LINEAR AND MULTILINEAR FRACTIONAL OPERATORS: WEIGHTED INEQUALITIES, SHARP BOUNDS, AND OTHER PROPERTIES

By

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To Andrea and Mom

Contents

Abstract

In this work we consider various fractional operators, including the classical fractional integral operators, related fractional maximal functions, multilinear fractional integral operators, and multisublinear fractional maximal functions. We characterize the weighted inequalities for the multilinear fractional operators, and examine more general two-weight inequalities giving sufficient conditions for their boundedness. For the classical fractional integral operator we obtain sharp bounds on the operator norm between weighted Lebesgue spaces in terms of the constant associated to the weight. We also introduce a more general fractional maximal operators, characterize their boundedness on weighted Lebsegue spaces, and obtain sharp bounds on the operator norms in terms of the weighted constants. Finally, we examine singular integral operators and fractional integral operators acting on mixed Lebesgue spaces with weights. We provide endpoint estimates for singular integrals and an off-diagonal extrapolation theorem.

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Introduction

Given an operator that is bounded on $L^p(\mathbb{R}^n)$, a natural problem arises when the measure is changed from Lebesgue measure to a general measure μ. More specifically, what conditions must one assume so that the same operator is bounded on $L^p(\mu)$? One approach to this problem is to consider measures that are absolutely continuous, i.e. $d\mu = w\,dx$ for some non-negative function or weight w. This is the essence of studying weighted inequalities, a subject that can probably be traced back to the beginning of integration.

Weighted inequalities are not mere generalizations, but also have far reaching applications. For instance, a weighted theory plays a big part in the study of boundary value problems for Laplace's equation on Lipschitz domains. Other applications include vector-valued operators and extrapolation of operators. Weighted inequalities for fractional operators have applications to potential theory and Quantum Mechanics.

Multilinear operators also appear naturally in fundamental problems and applications of harmonic analysis. Bilinear operators can be used as a tool to analyze nonlinearities where products of functions take place. The main content of this dissertation is concerned with weighted inequalities for various linear and multilinear fractional operators.

The groundbreaking work of Muckenhoupt [37] introduced the A_p class and used it to characterize the weighted inequalities for the Hardy-Littlewood maximal operator. This spurred the study of weighted estimates for other operators. Hunt, Muckenhoupt, and Wheeden [26] showed that weighted inequalities for the Hilbert transform are also characterized by the A_p class and Coifman and Fefferman [7] then extended the weighted theory to Calderón-Zygmund operators. Then, Muckenhoupt and Wheeden [38] showed the $A_{p,q}$ class characterizes the weighted inequalities for fractional operators.

In Chapter One we start with some basic facts about Lebesgue spaces that will be used in the following chapters. We introduce the main operators that pertain to this work. These operators include: maximal operators, operators that control the average of a function; fractional integral operators, operators that smooth a function; and singular integral operators, operators that are central to Calderón-Zygmund theory. We will be working with these operators or generalizations of these operators throughout the rest of this work. We conclude this chapter by introducing the corresponding class of weights for these operators and pointing out some of the basic properties of these weights.

Buckley [4] was the first person to consider the problem of finding sharp bounds on the operator norm in terms of the A_p constant. He found the sharp weighted bound for the Hardy-Littlewood maximal operator. Petermichl [42] ,[43], motivated by applications to partial differential equations (see Astala, Iwaniec, and Saksman [1]), found the sharp weighted bound on the operator norms of the Hilbert and Riesz transforms in terms of the A_p constant. Inspired by these results, Chapter Two is devoted to finding sharp bounds on the operator norms of fractional operators acting on weighted Lebesgue spaces. One of the main tools is a "sharp" off-diagonal extrapolation theorem of Harboure, Macias, and Segovia [24]. We use techniques of Sawyer and Wheeden [49] to obtain a sharp bound for a pair of exponents (p_0, q_0) and then extend this bound to a range of exponents (p, q). We also present some improved Sobolev estimates.

Some of the techniques in Chapter Two lead to a more general theory of maximal functions. In Chapter Three we consider maximal functions with respect to a general basis. We obtain one- and two-weight characterizations for the boundedness of the general maximal functions, extending the work of Jawerth [27]. We also discover a new testing condition for fractional maximal functions. As a consequence of our techniques we obtain sharp two-weight bounds on the operator norm of the Hardy-Littlewood maximal operator in terms of Sawyer's testing condition [47]. We also find a new sharp bound on a weighted maximal operator in terms of the Reverse Hölder constant.

The results contained in Chapter Four are multilinear versions of the fractional integral operator and fractional maximal function. Spurred by the work of Lerner, Ombrosi, Pérez, Torres, and Trujillo-Gonzalez [33] we develop a weighted theory for the multilinear fractional operators. We find new two-weight conditions for the multilinear operators that provide interesting contrast to the linear results. As a consequence of the two-weight theory we obtain the one-weight theory. Finally we end the chapter with some applications of the boundedness of the multilinear fractional integral operator including Sobolev inequalities for products of functions.

Chapter Five deals with operators acting on mixed Lebesgue spaces. We present some results for Calderón-Zygmund operators that are more general than those given by Stefanov and Torres, [50] and Kurtz [29]. The weak-type mixed norm endpoint for general Calderón-Zygmund operators is a new result. We also introduce a mixed $A_{p,q}$ class of weights and provide an off-diagonal extrapolation theorem for mixed Lebesgue spaces with product weights.

This dissertation contains results from the articles [35], [36], and [30] as well as some additional material. We present the work in a more comprehensive manner and sometimes, if possible, provide a different proof of a result than that contained in the articles. The material in Chapter Two is from our collaboration [30]. We have presented these results at the Analysis Seminar at Kansas State University, March 2009. The material in Chapter Three contains some results from the article [36]. These results were presented at the Eighth Prairie Analysis Seminar, Lawrence, November 2008. Finally, Chapter Four contains results from the article [35] and these results have been presented at the Eighth International Conference on Harmonic Analysis and Partial Differential Equations, El Escorial (Spain), June 2008, and the AMS-MAA Joint Mathematics Meetings, Washington, DC, January 2009.

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Chapter 1

Preliminaries

In this chapter we introduce the basic theory required for the later chapters. We will mainly be working on, although not limited to, Lebesgue spaces. They are examples of classic Banach spaces and are fundamental to analysis. Many of the basic definitions, propositions, and theorems will be presented without reference, although the interested reader may find more information in the books by Folland [14], Grafakos [18], or Rudin [46]. In this chapter we also introduce various classical operators including, maximal operators, fractional operators, and singular integral operators.

1.1 L^p spaces

Definition 1.1.1. Let $0 < p < \infty$ and $(X, \mathfrak{M}, \mu)$ be a measure space with associated σ-algebra $\mathfrak{M}$, and measure μ. Then $L^p(X, \mu)$ consists of all complex-valued measurable functions, f, on X such that

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f|^p \, d\mu\right)^{1/p} < \infty.$$

Furthermore $L^\infty(X,\mu)$ will denote the set of measurable functions, f, that are essentially bounded, that is

$$\|f\|_{L^\infty(X,\mu)} = \inf\{M \geq 0 : \mu(\{x : |f(x)| > M\}) = 0\} < \infty.$$

For $1 \leq p \leq \infty$, $\|\cdot\|_{L^p(X,\mu)}$ defines a complete norm on $L^p(X,\mu)$ making it a Banach space. For $1 < p < \infty$, p' will denote the dual or conjugate exponent defined by the equation

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

and we use the convention $1' = \infty$ and $\infty' = 1$.

For $1 < p < \infty$ the dual space of $L^p(X,\mu)$ is isometrically identified with $L^{p'}(X,\mu)$ under the pairing

$$\langle f, g\rangle = \int_X fg\, d\mu.$$

Furthermore,

$$\|f\|_{L^p(X,\mu)} = \sup\left|\int_X fg\, d\mu\right| \tag{1.1}$$

where the supremum is taken over all $g \in L^{p'}(X,\mu)$ with norm one. Another useful fact is the "layer-cake" principle

$$\|f\|^p_{L^p(X,\mu)} = p\int_0^\infty \lambda^{p-1}\mu(\{x \in X : |f(x)| > \lambda\})\, d\lambda.$$

Definition 1.1.2. When $0 < p < \infty$, $L^{p,\infty}(X,\mu)$ will denote the weak-$L^p(X,\mu)$ space, consisting of all measurable functions, f, that satisfy

$$\|f\|_{L^{p,\infty}(X,\mu)} = \sup_{\lambda>0} \lambda\mu(\{x \in X : |f(x)| > \lambda\})^{1/p} < \infty.$$

Note that $\|\cdot\|_{L^{p,\infty}(X,\mu)}$ is in general not a norm, however for $1 < p \leq \infty$ it is equivalent to a norm that makes $L^{p,\infty}(X,\mu)$ into a Banach space. Also notice that $L^p(X,\mu)$ is a proper subset of $L^{p,\infty}(X,\mu)$.

When $X = \mathbb{R}^n$ and the measure is Lebesgue measure, $d\mu = dx$, we will write L^p, $L^{p,\infty}$ for the respective spaces. When μ is absolutely continuous with respect to Lebesgue measure, i.e. $d\mu = w\,dx$, for some measurable function w, then we write $L^p(w)$ and $L^{p,\infty}(w)$. A non-negative locally integrable function will be called a weight. Given a measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes the Lebesgue measure of E, $w(E) = \int_E w\,dx$ is the weighted measure of E. Most of the time we will be working on $L^p(w)$ where w is a weight.

1.1.1 Operators on L^p spaces

Let (X,μ) and (Y,ν) be two measure spaces, and suppose that T is an operator defined on the space of all μ-measurable functions, taking values in the set of all ν-measurable functions. We say that T is linear if

$$T(\lambda f + g) = \lambda T f + T g$$

for all f, g and $\lambda \in \mathbb{C}$. The operator T is sublinear if for all f, g, and $\lambda \in \mathbb{C}$

$$\begin{aligned} |T(f+g)| &\leq |Tf| + |Tg| \\ |T(\lambda f)| &= |\lambda||Tf|. \end{aligned}$$

Given two Lebesgue spaces $L^p(X,\mu)$ and $L^q(Y,\nu)$ we say a linear or sublinear operator T is bounded from $L^p(X,\mu)$ to $L^q(Y,\nu)$, if there exists a constant $C = C_{p,q}$ such that for all functions $f \in L^p(X,\mu)$ we have

$$\|Tf\|_{L^q(Y,\nu)} \leq C\|f\|_{L^p(X,\mu)}. \tag{1.2}$$

Occasionally we will write $T : L^p(X,\mu) \to L^q(Y,\nu)$ to indicate T is bounded from $L^p(X,\mu)$ to $L^q(Y,\nu)$. The operator norm of T, denoted $\|T\|_{L^p(X,\mu)\to L^q(Y,\nu)}$ or simply $\|T\|$ when the ambient spaces are clear, is given by

$$\|T\|_{L^p(X,\mu)\to L^q(Y,\nu)} = \sup \|Tf\|_{L^q(X,\mu)}$$

where the supremum is taken over all $f \in L^p(X,\mu)$ of norm one. In light of (1.1) we may also compute an operator norm via,

$$\|T\|_{L^p(X,\mu)\to L^q(Y,\nu)} = \sup \left| \int_Y gTf \ d\nu \right|,$$

with the supremum taken over all $f \in L^p(X,\mu)$ and $g \in L^{q'}(Y,\nu)$ of norm one. An operator T is bounded from $L^p(X,\mu)$ to $L^{q,\infty}(Y,\nu)$ if

$$\|Tf\|_{L^{q,\infty}(Y,\nu)} \leq C\|f\|_{L^p(X,\mu)}.$$

In this case we will say T is weak (p,q) and write $T : L^p(X,\mu) \to L^{q,\infty}(Y,\nu)$. We will also consider the operator norm of weak (p,q) operators defined by

$$\|T\|_{L^p(X,\mu)\to L^{q,\infty}(Y,\nu)} = \sup \|Tf\|_{L^{q,\infty}(X,\mu)}$$

where the supremum is taken over all $f \in L^p(X,\mu)$ of norm one.

One important tool in the theory of L^p spaces is interpolation. The classical Marcienkiewicz interpolation theorem allows one to obtain strong boundedness of operators from two weak endpoints. We state the off-diagonal version here as it will be useful latter. The proof can be found in [18, p. 62].

Theorem 1.1.3. *Let $0 < p_0 \neq p_1 \leq \infty$ and $0 < q_0 \neq q_1 \leq \infty$ and suppose T is a sublinear operators defined on the space $L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$ taking values in the space of ν-measurable functions on Y. If*

$$T : L^{p_0}(X,\mu) \to L^{q_0,\infty}(Y,\nu)$$

$$T : L^{p_1}(X,\mu) \to L^{q_1,\infty}(Y,\nu),$$

then

$$T : L^p(X,\mu) \to L^q(Y,\nu)$$

for

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

where $\theta \in (0,1)$ and $p \leq q$.

1.2 The main operators

In this section we introduce the main operators we will be working with. They include maximal operators, fractional integral operators and Calderón-Zygmund operators.

1.2.1 Maximal operators

We will often use Q to denote a cube (either open, closed, or neither) in $\mathbb{R}^n$ with sides parallel to the axes and $B(x,r)$ will denote a ball in $\mathbb{R}^n$ centered at x with radius r. The side length of a cube will be $\ell(Q)$ and given a positive constant c, cQ will denote the concentric cube with Q that has side length $c\ell(Q)$. The set $\mathscr{D}$ is the set of all dyadic cubes, i.e. cubes of the form $2^k(m+[0,1)^n)$ with $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. In this section, and in the rest of this work we will use the notation $A \approx B$ to mean there exists positive constants c and C such that $cB \leq A \leq CB$.

Definition 1.2.1. Let f be a locally integrable function on $\mathbb{R}^n$ then the Hardy-Littlewood maximal operator with respect to the measure μ is defined by

$$M_\mu f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f| \, d\mu, \tag{1.3}$$

where the supremum is over all cubes, Q, with sides parallel to the axes that contain x. We use the convention that $M_\mu f(x) = 0$ if $\mu(Q) = 0$ for all cubes, Q, that contain x.

When μ is Lebesgue measure we drop the subscript and write M to denote the Hardy-Littlewood maximal function. Notice in this case we have

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

Notice that Lebesgue measure satisfies $|3Q| = 3^n|Q|$. It is this property that allows one to obtain the L^p boundedness of the Hardy-Littlewood maximal operator. General measures that satisfy this are called doubling measures.

Definition 1.2.2. We say a measure μ is doubling if there exists a positive constant C such that

$$\mu(3Q) \leq C\mu(Q) \tag{1.4}$$

for all cubes Q. The smallest constant C that satisfies (1.4) will be called the doubling constant of μ and denoted D_μ.

The maximal operator M_μ is always bounded on $L^\infty(\mu)$ and if the measure μ is doubling then M_μ is also weak (1,1). Using Theorem 1.1.3 one has the following result. A proof for the case of the Hardy-Littlewood maximal operator can be found in [18] or [10].

Theorem 1.2.3. *Suppose μ is a doubling measure with doubling constant D_μ,*

$$M_\mu : L^1(\mu) \to L^{1,\infty}(\mu) \qquad \text{and} \qquad M_\mu : L^p(\mu) \to L^p(\mu)$$

for $1 < p \leq \infty$. Furthermore, we have the following relationship between the operator norm of M_μ and the doubling constant D_μ,

$$\|M_\mu\|_{L^1(\mu)\to L^{1,\infty}(\mu)} \leq cD_\mu \qquad \text{and} \qquad \|M_\mu\|_{L^p(\mu)\to L^p(\mu)} \leq cD_\mu^{1/p}.$$

The proof of Theorem 1.2.3 is based on the following covering lemma due to Vitali. A proof Lemma 1.2.4 in the case of balls instead of cubes can be found in [46]. One can see this is where the doubling condition (1.4) come into play.

Lemma 1.2.4. *Let $\{Q_1, \dots, Q_n\}$ be a finite collection of cubes in $\mathbb{R}^n$. Then there exists a subset $S \subseteq \{1, \dots, n\}$ such that*

- *the collection $\{Q_i\}_{i\in S}$ is pairwise disjoint,*
- $\bigcup_{j=1}^n Q_j \subseteq \bigcup_{i\in S} 3Q_i$

We now discuss some variants of the Hardy-Littlewood maximal function. First, the dyadic version of M,

$$M_\mu^d f(x) = \sup_{x \in Q \in \mathscr{D}} \frac{1}{\mu(Q)} \int_Q |f| \, d\mu,$$

where the supremum is over all dyadic cubes that contain x. Using the fact that any two dyadic cubes are either disjoint or one is contained in the other we may write

$$\{x \in \mathbb{R}^n : M_\mu^d f(x) > \lambda\} = \bigcup_j Q_j,$$

where Q_j are maximal disjoint dyadic cubes that satisfy

$$\frac{1}{\mu(Q_j)} \int_{Q_j} |f| \, d\mu > \lambda.$$

It follows that

$$\mu(\{x : M_\mu^d f(x) > \lambda\}) \le \frac{1}{\lambda} \int_{\{M_\mu^d f > \lambda\}} |f| \, d\mu \le \frac{\|f\|_{L^1(\mu)}}{\lambda}.$$

Thus M_μ^d is weak $(1,1)$ with constant one and it follows that for $1 < p \le \infty$, $M_\mu : L^p(\mu) \to L^p(\mu)$ with norm that depends only on p. We also examine the centered maximal function,

$$M_\mu^c f(x) = \sup_{Q_x} \frac{1}{\mu(Q_x)} \int_{Q_x} |f| \, d\mu$$

where the supremum is over all cubes centered at x. Notice that when μ is Lebesgue measure $M^c \approx M$. We state another covering lemma due to Besicovitch.

Lemma 1.2.5. *Suppose E is a bounded subset of $\mathbb{R}^n$, for each $x \in E$, Q_x is a cube centered at x, and $E \subseteq \bigcup_{x\in E} Q_x$. Then there exists a subset of $\{Q_x\}_{x\in E}$, $\{Q_j\}$ such that*

$$E \subseteq \bigcup_j Q_j \quad \text{and} \quad \sum_j \chi_{Q_j} \leq C_n.$$

It follows that

$$\mu(\{x : M_\mu^c f(x) > \lambda\}) \leq \frac{C_n}{\lambda} \int_{\mathbb{R}^n} |f| \, d\mu.$$

Hence $M_\mu^c : L^p(\mu) \to L^p(\mu)$ for $1 < p \leq \infty$ with operator norm depending only on the dimension n, and p and not μ.

We examine one more variant of the maximal operators. This time a family of maximal functions,

$$M_{\mu,\alpha} f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)^{1-\alpha/n}} \int_Q |f| \, d\mu, \qquad 0 \leq \alpha < n.$$

Given $0 < \alpha < n$ we refer to $M_{\mu,\alpha}$ as the fractional maximal operator. The case $\alpha = 0$ corresponds to the operator from (1.3). By Hölder's inequality

$$\frac{1}{\mu(Q)^{1-\alpha/n}} \int_Q |f| \, d\mu \leq \left(\int_Q |f|^{n/\alpha} \, d\mu \right)^{\alpha/n},$$

which implies $M_{\alpha,\mu} : L^{n/\alpha}(\mu) \to L^\infty(\mu)$. If μ is doubling, then a similar argument to that given in the proof of Theorem 1.2.3 shows that $M_{\alpha,\mu}$ is weak $(1, (n/\alpha)')$. Thus, Theorem 1.1.3 implies

$$M_{\alpha,\mu} : L^p(\mu) \to L^q(\mu)$$

where $1 < p \leq n/\alpha$ and q is defined by the equation

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \tag{1.5}$$

If $M^d_{\alpha,\mu}$ denotes the dyadic fractional maximal operator and $M^c_{\alpha,\mu}$ denotes the centered maximal operator then similar arguments to the case $\alpha = 0$ show

$$M^d_{\alpha,\mu} : L^p(\mu) \to L^q(\mu)$$
$$M^c_{\alpha,\mu} : L^p(\mu) \to L^q(\mu)$$

with operator norms independent of μ and $p < q$ satisfing (1.5).

1.2.2 Fractional integral operators

We now introduce fractional integral operators. These are important operators in analysis pertaining to the smoothness of functions and Sobolev embedding theorems. More information can be found in the books by Grafakos [18] and Stein [51]. In order to define these operators we need to define some function spaces and distribution spaces. Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ be the space of Schwartz rapidly decreasing functions and its dual space $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ the space of all tempered distributions.

Definition 1.2.6. Let $f \in \mathscr{S}$ and define the Fourier transform of f by

$$\mathscr{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx,$$

and inverse Fourier transform

$$\mathscr{F}^{-1} f(x) = \check{f}(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi\cdot x} d\xi.$$

If $f \in \mathscr{S}$ then we have the inversion property $f = (\hat{f})\check{}$. Given a tempered distribution $u \in \mathscr{S}'$ we may define the Fourier transform and inverse Fourier transform by $\langle \hat{u}, f\rangle = \langle u, \hat{f}\rangle$ and $\langle \check{u}, f\rangle = \langle u, \check{f}\rangle$ respectively.

Definition 1.2.7. Let $0 < \alpha < n$, we define the fractional integral operator or Riesz potential by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

We notice that the function $|\cdot|^{\alpha-n}$ is locally integrable for $0 < \alpha < n$, so I_α is well defined by an absolutely convergent integral if, say, $f \in \mathscr{S}$. We may also define I_α on the Fourier transform side by

$$(I_\alpha f)\hat{}(\xi) = c_n|\xi|^{-\alpha}\hat{f}(\xi)$$

where c_n is an appropriate dimensional constant. In this sense I_α acts as the α-th order anti-derivative. The operator, I_α is also intimately related to M_α. First, it is a pointwise bigger operator, i.e.,

$$M_\alpha f \le cI_\alpha f$$

almost everywhere for all non-negative f. As we shall see later, the reverse inequality also holds in L^p norm. Furthermore, I_α has the same boundedness properties as M_α. We state the following theorem, a proof can be found in [18].

Theorem 1.2.8. *Suppose* $0 < \alpha < n$*, and* $1 \le p < q < \infty$ *satisfy* (1.5)*. Then*

$$I_\alpha : L^1 \to L^{q,\infty}$$

and

$$I_\alpha : L^p \to L^q$$

when $p > 1$.

One of the most important applications of the boundedness of I_α is the Sobolev Embedding Theorem. For $1 < p < \infty$ define the Sobolev space $W^{s,p}$, to be the space of all tempered distributions, u, with the property that

$$((1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi))^{\vee}$$

is in L^p. We may norm this space by

$$\|f\|_{W^{s,p}} = \|((1+|\cdot|^2)^{\frac{s}{2}}\hat{f})^{\vee}\|_{L^p},$$

and this norm makes $W^{s,p}$ into a Banach space, see [18], or [51] for details. A non-trivial fact, at least when $p \neq 2$, is that if $s = k$ for some non-negative integer k, then

$$\|f\|_{W^{k,p}} \approx \sum_{|s|\leq k} \|\partial^s f\|_{L^p} \approx \|f\|_{L^p} + \sum_{|s|=k} \|\partial^s f\|_{L^p}$$

with the convention that $\partial^{(0,\dots,0)} f = f$. In this case $W^{k,p}$ corresponds to the space of functions whose derivatives up to order k are in L^p. A proof of this can be found in [18]. We now state the Sobolev embedding Theorem. A proof, which depends heavily on the boundedness of I_α (Theorem 1.2.7) can be found also in [18].

Theorem 1.2.9. *1. Let $0 < \alpha < n$ and $1 < p < n/\alpha$. Then the Sobolev space $W^{\alpha,p}$ continuously embeds in L^q where q satisfies* (1.5).

2. *If $1 < p < \infty$ and $0 < \alpha = n/p$, then $W^{\alpha,p}$ continuously embeds in L^q for any $q > p$.*

3. *If $1 < p < \infty$ and $n/p < \alpha$, then every element of $W^{\alpha,p}$ can be modified on a set of measure zero so that it is uniformly continuous.*

1.2.3 Calderón-Zygmund operators

In this section we examine an important class of operators in analysis, Calderón-Zygmund operators. We say that a function K defined away from the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ is standard kernel if it satisfies the size condition

$$|K(x,y)| \leq C|x-y|^{-n} \tag{1.6}$$

and regularity conditions

$$|K(x,y+h)-K(x,y)|+|K(x+h,y)-K(x,y)| \leq C\frac{|h|^{\delta}}{|x-y|^{n+\delta}} \tag{1.7}$$

for some $0<\delta\leq 1$, whenever $|x-y| \geq 2|h|$.

Definition 1.2.10. An operator T is a Calderón-Zygmund operator if T is bounded on L^q for some $1<q<\infty$ and is associated with a standard kernel K, in the sense that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy,$$

whenever $f \in L^q$ has compact support and x is not in the support of f.

Some of the main examples of Calderón-Zygmund operators are ones that are given as convolution, $K(x,y)=k(x-y)$, where k is locally integrable away from zero and satisfies the corresponding estimates (1.6) and (1.7).

Example 1.2.11. Let $f \in \mathscr{S}(\mathbb{R})$, and define the Hilbert transform as

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy = \frac{1}{\pi}p.v.\int\frac{f(y)}{x-y}\,dy. \tag{1.8}$$

The higher dimensional versions of H are the Riesz transforms given by

$$R_j f(x) = c_n p.v. \int \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy \tag{1.9}$$

for $1 \leq j \leq n$ and $f \in \mathscr{S}(\mathbb{R}^n)$.

Notice that

$$(Hf)\hat{}(\xi) = -i \operatorname{sgn}(\xi) \hat{f}(\xi) \tag{1.10}$$

and if c_n is chosen correctly

$$(R_j f)\hat{}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi). \tag{1.11}$$

It follows from the Plancherel theorem that

$$H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

and

$$R_j : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

for $1 \leq j \leq n$. It can be easily checked that the kernels of the Riesz and Hilbert transform satisfy (1.6) and (1.7), and hence they are Calderón-Zygmund operators. As the following theorem states, these operators are also bounded on all L^p spaces for $1 < p < \infty$. A proof can be found in [18] and [10].

Theorem 1.2.12. *Suppose that T is a Calderón-Zygmund operator then*

$$T : L^1 \to L^{1,\infty}$$

and

$$T : L^p \to L^p$$

for $1 < p < \infty$.

1.3 Weights

Muckenhoupt [37] introduced the A_p class of weights and used it to characterize the boundedness of the Hardy-Littlewood maximal operator, on $L^p(w)$. In this section we introduce the Muckenhoupt or A_p weights and present some of the fundamental results concerning weighted inequalities for the operators introduced in the previous sections. A comprehensive guide to much of the material presented in this section can be found in the books by Duoandikoetxea [10] or Grafakos [19] .

A weight w belongs to the class A_p, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(y)\, dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'}\, dy \right)^{p-1} < \infty, \tag{1.12}$$

where the supremum is over all cubes, Q. We refer to $[w]_{A_p}$ as the A_p constant of w. For $p = 1$, we say $w \in A_1$ if

$$\frac{1}{|Q|} \int_Q w(y)\, dy \leq C \operatorname{essinf}_Q w,$$

where the smallest constant C will be denoted $[w]_{A_1}$. This is equivalent to saying

$$Mw(x) \leq [w]_{A_1} w(x)$$

for almost every $x \in \mathbb{R}^n$. We notice a few properties about the class A_p:

- $1 \leq [w]_{A_p}$, by Hölder's inequality .
- $A_p \subset A_q$, for $1 \leq p < q$.
- $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, with $[w]_{A_p} = [w^{1-p'}]_{A_{p'}}^{1/(p-1)}$.
- If $u, v \in A_1$, then $uv^{1-p} \in A_p$.
- If $w \in A_p$ then w is a doubling measure with $D_\mu \leq [w]_{A_p}$.

Example 1.3.1. The function $w(x) = |x|^a$ is in A_p if and only if $-n < a < n(p-1)$. Also, $w(x) = \log |x|$ for $|x| < 1/e$ and 1 otherwise is in A_1.

We notice that if $p \to \infty$ in (1.12) then we get

$$\frac{1}{|Q|} \int_Q w \, dy \leq C \exp\left(\frac{1}{|Q|} \int_Q \log w \, dy \right). \tag{1.13}$$

We will say say that $w \in A_\infty$ if inequality (1.13) holds. However, one may also define A_∞ in a number of other ways, including

$$A_\infty = \bigcup_{p>1} A_p,$$

which we will usually use as the definition of A_∞. The fact that these two definitions are equivalent can be found in [16].

The fundament result concerning A_p weights is due to Muckenhoupt [37].

Theorem 1.3.2. *Let $1 < p < \infty$. Then*

$$M : L^p(w) \to L^p(w)$$

if and only if $w \in A_p$.

For $1 \le p < \infty$, it is not difficult to show that $w \in A_p$ if and only $M : L^p(w) \to L^{p,\infty}(w)$. The key step in Muckenhoupt's approach to showing the strong inequality holds is showing that

$$A_p = \bigcup_{q<p} A_q. \tag{1.14}$$

From here it follows that if $w \in A_p$ then $w \in A_q$ for some $q < p$, hence $M : L^q(w) \to L^{q,\infty}(w)$. Since $M : L^\infty(w) \to L^\infty(w)$, we obtain Theorem 1.3.2. The equality in (1.14) follows from what is known as Reverse Hölder condition.

Theorem 1.3.3. *Let $w \in A_p$, $1 \le p < \infty$. The there exists a constant C and $r > 1$, depending on p and the A_p constant of w, such that for any cube Q,*

$$\left(\frac{1}{|Q|}\int_Q w^r\,dy\right)^{1/r} \le \frac{C}{|Q|}\int_Q w\,dy \tag{1.15}$$

When a weight w satisfies (1.15) we say w belongs to the class RH_r and write $w \in RH_r$. We give a short proof of Theorem 1.3.2 by Lerner [31] that avoids Theorem 1.3.3 and yields sharp constants. We give this proof because some of these techniques will be used later.

