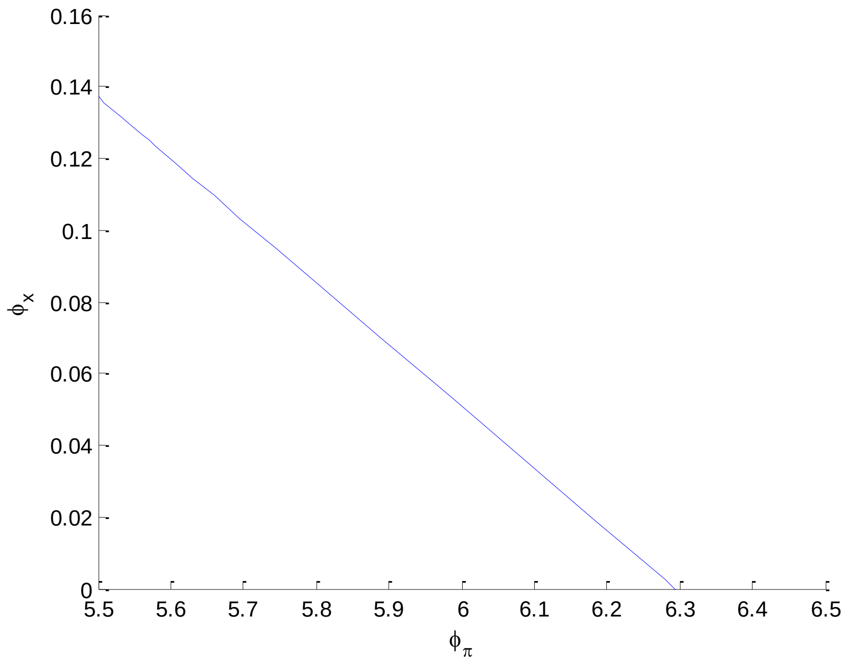


Figure (3.1.10) illustrates the bifurcation boundary for the parameter ϕ_π. In this case, along the bifurcation boundary, the values of bifurcation parameter ϕ_π lie between 5.5 and 6.3, which is a relatively small interval for the bifurcation to emerge. As the magnitude of the reaction of central bank to output gap, that is ϕ_x, increases, lower values of the parameter ϕ_π would be sufficient to cause a Period Doubling bifurcation.

3.2 Gali and Monacelli Model with Forward Looking Interest Rate Rule:

3.2.1 Under Forward Looking Taylor Rule:

Rational expectations based policy rules have been studied by many economists such as Evans and Honkapohja (2003a, b) and Branch and McGough (2009, 2010). Evans and Honkapohja (2003a) argue that expectations based rules can give rise to determinate and stable equilibrium under learning. McCallum (1999) criticizes the contemporaneous policy rules as being unrealistic in practice since the policy makers do not have complete information about the current situation. As Bullard and Mitra (2002) point out, together with backward looking policy rules, forward looking rules have been considered as an alternative to account for McCallum's (1999) critics. Batini and Haldane (1999) argue that inflation forecast based rules perform well in comparison with other simple rules. Forward looking approach lets the policy makers comprise the time lag between performing a certain policy and receiving its impacts on economy while. evaluating the future conditions of the economy in a more realistic setting based on the available information set.

Consider the following model in which the first two equations describe the economy while the third equation represents the monetary policy rule followed by the central bank:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t \tag{3.2.1}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma} \left(r_t - E_t \pi_{t+1} - \overline{r}_t \right) \tag{3.2.2}$$

$$r_t = \overline{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x E_t x_{t+1} \tag{3.2.3}$$

where x_t denotes the output gap, π_t is the inflation rate, and r_t is the nominal interest rate. E_t is the expectation operator. The policy parameters ϕ_π and ϕ_x represent the magnitude of the central bank's responses to the next period's expected inflation rate and expected output gap, respectively. As before, there is no exogenous shock.

Note that the policy rule (3.2.3) nests the standard Taylor rule as a special case. In this specification, the actual infation and output gap are replaced by the expected inflation and the expected output gap. The policy maker, however, looks only one quarter ahead while adjusting the nominal interest rate. Clarida, Gali and Gertler (2000) employ this approach in order to estimate a reaction function of the Federal Reserve Bank for the postwar US economy.

Substituting (3.2.3) into (3.2.2) for $r_t - \overline{r}_t$, we have a system of first order stochastic difference equations in terms of expected inflation and expected output gap, as follows:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t,$$

$$x_t = \left(1 - \frac{1+\alpha(\omega-1)}{\sigma}\phi_x\right)E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}(\phi_\pi - 1)E_t \pi_{t+1}.$$

As in the previous cases, $x_t = \pi_t = 0$ for all t is an equilibrium solution to the reduced system. Rearranging the terms, we have the reduced system in normal form $E_t y_{t+1} = C y_t$:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \dfrac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)\right)(\phi_\pi - 1)}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)} & \dfrac{(\phi_\pi - 1)\left(1+\alpha(\omega-1)\right)}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)} \\ -\dfrac{\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)}{\beta + \alpha\beta(\omega-1)} & \dfrac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}. \qquad (3.2.4)$$

Then, the characteristic polynomial of the Jacobian matrix is

$$p(\lambda) = \det(C - \lambda I) = \lambda^2 - a_1\lambda + a_0 = 0,$$

where

$$a_1 = \frac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)\right)(\phi_\pi - 1)}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)} + \frac{1}{\beta} \text{ and } a_0 = \frac{\sigma}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)},$$

which yields

$$\lambda_{1,2} = \left(\frac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)\right)(\phi_\pi - 1)}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)} + \frac{1}{\beta}\right)$$
$$\pm\sqrt{\left(-\frac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha(\omega-1)\right)\right)(\phi_\pi - 1)}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)} - \frac{1}{\beta}\right)^2 - 4\left(\frac{\sigma}{\beta\sigma - \beta\phi_x\left(1+\alpha(\omega-1)\right)}\right)}.$$

As before, we begin our analysis by examining the determinacy conditions of the system (3.2.4). Note that there exist two free endogenous variables, x_t and π_t. Then, following Blanchard and Kahn (1980), the equilibrium solution would be unique if and only if both eigenvalues are outside the unit circle. The following Proposition characterizes the necessary and sufficient conditions for the determinacy of the system (3.2.4).

Proposition 3.2.1: Under the monetary policy based on forward looking interest rate rule, the open economy New Keynesian model (3.2.4) has a unique stationary equilibrium if and only if

$$\phi_x < \frac{\sigma\left(1-\beta^{-1}\right)}{1+\alpha\left(\omega-1\right)}, \tag{3.2.5}$$

$$\left(1-\beta\right)\phi_x + \mu\left(\frac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right)\left(\phi_\pi-1\right) > 0, \tag{3.2.6}$$

and

$$\left(1+\beta\right)\phi_x + \mu\left(\frac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right)\left(\phi_\pi-1\right) < \frac{2\sigma\left(1+\beta\right)}{1+\alpha\left(\omega-1\right)}. \tag{3.2.7}$$

Proof: Following J.P. Lasalle (1986, p.28) and Bullard and Mitra (2002), both eigenvalues of the Jacobian matrix are outside the unit circle if and only if the following conditions are satisfied:

$|a_0| < 1$ and

$|a_1| < 1 + a_0$.

The first inequality implies the condition (3.2.5). The second inequality, on the other hand, leads to the conditions (3.2.6) and (3.2.7). □

The conditions (3.2.6) and (3.2.7) provide lower and upper boundaries, respectively, for the monetary policy to yield a unique stationary equilibrium. Therefore, conditions (3.2.6) and (3.2.7) are the necessary and sufficient condition for the Jacobian matrix *C* to have both eigenvalues outside the unit circle. For the baseline values of the parameters, this upper bound requires $(1+\beta)\phi_x + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) \geq 2.99$ in order to generate indeterminacy. At the point where $(1-\beta)\phi_x + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) = 0$, the system has a branching point that can be investigated by changing ϕ_π freely within a range as shown in Barnett and Duzhak (2008). We note that the range of determinacy varies as the degree of openness changes. The higher the parameters α and ω, the the lower the upper boundary. Consequently, the range of determinacy is smaller in the open economy framework and it gets smaller the larger the values of the parameters α and ω. It appears that, indeterminacy is more likely to occur in the open economy framework and becomes a more serious problem as the degree of openness increases. McKnight (2007) finds similar results for an open economy model with timing of money in consideration and points out that the range of indeterminacy is potentially greater the higher the degree of openness under forward looking monetary policy rules.

Using the calibration values of the parameters given in Gali and Monacelli (2005) and solving the equation $(1-\beta)\phi_x + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) = 0$ for ϕ_π, we found $\phi_\pi = 1$, approximately. That means that the system (3.2.4) will have a branching point at around $\phi_\pi = 1$. Therefore, we can say that in open economy framework the monetary policy instrument, which is the short term interest rate, should be slightly more reactionary in response to changes in expected inflation rate, accompanied by a small but positive response to the expected output gap.

Figure 3.2.1: Determinacy and indeterminacy regions under forward looking Taylor rule.

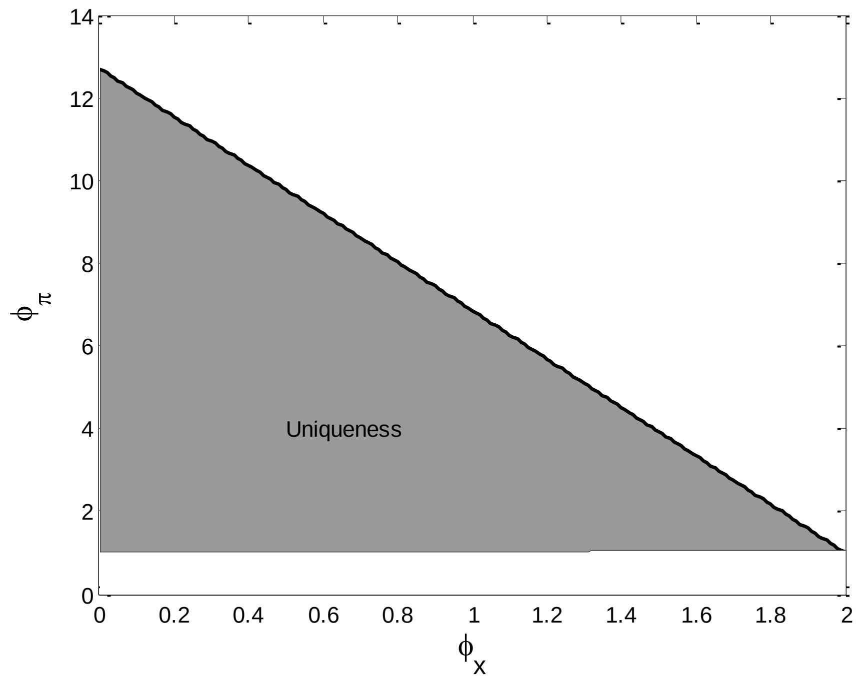


Figure (3.2.1) depicts the regions of determinacy and indeterminacy in (ϕ_x, ϕ_π)-space given the baseline values of the parameters. As it can be seen on Figure (3.2.1), high values of ϕ_π and/or ϕ_x cause the indeterminacy problem. Contrary to the current looking policy rule case, uniqueness of the equilibrium under the forward looking policy rule requires a mild response of the monetary authority to changes in inflation and/or in the output gap. Thus, the monetary authority should react neither "too strongly" nor "too weakly" to changes in the expected inflation and/or the expected output gap. Rules with $\phi_\pi > 1$ accompanied by a moderate reaction to expected output gap would be enough to acquire a unique equilibrium.

If the discriminant of the quadratic equation is strictly negative, that is if

$\Delta \equiv a_1^2 - 4a_0 < 0$,

then, the roots of the Jacobian matrix *C* will be complex conjugate in the form, $\lambda_{1,2} = a \pm ib$, where $a, b \in \mathbb{R}$, $b \neq 0$ is the real part, while $i = +\sqrt{-1}$ is the imaginary unit. Using the baseline values of the parameters, it is easy to verify that the inside of the square roots in both eigenvalues is negative, hence the discriminant Δ is strictly negative. Therefore, the eigenvalues of the system (3.2.4) are complex conjugate, $\lambda_{1,2} = a \pm ib$,

where

$$a = \frac{a_1}{2} = \frac{1}{2}\left(\frac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha\left(\omega-1\right)\right)\right)\left(\phi_\pi - 1\right)}{\beta\sigma - \beta\phi_x\left(1+\alpha\left(\omega-1\right)\right)} + \frac{1}{\beta} \right), \tag{3.2.8}$$

and

$$b = \frac{\sqrt{-\Delta}}{2} = \frac{1}{2}\sqrt{\frac{4\sigma}{\beta\sigma - \beta\phi_x\left(1+\alpha\left(\omega-1\right)\right)} - \left(\frac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha\left(\omega-1\right)\right)\right)\left(\phi_\pi - 1\right)}{\beta\sigma - \beta\phi_x\left(1+\alpha\left(\omega-1\right)\right)} + \frac{1}{\beta} \right)^2}. \tag{3.2.9}$$

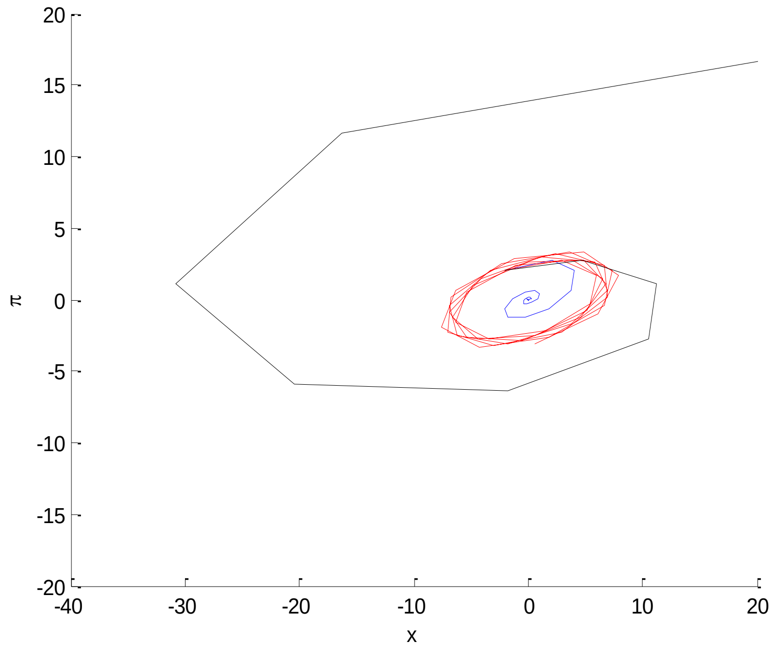
In fact, for the benchmark values of the parameters, the Jacobian matrix is

$$C = \begin{bmatrix} 0.9466 & 0.5772 \\ -0.3434 & 1.0101 \end{bmatrix}$$

having eigenvalues $\lambda_1 = 0.9784 + 0.4441\mathrm{i}$ and $\lambda_2 = 0.9784 - 0.4441\mathrm{i}$ with radius R=1.0745.

Figure (3.2.2) illustrates three trajectories constructed for different parameter settings.

Figure 3.2.2: Phase diagram of the system (3.2.4) for the baseline values of the parameters



Having a pair of complex conjugate eigenvalues, we may expect to see a Hopf bifurcation if certain conditions are satisfied. Using Theorem 1, the conditions for the existence of a Hopf bifurcation is presented in the following Proposition.

Proposition 3.2.2: The system (3.2.4) undergoes a Hopf bifurcation if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{(\beta - 1)}{\beta} \frac{\sigma}{1 + \alpha(\omega - 1)}. \tag{3.2.10}$$

Proof: Suppose the system (3.2.4) goes through a Hopf bifurcation at (y^*, ϕ_x^*), where

$y^* = (x^*, \pi^*)$. Then, we need to show that $\Delta < 0$ and $\phi_x^* = \frac{(\beta - 1)}{\beta} \frac{\sigma}{1 + \alpha(\omega - 1)}$.

The existence of a Hopf bifurcation, by definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative, that is $\Delta < 0$.

For the second part, note that the existence of a Hopf bifurcation requires $\text{mod}(\lambda_1) = \text{mod}(\lambda_2) = +\sqrt{a^2 + b^2} = 1$ from the first condition of Theorem 1. Re-writing the condition explicitly by substituting (3.2.8) and (3.2.9) into it, taking the square of both sides and solving for ϕ_x, we obtain the critical value of the parameter ϕ_x as in (3.2.10). Therefore, the first condition of Theorem 1 holds only if $\phi_x = \frac{(\beta - 1)}{\beta} \frac{\sigma}{1 + \alpha(\omega - 1)}$.

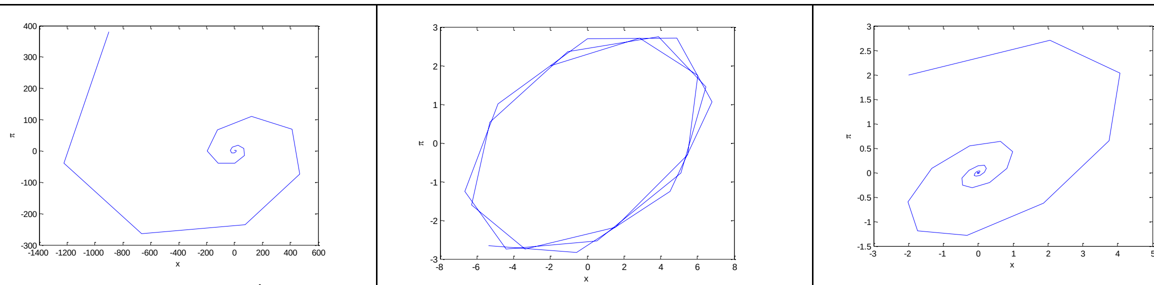
From the other side, suppose $\Delta < 0$ and $\phi_x = \frac{(\beta - 1)}{\beta} \frac{\sigma}{1 + \alpha(\omega - 1)}$. Substituting for ϕ_x^* into $\sqrt{a^2 + b^2}$ yields $\text{mod}(\lambda_1) = \text{mod}(\lambda_2) = 1$, which is the first condition in Theorem 1. In order to show that the critical value of the parameter ϕ_x is actually a Hopf bifurcation parameter, we check the second condition in Theorem 1, which gives

$$\left. \frac{d\left|\lambda_i(\phi_x)\right|}{d\phi_x} \right|_{\phi_x = \phi_x^*} = \frac{d}{d\phi_x}\left(\sqrt{a^2 + b^2}\right)\Bigg|_{\phi_x = \phi_x^*} = \frac{\beta\left(1 + \alpha(\omega - 1)\right)}{2\sigma} \neq 0 \text{ for } i = 1, 2.$$

Thus, both conditions of Theorem 1 are satisfied and we have $\phi_x^* = \frac{(\beta - 1)}{\beta} \frac{\sigma}{1 + \alpha(\omega - 1)}$. □

Proposition (3.2.2) shows that taking the parameter ϕ_x free to vary and keeping the other parameters constant, the system (3.2.4) is likely to undergo a Hopf bifurcation at ϕ_x^*. Therefore, theoccurrence of a Hopf bifurcation is theoretically possible as shown in Proposition (3.2.2). Figures (3.2.2) and (3.2.3) illustrate several phase diagrams which indicate theoccurrence of a Hopf bifurcation in Model (3.2.1).

Figure 3.2.3: Phase diagrams showing a Hopf bifurcation in Model (3.2.1)



Diverging trajectory at $\phi_\pi = 2.8$ and $\phi_x = 0.4$	Limit cycle at $\phi_\pi = 2.8$ and $\phi_x = 0$	Converging trajectory at $\phi_\pi = 2.8$ and $\phi_x = -0.4$

Numerical analysis indicates a Period Doubling bifurcation at $\phi_x = 1.913$ and a Hopf bifurcation at $\phi_x = -0.01$. Given the baseline values of the parameters, Hopf bifurcation occurs outside the feasible set of parameter values. Decreasing the value of ω results in a higher value of bifurcation parameter in absolute value, except when $\alpha = 0$ at which changes in ω does not make any difference. On the other hand, decreasing the value of α results in a lower value of bifurcation parameter in absolute value, except when $\omega = 1$ at which changes in α does not make any difference. Moreover, all bifurcations dissappear when $\alpha = 1$ and $\omega = 0$.

