NEW ESTIMATES IN HARMONIC ANALYSIS FOR MIXED LEBESGUE SPACES

By

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to my parents

Contents

Abstract

Mixed Lebesgue spaces are a generalization of L^p spaces that occur naturally when considering functions that depend on quantities with different properties, such as space and time. We first present mixed Lebesgue versions of several classical results, including the boundedness of Calderon-Zygmund operators, a Littlewood-Paley theorem, and some other vector-valued inequalities. As applications we present a Leibniz's rule for fractional derivatives in the context of mixed-Lebesgue spaces, some sampling theorems and a characterization of mixed Lebesgue spaces in terms of wavelet coefficients.

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Introduction

Traditionally, Fourier series and the Fourier transform have been used to analyze functions and signals by decomposing them in terms of the sine and cosine functions. In the 1980s, wavelets were introduced, and they provide an alternative. Wavelets, or a family of wavelets, provide an orthonormal basis for a variety of function spaces, as do sine and cosine, but wavelets are well localized in both the time and frequency domains. This additional localization provides insight into many properties of functions by considering the coefficients of their wavelet expansions. Wavelets have generated great interest in applied fields in addition to the mathematics community, as they have applications in in signal processing, digital imaging, and data compression. Many function spaces can be studied using wavelets. In this dissertation we add one more, using band-limited wavelets to characterize mixed Lebesgue spaces.

Mixed Lebesgue spaces are a generalization of Lebesgue spaces that arise naturally when considering functions that depend on independent quantities with different properties, like a function that depends on a spacial variable and on time. Rather than requiring the same level of control over all the variables of a function, mixed Lebesgue spaces consider the integrability of each variable separately. This flexibility allows these spaces to play an important role in the study of time-based partial differential equations. In this environment we will build the framework necessary to consider wavelets, by first considering the mixed Lebesgue analogues of several classical results.

In Chapter 1 we will present basic properties of Lebesgue and mixed Lebesgue spaces, as well as an overview of classical results for Lebesgue spaces. Several maximal functions will be introduced, and some preliminary results concerning them will be proved. The chapter ends with an overview of wavelets.

In Chapter 2 we develop analogues of classical results from Lebesgue spaces for mixed Lebesgue spaces. We begin with a generalization of Schur's test, then present a result concerning the boundedness of vector-valued Calderón-Zygmund operators in mixed Lebesgue spaces. A version of the Littlewood-Paley theorem in this setting is then proved.

We then develop a version of Leibniz's rule for fractional derivatives in mixed Lebesgue spaces as an application of these results. This can be found in Chapter 3. In the course of that discussion, a need for the Fefferman-Stein inequality (a vector-valued inequality for maximal functions) arises. We provide an extension of the established version for mixed Lebesgue spaces in Chapter 4.

We explore sampling theorems, in Chapter 5. We present a version of the Shannon Sampling theorem, which gives circumstances under which the reconstruction of a function from a sample is possible. A result concerning when the norm of a sample is equivalent to the norm of the function itself, analagous to the Plancherel-Polya inequalities is then proved.

In Chapter 6, we develop a characterization of mixed Lebesgue spaces in terms of wavelet coefficients by demonstrating that a normed expression in terms of those coefficients is equivalent to the norm of the mixed Lebesgue space itself. This result requires much of the theory developed in the previous chapters.

Some of this material has been presented at the Analysis Semiinar at Kansas State University in December 2009, and at the February Fourier Talks at the University of Maryland in February 2010. It constitutes the majority of the material contained in a

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Chapter 1

Preliminaries

In this chapter we will provide the basic definitions and results that will be necessary for later chapters. We will begin with basic facts about Lebesgue spaces and mixed Lebesgue spaces, and then discuss Calderón-Zygmund operators and maximal functions. A number of definitions and theorems will be stated here for reference; proofs and further discussion can be found in texts including Folland [6], Grafakos [10], and Duoandikoetxea[4].

1.1 Mixed Lebesgue spaces

Lebesgue spaces are an example of Banach spaces (when $p \geq 1$), and much of analysis is devoted to their study.

Definition 1.1.1. *Let $0 < p < \infty$. Then $L^p(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ such that*

$$||f||_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

$L^\infty(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ that are essentially bounded, i.e.

$$||f||_{L^\infty(\mathbb{R}^n)} = \inf\{M \geq 0 : \mu(\{x : |f(x)| > M\}) = 0\} < \infty.$$

For $1 \leq p < \infty$ the dual space of $L^p(\mathbb{R}^n)$ is isometrically identified with $L^{p'}(\mathbb{R}^n)$ under the pairing

$$\langle f, g \rangle = \int_{\mathbb{R}^n} fg dx$$

where p' is the conjugate exponent defined by the equation

$$\frac{1}{p} + \frac{1}{p'} = 1$$

for $1 < p < \infty$ and by the convention $1' = \infty$ and $\infty' = 1$.

Mixed Lebesgue spaces allow us to describe different amounts of control over different variables. Thus they arise naturally when considering functions that depend on a number of different quantitites. They were first described in detail by Benedek and Panzone in [1]; they were also explored by Rubio de Francia, Ruiz and Torrea in [18] . Since functions with a dependency on a time variable ($t \in \mathbb{R}$) and a space variable ($x \in \mathbb{R}^n$) are the most common application, we will primarily consider the spaces $L_t^p L_x^q(\mathbb{R}^{n+1})$:

Definition 1.1.2. *Let $0 < p, q < \infty$. Then $L_t^p L_x^q(\mathbb{R}^{n+1})$ consists of all measurable functions f on $\mathbb{R}^{n+1}$ such that*

$$||f||_{L_t^p L_x^q} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |f(t,x)|^q \, dx \right)^{p/q} dt \right)^{1/p} < \infty.$$

Unless specified, the results here also hold for spaces over $\mathbb{R}^m \times \mathbb{R}^n$. Benedek and Panzone presented many of the fundamental properties of these spaces [1]. Here we state a few that we will be using.

For $1 \leq p, q < \infty$, the dual space of $L_t^p L_x^q(\mathbb{R}^{n+1})$ is identified with $L_t^{p'} L_x^{q'}(\mathbb{R}^{n+1})$, where

p', q' are the conjugate exponents of p and q under the pairing

$$\langle f, g \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^n} fg\,dx\,dt.$$

Hölder's inequality for mixed Lebesgue spaces is obtained by applying the standard formulation of Hölder's inequality twice.

Theorem 1.1.3. *Let* $1 \le p, q, p_1, p_2, q_1, q_2 \le \infty$*, with* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$*. Then*

$$||fg||_{L_t^p L_x^q} \le ||f||_{L_t^{p_1} L_x^{q_1}} ||g||_{L_t^{p_2} L_x^{q_2}}.$$

We will also make use of the following:

Proposition 1.1.4. *For* $\lambda > 0$*,*

$$||f||_{L^p L^q} = ||f^{1/\lambda}||^{\lambda}_{L^{p\lambda} L^{q\lambda}}.$$

Proof.

$$\begin{aligned} ||f||_{L^p L^q} &= \left(\int \left(\int |f(t,x)|^q dx\right)^{p/q} dt\right)^{1/p} \\ &= \left(\int \left(\int |f^{1/\lambda}(t,x)|^{\lambda q} dx\right)^{\lambda p/\lambda q} dt\right)^{1/p} \\ &= ||f^{1/\lambda}||^{\lambda}_{L^{p\lambda} L^{q\lambda}}. \end{aligned}$$

□

We will represent points in $\mathbb{R}^{n+1}$ as $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, and (j, k) will denote an index in $\mathbb{Z}^{n+1}$ with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. When necessary, x_i and k_i will denote the i^{th} components of x and k.

Definition 1.1.5. *The Schwartz space, denoted $\mathscr{S}(\mathbb{R}^n)$ is the set of all C^∞ functions on $\mathbb{R}^n$ that, for a constant $C_{\alpha,\beta} < \infty$ for all α and β, satisfy*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta}$$

for all multi-indices α and β. $\mathscr{S}'(\mathbb{R}^n)$, the space of tempered distributions, is its dual.

Definition 1.1.6. *Let $f \in \mathscr{S}(\mathbb{R}^n)$. Then the Fourier transform of f is given by*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\, dx.$$

It extends to an isomorphism on $L^2(\mathbb{R}^n)$ and its inverse on $\mathscr{S}$ is given by

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{ix\cdot\xi}\, d\xi.$$

1.2 Calderón-Zygmund operators

An important class of operators are the Calderón-Zygmund operators.

Definition 1.2.1. *A function K, defined away from the diagonal of $\mathbb{R}^n\times\mathbb{R}^n$ is called a standard kernel if it that satisfies the size condition*

$$|K(x,y)| \leq \frac{C}{|x-y|^n} \tag{1.1}$$

and the regularity conditions

$$|K(x,y)-K(x,z)| < C\frac{|y-z|^\delta}{|x-y|^{n+\delta}} \qquad \text{for } |x-y| \geq 2|y-z| \tag{1.2}$$

$$|K(x,y) - K(w,y)| < C\frac{|x-w|^{\delta}}{|x-y|^{n+\delta}} \qquad \text{for } |x-y| \geq 2|x-w| \tag{1.3}$$

for some $\delta > 0$.

Definition 1.2.2. *An operator* T *is a Calderón-Zygmund operator if* T *is bounded on* L^q *for some* $1 < q < \infty$ *and is associated with a standard kernel* K*, in the sense that*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$$

and $x \notin \operatorname{supp} f$.

The most common examples of these operators are given by the principal value of a convolution. That is, K is a standard kernel that is given by a function $K(x,y) = k(x-y)$ to that $Tf(x) = p.v.(k * f)(x)$. The Hilbert transform H is one such operator:

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy = \frac{1}{\pi}\,p.v.\int\frac{f(y)}{x-y}\,dy$$

for $f \in \mathscr{S}(\mathbb{R})$. Note that the associated kernel is $K(x,y) = \frac{1}{\pi}\frac{1}{x-y}$, which satisfies (1.1), (1.2), (1.3). The higher dimensional analogues are the Riesz transforms:

$$R_j f(x) = C_n\,p.v.\int\frac{x_j - y_j}{|x-y|^{n+1}}f(y)\,dy$$

for $f \in \mathscr{S}(\mathbb{R}^n)$ and $1 \leq j \leq n$. The kernels of the Reisz transforms also are standard kernels. Moreover, these operators are bounded on L^2: notice that

$$\widehat{Hf}(\xi) = -i\,\operatorname{sgn}(\xi)\hat{f}(\xi)$$

and

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$$

if $C_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$. Thus, by the Plancherel theorem,

$$H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

and

$$R_j : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

for $1 \leq j \leq n$. Thus H and the R_j are Caderón-Zygmund operators. For these, as for all Calderón-Zygmund operators, the following holds:

Theorem 1.2.3. *Let T be a Calderón-Zygmund operator. Then T is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and it is also of weak type 1-1, i.e.*

$$|\{x : |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda}||f||_{L^1(\mathbb{R}^n)}.$$

We can extend the idea of Caderón-Zygmund operators to operators that take values in any reflexive, separable Banach space as follows. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be Banach spaces. We denote the space of all bounded linear operators from $\mathscr{B}_1$ to $\mathscr{B}_2$ by $\mathscr{L}(\mathscr{B}_1, \mathscr{B}_2)$. The norm on this space is the operator norm, and, for $T \in \mathscr{L}(\mathscr{B}_1, \mathscr{B}_2)$, is given by

$$||T||_{\mathscr{B}_1 \to \mathscr{B}_2} = \sup\{||Tx||_{\mathscr{B}_2} : ||x||_{\mathscr{B}_1} = 1\}.$$

Definition 1.2.4. *Consider a function $\overrightarrow{K} : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathscr{L}(\mathscr{B}_1, \mathscr{B}_2)$, i.e., $\overrightarrow{K}(u,v) : \mathscr{B}_1 \to \mathscr{B}_2$ for each $(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$. If $\overrightarrow{K}$ satisfies the size condition*

$$||\overrightarrow{K}(x,y)||_{\mathscr{B}_1 \to \mathscr{B}_2} \leq \frac{C}{|x-y|^n} \tag{1.4}$$

and the regularity conditions

$$||\overrightarrow{K}(x,y)-\overrightarrow{K}(x,z)||_{\mathscr{B}_1\to\mathscr{B}_2}\le C\frac{|y-z|^\delta}{|x-y|^{n+\delta}} \qquad \text{for } |x-y|\ge 2|y-z| \tag{1.5}$$

$$||\overrightarrow{K}(x,y)-\overrightarrow{K}(w,y)||_{\mathscr{B}_1\to\mathscr{B}_2}\le C\frac{|x-w|^\delta}{|x-y|^{n+\delta}} \qquad \text{for } |x-y|\ge 2|x-w| \tag{1.6}$$

for some $0<\delta<1$ *is a standard kernel.*

Definition 1.2.5. *An operator* $\overrightarrow{T}$ *is a vector-valued Calderón-Zygmund operator if* $\overrightarrow{T}$ *is bounded from* $L^q(\mathbb{R}^n,\mathscr{B}_1)$ *to* $L^q(\mathbb{R}^n,\mathscr{B}_2)$ *for some* $1<q<\infty$ *and is associated with a standard kernel* $\overrightarrow{K}$*, in the sense that*

$$\overrightarrow{T}f(x)=\int_{\mathbb{R}^m}\overrightarrow{K}(x,y)(f(y))dy$$

for $f\in L^\infty(\mathbb{R}^m,\mathscr{B}_1)$ *and* $x\notin \operatorname{supp} f$.

Note that a standard kernel $\overrightarrow{K}$ also satisfies the estimates

$$\int_{|x-y|\ge 2|y-z|}||\overrightarrow{K}(x,y)-\overrightarrow{K}(x,z)||_{\mathscr{B}_1\to\mathscr{B}_2}dx\le C \tag{1.7}$$

$$\int_{|x-y|\ge 2|w-x|}||\overrightarrow{K}(x,y)-\overrightarrow{K}(w,y)||_{\mathscr{B}_1\to\mathscr{B}_2}dy\le C. \tag{1.8}$$

Vector-valued Calderón-Zygmund operators have a parallel boundedness theorem.