Proof of Theorem 1.3.2. Let $1 < p < \infty$ and $w \in A_p$ with $\sigma = w^{1-p'}$. Notice the A_p condition for w can now be written as

$$\sup_Q \frac{w(Q)\sigma(Q)^{p-1}}{|Q|^p} < \infty.$$

We also notice that $M \approx M^c$, the centered Hardy-Littlewood maximal function. So it suffices to prove it for the centered maximal operator. Let $x \in \mathbb{R}^n$ and Q be any cube centered at x. Then

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f|\,dy &= \left(\frac{w(3Q)\sigma(Q)^{p-1}}{|Q|^p}\right)^{1/(p-1)} \frac{|Q|^{p'}}{w(Q)^{1/(p-1)}\sigma(3Q)}\frac{1}{|Q|}\int_Q |f|\,dy \\
&\leq 3^{np'}[w]_{A_p}^{1/(p-1)}\left(\frac{|Q|}{w(Q)}\left(\frac{1}{\sigma(3Q)}\int_Q |f|\sigma^{-1}\sigma\,dy\right)^{p-1}\right)^{1/(p-1)} \\
&\leq 3^{np'}[w]_{A_p}^{1/(p-1)}\left(\frac{1}{w(Q)}\int_Q M^c_\sigma(f\sigma^{-1})^{p/p'}w^{-1}w\,dy\right)^{p'/p}
\end{aligned}$$

Taking the supremum over all Q centered at x we have the following pointwise inequality,

$$M^c f(x) \leq 3^{np'}[w]_{A_p}^{1/(p-1)} M^c_w\{M^c_\sigma(f\sigma^{-1})^{p/p'}w^{-1})(x)^{p'/p}.$$

From the comments after Lemma 1.2.5 we have that $M^c_w : L^{p'}(w) \to L^{p'}(w)$ and $M^c_\sigma : L^p(\sigma) \to L^p(\sigma)$ with operators norms independent of w and σ respectively. Thus we have

$$\begin{aligned}
\|Mf\|_{L^p(w)} &\leq C\|M^c f\|_{L^p(w)} \\
&\leq C[w]_{A_p}^{1/(p-1)}\|M^c_w\{M^c_\sigma(f\sigma^{-1})^{p/p'}w^{-1})^{p'/p}\|_{L^p(w)} \\
&= C[w]_{A_p}^{1/(p-1)}\|M^c_w\{M^c_\sigma(f\sigma^{-1})^{p/p'}w^{-1})\|_{L^{p'}(w)}^{p'/p} \\
&\leq C\|M^c_w\|^{p'/p}[w]_{A_p}^{1/(p-1)}\|M^c_\sigma(f\sigma^{-1})^{p/p'}w^{-1}\|_{L^{p'}(w)}^{p'/p} \\
&= C\|M^c_w\|^{p'/p}[w]_{A_p}^{1/(p-1)}\|M^c_\sigma(f\sigma^{-1})\|_{L^{p'}(\sigma)} \\
&\leq C\|M^c_w\|^{p'/p}\|M^c_\sigma\|[w]_{A_p}^{1/(p-1)}\|f\sigma^{-1}\|_{L^p(\sigma)} \\
&= C\|M^c_w\|^{p'/p}\|M^c_\sigma\|[w]_{A_p}^{1/(p-1)}\|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of the Theorem. □

Notice that from Lerner's proof of Theorem 1.3.2 we have the following relationship between the operator norm of M and the A_p constant of w

$$\|M\|_{L^p(w)\to L^p(w)} \leq cpp'[w]_{A_p}^{1/(p-1)}. \tag{1.16}$$

We shall see later that (1.16) is sharp.

For Calderón-Zygmund operators the A_p class of weights is also the natural class of weights. Hunt, Muckenhoupt, and Wheeden [26] showed that A_p also characterizes the class of weights for which the Hilbert transform is bounded on $L^p(w)$. Then Coifman and Fefferman [7] extend the A_p theory to general Calderón-Zygmund operators.

For fractional operators which map off-diagonally, weighted inequalities are simplified by treating the weight as a multiplier rather than a measure. More specifically, the weighted inequalities we will be concerned with are

$$\left(\int_{\mathbb{R}^n} (wT_\alpha f)^q \, dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} (fw)^p \, dx\right)^{1/p}$$

where p and q satisfy (1.5) and T_α is either the fractional integral operator I_α or fractional maximal function M_α. The weights for these operators is the $A_{p,q}$ class of weights, w, that satisfy

$$[w]_{A_{p,q}} = \sup_Q \left(\frac{1}{|Q|}\int_Q w^q \, dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'} \, dx\right)^{q/p'} < \infty.$$

This class is defined for any $1 < p \leq q < \infty$, which is the case when p and q satisfy (1.5). For $p = 1$, $A_{1,q}$ is the class of weights w such that $w^q \in A_1$ and $[w]_{A_{1,q}} = [w^q]_{A_1}$.

We make a few observations:

- $1 \leq [w]_{A_{p,q}}$ since $p \leq q$, by Hölder's inequality .

- $w \in A_{p,q}$ if and only if $w^q \in A_{1+q/p'}$ with $[w]_{A_{p,q}} = [w^q]_{A_{1+q/p'}}$,
- $w \in A_{p,q}$ if and only if $w^{-1} \in A_{q',p'}$, with $[w]_{A_{p,q}} = [w^{-1}]_{A_{q',p'}}^{q/p'}$.
- $w \in A_{p,q}$ if and only if $w \in A_{1+1/p'} \cap RH_q$.

We provide a quick proof of the last point as it does not seem to be in the literature. Suppose $w \in A_{p,q}$ and set $r = 1 + 1/p'$, then for any cube Q

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-r'}\,dx\right)^{r-1} \leq\ &= \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\,dx\right)^{1/p'} \\
&\leq \left(\frac{1}{|Q|}\int_Q w^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w^{-p'}\,dx\right)^{1/p'} \\
&\leq [w]_{A_{p,q}}^{1/q} < \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q w^q\,dx\right)^{1/q} &= \left(\frac{1}{|Q|}\int_Q w^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w^{1/r}w^{-1/r}\,dx\right)^{r} \\
&\leq \left(\frac{1}{|Q|}\int_Q w^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-r'}\,dx\right)^{r-1} \\
&= \left(\frac{1}{|Q|}\int_Q w^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w^{-p'}\,dx\right)^{1/p'}\left(\frac{1}{|Q|}\int_Q w\,dx\right) \\
&\leq [w]_{A_{p,q}}^{1/q}\left(\frac{1}{|Q|}\int_Q w\,dx\right).
\end{aligned}$$

This shows $w \in A_{p,q}$ implies $w \in A_r \cap RH_q$. On the other hand, if $w \in A_r \cap RH_q$ then

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q w^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w^{-p'}\,dx\right)^{1/p'} &\leq C\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\,dx\right)^{1/p'} \\
&\leq C[w]_{A_{1+1/p'}} < \infty.
\end{aligned}$$

Muckenhoupt and Wheeden [38] characterized the weighted inequalities for the operators I_α and M_α in the following theorems below.

Theorem 1.3.4. *Suppose* $0 < \alpha < n$, *then*

$$I_\alpha : L^1(w) \to L^{n/(n-\alpha),\infty}(w^{n/(n-\alpha)})$$

if and only if $w \in A_{1,n/(n-\alpha)}$. *If* $1 < p < n/\alpha$ *and* q *is defined by* $1/q = 1/p - \alpha/n$ *then*

$$I_\alpha : L^p(w^p) \to L^q(w^q)$$

if and only if $w \in A_{p,q}$.

Theorem 1.3.5. *Suppose* $0 \leq \alpha < n$, *then*

$$M_\alpha : L^1(w) \to L^{n/(n-\alpha),\infty}(w^{n/(n-\alpha)})$$

if and only if $w \in A_{1,n/(n-\alpha)}$. *If* $1 < p < n/\alpha$ *and* q *is defined by* $1/q = 1/p - \alpha/n$ *then*

$$M_\alpha : L^p(w^p) \to L^q(w^q)$$

if and only if $w \in A_{p,q}$.

Chapter 2

Sharp weighted bounds for fractional operators

In this chapter we find sharp weighted bounds for the operators M_α and I_α. We use techniques similar to those developed in [31] for M_α. These techniques, in turn lead to a more general theory which will be presented in the next chapter.

Our main motivation for finding the sharp bound on the operator norm of I_α is the following result of Petermichl [42], [43]. If T is either the Hilbert transform (1.8), or the Riesz transform in $\mathbb{R}^n$ (1.9), then

$$\|T\|_{L^p(w)\to L^p(w)} \leq c[w]_{A_p}^{\max\{1,1/(p-1)\}}.$$

The problem of finding sharp bounds on the weighted operator norm of singular integral operators is also of interest because of applications to partial differential equations. More specifically it has applications to the regularity Beltrami equations in the plane see Astala, Iwaniec, and Saksman [1] and Petermichl and Volberg [44].

For I_α different techniques are used to find sharp bounds for operator norms. We use a dyadic decomposition to view the operator as a discrete operator. This decomposition lets us obtain a sharp bound for a fixed p_0 and q_0. We then use a sharp off-diagonal extrapolation theorem to obtain our results. We also present a weak extrapolation theorem and as an application we obtain sharp weak inequalities for I_α. This leads to an improved Sobolev estimate. Finally, we provide some examples to show that the bounds are indeed sharp.

For the most part, the content of this chapter overlaps with of the work [30], as originally started jointly with Pérez and Torres (see also Chapter 6). We include all of the Theorems and proofs with some of them similar to what appears in [30]. Our aim, however, is to provide a more comprehensive account of the work. We have included more detail and expanded in many places. Moreover, we have included different approaches or proofs when possible.

2.1 Sharp bounds for the fractional maximal operator

Theorem 2.1.1. *Suppose $0 \le \alpha < n$, $1 < p < n/\alpha$ and q is defined by the relationship $1/q = 1/p - \alpha/n$. If $w \in A_{p,q}$, then*

$$\|wM_\alpha f\|_{L^q} \le c[w]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})}\|wf\|_{L^p} \tag{2.1}$$

and the exponent $\frac{p'}{q}(1-\frac{\alpha}{n})$ is sharp.

Proof. First notice that $M_\alpha \approx M_\alpha^c$ where M_α^c is the centered version. Let $x \in \mathbb{R}^n$, Q a cube centered at x, $u = w^q$, $\sigma = w^{-p'}$ and $r = 1 + q/p'$. Noticing that $p'/q(1-\alpha/n) = r'/q$, we proceed as in [31] to obtain

$$\begin{aligned}
\frac{1}{|Q|^{1-\alpha/n}}\int_Q |f|\,dy &\le 3^{nr'/q}[w]_{A_{p,q}}^{p'/q(1-\alpha/n)}\left(\frac{|Q|}{u(Q)}\right)^{p'/q(1-\alpha/n)}\frac{1}{\sigma(3Q)^{1-\alpha/n}}\int_Q \frac{|f|}{\sigma}\sigma\,dy \\
&\le c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)}\left(\frac{1}{u(Q)}\int_Q M_{\alpha,\sigma}^c(f/\sigma)^{q/r'}\,dy\right)^{r'/q}.
\end{aligned}$$

Taking the supremum over all cubes centered at x we have the pointwise estimate

$$M^c_\alpha f(x) \leq c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)} M^c_u\{M^c_{\alpha,\sigma}(f/\sigma)^{q/r'} u^{-1}\}(x)^{r'/q}.$$

Using the fact that $M_u : L^{r'}(u) \to L^{r'}(u)$ and $M_\sigma : L^p(\sigma) \to L^q(\sigma)$ with operator norms independent of u and σ respectively, we get

$$\begin{aligned}
\|wM_\alpha f\|_{L^q} &\leq c\,\|M^c_\alpha f\|_{L^q(u)} \\
&\leq c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)} \|M^c_u\{M^c_{\alpha,\sigma}(f/\sigma)^{q/r'} u^{-1}\}\|_{L^{r'}(u)}^{r'/q} \\
&\leq c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)} \|fw\|_{L^p},
\end{aligned}$$

which is the desired estimate. □

We show that the bound (2.1) is sharp in Section 2.4.

2.2 Extrapolation

The celebrated extrapolation theorem of Rubio de Francia [45] is one of the most important theorems in the modern harmonic analysis. It allows one to obtain boundedness of an operator on a wide class of function spaces from a starting point. To obtain sharp bounds for singular integral operators on $L^p(w)$ (Petermichl [42], [43]) one only needs to obtain the bound for $p = 2$. The general case $p \neq 2$ then follows by the sharp version of the Rubio de Francia extrapolation theorem given by Dragičević, Grafakos, Pereyra, and Petermichl [9]. In this section we present an off-diagonal extrapolation theorem with sharp constants. The original off-diagonal extrapolation is due to Harboure, Macias, and Segovia [24]. We begin with a lemma about L^p space for $0 < p < 1$.

Lemma 2.2.1. *Let $f \geq 0$, $g > 0$ be measurable functions, $0 < s < 1$ and $s' = s/(s-1)$, then*

$$\int_X fg\, d\mu \geq \|f\|_{L^s(\mu)} \|g\|_{L^{s'}(\mu)}.$$

As a consequence,

$$\|f\|_{L^s(\mu)} = \inf \int_X fg\, d\mu$$

where the infimum is over all g with $\|g\|_{L^{s'}} = 1$, and the infimum is attained.

Proof. Since $0 < s < 1$, $1/s > 1$ so we may use Hölder's inequality with $1/s$ and $1/(1-s)$,

$$\left(\int_X f^s\, d\mu\right)^{1/s} = \left(\int_X (fg)^s g^{-s}\, d\mu\right)^{1/s} \leq \int_X fg\, d\mu \left(\int_X g^{s'}\, d\mu\right)^{-1/s'}.$$

Equality is attained by taking $g = f^{s-1}/\|f\|_{L^s(\mu)}^{s-1}$. □

The following result is due to Harboure, Macías, and Segovia [24], we repeat the proof to show the dependence on the constants.

Theorem 2.2.2. *Suppose that T is an operator defined on an appropriate class of functions such as $\mathscr{S}$, or $\bigcup_{w \in L^1_{\mathrm{loc}}} \bigcup_p L^p(w^p)$. Suppose further that p_0 and q_0 are exponents with $1 \leq p_0 \leq q_0 < \infty$, and*

$$\|wTf\|_{L^{q_0}(\mathbb{R}^n)} \leq c[w]_{A_{p_0,q_0}}^{\gamma} \|wf\|_{L^{p_0}(\mathbb{R}^n)}$$

for all $w \in A_{p_0,q_0}$ and some $\gamma > 0$. Then,

$$\|wTf\|_{L^q(\mathbb{R}^n)} \leq c[w]_{A_{p,q}}^{\gamma \max\{1, \frac{q_0}{p_0'} \frac{p'}{q'}\}} \|wf\|_{L^p(\mathbb{R}^n)}$$

holds for all p and q satisfying $1 < p \leq q < \infty$ *and*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0},$$

and all $w \in A_{p,q}$.

To prove Theorem 2.2.2 we need the following lemma whose proof can be found in [9].

Lemma 2.2.3. *Suppose that* $r > r_0$, $v \in A_r$, *and g is a non-negative function in* $L^{(r/r_0)'}(v)$. *Then, there exists a function G such that*

1. $G \geq g$,
2. $\|G\|_{L^{(r/r_0)'}(v)} \leq 2\|g\|_{L^{(r/r_0)'}(v)}$,
3. $Gv \in A_{r_0}$ *with* $[Gv]_{A_{r_0}} \leq c\,[v]_{A_r}$.

Proof of Theorem 2.2.2. First suppose $w \in A_{p,q}$ and $1 \leq p_0 < p$, which implies $q > q_0$. Then,

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |Tf|^q w^q\right)^{1/q} &= \left(\int_{\mathbb{R}^n} (|Tf|^{q_0})^{q/q_0} w^q\right)^{\frac{q_0}{q}\frac{1}{q_0}} \\ &= \left(\int_{\mathbb{R}^n} |Tf|^{q_0} g w^q\right)^{\frac{1}{q_0}}\end{aligned}$$

for some non-negative $g \in L^{(q/q_0)'}(w^q)$ with $\|g\|_{L^{(q/q_0)'}(w^q)} = 1$. Now, let $r = 1 + q/p'$ and $r_0 = 1 + q_0/p_0'$. Since $p > p_0$ we have $r > r_0$. Furthermore, by the relationship

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0},$$

we have $q/q_0 = r/r_0$. Hence by Lemma 2.2.3 and using that $w^q \in A_r$, there exists G with $G \geq g$, $\|G\|_{L^{(r/r_0)'}(w^q)} \leq 2$, $Gw^q \in A_{r_0}$, and $[Gw^q]_{A_{r_0}} \leq c\,[w^q]_{A_r} = c\,[w]_{A_{p,q}}$. Also, since $Gw^q \in A_{r_0}$ then $(Gw^q)^{1/q_0} \in A_{p_0,q_0}$ since,

$$\begin{aligned}
[(Gw^q)^{1/q_0}]_{A_{p_0,q_0}} &= \sup_Q \left(\frac{1}{|Q|}\int_Q (G^{1/q_0}w^{q/q_0})^{q_0}\right)\left(\frac{1}{|Q|}\int_Q (G^{1/q_0}w^{q/q_0})^{-p_0'}\right)^{q_0/p_0'} \\
&= \sup_Q \left(\frac{1}{|Q|}\int_Q Gw^q\right)\left(\frac{1}{|Q|}\int_Q (Gw^q)^{-p_0'/q_0}\right)^{q_0/p_0'} \\
&= [Gw^q]_{A_{r_0}}.
\end{aligned}$$

Then, we can proceed with

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |Tf|^q w^q\right)^{1/q} &= \left(\int_{\mathbb{R}^n} |Tf|^{q_0} g w^q\right)^{\frac{1}{q_0}} \\
&\leq \left(\int_{\mathbb{R}^n} |Tf|^{q_0} G w^q\right)^{\frac{1}{q_0}} \\
&= \left(\int_{\mathbb{R}^n} |Tf|^{q_0} (G^{1/q_0} w^{q/q_0})^{q_0}\right)^{\frac{1}{q_0}} \\
&\leq c\,[G^{1/q_0}w^{q/q_0}]^{\gamma}_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} |f|^{p_0} (G^{1/q_0} w^{q/q_0})^{p_0}\right)^{\frac{1}{p_0}} \\
&= c\,[Gw^q]^{\gamma}_{A_{r_0}} \left(\int_{\mathbb{R}^n} |f|^{p_0} w^{p_0} G^{p_0/q_0} w^{q/(p/p_0)'}\right)^{\frac{1}{p_0}} \\
&\leq c\,[w]^{\gamma}_{A_{p,q}} \left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} \left(\int_{\mathbb{R}^n} G^{(r/r_0)'} w^q\right)^{(p-p_0)/pp_0} \\
&\leq c\,[w]^{\gamma}_{A_{p,q}} \left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p},
\end{aligned}$$

where we have used the relationship

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}.$$

For the case $1 < p < p_0$, and hence $q < q_0$, notice that we can write

$$\left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} = \left(\int_{\mathbb{R}^n} (|fw^{p'}|^{p_0})^{p/p_0} w^{-p'}\right)^{1/p}.$$

Since $p/p_0 < 1$, by Lemma 2.2.1 exists a function $g \geq 0$ satisfying

$$\int_{\mathbb{R}^n} g^{p/(p-p_0)} w^{-p'} = 1$$

such that

$$\left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} = \left(\int_{\mathbb{R}^n} |fw^{p'}|^{p_0} g w^{-p'}\right)^{1/p_0}.$$

Let $h = g^{-p_0'/p_0}$, $r = 1 + p'/q$ and $r_0 = 1 + p_0'/q_0$, so that $r > r_0$. Notice that

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$$

implies $r/r_0 = p'/p_0'$, which in turn yields

$$\frac{p_0'}{p_0}\left(\frac{r}{r_0}\right)' = \frac{p}{p_0 - p}. \tag{2.2}$$

Hence,

$$\int_{\mathbb{R}^n} h^{(r/r_0)'} w^{-p'} = \int_{\mathbb{R}^n} g^{p/(p-p_0)} w^{-p'} = 1.$$

Observe that $w^{-p'} \in A_r$, so by Lemma 2.2.3 we obtain a function H such that $H \geq h$, $\|H\|_{L^{(r/r_0)'}(w^{-p'})} \leq 2$, and $Hw^{-p'} \in A_{r_0}$ with $[Hw^{-p'}]_{A_{r_0}} \leq c\,[w^{-p'}]_{A_r} = c\,[w]_{A_{p,q}}^{p'/q}$. Now, for $Hw^{-p'} \in A_{r_0}$ we claim that $(Hw^{-p'})^{-1/p_0'} \in A_{p_0,q_0}$ with $[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}} = [Hw^{p'}]_{A_{r_0}}^{q_0/p_0'}$. Indeed,

$$[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}} = \sup_Q \left(\frac{1}{|Q|}\int_Q (H^{-1/p_0'} w^{p'/p_0'})^{q_0}\right)\left(\frac{1}{|Q|}\int_Q (H^{-1/p_0'} w^{p'/p_0'})^{-p_0'}\right)^{q_0/p_0'}$$

$$
\begin{aligned}
&= \sup_Q \left(\frac{1}{|Q|}\int_Q (Hw^{-p'})^{-q_0/p_0'}\right)\left(\frac{1}{|Q|}\int_Q Hw^{-p'}\right)^{q_0/p_0'} \\
&= [Hw^{-p'}]_{A_{r_0}}^{q_0/p_0'}.
\end{aligned}
$$

Finally expressing g in terms for h and using (2.2), working backwards we have

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} &= \left(\int_{\mathbb{R}^n} |f|^{p_0} h^{-p_0/p_0'} w^{p'(p_0-1)}\right)^{1/p_0} \\
&\geq \left(\int_{\mathbb{R}^n} |f|^{p_0} H^{-p_0/p_0'} w^{p'(p_0-1)}\right)^{1/p_0} \\
&= \frac{[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}}^{\gamma}}{[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}}^{\gamma}} \left(\int_{\mathbb{R}^n} |f|^{p_0} (H^{-1/p_0'} w^{p'/p_0'})^{p_0}\right)^{1/p_0} \\
&\geq \frac{c}{[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}}^{\gamma}} \left(\int_{\mathbb{R}^n} |Tf|^{q_0} (H^{-1/p_0'} w^{p'/p_0'})^{q_0}\right)^{1/q_0} \\
&\geq \frac{c}{[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}}^{\gamma}} \left(\int_{\mathbb{R}^n} |Tf|^{q} w^{q}\right)^{1/q} \left(\int_{\mathbb{R}^n} H^{(r/r_0)'} w^{p'}\right)^{q-q_0/qq_0} \\
&\geq \frac{c}{[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}}^{\gamma}} \left(\int_{\mathbb{R}^n} |Tf|^{q} w^{q}\right)^{1/q}.
\end{aligned}
$$

In the second to last inequality we have used Lemma 2.2.1. Thus we have shown,

$$
\left(\int_{\mathbb{R}^n} |Tf|^q w^q\right)^{1/q} \leq c\,[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}}^{\gamma} \left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p}.
$$

From here we have

$$
\|T\| \leq c\,[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}}^{\gamma} = c\,[Hw^{-p'}]_{A_{r_0}}^{\gamma\frac{q_0}{p_0'}} \leq c\,[w^{-p'}]_{A_{1+p'/q}}^{\gamma\frac{q_0}{p_0'}} = c\,[w]_{A_{p,q}}^{\gamma\frac{q_0}{p_0'}\frac{p'}{q}}.
$$

This proves the theorem.

□

Using an idea of Grafakos and Martell [20] we may extend our extrapolation theorem to the weak case.

Corollary 2.2.4. *Suppose that for some* $1 \le p_0 \le q_0 < \infty$*, an operator* T *satisfies the weak-type* (p_0, q_0) *inequality*

$$\|Tf\|_{L^{q_0,\infty}(w^{q_0})} \le c[w]_{A_{p_0,q_0}}^{\gamma} \|wf\|_{L^{p_0}(\mathbb{R}^n)}$$

for every $w \in A_{p_0,q_0}$ *and some* $\gamma > 0$*. Then* T *also satisfies the weak-type* (p,q) *inequality,*

$$\|Tf\|_{L^{q,\infty}(w^q)} \le c[w]_{A_{p,q}}^{\gamma \max\{1, \frac{q_0}{p_0'} \frac{p'}{q}\}} \|wf\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p \le q < \infty$ *that satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$$

and all $w \in A_{p,q}$*.*

Proof. Note that Theorem 2.2.2 does not require T to be linear. We can simply apply the result to the operator $T_\lambda f = \lambda \chi_{\{|Tf|>\lambda\}}$. Fix $\lambda > 0$, then

$$\begin{aligned} \|wT_\lambda f\|_{L^{q_0}} &= \lambda w^{q_0}(\{x : |Tf(x)| > \lambda\})^{1/q_0} \\ &\le \|Tf\|_{L^{q_0,\infty}(w^{q_0})} \\ &\le c[w]_{A_{p_0,q_0}}^{\gamma} \|wf\|_{L^{p_0}}, \end{aligned}$$

with constant independent of λ. Hence by Theorem 2.2.2 if $w \in A_{p,q}$, T_λ maps $L^q(w^q) \to L^p(w^p)$ for all $1/p - 1/q = 1/p_0 - 1/q_0$ and with bound

$$\|wT_\lambda f\|_{L^q} \leq c\,[w]_{A_{p,q}}^{\gamma \max\{1, \frac{q_0}{p_0'}\frac{p'}{q}\}} \|fw\|_{L^p}.$$

with c independent of λ. Hence,

$$\|Tf\|_{L^{q,\infty}(w^q)} = \sup_{\lambda > 0} \|wT_\lambda f\|_{L^q} \leq c\,[w]_{A_{p,q}}^{\gamma \max\{1, \frac{q_0}{p_0'}\frac{p'}{q}\}} \|fw\|_{L^p}.$$

□

2.3 Sharp bounds for fractional integral operators

We now present our main result of this chapter, the sharp bounds for the operator norm of I_α. We need the following "packing condition" lemma see [49].

Lemma 2.3.1. *Suppose $\varepsilon > 0$, $c > 0$ and f is a locally integrable function. Let Q_0 be a cube and $\mathscr{G}$ be the dyadic grid associated to Q_0, then there exists a constant $C_\varepsilon > 0$ such that*

$$\sum_{\substack{Q \in \mathscr{G} \\ Q \subseteq Q_0}} |Q|^\varepsilon \int_{cQ} f(y)\, dy \leq C_\varepsilon |Q_0|^\varepsilon \int_{cQ_0} f(y)\, dy$$

Proof. Let $a = \ell(Q_0)$, then

$$\begin{aligned} \sum_{\substack{Q \in \mathscr{G} \\ Q \subseteq Q_0}} |Q|^\varepsilon \int_{cQ} f(y)\, dy &= \sum_{k=0}^{\infty} \sum_{\substack{Q \in \mathscr{G}, Q \subseteq Q_0 \\ \ell(Q) = 2^{-k}a}} |Q|^\varepsilon \int_{cQ} f(y)\, dy \\ &= \sum_{k=0}^{\infty} a^{\varepsilon n} 2^{-kn\varepsilon} \sum_{\substack{Q \in \mathscr{G}, Q \subseteq Q_0 \\ \ell(Q) = 2^{-k}a}} \int_{cQ} f(y)\, dy. \end{aligned}$$

Notice that at most c^n of the cubes cQ overlap if $Q \in \mathscr{G}$ with side length $c2^{-k}a$. It follows that

$$\sum_{\substack{Q\in\mathscr{G}, Q\subseteq Q_0\\ \ell(Q)=2^{-k}a}} \int_{cQ} f(y)\,dy \le C\int_{cQ_0} f(y)\,dy.$$

Proceeding we have,

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-kn\varepsilon} \sum_{\substack{Q\in\mathscr{G}, Q\subseteq Q_0\\ \ell(Q)=2^{-k}a}} \int_{cQ} f(y)\,dy &\le Ca^{\varepsilon n}\int_{cQ_0} f(y)\,dy \sum_{k=0}^{\infty} 2^{-kn\varepsilon}\\ &= C_\varepsilon |Q_0|^{\varepsilon}\int_{cQ_0} f(y)\,dy.\end{aligned}$$

□

Theorem 2.3.2. *Let $1 < p < n/\alpha$ and q be defined by the equation $1/q = 1/p - \alpha/n$, and let $w \in A_{p,q}$. Then,*

$$\|wI_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\,[w]_{A_{p,q}}^{\eta(p'/q)} \|wf\|_{L^p(\mathbb{R}^n)}, \tag{2.3}$$

where $\eta(x) = \min\{\max(1-\alpha/n, x), \max(1, (1-\alpha/n)x)\}$. The relationship $\|I_\alpha\| \le c\,[w]_{A_{p,q}}^{\eta(p'/q)}$ is sharp for p'/q in the range $(0, 1-\alpha/n] \cup [n/(n-\alpha), \infty)$ (see Figure 2.1).

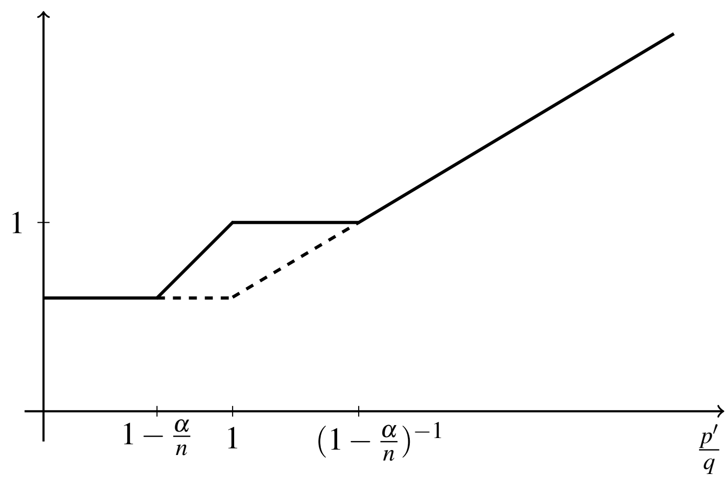

Figure 2.1: The graph of the function η.

Proof. We use Theorem 2.2.2 with base exponents $q_0/p_0' = 1-\alpha/n$. This along with the fact that $1/p_0 - 1/q_0 = \alpha/n$ yields

$$p_0 = \frac{2-\alpha/n}{\alpha/n-(\alpha/n)^2+1} \quad \text{and} \quad q_0 = \frac{2-\alpha/n}{1-\alpha/n}.$$

We will show the linear estimate

$$\|wI_\alpha f\|_{L^{q_0}} \leq c\,[w]_{A_{p_0,q_0}} \|wf\|_{L^{p_0}}. \tag{2.4}$$

Notice that (2.4) is equivalent to

$$\|I_\alpha(f\sigma)\|_{L^{q_0}(u)} \leq c\,[w]_{A_{p_0,q_0}} \|f\|_{L^{p_0}(\sigma)}, \tag{2.5}$$

where $u = w^{q_0}$ and $\sigma = w^{-p_0'}$. Moreover, by duality, showing (2.5) is equivalent to proving

$$\int_{\mathbb{R}^n} I_\alpha(f\sigma)gu\,dx \leq c\,[w]_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} f^{p_0}\sigma\,dx\right)^{1/p_0} \left(\int_{\mathbb{R}^n} g^{q_0'}u\,dx\right)^{1/q_0'}, \tag{2.6}$$

for all f and g non-negative bounded functions with compact support.

