COARSER CONNECTED TOPOLOGIES AND NON-NORMALITY POINTS

By

Lynne Yengulalp

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William Fleissner, Chairperson

Committee members

Arvin Agah

Jack Porter

Judith Roitman

Rodolfo Torres

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The Dissertation Committee for Lynne Yengulalp certifies
that this is the approved version of the following dissertation:

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Contents

Abstract **6**

Acknowledgments **7**

1 Background and preliminaries **8**

1.1 Outline 8

1.2 Topological terms 9

1.3 Filters 10

1.4 GCH and regular ultrafilters 12

2 Coarser connected topologies **14**

2.1 Introduction 14

2.2 Tools for defining coarser topologies 16

2.3 Coarser connected metric topology 19

3 Non-normality points **32**

3.1 Introduction 32

3.2 Discrete Spaces 34

3.3 Metric Spaces 36

4 Open questions **47**

4.1 Coarser connected topologies 47

4.2 Non-normality points 48

Bibliography **50**

Abstract

We investigate two topics, coarser connected topologies and non-normality points.

The motivating question in the first topic is:

Question 0.0.1. When does a space have a coarser connected topology with a nice topological property?

We will discuss some results when the property is Hausdorff and prove that if X is a non-compact metric space that has weight at least $\mathfrak{c}$, then it has a coarser connected metrizable topology.

The second topic is concerned with the following question:

Question 0.0.2. When is a point $y \in \beta X \setminus X$ a non-normality point of $\beta X \setminus X$?

We will discuss the question in the case that X is a discrete space and then when X is a metric space without isolated points. We show that under certain set-theoretic conditions, if X is a locally compact metric space without isolated points then every $y \in \beta X \setminus X$ a non-normality point of $\beta X \setminus X$.

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Chapter 1

Background and preliminaries

1.1 Outline

This chapter contains a development of the basic notions needed for the two topics of this dissertation. In Section 1.3, we introduce set ultrafilters, open ultrafilters and z-ultrafilters and discuss related terms. As with many results in general topology, the theorems in Chapter 3 contain set-theoretic assumptions. We explain the set-theoretic statements in Section 1.4 and talk about the consequences of including such assumptions in the hypotheses of theorems.

We start Chapter 2 with some history on the topic of coarser connected topologies (Section 2.1) and some examples illustrating the techniques for defining coarser connected topologies (Section 2.2). In Section 2.3 we present the main theorem, that any non-compact metric space with weight $> \mathfrak{c}$ has a coarser connected metrizable topology (Theorem 2.3.8).

We define non-normality points and butterfly points in Section 3.1 and present some background on the study of such points. We discuss the study of non-normality points in the Stone-Cech compactification of discrete spaces (Section 3.2) and then look at non-normality points in the Stone-Cech compactification of metric spaces (Section 3.3).

1.2 Topological terms

Nonstandard terms will be defined as needed. Other terms and notation will be consistent with Engelking [5] and Jech [12].

For set inclusion we write $\subset$ for $\subseteq$. We designate the Greek letters τ and σ for topologies and μ, ν and ρ for metrics. We write (X, τ) for a topological space X with topology τ. If X is a metric space and the topology τ is generated by the metric μ, we write (X, τ, μ). If (X, τ) is a space and $Y \subset X$ then we write $\tau|_Y$ for $\{U \cap Y : U \in \tau\}$, the topology on Y as a subspace of X. Let σ and τ be topologies on a space X. We say that σ is **coarser** than τ and that τ is **finer** than σ if $\sigma \subset \tau$.

The following notions will be used in Chapter 2.

Definition 1.2.1. A space (X, τ) is **minimal Hausdorff** if there is no Hausdorff topology, σ on X coarser than τ.

Proposition 1.2.2. *If a Hausdorff space X is compact then it is minimal Hausdorff.*

The following are definitions of cardinal function for a space (X, τ): **density**, $d(X)$, **extent**, $e(X)$ and **weight**, $w(X)$.

$d(X) = \inf\{|D| : D \text{ is dense in X}\}$

$e(X) = \sup\{|C| : C \text{ is closed discrete in X}\}$

$w(X) = \inf\{|\mathscr{B}| : \mathscr{B} \text{ is a base for } \tau\}$

Proposition 1.2.3. *If X is a metric space then $d(X) = e(X) = w(X)$.*

We now introduce some notation and a technical lemma that will be useful for Chapter 3.

Definition 1.2.4. Suppose that Y is a subspace of X. We say that Y is C^***-embedded** in X if every bounded continuous real valued function on Y can be extended to a continuous function on X.

Definition 1.2.5. A discrete subset D of a space X is called **strongly discrete** if there is a pairwise disjoint collection of open subsets of X separating the points of D.

For a collection $\mathscr{U}$ of subsets of a space X we write $\mathscr{U}^* = \bigcup \mathscr{U}$. We say a collection of subsets, $\mathscr{V}$, **densely refines** a collection $\mathscr{U}$ if $\mathrm{cl}_X(\mathscr{V}^*) = \mathrm{cl}_X(\mathscr{U}^*)$ and for all $V \in \mathscr{V}$ there is $U \in \mathscr{U}$ such that $V \subset U$.

Lemma 1.2.6. *Let $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$, and let $f : X \to Y$ be a closed map. If $f^{\leftarrow}[f[X_1]] = X_1$, (we say X_1 is a **full preimage**) then $f|_{X_1}$ is a closed map.*

Proof. Let $H \subset X_1$ be closed in X_1. There exists $H' \subset X$ closed in X such that $H = H' \cap X_1$. Since f is closed, $f[H']$ is a closed subset of Y. So, to show that $f[H]$ is closed in $f[X_1]$ we argue that $f[H] = f[H'] \cap f[X_1]$. First, $f[H] = f[H' \cap X_1] \subset f[H'] \cap f[X_1]$. For the other direction, let $y \in f[H'] \cap f[X_1]$. Since $y \in f[H']$, there is $x \in H'$ such that $f(x) = y$. Since $y \in f[X_1]$ and $f^{\leftarrow}[f[X_1]] = X_1$, $f^{-1}(y) \subset X_1$. Hence $x \in X_1 \cap H' = H$ and therefore $y \in f[H]$.

1.3 Filters

We introduce some basic concepts associated with filters to be used when discussing neighborhood bases in Chapter 2 and points in the Stone Cech compactification in Chapter 3.

Definition 1.3.1. An **filter** $\mathscr{U}$ on a set A is a collection of subsets of A that satisfies

1. $\emptyset \notin \mathscr{U}$
2. $U \in \mathscr{U}$ and $U \subset V$ then $V \in \mathscr{U}$
3. if $U, V \in \mathscr{U}$ then there is $W \in \mathscr{U}$ such that $W \subset U \cap V$

A filter $\mathscr{U}$ on A is an **ultrafilter** if it is maximal. In other words, for every filter $\mathscr{V}$ on A such that $\mathscr{U} \subset \mathscr{V}$ it must be that $\mathscr{U} = \mathscr{V}$. Equivalently, a filter $\mathscr{U}$ on A is an ultrafilter if for all $U \subset A$ either $U \in \mathscr{U}$ or $A \setminus U \in \mathscr{U}$.

If the set A has no topology, we refer to an ultrafilter on A as a set-ultrafilter. Suppose that the set A does have a topology, τ. A **zero-set (z-set)** is a set $Z \subset A$ for which there exists a continuous function $f : A \to \mathbb{R}$ such that $Z = f^{\leftarrow}(0) = \{a \in A : f(a) = 0\}$. Z-sets are closed and in a metric space, all closed sets are z-sets. Let $\mathscr{Z}$ be the collection of zero-sets in A.

Definition 1.3.2. An **open filter (z-filter)** $\mathscr{U}$ on a set A is a subset of τ (a subset of $\mathscr{Z}$) that satisfies

1. $\emptyset \notin \mathscr{U}$
2. if $V \in \tau$ (if $V \in \mathscr{Z}$), $U \in \mathscr{U}$ and $U \subset V$ then $V \in \mathscr{U}$
3. if $U, V \in \mathscr{U}$ then there is $W \in \mathscr{U}$ such that $W \subset U \cap V$

An open filter (z-filter) $\mathscr{U}$ on A is an **open ultrafilter (z-ultrafilter)** if for every open filter (z-filter) $\mathscr{V}$ on A such that $\mathscr{U} \subset \mathscr{V}$ it must be that $\mathscr{U} = \mathscr{V}$. Equivalently, $U \in \tau$ (if $U \in \mathscr{Z}$) then either $U \in \mathscr{U}$ or there is $V \in \tau \cap \mathscr{U}$ ($V \in \mathscr{Z} \cap \mathscr{U}$) such that $V \subset A \setminus U$.

The collection $\mathscr{N}$ of open neighborhoods of a point x in a topological space X is an open filter. However, $\mathscr{N}$ is usually not an open ultrafilter. For example in $\mathbb{R}$, the open set $(0,1)$ is not a neighborhood of 0 and there is no open neighborhood of zero disjoint from $(0,1)$.

A filter $\mathscr{U}$ is called **fixed** if the set of convergence points, $\bigcap\{\mathrm{cl}\, U : U \in \mathscr{U}\}$, is not empty. A filter $\mathscr{U}$ is called **free** if $\bigcap\{\mathrm{cl}\, U : U \in \mathscr{U}\}$ is empty.

The collection of set-ultrafilters on a topological space gives no information (other than cardinality) about the structure of the space. On the other hand, the z-sets and z-ultrafilters can give information about the topological structure of a space X. For example, one of the several constructions of the Stone Cech compactification, βX, is developed using z-ultrafilters [[10], Ch. 6].

$\square$

1.4 GCH and regular ultrafilters

The hypotheses of Theorem 3.3.4 and Corollary 3.3.6 contain set-theoretic assumptions that we discuss here. The generalized continuum hypothesis (GCH) is the statement:

$$\text{For all cardinals } \kappa, \quad 2^\kappa = \kappa^+$$

GCH is a generalization of Cantor's continuum hypothesis (CH) which states that there is no set whose cardinality lies strictly between that of the natural numbers and that of the real numbers. Symbolically, CH is the statement:

$$2^{\aleph_0} = \aleph_1$$

Gödel showed that GCH is consistent with the axioms of Zermelo and Frankel (ZF) which, together with the axiom of choice (AC), form the foundation for most of mathematics. In other words, it is safe to assume GCH is true. Topologists often aim to prove topological statements using only ZF and AC (abbreviated ZFC). A common first step towards proving a theorem in ZFC is to assume an extra consistent axiom (such as CH or GCH) to prove the statement. There is, however, no guarantee that the theorem one is trying to prove is not itself independent of ZFC. In this case, one may investigate the truth of the statement under different set-theoretic assumptions.

In Chapter 3, we will define **regular ultrafilter**, which is a special type of set-ultrafilter. Donder [3] showed that the assumption that all set-ultrafilters are regular is consistent with ZFC. In particular, assuming $V = L$, all set-ultrafilters are regular. To understand the strength of assuming all set-ultrafilters are regular, it is helpful to note that the existence of a non-regular set-ultrafilter is actually a hidden large cardinal axiom. A cardinal κ is measurable iff there is a uniform κ-complete ultafilter q on κ. As we will see later, κ-complete ultrafilters are non-regular. So, the assumption that all ultrafilters are regular implies there are no measurable cardinals.

The existence of measurable cardinals makes the statement of some theorems in general topology more complicated. For example, below are two ways of expressing a theorem of Mrowka.

Theorem 1.4.1. *(No measurable cardinals) Every regular paracompact space is realcompact.*

Theorem 1.4.2. *(ZFC) Every regular, paracompact space whose cardinality is less than the least measurable cardinal is realcompact.*

Chapter 2

Coarser connected topologies

2.1 Introduction

The general goal in this area is to find for a topological space (X,τ), a coarser topology $\sigma \subset \tau$ such that (X,σ) is connected. The coarsest topology on a set X is the indiscrete topology: $\{X,\emptyset\}$, which is always connected. However, the indiscrete topology is not even Hausdorff if X has more than one point. Therefore finding a coarser connected topology is only interesting when (X,σ) is required to have other nice properties; for example Hausdorff, regular, collectionwise normal, metrizable. Certainly, if (X,τ) is connected then it has a coarser connected topology, namely τ itself. Since compact spaces are minimal Hausdorff, only non-compact disconnected spaces can have strictly coarser connected Hausdorff topologies.

The study of coarser connected topologies was started by Tkacenko, Tkachuk, and Uspenskij in [21]. They developed some necessary and sufficient conditions for a topological space to have coarser connected Hausdorff or regular topologies. Continuing the develpment, Gruenhage, Tkachuk, Wilson showed that a metric space, X, has a coarser connected Hausdorff topology if and only if X is not compact [[11], Theorem 2.8].

We mentioned before that the collection of neighborhoods of a point x in a space X is a fixed open filter. Often, to define a topology on a space X, one describes the open neighborhood filter of each point $x \in X$. Likewise, to define a coarser topology, one can define coarser neighborhood filters of points of X. If there is a free open filter $\mathscr{U}$ on the space X, it can be used to define a coarser neighborhood filter of a point, $x \in X$: let $\mathscr{N}' = \{U \cup N : U \in \mathscr{U}, N \in \mathscr{N}\}$ where $\mathscr{N}$ is the collection of neighborhoods of x. Notice $\mathscr{N}' \subsetneq \mathscr{N}$. Informally, in the new topology, the point x is closer to the sets $U \in \mathscr{U}$.

For example in the disconnected subset of the real line $X = (0,1) \cup [2,3]$, the collection $\{(r,1) : r \in [0,1)\}$ is free open filter. Define a new topology for X by defining a coarser neighborhood base of the point 2 that consist of sets of the form $(r,1) \cup [2,2+r)$. This coarser topology makes X homeomorphic to a single interval, and is hence Hausdorff and connected.

