Some topics on the fractional Brownian motion and stochastic partial differential equations

By

Jian Song

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David Nualart, Chairperson

Yaozhong Hu, Chairperson

Committee members

Tyrone Duncan

Jin Feng

Paul D Koch

Date defended:

The Dissertation Committee for Jian Song certifies
that this is the approved version of the following dissertation:

Some topics on the fractional Brownian motion and stochastic partial differential equations

Committee:

David Nualart, Chairperson

Yaozhong Hu, Chairperson

Tyrone Duncan

Jin Feng

Paul D Koch

Date approved:

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Abstract

In this dissertation, we investigate some problems in fractional Brownian motion and stochastic partial differential partial differential equations driven by fractional Brownian motion and Hilbert space valued Wiener process.

This dissertation is organized as follows.

In Chapter 1, we introduce some preliminaries on fractional Brownian motion and Malliavin calculus, used in this research. Some main original results are also stated also in this chapter.

In Chapter 2, the notion of fractional martingale as the fractional derivative of order α of a continuous local martingale, where $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, is introduced. Then we show that it has a nonzero finite variation of order $\frac{2}{1+2\alpha}$, under some integrability assumptions on the quadratic variation of the local martingale. As an application, we achieve our objective, an extension of Lévy's characterization theorem to fractional Brownian motion.

Chapter 3 is concerned with the problem of exponential moments of the renormalized self-intersection local time of the d-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. We first apply Clark-Ocone formula to deduce an explicit integral representation for this random variable and then derive the existence of some exponential moments.

In Chapter 4, we establish a version of the Feynman-Kac formula for the multidimensional stochastic heat equation with a multiplicative fractional Brownian sheet. We use the techniques of Malliavin calculus to prove that the process defined by the Feynman-Kac formula is a weak solution of the stochastic heat equation. From the Feynman-Kac formula we establish the smoothness of the density of the solution, and the Hölder regularity of the solution in the space and time variables. We also derive a Feynman-Kac formula for the stochastic heat equation in the Skorohod sense and we obtain Feynman-Kac formula to each Wiener chaos of the solution.

In Chapter 5, A version of the Feynman-Kac formula for the multidimensional stochastic heat equation with spacially correlated noise is established. For a class of stochastic heat equations, we study the Hölder continuity of the solutions, and get an explicit expression for the Malliavin derivatives of the solutions by using the Feynman-Kac formula. Based on the above results and the result from the Malliavin calculus, we show that the law of the solution of the stochastic heat equation has smooth density.

Chapter 1

Introduction

1.1 Introduction of main results

This dissertation is based mainly on four papers which are my joint works with Yaozhong Hu and David Nualart. The first two papers have been published, the third one has been accepted by Annals of Probability, and the last one is in progress. I place these papers in Chapters 2, 3, 4 and 5 with few changes(I collected all the references together and put them at the end).

Chapter 2 is based on the paper *Fractional martingales and characterization of the fractional Brownian motion. Annals of Probability. 37 (2009), no. 6, 2404–2430.*

In the case of Brownian motion the famous Lévy's characterization theorem states that a continuous stochastic process $(B_t, t \geq 0)$ adapted to a right-continuous filtration $(\mathscr{F}_t, t \geq 0)$ is an $\mathscr{F}_t$-Brownian motion if and only if B is a local martingale and $\langle B \rangle_t = t$. A natural problem is the extension of Lévy's characterization theorem to the fractional Brownian motion.

The purpose of this paper is to introduce and study the notion of fractional martingale, and apply it to the above problem. Fix $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. If $M = (M_t, t \geq 0)$ is a continuous local martingale, we denote by $M^{(\alpha)} = (M_t^{(\alpha)}, t \geq 0)$ the stochastic process defined by

$$M_t^{(\alpha)} = \int_0^t (t-s)^{\alpha} dM_s, \tag{1.1}$$

provided this stochastic integral exists for all $t \geq 0$. The process $M^{(\alpha)}$ is called the Riemann-Liouville process of M. Notice that $M^{(\alpha)}$ is no longer a martingale and we prefer to call it as a fractional martingale.

We obtained the following result of β-variation of a fractional martingale.

Theorem 1.1.1. *Set $\beta = 2/(1+2\alpha)$. Consider a continuous local martingale of the form $M_t = \int_0^t \xi_s dW_s$, where $\xi = (\xi_t, t \geq 0)$ is a progressively measurable process such that for all $t \geq 0$*

$$\begin{cases} \int_0^t (E(|\xi_s|^{\beta}))^{\frac{\beta'}{\beta}} ds < \infty \quad \text{for some } \beta' > \beta & \text{if} \quad \alpha < 0, \\ \int_0^t (E(\xi_s^2))^{\frac{\beta}{2}} ds < \infty & \text{if} \quad \alpha > 0. \end{cases} \tag{1.2}$$

Then, the β-variation of $M^{(\alpha)}$ on any interval $[0,t]$ exists in L^1, and $\left\langle M^{(\alpha)} \right\rangle_{\beta,t} = c_\alpha \int_0^t |\xi_s|^{\beta} ds$, where $c_\alpha = c_H \kappa_H^{-\frac{1}{H}}$, $H = \frac{1}{2} + \alpha$, and κ_H is defined in (3.2).

By using the above theorem and analyzing the Höder continuity of some related stochastic processes, we obtained the following Lévy characterization theorem for fBm.

Theorem 1.1.2. *Fix $H \in (0,1)$, $H \neq \frac{1}{2}$. Suppose that $B = (B_t, t \geq 0)$ is a zero mean continuous stochastic process. The following two conditions are equivalent:*

(1) B is a fractional Brownian motion with Hurst parameter H.

(2) The process B satisfies the following conditions:

(i) The trajectories of B are Hölder continuous of order $H-\varepsilon$ for any $H-\varepsilon \in (0,H)$.

(ii) Let

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB_s\,. \tag{1.3}$$

Then M is a local martingale. Furthermore, if $H > \frac{1}{2}$, the quadratic variation of the martingale M is absolutely continuous with respect to the Lebesgue measure almost surely.

(iii) For any $t > 0$, the $\frac{1}{H}$-variation of B in the interval $[0,t]$ exists in L^1, and $\langle B \rangle_{\frac{1}{H},t} = c_H t$, where $c_H = E(|\xi|^{\frac{1}{H}})$ and ξ is a standard normal random variable.

Chapter 3 is based on the paper *Integral representation of renormalized self-intersection local times. J. Funct. Anal. 255 (2008), no. 9, 2507–2532.*

The purpose of this paper is to apply Clark-Ocone's formula to the renormalized self-intersection local time of the d-dimensional fractional Brownian motion B^H. As a consequence, we derive the existence of some exponential moments for this local time.

Let

$$L_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t^H - B_s^H) ds, \tag{1.4}$$

where p_ε denotes the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2\varepsilon}\right).$$

The following three theorems are the main results of this paper.

Theorem 1.1.3. *Suppose that $Hd < 1$. Then, the self-intersection local time $L = \int_0^T \int_0^t \delta(B_t^H - B_s^H) ds dr$ exists as the limit in L^2 of L_ε, as ε tends to zero, and for all integers $n \geq 1$ we have*

$$E(L^n) \leq C^n (n!)^{Hd},$$

for some constant C. As a consequence,

$$E(e^{L^p}) < \infty,$$

for any $p < \frac{1}{Hd}$, *and there exists a constant* $\lambda_0 > 0$ *such that* $E(e^{\lambda L^{\frac{1}{Hd}}}) < \infty$ *for all* $\lambda < \lambda_0$.

Theorem 1.1.4. *Suppose that* $H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$. *Then the renormalized self-intersection local time* $\widetilde{L} = L - E(L)$ *of the d-dimensional fractional Brownian motion* B^H *exists in* L^2 *and it has the following integral representation*

$$\widetilde{L} = -\sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_0^t \frac{A^i_{r,t,s}}{\sigma^2_{r,s,t}} p_{\sigma^2_{r,s,t}}(A^i_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right] ds dt \right) dW^i_r, \qquad (1.5)$$

where

$$A_{r,t,s} = E(B^H_t - B^H_s | \mathscr{F}_r)$$

and

$$\sigma^2_{r,s,t} = \mathrm{Var}(B^{H,i}_t - B^{H,i}_s | \mathscr{F}_r).$$

Theorem 1.1.5. *Assume* $\frac{1}{d} \leq H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$. *For any integer* $p < \frac{1}{2}\left[\left(\frac{1}{2} + H\right)\left(d - \frac{1}{2H}\right)\right]^{-1}$ *we have*

$$E(\exp |\widetilde{L}|^p) < \infty.$$

Chapter 4 is based on the submitted paper *Feynman-Kac formula for heat equation driven by fractional white noise*, to appear in Annals of Probability.

In this paper, we extend the Feynman-Kac formula to the heat equation with fractional noise. consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dfrac{\partial^{d+1}W}{\partial t\partial x_1\cdots\partial x_d} \\ u(0,x) = f(x), \end{cases} \tag{1.6}$$

where $W(t,x)$ is a fractional Brownian sheet with Hurst parameters H_0 in time and $(H_1,\ldots,H_d)$ in space, respectively. For this equation, we can prove the Feynman-Kac formula for the solution

$$u(t,x) = E^B\left[f(B_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy)\right)\right], \tag{1.7}$$

where E^B denotes the expectation with respect to the Brownian motion B_t^x, and δ denotes the Dirac delta function.

By using the above Feynman-Kac formula, we can prove the following consequences.

Theorem 1.1.6. *Suppose that $2H_0+\sum_{i=1}^d H_i > d+1$ and let $u(t,x)$ be the solution of Equation (1.1). Then $u(t,x)$ has a continuous modification such that for any $\rho\in\left(0,\frac{\kappa}{2}\right)$ (where κ has been defined in (3.9)), and any compact rectangle $I\subset\mathbb{R}_+\times\mathbb{R}^d$ there exists a positive random variable K_I such that almost surely, for any $(s,x),(t,y)\in I$ we have*

$$|u(t,y)-u(s,x)|\le K_I(|t-s|^\rho+|y-x|^{2\rho}).$$

Theorem 1.1.7. *Suppose that $2H_0+\sum_{i=1}^d H_i > d+1$. Fix $t>0$ and $x\in\mathbb{R}^d$. Assume that for any positive number p, $E|f(B_t+x)|^{-p}<\infty$. Then, the law of $u(t,x)$ has a smooth density.*

Chapter 5 is based on the paper *Some properties of the solutions to a class of stochastic partial differential equations* which is a work in progress.

Consider the heat equation,

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) = \dfrac{1}{2}\triangle V(t,x) + V\dot{M}(t,x) \\ V(x,0) = h(x) \end{cases} \tag{1.8}$$

where M is a semi-martingale, with quadratic variation
$\langle M(\cdot,x)M(\cdot,y)\rangle_t = f(t,x,y) = \int_0^t g(s,x,y)ds.$
We will use $\bar{f}(t,x)$ to denote $f(t,x,x)$ and use $\bar{g}(t,x)$ to denote $g(t,x,x)$.
We shall prove the following Feynman-Kac formula provided g satisfies some technical conditions,

$$V(t,x) = E^B\big\{h(x+B_t)\exp\big(\int_0^t M(dr, x+B_t-B_r) - \frac{1}{2}\int_0^t \bar{f}(dr, x+B_t-B_r)\big)\big\}$$

where B is a d-dimensional standard Brownian motion independent of M.

Now let's consider a general stochastic heat equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\triangle u + b(u) + \sigma(u)\dfrac{\partial W}{\partial t}(t,x), t \geq 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x) \end{cases}$$

where $W(t,x)$ is a Hilbert space valued Wiener process.

First we establish Hölder continuity of u. Then by using the Feynman-Kac formula the Hölder continuity of u and the Malliavin calculus, we can prove the smoothness of the density of $u(t,x)$ for all $t > 0$ and all $x \in \mathbb{R}^d$ under the nondegeneracy condition $\sigma(u(0,x_0)) > 0$ for some point $x_0 \in \mathbb{R}^d$.

1.2 Preliminaries

In this section, we will summarize some basic results on fractional Brownian motion and Malliavin calculus which will be used throughout this dissertation.

Some special classes of stochastic partial differential equations are involved in the last two chapters, where some preparation will be given. we will not present preliminaries for stochastic partial differential equations here and refer to [50] and [8] for some introduction of stochastic partial differential equations.

1.2.1 Fractional Brownian motion

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with covariance

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2.9}$$

This process is a Brownian motion when $H = \frac{1}{2}$. From the relation $E(|B_t^H - B_s^H|^2) = |t-s|^{2H}$, it follows that B^H has Hölder continuous trajectories of order $H - \varepsilon$, for any $\varepsilon > 0$.

Set $X_n = B_n^H - B_{n-1}^H, n \geq 1$. Then $\{X_n, n \geq 1\}$ is a Gaussian stationary sequence with covariance function

$$E(X_1 X_{n+1}) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right).$$

This implies that the two increments of the fractional Brownian motion are positively correlated if $H > \frac{1}{2}$ and they are negatively correlated if $H < \frac{1}{2}$.

When $H > \dfrac{1}{2}$, the sequence $\{X_n\}$ exhibits long range dependence, that is,

$$\lim_{n\to\infty} \frac{E(X_1 X_{n+1})}{H(2H-1)n^{2H-2}} = 1$$

and hence

$$\sum_{n=1}^{\infty} E(X_1 X_{n+1}) = \infty.$$

When $H < \dfrac{1}{2}$, we have

$$\sum_{n=1}^{\infty} E(X_1 X_{n+1}) < \infty.$$

On the other hand, the self-similarity of the fBm and the ergodic theorem imply that the fBm has $\frac{1}{H}$-variation on any time interval $[0,t]$ which equals to $c_H t$, where $c_H = (E|B_1^H|^{\frac{1}{H}})$ (see [40]). More Precisely,

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^p = \begin{cases} \infty, & p < \frac{1}{H}, \\ c_H t, & p = \frac{1}{H}, \\ 0, & p > \frac{1}{H}, \end{cases} \tag{2.10}$$

where $t_i = \dfrac{it}{n}$.

We can extend the underlying probability space in such a way that $(W_{-t}, t \geq 0)$ is a Brownian motion independent of W. Then, the process B^H defined by

$$B_t^H = \kappa_H \left(\int_0^t (t-s)^\alpha dW_s + \int_{-\infty}^0 ((t-s)^\alpha - (-s)^\alpha) dW_s \right),$$

is a fractional Brownian motion with Hurst parameter H (see Mandelbrot and Van Ness [33]).

The fractional Brownian motion B^H also has the following representation (see [18])

$$B_t^H = \int_0^t Z_H(t,s) dW_s, \tag{2.11}$$

where

$$Z_H(t,s) = \kappa_H \left[(\frac{t}{s})^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \tag{2.12}$$

with $\kappa_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}}$.

We refer to the monograph [19] and the review paper [37] for detailed accounts on the properties of the fBm.

1.2.2 Malliavin calculus

We introduce some preliminaries on the Malliavin calculus and refer to Malliavin [32] and Nualart [38] for a more detailed presentation of this theory.

Let us denote by H the Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle$. The norm of an element $h \in H$ will be denoted by $\| \cdot \|_H$. We say that a stochastic process $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathscr{F}, P)$ is an isonormal Gaussian process (or a Gaussian process on H) if W is a centered Gaussian family of random variables such that $E(W(h)W(g)) = \langle h, g \rangle$ for all $g, h \in H$.

Let $\mathscr{S}$ be the class of smooth and cylindrical random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where $n \geq 1$, $h_1, \ldots, h_n \in H$, and f is an infinitely differentiable function such that together with all its partial derivatives has at most polynomial growth order. The derivative operator of the random variable F is defined as

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n))h_j(t).$$

DF is a random variable with values in the Hilbert space H. The derivative is a closable operator on $L^2(\Omega)$ with values in $L^2(\Omega;H)$. We denote by $\mathbb{D}^{1,2}$ the Hilbert spaced defined as the completion of $\mathscr{S}$ with respect to the scalar product

$$\langle F,G\rangle_{1,2} = E(FG) + E\left(\langle DF,DG\rangle\right).$$

The divergence operator δ is the adjoint of the derivative operator D. The domain of δ, denoted by $Dom\delta$, is the set of H-valued sqaure integrable random variables $u \in L^2(\Omega;H)$ such that

$$|E\langle DF,u\rangle_H \leq c\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u. The operator δ is an unbounded operator from $L^2(\Omega;H)$ into $L^2(\Omega)$, and is determined by the duality relationship

$$E(\delta(u)F) = E(\langle u,DF\rangle_H),$$

for any $u \in Dom\delta$, and F in $\mathbb{D}^{1,2}$.

This concludes the concise introduction of Malliavin calculus for a general Hilbert space. In each section, we will provide more specific details necessary for our study .

1.2.3 Stochastic calculus with respect to fBm

In this section, we introduce some main results of stochastic calculus with respect to fBm. We refer to [38] for details.

Let $B = \{B_t, 0 \le t \le T\}$ be a fBm with Hurst parameter $H \in (0,1)$. We denote by $\mathscr{E}$ the set of step functions on $[0,T]$. Let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathscr{H}} = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

The process $\{B(\varphi), \varphi \in \mathscr{H}\}$ is an isonormal Gaussian process associated with the Hilbert space $\mathscr{H}$ (see the beginning of last section.) We will denote by D and δ the Malliavin derivative and divergence operators associated with this process.

Let $|\mathscr{H}|$ be the linear space of measurable function ψ on $[0,T]$ such that

$$\|\psi\|^2_{|\mathscr{H}|} = \alpha_H \int_0^T \int_0^T |\psi_r||\psi_u||r-u|^{2H-2} dr du < \infty$$

where $\alpha_H = H(2H-1)$.

Case $H > \frac{1}{2}$

Consider a measurable process $u = \{u_t, t \in [0,T]\}$ such that $\int_0^T |u_t| dt < \infty$ a.s. Let us define the approximation sequences of processes

$$(\pi^n u)_t = \sum_{i=0}^{n-1} \triangle_n^{-1} \left(\int_{t_i}^{t_{i+1}} u_s ds \right) I_{(t_i, t_{i+1}]}(t),$$

where $t_i = i\triangle_n, i = 0, \ldots, n$, and $\triangle_n = \dfrac{T}{n}$. Set

$$S^n = \sum_{i=0}^{n-1} \triangle_n^{-1} \left(\int_{t_i}^{t_{i+1}} u_s ds \right) (B_{t_{i+1}} - B_{t_i}).$$

Definition 1.2.1. *We say that a measurable process $u = \{u_t, t \in [0,T]\}$ such that $\int_0^T |u_t| dt < \infty$ a.s. is Stratonovich integrable with respect to the fBm if the sequence S^n converges in probability as $|\pi| \to 0$, and in this case the limit will be denoted by $\int_0^T u_t \circ dB_t$.*

Remark Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process which is continuous in the norm of $\mathbb{D}^{1,2}$ and satisfies the condition $\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2} dsdt < \infty$. Then the Riemann sums

$$\sum_{i=0}^{n-1} u_{s_i}(B_{t_{i+1}} - B_{t_i}),$$

where $t_i \leq s_i \leq t_{i+1}$, converge in probability to $\int_0^T u_s \circ dB_s$. In particular, the forward and backward integrals of u with respect to the fBm exists and they coincide with the stratonovich integral.

The following proposition establishes the relationship between the Stratonovich integral and the divergence integral.

Proposition 1.2.2. *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in the space $\mathbb{D}^{1,2}(|\mathscr{H}|)$. Suppose also that a.s.*

$$\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2} dsdt < \infty.$$

Then u is Statonovich integrable and we have

$$\int_0^T u_t \circ dB_t = \delta(u) + \alpha_H \int_0^T \int_0^T |D_s u_t||t-s|^{2H-2} dsdt.$$

We also have the following Itô formula if $\max\{|F(x)|,|F'(x)|,|F''(x)|\} \leq ce^{\lambda^2}$ where c and λ are positive constants such that $\lambda < \frac{1}{4T^{2H}}$,

$$
\begin{aligned}
F(B_t) =& F(0) + \int_0^t F'(B_s) \circ dB_s \\
=& F(0) + \int_0^t F'(B_s) dB_s + H \int_0^t F''(B_s) s^{2H-1} ds.
\end{aligned}
$$

Case $H < \frac{1}{2}$

In this case, it's more sophisticated to develop the same calculus than the case $H > \frac{1}{2}$. One fact that indicates the difficulty is the forward integral $\int_0^T B_t dB_t$ defined as the limit in L^2 of the Riemann sums

$$
\sum_{i=0}^{n-1} B_{t_i}(B_{t_{t+1}} - B_{t_i}),
$$

where $t_i = \frac{iT}{n}$, does not exist. We shall use the symmetric integral.

Definition 1.2.3. *The symmetric integral of a process u with integral paths with respect to the fBm is defined as the limit in probability of*

$$
(2\varepsilon)^{-1} \int_0^T u_s (B_{s+\varepsilon} - B_{s-\varepsilon}) ds,
$$

as $\varepsilon \downarrow 0$ *if it exists. We denote this limit by* $\int_0^T u_s \circ dB_s$.

Consider the following seminorm on the set $\mathscr{E}$ of step functions on $[0,T]$:

$$
\begin{aligned}
\|\varphi\|_K^2 =& \int_0^T \varphi^2(s)(T-s)^{2H-1} ds \\
&+ \int_0^T \left(\int_s^T |\varphi(t) - \varphi(s)|(t-s)^{H-\frac{3}{2}} \right)^2 ds.
\end{aligned}
$$

We denote by $\mathscr{H}_K$ the completion of $\mathscr{E}$ with respect to this seminorm.

We have the following proposition.

Proposition 1.2.4. *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in the space $\mathbb{D}^{1,2}(\mathscr{H}_K)$. Suppose that the trace defined as the limit in probability*

$$TrDu := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T \langle Du_s, I_{[s-\varepsilon, s+\varepsilon] \cap [0,T]} \rangle_{\mathscr{H}} ds$$

exists. Then the symmetric stochastic integral of u with respect to fBm exists and

$$\int_0^T u_t \circ dB_t = \delta(u) + TrDu.$$

Chapter 2

Lévy Characterization Theorem for fBm

2.1 Introduction

In the case of Brownian motion the famous Lévy's characterization theorem states that a continuous stochastic process $(B_t, t \geq 0)$ adapted to a right-continuous filtration $(\mathscr{F}_t, t \geq 0)$ is an $\mathscr{F}_t$-Brownian motion if and only if B is a local martingale and $\langle B \rangle_t = t$. A natural problem is the extension of Lévy's characterization theorem to the fractional Brownian motion.

The purpose of this chapter is to introduce and study the notion of fractional martingale, and apply it to the above problem. Fix $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. If $M = (M_t, t \geq 0)$ is a continuous local martingale, we denote by $M^{(\alpha)} = (M_t^{(\alpha)}, t \geq 0)$ the stochastic process defined by

$$M_t^{(\alpha)} = \int_0^t (t-s)^\alpha dM_s, \tag{1.1}$$

provided this stochastic integral exists for all $t \geq 0$. The process $M^{(\alpha)}$ is called the Riemann-Liouville process of M. Notice that $M^{(\alpha)}$ is no longer a martingale and we will say that it is a fractional martingale.

If $\alpha \in (0, \frac{1}{2})$, then the stochastic integral in (4.11) always exists, and $M_t^{(\alpha)} = \Gamma(1+\alpha) I_{0+}^{\alpha}(M)_t$, where I_{0+}^{α} is the left-sided fractional integral of order α. If $\alpha \in (-\frac{1}{2}, 0)$ and M has α'-Hölder continuous trajectories on any finite interval for some $\alpha' > -\alpha$, then $M_t^{(\alpha)}$ exists and $M_t^{(\alpha)} = \Gamma(1+\alpha) D_{0+}^{-\alpha}(M)_t$, where $D_{0+}^{-\alpha}$ is the left-sided fractional derivative of order $-\alpha$. We refer to Samko et al. [43] for the definition and properties of the fractional operators.

We are interested in the variation properties of fractional martingales. The process $M^{(\alpha)}$ has Hölder continuous trajectories of order γ on any finite interval, for any $\gamma < \frac{1}{2} + \alpha$, provided M has Hölder continuous trajectories of order $\frac{1}{2} - \varepsilon$ on any finite interval, for any $\varepsilon > 0$. Then, it is natural to expect that $M^{(\alpha)}$ has a finite and non zero variation of order $\beta = \left(\frac{1}{2} + \alpha\right)^{-1} = \frac{2}{1+2\alpha}$. We show that (see Theorem 2.2.6) if $d\langle M \rangle_t = \xi_t^2 dt$, then $M^{(\alpha)}$ has a finite β-variation $c_\alpha \int_0^t |\xi_s|^\beta ds$ under some integrability conditions on ξ, where c_α is a constant depending only on α. The proof of this result is based on the variation properties of the fractional Brownian motion.

The fractional Brownian motion B^H is not a martingale unless $H = \frac{1}{2}$. But the process

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB_s^H \tag{1.2}$$

is a martingale with respect to the filtration generated by the fBm, verifying $\langle M \rangle_t = d_H t^{2H}$ for some constant d_H (see Norros et al. [36]). We show that if $B = (B_t, t \geq 0)$ is a continuous square integrable centered process with $B_0 = 0$, then B is a fractional Brownian motion with Hurst parameter H if and only if the process B has the following properties:

(i) The sample paths of the process B are Hölder continuous of order γ for any $\gamma \in (0, H)$.

(ii) The process M defined in (1.2), where B^H is replaced by B, is a martingale with respect to the filtration generated by B. If $H > \frac{1}{2}$ we also assume that the quadratic variation of M is absolutely continuous with respect to the Lebesgue measure.

(iii) For any $t > 0$, the process B has $\frac{1}{H}$-variation (in the sense of Definition 2.3) which equals to $c_H t$ on the interval $[0,t]$.

In order to prove that the conditions (i), (ii), and (iii) imply that B is a fractional Brownian motion, it suffices to show that the martingale M satisfies $\langle M \rangle_t = d_H t^{2H}$ for some constant d_H, and this will be a consequence of the condition (iii) and the general result on the β-variation of a fractional martingale.

In a recent work [34], Mishura and Valkeila have proved another extension of Lévy characterization theorem, where condition (iii) is replaced by an assumption on the renormalized quadratic variation, and no restriction on the quadratic variation of M is required.

Theorem 2.1.1 (Mishura and Valkeila). *Assume that B is a continuous square integrable centered process with $B_0 = 0$. Then the following are equivalent:*

(a) The process B is a fractional Brownian motion with Hurst parameter $H \in (0,1)$.

(b) The process B satisfies the following properties:

(i) The process B has Hölder continuous sample paths of order γ for any $\gamma \in (0,H)$ in any finite interval.

(ii) The process M defined in (1.2), where B^H is replaced by B, is a martingale with respect to the filtration generated by B.

(iii) For any $t > 0$

$$\lim_{n\to\infty} n^{2H-1} \sum_{k=1}^{n} (B_{tk/n} - B_{t(k-1)/n})^2 = t^{2H},$$

in L^1.

The proof of this theorem uses different kind of techniques, and is based on the stochastic calculus with respect to the fractional Brownian motion.

The chapter is organized as follows. Section 2 is devoted to study the β-variation of fractional martingales, and Section 3 contains the proof of the Lévy characterization theorem for the fBm. Some technical lemmas are included in the appendix.

2.2 β-variation of fractional martingales

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space equipped with a right-continuous filtration $(\mathscr{F}_t, t \geq 0)$ such that $\mathscr{F}_0$ contains the P-null sets. Fix a parameter $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. We introduce the following notion.

Definition 2.2.1. *A continuous $\mathscr{F}_t$-adapted process $(M_t^{(\alpha)}, t \geq 0)$ is called a fractional martingale of order α if there is a continuous local martingale $(M_t, t \geq 0)$ such that for all $t \geq 0$*

$$\int_0^t (t-s)^{2\alpha} d\langle M\rangle_s < \infty, \tag{2.1}$$

almost surely, and

$$M_t^{(\alpha)} = \int_0^t (t-s)^{\alpha} dM_s. \tag{2.2}$$

Notice that by Fubini's theorem condition (4.10) holds true for almost all $t \geq 0$.

If $\alpha \in (0, \frac{1}{2})$, then (4.10) is always fulfilled. Moreover, an integration by parts implies that the integral appearing in (8.12) exists as a Riemann-Stieltjes integral and $M_t^{(\alpha)} = \Gamma(\alpha+1) I_{0+}^{\alpha}(M)_t$, where I_{0+}^{α} is the left-sided fractional integral of order α.

For any $\alpha \in (-\frac{1}{2}, 0)$ we introduce the following hypothesis:

(H) The trajectories of M are α'-Hölder continuous on finite intervals for some $\alpha' > -\alpha$.

Then we have the following result.

Lemma 2.2.2. *Fix $\alpha \in (-\frac{1}{2}, 0)$, and let M be a continuous local martingale satisfying condition (H). Then (4.10) holds, $M_t^{(\alpha)}$ exists as a Riemann-Stielejes integral and it coincides with $\Gamma(\alpha+1)D_{0+}^{-\alpha}(M)_t$, where $D_{0+}^{-\alpha}$ is the left-sided fractional derivative of order $-\alpha$.*

Proof Set

$$Z_t = |M_t| + \langle M \rangle_t + \sup_{0 \le s < u \le t} \frac{|M_s - M_u|}{|s-u|^{\alpha'}}.$$

For any integer $n \geq 1$ we define

$$T_N = \inf\{t \geq 0 : Z_t > N\}.$$

Then, T_N is an nondecreasing sequence of stopping times such that $T_N \uparrow \infty$. For any $s < t$ we can write

$$E\left(|\langle M \rangle_{t\wedge T_N} - \langle M \rangle_{s\wedge T_N}|^p\right) \le C_p E\left(|M_{t\wedge T_N} - M_{s\wedge T_N}|^{2p}\right) \le C_p N^{2p} |t-s|^{2p\alpha'}.$$

By Kolmogorov's continuity criterion the sample paths of $\langle M \rangle$ are Hölder continuous of order γ for any $\gamma < 2\alpha'$, on any finite inerval. This implies (4.10), and it is easy to check that the stochastic integral is a Riemann-Stieltjes integral and coincides with $\Gamma(\alpha+1)D_{0+}^{-\alpha}(M)_t$.

From fractional calculus, assuming condition **(H)** if $\alpha < 0$, we have $M_t = \frac{1}{\Gamma(\alpha+1)} I_{0+}^{-\alpha}(M^{(\alpha)})_t$, where $I^{-\alpha} = D^{\alpha}$ if $\alpha > 0$. Using the definition of left-sided fractional integral and derivative we have

$$M_t = \begin{cases} \frac{1}{\Gamma(1+\alpha)\Gamma(-\alpha)} \int_0^t (t-s)^{-1-\alpha} M_s^{(\alpha)} ds & if \quad \alpha < 0 \\ \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} dM_s^{(\alpha)} & if \quad \alpha > 0 \,. \end{cases} \tag{2.3}$$

In order to define the β-variation, let us first introduce some notation. Fix a time interval $[a,b]$, and consider the uniform partition

$$\pi^n = \{a = t_0^n < t_1^n < \cdots < t_n^n = b\},$$

where $t_i^n = a + \dfrac{i}{n}(b-a)$ for $i = 0, \dots, n$. Let $\beta \geq 1$ and let $X = (X_t, t \geq 0)$ be a continuous stochastic process.

Definition 2.2.3. *We define the β-variation of X on the interval $[a,b]$, denoted by $\langle X \rangle_{\beta,[a,b]}$, as the limit in probability of*

$$S_{\beta,n}^{[a,b]}(X) := \sum_{i=1}^{n} |\Delta_i^n X|^\beta \,, \tag{2.4}$$

if the limit exists, where $\Delta_i^n X = X_{t_i^n} - X_{t_{i-1}^n}$. We say that the β-variation of X on $[a,b]$ exist in L^1 if the above limit exists in L^1.

We also denote $\langle X \rangle_{\beta,[0,t]}$ by $\langle X \rangle_{\beta,t}$. For instance, a continuous local martingale has a finite 2-variation, denoted by $\langle M \rangle_t$, and the fractional Brownian motion B_t^H of Hurst parameter $H \in (0,1)$ has $\frac{1}{H}$-variation which is equal to $c_H t$, where $c_H = \left(E|B_1^H|\right)^{\frac{1}{H}}$.

A direct consequence of the above definition is that if $\langle X \rangle_{\beta,[a,c]}$ exists, then for any $a < b < c$, both $\langle X \rangle_{\beta,[a,b]}$ and $\langle X \rangle_{\beta,[b,c]}$ exist and

$$\langle X \rangle_{\beta,[a,c]} = \langle X \rangle_{\beta,[a,b]} + \langle X \rangle_{\beta,[b,c]} \,. \tag{2.5}$$

It is also easy to see that the following triangular inequality holds:

$$S_{\beta,n}^{[a,b]}(X+Y)^{\frac{1}{\beta}} \leq S_{\beta,n}^{[a,b]}(X)^{\frac{1}{\beta}} + S_{\beta,n}^{[a,b]}(Y)^{\frac{1}{\beta}}. \tag{2.6}$$

This inequality implies that if X and Y are two continuous stochastic processes such that $\langle X \rangle_{\beta,[a,b]}$ exists and $\langle Y \rangle_{\beta,[a,b]} = 0$, then

$$\langle X+Y \rangle_{\beta,[a,b]} = \langle X \rangle_{\beta,[a,b]}. \tag{2.7}$$

Let $W = (W_t, t \geq 0)$ be an $\mathscr{F}_t$-Brownian motion. We want to compute the β-variation of $M^{(\alpha)}$, where M is a martingale of the form $M_t = \int_0^t \xi_s dW_s$. We will denote by C a generic constant that may depend on α. Consider first the case where the martingale is just a standard Wiener process. We recal that

$$\beta = \frac{2}{1+2\alpha}.$$

Lemma 2.2.4. *Let* $(W_t, t \geq 0)$ *be a Wiener process, and set* $X_t = W_t^{(\alpha)} = \int_0^t (t-s)^\alpha dW_s$*. Then the* β*-variation of* X *exists in* L^1 *and* $\langle X \rangle_{\beta,[a,b]} = c_\alpha(b-a)$*, where* $c_\alpha = c_H \kappa_H^{-\frac{1}{H}}$*,* $H = \frac{1}{2}+\alpha$*,* $c_H = (E|B_1^H|^{\frac{1}{H}})$*, and*

$$\kappa_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}}. \tag{2.8}$$

Proof We can extend the underlying probability space in such a way that $(W_{-t}, t \geq 0)$ is a Brownian motion independent of W. Then, the process B^H defined by

$$B_t^H = \kappa_H \left(\int_0^t (t-s)^\alpha dW_s + \int_{-\infty}^0 ((t-s)^\alpha - (-s)^\alpha) dW_s \right),$$

is a fractional Brownian motion with Hurst parameter H (see Mandelbrot and Van Ness [33]). Hence,

$$X_t = \kappa_H^{-1} B_t^H - Z_t,$$

where $Z_t = \int_{-\infty}^0 ((t-s)^\alpha - (-s)^\alpha)\, dW_s$. From the $\frac{1}{H}$-variation property of fractional Brownian motion we know that $\langle B^H \rangle_{\beta,t} = c_H t$, in L^1, because $\beta = \frac{1}{H}$. Then, by (3.3) it suffices to show that $\lim_{n\to\infty} E\left(|S_{\beta,n}^{[0,t]}(Z)|\right) = 0$ for all $t \geq 0$. We have

$$\begin{aligned}
& \sum_{i=1}^n E\left(|Z_{t_i^n} - Z_{t_{i-1}^n}|^\beta\right) \\
= \;& C \sum_{i=1}^n \left(\int_{-\infty}^0 \left((t_i^n - s)^\alpha - (t_{i-1}^n - s)^\alpha\right)^2 ds\right)^{\frac{\beta}{2}} \\
= \;& C \sum_{i=1}^n \left(\int_0^\infty \left((t_{i-1}^n + \frac{t}{n} + s)^\alpha - (t_{i-1}^n + s)^\alpha\right)^2 ds\right)^{\frac{\beta}{2}} \\
\leq \;& C \left(\int_0^\infty \left((\frac{t}{n} + s)^\alpha - s^\alpha\right)^2 ds\right)^{\frac{\beta}{2}} + \frac{C}{n^\beta} \sum_{i=2}^n \left(\int_0^\infty (t_{i-1}^n + s)^{2\alpha-2} ds\right)^{\frac{\beta}{2}} \\
= \;& I_1 + I_2\,.
\end{aligned}$$

It is easy to see by the dominated convergence theorem that $I_1 \to 0$ as $n \to \infty$. On the other hand,

$$I_2 \leq Ctn^{-1} \sum_{i=2}^n (i-1)^{\frac{(2\alpha-1)\beta}{2}} \leq Ctn^{\frac{2\alpha-1}{2\alpha+1}} \to 0$$

since $\alpha < 1/2$. This proves the lemma.

We will make use of the following lemma.