We first discretize the operator I_α as follows. Given a non-negative function f,

$$\begin{aligned} I_\alpha f(x) &= \sum_{k\in\mathbb{Z}} \int_{2^{k-1}<|x-y|\le 2^k} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \\ &\le c\sum_k \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)=2^k}} \chi_Q(x)\frac{1}{\ell(Q)^{n-\alpha}}\int_{|x-y|\le \ell(Q)} f(y)\,dy \\ &\le c\sum_{Q\in\mathscr{D}} \chi_Q(x)\frac{|Q|^{\alpha/n}}{|Q|}\int_{3Q} f\,dy \end{aligned}$$

where the last inequality holds because if $x\in Q$, then $B(x,\ell(Q))\subseteq 3Q$.

One immediately gets then

$$\int_{\mathbb{R}^n} I_\alpha(f\sigma)gu\,dx \le c\sum_{\mathscr{D}} \frac{|Q|^{\alpha/n}}{|Q|}\int_{3Q} f\sigma\,dx\int_Q gu\,dx.$$

We may pass the sum to smaller set of dyadic cubes that are better suited for our calculations. We combine ideas from the work of Sawyer and Wheeden in [49], together with some techniques from Pérez [40].

Fix $a>2^n$. Since g is bounded with compact support, for each $k\in\mathbb{Z}$, one can construct a collection $\{Q_{k,j}\}_j$ of pairwise disjoint maximal dyadic cubes (maximal with respect to inclusion) with the property that

$$a^k < \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} gu\,dx.$$

By maximality, the above also gives

$$\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} gu\,dx \le 2^n a^k.$$

For a fixed k the family $\{Q_{k,j}\}_j$ is disjoint in j. If we define for each k the collection

$$\mathscr{C}^k = \{Q \in \mathscr{D} : a^k < \frac{1}{|Q|}\int_Q gu\,dx \le a^{k+1}\},$$

then each dyadic cube Q belongs to only one $\mathscr{C}^k$ or gu vanishes on it. Moreover, each $Q \in \mathscr{C}^k$ has to be contained in one of the maximal cubes Q_{k,j_0} and verifies for all $Q_{k,j}$

$$\frac{1}{|Q|}\int_Q gu\,dx \le a^{k+1} \le \frac{a}{|Q_{k,j}|}\int_{Q_{k,j}} gu\,dx.$$

Lemma 2.3.1 shows that for any dyadic cube Q_0,

$$\sum_{Q\in\mathscr{D}, Q\subset Q_0} |Q|^{\alpha/n}\int_{3Q} f\sigma\,dx \le c_\alpha |Q_0|^{\alpha/n}\int_{3Q_0} f\sigma\,dx.$$

Thus one easily deduces as in [49] that

$$\sum_{\mathscr{D}} \frac{|Q|^{\alpha/n}}{|Q|}\int_{3Q} f\sigma\,dx \int_Q gu\,dx \le a\,c_\alpha \sum_{k,j}\frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|}\int_{3Q_{k,j}} f\sigma\,dx\int_{Q_{k,j}} gu\,dx.$$

Notice also that,

$$[w]_{A_{p_0,q_0}} = \sup_Q \frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^{1-\alpha/n} < \infty,$$

so we can estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} I_\alpha(f\sigma) gu\,dx &\le c\sum_{k,j}\frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|}\int_{3Q_{k,j}} f\sigma\,dx\int_{Q_{k,j}} gu\,dx \\
&= c\sum_{k,j}\frac{1}{\sigma(5Q_{k,j})^{1-\alpha/n}}\int_{3Q_{k,j}} f\sigma\;dx\;\frac{1}{u(3Q_{k,j})}\int_{Q_{k,j}} gu\;dx \\
&\quad \times \frac{u(3Q_{k,j})}{|Q_{k,j}|}\left(\frac{\sigma(5Q_{k,j})}{|Q_{k,j}|}\right)^{1-\alpha/n}|Q_{k,j}| \\
&\le c\,[w]_{A_{p_0,q_0}}\sum_{k,j}\frac{1}{\sigma(5Q_{k,j})^{1-\alpha/n}}\int_{3Q_{k,j}} f\sigma\;dx\frac{1}{u(3Q_{k,j})}\int_{Q_{k,j}} gu\;dx\;|Q_{k,j}|,
\end{aligned}$$

(2.7)

where we have set up things to use, in a moment, certain centered maximal functions.

Before we do so, we need one last property about the Calderón-Zygmund cubes $Q_{k,j}$. We need to pass to a disjoint collection of sets $E_{k,j}$ each of which retains a substantial portion of the mass of the corresponding cube $Q_{k,j}$.

Define the sets

$$E_{k,j} = Q_{k,j} \cap \{x \in R^n : a^k < M^d(gu) \leq a^{k+1}\},$$

where M^d is the dyadic maximal function. The family $\{E_{k,j}\}_{k,j}$ is pairwise disjoint for all j and k. Moreover, set

$$\Omega_k = \{x : M^d(gu)(x) > a^k\}$$

so $\Omega_k = \bigcup_j Q_{k,j}$ where

$$a^k < \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} gu \, dx \leq 2^n a^k.$$

Then,

$$\begin{aligned}
|Q_{k,j} \cap \Omega_{k+1}| &= \sum_i |Q_{k,j} \cap Q_{k+1,i}| \\
&= \sum_{Q_{k+1,i} \subset Q_{k,j}} |Q_{k+1,i}| \\
&\leq \sum_{Q_{k+1,i} \subset Q_{k,j}} \frac{1}{a^{k+1}} \int_{Q_{k+1,i}} gu \, dx \\
&= \frac{|Q_{k,j}|}{a^{k+1}} \sum_{Q_{k+1,i} \subset Q_{k,j}} \frac{1}{|Q_{k,j}|} \int_{Q_{k+1,i}} gu \, dx \\
&\leq \frac{|Q_{k,j}|}{a^{k+1}} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} gu \, dx
\end{aligned}$$

$$\leq \quad \frac{2^n}{a}|Q_{k,j}|.$$

It follows that

$$|E_{k,j}| \geq (1 - \frac{2^n}{a})|Q_{k,j}|.$$

Recalling now that $1 = u^{\frac{n}{n-\alpha}}\sigma = u^{\frac{1}{q_0}\frac{n}{n-\alpha}}\sigma^{\frac{1}{q_0}}$, we can use Hölder's inequality to write

$$|Q_{k,j}| \approx |E_{k,j}| = \int_{E_{k,j}} u^{\frac{1}{q_0}\frac{n}{n-\alpha}}\sigma^{\frac{1}{q_0}} \leq u(E_{k,j})^{1/q_0'}\sigma(E_{k,j})^{1/q_0}, \tag{2.8}$$

since

$$\frac{q_0'}{q_0}\frac{n}{n-\alpha} = 1.$$

With (2.8) we go back to the string of inequalities to estimate $\int I_\alpha(f\sigma)\,gu\,dx$. Using the discrete version of Hölder's inequality, we can estimate in (2.7)

$$\begin{aligned}
&\leq \quad c\,[w]_{A_{p_0,q_0}} \left(\sum_{k,j}\left(\frac{1}{\sigma(5Q_{k,j})^{1-\alpha/n}}\int_{3Q_{k,j}} f\sigma\,dx\right)^{q_0}\sigma(E_{k,j})\right)^{1/q_0} \\
&\qquad \times \left(\sum_{k,j}\left(\frac{1}{u(3Q_{k,j})}\int_{Q_{k,j}} gu\,dx\right)^{q_0'} u(E_{k,j})\right)^{1/q_0'} \\
&\leq \quad c\,[w]_{A_{p_0,q_0}} \left(\sum_{k,j}\int_{E_{k,j}} (M^c_{\alpha,\sigma}f)^{q_0}\sigma\,dx\right)^{1/q_0} \left(\sum_{k,j}\int_{E_{k,j}} (M^c_u g)^{q_0'}u\,dx\right)^{1/q_0'} \\
&\leq \quad c\,[w]_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} (M^c_{\alpha,\sigma}f)^{q_0}\sigma\,dx\right)^{1/q_0} \left(\int_{\mathbb{R}^n} (M^c_u g)^{q_0'}u\,dx\right)^{1/q_0'} \\
&\leq \quad c\,[w]_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} f^{p_0}\sigma\,dx\right)^{1/p_0} \left(\int_{\mathbb{R}^n} g^{q_0'}u\,dx\right)^{1/q_0'}.
\end{aligned}$$

We have also used the boundedness of M_u^c and $M_{\alpha,\sigma}^c$ with operator norms independent of the corresponding measure. We obtain then the desired linear estimate

$$\|wI_\alpha f\|_{L^{q_0}} \leq c\,[w]_{A_{p_0,q_0}} \|wf\|_{L^{p_0}}. \tag{2.9}$$

From this estimate we can extrapolate (Theorem 2.2.2) to get,

$$\|wI_\alpha f\|_{L^q} \leq c\,[w]_{A_{p,q}}^{\max\{1,(1-\alpha/n)p'/q\}} \|wf\|_{L^p} \tag{2.10}$$

for all $1 < p < q < \infty$ with $1/p - 1/q = \alpha/n$. This proves one of the estimates in Theorem 2.3.2.

The estimate (2.10) is equivalent to saying that the linear operator

$$T(f) = wI_\alpha(fw^{-1})$$

is bounded $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ with bound $[w]_{A_{p,q}}^{\max\{1,(1-\alpha/n)p'/q\}}$. By duality, the transpose operator given by

$$T^t(f) = w^{-1}I_\alpha(fw)$$

is bounded $T^t : L^{q'}(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$. Furthermore the bound is the same as above, namely less than $[w]_{A_{p,q}}^{\max\{1,(1-\alpha/n)p'/q\}}$. Hence we have

$$\|w^{-1}I_\alpha f\|_{L^{p'}(\mathbb{R}^n)} \leq c\,[w]_{A_{p,q}}^{\max\{1,(1-\alpha/n)p'/q\}} \|w^{-1}f\|_{L^{q'}(\mathbb{R}^n)}. \tag{2.11}$$

Since $1/p - 1/q = 1/q' - 1/p' = \alpha/n$ we can replace this by

$$\|wI_\alpha f\|_{L^q(\mathbb{R}^n)} \leq c\,[w^{-1}]_{A_{q',p'}}^{\max\{1,(1-\alpha/n)q/p'\}} \|wf\|_{L^p(\mathbb{R}^n)}. \tag{2.12}$$

Thus we have

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le c[w]_{A_{p,q}}^{\min\{\max(1-\frac{\alpha}{n},\frac{p'}{q}),\max(1,(1-\frac{\alpha}{n})\frac{p'}{q})\}}.$$

The sharpness of the bounds obtained for $p'/q \in (0,1-\alpha/n]\cup[n/(n-\alpha),\infty)$ will be shown with an example in Section 2.4.

□

Remark 2.3.3. One may also take a different approach by examining the dyadic fractional operator defined in [49] by

$$I_\alpha^d f(x) = \sum_{Q\in\mathscr{D}} \chi_Q(x)\frac{|Q|^{\alpha/n}}{|Q|}\int_Q f\,dy.$$

Notice that for a given $f\ge 0$ if we let $\{a_Q(x)\}_{Q\in\mathscr{D}}$ be the function from $\mathbb{R}^n$ to the space of sequences indexed by $\mathscr{D}$, defined by

$$a_Q(x) = \chi_Q(x)\frac{|Q|^{\alpha/n}}{|Q|}\int_Q f\,dy.$$

Then

$$I_\alpha^d f(x) = \|a_Q(x)\|_{\ell^1(\mathscr{D})}$$

and

$$M_\alpha^d f(x) = \|a_Q(x)\|_{\ell^\infty(\mathscr{D})},$$

where $\ell^1(\mathscr{D})$ and $\ell^\infty(\mathscr{D})$ are the spaces of absolutely summable sequences index by $\mathscr{D}$ and bounded sequences indexed by $\mathscr{D}$ respectively. Minor modifications to the proof (using the boundedness of $M_{\alpha,\sigma}^d$ and M_u^d instead of $M_{\alpha,\sigma}^c$ and M_u^c) give

$$\|I_\alpha^d\|_{L^{p_0}(w^{p_0})\to L^{q_0}(w^{q_0})} \le c[w]_{A_{p_0,q_0}}.$$

From here one could use the shifting lemma in [49] similar to Lemma 3.2.6 below to conclude

$$\|I_\alpha\|_{L^{p_0}(w^{p_0})\to L^{q_0}(w^{q_0})} \le c[w]_{A_{p_0,q_0}}.$$

Continuing from here one obtains the results of the Theorem 2.3.

Remark 2.3.4. Examples indicate that the sharp bound for I_α should be

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le c[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}. \tag{2.13}$$

To prove (2.13) using extrapolation (Theorem 2.2.2) one would need to consider the case ${p_0}' = q_0$ and show that the estimate

$$\|I_\alpha\|_{L^{p_0}(w^{p_0})\to L^{q_0}(w^{q_0})} \le c[w]_{A_{p_0,q_0}}^{1-\frac{\alpha}{n}}$$

holds. We do not know if this approach can be modified to work. See again Chapter 6.

We also have the following theorem for the sharp bound on the weak operator norm of the fractional integral operator.

Theorem 2.3.5. *Let* $0 < \alpha < n$, $1/q_0 = 1 - \alpha/n$, *and* w *be a weight with* $u = w^{q_0}$

$$\|I_\alpha f\|_{L^{q_0,\infty}(u)} \le C\|f\|_{L^1((Mu)^{1-\frac{\alpha}{n}})}. \tag{2.14}$$

Remark 2.3.6. Estimate (2.14) is a fractional version of the Muckenhoupt-Wheeden conjecture. The Muckenhoupt-Wheeden conjecture states that given a weight and a Calderón-Zygmund operator T,

$$\|Tf\|_{L^{1,\infty}(w)} \le C\|f\|_{L^1(Mw)}. \tag{2.15}$$

This remains a difficult open problem in the theory of weights.

Another version of the Muckenhoupt-Wheeden conjecture for I_α is the following estimate

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \le c\|f\|_{L^1(M_\alpha w)}.$$

However, in general this estimate is false, see Carro, Pérez, F. Soria, and J. Soria [5] for a counter example. Since $w \in A_{1,q}$ implies $u = w^q \in A_1$ we have

$$M(w^q)^{1/q} = M(u)^{1/q} \le [u]_{A_1}^{1/q} u^{1/q} = [w]_{A_{1,q}}^{1/q} w.$$

Combining this with (2.14) we have

$$\|I_\alpha f\|_{L^{q_0,\infty}(w^{q_0})} \le [w]_{A_{1,q_0}}^{1-\alpha/n} \|f\|_{L^1(w)},$$

where $q_0 = n/(n-\alpha)$. From here we may apply Theorem 2.2.4 to obtain the following Theorem.

Theorem 2.3.7. *Suppose that* $1 \le p < n/\alpha$ *and that q satisfies* $1/q = 1/p - \alpha/n$. *Then*

$$\|I_\alpha f\|_{L^{q,\infty}(w^q)} \le c\,[w]_{A_{p,q}}^{1-\frac{\alpha}{n}} \|wf\|_{L^p(\mathbb{R}^n)} \tag{2.16}$$

and the exponent $1 - \frac{\alpha}{n}$ *is sharp.*

Remark 2.3.8. Theorem 2.3.7 is the fractional version of the "linear growth" conjecture for Calderón-Zygmund operators:

$$\|Tf\|_{L^{p,\infty}(w)} \le [w]_{A_p}\|f\|_{L^p(w)}.$$

This conjecture was formulated by Lerner, Ombrosi, and Pérez in [32] and is another unsolved problem in the theory of weights.

Proof of Theorem 2.3.5. In order to prove (2.14), we note that $\|\cdot\|_{L^{q_0,\infty}(u)}$ is equivalent to a norm since $q_0 > 1$. Hence, we may use Minkowski's integral inequality as follows

$$\|I_\alpha f\|_{L^{q_0,\infty}(u)} \leq c_q \int_{\mathbb{R}^n} |f(y)| \| |\cdot - y|^{\alpha-n}\|_{L^{q_0,\infty}(u)}\, dy. \tag{2.17}$$

We can finally calculate the inner norm by

$$\begin{aligned} \| |\cdot - y|^{\alpha-n}\|_{L^{q_0,\infty}(w^q)} &= \sup_{\lambda>0} \lambda u(\{x : |x-y|^{\alpha-n} > \lambda\})^{1/q_0} \\ &= (\sup_{t>0} \frac{1}{t^n} u(\{x : |x-y| < t\}))^{1/q_0} \\ &= cMu(y)^{1/q_0}. \end{aligned}$$

Once again, the sharpness of the exponent $1-\alpha/n$ will be shown with an example in Section 2.4.

□

2.4 Examples

As mentioned in the introduction, power functions such as $|x|^a$ with $-n < a < n(p-1)$ are important examples of A_p weights. It is with these examples that we will show Theorems 2.3.2, 2.3.7, and 2.1.1 are sharp. This technique was first used by Buckley [4], to show (1.16) is sharp. We state one lemma that will be used through out this section.

Lemma 2.4.1. *Suppose* $0 < \delta < 1$*, and* $w_\delta(x) = |x|^{(n-\delta)(p-1)}$*, then* $w_\delta \in A_p$ *with*

$$[w_\delta]_{A_p} \approx c\delta^{1-p}. \tag{2.18}$$

Proof. Let Q be a cube and let Q_0 denote its translate to the origin, i.e. $\ell(Q_0) = \ell(Q)$ and Q_0 is centered at the origin. Then either $2Q_0 \cap Q = \varnothing$ or $2Q_0 \cap Q \neq \varnothing$. Call these two cases, Case 1 and Case 2 respectively.

Case 1: In this case we have $\ell(Q) \leq |x|$ for all $x \in Q$. If we let x_0 be the center of Q, and x be any point in Q then

$$|x_0| - |x| \leq |x_0 - x| \leq \ell(Q),$$

so $|x| \geq |x_0|/2$. Furthermore,

$$|x| \leq |x - x_0| + |x_0| \leq \ell(Q) + |x_0| \leq 2|x_0|.$$

Combining these things we have

$$\frac{|x_0|}{2} \leq |x| \leq 2|x_0|$$

for all $x \in Q$. Thus

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |x|^{(n-\delta)(p-1)}\,dx\right)\left(\frac{1}{|Q|}\int_Q (|x|^{(n-\delta)(p-1)})^{1-p'}\,dx\right)^{p-1} \\ \leq\ & c|x_0|^{(n-\delta)(p-1)}|x_0|^{-(n-\delta)(p-1)} \\ \leq\ & c\delta^{1-p}, \end{aligned}$$

since $1 < \delta^{1-p}$.

Case 2: Let Q be a cube such that $2Q_0 \cap Q \neq \varnothing$ and set $a = \ell(Q)$. In this case $Q \subset B(0,5a)$ thus,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |x|^{(n-\delta)(p-1)}\,dx\right)\left(\frac{1}{|Q|}\int_Q (|x|^{(n-\delta)(p-1)})^{1-p'}\,dx\right)^{p-1} \\
&\leq c\left(\frac{1}{|B(0,5a)|}\int_{B(0,5a)} |x|^{(n-\delta)(p-1)}\,dx\right) \\
&\qquad \times\left(\frac{1}{|B(0,5a)|}\int_{B(0,5a))} (|x|^{(n-\delta)(p-1)})^{1-p'}\,dx\right)^{p-1} \\
&= c\left(\frac{1}{(5a)^n}\int_0^{5a} r^{p(n-\delta)+\delta-1}\,dr\right)\left(\frac{1}{(5a)^n}\int_0^{5a} r^{\delta-1}\,dr\right)^{p-1} \\
&= c\frac{1}{(5a)^n}\frac{(5a)^{p(n-\delta)+\delta}}{(pn-p\delta+\delta)}\frac{1}{(5a)^{pn-n}}\frac{(5a)^{\delta p-\delta}}{\delta^{p-1}} \\
&\leq c\frac{1}{p(n-1)+1}\delta^{1-p}.
\end{aligned}$$

This shows that

$$[w_\delta]_{A_p} \leq c\delta^{1-p}.$$

To show the reverse inequality we let $Q = [-1,1]^n$ then $B(0,1) \subset Q$ and hence

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |x|^{(n-\delta)(p-1)}\,dx\right)\left(\frac{1}{|Q|}\int_Q (|x|^{(n-\delta)(p-1)})^{1-p'}\,dx\right)^{p-1} \\
&\geq \left(\frac{1}{2^n}\int_{B(0,1)} |x|^{(n-\delta)(p-1)}\,dx\right)\left(\frac{1}{2^n}\int_{B(0,1)} |x|^{\delta-n}\,dx\right)^{p-1} \\
&= \frac{1}{2^{np}}\left(\int_0^1 r^{np-p+\delta-1}\,dx\right)\left(\int_0^1 r^{\delta-1}\,dx\right)^{p-1} \\
&= \frac{1}{2^{np}}\frac{1}{(np+\delta(1-p))}\delta^{1-p} \\
&\geq c\frac{1}{np}\delta^{1-p} \\
&= c\delta^{1-p}.
\end{aligned}$$

□

2.4.1 Sharpness of the strong bounds

Suppose again $0 < \alpha < n$ with

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Let $w_\delta(x) = |x|^{(n-\delta)/p'}$ by 2.4.1 we have $w_\delta \in A_{p,q}$, with

$$[w_\delta]_{A_{p,q}} = [w_\delta^q]_{A_{1+q/p'}} \approx \delta^{-q/p'}.$$

Then, if $f_\delta(x) = |x|^{\delta-n}\chi_B$, where B is the unit ball in $\mathbb{R}^n$, we have

$$\|w_\delta f_\delta\|_{L^p} \approx \delta^{-1/p}.$$

For $x \in B$,

$$M_\alpha f_\delta(x) \geq \frac{C}{|x|^{n-\alpha}} \int_{B(0,|x|)} |f_\delta(y)|\, dy \approx \frac{|x|^{\delta-n+\alpha}}{\delta},$$

and so we have

$$\int_{\mathbb{R}^n} w_\delta^q M_\alpha f_\delta(x)^q\, dx \geq \delta^{-q} \int_B |x|^{(\delta-n+\alpha)q} |x|^{(n-\delta)\frac{q}{p'}}\, dx \approx \delta^{-q-1}.$$

It follows that

$$\delta^{-1-1/q} \leq c\,\|w_\delta M f_\delta\|_{L^q} \leq c\,[w_\delta]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})} \|w_\delta f_\delta\|_{L^p} \approx \delta^{-(1-\frac{\alpha}{n})}\delta^{-1/p} = \delta^{-1-1/q}, \tag{2.19}$$

showing Theorem 2.1.1 is sharp.

Next we now show that the exponent in Theorem 2.3.2 is sharp for $p'/q \in (0, 1-\alpha/n] \cup [n/(n-\alpha), \infty)$. On could show this by simply observing $M_\alpha \leq cI_\alpha$. Since the sharp bound for M_α is

$$\|M_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le c[w]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})},$$

and is sharp for all p and q satisfying $1/p - 1/q = \alpha/n$. Thus it must be sharp for I_α in the range $[n/(n-\alpha),\infty)$, for in this range we have the same bound for $\|I_\alpha\|$ as $\|M_\alpha\|$. For the sharpness when $p'/q \in (0, 1-\alpha/n]$ a duality argument is needed (see below). However, we will show it by a direct example. Assume first that $p'/q \in [n/(n-\alpha),\infty)$. Let w_δ and f_δ as above and notice that

$$\begin{aligned}
\|I_\alpha(f_\delta)w_\delta\|_{L^q}^q &\ge \int_{(2B)^c}\left(\int_B \frac{|y|^{\delta-n}}{|x-y|^{n-\alpha}}\,dy\right)^q |x|^{(n-\delta)q/p'}\,dx \\
&\ge c\int_{(2B)^c}\left(\int_B |y|^{\delta-n}\,dy\right)^q |x|^{(\alpha-n)q}|x|^{(n-\delta)q/p'}\,dx \\
&= c\delta^{-q}\int_{(2B)^c} |x|^{(\alpha-n)q}|x|^{(n-\delta)q/p'}\,dx \\
&= c\delta^{-q}\int_2^\infty r^{-\delta q/p'-1}\,dr \\
&= c\delta^{-q-1}.
\end{aligned}$$

Thus,

$$c\delta^{-1-1/q} \le \|I_\alpha(f_\delta)w_\delta\|_{L^q} \le c\,[w_\delta]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})}\|w_\delta f_\delta\|_{L^p} \approx \delta^{-(1-\frac{\alpha}{n})}\delta^{-1/p} = \delta^{-1-1/q}.$$

For the case when $p'/q \in (0, 1-\alpha/n]$ we use the fact that I_α is self adjoint. Let $w \in A_{p,q}$ and suppose we had better estimate, say

$$\|I_\alpha\|_{L^p(u^p)\to L^q(u^q)} \le c\varphi([u]_{A_{p,q}}),$$

where $\varphi(t)$ grows slower than $t^{1-\alpha/n}$ as $t\to\infty$. If we let $w\in A_{q',p'}=A_{r,s}$ so $w^{-1}\in A_{p,q}$, with $[w]_{A_{r,s}}^{s/r'}=[w^{-1}]_{A_{p,q}}$. Now since $p'/q\in(0,1-\alpha/n]$ we have $r'/s\in[n/(n-\alpha),\infty)$ and

$$\begin{aligned}\|I_\alpha\|_{L^r(w^r)\to L^s(w^s)} &= \|I_\alpha^*\|_{L^r(w^r)\to L^s(w^s)}=\|I_\alpha\|_{L^p(w^{-p})\to L^p(w^{-q})}\\ &\leq C\varphi([w^{-1}]_{A_{p,q}})=C\varphi([w]_{A_{r,s}}^{s/r'}).\end{aligned}$$

This gives a better growth estimate than we saw possible for exponents in this range. Thus we have established sharpness for the range $p'/q\in(0,1-\alpha/n]\cup[n/(n-\alpha),\infty)$.

2.4.2 Sharpness of the weak bounds

Finally, we show that the exponent $1-\alpha/n$ in the estimate

$$\|I_\alpha f\|_{L^{q,\infty}(w^q)}\leq c\,[w]_{A_{p,q}}^{1-\alpha/n}\|fw\|_{L^p} \tag{2.20}$$

from Theorem 2.3.7 is sharp for $p\geq 1$. Notice that if we take a weight $u\in A_1$, then $u^{1/q}\in A_{p,q}$ with

$$[u^{1/q}]_{A_{p,q}}\leq[u]_{A_1}.$$

By (2.20) we have the following estimate in terms of the A_1 constant of u

$$\|I_\alpha f\|_{L^q(u)}\leq c[u^{1/q}]_{A_{p,q}}\|f\|_{L^p(u^{p/q})}\leq c[u]_{A_1}^{1-\frac{\alpha}{n}}\|f\|_{L^p(u^{p/q})}. \tag{2.21}$$

Since $\frac{p}{q}=1-\frac{p\alpha}{n}$, inequality (2.21) is equivalent to

$$\|I_\alpha(u^{\frac{\alpha}{n}}f)\|_{L^q(u)}\leq c[u]_{A_1}^{1-\frac{\alpha}{n}}\|f\|_{L^p(u)}. \tag{2.22}$$

Thus, a better bound in (2.20) would imply a better bound in (2.22). We now proceed to show that (2.22) is sharp, thus proving the sharpness for Theorem 2.3.7. Let $0 < x_\delta < 1$ be a parameter whose value will be chosen soon. We have

$$\begin{aligned}
\|I_\alpha(u^{\alpha/n}f)\|_{L^{q,\infty}(u)} &\geq \sup_{\lambda>0}\lambda\left(u\{x\in B(0,x_\delta):\int_B \frac{|y|^{(\delta-1)\alpha/n}}{|x-y|^{1-\alpha/n}}dy>\lambda\}\right)^{1/q}\\
&\geq \sup_{\lambda>0}\lambda\left(u\{x\in B(0,x_\delta):\int_{B\setminus B(0,|x|)} \frac{|y|^{(\delta-1)\alpha/n}}{|x-y|^{1-\alpha/n}}dy>\lambda\}\right)^{1/q}\\
&\geq \sup_{\lambda>0}\lambda\left(u\{x\in B(0,x_\delta):\int_{B\setminus B(0,|x|)} \frac{|y|^{(\delta-1)\alpha/n}}{(2|y|)^{1-\alpha/n}}dy>\lambda\}\right)^{1/q}\\
&= \sup_{\lambda>0}\lambda\left(u\{x\in B(0,x_\delta):\frac{c_{\alpha,n}}{\delta}(1-|x|^{\delta\alpha/n})>\lambda\}\right)^{1/q}\\
&\geq \frac{c_{\alpha,n}}{2\delta}\left(u\{x\in B(0,x_\delta):\frac{c_{\alpha,n}}{\delta}(1-|x|^{\delta\alpha/n})>\frac{c_{\alpha,n}}{2\delta}\}\right)^{1/q}\\
&= \frac{c_{\alpha,n}}{2\delta}u(B(0,x_\delta))^{1/q},
\end{aligned}$$

if we choose $x_\delta = (\frac{1}{2})^{n/\alpha\delta}$. It follows that for $0 < \delta < 1$,

$$\|I_\alpha(u^{\alpha/n}f)\|_{L^{q,\infty}(u)} \geq \frac{c}{\delta}\left(\frac{x_\delta^\delta}{\delta}\right)^{1/q} = c\frac{1}{\delta}\left(\frac{1}{\delta}\right)^{1/q}. \tag{2.23}$$

Using that $\frac{1}{q}-\frac{1}{p}=-\frac{\alpha}{n}$, we have that (2.22) is sharp, thus showing Theorem 2.3.7 is sharp.

2.5 Sobolev inequalities

In this section we use the weak bound from Theorem 2.3.7, to obtain a strong bound when a gradient operator is involved. Notice that if f is sufficiently smooth, one has the estimate

$$|f(x)| \leq CI_1(|\nabla f|)(x) \tag{2.24}$$

where ∇ is the gradient operator. Combining Theorem 2.3.2 with (2.24) one obtains the bound

$$\|wf\|_{L^q} \leq [w]_{A_{p,q}}^{\min\{\max(1/n',p'/q),\max(1,p'/(qn'))\}} \|w|\nabla f|\|_{L^p}$$

where $1/p - 1/q = 1/n$. Even if one had the conjectured bound 2.13 and combined it with (2.24), we would have

$$\|wf\|_{L^q} \leq [w]_{A_{p,q}}^{1/n' \max\{1,p'/q\}} \|w|\nabla f|\|_{L^p}.$$

However, we may exploit some of the properties of the gradient operator using ideas of Long and Nie [34] (see also Hajlasz [23]) to obtain a better estimate.