Theorem 1.2.6. *Let* $\overrightarrow{T}$ *be bounded from* $L^q(\mathbb{R}^n,\mathscr{B}_1)$ *to* $L^q(\mathbb{R}^n,\mathscr{B}_2)$ *for some* $1<q<\infty$ *and have associated kernel* $\overrightarrow{K}$*. If* $\overrightarrow{K}$ *satisfies* (1.7) *and* (1.8)*, then* $\overrightarrow{T}$ *is bounded from* $L^p(\mathbb{R}^n,\mathscr{B}_1)$ *to* $L^p(\mathbb{R}^n,\mathscr{B}_2)$ *for all* $1<p<\infty$ *and* $\overrightarrow{T}$ *is weak-type (1,1) in the sense that*

$$|\{x:||\overrightarrow{T}f(x)||_{\mathscr{B}_2}>\lambda\}|\le\frac{C}{\lambda}||f||_{L^1(\mathbb{R}^m,\mathscr{B}_1)}.$$

Using this theorem, one can prove the Littlewood-Paley characterization of L^p, which extends Plancherel's theorem to other Lebesgue spaces by stating that the norms of f and that of the square function are comperable in L^p for $1 < p < \infty$. We will use $\Psi_{2^{-j}}$ to denote $2^{jn}\Psi(2^j\cdot)$ for all $x \in \mathbb{R}^n$.

Theorem 1.2.7. *Suppose that Ψ is an integrable C^1 function on $\mathbb{R}^n$ with mean value zero that satisfies*

$$|\Psi(x)| + |\nabla\Psi(x)| \le \frac{B}{(1+|x|)^{n+1}}.$$

Then there exists a constant $C = C_n < \infty$ such that for all $1 < p < \infty$ and all $f \in L^p(\mathbb{R}^n)$ we have

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |f * \Psi_{2^{-j}}|^2 \right)^{1/2} \right|\right|_{L^p(\mathbb{R}^n)} \le C||f||_{L^p(\mathbb{R}^n)}.$$

There also exists a $C' = C'_n < \infty$ such that for all $f \in L^1(\mathbb{R}^n)$ we have

$$\left|\left| \left(\sum_{j\in\mathbb{Z}} |f * \Psi_{2^{-j}}|^2 \right)^{1/2} \right|\right|_{L^{1,\infty}(\mathbb{R}^n)} \le C||f||_{L^1(\mathbb{R}^n)}.$$

Conversely, suppose that Ψ is a Schwartz function that satisfies either

$$\sum_{j\in\mathbb{Z}} |\hat{\Psi}(2^{-j}\xi)|^2 = 1 \qquad \xi \in \mathbb{R}^n \setminus \{0\},$$

or

$$\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1 \qquad \xi \in \mathbb{R}^n \setminus \{0\},$$

and that f is a tempered distribution so that the function $(\sum_{j\in\mathbb{Z}} |f * \Psi_{2^{-j}}|^2)^{1/2}$ is in $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then there exists a unique polynomial Q such that the *distribution $f-Q$ coincides with an L^p function and we have*

$$||f-Q||_{L^p(\mathbb{R}^n)} \leq C \left|\left| \left(\sum_{j\in\mathbb{Z}} |f * \Psi_{2^{-j}}|^2 \right)^{1/2} \right|\right|_{L^p(\mathbb{R}^n)}$$

for some constant $C = C_{n,\Psi}$.

Calderón-Zygmund operators on mixed Lebesgue spaces have been the object of some study, in works that include papers by Fernandez [5], Kurtz [15], and Stefanov and Torres [20]. In particular, they proved the boundedness of Calderón-Zygmund operators on mixed Lebesgue spaces. The following version is from the work of Moen [17].

Theorem 1.2.8. *Let T be a Calderón-Zygmund operator in $\mathbb{R}^{m+n}$. Then T is bounded from L^pL^q to L^pL^q for all $1 < p,q < \infty$ and it also satisfies the weak-type bound*

$$|\{x \in \mathbb{R}^n : ||Tf(x,\cdot)||_{L^q(\mathbb{R}^m)} > \lambda\}| \leq \frac{C}{\lambda}||f||_{L^1_xL^q_y}.$$

In Section 2.2 we will prove a vector-valued theorem for mixed Lebesgue spaces. We will use that result to obtain a Littlewood-Paley characterization for mixed Lebesgue spaces in Section 2.3.

1.3 Maximal functions

Maximal functions arise from attempting to control various functions by more tractable expressions. We will consider the following such functions. Let g be a function on $\mathbb{R}\times\mathbb{R}^n$.

Definition 1.3.1. *The Harvey-Littlewood maximal function, Mg is defined as*

$$Mg(t,x) = \sup_{Q \ni (t,x)} \frac{1}{|Q|} \int_Q |g(u,v)| dv du$$

where Q is a cube in $\mathbb{R}^{n+1}$ with sides parallel to the axes which contains (t,x). A variation we will also use is the centered version for balls:

$$M_c g(t,x) = \sup_{r>0} \frac{1}{r^{n+1}} \int_{B_r(t,x)} |g(u,v)| dv du$$

where $B_r(t,x)$ is the ball of radius r centered at the point (t,x).

Kurtz [15] proved that the Hardy-LIttlewood maximal function is bounded on $L^p L^q$. Among M's other important properties is the Fefferman-Stein inequality.

Theorem 1.3.2. *Suppose that $\{f_j\}$ is a sequence of locally integrable functions. Whenever $1 < p, r < \infty$,*

$$\left\| \left(\sum_{P j \in \mathbb{Z}} |M(f_j)|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}$$

with C dependent only on n, p, and r. Furthermore, for $r < \infty$,

$$\left| \left\{ x : \left(\sum_j |M(f_j)^r \right)^{1/r} > \lambda \right\} \right| \leq \frac{C}{\lambda} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^r \right)^{1/r} \right\|_{L^1(\mathbb{R}^n)}.$$

In our wavelet characterization we will need some versions of the Peetre maximal function. For simplicity in the presentation, we will consider here the case of $\mathbb{R} \times \mathbb{R}$, but it can easily be extended to $\mathbb{R}^m \times \mathbb{R}^n$.

Definition 1.3.3. *Let $\lambda > 0$ be a real number. Then the maximal function g_λ^* is given by*

$$g_\lambda^*(t,x) = \sup_{(u,v)\in\mathbb{R}\times\mathbb{R}^n} \frac{|g(t-u,x-v)|}{(1+|(u,v)|)^\lambda}$$

for each $(t,x) \in \mathbb{R}\times\mathbb{R}^n$.

Definition 1.3.4. *For $\lambda > 0$ we write $\phi_{j,\lambda}^{**}g$ for the function given by*

$$(\phi_{j,\lambda}^{**}g)(t,x) = \sup_{(u,v)\in\mathbb{R}\times\mathbb{R}^n} \frac{|(\phi_{2^{-j}} * g)(t-u,x-v)|}{(1+2^j|(u,v)|)^\lambda}$$

where $\phi_r(x,y) = \frac{1}{r^2}\phi(\frac{t}{r},\frac{x}{r})$.

The following lemmas are known (see [11], for example). We include a proof here for completeness and to demonstrate that they do not depend on the product structure of $\mathbb{R}\times\mathbb{R}$.

Lemma 1.3.5. *Let $g \in L^\infty(\mathbb{R}^2)$ be such that $\hat{g}$ is compactly supported. (It is also sufficient for g to be of at most polynomial growth with $\hat{g}$ compactly supported.) Then, for any real $\lambda > 0$, there exists a C_λ such that*

$$(|\nabla g|)_\lambda^*(t.x) \leq C_\lambda g_\lambda^*(t.x)$$

for $(t,x) \in \mathbb{R}\times\mathbb{R}$.

Proof. Suppose that $\hat{g}$ is compactly supported, so that supp $(\hat{g}) \subset \{(\xi,\eta) \in \mathbb{R}^2 : |(\xi,\eta)| \leq L\}$ for some $L > 0$. Let $\gamma \in \mathscr{S}$ that satisfies $\hat{\gamma}(\xi,\eta) \equiv 1$ on the support of $\hat{g}$. Then $\hat{\gamma}\hat{g} = \hat{g}$ for all $(\xi,\eta) \in \mathbb{R}^2$. Moreover, $\gamma * g = g$ and $\nabla g = \nabla\gamma * g$. As a result, g is differentiable, and

$$
\begin{aligned}
&||\nabla g|((t,x)-(u,v))| \\
&\quad = \left| \int_{\mathbb{R}^2} |\nabla \gamma((t,x)-(u,v)-(z_1,z_2))| g(z_1,z_2) dz_2 dz_1 \right| \\
&\quad = \left| \int_{\mathbb{R}^2} |\nabla \gamma((w_1,w_2)-(u,v))| g((t,x)-(w_1,w_2)) dw_2 dw_1 \right| \\
&\quad \leq \int_{\mathbb{R}^2} ||\nabla \gamma(w-(u,v))||(1+|w-(u,v)|)^\lambda (1+|(u,v)|)^\lambda \frac{|g((t,x)-w)|}{(1+|w|)^\lambda} dw
\end{aligned}
$$

where $w=(w_1,w_2)$, with the last inequality holding because

$$
\begin{aligned}
1+|w| &= 1+(w_1,w_2) \\
&\leq 1+|(w_1,w_2)-(u,v)|+|(u,v)| \\
&\leq (1+|(w_1,w_2)-(u,v)|)(1+|(u,v)|).
\end{aligned}
$$

As a result we can write

$$
|\nabla g((t,x)-(u,v))| \leq g_\lambda^*(t,x)(1+|(u,v)|)^\lambda \int_{\mathbb{R}^2} |\nabla \gamma(w-(u,v))|(1+|w-(u,v)|)^\lambda dw.
$$

Notice that the integral is a finite constant independent of u and v, as $\nabla\gamma$ is in $\mathscr{S}$. We can therefore write

$$
|\nabla g((t,x)-(u,v))| \leq C_\lambda g_\lambda^*(t,x)(1+|(u,v)|)^\lambda
$$

which gives us the desired result. □

We say that a function is band-limited if its Fourier transform has compact support.

Lemma 1.3.6. *If g is a band-limited function on $\mathbb{R}\times\mathbb{R}$ with $g_\lambda^*(t,x)<\infty$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, then there exists a constant C_λ such that*

$$
g_\lambda^*(t,x) \leq C_\lambda \left(M_c(|g|^{1/\lambda})(t,x) \right)^\lambda
$$

for each $(t,x) \in \mathbb{R} \times \mathbb{R}$.

Proof. We reproduce the proof from [11], extending it to $\mathbb{R}^2$. The function g is band-limited, so by the Paley-Weiner theorem, g is also differentiable. As a result, we can consider the following. Let (t,x) and (u,v) be in $\mathbb{R}^2$ and let $0 < \delta < 1$, to be determined later. Choose a point $(z_1,z_2) \in \mathbb{R}^2$ so that $|((t,x)-(u,v))-(z_1,z_2)| < \delta$. Now apply the mean value theorem to g with end points $(t,x)-(u,v)$ and (z_1,z_2) to obtain

$$|g((t,x)-(u,v))| \le |g(z_1,z_2)| + \delta \sup_{(w_1,w_2):|((t,x)-(u,v))-(w_1,w_2)|<\delta} |\nabla g(w_1,w_2)|.$$

Raising everything to the $(1/\lambda)$th power and averaging over the region $B_\delta((t,x)-(u,v))$ in z_1 and z_2 gives us

$$\begin{aligned} |g((t,x)-(u,v))|^{1/\lambda} &\le \frac{c_\lambda}{\delta^2}\int_{B_\delta((t,x)-(u,v))} |g(z_1,z_2)|^{1/\lambda} dz_2 dz_1 \\ &\quad + c_\lambda \delta^{1/\lambda} \sup_{(w_1,w_2):|((t,x)-(u,v))-(w_1,w_2)|<\delta} ||\nabla g(w_1,w_2)|^{1/\lambda}. \end{aligned}$$

By noticing that $B_\delta((t,x)-(u,v))$ is contained in $B_{|(u,v)|+\delta}(t,x)$, we can write

$$\begin{aligned} \int_{B_\delta((t,x)-(u,v))} |g(z_1,z_2)|^{1/\lambda} dz_2 dz_1 &\le \int_{B_{|(u,v)|+\delta}(t,x)} |g(z_1,z_2)|^{1/\lambda} dz_2 dz_1 \\ &\le 2(\delta + |(u,v)|)^2 M(|g|^{1/\lambda})(t,x) \end{aligned}$$

and

$$\begin{aligned} &\sup_{(w_1,w_2):|((t,x)-(u,v))-(w_1,w_2)|<\delta} |\nabla g(w_1,w_2)|^{1/\lambda} \\ &\qquad \le \sup_{(w_1,w_2):|(t,x)-(w_1,w_2)|<|(u,v)|+\delta} |\nabla g(w_1,w_2)|^{1/\lambda} \\ &\qquad = \sup_{(y_1,y_2):|(y_1,y_2)|<|(u,v)|+\delta} |\nabla g((t,x)+(y_1,y_2))|^{1/\lambda} \\ &\qquad \le (1+|(u,v)|+\delta)[(\nabla g)^*_\lambda(t,x)]^{1/\lambda}. \end{aligned}$$

So we have

$$
\begin{aligned}
&|g((t,x)-(u,v))|^{1/\lambda} \\
&\qquad \leq c_\lambda \left(\tfrac{\delta+|(u,v)|}{\delta}\right)^2 M_c(|g|^{1/\lambda})(t,x) + c_\lambda \delta^{1/\lambda}(1+|(u,v)|+\delta)[(|\nabla|g)^*_\lambda(t,x)]^{1/\lambda} \\
&\qquad \leq c_\lambda \left(\delta^{-2} M_c(|g|^{1/\lambda})(t,x) + \delta^{1/\lambda}[(|\nabla|g)^*_\lambda(t,x)]^{1/\lambda}\right)(1+|(u,v)|+\delta)
\end{aligned}
$$

Because $\delta < 1$, we know that $(1+|(u,v)|+\delta) < 2(1+|(u,v)|)$, and so raising everything to the λth power gives us

$$
\frac{|g((t,x)-(u,v))|}{(1+|(u,v)|)^\lambda} \leq c_\lambda \left(\delta^{-2\lambda}[M_c(|g|^{1/\lambda})(t,x)]^\lambda + \delta[(\nabla g)^*_\lambda(t,x)]\right),
$$

that is,

$$
g^*_\lambda(t,x) \leq c_\lambda \left(\delta^{-2\lambda}[M_c(|g|^{1/\lambda})(t,x)]^\lambda + \delta[(\nabla g)^*_\lambda(t,x)]\right).
$$

Applying Lemma 1.3.5 and choosing δ small enough that $c_\lambda C_\lambda \delta < \frac{1}{2}$ (where C_λ is the constant from the Lemma) we have

$$
g^*_\lambda(t,x) \leq \frac{c_\lambda}{\delta^\lambda}[M_c(|g|^{1/\lambda})(t,x)]^\lambda + \frac{1}{2} g^*_\lambda(t,x).
$$

Since $g^*_\lambda(t,x) < \infty$ we have obtained the desired result. □

This result is used in sections 5.2 and 6.1.