Barnett and Duzhak (2010) report a Period Doubling bifurcation at $\phi_x = 2.994$ under forward looking Taylor rule for the closed economy structure. Comparing the results from Barnett and Duzhak (2010) with the open economy case, however, does not give us a clear statement about whether or not the openness makes the New Keynesian model more sensitive to bifurcations since Gali and Monacelli Model incorporates a wider set of parameters including some deep parameters relevant to the open economy. The fact that both studies use different set of benchmark values for the parameters makes a direct comparison even harder. Airaudo and Zanna (2005), using a money-in-utility function and assuming non-separability, show that cyclical and chaotic dynamics are more likely the more open the economy and the higher the exchange rate pass-through into import prices. They also show that the existence of cyclical and chaotic dynamics depend on some open economy features and is in general robust to different timings in the policy rule. Analyzing the effects of a change in the parameters α and ω in the Gali and Monacelli Model, on the other hand, does not seem to be indicative for more sensitivity in open economy environment.

Figure 3.2.4: Period Doubling bifurcation boundary for ϕ_x in Model (3.2.1)

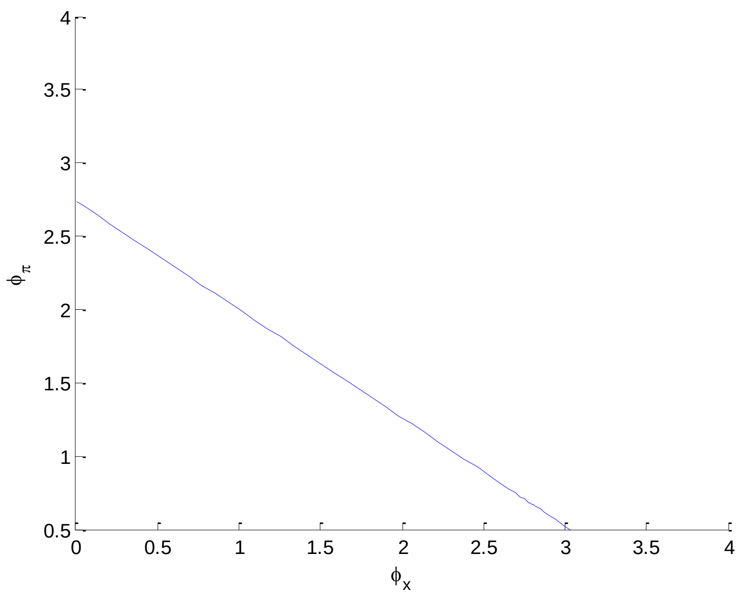


Figure (3.2.4) illustrates the boundaries of Period Doubling bifurcation under forward looking Taylor rule. Note that along the bifurcation boundary, the values of the bifurcation parameter ϕ_x lie between 0 and 2.8. As the weight of the reaction of central bank to expected inflation, that is ϕ_π, increases, the smaller values of parameter ϕ_x would be sufficient to cause a Period Doubling bifurcation.

Figure 3.2.5: Phase diagrams showing a periodic solution using two different number of iterations at $\phi_\pi = 2.8$ and $\phi_x = 0$ in Model (3.2.1)

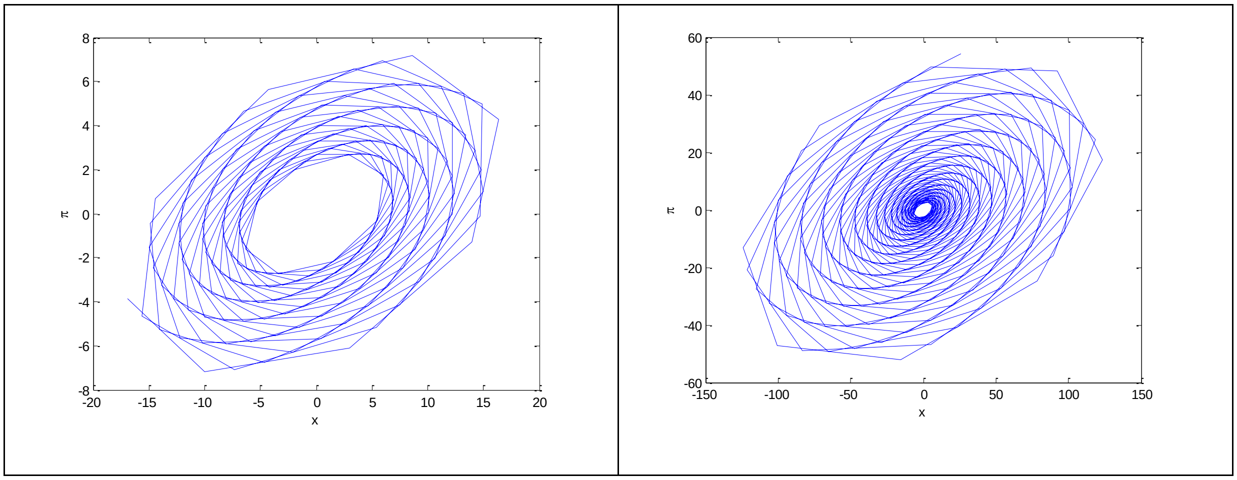


Figure (3.2.5) illustrates the phase diagrams constructed for two different number of iterations at $\phi_\pi = 2.8$ and $\phi_x = 0$. The system has a periodic solution at these parameter values. The origin is a stable spiral point. Any solution that starts around the origin in the phase plane will spiral toward the origin. Since the trajectories spiral inward, the origin is a stable sink.

3.2.2 Under Pure Forward Looking Inflation Targeting:

Most countries have been pursuing inflation targeting, with or without accompanied by an output gap variable, so as to reduce the high and volatile inflation risk. Therefore, in recent years, it is not surprising to see that the rules that set the policy rate in response to the expected/forecasted rate of inflation has been widely used by many central banks to practice "inflation-averse" monetary policies. Such policy rules have been favored from the perspective of policy effectiveness, historical and econometric evidence as well as central bank's performance about dealing with the observed time delay and driving private sector's inflation expectations.[10]

[10] See, among others, Huang and Meng (2007), Svensson (1997), Bernanke and Woodford (1997), Batini and Haldane (1999), Levin, Wieland, and Williams (2003), Orphanides and Williams (2005), and Bernanke and Woodford, eds. (2005).

Consider the following model in which the first two equations describe the economy while the third equation represents the monetary policy rule followed by the central bank:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t, \tag{3.2.1}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma} \left(r_t - E_t \pi_{t+1} - \overline{r}_t \right), \tag{3.2.2}$$

$$r_t = \overline{r}_t + \phi_\pi E_t \pi_{t+1}, \tag{3.2.11}$$

where x_t denotes the output gap, π_t is the inflation rate, and r_t is the nominal interest rate. E_t is the expectation operator. Equation (3.2.11) describes a pure forward looking interest rate rule where the policy parameter ϕ_π measures the extent of the policy rate's response to the next period's expected inflation. That said, nominal interest rate is determined by looking at the changes in the next period's expected inflation. As before, there is no exogenous shock.

Substituting (3.2.11) into (3.2.2) for $r_t - \overline{r}_t$, we have a two-equation first order stochastic difference equation system in terms of the expected inflation rate and the expected output gap:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma} \left(\phi_\pi - 1 \right) E_t \pi_{t+1}$$

As in the previous cases, $x_t = \pi_t = 0$ for all t is an equilibrium solution to the reduced system. Rearranging the terms, we have the following reduced system in normal form $E_t y_{t+1} = C y_t$:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1-\left(\dfrac{\mu}{\beta}+\dfrac{\varphi\mu\left(1+\alpha\left(\omega-1\right)\right)}{\beta\sigma}\right)\left(\phi_\pi-1\right) & \dfrac{\left(\phi_\pi-1\right)\left(1+\alpha\left(\omega-1\right)\right)}{\beta\sigma} \\ -\dfrac{\mu}{\beta}\left(\dfrac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right) & \dfrac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \quad (3.2.12)$$

Then, the characteristic polynomial of the Jacobian matrix is

$$p\left(\lambda\right)=\det\left(C-\lambda I\right)=\lambda^2-a_1\lambda+a_0=0$$

where,

$$a_1=1+\frac{\sigma-\mu\left(\sigma+\varphi\left(1+\alpha\left(\omega-1\right)\right)\right)\left(\phi_\pi-1\right)}{\beta\sigma} \text{ and } a_0=\frac{1}{\beta}.$$

The solution of the characteristic polynimal is given by

$$\lambda_{1,2}=\left(1+\frac{\sigma-\mu\left(\sigma+\varphi\left(1+\alpha\left(\omega-1\right)\right)\right)\left(\phi_\pi-1\right)}{\beta\sigma}\right)\pm\sqrt{\left(1+\frac{\sigma-\mu\left(\sigma+\varphi\left(1+\alpha\left(\omega-1\right)\right)\right)\left(\phi_\pi-1\right)}{\beta\sigma}\right)^2-4\left(\frac{1}{\beta}\right)}.$$

Figure 3.2.6: Phase space plot for $\beta = 1$ and $\phi_\pi = 8$ in Model (3.2.2)

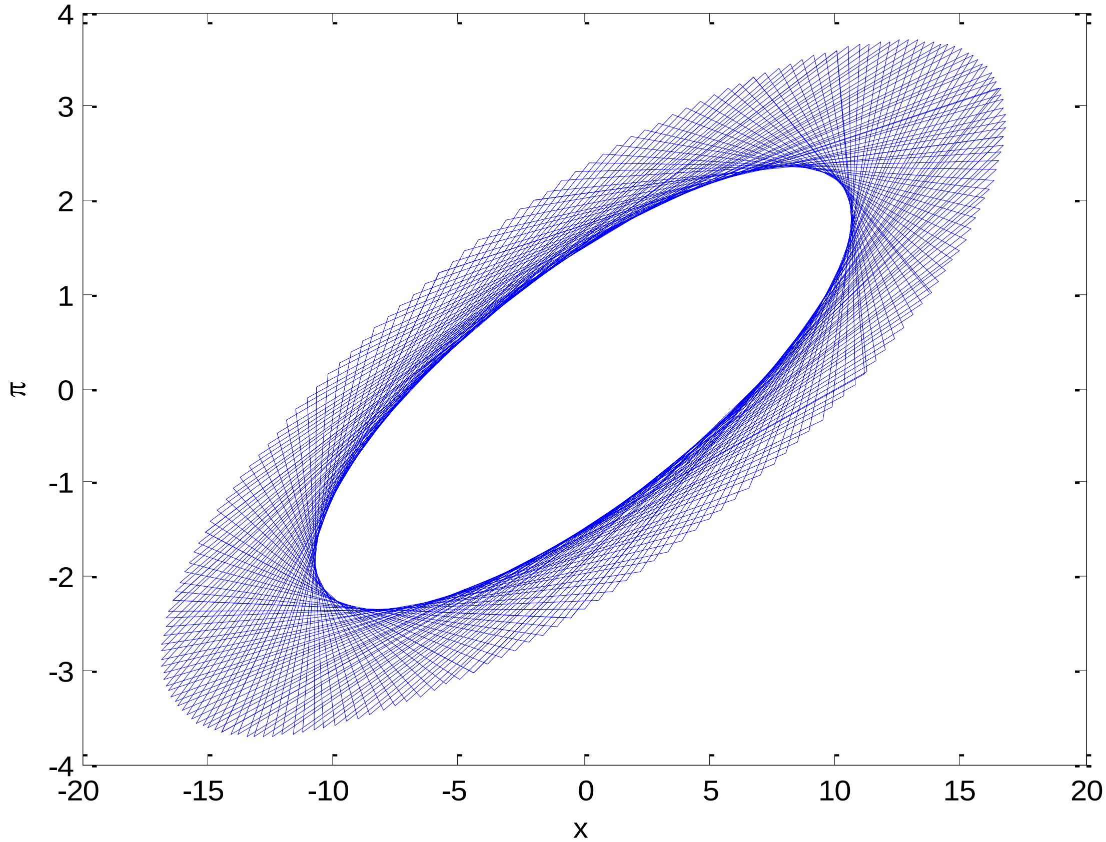


Figure (3.2.6) illustrates a solution path for $\beta = 1$ and $\phi_\pi = 8$. The solution path is periodic which oscillates around the origin without converging or diverging. The origin is a stable (but not asymptotically) center.

Note that there exist two free endogenous variables x_t and π_t. Then, following Blanchard and Kahn (1980), the equilibrium solution would be unique if and only if both eigenvalues are outside the unit circle. The following Proposition characterizes the necessary and sufficient conditions for having a determinate equilibrium solution to the system (3.2.12).

Proposition 3.2.3: Under the pure forward looking inflation targeting rule, the open economy New Keynesian model, specified in (3.2.12) has a unique stationary equilibrium if and only if

$$1 < \beta \tag{3.2.13}$$

and

$$\mu\left(\frac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right)\left(\phi_{\pi}-1\right)>0\ . \tag{3.2.14}$$

Proof: Following Lasalle (1986) and Bullard and Mitra (2002), both eigenvalues of the Jacobian matrix are outside the unit circle if and only if $|a_0|<1$, and $|a_1|<1+|a_0|$. From the first inequality, we derive (3.2.13). From the second inequality, we derive (3.2.14). □

Notice that the condition (3.2.13) is not satisfied since $\beta \in (0,1)$. Hence, for the given parameter values in Gali and Monacelli Model, the system does not guarantee the uniqueness of the equilibrium solution path. If the monetary policy is irresponsive to output, that is, if $\phi_x = 0$, then controlling the parameter value of inflation has an insufficient effect on the determinacy. In this case, no value of ϕ_{π} can ensure a determinate equilibrium unless $\beta > 1$. A discount factor β greater than unity is required to fix the indeterminacy problem, but this would be quite questionable from an empirical prespective. As we also verified numerically, $\beta = 1$ is a branching point separating a unique equilibrium and multiple equilibria.

One major drawback of pure forward-looking inflation targeting is that it often causes equilibrium indeterminacy. One way to prevent such policy-induced instability problem, some authors suggest that while setting the nominal interest rate, some other endogenous variables such as output gap need to be considered, beside the expected inflation. For instance, De Fiore and Liu (2005), and Carlstrom and Fuerst (2006) argue that the New Keynesian model with a pure forward-looking inflation targeting rule usually brings the equilibrium to instability. The failure of pure inflation targeting emphasizes the significance of policy response to output. They also claim that, under nominal rigidities, equilibrium indeterminacy problem cannot be solved just by having the nominal rate respond to both inflation and output gap. Huang and Meng (2007) show that interest rate policy rules that are unresponsive to output usually give rise to equilibrium indeterminacy. They argue that increasing the degree of price stickiness or allowing for policy response to current output render the determinacy of equilibrium solution. But the first method has a quantitatively negligible effect, while the second method's success is sensitive to the elasticity of labor supply and the degree of stickiness.

As the determinacy condition does not hold, there is a multiplicity of stable equilibria. Theoretically, any of these solution paths could be realized. In such cases, as Cochrane (2009) argues, the New Keynesian model has nothing to say about inflation, other than saying that anything can happen. That is why the non-uniqueness problem is important in modeling. Bernanke and Woodford (1997) argue that forecasted inflation targeting is inconsistent with a rational expectation equilibrium and prone to indeterminacy. They suggest that monetary authority should develop a structural model and monitor some other variables besides the inflation target. Batini and Haldane (1999) argue that even though the forward looking dimension makes the policy rule perform better than the standard Taylor rule, longer forecast horizons (longer than 3-6 quarters) risk macroeconomic stability. Giannoni (2000) argues that the presence of indeterminacy in a sticky price model under inflation targeting is possible for a reasonable subset of parameter values. He also shows that the indeterminacy vanishes when the central bank targets a price path. Dittmar and Gavin's (2004) findings support this argument in a flexible-price model and point out that the determinacy can be ensured once the central bank targets a path for the price level instead of inflation rate.

As we also checked numerically, one of the two real eigenvalues of the Jacobian matrix lies inside the unit circle, while the other one is outside. Given that both x_t and π_t are non-predetermined, the existence of an eigenvalue inside the unit circle implies the existence of multiple equilibria. Hence, there is no guarante that $x_t = \pi_t = 0$ will be the equilibrium solution.

Gali (2008), on the other hand, states that the following condition

$$1 < \phi_\pi < 1 + \frac{2\sigma_\alpha\left(1+\beta\right)}{\kappa_\alpha}, \qquad (3.2.15)$$

would be necessary and sufficient for determinacy. This condition suggests that, besides satisfying the Taylor principle ($\phi_\pi > 1$), the monetary authority should not adjust the nominal interest rate too aggressively in reaction to a change in expected inflation. When $\phi_\pi = 1$, a smooth and non-converging sunspot equilibrium emerges. When $\phi_\pi = 1 + \frac{2\sigma_\alpha\left(1+\beta\right)}{\kappa_\alpha}$, on the other hand, a cyclical and non-converging sunspot equilibrium appears. Nakagawa (2009) supports the same argument and states that an aggressive response to expected inflation would lead to equilibrium indeterminacy making the current economy fluctuate even though it stabilizes the expectations for the future economy.

Given the values of the parameters by Gali (2008), $\sigma_\alpha = 1$, $\beta = 1$, $\kappa_\alpha = 0.1275$ and $\phi_\pi = 32.2157$, Nakagawa (2009) finds sunspot equilibrium dynamics such that sunspot equilibria under $\phi_\pi > 32.2$ are oscillatory convergent. The sunspot equilibria under $\phi_\pi < 1$ smoothly approach to the steady state. If $\phi_\pi = 1$ or $\phi_\pi = 32.2$, sunspot dynamics stop converging.

If the discriminant of the quadratic equation is strictly negative, that is if $\Delta \equiv a_1^2 - 4a_0 < 0$, then the roots of the 2x2 matrix *C* will be complex conjugate numbers in the form $\lambda_{1,2} = a \pm ib$, with $a,b \in \mathbb{R}$, $b \neq 0$ is the real part, while $i = +\sqrt{-1}$ is the imaginary unit. Using the aforementioned numerical values of the parameters, it is easy to verify that, for a feasible set of parameter values, the inside of the square root in eigenvalues is negative, hence the discriminant Δ is strictly negative. Therefore, the eigenvalues of the system (3.2.12) are complex conjugate in the form $\lambda_{1,2} = a \pm ib$,

where

$$a = \frac{a_1}{2} = \left(\frac{1}{2} + 1 + \frac{\sigma - \mu\left(\sigma + \varphi\left(1 + \alpha\left(\omega - 1\right)\right)\right)\left(\phi_\pi - 1\right)}{2\beta\sigma} \right) \tag{3.2.16}$$

and

$$b = \frac{\sqrt{-\Delta}}{2} = \frac{1}{2}\sqrt{4\left(\frac{1}{\beta}\right) - \left(1 + \frac{\sigma - \mu\left(\sigma + \varphi\left(1 + \alpha\left(\omega - 1\right)\right)\right)\left(\phi_\pi - 1\right)}{\beta\sigma}\right)^2}. \tag{3.2.17}$$

Using the baseline values of the parameters, we can write the eigenvalues as follows:

$$\lambda_1 = \frac{233 - 34\phi_\pi}{198} + \frac{\sqrt{1156\phi_\pi^2 - 15844\phi_\pi + 14689}}{198}$$

and

$$\lambda_2 = \frac{233 - 34\phi_\pi}{198} - \frac{\sqrt{1156\phi_\pi^2 - 15844\phi_\pi + 14689}}{198}.$$

Then,

i. If $1156\phi_\pi^2 - 15844\phi_\pi + 14689 = 0$, then the discriminant is $\Delta = 0$, and the system has two real and equal roots. Hence, the system (3.2.12) has one solution. Solving for ϕ_π, we have $\phi_\pi = 1.0001$ and $\phi_\pi = 12.7$ that gives the multiple roots with multiplicity two.

ii. If $1156\phi_\pi^2 - 15844\phi_\pi + 14689 < 0$, then the system (3.2.12) has complex conjugate eigenvalues. Any value of the parameter ϕ_π between $1 < \phi_\pi \leq 12.7$ yields the complexity.

iii. If $1156\phi_\pi^2 - 15844\phi_\pi + 14689 > 0$, then the system (3.2.12) has two real and distinct eigenvalues. Any value of the parameter ϕ_π that is less than one and/or greater than 12.7 yields that result.

