Polynomials associated with graph coloring and orientations

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Abstract

We study colorings and orientations of graphs in two related contexts.

Firstly, we generalize Stanley's chromatic symmetric function X_G [17] using the k-balanced colorings of Pretzel [12] to create a new graph invariant X_G^k. We show that in fact X_G^k is a quasisymmetric function which has a positive expansion in the fundamental basis. We also define a graph invariant χ_G^k generalizing the chromatic polynomial χ_G for which we prove some analogous theorems.

Secondly, we examine graphs and graph colorings in the context of the combinatorial Hopf algebras of Aguiar, Bergeron and Sottile [2]. By doing so, we are able to obtain a new formula for the antipode of a Hopf algebra on graphs previously studied by Schmitt [15]. We also obtain new interpretations of evaluations of the Tutte polynomial.

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Introduction

A *coloring* of a graph G is some assignment of "colors" from the set of positive integers to the vertices of G so that none of the edges of G are monochromatic. This rather straightforward definition sprang from attempts to formalize the four color theorem in the 1800's, and in fact many of the "classical" results and tools related to graph colorings that were created in the 19th and 20th century were generated by attempts to solve the four color theorem.

Also quite important in graph theory is the idea of a *directed graph*—a graph whose edges also have a direction. This is a quite natural construction; if the vertices and edges of a graph are representative of objects and connections between those objects respectively, then a directed graph indicates that the relationships between objects are not symmetric. Such an idea appears implicitly in almost all mathematical concepts. An *orientation* of an undirected graph G is a directed graph created by assigning a direction to each edge of G.

The relationship between these two ideas is simple to state: a coloring of an undirected graph G *induces* an orientation of G by directing each edge toward the vertex of larger color. What's more, such an orientation is *acyclic*. That is, starting at any vertex and following the directions of the edges, there is no way to cycle back to your starting vertex.

In fact, there is a more subtle connection between the two; if $\chi_G(\lambda)$ counts the number of colorings of G using the colors $[\lambda] = \{1, 2, \ldots, \lambda\}$, then $\chi_G(\lambda)$ is a polynomial in λ, called the *chromatic polynomial of* G. Stanley [16, Cor 1.3] proved that evaluating $\chi_G(-1)$ gave the number of acyclic orientations of G up to a sign.

This dissertation focuses further on the relationship between graph colorings and graph orientations, by using tools from poset theory and Hopf algebra theory.

The relationship between a graph orientation and a poset is clear; directed graphs and posets both are defined by asymmetric relations between their objects. However, as a consequence of the fact that posets are often presented in a Hasse diagram, Pretzel [12] was led to make a definition of a *k-balanced orientation*—an orientation of G where in every cycle of G, there are k edges directed "clockwise" and k edges directed "counterclockwise".

In Chapter 2, we study k-balanced orientations as they relate to the well-known *chromatic symmetric function* X_G of Stanley [17], a refinement of the chromatic polynomial which enumerates colorings not only by the number of colors used but also by which colors are used.

Specifically, if we define a *k-balanced coloring* of a graph as a coloring which induces a k-balanced orientation, then we generalize X_G to a new quasisymmetric function X_G^k by the same enumeration over k-balanced colorings instead of over all colorings.

With this definition, we are able to show that X_G^k has positive coefficients in the fundamental basis for the quasisymmetric functions, as well as demonstrate some explicit calculations on cyclic graphs and complete bipartite graphs. Furthermore, in the same way that χ_G is a specialization of X_G, we define the *k-chromatic polynomial* of G as a specialization of X_G^k. This polynomial has the interesting property that $\chi_G^k(-1)$

counts the number of k-balanced orientations of G up to sign, much like the chromatic polynomial.

We also examine colorings and orientations in relation to Hopf algebras, a structure first investigated abstractly by Milnor and Moore in 1965 [10] in the context of homology on manifolds. The first major appearance of Hopf algebras in combinatorics was in the seminal paper of Joni and Rota in 1979 [9], where they demonstrated to the community the natural place that Hopf algebras serve in many combinatorial problems. The basic principle is that given a family of combinatorial objects, one often has natural methods to "join" two objects in the family and to decompose an object in the family into others. The Hopf algebra structure gives us a way to formalize this and investigate such families in an abstract context.

For example, in the context of graphs, the decomposition which we will investigate is partitioning the vertex set into two disjoint sets U, $\overline{U}$ and considering the graphs induced by U and U'. This decomposition turns out to be closely related to the idea of graph colorings.

The major inspiration for our investigation into Hopf algebras in this dissertation was the paper of Aguiar, Bergeron, and Sottile [2] that investigated the structure of *combinatorial Hopf algebras*—a certain class of Hopf algebras which models combinatorial problems very well—by examining a morphism from any combinatorial Hopf algebra to the quasisymmetric functions.

Our investigation in Chapter 3 follows much the same principle, except where they investigate morphisms to the quasisymmetric functions, we investigate morphisms to polynomials in one variable, which also have a combinatorial Hopf algebra structure. In particular, we study morphisms from a combinatorial Hopf algebra on graphs.

This Hopf algebra and its generalizations have been studied in the past, notably by Schmitt [15] and Takeuchi [22], who discovered independently an explicit formula for an important map on a Hopf algebra called the *antipode*. We find a different formula for the antipode with many fewer terms which allows us to make some calculations very efficiently. Our formula is an independently discovered specialization of an unpublished result of Aguiar and Ardila [1].

Finally, using the properties of the well-known Tutte polynomial, we are able to express it in combinatorial Hopf algebra terms. This allows us to note some interesting and previously unknown identities which it satisfies.

Chapter 1

Background

Throughout this dissertation, let $\mathbb{N}$ denote the set of positive integers and $[n] = \{1,2,\ldots,n\}$.

1.1 Posets

A *poset* is a set P together with a relation $<$ which is *antireflexive* ($x \not< x$), *antisymmetric* (if $x < y$ then $y \not< x$), and *transitive* (if $x < y$ and $y < z$ then $x < z$).

An element $y \in P$ *covers* $x \in P$ if $x < y$ and there does not exist a $z \in P$ such that $x < z < y$. There is a graph related to P called the *Hasse diagram* of P, whose vertices are elements of P and whose edges are the covering relations of P. Such diagrams are typically drawn so that the "larger" elements of the poset are above the "smaller" elements of the poset. For example, in Figure 1.1 we have the Hasse diagram of a poset P on [5], where $3 <_P 2$, $3 <_P 4$, etc.

For a permutation $\pi \in \mathfrak{S}_n$, *the ascent set of* π is

$$\operatorname{asc}(\pi) = \{i \in [n] : \pi(i) < \pi(i+1)\}.$$

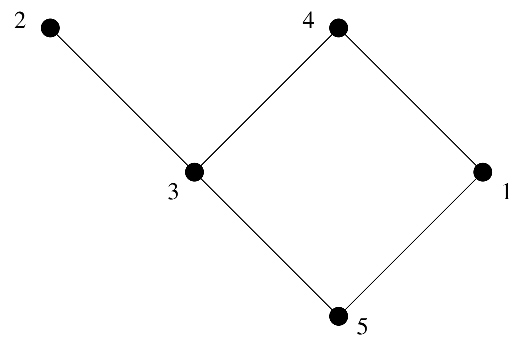


Figure 1.1: A Hasse diagram of a poset

We can then define *the composition associated to* π as

$$c(\pi) = \text{co}(\text{asc}(\pi)),$$

where $\text{co}(\text{asc}(\pi))$ is the composition associated to the set $\text{asc}(\pi)$ as in Section 1.2.1. The parts of $c(\pi)$ are thus the lengths of the maximal contiguous decreasing subsequences. For example, $c(52164783) = (3,2,1,2)$.

Given a poset P defined on $[n]$, we will call $\pi \in \mathfrak{S}_n$ a *linear extension* of P if $i <_P j$ implies that i precedes j in π. The set of all linear extensions of P will be written as $\mathscr{L}_P$. We will call P *naturally-labeled* if $12\ldots n \in \mathscr{L}_P$.

A *P-partition* is a strict order-preserving map $\tau : P \to [n]$, where P is a naturally labeled poset on $[n]$. That is, if $i <_P j$, then $\tau(i) < \tau(j)$. This form of a P-partition is sometimes called a strict P-partition, to differentiate it from a weak P-partition where the map τ needn't be strict. The reasoning behind this name comes from the fact that if P is the chain poset where $1 < 2 < \cdots < n$, then a weak P-partition corresponds to a partition of n parts.

Definition 1.1.1. Let $\pi \in \mathfrak{S}_n$. Then a function $f : [n] \to \mathbb{N}$ is π-*compatible* whenever

$$f(\pi_1) \leq f(\pi_2) \leq \cdots \leq f(\pi_n)$$

and

$$f(\pi_i) < f(\pi_{i+1}) \text{ if } \pi_i < \pi_{i+1}.$$

For all $f : [n] \to \mathbb{N}$, there exists a unique permutation $\pi \in \mathfrak{S}_n$ for which f is π-compatible. Specifically, if $\{i_1 < i_2 < \cdots < i_k\}$ is the image of f, then we obtain π by listing the elements of $f^{-1}(i_1)$ in increasing order, then the elements of $f^{-1}(i_2)$ in increasing order, and so on.

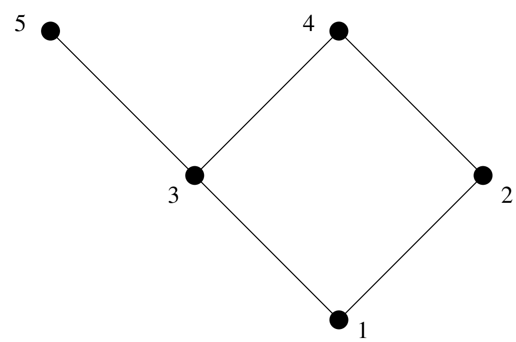


Figure 1.2: A naturally labeled poset

Example 1.1.2. Notice that the poset P in Figure 1.1 is not naturally labeled, as $2 >_P 3$ for instance. However, the poset P' in Figure 1.2 is naturally labeled. In fact, P' is a natural *relabeling* of P, as P is isomorphic to P'.

The linear extensions of P' are

$$\mathscr{L}_{P'} = \{12345, 12354, 13245, 13254, 13524\}.$$

Let f be defined by $f(1) = 3, f(2) = 4, f(3) = 2, f(4) = 2, and f(5) = 1$, which is not a P'-partition. Then f is compatible with the permutation 54312, which is not in $\mathscr{L}_{P'}$.

In fact, the situation of the previous example is standard, as the next proposition demonstrates.

Proposition 1.1.3 (Lemma 4.5.3 in [18])**.** *Let P be a naturally labeled poset on $[n]$. Then $\tau : P \to \mathbb{N}$ is a P-partition if and only if τ is π-compatible for some $\pi \in \mathscr{L}_P$.*

Proof. Given a P-partition τ, let π be the unique permutation of $[n]$ so that τ is π-compatible. Now if $i <_P j$, then $\tau(i) < \tau(j)$, and since τ is π-compatible, i must appear before j in π. Thus π is a linear extension of P.

On the other hand, given a π-compatible function τ with π a linear extension of P, if $i <_P j$, then i appears before j in π, and so $\tau(i) < \tau(j)$. □

We write S_π for the set of all π-compatible functions, and $\mathscr{A}(P)$ for the set of all P-partitions. Then from Proposition (1.1.3) we get the decomposition

$$\mathscr{A}(P) = \bigsqcup_{\pi \in \mathscr{L}_P} S_\pi. \tag{1.1}$$

1.2 Formal Power Series

1.2.1 Partitions and Compositions

A *partition* λ is a multiset $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell\}$ of positive integers. The *weight* of a partition λ is $|\lambda| = \sum \lambda_i$. If $|\lambda| = n$, we will say that λ is a *partition of n* and write $\lambda \vdash n$. The number ℓ is the *length* of λ.

Similarly, a *composition* α is a list $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of positive integers. The *weight* of a composition is $|\alpha| = \sum \alpha_i$. If $|\alpha| = n$, we will say that α is a *composition of n* and write $\alpha \models n$. The number ℓ is the *length* of α.

The following is a basic result about compositions.

Proposition 1.2.1. *There is a bijection between compositions of n and subsets of $[n-1]$.*

Proof. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of n, define $S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, n - \alpha_\ell\}$. For a subset $S = \{s_1 < s_2 < \ldots < s_m\}$ of $[n-1]$, define $\mathrm{co}(S) = (s_1, s_2 - s_1, \ldots, s_m - s_{m-1}, n - s_m)$.

Then,

$$\begin{aligned}\mathrm{co}(S_\alpha) &= (\alpha_1, (\alpha_1 + \alpha_2) - \alpha_1, \ldots, n - (n - \alpha_\ell)) \\ &= \alpha\end{aligned}$$

and

$$\begin{aligned}S_{\mathrm{co}(S)} &= \{s_1, s_1 + (s_2 - s_1), \ldots, n - (n - s_m)\} \\ &= S.\end{aligned}$$

□

The compositions of n are ordered by refinement: for $\alpha, \beta \models n$, $\alpha \prec \beta$ if and only if $S_\alpha \subsetneq S_\beta$. Notice that under the bijection above, this relation is set containment, so that this poset is isomorphic to the boolean poset of subsets of $[n-1]$.

For example, the composition $\beta = (1,1,4,2)$ of 8 refines the composition $\alpha = (2,6)$, since $\{2\} \subsetneq \{1,2,6\}$.

1.2.2 Symmetric and Quasisymmetric Functions

If p is a polynomial or formal power series and m is a monomial, then let $[m]p$ denote the coefficient of m in p.

A *symmetric function* is an element $F \in \mathbb{Q}[[x_1, x_2, \ldots]]$ with the property that

$$[x_{i_1}^{a_1} x_{i_2}^{a_2}, \ldots, x_{i_\ell}^{a_\ell}]F = [x_{j_1}^{a_1} x_{j_2}^{a_2}, \ldots, x_{j_\ell}^{a_\ell}]F$$

for any distinct set of i and distinct set of j. Let $\mathscr{S}$ denote the vector space of all symmetric functions and let $\mathscr{S}_n$ denote the vector space of all symmetric functions in degree n.

The most "natural" vector space basis of $\mathscr{S}_n$ is the set of *monomial symmetric functions* m_λ, where for a partition $\lambda \vdash n$ we let

$$m_\lambda = \sum_{i_1,\ldots,i_\ell \text{ distinct}} x_{i_1}^{\lambda_1} \ldots x_{i_\ell}^{\lambda_\ell}.$$

Symmetric functions in general are very important in combinatorics, appearing most notably in relation to Young tableaux and representation theory, see for example [14]. However, the only symmetric function which we will discuss is the chromatic symmetric function, introduced in Section 1.3.

A *quasisymmetric function* is an element $F \in \mathbb{Q}[[x_1, x_2, \ldots]]$ with the property that

$$[x_{i_1}^{a_1} x_{i_2}^{a_2}, \ldots, x_{i_\ell}^{a_\ell}]F = [x_{j_1}^{a_1} x_{j_2}^{a_2}, \ldots, x_{j_\ell}^{a_\ell}]F$$

whenever $i_1 < i_2 < \cdots < i_\ell$ and $j_1 < j_2 < \cdots < j_\ell$. The subspace of $\mathbb{Q}[[x_1, x_2, \ldots]]$ consisting of all quasisymmetric functions will be denoted $\mathscr{Q}$, and the vector space spanned by all quasisymmetric functions of degree n will be denoted $\mathscr{Q}_n$. The *standard basis* or *monomial basis* for $\mathscr{Q}_n$ is indexed by compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \models n$, and is given by

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}.$$

The standard basis is in fact a vector space basis since each monomial is precisely one of the monomials that can appear in a quasisymmetric function.