1.4 Wavelets

The Fourier transform allows us to view a function as being composed of oscillations of various frequencies by decomposing it into sine and cosine functions. A wavelet (or family of wavelets) provides an alternative basis against which a function can be expanded, and thus highlights different information than the Fourier transform does. In particular, wavelets are localized in both time and frequency domains. Here we present an overview of wavelets. Much more in-depth discussions are available in a number of texts, [11], [9], and [16] among them. If we begin with $\mathbb{R}$, we have the following.

Definition 1.4.1. *A function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if the collection $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, where*

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k),$$

that is, translations and dyadic, L^2-normalized dilations of ψ.

There are a number of ways to construct wavelets. One that brings the localization properties to the forefront is to build the wavelet from a multiresolution analysis.

Definition 1.4.2. *A multiresolution analysis (MRA) consisists of a sequence of closed subspaces of $L^2(\mathbb{R})$, V_j, $j \in \mathbb{Z}$, that satisfy the following conditions:*

1. $V_j \subset V_{j+1} \qquad j \in \mathbb{Z}$
2. $f \in V_j \iff f(2(\cdot)) \in V_{j+1} \qquad j \in \mathbb{Z}$
3. $\cap_{j\in\mathbb{Z}} V_j = \{0\}$
4. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$
5. *There exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.*

This φ is the scaling function of the MRA.

Then, a wavelet ψ can be constructed from φ with the following formula:

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} (-1)^k \overline{\alpha_k} \varphi(2x - (k-1))$$

where

$$\alpha_k = \frac{1}{2}\int_{\mathbb{R}} \varphi(2^{-1}x)\overline{\varphi(x+k)}\,dx.$$

One of the simplest wavelets is the Haar wavelet. Let V_j be the collection of all functions $f \in L^2(\mathbb{R})$ that are constant on intervals of the form $[2^{-j}k, 2^{-j}(k+1)]$ for $k \in \mathbb{Z}$. Then $\{V_j\}$ is an MRA if we take the scaling function $\varphi = \chi_{[-1,0]}$. The associated wavelet is then

$$\psi = \chi_{[-1,-\frac{1}{2})} - \chi_{[-\frac{1}{2},0)}.$$

Since mixed Lebesgue spaces are multi-variable spaces, we must consider wavelets for spaces that are at least two dimensional. To construct an orthonormal wavlet basis for $L^2(\mathbb{R}^n)$ if n is greater than 1 we need multiple wavelets (see, for example, [16]). These can be constructed in a number of ways; the simplest is via the tensor product of n one-dimesional wavelets, producing 2^{n-1} n-dimensional wavelets. For $L^2(\mathbb{R}^2)$, then, if ψ is a one-dimensional orthogonal wavelet generated by the orthogonal scaling function φ, then

$$\psi_1(t,x) = \psi(t)\varphi(x)$$

$$\psi_2(t,x) = \varphi(t)\psi(x)$$

$$\psi_3(t,x) = \psi(t)\psi(x)$$

are the corresponding two-dimensional tensor wavelets. If ψ is a band-limited orthonormal wavelet on $L^2(\mathbb{R})$, $\{\psi_1, \psi_2, \psi_3\}$ is a band-limited orthonormal wavelet on $L^2(\mathbb{R}^2)$ in the sense that $\{\psi_{1Q}, \psi_{2Q}, \psi_{3Q} : \nu, j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$ where $\psi_{iQ} = 2^{\nu}\psi_i(2^{\nu}t - k, 2^{\nu}x - j)$ and Q is the dyadic cube $Q = I_{\nu k} \times I_{\nu j} = [2^{-\nu}k, 2^{-\nu}k+1] \times [2^{-\nu}j, 2^{-\nu}j+1]$.

Definition 1.4.3. *Given functions f and ψ_1, ψ_2, ψ_3 for which $\langle f, \psi_i \rangle$ make sense, we define the operator $\mathscr{W}_\psi$, an expansion of f against the wavelets ψ_i:*

$$(\mathscr{W}_\psi f)(t,x) = \left(\sum_{i=1}^{3} \sum_{\nu,j,k \in \mathbb{Z}} |\langle f, \psi_{iQ} \rangle|^2 2^{2\nu} \chi_Q(t,x) \right)^{1/2}$$

where Q is as above.

If we further define T_ψ, the operator mapping f to the function that takes values in $\ell^2(\mathbb{Z} \times \mathbb{Z})$, given by

$$(T_\psi f)(t,x) = \{ \langle f, \psi_{iQ} \rangle 2^\nu \chi_Q(t,x) \}_{\substack{i=1,2,3 \\ \nu,j,k \in \mathbb{Z}}},$$

we have

$$(\mathscr{W}_\psi f)(t,x) = \sqrt{(T_\psi f)(t,x) \cdot (T_\psi f)(t,x)}$$

where $\cdot$ denotes the dot product in $\ell^2(\mathbb{Z} \times \mathbb{Z})$.

Chapter 2

Littlewood-Paley theory

2.1 Schur's test for Banach spaces

Before we proceed further, we will also need the following version of Schur's Test. The proof is parallel to that of the standard version, found, for example, in [6]. We have not been able to find this version in the literature, so we include a proof for completeness. (See another vector valued version in [13].)

Proposition 2.1.1. *Let* $\overrightarrow{K}$ *be an operator valued function from* $(X,\mu)\times(Y,\nu)$ *to* $\mathscr{L}(\mathscr{B}_1,\mathscr{B}_2)$ *where* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *are Banach spaces. If there exist* $A,B<\infty$ *such that*

$$\int_Y ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2} d\nu(y) = A \qquad \text{a.e. } x\in X \tag{2.1}$$

$$\int_X ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2} d\mu(x) = B \qquad \text{a.e. } y\in Y \tag{2.2}$$

then the operator $\overrightarrow{T}$ *given by*

$$\overrightarrow{T}F(x) = \int_Y \overrightarrow{K}(x,y)(F(y))d\nu(y)$$

is a bounded operator from $L^p(Y,\mathscr{B}_1)$ *into* $L^p(X,\mathscr{B}_2)$ *with norm at most* $A^{1-\frac{1}{p}}B^{\frac{1}{p}}$ *for* $1<p<\infty$.

Proof. Fix p, $1<p<\infty$ and let q be such that $\frac{1}{p}+\frac{1}{q}=1$. We can write

$$\begin{aligned} ||\overrightarrow{K}(x,y)(F(y))||_{\mathscr{B}_2} &\le ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}||F(y)||_{\mathscr{B}_1} \\ &= ||\overrightarrow{K}(x,y)||^{\frac{1}{q}}_{\mathscr{B}_1\to\mathscr{B}_2}\left(||\overrightarrow{K}(x,y)||^{\frac{1}{p}}_{\mathscr{B}_1\to\mathscr{B}_2}||F(y)||_{\mathscr{B}_1}\right). \end{aligned}$$

Using (2.1) and applying Holder's inequality gives us

$$\begin{aligned} \int_Y ||\overrightarrow{K}(x,y)(F(y))||_{\mathscr{B}_2}d\nu(y) &\le \left(\int_Y ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}d\nu(y)\right)^{\frac{1}{q}} \\ &\quad\cdot\left(\int_Y ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}||F(y)||^p_{\mathscr{B}_1}d\nu(y)\right)^{\frac{1}{p}} \\ &\le A^{\frac{1}{q}}\left(\int_Y ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}||F(y)||^p_{\mathscr{B}_1}d\nu(y)\right)^{\frac{1}{p}}. \end{aligned}$$

Raise both sides to the power p and integrate in x. Then apply Tonelli and use (2.2):

$$\begin{aligned} &\int_X\left(\int_Y ||\overrightarrow{K}(x,y)(F(y))||_{B_2}d\nu(y)\right)^p d\mu(x) \\ &\le A^{\frac{p}{q}}\int_X\int_Y ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}||F(y)||^p_{\mathscr{B}_1}d\nu(y)d\mu(x) \\ &\le A^{\frac{p}{q}}\int_Y ||F(y)||^p_{\mathscr{B}_1}\int_X ||\overrightarrow{K}(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}d\mu(x)d\nu(y) \\ &\le A^{\frac{p}{q}}B\int_Y ||F(y)||^p_{\mathscr{B}_1}d\nu(y). \end{aligned}$$

Note that the last integral is finite. Moreover, by Jensen's inequality,

$$\int_Y ||\overrightarrow{K}(x,y)(F(y))||_{\mathscr{B}_2}d\nu(y) \ge \left|\left|\int_Y \overrightarrow{K}(x,y)(F(y))d\nu(y)\right|\right|_{\mathscr{B}_2}.$$

Thus, by Fubini's theorem, $\overrightarrow{T}F(x)$ is well defined for a.e. $x\in X$, and

$$\int_X ||\overrightarrow{T}F(x)||^p_{\mathscr{B}_2}d\mu(x) \le A^{\frac{p}{q}}B\int_Y ||F(y)||^p_{\mathscr{B}_1}d\nu(y).$$

So taking pth roots we have

$$||\overrightarrow{T}F||_{L^p(X,\mathscr{B}_2)} \leq A^{1-\frac{1}{p}}B^{\frac{1}{p}}||F||_{L^p(Y,\mathscr{B}_1)}$$

as desired. □

2.2 Vector-valued Calderón-Zygmund theorem for mixed Lebesgue spaces

Now we can prove the boundedness of vector-valued Calderón-Zygmund operators over mixed Lebesgue spaces.

Theorem 2.2.1. *Let $\overrightarrow{T}$ be bounded from $L^q(\mathbb{R}^{n+1},\mathscr{B}_1)$ to $L^q(\mathbb{R}^{n+1},\mathscr{B}_2)$ for all q, $1 < q < \infty$ with associated kernel $\overrightarrow{K}$ that sastisfies (1.4), (1.5), and (1.6). Then $\overrightarrow{T}$ extends to a bounded operator from $L^p_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_1)$ to $L^p_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_2)$ for all $1 < p,q < \infty$ and it also satisfies the weak-type bound*

$$|\{t\in\mathbb{R} : ||\overrightarrow{T}f(t,\cdot)||_{L^q(\mathbb{R}^n,\mathscr{B}_2)} > \lambda\}| \leq \frac{C}{\lambda}||f||_{L^1_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_2)}.$$

Proof. We have $\overrightarrow{T} : L^q(\mathbb{R}^{n+1},\mathscr{B}_1) \to L^q(\mathbb{R}^{n+1},\mathscr{B}_2)$, with kernel $\overrightarrow{K}(u,v) : \mathscr{B}_1 \to \mathscr{B}_2$, $u,v\in\mathbb{R}^{n+1}$, so

$$\overrightarrow{T}f(u) = \int_{\mathbb{R}^{n+1}} \overrightarrow{K}(u,v)(f(v))dv$$

for u not in the support of f. But if we write $u = (t,x) \in \mathbb{R}\times\mathbb{R}^n$, $L^q_u = L^q_tL^q_x$, so we can also write

$$\overrightarrow{T} : L^q_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_1) \to L^q_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_2)$$

with associated kernel $\overrightarrow{K}[(t,x),(s,y)] : \mathscr{B}_1 \to \mathscr{B}_2$; $(t,x),(s,y) \in \mathbb{R}\times\mathbb{R}^n$ so that

$$\overrightarrow{T} f(t,x) = \int_R \int_{R^n} \overrightarrow{K}[(t,x),(s,y)](f(s,y))dyds$$

for (t,x) not in the support of f. We want to show that the above is true for exponents p and q when $p \neq q$. With that in mind, note that $L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}) = L_t^p(\mathbb{R}, L_x^q(\mathbb{R}^n,\mathscr{B}))$.

Given an operator $\overrightarrow{T}$ as above, it has an associated operator $\overrightarrow{\mathscr{T}}$ with associated kernel $\overrightarrow{\mathscr{K}}(u,v) : L_x^q(\mathbb{R}^n,\mathscr{B}_1) \to L_x^q(\mathbb{R}^n,\mathscr{B}_2)$, $u,v \in \mathbb{R}$. That is, for each $t \in \mathbb{R}$,

$$\overrightarrow{\mathscr{T}} F(t)(x) = \int_{\mathbb{R}} \overrightarrow{\mathscr{K}}(x,u)(F(t)(u))du$$

for x not in the support of $F(t)$, with $\overrightarrow{\mathscr{T}} F(t)(x) = \overrightarrow{T} f(t,x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Now: fix q, $1 < q < \infty$. Set $L^q(\mathbb{R}^n,\mathscr{B}_1) = \mathscr{B}_3$ and $L^q(\mathbb{R}^n,\mathscr{B}_2) = \mathscr{B}_4$. Note that

$$L_t^q(\mathbb{R}, L_x^q(\mathbb{R}^n,\mathscr{B})) = L_t^q L_x^q(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}) = L_{(t,x)}^q(\mathbb{R}^{n+1},\mathscr{B}).$$

Therefore, since $\overrightarrow{T}$ is bounded from $L^q(\mathbb{R}^{n+1},\mathscr{B}_1)$ to $L^q(\mathbb{R}^{n+1},\mathscr{B}_2)$, $\overrightarrow{T}$ is bounded from $L^q L^q(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_1)$ to $L^q L^q(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_2)$, and $\overrightarrow{\mathscr{T}}$ is bounded from $L^q(\mathbb{R},\mathscr{B}_3)$ to $L^q(\mathbb{R},\mathscr{B}_4)$.