Theorem 2.5.1. *Let $p \geq 1$ and let $w \in A_{p,q}$ with q satisfying $1/p - 1/q = 1/n$. Then, for any Lipschitz function f with compact support,*

$$\|wf\|_{L^q} \leq [w]_{A_{p,q}}^{1/n'} \|w|\nabla f|\|_{L^p}.$$

Proof. Since $|f(x)| \leq cI_1(|\nabla f|)(x)$ we can use Theorem 2.3.7 to obtain

$$\|f\|_{L^{q,\infty}(w^q)} \leq c[w]_{A_{p,q}}^{1/n'} \|\nabla f w\|_{L^p}. \tag{2.25}$$

From this weak-type estimate we can pass to a strong one with the procedure that follows. We use the so-called truncation method from [34].

Given a non-negative function g and $\lambda > 0$ we define its truncation about λ, $\tau_\lambda g$, to be

$$\tau_\lambda g(x) = \min\{g, 2\lambda\} - \min\{g, \lambda\} = \begin{cases} 0 & g(x) \le \lambda \\ g(x) - \lambda & \lambda < g(x) \le 2\lambda \\ \lambda & g(x) > 2\lambda \end{cases}.$$

Notice that since f is a Lipschitz function, so is $|f|$, with $|\nabla|f|| \le |\nabla f|$ a.e. Similarly, truncations of Lipschitz functions are again also Lipschitz functions. Define

$$\Omega_k = \{x : 2^k < |f(x)| \le 2^{k+1}\}$$

and let $u = w^q$. Then,

$$\begin{aligned} \left(\int_{\mathbb{R}^n} (|f(x)|w(x))^q \, dx\right)^{1/q} &\le \left(\sum_k \int_{\{2^{k+1} < |f(x)| \le 2^{k+2}\}} |f(x)|^q u(x) \, dx\right)^{1/q} \\ &\le c\left(\sum_k 2^{kq} u(\Omega_{k+1})\right)^{1/q} \\ &\le c\left(\sum_k 2^{kp} u(\Omega_{k+1})^{p/q}\right)^{1/p}. \end{aligned}$$

Let $\tau_k = \tau_{2^k}$, if $x \in \Omega_{k+1}$, $|f(x)| > 2^{k+1}$. Thus, $\tau_k|f(x)| = 2^k > 2^{k-1}$ and hence

$$\Omega_{k+1} \subseteq \{x : \tau_k|f(x)| > 2^{k-1}\}.$$

Notice that

$$\nabla \tau_k(|f|) = \begin{cases} 0 & |f| \ge 2^k \\ \nabla|f| & 2^k < |f| \le 2^{k+1} \\ 0 & |f| > 2^{k+1} \end{cases}$$

showing

$$|\nabla \tau_k |f|| = |\nabla |f|| \chi_{\Omega_k} \leq |\nabla f| \chi_{\Omega_k}, \quad \text{a.e.}$$

Continuing and using the weak-type estimate (2.25) we have

$$\begin{aligned}
&\leq \; c \left(\sum_k (2^k u(\{x : \tau_k |f|(x) > 2^{k-1}\})^{1/q})^p \right)^{1/p} \\
&\leq \; c\,[w]_{A_{p,q}}^{1/n'} \left(\sum_k \int_{\Omega_k} (|\nabla \tau_k |f|(x)| w(x))^p \, dx \right)^{1/p} \\
&\leq \; c\,[w]_{A_{p,q}}^{1/n'} \left(\int_{\mathbb{R}^n} (|\nabla f(x)| w(x))^p \, dx \right)^{1/p},
\end{aligned}$$

since $p < q$ and the sets Ω_k are disjoint. This finishes the proof of the theorem. □

Chapter 3

Weighted inequalities for general maximal operators

In this chapter we examine maximal operators with respect to a general basis. A collection of open sets in $\mathbb{R}^n$ will be denoted by $\mathscr{B}$ and referred to as a basis. Jawerth [27] considered a maximal function with respect to a basis $\mathscr{B}$ and characterized its boundedness on weighted Lebesgue spaces. As a corollary to Jawerth's results one obtains Theorem 1.3.2. The boundedness of the a maximal operator with respect to a basis $\mathscr{B}$ depends on the geometry of $\mathscr{B}$.

We have already encountered two bases: $\mathscr{Q}$ the base of all cubes in $\mathbb{R}^n$ and $\mathscr{D}$ the set of all dyadic cubes in $\mathbb{R}^n$. The way we have defined cubes in $\mathbb{R}^n$ does not make them open subsets. However since the boundary of any cube, ∂Q, has Lebesgue measure zero, and hence $\mu(\partial Q) = 0$ for any absolutely continuous measure μ, our calculations will be justified. We will not comment on this minor technicality any further.

We defined a fractional version of the general basis maximal function and characterize the one-weight and two-weight inequalities for it. We also introduce a new testing condition for the general basis fractional maximal function. Our techniques lead to sharp bounds on the operator norm when the basis is $\mathscr{Q}$.

3.1 Maximal operators with respect to a general basis

Given a basis $\mathscr{B}$ we say that w is a weight with respect to $\mathscr{B}$ if w is a non-negative function that satisfies $w(B) < \infty$ for all $B \in \mathscr{B}$. For the rest of this chapter we will refer to such a function w as simply a weight.

Definition 3.1.1. Let $\mathscr{B}$ be a basis for $\mathbb{R}^n$, w be a weight, and $0 \leq \alpha < n$. We define the weighted fractional maximal operator with respect to $\mathscr{B}$ as

$$M^{\mathscr{B}}_{\alpha,w} f(x) = \sup_{x \in B \in \mathscr{B}} \frac{1}{w(B)^{1-\alpha/n}} \int_B |f|\, w\, dy,$$

if $x \in \bigcup_{B \in \mathscr{B}} B$ and 0 otherwise. If $w = 1$ we drop the subscript w and write $M^{\mathscr{B}}_{\alpha}$ and if $\alpha = 0$ we simply write $M^{\mathscr{B}}_{w}$.

We give some examples of bases in $\mathbb{R}^n$ and the associated maximal operators (with $\alpha = 0$ and $w = 1$).

- Let $\mathscr{Q}$ be the basis of cubes, in this case $M^{\mathscr{Q}} = M$ the Hardy-Littlewood maximal operator.
- Let $\mathscr{D}$ be the basis of dyadic cubes, in this case we write $M^{\mathscr{D}} = M^d$ the dyadic version of the Hardy-Littlewood maximal operator.
- Let $\mathscr{B}_0$ be the basis of all balls, then $M^{\mathscr{B}_0}$ is a comparable operator to M and is also referred to as the Hardy-Littlewood maximal operator.
- Let $\mathscr{R}_0$ be the collection of all rectangles with sides parallel to the coordinate axes. Then $M^{\mathscr{R}_0} = M^s$, the "strong" maximal operator.
- Let $N > 1$, and let $\mathscr{R}_N$ be the set of all rectangles in $\mathbb{R}^n$ with $n-1$ sides of length h and one side of length Nh, for $h > 0$. Then $M^{\mathscr{R}_N} = \mathscr{K}_N$ the Kakeya maximal function.

- Let $\mathscr{R}$ be the collection of all rectangles with arbitrary orientation (not necessarily with sides parallel to the axes), then $M^{\mathscr{R}}$ is a larger maximal operator than M^s.

As mentioned above the boundedness of the maximal operators above depends greatly on the geometry of the basis $\mathscr{B}$. For instance, the operators $M^{\mathscr{Q}}$, $M^{\mathscr{D}}$, and $M^{\mathscr{B}_0}$ all map L^1 into $L^{1,\infty}$ and $L^p \to L^p$ for $1 < p \leq \infty$. The operator $M^{\mathscr{R}_0}$ is bounded on L^p for $1 < p \leq \infty$ but is not weak (1,1). The operator $M^{\mathscr{R}}$ is neither bounded on L^p nor weak (1,1). Finally, the operator $M^{\mathscr{R}_N} = \mathscr{K}_N$ is weak (1,1) and bounded on L^p for $1 < p \leq \infty$. This operator, the Kakeya maximal operator $\mathscr{K}_N$, is closely related to the Kakeya "needle problem", which arises in determining the minimum area of a set K in the plane that contains unit line segments in all possible directions. Much more about such maximal functions can be found in Stein [52, Chap. 10].

3.2 One-weight inequalities

Definition 3.2.1. The class $A_p^{\mathscr{B}}$ is composed of the weights w that satisfy

$$[w]_{A_p^{\mathscr{B}}} = \sup_{B\in\mathscr{B}} \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\right)^{p-1} < \infty.$$

Similarly for $1 < p \leq q < \infty$ we define $A_{p,q}^{\mathscr{B}}$ to be the class of weights w that satisfy

$$[w]_{A_{p,q}^{\mathscr{B}}} = \sup_{B\in\mathscr{B}} \left(\frac{1}{|B|}\int_B w(x)^q\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-p'}\,dx\right)^{q/p'} < \infty.$$

Theorem 3.2.2. *Suppose* $0 \leq \alpha < n$, $1 < p < n/\alpha$, *q is the number defined by* $1/q = 1/p - \alpha/n$, *and w is weight. Let* $u = w^q$, $\sigma = w^{-p'}$ *and* $r = 1 + q/p'$. *Then we have,*

$$M_\alpha^{\mathscr{B}} : L^p(w^p) \to L^q(w^q), \tag{3.1}$$

$$M_\alpha^{\mathscr{B}} : L^{q'}(w^{-q'}) \to L^{p'}(w^{-p'}), \tag{3.2}$$

$$M^{\mathscr{B}} : L^r(u) \to L^r(u) \tag{3.3}$$

$$M^{\mathscr{B}} : L^{r'}(\sigma) \to L^{r'}(\sigma), \tag{3.4}$$

if and only if $w \in A_{p,q}^{\mathscr{B}}$,

$$M_{\alpha,\sigma}^{\mathscr{B}} : L^p(\sigma) \to L^q(\sigma), \tag{3.5}$$

$$M_{\alpha,u}^{\mathscr{B}} : L^{q'}(u) \to L^{p'}(u), \tag{3.6}$$

$$M_\sigma^{\mathscr{B}} : L^r(\sigma) \to L^r(\sigma) \tag{3.7}$$

$$M_u^{\mathscr{B}} : L^{r'}(u) \to L^{r'}(u). \tag{3.8}$$

Furthermore, we have the following operator norm inequalities,

$$\|M_\alpha^{\mathscr{B}}\|_{L^p(w^p)\to L^q(w^q)} \le [w]_{A_{p,q}^{\mathscr{B}}}^{\frac{p'}{q}(1-\frac{\alpha}{n})} \|M_u^{\mathscr{B}}\|_{L^{r'}(u)\to L^{r'}(u)}^{\frac{p'}{q}(1-\frac{\alpha}{n})} \|M_{\alpha,\sigma}^{\mathscr{B}}\|_{L^p(\sigma)\to L^q(\sigma)}$$

and

$$\|M_\alpha^{\mathscr{B}}\|_{L^{q'}(w^{-q'})\to L^{p'}(w^{-p'})} \le [w^{-1}]_{A_{q',p'}^{\mathscr{B}}}^{\frac{q}{p'}(1-\frac{\alpha}{n})} \|M_\sigma^{\mathscr{B}}\|_{L^r(\sigma)\to L^r(\sigma)}^{\frac{q}{p'}(1-\frac{\alpha}{n})} \|M_{\alpha,u}^{\mathscr{B}}\|_{L^{q'}(u)\to L^{p'}(u)}.$$

Remark 3.2.3. Note that when $\alpha = 0$, and hence $q = p$, many of the conditions in Theorem 3.2.2 collapse. In such a case we have the following equivalent conditions (3.1)=(3.3), (3.2)=(3.4), (3.5)=(3.7), and (3.6)=(3.8). However, this is just the re-normalized ($w \mapsto w^p$) version of Jawerth's Theorem and hence we exclude this case from the proof of Theorem 3.2.2.

Remark 3.2.4. Since

$$w \in A_{p,q}^{\mathscr{B}} \iff w^q \in A_{1+q/p'}^{\mathscr{B}},$$

if we apply Jawerth's Theorem with exponent $r = 1 + q/p'$ (notice $1 < r < \infty$) we have the following equivalence

$$(3.3),(3.4) \iff (3.8),(3.7),\text{and, } w \in A^{\mathscr{B}}_{p,q}.$$

Here are some guidelines for the conditions in Theorem 3.2.2. We will show that

$$(3.5),(3.8),\text{and } w \in A^{\mathscr{B}}_{p,q} \Rightarrow (3.1)$$

and

$$(3.6),(3.7),\text{and } w \in A^{\mathscr{B}}_{p,q} \Rightarrow (3.2).$$

For the reverse implications, any of conditions (3.1), (3.2), (3.3), (3.4) imply that $w \in A^{\mathscr{B}}_{p,q}$ and for the maximal functions we will show

$$(3.1),(3.4) \Rightarrow (3.5)$$

and

$$(3.2),(3.8) \Rightarrow (3.6).$$

Proof of Theorem 3.2.2. Suppose $\alpha > 0$, we only prove that (3.5), (3.8), and $w \in A^{\mathscr{B}}_{p,q}$ implies (3.1); also (3.1) and (3.4) implies (3.5), as the other implications stated in Remark 3.2.4 are similar. We follow some ideas in [31]. Suppose that $M^{\mathscr{B}}_{\alpha,\sigma}$ and $M^{\mathscr{B}}_{u}$ are as in (3.5) and (3.8) with operator norms $\|M^{\mathscr{B}}_{\alpha,\sigma}\|_{p,q}$, $\|M^{\mathscr{B}}_{u}\|_{r'}$, and $w \in A^{\mathscr{B}}_{p,q}$. Notice we may write the $A^{\mathscr{B}}_{p,q}$ constant as

$$[w]_{A^{\mathscr{B}}_{p,q}} = \sup_{B} \frac{u(B)}{|B|} \left(\frac{\sigma(B)}{|B|} \right)^{q/p'}.$$

Let $x \in \mathbb{R}^n$ and $B \in \mathscr{B}$ be a set containing x. Let $r = 1 + q/p'$ so that $r' = 1 + p'/q$. Then using the equation $1 - \alpha/n = 1/q + 1/p'$, we have

$$\begin{aligned}
\frac{1}{|B|^{1-\alpha/n}} \int_B |f|\, dx &= \frac{u(B)^{\frac{p'}{q}(1-\alpha/n)} \sigma(B)^{1-\alpha/n}}{|B|^{(1+p'/q)(1-\alpha/n)}} \\
&\quad \times \frac{|B|^{(1+p'/q)(1-\alpha/n)}}{u(B)^{\frac{p'}{q}(1-\alpha/n)} \sigma(B)^{1-\alpha/n}} \frac{1}{|B|^{1-\alpha/n}} \int_B |f|\, dx \\
&\leq [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} \left(\frac{|B|}{u(B)} \left(\frac{1}{\sigma(B)^{1-\alpha/n}} \int_B |f| \sigma^{-1} \sigma\, dx \right)^{q/r'} \right)^{r'/q} \\
&\leq [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} \left(\frac{1}{u(B)} \int_B (M^{\mathscr{B}}_{\alpha,\sigma}(f\sigma^{-1})(x)^{q/r'} u^{-1} u\, dx \right)^{r'/q}.
\end{aligned}$$

Taking the supremum we have the pointwise estimate

$$M^{\mathscr{B}}_{\alpha} f(x) \leq [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} (M^{\mathscr{B}}_u (M^{\mathscr{B}}_{\alpha,\sigma}(f\sigma^{-1})^{q/r'} u^{-1})(x))^{r'/q}.$$

Hence,

$$\begin{aligned}
\|M^{\mathscr{B}}_{\alpha} f w\|_{L^q} &\leq [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} \left(\int_{\mathbb{R}^n} (M^{\mathscr{B}}_u (M^{\mathscr{B}}_{\alpha,\sigma}(f\sigma^{-1})^{q/r'} u^{-1})(x))^{r'} u\, dx \right)^{1/q} \\
&\leq [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} \|M^{\mathscr{B}}_u\|_{r'}^{r'/q} \left(\int_{\mathbb{R}^n} (M^{\mathscr{B}}_{\alpha,\sigma}(f\sigma^{-1})(x))^q u^{-r'} u\, dx \right)^{1/q} \\
&= [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} \|M^{\mathscr{B}}_u\|_{r'}^{r'/q} \left(\int_{\mathbb{R}^n} (M^{\mathscr{B}}_{\alpha,\sigma}(f\sigma^{-1})(x))^q \sigma\, dx \right)^{1/q} \\
&\leq [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} \|M^{\mathscr{B}}_u\|_{r'}^{\frac{p'}{q}(1-\alpha/n)} \|M^{\mathscr{B}}_{\alpha,\sigma}\|_{p,q} \left(\int_{\mathbb{R}^n} |f|^p \sigma^{-p} \sigma\, dx \right)^{1/p} \\
&= [w]_{A^{\mathscr{B}}_{p,q}}^{\frac{p'}{q}(1-\alpha/n)} \|M^{\mathscr{B}}_u\|_{r'}^{\frac{p'}{q}(1-\alpha/n)} \|M^{\mathscr{B}}_{\alpha,\sigma}\|_{p,q} \|fw\|_{L^p},
\end{aligned}$$

and we obtain (3.1) with the right bound. Suppose now M_α^B and $M^{\mathscr{B}}$ are bounded as in (3.1) and (3.3) respectively. Notice that for any $B \in \mathscr{B}$, by Hölder's inequality, we have

$$1 = \frac{1}{|B|}\int_B w^{q/r} w^{-q/r}\, dx \le \left(\frac{u(B)}{|B|}\right)^{1/r}\left(\frac{\sigma(B)}{|B|}\right)^{1/r'},$$

so

$$\left(\frac{|B|}{\sigma(B)}\right)^{r} \le \left(\frac{u(B)}{|B|}\right)^{r'}.$$

With similar computations as above we have,

$$\begin{aligned}
\left(\frac{1}{\sigma(B)^{1-\alpha/n}}\int_B f\sigma\, dx\right)^q &= \left(\frac{|B|^{1-\alpha/n}}{\sigma(B)^{1-\alpha/n}}\right)^q \left(\frac{1}{|B|^{1-\alpha/n}}\int_B f\sigma\, dx\right)^q \\
&= \left(\frac{|B|}{\sigma(B)}\right)^r \left(\frac{1}{|B|^{1-\alpha/n}}\int_B f\sigma\, dx\right)^q \\
&\le \left(\frac{u(B)}{|B|}\right)^{r'} \left(\frac{1}{|B|^{1-\alpha/n}}\int_B f\sigma\, dx\right)^q \\
&= \left(\frac{u(B)}{|B|}\left(\frac{1}{|B|^{1-\alpha/n}}\int_B f\sigma\, dx\right)^{q/r'}\right)^{r'} \\
&\le \left(\frac{1}{|B|}\int_B M_\alpha^{\mathscr{B}}(f\sigma)^{q/r'} u\, dx\right)^{r'}.
\end{aligned}$$

Taking the supremum over all $B \in \mathscr{B}$ with $x \in B$ we have

$$M_{\alpha,\sigma}^{\mathscr{B}} f(x)^q \le M^{\mathscr{B}}(M_\alpha^{\mathscr{B}}(f\sigma)^{q/r'} u)(x)^{r'}.$$

Hence,

$$\begin{aligned}
\|M_{\alpha,\sigma}^{\mathscr{B}} f\|_{L^q(\sigma)} &\le \|M^{\mathscr{B}}(M_\alpha^{\mathscr{B}}(f\sigma)^{q/r'} u)\|_{L^{r'}(\sigma)}^{r'/q} \\
&\le C\|M_\alpha^{\mathscr{B}}(f\sigma)\|_{L^q(u)} \\
&\le C\|f\sigma\|_{L^p(w^p)}
\end{aligned}$$

$$= C\|f\|_{L^p(\sigma)}.$$

Completing the proof of our theorem. □

When $\mathscr{B}=\mathscr{D}$, the operators $M^{\mathscr{D}}_{\sigma}$, $M^{\mathscr{D}}_{u}$, $M^{\mathscr{D}}_{\alpha,\sigma}$ and $M^{\mathscr{D}}_{\alpha,u}$ are all bounded with operator norms independent of w. Hence we have as a corollary the following dyadic version of the result found in [38].

Corollary 3.2.5. *Suppose $1<p<n/\alpha$, q is defined by $1/q=1/p-\alpha/n$. Then $M^{\mathscr{D}}_{\alpha}: L^p(w^p)\to L^q(w^q)$ if and only if $w\in A^{\mathscr{D}}_{p,q}$ with,*

$$\|M^{\mathscr{D}}_{\alpha}\|\le C[w]^{\frac{p'}{q}(1-\frac{\alpha}{n})}_{A^{\mathscr{D}}_{p,q}}.$$

Finally we state one more lemma that allows us to transfer results from the the basis $\mathscr{D}$ to the basis $\mathscr{Q}$. We state it without proof as the case $\alpha=0$ can be found in the book by Garcia-Cuerva and Rubio de Francia [16, p. 431] and it is based on the ideas of Fefferman and Stein [12]. The proof for general α is a straight forward generalization.

Lemma 3.2.6. *Let $0<q<\infty$, u be a non-negative function, and τ_t be the shift operator $\tau_t g(x)=g(x-t)$. Then*

$$\|M_\alpha f\|_{L^q(u)}\le C_n\sup_t\|\tau_{-t}\circ M^{\mathscr{D}}_{\alpha}\circ\tau_t f\|_{L^q(u)},$$

where C_n depends only on the dimension.

If we combine Corollary 3.2.5 with Lemma 3.2.6 and use the fact that if $w\in A_{p,q}$ then $\tau_t w\in A^{\mathscr{D}}_{p,q}$ with

$$\sup_t[\tau_t w]_{A^{\mathscr{D}}_{p,q}}\le[w]_{A_{p,q}},$$

we obtain a slightly different proof of the bound (2.1) in Theorem 2.1.1.

3.3 Two-weight inequalities

For the two-weight theory we seek conditions on pairs of weights (u,v) so that

$$\left(\int_{\mathbb{R}^n} M_\alpha^{\mathscr{B}} f(x)^q u(x)\, dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx\right)^{1/p}.$$

We have the following theorem analogous to the two-weight theorem of Jawerth.

Theorem 3.3.1. *Let $\mathscr{B}$ be a basis, $0 \le \alpha < n$, $1 < p \le q < \infty$, and (u,v) be a pair of weights, $\sigma = v^{1-p'}$, and suppose that $M_\sigma^{\mathscr{B}}$ is bounded on $L^p(\sigma)$. Then*

$$\|M_\alpha^{\mathscr{B}} f\|_{L^q(u)} \le C\|f\|_{L^p(v)} \tag{3.9}$$

holds for all $f \in L^p(v)$ if and only if the pair of weights satisfies the following testing condition: there exists $C > 0$ such that

$$\left(\int_G M_\alpha(\chi_G\sigma)(x)^q u(x)\, dx\right)^{1/q} \le C\left(\int_G \sigma(x)\, dx\right)^{1/p} \tag{3.10}$$

for all G that are the union of sets in $\mathscr{B}$. Furthermore, if we let

$$[u,v]_{S_{p,q}^{\mathscr{B}}} = \sup_G \frac{(\int_G M_\alpha(\chi_G\sigma)^q u\, dx)^{1/q}}{\sigma(G)^{1/p}} < \infty,$$

where the supremum is over all G that are the union of sets in $\mathscr{B}$, then

$$\|M_\alpha^{\mathscr{B}}\|_{L^p(v)\to L^q(u)} \le C[u,v]_{S_{p,q}^{\mathscr{B}}}\|M_\sigma^{\mathscr{B}}\|_{L^p(\sigma)\to L^p(\sigma)}.$$

Proof. We prove only the case $p < q$. The necessity of the testing condition follows from letting $f = \chi_G\sigma$.

Suppose that (u,v) are a pair of weights that satisfy the testing condition (3.10). Let $\Omega_k = \{x : 2^k < M_\alpha^{\mathscr{B}} f(x) \leq 2^{k+1}\}$ for $k \in \mathbb{Z}$. From the definition of $M_\alpha^{\mathscr{B}}$, for each k we get $\Omega_k \subseteq \bigcup_{j=1}^{\infty} B_{k,j}$, where $B_{k,j}$ satisfies

$$\frac{1}{|B_{k,j}|^{1-\alpha/n}} \int_{B_{k,j}} |f(y)|\, dy > 2^k.$$

For each k, let $E_{k,1} = B_{k,1} \cap \Omega_k$ and for $j > 1$, define

$$E_{k,j} = (B_{k,j} \setminus \bigcup_{i=1}^{j-1} B_{k,j}) \cap \Omega_k.$$

Notice that for each k the collection $\{E_{k,j}\}_j$ is disjoint. Furthermore, since the Ω_k's are disjoint, the $E_{k,j}$'s are disjoint for all k, j. Also $\Omega_k = \bigcup_j E_{k,j}$, and we may estimate $\|M_\alpha^{\mathscr{B}} f\|_{L^q(u)}$ as follows

$$\begin{aligned}
\int_{\mathbb{R}^n} (M_\alpha^{\mathscr{B}} f(x))^q u(x)\, dx &\leq C \sum_{k,j} u(E_{k,j}) \left(\frac{1}{|B_{k,j}|^{1-\alpha/n}} \int_{B_{k,j}} |f(y)|\, dy \right)^q \\
&= C \sum_{k,j} u(E_{k,j}) \left(\frac{\sigma(B_{k,j})}{|B_{k,j}|^{1-\alpha/n}} \right)^q \left(\frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |f|\sigma^{-1}\sigma\, dy \right)^q \\
&= C \int_X g\, d\mu,
\end{aligned}$$

where $X = \mathbb{Z} \times \mathbb{N}$, g is the function on X defined by

$$g(k,j) = \left(\frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |f|\sigma^{-1}\sigma\, dy \right)^q,$$

and μ is a discrete measure on X given by

$$\mu(k,j) = u(E_{k,j}) \left(\frac{\sigma(B_{k,j})}{|B_{k,j}|^{1-\alpha/n}} \right)^q.$$

Let

$$\Gamma_\lambda = \{(k,j) \in X : g(k,j) > \lambda\}$$

and

$$G_\lambda = \bigcup\{B_{k,j} : (k,j) \in \Gamma_\lambda\}.$$

We estimate $\mu(\Gamma_\lambda)$ using the testing condition (3.10). We have

$$\begin{aligned}
\mu(\Gamma_\lambda) &= \sum_{(k,j)\in\Gamma_\lambda} u(E_{k,j}) \left(\frac{\sigma(B_{k,j})}{|B_{k,j}|^{1-\alpha/n}} \right)^q \\
&\leq \sum_{(k,j)\in\Gamma_\lambda} \int_{E_{k,j}} M_\alpha^{\mathscr{B}}(\chi_{B_{k,j}}\sigma)^q u \, dx \\
&\leq \int_{G_\lambda} M_\alpha^{\mathscr{B}}(\chi_{G_\lambda}\sigma)^q u \, dx \\
&\leq [u,v]^q_{S^{\mathscr{B}}_{p,q}} \sigma(G_\lambda)^{q/p} \\
&\leq C[u,v]^q_{S^{\mathscr{B}}_{p,q}} \sigma\{x : M_\sigma^{\mathscr{B}}(f/\sigma)(x)^q > \lambda\}^{q/p}.
\end{aligned}$$

From here we proceed with estimating $\int_X g \, d\mu$. We have

$$\begin{aligned}
\int_X g \, d\mu &= \int_0^\infty \mu(\Gamma_\lambda) d\lambda \\
&\leq [u,v]^q_{S^{\mathscr{B}}_{p,q}} \int_0^\infty \sigma\{x : M_\sigma^{\mathscr{B}}(f/\sigma)(x)^q > \lambda\}^{q/p} d\lambda \\
&= [u,v]^q_{S^{\mathscr{B}}_{p,q}} \int_0^\infty (t\sigma\{x : M_\sigma^{\mathscr{B}}(f/\sigma)(x)^p > t\})^{q/p} \frac{dt}{t}.
\end{aligned}$$

For the case $p < q$, we use that the measure dt/t on $(0,\infty)$ is essentially a counting measure. Continuing,

$$\int_0^\infty (t\sigma\{x : M_\sigma^{\mathscr{B}} f(x)^p > t\})^{q/p} \frac{dt}{t}$$

$$
\begin{aligned}
&= \sum_{l\in\mathbb{Z}} \int_{2^l}^{2^{l+1}} (t\sigma\{x : M_\sigma^{\mathscr{B}} f(x)^p > t\})^{q/p} \frac{dt}{t} \\
&\le 2^{q/p} \log 2 \sum_{l\in\mathbb{Z}} (2^l \sigma\{x : M_\sigma^{\mathscr{B}}(f/\sigma)(x)^p > 2^l\})^{q/p} \\
&\le C \left(\sum_{l\in\mathbb{Z}} \sigma\{x : M_\sigma^{\mathscr{B}}(f/\sigma)(x)^p > 2^l\} 2^l \right)^{q/p} \\
&\le C \left(\sum_{l\in\mathbb{Z}} \int_{2^{l-1}}^{2^l} \sigma\{x : M_\sigma^{\mathscr{B}}(f/\sigma)(x)^p > t\} dt \right)^{q/p} \\
&= C \left(\int_0^\infty \sigma\{x : M_\sigma^{\mathscr{B}}(f/\sigma)(x)^p > t\} dt \right)^{q/p} \\
&\le C \|M_\sigma^{\mathscr{B}}\|^q \left(\int_{\mathbb{R}^n} |f|^p v \, dx \right)^{q/p}.
\end{aligned}
$$

This finishes the proof of Theorem 3.3.1, and if one keeps track of the constants, one can easily see that

$$\|M_\alpha^{\mathscr{B}}\| \le C[u,v]_{S_{p,q}^{\mathscr{B}}} \|M_\sigma^{\mathscr{B}}\|.$$

□

We state two corollaries of Theorem 3.3.1 for the bases $\mathscr{Q}$ and $\mathscr{D}$. We first start with the basis $\mathscr{D}$ and employ an argument similar to the one found in [16, p. 430]. We have the following dyadic version of Sawyer's Theorem.