An important concept for both symmetric and quasisymmetric functions is the idea of *specializations*, which are defined here as any substitution for the variables $x_1, x_2, \ldots$. Of particular interest is the *principal specialization* of F,

$$\mathrm{ps}_\lambda(F) = F(\underbrace{1,1,\ldots,1}_{\lambda \text{ 1's}},0,0,\ldots),$$

where λ is some nonnegative integer, which plays a role in both Chapter 2 and 3. For example, the principal specialization of an element M_α is

$$\mathrm{ps}_\lambda(M_\alpha) = \binom{\lambda}{|\alpha|},$$

where

$$\binom{\lambda}{\ell} = \frac{\lambda(\lambda-1)\ldots(\lambda-\ell+1)}{\ell!}.$$

Another basis for $\mathcal{Q}_n$ besides the monomial basis is the *fundamental basis*, whose elements are

$$L_\alpha = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} x_{i_2} \ldots x_{i_n}, \tag{1.2}$$

where $\alpha \models n$.

Working with the bijection between sets and compositions, and using the fact that the refinement poset is boolean, these bases are related by Möbius inversion as:

$$L_\alpha = \sum_{\beta \succeq \alpha} M_\beta, \tag{1.3}$$

$$M_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} L_\beta. \tag{1.4}$$

These equations demonstrate that, in fact, the fundamental basis is a basis for $\mathcal{Q}_n$.

Example 1.2.2. If $\alpha = (1,2,1,2)$ is a composition of 6, then $S_\alpha = \{1,3,4\} \subseteq [5]$. Refinements of α then correspond to subsets of [5] containing S_α:

$$\begin{aligned}
\{1,3,4\} &\leftrightarrow (1,2,1,2)\\
\{1,2,3,4\} &\leftrightarrow (1,1,1,1,2)\\
\{1,3,4,5\} &\leftrightarrow (1,2,1,1,1)\\
\{1,2,3,4,5\} &\leftrightarrow (1,1,1,1,1,1)
\end{aligned}$$

So that

$$L_{1212} = M_{1212} + M_{11112} + M_{12111} + M_{111111}$$

and

$$M_{1212} = L_{1212} - L_{11112} - L_{12111} + L_{111111}.$$

The relationship between the monomial symmetric functions m_λ and the monomial quasisymmetric functions M_α is given by

$$m_\lambda = \sum_\alpha M_\alpha \ ,$$

where the α range over permutations of the partition λ. Unfortunately, the fundamental basis has no such easily described relationship to a well-known symmetric function basis.

The fundamental basis gives us an important tool for understanding posets. Specifically, the form of the fundamental quasisymmetric basis given in equation 1.2 and the definition of π-compatibility 1.1.1 imply that

$$\sum_{\tau \in S_\pi} x^\tau = L_{\mathrm{co}(\pi)}(x).$$

Given a poset P, we define the *quasisymmetric function of a poset K_P* to be

$$K_P(x) = \sum_{\tau \in \mathscr{A}(P)} x^\tau.$$

In the case that P is naturally labelled, we also have from [19, Corollary 7.19.6] that

$$\begin{aligned} K_P(x) &= \sum_{\pi \in \mathscr{L}_P} \sum_{\tau \in S_\pi} x^\tau \\ &= \sum_{\pi \in \mathscr{L}_P} L_{\mathrm{co}(\pi)}(x), \end{aligned} \tag{1.5}$$

where the first equality here is from equation (1.1). Further, notice that for any two natural relabellings P', P'' of a poset P, we have $\mathscr{L}_{P'} = \mathscr{L}_{P''}$, and thus from equation (1.5), $K_{P'} = K_{P''}$. So, even though P may not be naturally labelled, we can use the above to calculate K_P.

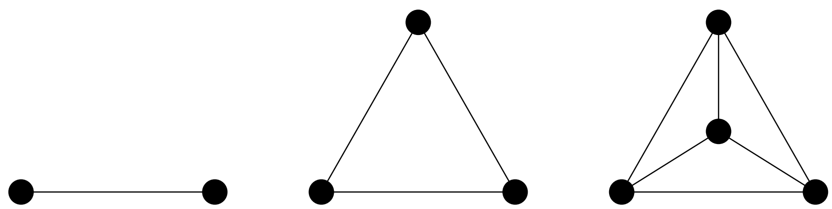
1.3 Graphs

1.3.1 Definitions

The results and definitions of this section can be found in most standard references on graph theory, for example Bollobás [4].

In the following, a *graph* G is a pair (V,E), where $V = [n]$ is the set of *vertices* of the graph and E is a multiset of unordered pairs of vertices, called *edges*. The *empty graph* $\emptyset$ is also a graph, with $V(\emptyset) = \emptyset$ and $E(\emptyset) = \emptyset$. A *directed graph* (or *digraph*) is similar, with the difference that E is a multiset of ordered pairs of vertices, called *directed edges*. We will write $n(G) = |V(G)|$ and $e(G) = |E(G)|$.

If there is an edge between two vertices, we will say those vertices are *adjacent*. A *path* in a graph is a sequence of vertices such that successive pairs are adjacent and such that no vertices are repeated. We will call a graph *connected* if for every pair of vertices u, v of G, there exists a path from u to v. A *connected component* of a graph is a maximal subset of the vertices which is connected. The number of connected components of G will be written as $c(G)$.



(a) K_2 (b) K_3 (c) K_4

Figure 1.3: Complete graphs

The *complete graph on n vertices K_n* is defined to be the n-vertex graph such that every pair of vertices forms an edge. The *complete bipartite graph $K_{m,n}$* is the graph with vertex set $X \cup Y$ where $|X| = m$, $|Y| = n$, and there is an edge between every pair $x \in X$, $y \in Y$.

A *cycle* in a graph is a sequence of vertices such that successive pairs are adjacent and such that no vertices are repeated except that the first and last vertex must be the same. The *length* of a cycle is the number of edges it contains (or equivalently the number of vertices it contains). The *girth* $g(G)$ (or simply g) of a graph G is the length of the smallest cycle of G.

A graph G is *acyclic* if G contains no cycles, and in that case we call G a *forest*. A connected, acyclic graph G is called a *tree*. The following is a basic result on trees.

Proposition 1.3.1. *Let G be a graph. Any two of the following conditions implies the third:*

1. *G is acyclic.*

2. *G is connected.*

3. *$e(G) = n(G) - 1$.*

Let $\mathrm{rk}(A)$, the *rank* of a subset A of edges of G, be defined as the size of a maximal acyclic subset of A. From this, we can see that any subset A of edges of a forest has $\mathrm{rk}(A) = |A|$, since any subset of an acyclic set is acylic. A subset $A \subseteq E(G)$ *spans* G if $\mathrm{rk}(A) = \mathrm{rk}(E)$. One can deduce the number of connected components of G from the rank of G, due to the fact that

$$c(G) = n(G) - \mathrm{rk}(G).$$

Given a subset $U \subset V(G)$ of vertices, we define $G|_U$, an *induced subgraph* of G, to be the graph with vertices U and edges

$$E(G|_U) = \{\{u, v\} \in E(G) | u, v \in U\}.$$

A *loop* is an edge which has a single vertex as both its endpoints. A *bridge* (or *coloop*) of a graph is an edge such that when it is removed, the graph becomes disconnected. There is a relationship between these two concepts related to taking the matroid dual (or planar dual, in the case that the graph is planar) which we will not investigate in this manuscript.

A *strong cycle* of a digraph is a sequence of vertices $v_1, \ldots, v_\ell, v_1$ such that $v_i \neq v_j$ for $i \neq j$ and also such that (v_i, v_{i+1}) is an edge of G for $i < \ell$ and (v_ℓ, v_1) is an edge of G. We define a *weak cycle* of a digraph to be the edges and vertices inherited from a cycle of the underlying undirected graph. A *source* of an orientation is a vertex in which all edges incident to the vertex are directed away from it, while a *sink* is a vertex with all incident edges directed towards it.

1.3.2 Orientations

An *orientation* of a graph G is a directed graph $\mathcal{O}$ with the same vertices, so that every edge $\{i, j\}$ of G corresponds to exactly one of (i, j) and (j,i) is an edge of $\mathcal{O}$. (If there are multiple edges $\{i, j\}$ of G, then $\mathcal{O}$ can have both edges of the form (i, j) and (j,i).) An orientation can be regarded as giving a direction to each edge of an undirected graph. An *acyclic* orientation of a graph G is an orientation $\mathcal{O}$ which posesses no strong cycles.

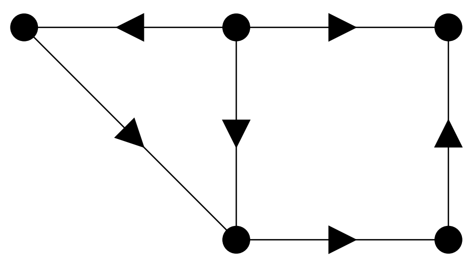


Figure 1.4: An acyclic orientation

Proposition 1.3.2. *Every loopless graph has an acyclic orientation.*

Proof. Let G be a graph with vertex set $[n]$. Choose any total ordering $<_{\mathcal{O}}$ of the vertex set. Then orient each edge i, j of G so that $(i, j) \in \mathcal{O}$ if and only if $i <_{\mathcal{O}} j$.

Let $v_1 v_2 \ldots v_\ell v_1$ be a weak cycle of $\mathcal{O}$. Notice that $\ell > 1$ since G is loopless. Now, it must be the case that this weak cycle is acyclically oriented, as otherwise WLOG $v_1 <_{\mathcal{O}} v_2 <_{\mathcal{O}} < \cdots <_{\mathcal{O}} v_\ell <_{\mathcal{O}} v_1$ which implies $v_1 <_{\mathcal{O}} v_1$, a contradiction. $\square$

We can generalize the idea of acyclicity in the following way:

Definition 1.3.3. For $k \geq 1$, an orientation $\mathscr{O}$ is *k-balanced* if, for every weak cycle $v_1 v_2 \dots v_\ell v_1$ of $\mathscr{O}$, there are at least k edges (v_i, v_{i+1}) and at least k edges (v_{j+1}, v_j) in $\mathscr{O}$ originating from the weak cycle.

This concept was published by Pretzel [12, p. 27], though he used the term *k-good*.

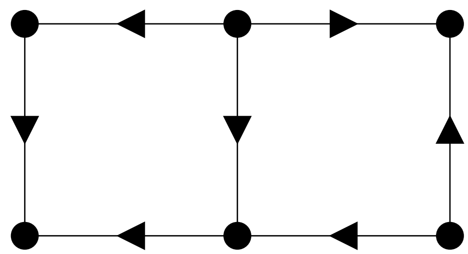


Figure 1.5: A 2-balanced orientation

Example 1.3.4. The orientation of Figure 1.4 is not 2-balanced, since in its 3-cycle, it does not have enough edges to satisfy the 2-balance condition. On the other hand, the orientation of Figure 1.5 is 2-balanced, which can be readily checked by examining each of its cycles.

From Definition 1.3.3, we see that being 1-balanced is equivalent to being acyclic. Thus, every loopless graph has a 1-balanced orientation by 1.3.2. However, it is not true that every loopless graph has a k-balanced orientation, for $k > 1$—for G to have a k-balanced orientation, it is necessary that its girth g satisfy $g \geq 2k$ since a cycle of length smaller than $2k$ cannot have k edges directed in each direction on the weak cycle. However, this is not enough—due to a result of Nešetřil and Rödl [11, Corollary 3], there exist graphs with arbitrarily high girth which are not 2-balanced. If $\mathscr{O}$ is k-balanced, then it is also k'-balanced for all $k' < k$. Thus, there are graphs of arbitrarily high girth which are not k-balanced for any $k > 1$.

We will give a concrete example of such a failure in Example 1.3.6, but first we require a proposition. The original inspiration for the definition of k-balance is due to the following idea, which originally appeared in Pretzel's paper.

Proposition 1.3.5. *A* 2*-balanced orientation of a graph G is equivalent to a poset which has G as its Hasse diagram.*

Proof. Let $\mathscr{O}$ be a 2-balanced orientation of G. Recall that this implies that $\mathscr{O}$ is an acyclic orientation. So, consider a relation $<$ on the vertices of G given by $v_1 < v_2$ if there is an edge (v_1, v_2) in $\mathscr{O}$ and its transitive closure. This relation is transitive by construction. It is antireflexive since if $v < v$, there exists a cycle in $\mathscr{O}$ with v as one of the vertices of the cycle. Similarly, it is antisymmetric because $v_1 < v_2$ and $v_2 < v_1$ implies there are two paths which, when concatenated, create a cycle in $\mathscr{O}$. Thus, $<$ gives a poset on V, and furthermore, the Hasse diagram of this poset is G.

To see this final claim, we must show the covering relations in the poset $(V, <)$ are exactly the edges of the graph G. If $v_1 < v_2$ is a covering relation, then it must be an edge of G, since all the order relations come from edges of $\mathscr{O}$ and their transitive closure; a covering relation can't come from a transitive closure (a multi-edge path in $\mathscr{O}$) and thus it must come from a single edge.

Assume then that (v_1, v_2) is an edge of $\mathscr{O}$ and there exists a $v_3 \in V$ such that $v_1 < v_3 < v_2$. Then there is a directed path from v_1 to v_2 which passes through v_3. Combining this path with the single edge (v_1, v_2), we obtain a weak cycle of G which is not 2-balanced. $\square$

Example 1.3.6. An example of a graph with girth $g = 4$ which is not 2-balanced is the Grötzsch graph Γ (Figure 1.6).

To see that Γ has no 2-balanced orientation, first notice that by the construction in Proposition 1.3.5, we can think of any 2-balanced orientation as a poset. Further, a

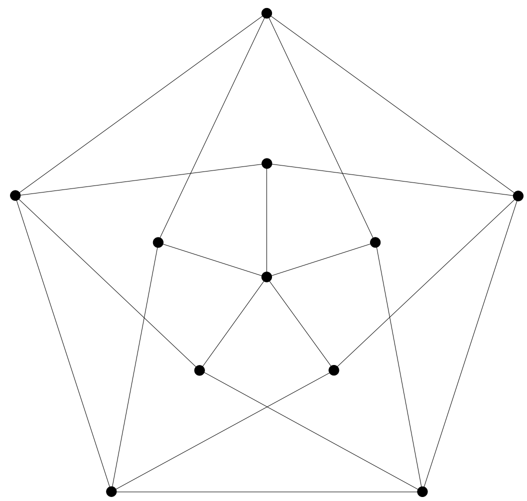


Figure 1.6: The Grötzsch graph

source of the orientation corresponds to a minimal element of the poset. Since every poset possesses a minimal element, from this we can see that any 2-balanced orientation contains a source. (In fact, any acyclic orientation contains a source, by the same construction; it does not matter that some edges of the graph may not appear in the Hasse diagram of the corresponding poset.)

In fact, if G is connected and has a k-balanced orientation $\mathscr{O}$, there is a k-balanced orientation $\mathscr{O}'$ of G with a specified vertex v as a unique source of $\mathscr{O}'$. To see this, let $u \neq v$ be a source of $\mathscr{O}$. We can create a new k-balanced orientation of G by changing u from a source to a sink while leaving all other edges directed the same—any weak cycle not involving u is unchanged, and any weak cycle involving u is still k-balanced since two adjacent edges have swapped directions. Continuing this process will eventually establish v as a source of G, and further transformations will eliminate all other sources of G.

Thus, to see that Γ has no 2-balanced orientation, without loss of generality we can assume for contradiction that there is a 2-balanced orientation with the central vertex (as drawn above) as the unique source. Then all the edges except those on the periphery must be directed outwards (since the central vertex is the *unique* source) and any choice of directions for the edges on the periphery will give a cycle of length four which is not 2-balanced due to the pigeonhole principle.