When a space has a large enough closed discrete set, there is a large number of free open filters. Therefore, one can define coarser neighborhood bases of many points in the space. For example, from [[7],Theorem 2]

Proposition 2.1.1. *Let K be Hausdorff and $X = K \oplus D$ where D is discrete. If $w(K) \leq 2^{|D|}$ then X has a coarser connected Hausdorff topology.*

If a space, X, has a closed discrete set of size $e(X)$ we say the extent of X is attained. A space with extent attained has a large closed discrete set and in some cases this is enough to define a coarser connected topology. For example, Druzhinina, [[4], Theorem 3.3] and Fleissner, Porter, Roitman [[8], Theorem 2.5] showed that a metric space, X, with $e(X)$ attained and $w(X) \geq \mathfrak{c}$ has a coarser connected metrizable topology.

When a space does not have attained extent, it may be more difficult to define a coarser connected topology. However, Fleissner, Porter and Roitman showed that any zero-dimensional metric space, X, with weight greater than or equal to $\mathfrak{c}$ has a coarser connected metrizable topology if and only if it is not compact [[8], Theorem 3.4]. On the other hand, if $w(X) < \mathfrak{c}$, X does not necessarily have a coarser connected topology. For example, the disjoint union of countably many Cantor sets is a separable, metrizable, disconnected space with no coarser connected regular topology [[21], Example 2.18].

2.2 Tools for defining coarser topologies

Given an appropriate connected space (Y, υ) and a set bijection, ϕ, from Y onto a subset of X, one can attempt to create a topology σ on X that make the maps id and ϕ continuous.

$$\begin{tikzcd}
(X,\tau) \arrow[dr, "\mathrm{id}"'] & (Y,\upsilon) \arrow[d, "\phi"] \\
& (X,\sigma)
\end{tikzcd}$$

If the identity map from (X, τ) to (X, σ) is continuous, then σ is coarser than τ. If ϕ is continuous, then $\phi[Y]$ is connected. So, if $\phi[Y]$ is either dense in (X, τ) or intersects each component of (X, τ), then (X, σ) is connected. Therefore, the task of defining a coarser connected topology for a space (X, τ) becomes a search for suitable connected space (Y, υ) and a map ϕ. Of course, both must be selected carefully to ensure that σ has a nice property like Hausdorff or metrizability. The hedgehog space and Bing's Tripod space are useful connected spaces for this purpose.

Example 2.2.1 ([5], pg. 381)**.** The point set of **Bing's space** is

$$\mathbb{Q} \cup \{q + r\sqrt{2} : q \in \mathbb{Q}, r \in \mathbb{Q}^+\}$$

For $q, r, \varepsilon \in \mathbb{R}$, let $I(q,r,\varepsilon) = (q + r\sqrt{2} - \varepsilon, q + r\sqrt{2} + \varepsilon) \cap \mathbb{Q}$. An open neighborhood of $q \in \mathbb{Q}$, is $(q+\varepsilon, q-\varepsilon) \cap \mathbb{Q}$ and an open neighborhood of $q + r\sqrt{2}$ is $\{q + r\sqrt{2}\} \cup I(q,r,\varepsilon) \cup I(q,-r,\varepsilon)$.

Bing's space is a countable connected Hausdorff space. The natural numbers $\mathbb{N} \subset \mathbb{Q}$ are a countable strongly discrete subset of $\mathbb{B}$.

Example 2.2.2 ([5], pg. 314)**.** Let κ be a cardinal number. The point set of **hedgehog space of spininess** κ is the set, Z_κ, defined by identifying all points $(0,\alpha)$ in $[0,1] \times \kappa$. The metric μ on Z is defined

$$\mu((x,\alpha),(y,\beta)) = \begin{cases} |x-y| & \text{if } \alpha = \beta \\ |x| + |y| & \text{if } \alpha \neq \beta \end{cases}$$

The space Z_κ is a connected metric space with density $\max\{\omega, \kappa\}$. For $\kappa \geq \omega$, the extent of Z_κ is attained by the closed discrete set $\{(1,\alpha) : \alpha \in \kappa\}$.

The following proposition gives two examples of defining a coarser connected topology via a bijection from Bing's space.

Proposition 2.2.3. *The following disconnected subsets of the real line have coarser connected Hausdorff topologies.*

1. $X = \bigcup\{[2n, 2n+1] : n \in \omega\}$
2. $X = \bigcup\{\mathbb{C}_n : n \in \omega\}$ *where* $\mathbb{C}_n$ *is a Cantor set in* $[2n, 2n+1]$ *such that* $2n \in \mathbb{C}_n$.

Proof. Let $\mathbb{B}$ be Bing's space and let $D = \{2n : n \in \omega\} \subset X$. In each case, X has the topology generated by the Euclidean metric, d with ε balls, $B_\varepsilon(x)$. For $x \neq y \in X$, let $\varepsilon(x,y) = \min(\{d(x,D), d(y,D), d(x,y)\} \setminus \{0\})/2$.

1. Let ϕ be a bijection from $\mathbb{B}$ to D and let σ be the topology coarser than the Euclidean topology on X that makes ϕ continuous. (X,σ) is connected since

$\phi[\mathbb{B}]$ intersects every component of X. We can also show that σ is Hausdorff. Let $x \neq y \in X$ and let $\varepsilon = \varepsilon(x,y)$. Let U_x and U_y be separating open neighborhoods of $\phi^{-1}(x)$ and $\phi^{-1}(y)$ in $\mathbb{B}$. For $p = x,y$, if $\phi^{-1}(p)$ is not defined, let $U_p = \emptyset$. Then $U = B_\varepsilon(x) \cup \bigcup\{B_\varepsilon(z) : \phi^{-1}(z) \in U_x\}$ and $V = B_\varepsilon(y) \cup \bigcup\{B_\varepsilon(z) : \phi^{-1}(z) \in U_y\}$ are σ-open sets separating x and y.

2. Let C be a countable strongly discrete subset of $\mathbb{B}$ and let ϕ be a bijection that takes $\mathbb{B} \setminus C$ to D and C to a countable dense subset of $X \setminus D$. Let $\mathscr{V} = \{V_c : c \in C\}$ be a pairwise disjoint open collection separating C. Let $x \neq y \in X$ and let $\varepsilon = \varepsilon(x,y)$. Let U_x and U_y be as in 1.with the extra condition that $|\{c \in C : c \in U_p\}| \leq 1$. That is, $U_p \cap C = \emptyset$ if $p \notin C$, otherwise $|U_p \cap C| = 1$. For $p = x,y$ we inductively define a σ neighborhood of p.

$$W_{p,0} = B_\varepsilon(p) \cup \bigcup\{B_\varepsilon(z) : \phi^{-1}(z) \in U_p\}$$

$$W_{p,n} = \bigcup\{B_\varepsilon : \phi^{-1}(z) \in U_c, c \in \phi^{\leftarrow}[B_\varepsilon(z')], z' \in W_{p,n-1} \cap D\}$$

Let $U = \bigcup\{W_{x,n} : n \in \omega\}$ and $V = \bigcup\{W_{y,n} : n \in \omega\}$. Then, U and V are σ-open neighborhoods separating x and y in X.

□

The strongly discrete subsets of D of X and C of $\mathbb{B}$ play an important role in the defining of coarser neighborhoods of points x in X. New neighborhoods are a union of intervals around points of a filter set on D. The filter is determined by a neighborhood of $\phi^{-1}(x)$ in $\mathbb{B}$. In the second example of Proposition 2.2.3, more and more points of $X \setminus D$ get picked up in the induction, but Hausdorff is maintained since these points are associated with the discrete set C, subsets of which are separated by open sets in $\mathbb{B}$.

Note, though, that the coarser topologies above are not metrizable. As mentioned before, $X = \bigcup\{\mathbb{C}_n : n \in \omega\}$ has no coarser connected metrizable topology.

2.3 Coarser connected metric topology

This bulk of this section comes from [23], submitted.

Lemma 2.3.1. *Suppose that Y is a subset of a metric space (X,τ,μ), $e(\mathrm{cl}\,Y) > \mathfrak{c}$ is attained by $C \subset \mathrm{int}(Y)$ and $\mathrm{diam}_\mu(\mathrm{cl}\,\mathrm{Y}) = \varepsilon < 1/2$. Then there is a coarser topology τ' on X with corresponding metric μ' such that*

i) $\tau'\big|_{\mathrm{cl}\,Y}$ *is connected,*

ii) $\mu\big|_{(X\setminus Y)^2} = \mu'\big|_{(X\setminus Y)^2}$ *and*

iii) $\mu' \leq \mu + 2\varepsilon$.

Proof. Let $e(\mathrm{cl}\,Y) = \kappa$. So $|C| = \kappa$, C is closed discrete in $\mathrm{cl}_\tau Y$ and therefore is closed discrete in X. By replacing C with a subset, we may assume that $|Y \setminus C| = \kappa$. Let $\mathscr{U} = \{U_c \in \tau : c \in C\}$ be a discrete collection such that $c \in U_c \subset cl(U_c) \subset Y$. For each $c \in C$ define a continuous function $f_c : X \to [0,\varepsilon]$ such that $f_c(c) = \{\varepsilon\}$ and $f_c[X \setminus U_c] = \{0\}$. For each $x,y \in X$ define $\mu^*(x,y) = \Sigma_{c\in C}|f_c(x) - f_c(y)| + \mu(x,y)$. It is easy to check that since C is closed discrete in X, μ^* generates τ, $\mu^*\big|_{(X\setminus Y)^2} = \mu\big|_{(X\setminus Y)^2}$ and $\mu^* \leq \mu + 2\varepsilon$.

Let (Z,ρ) be a hedgehog space with spininess κ, let $T = \{(\alpha,1) : \alpha \in \kappa\}$ and let $S = Z \setminus T$. Let $D \subset Y \setminus C$ be a dense subset of $Y \setminus C$ of size $d(Y) = e(Y) = \kappa$. Define a one-to-one map $f : Z \to \mathrm{cl}\,Y$ such that $f[T] = D$ and $f[S] = C$. For $x,y \in \mathrm{im}(f)$, let $\lambda(x,y) = \min\{\mu^*(x,y), \rho(f^{-1}(x), f^{-1}(y))\}$. For all other $x,y \in X$, let $\lambda(x,y) = \mu^*(x,y)$.

Define a metric μ' on X as follows:

$$\mu'(x,y) = \inf\{\lambda(x,x_1) + \lambda(x_1,x_2) + \cdots + \lambda(x_{n-1},x_{n-2}) + \lambda(x_n,y)\}$$

where $x_1,\ldots,x_n$ ranges over all finite sequences (including the empty sequence) of distinct elements of X. Since ρ and μ^* satisfy the triangle inequality, in defining μ' it suffices to consider sums of the form

$$\begin{aligned}\mu^*(x,x_1) + \rho(f^{-1}(x_1), f^{-1}(x_2)) + \cdots + \mu^*(x_{n-1},x_{n-2}) \\ + \rho(f^{-1}(x_n), f^{-1}(y))\end{aligned} \tag{2.1}$$

where the sum may start or end with either a ρ term or μ^*term and the terms of the sum alternate between ρ and μ^*. Note also that since $x_1,\ldots x_n$ are in the image of f, $x_1,\ldots x_n \in D \cup C$.

Since μ' is an infimum over all finite sequences in X, it satisfies the triangle inequality. That $\mu'(x,y) = \mu'(y,x)$ and $\mu'(x,x) = 0$ for all $x,y \in X$ is clear. It remains to show that $\mu'(x,y) = 0$ implies that $x = y$.

Claim (1)**.** Suppose that for a particular sum in the form of (2.1), $\lambda(x_i,x_{i+1}) < \varepsilon$ for each $i \in \{m, m+1, \ldots, n-1\}$. Then, if $\lambda(x_m,x_{m+1}) = \rho(f^{-1}(x_m), f^{-1}(x_{m+1}))$ and $x_m \in D$, then $x_{m+i} \in C$ for all i odd and $x_{m+i} \in D$ for all i even in $\{0,1,\ldots,n-m\}$.

Proof. Suppose that $\lambda(x_i,x_{i+1}) < \varepsilon$ for each $i \in \{m,m+1,\ldots,n-1\}$, $\lambda(x_m,x_{m+1}) = \rho(f^{-1}(x_m), f^{-1}(x_{m+1}))$ and $x_m \in D$. Fix i even such that $0 \leq i < n-m$, and suppose that $x_{m+i} \in D$. Since $x_{m+i} \in D$, $f^{-1}(x_{m+i}) \in T$. But since the ρ and μ terms of the sum alternate, $\lambda(x_{m+i},x_{m+i+1}) = \rho(f^{-1}(x_{m+i}), f^{-1}(x_{m+i+1})) < \varepsilon < 2$ and hence $f^{-1}(x_{m+i+1}) \in S$. By the definition of f, $x_{m+i+1} \in C$.

Now, since $\lambda(x_{m+i+1}, x_{m+i+2}) = \mu^*(x_{m+i+1}, x_{m+i+2}) < \varepsilon$ and $x_{m+i+1} \in C$, it must be that $x_{m+i+2} \in D$. So, since $x_m \in D$, by induction, $x_{m+i} \in C$ for all i odd and $x_{m+i} \in D$ for all i even in $\{0, 1, \ldots, n-m\}$. $\square$

Claim. $\mu'(x,y) = 0$ implies that $x = y$

Proof. Suppose there were $x, y \in X$ such that $\mu'(x,y) = 0$ but $x \neq y$. If defined, let $x' = f^{-1}(x)$ and $y' = f^{-1}(y)$. Let $x_1, x_2, \ldots x_n \in D \cup C$ be a sequence that yeilds an alternating μ^*, ρ sum between x and y that is less than

$$\delta = \min\{\varepsilon, \mu^*(x, C \setminus \{x\}), \mu^*(y, C \setminus \{y\}), \rho(x', T \setminus \{x'\}), \rho(y', T \setminus \{y'\})\}.$$

Since C is closed discrete in X and T is closed discrete in Y, δ is a postive real number.