Corollary 3.3.2. *Let* $0 \le \alpha < n$ *and* $1 < p \le q < \infty$ *and* (u,v) *be a pair of weights with* $\sigma = v^{1-p'}$. *Then the inequality*

$$\|M_\alpha^{\mathscr{D}} f\|_{L^q(u)} \le C\|f\|_{L^p(v)}$$

holds if and only if (u,v) *satisfies the testing condition*

$$[u,v]_{S^d_{p,q}} = \sup_{Q\in\mathscr{D}} \frac{\left(\int_Q M_\alpha(\chi_Q\sigma)(x)^q u(x)\,dx\right)^{1/q}}{\sigma(Q)^{1/p}} < \infty. \tag{3.11}$$

We have the following dependence on the operator norm,

$$\|M_\alpha^{\mathscr{D}}\| \le C[u,v]_{S^d_{p,q}}.$$

Proof. The necessity of the condition (3.11) is clear. Note that $M_\sigma^{\mathscr{D}}$ is bounded on $L^p(\sigma)$ with $\|M_\sigma^{\mathscr{D}}\| \le C_{n,p}$. We will show that (u,v) satisfies the testing condition

$$\left(\int_G M_\alpha^{\mathscr{D}}(\chi_G\sigma)^q u\,dx\right)^{1/q} \le c[u,v]_{S^d_{p,q}}\sigma(G)^{1/p} \tag{3.12}$$

for G a union of dyadic cubes, hence showing $[u,v]_{S^{\mathscr{D}}_{p,q}} \le c[u,v]_{S^d_{p,q}}$.

We will actually show this inequality for the truncated version of $M_\alpha^{\mathscr{D}}$. Let M_α^N be the same operator as $M_\alpha^{\mathscr{D}}$ except with supremum taken over all dyadic cubes with side length less or equal to 2^N. We show (3.12) with $M_\alpha^{\mathscr{D}}$ replaced by M_α^N and constant independent of N. Let G be a union of dyadic cubes. Using the same discretization as Theorem 3.3.1, we may write $\{x : M_\alpha^N(\chi_G\sigma)(x) > 2^k\} = \bigcup_j Q_{k,j}$ where $Q_{k,j}$ are maximal dyadic (hence disjoint for a fixed k) cubes with side length less or equal to 2^N that are contained in G and satisfy

$$\frac{\sigma(Q_{k,j})}{|Q_k,j|^{1-\alpha/n}} > 2^k.$$

If we let

$$E_{k,j} = Q_{k,j} \cap \{x : 2^k < M_\alpha^N(\chi_G\sigma) \le 2^{k+1}\},$$

then the $E_{k,j}$'s are disjoint for all k and j and

$$\{x : 2^k < M_\alpha^N(\chi_G\sigma) \le 2^{k+1}\} = \bigcup_j E_{k,j}.$$

Thus, continuing as in Theorem 3.3.1 we have

$$\int_G M_\alpha^N(\chi_G\sigma)^q u\, dx \le C \sum_{k,j} u(E_{k,j}) \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|^{1-\alpha/n}}\right)^q.$$

Since the $Q_{k,j}$'s are dyadic cubes with side length less that 2^N we can extract a maximally disjoint collection of them, say $\{Q_i\}$. We have

$$\begin{aligned}
\sum_{k,j} u(E_{k,j}) \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|^{1-\alpha/n}}\right)^q &\le \sum_i \sum_{Q_{k,j}\subseteq Q_i} u(E_{k,j}) \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|^{1-\alpha/n}}\right)^q \\
&\le \sum_i \sum_{Q_{k,j}\subseteq Q_i} \int_{E_{k,j}} M_\alpha(\chi_{Q_{k,j}}\sigma)^q u\, dx \\
&\le \sum_i \int_{Q_i} M_\alpha(\chi_{Q_i}\sigma)^q u\, dx \\
&\le [u,v]^q_{S^d_{p,q}} \sum_i \sigma(Q_i)^{q/p} \\
&\le [u,v]^q_{S^d_{p,q}} \sigma(G)^{q/p}.
\end{aligned}$$

□

Finally we may obtain the full version of Sawyer's theorem using Lemma 3.2.6. We have the following Corollary.

Corollary 3.3.3. *Suppose that $0 \le \alpha < n$, $1 < p \le q < \infty$, and (u,v) are a pair of weights with $\sigma = v^{1-p'}$. Then the inequality*

$$\|M_\alpha f\|_{L^q(u)} \le C\|f\|_{L^p(v)}$$

holds if and only if (u,v) *satisfies*

$$[u,v]_{S_{p,q}} = \sup_Q \frac{\left(\int_Q M_\alpha(\chi_Q\sigma)^q u\,dx\right)^{1/q}}{\sigma(Q)^{1/p}} < \infty, \tag{3.13}$$

and

$$\|M_\alpha\| \le C[u,v]_{S_{p,q}}.$$

Proof. First notice that if (u,v) satisfies condition (3.13), then $(\tau_t u, \tau_t v)$ satisfies the dyadic condition $S^d_{p,q}$ (3.11) with

$$\sup_t [\tau_t u \tau_t v]_{S^d_{p,q}} \le [u,v]_{S_{p,q}}.$$

Combining Corollary 3.3.2 and Lemma 3.2.6 we have

$$\begin{aligned}
\|M_\alpha f\|_{L^q(u)} &\le C\sup_t \|\tau_{-t}\circ M_\alpha \circ \tau_t f\|_{L^q(u)} \\
&\le C\sup_t [\tau_t u, \tau_t v]_{S^d_{p,q}} \|\tau_t f\|_{L^p(\tau_t v)} \\
&\le C[u,v]_{S_{p,q}} \|f\|_{L^p(v)}.
\end{aligned}$$

□

We now give a testing condition for $M^{\mathscr{B}}_\alpha$ that is more natural when $\alpha > 0$ and also yields sharp operator norms in the one-weight case. We have the following result.

Theorem 3.3.4. *Suppose that* $0 \le \alpha < n$, $1 < p,q < \infty$, *and* (u,v) *is a pair of weights such that* $M^{\mathscr{B}}_{\alpha,\sigma}$ *is bounded from* $L^p(\sigma)$ *to* $L^q(\sigma)$, *and that satisfy the testing condition*

$$\left(\int_Q M^{\mathscr{B}}(\chi_G\sigma)^{(1-\alpha/n)q} u\,dx\right)^{1/q} \le C\sigma(G)^{1/q} \tag{3.14}$$

for all G that are the union of sets in $\mathscr{B}$. If $[u,v]_{T_q^{\mathscr{B}}}$ denotes the smallest constant that satisfies (3.14) *for all such G, then*

$$\|M_\alpha^{\mathscr{B}}\|_{L^q(u)\to L^p(v)} \le C[u,v]_{T_q^{\mathscr{B}}}\|M_{\alpha,\sigma}^{\mathscr{B}}\|_{L^p(\sigma)\to L^q(\sigma)}.$$

Before we present the proof some remarks are in order. First, notice that condition (3.14) is just a sufficient condition for the boundedness of $M_\alpha^{\mathscr{B}}$. It is not known if this is also necessary since the testing condition is based on testing $M^{\mathscr{B}}$ and not $M_\alpha^{\mathscr{B}}$. When $\alpha = 0$ and $p = q$ the two conditions (3.10) and (3.14) are the same and thus we once again recover Jawerth's result. Further notice that we do not have the restriction $p \le q$ but we do need $M_{\alpha,\sigma}^{\mathscr{B}}$ to be bounded from $L^p(\sigma)$ to $L^q(\sigma)$, which usually happens when $1/p - 1/q = \alpha/n$.

Proof. We use the same basic discretization of Jawerth as in Theorem 3.3.1 to obtain,

$$\begin{aligned}\int_{\mathbb{R}^n} M_\alpha^{\mathscr{B}} f^q u\,dx &\le C\sum_{k,j} u(E_{k,j})\left(\frac{1}{|B_{k.j}|^{1-\alpha/n}}\int_{B_{k,j}}|f|\,dx\right)^q\\ &= C\sum_{k,j} u(E_{k,j})\left(\frac{\sigma(B_{k,j})}{|B_{k,j}|}\right)^{(1-\alpha/n)q}\left(\frac{1}{\sigma(B_{k,j})^{1-\alpha/n}}\int_{B_{k,j}}|f|\sigma^{-1}\sigma\right)^q\\ &= C\int_X g\,d\mu.\end{aligned}$$

Here X, g and μ are defined analogous to those in the proof Theorem 3.3.1. The definitions for Γ_λ and G_λ are also exactly as in the proof of Theorem 3.3.1. Then,

$$\begin{aligned}\mu(\Gamma_\lambda) &= \sum_{(k,j)\in\Gamma_\lambda} u(E_{k,j})\left(\frac{\sigma(B_{k,j})}{|B_{k,j}|}\right)^{(1-\alpha/n)q}\\ &\le \int_{G_\lambda} M(\chi_{G_\lambda}\sigma)^{(1-\alpha/n)q}u\,dx\\ &\le [u,v]_{T_q^{\mathscr{B}}}^q\sigma(G_\lambda)\end{aligned}$$

$$\leq \quad [u,v]^q_{T_q^{\mathscr{B}}}\sigma(\{x : M^{\mathscr{B}}_{\alpha,\sigma}(f/\sigma)(x)^q > \lambda\}).$$

Plugging this into the estimate for $M^{\mathscr{B}}_{\alpha}$ we have,

$$\begin{aligned}
\int_X g\,d\mu &= \int_0^\infty \mu(\Gamma_\lambda)\,d\lambda \\
&\leq [u,v]^q_{T_q^{\mathscr{B}}}\int_0^\infty \sigma(\{x : M^{\mathscr{B}}_{\alpha,\sigma}(f/\sigma)(x)^q > \lambda\})\,d\lambda \\
&= [u,v]^q_{T_q^{\mathscr{B}}}\int_{\mathbb{R}^n} M^{\mathscr{B}}_{\alpha,\sigma}(f/\sigma)^q\sigma\,dx \\
&\leq [u,v]^q_{T_q^{\mathscr{B}}}\|M^{\mathscr{B}}_{\alpha,\sigma}\|^q\left(\int_{\mathbb{R}^n}|f|^p v\,dx\right)^{q/p}.
\end{aligned}$$

□

We also note that if p and q are related by the equation $1/q = 1/p - \alpha/n$ and $\mathscr{B} = \mathscr{D}$, then $M^{\mathscr{D}}_{\alpha,\sigma} : L^p(\sigma) \to L^q(\sigma)$. Once again we may relax the testing conditions in the case $\mathscr{B} = \mathscr{D}$ or $\mathscr{Q}$. We obtain the following corollaries which are similar to Corollaries 3.3.2 and 3.13, and we state them without proof.

Corollary 3.3.5. *Suppose $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If (u,v) is a pair of weights that satisfies*

$$[u,v]_{T_q^d} = \sup_{Q\in\mathscr{D}} \frac{\left(\int_Q M(\chi_Q\sigma)^{1+q/p'}u\,dx\right)^{1/q}}{\sigma(Q)^{1/q}} < \infty,$$

then $M^{\mathscr{D}}_{\alpha}$ maps $L^p(v)$ into $L^q(u)$ with

$$\|M^{\mathscr{D}}_{\alpha}\| \leq C[u,v]_{T_q^d}.$$

Using Lemma 3.2.6 we may pass this result to the basis of cubes.

Corollary 3.3.6. *Suppose $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If (u,v) is a pair of weights that satisfies*

$$[u,v]_{T_q} = \sup_Q \frac{\left(\int_Q M(\chi_Q \sigma)^{1+q/p'} u\, dx\right)^{1/q}}{\sigma(Q)^{1/q}} < \infty, \tag{3.15}$$

then M_α maps $L^p(v)$ into $L^q(u)$ with

$$\|M_\alpha\| \leq C[u,v]_{T_q}. \tag{3.16}$$

When $\alpha > 0$, (3.15) is a new sufficient condition for the two-weight boundedness of M_α. Instead of testing M_α, one needs to test M, to obtain the two-weight boundedness of M_α. Clearly it is stronger that the testing condition (3.13), however it does give the sharp constant for the one-weight case (see below).

3.4 Sharp bounds

We remarked in the introduction that when $\mathscr{B} = \mathscr{Q}$ the sharp dependence on the operator norm of M_α is given by

$$\|M_\alpha\| \leq C[w]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})}. \tag{3.17}$$

This is shown in Chapter 2 using techniques similar to those in [31]. It should also be noted that (3.17) follows from combining Lemma 3.2.6 and inequality (3.2.5). We give a different proof of (3.17) using the two-weight dependence of Corollary 3.15. First, we examine the relationship between the two-weight T_q constant and the one-weight $A_{p,q}$ constant. We use a similar approach to that of Hunt, Kurtz, and Neugebauer [25].

Theorem 3.4.1. *Let* $0 \le \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, *and* w *be a weight for the basis* $\mathscr{Q}$. *Then*

$$[w]_{A_{p,q}} \le [w^q, w^p]_{T_q}^q \le C[w]_{A_{p,q}}^{p'(1-\alpha/n)}. \tag{3.18}$$

Proof. Let w be a weight, $u = w^q$, and $v = w^p$ so that $\sigma = w^{-p'}$. First notice that,

$$u(Q)\left(\frac{\sigma(Q)}{|Q|}\right)^{1+q/p'} \le \int_Q M(\chi_Q\sigma)^{1+q/p'} u\,dx \le [u,v]_{T_q}^q \sigma(Q).$$

This shows that $[w]_{A_{p,q}} \le [w^q, w^p]_{T_q}^q$. On the other hand, let Q be a cube and notice that

$$M(\chi_Q\sigma)(x) = \sup_{P \ni x} \frac{1}{|P|}\int_P \sigma\,dx,$$

where the supremum is over all cubes P containing x and that are contained in Q. Suppose $w \in A_{p,q}$, $x \in P \subseteq Q$. Then,

$$\begin{aligned}
\left(\frac{\sigma(P)}{|P|}\right)^{1+q/p'} &\le C_n[w]_{A_{p,q}}^{p'(1-\alpha/n)}\left(\frac{1}{u(3P)}\int_P u^{-1}\chi_Q u\right)^{1+p'/q} \\
&\le C[w]_{A_{p,q}}^{p'(1-\alpha/n)} M_u^c(\chi_Q u^{-1})(x)^{1+p'/q}.
\end{aligned}$$

It follows that $M(\chi_Q\sigma)(x)^{1+q/p'} \le C[w]_{A_{p,q}}^{p'(1-\alpha/n)} M_u^c(\chi_Q u^{-1})(x)^{1+p'/q}$ for all $x \in Q$. Plugging this into the testing condition and using the fact that M_u^c is bounded on $L^{1+p'/q}(u)$ with norm independent of u, we have

$$\begin{aligned}
\int_Q M(\chi_Q\sigma)^{1+q/p'} u\,dx &\le C[w]_{A_{p,q}}^{p'(1-\alpha/n)} \int_{\mathbb{R}^n} M_u^c(\chi_Q u^{-1})^{1+p'/q} u\,dx \\
&\le C_n[w]_{A_{p,q}}^{p'(1-\alpha/n)} \int_Q u^{-1-p'/q} u\,dx \\
&= C_n[w]_{A_{p,q}}^{p'(1-\alpha/n)} \sigma(Q).
\end{aligned}$$

□

From here we obtain a different proof of the bound (2.1). We have

$$\|M_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le C[w^q,w^p]_{T_p} \le [w]_{A_{p,q}}^{p'/q(1-\alpha/n)}.$$

We make some remarks about the consequences of Corollary 3.15 and Theorem 3.4.1 when $\alpha = 0$. In this case we have $p = q$ and the testing conditions T_p and S_p are the same. When we renormalize back to w ($w^p \mapsto w$) inequality (3.18) in Theorem 3.4.1 becomes

$$[w]_{A_p}^{1/p} \le [w]_{S_p} \le C[w]_{A_p}^{1/(p-1)}. \tag{3.19}$$

Inequality (3.19) has a few interesting consequences. First, it leads to a new proof of estimate (1.16). Indeed, using (3.16) in Corollary 3.15 and the second inequality in (3.19) we obtain

$$\|M\|_{L^p(w)\to L^p(w)} \le C[w]_{S_p} \le C[w]_{A_p}^{1/(p-1)}.$$

As noted in the introduction of this chapter, this is basically combining Sawyer's two-weight result with a variation of the arguments of Hunt, Kurtz, and Nuegebauer. Second, the operator norm dependence for the two-weight case is sharp.

Theorem 3.4.2. *Suppose* $1 < p < \infty$ *then*

$$\|M\|_{L^p(v)\to L^p(u)} \le C[u,v]_{S_p} \tag{3.20}$$

is sharp.

This follows from the one-weight case, since if we had a better bound in (3.20), then taking $u = v = w \in A_p$ and using (3.19) would imply a better bound in (1.16). Finally, the second inequality in (3.19) is sharp. Once again, a better bound in the second inequality in (3.19) would imply a sharper bound in (1.16).

3.5 Reverse Hölder class

In his Ph.D. dissertation [39], Pérez characterized the reverse holder class in terms of the $L^p(\mathbb{R}^n)$ boundedness of the weighted maximal operator, M_u, given by

$$M_u f(x) = \sup_{Q \ni x} \frac{1}{u(Q)} \int_Q |f(y)| u(y)\, dy.$$

As mentioned in Chapter One, a weight u is in RH_q if

$$[u]_{RH_q} = \sup_Q \left(\frac{u^q(Q)}{|Q|} \right)^{1/q} \frac{|Q|}{u(Q)} < \infty.$$

Pérez proved the following theorem.

Theorem 3.5.1. *The operator M_u is bounded on $L^p(\mathbb{R}^n)$ if and only if $u \in RH_{p'}$.*

We find the sharp constant for the operator norm of M_u^d, in terms of the $RH_{p'}^d$ constant of u. We have the following theorem.

Theorem 3.5.2. *Suppose $1 < p < \infty$ and $u \in RH_{p'}^d$, then M_u is bounded on L^p with*

$$\|M_u^d\| \le C[u]_{RH_{p'}^d}^{p'}$$

and this result is sharp.

Proof. Let $x \in \mathbb{R}^n$ and Q be a dyadic cube containing x. Then

$$\begin{aligned} \frac{1}{u(Q)} \int_Q |f| u\, dx &\le [u]_{RH_{p'}^d}^{p'} \left(\frac{u(Q)}{|Q|} \right)^{p'} \frac{|Q|}{u^{p'}(Q)} \frac{1}{u(Q)} \int_Q |f| u\, dx \\ &= [u]_{RH_{p'}^d}^{p'} \left(\frac{u(Q)}{|Q|} \left(\frac{1}{u^{p'}(Q)} \int_Q |f| u^{1-p'} u^{p'}\, dx \right)^{p-1} \right)^{p'-1} \\ &\le [u]_{RH_{p'}^d}^{p'} \left(\frac{1}{|Q|} \int_Q M_{u^{p'}}^d (f u^{1-p'})^{p-1} u\, dx \right)^{p'-1}. \end{aligned}$$

Taking the supremum over all dyadic cubes containing x we have the pointwise estimate,

$$M_u^d f(x) \leq [u]_{RH_{p'}^d}^{p'} M^d\{M_{u^{p'}}^d(fu^{1-p'})^{p-1}u\}(x)^{p'-1}.$$

Since $M_{u^{p'}}^d$ is bounded on $L^p(u^{p'})$ with operator norm depending only on the dimension and $M : L^{p'} \to L^{p'}$, we have

$$\begin{aligned}
\|M_u^d f\|_{L^p} &\leq [u]_{RH_{p'}^d}^{p'} \|M^d\{M_{u^{p'}}^d(fu^{1-p'})^{p-1}u\}\|_{L^{p'}}^{p'-1} \\
&\leq C[u]_{RH_{p'}^d}^{p'} \|M_{u^{p'}}^d(fu^{1-p'})\|_{L^p(u^{p'})} \\
&\leq C[u]_{RH_{p'}^d}^{p'} \|fu^{1-p'}\|_{L^p(u^{p'})} \\
&= C[u]_{RH_{p'}^d}^{p'} \|f\|_{L^p}.
\end{aligned}$$

We show the sharpness in dimension one to simplify matters. The n dimensional case is similar. We use the families of power weights $u_\delta(x) = |x|^{(\delta-1)/p'}$ and functions $f_\delta(x) = x^{(\delta-1)/p}\chi_{[0,1]}$. A calculation similar to that in Lemma 2.4.1 gives $[u_\delta]_{RH_{p'}} \approx \delta^{-1/p'}$. Furthermore, given $x \in [0,1]$, there exist a $k \in \mathbb{Z}$ such that $2^{-k} \leq x < 2^{-k+1}$. Then, x belongs to the dyadic cube $[0, 2^{-k+1}]$ with

$$u([0,2^{-k+1}]) = \int_0^{2^{-k+1}} |y|^{(\delta-1)/p'}\, dy \approx x^{(\delta-1)/p'+1}.$$

Hence,

$$\begin{aligned}
M_{u_\delta}^d f_\delta(x) &\geq \frac{1}{u([0,2^{-k+1}])} \int_0^{2^{-k+1}} f_\delta(y) u_\delta(y)\, dy \\
&\geq C\frac{1}{x^{(\delta-1)/p'+1}} \int_0^x y^{\delta-1}\, dy \\
&= C\frac{x^{(\delta-1)/p}}{\delta} = C\delta^{-1} f_\delta(x).
\end{aligned}$$

By the first part of the theorem we have $\|M_{u_\delta} f_\delta\|_{L^p}^p \le C[u_\delta]_{RH_{p'}}^{p'p} \|f_\delta\|_{L^p}^p \le C\delta^{-p}\|f_\delta\|_{L^p}^p$, and

$$\int_{\mathbb{R}} M_{u_\delta} f_\delta(x)^p \, dx \ge \int_0^1 M_{u_\delta} f_\delta(x)^p \, dx \ge C\delta^{-p}\|f_\delta\|_{L^p}^p.$$

Letting $\delta \to 0^+$ shows the result is sharp. $\square$

Chapter 4

Weighted inequalities for multilinear fractional operators

The question of characterizing the weighted inequalities for multilinear operators was first posed by Grafakos and Torres in [22], following their previous multilinear work [21]. Lerner, Ombrosi, Pérez, Torres, and Trujillo-Gonzalez [33] answered this question for multilinear singular integral operators by introducing a new multisublinear maximal operator. This chapter is devoted to weighted inequalities for multilinear versions of the fractional operators I_α and M_α. Multilinear fractional operators have been studied by Grafakos [17] and Kenig and Stein [28]. Much of multilinear weighted theory stems from the linear techniques in the article [40] and the multilinear weighted theory of [33]. This chapter essentially contains the material from the article [35].

4.1 Multilinear fractional operators

Throughout this chapter we use the notation, $\vec{f} = (f_1, \dots, f_m)$, for an m-tuple of functions. A multilinear (multisublinear) operator, T is an operator defined on vectors of functions

$\vec{f}$ such that for each $1 \leq i \leq m$ and fixed functions $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m$ the operators

$$T_i = T(f_1, \ldots, f_{i-1}, \cdot, f_{i+1}, \ldots, f_m)$$

are linear (sublinear). We write

$$T : X_1 \times \cdots \times X_m \to X$$

if for all $\vec{f} \in X_1 \times \cdots \times X_m$,

$$\|T(\vec{f})\|_X \leq C \prod_{i=1}^{m} \|f_i\|_{X_i}$$

with the smallest C denoted as $\|T\|_{X_1 \times \cdots \times X_m \to X}$ or simply $\|T\|$ when the ambient spaces are clear.

Definition 4.1.1. Let $m \geq 1$ and $0 < \alpha < nm$, we define the multilinear fractional operator,

$$\mathscr{I}_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{nm-\alpha}} \, d\vec{y}$$

where

$$d\vec{y} = dy_1 \cdots dy_m.$$

The operator $\mathscr{I}_\alpha$ is well defined for, say $\vec{f} \in \mathscr{S} \times \cdots \times \mathscr{S}$. We also notice that if we let

$$\mathscr{I}_{\alpha,p}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x-y_1|^p + \cdots + |x-y_m|^p)^{(nm-\alpha)/p}} \, d\vec{y}$$

for $1 < p < \infty$ and

$$\mathscr{I}_{\alpha,\infty}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(\sup_i |x-y_i|)^{nm-\alpha}} \, d\vec{y},$$

then

$$\mathscr{I}_\alpha(\vec{f}) \approx \mathscr{I}_{\alpha,p}(\vec{f}) \approx \mathscr{I}_{\alpha,\infty}(\vec{f}).$$

This follows from equivalence of norms on finite dimensional spaces. Next, notice that if α_i for $1 \le i \le m$ satisfy $0 < \alpha_i < n$ and $\alpha_1 + \cdots + \alpha_m = \alpha$, then

$$\mathscr{I}_\alpha(\vec{f}) \le \prod_{i=1}^{m} I_{\alpha_i} f_i.$$

Suppose $0 < \alpha < nm$, $1 < p_1, \ldots, p_m$ are exponents with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/m < p < n/\alpha$. Then if $1/q = 1/p - \alpha/n$, we can find $0 < \alpha_1, \ldots, \alpha_m < n$ with $\alpha_1 + \cdots + \alpha_m$ and $p_i < n/\alpha_i$. Setting $1/q_i = 1/p_i - \alpha/n$ we have

$$\|\mathscr{I}_\alpha(\vec{f})\|_{L^q} \le \|\prod_i I_{\alpha_i} f_i\|_{L^q} \le \prod_i \|I_{\alpha_i} f_i\|_{L^{q_i}} \le C \prod_i \|f_i\|_{L^{p_i}}.$$

Thus, for $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$ we have

$$\mathscr{I}_\alpha : L^{p_1} \times \cdots L^{p_m} \to L^q.$$

We now look at the multilinear version of M_α.

Definition 4.1.2. Let $\vec{f} \in L^1_{\text{loc}} \times \cdots \times L^1_{\text{loc}}$ and $0 \le \alpha < nm$, we define the multisublinear fractional maximal function (multilinear for brevity) by

$$\mathscr{M}_\alpha(\vec{f})(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)|\, dy_i.$$

Once again if $\alpha_1 + \cdots + \alpha_m = \alpha$ then

$$\mathscr{M}_\alpha(\vec{f}) \le \prod_i M_{\alpha_i} f_i.$$

A similar argument to that for $\mathscr{I}_\alpha$ shows

$$\mathscr{M}_\alpha : L^{p_1} \times \cdots \times L^{p_m} \to L^q$$

when $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$. Like the case $m = 1$ we have that $\mathscr{M}_\alpha$ is a pointwise smaller operator than $\mathscr{I}_\alpha$.

Proposition 4.1.3. *Let $0 < \alpha < nm$ then there exists a positive constant c such that*

$$\mathscr{M}_\alpha(\vec{f}) \leq c\mathscr{I}_\alpha(\vec{f})$$

for $f_i \geq 0$.

Proof. Let $x \in \mathbb{R}^n$ and Q be a cube containing x then

$$\begin{aligned} |Q|^{\alpha/n} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q f_i(y_i)\, dy_i &= \frac{1}{\ell(Q)^{nm-\alpha}} \int_{Q^m} f_1(y_1) \cdots f_m(y_m)\ d\vec{y} \\ &\leq c \int_{Q^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{nm-\alpha}}\, d\vec{y} \\ &\leq c\mathscr{I}_\alpha(\vec{f})(x). \end{aligned}$$

It follows that $\mathscr{M}_\alpha(\vec{f})(x) \leq c\mathscr{I}_\alpha(\vec{f})(x)$.

□

4.2 Banach function spaces

In this section we give a short introduction to Banach function spaces. The interested reader may find more information including additional concrete examples of Banach function spaces in the book by Bennet and Sharpley [3]. Let (R, μ) be a measure space and let $M^+(B)$ be the set of all non-negative measurable functions. A mapping $\rho : M^+(R) \to [0,\infty]$ is called a Banach function norm if satisfies

- $\rho(f) = 0$ if and only if $f = 0$ a.e., $\rho(af) = a\rho(f)$ for all $a \geq 0$, and $\rho(f+g) \leq \rho(f)+\rho(g)$
- $0 \leq g \leq f$ a.e. implies $\rho(g) \leq \rho(f)$
- If $0 \leq f_1 \leq \cdots \leq f_n \leq f_{n+1} \leq \cdots$ and $f_n \to f$ a.e. then $\rho(f_n) \to \rho(f)$
- $\mu(E) < \infty$ implies $\rho(\chi_E) < \infty$
- $\mu(E) < \infty$ implies $\int_E f \, d\mu \leq C_E \rho(f)$

If $M(R)$ is the collection of all measurable functions on R, then $X = X(\rho)$ will denote the collection of all functions $f \in M(R)$ such that $\rho(|f|) < \infty$. The norm $\|f\|_X = \rho(|f|)$ makes X into a Banach space of functions on R (hence the name). One of the most important properties of X is that there exists another Banach function space X' called the associate space of X for which the following generalized Hölder inequality holds:

$$\int_R |fg| \, d\mu \leq c\|f\|_X \|g\|_{X'}.$$

One of the main examples of a Banach function space is $L^p(\mu)$, in this case the associate is the dual space $L^{p'}(\mu)$. Other examples include Lorentz spaces and Orlicz spaces. The Orlicz spaces are defined as follows. A function $B : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex, increasing, $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. Moreover, we shall assume B is normalized so that $B(1) = 1$ and B satisfies the doubling condition, namely there exists constants C and N such that

$$B(2t) \leq CB(t)$$

for all $t \geq N$. For each such function B there exists a complementary Young function $\overline{B}$ such that

$$t \leq B^{-1}(t)\overline{B}^{-1}(t) \leq 2t$$

for $t > 0$. The Orlicz space L^B is all functions such that

$$\int_R B\left(\frac{|f(y)|}{\lambda}\right) d\mu < \infty$$

for some $\lambda > 0$. The Banach function norm ρ is given by

$$\rho(f) = \inf\left\{\lambda > 0 : \int_R B\left(\frac{f(y)}{\lambda}\right) d\mu \leq 1\right\}.$$

If $B(t) = t^p$ then $L^B = L^p$. Other examples are given by

$$B(t) = t^q \log(1+t)^{q-1+\delta}$$

and

$$B(t) = t^q \log(1+t)^{q-1} \log\log(1+t)^{q-1+\delta}$$

for $\delta > 0$. These spaces form the Zygmund "$L\log L$" spaces.

4.3 Multilinear weights

In the article [33] a multilinear weighted theory is developed for the operator $\mathscr{M} = \mathscr{M}_0$. They introduce the class of vector weights $A_{\vec{P}}$, given by $\vec{w} = (w_1, \ldots, w_m)$ that satisfy

$$[\vec{w}]_{A_{\vec{P}}} = \sup_Q \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\, dx\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\, dx\right)^{1/p_i'} < \infty,$$

where $v_{\vec{w}} = \Pi_i w_i^{p/p_i}$. In [33] they prove the following theorem.

Theorem 4.3.1. *Suppose* $1 < p_1, \ldots, p_m < \infty$, *and* $1/p = 1/p_1 + \cdots 1/p_m$, *then*

$$\mathcal{M} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(v_{\vec{w}})$$

if and only if $\vec{w} \in A_{\vec{P}}$.