So if we show that $\overrightarrow{\mathscr{K}}$ is bounded and satisfies the two estimates

$$\int_{|t-s|\geq 2|s-r|} ||\overrightarrow{\mathscr{K}}(t,s) - \overrightarrow{\mathscr{K}}(t,r)||_{\mathscr{B}_3\to\mathscr{B}_4} dt \leq C \tag{2.3}$$

$$\int_{|t-s|\geq 2|t-t|} ||\overrightarrow{\mathscr{K}}(t,s) - \overrightarrow{\mathscr{K}}(t,s)||_{\mathscr{B}_3\to\mathscr{B}_4} dt \leq C \tag{2.4}$$

then by Theorem 1.2.6 we will have the desired result (after translating appropriately).

We start with (2.3). Fix $t,s,r\in\mathbb{R}$ with $|t-s|\geq 2|s-r|$ and define and operator G as follows:

$$\begin{aligned} G(t,s,r)H(x) &= (\overrightarrow{\mathscr{K}}(t,s)-\overrightarrow{\mathscr{K}}(t,r))H(x) \\ &= \int_{\mathbb{R}^n}\left(\overrightarrow{K}[(t,x),(s,y)]-\overrightarrow{K}[(t,x),(r,y)]\right)H(y)dy \\ &= \int_{\mathbb{R}^n} g(t,s,r)(x,y)H(y)dy. \end{aligned}$$

Now consider the kernel $g(t,s,r)(x,y)$. Using (1.5)

$$\begin{aligned} ||g(t,s,r)(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2} &= ||\overrightarrow{K}[(t,x),(s,y)]-\overrightarrow{K}[(t,x),(r,y)]||_{\mathscr{B}_1\to\mathscr{B}_2} \\ &\leq C\frac{|(s,y)-(r,y)|^\delta}{|(t,x)-(s,y)|^{n+1+\delta}} \\ &= C\frac{|s-r|^\delta}{(|t-s|^2+|x-y|^2)^{(n+1+\delta)/2}}. \end{aligned}$$

If we integrate in x we have:

$$\begin{aligned} \int_{\mathbb{R}^n}||g(t,s,r)(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}dx &\leq C|s-r|^\delta\int_{\mathbb{R}^n}\frac{dx}{(|t-s|^2+|x-y|^2)^{(n+1+\delta)/2}} \\ &= C|s-r|^\delta|\mathbb{S}^{n-1}|\int_0^\infty\frac{\rho^{n-1}d\rho}{(|t-s|^2+\rho^2)^{(n+1+\delta)/2}} \\ &= C|s-r|^\delta\frac{1}{|t-s|^{n+1+\delta}}\int_0^\infty\frac{\rho^{n-1}d\rho}{(1+(\frac{\rho}{|t-s|})^2)^{(n+1+\delta)/2}} \\ &= C|s-r|^\delta\frac{|t-s|^n}{|t-s|^{n+1+\delta}}\int_0^\infty\frac{u^{n-1}du}{(1+u^2)^{(n+1+\delta)/2}} \\ &\leq C\frac{|s-r|^\delta}{|t-s|^{1+\delta}}. \end{aligned}$$

Likewise, if we integrate in y we get

$$\int_{\mathbb{R}^n}||g(t,s,r)(x,y)||_{\mathscr{B}_1\to\mathscr{B}_2}dy\leq C\frac{|s-r|^\delta}{|t-s|^{1+\delta}}.$$

These are the two estimates on $g(t,s,r)$ necessary to apply Theorem 2.1.1, so $G(t,s,r)$ maps $\mathscr{B}_3$ into $\mathscr{B}_4$ with norm

$$||G(t,s,r)||_{\mathscr{B}_3\to\mathscr{B}_4} = ||\overrightarrow{\mathscr{K}}(t,s) - \overrightarrow{\mathscr{K}}(t,r)||_{\mathscr{B}_3\to\mathscr{B}_4} \le C\frac{|s-r|^{\delta}}{|t-s|^{1+\delta}}$$

and therefore $\overrightarrow{\mathscr{K}}$ satisfies (2.3). By following the same steps in the other variable, $\overrightarrow{\mathscr{K}}$ satisfies (2.4). A similar argument shows that $\overrightarrow{\mathscr{K}}$ is bounded.

By Theorem 1.2.6, then, $\overrightarrow{\mathscr{T}}$ is bounded from $L^p(\mathbb{R},\mathscr{B}_3)$ to $L^p(\mathbb{R},\mathscr{B}_4)$ for all $1 < p < \infty$, and $\overrightarrow{\mathscr{T}}$ is weakly bounded on $L^1(\mathbb{R},\mathscr{B}_3)$ in the sense that

$$|\{t : ||\overrightarrow{\mathscr{T}}f(t)||_{\mathscr{B}_4} > \lambda\}| \le \frac{C}{\lambda}||F||_{L^1(\mathbb{R},\mathscr{B}_3)}.$$

Translating back to the desired setting, we have $\overrightarrow{T}$ bounded from $L^p_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_1)$ to $L^p_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_2)$ for all $1 < p,q < \infty$ and weakly bounded in the sense that

$$|\{t\in\mathbb{R} : ||\overrightarrow{T}f(t,\cdot)||_{L^q(\mathbb{R}^n,\mathscr{B}_2)} > \lambda\}| \le \frac{C}{\lambda}||f||_{L^1_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathscr{B}_2)}.$$

□

2.3 Littlewood-Paley characterization of mixed Lebesgue spaces

We will use this theorem to obtain a Littlewood-Paley theorem for mixed spaces.

Theorem 2.3.1. *Suppose that Ψ is an integrable C^1 function on $\mathbb{R}^{n+1}$ with mean value zero that satisfies*

$$|\Psi(t,x)| + |\nabla\Psi(t,x)| \leq \frac{B}{(1+|(t,x)|)^{n+2}}. \tag{2.5}$$

Let $f \in L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)$, $1 < p,q < \infty$. Then there exists a constant $C = C_{n,p,q}$ independent of f such that

$$\left|\left|\left(\sum_{j\in\mathbb{Z}} |f*\Psi_{2^{-j}}|^2\right)^{1/2}\right|\right|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)} \leq C||f||_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)}. \tag{2.6}$$

Conversely, if $\Psi \in \mathscr{S}(\mathbb{R}^{n+1})$ is such that $\operatorname{supp}\hat{\Psi} \supset \{\frac{\pi}{4} < |\xi| < \pi\}$ *and $\hat{\Psi} > c > 0$ on $\{\frac{\pi}{4}+\varepsilon < |\xi| < \pi - \varepsilon\}$,*

$$||f||_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)} \leq C\left|\left|\left(\sum_{j\in\mathbb{Z}} |f*\Psi_{2^{-j}}|^2\right)^{1/2}\right|\right|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)} \tag{2.7}$$

for some constant $C = C_{n,\Psi,p,q}$.

Proof. First note that when $p = q$, $L^pL^q(\mathbb{R}\times\mathbb{R}^n) = L^q(\mathbb{R}^{n+1})$, and so we apply Theorem 1.2.7 to obtain

$$\left|\left|\left(\sum_{j\in\mathbb{Z}} |f*\Psi_{2^{-j}}|^2\right)^{1/2}\right|\right|_{L^q(\mathbb{R}^{n+1})} \leq CBc_q||f||_{L^q(\mathbb{R}^{n+1})}$$

for $f \in L^q(\mathbb{R}^{n+1})$. This inequality tells us that the operator $\overrightarrow{T}$, acting on $f \in L^q(\mathbb{R}^{n+1})$ by

$$\overrightarrow{T}f(u) = \{f*\Psi_{2^{-j}}(u)\}_j$$

is bounded from $L^q(\mathbb{R}^{n+1},\mathbb{C})$ to $L^q(\mathbb{R}^{n+1},\ell^2)$. $\overrightarrow{K}$, the kernel associated to $\overrightarrow{T}$, is a bounded linear operator from $\mathbb{C}$ to ℓ^2 given by

$$\overrightarrow{K}(u,v)(a) = \{\Psi_{2^{-j}}(u-v)(a)\}_j$$

for each $u,v \in \mathbb{R}^{n+1}$. Moreover, $\overrightarrow{K}$ is the kernel that appears in the proof of Theorem 1.2.7, and so by the same argument there (see, for example, [10]) it is a standard kernel, and thus satisfies (1.4), (1.5), and (1.6). So by Theorem 2.2.1, $\overrightarrow{T}$ extends to a bounded operator from $L^p_x L^q_y(\mathbb{R}^m \times \mathbb{R}^n,\mathbb{C})$ to $L^p_x L^q_y(\mathbb{R}^m \times \mathbb{R}^n,\ell^2)$ for all $1 < p,q < \infty$. This, in turn, gives us the desired estimate (2.6).

The proof of (2.7) follows by duality. First, note that there exists Φ that satisfies the same conditions as Ψ such that

$$\sum_j \hat{\Phi}(2^{-j}\xi)\hat{\Psi}(2^{-j}\xi) = 1.$$

Further, denote by Δ_j^Ψ and $\Delta_j^\Phi$the operators

$$\Delta_j^\Psi(f) = f * \Psi_{2^{-j}}$$

and

$$\Delta_j^\Phi(f) = f * \Phi_{2^{-j}}.$$

Note that we can write

$$f = \sum_{j\in\mathbb{Z}} \Delta_j^\Phi \Delta_j^\Psi(f).$$

Let g be a Schwartz function, and consider

$$\begin{aligned} |\langle f,\overline{g}\rangle| &= \left|\langle \sum_{j\in\mathbb{Z}} \Delta_j^{\Phi}\Delta_j^{\Psi}(f),\overline{g}\rangle\right| \\ &= \left|\sum_{j\in\mathbb{Z}} \langle \Delta_j^{\Psi}(f),\overline{\Delta_j^{\Phi}(g)}\rangle\right| \\ &= \left|\int_{\mathbb{R}^m\times\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \Delta_j^{\Psi}(f)\overline{\Delta_j^{\Phi}(g)}dydx\right|. \end{aligned}$$

Now apply the Cauchy-Schwartz inequality in the sum:

$$\leq \int_{\mathbb{R}\times\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\Psi}(f)|^2\right)^{1/2} \left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\Phi}(g)|^2\right)^{1/2} dxdt,$$

The dual of L^pL^q is $L^{p'}L^{q'}$. Apply Hölder's inequality:

$$\leq \left\|\left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\Psi}(f)|^2\right)^{1/2}\right\|_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R}^n)} \left\|\left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\Phi}(g)|^2\right)^{1/2}\right\|_{L_t^{p'}L_x^{q'}(\mathbb{R}\times\mathbb{R}^n)}$$

and then (2.6) for the expression in g:

$$\leq \left\|\left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\Psi}(f)|^2\right)^{1/2}\right\|_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R}^n)} C||g||_{L_t^{p'}L_x^{q'}(\mathbb{R}\times\mathbb{R}^n)}.$$

Now taking the supremum over all g in $L_t^{p'}L_x^{q'}(\mathbb{R}\times\mathbb{R}^n)$ with norm at most one gives the desired inequality (2.7). □

Chapter 3

Leibniz's rule

3.1 Versions of Leibniz's rule

The machinery established in Chapter 2 is important to answering a number of questions in analysis. One of the places it is useful is in establishing a version of Leibniz's rule for fractional derivatives. In it's simplest form, Leibniz's rule is the product rule:

$$\frac{d}{dx}fg = f\frac{dg}{dx} + \frac{df}{dx}g.$$

To consider a version in Lebesgue spaces, we first extend the derivative to $s \in (0,1)$ by

$$|\nabla|^s f = (|\cdot|^s \hat{f})\check{}.$$

With that frame, Kato and Ponce [12] proved a version for Lebesgue spaces. See also [3].

Theorem 3.1.1. *Let $s \in (0,1)$, $1 < r, p_i, q_i < \infty$ and $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ for $i = 1,2$. Then*

$$||\,|\nabla|^s(fg)||_{L^r} \leq ||f||_{L^{p_1}}||\,|\nabla|^s g||_{L^{q_1}} + ||\,|\nabla|^s f||_{L^{p_2}}||g||_{L^{q_2}}.$$

Another range of exponents can be obtained using bilinear operators. Kenig, Ponce, and Vega proved a version for mixed Lebesgue spaces in the appendix of [14], with derivatives in only one variable. We succeed in considering derivatives in both x and t.

3.2 A Leibniz's rule for mixed Lebesgue spaces

We begin by stating a version of the Hormander-Mikhlm multiplier theorem for our setting.

Theorem 3.2.1. *Let $\alpha \geq 0$ be a multi-index with $|\alpha| \leq \left[\frac{n+1}{2}\right] + 1$ and suppose that the operator T_m is given by*

$$\widehat{T_m f} = m(\xi_1, \xi_2)\hat{f}(\xi_1, \xi_2)$$

for $\xi_1 \in \mathbb{R}$ and $\xi_2 \in \mathbb{R}^n$ and that m satisfies the condition

$$|\partial^\alpha m(\xi_1, \xi_2)| \leq \frac{C}{|(\xi_1, \xi_2)|^{|\alpha|}}.$$

Then $T_m : L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n) \to L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n)$ for $1 < p, q < \infty$.

Note that the kernel in question is still a Calderón-Zygmund kernel, as proved in the L^p case (Theorem 1.2.6). Thus it is bounded from L^p to L^p. By Theorem 2.2.1, then, it is bounded on mixed Lebesgue spaces.

Lemma 3.2.2. *For $\Psi \in \mathscr{S}(\mathbb{R}^{n+1})$ is such that* supp $\hat{\Psi} \supset \{\frac{\pi}{4} < |\xi| < \pi\}$ *and $\hat{\Psi} > c > 0$ on $\{\frac{\pi}{4} + \varepsilon < |\xi| < \pi - \varepsilon\}$and $\Delta_j = \Delta_j^\Psi$ as in Theorem 2.3.1, $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$,*

$$\| |\nabla|^s f \|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \approx \left\| \left(\sum_{j\in\mathbb{Z}} (2^{js}|\Delta_j(f)|)^2 \right)^{1/2} \right\|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)}$$

for $f \in L^pL^q$ with $|\nabla|^s f \in L^pL^q$.