1.3.3 Graph Colorings and the Tutte Polynomial

A *coloring* of a graph G is a map $\kappa : V(G) \to \mathbb{N}$ such that if $\kappa(i) = \kappa(j)$, then $\{i, j\}$ is not an edge of G.

Given any coloring κ of G, we can construct the *orientation induced by* κ $\mathscr{O}_\kappa$ by directing each edge of G towards the vertex with the larger color. That is, for each $\{u, v\} \in E(G)$, if $\kappa(u) > \kappa(v)$ then orient the edge as (v, u), and otherwise orient it as (u, v). See Figure 1.7 for an example of an orientation induced by a coloring.

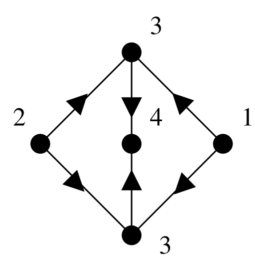


Figure 1.7: A coloring and its induced orientation

Orientations which are induced by colorings have the following convenient property.

Proposition 1.3.7. *Given a coloring κ of the graph G, the induced orientation $\mathscr{O}_\kappa$ is acyclic.*

Proof. The reasoning in this proof is similar to that of Proposition 1.3.2. Assume for contradiction that $v_1, \ldots, v_\ell, v_1$ is a strong cycle of $\mathscr{O}_\kappa$. Then we have

$$\kappa(v_1) < \kappa(v_2) < \cdots < \kappa(v_\ell) < \kappa(v_1),$$

a contradiction. $\square$

The *chromatic polynomial* of G is the function $\chi_G : V \to \mathbb{N}$ where $\chi(\lambda)$ equals the number of colorings κ of G such that $\kappa(V) \subseteq [\lambda]$.

Given a graph G and an edge e of G, the graph $G - e$ with vertex set $V(G-e) = V(G)$ and edge set $E(G-e) = E(G) \setminus e$ is called G *delete* e. If the endpoints of e are v and v', then we define G/e, G *contract* e, by $V(G/e) = V(G) \setminus \{v, v'\} \cup \{vv'\}$ and edge set given by

$$E(G/e) = \{\{u,u'\} \in E(G) | u,u' \neq v,v'\} \cup \{\{u,vv'\} | \{u,v\} \in E(G) \text{ or } \{u,v'\} \in E(G)\}.$$

Colloquially we think of identifying the vertices v and v' in G.

Using deletion and contraction, we can prove that the chromatic polynomial is a polynomial function of its argument by the following *deletion-contraction recurrence*.

Proposition 1.3.8. *The chromatic polynomial χ_G satisfies the relation*

$$\chi_G = \chi_{G-e} - \chi_{G/e}$$

for all edges $e \in E(G)$.

Proof. Let the endpoints of e be u, v. Then in a coloring κ of $G - e$, either u and v are colored differently or the same. In the former case, κ is a coloring of G also. In the latter case, κ induces a coloring of G/e in the obvious way. This process can be reversed also to get colorings of $G - e$ from either a coloring of G or a coloring of G/e.

Thus, we have $\chi_{G-e} = \chi_G + \chi_{G/e}$. $\square$

Corollary 1.3.9. *The chromatic polynomial χ_G is a polynomial with integer coefficients.*

Proof. We induct on the number of edges of G. Using 1.3.8, it is enough to show that edgeless graphs have χ_G a polynomial with integer coefficients, since both deletion and contraction reduce the number of edges of a graph. Finally, the number of colorings of an edgeless graph with λ colors is λ^n, an integer polynomial in λ. □

Example 1.3.10. For an example of a chromatic polynomial calculation, take the complete graph on n vertices K_n. Any coloring of K_n using λ colors must choose a different color for each vertex, and any such choice will give a coloring of K_n. Thus,

$$\chi_{K_n} = \lambda(\lambda-1)\dots(\lambda-n+1).$$

Given a coloring κ on G, let

$$x_\kappa = \prod_{v\in G} x_{\kappa(v)}.$$

There is a refinement of the chromatic polynomial introduced by Stanley ([17]) called the *chromatic symmetric function*, defined by

$$X_G = \sum_\kappa x_\kappa,$$

where the summation is over colorings of G.

To see that X_G is symmetric, consider the action of switching any two colors in a coloring of G. That is, given a coloring κ and some transposition $\sigma = (ij)$, define $\sigma\kappa$ as the coloring of G where all vertices colored i by κ are colored j by $\sigma\kappa$ and vice versa. This is still a coloring, as no edges have become monochromatic. Furthermore, σ is an involution, which demonstrates a bijection. Thus, if we allow σ to act on X_G in the obvious way, we have that $\sigma X_G = X_G$, which means that X_G is symmetric.

The chromatic symmetric function is a refinement of χ_G in the sense that X_G contains strictly more information than χ_G. That is, X_G not only enumerates colorings, but also records the particular colors which were used in those colorings. Thus, combinatorially we can recover χ_G from X_G. The situation is even nicer than that however—algebraically we recover χ_G by way of the principal specialization of X_G. That is,

$$\mathrm{ps}_\lambda(X_G) = \chi_G(\lambda). \tag{1.6}$$

Another invariant closely related to the chromatic polynomial is $T_G(x,y)$, the *Tutte polynomial* of a graph G, defined as

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}, \tag{1.7}$$

where $\mathrm{rk}(A)$ is the number of edges in a maximal acyclic subgraph of A. The importance of the Tutte polynomial is due to many factors—for a survey see [5]—but for our purposes the most important fact is that essentially any graph invariant which satisfies a deletion-contraction recurrence (as the chromatic polynomial does above) is some evaluation of the Tutte polynomial. This idea will be made more specific in Section 3.4.

One fact which we will use about the Tutte polynomial is that evaluations of it can give interesting information about a graph.

Example 1.3.11. The Tutte evaluation $T_G(1,1)$ counts the number of spanning forests of G. This is because in Equation 1.7, when $x = 1$, $y = 1$, all the terms are zero except those corresponding to edge sets whose rank is maximal and whose cardinality is equal to their rank. These are precisely the spanning forests.

As another example without proof, the Tutte evaluation $T_G(2,0)$ counts (up to sign) the number of acyclic orientations of G. In fact, it is true that

$$\chi_G(\lambda) = (-1)^{n-c(G)}\lambda^{c(G)}T_G(1-\lambda,0),$$

where $c(G)$ is the number of connected components of G.

1.4 Hopf algebras

The material from this section can be found in Sweedler's text ([21]) on Hopf algebras.

1.4.1 Algebras and Coalgebras

Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*algebra* is a $\mathbb{F}$-vector space $\mathscr{A}$ along with linear maps $M : \mathscr{A}\otimes\mathscr{A} \to \mathscr{A}$ called *multiplication* and $u : \mathbb{F} \to \mathscr{A}$ called the *unit* such that the following diagrams commute.

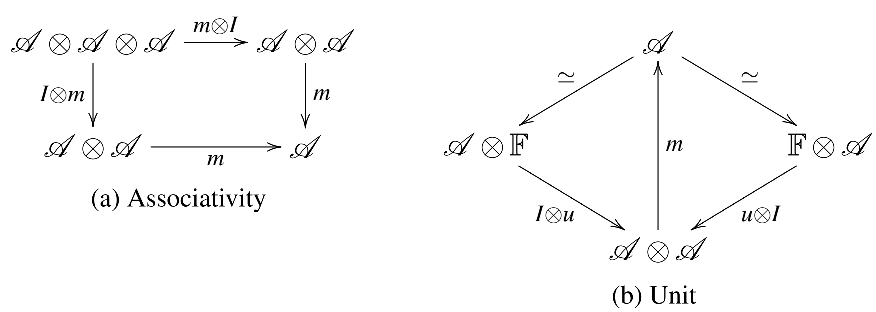


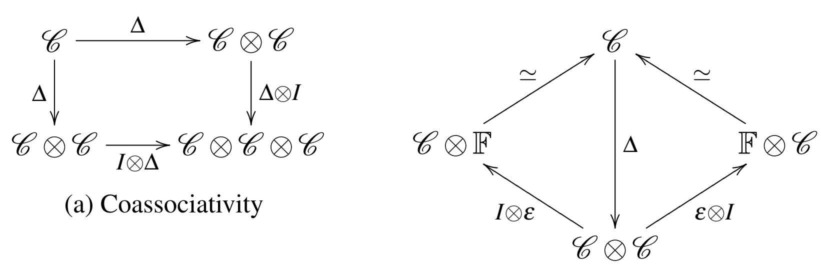
Figure 1.8: Algebra diagrams

(We will use I to denote the identity map.) Figure 1.8(a) shows the associative property, which is the standard associative property of algebras, and Figure 1.8(b) shows that the unit element $u(1)$ of $\mathscr{A}$, when multiplied with any element a, simply gives a. For convenience, we will sometimes write $M(a_1 \otimes a_2)$ as $a_1 \cdot a_2$.

Given two $\mathbb{F}$-algebras $\mathscr{A}$, $\mathscr{A}'$, an $\mathbb{F}$-*algebra morphism* (or simply *algebra morphism*) is a map $f : \mathscr{A} \to \mathscr{A}'$ such that $M_{\mathscr{A}'} \circ (f \otimes f) = f \circ M_{\mathscr{A}}$ and $f \circ u_{\mathscr{A}} = u_{\mathscr{A}'}$.

The definition of an algebra which is presented here may seem to include unnecessary formalism—definitions in most texts suppress any mention of tensor products (see for example [13, p. 31]) and simply define an algebra over a field as a vector space with a multiplication that is associative, distributive, and unital. However, the formalism here does serve an important role in that it allows us to more easily understand the symmetry in the following definition.

In a converse fashion to the definition of an algebra, we define an $\mathbb{F}$-*coalgebra* to be an $\mathbb{F}$-vector space $\mathscr{C}$ along with linear maps $\Delta : \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$ called *comultiplication* and $\varepsilon : \mathscr{C} \to \mathbb{F}$ called the *counit* such that the following diagrams commute.



(a) Coassociativity

(b) Counit

Figure 1.9: Coalgebra diagrams

Notice that the above maps are the same as those for an algebra, with all the arrows reversed in direction. The intuitive idea is that while an algebra gives us a way to take two objects and combine them, a coalgebra tells us how an object decomposes (which generally is not inverse to the amalgamation given by the algebra structure).

Figure 1.9(a) states that applying Δ twice always gives the same sum of tensors irrespective of the order of application, and figure 1.9(b) states that applying ε to either tensor component after applying Δ is essentially the identity map.

Stating these facts about coalgebras is simpler with *Sweedler notation* where we write

$$\Delta(c) = \sum c_1 \otimes c_2.$$

Since the result of a coproduct is always a summation of simple tensors (those of the form $c_1 \otimes c_2$) we ignore the specifics of how the coproduct decomposes c and pay attention to the form—the simple tensor $c_1 \otimes c_2$ is a representative of any summand of the coproduct. Thus, we can write coassociativity as

$$\sum \Delta(c_1) \otimes c_2 = \sum c_1 \otimes \Delta(c_2) = \sum c_1 \otimes c_2 \otimes c_3$$

and the counit property as

$$\sum \varepsilon(c_1) \otimes c_2 = \sum c_1 \otimes \varepsilon(c_2) = c.$$

Notice that the existence of coassociativity actually implies that the notation $c_1 \otimes c_2 \otimes c_3$ is well-defined.

Given two $\mathbb{F}$-coalgebras $\mathscr{C}$, $\mathscr{C}'$, an $\mathbb{F}$-*coalgebra morphism* (or simply *coalgebra morphism*) is a map $f : \mathscr{C} \to \mathscr{C}'$ such that $\Delta_{\mathscr{C}'} \circ f = (f \otimes f) \circ \Delta_{\mathscr{C}}$ and $\varepsilon_{\mathscr{C}'} \circ f = \varepsilon_{\mathscr{C}}$.

Example 1.4.1. Some examples of algebras and coalgebras are:

1. Given any group G, we can form the vector space $\mathbb{C}[G]$ of all formal linear combinations of elements of G over $\mathbb{C}$. To form an associative algebra, we define multiplication as given by the group operation, $g_1 \cdot g_2 = g_1 g_2$ and the unit as $u(1) = e$, the group identity. For a coalgebra structure, we take comultiplication as given by $\Delta(g) = g \otimes g$, and the counit as $\varepsilon(g) = 1$.

2. The vector space $\mathbb{F}[t]$ of polynomials in one variable has the standard algebraic structure, but it also has a coalgebraic structure given by

$$\Delta(p(t)) = p(t \otimes 1 + 1 \otimes t)$$

and $\varepsilon_0(p(t)) = p(0)$.

3. Let $\mathscr{R}$ be the vector space of all formal linear combinations over $\mathbb{C}$ of isomorphism classes of graded posets. (That is, posets with a maximal element $\hat{1}$, a minimal element $\hat{0}$, and with all maximal chains the same length.)

   Then if P_1, P_2 are representatives of isomorphism classes, we let $P_1 \cdot P_2 = P_1 \times P_2$, the Cartesian product of P_1, P_2 and we let $u(1) = \bullet$, the poset with one element. (Recall that the Cartesian product of two posets is the poset on $P_1 \times P_2$ where $(p_1, p_2) \leq (p_1', p_2')$ if and only if $p_1 \leq p_1'$ and $p_2 \leq p_2'$.)

   To get a coalgebraic structure, we define

$$\Delta(P) = \sum_{x \in P} [\hat{0}, x] \otimes [x, \hat{1}],$$

   where $[p, p'] = \{x \mid p \leq x \leq p'\}$, and we define

$$\varepsilon(P) = \begin{cases} 1, & P = \bullet \\ 0, & P \neq \bullet \end{cases}.$$

1.4.2 Bialgebras and Hopf Algebras

Given any two $\mathbb{F}$-linear maps $F, G : \mathscr{C} \to \mathscr{A}$ from a coalgebra to an algebra, we define the convolution product $F * G$ to be

$$F * G = M \circ (F \otimes G) \circ \Delta.$$

Proposition 1.4.2. *The set* $\mathrm{Hom}(\mathscr{C}, \mathscr{A})$ *of linear maps from* $\mathscr{C}$ *to* $\mathscr{A}$ *with the operation of convolution forms a monoid with identity element* $u_{\mathscr{A}} \varepsilon_{\mathscr{C}}$.

Proof. Given two such linear maps F, G, it's clear that $F * G$ is linear since it is a composition of linear maps. Convolution is associative due to the associativity and coassociativity of the underlying structures. If F, G, $H \in \mathrm{Hom}(\mathscr{C}, \mathscr{A})$, then

$$\begin{aligned}((F * G) * H))(c) &= \sum (F * G)(c_1) \cdot H(c_2) \\ &= \sum F(c_1) \cdot G(c_2) \cdot H(c_3) \\ &= \sum F(c_1) \cdot (G * H)(c_2) \\ &= (F * (G * H))(c).\end{aligned}$$

Finally, $u_{\mathscr{A}} \varepsilon_{\mathscr{C}}$ is the identity since

$$(F * u_{\mathscr{A}} \varepsilon_{\mathscr{C}})(c) = \sum F(c_1) \cdot u_{\mathscr{A}} \varepsilon_{\mathscr{C}}(c_2) = F(c) \cdot u(1) = F(c).$$

□

Let $\mathscr{H}$ be an $\mathbb{F}$-vector space with algebra maps M, u and coalgebra maps Δ, ε. If M, u are coalgebra morphisms, then we call $\mathscr{H}$ a *bialgebra*. (Notice that $\mathscr{A} \otimes \mathscr{A}$ has a coalgebra structure with comultiplication $\Delta \otimes \Delta$ and counit $\varepsilon \otimes \varepsilon$.) Equivalently, if Δ,

ε are algebra morphisms, then $\mathscr{H}$ is a bialgebra. (The diagrams for the two statements are exactly the same.)