The authors of [33] also prove that the $A_{\vec{P}}$ is the natural class of weights for multilinear Calderón-Zygmund operators. It is with this motivation we study weighted inequalities for multilinear fractional operators

4.4 Weights for multilinear fractional operators

We study weighted inequalities of the form

$$\|u\mathcal{T}_\alpha(\vec{f})\|_{L^q} \leq \prod_i \|fv_i\|_{L^{p_i}}$$

where $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$ and $\mathcal{T}_\alpha$ is one of the operators $\mathcal{I}_\alpha$ or $\mathcal{M}_\alpha$. We present the most general result and then state the corollaries that are more specific.

Throughout this section all Banach function spaces will be over $\mathbb{R}^n$ with Lebesgue measure. Let X be a Banach function space, for a function $f \in X$ and a cube $Q \subset \mathbb{R}^n$ we define the X average of f over Q to be

$$\|f\|_{X,Q} = \|\delta_{\ell(Q)}(f\chi_Q)\|_X,$$

where for $a > 0$, $\delta_a f(x) = f(ax)$. Observe that if $X = L^r$ then

$$\|f\|_{X,Q} = \left(\frac{1}{|Q|} \int_Q |f|^r \, dx \right)^{1/r}$$

and if $X = L^B$ then,

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f(y)|}{\lambda} \right) \, dy \leq 1 \right\}.$$

Following Pérez [41] we define the maximal operator associated to the Banach function space X to be

$$M_X f(x) = \sup_{Q \ni x} \|f\|_{X,Q}.$$

When X is the Orlicz space L^B we denote M_X by M_B. Notice that if M is the Hardy-Littlewood maximal operator, then $M_{L^1} = M$ and $M_{L^r} f(x) = M(f^r)^{1/r}$. If $Y_1, \ldots, Y_m$ are Banach function spaces we define the multisublinear maximal function to be

$$\mathscr{M}_{\vec{Y}} \vec{f}(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \|f_i\|_{Y_i,Q}.$$

Notice that $\mathscr{M}_{\vec{Y}} \vec{f}(x) \leq \prod_{i=1}^{m} M_{Y_i} f_i(x)$. Hence if $1 \leq p_1, \ldots, p_m \leq \infty$ and $M_{Y_i} : L^{p_i} \to L^{p_i}$ then by Hölders inequality

$$\mathscr{M}_{\vec{Y}} : L^{p_1} \times \cdots \times L^{p_m} \to L^p.$$

Theorem 4.4.1. *Suppose* $0 < \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$, *with* $1/p = 1/p_1 + \cdots + 1/p_m$, *and* $Y_1, \ldots, Y_m$ *are Banach function spaces over* $\mathbb{R}^n$ *such that*

$$\mathscr{M}_{\vec{Y}'} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \tag{4.1}$$

where $\mathscr{M}_{\vec{Y}'}$ is the multilinear maximal function associated to $Y_1', \ldots, Y_m'$. Let q be an exponent satisfying $1/m < p \leq q < \infty$. Suppose that one of the following two conditions holds.

i) $q > 1$, X is a Banach function space that satisfy

$$M_{X'} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n), \tag{4.2}$$

and $(u, \vec{v})$ are weights that satisfy

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \|u\|_{X,Q} \prod_{i=1}^m \|{v_i}^{-1}\|_{Y_i,Q} < \infty. \tag{4.3}$$

ii) $q \leq 1$ and $(u, \vec{v})$ are weights that satisfy

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \left(\frac{1}{|Q|} \int_Q u^q \, dx \right)^{1/q} \prod_{i=1}^m \|{v_i}^{-1}\|_{Y_i,Q} < \infty. \tag{4.4}$$

Then the inequality

$$\left(\int_{\mathbb{R}^n} (u|\mathscr{I}_\alpha \vec{f}|)^q \, dx \right)^{1/q} \leq C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx \right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

We delay the proof of Theorem 4.4.1 and state some consequences of it. For the Orlicz spaces, L^B, the boundedness of the corresponding maximal functions M_B has been developed by Pérez [40], [41]. He showed that

$$M_B : L^s \to L^s$$

if and only if there exists $c > 0$ such that

$$\int_c^\infty \frac{B(t)}{t^s}\frac{dt}{t} < \infty.$$

Thus we have the following theorem.

Theorem 4.4.2. *Suppose* $0 < \alpha < nm$, $1 < p_1, \dots, p_m < \infty$, *with* $1/p = 1/p_1 + \cdots + 1/p_m$, *$q$ is an exponent with* $1/m < p \le q < \infty$ *and* $\Psi, \Phi_1, \dots, \Phi_m$ *are Young functions that satisfy*

$$\int_c^\infty \frac{\Psi(t)}{t^q}\frac{dt}{t} < \infty \tag{4.5}$$

and

$$\int_c^\infty \frac{\Phi_i(t)}{t^{p_i'}}\frac{dt}{t} < \infty, \qquad i = 1, \dots, m \tag{4.6}$$

for some $c > 0$. Let q be an exponent satisfying $1/m < p \le q < \infty$ and assume that

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \|u\|_{\Psi,Q} \prod_{i=1}^m \|v_i{}^{-1}\|_{\Phi_i,Q} < \infty.$$

Then the inequality

$$\left(\int_{\mathbb{R}^n} (u|\mathscr{I}_\alpha \vec{f}|)^q \, dx\right)^{1/q} \le C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx\right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

We notice that the functions $\Psi(t) = t^q(\log(1+t))^{q-1+\delta}$ and $\Phi_i = t^{p_i'}(\log(1+t))^{p_i'-1+\delta}$ satisfy (4.5) and (4.6) respectively if $\delta > 0$. We also the following Theorem in the spirit of a Fefferman-Phong "bump" condition see also Chang, Wilson, and Wolff [6] and [49].

Theorem 4.4.3. *Suppose that* $0 < \alpha < nm$, $1 < p_1, \dots, p_m < \infty$ *and q is a number that satisfies* $1/m < p \le q < \infty$. *Suppose that one of the following two conditions holds.*

i) $q > 1$ and $(u, \vec{v})$ are weights that satisfy

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \left(\frac{1}{|Q|} \int_Q u^{qr}\, dx \right)^{1/qr} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q v_i{}^{-p_i' r}\, dx \right)^{1/p_i' r} < \infty \tag{4.7}$$

for some $r > 1$.

ii) $q \le 1$ and $(u, \vec{v})$ are weights that satisfy

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \left(\frac{1}{|Q|} \int_Q u^{q}\, dx \right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q v_i{}^{-p_i' r}\, dx \right)^{1/p_i' r} < \infty \tag{4.8}$$

for some $r > 1$.

Then the inequality,

$$\left(\int_{\mathbb{R}^n} (u |\mathscr{I}_\alpha \vec{f}|)^q\, dx \right)^{1/q} \le C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i}\, dx \right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Theorem 4.4.3 is a consequence of Theorem 4.4.1. If we let $X = L^{rq}$ and $Y_i = L^{rp_i'}$ for $r > 1$, then $X' = L^{(rq)'}$ and $Y_i' = L^{(rp_i')'}$. The operators $M_{L^{(rq)'}} f = M(|f|^{(rq)'})^{1/(rq)'}$, and $M_{L^{(rp_i')'}} f = M(|f|^{(rp_i)'})^{1/(rp_i')'}$ for $i = 1, \ldots, m$, satisfy

$$\begin{cases} M_{L^{(rq)'}} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n) \\ M_{L^{(rp_1')'}} : L^{p_1}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n) \\ \vdots \\ M_{L^{(rp_m')'}} : L^{p_m}(\mathbb{R}^n) \to L^{p_m}(\mathbb{R}^n). \end{cases} \tag{4.9}$$

Using the fact that A_∞ weights satisfy the Reverse Hölder condition (1.15), we have the following Corollary to 4.4.3.

Corollary 4.4.4. *Suppose that* $0 < \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$ *and* q *is such that* $1/m < p \le q < \infty$. *Further suppose that* $u, v_1, \ldots, v_m$ *are weights with* $u^q, {v_1}^{-p_1'}, \ldots, {v_m}^{-p_m'} \in A_\infty$, *that satisfy,*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \left(\frac{1}{|Q|}\int_Q u^q\, dx\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q {v_i}^{-p_i'}\, dx\right)^{1/p_i'} < \infty.$$

Then,

$$\left(\int_{\mathbb{R}^n} (u|\mathscr{I}_\alpha \vec{f}|)^q\, dx\right)^{1/q} \le C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i}\, dx\right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Proof of Theorem 4.4.1. We first treat the case $q > 1$. We wish to show that

$$\left(\int_{\mathbb{R}^n} (u|\mathscr{I}_\alpha \vec{f}|)^q\, dx\right)^{1/q} \le C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i}\, dx\right)^{1/p_i}.$$

Equivalently, since $\mathscr{I}_\alpha$ is a positive operator, it is enough to show that

$$\int_{\mathbb{R}^n} \mathscr{I}_\alpha \vec{f}(x) u(x) g(x)\, dx \le C \left(\int_{\mathbb{R}^n} g(x)^{q'}\, dx\right)^{1/q'} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (f_i(x) v_i(x))^{p_i}\, dx\right)^{1/p_i}$$

for all $g \in L^{q'}(\mathbb{R}^n)$, with $g \ge 0$, and all $f_i \ge 0$, bounded with compact support. We apply a discretization technique similar to that used in [40] for the operator $\mathscr{I}_\alpha$.

For a fixed $x \in \mathbb{R}^n$ and $l \in \mathbb{Z}$ there is a unique dyadic cube of side length 2^l that contains x. Hence we have

$$\begin{aligned} \mathscr{I}_\alpha \vec{f}(x) &= \sum_{\nu\in\mathbb{Z}} \int_{2^{\nu-1} < \sum_i |x-y_i| \le 2^\nu} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{nm-\alpha}} d\vec{y} \\ &= \sum_{\nu\in\mathbb{Z}} \sum_{\substack{Q\in\mathscr{D}(\mathbb{R}^n)\\ \ell(Q)=2^\nu}} \chi_Q(x) \int_{\ell(Q)/2 < \sum_i |x-y_i| \le \ell(Q)} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{nm-\alpha}} d\vec{y} \end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{Q\in\mathscr{D}(\mathbb{R}^n)} \frac{\ell(Q)^\alpha}{|Q|^m} \int_{\sum_i |x-y_i|\leq \ell(Q)} f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\, \chi_Q(x) \\
&\leq C \sum_{Q\in\mathscr{D}(\mathbb{R}^n)} \frac{\ell(Q)^\alpha}{|Q|^m} \int_{\sup_i |x-y_i|\leq \ell(Q)} f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\, \chi_Q(x) \\
&\leq C \sum_{Q\in\mathscr{D}} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\, \chi_Q(x).
\end{aligned}
$$

Let g be a non-negative function in $L^{q'}(\mathbb{R}^n)$, then

$$
\int_{\mathbb{R}^n} \mathscr{I}_\alpha \vec{f}(x) u(x) g(x)\, dx \leq C \sum_{Q\in\mathscr{D}} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} \int_Q u(x) g(x)\, dx.
$$

Define

$$
\mathscr{M}_{3\mathscr{D}}\vec{h}(x) = \sup_{x\in Q\in\mathscr{D}} \prod_{i=1}^m \frac{1}{|3Q|} \int_{3Q} |h_i(y_i)|\, dy_i,
$$

to be the maximal function with the basis of triples of dyadic cubes. Notice that $\mathscr{M}_{3\mathscr{D}}\vec{f} \leq \mathscr{M}\vec{f}$. Let $\|\mathscr{M}\|$ be the constant from the $L^1\times\cdots\times L^1 \to L^{1/m,\infty}$ inequality for $\mathscr{M}$, $a > 6^n\|\mathscr{M}\|$ and

$$
D_k = \{x\in\mathbb{R}^n : \mathscr{M}_{3\mathscr{D}}\vec{f}(x) > a^k\}.
$$

If D_k is non-empty we can find a dyadic cube Q with $x\in Q$ and

$$
\frac{1}{|3Q|^m} \int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} > a^k.
$$

Since f_i is bounded with compact support we can find a dyadic cube that satisfies this (4.4) and is maximal with respect to inclusion. Thus, we get $D_k = \bigcup_j Q_{k,j}$ where, for each k the cubes $Q_{k,j}$ are maximal, disjoint, dyadic cubes that satisfy

$$
a^k < \frac{1}{|3Q_{k,j}|^m} \int_{(3Q_{k,j})^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} \leq 2^{nm} a^k.
$$

Fix $Q_{k,j}$, we compute the part of $Q_{k,j}$ covered by D_{k+1}. We have,

$$Q_{k,j} \cap D_{k+1} = \{x \in Q_{k,j} : \mathscr{M}_{3\mathscr{D}}\vec{f}(x) > a^{k+1}\}.$$

Since $x \in Q_{k,j}$ the supremum in

$$\mathscr{M}_{3\mathscr{D}}\vec{f}(x) = \sup_{x \in P \in \mathscr{D}} \frac{1}{|3P|^m} \int_{(3P)^m} f_1(y_1) \cdots f_m(y_m)\, d\vec{y} > a^{k+1}.$$

is taken over all dyadic cubes that contain $Q_{k,j}$ or are contained in $Q_{k,j}$. But the maximality of $Q_{k,j}$ implies

$$\frac{1}{|3P|^m} \int_{(3P)^m} f_1(y_1) \cdots f_m(y_m)\, d\vec{y} \leq a^k$$

for all $P \supsetneq Q_{k,j}$. It now follows that if $x \in Q_{k,j}$ and $\mathscr{M}_{3\mathscr{D}}\vec{f}(x) > a^{k+1}$, then

$$\mathscr{M}_{3\mathscr{D}}(f_1\chi_{3Q_{k,j}}, \ldots, f_m\chi_{3Q_{k,j}})(x) > a^{k+1}.$$

We have,

$$\begin{aligned}
|Q_{k,j} \cap D_{k+1}| &= |\{x \in Q_{k,j} : \mathscr{M}_{3\mathscr{D}}\vec{f}(x) > a^{k+1}\}| \\
&\leq |\{x \in Q_{k,j} : \mathscr{M}_{3\mathscr{D}}(f_1\chi_{3Q_{k,j}}, \ldots, f_m\chi_{3Q_{k,j}})(x) > a^{k+1}\}| \\
&\leq |\{x \in \mathbb{R}^n : \mathscr{M}(f_1\chi_{3Q_{k,j}}, \ldots, f_m\chi_{3Q_{k,j}})(x) > a^{k+1}\}| \\
&\leq \left(\frac{\|\mathscr{M}\|}{a^{k+1}} \prod_{i=1}^{m} \int_{3Q_{k,j}} f_i(y_i)\, dy_i \right)^{1/m} \\
&\leq \left(\frac{\|\mathscr{M}\|}{a^{k+1}} \frac{1}{|3Q_{k,j}|^m} \int_{(3Q_{k,j})^m} f_1(y_1) \cdots f_m(y_m)\, d\vec{y} \right)^{1/m} |3Q_{k,j}| \\
&\leq \frac{6^n \|\mathscr{M}\|^{1/m}}{a^{1/m}} |Q_{k,j}|.
\end{aligned}$$

Thus,

$$|Q_{k,j} \cap D_{k+1}| \leq \beta |Q_{k,j}|$$

for some $0 < \beta < 1$. If $E_{k,j} = Q_{k,j} \backslash D_{k+1}$ then $\{E_{k,j}\}_{k,j}$ is a disjoint family of sets that satisfy

$$|Q_{k,j}| \leq C|E_{k,j}|$$

for some $C > 0$. Let,

$$\mathscr{C}^k = \{Q \in \mathscr{D} : a^k < \frac{1}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m) d\vec{y} \leq a^{k+1}\}.$$

Then,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathscr{I}_\alpha \vec{f}(x) u(x) g(x)\, dx \\
&\leq \; C \sum_{Q \in \mathscr{D}} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m)\, d\vec{y} \int_Q u(x) g(x)\, dx \\
&\leq \; C \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathscr{C}^k} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m)\, d\vec{y} \int_Q u(x) g(x)\, dx \\
&\leq \; C \sum_{k \in \mathbb{Z}} a^{k+1} \sum_{j \in \mathbb{Z}} \sum_{\substack{Q \in \mathscr{C}^k \\ Q \subset Q_{k,j}}} \ell(Q)^\alpha \int_Q u(x) g(x)\, dx \\
&\leq \; Ca \sum_{k \in \mathbb{Z}} a^k \sum_{j \in \mathbb{Z}} \ell(Q_{k,j})^\alpha \int_{Q_{k,j}} u(x) g(x)\, dx. \\
&\leq \; C \sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^m \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f_i(y_i) v_i(y) v_i^{-1}(y_i) dy_i \\
&\qquad \times \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} u(x) g(x)\, dx |Q_{k,j}|.
\end{aligned}$$

Using the generalized Hölder inequality for Banach function spaces, a discrete Hölder's inequality and replacing $Q_{k,j}$ with the disjoint $E_{k,j}$ we have,

$$
\begin{aligned}
&\leq C\sum_{k,j}\ell(3Q_{k,j})^{\alpha}\|u\|_{X,3Q_{k,j}}\|g\|_{X',3Q_{k,j}}\left(\prod_{i=1}^{m}\|f_iv_i\|_{Y_i',3Q_{k,j}}\|v_i^{-1}\|_{Y_i,3Q_{k,j}}\right)|Q_{k,j}| \\
&\leq CK\sum_{k,j}\|g\|_{X',3Q_{k,j}}\prod_{i=1}^{m}\|f_iv_i\|_{Y_i',3Q_{k,j}}|E_{k,j}|^{1/p+1/q'} \\
&\leq CK\left(\sum_{k,j}\|g\|_{X',3Q_{k,j}}^{p'}|E_{k,j}|^{p'/q'}\right)^{1/p'}\left(\sum_{k,j}\prod_{i=1}^{m}\|f_iv_i\|_{Y_i',3Q_{k,j}}^{p}|E_{k,j}|\right)^{1/p} \\
&\leq CK\left(\sum_{k,j}\|g\|_{X',3Q_{k,j}}^{q'}|E_{k,j}|\right)^{1/q'}\left(\sum_{k,j}\prod_{i=1}^{m}\|f_iv_i\|_{Y_i',3Q_{k,j}}^{p}|E_{k,j}|\right)^{1/p} \\
&\leq CK\left(\sum_{k,j}\int_{E_{k,j}}M_{X'}(g)^{q'}dx\right)^{1/q'}\left(\sum_{k,j}\int_{E_{k,j}}\mathscr{M}_{\vec{Y}'}(\vec{f})^{p}dx\right)^{1/p} \\
&\leq CK\left(\int_{R^n}M_{X'}(g)^{q'}dx\right)^{1/q'}\left(\int_{\mathbb{R}^n}\mathscr{M}_{\vec{Y}'}(\vec{f})^{p}dx\right)^{1/p} \\
&\leq CK\left(\int_{\mathbb{R}^n}g^{q'}dx\right)^{1/q'}\prod_{i=1}^{m}\left(\int_{\mathbb{R}^n}(f_iv_i)^{p_i}dx\right)^{1/p_i}
\end{aligned}
$$

where K is the constant from (4.3). Thus proving the case $q>1$. For the case $q\leq 1$ using the same discritization technique as above, we obtain

$$
\mathscr{I}_{\alpha}\vec{f}(x)^{q}\leq C\sum_{Q\in\mathscr{D}}\left(\frac{\ell(Q)^{\alpha}}{|3Q|^{m}}\int_{(3Q)^m}f_1(y_1)\cdots f_m(y_m)\ d\vec{y}\right)^{q}\chi_Q(x).
$$

Multiplying by u^q and integrating,

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n}(\mathscr{I}_{\alpha}\vec{f}(x)u(x))^{q}\ dx\right)^{1/q} \\
&\leq \left(C\sum_{Q\in\mathscr{D}}\left(\frac{\ell(Q)^{\alpha}}{|3Q|^{m}}\int_{(3Q)^m}f_1(y_1)\cdots f_m(y_m)\ d\vec{y}\right)^{q}\int_{Q}u(x)^{q}\ dx\right)^{1/q}.
\end{aligned}
$$

Performing the same decomposition as above we obtain $\{Q_{k,j}\}_{k,j}$ and construct $\{E_{k,j}\}$ satisfying the same properties. Thus,

$$\left(C \sum_{Q\in\mathscr{D}} \left(\frac{\ell(Q)^{\alpha}}{|3Q|^m} \int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\ d\vec{y} \right)^q \int_Q u(x)^q\, dx \right)^{1/q}$$
$$\leq C \left(C \sum_{k\in\mathbb{Z}} a^{(k+1)q} \sum_{j\in\mathbb{Z}} \sum_{\substack{Q\in C^k \\ Q\subset Q_{k,j}}} \ell(Q)^{\alpha q} \int_Q u(x)^q\, dx \right)^{1/q}$$
$$\leq C \left(\sum_{k,j} \frac{\ell(3Q_{k,j})^{\alpha q}}{|3Q_{k,j}|} \int_{3Q_{k,j}} u(x)^q\, dx \prod_{i=1}^m \left(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f_i(y_i) v_i(y_i) v_i^{-1}(y_i) dy_i \right)^q |Q_{k,j}| \right)^{1/q}. \tag{4.10}$$

Using the generalized Hölder inequality and condition (4.4), we continue with the estimates

$$\leq CK \left(\sum_{k,j} \prod_{i=1}^m \|f_i v_i\|_{Y_i',3Q_{k,j}} |Q_{k,j}|^{q/p} \right)^{1/q}.$$

Using $p \leq q$, replacing the $Q_{k,j}$'s with $E_{k,j}$'s

$$\begin{aligned}
&\leq CK \left(\sum_{k,j} \prod_{i=1}^m \|f_i v_i\|_{Y_i',3Q_{k,j}} |E_{k,j}| \right)^{1/p} \\
&\leq CK \left(\int_{\mathbb{R}^n} \mathscr{M}_{\vec{Y}'}(\vec{f})^p dx \right)^{1/p} \\
&\leq CK \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (f_i v_i)^{p_i}\ dy_i \right)^{1/p_i}.
\end{aligned}$$

This concludes the proof of Theorem 4.4.1.

□

A close examination of the above proof yeilds that the operator norm denoted $\|\mathscr{I}_\alpha\|$ has the dependence,

$$\|\mathscr{I}_\alpha\| \le c\|M_{\vec{Y}}\|K$$

where C is a dimensional constant and K is the constant from (4.4.1) or (4.4.1).

We now examine the two-weight inequalities for $\mathscr{M}_\alpha$. We start with a weak characterization notice the renormalization $u^q \mapsto u$ and $v_i^{p_i} \mapsto v_i$.

Theorem 4.4.5. *Suppose that $0 \le \alpha < nm$, $1 \le p_1, \dots, p_m < \infty$ and q is a number satisfying $1/m < p \le q < \infty$. Then the inequality,*

$$\|\mathscr{M}_\alpha \vec{f}\|_{L^{q,\infty}(u)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(v_i)}$$

holds if and only if the weights $(u, \vec{v})$ satisfy,

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \left(\frac{1}{|Q|}\int_Q u\, dx\right)^{1/q} \prod_{i=1}^{m} \left(\frac{1}{|Q|}\int_Q {v_i}^{1-p_i'}\, dx\right)^{1/p_i'} < \infty. \qquad (4.11)$$

Here $\left(\frac{1}{|Q|}\int_Q {v_j}^{1-p_j'}\, dx\right)^{1/p_j'}$ is understood as $(\inf_Q w_j)^{-1}$ when $p_j = 1$.

Proof. The proof is similar to that of the weak inequality given in [33]. We only present the case where $p_1, \dots, p_m > 1$ as a the case when some $p_j = 1$ is a minor modification of the linear case. Suppose that $\mathscr{M}_\alpha$ is weakly bounded i.e.

$$u(\{x \in \mathbb{R}^n : \mathscr{M}_\alpha \vec{f}(x) > \lambda\}) \le \left(\frac{C}{\lambda} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(v_i)}\right)^q$$

for all $\lambda > 0$. Let $f_i \ge 0$ and fix a cube Q with

$$\ell(Q)^\alpha \prod_i \frac{1}{|Q|}\int_Q f_i\, dy_i > 0.$$

Notice that for $x \in Q$ we have

$$\ell(Q)^{\alpha} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q f_i \leq \mathscr{M}_{\alpha}(f_1\chi_Q, \ldots, f_m\chi_Q)(x).$$

Hence, if $\lambda < \ell(Q)^{\alpha}\Pi_i |Q|^{-1} \int_Q f_i \leq \mathscr{M}_{\alpha}(f_1\chi_Q, \ldots, f_m\chi_Q)(x)$ we have

$$Q \subset \{x \in \mathbb{R}^n : \mathscr{M}_{\alpha}(f_1\chi_Q, \ldots, f_m\chi_Q)(x) > \lambda\}.$$

Thus,

$$u(Q) \leq u(\{x \in \mathbb{R}^n : \mathscr{M}_{\alpha}(f_1\chi_Q, \ldots, f_m\chi_Q)(x) > \lambda\}) \leq \left(\frac{C}{\lambda} \prod_{i=1}^{m} \left(\int_Q {f_i}^{p_i} v_i \right)^{1/p_i} \right)^{q}.$$

Since this holds for all

$$\lambda < \ell(Q)^{\alpha} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q f_i$$

it follows that

$$|Q|^{\alpha/n-m} u(Q)^{1/q} \prod_{i=1}^{m} \left(\int_Q f_i \right) \leq C \prod_{i=1}^{m} \left(\int_Q {f_i}^{p_i} v_i \right)^{1/p_i}.$$

If we set $f_i = {v_i}^{1-p_i'}$ we get

$$|Q|^{\alpha/n-m} u(Q)^{1/q} \prod_{i=1}^{m} \left(\int_Q {v_i}^{1-p_i'} \right) \leq C \prod_{i=1}^{m} \left(\int_Q {v_i}^{1-p_i'} \right)^{1/p_i}$$

which gives condition (4.11). Conversely, suppose that $(u, \vec{v})$ satisfies condition (4.11) and assume for the moment that for all $1 \leq i \leq m$ $\|f_i\|_{L^{p_i}(v_i)} = 1$. We will also use the centered fractional multilinear maximal function $\mathscr{M}_{\alpha}^{c}$ where the supremum is taken over all cubes centered at x. Clearly $\mathscr{M}_{\alpha} \approx \mathscr{M}_{\alpha}^{c}$.

Given x fix a cube, Q, centered at x. Then Hölder's inequality yields

$$\begin{aligned}
\ell(Q)^{\alpha}\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q|f_i| &= |Q|^{\alpha/n-m}\prod_{i=1}^{m}\int_Q|f_i|{v_i}^{1/p_i}{v_i}^{-1/p_i} \\
&\le |Q|^{\alpha/n-m}\prod_{i=1}^{m}\left(\int_Q|f_i|^{p_i}v_i\right)^{1/p_i}\left(\int_Q{v_i}^{1-p_i'}\right)^{1/p_i'} \\
&= |Q|^{\alpha/n-m}u(Q)^{1/q}\prod_{i=1}^{m}\left(\int_Q{v_i}^{1-p_i'}\right)^{1/p_i'}u(Q)^{-1/q}\prod_{i=1}^{m}\left(\int_Q|f_i|^{p_i}v_i\right)^{1/p_i} \\
&\le Cu(Q)^{-1/q}\prod_{i=1}^{m}\left(\int_Q|f_i|^{p_i}v_i\right)^{1/p_i}.
\end{aligned}$$

Now, since we are assuming that $\|f_i\|_{L^{p_i}(v_i)}=1$, we have

$$\prod_{i=1}^{m}\left(\int_Q|f_i|^{p_i}v_i\right)^{1/p_i}\le 1.$$

Moreover, since $p/q\le 1$ we have

$$\begin{aligned}
\ell(Q)^{\alpha}\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q|f_i| &\le C\frac{1}{u(Q)^{1/q}}\prod_{i=1}^{m}\left(\int_Q|f_i|^{p_i}v_i\right)^{1/p_i} \\
&\le C\frac{1}{u(Q)^{1/q}}\left(\prod_{i=1}^{m}\left(\int_Q|f_i|^{p_i}v_i\right)^{1/p_i}\right)^{p/q} \\
&= C\left(\frac{1}{u(Q)^{1/p}}\prod_{i=1}^{m}\left(\int_Q|f_i|^{p_i}v_i\right)^{1/p_i}\right)^{p/q} \\
&= C\left(\prod_{i=1}^{m}\left(\frac{1}{u(Q)}\int_Q|f_i|^{p_i}v_iu^{-1}u\right)^{1/p_i}\right)^{p/q} \\
&\le C\left(\prod_{i=1}^{m}M_u^c(|f_i|^{p_i}v_i/u)(x)^{1/p_i}\right)^{p/q}.
\end{aligned}$$

Hence,

$$\mathscr{M}_{\alpha}^{c}\vec{f}(x) \leq C\left(\prod_{i=1}^{m} M_u^c(|f_i|^{p_i}v_i/u)(x)^{1/p_i}\right)^{p/q}.$$

Using a weak-type Hölder's inequality we have,

$$\begin{aligned}
\|\mathscr{M}_{\alpha}^{c}\vec{f}\|_{L^{q,\infty}(u)} &\leq C\|(\Pi_i M_u^c(|f_i|^{p_i}v_i/u)^{1/p_i})^{p/q}\|_{L^{q,\infty}(u)} \\
&= C\|\Pi_i M_u^c(|f_i|^{p_i}v_i/u)^{1/p_i}\|_{L^{p,\infty}(u)}^{p/q} \\
&\leq C\left(\prod_{i=1}^{m}\|M_u^c(|f_i|^{p_i}v_i/u)\|_{L^{p_i,\infty}(u)}\right)^{p/q} \\
&= C\left(\prod_{i=1}^{m}\|M_u^c(|f_i|^{p_i}v_i/u)\|_{L^{1,\infty}(u)}^{1/p_i}\right)^{p/q} \\
&\leq C\left(\prod_{i=1}^{m}\||f_i|^{p_i}v_i/u\|_{L^{1}(u)}^{1/p_i}\right)^{p/q} \\
&= C\left(\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(v_i)}\right)^{p/q} = C.
\end{aligned}$$

For general f_i the result follows if we replace $f_i \to f_i/\|f_i\|_{L^{p_i}(v_i)}$. □

We notice that for the weak boundedness we do not need to invoke the Banach function space norms.