Thus, keeping the discount factor constant at $\beta = 0.99$ and varying the policy parameter ϕ_π over a feasible range, we find that the Jacobian of the system (3.2.12) has complex conjugate eigenvalues for $1 < \phi_\pi \leq 12.7$ with radius greater than unity implying stability. For $12.71 \leq \phi_\pi$, eigenvalues turn out to be real valued, one inside the unit circle and the other one outside the unit circle in absolute value making the equilibrium a saddle.

With the presence of a pair of complex conjugate eigenvalues, we may expect to see a Hopf bifurcation if the transversality conditions are satisfied. Using Theorem 1, the conditions for the existence of a Hopf bifurcation is presented in the following Proposition.

Proposition 3.2.4: The system (3.2.12) undergoes a Hopf bifurcation if and only if $\Delta < 0$ and

$$\beta^* = 1. \qquad (3.2.18)$$

Proof: Suppose the system (3.2.12) goes through a Hopf bifurcation at $\left(y^*, \beta^*\right)$, where $y^* = \left(x^*, \pi^*\right)$. Then we need to show that $\Delta < 0$ and $\beta^* = 1$.

The existence of a Hopf bifurcation, by definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative, that is $\Delta < 0$. For the second part, note that the existence of a Hopf bifurcation requires $\mathrm{mod}\left(\lambda_1\right) = \mathrm{mod}\left(\lambda_2\right) = +\sqrt{a^2 + b^2} = 1$ from the first part of Theorem 1.

Rewriting the condition explicitly by substituting (3.2.16) and (3.2.17) into it, and taking the square of both sides we obtain the critical value of the parameter as in (3.2.18). Therefore, the first condition of Theorem 1 holds only if $\beta = 1$.

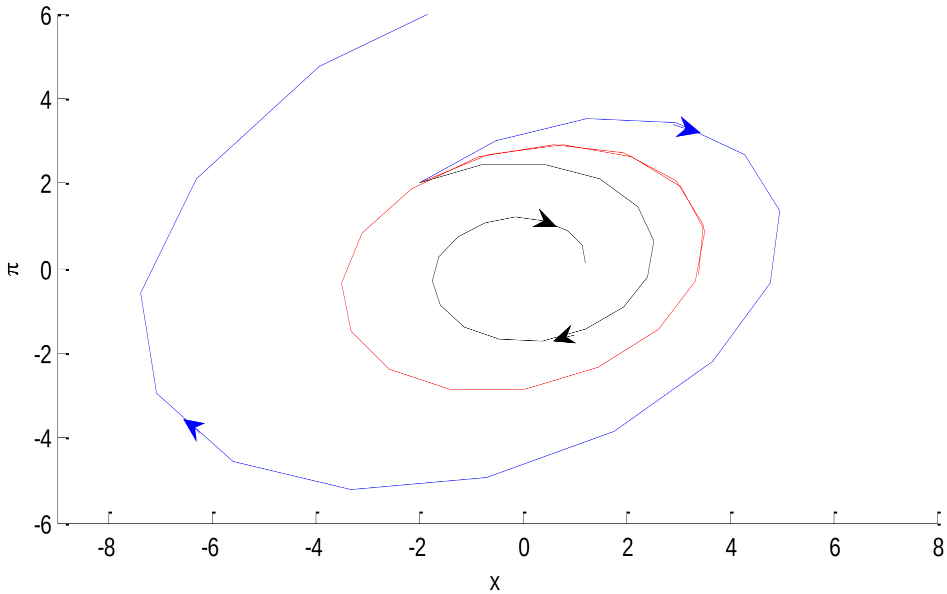
Conversely, suppose the system (3.2.12) possess' $\Delta < 0$ and $\beta = 1$. Substituting for $\beta = 1$ into $\sqrt{a^2 + b^2}$ yields $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = 1$, which is the first condition in Theorem 1. In order to show that $\beta = 1$ gives rise to a Hopf bifurcation, we check the second condition in Theorem 1, which yields

$$\left.\frac{d\left|\lambda_i(\beta)\right|}{d\beta}\right|_{\beta^*=1} = \left.\frac{d}{d\beta}\left(\sqrt{a^2 + b^2}\right)\right|_{\beta^*=1} = -\frac{1}{2} \neq 0 \text{ for } i = 1, 2.$$

Thus, both conditions of Theorem 1 are satisfied and we have $\beta^* = 1$. □

Proposition (3.2.4) shows that the occurrence of a Hopf bifurcation is theoretically possible within the functional structure of Model (3.2.2).

Figure 3.2.7: Phase plots for various values of the parameter β in (x, π)-space in Model (3.2.2)[11]



Phase plots in Figure (3.2.7) show an evidence of Hopf bifurcation in the system. Here, there is only one periodic solution and other solutions diverge from the periodic solution as t→∞. In this case, the periodic solution is called an unstable limit cycle.

This result shows that setting the discount factor equal to 1 puts the system on the Hopf bifurcation boundary and creates instability. The result is in line with the findings of Barnett and Duzhak (2008) for the closed economy case. We also numerically find a Period Doubling bifurcation at $\beta = -0.91$, which is not in the feasible subset of parameter space. Note that, Gali and Monacelli (2005) assumes $\beta = 0.99$. For the baseline values of the parameters and $\beta = 0.99$, the Jacobian matrix is

$$C = \begin{bmatrix} 0.8283 & 0.5051 \\ -0.3434 & 1.0101 \end{bmatrix}$$

[11] The outer trajectory is for $\beta = 0.9$ and $\phi_\pi = 1.5$, limit cycle is for $\beta = 1$ and $\phi_\pi = 1.5$, and the inner trajectory is for $\beta = 1.1$ and $\phi_\pi = 1.5$.

having eigenvalues $\lambda_1 = 0.9192 + 0.4064i$ and $\lambda_2 = 0.9192 - 0.4064i$ with radius R=1.0050.

For $\beta = 1$, the system has complex conjugate eigenvalues with radius of 1. For $\beta < 1$, the radius is greater than unity implying stability, while for $\beta > 1$, it is less than unity indicating an unstable equilibrium. Hence, $\beta = 1$ is the Hopf bifurcation value. Furthermore, Hopf bifurcation appears at $\beta = 1$ independent of the values of α and ω. Bifurcation analysis in open economy framework yields the same result as in the closed economy case under pure forward looking inflation targeting rule. Barnett and Duzhak (2010) report a Hopf bifurcation at $\beta = 1$ for the closed economy case.

Recall that the parameter β is the discount factor which appears in the IS curve equation and comes from the optimization problem. Although it could be in a range between 0 and 1, Rumler (2007) states that, reasonable estimates of the discount factor β are found to be in the range between 0.9 and 1, while values of β much closer to 1 are theoretically more reasonable since it suggests the quarterly subjective discount rate. For the sake of simplicity, some authors assume a discount factor β equal to one.[12] As Barnett and Duzhak (2008, 2010) point out, however, setting the discount factor equal to 1 can put Model (3.2.12) directly on the Hopf bifurcation boundary, and therefore can induce instability. That said, setting the discount factor at 1 is not an appropriate approach within the New Keynesian modeling whether it is an open or closed economy case.

[12] See, for example, Roberts (1995), Gali and Gertler (1999). Bjornland et al. (2010) empirically find an annual discount rate of 1.6% which implies a discount factor greater than one.

Figure 3.2.8: Hopf bifurcation in phase space of Model (3.2.2)

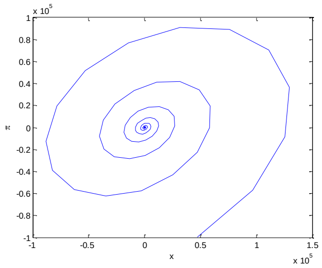
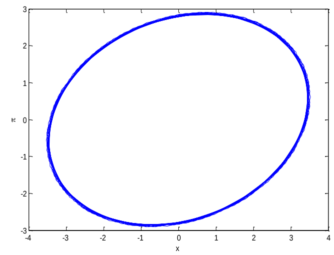
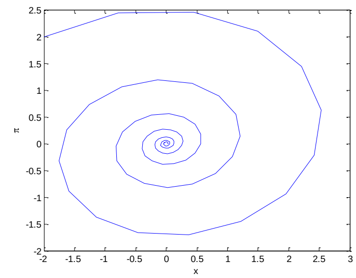
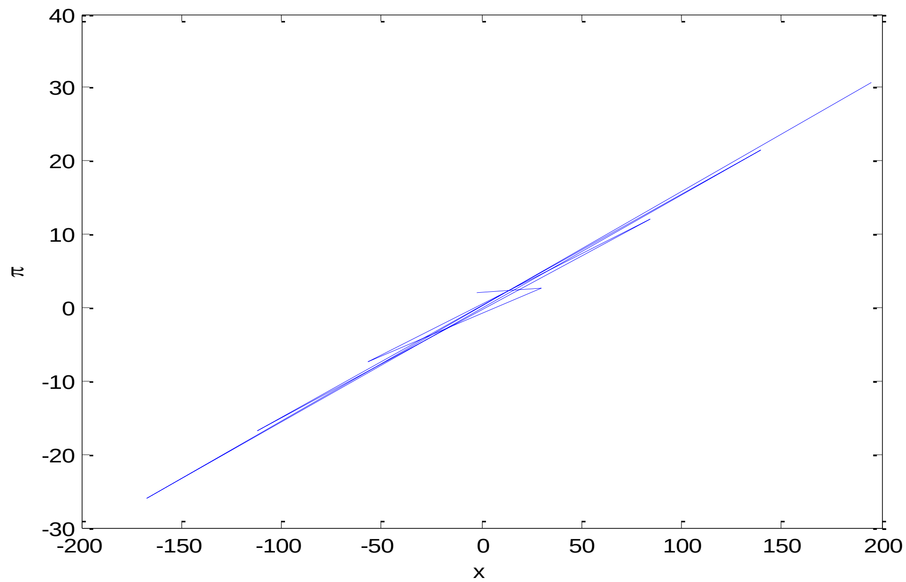
		
$\beta = 0.9$ and $\phi_\pi = 1.5$	$\beta = 1$ and $\phi_\pi = 1.5$	$\beta = 1.1$ and $\phi_\pi = 1.5$

Figure (3.2.8) illustrates different phase plots for the Model (3.2.2). In the first plot, the origin is a spiral sink (asym. stable), in the second plot it is a stable center while in the third one it is a spiral source.

If we vary the policy parameter ϕ_π while taking $\beta = 1$ and keeping the other parameters constant at their baseline values, we numerically find a Hopf bifurcation at $\phi_\pi = 1.0176$, a Period Doubling bifurcation at $\phi_\pi = 12.76$, as well as a branching point at $\phi_\pi = 1$. The phase space plot in Figure (3.2.9) illustrates the trajectory for $\beta = 1$ and $\phi_\pi = 12.764706$ at which a Period Doubling bifurcation occurs.

Figure 3.2.9: Phase space plot for $\beta = 1$ and $\phi_\pi = 12.764706$



Decreasing the value of ω results in a higher value of the Period Doubling bifurcation parameter ϕ_π in absolute value, except when $\alpha = 0$ at which changes in ω does not make any difference. On the other hand, decreasing the value of α results in a lower value of the bifurcation parameter ϕ_π in absolute value, except when $\omega = 1$ at which changes in α does not make any difference. Furthermore, Hopf bifurcation appears at $\beta = 1$ independent of the values of α and ω.

Figure 3.2.10: Phase space plots for various values of ϕ_π

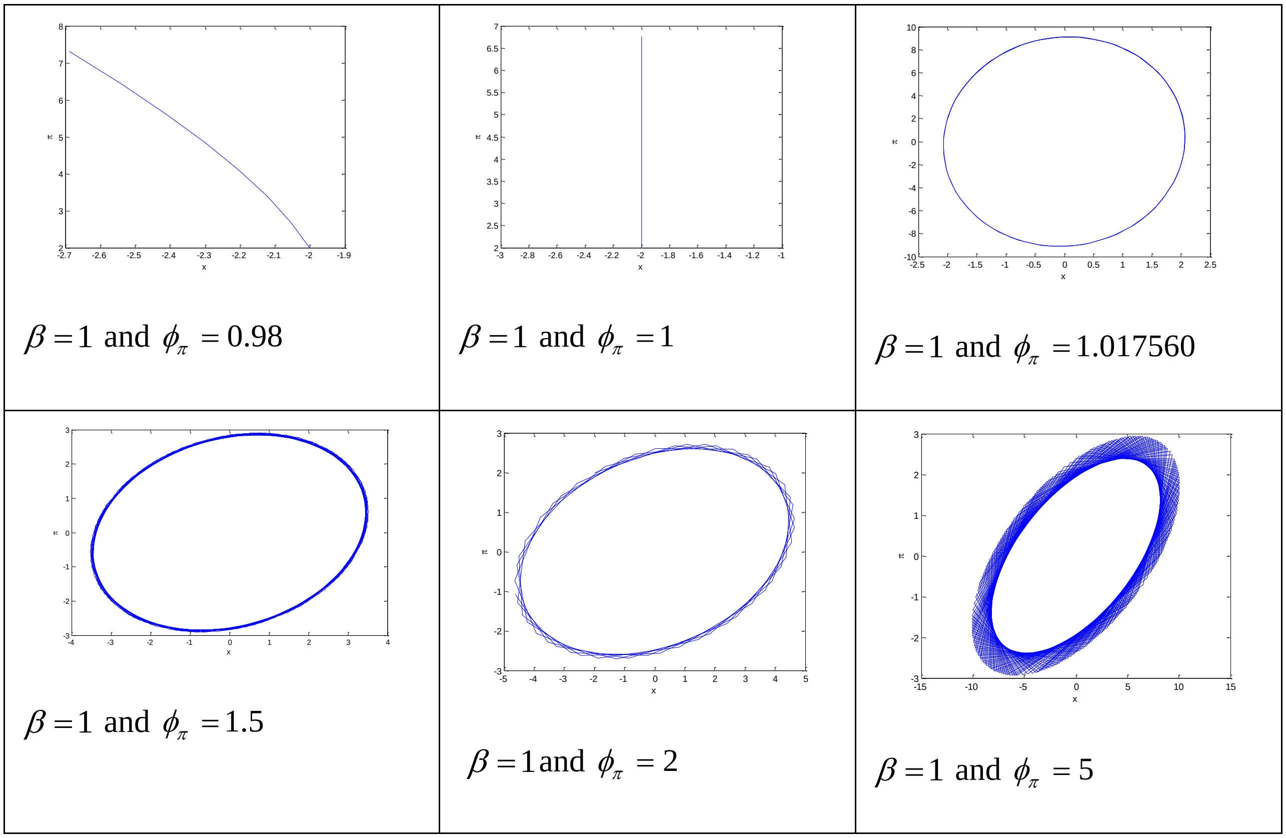


Figure (3.2.10) illustrates several phase plots constructed for different values of the parameter ϕ_π while taking $\beta = 1$.

3.3 Gali and Monacelli Model under Backward Looking Policy Rules:

3.3.1 Under Backward Looking Taylor Rule:

A backward looking approach enables monetary policy makers to consider the lagged information on output gap and inflation which can be obtained from the previous period while determining the current period’s policy rate. This has been considered a more realistic assumption than making decisions based on contemporaneous information. Carlstrom and Fuerst (2000) argue that following a backward looking policy rule would help the central bank reach a unique stationary equilibrium. See, for example, McCallum (1999) and Bullard and Mitra (2002) for a discussion of employing lagged data in monetary policy rules.

Consider the following model in which the first two equations describe the economy while the third equation the instrument rule employed by the central bank for the monetary policy:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \tag{3.3.1}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(r_t - E_t\pi_{t+1} - \overline{r}_t\right) \tag{3.3.2}$$

$$r_t = \overline{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1} \tag{3.3.3}$$

Moving equation (3.3.3) one period forward, adding expectations, and rearranging the terms, and defining $y_t = [x_t, \pi_t, r_t]'$ we can write the system in the standard form $E_t y_{t+1} = Cy_t$:

$$E_t y_{t+1} = Cy_t + \begin{bmatrix} -\frac{1+\alpha(\omega-1)}{\sigma}\overline{r}_t \\ 0_t \\ E_t \overline{r}_{t+1} \end{bmatrix} \tag{3.3.4}$$

where $C = \begin{bmatrix} \frac{\mu}{\beta}\left(1+\frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right)+1 & -\frac{1+\alpha(\omega-1)}{\beta\sigma} & \frac{1+\alpha(\omega-1)}{\sigma} \\ -\frac{\mu}{\beta}\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right) & \frac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & 0 \end{bmatrix}$.

The Jacobian matrix C has a third order characteristic polynomial,

$$p(\lambda) = \det(C - \lambda I_3) = \lambda^3 - a_2\lambda^2 + a_1\lambda + a_0 = 0$$

where,

$$a_0 = \phi_x \frac{1+\alpha(\omega-1)}{\beta\sigma} + \phi_\pi \mu\left(\frac{1}{\beta} + \varphi\frac{1+\alpha(\omega-1)}{\beta\sigma}\right), \tag{3.3.5}$$

$$a_1 = \frac{1}{\beta} - \phi_x \frac{1+\alpha(\omega-1)}{\sigma}, \tag{3.3.6}$$

and $a_2 = -\left(\frac{1}{\beta} + \frac{\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)+1\right)$. (3.3.7)

We begin our analysis by examining the determinacy conditions of the system (3.3.4). Note that there are three endogeneous variables; the rate of inflation, π_t, the output gap, x_t, and the nominal interest rate, r_t; and two predetermined variables; x_{t-1} and π_{t-1}. Following Blanchard and Kahn (1980), the system (3.3.4) has a unique, stationary equilibrium solution if and only if the number of eigenvalues outside the unit circle is equal to the number of forward looking (non-predetermined) variables, which is two ($E_t x_{t+1}$ and $E_t \pi_{t+1}$). Accordingly, two of the eigenvalues need to be outside the unit circle for uniqueness.

With the backward looking Taylor rule, the following Proposition characterizes the necessary and sufficient conditions to ensure that the economy described by the system (3.3.4) has a unique stationary equilibrium.

Proposition 3.3.1: Under the backward looking Taylor rule, the open economy New Keynesian model specified in (3.3.4) has a unique stationary equilibrium if and only if

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) + (1-\beta)\phi_x > 0, \tag{3.3.8}$$

and

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) + (1+\beta)\phi_x < \frac{2\sigma(1+\beta)}{1+\alpha(\omega-1)}. \tag{3.3.9}$$

Proof: Following Bullard and Mitra (2002), the number of the roots with the same sign can be confirmed by Descartes' rule of signs theorem as follows. In the positive case, we keep the signs on λ unchanged and count the sign changes in the characteristic polynomial, which is two:

$$p(\lambda) = +\lambda^3 - a_2\lambda^2 + a_1\lambda + a_0$$

Hence the maximum number of roots with positive sign is two, which might be real or complex conjugate.

In the negative case, by having changed the sign on λ, we look at $p(-\lambda)$:

$$p(-\lambda) = -\lambda^3 - a_2\lambda^2 - a_1\lambda + a_0$$

Notice that in this case we have only one sign change in the characteristic polynomial. This implies that the polynomial has at most one negative root, and therefore exactly one. Thus, by Descartes' rule of signs, the characteristic polynomial has one negative root besides having either two positive roots or a pair of complex conjugate roots.