Case (1). The alternating sum begins and ends with μ^* terms.

If $x_1 \in C$ then $\mu^*(x, x_1) \geq \mu^*(x, C \setminus \{x\}) \geq \delta$, which is a contradiction. Note that n is even. If $x_1 \in D$, then since $\lambda(x_i, x_{i+1}) < \varepsilon$ for each $1 \leq i < n$ and $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$, by Claim (1), $x_n \in C$. Hence $\mu^*(x_n, y) \geq \mu^*(y, C \setminus \{y\}) \geq \delta$, which is a contradiction.

Case (2). The sum begins with a μ^* term and ends with a ρ term. (or begins with ρ, ends with μ^*).

If $x_1 \in C$ then $\mu^*(x, x_1) \geq \mu^*(x, C \setminus \{x\}) \geq \delta$, which is a contradiction. Note that n is odd. If $x_1 \in D$, then since $\lambda(x_i, x_{i+1}) < \varepsilon$ for each $1 \leq i < n$ and $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$, by Claim (1), $x_n \in D$. Then $\rho(f^{-1}(x_n), y') \geq \rho(y', T \setminus \{y'\})$, which is a contradiction.

Case (3). The sum begins and ends with ρ terms.

Suppose $x_1 \in D$. Then $\rho(x', x_1) \geq \rho(x', T \setminus \{x'\}) \geq \delta$, which is a contradiction.

Suppose $x_1 \in C$. Since $\lambda(x_1,x_2) = \mu^*(x_1,x_2) < \varepsilon$, $x_2 \in D$. Note that n is even. Then since $\lambda(x_i,x_{i+1}) < \varepsilon$ for each $2 \le i < n$ and $\lambda(x_2,x_3) = \rho(f^{-1}(x_2), f^{-1}(x_3))$, by Claim (1), $x_n \in D$. Then $\rho(f^{-1}(x_n), y') \ge \rho(y', T \setminus \{y'\}) \ge \delta$, which is a contradiction. $\square$

So, μ' defines a metric on X. Let τ' be the topology on X generated by μ'. Since μ^* generates τ and $\mu' \le \mu^*$, τ' is coarser than τ. In order to show that $\tau'\big|_{\mathrm{cl}\,Y}$ is connected, we argue that $(D \cup C, \tau'|_{D\cup C})$ is continuous image of the connected space Z. Then, since $\mathrm{cl}_{\tau'} Y = \mathrm{cl}_\tau Y = \mathrm{cl}_\tau(D \cup C)$, we will have that $(\mathrm{cl}\,Y, \tau'\big|_{\mathrm{cl}\,Y})$ is connected. To do this, we show that μ' makes the map f continuous.

Claim. The map $f : (Z,\rho) \to (Y,\mu')$ is continuous.

Proof. Let $x \in Y$, $\delta > 0$ and let $U = B_{\mu'}(x,\delta)$ be the μ'-ball of radius δ about x. Suppose $z \in f^{\leftarrow}[U]$. Then $\mu'(f(z),x) = \delta' < \delta$. Let $\xi = \frac{\delta-\delta'}{3}$. Suppose $z' \in Z$ such that $\rho(z,z') < \xi$. We wish to show that $\mu'(f(z'),x) < \delta$ so that $B_\rho(z,\xi) \subset f^{\leftarrow}[U]$. Since $\mu'(x,f(z)) = \delta'$, there is a sequence $x_1,x_2,\ldots,x_n \in Y$ such that

$$\lambda(x,x_1) + \lambda(x_1,x_2) + \cdots + \lambda(x_n, f(z)) < \delta' + \xi$$

Adding the term $\rho(z,z') = \rho(f^{-1}(f(z)), f^{-1}(f(z')))$ to this sum

$$\lambda(x,x_1) + \lambda(x_1,x_2) + \cdots + \lambda(x_n, f(z)) + \rho(z,z') < \delta' + 2\xi < \delta$$

illustrates that the sequence $x_1,x_2,\ldots,x_n, f(z)$ yields a sum between x and $f(z')$ that is less than δ. Hence, $\mu'(x, f(z')) < \delta$ as desired. So, the map f is continuous and the claim is proven. $\square$

Therefore, $\tau'\big|_{\mathrm{cl}\,Y}$ is connected. Since $\mu' \le \mu^*$ and $\mu^* \le \mu + 2\varepsilon$, we have that $\mu' \le \mu + 2\varepsilon$.

We now show that $\mu\big|_{(X\setminus Y)^2} = \mu'\big|_{(X\setminus Y)^2}$. Let $x, y \notin Y$. We verify that $\mu'(x,y) = \mu(x,y)$. By definition, $\mu'(x,y) \leq \mu^*(x,y) = \mu(x,y)$ so we need only show that $\mu'(x,y) \geq \mu^*(x,y)$. Suppose for a contradiction that $\mu'(x,y) < \mu^*(x,y)$. In other words, there exist $x_1, x_2, \ldots, x_n \in D \cup C$ such that

$$\mu^*(x,x_1) + \rho(f^{-1}(x_1), f^{-1}(x_2)) + \cdots + \mu^*(x_n,y) < \mu^*(x,y)$$

Since $x, y \notin Y$, $f^{-1}(x)$ and $f^{-1}(y)$ are not defined. Hence the sum above must start and end with μ^* terms implying n is even. Also, since $x, y \notin Y$, $\mu^*(x,y) = \mu(x,y)$. Now, since $\mathrm{diam}_\mu(Y) = \varepsilon$, for any $z_1, z_2 \in Y$,

$$\mu(x,y) \leq \mu(x,z_1) + \varepsilon + \mu(z_2,y)$$

Hence,

$$\mu(x,y) \leq \mu(x,Y) + \varepsilon + \mu(y,Y)$$

Combining these inequalities we have

$$\begin{aligned} \mu^*(x,x_1) + \rho(f^{-1}(x_1), f^{-1}(x_2)) + \cdots + \mu^*(x_n,y) < \\ \mu(x,Y) + \varepsilon + \mu(y,Y) \end{aligned} \tag{2.2}$$

Now, since $x, y \notin Y$ and $x_1, x_n \in C \cup D \subset Y$, $\mu^*(x,x_1) \geq \mu(x,x_1) \geq \mu(x,Y)$ and $\mu^*(x_n,y) \geq \mu(x_n,y) \geq \mu(y,Y)$. Combining this with (2.2) gives:

$$\rho(f^{-1}(x_1), f^{-1}(x_2)) + \mu^*(x_2,x_3) + \cdots + \rho(f^{-1}(x_{n-2}), f^{-1}(x_{n-1})) < \varepsilon$$

$$\mu^*(x,x_1) < \mu(x,Y) + \varepsilon \tag{2.3}$$

and

$$\mu^*(x_n, y) < \mu(y, Y) + \varepsilon \tag{2.4}$$

Suppose $x_1 \in C$, then $\mu^*(x, x_1) = \mu(x, x_1) + \varepsilon \geq \mu(x, Y) + \varepsilon$ contradicting (2.3). So, $x_1 \notin C$. Similarly, $x_n \in C$ contradicts (2.4). So, $x_n \notin C$. So, we have shown that $x_1, x_n \in D$. Now, $\lambda(x_i, x_i + 1) < \varepsilon$ for each $1 \leq i < n$ and $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$. So, since $x_1 \in D$ and n is even, by Claim 1, $x_n \in C$, which is a contradiction. So,
$\mu'(x, y) = \mu^*(x, y) = \mu(x, y)$, and since $x, y \notin Y$, we have shown $\mu\big|_{(X \setminus Y)^2} = \mu'\big|_{(X \setminus Y)^2}$.
$\square$

The following lemma contains Lemma 1 from [6] and Theorem 3.2 from [8].

Lemma 2.3.2. *Let (X, τ) be a metric space with metric μ in which $e(X) = \kappa$ is not attained. Let K be the set of points x of X such that every neighborhood of x has extent κ. Then*

(1) κ is a singular cardinal of cofinality ω.

(2) K is a compact, nowhere dense subset of X.

(3) If U is an open subset of X such that $\mathrm{cl}_\tau U \cap K = \emptyset$, then $e(U) < \kappa$.

(4) K is nonempty.

Recall Konig's Lemma, $\mathrm{cf}(\mathfrak{c}) > \omega$. So, if (X, τ) is a metric space and $e(X) = \mathfrak{c}$, by Lemma 2.3.2 the extent of X must be attained.

Definition 2.3.3 ([8], proof of Theorem 3.2)**.** An open set V is called $e-$*homogeneous* if for every nonempty open subset V' of V, $e(V') = e(V)$. Also note that any nonempty open subset U of a metric space has a nonempty open e-homogeneous set V.

Remark 2.3.4. If the extent of an open subset, U, of a metric space is not attained, then as a consequence of Lemma 2.3.2 (2) and (4), there is $V \subset U$ with $e(V) < e(U)$. Therefore, if U is an e-homogeneous subset of a metric space, $e(U)$ is attained.

Lemma 2.3.5. *Let U be an e-homogeneous subset of a metric space (X,τ,μ) and $e(U) > \aleph_0$. Then $e(\mathrm{cl}_\tau U)$ is attained by some closed discrete $C \subset U$.*

Proof. Since U is e-homogeneous, $e(U) = \lambda$ is attained. Suppose that $\mathrm{cf}(\lambda) > \aleph_0$. Then, $e(U)$ is attained by some closed discrete (in U) subset $C' \subset U$ of cardinality λ. However, it may not be the case that C' is closed and discrete in $cl_\tau U$. Let $\mathscr{U}$ be a discrete collection of open subsets of V that witnesses that C' is closed discrete. Let $L = \mathrm{cl}_\tau(\bigcup \mathscr{U}) \setminus \bigcup_{V\in\mathscr{U}} \mathrm{cl}_\tau V$. Note that $L \subset \mathrm{cl}_\tau U \setminus U$. Let $C_n = \{c \in C' : \mu(c,L) \geq 1/n\}$ and note that $\bigcup C_n = C'$. Since $\mathrm{cf}(\lambda) > \aleph_0$ there is $n \in \omega$ so that $|C_n| = \lambda$. Set $C = C_n$. By construction, C is closed and $\mathrm{cl}_\tau C \cap L = \emptyset$, so C is closed discrete in $\mathrm{cl}_\tau U$, $C \subset U$ and $|C| = \lambda = e(\mathrm{cl}_\tau U)$ as desired. Now suppose that $\mathrm{cf}(\lambda) = \aleph_0$. Let λ_n be such that $\lambda = \sup_{n\in\omega} \lambda_n$. Let W be an open subset of U such that $\mathrm{cl}_\tau W \subset U$. Note that since U is e-homogeneous, $e(W) = e(\mathrm{cl}_\tau W) = e(U) = \lambda$. Since $e(\mathrm{cl}_\tau W) = \lambda > \aleph_0$ there is $C' \subset \mathrm{cl}_\tau W \subset U$ a countable closed discrete set in $\mathrm{cl}_\tau W$, hence closed discrete in U and $\mathrm{cl}_\tau U$. Let $\mathscr{U} = \{U_n : n \in \omega\}$ be a discrete collection of open subsets of U that witnesses C' is discrete in U. For each $n \in \omega$, $e(\mathrm{cl}_\tau U_n) = \lambda > \lambda_n$, so there is $C_n \subset \mathrm{cl}_\tau U_n$ a closed discrete subset of cardinality λ_n. Let $C = \bigcup C_n$. Since C_n is closed discrete in $\mathrm{cl}_\tau U_n$, it is closed discrete in $\mathrm{cl}_\tau U$. Moreover, since $\mathscr{U}$ is discrete, C is closed discrete in $\mathrm{cl}_\tau U$. Finally, $|C| = \lambda = e(\mathrm{cl}_\tau U)$ by construction and since $U_n \subset U$ for each n, $C \subset U$. $\square$

Lemma 2.3.6. *Let (X,τ) be a metric space with metric μ in which $e(X) = \kappa$ is not attained. Let K be the set of points x of X such that every neighborhood of x has extent κ. Then, for every open set U meeting K and every $\theta < \kappa$ there is an open subset V of U such that $\mathrm{cl}_\tau V \subset U$, $e(\mathrm{cl}_\tau V) > \theta$ is attained by $C \subset V$ and $\mathrm{cl}_\tau V \cap K = \emptyset$.*