If $\mathscr{H}$ is a bialgebra, then we can consider the convolution product of elements of $\mathrm{Hom}(\mathscr{H},\mathscr{H})$. In particular, if the identity map I on $\mathscr{H}$ has an inverse under convolution, we say that $\mathscr{H}$ is a *Hopf algebra* and that $S = I^{-1}$ is the *antipode* of $\mathscr{H}$. Using Sweedler notation, S is defined to be the unique map such that

$$\sum S(h_1)\cdot h_2 = \sum h_1 \cdot S(h_2) = u\varepsilon(h).$$

A Hopf algebra $\mathscr{H}$ is *graded* if $\mathscr{H} = \bigoplus_{n\geq 0}\mathscr{H}_n$ as vector spaces, and multiplication and comultiplication respect this decomposition, i.e.,

$$m(\mathscr{H}_i\otimes\mathscr{H}_j)\subseteq\mathscr{H}_{i+j} \quad \text{and} \quad \Delta(\mathscr{H}_k)\subseteq\sum_{i+j=k}\mathscr{H}_i\otimes\mathscr{H}_j.$$

In the case that $\mathscr{H}$ is graded, we will say that elements in $\mathscr{H}_n$ have *degree* n. Meanwhile, $\mathscr{H}$ is *connected* if $\dim(\mathscr{H}_0) = 1$. Most (if not all) of the Hopf algebras arising naturally in combinatorics are graded and connected. Usefully, if $\mathscr{H}$ is a graded and connected bialgebra, its antipode can be defined inductively as follows:

$$\begin{aligned} S(h) &= h, && h\in\mathscr{H}_0, \\ (m\circ(S\otimes I)\circ\Delta)(h) &= 0, && h\in\mathscr{H}_i, i>0. \end{aligned} \tag{1.8}$$

Example 1.4.3. The examples of bialgebras given in 1.4.1 are all Hopf algebras:

1. The algebra and coalgebra $\mathbb{C}[G]$ is actually a bialgebra, and further has an antipode given by $S(g) = g^{-1}$. To see this, note that $u\varepsilon(g) = e$ for all g, so that S must satisfy

$$S(g)g = gS(g) = e,$$

which means S must be as above. Thus $k[G]$ is a Hopf algebra.

2. The vector space of polynomials in one variable $\mathbb{F}[t]$ forms a bialgebra, and since it is graded and connected, it has an antipode. (The Hopf algebra grading on $\mathbb{F}[t]$ is simply the standard grading, where the homogeneous elements are monomials and their degree is their degree as polynomials.) However, we can describe the antipode in this case concretely—if we let $S(p(t)) = p(-t)$, then

$$(M \circ (S \otimes I) \circ \Delta)(p(t)) = p(-t \cdot 1 + 1 \cdot t) = p(0)$$

as desired. This Hopf algebra is called the *binomial Hopf algebra*.

3. The vector space $\mathscr{R}$ of linear combinations of isomorphism classes of graded posets is a graded and connected bialgebra, and thus has an antipode by the immediately preceding remark. This Hopf algebra is often called *Rota's Hopf algebra*.

The following proposition gives a useful general fact about graded and connected Hopf algebras.

Proposition 1.4.4. *Let $\mathscr{H}$ be a graded and connected Hopf algebra. Then for any homogeneous $h \in \mathscr{H}_n$, the coproduct Δ has the form*

$$\Delta(h) = h \otimes 1 + \sum h_1 \otimes h_2 + 1 \otimes h,$$

where the summation includes only simple tensors whose tensor factors have degrees strictly between 0 and n.

Proof. To see why this should be the case, recall the counit property from Figure 1.9(b), which gives that $\sum h_1 \otimes \varepsilon(h_2) = h$. This implies that one of the summands $h' \otimes h''$ of the coproduct $\Delta(h)$ must have $h' \in \mathscr{H}_n$. Gradedness of $\mathscr{H}$ then gives $h'' \in \mathscr{H}_0$, which is isomorphic to $\mathbb{F}$, by connectedness. Multilinearity of the tensor product then allows us to collect all such terms belonging to $\mathscr{H}_n \otimes \mathscr{H}_0$ into a single simple tensor, which must have the form $h \otimes c$, for some $c \in \mathbb{F}$. Since the counit is an algebra morphism in a bialgebra, and also since we have $\varepsilon(c) = 1$ by the counit property, this implies that $c = 1$. □

Now, define the reduced coproduct

$$\tilde{\Delta}(h) = \Delta(h) - (h \otimes 1 + 1 \otimes h).$$

The result of this operator only has summands with nontrivial tensor components. Notice that if $h \in \mathscr{H}_n$, then necessarily $\tilde{\Delta}^n(h) = 0$.

1.4.3 Combinatorial Hopf Algebras

A *character* of a Hopf algebra $\mathscr{H}$ is an algebra morphism $\phi : \mathscr{H} \to \mathbb{F}$. We denote by $\mathbb{X}(\mathscr{H})$ the set of all characters of $\mathscr{H}$. Since there is a coalgebra structure on $\mathscr{H}$, from (1.4.2) the set of characters forms a monoid under convolution. Further, since characters are algebra morphisms and not just ring homomorphisms, we have the following.

Proposition 1.4.5. *The characters of $\mathscr{H}$ form a group with identity ε and inverse given by $\phi^{-1} = \phi \circ S$.*

Proof. By (1.4.2), all we need show is that every character has an inverse. So, consider

$$\begin{aligned}(\phi * (\phi \circ S))(h) &= \sum \phi(h_1) \cdot \phi(S(h_2)) \\ &= \phi\left(\sum h_1 \cdot S(h_2)\right) \\ &= \phi(u(\varepsilon(h)) \\ &= \varepsilon(h)\end{aligned}$$

where the last equality is by the fact that $\phi(u(z)) = z\phi(u(1)) = z$ for all $z \in \mathbb{F}$. □

Notice that under the convolution product, powers of characters have an interesting characterization

$$\begin{aligned}\phi^k(h) &= \sum \phi(h_1) \cdot \dots \cdot \phi(h_k) \\ &= \phi\left(\sum h_1 \cdot \dots \cdot h_k\right) \\ &= \phi(\Delta^{k-1}(h)).\end{aligned}$$

Since $\mathscr{H}$ is graded, there is an involution $\overline{h} = (-1)^n h$ for $h \in \mathscr{H}_n$. This extends to an involution on $\mathbb{X}(\mathscr{H})$, given by $\overline{\phi}(h) = \phi(\overline{h})$.

Following Aguiar, Bergeron, and Sottile [2, SS2], we now define a *combinatorial Hopf algebra* (CHA) as a pair $(\mathscr{H}, \zeta)$, where $\mathscr{H}$ is a graded, connected Hopf algebra and ζ is a fixed character of $\mathscr{H}$. Give two CHAs $(\mathscr{H}_1, \zeta_1)$, $(\mathscr{H}_2, \zeta_2)$, a *CHA morphism* is a map $\Phi : \mathscr{H}_1 \to \mathscr{H}_2$ that is an algebra and coalgebra morphism which also has the property that $\zeta_2 \circ \Phi = \zeta_1$.

Example 1.4.6. Given compositions $\alpha \models m, \beta \models n$, define the *concatenation* of α and β to be

$$\alpha + \beta = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell).$$

Notice that $\alpha+\beta \models m+n$.

Now, recall that $\mathscr{Q}$ is the vector space of quasisymmetric functions over a field $\mathbb{F}$. Then $\mathscr{Q}$ has a Hopf algebra structure where multiplication is inherited from the algebra of formal power series, and the coproduct is defined by

$$\Delta(M_\alpha) = \sum_{\alpha_1+\alpha_2=\alpha} M_{\alpha_1} \otimes M_{\alpha_2}.$$

The counit then is the principal specialization ps_0 which sets every x_i to 0. Then $\mathscr{Q}$ is connected and it is graded, since its inherited multiplication is clearly graded and the comultiplication is graded due to the remark at the end of the previous paragraph.

To make $\mathscr{Q}$ into a CHA, Aguiar, Bergeron, Sottile used the character $\zeta_{\mathscr{Q}} = \mathrm{ps}_1$ that essentially detects whether M_α has α of length less than or equal to 1. The importance of $\mathscr{Q}$ in this context is that, given any CHA $(\mathscr{H}, \zeta)$, there is a CHA morphism Ψ : $\mathscr{H} \to \mathscr{Q}$. This morphism assigns any element of $\mathscr{H}$ to a quasisymmetric function, the coefficients of which are often quite interesting. Their main example involves an alternative proof of the generalized Dehn-Sommerville equations of [3].

The previous example was the inspiration for much of Chapter 3, using the binomial Hopf algebra with the character $\varepsilon_1(p(t)) = p(1)$ in place of the quasisymmetric functions. The following proposition plays the same role here as the main theorem of Aguiar, Bergeron, Sottile.

Proposition 1.4.7. *If $(\mathscr{H}, \zeta)$ is a CHA, then there exists a CHA morphism $\Phi : \mathscr{H} \to \mathbb{F}[t]$ where $\Phi(h) = P_{\zeta,h}(t)$ has the property that $P_{\zeta,h}(k) = \zeta^k(h)$ for all $k \in \mathbb{Z}$.*

Proof. Recall that $\mathscr{H}$ is graded and connected, so we know that the form of the coproduct given in Proposition 1.4.4 applies. In particular, we can use the reduced coproduct Δ', defined immediately after that proposition.

We will first show that, for $h \in \mathscr{H}_n$ and $k \geq 0$, $\zeta^k(h)$ is a polynomial in k.

Firstly, if $h \in \mathscr{H}_0$, then since ζ is a multiplicative linear map

$$\begin{aligned}\zeta^k(h) &= h \cdot \zeta^k(1) \\ &= h \sum \zeta(1) \cdot \dots \cdot \zeta(1) \\ &= h\end{aligned}$$

so that $\zeta^k(h)$ is a constant polynomial in k for $k \geq 0$.

Now assume that $h \in \mathscr{H}_n$ for $n > 0$. We have

$$\begin{aligned}\zeta^k(h) &= \sum \zeta(h_1) \cdot \dots \cdot \zeta(h_k) \\ &= \sum_{j=1}^{n} \binom{k}{j} \zeta(\tilde{\Delta}^{j-1}(h))\end{aligned}$$

where we let j count the components of the coproduct which do not belong to $\mathscr{H}_0$. Notice that the summation runs from 1 to n as h cannot be split into more than n nontrivial parts.

If D denotes the difference operator $Df(k) = f(k) - f(k-1)$, then since $\binom{k}{j} - \binom{k-1}{j} = \binom{k-1}{j-1}$, we have

$$D\zeta^k(h) = \sum_{j=1}^{n} \binom{k-1}{j-1} \zeta(\tilde{\Delta}^{j-1}(h))$$

and in general,

$$D^i\zeta^k(h) = \sum_{j=1}^{n} \binom{k-i}{j-i} \zeta(\tilde{\Delta}^{j-1}(h)).$$

Thus, $D^{n+1}\zeta^k(h) = 0$ for all $k \geq n+1$, so that $\zeta^k(h) = P_{\zeta,h}(k)$ is a polynomial for $k \geq 0$.

Thus, there is a well-defined map from $\mathscr{H}$ to $\mathbb{F}[t]$, which we will now show is a Hopf algebra morphism.

Firstly, Φ is a linear transformation; notice that for all $k \geq 0$, for h, $h' \in \mathscr{H}$,

$$\begin{aligned}\Phi(ah+bh')(k) &= \zeta^k(ah+bh') \\ &= a\zeta^k(h)+b\zeta^k(h') \\ &= (a\Phi(h)+b\Phi(h'))(k).\end{aligned}$$

Since these polynomials agree on all positive integers, they are necessarily the same polynomial. Similarly, to see that Φ is an algebra morphism, we have for all $k \geq 0$ that

$$\begin{aligned}\Phi(h\cdot h')(k) &= \zeta^k(h\cdot h') \\ &= \zeta^k(h)\zeta^k(h') \\ &= (\Phi(h)\Phi(h'))(k).\end{aligned}$$

To see that Φ is a coalgebra morphism, notice that for all k, $\ell \geq 0$,

$$\begin{aligned}\Phi(h)(k+\ell) &= \zeta^{k+\ell}(h) \\ &= \sum \zeta(h_1)\ldots\zeta(h_k)\zeta(h_{k+1})\ldots\zeta(h_{k+\ell}) \\ &= \sum \zeta((h_1)_1)\ldots\zeta((h_1)_k)\zeta((h_2)_1)\ldots\zeta((h_2)_\ell) \\ &= \sum \zeta^k(h_1)\zeta^\ell(h_2),\end{aligned}$$

where $(h_i)_j$ represents the j^{th} tensor component of the iterated coproduct on h_i. Notice that the 3rd equality follows from coassociativity. The above proves that $\Delta(\Phi(h)) = (\Phi\otimes\Phi)(\Delta(h))$.

Thus, Φ is a Hopf algebra morphism.

Finally,

$$\zeta(h) = P_{\zeta,h}(1) = \varepsilon_1(P_{\zeta,h}),$$

proving that Φ is a combinatorial Hopf algebra morphism. As a consequence, $\zeta^{-1}(h) = \varepsilon_1^{-1}(P_{\zeta,h})$. To find the inverse of this character, recall that the antipode in the binomial Hopf algebra is $S(f(t)) = f(-t)$. Thus,

$$\varepsilon_1^{-1}(f(t)) = \varepsilon_1(S(f(t))) = \varepsilon_1(f(-t)) = f(-1).$$

In the case of ζ, we then have

$$\zeta^{-k}(h) = P_{\zeta,h}(-k)$$

for all $k \geq 0$. □

Using the result of Aguiar, Bergeron, Sottile mentioned in Example 1.4.6 that there is a CHA morphism from any CHA $\mathscr{H}$ to $\mathscr{Q}$, there is a less lengthy proof than the above. Specifically, it can be shown that the map from $\mathscr{Q}$ to $\mathbb{F}[t]$ where $F \mapsto \mathrm{ps}_t(F)$ is a CHA morphism, and furthermore that for any CHA $\mathscr{H}$ and $h \in \mathscr{H}$,

$$ps_t(\Psi(h)) = \Phi(h).$$

From this the proposition follows.

Chapter 2

The k-chromatic Quasisymmetric Function of a Graph

In this chapter, we create a generalization of Stanley's chromatic symmetric function X_G by enumerating k-balanced colorings as opposed to all colorings. In so doing we are able to prove some generalizations of well-known results related to the chromatic symmetric function and the chromatic polynomial.

The material of this chapter has been published in [7].

2.1 Definitions

Keeping in mind that, by Proposition 1.3.7, colorings are closely related to acyclic orientations, we make the following definition.

Definition 2.1.1. Given a graph G, a *k-balanced coloring* κ of G is a coloring where the induced orientation $\mathscr{O}_\kappa$ is k-balanced.

In Figure 2.1 there is an example of a 2-balanced coloring along with the 2-balanced orientation that it induces.

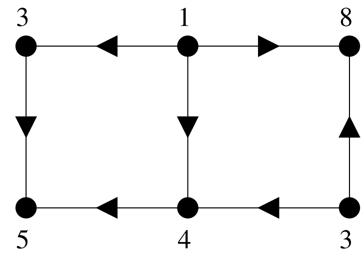


Figure 2.1: A 2-balanced coloring

Definition 2.1.2. Given a simple graph G with n vertices and any positive integer k, define the *k-balanced chromatic quasisymmetric function of G* by

$$X_G^k = X_G^k(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)},$$

the sum over all k-balanced colorings.

To see that X_G^k is indeed quasisymmetric, let κ be a k-balanced coloring and let $\tau : \mathbb{N} \to \mathbb{N}$ be an order-preserving injection. Then $\kappa' = \tau \circ \kappa$ is also a proper coloring, and since τ is order-preserving, every edge of $\mathscr{O}_{\kappa'}$ is oriented identically in $\mathscr{O}_{\kappa}$ so that κ' is also k-balanced. If τ^* is defined by $\tau^*(x_i) = x_{\tau(i)}$, then the previous implies that X_G^k is invariant under any τ^*, which is exactly the condition necessary for quasisymmetry.