Proof. Note that for $\xi = (\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^n$,

$$\begin{aligned} 2^{js}\Delta_j(f) &= 2^{js}(\hat{\Psi}(2^{-j}\xi)\hat{f})\check{} \\ &= 2^{js}(|\xi|^s|\xi|^{-s}\hat{\Psi}(2^{-j}\xi)\hat{f})\check{} \\ &= (\hat{\sigma}(2^{-j}\xi)\hat{f})\check{} \\ &= \Delta_j^{\sigma}(|\nabla|^s f). \end{aligned}$$

where $\hat{\sigma}(\xi) = |\xi|^{-s}\hat{\Psi}(\xi)$. Thus we have

$$\begin{aligned} \left|\left|\left(\sum_{j\in\mathbb{Z}} |2^{js}\Delta_j(f)|^2\right)^{1/2}\right|\right|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} &= \left|\left|\left(\sum_{j\in\mathbb{Z}} |\Delta_j^{\sigma}(|\nabla|^s f)|^2\right)^{1/2}\right|\right|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \\ &\approx C||\,|\nabla|^s f||_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \end{aligned}$$

by Theorem 2.3.1. □

We also need a mixed Lebesgue version of the Fefferman-Stein theorem. This version follows from Theorem 4.2 in Fernandez [5], and is sufficient for our purposes here.

Theorem 3.2.3. *Suppose that $\{f_j\}$ is a sequence of locally integrable functions. Whenever $1 < p, q, r < \infty$*

$$\left|\left|\left(\sum_j |M(f_j)|^r\right)^{1/r}\right|\right|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \leq C\left|\left|\left(\sum_j |f_j|^r\right)^{1/r}\right|\right|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)}$$

with C dependent only on n, p, q, r.

Given the above, we can prove a version of Liebnitz's rule in the same spirit of that given by Christ and Weinstein [3] for the Lebesgue case by using the mixed versions of the required theorems.

Theorem 3.2.4. *Let $s>0$, $1<p,q,p_i,q_i<\infty$ for $1=1,\dots 4$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$ and $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}+\frac{1}{q_4}$. Suppose that $f\in L^{p_1}L^{q_1}$, $|\nabla|^s f\in L^{p_3}L^{q_3}$, $g\in L^{p_4}L^{q_4}$, and $|\nabla|^s g\in L^{p_2}L^{q_2}$. Then $|\nabla|^s(fg)\in L^pL^q$ and*

$$||\,|\nabla|^s(fg)||_{L^pL^q}\le C||f||_{L^{p_1}L^{q_1}}||\,|\nabla|^s g||_{L^{p_2}L^{q_2}}+C||\,|\nabla|^s f||_{L^{p_3}L^{q_3}}||g||_{L^{p_4}L^{q_4}}.$$

Proof. Here we combine the proof of [3] with our mixed Lebesgue space results. Consider a $\Psi\in\mathscr{S}(\mathbb{R}^{n+1})$ such that $\hat{\Psi}\equiv 1$ on $\{\frac{1}{4}\le|\xi|\le 4\}$ and supp $\hat{\Psi}\subset\{\frac{1}{8}<|\xi|<8\}$. Let $\Delta_j=\Delta_j^{\Psi}$ as in the proof of Theorem 2.3.1, and let $\tilde{\Delta}_j$ denote the corresponding Δ_j^{Φ} so that $1=\sum_{j\in\mathbb{Z}}\tilde{\Delta}_j\Delta_j$. Now define

$$P_kf=\sum_{j\le k-3}\Delta_jf.$$

Note that for all f and g,

$$\Delta_kg\cdot P_kf=\Delta_k(\Delta_kg\cdot P_kf)$$

so we can write

$$\begin{aligned} fg &= \sum_k\Delta_kg\cdot P_kf+\sum_k\Delta_kf\cdot P_kg+\sum_{|i-j|\le 2}\Delta_if\cdot\Delta_jg \\ &= \sum_k\tilde{\Delta}_k(\Delta_kg\cdot P_kf)+\sum_k\tilde{\Delta}_k(\Delta_kf\cdot P_kg)+\sum_{|i-j|\le 2}\Delta_if\cdot\Delta_jg \end{aligned}$$

Then by Lemma 3.2.2 we have

$$\begin{aligned}
||\,|\nabla|^s(fg)||_{L^pL^q} &\approx \left\|\left(\sum_{j\in\mathbb{Z}}(2^{js}|\Delta_j(fg)|)^2\right)^{1/2}\right\|_{L^pL^q} \\
&\leq \left\|\left(\sum_j 2^{2js}|\Delta_j\left(\sum_k \tilde{\Delta}_k(\Delta_k g\cdot P_k f)\right)|^2\right)^{1/2}\right\|_{L^pL^q} \\
&\quad + \left\|\left(\sum_j 2^{2js}|\Delta_j\left(\sum_k \tilde{\Delta}_k(\Delta_k f\cdot P_k g)\right)|^2\right)^{1/2}\right\|_{L^pL^q} \\
&\quad + \left\|\left(\sum_k 2^{2ks}|\Delta_k\left(\sum_{|i-j|\leq 2}\Delta_i f\cdot\Delta_j g\right)|^2\right)^{1/2}\right\|_{L^pL^q} \\
&= I+II+III.
\end{aligned}$$

We consider each term individually. For the first, since the Fourier transform of $\tilde{\Delta}_k(\Delta_k g\cdot P_k f)$ is supported in $|\xi|\sim 2^k$,

$$\begin{aligned}
I &= \left\|\left(\sum_j 2^{2js}|\Delta_j\left(\sum_k \tilde{\Delta}_k(\Delta_k g\cdot P_k f)\right)|^2\right)^{1/2}\right\|_{L^pL^q} \\
&\leq \left\|\left(\sum_j 2^{2js}|C\Delta_j\tilde{\Delta}_j(\Delta_j g\cdot P_j f)|^2\right)^{1/2}\right\|_{L^pL^q} \\
&\leq \left\|\left(\sum_j |M(2^{js}\Delta_j g\cdot P_j f)|^2\right)^{1/2}\right\|_{L^pL^q}
\end{aligned}$$

because $\Delta_k h, \tilde{\Delta}_k h \leq CMh$. Then we have

$$\begin{aligned}
I &\leq C\left\|\left(\sum_j 2^{2js}(Mf)^2\cdot|\Delta_j g|^2\right)^{1/2}\right\|_{L^pL^q} \\
&\leq C||Mf||_{L^{p_1}L^{q_1}}\left\|\left(\sum_j 2^{2js}|\Delta_j g|^2\right)^{1/2}\right\|_{L^{p_2}L^{q_2}} \\
&\leq C||f||_{L^{p_1}L^{q_1}}|||\nabla|^s g||_{L^{p_2}L^{q_2}}
\end{aligned}$$

with the second to last line being by Hölder's inequality and the final line by Lemma 3.2.2 and the boundedness of M. Treating II in exactly the same way with the roles of f and g reversed gives

$$II \leq C|| \, |\nabla|^s f||_{L^{p_3}L^{q_3}} ||g||_{L^{p_4}L^{q_4}}.$$

□

For III, note that when $|i-j| \leq 2$, $\Delta_k(\Delta_i f \cdot \Delta_j g) \equiv 0$ except when $k \leq \max(i,j)+4$. So we have

$$\begin{aligned}
III &= \left|\left| \left(\sum_k 2^{2ks} |\Delta_k \left(\sum_{|i-j|\leq 2} \Delta_i f \cdot \Delta_j g \right) |^2 \right)^{1/2} \right|\right|_{L^pL^q} \\
&\leq C \left|\left| \left(\sum_k 2^{2ks} |\Delta_k \left(\sum_{\substack{|i-j|\leq 2 \\ \max(i,j)\geq k-4}} \Delta_i f \cdot \Delta_j g \right) |^2 \right)^{1/2} \right|\right|_{L^pL^q} \\
&\leq C \left|\left| \sum_{\ell \geq -6} \sum_{|m|\leq 2} \left(\sum_j 2^{2(j-\ell)s} |\Delta_{j-\ell}(\Delta_{j-m} f \cdot \Delta_j g)|^2 \right)^{1/2} \right|\right|_{L^pL^q} \\
&= C \left|\left| \sum_{\ell \geq -6} \sum_{|m|\leq 2} 2^{-\ell s} \left(\sum_j |\Delta_{j-\ell}(\Delta_{j-m} f \cdot 2^{js} \Delta_j g)|^2 \right)^{1/2} \right|\right|_{L^pL^q}
\end{aligned}$$

and we finish the estimate as we did with I. Assembling the three pieces we have

$$\begin{aligned}
|||\nabla|^s(fg)||_{L^pL^q} &\leq C||f||_{L^{p_1}L^{p_1}} |||\nabla|^s g||_{L^{p_2}L^{p_2}} \\
&\quad + C|||\nabla|^s f||_{L^{p_3}L^{p_3}} ||g||_{L^{p_4}L^{p_4}} \\
&\quad C||f||_{L^{p_1}L^{p_1}} |||\nabla|^s g||_{L^{p_2}L^{p_2}} \\
&= C||f||_{L^{p_1}L^{q_1}} |||\nabla|^s g||_{L^{p_2}L^{q_2}} + C|||\nabla|^s f||_{L^{p_3}L^{q_3}} ||g||_{L^{p_4}L^{q_4}}.
\end{aligned}$$

Chapter 4

More on Fefferman-Stein inequalities in mixed Lebesgue spaces

While the previously stated version of the Fefferman-Stein theorem is all that is needed for the above proof, we have proved the weak-type end-point result that is missing in the work of Fernandez (Theorem 3.2.3) but is present in the Lebesgue space version (Theorem 1.3.2). We succeed by adapting to our context some ideas from [19] (page 52).

Lemma 4.0.5. *Let $0 < p < q_1, q_2, r_1, r_2 < \infty$, and $\{T_j\}$ be a sequence of sublinear operators that map $L^{q_1}L^{q_2}(\mathbb{R}\times\mathbb{R}) \to L^{r_1}L^{r_2}(\mathbb{R}\times\mathbb{R})$. Set $s_1 = \frac{r_1}{r_1-p}$, $s_2 = \frac{r_2}{r_2-p}$, $s_3 = \frac{q_1}{q_1-p}$, and $s_4 = \frac{q_2}{q_2-p}$. If for each $u \in L^{s_1}L^{s_2}(\mathbb{R}\times\mathbb{R})$ there is a $U \in L^{s_3}L^{s_4}(\mathbb{R}\times\mathbb{R})$ such that*

$$||U||_{L^{s_3}L^{s_4}} \leq ||u||_{L^{s_1}L^{s_2}}$$

and

$$\sup_j \int |T_j(f)|^p u \, dydx \leq C^p \int |f|^p U \, dydx \tag{4.1}$$

for each $f \in L^{q_1}L^{q_2}$, then

$$\left|\left|\left(\sum_j |T_j(f_j)|^p\right)^{1/p}\right|\right|_{L^{r_1}L^{r_2}} \leq C\left|\left|\left(\sum_j |f_j|^p\right)^{1/p}\right|\right|_{L^{q_1}L^{q_2}}$$

for every $f_j \in L^{q_1}L^{q_2}$.

Proof. By Proposition 1.1.4, duality (as the dual of $L^{r_1/p}L^{r_2/p}$ is $L^{s_1}L^{s_2}$), and the assumption (4.1) we have

$$\begin{aligned}
\left|\left|\left(\sum_j |T_j(f_j)|^p\right)^{1/p}\right|\right|_{L^{r_1}L^{r_2}} &= \left|\left|\sum_j |T_j(f_j)|^p\right|\right|^{1/p}_{L^{r_1/p}L^{r_2/p}} \\
&= \sup_{||u||_{L^{s_1}L^{s_2}}\leq 1} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \sum_j |T_j(f_j)|^p u\, dydx\right)^{1/p} \\
&\leq \sup_{||u||_{L^{s_1}L^{s_2}}\leq 1} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \sum_j |f_j|^p U\, dydx\right)^{1/p} \\
&\leq \sup_{||u||_{L^{s_1}L^{s_2}}\leq 1} C\left|\left|\left(\sum_j |f_j|^p\right)\right|\right|^{1/p}_{L^{q_1/p}L^{q_2/p}} ||U||^{1/p}_{L^{s_3}L^{s_4}} \\
&\leq \sup_{||u||_{L^{s_1}L^{s_2}}\leq 1} C\left|\left|\left(\sum_j |f_j|^p\right)^{1/p}\right|\right|_{L^{q_1}L^{q_2}} ||U||^{1/p}_{L^{s_3}L^{s_4}} \\
&\leq C\left|\left|\left(\sum_j |f_j|^p\right)^{1/p}\right|\right|_{L^{q_1}L^{q_2}}
\end{aligned}$$

with the final lines by Hölder's inequality (note that $\frac{1}{p} = \frac{1}{q_i} + \frac{1}{\frac{pq_i}{q_i-p}}$) and Proposition 1.1.4, and then we obtain the final inequality by the assumption on the norms of u and U. □

Remark 4.0.6. *The requirement imposed by (4.1) may be a necessary condition. It is in the scalar-valued case.*

Lemma 4.0.7. *For $1 < q < \infty$ there is a C dependent only on q such that*

$$\int_{\mathbb{R}^n} M(f)^q g\, dx \leq C\int_{\mathbb{R}^n} f^q M(g)\, dx$$

for all $f,g \geq 0$ that are locally integrable on $\mathbb{R}^n$.

This fact is well known, see, for example, [10].

We begin by providing an alternate proof of Fernandez result for a limited range of exponents.

Theorem 4.0.8. *Suppose that $\{f_j\}$ is a sequence of locally integrable functions. Whenever $1 < r < p,q < \infty$,*

$$\left|\left|\left(\sum_j |M(f_j)|^r\right)^{1/r}\right|\right|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \leq C\left|\left|\left(\sum_j |f_j|^r\right)^{1/r}\right|\right|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)}$$

with C dependent only on n,p,q,r.

Proof. In Lemma 4.0.5, set all the $T_j = M$. Set $s_1 = \frac{p}{p-r}$ and $s_2 = \frac{q}{q-r}$. Given $u \in L^{s_1}L^{s_2}(\mathbb{R}\times\mathbb{R}^n)$ set $U = ||M||^{-1}_{L^{s_1}L^{s_2}\to L^{s_1}L^{s_2}}M(u)$. Then by Lemma 4.0.7,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^n} M(f)^p u\, dxdt &\leq C\int_{\mathbb{R}}\int_{\mathbb{R}^n} f^p M(u)\, dxdt \\ &\leq C\int_{\mathbb{R}}\int_{\mathbb{R}^n} |f|^p U\, dxdt.\end{aligned}$$

Also,

$$||U||_{L^{s_1}L^{s_2}} \leq ||M||^{-1}_{L^{s_1}L^{s_2}\to L^{s_1}L^{s_2}}||M(u)||_{L^{s_1}L^{s_2}} \leq ||u||_{L^{s_1}L^{s_2}}.$$

Thus, by Lemma 4.0.5, we have the desired result. □

Now we obtain the weak-type end point.