Theorem 4.4.6. *Suppose* $0 \leq \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$, *with* $1/p = 1/p_1 + \cdots + 1/p_m$, *and* q *is an exponent satisfying* $1/m < p \leq q < \infty$, *and* $Y_1, \ldots, Y_m$ *are are translation invariant Banach function spaces with*

$$\mathscr{M}_{\vec{Y}'} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

If $(u,\vec{v})$ *are weights that satisfy*

$$\sup_{Q} \ell(Q)^{\alpha}|Q|^{1/q-1/p}\left(\frac{1}{|Q|}\int_{Q} u^{q}\,dx\right)^{1/q}\prod_{i=1}^{m}\|v_i{}^{-1}\|_{Y_i,Q} < \infty \tag{4.12}$$

then the inequality

$$\left(\int_{\mathbb{R}^n}(u\mathscr{M}_{\alpha}\vec{f})^{q}\,dx\right)^{1/q} \leq C\prod_{i=1}^{m}\left(\int_{\mathbb{R}^n}(|f_i|v_i)^{p_i}\,dx\right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Proof. We first prove the boundedness for the dyadic version,

$$\mathscr{M}_{\alpha}^{d}\vec{f}(x) = \sup_{Q\in\mathscr{D}:x\in Q} \ell(Q)^{\alpha}\prod_{i=1}^{m}\frac{1}{|Q|}\int_{Q}|f_i(y_i)|dy_i.$$

Let a be a constant satisfying $a > 2^{nm}$ and let

$$D_k = \{x \in \mathbb{R}^n : \mathscr{M}_{\alpha}^{d}f(x) > a^k\}.$$

If D_k is non-empty then we can write $D_k = \bigcup_j Q_{k,j}$ where each $Q_{k,j}$ is a maximal dyadic cube satisfying

$$a^k < \ell(Q)^{\alpha}\prod_{i=1}^{m}\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} f_i(y_i)dy_i < 2^{mn-\alpha}a^k \leq 2^{mn}a^k.$$

Also, each $D_{k+1} \subseteq D_k$ and each $Q_{k+1,l}$ is contained in $Q_{k,j}$ for some j by properties of dyadic cubes and we have

$$|Q_{k,j} \cap D_{k+1}| \leq \frac{2^n}{a^{1/m}}|Q_{k,j}|.$$

Hence the sets $E_{k,j} = Q_{k,j} \backslash (Q_{k,j} \cap D_{k+1})$ are disjoint and satisfy

$$|Q_{k,j}| < \beta |E_{k,j}|$$

for some $\beta > 1$. Thus, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} (\mathscr{M}_\alpha^d \vec{f}(x) u(x))^q \, dx \right)^{1/q} \\
&= \left(\sum_k \int_{D_k \backslash D_{k+1}} (\mathscr{M}_\alpha^d \vec{f}(x) u(x))^q \, dx \right)^{1/q} \\
&\leq \left(\sum_k a^{(k+1)q} \int_{D_k} u^q(x) \, dx \right)^{1/q} \\
&\leq a \left(\sum_{k,j} a^{kq} \int_{Q_{k,j}} u^q(x) \, dx \right)^{1/q} \\
&\leq a \left(\sum_{k,j} \ell(Q_{k,j})^\alpha \left(\prod_{i=1}^m \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} f_i(y_i) v_i(y_i) v_i(y_i)^{-1} dy_i \right)^q \int_{Q_{k,j}} u^q(x) \, dx \right)^{1/q}.
\end{aligned}$$

This equation is the same as (4.10) in the proof of Theorem 4.4.3 and the dyadic version of the theorem follows. The non-dyadic version follows from the inequality

$$\mathscr{M}_\alpha^k \vec{f}(x)^q \leq \frac{C_{\alpha,n}}{|B_k|} \int_{B_k} (\tau_{-t} \circ \mathscr{M}_\alpha^d \circ \vec{\tau}_t)(\vec{f})(x)^q \, dt \tag{4.13}$$

for all $x \in \mathbb{R}^n$ and $f_i \geq 0$. Where $B_k = [-2^{k+2}, 2^{k+2}]^n$, $\mathscr{M}_\alpha^k \vec{f}$ is the maximal function with the supremum taken over cubes of side length less than 2^k, $\tau_t g(x) = g(x-t)$, $\vec{\tau}_t \vec{f} = (\tau_t f_1, \ldots, \tau_t f_m)$. The inequality (4.13) holds for all $0 < q < \infty$, and a proof for the linear case can be found in [16, p. 431] and the multilinear case is a slight modification.

From (4.13) it follows that

$$\|\mathscr{M}_\alpha \vec{f} u\|_{L^q} \le \sup_t \|\tau_t \circ \mathscr{M}_\alpha^d \circ \vec{\tau}_t \vec{f} u\|_{L^q}.$$

If $(u, \vec{v})$ satisfy condition (4.12), then $(\tau_t u, \vec{\tau}_t \vec{v})$ satisfy the condition (4.12) independent of t (since the Y_i are translation invarient). By the dyadic case we have,

$$\|(\tau_{-t} \circ \mathscr{M}_\alpha^d \circ \vec{\tau}_t)\vec{f} u\|_{L^q} = \|(\mathscr{M}_\alpha^d \circ \tau_t)\vec{f} \tau_t u\|_{L^q}$$
$$\le C \prod_{i=1}^m \|\tau_t f_i \tau_t v_i\|_{L^{p_i}} = C \prod_{i=1}^m \|f_i\|_{L^{p_i}(v_i)},$$

where the constant C is independent of t. It now follows that,

$$\|\mathscr{M}_\alpha \vec{f} u\|_{L^q} \le C \sup_t \|\tau_t \circ \mathscr{M}_\alpha^d \circ \vec{\tau}_t \vec{f} u\|_{L^q} \le C \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}}.$$

□

We now state the corollaries to Theorem 4.4.6 similar to those for $\mathscr{I}_\alpha$.

Theorem 4.4.7. *Suppose $0 < \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$, with $1/p = 1/p_1 + \cdots + 1/p_m$, q is an exponent with $1/m < p \le q < \infty$ and $\Phi_1, \ldots, \Phi_m$ are Young functions that satisfy*

$$\int_c^\infty \frac{\Phi_i(t)}{t^{p_i'}} \frac{dt}{t} < \infty, \qquad i = 1, \ldots, m \tag{4.14}$$

for some $c > 0$. Let q be an exponent satisfying $1/m < p \le q < \infty$.

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q - 1/p} \left(\frac{1}{|Q|} \int_Q u^q \, dx \right)^{1/q} \prod_{i=1}^m \|v_i{}^{-1}\|_{\Phi_i, Q} < \infty.$$

Then the inequality

$$\left(\int_{\mathbb{R}^n} (u\mathscr{M}_\alpha \vec{f})^q \, dx\right)^{1/q} \leq C \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx\right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Theorem 4.4.8. *Suppose* $0 \leq \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$, *and* q *is a number such that* $1/m < p \leq q < \infty$. *If* $(u, \vec{v})$ *are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \left(\frac{1}{|Q|}\int_Q u^q \, dx\right)^{1/q} \prod_{i=1}^{m} \left(\frac{1}{|Q|}\int_Q v_i^{-rp_i'} \, dx\right)^{1/rp_i'} < \infty, \tag{4.15}$$

for some $r > 1$, *then*

$$\left(\int_{\mathbb{R}^n} (u\mathscr{M}_\alpha \vec{f})^q \, dx\right)^{1/q} \leq C \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx\right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Corollary 4.4.9. *Suppose* $0 \leq \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$, *and* q *is a number such that* $1/m < p \leq q < \infty$. *If* $(u, \vec{v})$ *are weights with* $v_i^{-p_i} \in A_\infty$ *and*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \left(\frac{1}{|Q|}\int_Q u^q \, dx\right)^{1/q} \prod_{i=1}^{m} \left(\frac{1}{|Q|}\int_Q v_i^{-p_i'} \, dx\right)^{1/p_i'} < \infty, \tag{4.16}$$

for some $r > 1$, *then*

$$\left(\int_{\mathbb{R}^n} (u\mathscr{M}_\alpha \vec{f})^q \, dx\right)^{1/q} \leq C \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx\right)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

4.5 One-weight theory

We now turn our attention to the multilinear and multisublinear one vector weight case. Proposition 4.1.3 shows that $\mathscr{M}_\alpha$ is a pointwise smaller operator than $\mathscr{I}_\alpha$. However we also have the reverse inequality in norm. We obtain the following theorem relating $\mathscr{I}_\alpha$ and $\mathscr{M}_\alpha$ as an application of the extrapolation theorem of Cruz-Uribe, Pérez, and Martell [8] .

Theorem 4.5.1. *Suppose that $0 < \alpha < mn$, then for every $w \in A_\infty$ and all $0 < q < \infty$, we have,*

$$\int_{\mathbb{R}^n} |\mathscr{I}_\alpha \vec{f}(x)|^q w(x)\, dx \leq C \int_{\mathbb{R}^n} \mathscr{M}_\alpha \vec{f}(x)^q w(x)\, dx$$

for all functions $\vec{f}$ with f_i bounded with compact support.

Proof. In light of the extrapolation theorem in [8] we just need to show that the result holds for $q = 1$ and all $w \in A_\infty$. Using the same decomposition as in Theorem 4.4.3 with $g = 1$ we have,

$$\int_{\mathbb{R}^n} \mathscr{I}_\alpha \vec{f} w\, dx \leq c \sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^{m} \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f_i(y_i) dy_i w(Q_{k,j}). \tag{4.17}$$

Since $w \in A_\infty$ and $|Q_{k,j}| \leq C|E_{k,j}|$ we have,

$$w(Q_{k,j}) \leq C w(E_{k,j}).$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} \mathscr{I}_\alpha \vec{f}\, w\, dx &\leq c \sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^{m} \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f_i(y_i) dy_i w(E_{k,j}) \\ &\leq c \sum_{k,j} \int_{E_{k,j}} \mathscr{M}_\alpha \vec{f}\, w\, dx \end{aligned}$$

$$\leq \quad c\int_{\mathbb{R}^n} \mathscr{M}_\alpha \vec{f}\, w\, dx.$$

□

We notice that with a slight adaption of the proof of Theorem 4.5.2 without using an A_∞ condition we have the following corollary.

Corollary 4.5.2. *Suppose that* $0 < \alpha < mn$, *and* w *is a weight, then*

$$\int_{\mathbb{R}^n} |\mathscr{I}_\alpha \vec{f}(x)| w(x)\, dx \leq c \int_{\mathbb{R}^n} \mathscr{M}_\alpha \vec{f}(x) Mw(x)\, dx$$

for all functions $\vec{f}$ *with* f_i *bounded with compact support.*

Proof. From inequality 4.17, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathscr{I}_\alpha \vec{f}\, w\, dx &\leq c\sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^{m} \frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i(y_i) dy_i w(Q_{k,j}) \\
&\leq c\sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^{m} \frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i(y_i) dy_i \frac{w(Q_{k,j})}{|Q_{k,j}|}|Q_{k,j}| \\
&\leq c\sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^{m} \frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i(y_i) dy_i \frac{w(Q_{k,j})}{|Q_{k,j}|}|E_{k,j}| \\
&\leq c\int_{\mathbb{R}^n} \mathscr{M}_\alpha \vec{f}\, Mw\, dx.
\end{aligned}$$

□

If we assume, say $v_i^{-p_i'} \in A_\infty$, then the two-weight characterization for the fractional maximal function becomes,

$$\left(\int_{\mathbb{R}^n} (\mathscr{M}_\alpha(\vec{f})u)^q\, dx\right)^{1/q} \leq C\prod_{i=1}^{m}\left(\int_{\mathbb{R}^n} (|f_i|v_i)^{p_i}\, dx\right)^{1/p_i}$$

if and only if

$$\sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q u^q\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q v_i{}^{-p_i'}\right)^{1/p_i'} < \infty. \tag{4.18}$$

Notice that when we have the Sobolev relationship,

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},$$

we obtain from (4.18)

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u^q\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q v_i{}^{-p_i'}\right)^{1/p_i'} < \infty.$$

In this situation, the Lesbegue differentiation theorem gives then

$$u \le C \prod_{i=1}^m v_i.$$

With this motivation we define a one-weight condition as follows.

Definition 4.5.3. Let $1 < p_1, \dots, p_m < \infty$ and q be a number $1/m < p \le q < \infty$. We say that a vector of weights $\vec{w} = (w_1, \dots, w_m)$ is in the class $A_{\vec{P},q}$, or that it satisfies the $A_{\vec{P},q}$ condition, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q (\Pi_{i=1}^m w_i)^q\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i{}^{-p_i'}\right)^{1/p_i'} < \infty.$$

Notice that when $q = p$ we have the re-normalized $A_{\vec{P}}$ class of weights from [33].

Remark 4.5.4. If $p_i \leq q_i$ and $1/q = 1/q_1 + \cdots + 1/q_m$ then by Hölders inequality we have,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q (\Pi_i w_i)^q\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q {w_i}^{-p_i'}\right)^{1/p_i'} \\
&\leq \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q {w_i}^{q_i}\right)^{1/q_i} \left(\frac{1}{|Q|}\int_Q {w_i}^{-p_i'}\right)^{1/p_i'},
\end{aligned}$$

and hence

$$\bigcup_{q_1,\ldots,q_m} \prod_{i=1}^m A_{p_i,q_i} \subseteq A_{\vec{P},q}, \tag{4.19}$$

where the union is over all $q_i \geq p_i$ that satisfy $1/q = 1/q_1 + \cdots + 1/q_m$. However, in general this containment is strict. Take for example, $n = 1$, $m = 2$, $p_1 = p_2 = 2$, and $q = 3/2$. We use a similar example to the one given in [33] let

$$w_1(x) = \begin{cases} |x-1|^{-1/2} & x \in [0,2] \\ 1 & \text{otherwise} \end{cases}$$

and $w_2(x) = |x|^{-1/2}$. Then $(w_1 w_2)^q$ is in A_1 and $\inf_Q (w_1 w_2)^q \sim (\inf_Q w_1^q)(\inf_Q w_2^q)$ but for any power $r \geq 2$ $w_i^r \notin L^1_{\text{loc}}$ and hence cannot be in $A_{r,2}$ for any such r.

Theorem 4.5.5. *Suppose,* $1 < p_1, \ldots, p_m < \infty$, *and* $0 < \alpha < nm$ *and* $\vec{w} \in A_{\vec{P},q}$, *then*

$$(\Pi_{i=1}^m w_i)^q \in A_{mq} \quad \text{and} \quad w_i^{-p_i'} \in A_{mp_i'}.$$

Proof. Since $p \leq q$, if we let $q_i = qp_i/p$ then, $q_i \geq p_i$ and $1/q = 1/q_1 + \cdots + 1/q_m$. Moreover, we have

$$\frac{1}{q_1'} + \cdots + \frac{1}{q_m'} = m - \frac{1}{q},$$

and hence Hölders inequality with $r_i = (m-1/q)q_i'$ can be applied to get

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q (\Pi_i w_i)^q\right)^{\frac{1}{mq}} \left(\frac{1}{|Q|}\int_Q (\Pi_i w_i)^{-q/(mq-1)}\right)^{(mq-1)/mq} \\
\leq &\left(\frac{1}{|Q|}\int_Q (\Pi_i w_i)^q\right)^{\frac{1}{mq}} \left(\prod_i \left(\frac{1}{|Q|}\int_Q w_i^{-q_i'}\right)^{1/q_i'}\right)^{1/m}.
\end{aligned}$$

We now use Hölder's with $p_i'/q_i' > 1$ to get

$$\leq \left(\left(\frac{1}{|Q|}\int_Q (\Pi_i w_i)^q\right)^{\frac{1}{q}} \prod_i \left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)^{1/p_i'}\right)^{1/m}.$$

This shows that $\Pi_i w_i{}^q \in A_{mq}$. Now to show that $w_i^{-p_i'} \in A_{mp_i'}$, for this fix $1 \leq i \leq m$, then the $A_{mp_i'}$ condition is,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)^{1/mp_i'} \left(\frac{1}{|Q|}\int_Q w_i{}^{p_i'/(mp_i'-1)}\right)^{(mp_i'-1)/mp_i'} < \infty.$$

If we set

$$r_i = p(m-1+\frac{1}{p_i}) = p(m-\frac{1}{p_i'}) \quad \text{and} \quad r_j = \frac{p_j}{p_j-1}\frac{r_i}{p} = \frac{p_j'}{p} r_i \quad 1 \leq j \neq i \leq m.$$

Then notice $1 < r_j < \infty$ and

$$\sum_{j=1}^m \frac{1}{r_j} = \frac{1}{r_j}\left(1 + \sum_{1\leq j\neq i\leq m} \frac{p}{p_j'}\right) = 1.$$

Further

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q w_i{}^{p_i'/(mp_i'-1)}\right)^{(mp_i'-1)/mp_i'} &= \left(\frac{1}{|Q|}\int_Q w_i{}^{p/r_i}\right)^{r_i/mp} \\
&= \left(\frac{1}{|Q|}\int_Q (\Pi_{j=1}^m w_j)^{p/r_i} \Pi_{j\neq i} w_j^{-p/r_i}\right)^{r_i/mp}.
\end{aligned}$$

We use Hölder's inequality with exponents $r_1, \dots, r_m$ to get

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^{p/r_i}\Pi_{j\neq i} w_j^{-p/r_i}\right)^{r_i/mp} \\
\leq\ & \frac{1}{|Q|^{r_i/mp}}\left[\left(\int_Q (\Pi_j w_j)^p\right)^{1/r_i}\prod_{j\neq i}\left(\int_Q w_j{}^{-p_j'}\right)^{1/r_j}\right]^{r_i/mp} \\
=\ & \frac{1}{|Q|^{r_i/mp}}\left(\left(\int_Q (\Pi_j w_j)^p\right)^{1/p}\prod_{j\neq i}\left(\int_Q w_j{}^{-p_j'}\right)^{1/p_j'}\right)^{1/m} \\
=\ & \left(\left(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^p\right)^{1/p}\prod_{j\neq i}\left(\frac{1}{|Q|}\int_Q w_j{}^{-p_j'}\right)^{1/p_j'}\right)^{1/m} \\
\leq\ & \left(\left(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^q\right)^{1/q}\prod_{j\neq i}\left(\frac{1}{|Q|}\int_Q w_j{}^{-p_j'}\right)^{1/p_j'}\right)^{1/m}.
\end{aligned}$$

The second to last inequality follows since

$$\frac{r_i}{p} = m - \frac{1}{p_i'} = \frac{1}{p} + \sum_{1\leq j\neq i\leq m}\frac{1}{p_j'},$$

and the last inequality follows from Hölder's with q/p. Thus we arrive at the inequality,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)^{1/mp_i'}\left(\frac{1}{|Q|}\int_Q w_i{}^{p_i'/(mp_i'-1)}\right)^{(mp_i'-1)/mp_i'} \\
\leq\ & \left(\left(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^q\right)^{1/q}\prod_{j=1}^{m}\left(\frac{1}{|Q|}\int_Q w_j{}^{-p_j'}\right)^{1/p_j'}\right)^{1/m}.
\end{aligned}$$

This shows that $w_i^{-p_i'} \in A_{mp_i'}$. □

We now state the main theorem for these weights. In the one-weight situation we obtain necessary and sufficent conditions for the boundedness of $\mathscr{I}_\alpha$ and $\mathscr{M}_\alpha$.

Theorem 4.5.6. *Suppose that $0 < \alpha < nm$ and $1 < p_1, \ldots, p_m < \infty$ are exponents with $1/m < p < n/\alpha$ and q is the exponent defined by $1/q = 1/p - \alpha/n$. Then the inequality*

$$\left(\int_{\mathbb{R}^n} (|\mathscr{I}_\alpha \vec{f}|(\Pi_i w_i))^q \, dx\right)^{1/q} \leq C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| w_i)^{p_i} \, dx\right)^{1/p_i}$$

holds for every $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$ if and only if w satisfies the $A_{\vec{P},q}$ condition.

In light of Theorem 4.5.5 and Corollary 4.4.4 the sufficiency of the $A_{\vec{P},q}$ condition follows from the two-weight case with $u = \Pi_i w_i$ and $v_i = w_i$. The necessity of the $A_{\vec{P},q}$ condition follows from Theorem 4.4.5 and the fact that $\mathscr{I}_\alpha$ is a bigger operator than $\mathscr{M}_\alpha$.

Theorem 4.5.7. *Suppose that $0 < \alpha < nm$ and $1 < p_1, \ldots, p_m < \infty$ are exponents with $1/m < p < n/\alpha$ and q is the exponent defined by $1/q = 1/p - \alpha/n$. Then the inequality*

$$\left(\int_{\mathbb{R}^n} (\mathscr{M}_\alpha(\vec{f})(\Pi_i w_i))^q \, dx\right)^{1/q} \leq C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| w_i)^{p_i} \, dx\right)^{1/p_i}$$

holds for every $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$ if and only if $\vec{w}$ satisfies the $A_{\vec{P},q}$ condition.

Once again the sufficiency of the $A_{\vec{P},q}$ condition follows from the two-weight case Theorem 4.4.8 and the necessity follows from the weak characterization in Theorem 4.4.8. We do note, however, that Theorem 4.5.7 combined with Theorems 4.5.2 and 4.5.5 gives a different proof of the sufficiency of the $A_{\vec{P},q}$ condition in Theorem 4.5.6. When $\alpha = 0$ (so $p = q$) we recover the result from [33].

4.6 Multilinear Sobolev inequalities

One of the main applications of the boundedness of fractional integrals are Sobolev and Poincaré inequalities. Most of these follow from the fact that if $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$

and f is sufficiently smooth and compactly supported then

$$|f(x)| \leq cI_1(|\nabla f|)(x).$$

Details can be found in the book by Stein [51, p. 125]. Also, if f is sufficiently smooth and

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

then $I_2 = c(-\Delta)^{-1}$, which can be seen by taking the Fourier transform. Using the boundedness of I_1 and I_2 one obtains inequalities like

$$\|f\|_{L^q} \leq c\||\nabla f|\|_{L^p} \tag{4.20}$$

when $n \geq 2$, $1 < p < n$ and $1/q = 1/p - 1/n$. For Δ one has

$$\|f\|_{L^q} \leq c\|\Delta f\|_{L^p} \tag{4.21}$$

when $n \geq 3$, $1 < p < n/2$ and $1/q = 1/p - 2/n$. Weights also come into play for these when considering inequalities (4.20) and (4.21). For example Fefferman [11], asked what conditions on a weight u imply the inequality

$$\int_{\mathbb{R}^n} |f(x)|^2 u(x)\, dx \leq C \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx.$$

This is related to the "Uncertainty principle" in quantum mechanics. We refer the reader to [15, Chap. 9] for a more detailed account.

We notice that for a product of functions say f and g (non-negative without loss of generality) one has the estimate

$$|f(x)g(x)| \leq I_1(|\nabla(fg)|)(x) \leq I_1(g|\nabla f|)(x) + I_1(f|\nabla g|)(x).$$

Using the boundedness of the linear operator I_1 and then Hölder's inquality, if $1/p_1 + 1/p_2 = 1/p$ with $1 < p < n$ and $1/q = 1/p - 1/n$ we have

$$\begin{aligned} \|fg\|_{L^q} &\leq \|I_1(g|\nabla f|)\|_{L^q} + \|I_1(f|\nabla g|)\|_{L^q} \\ &\leq C(\|g|\nabla f|\|_{L^p} + \|f|\nabla g|)\|_{L^p}) \\ &\leq C(\||\nabla f|\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\||\nabla g|\|_{L^{p_2}}). \end{aligned}$$

This is a Sobolev inequality for products of functions.

However, using the multilinear theory we obtain better results, namely the p can be a number less than one. We present the following Sobolev inequalities for products of functions in the weighted case and for $m = 2$. The general $m > 2$ follows from an easy generalization. We also present the simplest case, when the weights are all in A_∞, thus the results below will be a consequence of Corollary 4.4.4. This will be suitable for our examples and we avoid the generality of the Banach function spaces. For the following theorems, given $1 < p_1, p_2 < \infty$, the exponent p will always be determined by the formula $1/p = 1/p_1 + 1/p_2$.

Theorem 4.6.1. *Suppose that $n \geq 1$, $1 < p_1, p_2 < \infty$, $1/2 < p \leq q < \infty$ and u, v, w are A_∞ weights. If the weights u, v and w also satisfy:*

$$\sup_Q \ell(Q)|Q|^{1/q-1/p}\left(\frac{1}{|Q|}\int_Q u^q\right)^{1/q}\left(\frac{1}{|Q|}\int_Q v^{-p_1'}\right)^{1/p_1'}\left(\frac{1}{|Q|}\int_Q w^{-p_2'}\right)^{1/p_2'} < \infty, \tag{4.22}$$

then

$$\|fgu\|_{L^q} \le C(\||\nabla f|v\|_{L^{p_1}}\|gw\|_{L^{p_2}} + \|fv\|_{L^{p_1}}\||\nabla g|w\|_{L^{p_2}})$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

Proof. The proof follows from the estimate

$$\begin{aligned} |f(x)g(x)| &\le C\int_{\mathbb{R}^{2n}} \frac{|\nabla_{2n} f(x-y_1)g(x-y_2)|}{|\vec{y}|^{2n-1}}\, d\vec{y} \\ &\le C(\mathscr{I}_1(|\nabla f|, |g|)(x) + \mathscr{I}_1(|f|, |\nabla g|)(x)) \end{aligned}$$

where ∇f and ∇g are the gradients of f and g in $\mathbb{R}^n$ and ∇_{2n} is the gradient in $\mathbb{R}^{2n}$. □

Suppose, $p = q$ and $w = v = 1$. Condition (4.22) becomes

$$\sup_Q \left(\frac{1}{|Q|^{1-p/n}}\int_Q u^p\right)^{1/p} < \infty. \tag{4.23}$$

If $n > 1$ and $1/2 < p < n$, one such weight that satisfies (4.23) is $u(x) = 1/|x|$ (an argument similar to that of Lemma 2.4.1 can be used). Thus we have the following corollary, which is related to the Uncertainty Principle when $p = 2$.

Corollary 4.6.2. *Suppose* $n > 1$, $1 < p_1, p_2 < \infty$ *and* $1/2 < p < n$ *then the following estimate holds*

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |f(x)g(x)|^p \frac{dx}{|x|^p}\right)^{1/p} &\le C\left(\int_{\mathbb{R}^n} |\nabla f|^{p_1}\, dx\right)^{1/p_1}\left(\int_{\mathbb{R}^n} |g|^{p_2}\, dx\right)^{1/p_2} \\ &\quad + C\left(\int_{\mathbb{R}^n} |f|^{p_1}\, dx\right)^{1/p_1}\left(\int_{\mathbb{R}^n} |\nabla g|^{p_2}\, dx\right)^{1/p_2} \end{aligned}$$

Finally we present one more theorem for the Laplace operator Δ. We state it without proof as it follows from the estimate

$$|f(x)g(x)| \leq C\mathscr{I}_2(|\Delta f|, |g|)(x) + C\mathscr{I}_2(|f|, |\Delta g|)(x).$$

Theorem 4.6.3. *Suppose that $n \geq 2$, $1 < p_1, p_2 < \infty$, $1/2 < p \leq q < \infty$ and u, v, w are A_∞ weights. If the weights u, v and w also satisfy:*

$$\sup_Q \ell(Q)^2 |Q|^{1/q-1/p} \left(\frac{1}{|Q|} \int_Q u^q \right)^{1/q} \left(\frac{1}{|Q|} \int_Q v^{-p_1'} \right)^{1/p_1'} \left(\frac{1}{|Q|} \int_Q w^{-p_2'} \right)^{1/p_2'} < \infty,$$

then

$$\|fgu\|_{L^q} \leq C(\|v\Delta f\|_{L^{p_1}} \|gw\|_{L^{p_2}} + \|fv\|_{L^{p_1}} \|w\Delta g\|_{L^{p_2}})$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

4.7 Multilinear *BMO*

Recall the space of functions that have bound mean oscillation, BMO, given by functions that satisfy

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where $f_Q = |Q|^{-1} \int_Q f \, dx$. The related sharp maximal function,

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx,$$

plays and important part in the theory of singular integrals see [10] and [19]. In this section we consider a multilinear version of the sharp maximal operator $M^\sharp$. This leads to a definition of a multilinear BMO, or $\mathscr{BMO}$ which we show is larger than BMO^m.

Definition 4.7.1. Let $\vec{f} \in L^1_{\text{loc}} \times \cdots \times L^1_{\text{loc}}$, and define the multisublinear sharp maximal operator $\mathscr{M}^\sharp$ by

$$\mathscr{M}^\sharp \vec{f}(x) = \mathscr{M}^\sharp(f_1, \ldots, f_m)(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i) - (f_i)_Q| dy_i.$$

We notice that

$$\mathscr{M}^\sharp \vec{f} \leq 2\mathscr{M}\vec{f}.$$

Hence $\mathscr{M}^\sharp$ is bounded whenever $\mathscr{M}$ is bounded. Further,

$$\mathscr{M}^\sharp \vec{f} \leq \prod_{i=1}^m M^\sharp f_i.$$

Definition 4.7.2. We say that $\vec{f} \in \mathscr{BMO}$ if $\mathscr{M}^\sharp \vec{f} \in L^\infty$. Equivalently, $\vec{f} \in \mathscr{BMO}$ if

$$\|\vec{f}\|_{\mathscr{BMO}} = \sup_Q \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i) - (f_i)_Q| dy_i < \infty.$$

Notice that if $f_i \in BMO$ for $1 \leq i \leq m$ then $\vec{f} \in \mathscr{BMO}$, with

$$\|\vec{f}\|_{\mathscr{BMO}} \leq \prod_{i=1}^m \|f_i\|_{BMO}.$$

In other words

$$BMO^m \subseteq \mathscr{BMO}. \tag{4.24}$$

However, the following example shows that we have strict containment in (4.24). We show the particular case $m = 2$ and minor modifications yield (4.24) for a general $m > 2$.