Lemma 2.2.5. *Fix $a > 0$. Let $X_t = \int_0^a (t-s)^\alpha dW_s$, where $W = (W_t, t \geq 0)$ is a Wiener process. Then, for all $t \geq 0$*

$$\lim_{n\to\infty} E\left(|S_{\beta,n}^{[0,t]}(X)|\right) = 0. \tag{2.9}$$

Proof Take $\beta = 2/(1+2\alpha)$. First we have

$$\begin{aligned}
&\sum_{i=1}^{n} E \left| \int_0^a \left[(t_i^n - s)^\alpha - (t_{i-1}^n - s)^\alpha \right] dW_s \right|^\beta \\
\leq\ & C \sum_{i=1}^{n} \left\{ \int_0^a \left[(t_i^n - s)^\alpha - (t_{i-1}^n - s)^\alpha \right]^2 ds \right\}^{\beta/2} .
\end{aligned}$$

Then we apply a similar argument as in the proof of Lemma 2.2.4.

The following theorem is the main result of this section.

Theorem 2.2.6. *Set $\beta = 2/(1+2\alpha)$. Consider a continuous local martingale $M = (M_t, t \geq 0)$ with $M_0 = 0$ and $\langle M \rangle_t = \int_0^t \xi_s^2 ds$, where $\xi = (\xi_t, t \geq 0)$ is a progressively measurable process such that for all $t \geq 0$*

$$\begin{cases} \int_0^t (E(|\xi_s|^\beta))^{\frac{\beta'}{\beta}} ds < \infty \quad \text{for some } \beta' > \beta & \text{if} \quad \alpha < 0, \\ \int_0^t (E(\xi_s^2))^{\frac{\beta}{2}} ds < \infty & \text{if} \quad \alpha > 0. \end{cases} \tag{2.10}$$

Then, the β-variation of $M^{(\alpha)}$ on any interval $[0,t]$ exists in L^1, and $\left\langle M^{(\alpha)} \right\rangle_{\beta,t} = c_\alpha \int_0^t |\xi_s|^\beta ds$, where $c_\alpha = c_H \kappa_H^{-\frac{1}{H}}$, $H = \frac{1}{2} + \alpha$, and κ_H is defined in (3.2).

Proof We can represent the martingale M as a stochastic integral $M_t = \int_0^t \xi_s dW_s$, where $W = (W_t, t \geq 0)$ is a Brownian motion defined on an extension $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{P})$ of our original probability space $(\Omega, \mathscr{F}, P)$. The space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{P})$ is the product of $(\Omega, \mathscr{F}, P)$, and another space $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{P})$ supporting a Brownian motion independent of M. Clearly, if the conclusion of the theorem holds in the extended space, it also holds in the original space.

Notice that if $\alpha < 0$, by Hölder's inequality condition (2.10) implies that

$$\int_0^t (t-s)^{-2\alpha} E(\xi_s^2) ds < \infty,$$

and (4.10) holds.

Suppose first that the process ξ has the form $\xi_t = YI_{(t_1,t_2]}(t)$, where $0 \le t_1 < t_2$ and Y is a bounded $\mathscr{F}_{t_1}$-measurable random variable. In this case the process $M^{(\alpha)}$, denoted by X, is given by

$$X_t = YI_{[t_1,\infty)}(t)\int_{t_1}^{t\wedge t_2}(t-s)^{\alpha}dW_s.$$

For $t \in [0,t_1]$, we clearly have $\langle X\rangle_{\beta,t} = 0$. For $t \in [t_1,t_2]$,

$$X_t = Y\int_0^t (t-s)^{\alpha}dW_s - Y\int_0^{t_1}(t-s)^{\alpha}dW_s,$$

and by lemmas 2.2.4 and 2.2.5, for any interval $[a,b] \subset [t_1,t_2]$, the β-variation of X exists in L^1, and

$$\langle X\rangle_{\beta,[a,b]} = c_{\alpha}|Y|^{\beta}(b-a).$$

Finally, by Lemma 2.2.5, for any interval $[a,b] \subset [t_2,\infty)$, $\langle X\rangle_{\beta,[a,b]} = 0$, in L^1. Hence, we have proved that

$$\langle X\rangle_{\beta,t} = c_{\alpha}|Y|^{\beta}(t\wedge t_2 - t_1)_+ = c_{\alpha}\int_0^t |\xi_s|^{\beta}ds.$$

Let us denote by $\mathscr{S}$ the space of step functions of the form

$$\xi_t = \sum_{i=1}^{n} Y_i I_{(t_{i-1},t_i]}(t),$$

where Y_i is $\mathscr{F}_{t_{i-1}}$ measurable and bounded, and $0 = t_0 < \cdots < t_n$. For $\xi \in \mathscr{S}$ we have $X_t = \sum_{i=1}^{n} X_t^i$, where $X_t^i = \int_0^t \xi_t^i (t-s)^{\alpha}dW_s$ and $\xi_t^i = Y_i I_{(t_{i-1},t_i]}(t)$. From (2.5) we have

$$\langle X\rangle_{\beta,t} = \sum_{i=1}^{n}\langle X\rangle_{\beta,[t_{i-1},t_i]\cap[0,t]}.$$

From the first part of the proof we see that

$$\langle X^j\rangle_{\beta,[t_{i-1},t_i]\cap[0,t]} = \begin{cases} c_\alpha\,|Y_i|^\beta\,(t_i\wedge t - t_{i-1})_+ & \text{if} \quad j=i \\ 0 & \text{if} \quad j\neq i \end{cases},$$

and applying the triangular inequality (2.6) we see then that

$$\langle X\rangle_{\beta,[t_{i-1},t_i]\cap[0,t]} = \langle X^i\rangle_{\beta,[t_{i-1},t_i]\cap[0,t]}\,.$$

Hence,

$$\langle X\rangle_{\beta,[0,t]} = c_\alpha\sum_{i=1}^n |Y_i|^\beta\,(t_i\wedge t - t_{i-1})_+ = c_\alpha\int_0^t |\xi_s|^\beta ds, \tag{2.11}$$

and this proves the result for step functions.

To complete the proof we use a density argument. Fix a time interval $[0,T]$. We can find a sequence of step functions $\left(\xi^k, k\geq 1\right)$ in $\mathscr{S}$ such that if $\alpha>0$, then

$$\lim_{k\to\infty}\int_0^T \left(E|\xi_s-\xi_s^k|^2\right)^{\frac{\beta}{2}} ds = 0,$$

and if $\alpha<0$, then

$$\lim_{k\to\infty}\int_0^T (E|\xi_s-\xi_s^k|^\beta)^{\frac{\beta'}{\beta}} ds = 0.$$

Define $X_t^k = \int_0^t (t-s)^\alpha \xi_s^k dB_s$ for $t \in [0,T]$. From the triangular inequality (2.6) and the Burkholder-Davis-Gundy inequality (see, for instance, [29]) we have, for all $t \in [0,T]$

$$
\begin{aligned}
&E\left(|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - S_{\beta,n}^{[0,t]}(X^k)^{\frac{1}{\beta}}|\right) \le E\left(\left(S_{\beta,n}^{[0,t]}(X-X^k)\right)^{\frac{1}{\beta}}\right) \\
&\le C\left(E\left(\sum_{i=1}^n \left|\int_0^{t_i^n} \left((t_i^n-s)^\alpha - (t_{i-1}^n-s)_+^\alpha\right)(\xi_s - \xi_s^k)dW_s\right|^\beta\right)\right)^{\frac{1}{\beta}} \\
&\le C\left(E\left(\sum_{i=1}^n \left|\int_0^{t_i^n} \left((t_i^n-s)^\alpha - (t_{i-1}^n-s)_+^\alpha\right)^2 (\xi_s - \xi_s^k)^2 ds\right|^{\frac{\beta}{2}}\right)\right)^{\frac{1}{\beta}}. \qquad (2.12)
\end{aligned}
$$

Now we will consider two cases depending on the sign of α.

(i) If $\alpha > 0$, namely, $\beta < 2$, then by the concavity of $x^{\beta/2}$ and Lemma 2.4.1, we have

$$
\begin{aligned}
&E\left(|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - S_{\beta,n}^{[0,t]}(X^k)^{\frac{1}{\beta}}|\right) \\
\le\ & C\left(\sum_{i=1}^n \left|\int_0^{t_i^n} \left((t_i^n-s)^\alpha - (t_{i-1}^n-s)_+^\alpha\right)^2 E\left(|\xi_s - \xi_s^k|^2\right) ds\right|^{\frac{\beta}{2}}\right)^{\frac{1}{\beta}} \\
\le\ & C\left(\int_0^t (E\left(|\xi_s - \xi_s^k|^2\right))^{\frac{\beta}{2}} ds\right)^{\frac{1}{\beta}}. \qquad (2.13)
\end{aligned}
$$

Then

$$
\begin{aligned}
&E\left(\left|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t |\xi_s|^\beta ds\right)^{\frac{1}{\beta}}\right|\right) \\
\le\ & E\left(\left|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - S_{\beta,n}^{[0,t]}(X^k)^{\frac{1}{\beta}}\right|\right) + E\left(\left|S_{\beta,n}^{[0,t]}(X^k)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t |\xi_s^k|^\beta ds\right)^{\frac{1}{\beta}}\right|\right) \\
& + c_\alpha^{\frac{1}{\beta}} E\left(\left|\left(\int_0^t |\xi_s^k|^\beta ds\right)^{\frac{1}{\beta}} - \left(\int_0^t |\xi_s|^\beta ds\right)^{\frac{1}{\beta}}\right|\right).
\end{aligned}
$$

From (4.3) and (4.4) we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty} E\left(\left|S^{[0,t]}_{\beta,n}(X)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t |\xi_s|^\beta ds\right)^{\frac{1}{\beta}}\right|\right) \\
\leq\ & C\left(\int_0^t (E|\xi_s - \xi_s^k|^2)^{\frac{\beta}{2}} ds\right)^{\frac{1}{\beta}} + c_\alpha^{\frac{1}{\beta}} E\left(\left|\left(\int_0^t |\xi_s^k|^\beta ds\right)^{\frac{1}{\beta}} - \left(\int_0^t |\xi_s|^\beta ds\right)^{\frac{1}{\beta}}\right|\right),
\end{aligned}
$$

and letting k tend to zero we prove the desired result.

(ii) If $\alpha < 0$, namely, $\beta > 2$, then applying Minkovski inequality in (3.6) and using Lemma 2.4.2, we have

$$
\begin{aligned}
& E\left(|S^{[0,t]}_{\beta,n}(X)^{\frac{1}{\beta}} - S^{[0,t]}_{\beta,n}(X^k)^{\frac{1}{\beta}}|\right) \\
\leq\ & C\left(\sum_{i=1}^n \left|\int_0^{t_i^n} \left((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha\right)^2 \left(E\left(|\xi_s - \xi_s^k|^\beta\right)\right)^{\frac{2}{\beta}} ds\right|^{\frac{\beta}{2}}\right)^{\frac{1}{\beta}} \\
\leq\ & C\left(\int_0^t (E|\xi_s - \xi_s^k|^\beta)^{\frac{\beta'}{\beta}} ds\right)^{\frac{1}{\beta'}}.
\end{aligned}
$$

Now in the same way as for the case $\alpha > 0$, we can show

$$
\lim_{n\to\infty} E\left(\left|S^{[0,t]}_{\beta,n}(X)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t |\xi_s|^\beta ds\right)^{\frac{1}{\beta}}\right|\right) = 0.
$$

This proves the theorem.

Remark 2.2.7. *If $\alpha > 0$ and $\int_0^t E(\xi_s^2) ds < \infty$, then $\int_0^t \left(E(\xi_s^2)\right)^{\frac{\beta}{2}} ds < \infty$, and the β-variation of the fractional martingale $M^{(\alpha)}$ exists in L^1, and $\langle M^{(\alpha)}\rangle_{\beta,t} = c_\alpha \int_0^t |\xi_s|^\beta ds$. Using a localization argument, we can prove that this result remains true with the convergence in probability, for any continuous local martingale of the form $M_t = \int_0^t \xi_s dW_s$. On the other hand, if $\alpha < 0$ and $\int_0^t E(|\xi_s|^{\beta'}) ds < \infty$ for all $t \geq 0$, and for some $\beta' > \beta$, then the β-variation of the fractional martingale $M^{(\alpha)}$ exists in L^1 and*

$\langle M^{(\alpha)}\rangle_{\beta,t} = c_\alpha \int_0^t |\xi_s|^\beta ds$. As a consequence, again by a localization argument, the result remains true with the convergence in probability, for any local martingale of the form $M_t = \int_0^t \xi_s dW_s$, assuming that $\int_0^t |\xi_s|^{\beta'} ds < \infty$ almost surely, for all $t \geq 0$, and for some $\beta' > \beta$.

Corollary 2.2.8. *Consider a continuous local martingale $M = (M_t, t \geq 0)$ with $M_0 = 0$ and $\langle M\rangle_t = \int_0^t \xi_s^2 ds$, where $\xi = (\xi_t, t \geq 0)$ is a progressively measurable process. Suppose that M satisfies (4.10) for some $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then there exists $C > 0$, such that*

$$\liminf_{n\to\infty} E(S_{\beta,n}^{[a,b]}(M^{(\alpha)})) \geq C \int_a^b E\left(|\xi_s|^\beta\right) ds.$$

Proof For each integer $N \geq 1$ let $\psi_N(x) = x$ if $|x| \leq N$ and $\psi_N(x) = \frac{N}{x}$ if $|x| > N$. Denote $M_t^{(\alpha),N} = \int_0^t (t-s)^\alpha \psi_N(\xi_s) dM_s$. An application of the Burkholder's inequality yields

$$\begin{aligned}
E\left(S_{\beta,n}^{[a,b]}(M^{(\alpha)})\right) &= E\left(\sum_{i=1}^n \left|\int_0^{t_i^n} \left((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha\right) dM_s\right|^\beta\right) \\
&\geq CE\left(\sum_{i=1}^n \left|\int_0^{t_i^n} \left((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha\right)^2 |\xi_s|^2 ds\right|^{\frac{\beta}{2}}\right) \\
&\geq CE\left(\sum_{i=1}^n \left|\int_0^{t_i^n} \left((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha\right)^2 (|\xi_s| \wedge N)^2 ds\right|^{\frac{\beta}{2}}\right) \\
&\geq CE(S_{\beta,n}^{[a,b]}(M^{(\alpha),N})).
\end{aligned}$$

By Theorem 2.2.6, $S_{\beta,n}^{[a,b]}(M^{(\alpha),N})$ converges to $\int_a^b (|\xi_s| \wedge N)^\beta ds$ in L^1 as n tends to infinity. So, $\lim_{n\to\infty} E(S_{\beta,n}^{[a,b]}(M^{(\alpha),N})) = \int_a^b E\left((|\xi_s| \wedge N)^\beta\right) ds$, and consequently $\liminf_{n\to\infty} E\left(S_{\beta,n}^{[a,b]}(M^{(\alpha)})\right) \geq C \int_a^b E|\xi_s|^\beta ds$.

So far we have considered continuous local martingales of the form $M_t = \int_0^t \xi_s dW_s$. The next result says that in the case $\alpha < 0$ if the quadratic variation of the martingale is not absolutely continuous with respect to the Lebesgue measure with positive probability, then the β-variation is infinite.

Proposition 2.2.9. *Fix* $-\frac{1}{2} < \alpha < 0$*. Suppose that* $M = (M_t, t \geq 0)$ *is a continuous local martingale, satisfying (4.10). Consider the Lebesgue decomposition of its quadratic variation given by* $\langle M \rangle_t = \mu_t + \nu_t$*, where* μ_t *and* ν_t *are continuous nondecreasing adapted processes such that* $d\mu_t$ *is absolutely continuous with respect to the Lebesgue measure, and* $d\nu_t$ *is singular. If* $P(d\nu_t \neq 0) > 0$*, then we have* $\lim_{n\to\infty} E(S^{[0,t]}_{\beta,n}(M^{(\alpha)})) = \infty$*, for all* $t \geq 0$*.*

Proof By Burkholder's inequality, we have

$$\begin{aligned} & E\left(\sum_{i=1}^{n} |M^{(\alpha)}_{t_i^n} - M^{(\alpha)}_{t_{i-1}^n}|^{\beta}\right) \\ \geq\ & C\sum_{i=1}^{n} E\left(\int_0^{t_i^n} \left((t_i^n - s)^{\alpha} - (t_{i-1}^n - s)_+^{\alpha}\right)^2 d\langle M \rangle_s\right)^{\frac{\beta}{2}} \\ \geq\ & C\sum_{i=1}^{n} E\left(\int_0^{t_i^n} \left((t_i^n - s)^{\alpha} - (t_{i-1}^n - s)_+^{\alpha}\right)^2 d\mu_s\right)^{\frac{\beta}{2}} \\ & \quad + C\sum_{i=1}^{n} E\left(\int_0^{t_i^n} \left((t_i^n - s)^{\alpha} - (t_{i-1}^n - s)_+^{\alpha}\right)^2 d\nu_s\right)^{\frac{\beta}{2}}. \end{aligned}$$

Then the result follows from the above inequality and Lemma 4.8.4, proved in the Appendix.

On the other hand, the next results says that in the case $\alpha \in (0, \frac{1}{4})$, the β-variation is zero if the quadratic variation of the martingale is singular.

Proposition 2.2.10. *Suppose that* $M = (M_t, t \geq 0)$ *is a continuous local martingale, such that almost surely the measure* $d\langle M \rangle_t$ *is singular with respect to the Lebesgue measure. Then, if* $\alpha \in (0, \frac{1}{4})$ *we have* $\lim_{n\to\infty} E(S^{[0,t]}_{\beta,n}(M^{(\alpha)})) = 0$*, for all* $t \geq 0$*.*

Proof The result is an immediate consequence of Lemma 4.8.4, proved in the Appendix.

2.3 Characterization of fractional Brownian motion

Suppose that B^H is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. The process B^H admits the following representation (see [18])

$$B_t^H = \int_0^t Z_H(t,s)dW_s, \tag{3.1}$$

where

$$Z_H(t,s) = \kappa_H \left[(\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du \right], \tag{3.2}$$

with κ_H defined in (3.2).

The next theorem is the main result of this chapter and provides an extension of Lévy characterization to the fractional Brownian motion.

Theorem 2.3.1. *Fix $H \in (0,1)$, $H \neq \frac{1}{2}$. Suppose that $B = (B_t, t \geq 0)$ is a zero mean continuous stochastic process. The following two conditions are equivalent:*

(1) B is a fractional Brownian motion with Hurst parameter H.

(2) The process B satisfies the following conditions:

(i) The trajectories of B are Hölder continuous of order $H-\varepsilon$ for any $H-\varepsilon \in (0,H)$.

(ii) Let

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}dB_s. \tag{3.3}$$

Then M is a local martingale. Furthermore, if $H > \frac{1}{2}$, the quadratic variation of the martingale M is absolutely continuous with respect to the Lebesgue measure almost surely.

(iii) For any $t > 0$, the $\frac{1}{H}$-variation of B in the interval $[0,t]$ exists in L^1, and $\langle B \rangle_{\frac{1}{H},t} = c_H t$, where $c_H = E(|\xi|^{\frac{1}{H}})$ and ξ is a standard normal random variable.

Remark 2.3.2. *Notice that condition (i) is always true if $H < \frac{1}{2}$, and the Riemann-Stieltjes integral in (3.3) exists by Proposition 2.4.6.*

Proof From the properties of the fractional Brownian motion we know that (1) implies (2). Suppose that (2) holds. Fix $H - \varepsilon \in (0, H)$, and $T > 0$. We are going to show that B is a fractional Brownian motion with Hurst parameter H in the time interval $[0,T]$. Denote by $\|B\|_{H-\varepsilon}$ the Hölder norm of order $H - \varepsilon$ on $[0,T]$ (see (4.2)). The proof is divided into several steps.

Step 1. From (3.3), we can solve the integral equation to express B as a functional of M. This can be done as in the proof of Theorem 5.2 of [36]. In this way we obtain

$$B_t = d_H \left[t^{H-\frac{1}{2}} R_t - (H - \frac{1}{2}) Y_t \right],$$

where $d_H = B(\frac{3}{2} - H, H + \frac{1}{2})^{-1}$,

$$R_t = \int_0^t (t-s)^{H-\frac{1}{2}} dM_s,$$

and

$$Y_t = \int_0^t \left(\int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right) dM_s.$$

Comparing with the representation formula (3.7) for the fractional Brownian motion it suffices to prove that

$$d\langle M\rangle_s = (\kappa_H d_H^{-1} s^{\frac{1}{2}-H})^2 ds\,, \tag{3.4}$$

because this implies that M is a Gaussian martingale, and B has the covariance of the fractional Brownian motion with Hurst parameter H. In order to show (3.4) we are going to compute the $\frac{1}{H}$-variation of R, from the decomposition

$$R_t = d_H^{-1} t^{\frac{1}{2}-H} B_t + (H-\frac{1}{2}) t^{\frac{1}{2}-H} Y_t. \tag{3.5}$$

Step 2. Fix $0<\varepsilon<H\wedge\frac{1}{2}\wedge(1-H)$ and suppose that $E\left(\|B\|_{H-\varepsilon}^{\frac{1}{H}}\right)<\infty$. We will first show that the $\frac{1}{H}$-variation of the process $Z_t = t^{\frac{1}{2}-H}B_t$ exists in L^1 in any interval $[0,t]\subset[0,T]$, and

$$\langle Z\rangle_{\frac{1}{H},t} = 2Hc_H t^{\frac{1}{2H}}. \tag{3.6}$$

An application of the triangular inequality yields

$$\begin{aligned} S^{[0,t]}_{\frac{1}{H},n}(Z) \;\le\; & \Bigg| \Big(\sum_{i=1}^n (t_i^n)^{\frac{1}{2H}-1} |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}} \Big)^H \\ & + \Big(\sum_{i=1}^n |(t_i^n)^{\frac{1}{2}-H} - (t_{i-1}^n)^{\frac{1}{2}-H}|^{\frac{1}{H}} |B_{t_{i-1}^n}|^{\frac{1}{H}} \Big)^H \Bigg|^{\frac{1}{H}}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} S^{[0,t]}_{\frac{1}{H},n}(Z) \;\ge\; & \Bigg| \Big(\sum_{i=1}^n (t_i^n)^{\frac{1}{2H}-1} |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}} \Big)^H \\ & - \Big(\sum_{i=1}^n |(t_i^n)^{\frac{1}{2}-H} - (t_{i-1}^n)^{\frac{1}{2}-H}|^{\frac{1}{H}} |B_{t_{i-1}^n}|^{\frac{1}{H}} \Big)^H \Bigg|^{\frac{1}{H}}. \end{aligned} \tag{3.8}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{n} |(t_i^n)^{\frac{1}{2}-H} - (t_{i-1}^n)^{\frac{1}{2}-H}|^{\frac{1}{H}} |B_{t_{i-1}^n}|^{\frac{1}{H}} \\
\leq\ & C\,\|B\|_{H-\varepsilon}^{\frac{1}{H}} \left(\frac{t}{n}\right)^{\frac{1}{2H}-\frac{\varepsilon}{H}} \sum_{i=2}^{n} (i-1)^{-\frac{1}{2H}-\frac{\varepsilon}{H}} \\
\leq\ & C\,\|B\|_{H-\varepsilon}^{\frac{1}{H}} t^{\frac{1}{2H}-\frac{\varepsilon}{H}} n^{1-\frac{1}{H}}, \qquad (3.9)
\end{aligned}$$

which converges in L^1 to 0 as n tends to infinity. From (3.7) to (3.9) we obtain,

$$\lim_{n\to\infty} S^{[0,t]}_{\frac{1}{H},n}(Z) = \lim_{n\to\infty} \sum_{i=1}^{n} (t_i^n)^{\frac{1}{2H}-1} |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}}, \qquad (3.10)$$

in L^1. Denote $I_j^m = (t_j^m, t_{j+1}^m]$ for $j = 1, 2, \ldots, m$. We divide every subinterval I_j^m into n parts, and we get a finer partition $0 = t_0^{mn} < \cdots < t_{mn}^{mn} = t$. Then, we have

$$\begin{aligned}
&|\sum_{i=1}^{mn} (t_i^{mn})^{\frac{1}{2H}-1} |B_{t_i^{mn}} - B_{t_{i-1}^{mn}}|^{\frac{1}{H}} - \sum_{j=1}^{m} c_H (t_j^m)^{\frac{1}{2H}-1} (t_j^m - t_{j-1}^m)| \\
=\ & \sum_{j=1}^{m} \Bigg(\sum_{i=(j-1)n+1}^{jn} ((t_i^{mn})^{\frac{1}{2H}-1} - (t_j^m)^{\frac{1}{2H}-1}) |B_{t_i^{mn}} - B_{t_{i-1}^{mn}}|^{\frac{1}{H}} \\
& \quad + (t_j^m)^{\frac{1}{2H}-1} (\sum_{i=(j-1)n+1}^{jn} |B_{t_i^{mn}} - B_{t_{i-1}^{mn}}|^{\frac{1}{H}} - c_H (t_j^m - t_{j-1}^m)) \Bigg) \\
\leq\ & \sum_{j=1}^{m} \left(\left(t_j^m\right)^{\frac{1}{2H}-1} - \left(t_{j-1}^m\right)^{\frac{1}{2H}-1} \right) \sum_{i=(j-1)n+1}^{jn} |B_{t_i^{mn}} - B_{t_{i-1}^{mn}}|^{\frac{1}{H}} \\
& \quad + (t_j^m)^{\frac{1}{2H}-1} \left| \sum_{i=(j-1)n+1}^{jn} |B_{t_i^{mn}} - B_{t_{i-1}^{mn}}|^{\frac{1}{H}} - c_H (t_j^m - t_{j-1}^m) \right| \Bigg).
\end{aligned}$$

Letting n tend to infinity and using Assumption (ii), we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} t_i^{\frac{1}{2H}-1} |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}} = 2Hc_H t^{\frac{1}{2H}},$$

in L^1, which shows (4.2).

Step 3. We claim that the $\frac{1}{H}$-variation of the process $V_t = t^{\frac{1}{2}-H} Y_t$ in L^1 is zero. The increment $|Y_t - Y_s|$ can be estimated by Lemma 2.4.7 in the Appendix with $\alpha = \frac{1}{2} - H$, f being a trajectory of the process B and $\beta = H - \varepsilon$. Notice that $\alpha + \beta = \frac{1}{2} - \varepsilon$, and $2\alpha + \beta = 1 - H - \varepsilon$. Hence, for any $s,t \in [0,T]$ we have

$$|Y_t - Y_s| \leq C \|B\|_{H-\varepsilon} (t^\beta - s^\beta).$$

Therefore, as in (3.7) we have

$$\begin{aligned}
E\left(S^{[0,t]}_{\frac{1}{H},n}(V)\right) &\leq C \sum_{i=1}^{n} (t_i^n)^{\frac{1}{2H}-1} E\left(|Y_{t_i^n} - Y_{t_{i-1}^n}|^{\frac{1}{H}}\right) \\
&\quad + C \sum_{i=1}^{n} \left((t_i^n)^{\frac{1}{2}-H} - (t_{i-1}^n)^{\frac{1}{2}-H}\right)^{\frac{1}{H}} E\left(|Y_{t_{i-1}^n}|^{\frac{1}{H}}\right) \\
&= A_n + B_n \,.
\end{aligned}$$

For the term A_n we have

$$\begin{aligned}
A_n &\leq C \|B\|_{H-\varepsilon}^{\frac{1}{H}} \sum_{i=1}^{n} (t_i^n)^{\frac{1}{2H}-1} ((t_i^n)^{H-\varepsilon} - \left(t_{i-1}^n\right)^{H-\varepsilon})^{\frac{1}{H}} \\
&= C \|B\|_{H-\varepsilon}^{\frac{1}{H}} (\frac{t}{n})^{\frac{1}{2H}-\frac{\varepsilon}{H}} \sum_{i=1}^{n} i^{\frac{1}{2H}-1} (i-1)^{1-\frac{\varepsilon}{H}-\frac{1}{H}} \\
&\leq C \|B\|_{H-\varepsilon}^{\frac{1}{H}} (\frac{t}{n})^{\frac{1}{2H}-\frac{\varepsilon}{H}} n^{-\frac{1}{2H}-\frac{\varepsilon}{H}+1}.
\end{aligned}$$

By Lemma 2.4.7, $\lim_{n\to\infty} E(A_n) = 0$. For the term B_n, using that $E\left(|Y_{t^n_{i-1}}|^{\frac{1}{H}}\right) \leq CE\left(\|B\|^{\frac{1}{H}}_{H-\varepsilon}\right)|t^n_{i-1}|^{1-\frac{\varepsilon}{H}}$, we obtain

$$\begin{aligned} E(B_n) &\leq CE\left(\|B\|^{\frac{1}{H}}_{H-\varepsilon}\right)\sum_{i=1}^{n}(t^n_{i-1})^{-\frac{1}{2H}-\frac{\varepsilon}{H}}\left(\frac{t}{n}\right)^{\frac{1}{H}} \\ &\leq CE\left(\|B\|^{\frac{1}{H}}_{H-\varepsilon}\right)\left(\frac{1}{n}\right)^{-1+\frac{1}{H}-\frac{\varepsilon}{H}} \to 0\,. \end{aligned}$$

Hence, $\langle Y\rangle_{\frac{1}{H},t} = 0$, in L^1, for all $t \in [0,T]$.

Step 4. From (4.2), (3.6), Step 3 and (3.3) we get that the $\frac{1}{H}$-variation of the process R in any interval $[0,t] \subset [0,T]$ exists in L^1, and

$$\langle R\rangle_{\frac{1}{H},t} = c_H d_H^{-\frac{1}{H}} 2Ht^{\frac{1}{2H}}\,. \tag{3.11}$$

On the other hand, since R_t is an $H-\frac{1}{2}$ martingale, Theorem 2.2.6 and Proposition 2.2.9 imply that if $H < 1/2$, the quadratic variation $d\langle M\rangle_s$ must be absolutely continuous with respect to the Lebesgue measure, almost surely. In the case $H > \frac{1}{2}$ this is true by the assumption (ii). This implies that $\langle M\rangle_t = \int_0^t \xi_s^2 ds$, where $\xi = (\xi_t, t \geq 0)$ is a progressively measurable process.

By Corollary 2.2.8, there is a positive constant C such that for any t_1, $t_2 \in [0,T]$, $C\int_{t_1}^{t_2} s^{\frac{1}{2H}-1}ds \geq \int_{t_1}^{t_2} E\left(|\xi_s|^{\frac{1}{H}}\right)ds$. Then $E\left(|\xi_s|^{\frac{1}{H}}\right) \leq Cs^{\frac{1}{2H}-1}$. Thus we can apply Theorem 2.2.6 to obtain $\langle R\rangle_{\frac{1}{H},t} = c_H \kappa_H^{-\frac{1}{H}} \int_0^t |\xi_s|^{\frac{1}{H}} ds$. Comparing this with (3.11), we obtain

$$|\xi_s| = \kappa_H d_H^{-1} s^{\frac{1}{2}-H}, 0 \leq s \leq t,$$

and (3.4) holds. This proves that B is a fractional Brownian motion with Hurst parameter H under the condition $E\left(\|B\|^{\frac{1}{H}}_{H-\varepsilon}\right) < \infty$.

Step 5. If $E\left(\|B\|_{H-\varepsilon}^{\frac{1}{H}}\right)$ is not necessarily finite, we can use a localization argument. Denote

$$T_K = \inf\{t \geq 0 : \|B\|_{t,H-\varepsilon} \geq K\} \wedge T.$$

and $B_t^K = B_{t\wedge T_K}$. Since $\sum_{i=1}^n |B_{t_i^n}^K - B_{t_{i-1}^n}^K|^{\frac{1}{H}} \leq \sum_{i=1}^n |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}} + (K\frac{t}{n})^{\frac{1}{H}}$, by dominated convergence theorem we can also get

$$\lim_n E\left(\left|\sum_{i=1}^n |B_{t_i^n}^K - B_{t_{i-1}^n}^K|^{\frac{1}{H}} - c_H(t\wedge T_K)\right|\right) = 0.$$

By modifying the proof in *Step 1* - *Step 4* slightly, we get

$$|\xi_s| = \kappa_H d_H^{-1} s^{\frac{1}{2}-H}, 0 \leq s \leq t \wedge T_K.$$

Clearly $\lim_{K\to\infty} T_K = T$, and then

$$|\xi_s| = \kappa_H d_H^{-1} s^{\frac{1}{2}-H}, 0 \leq s \leq T.$$

Remark 2.3.3. *Notice that in the case $H > \frac{1}{2}$ we have imposed the additional assumption that the martingale (3.3) has an absolutely continuous quadratic variation. This is true, for instance, if the filtration generated by the process B is included in the filtration generated by a Brownian motion. The next proposition shows that this condition is necessary at least in the case $H \in (\frac{1}{2}, \frac{3}{4})$.*

Proposition 2.3.4. *Suppose that $H \in (\frac{1}{2}, \frac{3}{4})$. There exists a process B, satisfying conditions (i) and (iii) of Theorem 3.1, such that the process M defined in (3.3) is a local martingale, and B is not a fractional Brownian motion.*

Proof Let B^H be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, \frac{3}{4})$. Define

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB_s^H.$$

Let $N_t = W_{\phi(t)}$, where W is a Brownian motion independent of B^H, and ϕ is a strictly increasing, Hölder continuous function of exponent $1-\varepsilon$, null at zero, such that the measure $d\phi(t)$ is singular with respect to the Lebesgue measure (for the existence of such function see Lemma 4.8 in the Appendix). Set

$$\widetilde{M}_t = M_t + N_t,$$

and

$$\widetilde{B}_t^H = B_t^H + Y_t,$$

where

$$Y_t = d_H \left(t^{H-\frac{1}{2}} \int_0^t (t-s)^{H-\frac{1}{2}} dN_s - (H - \frac{1}{2}) \int_0^t \left(\int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right) dN_s \right).$$

The process $\widetilde{B}^H$ clearly satisfies (i) and it is not a fractional Brownian motion. Finally, $\langle \widetilde{B}^H \rangle_{\frac{1}{H},t} = c_H t$ in L^1, because the $\frac{1}{H}$-variation of $\int_0^t (t-s)^{H-\frac{1}{2}} dN_s$ is zero by Proposition 2.2.10, and, by the same arguments as in the proof of Theorem 3.1 we can show that the $\frac{1}{H}$-variation of Y vanishes.