Theorem 4.0.9. *Suppose that $\{f_j\}$ is a sequence of locally integrable functions. For $1 < r,q < \infty$,*

$$\left|\{t : \left|\left|\left(\sum_j |M(f_j)|^r\right)^{1/r}\right|\right|_{L^q} > \lambda\}\right| \leq \frac{C}{\lambda}\left|\left|\left|\left|\left(\sum_j |f_j|^r\right)^{1/r}\right|\right|_{L^q}\right|\right|_{L^1}$$

with C depending only on n, q, r.

Proof. Let us write

$$|f(t)| = ||\sum f_j^r(t,x))^{1/r}||_{L_x^q}$$

We write $F(t) = \{f_j(t,x)\}_{j=1}^{\infty}$ and define

$$|F(t)|_X = ||(\sum (f_j(t,x))^r)^{1/r}||_{L_x^q}$$

and

$$||F||_{L_t^p} = \left(\int |F(t)_X^p \right)^{1/p}.$$

We further define

$$M_{r,q}(F) = ||(\sum (Mf_j(t,x))^r)^{1/r}||_{L_x^q}.$$

Now , given $F(t)$ we can find $Q_j \in \mathbb{R}$ disjoint so that

- $|F(t)|_X < \lambda$ if $t \notin \cup Q_j$
- $\sum |Q_j| \leq \frac{1}{\lambda} \int |F(t)|_X dt$
- $\frac{1}{|Q_j|} \int_{Q_j} |F(t)|_X \leq 2^n \lambda$.

Write $f_k(t,x) = g_k(t,x) + h_k(t,x)$ where $g_k(t,x) = f_k(t,x)\chi_{\mathbb{R}^n \setminus \cup Q_j}(t)$. Set $G = \{g_k\}$ and $H = \{h_k\}$. Then we have

$$\begin{aligned} \left|\left\{M_{r,q}(F)(t) > \lambda\right\}\right| &\lesssim \left|\left\{M_{r,q}(G)(t) > \lambda/2\right\}\right| + \left|\left\{M_{r,q}(H)(t) > \lambda/2\right\}\right| \\ &= I + II. \end{aligned}$$

For I we have the following for any q:

$$\begin{aligned}
I &\leq \frac{2^q}{\lambda^q}\int_{\mathbb{R}} |M_{r,q}(G)(t)|^q dt \\
&= \frac{2^q}{\lambda^q}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|\left(\sum (Mg_j(t,x))^r\right)^{1/r}\right|^q \, dx\, dt \\
&\leq \frac{2^q}{\lambda^q}\int_{\mathbb{R}} |G(t)|_X^q dt \\
&\leq \frac{2^q}{\lambda^q}\int_{\mathbb{R}\setminus\cup Q_j} |F(t)|_X^q dt \\
&= \frac{2^q\lambda^{q-1}}{\lambda^q}\int_{\mathbb{R}} |F(t)|_X dt \\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} \left(\sum |f_j(t,x)|^r\right)^{q/r} dx\right)^{1/q} dt \\
&\leq \frac{2}{\lambda}\left|\left|\left(\sum |f_j(t,x)|^r\right)^{1/r}\right|\right|_{L^1L^q}
\end{aligned}$$

with the third line by the Fefferman-Stein inequality in $\mathbb{R}^{n+1}$. To estimate II consider

$$H^0(t) = \sum F_j(t,x)\chi_j(t)$$

where

$$F_j = \frac{1}{|Q_j|}\int_{Q_j} F(t)dx = \left\{\frac{1}{|Q_j|}\int_{Q_j} f_k(t,x)dt\right\}_{k=1}^{\infty}$$

and note that $H^0 \equiv 0$ on $\mathbb{R}^n \setminus \cup Q_j$ and

$$H^0(t) = \left\{\sum_j \frac{1}{Q_j}\int_{Q_j} f_k(t,x)dt\chi_j(t)\right\}_{k=1}^{\infty}$$

Therefore, for $t \in Q_j$,

$$
\begin{aligned}
|H^0(t)|_X &= \left\| \left(\sum_k \left(\sum_j \frac{1}{Q_j} \int_{Q_j} f_k(t,x) dt \chi_j(t) \right)^r \right)^{1/r} \right\|_{L^q_x} \\
&= \left\| \left(\sum_k \left(\frac{1}{|Q_j|} \int_{Q_j} f_k(t,x) dt \right)^r \right)^{1/r} \right\|_{L^q_x} \\
&\leq \left\| \frac{1}{Q_j} \int_{Q_j} \left(\sum_k |f_k(t,x)|^r \right)^{1/r} dt \right\|_{L^q_x} \\
&\leq \frac{1}{Q_j} \int_{Q_j} \left\| \left(\sum_k |f_k|^r \right)^{1/r} \right\|_{L^q_x} dt \\
&\leq \frac{1}{|Q_j|} \int_{Q_j} |F(t)|_X dt \\
&\leq 2^n \lambda.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\left| \left\{ |M_{r,q} H^0|_X(t) > \lambda/2 \right\} \right| &\leq \frac{c}{\lambda^p} ||M_{r,q} H^0||^p_{L^p} \\
&\leq \frac{c}{\lambda^p} ||H^0||^p_{L^p} \\
&\leq \frac{c}{\lambda^p} ||H^0||^p_{L^\infty} |\cup Q_j| \\
&\lesssim \frac{\lambda^p}{\lambda^p} \frac{||F||_{L^1}}{\lambda} = \frac{||F||_{L^1}}{\lambda}.
\end{aligned}
$$

To conclude estimating II, it would be enough to show that

$$M_{r,q} H(t) \leq c M_{r,q} H^0(t)$$

when $t \notin \cup Q_j^*$ where Q_j^* is some fixed dilation of Q_j. Note that the measure of $\cup Q_j^*$ is still $O\left(\frac{||F||_{L^1}}{\lambda}\right)$. Moreover, it is enough to show that

$$M h_k(t_0, x_0) \leq c M H_k^0(t_0, x_0)$$

where $(t_0,x_0) \in (\cup Q_j^*)^C \times \mathbb{R}^n$ (with $\cup Q_j^* \subset \mathbb{R}$, as we are considering $L^pL^q(\mathbb{R}\times\mathbb{R}^n)$) and where H_k^0 is the kth component of H^0, i.e.

$$H_k^0 = \sum_j \frac{1}{Q_j}\int_{Q_j} f_k(t,x)dt\chi_j(t).$$

But if $(t_0,x_0) \in (\cup Q_j^*)^C \times \mathbb{R}^n$ and $Q\times R$ is a cube containing (t_0,x_0) in $\mathbb{R}\times\mathbb{R}^n$ such that $Q\cap\cup Q_j \neq \emptyset$ then there exists an appropriate dilation $\tilde{Q} \supset Q_j$ for each Q_j such that $Q\cap Q_j \neq \emptyset$. So let

$$J = \{j : Q_j \cap Q \neq \emptyset\}.$$

Now consider

$$\begin{aligned}
\frac{1}{|Q\times R|}\int_{Q\times R}|h_k(t,x)|dtdx &= \sum_{j\in J}\frac{1}{|Q|\,|R|}\int_R\int_{Q\cap Q_j}|f_k(t,x)|dtdx \\
&\leq \sum_{j\in J}\frac{1}{|Q|}\frac{1}{|R|}\int_R\int_{Q_j}|f_k(t,x)|dtdx \\
&= \sum_{j\in J}\frac{1}{|Q|}\frac{1}{|R|}\int_R\int_{Q_j}\frac{1}{|Q_j|}|f_k(y,x)|dx\chi_{Q_j}(y)dydx \\
&\leq \sum_{j\in J}\frac{1}{|Q|\,|R|}\int_R\int_{\tilde{Q}}\frac{1}{|Q_j|}|f_k(y,x)|dx\chi_{Q_j}(y)dydx \\
&\leq \frac{1}{|Q|\,|R|}\int_R\int_{\tilde{Q}}\sum_{j\in J}\frac{1}{|Q_j|}|f_k(y,x)|dx\chi_{Q_j}(y)dydx \\
&\leq C\frac{1}{|\tilde{Q}|\,|\tilde{R}|}\int_{\tilde{R}}\int_{\tilde{Q}}H_k^0(t,x)dtdx \\
&= C\frac{1}{|\tilde{Q}\times\tilde{R}|}\int_{\tilde{R}\times\tilde{Q}}H_k^0(t,x)dtdx \\
&\leq CMH_k^0(t_0,x_0).
\end{aligned}$$

Therefore

$$Mh_k(t_0,x_0) \leq CMH_k^0(t_0,x_0)$$

with C independent of k, and

$$M_{r,q}H(t) \leq cM_{r,q}H^0(t)$$

so we obtain the weak-type estimate. $\square$

Remark 4.0.10. *We note that from the weak-type and Fefferman-Stein inequalities we can also recover, by interpolation, Fernandez result for $p \leq q$, i.e.*

$$\left\| \left(\sum_j |M(f_j)|^r \right)^{1/r} \right\|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \leq C \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)}.$$

Chapter 5

Sampling theorems

Sampling theorems describe when and in what sense features of a function can be recovered from a subset of the function's information. In addition to their theoretical importance, these ideas have applications in data and signal analysis as technology moves from analog (continuous) to digital (discrete) data. The Shannon sampling theorem tells us that under certain circumstances we can recover the entirety of a function from a well-chosen sample. Under less stringent conditions, a sample allows us to retrieve other features of the original function. Here we will look at the Plancherel-Polya inequalities, which tell us that the norm of a suitable sample is equivalent to that of the function itself.

5.1 Shannon sampling theorem

We begin by recalling a version of the Shannon sampling theorem and include a proof for completeness.

Theorem 5.1.1. *Let* $f = f(t,x) \in L^2(\mathbb{R}^{n+1})$ *and* supp $\hat{f} \subseteq [-\pi,\pi] \times \cdots \times [-\pi,\pi] = [-\pi,\pi]^{n+1}$. *Then*

$$\begin{aligned} f(t,x) &= \sum_{(j,k)\in\mathbb{Z}^{n+1}} \left(f(j,k)\frac{\sin\pi(t-j)}{\pi(t-j)} \prod_{i=1}^{n} \frac{\sin\pi(x_i-k_i)}{\pi(x_i-k_i)} \right) \\ &= \sum_{(j,k)\in\mathbb{Z}^{n+1}} \left(f(j,k)\text{sinc }(t-j) \prod_{i=1}^{n} \text{sinc }(x_i-k_i) \right) \end{aligned}$$

in L^2 *and uniformly.*

Proof. Since $\hat{f}$ has compact support and $f \in L^2$, $\hat{f} \in L^1$ and thus we may assume that f is continuous. Let $D = [-\pi,\pi]^{n+1}$. We have, with convergence in L^2,

$$\hat{f}(\xi,\eta) = \left(\sum_{(j,k)\in\mathbb{Z}^{n+1}} c_{j,k} e^{-i(j,k)\cdot(\eta,\xi)} \right) \chi_D(\eta,\xi)$$

where

$$\begin{aligned} c_{j,k} &= \frac{1}{(2\pi)^{n+1}} \int_D \hat{f}(z_1,z_2) e^{i(j,k)\cdot(z_1,z_2)} dz_1 dz_2 \\ &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^2} \hat{f}(z_1,z_2) e^{i(j,k)\cdot(z_1,z_2)} dz_1 dz_2 \\ &= f(j,k) \end{aligned}$$

And so

$$\hat{f}(\xi,\eta) = \sum_{(j,k)\in\mathbb{Z}^{n+1}} f(j,k) e^{-i(j,k)\cdot(\eta,\xi)} \chi_D(\eta,\xi)$$

Thus

$$f(t,x) = \left(\check{\hat{f}}\right)(t,x) \overset{L^2}{=} \sum_{(j,k)\in\mathbb{Z}^{n+1}} f(j,k) \left(e^{-i(j,k)\cdot\bullet} \chi_D(\bullet) \right)^{\vee}$$

$$= \sum_{(j,k)\in\mathbb{Z}^{n+1}} \left(f(j,k)\text{sinc }(t-j) \prod_{i=1}^{n} \text{sinc }(x_i-k_i) \right)$$

Finally,

$$\left| f(t,x) - \sum_{j=-M_{n+1}}^{N_{n+1}} \left(\sum_{i=1}^{n} \sum_{k_i=-M_i}^{N_i} f(j,k)\text{sinc }(t-j) \prod_{i=1}^{n} \text{sinc }(x_i-k_i) \right) \right|$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^{n+1}} \Big| \int_D \hat{f}(z_1,z_2) e^{i(t,x)\cdot(z_1,z_2)} dz_1 dz_2 \\
&\quad - \int_D \sum_{j=-M_{n+1}}^{N_{n+1}} \left(\sum_{i=1}^{n} \sum_{k_i=-M_i}^{N_i} f(j,k) e^{-i(j,k)\cdot(z_1,z_2)} e^{i(z_1,z_2)\cdot(t,x)} dz_1 dz_2 \right) \Big| \\
&\leq \left\| \hat{f} - \sum_{j=-M_{n+1}}^{N_{n+1}} \left(\sum_{i=1}^{n} \sum_{k_i=-M_i}^{N_i} c_{j,k} e^{-(j,k)\cdot \bullet} \right) \right\|_{L^2} \to 0
\end{aligned}$$

as all of the $N_i, M_i, \to \infty$ by the convergence of the Fourier series in $L^2(D)$. □

Remark 5.1.2. *One can also prove convergence in L^∞. Moreover. the result is true at least in $\mathscr{S}'$ if $\hat{f} \subset (-\pi,\pi)^n$ [21].*

Remark 5.1.3. *If* supp $\hat{f} \subset B(0,(1-\varepsilon)\pi)$*, then f can be sampled by a smooth function $\tilde{\chi}_\varepsilon(\eta,\xi) \in C_0^\infty$ with $\widehat{\tilde{\chi}_\varepsilon}(\eta,\xi) \equiv 1$ on $B(0,(1-\varepsilon)\pi)$ with the formula*

$$f(t,x) = \sum_{(j,k)\in\mathbb{Z}^{n+1}} f(j,k)\tilde{\chi}_\varepsilon(t-j,x-k).$$

5.2 Plancherel-Polya theorem for mixed Lebesgue spaces

For Lebesgue spaces, we have the Plancherel-Polya inequalities (see the references in [21], in particular [2]).