We can also evaluate $p(x)$ at $x = 1$ for the roots with positive sign and at $x = -1$ for the roots with negative sign to see whether the negative and positive roots lie outside (or inside) the unit circle. Note that for $\lambda = 0$, we have

$$p(0) = \phi_x \frac{1+\alpha(\omega-1)}{\beta\sigma} + \phi_\pi \mu \left(\frac{1}{\beta} + \varphi \frac{1+\alpha(\omega-1)}{\beta\sigma} \right) > 0,$$

which is trivially satisfied as all the parameters here are assumed to be positive.

For $\lambda = -1$, after some manupilations, we have

$$p(-1) = \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right)(\phi_\pi - 1) + \phi_x (1+\beta) < \frac{2\sigma(\beta+1)}{1-\alpha+\alpha\omega}.$$

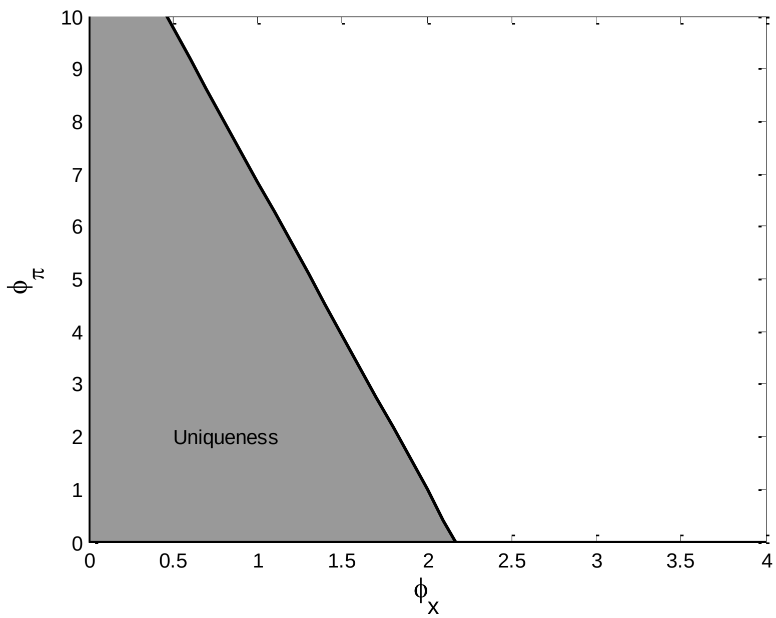
For $\lambda = 1$, after some manupilations, we have

$$p(1) = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) + \phi_x(1-\beta) > 0$$

for the given values of the parameters in Gali and Monacelli (2005).[13] □

Conditions (3.3.8) and (3.3.9) together imply that a sufficiently active policy rule with $\phi_\pi > 1$ accompanied by a small response to the output gap is sufficient to lead to a unique equilibrium. Eusepi (2005) argues that contrary to the forecast-based Taylor rules, the backward looking Taylor rule stabilizes the economy by leading to a uniquely learnable equilibrium. Figure (3.3.1) illustrates the regions of unique and multiple solutions.

Figure 3.3.1: Determinacy diagram for the backward looking Taylor rule



Using the calibration values of the parameters given by Gali and Monacelli (2005), determinacy of the equilibrium can be checked by computing the eigenvalues of the Jacobian matrix, that is:

[13] See also Bullard and Mitra (2001) for an alternative version of proof.

$$C = \begin{bmatrix} 1.3434 & -1.0101 & 1 \\ -0.3434 & 1.0101 & 0 \\ 0.1250 & 1.5 & 0 \end{bmatrix}$$

with the eigenvalues $\lambda_1 =$ 1.3518 + 0.0658i , $\lambda_2 =$1.3518 - 0.0658i , $\lambda_3 =$ -0.3502 . Note that one solution is real and inside the unit circle in absolute value, while the radius of the two complex conjugate solutions is greater than one with $R = 1.3534$. The number of eigenvalues outside the unit circle is equal to the number of forward looking variables, which is two. Hence, there exists a unique solution of the system.

In order for a 3-dimensional system to exhibit a Hopf bifurcation, it should have a real root and a pair of complex conjugate roots on the unit circle. The following Proposition states the conditions for the system (3.3.4) to exhibit a Hopf bifurcation.

Proposition 3.3.2: The system (3.3.4) undergoes a Hopf bifurcation if and only if the following transversality condition

$\left.\dfrac{\partial\left|\lambda_i(\phi)\right|}{\partial\phi_j}\right|_{\phi_j=\phi_j^*} \neq 0$ holds and the following conditions are satisfied:

$$i)\ \phi_x + \phi_\pi \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) - \frac{\beta\sigma}{1+\alpha(\omega-1)} < 0\,, \tag{3.3.10}$$

$$ii)\ \phi_x(\beta-1) + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(1-\phi_\pi) < 0\,, \tag{3.3.11}$$

$$\text{iii)}\ \left(\phi_x + \phi_\pi\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\mu\right)^2 + \left(\phi_x + \phi_\pi\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\mu\right)\xi_1 - \phi_x\xi_2 = \xi_3\,. \tag{3.3.12}$$

Proof: For the necessary part of the proof, we apply Theorem 2 to derive the conditions of the Proposition (3.3.2). From the condition (*i*) in Theorem 2, we obtain

$$\left|a_0\right| = \left|\phi_x \frac{1+\alpha(\omega-1)}{\beta\sigma} + \phi_\pi \mu \left(\frac{1}{\beta} + \varphi \frac{1+\alpha(\omega-1)}{\beta\sigma}\right)\right| < 1.$$

Using the definition of absolute value, we have either

$$\phi_x + \phi_\pi \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\mu - \frac{\beta\sigma}{1+\alpha(\omega-1)} < 0$$

if $\phi_x \frac{1+\alpha(\omega-1)}{\beta\sigma} + \phi_\pi \left(\frac{1}{\beta} + \varphi \frac{1+\alpha(\omega-1)}{\beta\sigma}\right)\mu \geq 0$, or

$$\phi_x + \phi_\pi \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\mu + \frac{\beta\sigma}{1+\alpha(\omega-1)} > 0$$

if $\phi_x \frac{1+\alpha(\omega-1)}{\beta\sigma} + \phi_\pi \left(\frac{1}{\beta} + \varphi \frac{1+\alpha(\omega-1)}{\beta\sigma}\right)\mu < 0$.

Since the parameters are assumed to be positive the second case is redundant and hence we only consider the first case as binding condition.

Using the condition (*ii*) in Theorem 2, we obtain

$$\left|\phi_x \frac{1+\alpha(\omega-1)}{\beta\sigma} + \left(1+\varphi \frac{1+\alpha(\omega-1)}{\sigma}\right)\left(\phi_\pi \mu\beta^{-1} - \mu\beta^{-1}\right) - \frac{1}{\beta} - 1\right| < 1 + \frac{1}{\beta} - \phi_x \frac{1+\alpha(\omega-1)}{\sigma}$$

Again, by definition of absolute value, we have either

$$\left(\phi_x - \frac{2\sigma}{1+\alpha(\omega-1)}\right)(1+\beta) + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) < 0$$

if $\phi_x \frac{1+\alpha(\omega-1)}{\beta\sigma} + \left(1+\varphi \frac{1+\alpha(\omega-1)}{\sigma}\right)\left(\phi_\pi \mu\beta^{-1} - \mu\beta^{-1}\right) - \frac{1}{\beta} - 1 \geq 0$, or

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(1-\phi_\pi) + \phi_x(\beta-1) < 0$$

if $\phi_x \frac{1+\alpha(\omega-1)}{\sigma} + \mu\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)(\phi_\pi - 1) - (\beta+1) < 0$.

Note that from the first part of the proof we have

$$\phi_x + \phi_\pi\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\mu - \frac{\beta\sigma}{1+\alpha(\omega-1)} < 0.$$

Then,

$$\phi_x \frac{1+\alpha(\omega-1)}{\sigma} + \mu\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)(\phi_\pi - 1) - (\beta+1) < 0$$

since we assume positive parameters. Thus the first case drops and we only consider the second case as binding condition.

Finally, using the condition (*iii*) in Theorem 2, we obtain (3.3.12):

$$\left(\phi_x + \phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\right)^2 + \left(\phi_x + \phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\right)\xi_1 - \phi_x\xi_2 = \xi_3,$$

where $\xi_1 = \left(\frac{\sigma}{1-\alpha+\alpha\omega} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) + \frac{\beta\sigma}{1-\alpha+\alpha\omega}\right)$, $\xi_2 = \frac{\beta^2\sigma}{1-\alpha+\alpha\omega}$ and

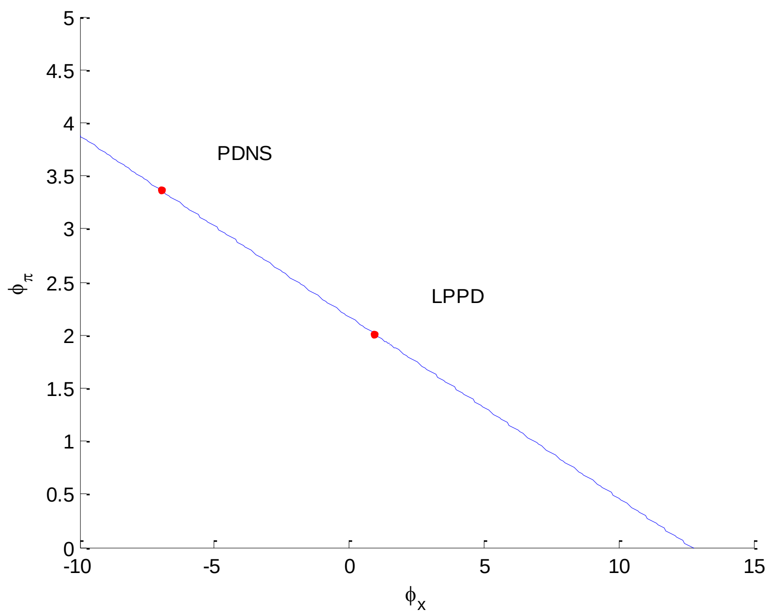
$$\xi_3 = \left(\frac{\beta\sigma}{1-\alpha+\alpha\omega}\right)^2 - \beta\left(\frac{\sigma}{1-\alpha+\alpha\omega}\right)^2. \quad (3.3.13)$$

For the sufficient part, assume the conditions (*i*)-(*iii*) hold true. Then, we need to show that the transversality condition holds which can be verified numerically. □

Therefore, the existence of a Hopf bifurcation in the New Keynesian model under backward looking Taylor rule (3.3.4) is theoretically possible in an open economy environment as shown in Proposition (3.3.2).

Furthermore, we numerically detect a Period Doubling bifurcation at $\phi_x = 1.91$. Over the interval $0.1464 < \phi_x < 1.9145$, one eigenvalue is inside the unit circle while the other two are outside the unit circle. At $\phi_x = 1.9145$, one eigenvalue is on the unit circle while the other two are outside the unit circle. For $1.9145 < \phi_x$, all eigenvalues are outside the unit circle, one of them is in absolute value. Starting from the point $\phi_x = 1.91$, we construct the Period Doubling bifurcation boundary by varying ϕ_x and ϕ_π simultaneously, as shown in Figure (3.3.2). Note that along the bifurcation boundary, the positive values of the bifurcation parameter ϕ_x lie between 0 and around 13. As the magnitude of the reaction of central bank to inflation, that is ϕ_π, increases, even the smaller values of parameter ϕ_x would be sufficient to cause Period Doubling bifurcation under backward looking Taylor rule.

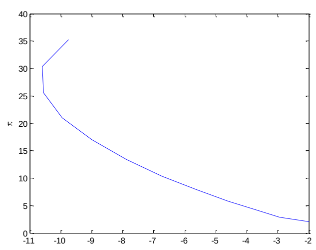
Figure 3.3.2: Period Doubling bifurcation boundary for ϕ_x in Model (3.3.1).



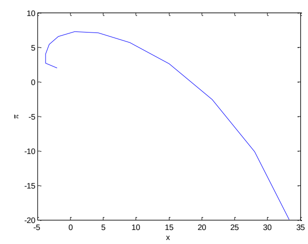
At this second round, while varying both parameters ϕ_x and ϕ_π simultaneously, our numerical analysis detect a codimension-2 Fold-Flip bifurcation at $(\phi_x, \phi_\pi) = (0.94, 2.01)$ and a Flip-Hopf bifurcation at $(\phi_x, \phi_\pi) = (-6.98, 3.36)$.

Decreasing the value of ω results in a higher value of the Period Doubling bifurcation parameter, except when $\alpha = 0$ at which changes in ω does not make any difference. On the other hand, decreasing the value of α results in a lower value of the bifurcation parameter in absolute value, except when $\omega = 1$ at which changes in α does not make any difference. Hence lowering ω and raising α increase the value of the bifurcation parameter. Figure (3.3.3) illustrates three phase plots for different set of parameter values in Model (3.3.1).

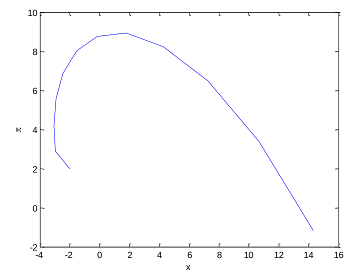
Figure 3.3.3: Phase diagrams for various values of parameters in Model (3.3.1)



$(\alpha, \omega, \phi_x, \phi_\pi) = (1, 0.5, 0.125, 1.5)$



$(\alpha, \omega, \phi_x, \phi_\pi) = (0.4, 1, -0.5, 1.5)$



$(\alpha, \omega, \phi_x, \phi_\pi) = (1, 1.5, -0.5, 1.5)$

On the other hand, considering the policy parameter ϕ_π as the potential source of bifurcation while keeping the other parameters constant at their benchmark values, our numerical analysis indicates a Period Doubling bifurcation at $\phi_\pi = 11.87$. We find Period Doubling bifurcation at relatively larger values of the parameter ϕ_π, but still close to the feasible subset of parameter space according to Bullard and Mitra (2002). Lowering ω and raising α increase the value of the bifurcation parameter ϕ_π.

3.3.2 Under Pure Backward Looking Inflation Targeting Rule:

Consider the following model in which the first two equations describe the economy while the third equation is the instrument rule employed by the central bank for the monetary policy:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \tag{3.3.1}$$

$$x_t = E_t x_{t+1} - \frac{1-\alpha+\alpha\omega}{\sigma}\left(r_t - E_t\pi_{t+1} - \overline{r}_t\right) \tag{3.3.2}$$

$$r_t = \overline{r}_t + \phi_\pi \pi_{t-1} \tag{3.3.14}$$

Equation (3.3.14) formulates the policy rule as a pure backward looking inflation targeting in which the nominal interest rate is set according to the inflation rate realized in period $t-1$. Moving the equation (3.3.14) one period forward, adding expectations, rearranging the terms, and then defining $y_t = \left[x_t, \pi_t, r_t\right]'$, we can write the system in normal form $E_t y_{t+1} = Cy_t$,

$$E_t y_{t+1} = Cy_t + \begin{bmatrix} -\dfrac{1+\alpha(\omega-1)}{\sigma}\overline{r}_t \\ 0_t \\ E_t\overline{r}_{t+1} \end{bmatrix} \tag{3.3.15}$$

where

$$C = \begin{bmatrix} \mu\left(\dfrac{1}{\beta} + \varphi\dfrac{1+\alpha(\omega-1)}{\beta\sigma}\right)+1 & -\dfrac{1+\alpha(\omega-1)}{\beta\sigma} & \dfrac{1+\alpha(\omega-1)}{\sigma} \\ -\dfrac{\mu}{\beta}\left(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) & \dfrac{1}{\beta} & 0 \\ 0 & \phi_\pi & 0 \end{bmatrix}.$$

It is the Jacobian matrix C that is relevant for determinacy and bifurcation analysis. The characteristic polynomial of the Jacobian matrix C is,

$$p(\lambda) = \det\left(C - \lambda I_3\right) = \lambda^3 - a_2\lambda^2 + a_1\lambda + a_0$$

where

$$a_0 = \frac{\phi_\pi \mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right), \tag{3.3.16}$$

$$a_1 = \frac{1}{\beta}, \tag{3.3.17}$$

and $$a_2 = -\left(\frac{1}{\beta}+\frac{\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)+1\right). \tag{3.3.18}$$

As usual, we begin our analysis by examining the determinacy conditions of the system (3.3.15). Note that there are three endogeneous variables (the rate of inflation, π_t, the output gap, x_t, and the nominal interest rate, r_t) and one pre-determined variable (π_{t-1}). Following Blanchard and Kahn (1980), the system has a unique, stationary equilibrium solution if and only if the number of eigenvalues outside the unit circle is equal to the number of forward looking (non-predetermined) variables, which is two ($E_t x_{t+1}$ and $E_t \pi_{t+1}$) in our case. Proposition (3.3.3) characterizes the necessary and sufficient conditions for the determinacy.

Proposition 3.3.3: Under pure backward looking inflation targeting rule, the open economy New Keynesian model (3.3.15) has a unique stationary equilibrium if and only if

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(\phi_\pi - 1) > 0, \tag{3.3.19}$$

and

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(\phi_\pi - 1) < \frac{2(1+\beta)\sigma}{1-\alpha+\alpha\omega}. \tag{3.3.20}$$

Proof: Following Bullard and Mitra (2002), the number of the roots with the same sign can be checked by Descartes' rule of signs theorem. In the positive case, we keep the signs on λ unchanged and count the sign changes in the characteristic polynomial from one term to the next, which is two:

$$p(\lambda)=+\lambda^3-a_2\lambda^2+a_1\lambda+a_0$$

Hence the maximum number of positive roots for the characteristic polynomial is two, which might be complex conjugate.

In the negative case, by changing the sign on λ, we look at $p(-\lambda)$ which gives,

$$p(-\lambda)=-\lambda^3-a_2\lambda^2-a_1\lambda+a_0.$$

Notice that we have only one sign change in the negative case, which implies that there is at most one negative root, and therefore exactly one.

Thus, by Descartes' rule of signs, the characteristic polynomial has a negative root besides having either two positive roots or a pair of complex conjugate roots.

We can also evaluate $p(x)$ at $x=1$ for the roots with positive sign and at $x=-1$ for the roots with negative sign to see whether the negative and positive roots lie outside (or inside) the unit circle. Note that for $\lambda=0$, we have

$$p(0)=\frac{\phi_\pi\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)>0,$$

which is trivially satisfied since all the parameters are assumed to be positive.

For $\lambda=-1$, we have

$$p(-1)=-2\left(1+\frac{1}{\beta}\right)+(\phi_\pi-1)\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\frac{1-\alpha+\alpha\omega}{\beta\sigma}<0,$$

for the given values of the parameters in Gali and Monacelli (2005).

For $\lambda=1$, we have

$$p(1)=\mu\left(\frac{1}{\beta}+\varphi\frac{1+\alpha(\omega-1)}{\beta\sigma}\right)(\phi_\pi-1)>0,$$

for the given values of the parameters in Gali and Monacelli (2005). Hence,

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(\phi_\pi-1)>0\,. \qquad \square$$

Conditions (3.3.19) and (3.3.20) show that a sufficiently active policy rule with $\phi_\pi > 1$ leads to a determinate equilibrium. We can numerically verify whether Proposition (3.3.3) holds for the given values of the parameters in Gali and Monacelli Model. The uniqueness of a solution can be easily checked by computing the Jacobian matrix, which is

$$C=\begin{bmatrix} 1.3434 & -1.0101 & 1 \\ -0.3434 & 1.0101 & 0 \\ 0 & 1.5 & 0 \end{bmatrix}$$

with eigenvalues $\lambda_1 =$ 1.3217 + 0.1720i , $\lambda_2 =$1.3217 - 0.1720i and $\lambda_3 =$ -0.2900 . Note that one solution is real and inside the unique circle in absolute value, while the radius of the two complex conjugate solutions are outside the unit circle with $R = 1.3534$. Recalling Blanchard and Kahn (1980), since the number of eigenvalues outside the unit circle is equal to the number of forward looking variables, the system (3.3.15) has a unique, stationary equilibrium solution.

In order for a 3-dimensional system to exhibit a Hopf bifurcation, it should have a real root and a pair of complex conjugate roots on the unit circle. Using Theorem 2, with respect to the parameter ϕ_π , the conditions for the existence of a Hopf bifurcation is presented in the following Proposition.