Proof. Let $\mathscr{V}$ be a maximal pairwise disjoint collection of e-homogeneous subsets V of $U \setminus K$ such that $\mathrm{cl}_\tau V \cap K = \emptyset$. Note that $\mathrm{cl}_\tau V \cap K = \emptyset$ implies that $e(V) < \kappa$, by Lemma 0.2 (3). Suppose that for some $V \in \mathscr{V}$, $e(\mathrm{cl}_\tau V) = e(V) > \theta$. By Lemma 2.3.5, $e(\mathrm{cl}_\tau V)$ is attained by some $C \subset V$ and we are done. Suppose on the other hand that $e(V') \leq \theta$ for all $V' \in \mathscr{V}$. Note that $\bigcup\mathscr{V}$ is dense in U since $\mathscr{V}$ is maximal, K is nowhere dense and every open subset of X has an e-homogeneous subset. Since X is metric, $e(W) = d(W)$ for any open subset W. Suppose that $|\mathscr{V}| = \lambda < \kappa$. Then, $d(U) = d(\bigcup\mathscr{V}) \leq \lambda \cdot \theta < \kappa$ which is a contradiction. So, $|\mathscr{V}| = \kappa$. Since $\mu(V',K) > 0$ for all $V' \in \mathscr{V}$ and $\mathrm{cf}(\kappa) = \omega$, there is $n \in \omega$ such that $|\{V' \in \mathscr{V} : \mu(K,V') > 1/n\}| > \theta$. Set $\mathscr{V}' = \{V' \in \mathscr{V} : \mu(K,V') > 1/n\}$. We now refine $\mathscr{V}'$ to a discrete collection of size $> \theta$. Let $L = \mathrm{cl}_\tau(\bigcup\mathscr{V}') \setminus \bigcup\limits_{\mathscr{V}'}\mathrm{cl}_\tau V$ and let $L_\varepsilon = \{x \in U : \mu(x,L) < \varepsilon\}$. Since $\bigcap\limits_{\varepsilon>0} L_\varepsilon = L$ and $V' \cap L = \emptyset$ for all $V' \in \mathscr{V}'$, there is $m \in \omega$ such that $|\{V' \in \mathscr{V}' : V' \setminus L_{1/m} \neq \emptyset\}| > \theta$. Set $\mathscr{V}'' = \{V' \in \mathscr{V}' : V' \setminus L_{1/m} \neq \emptyset\}$. For each $V' \in \mathscr{V}''$, let $W(V') = V' \setminus L_{1/m}$. The collection $\mathscr{W} = \{W(V') : V' \in \mathscr{V}'\}$ by construction is discrete and has cardinality $> \theta$. Set $V = \bigcup\mathscr{W}$. For each $W \in \mathscr{W}$, choose $x_W \in W$. Let $C = \{x_W : W \in \mathscr{W}\}$. C is closed discrete in $\mathrm{cl}_\tau V$ since $\mathscr{W}$ is discrete in X, and $|C| = |\mathscr{W}| > \theta$. Since $e(\mathrm{cl}_\tau V) = e(V) \leq \sup\{e(W) : W \in \mathscr{W}\} \cdot |\mathscr{W}| \leq \theta \cdot |\mathscr{W}| = |C|$, we have that $e(\mathrm{cl}_\tau V) > \theta$ is attained by $C \subset V$. □

Lemma 2.3.7. *Let (Z,τ) be a compact metric space and let $\mathscr{U} \in [\tau]^{<\omega}$ be a pairwise disjoint collection such that $\bigcup\mathscr{U}$ is dense in X. If $\varepsilon > 0$ and $\mathscr{V}$ is the collection of open subsets of X with diameter less than ε, then there exists a pairwise disjoint $\mathscr{V}' \in [\mathscr{V}]^{<\omega}$ such that $\mathscr{V}'$ refines $\mathscr{U}$ and $\bigcup\mathscr{V}'$ is dense in X.*

Proof. Since $\mathscr{V}$ covers Z, compact, there is $n \in \omega$ and $V_1, V_2, \ldots, V_n \in \mathscr{V}$ such that $Z = \bigcup_{1\leq i\leq n} V_i$. Define $\hat{V}_1 = V_1$ and for $1 < i \leq n$ let $\hat{V}_i = V_i \setminus \mathrm{cl}(\bigcup_{1\leq j<i} V_j)$. Let $\hat{\mathscr{V}} = \{\hat{V}_i : 1 \leq i \leq n\}$. Note, $\hat{\mathscr{V}}$ is pairwise disjoint and $\bigcup\hat{\mathscr{V}}$ is dense in X. Now define $\mathscr{V}' = \{V \cap U : V \in \hat{\mathscr{V}}, U \in \mathscr{U}\}$. Since $V \cap U \subset U$ for each $V \in \hat{\mathscr{V}}$ and $U \in \mathscr{U}$, $\mathscr{V}'$ refines $\mathscr{U}$. Since $\mathrm{diam}(V \cap U) \leq \mathrm{diam}(V) \leq V_i$ for some $1 \leq i \leq n$, $\mathrm{diam}(V') < \varepsilon$ for all $V' \in \mathscr{V}'$. Hence $\mathscr{V}' \subset \mathscr{V}$. Since $\hat{\mathscr{V}}$ is pairwise disjoint, $\mathscr{V}'$ is as well. Finally,

$\bigcup \mathscr{V}' = \bigcup \hat{\mathscr{V}} \cap \bigcup \mathscr{U}$ and therefore $\bigcup \mathscr{V}'$ is dense in Z, since $\bigcup \mathscr{V}'$ and $\bigcup \mathscr{U}$ are open and dense in X. □

Theorem 2.3.8. *If (X, τ, μ) is a metric space and $e(X) = \kappa > \mathfrak{c}$ is not attained, then there is σ, a topology on X coarser than τ, such that (X, σ) is connected and metrizable.*

Proof. Re-scale μ so that $\mathrm{diam}_\mu(X) < 1/2$ by replacing it with $\frac{\mu}{2(1+\mu)}$. Let K be the set of points x of X such that every neighborhood of x has extent κ. By Lemma 2.3.2, K is compact.

Let $\mathscr{C}_0^* = \{K\}$. For each $n \in \omega \setminus \{0\}$, define $\mathscr{C}_n^* \subset \tau|_K$, a pairwise disjoint finite collection with the following properties:

- $\mathrm{cl}(\bigcup \mathscr{C}_n^*) = K$.
- $\mathscr{C}_n^*$ refines $\mathscr{C}_{n-1}^*$
- $B \in \mathscr{C}_n^*$ implies $\mathrm{diam}(B) < 1/2^n$

Let $n \in \omega \setminus \{0\}$. Apply Lemma 2.3.7 with $Z = K$, τ, $\mathscr{U} = \mathscr{C}_{n-1}^*$ and $\varepsilon = 1/2^n$ to get $\mathscr{V}'$, a pairwise disjoint collection of open sets with diameter less that $1/2^n$ that refines $\mathscr{C}_{n-1}^*$ and whose union is dense in K. Set $\mathscr{C}_n^* = \mathscr{V}'$.

Definition of B_i's

For $n \in \omega$ enumerate the elements of $\mathscr{C}_n^*$ as $\mathscr{C}_n^* = \{B_i^* : i_n \leq i < i_{n+1}\}$ with an increasing sequence of integers, i_n. For each $i \in \omega$, let $L_i = \{x \in X : \mu(x, B_i^*) \leq \mu(x, K \setminus B_i^*)\}$. Fix $n \in \omega$ and let $B_{i_n} = L_{i_n}$ and for $i_n < i < i_{n+1}$, let $B_i = L_i \setminus \bigcup_{i_n \leq j < i} \mathrm{cl}(B_j)$. Note that $B_j^* \subset B_i^*$ implies $L_j \subset L_i$.

We define $\mathscr{C}_n = \{B_i : i_n \leq i < i_{n+1}\}$ and verify the following.

i) For each $n \in \omega$, $\mathscr{C}_n$ is pairwise disjoint.

ii) For each $n \in \omega$, $\bigcup \mathscr{C}_n$ is dense in X.

iii) For all $i \in \omega$, $\operatorname{int}(B_i) \cap K \neq \emptyset$.

From the definition of B_i and $\mathscr{C}_n$, i) is clear. Towards ii), let $n \in \omega$ and let $x \in X$. Since $\bigcup \mathscr{C}_n^*$ is dense in K, $\mu(x,K) = \mu(x, \bigcup \mathscr{C}_n^*)$. Since $\mathscr{C}_n^*$ is finite, there is i such that $i_n \leq i < i_{n+1}$ and $\mu(x,K) = \mu(x,B_i^*)$. Therefore, $\mu(x,B_i^*) \leq \mu(x, K \setminus B_i^*)$ and either $x \in \operatorname{cl}(B_i)$ or $x \in \operatorname{cl}(\bigcup_{i_n \leq j < i} B_j)$. In either case, $x \in \operatorname{cl}(\bigcup \mathscr{C}_n)$. For iii), note that $\operatorname{int}(B_i) \cap K = B_i^* \setminus \operatorname{cl}(\bigcup_{i_n \leq j < i} B_j^*)$ which is nonempty since $\mathscr{C}_n^*$ is pairwise disjoint.

Let $U_i = \{x \in X : \mu(x,K) < \frac{1}{2^{i+1}}\}$.

Claim (2)**.** For $n \in \omega$, $i_n \leq i < i_{n+1}$, $\operatorname{diam}_\mu(L_i \cap U_m) < \frac{3}{2^n}$ for any $m \geq n-1$.

Proof. Note that if $x \in L_i$, $\mu(x,B_i^*) \leq \mu(x,K \setminus B_i^*)$ which implies $\mu(x,K) = \mu(x,B_i^*)$. Hence for $i_n \leq i < i_{n+1}$, $m \geq n-1$, $x,y \in L_i \cap U_m$ and $\varepsilon > 0$ there exists $x_0, y_0 \in B_i^*$ such that $\mu(x,x_0), \mu(y,y_0) < 1/2^{m+1} + \varepsilon/2 \leq 1/2^n + \varepsilon/2$. So,

$$\mu(x,y) \leq \mu(x,x_0) + \mu(y,y_0) + \mu(x_0,y_0) \leq 2/2^n + \varepsilon + 1/2^n \leq 3/2^n + \varepsilon$$

Hence, $\operatorname{diam}_\mu(L_i \cap U_m) < \frac{3}{2^n}$. □

Definition of W_i's

We define $\mathscr{W} = \{W_i : i \in \omega\}$, a pairwise disjoint collection of open subsets of X such that

i) $\operatorname{cl}(W_i) \cap K = \emptyset$,

ii) $e(\operatorname{cl}(W_i)) > \mathfrak{c}$ is attained by $C_i \subset W_i$,

iii) for $i_n \leq i < i_{n+1}$, $\varepsilon_i = \operatorname{diam}_\mu(W_i) < \frac{3}{2^n}$ and

iv) $\bigcup \mathscr{W}$ is dense in X.

Let $\hat{W}_0 = \emptyset$. Apply Lemma 2.3.6 with $\theta = \mathfrak{c}$ and $U = \text{int}(B_0) = X$ to get an open subset V of X such that $e(\text{cl}_\tau V) > \theta = \mathfrak{c}$ is attained by $C_0 \subset V$ and $\text{cl}_\tau V \cap K = \emptyset$. Set $W_0 = V$. Set $S_0 = V$. Let $k_0 = \min\{k : S_0 \subset X \setminus U_k\} \geq 1$. By definition W_0 is open, and since $\text{cl} V \cap K = \text{cl}(W_0) \cap K = \emptyset$ i) holds. By Lemma 2.3.6, ii) holds and iii) is trivial since $\text{diam}(X) \leq 1/2$.

Suppose we have defined W_i for all $0 < i < i_n$ so that i), ii), and iii) are satisfied. Also suppose that $S_m = \bigcup\{W_i : i < i_{m+1}\}$ is dense in $X \setminus \text{cl}(U_{k_{m-1}+1})$ and that $k_m = \min\{k : S_m \subset X \setminus U_k\} \geq m$ for all $0 < m < n$.

Let i be such that $i_n \leq i < i_{n+1}$. Let $\hat{W}_i = B_i \setminus (\text{cl}(U_{k_{n-1}+1} \cup S_{n-1}))$. Since $\text{int}(B_i) \cap K \neq \emptyset$ and $K \subset X \setminus \text{cl}(S_{n-1})$, $\text{int}(B_i) \setminus cl(S_{n-1})$ is an open set meeting K. So, apply Lemma 2.3.6 with $\theta = \max\{e(\hat{W}_i), \mathfrak{c}\}$ and $U = \text{int}(B_i) \setminus \text{cl}(S_{n-1})$ to get an open subset V of U such that $\text{cl}_\tau V \subset U$, $e(\text{cl}_\tau V) > \theta$ is attained by $C_i \subset V$ and $\text{cl}_\tau V \cap K = \emptyset$. Set $W_i = \hat{W}_i \cup V$. By the lemma, W_i satisfies (i) (ii) for each i such that $i_n \leq i < i_{n+1}$. Set $S_n = \bigcup\{W_i : i_n \leq i < i_{n+1}\} \cup S_{n-1}$. Let $k_n = \min\{k : S_n \subset X \setminus U_k\} \geq n$. By Claim (2.3), $\text{diam}(B_i \cap U_{n-1}) < \frac{3}{2^n}$ since $B_i \subset L_i$ and $i_n \leq i < i_{n+1}$. Also, S_{n-1} is dense in $X \setminus U_{k_{n-2}+1}$. Therefore $W_i = \hat{W}_i \cup V \subset B_i \setminus \text{cl}(S_{n-1}) \subset B_i \cap U_{k_{n-2}+1} \subset B_i \cap U_{n-1}$, since $k_{n-2} \geq n-2$. Hence $\text{diam}_\mu(W_i) < \frac{3}{2^n}$ and iii) is satisfied.

Towards iv), since K is nowhere dense in X, we only show that $\bigcup \mathscr{W}$ is dense in $X \setminus K$. Let $x \in X \setminus K$ and let $n \in \omega$ be such that $x \in X \setminus \text{cl}(U_{k_{n-1}+1})$. Then either $x \in \text{cl}(S_{n-1}) \subset \text{cl}(S_n)$ or $x \in X \setminus \text{cl}(U_{k_{n-1}+1} \cup S_{n-1})$. If $x \in X \setminus \text{cl}(U_{k_{n-1}+1} \cup S_{n-1})$ then since $\{B_i : i_n \leq i < i_{n+1}\}$ is dense in X, $x \in \text{cl}(B_i \setminus (\text{cl}(U_{k_{n-1}+1} \cup S_{n-1})) \subset \text{cl}(W_i) \subset \text{cl}(S_n)$ for some i. In either case $x \in \text{cl}(S_n)$. So, $\text{cl}(\bigcup\{W_i : i \in \omega\}) = \text{cl}(\bigcup\{S_n : n \in \omega\}) \supset (X \setminus \text{cl}(U_{k_m+1}))$ for each $m \in \omega$. But, $X \setminus K = \bigcup X \setminus \text{cl}(U_{k_m+1})$. Hence $\bigcup\{W_i : i \in \omega\}$ is dense in $X \setminus K$.