Theorem 5.2.1. *Let $f \in \mathscr{S}'(\mathbb{R}^{n+1})$.*

1. *If* supp $\hat{f} \subset \overline{B(0,\pi)}$*, then for $0 < p \leq \infty$,*

$$||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^p(\mathbb{Z}^{n+1})} \leq c_p ||f||_{L^p(\mathbb{R}^{n+1})}.$$

2. *Let* supp $\hat{f} \subset B(0,(1-\varepsilon)\pi)$, $\varepsilon > 0$. *Then for* $0 < p \leq \infty$,

$$||f||_{L^p(\mathbb{R}^{n+1})} \leq c_{p,\varepsilon}||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^p(\mathbb{Z}^{n+1})}$$

In this section, we will prove a version for mixed Lebesgue spaces.

Theorem 5.2.2. *Let* $f \in \mathscr{S}'(\mathbb{R}^{n+1})$.

1. *If* supp $\hat{f} \subset \overline{B(0,\pi)}$, *then for* $0 < p,q < \infty$,

$$||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l_j^p l_k^q(\mathbb{Z}^{n+1})} \leq c_{p,q}||f||_{L_t^p L_x^q(\mathbb{R}^{n+1})}.$$

2. *If* supp $\hat{f} \subset B(0,(1-\varepsilon)\pi)$, $\varepsilon > 0$,*then for* $1 < p,q < \infty$,

$$||f||_{L_t^p L_x^q(\mathbb{R}^{n+1})} \leq c_{p,q,\varepsilon}||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l_j^p l_k^q(\mathbb{Z}^{n+1})}.$$

Proof. For the first inequality, let $Q_{(j,k)} = \{(t,x) : j \leq t < j+1, k_i \leq x_i < k_i+1\}$. Then for $(t,x) \in Q_{(j,k)}$,

$$|f(j,k)| \leq |f(t,x)| + c \sup_{w\in B((t,x),d)} |\nabla f|(w)$$

where d is fixed large enough so that $Q_{(j,k)} \subset B((t,x),d)$ (d depends only on the dimension). Let $w = (w_1,w_2)$. Then we have

$$\begin{aligned}
|f(j,k)| &\leq |f(t,x)| + c \sup_{w\in B(0,d)} |\nabla f|(t-w_1,x-w_2) \\
&\leq |f(t,x)| + c' \sup_{w\in B(0,d)} \frac{|\nabla f|(t-w_1,x-w_2)}{(1+|w|)^{2/r}} \\
&\leq |f(t,x)| + c'(|\nabla f|)_r^*(t,x) \\
&\leq |f(t,x)| + \tilde{c}(M_c(|f|^r))^{1/r}(t,x),
\end{aligned}$$

for all $r > 0$, with the last two lines by Definition 1.3.3 and Lemma 1.3.6. Thus we have

$$|f(j,k)| \leq |f(t,x)| + \tilde{c}M_c(|f|^r))^{1/r}(t,x).$$

Raise to the power q:

$$|f(j,k)|^q \leq c\left(|f(t,x)|^q + [M_c(|f|^r))^{1/r}(t,x)]^q\right).$$

Write $Q_{(j,k)} = Q_j^t \cdot Q_k^x$ and average over Q_k^x in x (note that the cube has volume 1):

$$|f(j,k)|^q \leq c\left(\int_{Q_k^x} |f(t,x)|^q dx + \int_{Q_k^x} [M_c(|f|^r))^{1/r}(t,x)]^q\right)$$

and sum over all $k \in \mathbb{Z}^n$:

$$\sum_{k\in\mathbb{Z}^n} |f(j,k)|^q \leq c\left(\int_{\mathbb{R}^n} |f(t,x)|^q dx + \int_{\mathbb{R}^n} [M_c(|f|^r))^{1/r}(t,x)]^q\right).$$

Now raise to the power $\frac{p}{q}$:

$$\left(\sum_{k\in\mathbb{Z}^n} |f(j,k)|^q\right)^{p/q} \leq c\left[\left(\int_{\mathbb{R}^n} |f(t,x)|^q dx\right)^{p/q} + \left(\int_{\mathbb{R}^n} [M_c(|f|^r))^{1/r}(t,x)]^q\right)^{p/q}\right]$$

then average over Q_j^t in t (note that the interval has length 1):

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}^n} |f(j,k)|^q\right)^{p/q} \quad \leq \quad & c\int_{Q_j^t} \left(\int_{\mathbb{R}^n} |f(t,x)|^q dx\right)^{p/q} \\ & +c\int_{Q_j^t} \left(\int_{\mathbb{R}^n} [M_c(|f|^r))^{1/r}(t,x)]^q\right)^{p/q}\end{aligned}$$

and sum over all $j \in \mathbb{Z}$:

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}^n}|f(j,k)|^q\right)^{p/q} &\leq c\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|f(t,x)|^q dx\right)^{p/q} \\ &\quad +c\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}[M_c(|f|^r))^{1/r}(t,x)]^q\right)^{p/q}\end{aligned}$$

Now raise to the power $\frac{1}{p}$:

$$\begin{aligned}\left(\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}^n}|f(j,k)|^q\right)^{p/q}\right)^{1/p} &\leq c\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|f(t,x)|^q dx\right)^{p/q}\right)^{1/p} \\ &\quad +c\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}[M_c(|f|^r))^{1/r}(t,x)]^q\right)^{p/q}\right)^{1/p}.\end{aligned}$$

That is:

$$\begin{aligned}||\{f(j,k)\}||_{l_j^p l_k^q(\mathbb{Z}^{n+1})} &\leq c||f||_{L_t^p L_x^q(\mathbb{R}^{n+1})} \\ &\quad +c\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}[M_c(|f|^r))^{1/r}(t,x)]^q\right)^{p/q}\right)^{1/p}.\end{aligned}$$

The second term is controlled by $||f||_{L_t^p L_x^q(\mathbb{R}^{n+1})}$ as follows: let $r < \min(p,q)$. Then $\frac{p}{r}, \frac{q}{r} > 1$, so

$$\begin{aligned}\left(\int\left(\int(M_c|f|^r(t,x))^{(1/r)q}\right)^{p/q}\right)^{1/p} &= \left[\left(\int\left(\int(M_c|f|^r(t,x))^{q/r}\right)^{(p/r)/(q/r)}\right)^{r/p}\right]^{1/r} \\ &= ||M_c|f|^r||_{L^{p/r}L^{q/r}}^{1/r} \\ &\leq c||\,|f|^r||_{L^{p/r}L^{q/r}}^{1/r} \\ &= c||f||_{L^pL^q}\end{aligned}$$

so we have $(0 < p,q < \infty)$

$$||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^p_j l^q_k(\mathbb{Z}^{n+1})} \leq c_{p,q}||f||_{L^p_t L^q_x(\mathbb{R}^{n+1})}.$$

Now to prove the second inequality, use Remark 5.1.3 to write

$$f(t,x) = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} f(j,k)\tilde{\chi}_\varepsilon(t-j,x-k)$$

and thus

$$\begin{aligned}
|\langle f(t,x),\varphi(t,x)\rangle| &= \left|\langle\sum_{j,k} f(j,k)\tilde{\chi}_\varepsilon(t-j,x-k),\varphi(t,x)\rangle\right| \\
&\leq \left|\langle\left(\sum_{j,k} f(j,k)\tilde{\chi}_\varepsilon(t-j,x-k)\right)\hat{,}\widehat{\varphi(t,x)}\rangle\right| \\
&= \left|\langle\sum_{j,k}\left(f(j,k)\widehat{\tilde{\chi}_\varepsilon}(\eta,\xi)e^{ik_i\xi_i}\right),\hat{\varphi}(\eta,\xi)\rangle\right| \\
&\leq c||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^p_j l^q_k(\mathbb{Z}^{n+1})}||\{(\varphi*\tilde{\chi}_\varepsilon)(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^{p'}_j l^{q'}_k(\mathbb{Z}^{n+1})} \\
&= c||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^p_j l^q_k(\mathbb{Z}^{n+1})}||\varphi*\tilde{\chi}_\varepsilon||_{L^{p'}_t L^{q'}_x(\mathbb{R}^{n+1})},
\end{aligned}$$

with the last by Hölder for sequences.Note that $\tilde{\chi}_\varepsilon$ is a multiplier in L^pL^q for all $p,q > 1$. And hence, by Theorem 3.2.1,

$$c||\varphi*\tilde{\chi}_\varepsilon||_{L^{p'}_t L^{q'}_x(\mathbb{R}^{n+1})} \leq c_\varepsilon||\varphi||_{L^{p'}_t L^{q'}_x(\mathbb{R}^{n+1})}.$$

It follows that

$$\begin{aligned}
|\langle f(t,x),\varphi(t,x)\rangle| &\leq c_\varepsilon||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^p_j l^q_k(\mathbb{Z}^{n+1})}||\varphi*\tilde{\chi}_\varepsilon||_{L^{p'}_t L^{q'}_x(\mathbb{R}^{n+1})} \\
&\leq c_\varepsilon||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l^p_j l^q_k(\mathbb{Z}^{n+1})}||\varphi||_{L^{p'}_t L^{q'}_x(\mathbb{R}^{n+1})}
\end{aligned}$$

and therefore, for $1 < p,q < \infty$,

$$||f||_{L_t^p L_x^q(\mathbb{R}^{n+1})} \leq c_\varepsilon ||\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}||_{l_j^p l_k^q(\mathbb{Z}^{n+1})}.$$

□

Chapter 6

Wavelet Characterization of mixed Lebesgue spaces

We now have the tools required to characterize mixed Lebesgue spaces in terms of wavelet coefficients. This is well-established for Lebesgue spaces; see [16], [7] and [8]. We proceed following the detailed account in [11] (see also the references there). We will proceed via a characterization in terms of the maximal functions $\phi^{**}_{j,\lambda} f$.

6.1 Preliminary characterizations of mixed Lebesgue spaces

We will need the folowing results to proceed.

Lemma 6.1.1. *If g is a band-limited function on $\mathbb{R}\times\mathbb{R}$ and $g \in L^pL^q(\mathbb{R}\times\mathbb{R})$ for $1 < p,q < \infty$, then $g \in L^\infty(\mathbb{R}^2)$.*

Proof. Since g is band-limited, there is an L such that supp $\hat{g} \supset \{x : |x| < L\}$. Thus there exists a function $\gamma \in \mathscr{S}$ such that $\hat{\gamma} \equiv 1$ on supp $\hat{g}$, and supp $\hat{\gamma} \subset \{x : |x| < L+\varepsilon\}$. Then $\hat{g} = \hat{g}\hat{\gamma}$, and so

$$\begin{aligned} |g(t,x)| &\leq \int_{R^2} |g(t-s,x-y)|\,|\gamma(s,y)|dsdy \\ &\leq ||g||_{L^pL^q}||\gamma||_{L^{p'}L^{q'}} \\ &\leq \infty. \end{aligned}$$

That is, $g \in L^\infty(\mathbb{R}^2)$. $\square$

Lemma 6.1.2. *Suppose that ϕ is a band-limited function, $f \in \mathscr{S}'$ and $1 < p, q < \infty$ are such that $\phi_{2^{-j}} * f \in L_t^p L_x^q(\mathbb{R} \times \mathbb{R})$ for all $j \in \mathbb{Z}$. Then for $\lambda > 0$, there exists a constant C_λ such that*

$$(\phi_{j,\lambda}^{**} f)(t,x) \leq C_\lambda \left(M_c(|\phi_{2^{-j}} * f|^{1/\lambda})(t,x) \right)^\lambda$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}$.

Proof. We adapt the arguments in [11]. Set $g(t,x) = \left(\phi_{2^{-j}} * f\right)(2^{-j}(t,x))$, and note that $g \in L^p L^q(\mathbb{R} \times \mathbb{R})$. By Lemma 6.1.1, $g \in L^\infty(\mathbb{R}^2)$, so $g_\lambda^* < \infty$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$. Furthermore, since ϕ is band-limited, g is as well. If we apply Lemma 1.3.6 we can write

$$g_\lambda^*(t,x) \leq C_\lambda \left(M_c(|g|^{1/\lambda})(t,x) \right)^\lambda. \tag{6.1}$$

However, we have that

$$\begin{aligned}
g_\lambda^*(t,x) &= \sup_{(u,v) \in \mathbb{R} \times \mathbb{R}} \frac{|g(t-u, x-v)|}{(1+|(u,v)|)^\lambda} \\
&= \sup_{(u,v) \in \mathbb{R} \times \mathbb{R}} \frac{|(\phi_{2^{-j}} * f)(2^{-j}t - 2^{-j}u, 2^{-j}x - 2^{-j}v)|}{(1+|(u,v)|)^\lambda} \\
&= \sup_{(u',v') \in \mathbb{R} \times \mathbb{R}} \frac{|(\phi_{2^{-j}} * f)(2^{-j}t - u', 2^{-j}x - v')|}{(1+2^j|(u',v')|)^\lambda} \\
&= (\phi_{j,\lambda}^{**} f)(2^{-j}t, 2^{-j}x)
\end{aligned}$$

and

$$\begin{aligned}
M_c(|g|^{1/\lambda})(t,x) &= \sup_{r>0} \frac{1}{r^2} \int_{B_r(t,x)} |(\phi_{2^{-j}} * f)(2^{-j}u, 2^{-j}v)|^{1/\lambda} dv du \\
&= \sup_{r>0} \frac{2^{2j}}{r^2} \int_{B_{2^{-j}r}(2^{-j}t, 2^{-j}x)} |(\phi_{2^{-j}} * f)(u',v')|^{1/\lambda} dv' du' \\
&= M_c(|\phi_{2^{-j}} * f|^{1/\lambda})(2^{-j}t, 2^{-j}x)
\end{aligned}$$

Rewriting (6.1) with these two inequalities yields the desired result. □

We now can characterize the mixed Lebesgue spaces in terms of the maximal functions $\phi_{j,\lambda}^{**} f$ for certain ϕ.