Proposition 3.3.4: The system (3.3.15) exhibits a Hopf bifurcation if and only if the following transversality condition

$\left.\frac{\partial|\lambda_i(\phi)|}{\partial\phi_j}\right|_{\phi_j=\phi_j^*} \neq 0$ holds and the following conditions are satisfied:

i) $\phi_\pi \mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)-\frac{\beta\sigma}{1+\alpha(\omega-1)}<0$, (3.3.21)

ii) $\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(1-\phi_\pi)<0$, (3.3.22)

iii) $\phi_\pi^2\xi_0+\phi_\pi\xi_1=\xi_2$. (3.3.23)

Proof: For the necessary part of the proof, we apply the conditions in Theorem 2. Using the condition (*i*) in Theorem 2, we obtain

$$|a_0|=\left|\frac{\phi_\pi\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)\right|<1.$$

Using the definition of absolute value, we have either

$$\phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)-\frac{\beta\sigma}{1+\alpha(\omega-1)}<0$$

if $\frac{\phi_\pi\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)\geq 0$, or

$$\phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)+\frac{\beta\sigma}{1+\alpha(\omega-1)}>0$$

if $\frac{\phi_\pi\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)<0$.

Since the parameters are assumed to be positive the second case is redundant and we only consider the first case as a binding condition.

Using the condition (*ii*) in Theorem 2, we obtain

$$|a_0+a_2|<1+a_1.$$

Again, by definition of absolute value, we have either

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(\phi_\pi-1)-\frac{2\sigma(1+\beta)}{1+\alpha(\omega-1)}<0$$

if $\frac{\phi_\pi\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)-\left(\frac{1}{\beta}+\frac{\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)+1\right)\geq 0$, or

$$\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(1-\phi_\pi)<0$$

if $\frac{\phi_\pi\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)-\left(\frac{1}{\beta}+\frac{\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)+1\right)<0$.

Note that from the first part of the proof we have $\phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)-\frac{\beta\sigma}{1+\alpha(\omega-1)}<0$.

Hence $\frac{\phi_\pi\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)-\left(\frac{1}{\beta}+\frac{\mu}{\beta}\left(1+\varphi\frac{1+\alpha(\omega-1)}{\sigma}\right)+1\right)<0$ since we assume positive values for the parameters. Thus the first case is redundant and we only consider the second case as a binding condition.

Finally, using the condition (*iii*) in Theorem 2, we obtain

$$a_1-a_0a_2=1-a_0^2,$$

which yields

$$\phi_\pi^2\xi_0+\phi_\pi\xi_1=\xi_2$$

where

$\xi_0=\mu^2\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)^2$, $\xi_1=\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\left(\sigma_\alpha+\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)+\beta\sigma_\alpha\right)$ and

$$\xi_2=\beta^2\left(\frac{\sigma}{1-\alpha+\alpha\omega}\right)^2-\beta\left(\frac{\sigma}{1-\alpha+\alpha\omega}\right)^2.$$

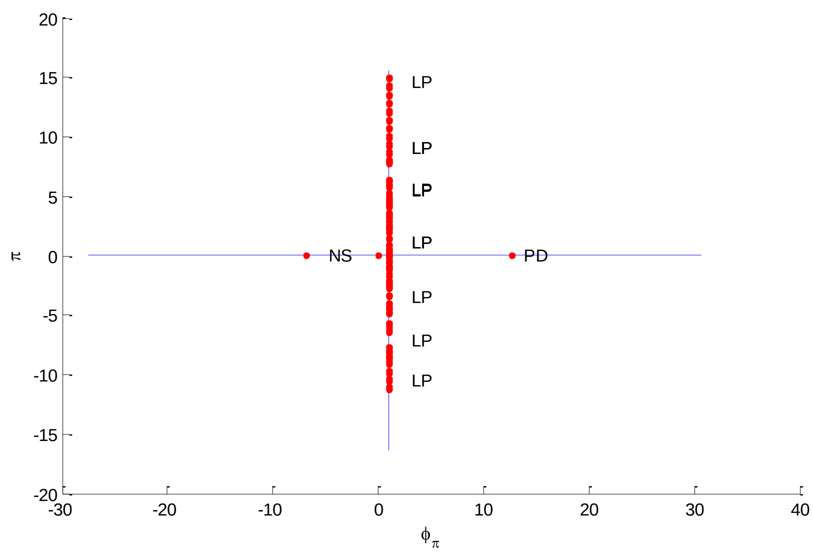
For the sufficient part, assume that the conditions (*i*)-(*iii*) hold true for the parameter ϕ_π^*.

Then, we need to show that the transversality condition also holds at $\phi_\pi = \phi_\pi^*$, which can be verified numerically. □

Therefore, the existence of a Hopf bifurcation in the New Keynesian model under pure backward looking inflation targeting rule (3.3.14) is theoretically possible in an open economy environment, as shown in Proposition (3.3.4).

For the numerical analysis, we examine the Jacobian matrix *C* while keeping the structural parameters constant at their baseline values and altering the policy parameter ϕ_π over a certain range. Our numerical analysis indicates a Period Doubling bifurcation at $\phi_\pi = 12.7$ and a branching point at $\phi_\pi = 1$ given the benchmark values of the parameters. We also numerically detected a series of Limit Point bifurcations emerging for different values of inflation at $\phi_\pi = 1$ as illustrated in Figure (3.3.4).

Figure 3.3.4: Period Doubling and Limit Point bifurcations in Model (3.3.2)



On the other hand, lowering ω and raising α increase the value of the bifurcation parameter ϕ_π. But, lowering ω while α = 1 makes bifurcations dissapear. We also found that there is no bifurcation in any type at $(\omega, \alpha) = (0, 1)$.

3.3.3 Under Backward Looking Taylor Rule with Interest Rate Smoothing:

Consider the following model in which the first two equations describe the economy while the third equation is the instrument rule followed by the central bank for the monetary policy:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t \tag{3.3.1}$$

$$x_t = E_t x_{t+1} - \frac{1-\alpha+\alpha\omega}{\sigma} \left(r_t - E_t \pi_{t+1} - \overline{r}_t \right) \tag{3.3.2}$$

$$r_t = \overline{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1} + \phi_r r_{t-1} \tag{3.3.24}$$

Equation (3.3.24) describes the policy rule as a backward looking policy rule in which the nominal interest rate is set according to the previous period's inflation rate, output gap and policy rate. Moving the equation (3.3.24) one period forward, adding expectations, rearranging the terms, and then defining $y_t = [x_t, \pi_t, r_t]'$, the system can be written in the form $E_t y_{t+1} = C y_t$,

$$E_t y_{t+1} = C y_t + \begin{bmatrix} -\frac{1-\alpha+\alpha\omega}{\sigma} \overline{r}_t \\ 0 \\ E_t \overline{r}_{t+1} \end{bmatrix}, \tag{3.3.25}$$

where $C = \begin{bmatrix} \mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+1 & -\frac{1-\alpha+\alpha\omega}{\beta\sigma} & \frac{1-\alpha+\alpha\omega}{\sigma} \\ -\mu\left(1+\varphi\frac{1-\alpha+\alpha\omega}{\sigma}\right) & \frac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & \phi_r \end{bmatrix}$.

It is the coefficient matrix *C* that is relevant for determinacy and bifurcation analysis. The characteristic polynomial for the system (3.3.25) is in the following form:

$$p(\lambda) = \det(C - \lambda I_3) = \lambda^3 - a_2\lambda^2 + a_1\lambda + a_0 = 0$$

where

$$a_0 = \phi_x \frac{1-\alpha+\alpha\omega}{\beta\sigma} - \phi_r \frac{1}{\beta} + \phi_\pi \mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right),$$

$$a_1 = \phi_r\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right) - \phi_x \frac{1-\alpha+\alpha\omega}{\sigma} + \frac{1}{\beta},$$

and $a_2 = -\left(\phi_r + \frac{1+\mu}{\beta} + \varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)$.

Following Farebrother (1973) and Gandolfo (1996), a third order dynamical system whose characteristic polynomial is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where $a_i \in \mathbb{R}$ for all $i = 1,2,3$ is stable if and only if

$$1 + a_2 + a_1 + a_0 > 0,$$
$$1 - a_2 + a_1 - a_0 > 0,$$
$$1 - a_1 + a_2 a_0 - a_0^2 > 0.$$

Using the baseline values of the parameters given in Gali and Monacelli (2005), as it could be checked numerically, the system (3.3.25) is found to be stable if the third condition satisfies $\phi_r > 2.7795$. Otherwise, the system (3.3.25) is not stable given the baseline values of parameters.

Based on Theorem 2, the following Proposition states the conditions for the system (3.3.25) to exhibit a Hopf bifurcation.

Proposition 3.3.5: The system (3.3.25) undergoes a Hopf bifurcation if and only if the transversality condition $\left.\frac{\partial\left|\lambda_i(\phi)\right|}{\partial\phi_j}\right|_{\phi_j=\phi_j^*} \neq 0$ holds and the following conditions are satisfied:

i) $$\left|\frac{\phi_x - \phi_r\frac{\sigma}{1-\alpha+\alpha\omega} + \phi_\pi\left(\frac{\sigma\mu}{1+\alpha(\omega-1)} + \varphi\mu\right)}{\frac{\beta\sigma}{1-\alpha+\alpha\omega}}\right| < 1,$$

1. $\phi_x - \phi_r\xi_2 + \phi_\pi\xi_3 < \frac{\beta\sigma}{1-\alpha+\alpha\omega}$,

2. $\phi_r < \phi_x\xi_2 + \phi_\pi\xi_1 + \beta$,

ii)

$$\left|\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma} - \phi_r\frac{1}{\beta} + \phi_\pi\mu\left(\frac{1}{\beta} + \varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right) - \left(\phi_r + \frac{1+\mu}{\beta} + \varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma} + 1\right)\right| < 1 + \phi_r\left(\frac{1+\mu}{\beta} + \varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma} + 1\right) - \phi_x\frac{1-\alpha+\alpha\omega}{\sigma} + \frac{1}{\beta}$$

1. $\phi_x\xi_2 + \phi_\pi\xi_1 - (1+\phi_r)\xi_0 < 0$,

2. $\phi_x\xi_3 - \xi_4(\phi_\pi + \phi_r - 1) < 0$,

iii)

$$\phi_r\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)-\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1}{\beta}+\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\left(\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right)\left(\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)$$
$$=1-\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\left(\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right)^2.$$

Proof: For the necessary part, we apply Theorem 2.

By the condition *(i)* in Theorem 2, we obtain

$$|a_0|=\left|\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right|<1.$$

Using the definition of absolute value, we have either

1. $\phi_x-\phi_r\dfrac{\sigma}{1-\alpha+\alpha\omega}+\phi_\pi\left(\dfrac{\sigma\mu}{1+\alpha(\omega-1)}+\varphi\mu\right)<\dfrac{\beta\sigma}{1-\alpha+\alpha\omega}$

if $\phi_x\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\dfrac{1}{\beta}+\phi_\pi\mu\left(\dfrac{1}{\beta}+\varphi\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\geq 0$, or

2. $\phi_r<\phi_x\dfrac{1-\alpha+\alpha\omega}{\sigma}+\phi_\pi\left(\mu+\varphi\mu\dfrac{1-\alpha+\alpha\omega}{\sigma}\right)+\beta$

if $\phi_x\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\dfrac{1}{\beta}+\phi_\pi\mu\left(\dfrac{1}{\beta}+\varphi\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}\right)<0$.

From the condition *(ii)* in Theorem 2, we obtain

$|a_0+a_2|<1+a_1$.

Then, by the definition of absolute value, we have either

1. $\phi_x(1+\beta)+\phi_\pi\left(\dfrac{\sigma\mu}{1+\alpha(\omega-1)}+\varphi\mu\right)-(1+\phi_r)\left(\dfrac{\sigma(2+2\beta+\mu)}{1-\alpha+\alpha\omega}+\varphi\mu\right)<0$

if $\phi_x\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\left(\dfrac{1}{\beta}+1\right)+\phi_\pi\left(\dfrac{\mu}{\beta}+\varphi\mu\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}\right)-\left(\dfrac{1+\mu}{\beta}+\varphi\mu\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)\geq 0$, or

2. $\phi_x(1+\beta)-\left(\frac{\sigma\mu}{1+\alpha(\omega-1)}+\varphi\mu\right)(\phi_\pi+\phi_r-1)<0$

if $\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\left(\frac{1}{\beta}+1\right)+\phi_\pi\left(\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)-\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)<0$.

Finally, by the condition *(iii)* in Theorem 2, we have

$a_1-a_0a_2=1-a_0^2$,

which implies

$$\phi_r\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)-\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1}{\beta}+\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right)\left(\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)$$
$$=1-\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\left(\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right)^2.$$

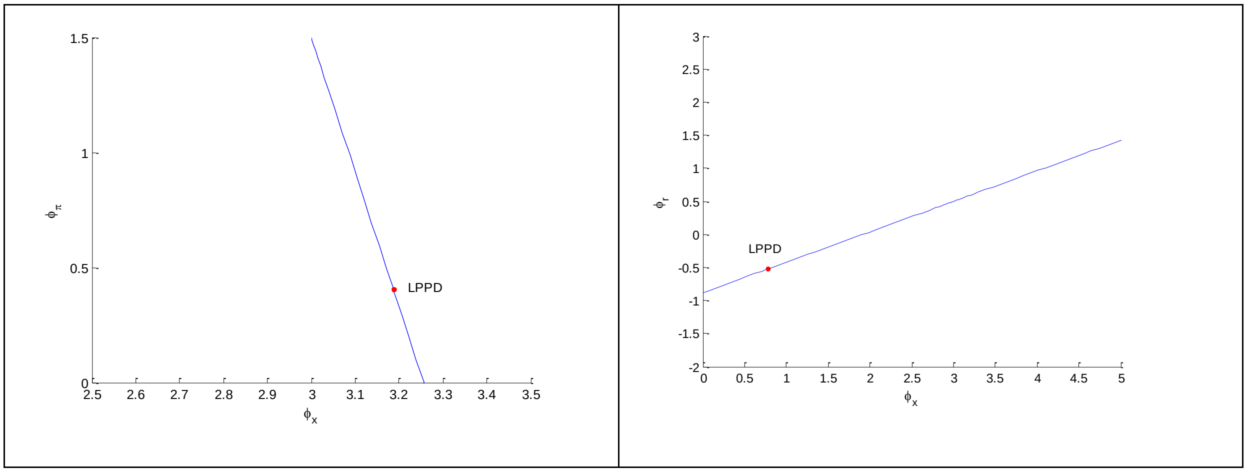
For the sufficient part, assume the conditions *(i)*-*(iii)* hold true. Then, we need to show that the transversality condition holds which can be verified numerically. □

Proposition (3.3.5) proves that the Gali and Monacelli Model under the backward looking Taylor rule with interest rate smoothing is likely to undergo a Hopf bifurcation. In order to determine numerically whether there are parameter values such that a bifurcation is possible, we examine the Jacobian matrix *C*. Given the benchmark values of the parameters and taking $\phi_r=0.5$, a Period Doubling bifurcation is detected numerically at $\phi_x=3$. When $\phi_r=1$, on the other hand, the Period Doubling bifurcation occurs at $\phi_x=4.09$.

For the closed economy case under the same policy rule, Barnett and Duzhak (2010) report a Period Doubling bifurcation at $\phi_x=5.7$, assuming $\phi_r=0.9$. Comparing the results from Barnett and Duzhak (2010) with the open economy case, however, does not give us a clear statement about whether or not the openness makes the New Keynesian model more sensitive to bifurcations since Gali and Monacelli Model incorporates a wider set of parameters including some deep parameters relevant to the open economy. The fact that both studies use different set of calibration values for the parameters makes a direct comparison even harder. Airaudo and Zanna (2005), using a money-in-utility function and assuming non-separability, show that cyclical and chaotic dynamics are more likely the more open the economy and the higher the exchange rate pass-through into import prices. They also show that the existence of cyclical and chaotic dynamics depend on some open economy features and is in general robust to different timings in the policy rule. Examining the effects of a change in the parameters α and ω in the Gali and Monacelli Model, on the other hand, does not support that argument in our model.

Starting from this bifurcation point, we construct the bifurcation boundary by varying ϕ_x and ϕ_π, and then ϕ_x and ϕ_r simultaneously as shown in Figure (3.3.5). Note that in (ϕ_x, ϕ_π)-space, the bifurcation boundary lies between $\phi_x = 3$ and $\phi_x = 3.25$, which implies that Period Doubling bifurcation occurs for a very limited set of values of parameter ϕ_x no matter what value parameter ϕ_π takes. This is not the case in (ϕ_r, ϕ_x)-space, as shown in the second part of the Figure (3.3.5). Bifurcation parameter ϕ_x varies more elastically in response to changes in parameter ϕ_r to give rise to a Period Doubling bifurcation.

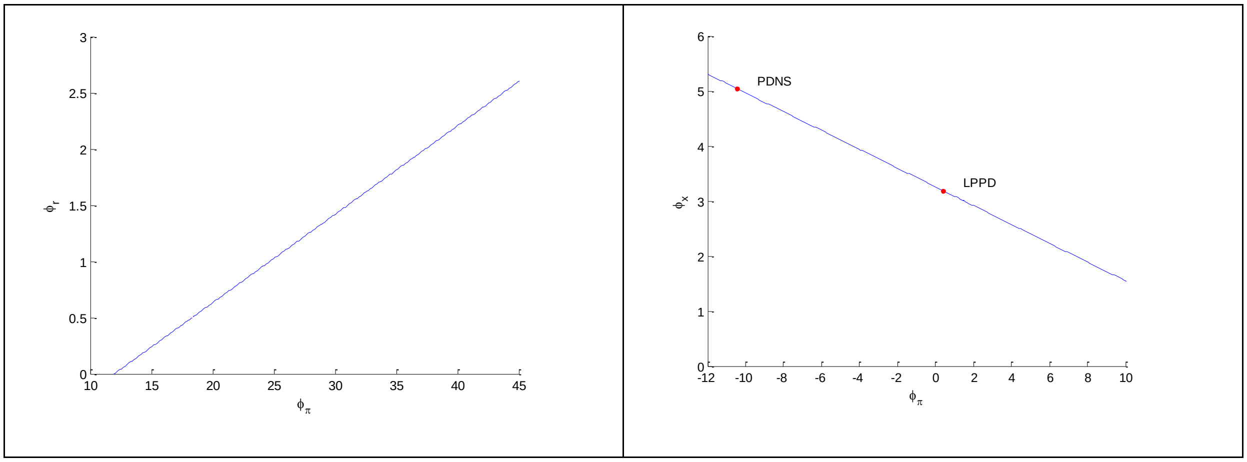
Figure 3.3.5: Period Doubling bifurcation boundary diagrams for ϕ_x in (ϕ_r, ϕ_x) and (ϕ_x, ϕ_π)-spaces in Model (3.3.3)



At this second step, while varying two parameters ϕ_x and ϕ_π or ϕ_r simultaneously, our numerical analysis indicates codimension-2 Fold-Flip bifurcations at $(\phi_x, \phi_\pi) = (0.41, 3.19)$ and at $(\phi_x, \phi_r) = (0.78, -0.52)$ as well as Flip-Hopf bifurcations at $(\phi_x, \phi_\pi) = (\text{-}10.44, 5.04)$ and $(\phi_x, \phi_r) = (\text{-}0.74, \text{-}1.23)$. For ϕ_x being the bifurcation parameter, lowering ω and raising α increase the value of the bifurcation parameter. The bifurcation dissapears at $(\alpha, \omega) = (1, 0)$.