Linking the W_i's

Suppose that if $n, i, j \in \omega$ are such that $i_n \leq i < i_{n+1} \leq j < i_{n+2}$ and $B_j^* \subset B_i^*$. Then, $W_i \subset B_i \cap U_{n-1} \subset L_i \cap U_{n-1}$ and $W_j \subset B_j \cap U_n \subset L_j \cap U_n \subset L_i \cap U_{n-1}$. Hence by Claim (2), $x \in W_i$ and $y \in W_j$ implies $\mu(x,y) \leq \varepsilon_i$. For each $j \in \omega$ choose $x_j \in W_j \setminus C_i$ arbitrarily. For $i_n \leq i < i_{n+1}$, let $J_i = \{j : i_{n+1} \leq j < i_{n+2}, B_j^* \subset B_i^*\}$and let $X_i = \{x_j : j \in J_i\}$. Notice that $\operatorname{diam}(W_i \cup X_i) \leq \varepsilon_i$ $C_i \subset W_i \subset \operatorname{int}(W_i \cup X_i)$ and that by the definition of $\mathscr{B}^*$, $\bigcup_{i \in \omega} J_i = \omega \setminus \{0\}$.

Defining the connected topology on X

We define a sequence of metrics μ_n on X such that $\nu = \lim \mu_n$ is a well defined metric that generates a coarser connected topology on X. We define μ_n by induction. Apply Lemma 2.3.1 with $Y = W_0 \cup X_0$, $C = C_0$, $\varepsilon = 3/2$ and $\mu = \mu^*$ to get τ_0 and μ_0 such that

i) $\tau_0\big|_{\operatorname{cl} W_0 \cup X_0}$ is connected

ii) $\mu^*\big|_{(X \setminus (W_0 \cup X_0))^2} = \mu_0\big|_{(X \setminus (W_0 \cup X_0))^2}$,

iii) $\operatorname{diam}_{\mu_0}(\operatorname{cl} W_0 \cup X_0) \leq 9/2$ and

iv) $\mu_0 \leq \mu^* + 6/2$.

Fix n and suppose we have for each $1 \leq m < n$, μ_m defined on X such that μ_m is a metric that generates a coarser topology $\tau_m \subset \tau_{m-1}$ in which S_m is a connected subset of X.

Set $\sigma_0 = \tau_{n-1}$ and $\rho_0 = \mu_{n-1}$. For $i_n \leq i < i_{n+1}$, let $j = i - i_n$ and apply Lemma 2.3.1 with $X = X$, $C = C_i$, $\tau = \sigma_j$, $\mu = \rho_j$, $Y = W_i \cup X_i$ and $\varepsilon = \varepsilon_i$to get σ_{j+1} and ρ_{j+1} such that

i) $\sigma_{j+1}\big|_{\operatorname{cl} W_i \cup X_i}$ is connected

ii) $\rho_j\big|_{(X\setminus(W_i\cup X_i))^2} = \rho_{j+1}\big|_{(X\setminus(W_i\cup X_i))^2}$

iii) $\mathrm{diam}_{\rho_{j+1}}(\mathrm{cl}\,W_i\cup X_i)\leq 3\varepsilon_i\leq \frac{9}{2^n}$ and

iv) $\rho_{j+1}\leq\rho_j+2\varepsilon_i\leq\rho_j+\frac{6}{2^n}$

Let $\mu_n=\rho_{m_n}$ and $\tau_n=\sigma_{m_n}$. As a consequence of ii) and iv), $\mu_n\leq\mu^*+\frac{6}{2^n}$ on $\bigcup_{i_n\leq i<i_{n+1}} W_i$, $\mu_n=\mu_{n-1}$ on $X\setminus\bigcup_{i_n\leq i<i_{n+1}} W_i$ and $\tau_n\big|_{\mathrm{cl}\,W_i\cup X_i}$ is connected for each $i_n\leq i<i_{n+1}$.

Define $\nu(x,y)=\lim\mu_n(x,y)$. This map is a well defined metric since for any $x,y\in X$, $\nu(x,y)=\mu_m(x,y)$ for all $m\geq\max\{n:i_n\leq i<i_{n+1}\text{and } x\in W_i \text{ or } y\in W_i\}\cup\{0\}$. Let τ' be the topology generated by ν. To show that $\tau'\subset\tau$ we show that ν preserves convergent μ sequences. Suppose that $\{x_n:n\in\omega\}$ and x are such that $\lim_{i\to\infty}\mu(x,x_i)=0$. If $x\notin K$ there is $n,m\in\omega$ such that $x,x_i\in X\setminus\mathrm{cl}(U_{k_n+1})$ for all $i\geq m$. Then, $\nu(x,x_i)=\mu_n(x,x_i)$ for all $i\geq m$, hence $\lim_{i\to\infty}\nu(x,x_i)=\lim_{i\to\infty}\mu_n(x,x_i)=0$, since μ_n preserves μ convergent sequences. Now suppose that $x\in K$. For each $n\in\omega$, there is m_n such that $x_n\in U_{m_n}$ and since $\mu^*(x,x_n)\to 0$, $m_n\to\infty$. If there is $i\in\omega$ such that $x_n\in W_i$ then it must be that $i\geq m_n$ since $x_n\in U_{m_n}$. In this case, by the consequence of ii) and iv), $\nu(x,x_n)\leq\mu^*(x,x_n)+\frac{6}{2^i}\leq\mu^*(x,x_n)+\frac{6}{2^{m_n}}$. If $x\notin W_i$ for all $i\in\omega$ then $\nu(x,x_n)=\mu^*(x,x_n)$. In either case $\nu(x,x_n)\leq\mu^*(x,x_n)+\frac{6}{2^{m_n}}$ and therefore $\nu(x,x_n)\to 0$.

Notice $\nu\Big|_{W_i\cup X_i}=\tau_n\Big|_{W_i\cup X_i}$ for $i_n\leq i<i_{n+1}$. Hence $W_i\cup X_i$ is connected in τ'. Notice that $X_i\cap W_j\neq\emptyset$ for each $j\in J_i$ so that $W_i\cup\bigcup_{j\in X_i}W_j$ is connected as well. This means that W_i is 'linked' to W_0 for every $1=i_1\leq i<i_2$ and since $\bigcup_{i\in\omega}J_i=\omega\setminus\{0\}$, any later W_i is 'linked' to W_0. Therefore any τ'-clopen subset, Z, of X would have to be empty, or contain W_i for all i. Since $\bigcup_{i\in\omega}W_i$ is dense in X, Z is trivial. Hence τ' is connected. $\square$

Chapter 3

Non-normality points

3.1 Introduction

Definition 3.1.1. 1. A point y in a space X is called a **non-normality point of** X if $X \setminus \{y\}$ is not normal.

2. A point y in a space X is called a **butterfly point of** X if there are closed subsets H, K of X such that $\{y\} = \mathrm{cl}(H \setminus \{y\}) \cap \mathrm{cl}(K \setminus \{y\})$.

If a point y is a non-normality point of a normal space X, then a pair of disjoint closed sets H, K that cannot be separated in $X \setminus \{y\}$ actually demonstrate that y is a butterfly point of X. It is not always the case, however, that a butterfly point in a normal space is a non-normality point. For example, any point x in $\mathbb{R}$, the real line, is a butterfly point via the sets $[x, x+1]$ and $[x-1, x]$. However, x is not a non-normality point of $\mathbb{R}$ since $\mathbb{R}$ is hereditarily normal.

One may ask the following questions for a topological space X.

Question 3.1.2. Which points $y \in \beta X \setminus X$ are non-normality (butterfly) points of $\beta X \setminus X$.

Question 3.1.3. Under what set-theoretic conditions are all points $y \in \beta X \setminus X$ non-normality (butterfly) points of $\beta X \setminus X$.

Lemma 3.1.4. *Let y be an element of a normal space X and suppose $X \setminus \{y\}$ is C^*-embedded in X. If y is a butterfly point of X then y is a non-normality point of X.*

Proof. Let y be a butterfly point of X such that $X \setminus \{y\}$ is C^*-embedded in X. Suppose that $X \setminus \{y\}$ is normal. Since y is a butterfly point, there are closed sets H and K in X such that $\{y\} = \mathrm{cl}(H \setminus \{y\}) \cap \mathrm{cl}(K \setminus \{y\})$. Let $H' = H \setminus \{y\}$ and $K' = K \setminus \{y\}$. Since $X \setminus \{y\}$ is normal and H' and K' are closed in $X \setminus \{y\}$, there is a continuous function $f : X \setminus \{y\} \to [0,1]$ such that $H' = f^{\leftarrow}[\{1\}]$ and $K' = f^{\leftarrow}[\{0\}]$. Since $X \setminus \{y\}$ is C^*-embedded in X, there is a continuous extension, g, of f to X. However, since $y \in \mathrm{cl}(H \setminus \{y\}) \cap \mathrm{cl}(K \setminus \{y\})$ it must be that $0 = g(y) = 1$, a contradiction. Hence $X \setminus \{y\}$ is not normal and therefore y is a non-normality point of X. □

As the next example shows, a butterfly point of $\beta X \setminus X$ is not necessarily a non-normality point of $\beta X \setminus X$.

Example 3.1.5. Let $X = (\omega + 1) \times \omega_1$. Then $\beta X = (\omega + 1) \times (\omega_1 + 1)$. Notice that $\beta X \setminus X$ is a convergent sequence $\{(n, \omega_1) : n \in \omega\} \cup \{(\omega, \omega_1)\}$. Each (n, ω_1) is neither a buttterfly point nor a non-normality point of $\beta X \setminus X$. However, the point (ω, ω_1) is a butterfly point of $\beta X \setminus X$ via the sets $H = \{(2n, \omega_1) : n \in \omega\}$ and
$K = \{(2n+1, \omega_1) : n \in \omega\}$. Even though the point (ω, ω_1) is a non-normality point of βX, it is not a non-normality point of $\beta X \setminus X$; the subspace $\{(n, \omega_1) : n \in \omega\}$ is normal.

Because of the previous example, when aiming for non-normality points in $\beta X \setminus X$, we may restrict our attention to a special class of spaces. In particular, we focus not on arbitrary Tychonoff spaces X, but for discrete, or more generally, metrizable spaces.

We will use p and q for set-ultrafilters or ultrafilters on a discrete space and y for a z-ultrafilter on a metric space. When considering the Stone Cech compactification, βX, we view the points $y \in \beta X$ as z-ultrafilters on X. In a Tychonoff space X, a z-ultrafilter can have at most one point of convergence. Associating each point x of X to the fixed z-ultrafilter of all z-sets containing x gives an embedding of X into βX.

For an infinite cardinal κ we write $D(\kappa)$ for the discrete space of cardinality κ. All subsets of $D(\kappa)$ are clopen z-sets. Therefore, any ultrafilter on $D(\kappa)$ is also an open ultrafilter and a z-ultrafilter. We would like to extend some notions defined for set-ultrafilters to z-ultrafilters.

Definition 3.1.6.

1. An ultrafilter p on $D(\kappa)$ is called **uniform** if $|A| = \kappa$ for each $A \in p$.
2. A point $y \in \beta X \setminus X$ is **uniform** if $w(Z) = w(X)$ for all $Z \in y$.

For an infinite cardinal κ, we denote the set of uniform ultrafilters on the discrete space of size κ by $U(\kappa)$ and the set of non-uniform ultrafilters by $NU(\kappa)$.

Definition 3.1.7.

1. A uniform ultrafilter, p, on a $D(\kappa)$ is $(\aleph_0, \kappa)$**-regular** (or just **regular**) if there exists $\{S_\alpha : \alpha \in \kappa\} \subset p$ such that for all $A \in [\kappa]^{\aleph_0}$, $\bigcap\{S_\alpha : \alpha \in A\} = \emptyset$.
2. A uniform z-filter, y, on a metric space with weight κ is **regular** if there exists $\mathscr{Z} \subset y$ such that $|\mathscr{Z}| = \kappa$ and $\mathscr{Z}$ is a locally finite collection in X.

3.2 Discrete Spaces

A direct way of showing that a point y in a space Y is a non-normality point of Y, is to exhibit two closed subsets of $Y \setminus \{y\}$ that cannot be separated.

Blaszczyk and Szymanski [2] showed that if κ is regular and $p \in \beta D(\kappa) \setminus D(\kappa)$ is in the closure of a strongly discrete subset of $\beta D(\kappa) \setminus D(\kappa)$ then p is a non-normality point of $\beta D(\kappa) \setminus D(\kappa)$. They used the closure of the strongly discrete set as one of the two closed sets that cannot be separated in $\beta D(\kappa)\setminus D(\kappa)\setminus\{p\}$. Blaszczyk and Szymanski's result addresses a question of the form 3.1.2, which points are non-normality points? Notice that they did not assume extra axioms of set theory.

When the conclusion is strengthened to: all points are non-normality points, as in 3.1.3, the hypotheses usually include a set-theoretic assumption. For example, assuming CH, any point $p\in\beta\omega\setminus\omega$ is a non-normality point of $\beta\omega\setminus\omega$. This theorem was proven in two parts. Gillman, [9], showed that under CH, a certain class of points $p\in\beta\omega\setminus\omega$ are non-normality points of $\beta\omega\setminus\omega$. Then, Rajagopalan [18] and Warren [22] showed all other points $p\in\beta\omega\setminus\omega$ are non-normality points. Since every free ultrafilter on ω is uniform, this result can be phrased: CH implies every $p\in U(\omega)$ is a non-normality point of $U(\omega)$.

Note that closed subspaces of normal spaces are normal. So, an indirect way of showing that a point y in a space Y is a non-normality point, is to embed a non-normal space Z as a closed subspace of $Y\setminus\{y\}$. Warren [22] showed that $NU(\omega_1)$ is not normal. Then she showed, assuming CH, that $NU(\omega_1)\simeq(\beta\omega\setminus\omega)\setminus\{y\}$, completing the proof that y is a non-normality point.