2.4 Appendix

2.4.1 Some technical lemmas

Lemma 2.4.1. *Let $\alpha \in (0, \frac{1}{2})$. Fix an interval $[0,t]$. For any natural number m we define $t_i^m = \dfrac{i}{m}t$, $0 \le i \le m$. Let g be a measurable function on $[0,\infty)$ such that for all $t \ge 0$, $\int_0^t |g(s)|ds < \infty$. Then there exists a function $C(t) > 0$ satisfying*

$$\limsup_{m\to\infty} \sum_{i=1}^m \left(\int_0^{t_i^m} \left((t_i^m - s)^\alpha - (t_{i-1}^m - s)_+^\alpha\right)^2 |g(s)|ds \right)^{\frac{\beta}{2}} \le C(t) \int_0^t |g(s)|^{\frac{\beta}{2}} ds.$$

Proof Set

$$A_m = \sum_{i=1}^m \left(\int_0^{t_i^m} \left((t_i^m - s)^\alpha - (t_{i-1}^m - s)_+^\alpha\right)^2 |g(s)|ds \right)^{\frac{\beta}{2}}.$$

We have $A_m \le C(A_{1,m} + A_{2,m} + A_{3,m})$, where

$$A_{1,m} = \sum_{i=3}^m \left(\int_0^{t_{i-2}^m} \left((t_i^m - s)^\alpha - (t_{i-1}^m - s)^\alpha\right)^2 |g(s)|ds \right)^{\frac{\beta}{2}},$$

$$A_{2,m} = \sum_{i=2}^m \left(\int_{t_{i-2}^m}^{t_{i-1}^m} \left((t_i^m - s)^\alpha - (t_{i-1}^m - s)^\alpha\right)^2 |g(s)|ds \right)^{\frac{\beta}{2}}$$

and

$$A_{3,m} = \sum_{i=1}^m \left(\int_{t_{i-1}^m}^{t_i^m} (t_i^m - s)^{2\alpha} |g(s)|ds \right)^{\frac{\beta}{2}}.$$

Let $\phi_m(x) = \left((x+\frac{t}{m})^\alpha - x^\alpha\right)^2$. The $\phi(x)$ is a non-increasing of x when $x \geq 0$. As a consequence,

$$\begin{aligned} A_{1,m} &= \frac{m}{t}\sum_{i=3}^{m}\int_{t_{i-2}^m}^{t_{i-1}^m}\left(\int_0^{t_{i-2}^m}\left((t_i^m - s)^\alpha - (t_{i-1}^m - s)^\alpha\right)^2 |g(s)|ds\right)^{\frac{\beta}{2}} du \\ &= \frac{m}{t}\sum_{i=3}^{m}\int_{t_{i-2}^m}^{t_{i-1}^m}\left(\int_0^{t_{i-2}^m} \phi_m(t_{i-1}^m - s)|g(s)|ds\right)^{\frac{\beta}{2}} du \\ &\leq \frac{m}{t}\int_0^t\left(\int_0^u \phi_m(u-s)|g(s)|ds\right)^{\frac{\beta}{2}} du. \end{aligned}$$

Using Hölder inequality we obtain

$$\begin{aligned} \left(\int_0^u \phi_m(u-s)|g(s)|ds\right)^{\frac{\beta}{2}} &\leq \left(\int_0^u \phi_m(u-s)ds\right)^{\frac{\beta}{2}-1}\int_0^u \phi_m(u-s)|g(s)|^{\frac{\beta}{2}}ds \\ &\leq \left(\int_0^t \phi_m(s)ds\right)^{\frac{\beta}{2}-1}\int_0^t \phi_m(u-s)|g(s)|^{\frac{\beta}{2}}ds\,. \end{aligned}$$

Integrating in the variable u yields

$$A_{1,m} \leq \frac{m}{t}\left(\int_0^t \phi_m(s)ds\right)^{\frac{\beta}{2}}\int_0^t |g(s)|^{\frac{\beta}{2}}ds\,. \tag{4.1}$$

Therefore

$$\lim_{m\to\infty} A_{1,m} = t^{-1}\left(\int_0^\infty ((x+t)^\alpha - x^\alpha)^2 dx\right)^{\beta/2}\int_0^t |g(s)|^{\frac{\beta}{2}}du.$$

For the term $A_{3,m}$ we can write

$$A_{3,m} \leq \left(\frac{t}{m}\right)^{\alpha\beta} \sum_{i=1}^{m} \left(\int_{t_{i-1}^m}^{t_i^m} |g(s)| ds \right)^{\frac{\beta}{2}}$$
$$= \sum_{i=1}^{m} \left(\frac{m}{t} \int_{t_{i-1}^m}^{t_i^m} |g(s)| ds \right)^{\frac{\beta}{2}} \frac{t}{m}.$$

The functions

$$g_m(s) = \frac{m}{t} \sum_{i=1}^{m} \left(\int_{t_{i-1}^m}^{t_i^m} |g(s)| ds \right) I_{(t_{i-1}^m, t_i^m]}(s)$$

converge almost everywhere to $|g|$, and they are bounded in $L^1([0,t])$. Hence $|g(s)|^{\frac{\beta}{2}}$ is uniformly integrable on $[0,t]$. Therefore,

$$\limsup_{m\to\infty} A_{3,m} \leq \lim_{m\to\infty} \int_0^t |g_m(s)|^{\frac{\beta}{2}} ds = \int_0^t |g(s)|^{\frac{\beta}{2}} ds.$$

From the fact that $|x^\alpha - y^\alpha| \leq |x-y|^\alpha$, we see that

$$A_{2,m} \leq \sum_{i=2}^{m} \left(\int_{t_{i-2}^m}^{t_{i-1}^m} |t_i^m - t_{i-1}^m|^{2\alpha} |g(s)| ds \right)^{\beta/2}.$$

Thus in the same way as for $A_{3,m}$ we have

$$\limsup_{m\to\infty} A_{2,m} \leq 2 \int_0^t |g(s)|^{\frac{\beta}{2}} ds.$$

Lemma 2.4.2. *Let $\alpha \in (-\frac{1}{2}, 0)$. Fix an interval $[0,t]$. For any natural number m we define $t_i^m = \frac{i}{m} t$, $0 \leq i \leq m$. Let g be a measurable function on $[0,\infty)$ such that for all $t \geq 0$, $\int_0^t |g(s)|^{\frac{\beta'}{2}} ds < \infty$ for some $\beta' > \beta$. Then there exists a constant C depending on* *t such that:*

$$\sum_{i=1}^{m}\left(\int_0^{t_i^m}\left((t_i^m-s)^{\alpha}-(t_{i-1}^m-s)_+^{\alpha}\right)^2|g(s)|ds\right)^{\frac{\beta}{2}}\leq C\left(\int_0^t|g(s)|^{\frac{\beta'}{2}}ds\right)^{\frac{\beta}{\beta'}}.$$

Proof Consider the decomposition given in the proof of Lemma 2.4.1. For the first term we can write, from the inequality (4.4)

$$\begin{aligned}A_{1,m} &\leq C\frac{m}{t}\left(\int_0^t\left(s^{2\alpha}-(s+\frac{t}{m})^{2\alpha}\right)ds\right)^{\frac{\beta}{2}}\int_0^t|g(s)|^{\frac{\beta}{2}}ds\\ &\leq C\frac{m}{t}\left(t^{1+2\alpha}+(\frac{t}{m})^{1+2\alpha}-\left(t+\frac{t}{m}\right)^{1+2\alpha}\right)^{\frac{\beta}{2}}\int_0^t|g(s)|^{\frac{\beta}{2}}ds\\ &\leq C\frac{m}{t}(\frac{t}{m})^{(1+2\alpha)\frac{\beta}{2}}\int_0^t|g(s)|^{\frac{\beta}{2}}ds\\ &\leq C\int_0^t|g(s)|^{\frac{\beta}{2}}ds\leq C\left(\int_0^t|g(s)|^{\frac{\beta'}{2}}ds\right)^{\frac{\beta'}{\beta}}\end{aligned}$$

Let $2\alpha p>-1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $\beta'=2q>\beta$, and applying Hölder's inequaliy we can write

$$\begin{aligned}A_{3,m} &\leq \sum_{i=1}^{m}\left(\int_{t_{i-1}^m}^{t_i^m}(t_i^m-s)^{2\alpha p}ds\right)^{\frac{\beta}{2p}}\left(\int_{t_{i-1}^m}^{t_i^m}|g(s)|^q ds\right)^{\frac{\beta}{2q}}\\ &\leq C\sum_{i=1}^{m}(\frac{t}{m})^{\frac{1+2\alpha p}{p}\frac{\beta}{2}}\left(\int_{t_{i-1}^m}^{t_i^m}|g(s)|^q ds\right)^{\frac{\beta}{2q}}\\ &\leq Ct^{\frac{1+2\alpha p}{p}\frac{\beta}{2}}\left(\int_0^t|g(s)|^q ds\right)^{\frac{\beta}{2q}}.\end{aligned}$$

For the term $A_{2,m}$, with the same notation as above we can wite

$$\begin{aligned} A_{2,m} &\leq C\sum_{i=2}^{m}\left(\int_{t_{i-2}^{m}}^{t_{i-1}^{m}}(t_{i-1}^{m}-s)^{2\alpha}|g(s)|ds\right)^{\frac{\beta}{2}} \\ &\leq C\sum_{i=2}^{m}\left(\frac{t}{m}\right)^{\frac{1+2\alpha p}{p}\frac{\beta}{2}}\left(\int_{t_{i-2}^{m}}^{t_{i-1}^{m}}|g(s)|^{q}ds\right)^{\frac{\beta}{2q}} \\ &\leq Ct^{\frac{1+2\alpha p}{p}\frac{\beta}{2}}\left(\int_{0}^{t}|g(s)|^{q}ds\right)^{\frac{\beta}{2q}}. \end{aligned}$$

Lemma 2.4.3. *Suppose that ν is a measure on an interval $[0,t]$, which is singular with respect to the Lebesgue measure. We have*

(i) *If $\alpha\in(-\frac{1}{2},0)$, then*

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\int_{0}^{t_i^n}\left((t_i^n-s)^{\alpha}-(t_{i-1}^n-s)_{+}^{\alpha}\right)^{2}d\nu_s\right)^{\frac{\beta}{2}}=\infty.$$

(ii) *If $\alpha\in(0,\frac{1}{4})$, then*

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\int_{0}^{t_i^n}\left((t_i^n-s)^{\alpha}-(t_{i-1}^n-s)_{+}^{\alpha}\right)^{2}d\nu_s\right)^{\frac{\beta}{2}}=0.$$

Proof Denote $\triangle_i^n:=(t_{i-1}^n,t_i^n]$. Set

$$A_n=\sum_{i=1}^{n}\left(\int_{0}^{t_i^n}\left((t_i^n-s)^{\alpha}-(t_{i-1}^n-s)_{+}^{\alpha}\right)^{2}d\nu_s\right)^{\frac{\beta}{2}}.$$

(i) If $\alpha \in (-\frac{1}{2}, 0)$, then

$$
\begin{aligned}
A_n &\geq \sum_{i=1}^{n} \left(\int_{t_{i-1}^n}^{t_i^n} (t_i^n - s)^{2\alpha} d\nu_s \right)^{\frac{\beta}{2}} \\
&\geq \; C(\frac{t}{n})^{\alpha\beta} \sum_{i=1}^{n} (\nu(\triangle_i^n))^{\frac{\beta}{2}} \geq \sum_{i=1}^{n} C(\frac{t}{n}) \left(\frac{\nu(\triangle_i^n)}{m(\triangle_i^n)} \right)^{\frac{\beta}{2}},
\end{aligned}
$$

where m denotes the Lebesgue measure. Suppose that $\mathscr{F}_n$ is the σ-field of subsets of the interval $[0,t]$ generated by the partition $\{\triangle_i^n, i = 1, \ldots, n\}$. Denote by ν_n and m_n the restrictions of the measures ν and m to the σ-field $\mathscr{F}_n$. Set

$$
X_n = \sum_{i=1}^{n} \frac{\nu(\triangle_i^n)}{m(\triangle_i^n)} I_{\triangle_i^n}.
$$

Then $A_n \geq CE(X_n^{\frac{\beta}{2}})$. The sequence $(X_{2^k}, k \geq 0)$ is a martingale with respect to the filtration $\mathscr{F}_{2^k}$. As a consequence (see for instance, Theorem 3.3 in [11]), we have $\lim_{n\to\infty} X_{2^k} = X$ $(m+\nu)$-a.e. Since $\nu \perp m, X = 0$ m-a.e. If $\lim_{k\to\infty} E(X_{2^k}^{\frac{\beta}{2}}) < \infty$, then $(X_{2^k}, k \geq 0)$ would be a uniformly integrable martingale and hence $X_{2^k} = E(X|\mathscr{F}_{2^k}) = 0$, which is a contradiction.

(ii) If $\alpha \in (0, \frac{1}{4})$, then

$$
\begin{aligned}
A_n &= \sum_{i=1}^{n} \left(\int_0^{t_{i-1}} \left((t_i^n - s)^{\alpha} - (t_{i-1}^n - s)^{\alpha} \right)^2 d\nu_s \right)^{\frac{\beta}{2}} + \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i^n} (t_i^n - s)^{2\alpha} d\nu_s \right)^{\frac{\beta}{2}} \\
&= B_n + C_n.
\end{aligned}
$$

For the term C_n we have

$$
C_n \leq \left(\frac{t}{n} \right)^{\alpha\beta} \sum_{i=1}^{n} (\nu(\triangle_i^n))^{\frac{\beta}{2}} = t^{\alpha\beta} \sum_{i=1}^{n} \frac{1}{n} (\nu(\triangle_i^n) n)^{\frac{\beta}{2}} = t^{\alpha\beta} E(X_n^{\frac{\beta}{2}})
$$

Since $E(X_n) = \nu([0,t]) < \infty, \dfrac{\beta}{2} < 1$, and $X_n \to 0$ a.e. we have $\lim_{n\to\infty} C_n = 0$. On the other hand,

$$
\begin{aligned}
B_n &\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \left((t_i^n - s)^{\alpha} - (t_{i-1}^n - s)^{\alpha} \right)^2 d\nu_s \right)^{\frac{\beta}{2}} \\
&\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{i-1} \left(\frac{t}{n}\right)^{2\alpha} (i^{\alpha} - (i-1)^{\alpha})^2 \nu(\triangle_j^n) \right)^{\frac{\beta}{2}} \\
&\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{i-1} \left(\frac{t}{n}\right)^{\alpha\beta} (i^{\alpha} - (i-1)^{\alpha})^{\beta} \nu(\triangle_j^n)^{\frac{\beta}{2}} \right) \\
&\leq \left(\frac{t}{n}\right)^{\alpha\beta} \sum_{i=1}^{n} (i^{\alpha} - (i-1)^{\alpha})^{\beta} \sum_{j=1}^{n} \nu(\triangle_j^n)^{\frac{\beta}{2}}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\sum_{i=1}^{n} (i^{\alpha} - (i-1)^{\alpha})^{\beta} \leq C + \sum_{i=2}^{n} (i^{\alpha} - (i-1)^{\alpha})^{\beta} \\
\leq &C + \sum_{i=2}^{n} (i-1)^{(\alpha-1)\beta} = C + O(n^{\alpha\beta - \beta + 1}),
\end{aligned}
$$

where $C > 0$. If $\alpha \in (0, \dfrac{1}{4})$, we have $\alpha\beta - \beta + 1 < 0$ and then $\sup_n \sum_{i=1}^{n} (i^{\alpha} - (i-1)^{\alpha})^{\beta} < \infty$. Then similarly, $\lim_n A_n = 0$.

2.4.2 Transformations of Hölder continuous functions

Let $\beta \in (0,1]$. We denote by $C^{\beta}([0,T])$ the set of Hölder continuous functions on $[0,T]$. For any function f in $C^{\beta}([0,T])$ and any $0 \leq a < b \leq T$ we will write

$$
\|f\|_{\beta,a,b} = \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t-s|^{\beta}}. \tag{4.2}
$$

We also set $\|f\|_{\beta} = \|f\|_{\beta,0,T}$

Lemma 2.4.4. *Suppose that* $f \in C^{\beta}([0,T])$*, and assume that* $0 \leq a < b < v \leq T$*. Let,* $\gamma \geq 0$ *and* $\alpha+\beta \neq 0$*. Then*

$$\left|\int_a^b s^{\gamma}(v-s)^{\alpha} df(s)\right| \leq \|f\|_{\beta}\left(2+\left|\frac{\alpha}{\alpha+\beta}\right|\right) b^{\gamma}\left((v-b)^{\alpha+\beta}+(v-a)^{\alpha+\beta}\right).$$

Proof Suppose first $\gamma > 0$. Integrating by parts yields

$$\begin{aligned}
& \left|\int_a^b s^{\gamma}(v-s)^{\alpha} df(s)\right| \\
= & \Big| b^{\gamma}(v-b)^{\alpha}(f(b)-f(v)) - a^{\gamma}(v-a)^{\alpha}(f(a)-f(v)) \\
& - \int_a^b (f(s)-f(v))[s^{\gamma}(v-s)^{\alpha}]' ds \Big| \\
\leq & \|f\|_{\beta,a,v}\Big(b^{\gamma}(v-b)^{\alpha+\beta} + Ka^{\gamma}(v-a)^{\alpha+\beta} \\
& + \gamma\int_a^b (v-s)^{\alpha+\beta}s^{\gamma-1}ds + \alpha\int_a^b (v-s)^{\alpha+\beta-1}s^{\gamma}ds\Big) \\
\leq & \|f\|_{\beta,a,v}\Big[b^{\gamma}(v-b)^{\alpha+\beta} + b^{\gamma}(v-a)^{\alpha+\beta} \\
& + \max\{(v-a)^{\alpha+\beta},(v-b)^{\alpha+\beta}\}(b^{\gamma}-a^{\gamma}) \\
& + b^{\gamma}\left|\frac{\alpha}{\alpha+\beta}\right|\left((v-a)^{\alpha+\beta}-(v-b)^{\alpha+\beta}\right)\Big] \\
\leq & \|f\|_{\beta,a,v}\left(2+\left|\frac{\alpha}{\alpha+\beta}\right|\right) b^{\gamma}\left((v-b)^{\alpha+\beta}+(v-a)^{\alpha+\beta}\right).
\end{aligned}$$

The case $\gamma = 0$ is proved in a similar way.

Lemma 2.4.5. *Suppose that* $f \in C^{\beta}([0,T])$*, and suppose* $\alpha < 0, \alpha+\beta > 0$*. Let* $g(t) = \int_0^t s^{\alpha} df(s)$*. Then,* $g \in C^{\alpha+\beta}([0,T])$*, and*

$$\|g\|_{\alpha+\beta} \leq \frac{\beta}{\alpha+\beta}\|f\|_{\beta}.$$

Proof Fix $0 \le a < b \le T$. Integrating by parts yields

$$
\begin{aligned}
|g(b)-g(a)| &= \left| \int_a^b s^\alpha d[f(s)-f(a)] \right| \\
&= \left| b^\alpha [f(b)-f(a)] + \alpha \int_a^b [f(s)-f(a)] s^{\alpha-1} ds \right| \\
&\le \|f\|_\beta \, b^\alpha |b-a|^\beta + |\alpha| \int_a^b |f(s)-f(a)| (s-a)^{\alpha-1} ds \\
&\le \|f\|_\beta \left(|b-a|^{\alpha+\beta} + |\alpha| \int_a^b (s-a)^{\alpha+\beta-1} ds \right) \\
&\le \|f\|_\beta \, \frac{\beta}{\alpha+\beta} \, |b-a|^{\alpha+\beta},
\end{aligned}
$$

which give the desired result.

Proposition 2.4.6. *Fix* $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ *and* $\beta \in (0,1]$ *such that* $0 < \alpha + \beta \le 1$. *Suppose that* $f \in C^\beta([0,T])$, *and let* $g(t) = \int_0^t s^\alpha (t-s)^\alpha d f_s$. *Then,*

1. *If* $\alpha > 0$, $g \in C^{\alpha+\beta}([0,T])$ *and for any* $0 \le a < b \le T$ *we have*

$$
|g(b)-g(a)| \le C \, \|f\|_\beta \, b^\alpha (b-a)^{\alpha+\beta}. \tag{4.3}
$$

2. *If* $\alpha < 0$ *and* $0 < 2\alpha + \beta \le 1$, *then* $g \in C^{2\alpha+\beta}([0,T])$ *and*

$$
|g(b)-g(a)| \le C \, \|f\|_\beta \, (b-a)^{2\alpha+\beta}.
$$

Proof We can write

$$
\begin{aligned}
g(b)-g(a) &= \int_0^a s^\alpha \left((b-s)^\alpha - (a-s)^\alpha\right) d f_s + \int_a^b s^\alpha (b-s)^\alpha d f_s \\
&= \alpha \int_a^b \left(\int_0^a s^\alpha (v-s)^{\alpha-1} d f_s \right) dv + \int_a^b s^\alpha (b-s)^\alpha d f_s \\
&= A + B.
\end{aligned}
$$

If $\alpha > 0$ using Lemma 2.4.4 yields

$$\begin{aligned} |A| &\leq C\|f\|_\beta a^\alpha \int_a^b \left((v-a)^{\alpha+\beta} + v^{\alpha+\beta}\right) dv \\ &= C\|f\|_\beta a^\alpha \left[(b-a)^{\alpha+\beta} + b^{\alpha+\beta} - a^{\alpha+\beta}\right], \end{aligned}$$

and

$$|B| \leq C\|f\|_\beta b^\alpha (b-a)^{\alpha+\beta},$$

which implies (3.1) follows. On the other if $\alpha < 0$, the function $h(t) = \int_0^t s^\alpha d f_s$ is $(\alpha+\beta)$-Hölder continuous by Lemma 2.4.5, and $\|h\|_{\alpha+\beta} \leq C\|f\|_\beta$. Then, applying Lemma 2.4.4 to the function h we obtain the estimates

$$\begin{aligned} |A| &\leq \alpha \left| \int_a^b \left(\int_0^a (v-s)^{\alpha-1} dh_s \right) dv \right| \\ &\leq C\|f\|_\beta \int_a^b \left[(v-a)^{2\alpha+\beta-1} + v^{2\alpha+\beta-1}\right] dv \\ &\leq C\|f\|_\beta \left[(b-a)^{2\alpha+\beta} + b^{2\alpha+\beta} - a^{2\alpha+\beta}\right], \end{aligned}$$

and

$$|B| \leq \left| \int_a^b (b-s)^\alpha dh_s \right| \leq C\|f\|_\beta (b-a)^{2\alpha+\beta}.$$

The proof is complete.

Lemma 2.4.7. *Fix* $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ *and* $\beta \in (0,1]$ *such that* $0 < \alpha+\beta \leq 1$ *and* $0 < 2\alpha + \beta \leq 1$. *Suppose that* $f \in C^\beta([0,T])$, *and let* $g(t) = \int_0^t s^\alpha (t-s)^\alpha d f_s$. *Set*

$$h(t) = \int_0^t u^{-\alpha-1} \left(\int_0^u (u-s)^{-\alpha} dg_s \right) du.$$

Then for any $0 \le a < b \le T$ *we have*

$$|h(b) - h(a)| \le C \| f \|_\beta (b^\beta - a^\beta).$$

Proof We have

$$|h(b) - h(a)| \le \int_a^b u^{-\alpha-1} \left| \int_0^u (u-s)^{-\alpha} dg_s \right| du. \tag{4.4}$$

Suppose first that $\alpha < 0$. Then, $\|g\|_{2\alpha+\beta} \le C \| f \|_\beta$, and Lemma 2.4.4 yields

$$\left| \int_0^u (u-s)^{-\alpha} dg_s \right| \le C \| f \|_\beta u^{\alpha+\beta}. \tag{4.5}$$

Substituting (4.5) into (4.4) yields the results. In the case $\alpha > 0$, the Hölder norm $\|g\|_{\alpha+\beta}$ in an interval $[0,u]$ is bounded by $Cu^\alpha \| f \|_\beta$, and Lemma 2.4.4 yields

$$\left| \int_0^u (u-s)^{-\alpha} dg_s \right| \le C \| f \|_\beta u^{\beta+\alpha}.$$

This completes the proof of the lemma.

2.4.3 Existence of singular Hölder continuous distribution functions

The following lemma implies the existence of finite measures on the real line which are singular with respect to the Lebesgue measure, and whose distribution function is Hölder continuous of order γ, for any $\gamma < 1$.

Lemma 2.4.8. *Let* $(B_t^H, t \geq 0)$ *be a fractional Brownian motion with Hurst parameter* $H \in (0,1)$. *Then, there exists a version of its local time* $L(t,x)$*, jointly continuous in t and x, with the following properties*

(i) *For each* $x \in \mathbb{R}$, $L(t,x)$ *is Hölder continuous of order* $1-H$ *with respect to* t*, on any finite interval.*

(ii) $L(t,x)$ *is a non decreasing function of* t.

(iii) *For each* $x \in \mathbb{R}$*, the support of the measure* $L(dt,x)$ *is the set* $\{s, B_s^H = x\}$*, which has a Lebesgue measure* 0.

Proof Property (i) follows from [15, Section 30]. From Theorem 6.4, page 11, in [15] it follows that for each $x \in \mathbb{R}$ the support of the measure $L(dt,x)$ is the set $\Lambda_x = \{s, B_s^H = x\}$. Finally, to show that Λ_x has a Lebesgue measure 0, we write

$$E\int_0^T \mathbf{1}_{\Lambda_x}(s)ds = \int_0^T E(\mathbf{1}_{B_s^H = x})ds = 0,$$

which implies that $\int_0^T \mathbf{1}_{\Lambda_x}(s)ds = 0$ almost surely. This completes the proof of the lemma.

Chapter 3

Integral representation of renormalized self-intersection local times

3.1 Introduction

The purpose of this chapter is to apply Clark-Ocone's formula to the renormalized self-intersection local time of the d-dimensional fractional Brownian motion. As a consequence, we derive the existence of some exponential moments for this local time.

A well-known result in Itô's stochastic calculus asserts that any square integrable random variable in the filtration generated by a d-dimensional Brownian motion $W = \{W_t, t \geq 0\}$ can be expressed as the sum of its expectation plus the stochastic integral of a square integrable adapted process:

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty u^i(t) dW_t^i.$$

The process u is determined by F, except on sets of measure zero. In this context, Clark-Ocone formula provides an explicit representation of u in terms of the derivative operator in the sense of Malliavin calculus. More precisely, if F belongs to the Sobolev space $\mathbb{D}^{1,2}$, then $u^i(t) = E(D_t^i F | \mathscr{F}_t)$, where D^i denotes the derivative with respect to the

ith component of the Brownian motion and $\{\mathscr{F}_t, t \geq 0\}$ is the filtration generated by the Brownian motion. Extensions of this formula have been developed by Üstünel in [46], and by Karatzas, Ocone and Li in [28]. Clark-Ocone formula has proved to be a useful tool in finding hedging portfolios in mathematical finance (see, for instance, [27]).

The fractional Brownian motion on $\mathbb{R}^d$ with Hurst parameter $H \in (0,1)$ is a d-dimensional Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with zero mean and covariance function given by

$$E(B_t^{H,i} B_s^{H,j}) = \frac{\delta_{ij}}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \tag{1.1}$$

where $i, j = 1, \ldots, d$, $s, t \geq 0$, and

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \quad\quad i \neq j \end{cases}$$

is the Kronecker symbol. Assume $d \geq 2$. The *self-intersection local time* of B^H is formally defined as

$$L = \int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds,$$

where δ_0 is the Dirac delta function. It measures the amount of time that the process spends intersecting itself on the time interval $[0,T]$. Rigorously, L is defined as the limit in L^2, if it exists, of $L_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t^H - B_s^H) dsdt$, as ε tends to zero, where p_ε denotes the heat kernel.

For $H = \frac{1}{2}$, the process B^H is a classical Brownian motion and its self-intersection local time has been studied by many authors (see Albeverio et al. [1], Calais and Yor [6], He et al. [16], Hu [18], Imkeller et al. [26], Varadhan [47], Yor[52], and the references therein). In this case, if $d = 2$, Varadhan [47] has proved that L_ε does not converge in L^2, but it can be renormalized so that $L_\varepsilon - E(L_\varepsilon)$ converges in L^2 as ε

tends to zero to a random variable that we denote by $\widetilde{L}$. This result has been extended by Rosen [41] to the case $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ (still when $d = 2$), and by Hu and Nualart in [23], where they have obtained the following complete result on the existence of the self-intersection local time of the fractional Brownian motion:

(i) The self-intersection local time L exists if and only if $Hd < 1$.

(ii) If $Hd \geq 1$, the renormalized self-intersection local time $\widetilde{L}$ exists if and only if $Hd < \frac{3}{2}$.

An important question is the existence of moments and exponential moments for the (renormalized) self-intersection local time. Along this direction, Le Gall [31] proved that for the planar Brownian motion, there is a critical exponent λ_0, such that $E\left(\exp \lambda \widetilde{L}\right) < \infty$ for all $\lambda < \lambda_0$, and $E\left(\exp \lambda \widetilde{L}\right) = \infty$ if $\lambda > \lambda_0$. Using the theory of large deviations, Bass and Chen proved in [4] that the critical exponent λ_0 coincides with A^{-4}, where A is the best constant in the Gagliardo-Nirenberg inequality.

Clark-Ocone formula seems to be a suitable tool to analyze the renormalized self-intersection local time, because in this formula we do not take into account the expectation of the random variable. The fractional Brownian motion can be expressed as the stochastic integral

$$B_t^H = \int_0^t K_H(t,s) dW_s$$

of a square integrable kernel $K_H(t,s)$ with respect to an underlying Brownian motion W. In this way the renormalized self-intersection local time $\widetilde{L}$ is a functional of the Brownian motion W, and we can obtain an explicit integral representation $\widetilde{L}$, in the general case $Hd < \frac{3}{2}$. This formula allows us to obtain some exponential moments for the renormalized self-intersection local time, using the method of moments.

The chapter is organized as follows. In Section 2 we present some preliminaries on Malliavin calculus and Clark-Ocone formula. Section 3 is devoted to derive estimates for the moments of the self-intersection local time in the case of a general d-dimensional Gaussian process, using the method of moments. In the case of the fractional Brownian motion, this provides the existence of exponential moments in the case $Hd < 1$. Section 4 contains the main result, which is the integral representation of the renormalized self-intersection local time of the fractional Brownian motion in the case $H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$. As an application we show that $E\left(\exp\left|\widetilde{L}\right|^p\right) < \infty$ if $p < \frac{1}{2}\left[\left(\frac{1}{2}+H\right)\left(\frac{d}{2}-\frac{1}{4H}\right)\right]^{-1}$. A crucial tool is the local nondeterminism property introduced by Berman in [5] and developed by many authors (see Xiao [51] and the references therein).

3.2 Preliminaries

We need some preliminaries on the Malliavin calculus for the d-dimensional Brownian motion $W = \{W_t, t \geq 0\}$. We refer to Malliavin [32] and Nualart [38] for a more detailed presentation of this theory.

We assume that W is defined in a complete probability space $(\Omega, \mathscr{F}, P)$, and the σ-field $\mathscr{F}$ is generated by W. Let us denote by H the Hilbert space $L^2(\mathbb{R}_+; \mathbb{R}^d)$, and for any function $h \in H$ we set

$$W(h) = \sum_{i=1}^{d} \int_0^\infty h^i(t) dW_t^i.$$

Let $\mathscr{S}$ be the class of smooth and cylindrical random variables of the form

$$F = f(W(h_1), \dots, W(h_n)),$$

where $n \geq 1$, $h_1, \ldots, h_n \in H$, and f is an infinitely differentiable function such that together with all its partial derivatives has at most polynomial growth order. The derivative operator of the random variable F is defined as

$$D_t^i F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n))h_j^i(t),$$

where $i = 1, \ldots, d$ and $t \geq 0$. In this way, we interpret DF as a random variable with values in the Hilbert space H. The derivative is a closable operator on $L^2(\Omega)$ with values in $L^2(\Omega; H)$. We denote by $\mathbb{D}^{1,2}$ the Hilbert spaced defined as the completion of $\mathscr{S}$ with respect to the scalar product

$$\langle F, G \rangle_{1,2} = E(FG) + E\left(\sum_{i=1}^{d} \int_0^\infty D_t^i F D_t^i G dt\right).$$

The divergence operator δ is the adjoint of the derivative operator D. The operator δ is an unbounded operator from $L^2(\Omega; H)$ into $L^2(\Omega)$, and is determined by the duality relationship

$$E(\delta(u)F) = E(\langle u, DF \rangle_H),$$

for any u in the domain of δ, and F in $\mathbb{D}^{1,2}$. Gaveau and Trauber [14] proved that δ is an extension of the classical Itô integral in the sense that any d-dimensional square integrable adapted process belongs to the domain of δ, and $\delta(u)$ coincides with the Itô integral of u:

$$\delta(u) = \sum_{i=1}^{d} \int_0^\infty u^i(t) dW_t^i.$$

It is well-known that any random variable $F \in L^2(\Omega)$, possesses a stochastic integral representation of the form

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty u^i(t) dW_t^i,$$

for some d-dimensional square integrable adapted process u. Clark-Ocone formula says that if $F \in \mathbb{D}^{1,2}$, then

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty E(D_t^i F | \mathscr{F}_t) dW_t^i. \tag{2.1}$$

3.3 Exponential integrability of the self-intersection local time

Suppose that $W = \{W_t, t \geq 0\}$ is a d-dimensional standard Brownian motion, defined in a complete probability space $(\Omega, \mathscr{F}, P)$. Suppose that $\mathscr{F}$ is generated by W. We denote by $\{\mathscr{F}_t, t \geq 0\}$ the filtration generated by W and the sets of probability zero. Consider a d-dimensional Gaussian processs of the form

$$B_t = \int_0^t K(t,s) dW_s, \tag{3.1}$$

where $K(t,s)$ is a measurable kernel satisfying $\int_0^t K(t,s)^2 ds < \infty$ for all $t \geq 0$. We will assume that $K(t,s) = 0$ if $s > t$.

Fix a time interval $[0,T]$. We will make use of the following property on the kernel $K(t,s)$:

(H1) *For any* $s,t \in [0,T]$, $s < t$ *we have*

$$\int_s^t K(t,\theta)^2 d\theta \geq k_1 (t-s)^{2H} \tag{3.2}$$

for some constants $k_1 > 0$, *and* $H \in (0,1)$.

Notice that $\mathrm{Var}\left(B_t^i | \mathscr{F}_s\right) = \int_s^t K(t,\theta)^2 d\theta$, so condition **(H1)** is equivalent to say that $\mathrm{Var}\left(B_t^i | \mathscr{F}_s\right) \geq k_1 (t-s)^{2H}$, for each component $i = 1, \dots, d$. This property is satisfied, for instance, in the following two examples:

Example 1 Suppose that $K(t,s) = (t-s)^{H-\frac{1}{2}}$. Then, we have equality in (4.14) with $k_1 = \frac{1}{2H}$.

Example 2 Condition **(H1)** is satisfied by the kernel of the fractional Brownian motion, as a consequence of the local nondeterminism property (see (4.1) below).

We will denote by C a generic constant depending on T, the dimension d, and the constants appearing in the hypothesis such as H and k_1.

The *self-intersection local time* of the process B in the time interval $[0,T]$, denoted by L, is defined as the limit in L^2 as ε tends to zero of

$$L_\varepsilon = \int_0^T \int_0^t p_\varepsilon (B_t - B_s) ds, \tag{3.3}$$

where p_ε denotes the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2\varepsilon}\right).$$

The next theorem asserts that L exists if $Hd < 1$, and it has exponential moments of order $\frac{1}{Hd}$.

Theorem 3.3.1. *Suppose that $Hd < 1$. Then, the self-intersection local time L exists as the limit in L^2 of L_ε, as ε tends to zero, and for all integers $n \geq 1$ we have*

$$E(L^n) \leq C^n (n!)^{Hd},$$

for some constant C. As a consequence,

$$E(e^{L^p}) < \infty,$$

for any $p < \frac{1}{Hd}$, and there exists a constant $\lambda_0 > 0$ such that $E(e^{\lambda L^{\frac{1}{Hd}}}) < \infty$ for all $\lambda < \lambda_0$.

Proof From the equality

$$p_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(i \langle \xi, x \rangle - \frac{\varepsilon |\xi|^2}{2} \right) d\xi$$

and the definition of L_ε, we obtain

$$L_\varepsilon = \frac{1}{(2\pi)^d} \int_0^T \int_0^t \int_{\mathbb{R}^d} \exp\left(i \langle \xi, B_t - B_s \rangle - \frac{\varepsilon |\xi|^2}{2} \right) d\xi ds dt.$$

This expression allows us to compute the moments of L_ε. Fix an integer $n \geq 1$. Denote by T_n the set $\{0 < s < t < T\}^n$. Then

$$\begin{aligned} E(L_\varepsilon^n) &= \frac{1}{(2\pi)^{nd}} \int_{T_n} \int_{\mathbb{R}^{nd}} E\left[\exp\left(i \langle \xi_1, B_{t_1} - B_{s_1} \rangle + \cdots + i \langle \xi_n, B_{t_n} - B_{s_n} \rangle\right)\right] \\ &\quad \times \exp\left(-\frac{\varepsilon}{2} \sum_{j=1}^n |\xi_j|^2 \right) d\xi_1 \cdots d\xi_n ds dt, \end{aligned} \tag{3.4}$$

where $s=(s_1,\ldots,s_n)$ and $t=(t_1,\ldots,t_n)$. Notice that

$$\begin{aligned}
& \int_{\mathbb{R}^{nd}} E\left[\exp\left(i\langle\xi_1, B_{t_1}-B_{s_1}\rangle+\cdots+i\langle\xi_n, B_{t_n}-B_{s_n}\rangle\right)\right] \\
& \times e^{-\frac{\varepsilon}{2}\sum_{j=1}^n|\xi_j|^2} d\xi_1\cdots d\xi_n \\
= \ & \int_{\mathbb{R}^{nd}} \exp\left(-\frac{1}{2}E\left[\left(\langle\xi_1, B_{t_1}-B_{s_1}\rangle+\cdots+\langle\xi_n, B_{t_n}-B_{s_n}\rangle\right)^2\right]\right) \\
& \times e^{-\frac{\varepsilon}{2}\sum_{j=1}^n|\xi_j|^2} d\xi_1\cdots d\xi_n \\
= \ & \left(\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\xi^T Q\xi\right) e^{-\frac{\varepsilon}{2}|\xi|^2} d\xi\right)^d, \qquad (3.5)
\end{aligned}$$

where Q is the covariance matrix of the n-dimensional random vector $(B^1_{t_1}-B^1_{s_1},\ldots,B^1_{t_n}-B^1_{s_n})$. Substituting (3.5) into (3.4) yields

$$E(L^n_\varepsilon)=\frac{1}{(2\pi)^{nd}}\int_{T_n}\left(\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\xi^T Q\xi\right)e^{-\frac{\varepsilon}{2}|\xi|^2}d\xi\right)^d dsdt,$$

and $E(L^n_\varepsilon)$ converges as ε tends to zero to

$$\begin{aligned}
\alpha_n &= \frac{1}{(2\pi)^{nd}}\int_{T_n}\left(\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\xi^T Q\xi\right)d\xi\right)^d dsdt \\
&= \frac{1}{(2\pi)^{\frac{nd}{2}}}\int_{T_n}(\det Q)^{-\frac{d}{2}}\, dsdt,
\end{aligned}$$

provided α_n is finite.

If $\alpha_2<\infty$, then in the same way as before we obtain

$$\lim_{\varepsilon,\delta\downarrow 0}E(L_\varepsilon L_\delta)=\alpha_2,$$

which implies that L_ε converges in L^2 as ε tends to zero. Furthermore, if α_n is finite for all $n\geq 1$, then we deduce the convergence in L^p for any $p\geq 2$ of L_ε as ε tends to zero. The limit, denoted by L, will be, by definition, the self-intersection local time of the process B in the time interval $[0,T]$. To complete the proof of the theorem it suffices to show that α_n is bounded by $C^n(n!)^{Hd}$, for some constant C.