When we take the parameter ϕ_π into consideration as a potential source of bifurcation, we find a branching point at $\phi_\pi = 0.47$ and a Period Doubling bifurcation at $\phi_\pi = 18.3$. Empirically, this is not a feasible value for a bifurcation. It requires an extreme response of the Central Bank to changes in inflation rate which is beyond any convention. The bifurcation boundary diagram for the parameter ϕ_π is shown in Figure (3.3.6). Notice that in (ϕ_r, ϕ_π)-space, as shown in the first part of the Figure (3.3.6), bifurcation parameter ϕ_π varies in the same direction in response to changes in parameter ϕ_r to generate a Period Doubling bifurcation. The positive relationship between ϕ_π and ϕ_r is different than our previous findings. In (ϕ_x, ϕ_π)-space, the bifurcation boundary shows a negative relationship between the parameters ϕ_π and ϕ_x, implying that for higher values of ϕ_x even the smaller values of the parameter ϕ_π would be sufficient to yield a Period Doubling bifurcation.

Figure 3.3.6: Period Doubling bifurcation boundary for ϕ_π in (ϕ_r, ϕ_π) and (ϕ_x, ϕ_π)-spaces in Model (3.3.3)



Again, at the second step, while varying both parameters ϕ_r and ϕ_π simultaneously in order to construct the bifurcation boundary diagram, our numerical analysis indicates a codimension-2 Fold-Flip bifurcation at $(\phi_r, \phi_\pi) = (1.797, -0.8)$ and a Flip-Hopf bifurcation at $(\phi_r, \phi_\pi) = (-0.296, -0.966)$.

Varying ϕ_x and ϕ_π simultaneously, on the other hand, indicates a codimension-2 Fold-Flip bifurcation at $(\phi_x, \phi_\pi) = (0.4, 3.19)$ as well as a Flip-Hopf bifurcation at $(\phi_x, \phi_\pi) = (-10.44, -5.04)$. Lowering ω and raising α increase the value of the bifurcation parameter ϕ_π.

3.4 Gali and Monacelli Model under Hybrid Policy Rules:

3.4.1 Under Hybrid Taylor Rule:

In this specification, the current rate of inflation variable in the standard Taylor rule is replaced by the next period's forecasted/expected rate of inflation. Barnett and Duzhak (2008, 2010) examine this rule in the bifurcation analysis of closed economy New Keynesian models. Clarida, Gali and Gertler (2000) employ this version of the policy rule, among others, in order to analyze the pre-Volcker and Volcker-Greenspan era. Thurston (2010), however, claims that this modification has only minor effects through an additional condition on uniqueness and lifting the loci of optimizing ϕ_π, which ensure an intercept on the ϕ_π axis greater than unity. Bofinger and Mayer (2006), on the other hand, argue that the hybrid Taylor rule lacks the simplicity of simple policy rules and should be refused for practical reasons.

Consider the following model in which the first two equations describe the economy while the third equation represents the monetary policy rule followed by the central bank:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \tag{3.4.1}$$

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(r_t - E_t\pi_{t+1} - \overline{r}_t\right) \tag{3.4.2}$$

$$r_t = \overline{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x x_t \tag{3.4.3}$$

Equation (3.4.3) describes the policy rule in a way that the nominal interest rate is set according to the expected inflation rate and the current output gap. Substituting (3.4.3) for $r_t - \overline{r}_t$ into (3.4.2), we obtain a reduced system of first order difference equations in terms of inflation and output gap, which could be written in normal form $E_t y_{t+1} = C y_t$, as follows:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = C \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \tag{3.4.4}$$

where

$$C = \begin{bmatrix} \dfrac{\beta\phi_x + \mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(1-\phi_\pi)}{\dfrac{\beta\sigma}{1+\alpha(\omega-1)}} + 1 & \dfrac{(\phi_\pi - 1)(1+\alpha(\omega-1))}{\beta\sigma} \\ -\dfrac{\mu}{\beta}\left(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) & \dfrac{1}{\beta} \end{bmatrix}.$$

Clearly, $x_t = \pi_t = 0$ for all *t* constitutes an equilibrium solution to the system (3.4.4).

The eigenvalues of the coefficient matrix *C*, which is also the Jacobian matrix of the system (3.4.4), are the roots of the characteristic polynomial

$$p(\lambda) = \det(C - \lambda I) = \lambda^2 - a_1\lambda + a_0 = 0,$$

where

$$a_0 = \frac{(\alpha\omega - \alpha + 1)\phi_x}{\beta\sigma} + \frac{1}{\beta},$$

and $a_1 = 1 + \dfrac{1+\mu-\mu\phi_\pi}{\beta} + \dfrac{(\alpha\omega-\alpha+1)(\beta\phi_x + \mu\varphi - \mu\varphi\phi_\pi)}{\beta\sigma}$,

which yields

$$\lambda_{1,2} = \left(1 + \frac{1+\mu-\mu\phi_\pi}{\beta} + \frac{(\alpha\omega-\alpha+1)(\beta\phi_x + \mu\varphi - \mu\varphi\phi_\pi)}{\beta\sigma}\right)$$
$$\pm\sqrt{\left(1 + \frac{1+\mu-\mu\phi_\pi}{\beta} + \frac{(\alpha\omega-\alpha+1)(\beta\phi_x + \mu\varphi - \mu\varphi\phi_\pi)}{\beta\sigma}\right)^2 - 4\left(\frac{(\alpha\omega-\alpha+1)\phi_x}{\beta\sigma} + \frac{1}{\beta}\right)}.$$

The following Proposition characterizes the necessary and sufficient conditions for the determinacy, following Bullard and Mitra (2002).

Proposition 3.4.1: Under the Hybrid Taylor Rule as specified in (3.4.3), the open economy New Keynesian model (3.4.4) has a unique stationary equilibrium if and only if

$$\phi_x > \frac{\sigma(\beta-1)}{1+\alpha(\omega-1)} \tag{3.4.5}$$

and

$$(1-\beta)\phi_x + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) > 0 \tag{3.4.6}$$

Proof: Following the methodology suggested by Bullard and Mitra (2002), both eigenvalues are outside the unit circle if and only if

$|a_0| < 1$ and

$|a_1| < 1 + |a_0|$.

From the first inequality, we derive

$$1 - \left(\frac{\frac{\sigma}{1+\alpha(\omega-1)} + \phi_x}{\frac{\beta\sigma}{1+\alpha(\omega-1)}}\right) > 0,$$

which gives

$$\phi_x > \frac{\sigma(1-\beta)}{1+\alpha(\omega-1)}.$$

From the second inequality, we have

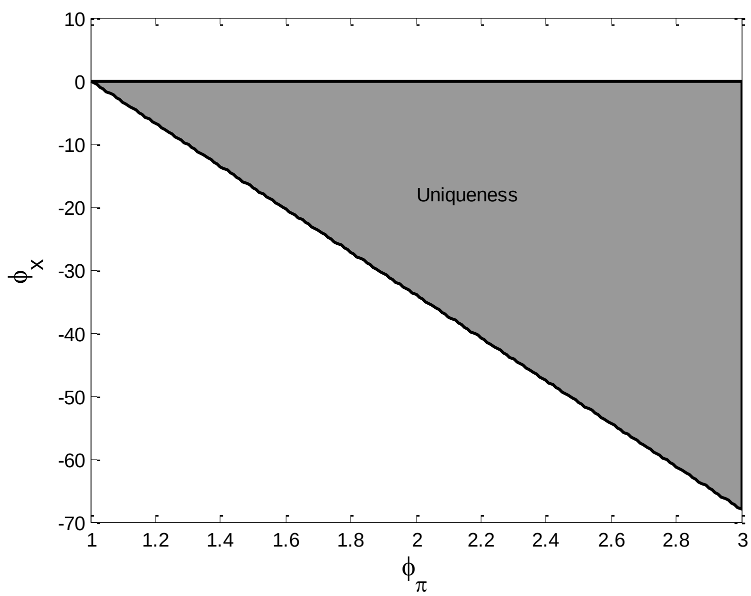
$$1 - a_1 + a_0 > 0,$$

which implies

$$(1-\beta)\phi_x + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(\phi_\pi - 1) > 0. \qquad \square$$

The Proposition (3.4.1) suggests that the policy maker is able to reach the uniquely determined stationary equilibrium by choosing feasible values for the policy parameters. Since $\beta \in (0,1)$, the condition (3.4.5) can be easily satisfied for positive values of the parameter ϕ_x, which makes it redundant. Hence, the condition (3.4.6) is the critical one regarding determinacy. Any value of the inflation parameter greater than unity, that is $\phi_\pi > 1$, accompanied by a non-negative output parameter ϕ_x would be sufficient to satisfy the condition (3.4.6). Nevertheless, as Thurston (2010) points out, a negative ϕ_x may sometimes be consistent with uniqueness and optimality, even though a negative value is not necessary for that purpose. Figure (3.4.1) is constructed based on the condition (3.4.6) and shows the regions of unique and multiple equilibria.

Figure 3.4.1: Determinacy diagram for the Hybrid Taylor rule



As the condition (3.4.6) is the critical one regarding determinacy, both eigenvalues will be outside the unit circle if and only if $(1-\beta)\phi_x + (\phi_\pi - 1)\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)\mu > 0$. Note that, since $\phi_x, \phi_\pi > 0$ by assumption, $\phi_\pi > 1$ would be sufficient for the condition (3.4.6) to hold. This implies the Taylor rule. If the central bank raises the nominal interest rate up more than one-for-one when inflation rises, the real interest rate also increases and that would be sufficient to achieve a uniquely determined stationary equilibrium.

As shown in Gandolfo (1996), in order to examine the nature of the eigenvalues we first check the sign of the discriminant $\Delta \equiv a_1^2 - 4a_0$. If the discriminant of the quadratic equation is strictly negative, that is if

$$\Delta \equiv a_1^2 - 4a_0 = \left(\frac{\beta + \mu - \mu\phi_\pi + 1}{\beta} - \frac{(\alpha\omega - \alpha + 1)(\beta\phi_x + \varphi\mu - \varphi\mu\phi_\pi)}{\beta\sigma} \right)^2 - 4\left(\frac{1}{\beta} + \frac{(\alpha\omega - \alpha + 1)\phi_x}{\beta\sigma} \right) < 0,$$

then the roots of the Jacobian matrix *C* will be complex conjugate in the form $\lambda_{1,2} = a \pm ib$, with $a, b \in \mathbb{R}$, $b \neq 0$ is the real part, and $i = +\sqrt{-1}$ is the imaginary unit.

It is algebraically quite cumbursome to identify the sign of the modulus of the eigenvalues. Using the baseline values of the parameters, however, we can numerically verify that the inside of the square roots in both eigenvalues is negative, hence the discriminant Δ is strictly negative. Therefore, the eigenvalues of the system (3.4.4) are complex conjugates,

$\lambda_{1,2} = a \pm ib$,

where

$$a = \frac{a_1}{2} = \frac{1}{2}\left(\frac{\beta + \mu - \mu\phi_\pi + 1}{\beta} - \frac{(\alpha\omega - \alpha + 1)(\beta\phi_x + \varphi\mu - \varphi\mu\phi_\pi)}{\beta\sigma} \right) \tag{3.4.7}$$

and

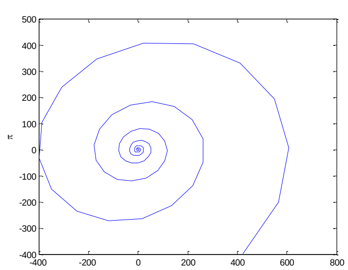
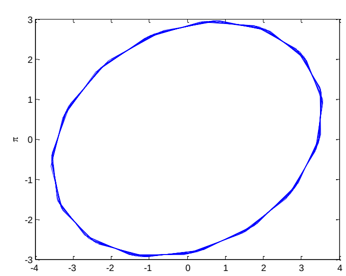
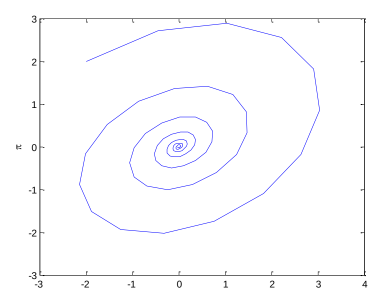
$$b=\frac{\sqrt{-\Delta}}{2}=\frac{1}{2}\sqrt{4\left(\frac{1}{\beta}+\frac{(\alpha\omega-\alpha+1)\phi_x}{\beta\sigma}\right)-\left(\frac{\beta+\mu-\mu\phi_\pi+1}{\beta}-\frac{(\alpha\omega-\alpha+1)(\beta\phi_x+\varphi\mu-\varphi\mu\phi_\pi)}{\beta\sigma}\right)^2}. \qquad (3.4.8)$$

In fact, for the benchmark values of the parameters, we numerically obtain the Jacobian matrix as

$$C=\begin{bmatrix} 0.9533 & 0.5051 \\ -0.3434 & 1.0101 \end{bmatrix}$$

having eigenvalues $\lambda_1 = 0.9817 + 0.4155\mathrm{i}$ and $\lambda_2 = 0.9817 - 0.4155\mathrm{i}$, and with modulus $R=\sqrt{(0.9817)^2+(0.4155)^2}=1.0660$. The Jacobian matrix C has complex conjugate eigenvalues with a radius greater than unity, implying that the system (3.4.4) has a unique, stationary equilibrium. Figure (3.4.2) illustrates various phase plots for different values of the parameter ϕ_x.

Figure 3.4.2: Phase diagrams for various values of ϕ_x in Model (3.4.1)

 $\phi_x = 0.1$ Divergent spiral trajectory	 $\phi_x = -0.01$ Limit cycle	 $\phi_x = -0.1$ Convergent spiral trajectory which starts at (2,-2) and ends at (0,0).

Having a pair of complex conjugate eigenvalues, we may expect to see a Hopf bifurcation if certain conditions are satisfied. Using Theorem 1, with respect to the policy parameter ϕ_x, the conditions for the existence of a Hopf bifurcation is stated in the following Proposition.

Proposition 3.4.2: The system (3.4.4) exhibits a Hopf bifurcation if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\sigma(\beta-1)}{1+\alpha(\omega-1)}. \tag{3.4.9}$$

Proof: Suppose the system (3.4.4) undergoes a Hopf bifurcation at (y^*, ϕ_x^*), where

$y^* = (x^*, \pi^*)$. Then, we need to show that $\Delta < 0$ and $\phi_x^* = \dfrac{\sigma(\beta-1)}{1+\alpha(\omega-1)}$. The existence of a Hopf bifurcation, by definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugates, the discriminant must be strictly negative, that is $\Delta < 0$. Note also that the existence of a Hopf bifurcation requires $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = +\sqrt{a^2+b^2} = 1$ by the first condition of Theorem 1. Rewriting the condition explicitly by substituting (3.4.7) and (3.4.8) into it, taking the square of both sides, and then solving for ϕ_x, we obtain the critical value of the parameter ϕ_x as in (3.4.9). Therefore, the first condition of Theorem 1 holds only if $\phi_x = \dfrac{\sigma(\beta-1)}{\alpha\omega-\alpha+1}$.

From the other side, suppose $\Delta < 0$ and $\phi_x = \dfrac{\sigma(\beta-1)}{\alpha\omega-\alpha+1}$. Substituting for ϕ_x^* into $\sqrt{a^2+b^2}$ yields $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = 1$, which is the first condition in Theorem 1. In order to show that the critical value of the parameter ϕ_x is actually a Hopf bifurcation parameter, we check the second condition in Theorem 1, which yields,

$$\left.\frac{d\left|\lambda_i\left(\phi_x\right)\right|}{d\phi_x}\right|_{\phi_x=\phi_x^*}=\left.\frac{d}{d\phi_x}\left(\sqrt{a^2+b^2}\right)\right|_{\phi_x=\phi_x^*}=\frac{1+\alpha\left(\omega-1\right)}{2\beta\sigma}\neq 0 \text{ for } i=1,2.$$

Thus, both conditions of Theorem 1 are satisfied and we have $\phi_x^*=\frac{\sigma\left(\beta-1\right)}{1+\alpha\left(\omega-1\right)}$. □

Proposition (3.4.2) shows formally that taking the policy parameter ϕ_x free to vary while keeping the other parameters constant at their baseline values, the system (3.4.4) is likely to go through a Hopf bifurcation. Therefore, the occurrence of a Hopf bifurcation is theoretically possible as shown in Proposition (3.4.2).

For the numerical analysis, we need to examine the Jacobian matrix *C* keeping the structural parameters constant at their baseline values while altering the policy parameter ϕ_x over a certain range. Our numerical analysis indicates a Period Doubling bifurcation at $\phi_x=-1.92$ as well as a Hopf bifurcation at $\phi_x=-0.01$, given the benchmark values of the system parameters. Under the hybrid Taylor rule, values of the bifurcation parameters are outside the feasible region of parameter space as we assume positive values for policy parameters. This also implies that the feasible set of parameter values for ϕ_x does not have a risk of bifurcation.

Figure 3.4.3: Period Doubling bifurcation boundary for ϕ_x in Model (3.4.1)

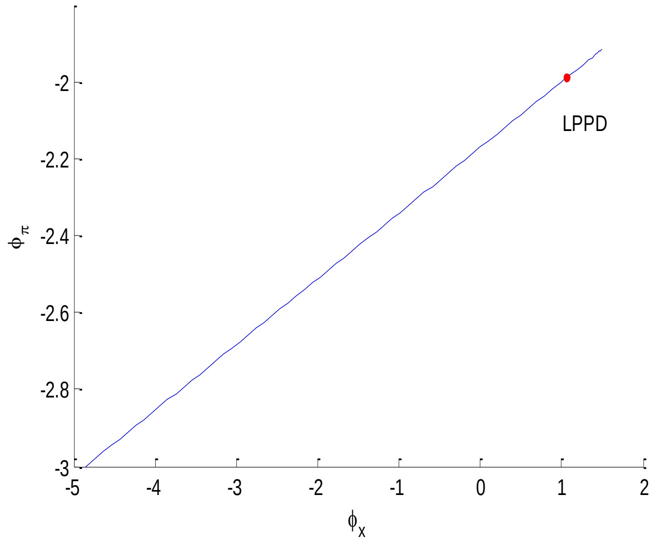
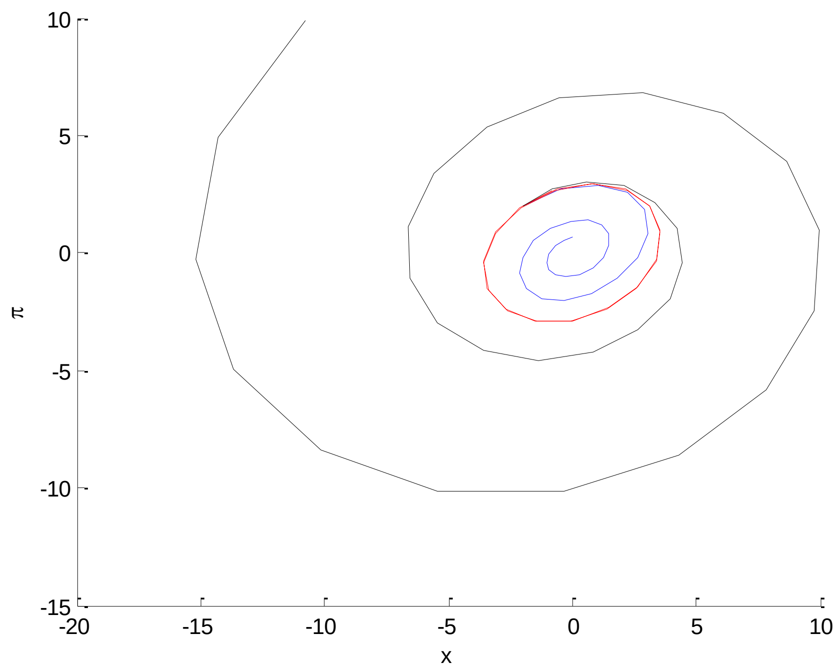


Figure (3.4.3) illustrates the combinations of parameters ϕ_x and ϕ_π that form the bifurcation boundary. Notice that, in (ϕ_π, ϕ_x)-space, bifurcation parameter ϕ_x varies in the same direction in response to changes in parameter ϕ_π to produce a Period Doubling bifurcation. As the policy maker's choice for ϕ_π increases, the higher values of ϕ_x are required to yield a Period Doubling bifurcation.