Malyhin [17] showed the following.

Lemma 3.2.1. *1. If θ is singular then $NU(\theta)$ is not normal.*

2. If θ is regular and not a strong limit cardinal, then $NU(\theta)$ is not normal.

Kunen and Parsons [[14], Theorem 1.11] then showed.

Lemma 3.2.2. *The space $NU(\theta)$ is not normal if and only if θ is regular and not weakly compact.*

Beslagic and van Douwen [[1], Theorem 1.1] generalized the results for ω with the following theorem.

Theorem 3.2.3. *($2^\kappa = \kappa^+$) Every point $p \in U(\kappa)$ is a non-normality point of $U(\kappa)$.*

The set-theoretic assumptions of their theorem are, in fact, weaker than $2^\kappa = \kappa^+$; the reaping number of κ is equal to 2^κ and $\sup\{2^\lambda : \lambda < \mathrm{cf}(2^\kappa)\} = 2^\kappa$. They showed that $NU(\mathrm{cf}(2^\kappa))$ embeds as a closed subspace of $U(\kappa) \setminus \{p\}$ for any $p \in U(\kappa)$. Note that since $U(\kappa)$ is a closed subset of $\beta D(\kappa) \setminus D(\kappa)$, if p is a non-normality point of $U(\kappa)$, it is a non-normality point of $\beta D(\kappa) \setminus D(\kappa)$.

Corollary 3.2.4. *(GCH) Every point $p \in \beta D(\kappa) \setminus D(\kappa)$ is a non-normality point of $\beta D(\kappa) \setminus D(\kappa)$.*

Proof. Let $p \in \beta D(\kappa) \setminus D(\kappa)$ and let $A \in p$ be such that $|A| = \min\{|A'| : A' \in p\}$. Since $D(\kappa)$ is discrete, A is C^*-embedded in $\beta D(\kappa)$, and so $\mathrm{cl}_{\beta D(\kappa)} A = \beta A$. Moreover, βA is a clopen subset of $\beta D(\kappa) \setminus D(\kappa)$. Since $|A|$ is minimum, $p|_A$ is a uniform on A. Therefore, by Theorem 3.2.3, p is a non-normality point of $\beta A \setminus A$. Since $\beta A \setminus A$ is closed in $\beta D(\kappa) \setminus D(\kappa)$, p is a non-normality point of $\beta D(\kappa) \setminus D(\kappa)$. $\square$

3.3 Metric Spaces

Since 2000, the study of non-normality points in $\beta X \setminus X$ has expanded to non-discrete spaces X (see [16] and [20]). We start with a result proved by Logunov [15] and Terasawa [19] independently.

Theorem 3.3.1. *If X is metrizable, non-compact and has no isolated points, then every point y in $\beta X \setminus X$ is a non-normality point of βX.*

They showed that any $y \in \beta X \setminus X$ is a butterfly point of βX. Because $X \subset \beta X \setminus \{y\} \subset \beta X$, we have that $\beta X \setminus \{y\}$ is C*-embedded in βX. Then by Lemma 3.1.4, if y is a butterfly point of βX, it is also a non-normalilty point. Since $\beta X \setminus (X \cup \{y\})$

is not necessarily C^*-embedded in βX, to be a non-normality point of the remainder, $\beta X \setminus X$, it does not suffice that p is a butterfly point. Our goal is to strengthen the conclusion of Theorem 3.3.1 to y in $\beta X \setminus X$ is a non-normality point of $\beta X \setminus X$. To do this we will add set-theoretic hypotheses.

Before proving the main theorem we develop two tools; a special π-base and an ultrapower. Terasawa [19] constructs a special π-base for a metric space without isolated points by modifying Arhangelski's regular base [[5], pg. 411]. The following π-base is the same, but we assume the metric space to be locally compact and get more structure (specifically, each B is split into four pieces).

Lemma 3.3.2. *Let X be a locally compact metric space without isolated points. There exists a collection $\mathscr{B} = \bigcup_{n\in\omega} \mathscr{B}_n$ of open subsets of X such that*

1. $\mathrm{cl}_X B$ *is compact for each* $B \in \mathscr{B}$, $\mathscr{B}_n$ *is pairwise disjoint, locally finite and* $cl(\mathscr{B}_n^*) = X$.
2. $\mathscr{B}_{n+1}$ *refines* $\mathscr{B}_n$ *and* $|\{B' \in \mathscr{B}_{n+1} : B' \subset B\}| = 4$ *for all* $B \in \mathscr{B}_n$.
3. *For* $B \in \mathscr{B}_n$ *there are* $B^0, B^1 \in \mathscr{B}_{n+1}$ *such that* $\mathrm{cl}\, B^0 \cap \mathrm{cl}\, B^1 = \emptyset$ *and* $\mathrm{cl}\, B^0, \mathrm{cl}\, B^1 \subset B$.
4. *If* $\mathscr{U} = \{U, V\}$ *is an open cover of X, there is a pairwise disjoint locally finite collection* $\mathscr{V} \subset \mathscr{B}$ *densely refining* $\mathscr{U}$.
5. *For each* $n \in \omega$, $|\mathscr{B}_n| = w(X)$.

Proof. Let $\mathscr{O}$ be an open cover of X consisting of sets U such that $\mathrm{cl}\, U$ is compact. Let $\mathscr{B}_0'$ be a locally finite open refinement of size $\leq w(X)$. In fact, it must be that $|\mathscr{B}_0'| = w(X)$. Otherwise, since $\mathrm{cl}\, B$ is compact metric for each $B \in \mathscr{B}_0'$, there is a countable collection of open subsets of X that is a base for points in $\mathrm{cl}\, B$. Since $\mathscr{B}_0'$ covers, if $|\mathscr{B}_0'| < w(X)$ the union of each of these bases would be a basis for X of size $< w(X)$, a contradiction. Let $\kappa = w(X)$. Well order $\mathscr{B}_0'$ as $\{B_\alpha' : \alpha \in \kappa\}$. Define $B_\alpha = B_\alpha' \setminus \bigcup_{\gamma<\alpha} \mathrm{cl}\, B_\gamma$ and set $\mathscr{B}_0 = \{B_\alpha : \alpha \in \kappa\}$. Notice that since $\mathscr{B}_0'$ is locally finite, $\mathscr{B}_0$ is locally finite as well and each B_α is open. Furthermore, since $\mathrm{cl}\, B_\alpha \subset \mathrm{cl}\, B_\alpha'$, $\mathrm{cl}\, B_\alpha$ is compact.

Fix $\alpha \in \kappa$. Since $\mathrm{cl}\, B_\alpha$ is compact and metric, there is a countable base for $\mathrm{cl}\, B_\alpha$. Let $\mathscr{A}_\alpha = \{A_i \subset \mathrm{cl}\, B_\alpha : i \in \omega\}$ be such a base such that $A_0 = \mathrm{cl}\, B_\alpha$ and A_i is open with respect to $\mathrm{cl}\, B_\alpha$. Notice that $\mathrm{int}(A_i) \neq \emptyset$ for all $i \in \omega$. Let $\mathscr{W}_\alpha^0 = \{B_\alpha\}$. Assume we have defined for each $i \leq n$ a collection $\mathscr{W}_\alpha^i$ of open (w.r.t. X) subsets of B_α such that:

i) $\mathscr{W}_\alpha^i$ is a pairwise disjoint finite collection such that $\mathrm{cl}(\bigcup \mathscr{W}_\alpha^n) = \mathrm{cl}\, B_\alpha$.

ii) $\mathscr{W}_\alpha^{i+1}$ refines $\mathscr{W}_\alpha^i$ and $|\{B' \in \mathscr{W}_\alpha^{i+1} : B' \subset B\}| = 4$ for all $B \in \mathscr{W}_\alpha^i$.

iii) For $B \in \mathscr{W}_\alpha^i$ there are $B^0, B^1 \in \mathscr{W}_\alpha^{n+1}$ such that $\mathrm{cl}\, B^0 \cap \mathrm{cl}\, B^1 = \emptyset$ and $\mathrm{cl}\, B^0, \mathrm{cl}\, B^1 \subset B$.

iv) For each $B \in \mathscr{W}_\alpha^i$, either $B \subset A_i$ or $B \subset B_\alpha \setminus \mathrm{cl}\, A_i$.

Fix $W \in \mathscr{W}_\alpha^n$.

Case (1). $W \cap A_{n+1} = \emptyset$ or $W \setminus A_{n+1} = \emptyset$.

Because X has no isolated points, we can find B^0 and B^1, non-empty open subsets of W, such that $\mathrm{cl}\, B^0 \cap \mathrm{cl}\, B^1 = \emptyset$ and $\mathrm{cl}\, B^0 \cup \mathrm{cl}\, B^1 \subsetneq W$. Then let B^2 and B^3 be non-empty open subsets of W such that $B^2 \cup B^3$ is dense in $W \setminus (\mathrm{cl}\, B^0 \cup \mathrm{cl}\, B^1)$.

Case (2). $W \cap A_{n+1} \neq \emptyset$ and $W \setminus A_{n+1} \neq \emptyset$.

Let B^0 be a non-empty open subset of W such that $\mathrm{cl}\, B^0 \subsetneq W \cap A_{n+1}$ and let $B^2 = (W \cap \mathrm{int}(A_{n+1})) \setminus \mathrm{cl}\, B^0$. Then let B^1 be a non-empty open subset of W such that $\mathrm{cl}\, B^1 \subsetneq W \setminus A_{n+1}$ and let $B^3 = W \setminus (\mathrm{cl}\, A_{n+1} \cup \mathrm{cl}\, B^1)$. Again, since X has no isolated points, this can be done.

Set $\mathscr{W}_\alpha^{n+1} = \{B^i : i = 0,1,2,3\}$. By construction, $\mathscr{W}_\alpha^{n+1}$ has properties (i) - (iv). Let $\mathscr{B}_n = \bigcup_{\alpha\in\kappa} \mathscr{W}_\alpha^n$. Properties (i)-(iii) for $\mathscr{W}_\alpha^n$ imply properties (1)-(3) for $\mathscr{B}_n$. It remains to show that (4) holds. Let $\{U,V\}$ be an open cover of X. Fix $\alpha \in \kappa$. If $B_\alpha \subset U$ or $B_\alpha \subset V$ then let $\mathscr{V}_\alpha = \mathscr{W}_\alpha^0 = \{B_\alpha\}$. Consider $\mathscr{U} = \{A_i \in \mathscr{A}_\alpha : A_i \subset V\}$. Since $\mathrm{cl}B_\alpha \setminus U \subset V$, $\mathscr{U}$ is an open (w.r.t. B_α) cover of the compact set $\mathrm{cl}B_\alpha \setminus U$, it has a finite subcover $\{A_{i_k} : k = 1,\dots,m\}$. Let $n = \max\{i_k : k = 1,\dots,m\}$. Then, $\mathscr{W}_\alpha^n$ has the property that for all $W \in \mathscr{W}_\alpha^n$, $W \subset A_i$ or $W \subset B_\alpha \setminus \mathrm{cl}A_i$ for all $i \le n$. So, either there exists i_k for some $k = 1,\dots,m$ such that $W \subset A_{i_k} \subset V$, or $W \subset \bigcap\{B_\alpha \setminus \mathrm{cl}A_{i_k} : k = 1,\dots,m\} \subset U$. Let $\mathscr{V}_\alpha = \mathscr{W}_\alpha^n$.

Now, let $\mathscr{V} = \bigcup_{\alpha\in\kappa} \mathscr{V}_\alpha$. Since $\mathscr{V}_\alpha = \mathscr{W}_\alpha^n$, it is finite. Moreover, since $\bigcup \mathscr{V}_\alpha \subset B_\alpha$ and $\mathscr{B}_0$ is locally finite, $\mathscr{V}$ is locally finite. Since $\mathrm{cl}(\bigcup \mathscr{W}_\alpha^n) = B_\alpha$ and $\mathrm{cl}(\bigcup \mathscr{B}_0) = X$, $\mathrm{cl}(\bigcup \mathscr{V}) = X$. Finally, $\mathscr{V}$ refines $\{U,V\}$ by construction. $\square$

In the main theorem we will embed a non-normal space $NU(\theta)$ into $\beta X \setminus X$. The cardinal, θ, will be the cofinality order of an ultrapower that we will construct now.

Let κ be an infinite cardinal and let ${}^\kappa\omega$ be the collection of functions from κ to ω. Given a filter p on $D(\kappa)$, we define an equivalence relation $\sim_p$ as follows. For $f,g \in {}^\kappa\omega$, $f \sim_p g$ if $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in p$. We define a partial order, $<_p$, on ${}^\kappa\omega$ as follows. For $f,g \in {}^\kappa\omega$, $f <_p g$ if $\{\alpha \in \kappa : f(\alpha) < g(\alpha)\} \in p$. We write ${}^\kappa\omega/p$ for ${}^\kappa\omega/\sim_p$ and $[f]$ for the equivalence class of f in ${}^\kappa\omega/p$. For $[f],[g] \in {}^\kappa\omega/p$ such that $f <_p g$, if $f' \in [f]$ and $g' \in [g]$ then it is easy to see that $f' <_p g'$. So, $<_p$ induces a partial order, $<$ on ${}^\kappa\omega/p$.

If p is an ultrafilter, for any $f,g \in {}^\kappa\omega$ one of $\{\alpha \in \kappa : f(\alpha) < g(\alpha)\}$, $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}$ or $\{\alpha \in \kappa : f(\alpha) > g(\alpha)\}$ is in p. Hence $<$ is a linear order on ${}^\kappa\omega/p$.