We can write

$$\alpha_n = \frac{n!}{(2\pi)^{\frac{nd}{2}}} \int_{T_n \cap \{t_1 < \cdots < t_n\}} (\det Q)^{-\frac{d}{2}} \, dsdt.$$

For each $i = 1, \ldots, n$ we denote by τ_i the point in the set $\{s_i, s_{i+1}, \ldots, s_n, t_{i-1}\}$ which is closer to t_i from the left. Then, by **(H1)** and the fact that $s_i < t_i$, $i = 1, \ldots, n$, we obtain, using Lemma 3.5.1 in the Appendix,

$$\begin{aligned}
\det Q &= \mathrm{Var}(B^1_{t_1} - B^1_{s_1})\mathrm{Var}(B^1_{t_2} - B^1_{s_2} | B^1_{t_1} - B^1_{s_1}) \\
&\quad \times \cdots \times \mathrm{Var}(B^1_{t_n} - B^1_{s_n} | B^1_{t_1} - B^1_{s_1}, \ldots, B^1_{t_{n-1}} - B^1_{s_{n-1}}) \\
&\geq \mathrm{Var}(B^1_{t_1} | B^1_{s_1})\mathrm{Var}(B^1_{t_2} | B^1_{t_1}, B^1_{s_1}, B^1_{s_2}) \\
&\quad \times \cdots \times \mathrm{Var}(B^1_{t_n} | B^1_{t_1}, B^1_{s_1}, \ldots, B^1_{t_{n-1}}, B^1_{s_{n-1}}, B^1_{s_n}) \\
&\geq \mathrm{Var}(B^1_{t_1} | \mathscr{F}_{\tau_1})\mathrm{Var}(B^1_{t_2} | \mathscr{F}_{\tau_2}) \cdots \mathrm{Var}(B^1_{t_n} | \mathscr{F}_{\tau_n}) \\
&\geq k_1^n (t_1 - \tau_1)^{2H} (t_2 - \tau_2)^{2H} \cdots (t_n - \tau_n)^{2H}.
\end{aligned}$$

As a consequence,

$$\alpha_n \leq \frac{n!}{(2\pi)^{\frac{nd}{2}}} k_1^{-\frac{nd}{2}} \int_{T_n \cap \{t_1 < \cdots < t_n\}} \prod_{i=1}^{n} (t_i - \tau_i)^{-Hd} dsdt.$$

If we fix the points $t_1 < \cdots < t_n$, there are $3 \times 5 \times \cdots \times (2n-1) = (2n-1)!!$ posible ways to place the points $s_1, \ldots, s_n$. In fact, s_1 must be in $(0, t_1)$. For s_2 we have three choices: $(0, s_1)$, (s_1, t_1) and (t_1, t_2). By a recursive argument it is clear that we have $(2i-1)$ possible choices for s_i, given $s_1, \ldots, s_{i-1}$. In this way, up to a set of measure zero, we can decompose the set $T_n \cap \{t_1 < \cdots < t_n\}$ into the union of $(2n-1)!!$ disjoint subsets. The integral of $\prod_{i=1}^n (t_i-\tau_i)^{-Hd}$ on each one of these subset can be expressed as

$$\Phi_\sigma = \int_{\{0<z_1<\cdots<z_{2n}<T\}} \prod_{i=1}^n (z_{\sigma(i)} - z_{\sigma(i)-1})^{-Hd} dz,$$

where $\sigma(1)<\cdots<\sigma(n)$ are n elements in $\{1,2,\ldots,2n\}$, and $z=(z_1,\ldots,z_{2n})$. Making the change of variables $y_i = z_i - z_{i-1},\ i=1,\ldots,2n$ (with the convention $z_0=0$) we obtain

$$\begin{aligned}\Phi_\sigma &= \int_{\{0<y_1+\cdots+y_{2n}<T\}} \prod_{i=1}^n y_{\sigma(i)}^{-Hd} dy \le \frac{T^n}{n!} \int_{\{0<y_1+\cdots+y_n<T\}} \prod_{i=1}^n y_i^{-Hd} dy \\ &= \frac{1}{n!} T^{n(2-Hd)+Hd} \frac{\Gamma(1-Hd)^{n-1}}{\Gamma(n(1-Hd)+Hd+1)}.\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_n &\le \frac{k_1^{-\frac{nd}{2}}(2n-1)!!T^{n(2-Hd)+Hd}\Gamma(1-Hd)^{n-1}}{(2\pi)^{\frac{nd}{2}}\Gamma(n(1-Hd)+Hd+1)} \\ &= C_1 C_2^n \frac{(2n-1)!!}{\Gamma(n(1-Hd)+Hd+1)},\end{aligned}$$

with $C_1 = T^{Hd}\Gamma(1-Hd)^{-1}$, and $C_2 = \frac{k_1^{-\frac{d}{2}}\Gamma(1-Hd)T^{2-Hd}}{(2\pi)^{\frac{d}{2}}}$. Taking into account that $(2n-1)!! \le 2^{n-1}n!$, and that

$$\Gamma(n(1-Hd)+Hd+1) \ge C^n (n!)^{1-Hd},$$

for some constant C, we obtain the desired estimate.

If $Hd \ge 1$, the above result is no longer true. In that case the expectation of L_ε blows up as ε tends to zero. In fact, if we denote $\sigma^2(s,t) = \mathrm{Var}(B_t^1 - B_s^1)$, for $s<t$,

then

$$E(L_\varepsilon) = \int_0^T \int_0^t p_{\varepsilon+\sigma^2(s,t)}(0)dsdt = (2\pi)^{-\frac{d}{2}} \int_0^T \int_0^t (\varepsilon + \sigma^2(s,t))^{-\frac{d}{2}} dsdt,$$

which converges to

$$(2\pi)^{-\frac{d}{2}} \int_0^T \int_0^t \sigma^2(s,t)^{-\frac{d}{2}} dsdt \geq (2\pi)^{-\frac{d}{2}} k_1^{-\frac{d}{2}} \int_0^T \int_0^t (t-s)^{-Hd} dsdt = \infty.$$

In this case, one can study the existence of the renormalized self-intersection local time defined as the limit as ε tends to zero of $L_\varepsilon - E(L_\varepsilon)$. In the next section we discuss the existence and exponential moments of the renormalized self-intersection local time, using Clark-Ocone formula, in the case of the fractional Brownian motion.

3.4 Renormalized self-intersection local time of the fBm

The fractional Brownian motion on $\mathbb{R}^d$ with Hurst parameter $H \in (0,1)$ is a d-dimensional Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with zero mean and covariance function given by (1.1). We will assume that $d \geq 2$.

It is well-known that B^H possesses the following integral representation

$$B_t^H = \int_0^t K_H(t,s) dW_s,$$

where $W = \{W_t, t \geq 0\}$ is a d-dimensional Brownian motion, and $K_H(s,t)$ is the square integrable kernel given by

$$K_H(t,s) = C_{H,1} s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

if $H > \frac{1}{2}$, and by

$$K_H(t,s) = C_{H,2}\left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right],$$

if $H < \frac{1}{2}$, for any $s < t$, where the constants are $C_{H,1} = \left[\frac{H(2H-1)}{B(2-2H,H-\frac{1}{2})}\right]^{\frac{1}{2}}$, and $C_{H,2} = \left[\frac{2H}{(1-2H)b(1-2H,H+\frac{1}{2})}\right]^{\frac{1}{2}}$, where $B(\alpha,\beta)$ denotes th beta function.

The processes B^H and W generate the same filtration, that is, $\mathscr{F}_t = \sigma\{W_s, 0 \le s \le t\} = \sigma\{B^H_s, 0 \le s \le t\}$.

The fractional Brownian motion satisfies the following local nondeterminism property:

(LND) *There exists a constant* $k_2 > 0$*, depending only on* H *and* T*, such that for any* $t \in [0,T]$, $0 < r < t \wedge (T-t)$ and for $i = 1, \dots, d$,

$$\mathrm{Var}(B_t^{H,i} | B_s^{H,i} : |s-t| \ge r) \ \ge k_2 \, r^{2H}. \tag{4.1}$$

Consider the approximated self-intersection local time L_ε introduced in (3.1). From the general result proved in Section 2 it follows that if $Hd < 1$, then L_ε converges in L^2 to the self-intersection local time L, and the random variable L has exponential moments. If $Hd \ge 1$, this result is no longer true, and one considers the renormalization of the self-intersection local time, introduced by Varadhan.

The purpose of this section is to apply the Clark-Ocone formula to provide a stochastic integral representation for the renormalized self-intersection local time $\widetilde{L}$. As a consequence, we will prove the existence of some exponential moments for the random variable $\widetilde{L}$.

Theorem 3.4.1. *Suppose that $H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$. Then the renormalized self-intersection local time of the d-dimensional fractional Brownian motion B^H exists in L^2 and it has the following integral representation*

$$\widetilde{L} = -\sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_0^t \frac{A^i_{r,t,s}}{\sigma^2_{r,s,t}} p_{\sigma^2_{r,s,t}}(A^i_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right] dsdt \right) dW^i_r, \tag{4.2}$$

where

$$A_{r,t,s} = E(B^H_t - B^H_s | \mathscr{F}_r)$$

and

$$\sigma^2_{r,s,t} = \mathrm{Var}(B^{H,i}_t - B^{H,i}_s | \mathscr{F}_r).$$

Proof The proof will be done in several steps.

Step 1 We are going to apply Clark-Ocone formula to the random variable L_ε. It is clear that L_ε belongs to $\mathbb{D}^{1,2}$, and its derivative can be computed as follows

$$D^i_r L_\varepsilon = \int_0^T \int_0^t \frac{\partial p_\varepsilon}{\partial x_i}(B^H_t - B^H_s) D^i_r \left(B^{H,i}_t - B^{H,i}_s\right) dsdt,$$

where $r \in [0,T]$, and $i = 1, \ldots, d$. Using

$$D^i_r \left(B^{H,i}_t - B^{H,i}_s\right) = \left[K_H(t,r) - K_H(s,r)\right] \mathbf{1}_{[0,t]}(r),$$

we obtain

$$D^i_r L_\varepsilon = \int_r^T \int_0^t \frac{\partial p_\varepsilon}{\partial x_i}(B^H_t - B^H_s) \left[K_H(t,r) - K_H(s,r)\right] dsdt. \tag{4.3}$$

The next step is to compute the conditional expectation $E(D^i_r L_\varepsilon | \mathscr{F}_r)$. The conditional law of $B^H_t - B^H_s$ given $\mathscr{F}_r$ is normal with mean $A_{r,t,s}$ and covariance matrix $\sigma^2_{r,s,t} I_d$, where I_d is the d-dimensional identity matrix. Hence, the conditional expectation

$E\left(\frac{\partial p_\varepsilon}{\partial x_i}(B_t^H - B_s^H)|\mathscr{F}_r\right)$ is given by

$$\begin{aligned}
E\left(\frac{\partial p_\varepsilon}{\partial x_i}(B_t^H - B_s^H)|\mathscr{F}_r\right) &= \int_{\mathbb{R}^d} \frac{\partial p_\varepsilon}{\partial x_i}(y) p_{\sigma^2_{r,s,t}}(y - A_{r,t,s}) dy \\
&= \frac{\partial p_{\varepsilon+\sigma^2_{r,s,t}}}{\partial x_i}(A_{r,t,s}) \\
&= -\frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}).
\end{aligned}$$

As a consequence, from (3.2) we obtain

$$E\left(D_r^i L_\varepsilon|\mathscr{F}_r\right) = -\int_r^T \int_0^t \frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right] dsdt,$$

and this leads to the following integral representation for $L_\varepsilon - E(L_\varepsilon)$

$$\begin{aligned}
& L_\varepsilon - E(L_\varepsilon) \\
= & -\sum_{i=1}^d \int_0^T \left(\int_r^T \int_0^t \frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right] dsdt\right) dW_r^i.
\end{aligned}$$

Step 2 In order to pass to the limit as ε tends to zero we proceed as follows. Set

$$\Sigma^i_\varepsilon(r,t,s) = \frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right]. \qquad (4.4)$$

Clearly, $\Sigma^i_\varepsilon(r,t,s)$ converges pointwise as ε tends to zero to

$$\Sigma^i(r,t,s) = \frac{A^i_{r,t,s}}{\sigma^2_{r,s,t}} p_{\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right].$$

In order to establish the convergence of the integrals in the variables s and t, we will first decompose the interval $[0,t]$ into the disjoint union of $[r,t]$ and $[0,r)$. In this way we obtain

$$L_\varepsilon - E(L_\varepsilon) = L_\varepsilon^{(1)} + L_\varepsilon^{(2)},$$

where

$$L_\varepsilon^{(1)} = -\sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_r^t \Sigma_\varepsilon^i(r,t,s) ds dt \right) dW_r^i,$$

and

$$L_\varepsilon^{(2)} = -\sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_0^r \Sigma_\varepsilon^i(r,t,s) ds dt \right) dW_r^i.$$

Step 3 We claim that the random field $\Sigma_\varepsilon^i(r,t,s)$ is uniformly bounded on the set $0 < r < s < t$ by an integrable function not depending on ε. In fact, using the local nondeterminism property **(LND)**, and Lemma 3.5.1 in the Appendix, we obtain the following lower bound for the conditional variance $\sigma_{r,s,t}^2 = \mathrm{Var}(B_t^{H,i} - B_s^{H,i} | \mathscr{F}_r)$:

$$\sigma_{r,s,t}^2 \geq \mathrm{Var}(B_t^{H,i} - B_s^{H,i} | \mathscr{F}_s) = \mathrm{Var}(B_t^{H,i} | \mathscr{F}_s) \geq k_2 (t-s)^{2H}. \tag{4.5}$$

We can get rid off the factor $A_{r,t,s}^i$ in the expression (4.4) of $\Sigma_\varepsilon^i(r,t,s)$ using the inequality

$$p_t(x) \leq C \frac{t^{-\frac{d}{2}+\frac{1}{2}}}{|x|} e^{-\frac{|x|^2}{4t}} \leq C \frac{t^{-\frac{d}{2}+\frac{1}{2}}}{|x|}, \tag{4.6}$$

for some constant $C > 0$. In this way we obtain, using (5.1) and (5.2)

$$\left| \Sigma_\varepsilon^i(r,t,s) \right| \leq C (t-s)^{-Hd-H} \left| K_H(t,r) - K_H(s,r) \right|, \tag{4.7}$$

for some constant $C > 0$, and by Lemma 3.5.3 in the Appendix we obtain that

$$\int_r^T \int_r^t (t-s)^{-Hd-H} \left| K_H(t,r) - K_H(s,r) \right| ds dt \leq C (r^{\frac{1}{2}-H} \vee 1). \tag{4.8}$$

By dominated convergence we deduce the convergence of the integrals

$$\lim_{\varepsilon\downarrow 0}\int_r^T\int_r^t \Sigma_\varepsilon^i(r,t,s)dsdt=\int_r^T\int_r^t \Sigma^i(r,t,s)dsdt$$

for all $(r,\omega)\in[0,T]\times\Omega$, and a second application of the dominated convergence theorem yields that $\int_r^T\int_r^t \Sigma_\varepsilon^i(r,t,s)dsdt$ converges in $L^2([0,T]\times\Omega)$ to $\int_r^T\int_r^t \Sigma^i(r,t,s)dsdt$. This implies the convergence of $L_\varepsilon^{(1)}$ to

$$-\sum_{i=1}^d\int_0^T\left(\int_r^T\int_r^t \Sigma^i(r,t,s)dsdt\right)dW_r^i$$

in $L^2(\Omega)$ as ε tends to zero.

Step 4 Consider now the case $s<r<t$. In this case the integral of the term $\Sigma_\varepsilon^i(r,t,s)$ is not necessarily bounded, and in order to show the convergence of $L_\varepsilon^{(2)}$ we will prove uniform bounds in ε for the expectation $E\left(\int_r^T\int_r^t \left|\Sigma_\varepsilon^i(r,t,s)\right|^p dsdt\right)$, for some $p>1$. We can write for $s<r<t$, using the first inequality in (5.2)

$$\begin{aligned}\left|\Sigma_\varepsilon^i(r,t,s)\right| &\le \frac{|A_{r,t,s}|}{\left(\varepsilon+\sigma_{r,s,t}^2\right)}p_{\varepsilon+\sigma_{r,s,t}^2}(A_{r,t,s})\,|K_H(t,r)|\\ &= (2\pi)^{-\frac{d}{2}}\frac{|A_{r,t,s}|}{\left(\varepsilon+\sigma_{r,s,t}^2\right)^{1+\frac{d}{2}}}\exp\left(-\frac{|A_{r,t,s}|^2}{2(\varepsilon+\sigma_{r,s,t}^2)}\right)|K_H(t,r)|\\ &\le C\left(\varepsilon+\sigma_{r,s,t}^2\right)^{-\frac{d+1}{2}}\exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon+\sigma_{r,s,t}^2)}\right)|K_H(t,r)|\,, \qquad (4.9)\end{aligned}$$

for some constant $C>0$. If $s<r<t$, using the local nondeterminism property **(LND)** we obtain the following lower bound for the conditional variance $\sigma_{r,s,t}^2$:

$$\sigma_{r,s,t}^2=\mathrm{Var}(B_t^{H,i}-B_s^{H,i}|\mathscr{F}_r)=\mathrm{Var}(B_t^{H,i}|\mathscr{F}_r)\ge k_2(t-r)^{2H}. \qquad (4.10)$$

On the other hand, if $s < r < t$

$$\begin{aligned}\sigma_{r,s,t}^2 &= \mathrm{Var}(B_t^{H,i} - B_s^{H,i} | \mathscr{F}_r) = \mathrm{Var}(B_t^{H,i} - B_r^{H,i} | \mathscr{F}_r) \\ &\leq \mathrm{Var}(B_t^{H,i} - B_r^{H,i}) = (t-r)^{2H}. \end{aligned} \tag{4.11}$$

Also we will make use of the estimate (see [20])

$$|K_H(t,r)| \leq k_3 (t-r)^{H-\frac{1}{2}} r^{\frac{1}{2}-H}. \tag{4.12}$$

Substituting the estimates (4.10), (4.11) and (4.12) into (4.9) yields

$$\left|\Sigma_\varepsilon^i(r,t,s)\right| \leq C r^{\frac{1}{2}-H} \Psi_\varepsilon(r,t,s), \tag{4.13}$$

for some constant C, where

$$\Psi_\varepsilon(r,t,s) = \left(\varepsilon + k_2 (t-r)^{2H}\right)^{-\frac{d+1}{2}} (t-r)^{H-\frac{1}{2}} \exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon + (t-r)^{2H})}\right). \tag{4.14}$$

Notice that if $Hd < \frac{1}{2}$, then $\left|\Sigma_\varepsilon^i(r,t,s)\right|$ is uniformly bounded by the integrable function $C r^{\frac{3}{2}-H} \, (t-r)^{-Hd-\frac{1}{2}}$, and we can conclude as in Step 3. For this reason, we can assume that $Hd \geq \frac{1}{2}$.

We claim that for some $p > 1$, we have

$$\sup_{\varepsilon > 0} E\left(\int_r^T \int_0^r \Psi_\varepsilon^p(r,t,s) ds dt\right) < \infty. \tag{4.15}$$

To show this estimate we first derive a lower bound for the expectation of $|A_{r,t,s}^1|^2 = \left[E(B_t^{H,1} - B_s^{H,1} | \mathscr{F}_r)\right]^2$. The main idea is to add and substract the term $B_r^{H,1}$, and then neglect the expectation $E\left(\left(\left(E(B_t^{H,1} | \mathscr{F}_r) - B_r^{H,1}\right)^2\right)\right)$. This argument will be used later to find a lower bound for the covariance matrix of the vector $\left(E(B^{H,1}_{t_i}-B^{H,1}_{s_i}|\mathscr{F}_r), 1\le i\le n\right)$.

$$
\begin{aligned}
E\left(|A^1_{r,t,s}|^2\right) &= E\left(\left(E(B^{H,1}_t-B^{H,1}_s|\mathscr{F}_r)\right)^2\right)\\
&= E\left(\left(E(B^{H,1}_t|\mathscr{F}_r)-B^{H,1}_r\right)^2\right)\\
&\quad +2E\left(\left(E(B^{H,1}_t|\mathscr{F}_r)-B^{H,1}_r\right)\left(B^{H,1}_r-B^{H,1}_s\right)\right)+E\left(\left(B^{H,1}_r-B^{H,1}_s\right)^2\right)\\
&\ge 2E\left(\left(B^{H,1}_t-B^{H,1}_r\right)\left(B^{H,1}_r-B^{H,1}_s\right)\right)+E\left(\left(B^{H,1}_r-B^{H,1}_s\right)^2\right)\\
&= E\left(\left(B^{H,1}_t-B^{H,1}_s\right)^2\right)-E\left(\left(B^{H,1}_t-B^{H,1}_r\right)^2\right)\\
&= (t-s)^{2H}-(t-r)^{2H}.
\end{aligned}
$$

As a consequence, we obtain, assuming $p<2$

$$
\begin{aligned}
&E\left(\exp\left(-\frac{p|A_{r,t,s}|^2}{4(\varepsilon+(t-r)^{2H})}\right)\right)\\
&\quad= \left(1+\frac{p}{2}(\varepsilon+(t-r)^{2H})^{-1}E\left(|A^1_{r,t,s}|^2\right)\right)^{-\frac{d}{2}}\\
&\quad\le \left(1+\frac{p}{2}(\varepsilon+(t-r)^{2H})^{-1}\left[(t-s)^{2H}-(t-r)^{2H}\right]\right)^{-\frac{d}{2}}\\
&\quad= (\varepsilon+(t-r)^{2H})^{\frac{d}{2}}\\
&\qquad\times\left(\varepsilon+\left(1-\frac{p}{2}\right)(t-r)^{2H}+\frac{p}{2}(t-s)^{2H}\right)^{-\frac{d}{2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&E\left(\exp\left(-\frac{p|A_{r,t,s}|^2}{4(\varepsilon+(t-r)^{2H})}\right)\right)\\
&\quad\le C(\varepsilon+(t-r)^{2H})^{\frac{d}{2}}(t-r)^{-2H\alpha}(t-s)^{-2H\beta},
\end{aligned}
\tag{4.16}
$$

where $\alpha+\beta=\frac{d}{2}$. Substituting (4.16) into (4.14) yields

$$
\begin{aligned}
E\left(\int_r^T\int_0^r \Psi_\varepsilon^p(r,t,s)dsdt\right) &\le C\int_r^T\int_0^r \left(\varepsilon+(t-r)^{2H}\right)^{-\frac{d+1}{2}p+\frac{d}{2}-\alpha} \\
&\quad \times (t-r)^{\left(H-\frac{1}{2}\right)p}(t-s)^{-\beta 2H}dsdt \\
&\le C\int_r^T\int_0^r (t-r)^{-pHd-\frac{p}{2}+2H\beta}\,(t-s)^{-2H\beta}dsdt.
\end{aligned}
$$

If $Hd>1$, we can choose β such that $2H\beta>1$, and integrating in the variable s, the above integral is bounded by

$$
C\int_r^T (t-r)^{-pHd-\frac{p}{2}+1}dt,
$$

which is finite it $p>1$ satisfyes $\left(Hd+\frac{1}{2}\right)p<2$ (this is possible because $Hd+\frac{1}{2}<2$). If $Hd\le 1$, we can choose β such that $2H\beta=Hd-\delta$, for any $\delta>0$, and we obtain the bound

$$
C\int_r^T (t-r)^{-pHd-\frac{p}{2}+Hd-\delta}dt,
$$

which is again finite if $p>1$ is close to one, and $\delta>0$ is small enough.

As a consequence, from (4.13) and (4.15), for any fixed $r\in[0,T]$, the family of functions $\left\{\Sigma_\varepsilon^i(r,t,s),\varepsilon>0\right\}$, is uniformly integrable in $[r,T]\times[0,r]$, so it converges in $L^1([r,T]\times[0,r])\times\Omega$ to $\Sigma^i(r,t,s)$, for $i=1,\dots,d$. This implies the convergence of the integrals

$$
\lim_{\varepsilon\downarrow 0}\int_r^T\int_0^r \Sigma_\varepsilon^i(r,t,s)dsdt=\int_r^T\int_0^r \Sigma^i(r,t,s)dsdt,
$$

for each fixed $r\in[0,T]$ in $L^1(\Omega)$.

Finally, we claim that this convergence also holds in $L^2([0,T]\times\Omega)$, and this implies the convergence of $L_\varepsilon^{(2)}$ to

$$-\sum_{i=1}^{d}\int_0^T\left(\int_r^T\int_0^r \Sigma^i(r,t,s)dsdt\right)dW_r^i$$

in $L^2(\Omega)$ as ε tends to zero. To show the convergence in $L^2([0,T]\times\Omega)$ of the integrals

$$Y_\varepsilon^i(r)=\int_r^T\int_0^r \Sigma_\varepsilon^i(r,t,s)dsdt$$

it suffices to prove that

$$\sup_{\varepsilon>0}\int_0^T E\left(\left|Y_\varepsilon^i(r)\right|^p\right)dr<\infty \qquad (4.17)$$

for all $i=1,\dots,d$ and for some $p>2$. The proof of (4.17) will be the last step in the proof of this theorem.

Step 5 Suppose first that $Hd<1$. Then, from (4.13) we obtain

$$\int_0^T E\left(\left|Y_\varepsilon^i(r)\right|^p\right)dr\le C\int_0^T E\left[\left(\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^p\right]r^{p\left(\frac{1}{2}-H\right)}dr.$$

Using (4.14) and Minkowski's inequality yields

$$\begin{aligned}\left\|\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right\|_p \;&\le\; \int_r^T\int_0^r\left(\varepsilon+k_2\,(t-r)^{2H}\right)^{-\frac{d+1}{2}}(t-r)^{H-\frac{1}{2}}\\ &\quad\times\left\|\exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon+(t-r)^{2H})}\right)\right\|_p dsdt, \qquad (4.18)\end{aligned}$$

and from (4.16), choosing $\beta=\frac{d}{2}$, we get

$$\left\|\exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon+(t-r)^{2H})}\right)\right\|_p\le C(\varepsilon+(t-r)^{2H})^{\frac{d}{2p}}(t-s)^{-\frac{Hd}{p}}. \qquad (4.19)$$

Substituting (4.19) into (4.18) yields

$$\left\| \int_r^T \int_0^r \Psi_\varepsilon(r,t,s)dsdt \right\|_p \leq C \int_r^T (t-r)^{-Hd-\frac{1}{2}+\frac{Hd}{p}} dr,$$

which is finite if we choose $p > 2$ such that $p < \frac{2Hd}{2Hd-1}$. Finally, if $p\left(\frac{1}{2} - H\right) > -1$ we complete the proof of (4.17) in the case $Hd < 1$.

In the case $Hd \geq 1$ we cannot apply the previous arguments, and the proof of (4.17) follows from the moment estimates given in Proposition 3.4.2.

Remark 1 Theorem 3.4.1 also provides an alternative proof of the existence of the self-intersection local time in the case $H \in [\frac{1}{d}, \min(\frac{3}{2d}, \frac{2}{d+1}))$, which was proved by Hu and Nualart in [23] in the general case $Hd < \frac{3}{2}$. Notice that for $d \geq 3$, the condition $H \in [\frac{1}{d}, \min(\frac{3}{2d}, \frac{2}{d+1}))$ is equivalent to $1 \leq Hd < \frac{3}{2}$, and for $d = 2$ we require $H < \frac{2}{3}$, instead of the more general condition $H < \frac{3}{4}$, that guarantees the existence of the renormalized local time (see [41] and [23]).

The next Proposition contains the basic estimates on the moments of the quadratic variation of the stochastic integral appearing in the representation of the renormalized self-intersection local time.

Proposition 3.4.2. *Assume* $1 \leq Hd < \frac{3}{2}$. *Set*

$$\Lambda_\varepsilon(r) = \int_r^T \int_0^r \Psi_\varepsilon(r,t,s)dsdt,$$

where $\Psi_\varepsilon(r,t,s)$ *has been defined in (4.14). Then, for any integer* $n \geq 1$,

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq C^n (n!)^\gamma,$$

for some constant $C > 0$, where

$$\gamma > \left(\frac{1}{2} + H\right)\left(d - \frac{1}{2H}\right).$$

Proof Set $g_\varepsilon(t-r) = \left(\varepsilon + k_2\,(t-r)^{2H}\right)^{-\frac{d+1}{2}}(t-r)^{H-\frac{1}{2}}$. We have

$$\begin{aligned}
E\left(\Lambda_\varepsilon^n(r)\right) &= E\left[\left(\int_r^T \int_0^r g_\varepsilon(t-r)\exp\left(-\frac{|A_{r,s,t}|^2}{4(\varepsilon+(t-r)^{2H})}\right)dsdt\right)^n\right] \\
&= n!\int_{[r,T]^n}\int_{S_n}\prod_{i=1}^n g_\varepsilon(t_i-r) \\
&\quad \times\left(E\left(\exp\left(-\sum_{i=1}^n \frac{|A^1_{r,s_i,t_i}|^2}{4(\varepsilon+(t_i-r)^{2H})}\right)\right)\right)^d dsdt, \qquad (4.20)
\end{aligned}$$

where $S_n = \{0 < s_1 < \cdots < s_n < r\}$, $s = (s_1,\ldots,s_n)$ and $t = (t_1,\ldots,t_n)$.

We denote by Q the covariance matrix of the vector

$$\left(E(B^{H,1}_{t_1} - B^{H,1}_{s_1}|\mathscr{F}_r),\ldots,E(B^{H,1}_{t_n} - B^{H,1}_{s_n}|\mathscr{F}_r)\right).$$

Then, a well-known formula for Gaussian random variables implies that

$$\begin{aligned}
E\left[\exp\left(-\sum_{i=1}^n \frac{|A^1_{r,s_i,t_i}|^2}{4(\varepsilon+(t_i-r)^{2H})}\right)\right] &= \det\left(I+\frac{1}{2}QD^{-1}\right)^{-\frac{1}{2}} \\
&= 2^{\frac{n}{2}}\prod_{i=1}^n\sqrt{a_i}\det(2D+Q)^{-\frac{1}{2}}, \qquad (4.21)
\end{aligned}$$

where D denotes the $n\times n$ diagonal matrix with entries $a_i=\varepsilon+(t_i-r)^{2H}$. As in the computation of $E\left(|A^1_{r,t,s}|^2\right)$, adding and substracting the term $B^{H,1}_r$ yields

$$\begin{aligned} Q_{ij} &= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{s_i}|\mathscr{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{s_j}|\mathscr{F}_r)\right) \\ &= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{r}|\mathscr{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{r}|\mathscr{F}_r)\right) \\ &\quad +E\left((B^{H,1}_{r}-B^{H,1}_{s_i})(B^{H,1}_{t_j}-B^{H,1}_{r})\right)+E\left((B^{H,1}_{t_i}-B^{H,1}_{r})(B^{H,1}_{r}-B^{H,1}_{s_j})\right) \\ &\quad +E\left((B^{H,1}_{r}-B_{s_i})(B^{H,1}_{r}-B^{H,1}_{s_j})\right) \\ &= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{r}|\mathscr{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{r}|\mathscr{F}_r)\right) \\ &\quad -E\left((B^{H,1}_{t_i}-B^{H,1}_{r})(B^{H,1}_{t_j}-B^{H,1}_{r})\right)+E\left((B^{H,1}_{t_i}-B^{H,1}_{s_i})(B^{H,1}_{t_j}-B^{H,1}_{s_j})\right). \end{aligned}$$

Hence, we obtain

$$Q=R-N+M,$$

where

$$\begin{aligned} R_{ij} &= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{r}|\mathscr{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{r}|\mathscr{F}_r)\right), \\ M_{ij} &= E\left((B^{H,1}_{t_i}-B^{H,1}_{s_i})(B^{H,1}_{t_j}-B^{H,1}_{s_j})\right), \\ N_{ij} &= E\left((B^{H,1}_{t_i}-B^{H,1}_{r})(B^{H,1}_{t_j}-B^{H,1}_{r})\right). \end{aligned}$$

All these matrices are nonnegative definite. The main idea will be to get rid off the matrix R, and control the matrix N by its diagonal elements which are

$$N_{ii}=(t_i-r)^{2H}.$$

Indeed, the matrix N is nonnegative definite and, hence, it safisties the inequality

$$N \leq nD_N, \tag{4.22}$$

where D_N is a diagonal matrix whose entries are N_{ii}. Therefore,

$$Q \geq -N + M \geq -nD_N + M,$$

and for any $1 \leq \delta < 2$, we can write

$$\det(2D + Q) \geq \det(2D + \frac{2-\delta}{n}Q) \leq \det(2D - (2-\delta)D_N + \frac{2-\delta}{n}M). \tag{4.23}$$

The entries of the diagonal matrix $D_1 = 2D - (2-\delta)D_N$ are the positive numbers

$$2\varepsilon + \delta(t_i - r)^{2H} > 0.$$

From (4.9), (4.10) and (4.11) we obtain

$$\begin{aligned} E\left(\Lambda_\varepsilon^n(r)\right) &\leq 2^{\frac{nd}{2}} n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^n \left(g_\varepsilon(t_i - r) a_i^{\frac{d}{2}} \right) \\ &\quad \times \det(D_1 + \frac{2-\delta}{n}M)^{-\frac{d}{2}} ds dt. \end{aligned}$$

We have

$$\det(D_1 + \frac{2-\delta}{n}M)^{-\frac{d}{2}} \leq \left(\frac{n}{2-\delta}\right)^{n\beta} (\det D_1)^{-\alpha} (\det M)^{-\beta},$$

where $\alpha+\beta=\frac{d}{2}$. Hence,

$$E\left(\Lambda_\varepsilon^n(r)\right) \le \left(\frac{n}{2-\delta}\right)^{n\beta} 2^{\frac{nd}{2}} n! \int_{[r,T]^n}\int_{S_n}\prod_{i=1}^n\left(g_\varepsilon(t_i-r)a_i^{\frac{d}{2}}\left(2\varepsilon+\delta(t_i-r)^{2H}\right)^{-\alpha}\right) \times(\det M)^{-\beta}dsdt.$$

Then,

$$\begin{aligned} & g_\varepsilon(t_i-r)a_i^{\frac{d}{2}}\left(2\varepsilon+2(t_i-r)^{2H}\right)^{-\alpha} \\ = & \left(\varepsilon+k_2(t_i-r)^{2H}\right)^{-\frac{d+1}{2}}(t_i-r)^{H-\frac{1}{2}}\left(\varepsilon+(t_i-r)^{2H}\right)^{\frac{d}{2}}\left(2\varepsilon+2(t_i-r)^{2H}\right)^{-\alpha} \\ \le & C(t_i-r)^{-\frac{1}{2}-2H\alpha}, \end{aligned}$$

for some constant $C>0$. Thus

$$E\left(\Lambda_\varepsilon^n(r)\right) \le C^n n^{\beta n} n! \int_{[r,T]^n}\int_{S_n}\prod_{i=1}^n (t_i-r)^{-\frac{1}{2}-2H\alpha}(\det M)^{-\beta}dsdt, \tag{4.24}$$

for some constant $C>0$.

Applying Lemma 3.5.1 in the Appendix and the local nondeterminism property of the fractional Brownian motion we obtain

$$\begin{aligned} \det M &= \mathrm{Var}(B_{t_n}-B_{s_n})\mathrm{Var}(B_{t_{n-1}}-B_{s_{n-1}}|B_{t_n}-B_{s_n}) \\ &\quad \times\cdots\times\mathrm{Var}(B_{t_1}-B_{s_1}|B_{t_2}-B_{s_2},\ldots,B_{t_n}-B_{n}) \\ &= (t_n-s_n)^{2H}\mathrm{Var}(B_{s_{n-1}}|B_{t_{n-1}},B_{t_n},B_{s_n}) \\ &\quad \times\cdots\times\mathrm{Var}(B_{s_1}|B_{t_1},\ldots,B_{t_n},B_{s_1},\ldots,B_{s_{n-1}}) \\ &\ge k_2^{n-1}(r-s_n)^{2H}\left((s_n-s_{n-1})\wedge s_{n-1}\right)^{2H}\cdots\left((s_2-s_1)\wedge s_1\right)^{2H}. \end{aligned} \tag{4.25}$$

Substituting (4.3) into (4.4), and choosing α such that $\alpha < \frac{1}{4H}$ (this is possible because $Hd \geq 1$) yields

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq C^n n^{\beta n} n! \int_{S_n} \left[(r - s_n)\left((s_n - s_{n-1}) \wedge s_{n-1}\right) \cdots \left((s_2 - s_1) \wedge s_1\right)\right]^{-2\beta H} ds.$$

Finally, by Lemma 3.5.4 in the Appendix we obtain

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq \frac{C^n n^{\beta n} n!}{\Gamma(n(1-2H\beta)+1)}.$$

Notice that $\beta = \frac{d}{2} - \alpha > \frac{d}{2} - \frac{1}{4H}$. And hence,

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq C^n (n!^{\beta + 2H\beta},$$

where

$$\beta(1+2H) > \frac{d}{2} - \frac{1}{4H} + Hd - \frac{1}{2} = \left(\frac{1}{2} + H\right)\left(d - \frac{1}{2H}\right).$$

This concludes the proof.

Using the above proposition we can deduce the following integrability results for the renormalized self-intersection local time.