The function X_G^k is naturally expressible in the monomial basis for $\mathscr{Q}$. That is, the coefficient c_α of M_α is the number of k-balanced colorings κ of G such that α_1 vertices of G are colored with the smallest color of κ, α_2 vertices with the next smallest, and so on.

In the case that $k = 1$, X_G^1 is symmetric. In particular, a 1-balanced coloring is exactly what we have defined as a coloring, so X_G^1 is Stanley's chromatic symmetric function X_G. In general, however, X_G^k is not symmetric. For example, in Example 2.3.5 we will show that $[M_{2121}]X_{K_{3,3}}^2 = 36$, but $[M_{2112}]X_{K_{3,3}}^2 = 18$. In fact, this is the smallest example—a graph G with fewer than 6 vertices has X_G^k symmetric.

The girth g of a graph G plays an important role in determining X_G^k, as must be expected from the remarks about girth preceding 1.3.5 . That is, if $k > \frac{g}{2}$, $X_G^k = 0$. As a special case, if G has a triangle, then $g = 3$ and so $X_G^k = 0$ for $k > 1$. Alternately, if $g = \infty$ (that is, G is a forest), then the condition that weak cycles are k-balanced is vacuous, so that $X_G^k = X_G$.

Directly calculating X_G^k is difficult, so to simplify the process we have created a set of Maple procedures which can be found in Appendix A.

2.2 L-positivity

We have discussed the k-balanced chromatic quasisymmetric function in the standard monomial basis, where the coefficients count colorings. As we now show, X_G^k has a natural positive expansion in the fundamental basis $\{L_\alpha\}$. The idea of the proof is to interpret colorings as certain P-partitions.

Corollary 2.2.1. *For all graphs G and for all k, X_G^k is L-positive.*

Proof. Let $\mathcal{O}$ be any k-balanced orientation of G, and define $P_{\mathcal{O}}$ to be the poset induced by $\mathcal{O}$ in the sense of Proposition 1.3.5.

Choose an arbitrary natural relabelling $P'_{\mathcal{O}}$ of $P_{\mathcal{O}}$. Now a $P'_{\mathcal{O}}$-partition is just an order-preserving map $f : P'_{\mathcal{O}} \to \mathbb{N}$. If we consider f as a function on the undirected graph G, then f is a coloring of G and $\mathcal{O}_f = \mathcal{O}$. That is to say, f is a k-balanced coloring of G. Thus, any $P'_{\mathcal{O}}$-partition is a k-balanced coloring of G.

Conversely, any k-balanced coloring κ is a $P'_{\mathcal{O}_\kappa}$-partition for the appropriate natural relabelling $P'_{\mathcal{O}_\kappa}$ of $P_{\mathcal{O}_\kappa}$.

Thus,

$$X_G^k = \sum_{\mathscr{O}} K_{P_{\mathscr{O}}}$$
$$= \sum_{\mathscr{O}} \sum_{\pi \in \mathscr{L}_{P'_{\mathscr{O}}}} L_{c(\pi)},$$

where the sum is over all k-balanced orientations $\mathscr{O}$ of G. □

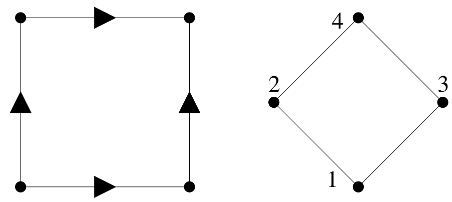


Figure 2.2: $\mathscr{O}_1$

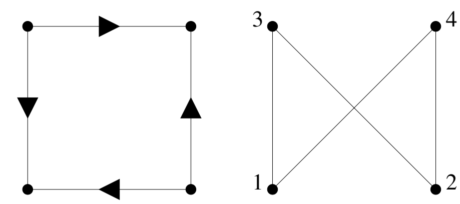


Figure 2.3: $\mathscr{O}_2$

For an example of how the formula in the the proof works in practice, consider the 4-cycle C_4. Up to relabeling the vertices, the 2-balanced orientations of C_4 are of two types: $\mathscr{O}_1$ pictured in Figure 2.2, and $\mathscr{O}_2$ pictured in Figure 2.3. There are 4 orientations of the form $\mathscr{O}_1$ and 2 orientations of the form $\mathscr{O}_2$. We then calculate the linear extensions for $P_{\mathscr{O}_1}$ as $\{1234, 1324\}$ and for $P_{\mathscr{O}_2}$ as $\{1234, 1243, 2134, 2143\}$. Thus,

$$X_{C_4}^2 = K_{P_{\mathscr{O}_1}} + K_{P_{\mathscr{O}_2}}$$
$$= 4(L_{1111} + L_{121}) + 2(L_{1111} + L_{112} + L_{211} + L_{22})$$
$$= 6L_{1111} + 2L_{211} + 4L_{121} + 2L_{112} + 2L_{22}.$$

This is in practice a much quicker way to compute X_G^k than the original definition, which requires one to check the k-balance of every proper coloring of G, which in turn amounts to checking each weak cycle of the graph for each proper coloring. However, this form is still computationally intensive, as it requires one to first find every k-balanced orientation of G. Using the Maple routines listed in Appendix A with graphs with more than 10 edges is still very slow, despite applying the formula in the proof.

2.3 X_G^k on Special Classes of Graphs

2.3.1 Cycles

Let the cyclic graph on n vertices be denoted by C_n. The colorings of C_n which are not 2-balanced are easy to describe—they are the colorings with n distinct colors arranged in order around the cycle. We can use this to obtain the following proposition.

Proposition 2.3.1. *For the cyclic graph C_n,*

$$X_{C_n}^2 = X_{C_n} - 2nM_{11\ldots1}.$$

In particular, $X_{C_n}^2$ is symmetric for all n. In fact, although we do not have an explicit formula for $k \geq 3$, this fact holds for all values of k.

Proposition 2.3.2. *For the cyclic graph C_n, $X_{C_n}^k$ is symmetric for all k.*

Proof. We will show that $X_{C_n}^k$ is invariant under interchanging x_i with x_{i+1}. Since the adjacent transpositions generate the symmetric group, this will demonstrate that $X_{C_n}^k$ is invariant under the symmetric group action that permutes its variables, which will show that it is symmetric.

If $\alpha = (\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_\ell)$, let $\widetilde{\alpha} = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_\ell)$. Then if $\mathscr{C}_\alpha$ denotes the set of k-balanced colorings contributing to the monomial M_α, we want a bijection $\varphi : \mathscr{C}_\alpha \to \mathscr{C}_{\widetilde{\alpha}}$.

Let $\kappa \in \mathscr{C}_\alpha$. The graph induced by the inverse image $\kappa^{-1}(\{i, i+1\})$ is either the entire cycle or a disjoint union of paths. In the former case, we set $\varphi(\kappa)(v) = i$ when $\kappa(v) = i+1$ and vice versa. This preserves k-balance since it reverses all edges.

If $\kappa^{-1}(\{i, i+1\})$ induces a collection of paths, let $\varphi(\kappa)(v)$ swap i and $i+1$ if v is in such a path of odd length and otherwise set $\varphi(\kappa)(v) = \kappa(v)$. We claim that the orientation induced by $\varphi(\kappa)$ is k-balanced. Firstly, if $j \neq i, i+1$, then $j > i$ if and only if $j > i+1$. Thus, no edges outside of the odd length paths will be reoriented. Secondly, there are an even number of edges in each odd length path, with exactly half pointing each direction. The effect of φ is to reverse all of these edges, which does not affect k-balance.

In either case, φ is an involution between $\mathscr{C}_\alpha$ and $\mathscr{C}_{\tilde{\alpha}}$, so we have the desired bijection.

□

2.3.2 Complete bipartite graphs

For a general simple graph G, the coefficients of X_G^k in the monomial basis directly count k-balanced colorings of G. However, in the case where G is the complete bipartite graph $K_{m,n}$ and $k = 2$, there is a more direct description of the coefficient of M_α.

Definition 2.3.3. Let $i_1, \ldots, i_k$ be positive integers. *The complete ranked poset* $Q_{i_1, i_2, \ldots, i_k}$ is the poset on $\bigcup_{j=1}^k R_j$, where $|R_j| = i_j$ and each element in R_j is covered by each element in R_{j+1}.

Figure 2.4 is an example of the complete ranked poset $Q_{2,2,3}$. Notice that the Hasse diagram is isomorphic to $K_{5,2}$. In fact, any 2-balanced orientation of $K_{m,n}$ will correspond to a complete ranked poset, which is part of the proof of the following theorem.

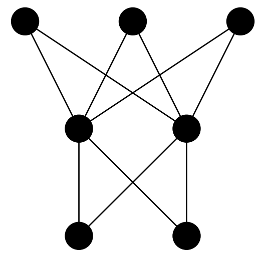


Figure 2.4: The complete ranked poset $Q_{2,2,3}$

Theorem 2.3.4. *For the complete bipartite graph $K_{m,n}$, we have*

$$X^2_{K_{m,n}} = \sum_{\alpha \in \mathrm{comp}(m+n)} \frac{m!n!}{\alpha!} r(\alpha; m,n) M_\alpha$$

where

$$r(\alpha;m,n) = \left| \left\{ (i,j) \,|\, 1 < i \leq j \leq \ell(\alpha) \text{ and } \sum_{t=i}^{j} \alpha_t = m \right\} \right| + \left| \left\{ (i,j) \,|\, 1 < i \leq j \leq \ell(\alpha) \text{ and } \sum_{t=i}^{j} \alpha_t = n \right\} \right|$$

and

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_{\ell(\alpha)}!.$$

Proof. A 2-balanced orientation of a graph is precisely a realization of that graph as a Hasse diagram by Proposition 1.3.5. So, we consider the posets which have Hasse diagram isomorphic to $K_{m,n}$. No such poset can have a chain with 4 vertices, since in such a chain there must be an edge from the greatest to the smallest element, due to the fact that they lie in different partite sets. This edge will create a cycle that is not 2-balanced.

Further, it is not hard to see that each of the complete ranked posets $Q_{i,m,n-i}$ for $0 < i \leq n$ or $Q_{i,n,m-i}$ for $0 < i \leq m$ have $K_{m,n}$ as their underlying graph. Thus, every 2-balanced orientation of $K_{m,n}$ comes from one of these posets.

We associate each 2-balanced coloring κ with a composition α, where α_i is the number of vertices colored with the i^{th} smallest color. Since κ induces a 2-balanced orientation $\mathscr{O}_\kappa$ which corresponds to a complete ranked poset, no vertices of different ranks in the poset may have the same color. So, κ will be feasible if and only if its associated composition can be written as a concatenation $\alpha = \alpha' + \alpha'' + \alpha'''$, where $\alpha', \alpha'', \alpha'''$ are compositions of magnitudes either $i, m, n-i$ or $i, n, m-i$ respectively. That is, α comes from a feasible coloring if and only if there is a partial sum $\alpha_i + \cdots + \alpha_j$ which equals m or n. So $r(\alpha; m, n)$ counts the number of feasible colorings associated with α up to the number of vertices of each color.

In the case of $Q_{i,m,n-i}$, the bottom rank can be colored in $\binom{i}{\alpha'}$ ways, the middle rank colored in $\binom{m}{\alpha''}$ ways, and the top rank colored in $\binom{n-i}{\alpha'''}$ ways. Further, we must choose i elements from the partite set with n elements to lie in the bottom rank. Thus, the number of colorings on the poset $Q_{i,m,n-i}$ with composition type α is

$$\binom{n}{i} \frac{i!}{\alpha'!} \frac{m!}{\alpha''!} \frac{(n-i)!}{\alpha'''!} = \frac{m!n!}{\alpha!}.$$

A similar calculation on $Q_{i,n,m-i}$ gives the same result, so that the number of 2-balanced colorings of $K_{m,n}$ with composition type α is

$$\frac{m!n!}{\alpha!} r(\alpha; m, n)$$

as desired.

□

α	$r(\alpha;3)$	α	$r(\alpha;3)$	α	$r(\alpha;3)$
111111	3	2121	2	213	1
21111	2	2112	1	132	1
12111	2	1221	1	123	1
11211	2	1212	2	51	0
11121	2	1122	1	15	0
11112	2	411	0	222	0
3111	1	141	0	42	0
1311	1	114	0	24	0
1131	1	321	1	33	1
1113	1	312	1	6	0
2211	1	231	1		

Table 2.1: Values of $r(\alpha;3)$

Example 2.3.5. For an example of how this theorem works, consider the complete bipartite graph $K_{3,3}$. Table 2.1 gives values of $r(\alpha;3) = \frac{1}{2}r(\alpha;3,3)$ for all compositions $\alpha \models 6$. Using the values in the table, it is straightforward to calculate $X^2_{K_{3,3}}$. For example, the coefficient of M_{2121} is

$$\frac{3!3!}{2!1!2!1!} \cdot 2r(2121;3) = 36,$$

whereas the coefficient of M_{2112} is

$$\frac{3!3!}{2!1!1!2!} \cdot 2r(2112;3) = 18.$$

2.4 The k-balanced Chromatic Polynomial

We now study the k-balanced versions of the chromatic polynomial of a graph. Stanley's theorem [16, Thm 1.2] enumerating acyclic orientations via the chromatic polynomial turns out to have a natural generalization to the k-balanced setting.

Definition 2.4.1. Given any graph G, the map $\chi_G^k : \mathbb{N} \to \mathbb{N}$ is defined by letting $\chi_G^k(\lambda)$ be the number of k-balanced colorings of G with λ colors.

Note that χ_G^k is the principal specialization of X_G^k, precisely the way that χ_G is the principal specialization of X_G (see Equation 1.6). That is,

$$\chi_G^k(\lambda) = \mathrm{ps}_\lambda(X_G^k) = X_G^k(\underbrace{1,1,\ldots,1}_{\lambda \text{ 1's}},0,0,\ldots).$$

This allows us to prove the following fact.

Proposition 2.4.2. *The map $\chi_G^k(\lambda)$ is a polynomial in λ with rational coefficients.*

Proof. We observe that

$$\begin{aligned}\chi_G^k(\lambda) &= X_G^k(\underbrace{1,1,\ldots,1}_{\lambda \text{ 1's}},0,0,\ldots)\\ &= \sum_{i=1}^{n}\sum_{\ell(\alpha)=i} c_\alpha M_\alpha(\underbrace{1,1,\ldots,1}_{\lambda \text{ 1's}},0,0,\ldots)\\ &= \sum_{i=1}^{n}\sum_{\ell(\alpha)=i} c_\alpha \binom{\lambda}{i},\end{aligned}$$

where the c_α are the integer coefficients of M_α. In particular, χ_G^k is a polynomial in λ with rational coefficients. □

With the above proposition, we will call χ_G^k the *k-balanced chromatic polynomial* of G.

It is well-known that the chromatic polynomial χ_G^1 has integer coefficients. However, this property does not hold for $k > 1$ for most graphs.

Theorem 2.4.3. *For $k > 1$, χ_G^k has integer coefficients if and only if G is a forest or G has no k-balanced coloring.*

Proof. If G is a forest, then since G has no weak cycles, every orientation is k-balanced for all k . Thus, $\chi_G^k = \chi_G^1$ has integer coefficients. Alternatively, if G has no k-balanced coloring, then $\chi_G^k = 0$.