Lemma 3.3.3. *If p is a regular ultrafilter on $D(\kappa)$ then* $\mathrm{cf}({}^\kappa\omega/p) > \kappa$

Proof. Let p be a regular ultrafilter on $D(\kappa)$. Let $\{S_\alpha : \alpha \in \kappa\} \subset p$ be such that for all $A \in [\kappa]^{\aleph_0}$, $\bigcap\{S_\alpha : \alpha \in A\} = \emptyset$. In other words, for each $\gamma \in \kappa$, $I_\gamma = \{\alpha : \gamma \in S_\alpha\}$ is finite. Let $\{f_\alpha : \alpha \in \kappa\}$ be representatives from an increasing sequence in ${}^\kappa\omega/p$. For $\gamma \in \kappa$ define $f(\gamma) = \max\{f_\alpha(\gamma) : \alpha \in I_\gamma\} + 1$. We wish to show that $f_\alpha <_p f$ for all $\alpha \in \kappa$. Let $\alpha \in \kappa$ and $\delta \in S_\alpha$. Since $\alpha \in I_\delta$, $f(\delta) > f_\alpha(\delta)$ and therefore $S_\alpha \subset \{\delta : f(\delta) > f_\alpha(\delta)\}$. Since $S_\alpha \in p$, $f_\alpha <_p f$. Hence, $\{[f_\alpha] : \alpha \in \kappa\}$ cannot be cofinal in ${}^\kappa\omega/p$. $\square$

From now on, X is a locally compact metric space without isolated points, $\kappa = w(X)$ and $\mathscr{B} = \bigcup\{\mathscr{B}_i : i \in \omega\}$ is a π-base as in 3.3.2. For a uniform $y \in \beta X \setminus X$ we will define a uniform ultrafilter p_y on κ. Let $Z \in y$, let $\mathscr{U}_0(Z) = \{B \in \mathscr{B}_0 : \mathrm{cl}_X B \cap Z \neq \emptyset\}$ and let $\hat{\mathscr{N}_0} = \{\mathscr{U}_0(Z) : Z \in y\}$. Extend $\hat{\mathscr{N}_0}$ to an ultrafilter, $\mathscr{N}_0$, on $\mathscr{B}_0$. Notice that $\mathrm{cl}_X \mathscr{U}^* \in y$ for all $\mathscr{U} \in \mathscr{N}_0$. If not, there would be $Z \in y$ such that $Z \cap \mathrm{cl}_X \mathscr{U}^* = \emptyset$. But $\mathscr{U}_0(Z) \in \mathscr{N}_0$, hence $\mathscr{U}_0(Z) \cap \mathscr{U} \neq \emptyset$ and so $\mathscr{U}^* \cap Z \neq \emptyset$, a contradiction.

List $\mathscr{B}_0 = \{B_{\alpha\emptyset} : \alpha \in \kappa\}$ and $\mathscr{B}_i = \{B_{\alpha\sigma} : \alpha \in \kappa, \sigma \in {}^i 4\}$ such that $B_{\alpha\sigma} \subset B_{\alpha\nu}$ if σ extends ν. We may assume that for $\alpha \in \kappa$ and $\sigma \in {}^i 4$, $\mathrm{cl}_X B_{\alpha\sigma^\frown 0} \cap \mathrm{cl}_X B_{\alpha\sigma^\frown 1} = \emptyset$ and $\mathrm{cl}_X B_{\alpha\sigma^\frown 0}, \mathrm{cl}_X B_{\alpha\sigma^\frown 1} \subset B_{\alpha\sigma}$. For any $\mathscr{V} \subset \mathscr{B}$, let
$S(\mathscr{V}) = \{\alpha \in \kappa :$there is σ such that $B_{\alpha\sigma} \in \mathscr{V}\}$. Since $\mathscr{N}_0$ is an ultrafilter,
$p_y = \{S(\mathscr{V}) : \mathscr{V} \in \mathscr{N}_0\}$ is an ultrafilter on κ. Moreover, since y is uniform, for any $Z \in y$, $w(Z) = \kappa$. Hence $|\mathscr{U}_0(Z)| = \kappa$ and therefore p_y is uniform.

For a uniform $y \in \beta X \setminus X$, let $\theta_y = \mathrm{cf}({}^\kappa\omega/p_y)$.

Theorem 3.3.4. *(GCH) Let X be a locally compact metric space with no isolated points. Then for each uniform $y \in \beta X \setminus X$ there is a closed copy of $NU(\theta_y)$ embedded into $\beta X \setminus X \setminus \{y\}$.*

Proof. Let $y \in \beta X \setminus X$ be uniform and let $\theta = \theta_y$ and $p = p_y$. Let $\{[f_\alpha] : \alpha \in \theta\}$ be an unbounded sequence in ${}^\kappa\omega/p$. Denote $f_\alpha(\gamma)$ by $n(\alpha, \gamma)$.

Following Terasawa [19], we defines a sequence of locally finite collections in $\mathscr{B}$, $\{\xi_\alpha : \alpha \in \theta\}$, and sequence of closed sets intersecting to y, $\{H_\alpha : \alpha \in \theta\}$.

For $i \in \omega \setminus \{0\}$, let $\xi_i = \mathscr{B}_i$ and for $\omega \leq \alpha < \theta$ let $\xi_\alpha = \{B_{\gamma\sigma} : \gamma \in \kappa, \sigma \in {}^{n(\alpha,\gamma)}4\}$. We inductively define $\mathscr{N}_\alpha$, an ultrafilter on ξ_α. Suppose that $\mathscr{N}_\eta$ has been defined and that $\mathrm{cl}_X\, \mathscr{U} \in y$ for all $\eta < \alpha$ and $\mathscr{U} \in \mathscr{N}_\eta$. Let $Z \in y$ and $\mathscr{U}_\alpha(Z) = \{B \in \xi_\alpha : \mathrm{cl}_X\, B \cap Z \neq \emptyset\}$. Define $\mathscr{N}_\alpha^\sharp = \{\mathscr{U} \subset \xi_\alpha : \mathrm{cl}_X\, \mathscr{U}^* = \mathrm{cl}_X\, \mathscr{V}^*$ for some $\eta < \alpha$ and $\mathscr{V} \in \mathscr{N}_\eta\}$ and let $\hat{\mathscr{N}_\alpha} = \{\mathscr{U}_\alpha(Z) : Z \in y\} \cup \mathscr{N}_\alpha^\sharp$. The collection $\hat{\mathscr{N}_\alpha}$ has the finite intersection property by the induction hypothesis. Extend $\hat{\mathscr{N}}_\alpha$ to an ultrafilter, $\mathscr{N}_\alpha$, on ξ_α. As with $\mathscr{N}_0$, since $\mathscr{U}_\alpha(Z) \in \mathscr{N}_\alpha$ for each $Z \in y$, we have that $\mathscr{U}^* \in y$ for each $\mathscr{U} \in \mathscr{N}_\alpha$.

For $S \in p$ and $\alpha \in \theta$, let $\xi_\alpha(S) = \bigcup_{\gamma \in S}\{B_{\gamma\sigma} : \sigma \in {}^{n(\alpha,\gamma)}4\}$. Note that for any $\alpha, \delta \in \theta$, $\mathrm{cl}_X(\xi_\alpha(S))^* = \mathrm{cl}_X(\xi_\delta(S))^*$. From the definition of p, if $S \in p$ then $\xi_0(S) \in \mathscr{N}_0$. Therefore, $\xi_\alpha(S) \in \mathscr{N}_\alpha^\sharp \subset \mathscr{N}_\alpha$.

Claim. For $\alpha < \beta < \theta$ and $\mathscr{U} \in \mathscr{N}_\alpha$, there is $\mathscr{U}' \in \mathscr{N}_\beta$ such that $\mathrm{cl}_X\, \mathscr{U}'^* \subset \mathrm{cl}_X\, \mathscr{U}^*$.

Proof. For $\alpha < \beta \in \theta$, $f_\alpha <_p f_\beta$ and hence there is $S \in p$ such that $f_\alpha(\gamma) < f_\beta(\gamma)$ for all $\gamma \in S$. If $\mathscr{U} \in \mathscr{N}_\alpha$, then ξ_β refines $\mathscr{V} = \mathscr{U} \cap \xi_\alpha(S)$. Let $Z = cl_X \mathscr{V}^*$. Since $\mathscr{V} \in \mathscr{N}_\alpha$, $Z \in y$. Moreover, $\mathscr{U}_\beta(Z) \in \mathscr{N}_\beta$. Now, since ξ_β refines $\mathscr{V}$, $\mathrm{cl}_X(\mathscr{U}_\beta(Z))^* \subset \mathrm{cl}_X\, \mathscr{V}^*$. Let $\mathscr{U}' = \mathscr{U}_\beta(Z)$. □

Defining H_α.

Let $H_\alpha = \bigcap\{\mathrm{cl}_{\beta X}(\mathscr{U}^*) : \mathscr{U} \in \mathscr{N}_\alpha\}$.

Claim. If $\alpha < \beta$ then $H_\beta \subset H_\alpha$.

Proof. Suppose $\alpha < \beta$. Let $\mathscr{U} \in \mathscr{N}_\alpha$ be arbitrary. To prove the claim, we show that $H_\beta \subset \mathrm{cl}_{\beta X}(\mathscr{U}^*)$. From the previous claim, there is $\mathscr{U}' \in \mathscr{N}_\beta$ such that $\mathrm{cl}_X(\mathscr{U}'^*) \subset \mathrm{cl}_X(\mathscr{U}^*)$. Therefore, $\mathrm{cl}_{\beta X}(\mathscr{U}'^*) \subset \mathrm{cl}_{\beta X}(\mathscr{U}^*)$. Since H_β is the intersection over all such $\mathscr{U}'$, we have that $H_\beta \subset \mathrm{cl}_{\beta X}(\mathscr{U}^*)$. So, $H_\beta \subset H_\alpha$ and the claim is proven.

$\square$

Since $\{H_\alpha : \alpha \in \theta\}$ is a nested sequence of closed subsets of the compact space βX, we have that $\bigcap\{H_\alpha : \alpha \in \theta\} \neq \emptyset$.

Claim. $\bigcap\{H_\alpha : \alpha \in \theta\} = \{y\}$.

Proof. Let $W' \in \tau_y$. We will find $\alpha \in \theta$ such that $H_\alpha \subset W'$. Let $V', U' \in \tau_y$ be such that $\mathrm{cl}_{\beta X} V' \subset U' \subset \mathrm{cl}_{\beta X} U' \subset W'$. Let $U = X \cap U'$ and $V = X \setminus \mathrm{cl}_{\beta X} V'$. Then, $\{U,V\}$ is an open cover of X. Let $\xi \subset \mathscr{B}$ be a refinement as guaranteed in Lemma 0.3 (4). For each $\alpha \in \kappa$, since ξ is locally finite and $\mathrm{cl}_X(B_\alpha)$ is compact, $\{B \in \xi : B \subset B_\alpha\}$ is finite. So, $A_\alpha = \{n \in \omega : \text{there is } \sigma \in {}^n4 \text{ such that } B_{\alpha\sigma} \in \xi\}$ is also finite. Define $g(\alpha) = \max(A_\alpha \cup \{0\})$ and let $\xi' = \bigcup\{B_{\gamma\sigma} : \sigma \in {}^{g(\gamma)}4, \gamma \in \kappa\}$. Note, ξ' refines ξ and in turn refines $\{U,V\}$. Also, $g \in {}^\kappa\omega$ and hence there is $\alpha \in \theta$ such that $f_\alpha >_p g$. So, there is $S \in p$ such that $f_\alpha(\gamma) > g(\gamma)$ for all $\gamma \in S$. In other words, $\xi_\alpha(S)$ refines $\bigcup\{B_{\gamma\sigma} : \sigma \in {}^{g(\gamma)}4, \gamma \in S\} \subset \xi'$. Now, $Z = \mathrm{cl}_{\beta X} V' \cap X = X \setminus V \in y$, hence $\mathscr{U}_\alpha(Z) \in \mathscr{N}_\alpha$. Since $S \in p$, as noted before, $\xi_\alpha(S) \in \mathscr{N}_\alpha$, so, $\mathscr{V} = \mathscr{U}_\alpha(Z) \cap \xi_\alpha(S) \in \mathscr{N}_\alpha$. Let $B \in \mathscr{V}$. Then, since $B \in \mathscr{U}_\alpha(Z)$, $B \cap Z = B \cap (X \setminus V) \neq \emptyset$. But since $B \in \xi_\alpha(S)$, there is $B' \in \xi'$ such that $B \subset B'$. Therefore $B' \cap (X \setminus V) \neq \emptyset$ and since ξ' refines $\{U,V\}$, it must be the case that $B' \subset U$. Hence $B \subset U \subset U'$ and therefore $\mathscr{V}^* \subset U' \subset \mathrm{cl}_{\beta X} U'$. Finally, since $\mathscr{V} \in \mathscr{N}_\alpha$, $H_\alpha \subset \mathrm{cl}_{\beta X} \mathscr{V}^* \subset \mathrm{cl}_{\beta X} U' \subset W'$. $\square$

Defining the $\mathscr{L}^i_\alpha$'s

For $\alpha \in \theta$ and $i = 0,1$ define $\mathscr{L}^i_\alpha = \{B_{\gamma\sigma^\frown i} : \gamma \in \kappa, \sigma \in {}^{n(\alpha,\gamma)}4\}$.

Claim. For all $\alpha \in \kappa^+$, $\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^0_\alpha) \cap \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^1_\alpha) = \emptyset$.