Theorem 3.4.3. *Assume* $\frac{1}{d} \leq H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$*. For any integer* $p < \frac{1}{2}\left[\left(\frac{1}{2} + H\right)\left(d - \frac{1}{2H}\right)\right]^{-1}$ *we have*

$$E(\exp|\widetilde{L}|^p) < \infty.$$

Proof Taking into account Lemma 3.5.2 in the Appendix, it suffices to show that

$$E\left(\exp\left\langle \widetilde{L} \right\rangle^p\right) < \infty,$$

where

$$\left\langle \widetilde{L} \right\rangle = \sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_0^t \Sigma^i(r,t,s) ds dt \right)^2 dr.$$

As in the proof of Theorem 3.4.1 we make the decomposition

$$\int_r^T \int_0^t \Sigma^i(r,t,s) ds dt = \int_r^T \int_r^t \Sigma^i(r,t,s) ds dt + \int_r^T \int_0^r \Sigma^i(r,t,s) ds dt.$$

From (4.7) and (4.8) we know that

$$\left| \int_r^T \int_r^t \Sigma^i(r,t,s) ds dt \right| \leq C(r^{\frac{1}{2}-H} \vee 1).$$

Therefore, applying Fatou's lemma and the estimate (4.13) yields

$$\begin{aligned}
E(\exp \left\langle \widetilde{L} \right\rangle^p) &\leq CE\left(\exp\left(\left| \sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_0^r \Sigma^i(r,t,s) ds dt \right)^2 dr \right|^p \right) \right) \\
&\leq C \liminf_{\varepsilon \downarrow 0} E\left(\exp\left(\left| \sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_0^r \Sigma^i_\varepsilon(r,t,s) ds dt \right)^2 dr \right|^p \right) \right) \\
&\leq C \liminf_{\varepsilon \downarrow 0} E\left(\exp\left(C \left| \int_0^T r^{1-2H} \left(\int_r^T \int_0^r \Psi_\varepsilon(r,t,s) ds dt \right)^2 dr \right|^p \right) \right).
\end{aligned}$$

Applying Hölder and Jensen inequalities we obtain

$$\begin{aligned}
E(\exp \left\langle \widetilde{L} \right\rangle^p) &\leq C \liminf_{\varepsilon \downarrow 0} E\left(\exp\left(C \int_0^T r^{1-2H} \left(\int_r^T \int_0^r \Psi_\varepsilon(r,t,s) ds dt \right)^{2p} dr \right) \right) \\
&\leq C \liminf_{\varepsilon \downarrow 0} \int_0^T r^{1-2H} E\left(\exp\left(C \left(\int_r^T \int_0^r \Psi_\varepsilon(r,t,s) ds dt \right)^{2p} \right) \right) dr.
\end{aligned}$$

Finally,

$$
\begin{aligned}
& E\left(\exp\left(C\left(\int_r^T \int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^{2p}\right)\right) \\
= & \sum_{n=1}^{\infty} \frac{C^n}{n!} E\left(\left(\int_r^T \int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^{2np}\right) \\
\leq & \sum_{n=1}^{\infty} \frac{C^n}{n!}(([2np]+1)!)^\gamma,
\end{aligned}
$$

and it suffices to apply Proposition 3.4.2 to conclude the proof.

Remark 2 The exponent $p_0 = \frac{1}{2}\left[\left(\frac{1}{2}+H\right)\left(d-\frac{1}{2H}\right)\right]^{-1}$ is not optimal. For instance, if $Hd = 1$, then $p_0 = \frac{2H}{1+2H}$ and we know that for $Hd < 1$, then $p_0 = \frac{1}{Hd}$. In particular, if $H = \frac{1}{2}$ and $d = 2$ we obtain $p_0 = \frac{1}{2}$, and we know that in this case the critical exponent is $p_0 = 1$. The lack of optimality is due to the factor n in the estimation of the positive definite matrix N by its diagonal elements given in (4.22). Without this factor n we would get the critical exponent $\frac{1}{2Hd-1}$, but our method does not allow to get this value.

Remark 3 In the case of the planar Brownian motion $B = \{B_t, t \geq 0\}$ (that is, $d = 2$, and $H = \frac{1}{2}$), formula (4.2) yields

$$
\widetilde{L} = -\frac{1}{2\pi}\sum_{i=1}^{2}\int_0^T \left(\int_r^T \int_0^r \frac{B_r^i - B_s^i}{(t-r)^2}\exp\left(-\frac{|B_r - B_s|^2}{2(t-r)}\right)dsdt\right)dB_r^i. \tag{4.26}
$$

The quadratic variation of this stochastic integral is

$$\begin{aligned}
\left\langle \widetilde{L} \right\rangle &= \frac{1}{4\pi^2} \sum_{i=1}^{2} \int_0^T \left(\int_r^T \int_0^r \frac{B_r^i - B_s^i}{(t-r)^2} \exp\left(-\frac{|B_r - B_s|^2}{2(t-r)} \right) dsdt \right)^2 dr \\
&\leq \frac{1}{4\pi^2} \int_0^T \left(\int_r^T \int_0^r \frac{|B_r - B_s|}{(t-r)^2} \exp\left(-\frac{|B_r - B_s|^2}{2(t-r)} \right) dsdt \right)^2 dr \\
&= \frac{1}{\pi^2} \int_0^T \left(\int_0^r \frac{1}{|B_r - B_s|} \exp\left(-\frac{|B_r - B_s|^2}{2(T-r)} \right) ds \right)^2 dr \\
&\leq \frac{1}{\pi^2} \int_0^T \left(\int_0^r \frac{ds}{|B_r - B_s|} \right)^2 dr.
\end{aligned}$$

From Itô's calculus we know that

$$\int_0^r \frac{ds}{|B_r - B_s|} = \frac{1}{d-1} \left(X_r - b_r \right),$$

where X_r has the law of the modulus of a d-dimensional Brownian motion at time r (Bessel process), and b_r has a normal $N(0,r)$ law. We can write

$$\exp\left(\lambda \left\langle \widetilde{L} \right\rangle \right) \leq \frac{1}{T} \int_0^T \exp\left(\frac{T\lambda}{\pi^2} \left(\int_0^r \frac{ds}{|B_r - B_s|} \right)^2 \right) dr,$$

which clearly imply the existence of some λ_0 such that $E\left(\exp\left(\lambda \left\langle \widetilde{L} \right\rangle \right) \right) < \infty$ for all $\lambda < \lambda_0$. From Lemma 3.5.2 we get that there exists β_0 such that $E\left(\exp\left(\beta \left| \widetilde{L} \right| \right) \right) < \infty$ for all $\beta < \beta_0$. This method does not allows us to obtain the critical exponent, just the existence of exponential moments.

Remark 4 The above results remain true if we replace the fractional Brownian motion with Hurst paramter H, by an arbitrary centered Gaussian process of the form (3.1) satisfying the local nondeterminism property **(LND)** and following properties:

(C1) For any $s,t \in [0,T]$, $s<t$, there exist constants k_3 and k_4 such that

$$k_3(t-s)^{2H} \le E(|B_t^i - B_s^i|^2) \le k_4(t-s)^{2H}.$$

(C2) The kernel $K(t,s)$ satisfies the estimates

$$|K(t,s)| \le k_5(t-s)^{H-\frac{1}{2}} s^{\frac{1}{2}-H},$$

for all $s<t$, and

$$\int_r^T \int_r^t (t-s)^{-Hd-H} |K(t,r)-K(s,r)|\, dsdt \le \psi(r),$$

where $\int_0^T \psi(r)^2 dr < \infty$.

3.5 Appendix

In this Appendix we will first state and prove some elementary lemmas. The first one is well-known.

Lemma 3.5.1. *Suppose that $\mathscr{G}_1 \subset \mathscr{G}_2$ are two σ-fields contained in $\mathscr{F}$. Then, for any square integrable random variable F we have*

$$\mathrm{Var}(F|\mathscr{G}_1) \ge \mathrm{Var}(F|\mathscr{G}_2).$$

Let $M = \{M_t, t \ge 0\}$ be a continuous local martingale such that $M_0 = 0$. Then, the following maximal exponential inequality is well-known

$$P\left(\sup_{0\le t\le T} |M_t| \ge \delta, \langle M\rangle_T < \rho\right) \le 2\exp\left(-\frac{\delta^2}{2\rho}\right).$$

As a consequence of this inequality we can obtain exponential moments for M_T from exponential moments of the quadratic variation $\langle M\rangle_T$

Lemma 3.5.2. *Suppose that for some* $\alpha > 0$ *and* $p \in (0,1]$ *we have* $E(e^{\alpha\langle M\rangle_T^p}) < \infty$. *Then,*

(i) if $p = 1$, *for any* $\lambda < \sqrt{\frac{\alpha}{2}}$, $E(e^{\lambda|M_T|}) < \infty$, *and*

(ii) if $p < 1$, $E(e^{\lambda|M_T|^p}) < \infty$ *for all* $\lambda > 0$.

Proof Set $X = |M_T|^p$. For any constant $c > 0$ we can write

$$\begin{aligned} E(e^{\lambda X}) &= \int_0^\infty P(X \geq y)\lambda e^{\lambda y}dy \\ &= \int_0^\infty \left[P(X \geq y, \langle M\rangle_T^p < cy) + P(X \geq y, \langle M\rangle_T^p \geq cy)\right]\lambda e^{\lambda y}dy \\ &\leq \int_0^\infty 2\exp\left(-\frac{y^{\frac{1}{p}}}{2c^{\frac{1}{p}}}\right)\lambda e^{\lambda y}dy + \int_0^\infty P\left(\frac{\langle M\rangle_T^p}{c} \geq y\right)\lambda e^{\lambda y}dy \\ &= \int_0^\infty 2\lambda\exp\left(\lambda y - \frac{y^{\frac{1}{p}}}{2c^{\frac{1}{p}}}\right)dy + E(e^{\frac{\lambda}{c}\langle M\rangle_T^p}). \end{aligned}$$

Then it suffices to choose $c = \frac{\lambda}{\alpha}$ to complete the proof.

The next two results are technical lemmas used in the chapter.

Lemma 3.5.3. *Suppose that* $H < \min(\frac{2}{d+1}, \frac{3}{2d})$. *Then, we have*

$$\int_r^T \int_r^t (t-s)^{-Hd-H}|K_H(t,r) - K_H(s,r)|\,dsdt \leq C\left(r^{\frac{1}{2}-H} \vee 1\right),$$

for some constant C.

Proof We know that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\left(H - \frac{1}{2}\right)\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

Then

$$
\begin{aligned}
I &:= \int_r^T \int_r^t (t-s)^{-Hd-H} |K_H(t,r) - K_H(s,r)| \, dsdt \\
&\le C \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} \left(\frac{\theta}{r}\right)^{H-\frac{1}{2}} (\theta - r)^{H-\frac{3}{2}} d\theta dsdt.
\end{aligned}
$$

If $H < \frac{1}{2}$, then, $\left(\frac{\theta}{r}\right)^{H-\frac{1}{2}} \le 1$, and if $H > \frac{1}{2}$, then $\left(\frac{\theta}{r}\right)^{H-\frac{1}{2}} \le Cr^{\frac{1}{2}-H}$. Hence, the above integral is bounded by

$$
C(r^{\frac{1}{2}-H} \vee 1) \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} (\theta - r)^{H-\frac{3}{2}} d\theta dsdt.
$$

From the decomposition

$$
\begin{aligned}
\frac{3}{2} - H &= \alpha + \beta, \\
Hd + H &= \gamma + \delta,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
& \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} (\theta - r)^{H-\frac{3}{2}} d\theta dsdt \\
= & \int_r^T \int_r^t \int_s^t (s-r)^{-\alpha} (\theta - s)^{-\beta-\gamma} (t-\theta)^{-\delta} d\theta dsdt.
\end{aligned}
$$

Finally, it suffices to show the parameters α, β, γ and δ in such a way that $\alpha < 1$, $\delta < 1$ and $\beta + \gamma < 1$. This leads to the condition

$$
\frac{1}{2} + Hd < \min(1, \frac{3}{2} - H) + \min(1, Hd + H),
$$

which is satisfied if $H < \min(\frac{2}{d+1}, \frac{3}{2d})$.

Lemma 3.5.4. *Let $a < 1$. Fix an interval $[0,T]$. For each integer $n \geq 1$ we have*

$$\begin{aligned}&\int_{\Delta_n(T)} [((T-s_n)\wedge s_n)((s_n-s_{n-1})\wedge s_{n-1})\cdots((s_2-s_1)\wedge s_1)]^{-a} ds \\ &\leq \frac{T^{n(1-a)}}{\Gamma(n(1-a)+1)} C^n,\end{aligned} \tag{5.1}$$

where $\Delta_n(T) = \{0 < s_1 < \cdots < s_n < T\}$

Proof We proceed by induction on n. For $n = 1$ we can write

$$\begin{aligned}\int_0^T ((T-s_1)\wedge s_1)^{-a} ds_1 &= \int_0^{\frac{T}{2}} s_1^{-a} ds_1 + \int_{\frac{T}{2}}^T (T-s_1)^{-a} ds_1 \\ &= \frac{2}{1-a}\left(\frac{T}{2}\right)^{1-a},\end{aligned}$$

which implies (4.6) with $C = \frac{\Gamma(2-a)}{1-a} 2^a$.

Suppose that the result holds for $n-1$. Then,

$$\begin{aligned}I_n &= \int_{\Delta_n(T)} [((T-s_n)\wedge s_n)((s_n-s_{n-1})\wedge s_{n-1})\cdots((s_2-s_1)\wedge s_1)]^{-a} ds \\ &= \int_0^T ((T-s_n)\wedge s_n)^{-a} \\ &\quad \times \left(\int_{\Delta_{n-1}(s_n)} [((s_n-s_{n-1})\wedge s_{n-1})\cdots((s_2-s_1)\wedge s_1)]^{-a} ds_1 \cdots ds_{n-1}\right) ds_n.\end{aligned}$$

By the induction hypothesis we can write

$$
\begin{aligned}
I_n &\leq \frac{C^{n-1}}{\Gamma(n-a)} \int_0^T ((T-s_n)\wedge s_n)^{-a} s_n^{(n-1)(1-a)} ds_n \\
&= \frac{C^{n-1}}{\Gamma((n-1)(1-a)+1)} \\
&\quad \times \left(\int_0^{\frac{T}{2}} s_n^{(n-1)(1-a)-a} ds_n + \int_{\frac{T}{2}}^T (T-s_n)^{-a} s_n^{(n-1)(1-a)} ds_n \right) \\
&\leq \frac{C^{n-1}}{\Gamma(n(1-a)+a)} \\
&\quad \times \left(\frac{1}{n(1-a)} \left(\frac{T}{2}\right)^{n(1-a)} + T^{n(1-a)} \int_0^1 (1-x)^{-a} x^{(n-1)(1-a)} dx \right) \\
&\leq \frac{T^{n(1-a)} C^{n-1}}{\Gamma(n(1-a)+a)} \left(\frac{1}{n(1-a)} + \frac{\Gamma(1-a)\Gamma((n-1)(1-a)+1)}{\Gamma(n(1-a)+1)} \right) \\
&= T^{n(1-a)} C^{n-1} \left(\frac{1}{n(1-a)\Gamma(n(1-a)+a)} + \frac{\Gamma(1-a)}{\Gamma(n(1-a)+1)} \right).
\end{aligned}
$$

Using the relation $\Gamma(n+1) = n\Gamma(n)$ we obtain

$$
n(1-a)\Gamma(n(1-a)+a) \geq n(1-a)\Gamma(n(1-a)) = \Gamma(n(1-a)+1),
$$

and, as a consequence

$$
I_n \leq T^{n(1-a)} C^{n-1} (1+\Gamma(1-a)) \frac{1}{\Gamma(n(1-a)+1)},
$$

and it suffices to take $C \geq \max\left(\frac{\Gamma(2-a)}{1-a} 2^a, 1+\Gamma(1-a)\right)$.

Chapter 4

Feynman-Kac formula for heat equation driven by fractional white noise

4.1 Introduction

Consider the following heat equation on $\mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u + c(t,x)u \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

where f is a bounded measurable function. If $c(t,x)$ is a continuous function of $(t,x) \in [0,\infty) \times \mathbb{R}^d$, then we have the well-known Feynman-Kac formula (see [13]) for the solution to above equation

$$u(t,x) = E\left[f(B_t^x) \exp\left(\int_0^t c(t-s, B_s^x) ds \right) \right],$$

where $B_t^x = B_t + x$ is a d-dimensional Brownian motion starting from the point x.

In this chapter, we shall extend the above Feynman-Kac formula to the heat equation with fractional noise

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \dfrac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d} \\ u(0,x) = f(x), \end{cases} \tag{1.2}$$

where $W(t,x)$ is a fractional Brownian sheet with Hurst parameters H_0 in time and $(H_1, \dots, H_d)$ in space, respectively. The difference between (1.1) and (1.1) is that $\frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}$ is no longer a function of t and x but a generalized (random) function. For this equation, we can still formally write down the Feynman-Kac formula

$$u(t,x) = E^B \left[f(B_t^x) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr,dy) \right) \right], \tag{1.3}$$

where E^B denotes the expectation with respect to the Brownian motion B_t^x, and δ denotes the Dirac delta function.

The aim of this chapter is to justify the above formula (5.3.1), to show that the process $u(t,x)$ is a weak solution to Equation (1.1) and to establish some properties of this process. First, we shall show that $V_{t,x} := \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr,dy)$ is a well-defined random variable. This will be done in Section 2 using a suitable approximation of the Dirac delta function, assuming that the Hurst parameters satisfy $2H_0 + \sum_{i=1}^d H_i > d+1$, $H_0 \geq \frac{1}{2}$, and $H_i > \frac{1}{2}$ for $1 \leq i \leq d$.

After the definition of the random variable $V_{t,x}$, the next problem is to show its exponential integrability. With the use of the covariance structure of the fractional Brownian sheet $W(t,x)$, we show that $u(t,x)$ has exponential moments provided

$$E \exp\left[\lambda \int_0^1 \int_0^1 |r-s|^{2H_0-2} \prod_{i=1}^d |B_r^i - B_s^i|^{2H_i-2} drds \right] < \infty, \tag{1.4}$$

for any $\lambda \in \mathbb{R}$. To show that (1.4) is true we use a method introduced by Le Gall in [31] to derive the exponential integrability of the renormalized self-intersection local time of the planar Brownian motion, together with the self-similarity of the fractional Brownian sheet and several other techniques. This is done in Section 3.

Another main point of this chapter is to show that $u(t,x)$ defined by (5.3.1) is a weak solution to (1.1). Instead of following the classical approach based on Itô's formula, which seems complicated in our situation, we use again the approximation technique together with Malliavin calculus. The main ingredient is to express the Stratonovich integral as the sum of a Skorohod integral plus a correction term involving Malliavin derivatives. This is a new methodology which is is developed in Section 4.

The Feynman-Kac formula gives an explicit form of a weak solution to Equation (1.1) which turns to be very useful to obtain regularity properties. Several consequences of this expression are derived in Section 5. First, we obtain some Hölder continuity properties of the solution $u(t,x)$ with respect to t and x, and afterwards we establish the smoothness of the density of the probability law of $u(t,x)$ (with respect to the Lebesgue measure) using techniques of Malliavin calculus.

In the above equation (1.1) the solution and the noise are multiplied using the ordinary product. This gives rise to the Stratonovitch integral when we interpret the equation in its integral form. There is a number of papers where the Wick product between the solution and the noise is used, which corresponds to the Skorohod integral. The Stratonovich integral is more difficult to handle but it is the right choice if we want to represent a physical model. Applying a Wiener chaos technique pioneered by Dawson and Salehi in [10], and used in several other papers (see, for instance, the work [22] on the relation between moments of the solution and self-intersection local times), one can show that there exists a unique mild solution to the Skorohod-type equation. We discuss this result in Section 7 and using Wiener chaos expansions we obtain a Feynman-Kac formula for this solution.

The above techniques work for $H_i > 1/2$, $i = 1, 2, \dots, d$. From the condition $2H_0 + \sum_{i=1}^{d} H_i > d + 1$ it follows that H_0 must be greater than $1/2$ and we cannot allow more than one of the $H_1, \dots, H_d$ to be less than or equal to $1/2$. Thus if we want to remove the condition $H_i > 1/2$, $i = 1, 2, \dots, d$, we need $d = 1$. We show in Section 7 that if $d = 1$, $H_1 = \frac{1}{2}$ and $H_0 > \frac{3}{4}$ then all previous results hold. When $d = 1$, we can also handle the case $H_0 < 1/2$, assuming that the process has a regular spacial covariance. This has been done in the companion paper [21] using different techniques. Finally, the appendix contains some technical results used along the chapter

We would like to close this introduction with some remarks about the motivation of our work and its connection with other related results. The existence of a Feynman-Kac formula like the one we have derived here was mentioned as a conjecture in a paper by Mocioalca and Viens (see [35]), although this problem has been circulating long before that. In the lectures by Walsh in Saint Flour (see [50]) it was stated that the one-dimensional equation in the Itô sense driven by a space-time white noise cannot have a Feynman-Kac formula because the Itô-Stratonovich correction term is infinite. In a pervious work [22] one considered a Skorohod-type equation assuming $H_i = \frac{1}{2}$ for $i = 1, \dots, d$. In this case, there exists a unique mild solution obtained by means of the Wiener chaos method if $H_0 > \frac{1}{2}$ and $d = 1$ or $d = 2$ and t is small enough, although the Feynman-Kac formula is not available unless $d = 1$ and $H_0 > \frac{3}{4}$ (see Section 7).

A process similar to (5.3.1) was studied by Viens and Zhang in [49], although it does no have a relation with a stochastic heat equation, and most likely the asymptotic results obtained in [49] can be extended to the process (5.3.1).

Recently, Hinz obtained in [17] a Feynman-Kac formula for the stochastic heat equation with a Gaussian multiplicative noise of the form $\frac{\partial W}{\partial t}(t, x)$, where W is a fractional Brownian sheet with Hurst parameter $H > \frac{1}{2}$ in time and $K \in (0,1)$ in space, and he used this formula to solve a stochastic Burguess equation by means of the Hopf-Cole transformation. In this paper the noise is more regular in space, and this allows the author to use techniques of classical fractional calculus together with curvilinear integrals.

4.2 Preliminaries

Fix a vector of Hurst parameters $H = (H_0, H_1, \ldots, H_d)$, where $H_i \in \left(\frac{1}{2}, 1\right)$. Suppose that $W = \{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a zero mean Gaussian random field with the covariance function

$$E(W(t,x)W(s,y)) = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i),$$

where for any $H \in (0,1)$ we denote by $R_H(s,t)$, the covariance function of the fractional Brownian motion with Hurst parameter H, that is,

$$R_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

In other words, W is a fractional Brownian sheet with Hurst parameters H_0 in time variable and H_i in space variables, $i = 1, \ldots, d$.

Denote by $\mathscr{E}$ the linear span of the indicator functions of rectangles of the form $(s,t] \times (x,y]$ in $\mathbb{R}_+ \times \mathbb{R}^d$. Consider in $\mathscr{E}$ the inner product defined by

$$\langle I_{(0,s]\times(0,x]}, I_{(0,t]\times(0,y]} \rangle_{\mathscr{H}} = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i).$$

In the above formula, if $x_i < 0$ we assume by convention that $I_{(0,x_i]} = -I_{(-x_i,0]}$. We denote by $\mathscr{H}$ the closure of $\mathscr{E}$ with respect to this inner product. The mapping W :

$I_{(0,t]\times(0,x]} \to W(t,x)$ extends to a linear isometry between $\mathscr{H}$ and the Gaussian space spanned by W. We will denote this isometry by

$$W(\phi)=\int_0^\infty\int_{\mathbb{R}^d}\phi(t,x)W(dt,dx),$$

if $\phi\in\mathscr{H}$. Notice that if ϕ and ψ are functions in $\mathscr{E}$, then

$$\begin{aligned}&E\left(W(\phi)W(\psi)\right)=\langle\phi,\psi\rangle_{\mathscr{H}}=\alpha_H\\&\times\int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}}\phi(s,x)\psi(t,y)|s-t|^{2H_0-2}\prod_{i=1}^d|x_i-y_i|^{2H_i-2}dsdtdxdy,\end{aligned}\tag{2.1}$$

where $\alpha_H=\prod_{i=0}^d H_i(2H_i-1)$. Furthermore, $\mathscr{H}$ contains the class of measurable functions ϕ on $\mathbb{R}_+\times\mathbb{R}^d$ such that

$$\int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}}|\phi(s,x)\phi(t,y)||s-t|^{2H_0-2}\prod_{i=1}^d|x_i-y_i|^{2H_i-2}dsdtdxdy<\infty.\tag{2.2}$$

We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F=f(W(\phi_1),\dots,W(\phi_n)),$$

$\phi_i\in\mathscr{H}$, $f\in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), then DF is the $\mathscr{H}$-valued random variable defined by

$$DF=\sum_{j=1}^n\frac{\partial f}{\partial x_j}(W(\phi_1),\dots,W(\phi_n))\phi_j.$$

The operator D is closable form $L^2(\Omega)$ into $L^2(\Omega;\mathscr{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|^2_{\mathscr{H}})}.$$

We denote by δ the adjoint of the derivative operator, given by duality formula

$$E(\delta(u)F) = E\left(\langle DF, u\rangle_{\mathscr{H}}\right), \tag{2.3}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathscr{H})$ in the domain of δ. The operator δ is also called the Skorohod integral because in the case of the Brownian motion it coincides with an extension of the Itô integral introduced by Skorohod. We refer to Nualart [38] for a detailed account on the Malliavin calculus with respect to a Gaussian process. If DF and u are almost surely measurable functions on $\mathbb{R}_+ \times \mathbb{R}^d$ verifying condition (2.2), then the duality formula (2.1) can be written using the expression of the inner product in $\mathscr{H}$ given in (2.1):

$$\begin{aligned} &E\left(\delta(u)F\right) = \alpha_H \\ &\times E\left(\int_0^\infty \int_{\mathbb{R}^d} D_{s,x}Fu(t,y)|s-t|^{2H_0-2}\prod_{i=1}^d |x_i - y_i|^{2H_i-2} dsdtdxdy\right). \end{aligned}$$

We recall the following formula, which will be used in the chapter

$$FW(\phi) = \delta(F\phi) + \langle DF, \phi\rangle_{\mathscr{H}}, \tag{2.4}$$

for any $\phi \in \mathscr{H}$ and any random variable F in the Sobolev space $\mathbb{D}^{1,2}$.

Along the chapter C will denote a positive constant which may vary from one formula to another one.

4.3 Definition and exponential integrability of $\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)$

For any $\varepsilon > 0$ we denote by $p_\varepsilon(x)$ the d-dimensional heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}},\ x \in \mathbb{R}^d.$$

On the other hand, for any $\delta > 0$ we define the function

$$\varphi_\delta(x) = \frac{1}{\delta} I_{[0,\delta]}(x).$$

Then, $\varphi_\delta(t) p_\varepsilon(x)$ provides an approximation of the Dirac delta function $\delta(t,x)$ as ε and δ tend to zero. We denote by $W^{\varepsilon,\delta}$ the approximation of the fractional Brownian sheet $W(t,x)$ defined by

$$W^{\varepsilon,\delta}(t,x) = \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s) p_\varepsilon(x-y) W(s,y) ds dy. \tag{3.1}$$

Fix $x \in \mathbb{R}^d$ and $t > 0$. Suppose that $B = \{B_t, t \geq 0\}$ is a d-dimensional standard Brownian motion independent of W. We denote by $B_t^x = B_t + x$ the Brownian motion starting at the point x. We are going to define the random variable $\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)$ by approximating the Dirac delta function $\delta(B_{t-r}^x - y)$ by

$$A_{t,x}^{\varepsilon,\delta}(r,y) = \int_0^t \varphi_\delta(t-s-r) p_\varepsilon(B_s^x - y) ds. \tag{3.2}$$

We will show that for any $\varepsilon > 0$ and $\delta > 0$ the function $A^{\varepsilon,\delta}_{t,x}$ belongs to the space $\mathscr{H}$ almost surely, and the family of random variables

$$V^{\varepsilon,\delta}_{t,x} = \int_0^t \int_{\mathbb{R}^d} A^{\varepsilon,\delta}_{t,x}(r,y) W(dr,dy)\,. \tag{3.3}$$

converges in L^2 as ε and δ tend to zero.

The reason we choose (3.3) as our approximation of $\int_0^t \int_{\mathbb{R}^d} \delta(B^x_{t-r} - y) W(dr,dy)$ is that the time variable t is in a finite interval and the space variable x is in the whole space and this approximation has the useful properties proved in Lemmas 4.8.2 and 4.8.3. We could have used other types of approximation schemes with similar results. Moreover, we can restrict ourselves to the special case $\delta = \varepsilon$, but the slightly more general case considered here does not need any additional effort.

Along the chapter we denote by $E^B(\Phi(B,W))$ (resp. by $E^W(\Phi(B,W))$) the expectation of a functional $\Phi(B,W)$ with respect to B (resp. with respect to W). We will use E for the composition $E^B E^W$, and also in case of a random variable depending only on B or W.

Theorem 4.3.1. *Suppose that* $2H_0 + \sum_{i=1}^d H_i > d+1$. *Then, for any* $\varepsilon > 0$ *and* $\delta > 0$, $A^{\varepsilon,\delta}_{t,x}$ *defined in (3.2) belongs to* $\mathscr{H}$ *and the family of random variables* $V^{\varepsilon,\delta}_{t,x}$ *defined in (3.3) converges in* L^2 *to a limit denoted by*

$$V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta(B^x_{t-r} - y) W(dr,dy)\,. \tag{3.4}$$

Conditional to B, $V_{t,x}$ *is a Gaussian random variable with mean* 0 *and variance*

$$\mathrm{Var}^W(V_{t,x}) = \alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d \left|B^i_r - B^i_s\right|^{2H_i-2} drds\,. \tag{3.5}$$

Proof Fix ε, ε', δ and $\delta' > 0$. Let us compute the inner product

$$\left\langle A_{t,x}^{\varepsilon,\delta}, A_{t,x}^{\varepsilon',\delta'} \right\rangle_{\mathscr{H}} = \alpha_H \int_{[0,t]^4} \int_{\mathbb{R}^{2d}} p_\varepsilon(B_s^x - y) p_{\varepsilon'}(B_r^x - z) \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v) \times |u-v|^{2H_0-2} \prod_{i=1}^d |y_i - z_i|^{2H_i-2} dydzdudvdsdr. \tag{3.6}$$

By Lemmas 4.8.2 and 4.8.3 we have the estimate

$$\begin{aligned} &\int_{[0,t]^2} \int_{\mathbb{R}^{2d}} p_\varepsilon(B_s^x - y) p_{\varepsilon'}(B_r^x - z) \\ &\times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2} \prod_{i=1}^d |y_i - z_i|^{2H_i-2} dydzdudv \\ \leq\ & C|s-r|^{2H_0-2} \prod_{i=1}^d \left|B_s^i - B_r^i\right|^{2H_i-2}, \end{aligned} \tag{3.7}$$

for some constant $C > 0$. The expectation of this random variable is integrable in $[0,t]^2$ because

$$\begin{aligned} & E^B \int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d \left|B_s^i - B_r^i\right|^{2H_i-2} dsdr \\ =\ & \prod_{i=1}^d E|\xi|^{2H_i-2} \int_0^t \int_0^t |s-r|^{2H_0+\sum_{i=1}^d H_i - d - 2} dsdr \\ =\ & \frac{2\prod_{i=1}^d E|\xi|^{2H_i-2} t^{\kappa+1}}{\kappa(\kappa+1)} < \infty, \end{aligned} \tag{3.8}$$

where

$$\kappa = 2H_0 + \sum_{i=1}^d H_i - d - 1 > 0. \tag{3.9}$$

and ξ is a $N(0,1)$ random variable.

As a consequence, taking the mathematical expectation with respect to B in Equation (3.6), letting $\varepsilon=\varepsilon'$ and $\delta=\delta'$ and using the estimates (3.7) and (3.8) yields

$$E^B\left\|A^{\varepsilon,\delta}_{t,x}\right\|^2_{\mathscr{H}}\leq C.$$

This implies that almost surely $A^{\varepsilon,\delta}_{t,x}$ belongs to the space $\mathscr{H}$ for all ε and $\delta>0$. Therefore, the random variables $V^{\varepsilon,\delta}_{t,x}=W(A^{\varepsilon,\delta}_{t,x})$ are well defined and we have

$$E^BE^W(V^{\varepsilon,\delta}_{t,x}V^{\varepsilon',\delta'}_{t,x})=E^B\left\langle A^{\varepsilon,\delta}_{t,x},A^{\varepsilon',\delta'}_{t,x}\right\rangle_{\mathscr{H}}.$$

For any $s\neq r$ and $B_s\neq B_r$, as ε, ε', δ and δ' tend to zero, the left-hand side of the inequality (3.7) converges to $|s-r|^{2H_0-2}\prod_{i=1}^d\left|B^i_s-B^i_r\right|^{2H_i-2}$. Therefore, by dominated convergence theorem we obtain that $E^BE^W(V^{\varepsilon,\delta}_{t,x}V^{\varepsilon',\delta'}_{t,x})$ converges to Σ_t as ε, ε', δ and δ' tend to zero, where

$$\Sigma_t=\frac{2\alpha_H\prod_{i=1}^dE|\xi|^{2H_i-2}t^{\kappa+1}}{\kappa(\kappa+1)}.$$

Thus we obtain

$$E\left(V^{\varepsilon,\delta}_{t,x}-V^{\varepsilon',\delta'}_{t,x}\right)^2=E\left(V^{\varepsilon,\delta}_{t,x}\right)^2-2E\left(V^{\varepsilon,\delta}_{t,x}V^{\varepsilon',\delta'}_{t,x}\right)+E\left(V^{\varepsilon',\delta'}_{t,x}\right)^2\to 0\,.$$

This implies that $V^{\varepsilon_n,\delta_n}_{t,x}$ is a Cauchy sequence in L^2 for any sequences ε_n and δ_n converging to zero. As a consequence, $V^{\varepsilon_n,\delta_n}_{t,x}$ converges in L^2 to a limit denoted by $V_{t,x}$, which does not depend on the choice of the sequences ε_n and δ_n. Finally, by a similar argument we show (3.5).

Condition $2H_0+\sum_{i=1}^dH_i>d+1$ is sharp and it cannot be improved. In fact, if this condition does not hold, then almost surely $(r,y)\mapsto\delta(B^x_{t-r}-y)$ is not an element of the space $\mathscr{H}$, as it follows from the next proposition.

Proposition 4.3.2. *Suppose $H_i > 1/2$, $i = 0,1,\dots,d$ and $2H_0 + \sum_{i=1}^d H_i \le d+1$. Then, conditionally to B the family $V_{t,x}^{\varepsilon,\delta}$ does not converge in probability as ε and δ tend to zero, for almost all trajectories of B.*

Proof Given B, $V_{t,x}^{\varepsilon,\delta}$ is a Gaussian family of random variables, and it suffices to show that they do not converge in L^2. This follows form the fact that the variance limit is infinite almost surely. In fact, from the Lévy modulus of continuity of the Brownian motion, it is easy to show that if $2H_0 + \sum_{i=1}^d H_i \le d+1$, then

$$\int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d \left|B_s^i - B_r^i\right|^{2H_i-2} dsdr = \infty$$

almost surely.

The next result provides the exponential integrability of the random variable $V_{t,x}$ defined in (3.4).

Theorem 4.3.3. *Suppose that $2H_0 + \sum_{i=1}^d H_i > d+1$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$E \exp\left(\lambda \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)\right) < \infty. \tag{3.10}$$

Proof The proof will be done in several steps.

Step 1 From (3.5) we obtain

$$Ee^{\lambda V_{t,x}} = E^B \exp\left(\frac{\lambda^2}{2}\alpha_H \int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2} dsdr\right),$$

and the scaling property of the Brownian motion yields

$$Ee^{\lambda V_{t,x}} = Ee^{\mu Y}, \tag{3.11}$$

where $\mu = \frac{\lambda^2}{2}\alpha_H t^{\kappa+1}$, where κ has been defined in (3.9), and

$$Y = \int_0^1 \int_0^1 |s-r|^{2H_0-2} \prod_{i=1}^{d} |B_s^i - B_r^i|^{2H_i-2} dsdr. \tag{3.12}$$

Then, it suffices to show that the random variable Y has exponential moments of all orders.

Step 2 Our approach to prove that $E\exp(\lambda Y) < \infty$ for any $\lambda \in \mathbb{R}$ is motivated by the method of Le Gall [31]. For $k = 1, \ldots, 2^{n-1}$ we denote $A_{n,k} = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]$ and define

$$\alpha_{n,k} = \int_{A_{n,k}} |s-r|^{2H_0-2} \prod_{i=1}^{d} |B_s^i - B_r^i|^{2H_i-2} dsdr.$$

The random variables $\alpha_{n,k}$ have the following two properties:

(i) For every $n \geq 1$, the variables $\alpha_{n,1}, \ldots, \alpha_{n,2^{n-1}}$ are independent.

(ii) $\alpha_{n,k} \stackrel{d}{=} 2^{-n(\kappa+1)}\alpha_0$, where

$$\alpha_0 = \int_0^1 \int_0^1 (s+r)^{2H_0-2} \prod_{i=1}^{d} |B_s^i - \tilde{B}_r^i|^{2H_i-2} dsdr,$$

and $\tilde{B}$ is a standard Brownian motion independent of B.