Recall that the transition from a static spiral fixed point to a periodic equilibrium or limit cycle as a parameter is smoothly varied is known as a Hopf bifurcation. At $\phi_x = -0.01$, while the system still have complex conjugate eigenvalues, $\lambda_1 = 0.9142 + 0.4053\text{i}$ and $\lambda_2 = 0.9142 - 0.4053\text{i}$, radius reaches unity, that is $R = \sqrt{(0.9142)^2 + (0.4053)^2} = 1$. For $\phi_x < -0.01$, the system has complex conjugate eigenvalues with $R < 1$, implying an unstable spiral. For $\phi_x > -0.01$, the system has complex conjugate eigenvalues with $R > 1$, implying a stable spiral. Thus, at $\phi_x = -0.01$ the system switches from a stable steady state to an unstable one.

Decreasing the value of ω results in a higher value of the Period Doubling bifurcation parameter in absolute value, except when $\alpha = 0$ at which changes in ω does not make any difference. On the other hand, decreasing the value of α results in a lower value of the bifurcation parameter in absolute value, except when $\omega = 1$ at which changes in α does not make any difference. Figure (3.4.4) illustrates different solution paths with different stability properties indicating a Hopf bifurcation. The inner spiral trajectory is converging to the equilibrium point, while the outer spiral is diverging. The limit cycle, thus, is unstable (non-attractive).

Figure 3.4.4: Phase diagram indicating a Hopf bifurcation under the hybrid Taylor rule.[14]



3.4.2 Under Hybrid Monetary Policy Rule and Interest Rate Smoothing:

Based on empirical studies, there is a general consensus that the monetary policy rule which takes the lagged nominal interest rate into account performs better in estimating the actual policy

[14] Limit cycle for $\phi_\pi = 1.5$ and $\phi_x = -0.01$, inner spiral for $\phi_\pi = 1.5$ and $\phi_x = -0.1$, outer spiral for $\phi_\pi = 1.5$ and $\phi_x = 0.1$.

rate employed by the central bank. In fact, when regressing the nominal interest rate on inflation and output gap, the lagged nominal interest rate variable is found statistically significant with large coefficients. This suggests that the monetary policy authority adjusts the policy rate gradually to changes in the output gap and the inflation rate. On the other hand, Taylor (1999) claims that policy rules with lagged interest rate work most poorly in models without rational expectations. For a discussion of the significance of lagged interest rate in the estimation of the monetary policy rules, see for example, English, Nelson and Sack (2003).

Consider the following model in which the first two equations describe the economy while the third equation represents the monetary policy rule followed by the central bank:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega - 1)} + \varphi \right) x_t \tag{3.4.1}$$

$$x_t = E_t x_{t+1} - \frac{1-\alpha+\alpha\omega}{\sigma} \left(r_t - E_t \pi_{t+1} - \overline{r}_t \right) \tag{3.4.2}$$

$$r_t = \overline{r}_t + (1-\phi_r)(\phi_\pi \pi_{t+1} + \phi_x x_t) + \phi_r r_{t-1} \tag{3.4.10}$$

Equation (3.4.10) describes the policy rule as a hybrid version of the Taylor rule in which the nominal interest rate is set according to the expected inflation rate, the current output gap, and the previous period's nominal interest rate. Rearranging the terms, and defining $y_t = [x_t, \pi_t, r_t]'$, we can write the system in normal form $E_t y_{t+1} = C y_t$,

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ E_t r_{t+1} \end{bmatrix} = C \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \overline{r}_t \tag{3.4.11}$$

where $C = \begin{bmatrix} \frac{\mu}{\beta}\left(1+\varphi\frac{1-\alpha+\alpha\omega}{\sigma}\right)+1 & -\frac{1-\alpha+\alpha\omega}{\beta\sigma} & \frac{1-\alpha+\alpha\omega}{\sigma} \\ -\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\frac{\mu}{\beta} & \frac{1}{\beta} & 0 \\ (1-\phi_r)(\phi_x-\phi_\pi)\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\frac{\mu}{\beta} & \frac{(1-\phi_r)\phi_\pi}{\beta} & \phi_r \end{bmatrix}$.

The Jacobian matrix C has a third order characteristic polynomial,

$$p(\lambda) = \det(C-\lambda I_3) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

where,

$$a_0 = \frac{\phi_x - \phi_r\sigma_\alpha - \phi_r\phi_x}{\beta\sigma_\alpha},$$

$$a_1 = \frac{\kappa_\alpha}{\beta\sigma_\alpha}\left((1-\phi_r)\phi_\pi + \phi_r\right) - \frac{\phi_x(1-\phi_r)}{\sigma_\alpha} + \frac{1+\phi_r}{\beta} + \phi_r,$$

and $a_2 = -\left(1+\phi_r+\frac{1}{\beta}+\frac{\kappa_\alpha}{\beta\sigma_\alpha}\right)$,

with $\sigma_\alpha = \frac{\sigma}{1-\alpha+\alpha\omega}$ and $\kappa_\alpha = \mu(\sigma_\alpha+\varphi) = \mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)$.

Following Farebrother (1973) and Gandolfo (1996), a third order dynamical system whose characteristic polynomial is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where $a_i \in \mathbb{R}$ for all $i = 1,2,3$ is stable if and only if

$$1 + a_2 + a_1 + a_0 > 0,$$
$$1 - a_2 + a_1 - a_0 > 0,$$
$$1 - a_1 + a_2a_0 - a_0^2 > 0.$$

Using the benchmark values of the parameters given in Gali and Monacelli (2005) and taking $\phi_r = 0.5$, it can be seen that the third condition is not satisfied. On the other hand, taking $\phi_r = 1$ leads to the first condition to fail. Thus, for the given values of the parameters, the system (3.4.11) is found unstable.

The following Proposition states the conditions for the system (3.4.11) to undergo a Hopf bifurcation:

Proposition 3.4.3: The system (3.4.11) undergoes a Hopf bifurcation if and only if the transversality condition $\left.\frac{\partial\left|\lambda_i(\phi)\right|}{\partial\phi_j}\right|_{\phi_j=\phi_j^*} \neq 0$ holds and the following conditions are satisfied:

i) $\left|\left(\phi_x - \phi_r\phi_x\right)\frac{1-\alpha+\alpha\omega}{\beta\sigma} - \phi_r\frac{1}{\beta\sigma}\right| < 1$,

1. $\phi_x\left(1-\phi_r\right) - \frac{\sigma}{1-\alpha+\alpha\omega}\left(\phi_r+\beta\right) < 0$,

2. $\phi_r - \beta < 0$,

ii)

$$\left|\left(\phi_x - \phi_r\phi_x - \varphi\mu\right)\frac{1-\alpha+\alpha\omega}{\beta\sigma} - \frac{\phi_r+\mu+1}{\beta} - 1 - \phi_r\right|$$
$$< 1 + \mu\left(\frac{1}{\beta} + \varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\left(\left(1-\phi_r\right)\phi_\pi + \phi_r\right) - \phi_x\left(1-\phi_r\right)\frac{1-\alpha+\alpha\omega}{\sigma} + \frac{1+\phi_r}{\beta} + \phi_r,$$

1.

$$-\phi_r\left(1-\phi_x\right)\frac{1-\alpha+\alpha\omega}{\sigma} - 2\left(1+\phi_r+\frac{1+\phi_r}{\beta}\right) + \left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma} - \mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\phi_\pi\right)\left(1-\phi_r\right)$$
$$-\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\left(1+\phi_r\right) < 0,$$

2.

$$\frac{1-\alpha+\alpha\omega}{\beta\sigma}\left(\phi_r\phi_x-\phi_x+\varphi\mu\right)+\frac{\mu}{\beta}-\left(\left(1-\phi_r\right)\phi_\pi\mu+\phi_r\mu\right)\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+\phi_x\left(1-\phi_r\right)\frac{1-\alpha+\alpha\omega}{\sigma}<0,$$

iii)

$$\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\left(\left(1-\phi_r\right)\phi_\pi+\phi_r\right)-\phi_x\left(1-\phi_r\right)\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1+\phi_r}{\beta}+\phi_r+\left(\left(\phi_x-\phi_r\phi_x\right)\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}\right)\left(1+\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)$$
$$=1-\left(\left(\phi_x-\phi_r\phi_x\right)\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}\right)^2.$$

Proof: For the necessary part, we apply Theorem 2. From the condition (*i*) in Theorem 2, we have

$$\left|a_0\right|=\left|\left(\phi_x-\phi_r\phi_x\right)\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta\sigma}\right|<1.$$

Then, using the definition of absolute value, we have either $\phi_x\left(1-\phi_r\right)-\dfrac{\sigma}{1-\alpha+\alpha\omega}\left(\phi_r+\beta\right)<0$,

or $\phi_r-\beta<0$. Note that the first expression always holds as the parameters ϕ_r and β are assumed to be positive. Hence the second condition is the binding one for the existence of a Hopf bifurcation.

From the condition (*ii*) in Theorem 2, we obtain

$\left|a_0+a_2\right|<1+a_1$,

which yields

$$\left|\left(\phi_x-\phi_r\phi_x-\varphi\mu\right)\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\frac{\phi_r+\mu+1}{\beta}-1-\phi_r\right|<1+\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\left(\left(1-\phi_r\right)\phi_\pi+\phi_r\right)-\phi_x\left(1-\phi_r\right)\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1+\phi_r}{\beta}+\phi_r.$$

Then, by the definition of absolute value, we have either

$$-\phi_r\left(1-\phi_x\right)\frac{1-\alpha+\alpha\omega}{\sigma}-2\left(1+\phi_r+\frac{1+\phi_r}{\beta}\right)+\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\phi_\pi\right)\left(1-\phi_r\right)-\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\left(1+\phi_r\right)<0$$

or

$$\frac{1-\alpha+\alpha\omega}{\beta\sigma}(\phi_r\phi_x-\phi_x+\varphi\mu)+\frac{\mu}{\beta}-\left((1-\phi_r)\phi_\pi\mu+\phi_r\mu\right)\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+\phi_x(1-\phi_r)\frac{1-\alpha+\alpha\omega}{\sigma}<0\cdot$$

Finally, from the condition (*iii*) in Theorem 2, we have

$$a_1-a_0a_2=1-a_0^2,$$

which yields

$$\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\left((1-\phi_r)\phi_\pi+\phi_r\right)-\phi_x(1-\phi_r)\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1+\phi_r}{\beta}+\phi_r+\left((\phi_x-\phi_r\phi_x)\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}\right)\left(1+\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)$$
$$=1-\left((\phi_x-\phi_r\phi_x)\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}\right)^2.$$

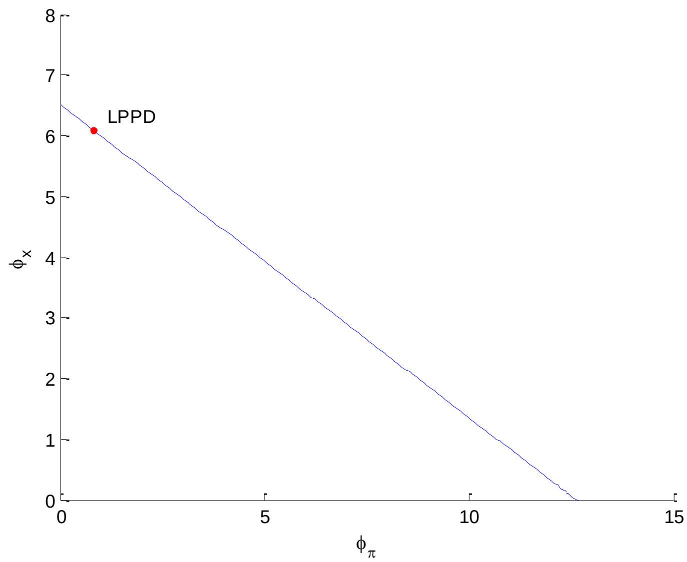
For the sufficient part, suppose the conditions (*i*)-(*iii*) hold true. Once the conditions (*i*)-(*iii*) are satisfied, a pair of complex conjugate eigenvalues lie on the unit circle while a real-valued eigenvalue lies outside the unit circle.

Finally, we need to show that the transversality condition holds for the bifurcation parameter which can be verified numerically. □

Proposition (3.4.3) proves that the Gali and Monacelli Model under the monetary policy rule specified in (3.4.10) is likely to experience a Hopf bifurcation. In order to determine numerically whether there are parameter values such that a bifurcation is possible, we examine the Jacobian matrix *C* keeping the structural parameters and the policy parameters ϕ_x and ϕ_r constant while varying ϕ_π over a certain range. We numerically find a Period Doubling bifurcation at $\phi_\pi = 12.38$ and a branching point at $\phi_\pi = 0.98$.

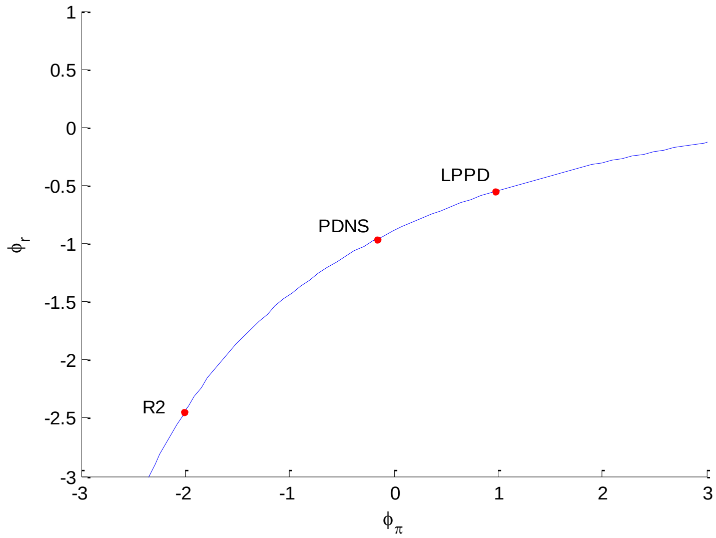
Starting at this bifurcation point, we construct the bifurcation boundary diagram in (ϕ_π, ϕ_x) space by changing ϕ_π and ϕ_x simultaneously while holding the other parameters constant. The bifurcation boundary diagram in (ϕ_π, ϕ_x) space is shown in Figure (3.4.5). At this stage, we also find a codimension-2 type Fold-Flip bifurcation at $(\phi_\pi, \phi_x) = (0.8, 6.1)$.

Figure 3.4.5: Period Doubling bifurcation boundary for $\phi_\pi = 12.38$ in the (ϕ_π, ϕ_x) space in Model (3.4.2)



Starting again from the Period Doubling bifurcation point $\phi_\pi = 12.38$, but this time varying ϕ_π and ϕ_r simultaneously and holding the other parameters constant, we construct the bifurcation boundary diagram in the (ϕ_π, ϕ_r)-space which is shown in Figure (3.4.6). Once simultaneously varying parameters ϕ_π and ϕ_r, we find a codimension-2 type Fold-Flip bifurcation at $(\phi_\pi, \phi_r) = (0.98, -0.55)$ and a Flip-Hopf bifurcation at $(\phi_\pi, \phi_r) = (-0.16, -0.96)$.

Figure 3.4.6: Period Doubling bifurcation boundary for $\phi_\pi = 12.38$ in the (ϕ_π, ϕ_r)-space in Model (3.4.2)



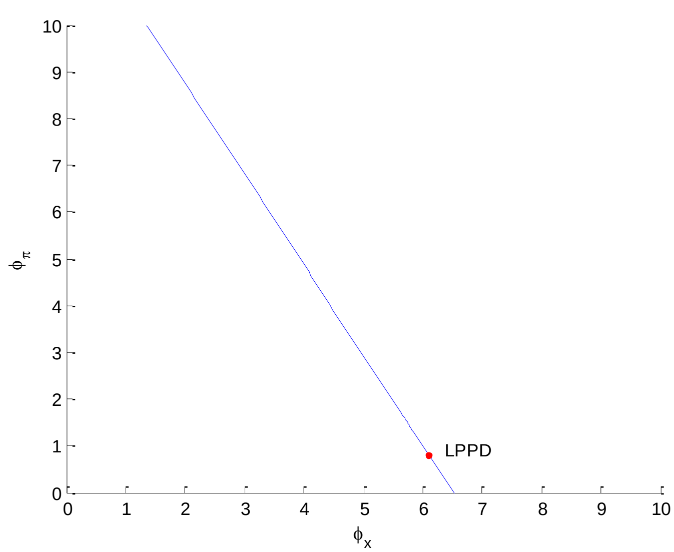
Decreasing the value of ω results in a higher value of the bifurcation parameter in absolute value. On the other hand, decreasing the value of α results in a lower value of the bifurcation parameter in absolute value, except when $\omega = 1$ at which changes in α does not make any difference. The Period Doubling bifurcation dissappears when $\alpha = 1$ and $\omega = 0$.

Secondly, we consider ϕ_x as the bifurcation parameter and keep the other parameters constant at their baseline values. We then numerically find a Period Doubling bifurcation at $\phi_x = 5.74$. For the close economy case, Barnett and Duzhak (2010) report a Period Doubling bifurcation at $\phi_x = 0.2831$. Under hybrid Taylor rule with interest rate smoothing we came across a higher value for Period Doubling bifurcation in open economy case. Comparing the results from Barnett and Duzhak (2010) with the open economy case, however, does not give us a clear statement about whether or not the openness makes the New Keynesian model more sensitive or resilient to bifurcations since Gali and Monacelli Model incorporates a wider set of parameters including some deep parameters relevant to the open economy. Furthermore, both studies use different set of benchmark values for the parameters. Hence, a direct comparison becomes even harder.

Starting from the Period Doubling bifurcation point $\phi_x = 5.74$, we construct the bifurcation boundary diagram in the (ϕ_π, ϕ_x)-space by varying ϕ_π and ϕ_x simultaneously while holding the other parameters constant. At this stage, we find a codimension-2 type Fold-Flip bifurcation at $(\phi_\pi, \phi_x) = (0.8, 6.09)$.

Figure (3.4.7) illustrates the Period Doubling bifurcation boundary for $\phi_x = 5.74$ in (ϕ_π, ϕ_x)-space. Note that along the bifurcation boundary, the values of the bifurcation parameter ϕ_x lie between $\phi_x = 1.2$ and $\phi_x = 6.8$ approximately. As the magnitude of the reaction of central bank to inflation, that is ϕ_π, increases, the smaller values of parameter ϕ_x would be sufficient to cause Period Doubling bifurcation.

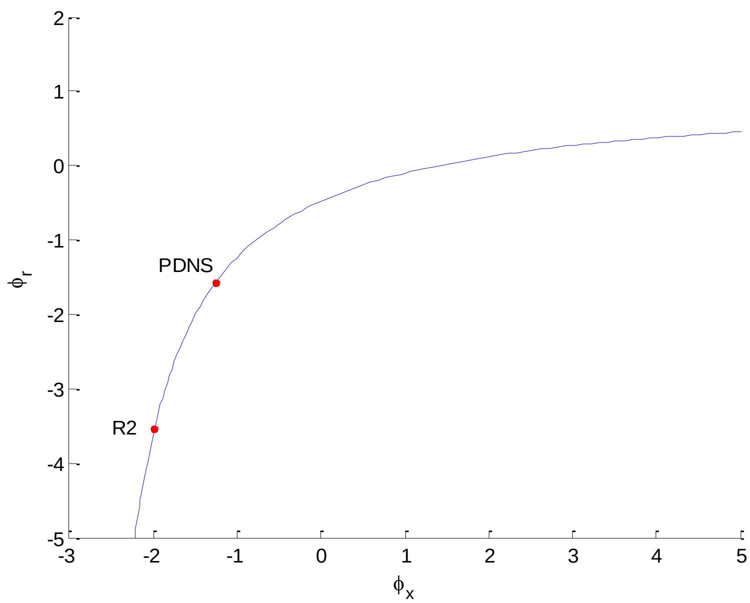
Figure 3.4.7: Period Doubling bifurcation boundary for $\phi_x = 5.74$ in the (ϕ_π, ϕ_x) space in Model (3.4.2)



Starting again from the Period Doubling bifurcation point $\phi_x = 5.74$, we construct the bifurcation boundary diagram in the (ϕ_x, ϕ_r)-space by holding the other parameters constant while varying ϕ_x and ϕ_r simultaneously. At this stage, we find a codimension-2 type Flip-Hopf bifurcation at $(\phi_x, \phi_r) = (-1.26, -1.57)$, which is outside the feasible parameter space.