On the other hand, if G has a cycle C and a k-balanced coloring, then the leading coefficient of $\chi_g^k(\lambda)$ is

$$\begin{aligned}[\lambda^n]\chi_G^k(\lambda) &= [\lambda^n]\sum_{i=1}^{n}\sum_{\ell(\alpha)=i} c_\alpha \binom{\lambda}{i} \\ &= \frac{c_{11\ldots1}}{n!}.\end{aligned}$$

Notice that $c_{11\ldots1}$ is precisely the number of k-balanced colorings of G with distinct colors. Since G contains a cycle, there exist colorings of G with distinct colors that are not k-balanced. That is, if C consists of the vertices $v_1, v_2, \ldots, v_t$ in order, assign them the colors $1, 2, \ldots, t$ respectively while arbitrarily assigning the remaining $n - t$ colors to the remaining vertices. This coloring will not be k-balanced, for $k \geq 2$. Thus, $c_{11\ldots1} < n!$.

Since G possesses a k-balanced coloring κ, it possesses a k-balanced coloring κ' with distinct colors—a natural relabelling of the induced orientation $\mathscr{O}_\kappa$ will give such a coloring—so that $c_{11\ldots1} > 0$. Thus, the leading coefficient of χ_G^k is not an integer. □

The following theorem is a generalization of a theorem of Stanley [16, Thm 1.2], who proved the theorem when $k = 1$—that is, when the orientations in question are acyclic and the colorings are simply proper colorings. The proof carries over directly to the $k > 1$ setting.

Theorem 2.4.4. *$(-1)^n\chi_G^k(-\lambda)$ is the number of pairs $(\kappa, \mathscr{O})$ where*

- *$\mathscr{O}$ is a k-balanced orientation of G;*

- *κ is a proper coloring of G with λ colors;*

- *$\kappa(i) \leq \kappa(j)$ if and only if (i,j) is an edge of $\mathscr{O}$.*

Proof. Let $\Omega(P,\lambda)$ be the *order polynomial* of P — the number of order-preserving maps from P to $[\lambda]$ — and let $\overline{\Omega}(P,\lambda)$ be the *strict order polynomial* of P — the number of strict order-preserving maps from P to $[\lambda]$.

Since a strict-order preserving map is just a P-partition, we find that

$$K_{P_{\mathscr{O}}}(\underbrace{1,1,\ldots,1}_{\lambda\text{ 1's}},0,0,\ldots) = \overline{\Omega}(P_{\mathscr{O}},\lambda).$$

Then, since $X_G^k = \sum_{\mathscr{O}} K_{P_{\mathscr{O}}}$,

$$\chi_G^k(\lambda) = \sum_{\mathscr{O}} \overline{\Omega}(P_{\mathscr{O}},\lambda),$$

where the sum is over all k-balanced orientations $\mathscr{O}$.

Now we can use the fact from [16, p. 174] that $\overline{\Omega}(P,-\lambda) = (-1)^{|P|}\Omega(P,\lambda)$ to get

$$\chi_G^k(-\lambda) = (-1)^n \sum_{\mathscr{O}} \Omega(P_{\mathscr{O}},\lambda).$$

Lastly, an order-preserving map from $P_{\mathscr{O}}$ to $[\lambda]$ can be regarded as a coloring of G which agrees with the orientation $\mathscr{O}$ in the above sense. □

Corollary 2.4.5. *$(-1)^n\chi_G^k(-1)$ is the number of k-balanced orientations of G.*

Remark 2.4.6. Since χ_G^k is a generalization of χ_G, it is natural to ask if there is a deletion-contraction recurrence that can be used to calculate χ_G^k, as in Proposition 1.3.8. Unfortunately, we conjecture that there is no such recurrence, due to the fact that a k-balanced coloring is not "local" in the same sense that a coloring is. That is, to determine if a map κ is a coloring, one need only examine all neighbors of all vertices.

To determine if κ is a k-balanced coloring, however, one must additionally examine the orientation induced by κ. This seems to indicate that simply understanding what happens when a single edge is deleted or contracted is not enough to deduce the global characteristics of an orientation of G.

Another nice property of χ_G is that its coefficients alternate in sign, which can be proved easily by induction on the number of vertices using the deletion-contraction recurrence. All examples which we have calculated seem to indicate that χ_G^k also shares the property that its coefficients alternate in sign, however, because χ_G^k does not have a deletion-contraction recurrence, the proof does not generalize.

2.5 Further Questions

In Stanley's original paper on the chromatic symmetric function [17], he presented two graphs with the same chromatic symmetric function. However, we were unable to find two graphs G, G' for which $X_G^k = X_{G'}^k \neq 0$ for $k > 1$. Thus, it is unknown whether X_G^k actually distinguishes between graphs. Notice also that the question of whether X_G^k distinguishes trees (a question posed by Stanley for X_G) is equivalent to the original question, as $X_G^k = X_G$ for all k for acyclic G.

An interesting characterization of k-balance can be seen by considering the hyperplane associated to a graph. In that context, a k-balanced orientation is a region of the hyperplane arrangement which lies on the positive side of at least k hyperplanes and the negative side of at least k hyperplanes. Unfortunately, there is no way to extend the concept of a coloring of a graph to this arena, but there may be a way to study k-balanced orientations of a graph by examining such regions.

Chapter 3

The Graph Hopf Algebra

In this chapter we examine graphs in the context of combinatorial Hopf algebras, using the result from Section 1.4.3 which shows that any element of a CHA corresponds to a one-variable polynomial. In the process we will prove a new antipode formula for a known Hopf algebra on graphs, and use these results to prove new facts about the Tutte polynomial.

The material of this chapter is also treated in a preprint [8] with Jeremy Martin.

3.1 Definitions

The main object of study in this chapter is the *graph Hopf algebra*, studied by Schmitt ([15, SS12]) and Aguiar, Bergeron, and Sottile ([2, Example 4.5]) among others. Let $\mathscr{G}$ be the vector space of all linear combinations of isomorphism classes of graphs over

$\mathbb{C}$, along with the algebra and coalgebra maps

$$M(G \otimes H) = G \cdot H = G \oplus H,$$

$$u(1) = \emptyset,$$

$$\Delta(G) = \sum_{T \subseteq V} G|_T \otimes G|_{\overline{T}},$$

$$\varepsilon(G) = \begin{cases} 1, & G = \emptyset, \\ 0, & G \neq \emptyset, \end{cases}$$

where $\emptyset$ is the empty graph with no vertices, $G|_T$ is the subgraph of G induced by T, and $G \oplus H$ is the disjoint union of G and H.

As noted by Schmitt [15, Equation 12.1] and Takeuchi [22, Lemma 14], there is an explicit formula for the antipode of $\mathscr{G}$ given by

$$S(G) = \sum_{\pi} (-1)^{|\pi|} |\pi|! G_{\pi}$$

where the sum runs over all ordered partitions π of $V(G)$ into nonempty sets (or "blocks"), and G_π is the disjoint union of the induced subgraphs on the blocks. In 3.2, we prove a different explicit formula which has the advantage of having fewer terms which do not cancel with one another. The formula we discover is a specialization, discovered by us independently, of a more general unpublished result of Aguiar and Ardila in the category of Hopf monoids [1].

We will principally be studying $\mathscr{G}$ by assigning some character from $\mathbb{X}(\mathscr{G})$ to create a CHA, then applying Proposition 1.4.7 to obtain a CHA morphism from $\mathscr{G}$ to $\mathbb{C}[t]$.

For example, the original character which was studied in [2] was

$$\zeta(G) = \begin{cases} 1, & G \text{ edgeless}, \\ 0, & \text{otherwise}. \end{cases}$$

In that case, we have the following. Recall that $P_{\zeta,G}(t)$ is the polynomial such that $P_{\zeta,G}(k) = \zeta^k(G)$ for all $k \in \mathbb{Z}$.

Proposition 3.1.1. *For all graphs G, $P_{\zeta,G} = \chi_G$, the chromatic polynomial of G.*

Proof. Notice that for $n > 0$,

$$\zeta^k(G) = \sum \zeta(G|_{V_1})\zeta(G|_{V_2})\dots\zeta(G|_{V_k})$$

so that each summand will be nonzero if and only if each of the V_i are independent sets. However, this describes a coloring of G with n colors—that is, assign the color i to the vertices of block V_i. Since $P_{\zeta,G}$ is a polynomial which agrees with χ_G on all positive integers, the two must be equal. □

Remark 3.1.2. An astute reader may now ask if there is a character ζ_ℓ on $\mathscr{G}$ so that $P_{\zeta_\ell,G} = \chi_G^\ell$, the invariant discussed in Section 2.4. Unfortunately, this does not seem to be the case. As mentioned in Remark 2.4.6, k-balance seems to be a global property where acyclicity is local. In the context of the coproduct in $\mathscr{G}$, knowing that we have decomposed the graph into independent sets is enough to tell us that we have constructed a coloring, but to determine if that coloring is k-balanced, we must know also how those vertices are connected—information which the coproduct of $\mathscr{G}$ does not provide.

That naturally leads one to ask if there is a way in which we could modify the coproduct of $\mathscr{G}$ to obtain a new Hopf algebra $\mathscr{G}'$ so that there exists a character ζ_ℓ on

$\mathscr{G}'$ with the desired property. This may be possible, but due to the necessity of retaining that global information through the decomposition by the coproduct, we conjecture that one would be able to reconstruct G from any summand of the coproduct. We cannot imagine that such a Hopf algebra would give any additional insight.

3.2 An Alternate Antipode Formula

If G is a graph, then recall that given $A \subseteq E(G)$, we define $\text{rk}(A)$, the *rank* of A, to be the size of a maximal acyclic subset of A. A *flat* A of a graph is a subset of the edge set with the property that $\text{rk}(A \cup e) = r(A) + 1$ for all $e \in E - A$. One can construct a flat of a graph by partitioning the vertex set as $V = V_1 \uplus \cdots \uplus V_k$ and letting $A = \bigcup_{i=1}^{k} E(G|_{V_i}$, which explains the presence of flats in the follow theorem.

Theorem 3.2.1. *Let G be a graph with vertex set $[n]$ and edge set E. Then*

$$S(G) = \sum_{\substack{F \subseteq E \\ F \text{ is a flat}}} (-1)^{n-\text{rk}(F)} a(G/F) G_{V,F}$$

where $a(G)$ is the number of acylic orientations of G, $r(F)$ is the rank of the flat F, and $G_{V,F}$ is the graph with vertices V and edges F.

Proof. We proceed by induction on the number of vertices of G. If $G = \emptyset$, then the only flat of G is the empty flat. Our formula then gives $S(\emptyset) = \emptyset$ as desired.

If G has at least one vertex, then using the fact that a graded connected Hopf algebra antipode has the form given in Equation 1.8, we write

$$\begin{aligned} S(G) &= -\sum_{\emptyset \neq T \subseteq V} G|_T \cdot S(G|_{\overline{T}}) \\ &= -\sum_{\emptyset \neq T \subseteq V} G|_T \sum_{\substack{F \subseteq E \\ F \text{ is a flat of } G|_{\overline{T}}}} (-1)^{n-|T|-r(F)} a(G|_{\overline{T}}/F) G_{\overline{T},F} \\ &= -\sum_{\emptyset \neq T \subseteq V} G|_T \sum_{\substack{F \subseteq E \\ F \text{ is a flat of } G|_{\overline{T}}}} \sum_{\substack{\text{Acyclic orientations } \mathcal{O} \\ \text{of } G|_{\overline{T}}/F}} (-1)^{n-|T|-r(F)} G_{\overline{T},F} \end{aligned} \tag{3.1}$$

Now we establish a bijection between two triples

$$(T, F, \mathcal{O}) \longleftrightarrow (F', \mathcal{O}', T') \tag{3.2}$$

where T is a nonempty set of vertices of G, F is a flat of the induced subgraph $G|_{\overline{T}}$, $\mathcal{O}$ is an acyclic orientation of $G|_{\overline{T}}/F$, F' is a flat of G, $\mathcal{O}'$ is an acyclic orientation of G/F', and T' is a nonempty subset of the set of sources $S_{\mathcal{O}'}$ of $\mathcal{O}'$.

Given a triple $(T, F, \mathcal{O})$, let F' be the flat comprised of the edges of $G|_T$ together with the edges of F. Then we can obtain an acyclic orientation of $\mathcal{O}'$ of G/F' by orienting all edges in $[\overline{T}, \overline{T}]$ as in $\mathcal{O}$, and orienting all edges in $[T, \overline{T}]$ towards $\overline{T}$. Further, the image of T under the contraction of F' is a set of vertices T' such that $\emptyset \neq T'$ and T' is a subset of $S_{\mathcal{O}'}$, the set of sources of $\mathcal{O}'$.

Conversely, given a triple $(F', \mathcal{O}', T)'$, let T be the inverse image of T' under contraction of F'. Then we can obtain a flat F of $G|_{\overline{T}}$ by deleting from F' all edges of $G|_T$, and we can obtain an orientation $\mathcal{O}$ of $G|_{\overline{T}}/F$ by orienting all edges as in $\mathcal{O}'$.

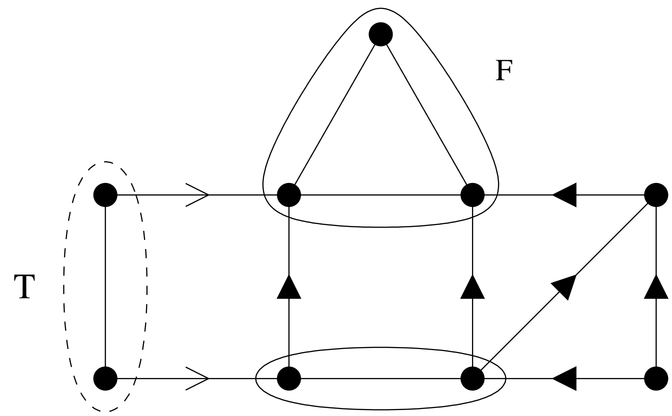


Figure 3.1: The relationship between $(T, F, \mathcal{O})$ and $(F', \mathcal{O}', T')$

A graphical representation of the bijection is in Figure 3.1, where the solid line enclosure represents F and the dashed line enclosure represents T. Notice that the non-solid arrows are directed edges which belong to $\mathcal{O}'$ but not $\mathcal{O}$.

It is straightforward to see that the maps between these triples are inverse. Furthermore, notice that under these operations, $|T'|$ is the number of connected components of $G|_T$. Thus, $|T| - |T'|$ is the rank of $G|_T$, which is equal to $\mathrm{rk}(F') - \mathrm{rk}(F)$. This gives $|T| + \mathrm{rk}(F) = |T'| + \mathrm{rk}(F')$. Also, $G|_T \cdot G_{\overline{T},F} = G_{V,F'}$.

Using the bijection in equation 3.2 in equation 3.1, we then have

$$\begin{aligned}
S(G) &= -\sum_{F'} \sum_{\mathcal{O}'} \sum_{\emptyset \neq T' \subseteq S_{\mathcal{O}'}} (-1)^{n - |T'| - \mathrm{rk}(F')} G_{V,F'} \\
&= -\sum_{F'} (-1)^{n - \mathrm{rk}(F')} G_{V,F'} \sum_{\mathcal{O}'} \sum_{\emptyset \neq T' \subseteq S_{\mathcal{O}'}} (-1)^{|T'|} \\
&= \sum_{F'} (-1)^{n - \mathrm{rk}(F')} a(G/F') G_{V,F'}
\end{aligned}$$

as desired. □

3.3 Inversion of Characters

Given our Theorem 3.2.1 and the fact from 1.4.2 that the inverse of a character is obtained by composition with the antipode, we can write a formula for the inverse of any character in the graph Hopf algebra.