The condition $2H_0 + \sum_{i=1}^d H_i > d+1$ implies that $E\alpha_0 < \infty$ and we deduce that

$$Y = 2\sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \alpha_{n,k},$$

where the series converges in the L^1 sense.

Step 3 For any integer $n \geq 1$, we claim that

$$E\alpha_0^n \leq E\left(C\int_0^1 \prod_{i=1}^d |B_s^i|^{2H_i-2} ds\right)^n, \tag{3.13}$$

for some constant $C > 0$. In fact, we have

$$E\alpha_0^n = E\int_{[0,1]^{2n}} \prod_{j=1}^n \prod_{i=1}^d (|s_j + t_j|^{2H_0-2} |B_{s_j}^i - \tilde{B}_{t_j}^i|^{2H_i-2}) ds dt. \tag{3.14}$$

Using the formula

$$c^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-c\tau} \tau^{z-1} d\tau,$$

we obtain for each $i = 1, \dots, d$,

$$\begin{aligned} E\prod_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^{2H_i-2} &= \Gamma(1-H_i)^{-n} \\ &\times \int_{[0,\infty)^n} E\exp\left(-\sum_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^2 \tau_j\right) \prod_{j=1}^n \tau_j^{-H_i} d\tau. \end{aligned} \tag{3.15}$$

For any $\tau_1, \dots, \tau_n > 0$ and $s_1, t_1, \dots, s_n, t_n \in (0,1)$, we denote

$$Q_1 = \left(E(B_{s_j}^i B_{s_k}^i)\sqrt{\tau_j \tau_k}\right)_{n\times n}, \quad Q_2 = \left(E(\tilde{B}_{t_j}^i \tilde{B}_{t_k}^i)\sqrt{\tau_j \tau_k}\right)_{n\times n}.$$

We know that

$$E\exp\left(-\sum_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^2 \tau_j\right) = \det(I + 2Q_1 + 2Q_2)^{-\frac{1}{2}}. \tag{3.16}$$

Substituting (3.16) into (3.15) yields

$$
\begin{aligned}
& E\prod_{j=1}^{n}|B^i_{s_j}-\tilde{B}^i_{t_j}|^{2H_i-2} = \Gamma(1-H_i)^{-n} \\
& \times\int_{[0,\infty)^n}\det(I+2Q_1+2Q_2)^{-\frac{1}{2}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau \\
\leq\ & \Gamma(1-H_i)^{-n}\int_{[0,\infty)^n}\det(I+2Q_1)^{-\frac{1}{4}}\det(I+2Q_2)^{-\frac{1}{4}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau \\
\leq\ & \Gamma(1-H_i)^{-n}\left[\int_{[0,\infty)^n}\det(I+2Q_1)^{-\frac{1}{2}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau\right]^{\frac{1}{2}} \\
& \times\left[\int_{[0,\infty)^n}\det(I+2Q_2)^{-\frac{1}{2}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau\right]^{\frac{1}{2}} \\
=\ & \left[E\prod_{j=1}^{n}|B^i_{s_j}|^{2H_i-2}E\prod_{j=1}^{n}|\tilde{B}^i_{t_j}|^{2H_i-2}\right]^{\frac{1}{2}}, \qquad (3.17)
\end{aligned}
$$

where in the above first inequality, we have used the estimates

$$
(I+2Q_1+2Q_2)\geq\frac{1}{2}[(I+2Q_1)+(I+2Q_2)]\geq(I+2Q_1)^{\frac{1}{2}}(I+2Q_2)^{\frac{1}{2}}.
$$

Substituting (3.17) into (3.14), and using the inequality $(s_j+t_j)^{2H_0-2}\leq s_j^{H_0-1}t_j^{H_0-1}$, we obtain

$$
\begin{aligned}
E\alpha_0^n \ \leq\ & \int_{[0,1]^{2n}}\prod_{j=1}^{n}(s_j+t_j)^{2H_0-2}\prod_{i=1}^{d}\left[E\prod_{j=1}^{n}|B^i_{s_j}|^{2H_i-2}E\prod_{j=1}^{n}|\tilde{B}^i_{t_j}|^{2H_i-2}\right]^{\frac{1}{2}}dsdt \\
\leq\ & \left(\int_{[0,1]^n}\prod_{j=1}^{n}s_j^{H_0-1}\left(E\prod_{j=1}^{n}\prod_{i=1}^{d}|B^i_{s_j}|^{2H_i-2}\right)^{\frac{1}{2}}ds\right)^2.
\end{aligned}
$$

Finally, using Hölder's inequality with $\frac{1}{H_0} < p < 2$ we get

$$\begin{aligned} E\alpha_0^n &\leq C^n \left(\int_{[0,1]^n} \left(E \prod_{i=1}^{d} \prod_{j=1}^{n} |B_{s_j}^i|^{2H_i-2} \right)^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \\ &\leq C^n \int_{[0,1]^n} E \prod_{i=1}^{d} \prod_{j=1}^{n} |B_{s_j}^i|^{2H_i-2} ds \\ &= E \left(C \int_0^1 \prod_{i=1}^{d} |B_s^i|^{2H_i-2} ds \right)^n . \end{aligned}$$

This completes the proof of (3.13).

Step 4 For any $\lambda > 0$, using (3.13) and Lemma 4.8.5 in the Appendix we obtain

$$Ee^{\lambda\alpha_0} \leq E \exp\left(C\lambda \int_0^1 \prod_{i=1}^{d} |B_s^i|^{2H_i-2} ds \right) < \infty, \tag{3.18}$$

because $\rho < 1$.

Step 5 Define $\varphi(\lambda) = E(e^{\lambda(\alpha_0 - E\alpha_0)})$. By (3.18), $\varphi(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$. Since $\varphi'(0) = 0$, for every $K > 0$ we can find a positive constant C_K such that for all $\lambda \in [0,K]$

$$\varphi(\lambda) \leq 1 + C_K \lambda^2.$$

Define $\overline{\alpha}_{n,k} = \alpha_{n,k} - E(\alpha_{n,k})$. Fix $K > 0$ and $a \in (0, \kappa+1)$, where κ has been introduced in (3.9). Recall that by property (ii) in Step 3, $\overline{\alpha}_{n,k} \stackrel{d}{=} 2^{-n(\kappa+1)}\overline{\alpha}_0$. For every $N \geq 2$ set $b_N = 2K \prod_{j=2}^{j=N} (1 - 2^{-a(j-1)})$ and set $b_1 = 2K$. Then by Hölder's inequality and properties (i) and (ii) of $\alpha_{n,k}$, we have for $N \geq 2$,

$$
\begin{aligned}
&E\exp\left(b_N \sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}} \overline{\alpha}_{n,k}\right) \\
&\leq \left[E\exp\left(\frac{b_N}{1-2^{-a(N-1)}}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}} \overline{\alpha}_{n,k}\right)\right]^{1-2^{-a(N-1)}} \\
&\quad\times \left[E\exp\left(2^{a(N-1)} b_N \sum_{k=1}^{2^{N-1}} \overline{\alpha}_{N,k}\right)\right]^{2^{-a(N-1)}} \\
&\leq E\exp\left(b_{N-1}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}} \overline{\alpha}_{n,k}\right)\varphi(b_N 2^{a(N-1)-(\kappa+1)N})^{2^{(1-a)(N-1)}}.
\end{aligned}
$$

Notice that $b_N 2^{a(N-1)-(\kappa+1)N} \leq 2K$. It follows that,

$$
\begin{aligned}
\varphi(b_N 2^{a(N-1)-(\kappa+1)N})^{2^{(1-a)(N-1)}} &\leq \left(1 + C_K b_N^2 2^{2((a-\kappa-1)N-a)}\right)^{2^{(1-a)(N-1)}} \\
&\leq \exp(C 2^{(a+1-2(\kappa+1))N})
\end{aligned}
$$

for a constant C independent of N. By induction we get

$$
\begin{aligned}
E\exp\left(b_N\sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}}\overline{\alpha}_{n,k}\right) &\leq \exp\left(C\sum_{n=2}^{N} 2^{(a+1-2(\kappa+1))n}\right) E\exp\left(b_1\overline{\alpha}_{1,1}\right) \\
&\leq \exp\left(C(1-2^{a+1-2(\kappa+1)})^{-1}\right)\varphi(K).
\end{aligned}
$$

Letting N tend to infinity and using Fatou's lemma, we obtain

$$
E\exp\left(b_\infty(Y - EY)/2\right) < \infty,
$$

where $b_\infty = 2K\prod_{j=1}^{\infty}(1-2^{-aj}) > 0$. Since $K > 0$ is arbitrary, we conclude that $E\exp(\lambda Y) < \infty$ for all $\lambda \in \mathbb{R}$. This completes the proof, in view of (3.11).

4.4 Feynman-Kac formula

We recall that W is a fractional Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}^d$ with Hurst parameters $(H_0, H_1, \dots, H_d)$ where $H_i \in (\frac{1}{2}, 1)$ for $i = 0, \dots, d$. For any ε, $\delta > 0$ we define

$$\dot{W}^{\varepsilon,\delta}(t,x) := \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s) p_\varepsilon(x-y) W(ds,dy).$$

In order to give a notion of solution for the heat equation with fractional noise (1.1) we need the following definition of the Stratonovitch integral, which is equivalent to that of Russo-Vallois in [42].

Definition 4.4.1. *Given a random field $v = \{v(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that*

$$\int_0^T \int_{\mathbb{R}^d} |v(t,x)| dx dt < \infty$$

almost surely for all $T > 0$, the Stratonovitch integral $\int_0^T \int_{\mathbb{R}^d} v(t,x) W(dt,dx)$ is defined as the following limit in probability if it exists

$$\lim_{\varepsilon,\delta \downarrow 0} \int_0^T \int_{\mathbb{R}^d} v(t,x) \dot{W}^{\varepsilon,\delta}(t,x) dx dt.$$

We are going to consider the following notion of solution for Equation (1.1).

Definition 4.4.2. *A random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a weak solution to Equation (1.1) if for any C^∞ function φ with compact support on $\mathbb{R}^d$, we have*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} u(s,x)\varphi(x) W(ds,dx),\end{aligned}$$

almost surely, for all $t \geq 0$, where the last term is a Stratonovitch stochastic integral in the sense of Definition 4.4.1.

The following is the main result of this section.

Theorem 4.4.3. *Suppose that $2H_0 + \sum_{i=1}^d H_i > d+1$ and that f is a bounded measurable function. Then the process*

$$u(t,x) = E^B\left(f(B_t^x)\exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y)W(dr,dy)\right)\right) \tag{4.1}$$

is a weak solution to Equation (1.1).

Proof Consider the approximation of the Equation (1.1) given by the following heat equation with a random potential

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{1}{2}\Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta}\dot{W}_{t,x}^{\varepsilon,\delta} \\ u^{\varepsilon,\delta}(0,x) = f(x). \end{cases} \tag{4.2}$$

From the classical Feynman-Kac formula we know that

$$u^{\varepsilon,\delta}(t,x) = E^B\left(f(B_t^x)\exp\left(\int_0^t \dot{W}^{\varepsilon,\delta}(t-s,B_s^x)ds\right)\right),$$

where B_t^x is a d-dimensional Brownian motion independent of W starting at x. By Fubini's theorem we can write

$$\begin{aligned} \int_0^t \dot{W}^{\varepsilon,\delta}(t-s,B_s^x)ds &= \int_0^t \left(\int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s-r)p_\varepsilon(B_s^x - y)W(dr,dy)\right)ds \\ &= \int_0^t \int_{\mathbb{R}^d} \left(\int_0^t \varphi_\delta(t-s-r)p_\varepsilon(B_s^x - y)ds\right)W(dr,dy) \\ &= V_{t,x}^{\varepsilon,\delta}, \end{aligned}$$

where $V_{t,x}^{\varepsilon,\delta}$ is defined in (3.3). Therefore,

$$u^{\varepsilon,\delta}(t,x) = E^B\left(f(B_t^x)\exp\left(V_{t,x}^{\varepsilon,\delta}\right)\right).$$

Step 1 We will prove that for any $x \in \mathbb{R}^d$ and any $t > 0$, we have

$$\lim_{\varepsilon,\delta\downarrow 0} \ E^W|u^{\varepsilon,\delta}(t,x) - u(t,x)|^p = 0, \tag{4.3}$$

for all $p \geq 2$, where $u(t,x)$ is defined in (4.1). Notice that

$$\begin{aligned} E^W|u^{\varepsilon,\delta}(t,x) - u(t,x)|^p &= E^W\left|E^B\left(f(B_t^x)\left[\exp\left(V_{t,x}^{\varepsilon,\delta}\right) - \exp\left(V_{t,x}\right)\right]\right)\right|^p \\ &\leq \|f\|_\infty^p E\left|\exp\left(V_{t,x}^{\varepsilon,\delta}\right) - \exp\left(V_{t,x}\right)\right|^p, \end{aligned}$$

where $V_{t,x}$ is defined in (3.4). Since $\exp\left(V_{t,x}^{\varepsilon,\delta}\right)$ converges to $\exp(V_{t,x})$ in probability by Theorem 4.3.1, to show (4.3) it suffices to prove that for any $\lambda \in \mathbb{R}$

$$\sup_{\varepsilon,\delta} E\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right) < \infty. \tag{4.4}$$

The estimate (4.4) follows from (3.3), (3.7), and (3.10):

$$\begin{aligned} E\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right) &= E\exp\left(\frac{\lambda^2}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathscr{H}}^2\right) \\ &\leq E\exp\left(\frac{\lambda^2}{2}C\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d \left|B_r^i - B_s^i\right|^{2H_i-2} drds\right) \\ &< \infty. \end{aligned} \tag{4.5}$$

Step 2 Now we prove that $u(t,x)$ is a weak solution to Equation (1.1) in the sense of Definition 4.4.2. Suppose φ is a smooth function with compact support. We know that,

$$\begin{aligned}\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\Delta\varphi(x)dxds \\ &\quad + \int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx. \end{aligned} \tag{4.6}$$

Therefore, it suffices to prove that

$$\lim_{\varepsilon,\delta\downarrow 0}\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx = \int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx),$$

in probability. From (4.6) and (4.3) it follows that $\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx$ converges in L^2 to the random variable

$$G = \int_{\mathbb{R}^d} u(t,x)\varphi(x)dx - \int_{\mathbb{R}^d} f(x)\varphi(x)dx - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(t,x)\Delta\varphi(x)dxds$$

as ε and δ tend to zero. Hence, if

$$B_{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}^d}(u^{\varepsilon,\delta}(s,x) - u(s,x))\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx$$

converges in L^2 to zero, then

$$\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}dsdx = \int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}dsdx - B_{\varepsilon,\delta}$$

converges to G in L^2. Thus $u(s,x)\varphi(x)$ will be Stratonovitch integrable and

$$\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx) = G,$$

which will complete the proof. In order to show the convergence to zero of $B_{\varepsilon,\delta}$, we will express the product $(u^{\varepsilon,\delta}(s,x)-u(s,x))\dot{W}^{\varepsilon,\delta}(s,x)$ as the sum of a divergence integral plus a trace term (see (2.4))

$$
\begin{aligned}
&(u^{\varepsilon,\delta}(s,x)-u(s,x))\dot{W}^{\varepsilon,\delta}(s,x)\\
&\quad=\int_0^t\int_{\mathbb{R}^d}(u^{\varepsilon,\delta}(s,x)-u(s,x))\varphi_\delta(s-r)p_\varepsilon(x-z)\delta W_{r,z}\\
&\qquad+\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}}\,.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
B_{\varepsilon,\delta} &= \int_0^t\int_{\mathbb{R}^d}\phi_{r,z}^{\varepsilon,\delta}\delta W_{r,z}\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\varphi(x)\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}}dsdx\\
&= B^1_{\varepsilon,\delta}+B^2_{\varepsilon,\delta},
\end{aligned}
\tag{4.7}
$$

where

$$
\phi_{r,z}^{\varepsilon,\delta}=\int_0^t\int_{\mathbb{R}^d}(u^{\varepsilon,\delta}(s,x)-u(s,x))\varphi(x)\varphi_\delta(s-r)p_\varepsilon(x-z)dsdx,
$$

and $\delta(\phi^{\varepsilon,\delta})=\int_0^t\int_{\mathbb{R}^d}\phi_{r,z}^{\varepsilon,\delta}\delta W_{r,z}$ denotes the divergence or the Skorohod integral of $\phi^{\varepsilon,\delta}$.

Step 3 For the term $B^1_{\varepsilon,\delta}$ we use the following L^2 estimate for the Skorohod integral

$$
E[(B^1_{\varepsilon,\delta})^2]\le E(\|\phi^{\varepsilon,\delta}\|^2_{\mathscr{H}})+E(\|D\phi^{\varepsilon,\delta}\|^2_{\mathscr{H}\otimes\mathscr{H}})\,. \tag{4.8}
$$

The first term in (4.8) is estimated as follows

$$
\begin{aligned}
E(\|\phi^{\varepsilon,\delta}\|^2_{\mathscr{H}}) &= \int_0^t\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}E\left[(u^{\varepsilon,\delta}(s,x)-u(s,x))(u^{\varepsilon,\delta}(r,y)-u(r,y))\right]\\
&\quad\times\varphi(x)\varphi(y)\langle\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot),\varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathscr{H}}dsdxdrdy.
\end{aligned}
\tag{4.9}
$$

Using lemmas 4.8.2 and 4.8.3 we can write

$$\begin{aligned}
& \langle \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot), \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathscr{H}} \\
= \ & \alpha_H \left(\int_{[0,t]^2} \varphi_\delta(s-\sigma)\varphi_\delta(r-\tau)|\sigma-\tau|^{2H_0-2} d\sigma d\tau \right) \\
& \times \left(\int_{\mathbb{R}^{2d}} p_\varepsilon(x-z)p_\varepsilon(y-w) \prod_{i=1}^{d} |z_i-w_i|^{2H_i-2} dz dw \right) \\
\leq \ & C|s-r|^{2H_0-2} \prod_{i=1}^{d} |x-y|^{2H_i-2}, \qquad (4.10)
\end{aligned}$$

for some constant $C > 0$. As a consequence, the integrand on the right-hand side of Equation (4.9) converges to zero as ε and δ tend to zero for any s, r, x, y due to (4.3). From (4.5) we get

$$\sup_{\varepsilon,\delta} \sup_{x\in\mathbb{R}^d} \sup_{0\leq s\leq t} E\left(u^{\varepsilon,\delta}(s,x)\right)^2 \leq \|f\|_\infty^2 \sup_{\varepsilon,\delta} \sup_{x\in\mathbb{R}^d} \sup_{0\leq s\leq t} E\exp\left(2V_{s,x}^{\varepsilon,\delta}\right) < \infty. \qquad (4.11)$$

Hence, from (4.10) and (4.11) we get that the integrand on the right-hand side of Equation (4.9) is bounded by $C|s-r|^{2H_0-2}\prod_{i=1}^d |x_i-y_i|^{2H_i-2}$, for some constant $C > 0$. Therefore, by dominated convergence we get that $E(\|\phi^{\varepsilon,\delta}\|_{\mathscr{H}}^2)$ converges to zero as ε and δ tend to zero.

Step 4 On the other hand, we have

$$D(u^{\varepsilon,\delta}(t,x)) = E^B\left[f(B_t+x)\exp(V_{t,x}^{\varepsilon,\delta})A_{t,x}^{\varepsilon,\delta} \right],$$

where $A_{t,x}^{\varepsilon,\delta}$ is defined in (3.2). Therefore,

$$
\begin{aligned}
& E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathscr{H}} \\
= \; & E^W E^B\Big(f(B_t^1+x) f(B_t^2+x) \\
& \times \exp(V_{t,x}^{\varepsilon,\delta}(B^1)+V_{t,x}^{\varepsilon,\delta}(B^2))\langle A_{t,x}^{\varepsilon,\delta}(B^1), A_{t,x}^{\varepsilon',\delta'}(B^2)\rangle_{\mathscr{H}}\Big), \qquad (4.12)
\end{aligned}
$$

where B^1 and B^2 are two independent d-dimensional Brownian motions, and here E^B denotes the expectation with respect to (B^1, B^2). Then from the previous results it is easy to show that

$$
\begin{aligned}
& \lim_{\varepsilon,\delta\downarrow 0} E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathscr{H}} \\
= \; & E\Big[f(B_t^1+x) f(B_t^2+x) \\
& \times \exp\left(\frac{\alpha_H}{2}\sum_{j,k=1}^{2}\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |B_s^{j,i}-B_r^{k,i}|^{2H_i-2} ds dr\right) \\
& \times \alpha_H \int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |B_s^{1,i}-B_r^{2,i}|^{2H_i-2} ds dr\Big]. \qquad (4.13)
\end{aligned}
$$

This implies that $u^{\varepsilon,\delta}(t,x)$ converges in the space $\mathbb{D}^{1,2}$ to $u(t,x)$ as $\delta\downarrow 0$ and $\varepsilon\downarrow 0$. Letting $\varepsilon'=\varepsilon$ and $\delta'=\delta$ in (4.12) and using the same argument as for (4.11), we obtain

$$
\sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le t} E\left\|D(u^{\varepsilon,\delta}(s,x))\right\|_{\mathscr{H}}^2 < \infty.
$$

Then

$$
\begin{aligned}
E\|D\phi^{\varepsilon,\delta}\|_{\mathscr{H}\otimes\mathscr{H}}^2 \; = \; & \int_0^t\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}} E\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)), D(u^{\varepsilon,\delta}(r,y)-u(r,y))\rangle_{\mathscr{H}} \\
& \times\varphi(x)\varphi(y)\langle\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot), \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathscr{H}} ds dx dr dy
\end{aligned}
$$

converges to zero as ε and δ tend to zero. Hence, by (4.8) $B^1_{\varepsilon,\delta}$ converges to zero in L^2 as ε and δ tend to zero.

Step 5 The second summand in the right-hand side of (4.7) can be written as

$$\begin{aligned} B^2_{\varepsilon,\delta} &= \int_0^t \int_{\mathbb{R}^d} \varphi(x)\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}} dsdx \\ &= \int_0^t \int_{\mathbb{R}^d} \varphi(x)E^B\left(f(B^x_s)\exp\left(V^{\varepsilon,\delta}_{s,x}\right)\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}}\right)dsdx \\ &\quad - \int_0^t \int_{\mathbb{R}} \varphi(x)E^B\left(f(B^x_s)\exp\left(V_{s,x}\right)\langle \delta(B^x_{s-\cdot}-\cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}}\right)dsdx \\ &= B^3_{\varepsilon,\delta} - B^4_{\varepsilon,\delta} \end{aligned}$$

where

$$\begin{aligned} \langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}} &= \alpha_H \int_{[0,s]^3}\int_{\mathbb{R}^{2d}} |r-v|^{2H_0-2}\prod_{i=1}^d |y_i-z_i|^{2H_i-2} \\ &\quad \times \varphi_\delta(s-r)p_\varepsilon(B^x_r-y) \\ &\quad \times \varphi_\delta(s-v)p_\varepsilon(x-z)dydyzdrdrdv, \end{aligned}$$

and

$$\begin{aligned} &\langle \delta(B^x_{s-\cdot}-\cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}} \\ = \;& \alpha_H \int_{[0,s]^2}\int_{\mathbb{R}^d} v^{2H_0-2}\prod_{i=1}^d |B^{x_i}_r-y_i|^{2H_i-2}\varphi_\delta(r-v)p_\varepsilon(x-y)dydvdr. \end{aligned}$$

Lemma 4.8.2 and Lemma 4.8.3 imply that

$$\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}} \le C\int_0^s r^{2H_0-2}\prod_{i=1}^d |B^i_r|^{2H_i-2}dr, \qquad (4.14)$$

and

$$\langle \delta(B^x_{s-\cdot} - \cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathscr{H}} \leq C\int_0^s r^{2H_0-2}\prod_{i=1}^d |B^i_r|^{2H_i-2}dr, \tag{4.15}$$

for some constant $C>0$. Then, from (4.14) and (4.15) and from the fact that the random variable $\int_0^s r^{2H_0-2}\prod_{i=1}^d |B^i_r|^{2H_i-2}dr$ is square integrable because of Lemma 4.8.4, we can apply the dominated convergence theorem and get that $B^3_{\varepsilon,\delta}$ and $B^4_{\varepsilon,\delta}$ converge both in L^2 to

$$\alpha_H\int_0^t\int_{\mathbb{R}^d}\varphi(x)E^B\left(f(B^x_s)\exp\left(V_{s,x}\right)\int_0^s r^{2H_0-2}\prod_{i=1}^d|B^i_r|^{2H_i-2}dr\right)dsdx,$$

as ε and δ tend to zero. Therefore $B^2_{\varepsilon,\delta}$ converges in L^2 to zero as ε and δ tend to zero. This completes the proof.

We can also show that the process $u(t,x)$ given in (4.1) is a mild solution to Equation (1.1), in the sense that the following equation holds

$$u(t,x) = p_t f(x) + \int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)u(s,y)dW_{s,y},$$

where p_t denotes the heat kernel and $p_t f(x) = \int_{\mathbb{R}^d} p_t(x-y)f(y)dy$. In fact, as in the proof of Theorem 4.4.3 we need to show that

$$\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)(u(s,y)-u^{\varepsilon,\delta}(s,y))dW^{\varepsilon,\delta}_{s,y}$$

converges to zero in L^2. This can be proved by the same arguments as in the proof of Theorem 4.4.3 replacing φ by the heat kernel. For instance, instead of the Estimate

(4.10) we obtain

$$\int_0^t \int_0^t \int_{\mathbb{R}^{2d}} p_{t-r}(x-y)p_{t-s}(x-z)|s-r|^{2H_0-2} \prod_{i=1}^d |y-z|^{2H_i-2} dydzdrds$$
$$= \int_0^t \int_0^t |s-r|^{2H_0-2} E\left(\prod_{i=1}^d |B_{t-r}^{1,i} - B_{t-s}^{2,i}|^{2H_i-2} \right) drds < \infty.$$

Remark 4.4.4. *The uniqueness of the solution remains to be investigated in a future work. The definition of the Stratonovich integral as a limit in probability makes the uniqueness problem nontrivial, and it is not clear how to proceed.*

As a corollary of Theorem 4.4.3 we obtain the following result.

Corollary 4.4.5. *Suppose* $2H_0 + \sum_{i=1}^d H_i > d+1$*. Then the solution* $u(t,x)$ *given by (4.1) has finite moments of all orders. Moreover, for any positive integer* p*, we have*

$$E\left(u(t,x)^p\right) = E\left(\prod_{j=1}^p f(B_t^j + x) \right. \tag{4.16}$$
$$\left. \times \exp\left[\frac{\alpha_H}{2} \sum_{j,k=1}^p \int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^{j,i} - B_r^{k,i}|^{2H_i-2} dsdr \right] \right),$$

where $B_1, \ldots, B_p$ *are independent* d*-dimensional standard Brownian motions.*

Remark 4.4.6. *In the previous work [22] a formula similar to (4.16) was obtained in special case* $H_1 = \cdots = H_d = \frac{1}{2}$*, without condition* $2H_0 + \sum_{i=1}^d H_i > d+1$*. This type of formula was proved if* $d = 1$ *and* $H_0 > \frac{3}{4}$*. In the case of the Skorohod-type equation a formula for the moments of the solution similar to (4.16) was established in [22] if* $H_0 > \frac{1}{2}$*, and* $d = 1$ *or* $d = 2$ *and* t *is small enough.*

4.5 Behavior of the Feynman-Kac formula

In this section we present two straightforward applications of the Feynman-Kac formula.

4.5.1 Hölder continuity of the solution

In this subsection, we study the Hölder continuity of the solution to the equation (1.1). The main result of this section is the following theorem.

Theorem 4.5.1. *Suppose that* $2H_0 + \sum_{i=1}^d H_i > d+1$ *and let* $u(t,x)$ *be the solution of Equation (1.1). Then* $u(t,x)$ *has a continuous modification such that for any* $\rho \in \left(0, \frac{\kappa}{2}\right)$ *(where* κ *has been defined in (3.9)), and any compact rectangle* $I \subset \mathbb{R}_+ \times \mathbb{R}^d$ *there exists a positive random variable* K_I *such that almost surely, for any* $(s,x),(t,y) \in I$ *we have*

$$|u(t,y) - u(s,x)| \leq K_I(|t-s|^\rho + |y-x|^{2\rho}).$$

Proof The proof will be done in several steps.

Step 1 Recall that $V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y)W(dr,dy)$ denotes the random variable introduced in (3.4) and

$$u(t,x) = E^B\left(f(B_t^x)\exp\left(V_{(t,x)}\right)\right).$$

Set $V = V_{s,x}$ and $\tilde{V} = V_{t,y}$. Then we can write

$$\begin{aligned} E^W|u(s,x) - u(t,y)|^p &= E^W|E^B(e^V - e^{\tilde{V}})|^p \\ &\le E^W\left(E^B[|\tilde{V} - V|e^{\max(V,\tilde{V})}]\right)^p \\ &\le E^W\left[\left(E^B e^{2\max(V,\tilde{V})}\right)^{p/2}\left(E^B(\tilde{V} - V)^2\right)^{p/2}\right] \\ &\le \left[E^W E^B e^{2p\max(V,\tilde{V})}\right]^{\frac{1}{2}}\left[E^W\left(E^B(\tilde{V} - V)^2\right)^p\right]^{\frac{1}{2}}. \end{aligned}$$

Applying Minkowski's inequality, the equivalence between the L^2 norm and the L^p norm for a Gaussian random variable, and using the exponential integrability property (3.10) we obtain

$$\begin{aligned} E^W|u(s,x) - u(t,y)|^p &\le C\left[E^W\left(E^B(\tilde{V} - V)^2\right)^p\right]^{\frac{1}{2}} \\ &\le C_p\left[E^B E^W|\tilde{V} - V|^2\right]^{p/2}. \end{aligned} \tag{5.1}$$

In a similar way to (3.5) we can deduce the following formula for the conditional variance of $\tilde{V} - V$

$$\begin{aligned} E^W|\tilde{V} - V|^2 = \alpha_H E^B \Bigg(&\int_0^s \int_0^s |r - v|^{2H_0 - 2} \prod_{i=1}^d |B^i_{s-r} - B^i_{s-v}|^{2H_i - 2} drdv \\ &+ \int_0^t \int_0^t |r - v|^{2H_0 - 2} \prod_{i=1}^d |B^i_{t-r} - B^i_{t-v}|^{2H_i - 2} drdv \\ &- 2\int_0^s \int_0^t |r - v|^{2H_0 - 2} \prod_{i=1}^d |B^i_{s-r} - B^i_{t-v} + x_i - y_i|^{2H_i - 2} drdv \Bigg) \\ &:= \alpha_H C(s,t,x,y). \end{aligned} \tag{5.2}$$

Step 2 Fix $1 \le j \le d$. Let us estimate $C(s,t,x,y)$ when $s = t$, and $x_i = y_i$ for all $i \ne j$. We can write

$$C(t,t,x,y) = 2\int_0^t \int_0^t |r-v|^{\kappa-1} \prod_{i\ne j}^{d} E\left(|\xi|^{2H_i-2}\right) E\left(|\xi|^{2H_j-2} - |z+\xi|^{2H_j-2}\right) drdv, \quad (5.3)$$

where $z = \frac{x_j - y_j}{\sqrt{|r-v|}}$ and ξ is a standard normal variable. Set $\beta_j = 2H_j + 1 > 2$. By Lemma 4.8.6 the factor $E\left(|\xi|^{2H_j-2} - |z+\xi|^{2H_j-2}\right)$ can be bounded by a constant if $|r-v| \le (x_j - y_j)^2$, and it can be bounded by $C|x_j - y_j|^{\beta_j} |r-v|^{-\beta_j/2}$ if $|r-s| > (x_j - y_j)^2$. In this way we obtain

$$\begin{aligned} C(t,t,x,y) \le & C\int_{\{0<r,v<t,|r-v|\le(x_j-y_j)^2\}} |r-v|^{\kappa-1} drdv \\ & + C|x_j - y_j|^{\beta_j} \int_{\{0<r,v<t,|r-v|>(x_j-y_j)^2\}} |r-v|^{\kappa-1-\beta_j/2} drdv \\ & \le C|x_j - y_j|^{2\kappa}. \end{aligned}$$

So, from (5.1) we have

$$E^W |u(t,x) - u(t,y)|^p \le C|x_j - y_j|^{\kappa p}. \quad (5.4)$$

Step 3 Suppose now that $s < t$, and $x = y$. Set $\delta = \sum_{i=1}^d H_i - d$. We have

$$\begin{aligned} & C(s,t,x,x) \\ = & C\Bigg[\int_s^t \int_s^t |r-v|^{\kappa-1} drdv \\ & + \int_0^s \int_0^t |r-v|^{2H_0-2} \left(|r-v|^{\delta} - |r-v+t-s|^{\delta}\right) drdv \Bigg]. \end{aligned}$$

The first integral is $O((t-s)^{\kappa+1})$, when $t-s$ is small. For the second integral we use the change of variable $\sigma = r-v, v=\tau$, and we have

$$
\begin{aligned}
&\int_0^s\int_0^t |r-v|^{2H_0-2}\big(|r-v|^{\delta}-|r-v+t-s|^{\delta}\big)drdv\\
\leq&\int_0^t d\tau\int_{-t}^{s}|\sigma|^{2H_0-2}\big||\sigma|^{\delta}-|\sigma+t-s|^{\delta}\big|d\sigma\\
=&t\Bigg[\int_0^s \sigma^{2H_0-2}\big(\sigma^{\delta}-(\sigma+t-s)^{\delta}\big)d\sigma\\
&+\int_{-t}^{s-t}(-\sigma)^{2H_0-2}\big((-\sigma-t+s)^{\delta}-(-\sigma)^{\delta}\big)d\sigma\\
&+\int_{s-t}^{0}(-\sigma)^{2H_0-2}\big|(-\sigma)^{\delta}-(\sigma+t-s)^{\delta}\big|d\sigma\Bigg]\\
=&t[A'+B'+C'].
\end{aligned}
$$

For the first term in the above decomposition we can write

$$
\begin{aligned}
A' &= (t-s)^{\kappa-1}\int_0^{\frac{t_1}{t-s}}\sigma^{2H_0-2}\big(\sigma^{\delta}-(\sigma+1)^{\delta}\big)d\sigma\\
&\leq (t-s)^{\kappa-1}\int_0^{\infty}\sigma^{2H_0-2}\big(\sigma^{\delta}-(\sigma+1)^{\delta}\big)d\sigma\\
&\leq C(t-s)^{\kappa},
\end{aligned}
$$

because $2H_0+\sum_{i=1}^{d}-d-3<-1$. Similarly we can get that

$$
B' \leq (t-s)^{\kappa}\int_1^{\infty}\sigma^{2H_0-2}\big(\sigma^{\delta}-(\sigma+1)^{\delta}\big)d\sigma.
$$

At last,

$$
C' \leq \int_0^{t-s}\sigma^{2H_0-2}\big(\sigma^{\delta}+(t-s-\sigma)^{\delta}\big)d\sigma = C(t-s)^{\kappa}.
$$

So we have

$$
E^W|u(s,x)-u(t,y)|^p \leq C(t-s)^{\frac{\kappa}{2}p}. \tag{5.5}
$$

Step 4 Combining Equation 5.4 and Equation 5.5 with the estimates (5.1) and (5.2), the result of this theorem now can be concluded from Theorem 1.4.1 in Kunita [30] if we choose p large enough.

4.5.2 Regularity of the density

In this subsection we shall use the Feynman-Kac formula established in the previous section to show that for any t and x, the probability law of the solution $u(t,x)$ of Equation (1.1) has a smooth density with respect to the Lebesgue measure. To this end we shall show that $\|Du(t,x)\|_{\mathscr{H}}$ has negative moments of all orders.