Example 4.7.3. Consider $\vec{f} = (f_1, f_2)$ with f_1 and f_2 functions on $\mathbb{R}$ given by $f_1(x) = \chi_{(-2,-1)}(x)$ and $f_2(x) = \log(1/x)\chi_{(0,1)}(x)$. Then notice that $f_2 \notin BMO$ since for $0 <$

$\varepsilon < 1$

$$(f_2)_{(-\varepsilon,\varepsilon)} = \frac{1}{2\varepsilon}\int_0^{\varepsilon} \log(1/y)dy = \frac{\log(1/\varepsilon)+1}{2}$$

and so

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |f_2(y) - (f_2)_{(-\varepsilon,\varepsilon)}|dy \geq \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0} |(f_2)_{(-\varepsilon,\varepsilon)}|dy = \frac{\log(1/\varepsilon)+1}{4}.$$

If we let $\varepsilon \to 0^+$ we see that the right side of the inequality is unbounded. Thus $\vec{f} \notin (BMO)^2$. However, $\vec{f} \in \mathscr{BMO}$. Indeed, if I is an interval with $|I| < 1$ then

$$\left(\frac{1}{|I|}\int_I |f_1(x) - (f_1)_I|dx\right)\left(\frac{1}{|I|}\int_I |f_2(y) - (f_2)_I|dy\right) = 0,$$

since $I \cap (\mathrm{supp} f_1) \cap (\mathrm{supp} f_2) = \varnothing$ if $|I| < 1$. On the other hand, if $|I| \geq 1$ then

$$\frac{1}{|I|}\int_I f_i \leq \int_{\mathbb{R}} f_i = 1 \quad i = 1,2.$$

Using these estimates we have

$$\begin{aligned}
\left(\frac{1}{|I|}\int_I |f_1(x) - (f_1)_I|dx\right)\left(\frac{1}{|I|}\int_I |f_2(y) - (f_2)_I|dy\right) &\leq \left(\frac{2}{|I|}\int_I f_1\, dx\right)\left(\frac{2}{|I|}\int_I f_2\, dx\right) \\
&\leq 4\left(\int_{\mathbb{R}} f_1\, dx\right)\left(\int_{\mathbb{R}} f_2\, dx\right) \\
&< \infty.
\end{aligned}$$

Chapter 5

Operators on mixed Lebesgue spaces

In this section we present some results for the classical operators on mixed-norm Lebesgue spaces or simply mixed Lebesgue spaces. These spaces naturally show up when considering vector valued functions or considering functions of two or more independent variables, say, time and space. It is in this regard that mixed Lebesgue spaces contribute to the theory of partial differential equations. The solution of a certain partial differential equation may be in L^p in the time variable and L^q in the space variable. Such examples arise when considering Strichartz estimates for the wave equation [50]. Much of the ground work for the mixed Lebesgue spaces can be found in the article by Benedek and Panzone [2].

Calderón-Zygmund operators on mixed Lebesgue space have been considered by Fernandez [13], Kurtz [29] and Stefanov and Torres [50] among others. The authors of [50] show that convolution type Calderón-Zygmund operators with certain regularity properties are bounded on mixed Lebesgue spaces. Kurtz [29] then showed that more general operators are bounded on mixed Lebesgue spaces by developing a weighted theory and an extrapolation theorem.

In this chapter we look again at general Calderón-Zygmund operators on mixed Lebesgue spaces. We use essentially the same techniques of [50] to obtain boundedness of more general operators. We obtain a weak endpoint that is not in [29]. We also study the fractional integral operator on mixed Lebesgue spaces. We introduce a new class of weights $A_{p_0,q_0}A_{p_1,q_1}$ and provide and off-diagonal extrapolation theorem for mixed Lebesgue spaces. As a corollary we obtain weighted inequalities for the fractional integral operator on mixed Lebesgue spaces.

5.1 Preleminaries

In the following section we present the definition of $L^p_xL^q_y$ for $1 \leq p,q \leq \infty$. The interested reader may generalize this to the situation (X_i,μ_i) for $1 \leq i \leq m$ and $L^{p_1}_{x_1}\cdots L^{p_m}_{x_m}$ (see also [2]).

Definition 5.1.1. Given two measure spaces (X,μ) and (Y,ν), we say that a $X \times Y$ measurable function f is in $L^p_xL^q_y(X \times Y, \mu \times \nu)$ if the norm

$$\|f\|_{L^pL^q(\mu\times\nu)} = \left(\int_X \left(\int_Y |f(x,y)|^q \, d\nu(y)\right)^{p/q} \, d\mu(x)\right)^{1/p}$$

is finite.

When both $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, and both μ and ν are Lebesgue measure, we simply write $L^p_xL^q_y$. Given a non-negative measurable function w we say $f \in L^p_xL^q_y(w)$ if

$$\|f\|_{L^pL^q(w)} = \left(\int_X \left(\int_Y |f(x,y)|^q w(x,y) \, dy\right)^{p/q} \, dx\right)^{1/p} < \infty.$$

We are going to be concerned with the classical operators: maximal functions, Calderón-Zygmund operators, and fractional integral operators. However, sometimes we have a more suitable versions of these operators for the spaces $L^p_xL^q_y$.

Definition 5.1.2. We define the strong maximal function M_S for a measurable function f on $\mathbb{R}^{n+m}$, and $(x,y) \in \mathbb{R}^{n+m}$ by

$$M_S f(x,y) = \sup_{Q\times P \ni (x,y)} \frac{1}{|Q\times P|} \int_{Q\times P} |f(s,t)|\, ds\, dt,$$

where the supremum is over all cubes $Q \subset \mathbb{R}^n$ and cubes $P \subset \mathbb{R}^m$ such that $(x,y) \in Q \times P$.

We notice that M_S is a bigger operator than M, the Hardy-Littlewood maximal operator on $\mathbb{R}^{n+m}$. For $L^p_x L^q_y$ there is also a class of weights, the class $A_p A_q$ introduced by Kurtz [29]. We say a non-negative locally integrable function on $\mathbb{R}^{n+m}$ w is in $A_p A_q$ if

$$\left(\int_Q \left(\int_P w(x,y)\, dy\right)^{p/q} dx\right) \left(\int_Q \left(\int_P w(x,y)^{1-q'}\, dy\right)^{p'/q'} dx\right)^{p-1} \leq C|Q\times P|^p \tag{5.1}$$

for all cubes $Q \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^m$. Notice that $A_p A_p = A_p^{\mathscr{R}}(\mathbb{R}^{n+m})$ where $\mathscr{R}$ is the basis of all $Q \times P$ with cubes $Q \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^m$. More properties of the $A_p A_q$ class can be found in [29].

In [29], it was show that if $w(x,y) = u(x)v(y)$ then $M_S : L^p_x L^q_y(w) \to L^p_x L^q_y(w)$ if and only if $w \in A_p A_q$. However it remains an open problem to show that for a non-product weight $w \in A_p A_q$ implies $M_S : L^p_x L^q_y(w) \to L^p_x L^q_y(w)$. More generally the following mixed Lebesgue extrapolation theorem is shown in [29].

Theorem 5.1.3. *Suppose* $1 \leq s < \infty$ *and* T *is an operator such that*

$$T : L^s(\mathbb{R}^{n+m}, w) \to L^s(\mathbb{R}^{n+m}, w)$$

for every $w \in A_s^{\mathscr{R}}(\mathbb{R}^{n+m})$. *Then given any* $1 < p,q < \infty$ *and* $w(x,y) = u(x)v(y)$ *with* $w \in A_pA_q$,

$$T : L_x^pL_y^q(w) \to L_x^pL_y^q(w).$$

5.2 Calderón-Zygmund operators on $L_x^pL_y^q$

Given a Calderón-Zygmund operator in $\mathbb{R}^{n+m}$, T, with standard kernel $K = K(x,y,s,t)$ where $x,s \in \mathbb{R}^n$ and $y,t \in \mathbb{R}^m$. The standard kernel estimates (1.6) and (1.7) for K translate to

$$|K(x,y,s,t)| \leq C(|x-s|^2 + |y-t|^2)^{-(n+m)/2} \tag{5.2}$$

and

$$\begin{aligned} |K(x+h,y+k,s,t) - K(x,y,s,t)| + |K(x,y,s+h,t+k) - K(x,y,s,t)| \\ \leq C\frac{(|h|^2+|k|^2)^{\delta/2}}{(|x-s|^2+|y-t|^2)^{(n+m+\delta)/2}} \end{aligned} \tag{5.3}$$

for some $\delta \in (0,1]$ and when $4(|h|^2+|k|^2) \leq |x-s|^2+|y-t|^2$. Thus, for $f = f(s,t)$ in $L^2(\mathbb{R}^{n+m})$ with compact support and $(x,y) \notin \operatorname{supp}(f)$

$$Tf(x,y) = \int_{\mathbb{R}^{n+m}} K(x,y,s,t)f(s,t)\, ds\, dt.$$

We notice that since $A_2^{\mathscr{R}}(\mathbb{R}^{n+m}) \subset A_2(\mathbb{R}^{n+m})$ and Calderón-Zygmund operators are bounded on $L^2(w)$ for $w \in A^2$, we may apply the extrapolation Theorem 5.3.2 with $w = 1$ to conclude the following Theorem from [29], see [50] for the convolution case.

Theorem 5.2.1. *Suppose* T *is a Calderón-Zygmund operator in* $\mathbb{R}^{n+m}$ *and* $1 < p,q < \infty$, *then*

$$T : L^pL^q \to L^pL^q$$

Theorem 5.2.1 shows that Calderón-Zygmund operators are bounded on $L^p_x L^q_y$ for $1 < p,q < \infty$. However, one cannot conclude any weak inequalities when $p = 1$ from the extrapolation theorem. We provide a proof of Theorem 5.2.1 that includes the weak endpoints. We first need the following vector valued version of Calderón-Zygmund theory. A proof can be found in [18] or [10]. Let B be a Banach space and $\mathscr{L}(B)$ be the space of all bounded linear operators on B. The space $L^p(\mathbb{R}^d, B) = L^p(B)$ consists of all measurable functions that satisfy

$$\|f\|_{L^p(B)} = \left(\int_{\mathbb{R}^d} \|f(x)\|_B^p \, dx \right)^{1/p} < \infty.$$

Lemma 5.2.2. *Suppose $K = K(x,y)$ is a kernel defined away from $x = y$ taking values $\mathscr{L}(B)$ and $\vec{T}$ is an operator that is bounded on $L^q(\mathbb{R}^d, B)$ associated to K,*

$$\vec{T} f(x) = \int_{\mathbb{R}^m} K(x,y) \cdot f(y) \, dy, \qquad x \notin \operatorname{supp}(f).$$

Also suppose K satisfies the conditions

$$\|K(x,y)\|_{\mathscr{L}(B)} \leq C|x-y|^{-d} \tag{5.4}$$

and

$$\|K(x+h,y) - K(x,y)\|_{\mathscr{L}(B)} + \|K(x,y+h) - K(x,y)\|_{\mathscr{L}(B)} \leq C \frac{|h|^\delta}{|x-y|^{d+\delta}}, \tag{5.5}$$

for some $\delta \in (0,1]$, whenever $|x-y| \geq 2|h|$. Then $\vec{T} : L^p(B) \to L^p(B)$, $1 < p < \infty$ and $\vec{T}$ is weak $(1,1)$, that is,

$$|\{x \in \mathbb{R}^d : \|\vec{T} f(x)\|_B > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} \|f(x)\|_B \, dx = \|f\|_{L^1(B)}.$$

Using the techniques from [50] we modify the authors proof to extend the following weak endpoint result to the non-convolution case.

Theorem 5.2.3. *If T is a Calderón-Zygmund operator in $\mathbb{R}^{n+m}$, then T is bounded on $L^p_xL^q_y$, $1<p,q<\infty$ and T is weakly bounded in the sense*

$$|\{x\in\mathbb{R}^n : \|Tf(x,\cdot)\|_{L^q(\mathbb{R}^m)}>\lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1_xL^q_y}.$$

Proof. Since T is a Calderón-Zygmund operator in $\mathbb{R}^{n+m}$, it is bounded on all $L^q(\mathbb{R}^{n+m})$ for all $1<q<\infty$. Fix now a $1<q<\infty$. We proceed using Lemma 5.2.2 with $B=L^q(\mathbb{R}^n)$ and exploit the fact that $L^q_xL^q_y(\mathbb{R}^{n+m})=L^q(\mathbb{R}^m,B)$. Define the operator kernel k for $x\neq s$ by

$$(k(x,s)h)(y)=\int_{\mathbb{R}^n}K(x,y,s,t)h(t)\ dt.$$

Then for $x\neq s$, $k(x,s)$ is bounded on every $L^q(\mathbb{R}^n)$ since its kernel is integrable in each variable y and t by (5.2). Notice there is a one to one correspondence between functions functions in $L^p_xL^q_y(\mathbb{R}^{n+m})$ and functions in $L^p(\mathbb{R}^n,B)$. We define the vector valued operator $\vec{T}$

$$\vec{T}f(x)(\cdot)=Tf(x,\cdot).$$

Then $\vec{T}$ is bounded on $L^q(\mathbb{R}^m,B)=L^q_xL^q_y(\mathbb{R}^{n+m})$ and is associated with the kernel k since

$$\vec{T}F(x)(y)=\int_{\mathbb{R}^{n+m}}K(x,y,s,t)f(s,t)\ ds\ dt=\int_{\mathbb{R}^m}(k(x,s)F(t))(y)\ dt.$$

We now show that k satisfies (5.4) and (5.5), thus proving the conclusion of the theorem. First, we show inequality (5.4) holds. We need to calculate $\|k(x,s)\|_{L^q(\mathbb{R}^n)}$ we do this by showing that the kernel of $k(x,s)$ which is a function of y and t is in $L^1(\mathbb{R}^m)$ in each variable. For a fixed $x \neq s$ the kernel of $k(x,s)$ is $K(x,\cdot,s,\cdot)$, and

$$\begin{aligned}
\int_{\mathbb{R}^m} |K(x,y,s,t)|\, dy &\leq C \int_{\mathbb{R}^m} (|x-s|^2 + |y-t|^2)^{-(n+m)/2}\, dy \\
&= C \int_{\mathbb{R}^m} (|x-s|^2 + |y|^2)^{-(n+m)/2} dy \\
&\leq C|x-s|^{-(n+m)} \int_{\mathbb{R}^m} \left(1 + \frac{|y|^2}{|x-s|^2}\right)^{-(n+m)/2} dy \\
&\leq C|x-s|^{-n} \int_{\mathbb{R}^m} (1+|y|^2)^{-(n+m)/2}\, dy \\
&\leq C|x-s|^{-n}.
\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^m} |K(x,y,s,t)|\, dt \leq C|x-s|^{-n}.$$

It follows (by Schur's test) that $k(x,s) : L^q(\mathbb{R}^m) \to L^q(\mathbb{R}^m)$ with

$$\|k(x,s)\|_{L^q \to L^q} \leq C|x-s|^{-n}.$$

Also fix $|x-s| \geq 2|h|$ then the kernel of the operator

$$(k(x+h,s) - k(x,s))h(y) = \int_{\mathbb{R}^m} (K(x+h,y,s,t) - K(x,y,s,t))h(t)\, dt$$

is

$$K(x+h,y,s,t) - K(x,y,s,t).$$

A similar calculation to the above yields

$$\int_{\mathbb{R}^m} |K(x+h,y,s,t) - K(x,y,s,t)|\, dy \leq C \frac{|h|^\delta}{|x-s|^{n+\delta}}$$

and

$$\int_{\mathbb{R}^m} |K(x+h,y,s,t) - K(x,y,s,t)| \, dt \le C \frac{|h|^\delta}{|x-s|^{n+\delta}}$$

showing that

$$\|k(x+h,s) - k(x,s)\|_{L^q \to L^q} \le C \frac{|h|^\delta}{|x-s|^{n+\delta}}.$$

The calculation for

$$\|k(x,s+h) - k(x,s)\|_{L^q \to L^q} \le C \frac{|h|^\delta}{|x-s|^{n+\delta}}$$

is similar. Thus by Lemma 5.2.2 we have that $\vec{T} : L^p(B) \to L^p(B)$ for $1 < p < \infty$ and

$$|\{x \in \mathbb{R}^n : \|TF(x)\|_B > \lambda\}| \le \frac{C}{\lambda} \|F\|_{L^1(B)}.$$

Translating this into $L^p_x L^q_y$ and $L^{1,\infty}_x L^q_y$ we have the conclusion of the theorem. □

5.3 An off-diagonal extrapolation theorem for $L^p_x L^q_y$ spaces

Notice that for $(x,y) \in \mathbb{R}^{n+m}$ and $0 < \alpha < n+m$ we may write the fractional integral operator I_α as

$$I_\alpha f(x,y) = \int_{\mathbb{R}^{n+m}} \frac{f(s,t)}{|(x,y)-(s,t)|^{n+m-\alpha}} \, ds \, dt.$$

If $0 < \alpha_0 < n$, $0 < \alpha_1 < m$ with $\alpha_0 + \alpha_1 = \alpha$ and p_0, p_1, q_0 and q_1 are given by

$$1 < p_0 < n/\alpha_0, \quad \frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha_0}{n} \tag{5.6}$$

$$1 < p_1 < m/\alpha_1, \quad \frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha_1}{m}. \tag{5.7}$$

Then by Minkowski's integral inequality,

$$\begin{aligned}
\|I_\alpha f\|_{L^{q_0}L^{q_1}} &\le \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^{n+m}}\frac{|f(s,t)|}{|(x,y)-(s,t)|^{n+m-\alpha}}\,ds\,dt.\right)^{q_1} dy\right)^{q_0/q_1} dx\right)^{1/q_0}\\
&\le \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{1}{|x-s|^{n-\alpha_0}}\|I_{\alpha_1}f(s,\cdot)\|_{L^{q_1}}\,ds\right)^{q_0} dx\right)^{1/q_0}\\
&\le \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{1}{|x-s|^{n-\alpha_0}}\|f(s,\cdot)\|_{L^{p_1}}\,ds\right)^{q_0} dx\right)^{1/q_0}\\
&\le \|f\|_{L^{p_0}L^{p_1}}.
\end{aligned}$$

When $p_0 = p_1 = p$ and $q_0 = q_1 = q$ this case corresponds to $\alpha_0 = n\alpha/(n+m)$ and $\alpha_1 = m\alpha/(n+m)$. Then $\alpha_0+\alpha_1 = \alpha$ and $1/q = 1/p-\alpha/(n+m)$ and this corresponds to Theorem 1.2.8.

We now examine the weighted inequalities for I_α on mixed Lebesgue space. We do so through an off-diagonal mixed Lebesgue space extrapolation theorem. We first define $A_{p_0,q_0}A_{p_1,q_1}$ class as follows.

Definition 5.3.1. Given a weight $w = w(x,y)$, $1 < p_0 \le q_0 < \infty$, and $1 < p_1 \le q_1 < \infty$ we say that $w \in A_{p_0,q_0}A_{p_1,q_1}$ if

$$\begin{aligned}
[w]_{A_{p_0,q_0}A_{p_1,q_1}} = \sup_{Q\times P}&\left(\frac{1}{|Q|}\int_Q\left(\frac{1}{|P|}\int_P w(x,y)^{q_1}\,dy\right)^{q_0/q_1} dx\right)^{1/q_0}\\
&\times\left(\frac{1}{|Q|}\int_Q\left(\frac{1}{|P|}\int_P w(x,y)^{-p_1'}\,dy\right)^{p_0'/p_1'} dx\right)^{1/p_0'} < \infty
\end{aligned}$$

where the supremum is over all $Q\times P$ with $Q\subset\mathbb{R}^n$ and $P\subset\mathbb{R}^m$ are cubes.

We make a few observations about the class $A_{p_0,q_0}A_{p_1,q_1}$ whose proofs follow from similar calculations to those in [29].

- If $p_0 = p_1 = p$ and $q_0 = q_1 = q$ then $A_{p,q}A_{p,q} = A^{\mathscr{R}}_{p,q}(\mathbb{R}^{n+m})$.

- If $w \in A_{p_0,q_0}A_{p_1,q_1}$ then $w(\cdot,y) \in A_{p_0,q_0}(\mathbb{R}^n)$ for a.e. y and $w(x,\cdot) \in A_{p_1,q_1}(\mathbb{R}^m)$ for a.e. x.

- If $w(x,y) = u(x)v(y)$ then $w \in A_{p_0,q_0}A_{p_1,q_1}$ if and only if $u \in A_{p_0,q_0}(\mathbb{R}^n)$ and $v \in A_{p_1,q_1}(\mathbb{R}^m)$.

- If $1/p_0 - 1/q_0 = 1/p_1 - 1/q_1$ then $w \in A_{p_0,q_0}A_{p_1,q_1}$ if and only if $w^{q_0} \in A_{1+q_0/p_0'}A_{1+q_1/p_1'}$.

We have the following off-diagonal mixed Lebesgue extrapolation theorem.

Theorem 5.3.2. *Suppose* $1 \leq r \leq s < \infty$ *and* T *is an operator with*

$$T : L^r(\mathbb{R}^{n+m}, w^r) \to L^s(\mathbb{R}^{n+m}, w^s)$$

for all $w \in A^{\mathscr{R}}_{s,r}(\mathbb{R}^{n+m})$. *Then*

$$T : L^{p_0}_x L^{p_1}_y(w^{p_1}) \to L^{q_0}_x L^{q_1}_y(w^{q_1})$$

for all product weights $w(x,y) = u(x)v(y)$ *with* $w \in A_{p_0,q_0}A_{p_1,q_1}$, *and all* $1 < p_0 \leq q_0 < \infty$ *and* $1 < p_1 \leq q_1 < \infty$ *that satisfy*

$$\frac{1}{r} - \frac{1}{s} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1}.$$

Proof. First we may use a rectangular version of Theorem 2.2.2 to extrapolate in $\mathbb{R}^{n+m}$ so that

$$T : L^p(\mathbb{R}^{n+m}, w^p) \to L^q(\mathbb{R}^{n+m}, w^q)$$

for all $w \in A^{\mathscr{R}}_{p,q}$ with $1/p - 1/q = 1/r - 1/s$. Now assume $q_0/q_1 > 1$ so that $p_0/p_1 > 1$ and $w(x,y) = u(x)v(y)$ with $w \in A_{p_0,q_0}A_{p_1,q_1}$. There exists a non-negative function

$g \in L^{(q_0/q_1)'}(\mathbb{R}^n, u^{q_0})$ such that

$$\begin{aligned} &\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|Tf(x,y)|^{q_1}v(y)^{q_1}\,dy\right)^{q_0/q_1}u(x)^{q_0}\,dx\right)^{1/q_0} \\ &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|Tf(x,y)|^{q_1}v(y)^{q_1}g(x)u(x)^{q_0}dy\,dx\right)^{1/q_1}. \end{aligned}$$

Set $r_1 = 1 + q_1/p_1'$ and $r_0 = 1 + q_0/p_0'$, so that $r_1 < r_0$ and

$$\frac{r_0}{r_1} = \frac{q_0}{q_1}\left(\frac{1/q_0 + 1/p_0'}{1/q_1 + 1/p_1'}\right) = \frac{q_0}{q_1}.$$

By the rectangular version of 2.2.3, there exists a function G on $\mathbb{R}^n$ such that $G \geq g$, $\|G\|_{L^{(r_0/r_1)'}(u^{q_0})} \leq 2$ and $Gu^{q_0} \in A_{r_1}$ which in turn implies $(Gu^{q_0})^{1/q_1} \in A_{p_1,q_1}(\mathbb{R}^n)$. By the above observations $v(Gu^{q_0})^{1/q_1} \in A^{\mathscr{R}}_{p_1,q_1}(\mathbb{R}^{n+m})$. Hence we may proceed using that

$$T : L^{q_1}(\mathbb{R}^{n+m}, w^{q_1}) \to L^{p_1}(\mathbb{R}^{n+m}, w^{p_1}).$$

We have

$$\begin{aligned} \|Tf\|_{L^{q_0}L^{q_1}(w^{q_1})} &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|Tf(x,y)|^{q_1}v(y)^{q_1}g(x)u(x)^{q_0}dy\,dx\right)^{1/q_1} \\ &\leq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|Tf(x,y)|^{q_1}v(y)^{q_1}G(x)u(x)^{q_0}dy\,dx\right)^{1/q_1} \\ &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|Tf(x,y)|^{q_1}(v(y)(G(x)u(x)^{q_0})^{1/q_1})^{q_1}dy\,dx\right)^{1/q_1} \\ &\leq C\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|f(x,y)|^{p_1}(v(y)(G(x)u(x)^{q_0})^{1/q_1})^{p_1}dy\,dx\right)^{1/p_1} \\ &\leq C\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|f(x,y)|^{p_1}v(y)^{p_1}\,dy\right)^{p_0/p_1}u(x)^{p_0}\,dx\right)^{1/p_0} \\ &\qquad \times\|G\|_{L^{(r_0/r_1)'}(u^{p_0})} \\ &\leq C\|f\|_{L^{p_0}L^{p_1}(w^{p_1})}. \end{aligned}$$

Now suppose $q_0/q_1 < 1$, hence $p_0/p_1 < 1$ and $w(x,y) = u(x)v(y)$ with $w \in A_{p_0,q_0}A_{p_1,q_1}$. Write

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|f(x,y)|^{p_1}v(y)^{p_1}\,dy\right)^{p_0/p_1}u(x)^{p_0}\,dx\right)^{1/p_0}\\
&\qquad = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|f(x,y)u(x)^{p_0'}|^{p_1}v(y)^{p_1}\,dy\right)^{p_0/p_1}u(x)^{-p_0'}\,dx\right)^{1/p_0}\\
&\qquad = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|f(x,y)u(x)^{p_0'}|^{p_1}v(y)^{p_1}\,dy\right)g(x)u(x)^{-p_0'}\,dx\right)^{1/p_1}
\end{aligned}$$

for some $g \in L^{p_0/(p_0-p_1)}(\mathbb{R}^n, u^{-p_0'})$ with $\|g\|_{L^{p_0/(p_0-p_1)}(u^{-p_0'})} = 1$. Set $h(x) = g(x)^{-p_1'/p_1}$, $r_0 = 1 + p_0'/q_0$, and $r_1 = 1 + p_1'/q_1$ so that $r_1 < r_0$. Notice (as in the proof of Theorem 2.2.2) that

$$-\frac{p_1'}{p_1}\left(\frac{r_0}{r_1}\right)' = \frac{p_0}{p_0 - p_1}.$$

thus, $\|h\|_{L^{(r_0/r_1)'}(u^{-p_0})} = \|g\|^{-p_1'/p_1}_{L^{p_0/(p_0-p_1)}(u^{-p_0'})} = 1$. Now $u^{-p_0'} \in A_{r_0}$ by Lemma 2.2.3 there exists a function H with $H \geq h$, $1 \leq \|H\| \leq 2$, and $Hu^{-p_0'} \in A_{r_1}$. This makes $(Hu^{-p_0'})^{-1/p_1'} \in A_{p_1,q_1}$ and $v(Hu^{-p_0'})^{-1/p_1'} \in A^{\mathscr{R}}_{p_1,q_1}(\mathbb{R}^{n+m})$. Working backwards

$$\begin{aligned}
\|f\|_{L^{p_0}L^{p_1}(w^{p_1})} &= \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|f(x,y)u(x)^{p_0'}|^{p_1}v(y)^{p_1}\,dy\right)h(x)^{-p_1/p_1'}u(x)^{-p_0'}\,dx\right)^{1/p_1}\\
&\geq \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|f(x,y)u(x)^{p_0'}|^{p_1}v(y)^{p_1}\,dy\right)H(x)^{-p_1/p_1'}u(x)^{-p_0'}\,dx\right)^{1/p_1}\\
&= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|f(x,y)|^{p_1}(v(y)H(x)^{-1/p_1'}u(x)^{p_0'/p_1'})^{p_1}\,dx\right)^{1/p_1}\\
&\geq C\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}|Tf(x,y)|^{q_1}(v(y)H(x)^{-1/p_1'}u(x)^{p_0'/p_1'})^{q_1}\,dy\,dx\right)^{1/q_1}\\
&\geq C\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|Tf(x,y)|^{q_1}v(y)^{q_1}\,dy\right)^{q_0/q_1}u(x)^{q_0}\,dx\right)^{1/q_0}\\
&= C\|Tf\|_{L^{q_0}L^{q_1}(w^{q_1})}.
\end{aligned}$$

□

Corollary 5.3.3. *Suppose* $0 < \alpha < n+m$, $1/p_0 - 1/q_0 = \alpha/(n+m)$, $1/p_1 - 1/q_1 = \alpha/(n+m)$, *and* $w(x,y) = u(x)v(y)$ *is in* $A_{p_0,q_0}A_{p_1,q_1}$. *Then*

$$I_\alpha : L_x^{p_0} L_y^{p_1}(w) \to L_x^{q_0} L_y^{q_1}(w).$$

Remark 5.3.4. We note that for the unweighted case, $w \equiv 1$, we have a wider range of exponents (5.6) and (5.7). Corollary 5.3.3 corresponds to the case $\alpha_0 = n\alpha/(n+m)$ and $\alpha_1 = m\alpha/(n+m)$.

Chapter 6

Conclusions

The main theme of this dissertation is the study of weighted inequalities for fractional operators. We have presented many new results pertaining to fractional operators. These results include: sharp bounds, general basis maximal functions, multilinear weighted theory, and extrapolation on mixed norm spaces.

The first chapter is an introduction to Lebesgue spaces, classical operators in harmonic analysis, and weighted inequalities. We provide the basic facts about Lebesgue spaces (without proof) that will be used in the later chapters. We also introduced the main operators and weighted inequalities for these operators.

Chapter Two is devoted to finding the sharp weighted bound for the fractional integral operator. Theorem 2.3.2 does not contain the full range of exponents p and q. A different approach to this problem is to consider the deep two-weight theory of fractional integrals by Sawyer [48]. The idea is to find the sharp two-weight bound on the operator norm of I_α and then show that the two-weight bound gives the correct power of the one-weight constant. At the time this dissertation was finished we started this different approach with Lacey, Pérez, and Torres [30].

In Chapter Three we examine maximal operators with respect to a general basis. We characterized the one- and two-weight inequalities for these operators. Our methods lead to sharp bounds in the case of a basis of cubes. Using similar techniques we are also able to find a sharp bound on the dyadic weighted maximal operator in terms of the Reverse Hölder constant of the weight.

Chapter Four contains a weighted theory for multilinear fractional operators. We invoke the theory of Banach function spaces to obtain general sufficient two-weight conditions for the boundedness of these operators. When there is no fractional part we have new two-weight results for the multilinear maximal operator considered by Lerner, Ombrosi, Pérez, Torres, and Trujillo-Gonzalez [33]. As a consequence of the two-weight we develop the one-weight theory for the multilinear fractional operators, completely characterizing there boundedness. As an application we obtain Sobolev inequalities for products of functions. A possible extension of this work is to generalize these results to spaces of homogeneous type and even non-homogeneous type. Another continuation of this multilinear weighted theory is to find the sharp constants for these operators. As in the linear theory [4], a good starting point would be to find the sharp bound on $\|\mathscr{M}\|$ in terms of $[\vec{w}]_{A_{\vec{p}}}$.

Chapter Five deals with operators on mixed norm Lebesgue spaces. We extend the weak endpoint result for convolution Calderón-Zygmund operators on mixed Lebesgue spaces to the non-convolution case. We also introduce a new class of weights and provide an off-diagonal mixed Lebesgue space extrapolation theorem. The extrapolation theorem provides weighted inequalities for the fractional integral operator on mixed Lebesgue spaces.

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