Figure (3.4.8) illustrates the Period Doubling bifurcation boundary for $\phi_x = 5.74$ in the (ϕ_x, ϕ_r)-space. Notice that values of the parameter $\phi_r > 0$ do not play a significant role in determining the bifurcation value of the parameter ϕ_x. Bifurcation boundary becomes almost horizontal at $\phi_r = 0$.

Figure 3.4.8: Period Doubling bifurcation boundary for $\phi_x = 5.74$ in the (ϕ_x, ϕ_r)-space in Model (3.4.2)



Decreasing the value of ω results in a higher value of the bifurcation parameter in absolute value. On the other hand, decreasing the value of α results in a lower value of the bifurcation parameter in absolute value, except when $\omega = 1$ at which changes in α does not make any difference. All bifurcation points dissappear when $\alpha = 1$ and $\omega = 0$.

We run a similar analysis for the last policy parameter ϕ_r and find a Hopf bifurcation at $\phi_r = 1.36$ and a Period Doubling bifurcation at $\phi_r = -0.41$ as well as a branching point at $\phi_r = 1$.

3.5 Gali and Monacelli Model with AR(1) Policy Rule:

Consider the following model in which the first two equations describe the economy while the third equation represents the monetary policy rule followed by the central bank:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) x_t \tag{3.5.1}$$

$$x_t = E_t x_{t+1} - \frac{1-\alpha+\alpha\omega}{\sigma} \left(r_t - E_t \pi_{t+1} - \overline{r}_t \right) \tag{3.5.2}$$

$$r_t = \overline{r}_t + \phi_r r_{t-1} \tag{3.5.3}$$

In this specification, the nominal interest rate is an exogenous AR(1) process. The current policy rate is set according to the previous policy rate, independent of the endogenous variables, output gap and inflation, that is $\phi_\pi = \phi_x = 0$.

Defining $y_t = \left[x_t, \pi_t, r_t \right]'$, we can write the system in normal form $E_t y_{t+1} = C y_t$, as

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ r_t \end{bmatrix} = A \begin{bmatrix} x_t \\ \pi_t \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} -\dfrac{1-\alpha+\alpha\omega}{\sigma} \\ 0 \\ 1 \end{bmatrix} \overline{r}_t , \tag{3.5.4}$$

where $C = \begin{bmatrix} \left(\frac{\mu}{\beta} + \varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+1 & -\frac{1-\alpha+\alpha\omega}{\beta\sigma} & \frac{1-\alpha+\alpha\omega}{\sigma} \\ -\frac{\mu}{\beta}\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right) & \frac{1}{\beta} & 0 \\ 0 & 0 & \phi_r \end{bmatrix}$.

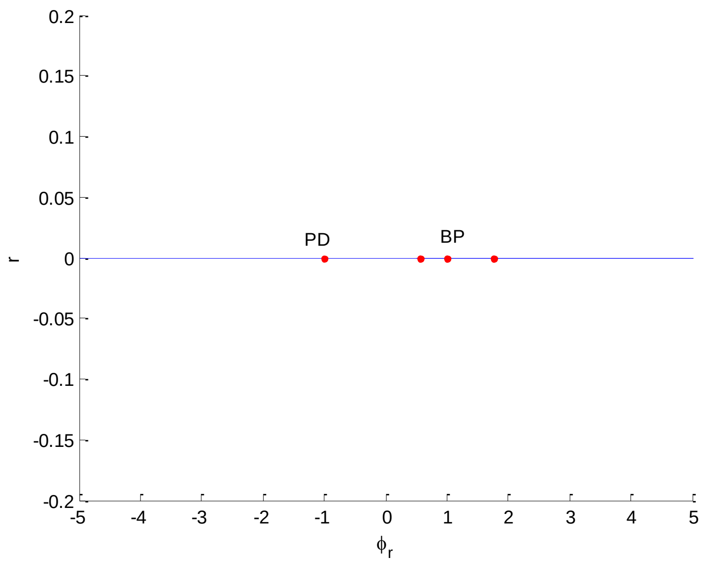
Following Blanchard and Kahn (1980), the system (3.5.4) has a unique, stationary equilibrium solution if and only if the number of eigenvalues outside the unit circle is equal to the number of forward looking (non-predetermined) variables, which is two ($E_t x_{t+1}$ and $E_t \pi_{t+1}$). One eigenvalue is ϕ_r that is inside the unit circle since $\phi_r \in (0,1)$ by assumption. Thus, in a similar way to the closed economy case as shown in Walsh (2003, p.246), stability and uniqueness could be achieved if both eigenvalues of the matrix

$$\begin{bmatrix} \left(\frac{\mu}{\beta} + \varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)+1 & -\frac{1-\alpha+\alpha\omega}{\beta\sigma} \\ -\frac{\mu}{\beta}\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right) & \frac{1}{\beta} \end{bmatrix}$$

are outside the unit circle which does not hold since the smaller eigenvalue of this matrix lies inside the unit circle. As Walsh (2003) points out, this implies the existence of multiple equilibria and the possibility of stationary sunspot equilibria. Thus, an exogenous policy rule that is independent of the endogenous variables of the system (3.5.4), inflation and output gap, might give rise to multiple equilibria.

Keeping the parameters constant at their baseline values while varying ϕ_r, our numerical analysis indicates a branching point at $\phi_r = 1$ and a Period Doubling bifurcation at $\phi_r = -1$ as illustrated in Figure (3.5.1).

Figure 3.5.1: Period Doubling bifurcation and branching point under the AR(1) policy rule



At the second step, while varying two parameters, ϕ_r and one of the structural parameters σ_α, β or κ_α simultaneously in order to construct the bifurcation boundary diagram, our numerical analysis indicates codimension-2 Fold-Flip and Flip-Hopf bifurcations. But, in each case, we obtain a bifurcation boundary that is parallel to the horizontal line implying that the bifurcation parameter is not affected by the changes in the structural parameters. That means, varying ω and α does not affect the value of the bifurcation parameter, unlike the previous cases.

3.6 An Extension: Clarida, Gali and Gertler (2002) Model:

Clarida, Gali and Gertler (2002) developed a two-country version of small open economy model which is basically based on Clarida, Gali and Gertler (2001) and Gali and Monacelli (1999). Following Walsh (2003, pages 539, 540), the model of Clarida, Gali and Gertler (2002) can be rewritten in the reduced form as follows:

$$\pi_t^h = \beta E_t \pi_{t+1}^h + \mu\left[\sigma + \eta + \left(\frac{\gamma\sigma}{1+w}\right)\right]x_t \tag{3.6.1}$$

$$x_t = E_t x_{t+1} - \left(\frac{1+w}{\sigma}\right)\left(r_t - E_t \pi_{t+1}^h - \overline{r}_t\right) \tag{3.6.2}$$

$$r_t = \overline{r}_t + \phi_\pi \pi_t^h + \phi_x x_t \tag{3.6.3}$$

where $\mu = \left[(1-\theta)(1-\beta\theta)\right]/\theta$, θ is the probability that a firm holds its price unchanged in a given period of time, while $(1-\theta)$ is the probability that a firm resets its price. The parameter w denotes the growth rate of nominal wages and γ denotes the population size in the foreign country, with $1-\gamma$ being the population size of the home country. Wealth effect is captured by the term $\gamma\sigma$.

Equation (3.6.1) is an inflation adjustment equation for the aggregate price of domestically produced goods. Equation (3.6.2) is the dynamic IS curve which is derived from the Euler condition of the consumers' optimization problem. The monetary policy rule (3.6.3) is a domestic inflation based current looking Taylor rule and completes the model.

Substituting (3.6.3) for $r_t - \overline{r}_t$ into the equation (3.6.2), we can reduce the system to a first order dynamic system in two equations for domestic inflation and output gap, given by:

$$\pi_t^h = \beta E_t \pi_{t+1}^h + \delta\left[\sigma + \eta + \left(\frac{\nu\sigma}{1+w}\right)\right]x_t,$$

$$x_t = E_t x_{t+1} - \left(\frac{1+w}{\sigma}\right)\left(\phi_\pi \pi_t^h + \phi_x x_t - E_t \pi_{t+1}^h\right).$$

Clearly, $x_t = \pi_t = 0$ for all t constitutes a solution (equilibrium) to the system. We can write the system in the standard form $AE_t y_{t+1} = By_t$ as follows:

$$A\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1}^h \end{bmatrix} = B\begin{bmatrix} x_t \\ \pi_t^h \end{bmatrix}, \tag{3.6.4}$$

where $A = \begin{bmatrix} 0 & \beta \\ 1 & \dfrac{1+w}{\sigma} \end{bmatrix}$ and $B = \begin{bmatrix} -\delta\left[\sigma + \eta + \left(\dfrac{\nu\sigma}{1+w}\right)\right] & 1 \\ 1 + \dfrac{(1+w)\phi_x}{\sigma} & \dfrac{(1+w)\phi_\pi}{\sigma} \end{bmatrix}$.

Then, premultiplying the terms on the right hand side by the inverse of the matrix A, the system can be reduced to the form $E_t y_{t+1} = C y_t$, where $C = A^{-1}B$,

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1}^h \end{bmatrix} = C\begin{bmatrix} x_t \\ \pi_t^h \end{bmatrix} \tag{3.6.5}$$

where $C = \begin{bmatrix} 1 + \dfrac{(1+w)\phi_x}{\sigma} + \delta(1+w)\left(\sigma + \eta + \left(\dfrac{\nu\sigma}{1+w}\right)\right)\dfrac{1}{\beta\sigma} & \dfrac{(1+w)\phi_\pi}{\sigma} - \dfrac{(1+w)}{\beta\sigma} \\ -\delta\left(\sigma + \eta + \left(\dfrac{\nu\sigma}{1+w}\right)\right)\dfrac{1}{\beta} & \dfrac{1}{\beta} \end{bmatrix}$.

The system (3.6.5) is in normal form in the sense that each equation has only one unknown variable evaluated at time $t+1$. Note that there were no disturbance term included in the model, hence $\varepsilon_t = 0$. For the uniqueness and stability of the equilibrium, both eigenvalues must be outside the unit circle.

The characteristic polynomial of the coefficient matrix C is given by

$p(\lambda) = \det(C - \lambda I) = \lambda^2 - a_1\lambda + a_0 = 0$,

where

$a_0 = (1+w)(\phi_x + \phi_\pi\delta\eta)\dfrac{1}{\beta\sigma} + (\phi_\pi\delta(1+\nu+w)+1)\dfrac{1}{\beta}$,

and

$$a_1 = (1+w)(\delta\eta + \phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta} + 1,$$

which yields

$$\lambda_{1,2} = \left((1+w)(\delta\eta + \phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta} + 1 \right)$$
$$\pm \left(\left((1+w)(\delta\eta + \phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta} + 1 \right)^2 - 4\left((1+w)(\phi_x + \phi_\pi\delta\eta)\frac{1}{\beta\sigma} + (\phi_\pi\delta(1+v+w)+1)\frac{1}{\beta} \right) \right)^{\frac{1}{2}}.$$

As shown in Gandolfo (1996), in order to examine the nature of the eigenvalues we need to check the sign of the discriminant $\Delta \equiv a_1^2 - 4a_0$. If the discriminant of the quadratic equation is strictly negative, that is if

$$\Delta \equiv a_1^2 - 4a_0 = \left((1+w)(\delta\eta + \phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta} + 1 \right)^2 - 4\left((1+w)(\phi_x + \phi_\pi\delta\eta)\frac{1}{\beta\sigma} + (\phi_\pi\delta(1+v+w)+1)\frac{1}{\beta} \right) < 0$$

then, the roots of the coefficient matrix *C* will be complex conjugate numbers in the form $\lambda_{1,2} = a \pm ib$, with $a,b \in \mathbb{R}$, $b \neq 0$ is the real part, and $i = +\sqrt{-1}$ is the imaginary unit.

Regarding the system (3.6.5), it is algebraically quite cumbursome to identify the sign of the discriminant. Therefore, we simply assume that the eigenvalues of the system (3.6.5) are complex conjugates, $\lambda_{1,2} = a \pm ib$,

where

$$a = \frac{a_1}{2} = \left((1+w)(\delta\eta + \phi_x\beta)\frac{1}{2\beta\sigma} + (1+\delta(1+v+w))\frac{1}{2\beta} + \frac{1}{2} \right) \tag{3.6.6}$$

and

$$b = \frac{\sqrt{-\Delta}}{2} = \frac{1}{2}\sqrt{4\left((1+w)(\phi_x + \phi_\pi\delta\eta)\frac{1}{\beta\sigma} + (\phi_\pi\delta(1+v+w)+1)\frac{1}{\beta} \right) - \left((1+w)(\delta\eta + \phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta} + 1 \right)^2}. \tag{3.6.7}$$

With the assumption of a pair of complex conjugate eigenvalues, we may expect to see a Hopf bifurcation if the transversality conditions are satisfied. Using Theorem 1, the conditions for the existence of a Hopf bifurcation is stated in the following Proposition.

Proposition 3.6: The system (3.6.5) undergoes a Hopf bifurcation if and only if $\Delta < 0$ and

$$\phi_x^* = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right). \tag{3.6.8}$$

Proof: Suppose the system (3.6.5) goes through a Hopf bifurcation at (y^*, ϕ_x^*), where

$y^* = (x^*, \pi^*)$. Then, we need to show that $\Delta < 0$ and $\phi_x^* = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right)$.

The existence of a Hopf bifurcation, by the definition, requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative, that is $\Delta < 0$.

For the second part, note that the existence of a Hopf bifurcation requires $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = +\sqrt{a^2 + b^2} = 1$ by the first condition of Theorem 1. Rewriting the condition explicitly by substituting (3.6.6) and (3.6.7) into it, taking the square of both sides and solving for ϕ_x, we obtain (3.6.8). Therefore, the first condition of Theorem (1) holds only if

$$\phi_x = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right).$$

From the other side, suppose $\Delta < 0$ and $\phi_x = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right)$.

Substituting for ϕ_x^* into $\sqrt{a^2 + b^2}$ yields $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = 1$, which is the first condition in Theorem 1. In order to show that the critical value of the parameter ϕ_x is actually a Hopf bifurcation parameter, we check the second condition in Theorem 1, which yields

$$\left.\frac{d\left|\lambda_i\left(\phi_x\right)\right|}{d\phi_x}\right|_{\phi_x=\phi_x^*} = \left.\frac{d}{d\phi_x}\left(\sqrt{a^2+b^2}\right)\right|_{\phi_x=\phi_x^*} = \frac{1+w}{2\beta\sigma} \neq 0 \text{ for } i=1,2.$$

Thus, both conditions of Theorem 1 are satisfied and we have

$$\phi_x^* = \frac{\beta\sigma-1}{1+w} - \phi_\pi\left(\frac{\delta\sigma\left(1+v+w\right)}{1+w}+\delta\eta\right). \qquad \square$$

Proposition (3.6) shows formally that taking the parameter ϕ_x free to vary and keeping the other parameters constant, the model of Clarida, Gali and Gertler (2002) is likely to undergo a Hopf bifurcation at ϕ_x^*.

Note that, the model of Clarida, Gali and Gertler (2002) differs from the Gali and Monacelli Model in several aspects. Additional paramaters exist in the former one. Parameters w, v, and δ play a role in determining the critical value of the bifurcation parameter. To what extend both models differ depends on the parameter settings.

CHAPTER IV:

CONCLUSION

In this study, we first reviewed the bifurcation phenomenon in dynamic economic systems and pointed out the significance of bifurcations in dynamic systems. We then briefly mentioned bifurcation studies in economics literature in conjuction with a summary of the common types of bifurcations that have been encountered in dynamical economic systems. Bifurcation analysis has been widely used to examine and classify the dynamic behavior of a wide variety of economic models in economic literature. We restricted our attention to local, codimension-1 bifurcations which require variations in only one control paramater in a small neighourhood of a fixed point.

Despite the growing research interest in exploring the bifurcation phenomena in economic systems, literature on this subject is still sparse. In this study, we run bifurcation and determinacy analyses on an open economy New Keynesian model developed by Gali and Monacelli (2005). We have shown that in a broad class of open economy New Keynesian models, the degree of openness has a significant role in equilibrium determinacy and emergence of bifurcations under various form and timing of monetary policy rules. We found that the open economy framework brings about more complex dynamics, a wider variety of qualitative behaviors and policy responses. The conditions for the uniqueness and local stability of the equilibrium points are established for each model and were evaluated using the numerical analysis results. Even though some of the models that we considered here were analysed before by other authors in terms of determinacy, we reestablished the determinacy conditions in the open economy framework and for the sake of completeness of our study. Determinacy diagrams are also constructed to show the regions of unique and multiple equilibria. We then established the conditions of a Hopf bifurcation for each model, based on the Hopf Bifurcation Theorem. Numerical analyses are performed to confirm the theoretical results. The numerical simulations showed that limit cycles and periodic behaviors are possible but in some cases only for unrealistic parameter values. Our numerical analyses also indicate the existence of the Period Doubling bifurcations in the models we examined. We then numerically constructed, for each circumstance, the corresponding bifurcation boundary diagram.

The most important finding of this study is about the effects of the openness of economy on the value of bifurcation parameter. Under the monetary policy rules we studied, degree of openness in New Keynesian models changes the value of bifurcation parameter and makes the stratification of the confidence region by bifurcations still a serious issue. This suggests that the central bank should react to changes in the rate of inflation and the output gap cautiously. Thus, the stratification of the confidence region, as often seen in closed economy New Keynesian models examined by Barnett and Duzhak (2008, 2010), is still an important risk to be considered in the context of the open economy New Keynesian functional structures.

Comparing the results from Barnett and Duzhak's (2010) closed economy analysis with the open economy case, however, does not give us a clear statement about whether or not the openness makes the New Keynesian model more sensitive to bifurcations. One reason is the fact that the Gali and Monacelli Model incorporates a wider set of parameters including some deep parameters relevant to the open economy. The fact that both studies use different set of benchmark values for the parameters makes a direct comparison even harder. Analyzing the effects of a change in the parameters α and ω in the Gali and Monacelli Model, on the other hand, does not seem to be indicative for more sensitivity in open economy environment.

Lowering the composite parameter ω and raising the parameter α usually increase the value of the bifurcation parameter.

Notice that our analysis is restricted to certain special cases within the framework of open economy New Keynesian structure which closely follows Gali and Monacelli (2005). Therefore, we must be cautious when applying our conclusions to a more general environment. Econometrics and optimal policy design become more complex with an open economy. Dynamical inferences need to be qualified by the increased risk of bifurcation boundaries crossing the confidence regions and policy design needs to take into consideration that a drastic change in monetary policy can produce an unanticipated bifurcation, if the econometrics research was not adequate. However, generalizing our results to real economies would be an overstatement and we must be cautious when applying our findings to a more general environment. Notice also that our research is not about endogenous bifurcations and our model's parameters are fixed and do not move on their own.

While simultaneously varying the policy parameters in order to construct bifurcation boundaries, we also encountered codimension-2 type Fold-Flip and Flip-Hopf bifurcations. In discrete-time dynamical systems, various types of codimension-2 bifurcations are possible. Detection of codimension-2 equilibrium bifurcations in a dynamic system allows us to predict the global behavior of the dynamic system by means of algebraic and numerical computations which could lead us to more complex dynamic behaviors and even to chaos. We left the analysis of codimension-2 bifurcations in New Keynesian models as a subject of future research.

In this study, we acquired a sense of quantitative importance of bifurcation phenomenon in open economy structure based on Gali and Monacelli (2005) Model through analytical and numerical investigations. It would be interesting to extend this study by testing our findings using real time data from various countries, which would give us important information about the existence and implications of bifurcations and provide us with empirical evidence regarding bifurcation phenomena in the real economy. This can be an insightful subject for future research.

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