Theorem 4.5.2. *Suppose that* $2H_0+\sum_{i=1}^d H_i > d+1$. *Fix* $t>0$ *and* $x \in \mathbb{R}^d$. *Assume that for any positive number* p, $E|f(B_t+x)|^{-p}<\infty$. *Then, the law of* $u(t,x)$ *has a smooth density.*

Proof From Theorem 4.3 we can write

$$u(t,x) = E^B\left[f(B_t^x)\exp\left(V_{t,x}\right)\right] .$$

The Malliavin derivative of the solution is given by

$$D_{r,y}u(t,x) = E^B\left[f(B_t^x)\exp\left(V_{t,x}\right)\delta(B_{t-r}^x-y)\right] .$$

It is not difficult to show that $u(t,x)\in\mathbb{D}^\infty$. Thus, by the general criterion for the smoothness of densities (see [38]), it suffices to show that $E\left(\|Du(t,x)\|_{\mathscr{H}}^{-2p}\right)<\infty$ for any $t>0$

and $x\in\mathbb{R}^d$. We have

$$\begin{aligned}\|Du(t,x)\|^2_{\mathscr{H}} &= E^B\Big[f(B^1_t+x)f(B^2_t+x)\exp\left(V_{t,x}(B^1)+V_{t,x}(B^2)\right)\\ &\quad\times\langle\delta(B^{1,x}_{t-r}-y),\delta(B^{2,x}_{t-r}-y)\rangle_{\mathscr{H}}\Big]\\ &= \alpha_H E^B\Big[f(B^1_t+x)f(B^2_t+x)\exp\left(V_{t,x}(B^1)+V_{t,x}(B^2)\right)\\ &\quad\times\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d |B^{1,i}_{t-r}-B^{2,i}_{t-s}|^{2H_i-2}drds\Big],\end{aligned}$$

where B^1 and B^2 are independent d-dimensional Brownian motions. By Jensen's inequality, we have for any $p>0$,

$$\begin{aligned}&\|Du(t,x)\|^{-2p}_{\mathscr{H}}\\ \le\;& (\alpha_H)^{-p}E^B\Big[|f(B^1_t+x)f(B^2_t+x)|^{-p}\exp\left(-p\left[V_{t,x}(B^1)+V_{t,x}(B^2)\right]\right)\\ &\times\left(\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d|B^{i,1}_{t-r}-B^{2,i}_{t-s}|^{2H_i-2}drds\right)^{-p}\Big].\end{aligned}$$

Hence by Hölder's inequality, we obtain

$$\begin{aligned}&E\,\|Du(t,x)\|^{-2p}_{\mathscr{H}}\\ \le\;& (\alpha_H)^{-p}\left(E|f(B^1_t+x)f(B^2_t+x)|^{-pp_1}\right)^{\frac{1}{p_1}}\\ &\times\left(E\exp\left(-pp_2\left[V_{t,x}(B^1)+V_{t,x}(B^2)\right]\right)\right)^{\frac{1}{p_2}}\\ &\times\left(E\left(\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d|B^{1,i}_{t-r}-B^{2,i}_{t-s}|^{2H_i-2}drds\right)^{-pp_3}\right)^{\frac{1}{p_3}}\\ =\;& I_1I_2I_3,\end{aligned}$$

where $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. The first factor I_1 is finite by the assumption on f and Hölder's inequality. The second factor is finite by Theorem 4.3.3. Finally, from Jensen's inequality, we have

$$\begin{aligned} I_3^{p_3} &= E\left[t^{-2pp_3}\left\{\frac{1}{t^2}\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{2H_i-2}drds\right\}^{-pp_3}\right] \\ &\leq E\left[t^{-2pp_3-2}\left\{\int_0^t\int_0^t |r-s|^{-(2H_0-2)pp_3}\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{-(2H_i-2)pp_3}drds\right\}\right] \\ &\leq C\int_0^t\int_0^t |r-s|^{-(2H_0-2)pp_3}E\left\{\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{-(2H_i-2)pp_3}\right\}drds \\ &< \infty. \end{aligned}$$

This completes the proof.

4.6 Case $H_0 > \frac{3}{4}, H_1 = \frac{1}{2}$ and $d = 1$

4.6.1 Preliminaries

In this case, all the setup is the same as before except that if ϕ and ψ are functions in $\mathscr{E}$, then

$$\begin{aligned} E\left(W(\phi)W(\psi)\right) = \langle \phi, \psi\rangle_{\mathscr{H}} = \alpha_{H_0} \\ \times \int_0^\infty\int_0^\infty\int_{\mathbb{R}} \phi(s,x)\psi(t,x)|s-t|^{2H_0-2}dsdtdx, \end{aligned}$$

where $\alpha_{H_0} = H_0(2H_0-1)$.

4.6.2 Definition and exponential integrability of $\int_0^t\int_{\mathbb{R}}\delta(B_{t-r}^x-y)W(dr,dy)$

Similarly we have the following theorem as well.

Theorem 4.6.1. *Suppose that $H_1 = 1/2$ and $H_0 > 3/4$. Then, for any $\varepsilon > 0$ and $\delta > 0$, $A^{\varepsilon,\delta}_{t,x}$ defined in (3.2) belongs to $\mathscr{H}$ and the family of random variables $V^{\varepsilon,\delta}_{t,x}$ defined in (3.3) converges in L^2 to a limit denoted by*

$$V_{t,x} = \int_0^t \int_{\mathbb{R}} \delta(B^x_{t-r} - y) W(dr,dy)\,. \tag{6.6}$$

Conditional to B, $V_{t,x}$ is a Gaussian random variable with mean 0 and variance

$$\mathrm{Var}^W(V_{t,x}) = \alpha_{H_0} \int_0^t \int_0^t |r-s|^{2H_0-2} \delta\left(B_r - B_s\right) drds\,. \tag{6.7}$$

Proof Fix ε, ε', δ and $\delta' > 0$.

$$\begin{aligned}
E^B E^W \left(V^{\varepsilon,\delta}_{t,x}, V^{\varepsilon',\delta'}_{t,x}\right) &= E^B \left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon',\delta'}_{t,x} \right\rangle_{\mathscr{H}} \\
&= \alpha_{H_0} E^B \Bigg(\int_{[0,t]^4} \int_{\mathbb{R}} p_\varepsilon(B^x_s - y) p_{\varepsilon'}(B^x_r - y) \\
&\quad \times \varphi_\delta(t-s-u) \varphi_{\delta'}(t-r-v) |u-v|^{2H_0-2} dydudvddsdr \Bigg) \\
&= \alpha_{H_0} \Bigg(\int_{[0,t]^4} E^B p_{\varepsilon+\varepsilon'}(B_s - B_r) \\
&\quad \times \varphi_\delta(t-s-u) \varphi_{\delta'}(t-r-v) |u-v|^{2H_0-2} dudvdsdr \Bigg) \\
&= \alpha_{H_0} \Bigg(\int_{[0,t]^4} \frac{1}{\sqrt{2\pi}} \left(\varepsilon + \varepsilon' + |s-r|\right)^{-1/2} \\
&\quad \times \varphi_\delta(t-s-u) \varphi_{\delta'}(t-r-v) |u-v|^{2H_0-2} dudvdsdr \Bigg).
\end{aligned}$$

By Lemma 4.8.3,

$$\int_{[0,t]^2} \left(\varepsilon+\varepsilon'+|s-r|\right)^{-1/2} \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2} ddudv$$
$$\leq C|s-r|^{2H_0-5/2}.$$

Then by the dominated convergence theorem, $E^B E^W \left(V_{t,x}^{\varepsilon,\delta}, V_{t,x}^{\varepsilon',\delta'}\right)$ converges to

$$\frac{\alpha_{H_0}}{\sqrt{2\pi}} \int_{[0,t]^2} |s-r|^{2H_0-5/2} dsdr$$

as ε, ε', δ, and δ' tend to zero. This implies that $V_{t,x}^{\varepsilon,\delta}$ converges in L^2 as ε and δ tend to zero to a limit denoted by $V_{t,x}$. On the other hand, from the above computations

$$E^W\left[\left(V_{t,x}^{\varepsilon,\delta}\right)^2\right] = \alpha_{H_0} \int_{[0,t]^4} p_{2\varepsilon}(B_s - B_r) \times \varphi_\delta(t-s-u)\varphi_\delta(t-r-v)|u-v|^{2H_0-2} dudvdsdr.$$

and this expression converges to right-hand side of Equation (6.7) almost surely. Moreover, because of the above arguments, the convergence is also in L^1, and this implies Equation (6.7).

4.6.3 Feynman-Kac formula

By Proposition 3.3 and Theorem 6.2 in [22], we have the following theorem.

Theorem 4.6.2. *Suppose that $H_1 = 1/2$ and $H_0 > 3/4$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$E \exp\left(\lambda \int_0^t \int_{\mathbb{R}} \delta(B_{t-r}^x - y) W(dr, dy)\right) < \infty,$$

and for any measurable and bounded function f the process

$$u(t,x) = E^B\left(f(B_t^x)\exp\left(\int_0^t\int_{\mathbb{R}}\delta(B_{t-r}^x - y)W(dr,dy)\right)\right) \tag{6.8}$$

is a weak solution to Equation (1.1).

4.6.4 Hölder continuity

We also have the following theorem, whose proof is similar to that of Theorem 6.1.

Theorem 4.6.3. *Suppose $H_1 = 1/2, H_0 > 3/4$ and let $u(t,x)$ be the solution of Equation (1.1). Then $u(t,x)$ has a continuous modification such that for any $\rho \in (0, H_0 - 3/4)$ and any compact rectangle $I \subset \mathbb{R}_+ \times \mathbb{R}$ there exists a positive random variable K_I such that almost surely, for any $(t_1,x_1),(t_2,x_2) \in I$ we have*

$$|u(t_2,x_2) - u(t_1,x_1)| \le K_I(|t_2 - t_1|^\rho + |x_2 - x_1|^{2\rho}).$$

Proof As in the proof of Theorem 6.1, we have

$$E^W|u(s,x) - u(t,y)|^p \le C_p\left[E^B E^W|\tilde{V} - V|^2\right]^{p/2},$$

where $V = \int_0^t\int_{\mathbb{R}}\delta(B_{t-r}^x - z)W(dr,dz)$ and $\tilde{V} = \int_0^s\int_{\mathbb{R}}\delta(B_{s-r}^y - z)W(dr,dz)$. If $s = t$, we can write

$$\begin{aligned} E^B E^W|\tilde{V} - V|^2 &= 2\int_0^t\int_0^t |r - v|^{2H_0 - 2} \\ &\quad \times E\left[\delta(B_r - B_v) - \delta(B_r - B_v + x - y)\right] drdv \\ &= \frac{2}{\sqrt{2\pi}}\int_0^t\int_0^t |r - v|^{2H_0 - 5/2}(1 - e^{-\frac{(x-y)^2}{2|r-v|}})drdv. \end{aligned}$$

For any $2\rho < \gamma < 2H_0 - 3/2$, we have $1 - e^{-\frac{(x-y)^2}{2|r-v|}} \leq \left(\frac{(x-y)^2}{2|r-v|}\right)^\gamma$. Thus $E^B E^W|\tilde{V} - V|^2 \leq C_\gamma |x-y|^{2\gamma}$. Consequently, we have

$$E^W|u(t,x) - u(t,y)|^p \leq C|x-y|^{\gamma p}. \tag{6.9}$$

On the other hand, if $x = y$,

$$E^B E^W|\tilde{V} - V|^2 = C\left[\int_s^t \int_s^t |r-v|^{2H_0-5/2} drdv \right.$$
$$\left. + \int_0^s \int_0^t |r-v|^{2H_0-2}\left(|r-v|^{-1/2} - |r-v+t-s|^{-\frac{1}{2}}\right) drds\right],$$

and by a similar computation as step 3 before, we can get

$$E^W|u(s,x) - u(t,x)|^p \leq C(t-s)^{(H_0-3/4)p}. \tag{6.10}$$

Combining (6.9) and (6.10) we prove the theorem.

4.6.5 Regularity of the density

We can also show the following result.

Theorem 4.6.4. *Suppose $d = 1$, $H_1 = 1/2$ and $H_0 > 3/4$. Fix $t > 0$ and $x \in \mathbb{R}$. Assume that for any positive number p, $E|f(B_t + x)|^{-p} < \infty$. Then, the law of $u(t,x)$ has a smooth density.*

Proof The proof is similar to that of Theorem 4.5.2, using the existence of finite moments of all orders for the self-intersection local time of the Brownian motion proved in the Appendix (see Proposition 4.8.7).

4.7 Skorohod type equations and chaos expansion

In this section we consider the following heat equation on $\mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamond \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} W \\ u(0,x) = f(x) \,. \end{cases} \tag{7.1}$$

The difference between the above equation and Equation (1.1) is that here we use the Wick product $\diamond$ (see [25], for example). This equation is studied in [22] in the case $H_1 = \cdots = H_d = 1/2$. As in that paper, we can define the following notion of mild solution.

Definition 4.7.1. *An adapted random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that $E(u^2(t,x)) < \infty$ for all (t,x) is a mild solution to Equation (7.1) if for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s), s \geq 0, y \in \mathbb{R}^d\}$ is Skorohod integrable, and the following equation holds*

$$u(t,x) = p_t f(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y) \delta W_{s,y}, \tag{7.2}$$

where $p_t(x)$ denotes the heat kernel and $p_t f(x) = \int_{\mathbb{R}^d} p_t(x-y) f(y) dy$.

As in the paper [22] the midl solution $u(t,x)$ to (7.1) admits the following Wiener chaos expansion

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{7.3}$$

where I_n denotes the multiple stochastic integral with respect to W and $f_n(\cdot, t, x)$ is a symmetric element in $\mathscr{H}^{\otimes n}$, defined explicitly as

$$f_n(s_1, y_1, \ldots, s_n, y_n, t, x) = \frac{1}{n!} \tag{7.4}$$
$$\times p_{t-s_{\sigma(n)}}(x - y_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)} - y_{\sigma(1)}) p_{s_{\sigma(1)}} f(y_{\sigma(1)}).$$

In the above equation σ denotes a permutation of $\{1, 2, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Moreover, the solution if it exist, it will be unique because the kernels in the Wiener chaos expansion are uniquely determined.

The following is the main result of this section.

Theorem 4.7.2. *Suppose that* $2H_0 + \sum_{i=1}^d H_i > d + 1$ *and that* f *is a bounded measurable function. Then the process*

$$\begin{aligned} u(t,x) &= E^B \Bigg[f(B_t^x) \exp\Bigg(\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy) \\ &\quad - \frac{1}{2} \alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d \left| B_r^i - B_s^i \right|^{2H_i-2} dr ds \Bigg) \Bigg] \end{aligned} \tag{7.5}$$

is the unique mild solution to Equation (1.1).

Proof From Theorem 3.2 we obtain that the expectation E^B in Equation (7.5) is well defined. Then, it suffices to show that the random variable $u(t,x)$ has the Wiener chaos expansion (7.3). This can be easily proved by expanding the exponential and then taken the expectation with respect to B.

Theorem 3.1 implies that almost surely $\delta(B_{t-\cdot}^x - \cdot)$ is and element of $\mathscr{H}$ with a norm given by (3.4). As a consequence, almost surely with respect to the Brownian motion

B, we have the following chaos expansion for the exponential factor in Equation (7.5)

$$\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(B^x_{t-r}-y)W(dr,dy)\right.$$
$$\left.-\frac{1}{2}\alpha_H\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d\left|B^i_r-B^i_s\right|^{2H_i-2}drds\right)=\sum_{n=0}^{\infty}I_n(g_n)\,,$$

where g_n is the symmetric element in $\mathscr{H}^{\otimes n}$ given by

$$g_n(s_1,y_1,\dots,s_n,y_n,t,x)=\frac{1}{n!}\,\delta(B^x_{t-s_1}-y_1)\cdots\delta(B^x_{t-s_n}-y_n)\,. \tag{7.6}$$

Thus the right hand side of (7.5) admits the following chaos expansion

$$u(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h_n(\cdot,t,x))\,, \tag{7.7}$$

with

$$h_n(t,x)=E^B\left[f(B^x_t)\delta(B^x_{t-s_1}-y_1)\cdots\delta(B^x_{t-s_n}-y_n)\right]\,. \tag{7.8}$$

This can be regarded as a Feynman-Kac formula for the coefficients of chaos expansion of the solution of (7.1). To compute the above expectation we shall use the following

$$\begin{aligned} E^B\left[f(B^x_t)\delta(B^x_t-y)\middle|\mathscr{F}_s\right] &= \int_{\mathbb{R}^d}p_{t-s}(B^x_s-z)f(z)\delta(z-y)dz \\ &= p_{t-s}(B^x_s-y)f(y)\,. \end{aligned} \tag{7.9}$$

Assume that $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t$ for some permutation σ of $\{1,2,\cdots,n\}$. Then conditioning with respect to $\mathscr{F}_{t-s_{\sigma(1)}}$ and using the Markov property of the Brownian

motion we have

$$\begin{aligned} h_n(t,x) &= E^B\{E^B[\delta(B^x_{t-s_{\sigma(n)}}-y_{\sigma(n)}) \\ &\quad \times\cdots\delta(B^x_{t-s_{\sigma(1)}}-y_{\sigma(1)})f(B^x_t)\big|\mathscr{F}_{t-s_{\sigma(1)}}]\} \\ &= E^B\left[\delta(B^x_{t-s_{\sigma(n)}}-y_{\sigma(n)})\cdots\delta(B^x_{t-s_{\sigma(1)}}-y_{\sigma(1)})p_{s_{\sigma(1)}}f(B^x_{t-s_{\sigma(1)}})\right]. \end{aligned}$$

Conditioning with respect to $\mathscr{F}_{t-s_{\sigma(2)}}$ and using (7.9), we have

$$\begin{aligned} h_n(t,x) &= E^B\{E^B[\delta(B_{t-s^x_{\sigma(n)}}-y_{\sigma(n)}) \\ &\quad \times\delta(B^x_{t-s_{\sigma(1)}}-y_{\sigma(1)})p_{s_{\sigma(1)}}f(B^x_{t-s_{\sigma(1)}})]\big|\mathscr{F}_{t-s_{\sigma(2)}}\} \\ &= E^B\Bigg\{\delta(B_{t-s^x_{\sigma(n)}}-y_{\sigma(n)})\cdots\delta(B^x_{t-s_{\sigma(2)}}-y_{\sigma(2)}) \\ &\qquad \times E^B\left[\delta(B^x_{t-s_{\sigma(1)}}-y_{\sigma(1)})p_{s_{\sigma(1)}}f(B^x_{t-s_{\sigma(1)}})\big|\mathscr{F}_{t-s_{\sigma(2)}}\right]\Bigg\} \\ &= E^B\Bigg[\delta(B_{t-s^x_{\sigma(n)}}-y_{\sigma(n)})\cdots\delta(B^x_{t-s_{\sigma(2)}}-y_{\sigma(2)}) \\ &\qquad \times p_{s_{\sigma(2)}-s_{\sigma(1)}}(B^x_{t-s_{\sigma(2)}}-y_{\sigma(1)})p_{s_{\sigma(1)}}f(y_{\sigma(1)})\Bigg]. \end{aligned}$$

Continuing this way we shall find out that

$$h_n(t,x)=p_{t-s_{\sigma(n)}}(x-y_{\sigma(n)})\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)}-y_{\sigma(1)})p_{s_{\sigma(1)}}f(y_{\sigma(1)})$$

which is the same as (7.4).

Remark 4.7.3. *The method of this section can be applied to obtain a Feynman-Kac formula for the coefficients of the chaos expansion of the solution to Equation (1.1):*

$$u(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h_n(\cdot,t,x)),$$

with

$$
\begin{aligned}
h_n(t,x) &= E^B\Bigg[f(B_t^x)\delta(B_{t-s_1}^x - y_1)\cdots\delta(B_{t-s_n}^x - y_n) \\
&\quad \times \exp\left(\frac{1}{2}\alpha_H \int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d \left|B_r^i - B_s^i\right|^{2H_i-2} drds\right)\Bigg].
\end{aligned}
\tag{7.10}
$$

Remark 4.7.4. *We can also consider Equation (1.1) when* $d=1$, $H_1 = 1/2$ *and* $H_0 > 3/4$. *In this case we see easily that the solution* $u(t,x)$ *admits the following chaos expansion*

$$
u(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(\cdot,t,x)),
$$

with

$$
\begin{aligned}
h_n(t,x) &= E^B\Bigg[f(B_t^x)\delta(B_{t-s_1}^x - y_1)\cdots\delta(B_{t-s_n}^x - y_n) \\
&\quad \times \exp\left(\frac{1}{2}\alpha_{H_0} \int_0^t\int_0^t |r-s|^{2H_0-2}\delta\left(B_r - B_s\right) drds\right)\Bigg].
\end{aligned}
\tag{7.11}
$$

From the Feyman-Kac formula we can derive the following formula for the moments of the solution analogous to (4.16), which can be compared with the formulas obtained in [22] in the case $H_1 = \cdots = H_d = \frac{1}{2}$

$$
\begin{aligned}
E\left(u(t,x)^p\right) = E\Bigg(&\prod_{j=1}^p f(B_t^j + x) \\
&\times \exp\left[\alpha_H \sum_{j,k=1, j<k}^p \int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |B_s^{j,i} - B_r^{k,i}|^{2H_i-2} dsdr\right]\Bigg),
\end{aligned}
$$

where $p \geq 1$ is an integer, and $B^j, 1 \leq j \leq d$, are independent d-dimensional Brownian motions.

4.8 Appendix

Lemma 4.8.1. *Suppose* $0 < \alpha < 1, \varepsilon > 0, x > 0$, *and that* X *is a standard normal random variable. Then there is a constant* C *independent of* x *and* ε *(it may depend on* α*) such that*

$$E|x+\varepsilon X|^{-\alpha} \leq C \min(\varepsilon^{-\alpha}, x^{-\alpha}) .$$

Proof It is straightforward to check that $K = \sup_{z\geq 0} E|z+X|^{-\alpha} < \infty$. Thus

$$E|x+\varepsilon X|^{-\alpha} = \varepsilon^{-\alpha} E|\frac{x}{\varepsilon} + X|^{-\alpha} \leq K\varepsilon^{-\alpha} . \tag{8.12}$$

On the other hand,

$$\begin{aligned} E|x+\varepsilon X|^{-\alpha} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x+\varepsilon y|^{-\alpha} e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{\{|x+\varepsilon y| > \frac{x}{2}\}} |x+\varepsilon y|^{-\alpha} e^{-\frac{y^2}{2}} dy \right. \\ &\quad \left. + \int_{\{|x+\varepsilon y| \leq \frac{x}{2}\}} |x+\varepsilon y|^{-\alpha} e^{-\frac{y^2}{2}} dy \right) . \end{aligned}$$

It is easy to see that the first integral is bounded by $Cx^{-\alpha}$ for some constant C. The second integral, denoted by B is bounded as follows.

$$\begin{aligned} B &= C\frac{1}{\varepsilon} \int_{|z|<\frac{x}{2}} |z|^{-\alpha} e^{-\frac{(z-x)^2}{2\varepsilon^2}} dz \leq C\frac{1}{\varepsilon} \int_{|z|<\frac{x}{2}} |z|^{-\alpha} e^{-\frac{x^2}{8\varepsilon^2}} dz \\ &= C\frac{x}{\varepsilon} e^{-\frac{x^2}{8\varepsilon^2}} x^{-\alpha} \leq Cx^{-\alpha} . \end{aligned}$$

Thus we have $E|x+\varepsilon X|^{-\alpha} \le C|x|^{-\alpha}$. Combining this with (8.12), we obtain the lemma.

Lemma 4.8.2. *Suppose* $\alpha \in (0,1)$. *There exists a constant* $C > 0$ *, such that*

$$\sup_{\varepsilon,\varepsilon'} \int_{\mathbb{R}^2} p_\varepsilon(x_1+y_1) p_{\varepsilon'}(x_2+y_2) |y_1-y_2|^{-\alpha} dy_1 dy_2 \le C|x_1-x_2|^{-\alpha}.$$

Proof We can write

$$\int_{\mathbb{R}^2} p_\varepsilon(x_1+y_1) p_{\varepsilon'}(x_2+y_2) |y_1-y_2|^{-\alpha} dy_1 dy_2 = E \left(|\varepsilon X_1 - x_1 - \varepsilon' X_2 + x_2|^{-\alpha} \right) .$$

Thus Lemma 4.8.2 follows directly from Lemma 4.8.1.

Lemma 4.8.3. *Suppose* $\alpha \in (0,1)$. *There exists a constant* $C > 0$ *, such that*

$$\sup_{\delta,\delta'} \int_0^t \int_0^t \varphi_\delta(t-s_1-r_1) \varphi_{\delta'}(t-s_2-r_2) |r_1-r_2|^{-\alpha} dr_1 dr_2 \le C|s_1-s_2|^{-\alpha}$$

Proof Since

$$p_\delta(x) \ge p_\delta(x) I_{[0,\sqrt{\delta}]}(x) = \frac{1}{\sqrt{2\pi\delta}} e^{-\frac{x^2}{2\delta}} I_{[0,\sqrt{\delta}]}(x) \ge \frac{1}{\sqrt{2\pi e}} \varphi_{\sqrt{\delta}}(x) ,$$

the lemma follows from Lemma 4.8.2.

Lemma 4.8.4. *Suppose that* $2H_0 + \sum_{i=1}^d H_i > d+1$. *Let* $B^1, \dots, B^d$ *be independent one-dimensional Brownian motions. Then we have*

$$E\left(\int_0^t s^{2H_0-2} \prod_{i=1}^d |B_s^i|^{2H_i-2} ds \right)^2 < \infty.$$

Proof We can write

$$E\left(\int_0^t s^{2H_0-2}\prod_{i=1}^d |B_s^i|^{2H_i-2}ds\right)^2 = 2\int_0^t\int_0^s (sr)^{2H_0-2}$$
$$\times\prod_{i=1}^d E(|B_s^i|^{2H_i-2}|B_r^i|^{2H_i-2})drds$$

Let X be a standard normal random variable. From Lemma 4.8.1, taking into account that $2-2H_i<1$, we have when $r<s$,

$$\begin{aligned} E(|B_r^i|^{2H_i-2}|B_s^i|^{2H_i-2}) &= E[|B_r^i|^{2H_i-2}E[|\sqrt{s-r}X+x|^{2H_i-2}|_{x=B_r^i}]] \\ &\le CE[|B_r^i|^{2H_i-2}(s-r)^{H_i-1}) \\ &\le Cr^{H_i-1}(s-r)^{H_i-1}. \end{aligned} \tag{8.13}$$

As a consequence, the conclusion of the lemma follows from the fact that

$$\int_0^t\int_0^s r^{2H_0+\sum_{i=1}^d H_i-d-2}s^{2H_0-2}(s-r)^{\sum_{i=1}^d H_i-d}drds<\infty,$$

because $2H_0+\sum_{i=1}^d H_i-d-2>-1$ and $\sum_{i=1}^d H_i-d>-1$.

Lemma 4.8.5. *Let $B^1,\dots,B^d$ be independent one-dimensional Brownian motions. If $\alpha_i\in(-1,0), i=1,\dots d$, and $\sum_{i=1}^d \alpha_i>-2$, then $E\exp\left(\lambda\int_0^1\prod_{i=1}^d |B_s^i|^{\alpha_i}ds\right)<\infty$ for all $\lambda>0$.*

Proof The proof is based on the method of moments. We can write

$$\begin{aligned} E\exp\left(\lambda\int_0^1\prod_{i=1}^d |B_s^i|^{\alpha_i}ds\right) &= \sum_{n=1}^\infty \frac{\lambda^n}{n!}E\int_{[0,1]^n}\prod_{k=1}^n\prod_{i=1}^d |B_{s_k}^i|^{\alpha_i}ds \\ &= \sum_{n=1}^\infty \lambda^n\int_{[0<s_1<\cdots<s_n<1]}\prod_{i=1}^d E\left(\prod_{k=1}^n |B_{s_k}^i|^{\alpha_i}\right)ds. \end{aligned}$$

From Lemma 9.1 since $\alpha_i \in (-1,0)$ we obtain

$$E\left[|B^i_{s_k}|^{\alpha_i}|\mathscr{F}^i_{s_{k-1}}\right] = E\left[|B^i_{s_k} - B^i_{s_{k-1}} + B^i_{s_{k-1}}|^{\alpha_i}|\mathscr{F}^i_{s_{k-1}}\right] \leq C(s_k - s_{k-1})^{\alpha_i/2},$$

where $\mathscr{F}_t$ is the filtration generated by the Brownian motion B^i. As a consequence, taking the conditional expectation of $\prod_{k=1}^n |B^i_{s_k}|^{\alpha_i}$ with respect to the σ-fields $\mathscr{F}^i_{s_{n-1}}, \mathscr{F}^i_{s_{n-2}}, \ldots, \mathscr{F}^i_{s_1}$ and $\mathscr{F}^i_0$, we get

$$E\left(\prod_{k=1}^n |B^i_{s_k}|^{\alpha_i}\right) \leq C^n (s_n - s_{n-1})^{\alpha_i/2} \ldots (s_2 - s_1)^{\alpha_i/2} s_1^{\alpha_i/2}.$$

Let $\alpha = \sum_{i=1}^d \alpha_i$, then we have

$$E \exp\left(\lambda \int_0^1 \prod_{i=1}^d |B^i_s|^{\alpha_i} ds\right) \leq \sum_{n=1}^\infty (C\lambda)^n \times \int_{[0<s_1<\cdots<s_n<1]} (s_n - s_{n-1})^{\alpha/2} \ldots (s_2 - s_1)^{\alpha/2} s_1^{\alpha/2} ds.$$

Since $\alpha > -2$, the integrals on the right side are equal to $\frac{(\Gamma(\alpha/2+1))^n}{(n+n\alpha/2)\Gamma(n+n\alpha/2)}$, and the series converges for any $\lambda > 0$.

Lemma 4.8.6. *For any* $0 < \alpha < 1$ *define*

$$C_\alpha(y) = E\left(|\xi|^{-\alpha} - |y+\xi|^{-\alpha}\right),$$

where $y > 0$ *and* ξ *is a standard normal random variable. Then*

$$C_\alpha(y) \leq C \min\left(1, (y^2 + y^{3-\alpha})\right),$$

for some constant $C > 0$.

Proof Notice first that $C_\alpha(y) < C$ where $C > 0$ is a constant, since $\lim_{y\to\infty} E|y+\xi|^{-\alpha} = 0$. On the other hand, we can decompose the function $C_\alpha(y)$ as follows

$$\begin{aligned}
C_\alpha(y) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(|x|^{-\alpha} - |y+x|^{-\alpha}\right) e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{\{x\geq 0\}\cup\{x\leq -y\}} \left(|x|^{-\alpha} - |y+x|^{-\alpha}\right) e^{-\frac{x^2}{2}} dx \right. \\
&\quad \left. + \int_{\{-y<x<0\}} \left(|x|^{-\alpha} - |y+x|^{-\alpha}\right) e^{-\frac{x^2}{2}} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} (A+B),
\end{aligned}$$

where A and B denote the first and second integrals, respectively, in the last second equation. For integral A we can write

$$\begin{aligned}
A &= \int_0^\infty \left(x^{-\alpha} - (x+y)^{-\alpha}\right) \left(e^{-x^2/2} - e^{-(x+y)^2/2}\right) dx \\
&\leq \int_0^\infty x^{-\alpha}(x+y)^{1-\alpha}[(x+y)^\alpha - x^\alpha] y e^{-\frac{x^2}{2}} dx.
\end{aligned}$$

Therefore,

$$A \leq \int_0^\infty x^{1-2\alpha}[(x+y)^\alpha - x^\alpha] y e^{-\frac{x^2}{2}} dx + \int_0^\infty x^{-\alpha}[(x+y)^\alpha - x^\alpha] y^{2-\alpha} e^{-\frac{x^2}{2}} dx.$$

For the first integral in the above expression we use the estimate $(x+y)^\alpha - x^\alpha \leq \alpha y x^{\alpha-1}$ and for the second we use $(x+y)^\alpha - x^\alpha \leq y^\alpha$. In this way we obtain

$$A \leq Cy^2,$$

for some constant $C > 0$. On the other hand,

$$B = \int_0^y x^{-\alpha}(e^{-\frac{x^2}{2}} - e^{-\frac{(x+y)^2}{2}})dx \le \int_0^y x^{-\alpha}(x+y)ydx \le Cy^{3-\alpha},$$

for some constant $C > 0$, which completes the proof of the lemma.

Proposition 4.8.7. *Let B be a one-dimensional standard Brownian motion. Then, for any $p > 0$,*

$$E\left|\int_0^1\int_0^1 \delta(B_t - B_s)dsdt\right|^{-p} < \infty.$$

Proof For $k = 1, \ldots, 2^{n-1}$ we denote $A_{n,k} = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]$ and define

$$\alpha_{n,k} = \int_{A_{n,k}} \delta(B_t - B_s)dsdt.$$

The random variables $\alpha_{n,k}$ have the following two properties:

(i) For every $n \ge 1$, the variables $\alpha_{n,1}, \ldots, \alpha_{n,2^{n-1}}$ are independent.

(ii) $\alpha_{n,k} \overset{d}{=} 2^{-n/2}\int_0^1\int_0^1 \delta(B_t - \tilde{B}_s)dsdt$. and $\tilde{B}$ is a standard Brownian motion independent of B.

For any $p > 0$, we may choose a integer $n > 0$ such that $p2^{1-n} < 1/3$. Then, we can write

$$E\left|\int_0^1\int_0^1 \delta(B_t - B_s)dsdt\right|^{-p} \le E\left|\sum_{k=1}^{2^{n-1}} \alpha_{n,k}\right|^{-p} \le E\left|\prod_{k=1}^{2^{n-1}} \alpha_{n,k}\right|^{-p2^{1-n}},$$

and it suffices to show that $E\left|\int_0^1\int_0^1 \delta(B_t - \tilde{B}_s)dsdt\right|^{-p} < \infty$, for some $p > 0$. Notice that

$$L := \int_0^1\int_0^1 \delta(B_t - \tilde{B}_s)dsdt = \int_{\mathbb{R}} L_1^x \tilde{L}_1^x dx,$$

where L_t^x (resp. $L_t^{\tilde{x}}$) denotes the local time of the Brownian motion B (resp. $\tilde{B}$). As a consequence, for any $0 < \alpha < 1$

$$
\begin{aligned}
P(L<\varepsilon) &\leq P\left(\int_0^{\varepsilon^{4/5}} L_1^x \tilde{L}_1^x dx\right) \\
&\leq P\left(L_1^0 \tilde{L}_1^0 < \frac{1}{2}\varepsilon^{1/5}\right) + P\left(\int_0^{\varepsilon^{4/5}} |L_1^0 \tilde{L}_1^0 - L_1^x \tilde{L}_1^x| dx \geq \frac{\varepsilon}{2}\right) \\
&\leq \frac{1}{\sqrt{2}} \varepsilon^{1/10} (E(L_1^0)^{-1/2})^2 + \frac{2}{\varepsilon} \int_0^{\varepsilon^{4/5}} E|L_1^0 \tilde{L}_1^0 - L_1^x \tilde{L}_1^x| dx \\
&\leq C\varepsilon^{1/10},
\end{aligned}
$$

which implies that $E(L^{1/10}) < \infty$.

Chapter 5

Some properties of the solutions to a class of stochastic partial differential equations

5.1 Introduction

In this paper we consider a general (nonlinear) stochastic heat equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\triangle u + b(u) + \sigma(u)\dfrac{\partial W}{\partial t}(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^d \\ u(0,x) = u_0(x)\,, \end{cases} \tag{1.1}$$

where $W(t,x)$ is a Gaussian random field which is a Brownian motion in time t and has a covariance structure $q(x,y)$and b and σ are globally Lipschitz continuous functions. First, we shall prove that under some mild condition on $q(x,y)$ the solution is Hölder continuous with certain explicit Hölder exponent. This result generalizes the main result in [44] for spatially homogeneous case ($q(x,y) = q(x-y)$) to general case with a different method. The second aim of the paper is to show that under the conditions b and σ are smooth and $|\sigma(u(0,y))| > 0$ for some $y \in \mathbb{R}^d$, the law of $u(t,x)$ is absolutely continuous with respect to Lebesgue measure and the density is C^∞. This result gives an answer to a problem that researchers were trying to clarify. The explain this point, let us recall the following known results:

- If $|\sigma(x)| > c > 0$ for all $x \in \mathbb{R}$, then the law of $u(t,x)$ has a C^∞ density (see [9] for details).

- If $|\sigma(u(0,y))| > 0$ for some y, then the law of $u(t,x)$ is absolutely continuous for all $t > 0$ and all x. Researchers are wondering if this is sufficient to guarantee the smoothness of the density. However, until before this paper, this problem has not been answered (see [39] for details). The present paper gives an affirmative answer to the above problem.

To show smoothness of the density, a standard technique is to use Malliavin calculus, where the main difficulty is to show that the Malliavin covariance matrix has all negative moment. To overcome this difficulty, we apply the Malliavin derivative operator $D_{s,y}$ in the equation (1.1) to obtain (denote $Z_{t,x} = D_{s,y}u(t,x)$)

$$\begin{cases} \frac{\partial}{\partial t} Z = \frac{1}{2} \triangle Z + b'(u)Z + \sigma'(u)Z \frac{\partial W}{\partial t}(t,x) + \sigma(u(s,y)), \quad t \geq 0, x \in \mathbb{R}^d \\ Z(0,x) = 0\,. \end{cases}$$

With a technique of factorization, one can relate the Malliavin covariance matrix to $V_{s,\xi}(t,x)$, which satisfies, as a function of t and x for fixed (s,ξ)

$$\frac{\partial V_{s,\xi}}{\partial t} = \frac{1}{2} \triangle V_{s,\xi} + b'(u)V_{s,\xi} + \sigma'(u)V_{s,\xi} \frac{\partial W}{\partial t}(t,x)\,, \quad t \geq s, x \in \mathbb{R}^d\,. \tag{1.2}$$

This is a linear stochastic partial differential equation driven by a semimartingale. To show the existence of all negative moments, we first develop a Feynman-Kac formula to represent $V_{s,\xi}(t,x)$. Since the Feynman-Kac formula is important itself, we develop it in a more general context. Namely, we consider a more general stochastic heat equation,

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) = \dfrac{1}{2}\triangle V(t,x) + V\dot{M}(t,x) \\ V(x,0) = h(x) \end{cases} \tag{1.3}$$

where M is a semi-martingale, with quadratic variation

$$\langle M(t,x), M(t,y)\rangle = f(t,x,y) = \int_0^t g(s,x,y)ds,$$

and the product between V and $\dot{M}(t,x)$ in (1.3) is Itô differential (intergral). Thus one of the main objectives of this paper is to establish the following Feynman-Kac formula under some condition on g

$$V(t,x) = E^B\big\{h(x+B_t)\exp\big(\int_0^t M(dr, x+B_t-B_r) - \frac{1}{2}\int_0^t \bar{g}(dr, x+B_t-B_r)\big)\big\}, \tag{1.4}$$

where B is a d-dimensional standard Brownian motion independent of M and $\bar{g}(t,x) = g(t,x,x)$. For the stratonovich case, we refer to section 4.4 in [8] for another Feynman-Kac formula, where the condition for the noises are different. Let us also mention that a Feynman-Kac formula for heat equation driven by fractional white noise for both stratonovich and skorohod cases in [24] (see the references therein for related work). In that paper, the formula is proved by approximating the noise with smooth ones. Presumably, one may intend to use the same idea to establish (1.4). However, although the approximation of the stochastic partial differential equations of smooth noises has been studied for example in [3], it is quite complex. We shall use a generalized Itô formula ([30]).