Corollary 3.3.1. *If $\phi \in \mathbb{X}(\mathscr{G})$, then*

$$\phi^{-1}(G) = \sum_{\substack{F \subseteq E \\ F \text{ is a flat}}} (-1)^{n-\mathrm{rk}(F)} a(G/F)\phi(G_{V,F}).$$

In particular, if $\mathscr{F}$ is a family of graphs which has the property that $\emptyset \in \mathscr{F}$ and for any nonempty graph $G \in \mathscr{F}$, every connected component of G is also in $\mathscr{F}$, then the linear map

$$\iota_{\mathscr{F}}(G) = \begin{cases} 1, & G \in \mathscr{F} \\ 0, & G \notin \mathscr{F} \end{cases}$$

is multiplicative and thus a character. Then the above Corollary 3.3.1 simplifies to

$$\iota_{\mathscr{F}}^{-1}(G) = \sum_{F \subseteq E} (-1)^{n-\mathrm{rk}(F)} a(G/F)$$

where the summation is over flats F of G such that $G_{V,F} \in \mathscr{F}$.

Example 3.3.2. Let S be some graph, and consider the family $\mathscr{F}_S$ of graphs G which are S-free—that is, G has no subgraph isomorphic to S. For example, a graph G is $K_{n,1}$-free if and only if the maximum degree of G is less than n. This is very different from the property of not having S as an induced subgraph—e.g. K_3 has no induced $K_{2,1}$ subgraph, but it is not $K_{2,1}$-free.

Notice that $\mathscr{F}_S$ has the property that for $G \in \mathscr{F}_S$, connected components of G are also in $\mathscr{F}_S$. Thus, the associated characteristic, denoted by η_S, is a character. Then from Corollary 3.3.1 we have

$$\eta_S^{-1}(G) = \sum_{F \subseteq E} (-1)^{n-\mathrm{rk}(F)} a(G/F)$$

where the sum is over S-free flats of G.

As a specific example, let $S = K_2$. Then $\mathscr{F}_{K_2}$ includes all graphs which do not contain an edge, so that in fact $\eta_{K_2} = \zeta$, the character mentioned in 3.1. So, we have that

$$\zeta^{-1}(G) = (-1)^n a(G),$$

since any graph has only one edgeless flat. By Proposition 3.1.1 we know that $\zeta^k(G) = \chi_G(k)$ for all $k \in \mathbb{Z}$, so this reproduces the well-known result of Stanley [16, Cor 1.3].

Example 3.3.3. Let $\mathscr{F}$ be the family of acyclic graphs, and define $\alpha = \iota_{\mathscr{F}}$. That is, α detects whether or not a graph is a forest. Since the rank of an acyclic edge set is just its cardinality, the inverse simplifies to

$$\alpha^{-1}(G) = \sum_{F \subseteq E} (-1)^{n-|F|} a(G/F)$$

where F ranges over the acyclic flats of G.

For example, for a cycle C_n, acyclic flats are precisely the edge set of size less than or equal to $n-2$. Furthermore, if A is an edge set of size j inside C_n then $C_n/A \simeq C_{n-j}$, which has $2^{n-j}-2$ acyclic orientations. So,

$$\begin{aligned}
\alpha^{-1}(C_n) &= \sum_{j=0}^{n-2}\binom{n}{j}(-1)^{n-j}(2^{n-j}-2) \\
&= \sum_{j=2}^{n}\binom{n}{j}(-1)^{j}(2^{j}-2) \\
&= \sum_{j=2}^{n}\binom{n}{j}(-2)^{j} - 2\sum_{j=2}^{n}\binom{n}{j}(-1)^{j} \\
&= ((-1)^n - n(-2) - 1) - 2(0^n - n(-1) - 1) \\
&= (-1)^n + 1
\end{aligned}$$

(which is coincidentally the Euler characteristic of an n-sphere).

As another example, consider the complete graph K_n. Acyclic flats of K_n are matchings—subsets of the edge set such that no two edges share a common vertex. If a flat F did have edges $e_1,/e_2$ with $v \in e_1,/e_2$, then because K_n is complete, the triangle with $e_1,/e_2$ and a third edge would necessarily be in the flat F.

To calculate the number of k-edge matchings for $k < \frac{n}{2}$, we think of a k-edge matching as a permutation of $2k$ elements of $[n]$ where consecutive numbers are paired. That is, the permutation

$$136492$$

of $[9]$ is paired as

$$(13)(64)(92),$$

giving us the edges of the matching. It's clear that the order in each pairing does not matter, and also that the ordering of the pairs themselves does not matter. Thus, the number of k-edge matchings of K_n is

$$\frac{n!}{2^k(n-2k)!k!}.$$

We can also say what acyclic orientations of K_n/F look like—the contraction of a k-edge matching F is isomorphic to K_{n-k}, where some edges will be repeated. However, in the context of counting acyclic orientations, multiple edges can be treated as a single edge, since in an acyclic orientation they must all be directed the same. Thus, K_n/F has $(n-k)!$ acyclic orientations, corresponding to orderings of the vertices of K_{n-k}.

Thus,

$$\alpha^{-1}(K_n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} \frac{n!}{2^k(n-2k)!k!}(n-k)!.$$

Starting at $n = 1$, these numbers are as follows:

$$-1,\ 1,\ 0,\ -6,\ 30,\ -90,\ 0,\ 2520,\ -22680,\ 113400,\ 0,\ -7484400,\ \ldots.$$

This is sequence A009775 in the Encylopedia of Integer Sequences [6], for which the generating function is $-\tanh(\ln(1+x))$.

3.4 Tutte characters

As mentioned in 1.3.3, the Tutte polynomial of a graph G is a very powerful invariant defined by

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}.$$

The principal property of T_G of concern in this section is that it has a so-called "Universal Form," given by Bollobás [4, p. 340]. Let E_n be the edgeless graph on n vertices,

$c(G)$ be the number of connected components of G, and let $\mathbf{G}$ denote the set of all graphs.

Proposition 3.4.1 (Universal Form of the Tutte Polynomial)**.** *There is a unique map* $U : \mathbf{G} \to \mathbb{Z}[x, y, \alpha, \sigma, \tau]$ *such that*

$$U(E_n) = \alpha^n$$

for all $n \geq 1$*, and for every* $e \in E(G)$ *we have*

$$U(G) = \begin{cases} xU(G-e) & \text{if } e \text{ is a bridge,} \\ yU(G-e) & \text{if } e \text{ is a loop,} \\ \sigma U(G-e) + \tau U(G/e) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

Furthermore,

$$U(G) = \alpha^{c(G)} \sigma^{n(G)} \tau^{\mathrm{rk}(G)} T_G(\alpha x/\tau, y/\sigma).$$

In the above sense, all graph invariants which arise from deletion-contraction recurrences actually come from the Tutte polynomial.

In the context of this manuscript, we would like to view the Tutte polynomial as a character:

Definition 3.4.2. The *Tutte character* $\tau_{x,y}$ is a two-parameter character in the graph algebra where

$$\tau_{x,y}(G) = T_G(x, y).$$

Thus, since $T_G(2,0)$ counts the number of acyclic orientations of G, we can write that $\tau_{2,0}(G) = \overline{\zeta}^{-1}(G)$.

Unfortunately, the Tutte character is inconvenient in the context of the universal form, due to the fact that corank is nonadditive (see 3.4.5). It turns out to be useful to use a closely related invariant called the *rank-nullity polynomial*, given by

$$R_G(x,y) = \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(A)} (y-1)^{|A|-\mathrm{rk}(A)}.$$

This contains the same information as the Tutte polynomial and can be made into a character in much the same way that the Tutte polynomial can.

Definition 3.4.3. The *rank-nullity character* $\rho_{x,y}$ is a two-parameter character in the graph algebra where

$$\rho_{x,y}(G) = R_G(x,y).$$

Because T_G and R_G have similar forms, there is a nice relationship between ρ and τ, which we will use later:

$$\tau_{x,y} = (x-1)^{\mathrm{rk}(G)} \rho_{x/(x-1),y}, \qquad \rho_{x,y} = (x-1)^{\mathrm{rk}(G)} \tau_{x/(x-1),y}. \tag{3.3}$$

In particular,

$$\tau_{0,y} = \widetilde{\overline{\rho_{0,y}}} \quad \text{and} \quad \tau_{2,y} = \rho_{2,y}, \tag{3.4}$$

where we define $\tilde{\varphi}(G) = (-1)^{\mathrm{rk}(G)} \varphi(G)$ and recall that $\overline{\varphi}(G) = (-1)^{n(G)} \varphi(G)$.

3.4.1 The Polynomial Associated to the Rank-Nullity Character

The main theorem of this section gives the characterization of the polynomial associated to the rank-nullity character. We will write this polynomial as $P_{x,y}(G;k)$ where in our previous notation we would write $P_{\rho_{x,y},G}(k)$.

Theorem 3.4.4. *We have*

$$\rho^k_{x,y}(G) = k^{c(G)} (x-1)^{\mathrm{rk}(G)} T_G\left(\frac{k+x-1}{x-1}, y\right).$$

Proof. We have

$$P_{x,y}(G;k) = \rho^k_{x,y}(G) = \sum_{V_1 \uplus \cdots \uplus V_k = V} \prod_{i=1}^{k} \rho_{x,y}(G|_{V_i}) \tag{3.5a}$$

$$= \sum_{V_1 \uplus \cdots \uplus V_k} \prod_{i=1}^{k} \sum_{A_i \subseteq E(G|_{V_i})} (x-1)^{\mathrm{rk}(A_i)}(y-1)^{\mathrm{null}(A_i)} \tag{3.5b}$$

$$= \sum_{f:V\to[k]} \prod_{i=1}^{k} \sum_{A_i \subseteq f^{-1}(i)} (x-1)^{\mathrm{rk}(A_i)}(y-1)^{\mathrm{null}(A_i)} \tag{3.5c}$$

$$= \sum_{f:V\to[k]} \sum_{A \subseteq M(f,G)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)} \tag{3.5d}$$

where $M(f,G)$ denotes the set of edges $e = uv \in E(G)$ such that $f(u) = f(v)$ (including, in particular, all loops). Here the sum is over all ordered partitions of $V(G)$ into pairwise disjoint subsets (possibly empty). In order to write $P_{x,y}(G;k)$ as a Tutte specialization, we need to know its value on edgeless graphs, and how it behaves with respect to deleting a loop, deleting a coloop, or deletion and contraction of an edge which is neither a loop nor a coloop.

Step 1: Edgeless graphs. If $E(G) = \emptyset$, then $R_H(x,y) = 1$ for every subgraph $H \subseteq G$. Therefore, every summand in (3.5a) is 1, so $P_{x,y}(G;k)$ is just the number of ordered partitions with $n = |V(G)|$ parts, that is:

$$P_{x,y}(E_n;k) = k^n. \tag{3.6}$$

Step 2: Loops. Suppose G has a loop ℓ. For every ordered partition $V_1 \uplus \cdots \uplus V_k$, let V_i be the part that contains the endpoint of ℓ. Then $\rho_{2,y}(G|_{V_i}) = y\rho_{2,y}((G-\ell)|_{V_i})$, and we conclude that

$$P_{x,y}(G;k) = y \cdot P_{x,y}(G-\ell;k). \tag{3.7}$$

Step 3: Nonloop edges. Suppose G has a nonloop edge e (possibly a coloop) with endpoints u, v. For a function $f : V \to [k]$, if $f(u) \neq f(v)$ then $M(f, G-e) = M(f, G)$, while if $f(u) = f(v)$ then $M(f, G-e) = M(f, G) - \{e\}$. For every edge set $A \subseteq M(f, G)$ containing e, the edge set $B = A - \{e\} \subseteq M(f, G/e)$ satisfies $\mathrm{null}(B) = \mathrm{null}(A)$ and $\mathrm{rk}(B) = \mathrm{rk}(A) - 1$; moreover, the correspondence between A and B is a bijection. Therefore,

$$\begin{aligned}
P_{x,y}(G;k) &= \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G)} (x-1)^{\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)}, \\
P_{x,y}(G-e;k) &= \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G-e)} (x-1)^{\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)}, \\
P_{x,y}(G;k) - P_{x,y}(G-e;k) &= \sum_{\substack{f:V\to[k] \\ e\in M(f,G)}} \sum_{\substack{A\subseteq M(f,G) \\ e\in A}} (x-1)^{\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)} \\
&= \sum_{\substack{f:V\to[k] \\ f(u)=f(v)}} \sum_{B\subseteq M(f,G/e)} (x-1)^{\mathrm{rk}(B)+1} (y-1)^{\mathrm{null}(B)} \\
&= (x-1) P_{x,y}(G/e;k).
\end{aligned}$$

To put this recurrence in a more familiar form,

$$P_{x,y}(G;k) = P_{x,y}(G-e;k) + (x-1) P_{x,y}(G/e;k). \tag{3.8}$$

Step 4: Coloops. Now suppose that $e = uv$ is a coloop. We have

$$P_{x,y}(G-e;k) = \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G-e)} (x-1)^{\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)}$$

and

$$P_{x,y}(G/e;k) = \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G/e)} (x-1)^{\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)}.$$

Let H be the connected component of $G-e$ containing u, and let $H' = G-H$. Then we have $E(G-e) = E(H) \cup E(H')$. The cyclic group $\mathbb{Z}_k$ acts on colorings f by cycling the colors of vertices in H and fixing the colors of vertices in H'; i.e., if ζ is a generator of $\mathbb{Z}_k$, then $(\zeta f)(w) f(w) + \zeta$ for $w \in V(H)$, while $(\zeta f)(w) = f(w)$ for $w \in V(H')$. Then the set $M(f, G-e)$ is invariant under the action of $\mathbb{Z}_k$; moreover, each orbit has size k and has exactly one coloring for which $f(u) = f(v)$. In that case, contracting the edge e does not change the nullity or rank. Therefore, $P_{x,y}(G/e;k) = k^{-1} P_{x,y}(G-e;k)$, which when combined with (3.8) yields

$$\begin{aligned} P_{x,y}(G;k) &= P_{x,y}(G-e;k) + (x-1)P_{x,y}(G-e;k)/k \\ &= \left(\frac{k+x-1}{k}\right) P_{x,y}(G-e;k). \end{aligned} \tag{3.9}$$

Now combining (3.6), (3.7), (3.8), and (3.9) with the universal form 3.4.1 (replacing Bollobás' $x, y, \alpha, \sigma, \tau$ with $(k+x-1)/k$, y, k, 1, $x-1$ respectively) gives the desired result. □

Remark 3.4.5. In the preceding proof, notice that the equality of line 3.5d uses the fact that for $A_1, A_2 \subseteq E(G)$ which do not have any vertices in common,

$$\operatorname{rk}(A_1 \cup A_2) = \operatorname{rk}(A_1) + \operatorname{rk}(A_2).$$

If one attempts to apply the reasoning of the proof to the Tutte character $\tau_{x,y}$ instead of the rank-nullity character $\rho_{x,y}$, then this step fails; the corank of A in G, defined as $\operatorname{rk}(E) - \operatorname{rk}(A)$, is not additive in the following sense: if $V = V_1 \uplus V_2$ with $A_1 \subseteq E(G|_{V_1})$, $A_2 \subseteq G|_{V_2}$, then

$$\operatorname{rk}(G) - \operatorname{rk}(A_1 \cup A_2) \neq (\operatorname{rk}(G|_{V_1}) - \operatorname{rk}(A_1)) + (\operatorname{rk}(G|_{V_2}) - \operatorname{rk}(A_2)).$$

For an example of this phenomenon, let $G = K_4$ on the vertex set $[4]$, with $V_1 = \{1,2\}$, $V_2 = \{3,4\}$, $A_1 = \{\{1,2\}\}$, and $A_2 = \{\{3,4\}\}$. Then the coranks of A_1 and A_2 inside $G|_{V_1}$ and $G|_{V_2}$ respectively are both 0. However, since $\operatorname{rk}(G) = 3$ and $\operatorname{rk}(A_1 \cup A_2) = 2$, the corank of $A_1 \cup A_2$ inside G is 1.

3.4.2 Applications to Tutte Polynomial Evaluations

As stated previously, Tutte polynomial evaluations can reveal information about the graph in question. We obtain some previously unknown identities by using the following corollary, which takes advantage of the fact that the rank-nullity character is essentially the Tutte character when $x = 0$ or $x = 2$.