Definition 6.1.3. *Following [11], we say that a function ϕ defined on $\mathbb{R}\times\mathbb{R}$ belongs to the regularity class $\mathfrak{R}^0$ if there are constants C_0, C_1, and $\varepsilon > 0$ so that ϕ satisfies the following conditions:*

$$\int_{\mathbb{R}\times\mathbb{R}} \phi(t,x)dxdt = 0$$

$$|\phi(t,x)| \leq \frac{C_0}{(1+|(t,x)|)^{2+\varepsilon}} \qquad \textit{for all } (t,x)\in\mathbb{R}\times\mathbb{R}$$

$$|\nabla\phi(t,x)| \leq \frac{C_1}{(1+|(t,x)|)^{2+\varepsilon}} \qquad \textit{for all } (t,x)\in\mathbb{R}\times\mathbb{R}.$$

Theorem 6.1.4. *Let $\phi\in\mathfrak{R}^0$ be a band-limited function. Given $\lambda\geq 1$ and $1<p,q<\infty$ there exists a constant $B = B_{p,q,\lambda}$ such that*

$$\left|\left|\left(\sum_{j\in\mathbb{Z}} |\phi_{j,\lambda}^{**} f|^2\right)^{1/2}\right|\right|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \leq B||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})}.$$

Proof. If $f\in L_t^p L_x^q(\mathbb{R}\times\mathbb{R})$, then since $\phi_{2^{-j}}\in L^1(\mathbb{R}^2)$, $\phi_{2^{-j}} * f \in L_t^p L_x^q(\mathbb{R}\times\mathbb{R})$ for all $j\in\mathbb{Z}$. We can thus apply Lemma 6.1.2, and so have

$$\begin{aligned}\left|\left|\left(\sum_{j\in\mathbb{Z}} |\phi_{j,\lambda}^{**} f|^2\right)^{1/2}\right|\right|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} &\leq C_\lambda \left|\left|\left(\sum_{j\in\mathbb{Z}} [M_c(|\phi_{2^{-j}} * f|^{1/\lambda})]^{2\lambda}\right)^{1/2}\right|\right|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \\ &\leq C_\lambda \left|\left|\left(\sum_{j\in\mathbb{Z}} [M_c(|\phi_{2^{-j}} * f|^{1/\lambda})]^{2\lambda}\right)^{1/2\lambda}\right|\right|^{\lambda}_{L_t^{p\lambda} L_x^{q\lambda}(\mathbb{R}\times\mathbb{R})}.\end{aligned}$$

Then applying Theorem 3.2.3 gives

$$
\begin{aligned}
C_\lambda \left\| \left(\sum_{j\in\mathbb{Z}} [M_c(|\phi_{2^{-j}} * f|^{1/\lambda})]^{2\lambda} \right)^{1/2\lambda} \right\|^{\lambda}_{L_t^{p\lambda} L_x^{q\lambda}(\mathbb{R}\times\mathbb{R})}
&\leq C_{\lambda,p,q} \left\| \left(\sum_{j\in\mathbb{Z}} |\phi_{2^{-j}} * f|^{2\lambda/\lambda} \right)^{1/2\lambda} \right\|^{\lambda}_{L_t^{p\lambda} L_x^{q\lambda}(\mathbb{R}\times\mathbb{R})} \\
&= C_{\lambda,p,q} \left\| \left(\sum_{j\in\mathbb{Z}} |\phi_{2^{-j}} * f|^{2} \right)^{1/2} \right\|_{L_t^{p} L_x^{q}(\mathbb{R}\times\mathbb{R})} \\
&\leq B_{p,q,\lambda} ||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})}
\end{aligned}
$$

with the final inequality by Theorem 2.3.1. □

Remark 6.1.5. *Note that we can get the other side of the characterization if we add the assumption that $\phi \in \mathscr{S}(\mathbb{R}^{n+1})$ is such that* supp $\hat{\phi} \supset \{\frac{\pi}{4} < |\xi| < \pi\}$ *and $\hat{\phi} > c > 0$ on $\{\frac{\pi}{4} + \varepsilon < |\xi| < \pi - \varepsilon\}$, i.e. there exist constants $A = A_{p,q,\lambda}$ and $B = B_{p,q,\lambda}$ such that*

$$
A||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \leq \left\| \left(\sum_{j\in\mathbb{Z}} |\phi^{**}_{j,\lambda} f|^2 \right)^{1/2} \right\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \leq B||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})}.
$$

For the left-hand inequality, note that by the defnition of $\phi^{**}_{j,\lambda} f$, $|(\phi_{2^{-j}} * f)(t,x)| \leq (\phi^{**}_{j,\lambda} f)(t,x)$. Then the result follows from Theorem 2.3.1.

Theorem 6.1.6. *Let $\psi \in \mathfrak{R}^0$ be a band limited function. Then for $1 < p,q < \infty$ and $f \in L_t^p L_x^q(\mathbb{R}\times\mathbb{R})$,*

$$
\left\| \left(\sum_{\nu,j,k\in\mathbb{Z}} |\langle f, \psi_Q \rangle|^2 2^{2\nu} \chi_Q \right)^{1/2} \right\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \leq C||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})}
$$

where $Q = I_{\nu k} \times I_{\nu j} = [2^{-\nu}k, 2^{-\nu}k+1] \times [2^{-\nu}j, 2^{-\nu}j+1]$.

Proof. Note that $\psi \in L_t^{p'} L_x^{q'}(\mathbb{R}\times\mathbb{R})$, so the numbers given by $\langle f, \psi_{j,k}\rangle$ make sense. Moreover, if we write $\tilde{\psi}(t,x) = \overline{\psi(-t,-x)}$, we can write

$$\begin{aligned}
|\langle f, \psi_Q\rangle| &= 2^{\nu}\left|\int_{\mathbb{R}\times\mathbb{R}} f(t,x)\overline{\psi(2^{\nu}t-k, 2^{\nu}x-j)}dxdt\right| \\
&= 2^{\nu}2^{-2\nu}\left|\int_{\mathbb{R}\times\mathbb{R}} f(t,x)\overline{\psi_{2^{-\nu}}(t-2^{-\nu}k, x-2^{-\nu}j)}dxdt\right| \\
&= 2^{-\nu}\left|\int_{\mathbb{R}\times\mathbb{R}} f(t,x)\tilde{\psi}_{2^{-\nu}}(2^{-\nu}k-t, 2^{-\nu}j-x)dxdt\right| \\
&= 2^{-\nu}|(\tilde{\psi}_{2^{-\nu}} * f)(2^{-\nu}k, 2^{-\nu}j)dxdt| \\
&= 2^{-\nu}\sup_{(u,v)\in Q}|(\tilde{\psi}_{2^{-\nu}} * f)(u,v)|.
\end{aligned}$$

Consider a fixed $\nu \in \mathbb{Z}$. Then

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_Q\rangle|^2 2^{2\nu}\chi_Q(t,x) \\
&\qquad \le \sum_{j,k\in\mathbb{Z}}\left(\sup_{(u,v)\in Q}|(\tilde{\psi}_{2^{-\nu}} * f)(u,v)|\right)^2 \chi_Q(t,x) \\
&\qquad \le \left(\sup_{|(z_1,z_2)|<2^{-\nu}}|(\tilde{\psi}_{2^{-\nu}} * f)((t-z_1, x-z_2)|\right)^2 \\
&\qquad = \sup_{|(z_1,z_2)|<2^{-\nu}}\left(\frac{|(\tilde{\psi}_{2^{-\nu}} * f)((t-z_1, x-z_2)|}{(1+2^{\nu}|(z_1,z_2)|)^{\lambda}}\right)^2 (1+2^{\nu}|(z_1,z_2)|)^{2\lambda} \\
&\qquad \le 2^{2\lambda}\left((\psi^{**}_{\nu,\lambda} f)(t,x)\right)^2.
\end{aligned}$$

whenever $\lambda > 0$. By Theorem 6.1.4 with $\lambda \ge 1$, then, we have the desired inequality. □

6.2 Wavelet characterization of mixed Lebesgue spaces

We are now able to prove a characterization of $L_t^p L_x^q(\mathbb{R}^2)$ in terms of wavelet coefficients.

Theorem 6.2.1. *Let $\{\psi_1, \psi_2, \psi_3\}$ be in $\mathfrak{R}^0$ and band-limited wavelets for $L^2(\mathbb{R}^2)$. Given $1 < p,q < \infty$, there exist two constants $0 < A_{p,q} \le B_{p,q} < \infty$ such that*

$$A_{p,q}||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \le ||\mathscr{W}_\psi f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \le B_{p,q}||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})}$$

for all $f \in L_t^p L_x^q(\mathbb{R}\times\mathbb{R})$.

Proof. Observe that the numbers $\langle f, \psi_{iQ}\rangle$ are well defined since $\psi_i \in L_x^{p'} L_y^{q'}(\mathbb{R}\times\mathbb{R})$. By applying Theorem 6.1.6 for each ψ_i we have the right-hand inequality—that is, we have $B_{p,q} < \infty$ such that

$$||\mathscr{W}_\psi f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \le B_{p,q}||f||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})}. \tag{6.2}$$

Note that if $p = q = 2$ we have equality with $B_{p.q} = 1$, because $\{\psi_1, \psi_2, \psi_3\}$ is an orthonormal wavelet:

$$\begin{aligned}
\int_{\mathbb{R}\times\mathbb{R}} (T_\psi f)(t,x)\cdot(T_\psi f)(t,x)dxdt &= ||\mathscr{W}_\psi f||^2_{L_t^2 L_x^2(\mathbb{R}\times\mathbb{R})} \\
&= ||\mathscr{W}_\psi f||^2_{L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \sum_{i=1}^{3} \sum_{\nu,j,k\in\mathbb{Z}} |\langle f, \psi_{iQ}\rangle|^2 2^{2\nu}\chi_Q(t,x)dxdt \\
&= \int_{\mathbb{R}} \sum_{i=1}^{3} \sum_{\nu,j,k\in\mathbb{Z}} |\langle f, \psi_{iQ}\rangle|^2 \\
&= ||f||^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

From this, the polarization identity, and a density argument, we have

$$\int_{\mathbb{R}\times\mathbb{R}} f(t,x)g(t,x)dxdt = \int_{\mathbb{R}\times\mathbb{R}} (T_\psi f)(t,x)\cdot(T_\psi g)(t,x)dxdt$$

for $f \in L_t^p L_x^q(\mathbb{R}\times\mathbb{R})$ and $g \in L_t^{p'} L_x^{q'}(\mathbb{R}\times\mathbb{R})$.

From this, via duality, followed by Hölder's inequality and (6.2) for $L_t^{p'}L_x^{q'}(\mathbb{R}\times\mathbb{R})$, we obtain

$$\begin{aligned}
||f||_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R})} &= \sup_{||g||_{L_t^{p'}L_x^{q'}}\leq 1}\left|\int_{\mathbb{R}\times\mathbb{R}} f(t,x)g(t,x)dxdt\right| \\
&= \sup_{||g||_{L_t^{p'}L_x^{q'}}\leq 1}\left|\int_{\mathbb{R}\times\mathbb{R}} (T_\psi f)(t,x)\cdot(T_\psi g)(t,x)dxdt\right| \\
&\leq \sup_{||g||_{L_t^{p'}L_x^{q'}}\leq 1} ||\mathscr{W}_\psi f||_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R})}||\mathscr{W}_\psi g||_{L_t^{p'}L_x^{q'}(\mathbb{R}\times\mathbb{R})} \\
&\lesssim ||\mathscr{W}_\psi f||_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R})}\sup_{||g||_{L_t^{p'}L_x^{q'}}\leq 1}||g||_{L_t^{p'}L_x^{q'}(\mathbb{R}\times\mathbb{R})} \\
&\leq B_{p',q'}||\mathscr{W}_\psi f||_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R})}.
\end{aligned}$$

This gives us the left-hand inequality. □

Remark 6.2.2. *The characterization constructed here can be repeated for higher dimensions, if the apropriate adjustments are made to* $\mathscr{W}$ *to accommodate the necessary family of* 2^{n-1} *wavelets.*

Chapter 7

Conclusions

Throughout this dissertation we explore mixed Lebesgue spaces, and establish results that parallel those that are available in Lebesgue spaces. These include considering how various operators behave on these spaces, characterizations of the spaces, and tools for working with these spaces, including a Leibniz's rule and sampling theorems.

After an introduction to the relevant terms and theorems in Chapter 1, in Chapter 2 we prove a Littlewood-Paley characterization (Theorem 2.3.1). This proof relies on the boundedness of vector-valued Calderón-Zygmund operators, which is demonstrated in Theorem 2.2.1.

This approach to mixed Lebesgue spaces has numerous potential applications, one of which is the Leibniz's Rule presented in Chapter 3. In Lebesgue spaces, there is an alternate proof of the corresponding fact by way of multilinear operators. This is one example that suggests that considering multilinear operators in L^pL^q could be fruitful.

In Chapter 4 we provide an alternate proof for Fernandez's version of the Fefferman-Stein inequality (Theorem 3.2.3) for a restricted range of exponents in Theorem 4.0.8. We then supply the new weak end point estimate in Theorem 4.0.9. The question posed in Remark 4.0.6 is an avenue for further investigation. We also plan to investigate related weighted estimates.

We consider sampling theorems in Chapter 5. We consider the relevant version of the Shannon sampling theorem (Theorem 5.1.1) and adapt the Plancherel-Polya inequalities to this setting in Theorem 5.2.2. These demonstrate the potential of considering sampling questions in L^pL^q, as there are a number of other sampling results that could be considered in this setting.

In Chapter 6 we characterize the mixed Lebesgue spaces in terms of the wavelet coefficients of band-limited wavelets. When considering one-dimensional wavelets for L^p spaces, the requirement that the wavelets be band-limited can be removed; exploring that possibility in L^pL^q is one possible extension of this work. Another is to consider the φ-transform (as defined in [8]) for mixed Lebesgue spaces. There, the requirement that the basis be orthonormal is relaxed, and thus a single function (rather than multiple wavelets) can be used to characterize $L^2(\mathbb{R})$.

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