Section 2 contains some preliminary material. Section 3 is devoted to the Feynman-Kac formula. Section 4 deals with the Hölder continuity. Finally in Section 5, the smoothness of the density is presented.

5.2 Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be a basic probability space. The expectation on this probability space is denoted by E. Let $\{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ be a mean zero Gaussian random field which is white in time and has the covariance structure determined by $q(x,y)$:

$$E\left[W(t,x)W(s,y)\right] = (s \wedge t) \int_0^x \int_0^y q(u,v)dudv\,.$$

For any two (deterministic) functions $f(s,x), g(s,x) \in L^2([0,T] \times \mathbb{R}^d)$ we can define a scalar product

$$\langle f, g\rangle_{\mathscr{H}} := \int_0^T \int_{\mathbb{R}^{2d}} f(s,x)g(s,y)q(x,y)dxdyds$$

and a Hilbert space $\mathscr{H}$. For any element f in $\mathscr{H}$, we may define the stochastic integral $\int_0^T \int_{\mathbb{R}^d} f(s,x)W(ds,x)dx$ can be defined in the standard way and it is easy to obtain

$$E\left(\int_0^T \int_{\mathbb{R}^d} f(s,x)W(ds,x)dx \int_0^T \int_{\mathbb{R}^d} g(s,x)W(ds,x)dx\right) = \langle f, g\rangle\,.$$

We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \ldots, W(\phi_n)),$$

$\phi_i \in \mathscr{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), then DF is the $\mathscr{H}$-valued random variable defined by

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j.$$

The operator D is closable form $L^2(\Omega)$ into $L^2(\Omega;\mathscr{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathscr{H}}^2)}.$$

We denote by δ the adjoint of the derivative operator, given by duality formula

$$E(\delta(u)F) = E\left(\langle DF, u\rangle_{\mathscr{H}}\right), \tag{2.1}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega;\mathscr{H})$ in the domain of δ. The operator δ is also called the Skorohod integral. The higher Malliavin derivatives can be defined in similar way and we can define $\mathbb{D}^{k,p}$. To obtain the smoothness of the density, we need to use the following

Theorem 5.2.1. *Let $F : \Omega \to \mathbb{R}$ be a random variable. If $F \in \bigcap_{k\geq 1, p\geq 2} \mathbb{D}^{k,p}$ and $E\left[\|DF\|_{\mathscr{H}}^{-p}\right] < \infty$ for all $p > 1$, then the law of F is absolutely continuous with respect to the Lebesgue measure and the density is smooth.*

For the proof of this result and some other results on the Malliavin calculus we refer to [38] and the references therein.

To show the solution to (1.1) has smooth density, we shall show that the Malliavin covariance matrix $\|DF\|_{\mathscr{H}}$ has all the negative moments. This will be done by using the Feynman-Kac formula for heat equation driven by general semimartingales. To establish such formula, we shall use the following generalized Itô formula ([30]).

Theorem 5.2.2 (Generalized Itô formula). *Let $F(t,x), x \in \mathbf{R}^d$ be a random field. Assume that*

(i) For fixed $t \in [0,T]$, $F(t,x)$ is twice differentiable with respect to x almost surely.

(ii) For fixed $x \in \mathbb{R}^d$, $F(t,x)$ is a semimartingale with respect to t.

Let X_t be $\mathbf{R}^d$-valued continuous semi-martingale. Then we have

$$\begin{aligned} F(t,X_t) =& F(0,X_0) + \int_0^t F(dr,X_r) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(r,X_r)dX_r^i \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(r,X_r)d\langle X^i,X^j\rangle_r + \sum_{i=1}^d \langle \int_0^\cdot \frac{\partial F}{\partial x_i}(dr,X_r),X^i\rangle_t . \end{aligned} \tag{2.2}$$

5.3 Feynman-Kac formula

By the generalized Itô formula, we can get the Feynman-Kac formula to heat equation (1.3).

Theorem 5.3.1 (Feynman-Kac Formula). *Assume that $g(t,x,y)$ is continuous in (t,x,y), $g(t,x,y) \le C(1+|x|^\alpha + |y|^\alpha + |x-y|^\beta)$ for $t \in [0,T]$, where $0 \le \alpha < 2$ and $\beta > -d$. Let $h(x)$ be continuous and of polynomial growth. Then*

$$V(t,x) = E^B\big\{h(x+B_t)\exp\big(\int_0^t M(dr,x+B_t-B_r) - \frac{1}{2}\int_0^t \bar{f}(dr,x+B_t-B_r)\big)\big\}$$

is a solution to the heat equation (1.3), where B is a d-dimensional standard Brownian motion independent of M.

Remark 5.3.2. *It is easy to show that (1.3) has a unique solution. Thus the solution to (1.3) can also represented by the above Feynman-Kac formula.*

Proof We divide the proof into three steps.

Step 1. Suppose M is C^2 in x and a semi-martingale in t, $f(t,x,y)$ is twice continuously differentiable in x and y, and $h(x)$ is twice continuously differentiable. For fixed x, let

$$F(t,y) = \int_0^t M(dr, x+y-B_r) - \frac{1}{2}\int_0^t \bar{f}(dr, x+y-B_r).$$

From the generalized Itô formula (2.2) it follows

$$\begin{aligned}
dF(t,B_t) &= M(dt,x) - \frac{1}{2}\bar{f}(dt,x) + \sum_{i=1}^d \int_0^t \frac{\partial M}{\partial x_i}(dr, x+B_s-B_r)dB_s^i \\
&\quad - \frac{1}{2}\sum_{i=1}^d \int_0^t \frac{\partial \bar{f}}{\partial x_i}(dr, x+B_s-B_r)dB_s^i + \frac{1}{2}\sum_{i=1}^d \int_0^t \frac{\partial^2 M}{\partial x_i^2}(dr, x+B_s-B_r)ds \\
&\quad - \frac{1}{4}\sum_{i=1}^d \int_0^t \frac{\partial^2 \bar{f}}{\partial x_i^2}(dr, x+B_s-B_r)ds
\end{aligned}$$

Denote $Y_t = F(t,B_t)$. Since M and B are independent we have

$$d\langle Y\rangle_t = f(dt,x) + \sum_{i=1}^d \Big[\int_0^t \frac{\partial M}{\partial x_i}(dr, x+B_t-B_r) - \frac{1}{2}\int_0^t \frac{\partial f}{\partial x_i}(dr, x+B_t-B_r)\Big]^2 dt\,.$$

We further assume that

$$E^B \frac{\partial h}{\partial x_i}(x+B_t)^2 < \infty, i = 1, \cdots, d; \tag{3.1}$$

$$E^B \int_0^t e^{4Y_s} ds < \infty; \tag{3.2}$$

$$E^B \int_0^t \sum_{i,j=1}^d \int_0^s \frac{\partial^2 g}{\partial x_i \partial y_j}(r, x+B_s-B_r, x+B_s-B_r) dr ds < \infty; \tag{3.3}$$

$$E^B \int_0^t \left(\sum_{i=1}^d \int_0^s \frac{\partial \bar{f}}{\partial x_i}(dr, x+B_s-B_r)\right)^2 ds < \infty; \tag{3.4}$$

Notice that in (3.3), the integrand function

$$\sum_{i,j=1}^{d}\int_0^s \frac{\partial^2 g}{\partial x_i \partial y_j}(r, x+B_s-B_r, x+B_s-B_r)dr = \langle \sum_{i=1}^{d}\int_0^s \frac{\partial M}{\partial x_i}(dr, x+B_s-B_r)\rangle_s.$$

Applying Theorem 5.2.2 again to $h(x+B_t)e^{Y_t}$ and then taking expectation with respect to B, we have

$$\begin{aligned} E^B[h(x+B_t)e^{Y_t}] =& h(x)+E^B\Bigg(\int_0^t h(x+B_s)e^{Y_s}dY_s + \frac{1}{2}\sum_{i=1}^{d}\int_0^t \frac{\partial^2 h}{\partial x_i^2}(x+B_s)e^{Y_s}ds \\ &+\frac{1}{2}\int_0^t h(x+B_s)e^{Y_s}d\langle Y\rangle_s + \sum_{i=1}^{d}\int_0^t \frac{\partial h}{\partial x_i}(x+B_s)e^{Y_s}d\langle B^i, Y\rangle_s\Bigg). \end{aligned}$$

If we denote $\hat{V}(t,x) = h(x+B_t)e^{Y_t}$, then we have

$$\begin{aligned} &E^B\hat{V}(t,x) = h(x) + \int_0^t E^B\hat{V}(s,x)M(ds,x) \\ &+\frac{1}{2}\sum_{i=1}^{d}E^B\Bigg(\int_0^t \hat{V}(s,x)\Bigg\{\int_0^s \frac{\partial^2 M}{\partial x_i^2}(dr, x+B_s-B_r) - \frac{1}{2}\int_0^s \frac{\partial^2 \overline{f}}{\partial x_i^2}(dr, x+B_s-B_r) \\ &+\Bigg[\int_0^s \frac{\partial M}{\partial x_i}(dr, x+B_s-B_r) - \frac{1}{2}\int_0^s \frac{\partial \overline{f}}{\partial x_i}(dr, x+B_s-B_r)\Bigg]^2\Bigg\}ds \\ &+\int_0^t \frac{\partial^2 h}{\partial x_i^2}(x+B_s)e^{Y_s}ds \\ &+2\int_0^t \frac{\partial h}{\partial x_i}(x+B_s)e^{Y_s}[\int_0^s \frac{\partial M}{\partial x_i}(dr, x+B_s-B_r) - \frac{1}{2}\int_0^s \frac{\partial \overline{f}}{\partial x_i}(dr, x+B_s-B_r)]ds\Bigg) \end{aligned}$$

Notice that

$$\frac{\partial Y_s}{\partial x_i} = \int_0^s \frac{\partial M}{\partial x_i}(dr, x+B_s-B_r) - \frac{1}{2}\int_0^s \frac{\partial \overline{f}}{\partial x_i}(dr, x+B_s-B_r),$$

and denote $V(s,x)=E^B\left[\hat{V}(s,x)\right]$, we get

$$V(t,x)=h(x)+\int_0^t V(s,x)M(ds,x)+\frac{1}{2}\sum_{i=1}^d\int_0^t \frac{\partial^2 V}{\partial x_i^2}(s,x)ds.$$

Step 2. For general M and h, let $M^\varepsilon(t,x)=\int_{\mathbb{R}^d}M(t,y)p_\varepsilon(x-y)dy$, and $h^\varepsilon(x)=\int_{\mathbb{R}^d}h(y)p_\varepsilon(x-y)dy$, where $p_\varepsilon(x)=(2\pi\varepsilon)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2\varepsilon}}$. It is easy to check that h^ε satisfies (3.1). We construct Y_t^ε as Y_t in Step 1 and we shall show that Y_t^ε satisfies (3.2)-(3.4). The bracket of M^ε is given by

$$f^\varepsilon(t,x,y)=\int_0^t\int_{\mathbb{R}^{2d}}g(s,x-z_1,y-z_2)p_\varepsilon(z_1)p_\varepsilon(z_2)dz_1dz_2ds..$$

For any $p>0$, we have

$$\begin{aligned}
&Ee^{p\int_0^t \bar{f}^\varepsilon(dr,x+B_t-B_r)}\\
=&Ee^{p\int_0^t\int_{\mathbb{R}^{2d}}g(r,x+B_t-B_r-z_1,x+B_t-B_r-z_2)p_\varepsilon(z_1)p_\varepsilon(z_2)dz_1dz_2dr}\\
\le&Ee^{p\int_0^t\int_{\mathbb{R}^{2d}}C(1+|x+B_t-B_r-z_1|^\alpha+|x+B_t-B_r-z_2|^\alpha+|z_1-z_2|^\beta)p_\varepsilon(z_1)p_\varepsilon(z_2)dz_1dz_2dr}\\
\le&e^{p\int_0^t\int_{\mathbb{R}^{2d}}|z_1-z_2|^\beta p_\varepsilon(z_1)p_\varepsilon(z_2)dz_1dz_2dr}\\
&\times Ee^{Cp\int_0^t\int_{\mathbb{R}^{2d}}(1+|x+B_t-B_r-z_1|^\alpha+|x+B_t-B_r-z_2|^\alpha)p_\varepsilon(z_1)p_\varepsilon(z_2)dz_1dz_2dr}\\
\le&C\int_{\mathbb{R}^{2d}}Ee^{p\int_0^t(|x+B_t-B_r-z_1|^\alpha+|x+B_t-B_r-z_2|^\alpha)dr}p_\varepsilon(z_1)p_\varepsilon(z_2)dz_1dz_2
\end{aligned}\tag{3.5}$$

The above last inequality follows from the assumptions on $g(t,x,y)$ and a Fernique's theorem [12]. Now we have

$$
\begin{aligned}
&Ee^{4\int_0^t M^\varepsilon(dr,x+B_t-B_r)-2\int_0^t \bar{f}^\varepsilon(dr,x+B_t-B_r)}\\
=&Ee^{4\int_0^t M^\varepsilon(dr,x+B_t-B_r)-16\int_0^t \bar{f}^\varepsilon(dr,x+B_r-B_r)+14\int_0^t \bar{f}^\varepsilon(dr,x+B_r-B_r)}\\
\le&\left(Ee^{8\int_0^t M^\varepsilon(dr,x+B_t-B_r)-32\int_0^t \bar{f}^\varepsilon(dr,x+B_r-B_r)}\right)^{\frac{1}{2}}\left(e^{28\int_0^t \bar{f}^\varepsilon(dr,x+B_r-B_r)}\right)^{\frac{1}{2}}\\
\le&\left(e^{28\int_0^t \bar{f}^\varepsilon(dr,x+B_r-B_r)}\right)^{\frac{1}{2}}
\end{aligned}
$$

which is finite. This proves that Y^ε satisfies (3.2). Inequalities (3.3) and (3.4)can be verified easily.

From Step 1 we see that

$$
V^\varepsilon(t,x)=E^B\big\{h^\varepsilon(x+B_t)\exp\big(\int_0^t M^\varepsilon(dr,x+B_t-B_r)-\frac{1}{2}\int_0^t \bar{f}^\varepsilon(dr,x+B_t-B_r)\big)\big\}
$$

is the solution to

$$
\begin{cases}
\dfrac{\partial V^\varepsilon}{\partial t}(t,x)=\dfrac{1}{2}\triangle V^\varepsilon(t,x)+V^\varepsilon \dot{M}^\varepsilon(t,x)\\
V^\varepsilon(x,0)=h^\varepsilon(x)
\end{cases}
\tag{3.6}
$$

Step 3. Now we show that V^ε converges to V in L^p, for any $p>0$, and as a consequence, we have the Feynman-Kac formula of V.

For any fixed r and the random sample B_t-B_r, we see that $\int_{\mathbb{R}^d}\bar{g}^\varepsilon(r,x+B_t-B_r-z)p_\varepsilon(z)dz$ converges to $\bar{g}(r,x+B_t-B_r)$ and

$$
\left|\int_{\mathbb{R}^d}\bar{g}^\varepsilon(r,x+B_t-B_r-z)p_\varepsilon(z)dz\right|\le\bar{g}(r,x+B_t-B_r)
$$

which is L_p for all $p>1$. By dominated convergence theorem, we have $\int_0^t\int_{\mathbb{R}^d}\overline{g}^\varepsilon(r,x+B_t-B_r-z)p_\varepsilon(z)dzdr$ converges to $\int_0^t\int_{\mathbb{R}^d}\overline{g}(r,x+B_t-B_r)dr$. Hence, $\int_0^t M^\varepsilon(dr,x+B_t-B_r)$ converges to $\int_0^t M(dr,x+B_t-B_r)$ in L^p for all $p>1$.

As in (3.4), it is easy to show that

$$\sup_{\varepsilon>0} E\exp\left(p\int_0^t \overline{f}^\varepsilon(dr,x+B_t-B_r)\right)<\infty, \quad \forall\, p\in\mathbb{R}\,.$$

This can be used to prove that V^ε converges to V in L^p for all $p>0$.

Since V^ε is a (strong) solution to (3.6), it is also a mild solution to (3.6), namely,

$$V^\varepsilon=\int_{\mathbb{R}^d}p_t(x-z)h^\varepsilon(z)dz+\int_0^t\int_{\mathbb{R}^d}p_{t-r}(x-z)V^\varepsilon(r,z)M^\varepsilon(dr,z)dz$$

Letting ε go to 0, we obtain

$$V=\int_{\mathbb{R}^d}p_t(x-z)h(z)dz+\int_0^t\int_{\mathbb{R}^d}p_{t-r}(x-z)V(r,z)M(dr,z)dz\,.$$

This completes the proof of the theorem.

5.4 Hölder continuity of the solution

Consider the following nonlinear stochastic partial differential equation:

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{1}{2}\triangle u+b(u)+\sigma(u)\dfrac{\partial W}{\partial t}(t,x)\,, \quad t\geq 0\,, \quad x\in\mathbb{R}^d\\ u(0,x)=u_0(x)\,.\end{cases} \tag{4.1}$$

Let us recall that an adapted random field $\{u(t,x)\,,t\geq 0\,,x\in\mathbb{R}^d\}$ is called a mild solution to the above equation (4.1) if u satisfies the following integral equation.

$$\begin{aligned}u(t,x) = &\int_{\mathbb{R}^d} p_t(x-z)u_0(z)dz + \int_0^t\int_{\mathbb{R}^d} p_{t-r}(x-z)b(u(r,z))dzdr\\ &+\int_0^t\int_{\mathbb{R}^d} p_{t-r}(x-z)\sigma(u(r,z))W(dr,z)dz\,.\end{aligned}\tag{4.2}$$

Theorem 5.4.1. *Suppose that b and σ are globally Lipschitz continuous functions and assume*

(i) $\displaystyle\sup_{x\in\mathbb{R}^d} E\int_0^t\int_{\mathbb{R}^{2d}} p_{t-s}(x-z_1)p_{t-s}(,x-z_2)q(z_1,z_2)dz_1dz_2<\infty,$

(ii) $u_0(x)$ *is bounded in* $\mathbb{R}^d$.

Then there exists a unique adapted process $u=\{u(t,x),t\in[0,T],x\in\mathbb{R}^d\}$ *satisfying (4.2). Moreover,*

$$\sup_{t\in[0,T],x\in\mathbb{R}^d} E|u(t,x)|^p<\infty,\quad \forall\, p\geq 2\,.\tag{4.3}$$

Proof Let $\mathbb{B}_p$ be the Banach space of all adapted random field u such that $\|u\|_p<\infty$, where $\|u\|_p^p=\sup_{t\in[0,T],x\in\mathbb{R}^d} E|u(t,x)|^p$. On $\mathbb{B}_p$, define the following mapping

$$\begin{aligned}\Psi(u)(t,x) := &\int p_t(x-z)u_0(z)dz + \int_0^t\int_{\mathbb{R}^d} p_{t-r}(x-z)b(u(r,z))dzdr\\ &+\int_0^t\int_{\mathbb{R}^d} p_{t-r}(x-z)\sigma(u(r,z))W(dr,z)dz\,.\end{aligned}$$

We shall show that Ψ is a contraction mapping on $\mathbb{B}$ when T is sufficiently small. We have

$$E\left|\Psi(u)-\Psi(v)\right|^p(t,x) \le C\Bigg[E\left(\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-z)|u(s,z)-v(s,z)|dzds\right)^p$$
$$+E\Bigg\{\int_0^t\int_{\mathbb{R}^{2d}} p_{t-s}(x-z_1)|u(s,z_1)-v(s,z_1)|p_{t-s}(x-z_2)|u(s,z_2)-v(s,z_2)|$$
$$q(z_1,z_2)dz_1dz_2ds\Bigg\}^{p/2}\Bigg].$$

Taking the superium with respect to t and x, we have

$$\|\Psi(u)-\Psi(v)\|_p^p \le C\int_0^T \|u-v\|_p^p ds \le CT\|u-v\|_p^p.$$

Consequently, Ψ is a contraction mapping on $\mathbb{B}_p$ when T sufficiently small. This proves the existence and uniqueness of the solution when some small T. From the above argument it is clear that the T such that Ψ is a contraction is independent of the initial value of the solution. This can be used to to show the existence and uniqueness of the solution for any T. The inequality (4.3) follows also.

Now we apply the factorization method to obtain the Hölder continuity of u. Fix an arbitarry $\alpha \in (0,1)$ and denote

$$Y_\alpha(r,z) = \int_0^r\int_{\mathbb{R}^d} p_{r-s}(z-y)\sigma(u(s,y))(r-s)^{-\alpha}W(ds,y)dy.$$

Then because of the semigroup property of K and the stochastic Fubini's theorem, we have

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u(s,y))W(ds,y)dy\\=&\frac{\sin(\pi\alpha)}{\pi}\int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)(t-r)^{\alpha-1}Y_\alpha(r,z)dzdr.\end{aligned} \tag{4.4}$$

Lemma 5.4.2. *Let the assumptions of Theorem 5.4.1 be satisfied. Assume*

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} p_t(x-z_1)p_t(x-z_2)q(z_1,z_2)dz_1dz_2 < Ct^\gamma \quad \text{for some } \gamma > -1. \tag{4.5}$$

Then for any fixed $T > 0, p \geq 1, \alpha \in (0, \frac{1+\gamma}{2})$*, we have*

$$\sup_{r\in[0,T],z\in\mathbb{R}^d} E(|Y_\alpha(r,z)|^p) < \infty.$$

Proof Since $\sup_{r\in[0,T],z\in\mathbb{R}^d} E(|u(r,z)|^p) < \infty$ from Theorem 5.4.1 and σ is Lipschitz continuous, we have

$$\sup_{r\in[0,T],z\in\mathbb{R}^d} E(|\sigma(u(r,z))|^p) < \infty.$$

$$\begin{aligned}E|Y_\alpha(r,z)|^p \leq& C\Bigg(E\int_0^r \int_{\mathbb{R}^{2d}} p_{r-s}(z-y_1)p_{r-s}(z-y_2)\\&\qquad \sigma(u(s,y_1))\sigma(u(s,y_2))(r-s)^{-2\alpha}q(y_1,y_2)dy_1dy_2ds\Bigg)^{\frac{p}{2}}\\\leq& C\sup_{r\in[0,T],z\in\mathbb{R}^d} E(|\sigma(u(r,z))|^p)\\&\left(\int_0^r \int_{\mathbb{R}^{2d}} p_{r-s}(z-y_1)p_{r-s}(z-y_2)(r-s)^{-2\alpha}q(y_1,y_2)dy_1dy_2ds\right)^{\frac{p}{2}}\\\leq& C\left(\int_0^r (r-s)^{\gamma-2\alpha}ds\right)^{\frac{p}{2}} < \infty.\end{aligned}$$

The equation (4.4) and Lemma (5.4.2) are the main ingredients to prove the following theorem concerning the Hölder continuity of the solution u.

Theorem 5.4.3. *Suppose that b and σ are globally Lipschitz continuous. Assume*

(i) $\sup_{x\in\mathbb{R}^d} E\int_{\mathbb{R}^{2d}} p_t(x-z_1)p_t(x-z_2)q(z_1,z_2)dz_1dz_2 < Ct^\gamma$ *for some* $\gamma > -1$.

((ii) $u_0(x)$ *is bounded in* $\mathbb{R}^d$ *and* ρ*-Hölder continuous.*

Then the sample paths of the solution u to (4.1) are a.s. β_1-Hölder continuous in time and β_2-Hölder continuous in space for any $\beta_1 \in (0, \frac{1}{2}[\rho\wedge(1+\gamma)])$ and $\beta_2 \in (0, \rho \wedge (1+\gamma))$.

Proof We shall follow the proof of Theorem 2.1 in [44].

Here are two examples.

Example 5.4.4. *The same Hölder continuity result was obtained in [44] under the assumption $q(x,y) = q(x-y)$ satisfying*

$$\int_{\mathbb{R}^d} \frac{\overline{\mathscr{F}q}(\xi)}{(1+|\xi|^2)^\eta} d\xi < \infty \quad \textit{for some } \eta \in (0,1),$$

where $\mathscr{F}q$ denotes the Fourier transform of q. This is consequence of the above theorem. Thus Theorem 5.4.3 extends the result of [44] to more general cases.

In fact, we see

$$\int_{\mathbb{R}^{2d}} p_t(x-z_1)p_t(x-z_2)q(z_1-z_2)dz_1dz_2 = \int_{\mathbb{R}^{2d}} p_t(x-z-z_2)p_t(x-z_2)q(z)dz_2dz$$
$$= \int_{\mathbb{R}^d} p_{2t}(z)q(z)dz = \int_{\mathbb{R}^d} \overline{\mathscr{F}p}_{2t}(\xi)\overline{\mathscr{F}q}(\xi)d\xi\,.$$

It is well-known that $\overline{\mathscr{F} p_{2t}}(\xi) = e^{-2t\xi^2}$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} p_t(x-z_1)p_t(x-z_2)q(z_1-z_2)dz_1dz_2 = \int_{\mathbb{R}^d} e^{-2t\xi^2}\overline{\mathscr{F}q}(\xi)d\xi \\
\leq\ & \int_{\mathbb{R}^d} e^{-2t(\xi^2+1)}\overline{\mathscr{F}q}(\xi)d\xi \leq Ct^{-\eta}\int_{\mathbb{R}^d} \frac{\overline{\mathscr{F}q}(\xi)}{(1+|\xi|^2)^\eta}d\xi\,.
\end{aligned}$$

Example 5.4.5. *Consider the bifractional Brownian sheet $B^{H,K} = B^{H,K}(t,x), t \geq 0, x \in \mathbb{R}, H \in (0,1), K \in (0,1]$, which is Gaussian random field with mean 0 and the following covariance structure*

$$E(B^{H,K}(s,x)B^{H,K}(t,y)) = (t \wedge s)2^{-K}((|x|^{2H} + |y|^{2H})^K - |x-y|^{2HK}).$$

If $W(t,x) = \frac{\partial}{\partial x}B^{H,K}(x,t)$, then

$$\begin{aligned}
q(x,y) =& 2^{-K}\frac{\partial^2}{\partial x \partial y}((|x|^{2H} + |y|^{2H})^K - |x-y|^{2HK}) \\
\leq& C[|x|^{2HK-2} + |y|^{2HK-2} + |x-y|^{2HK-2}]
\end{aligned}$$

and $\gamma = HK - 1 \in (-1,0)$.

5.5 Regularity of the density of the solution

Theorem 5.5.1. *Assume that $q(x_1,x_2)$ is γ-Hölder continuous and*

$$q(x_1,x_2) \leq C(1 + |x_1|^\alpha + |x_2|^\alpha + |x_2 - x_1|^\beta) \quad \textit{for some } \alpha \in [0,2), \beta > -d\,. \tag{5.1}$$

Let $q(y_1,y_2) = \int_{\mathbb{R}^d} c(\xi,y_1)c(\xi,y_2)d\xi$ for some $c(\xi,y)$ which has polynomial growth. Let b' and σ' be continuous and bounded. Let u_0 be bounded and ρ-Hölder continuous.

If there is a $x_0 \in \mathbb{R}^d$ such that $q(x_0,x_0) > 0$, $\sigma(u(0,x_0)) > 0$, then $u(t,x)$ has a smooth density for all $t > 0$ and $x \in \mathbb{R}^d$, where u is the solution to the equation (4.1).

Proof Denote

$$F := \|Du(t,x)\|^2_{\mathscr{H}} = \int_0^t \int_{\mathbb{R}^{2d}} D_{s,y_1}u(t,x)D_{s,y_2}u(t,x)q(y_1,y_2)dy_1dy_2ds\,,$$

where $\|\cdot\|_{\mathscr{H}}$ is the Hilbert norm introduced in Section 2. We need to prove that $E\left[F^{-p}\right] < \infty$ for any $p > 1$. We divide the proof into two steps.

Step 1. Introduce $V_{s,\xi}(t,x) = \int_{\mathbb{R}^d} c(\xi,y)D_{s,y}u(t,x)dy$. Then we can write

$$F = \int_0^t \int_{\mathbb{R}^d} V_{s,\xi}(t,x)^2 d\xi ds\,.$$

Applying the Malliavin derivative $D_{s,y}$ to (4.2) we obtain

$$\begin{aligned} D_{s,y}u(t,x) =& p_{t-s}(x-y)\sigma(u(s,y)) + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)b'(u(r,z))D_{s,y}u(r,z)dzdr \\ &+ \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,z)dz\,. \end{aligned}$$

This implies that $V_{s,\xi}(t,x)$ satisfies the following linear heat equation:

$$\begin{cases} \dfrac{\partial V_{s,\xi}}{\partial t} = \dfrac{1}{2}\triangle V_{s,\xi} + b'(u)V_{s,\xi} + \sigma'(u)V_{s,\xi}\dfrac{\partial W}{\partial t}(t,x), t \geq s, x \in \mathbb{R}^d, \\ V_{s,\xi}(s,x) = c(\xi,x)\sigma(u(s,x))\,. \end{cases}$$

Theorem 5.2.2 gives an explicit Feynman-Kac formula for the above equation. This means that we have

$$V_{s,\xi}(t,x) = E^B\Big[c(\xi,x+B_{t-s})\sigma(u(s,x+B_{t-s}))\\ \exp\Big\{\int_s^t M(dr,x+B_{t-s}-B_{r-s}) - \frac{1}{2}\int_s^t f(dr,B_{t-s}-B_{r-s})\Big\}\Big],$$

where

$$\begin{aligned} M(t,x) &= \int_0^t b'(u(r,x))dr + \int_0^t \sigma'(u(r,x))w(dr,x) \qquad \text{and} \\ f(t,x,y) &= \langle M(\cdot,x), M(\cdot,y)\rangle_t = \int_0^t \sigma'(u(r,x))\sigma'(u(r,y))q(x,y)dr\,. \end{aligned}$$

Step 2. Let $Y(s,t;B) = \int_s^t M(dr,x+B_{t-s}-B_{r-s}) - \frac{1}{2}\int_s^t f(dr,B_{t-s}-B_{r-s})$. Then

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} |V_{s,\xi}(t,x)|^2 d\xi ds = &\int_0^t \int_{\mathbb{R}^d} E^{B,\tilde{B}}\Big[c(\xi,x+B_{t-s})c(\xi,x+\tilde{B}_{t-s}) \\ &\sigma(u(s,x+B_{t-s}))\sigma(u(s,x+\tilde{B}_{t-s}))\exp\{Y(s,t;B)+Y(s,t;\tilde{B})\}\Big]d\xi ds \\ = &\int_0^t E^{B,\tilde{B}}\Big[q(x+B_{t-s},x+\tilde{B}_{t-s}) \\ &\sigma(u(s,x+B_{t-s}))\sigma(u(s,x+\tilde{B}_{t-s}))\exp\{Y(s,t;B)+Y(s,t;\tilde{B})\}\Big]ds\,, \end{aligned}$$

where $\tilde{B}$ is a standard Brownian motion, independent of B. Let $H(s)$ denote the integrand of the above last integral. If we can show that $E(H(0)^{-p}) < \infty$ for all $p > 1$, and $H(s)$ is Hölder continuous, then by Lemma 5.5.2 below we prove

$$E\left(\int_0^t \int_{\mathbb{R}^d} |V_{s,\xi}(t,x)|^2 d\xi ds\right)^{-p} = E\left(\int_0^t H(s)ds\right)^{-p} < \infty, \qquad \forall p > 1\,.$$

The Hölder continuity of $H(s)$ can be verified from the following inequality:

$$E|H(s_1)-H(s_2)|^p \le C|s_2-s_1|^{\frac{p}{2}\min\{\rho,\gamma\}}, 0<s_1<s_2<t,$$

where C is determined by the
$\sup_{s\in[0,t]}\left\{E|q(x+B_{t-s},x+\tilde{B}_{t-s})|^{8p}, E|\sigma(u(s,x+B_{t-s}))|^{8p}, E\exp\{8pY(s,t;B)\}\right\}.$

It remains to show that $E(H(0)^{-p})<\infty$. Choose $\varepsilon>0$ such that if $|y-x_0|<\varepsilon, \sigma(u(0,y))>\frac{1}{2}\sigma(u(0,x_0))$. Let $A=\{(\omega,\omega'),|x+B_t-x_0|<\varepsilon,|x+\tilde{B}_t-x_0|<\varepsilon\}$. We see

$$\left|\sigma(u(0,x+B_t))\sigma(u(0,x+\tilde{B}_t))I_A\right| \ge \frac{1}{4}\sigma(u(0,x_0))^2 I_A.$$

Now we have

$$\begin{aligned}
& E\left(E^{B,\tilde{B}}\left[q(x+B_t,x+\tilde{B}_t)\sigma(u(0,x+B_t))\sigma(u(0,x+\tilde{B}_t))\exp\{Y(0,t;B)+Y(0,t;\tilde{B})\}\right]\right)^{-p} \\
\le & E\left(E^{B,\tilde{B}}\left[I_A\left|q(x+B_t,x+\tilde{B}_t)\sigma(u(0,x+B_t))\sigma(u(0,x+\tilde{B}_t))\right|\exp\{Y(0,t;B)+Y(0,t;\tilde{B})\}\right]\right)^{-p} \\
\le & \left[\frac{1}{4}\sigma(u(0,x_0))^2\right]^{-p} \times E\left(E^{B,\tilde{B}}\left[I_A\left|q(x+B_t,x+\tilde{B}_t)\right|\exp\{Y(0,t;B)+Y(0,t;\tilde{B})\}\right]\right)^{-p} \\
\le & a^{-p-1}\left[\frac{1}{4}\sigma(u(0,x_0))^2\right]^{-p} EE^{B,\tilde{B}}\left[I_A\left|q(x+B_t,x+\tilde{B}_t)\right|\exp\{-p\left(Y(0,t;B)+Y(0,t;\tilde{B})\right)\}\right],
\end{aligned}$$

where $a=E^{B,\tilde{B}}\left[I_A\left|q(x+B_t,x+\tilde{B}_t)\right|\right]>0$. This implies that $E(H(0)^{-p})<\infty$ for all $p>1$.

Lemma 5.5.2. *Let $\{S_t, 0\le t\le 1\}$ be a positive stochastic process. If $ES_0^{-a}<\infty$ for some $a>0$, and $\sup_{0\le s\le t}|S_s-S_0|\le Gt^\gamma$ where G is a positive random variable with $EG^b<\infty$ for some $b>0$, then we have*

$$E\left|\int_0^1 S_t dt\right|^{-p}<\infty, \ \textit{for} \ \ 0<p<ab\gamma/(a+b+b\gamma).$$

Particularly, if a and b can be arbitrarily large, then p can also be chosen arbitrarily large.

Proof Let $\alpha, \beta > 0$, where $\alpha + \beta < 1$ and $b\beta\gamma - b\alpha \geq a\alpha$, and $0 < \varepsilon < 2^{\alpha+\beta-1}$. We have

$$
\begin{aligned}
P\left[\int_0^1 S_t dt < \varepsilon\right] &\leq P\left[\int_0^{\varepsilon^\beta} S_t dt < \varepsilon, \quad S_0 > \varepsilon^\alpha\right] + P\left[S_0 < \varepsilon^\alpha\right] \\
\leq & P\left[\sup_{0 \leq t \leq \varepsilon^\beta} |S_t - S_0| > \frac{1}{2}\varepsilon^\alpha\right] + P\left[S_0^{-a} > \varepsilon^{-a\alpha}\right] \\
\leq & 2^b \varepsilon^{-b\alpha} E\left(\sup_{0 \leq t \leq \varepsilon^\beta} |S_t - S_0|^b\right) + \varepsilon^{a\alpha} E S_0^{-a} \leq C\left(\varepsilon^{b\beta\gamma - b\alpha} + \varepsilon^{a\alpha}\right) \leq C\varepsilon^{a\alpha}.
\end{aligned}
$$

Then $E\left|\int_0^1 S_t dt\right|^{-p} < \infty$, for $0 < p < a\alpha$. The lemma follows with the choice of α and β such that $\alpha < b\gamma/(a+b+b\gamma)$ and $\beta = (a+b)/(a+b+b\gamma)$.

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