Corollary 3.4.6. *For $k \in \mathbb{Z}$ and y arbitrary, the Tutte characters $\tau_{2,y}$ and $\tau_{0,y}$ satisfy the identities*

$$\left(\tau_{2,y}\right)^k (G) = k^{c(G)} T_G(k+1, y), \tag{3.10}$$

$$\left(\widetilde{\tau_{0,y}}\right)^k (G) = k^{c(G)} (-1)^{\operatorname{rk}(G)} T_G(1-k, y). \tag{3.11}$$

In particular, $(\widetilde{\tau_{0,y}})^{-1} = \overline{\tau_{2,y}}$.

Proof. By 3.4, we have $\tau_{2,y} = \rho_{2,y}$ and $\tau_{0,y} = \widetilde{\rho_{2,y}}$. Letting $x = 0$ in Theorem 3.4.4 gives

$$\left(\widetilde{\tau_{0,y}}\right)^k (G) = \left(\rho_{0,y}\right)^k (G) = P_{0,y}(G;k) = k^{c(G)} (-1)^{\operatorname{rk}(G)} T_G(1-k, y)$$

and letting $x = 2$ gives

$$\left(\tau_{2,y}\right)^k (G) = \left(\rho_{2,y}\right)^k (G) = P_{2,y}(G;k) = k^{c(G)} T_G(k+1, y).$$

Now if we substitute $k = -1$ into the first equation, we find

$$\left(\widetilde{\tau_{0,y}}\right)^{-1}(G) = (-1)^{c(G)}(-1)^{\mathrm{rk}(G)}T_G(2,y) = (-1)^{n(G)}T_G(2,y) = \overline{\tau_{2,y}}(G)$$

□

This corollary gives concrete interpretations of Tutte evaluations which do not appear in the comprehensive survey article by Brylawski and Oxley [5] nor seemingly anywhere else in the literature. For example, since

$$\tau_{2,2}^2(G) = 2^{c(G)}T_G(3,2),$$

and further since $T_G(2,2)$ simply counts all subsets of the edge set of G, we have

$$\begin{aligned} T_G(3,2) &= \frac{1}{2^{c(G)}}\sum_{U\subseteq V}\tau_{2,2}(G|_U)\tau_{2,2}(G|_{\overline{U}}) \\ &= \frac{1}{2^{c(G)}}\sum_{U\subseteq V}2^{e(G|_U)+e(G|_{\overline{U}})}. \end{aligned}$$

For G connected, we have $c(G) = 1$, and we can interpret $T_G(3,2)$ as counting pairs (f,A), where $f : V \to \{1,2\}$ and $A \subseteq E$ is a subset of of the set of edges $\{u,v\}$ with $f(u) = f(v)$.

Even further, using the original definition of the Tutte polynomial, we can write

$$\begin{aligned} 2^{c(G)}\sum_{A\subseteq E}2^{\mathrm{rk}(G)-\mathrm{rk}(A)} &= \sum_{U\subseteq V}2^{e(G|_U)+e(G|_{\overline{U}})} \\ 2^{n(G)}\sum_{A\subseteq E}2^{-\mathrm{rk}(A)} &= \sum_{U\subseteq V}2^{e(G|_U)+e(G|_{\overline{U}})}, \end{aligned}$$

which is a novel identity.

For a similar example, consider

$$\tau_{2,0}^{2}(G) = 2^{c(G)} T_G(3,0).$$

Recalling that $T_G(2,0) = a(G)$ as mentioned in Example 1.3.11, we have

$$T_G(3,0) = \frac{1}{2^{c(G)}} \sum_{U \subseteq V} a(G|_U) a(G|_{\overline{U}})$$

so that we can regard $T_G(3,0)$ as counting (up to a power of 2) the number of pairs $(U, \mathscr{O})$, where $U \subseteq V$ and $\mathscr{O}$ is an acyclic orientation of G where all edges from U to $\overline{U}$ point towards $\overline{U}$.

3.5 Further Questions

The Hopf algebra structure on graphs generalizes in a natural way to a Hopf algebra structure on simplicial complexes (equivalently hypergraphs), where the coproduct partitions the vertex set and looks at the induced simplicial complexes on those blocks. This new Hopf algebra is still graded and connected, so it would make sense to examine it from the point of view of combinatorial Hopf algebras.

Formula 3.4.4 has the interesting property that the second coordinate stays fixed through the transformation. If we want to look at classifying more Tutte evaluations, then one feasible approach seems to be to consider planar duals of graphs, which exchange the first and second argument of the Tutte polynomial. In this way it should be possible to express Tutte evaluations as powers of characters applied to the dual of the graph.

Appendix A

Maple Routines for Calculating X_G^k

The following routines use Stembridge's `poset` package for Maple. [20]

```
with(networks):
with(combinat):
with(ListTools):
read "posets.txt": #Stembridge's poset package
withposets():

CoFromSet := proc(T::set,n::integer)::list:

# Given a set from [n-1], generates the corresponding composition

   local i::integer, A::list:
   if T = {} then
      A:=[n]:
      return A:
   fi:
   A := [T[1]]:
   for i from 2 to nops(T) do A:=[op(A),T[i]-T[i-1]] od:
   A := [op(A), n-T[nops(T)]]:
   return A:
end proc:

SetFromCo := proc(A::list)::[set,integer]:

# Given a composition of n, returns the corresponding
# subset of [n-1], along with n

   local i::integer, T::set, j::integer:
   T:={}:
```

```
   for i from 1 to nops(A)-1 do
      T := T union {sum(A[j],j=1..i)}:
   od:
   return [T, sum(A[j],j=1..nops(A))]:
end proc:

MtoL := proc(P::polynom)::polynom:

# Given a quasisymmetric function in the M base, returns
# the same in the L base

   local Q::polynom, R::polynom, i::integer, c::integer,
    alpha::list, S::set, n::integer, J::list, T::set:
   Q:=0:
   R:=0:
   if type(P,monomial)=false then
      for i from 1 to nops(P) do
         Q := Q + MtoL(op(P)[i]):
      od:
      return Q:
   fi:
   c:=lcoeff(P):
   R:=P/c:
   alpha := [op(R)]:
   J := SetFromCo(alpha):
   S := J[1]:
   n := J[2]:
   for T in powerset({seq(i,i=1..n-1)} minus S) do
      Q := Q + (-1)^nops(T)*c*L[op(CoFromSet(T union S, n))]:
   od:
   return Q:
end proc:

LtoM := proc(P::polynom)::polynom:

# Given a QSym function in the L base,
# returns the same in the M base

   local Q::polynom, R::polynom, i::integer, c::integer,
   alpha::list, S::set, n::integer, J::list, T::set:
   Q:=0:
   R:=0:
   if type(P,monomial)=false then
      for i from 1 to nops(P) do
         Q := Q + LtoM(op(P)[i]):
```

```
      od:
      return Q:
   fi:
   c:=lcoeff(P):
   R:=P/c:
   alpha := [op(R)]:
   J := SetFromCo(alpha):
   S := J[1]:
   n := J[2]:
   for T in powerset({seq(i,i=1..n-1)} minus S) do
      Q := Q + c*M[op(CoFromSet(T union S, n))]:
   od:
   return Q:
end proc:

PermtoCo := proc(Perm::list)::list:

# Given a permutation, returns the composition
# associated with that perm

   local A::set, i::integer:
   A := {}:
   for i from 1 to nops(Perm)-1 do
      if Perm[i] < Perm[i+1] then A := A union {i} fi:
   od:
   return CoFromSet(A, nops(Perm)):

end proc:

QSymAntipode := proc(P::polynom)::polynom:

# Given a quasisymmetric function in the monomial basis,
# returns the Hopf antipode in the monomial basis.

local r::polynom, Q::polynom, L::list, S::set, K::list,
 T::set, J::list, i::integer, R::polynom:
Q := 0:
R := P:
R := R + M[1]:
for r in R do
  S := {}:
  if coeffs(r) = 1 then L := [op(r)] else L:=[op(op(r)[2])] fi:
  for T in choose(nops(L)-1) do
    J := []:
    for i from 1 to nops(L) do
```

```
      if i-1 in T then J := [op(1..nops(J)-1,J),J[nops(J)]+L[i]]
      else J := [op(J),L[i]] fi:
    od:
    S := S union {J}:
  od:
  for K in S do
#    print(Q):
    Q := Q + (-1)^nops(L) * coeffs(r) * M[op(Reverse(K))]:
#    print(Q):
  od:
od:
Q := Q + M[1]:
return Q:
end proc:

Cycles := proc(G::graph)::set;
   # Given any undirected simple graph, returns the set of
   # cycles as a set of lists where the order is given
   # by the order that the edges appear in the cycle.

   local B::set, S::set, C::set, T::set, L::list, i::integer,
   j::integer, r::integer, c::integer, D::list, flag::boolean:

   i := 0:
   B := cyclebase(G):
   C := {}:
   for S in powerset(B) minus {{}} do
      T := symmdiff(op(S)): # Symmdiff's of cycle's in a cycbase
                            # are cycles or sets of cycles
      D := degreeseq(induce(T,G)):
      flag := false:
      for i from 1 to nops(D) do if D[i] > 2
      then flag := true fi: od: # Throws out symmdiff's which are
                                # 2+ cycles joined at a vertex
      if flag then next fi:
      if nops(components(induce(T,G))) > 1 then next fi:
      # Throws out symmdiff's which are 2+ cycles not
      # joined at a vertex

      T := ends(T,G):
      r := T[1][1]: # First vertex in the cycle
      T := T minus {T[1]}: # Gets rid of the first edge, then the
                           # while loop traverses that path in order
      L := [r]:
      while T <> {} do
```

```
         for i from 1 to nops(T) do
            if T[i][1] = r then j := 2: break:
            elif T[i][2] = r then j := 1: break: fi:
         od:
         r := T[i][j]:
         L := [op(L),r]:
         T := T minus {T[i]}:
      od:
      C := C union {L}:
   od:
   return C:
end proc:

Undir := proc(G::graph)::graph:

# Given a directed simple graph G,
# returns its underlying undirected graph

   local H::graph, E::list:
   H := graph(vertices(G), {}):
   for E in ends(G) do
      addedge({op(E)}, H):
   od:
   return H:
end proc:


KGood := proc(G::graph, k::integer, T::set)::boolean:

# Given a directed simple graph G, returns true if
# all cycles have at least k edges oriented each way,
# else false

# T is a set of lists which contains the cycles of the underlying
# simple graph of G. This is to keep KGoodChrQSymFunc from taking
# forever. If this is for a one-off function,
# do KGood(G,k,Cycles(Undir(G)))

   local L::list, i::integer, fwd::integer:
   for L in T do
      fwd := 0:
      for i from 1 to nops(L)-1 do
         if [L[i], L[i+1]] in ends(G) then fwd := fwd + 1 fi:
      od:
      if [L[nops(L)], L[1]] in ends(G) then fwd := fwd + 1 fi:
```

```
      if fwd < k or (nops(L) - fwd) < k then return false fi:
   od:
   return true:
end proc:

KGoodChrQSymFunc := proc(G::graph, k::integer)::polynom:

# Calculates the k-good chromatic quasisymmetric function
# of a graph G on vertices 1..n

   local Ornts::set, O::set, E::set, Tmp::set, Cyc::set,
   P::polynom, Cov::set, Exts::set, Perm::list:

# Get all orientations of G
   Ornts := {{}}:
   for E in ends(G) do
      Tmp := {}:
      for O in Ornts do
         Tmp := Tmp union {{op(O), [E[1], E[2]]},
                           {op(O), [E[2], E[1]]}}:
      od:
      Ornts := Tmp:
   od:

# Eliminate the ones which aren't k-good
   Cyc := Cycles(G):
   for O in Tmp do
      if not KGood(graph(vertices(G),O),k,Cyc)
      then Ornts := Ornts minus {O} fi:
   od:

# Get the corresp posets and their QSym functions in L
   P := 0:
   for O in Ornts do
      Cov := covers(O):
      Exts := {op(extensions(canon(Cov, 'natural')))}:
      for Perm in Exts do
         P := L[op(PermtoCo(Perm))] + P:
      od:
   od:

# Convert L to M
   return LtoM(P):

end proc:
```

```
KGQSFEval := proc(P::polynom)::integer:

# Given a quasisym function in the M basis,
# evaluates the corresponding polynomial at -1

local i::integer, q::integer, c::integer, R::polynom:
q:=0:
if type(P,monomial)=false then
  for i from 1 to nops(P) do
    q := q + KGQSFEval(op(P)[i]):
  od:
  return q:
fi:
c:=lcoeff(P):
R:=P/c:
q:=c*(-1)^nops(R):
return q:
end proc:

KGQSFCompBipart := proc(m::integer, n::integer)::polynom:
option remember:

# Finds the KGQSF of the complete bipartite graph K_{m,n}
# in the M basis

local Q::polynom, C::set, i::integer, A::list, r::integer,
s::integer, S::set, d::integer, j::integer:
Q:=0:
C:={}:
for i from 1 to m+n do C := C union composition(m+n, i) od:
for A in C do
  S:={}:
  r:=0:
  s:=0:
  d:=1:
  for i from 1 to nops(A) do S := S union {sum(A[j], j=1..i)} od:
  for i in S do
    if i + m in S then r := r + 1 fi:
    if i + n in S then s := s + 1 fi:
  od:
  for i from 1 to nops(A) do d := d * A[i]! od:
  Q := Q + M[op(A)] * m! * n! / d * (r + s):
od:
return Q;
```

```
end proc:

PrinSpec := proc(P::polynom)::polynom:

# Given a poly in the M basis, takes the principal specialization
# ps_x
# That is, returns a one-variable polynomial in x

local i::integer, Q::polynom, c::integer, R::polynom:
Q:=0:

if type(P,monomial)=false then
  for i from 1 to nops(P) do
    Q := Q + PrinSpec(op(P)[i]):
  od:
  return Q:
fi:

c := lcoeff(P):
R := P/c:

return expand(c*binomial(x, nops(R))):

end proc:


G := cycle(4):

KGoodChrQSymFunc(G, 2);


  4 M[1, 2, 1] + 16 M[1, 1, 1, 1] + 2 M[2, 2] + 4 M[1, 1, 2]

                 + 4 M[2, 1, 1]


MtoL(KGoodChrQSymFunc(G, 2));


  4 L[1, 2, 1] + 6 L[1, 1, 1, 1] + 2 L[2, 2] + 2 L[2, 1, 1]

                 + 2 L[1, 1, 2]


PrinSpec(KGoodChrQSymFunc(G,2));
```

$$-2 x^3 + \frac{7}{3} x^2 - x + \frac{2}{3} x^4$$

```
KGQSFCompBipart(3, 3);
```

```
2 M[3, 3] + 6 M[1, 2, 3] + 6 M[1, 3, 2] + 6 M[2, 1, 3]
          + 6 M[2, 3, 1] + 6 M[3, 1, 2] + 6 M[3, 2, 1]
          + 12 M[1, 1, 1, 3] + 18 M[1, 1, 2, 2]
          + 12 M[1, 1, 3, 1] + 36 M[1, 2, 1, 2]
          + 18 M[1, 2, 2, 1] + 12 M[1, 3, 1, 1]
          + 18 M[2, 1, 1, 2] + 36 M[2, 1, 2, 1]
          + 18 M[2, 2, 1, 1] + 12 M[3, 1, 1, 1]
          + 72 M[1, 1, 1, 1, 2] + 72 M[1, 1, 1, 2, 1]
          + 72 M[1, 1, 2, 1, 1] + 72 M[1, 2, 1, 1, 1]
          + 72 M[2, 1, 1, 1, 1] + 216 M[1, 1, 1, 1, 1, 1]
```

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