Proof. For each $\gamma \in \kappa$ and $\sigma \in {}^i4$, $\mathrm{cl}_X B_{\gamma\sigma^\frown 0} \cap \mathrm{cl}_X B_{\gamma\sigma^\frown 1} = \emptyset$. Also, $B_{\gamma\sigma} \cap B_{\gamma\beta} = \emptyset$ for $\sigma \neq \beta \in {}^{n(\alpha,\gamma)}4$, and for $i = 0,1$ we have $\mathrm{cl}_X B_{\gamma\sigma^\frown i} \subset B_{\gamma\sigma}$ and $\mathrm{cl}_X B_{\gamma\beta^\frown i} \subset B_{\gamma\beta}$. Therefore

$\mathrm{cl}_X B_{\gamma\sigma^\frown i} \cap \mathrm{cl}_X B_{\gamma\beta^\frown j} = \emptyset$ for $i, j = 0, 1$. So,

$\bigcup\{\mathrm{cl}_X B_{\gamma\sigma^\frown 0} : \sigma \in {}^{n(\alpha,\gamma)}4\} \cap \bigcup\{\mathrm{cl}_X B_{\gamma\sigma^\frown 0} : \sigma \in {}^{n(\alpha,\gamma)}4\} = \emptyset$. Now, since $\{B_{\gamma\emptyset} : \gamma \in \kappa\}$ is a locally finite family and since $\mathrm{cl}_X B_{\gamma\sigma^\frown i} \subset B_{\gamma\emptyset}$ for each $\sigma \in \bigcup_{n\in\omega} {}^n 4$ and $i = 0, 1$, we have that $\mathrm{cl}_X(\bigcup \mathscr{L}^0_\alpha) \cap \mathrm{cl}_X(\bigcup \mathscr{L}^1_\alpha) =$

$\bigcup\{\mathrm{cl}_X B_{\gamma\sigma^\frown 0} : \sigma \in {}^{n(\alpha,\gamma)}4, \gamma \in \kappa\} \cap \bigcup\{\mathrm{cl}_X B_{\gamma\sigma^\frown 0} : \sigma \in {}^{n(\alpha,\gamma)}4, \gamma \in \kappa\} = \emptyset$. Finally, since $\mathrm{cl}_X(\bigcup \mathscr{L}^0_\alpha) \cap \mathrm{cl}_X(\bigcup \mathscr{L}^1_\alpha) = \emptyset$, $\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^0_\alpha) \cap \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^1_\alpha) = \emptyset$. $\square$

Since $\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^0_\alpha) \cap \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^1_\alpha) = \emptyset$, y can be in at most one of $\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^0_\alpha)$ or $\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^1_\alpha)$. Without loss of generality, assume $y \notin \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^0_\alpha)$ for each $\alpha \in \theta$.

A special case of the following claim, in particular when Φ is constant, is proven in [[19], Lemma 3].

Claim. For any $\alpha < \theta$ and $\Phi : D \subset [\alpha, \theta) \to 2$, the collection $\{H_\alpha\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^{\Phi(\gamma)}_\gamma) : \gamma \in D\}$ has nonempty intersection.

Proof. Let $\alpha < \theta$ and $\Phi : D \to 2$ for some $D \subset [\alpha, \theta)$. To prove the claim we show that $\{\mathrm{cl}_{\beta X} \mathscr{U}^* : \mathscr{U} \in \mathscr{N}_\alpha\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^{\Phi(\gamma)}_\gamma) : \gamma \geq \alpha\}$ has the finite intersection property. Let $\mathscr{U}_1, \dots, \mathscr{U}_n \in \mathscr{N}_\alpha$ and let $\gamma_1, \dots, \gamma_m \in D$ be such that $\gamma_m \geq \cdots \geq \gamma_1 \geq \alpha$. Since $\mathscr{N}_\alpha$ is an ultrafilter, there is $\mathscr{U} \in \mathscr{N}_\alpha$ such that $\mathscr{U} \subset \bigcap\{\mathscr{U}_i : 1 \leq i \leq n\}$. Since $f_\alpha <_p f_{\gamma_1} <_p \cdots <_p f_{\gamma_m}$, there is $S \in p$ such that $f_\alpha(\mu) < f_{\gamma_1}(\mu) < \cdots < f_{\gamma_m}(\mu)$ for all $\mu \in S$, in other words $n(\alpha, \mu) < n(\gamma_1, \mu) < \cdots < n(\gamma_m, \mu)$. Since $\xi_\alpha(S) \in \mathscr{N}_\alpha$, $\xi_\alpha(S) \cap \mathscr{U} \neq \emptyset$. Hence there is $\mu \in S$ and $\sigma \in {}^{n(\alpha,\mu)}4$ such that $B_{\mu\sigma} \in \xi_\alpha(S) \cap \mathscr{U}$. Define $\sigma' \in {}^{n(\gamma_m,\mu)+1}4$ as follows: $\sigma'|_{n(\alpha,\mu)} = \sigma$, $\sigma'(n(\gamma_i, \mu) + 1) = \Phi(\gamma_i)$ for each $1 \leq i \leq m$ and $\sigma'(k) = 0$ otherwise. Then, $B_{\mu\sigma'} \subset B_{\mu\sigma}$, since σ' extends σ hence $B_{\mu\sigma'} \subset \mathscr{U}^*$. Furthermore, $B_{\mu\sigma} \subset \bigcup \mathscr{L}^{\Phi(\gamma_i)}_{\gamma_i}$ since σ' extends $\sigma'|_{n(\gamma_i,\mu)+1} = \sigma'|_{n(\gamma_i,\mu)}{}^\frown \Phi(\gamma_i)$ and $B_{\mu,\sigma'|_{n(\gamma_i,\mu)}{}^\frown \Phi(\gamma_i)} \in \mathscr{L}^{\Phi(\gamma_i)}_{\gamma_i}$. $\square$

We follow the argument found in [1] to embed $NU(\theta)$ into $\beta X \setminus (X \cup \{y\})$, using the $\mathscr{L}_\alpha$'s to play the role of the reaping sets.

The induction

Denote by θ the discrete space of size θ. We define an embedding, g, of θ into $\beta X \setminus X$ such that

1. $y \in \mathrm{cl}_{\beta X}\, g[A]$ if and only if $|A| = \theta$.

2. If $A, B \in [\theta]^{<\theta}$ and $A \cap B = \emptyset$ then $\mathrm{cl}_{\beta X}\, g[A] \cap \mathrm{cl}_{\beta X}\, g[B] = \emptyset$.

Then, we extend g to $\beta g : \beta\theta \to \beta X \setminus X$ and prove that $U(\theta) = g^{\leftarrow}[\{y\}]$. Therefore $\beta X \setminus (X \cup \{y\})$ contains a closed copy of $NU(\theta)$.

Since we assume GCH we have that $\theta^{<\theta} = \theta$. List $\theta \cup \{(A,B) : A, B \in [\theta]^{<\theta}$ and $A \cap B = \emptyset\}$ as $\{T_\eta : \eta \in \theta\}$ in such a way that if $T_\eta = (A,B)$, then $\eta \geq \sup(A \cup B)$ and if $T_\eta \in \theta$, then $\eta \geq T_\eta$.

For $\alpha \in \theta$ let $D_\alpha = \{\eta : T_\eta = (A,B)$ and $\alpha \in A \cup B\} \cup \{\eta : \alpha \in T_\eta\}$. Note that $D_\alpha \subset [\alpha, \theta)$.

For each $\alpha \in \theta$ we define $\Phi_\alpha : D_\alpha \to 2$ and choose $g(\alpha)$ to be any element of $\bigcap(\{H_\alpha\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}_\gamma^{\Phi_\alpha(\gamma)}) : \gamma \in D\})$. We define Φ_α by induction.

Let $\eta \in \theta$ and assume we have defined $\Phi_\alpha|_{\eta \cap D_\alpha}$. If $T_\eta \in \theta$, let $\Phi_\beta(\eta) = 0$ for all $\beta < T_\eta$. If $T_\eta = (A,B)$, let $\Phi_\beta(\eta) = 0$ for all $\beta \in A$ and let $\Phi_\beta(\eta) = 1$ for all $\beta \in B$.

Let $K_\alpha = \bigcap(\{H_\alpha\} \cup \{\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}_\gamma^{\Phi_\alpha(\gamma)}) : \gamma \in D_\alpha\}) = \emptyset$. By the claim, $K_\alpha \neq \emptyset$ for each $\alpha \in \theta$, so we may choose $g(\alpha) \in K_\alpha$.

To show 1., let $A \subset \theta$ be such that $|A| < \theta$. There is $\gamma \in \theta$ such that $A \subset [0, \gamma)$. Let η be such that $T_\eta = \gamma$. Note, $\eta \geq \gamma$. For any $\alpha < \gamma = T_\eta$, $\Phi_\alpha(\eta) = 0$. So, for $\alpha \in A$, $K_\alpha \subset \mathscr{L}_\eta^0$. But, $y \notin \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}_\eta^0)$. Hence, $y \notin \mathrm{cl}_{\beta X}\, g[A]$. For the other direction, let $A \subset \theta$ be such that $|A| = \theta$. Since θ is regular, A is unbounded in θ. Let $U \in \mathscr{N}$. There is

$\gamma \in \theta$ such that $H_\gamma \subset U$. For $\alpha \in A$ such that $\alpha \geq \gamma$, $g(\alpha) \in H_\alpha \subset H_\gamma \subset U$. Hence $y \in \mathrm{cl}_{\beta X}\, g[A]$.

To show 2., let $A, B \in [\theta]^{<\theta}$ be such that $A \cap B = \emptyset$. Let η be such that $T_\eta = (A, B)$. Then, for each $\alpha \in A$, $\Phi_\alpha(\eta) = 0$ and for each $\alpha \in B$, $\Phi_\alpha(\eta) = 1$. Hence $g(\alpha) \in K_\alpha \subset \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^0_\eta)$ for $\alpha \in A$ and $g(\alpha) \in K_\alpha \subset \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^1_\eta)$ for $\alpha \in B$. But, $\mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^0_\eta) \cap \mathrm{cl}_{\beta X}(\bigcup \mathscr{L}^1_\eta) = \emptyset$. Hence $\mathrm{cl}_{\beta X}\, g[A] \cap \mathrm{cl}_{\beta X}\, g[B] = \emptyset$. Note, 2. implies g is one-to-one.

Since θ is discrete, g is continuous. Extend g to $\beta g : \beta\theta \to \beta X \setminus X$.

Since $\beta\theta$ is compact, βg is a closed map. In order to show that βg maps $NU(\theta)$ homeomorphically to a closed subset of $\beta X \setminus (X \cup \{y\})$, we must verify the following:

1. $\beta g[\beta\theta] \setminus \{y\} = \beta g[NU(\theta)]$

2. $\beta g[U(\theta)] \subset \{y\}$

3. $\beta g|_{NU(\theta)}$ is one-to-one

If 1. holds, $NU(\theta)$ is mapped onto a closed subset of $\beta X \setminus (X \cup \{y\})$. If 1. and 2. hold, then $NU(\theta)$ is a full preimage. Since βg is a closed continuous map by 1.2.6, $\beta g|_{NU(\theta)}$ is a closed continuous map. Therefore if 3. holds, $\beta g|_{NU(\theta)}$ is a homeomorphism.

1. Let $q \in NU(\theta)$. There is $A \subset \theta$ such that $|A| < \theta$ and $A \in q$. Since βg is continuous, $g(q) \in \mathrm{cl}_{\beta X}\, g[A]$. Hence, $g(q) \neq y$. Let $z \in \beta g[\beta\theta] \setminus \{y\}$. Let U be an open neighborhood of z such that $y \notin \mathrm{cl}_{\beta X} U$. Since βg is continuous, $A' = U \cap \beta g[\theta] \neq \emptyset$. Let $A = g^{\leftarrow}[A']$. Since $y \notin \mathrm{cl}_{\beta X} U$, $|A| < \theta$, otherwise $y \in \mathrm{cl}_{\beta X} A' \subset \mathrm{cl}_{\beta X} U$. If $q \in g^{\leftarrow}(z)$ then $q \in \mathrm{cl}_{\beta\theta} A$. Hence $q \in NU(\theta)$ and therefore $z \in \beta g[NU(\theta)]$.

2. The preceding argument also shows that for any $q \in \beta\theta$, if $\beta g(q) \neq y$, then $q \in NU(\theta)$. Hence $\beta g[U(\theta)] \subset \{y\}$.

3. Let $q \neq q' \in NU(\theta)$. There are $A, B \in [\theta]^{<\theta}$ such that $A \cap B = \emptyset$ and $q \in \mathrm{cl}_{\beta\theta} A$ and $q' \in \mathrm{cl}_{\beta\theta} B$. By continuity, $g(q) \in \mathrm{cl}_{\beta X} g[A]$ and $g(q') \in \mathrm{cl}_{\beta X} g[B]$. But, by 2. $\mathrm{cl}_{\beta X} g[A] \cap \mathrm{cl}_{\beta X} g[B] = \emptyset$. Hence $g(q) \neq g(q')$.

$\square$

Corollary 3.3.5. *(GCH) Let X be a locally compact metric space with no isolated points. If p_y is regular, then each uniform $y \in \beta X \setminus X$ is a non-normality point of $\beta X \setminus X$.*

Proof. If p_y is regular, by lemma 3.3.3 $\theta_y = \mathrm{cf}({}^{\kappa}\omega/p_y) > \kappa$. Certainly, $\mathrm{cf}({}^{\kappa}\omega/p_y) \leq 2^{\kappa}$, so by GCH, $\theta_y = \kappa^+ = 2^{\kappa}$ and hence θ_y is regular and not a strong limit. By 3.2.1, $NU(\theta_y)$ is not normal. Hence, by the theorem, y is a non-normality point of $\beta X \setminus X$. $\square$

Corollary 3.3.6. *(GCH) Suppose all ultrafilters are regular. Let X be a locally compact metric space with no isolated points. Then each $y \in \beta X \setminus X$ is a non-normality point of $\beta X \setminus X$.*

Proof. Suppose all ultrafilters are regular. Then p_y is regular for all $y \in \beta X \setminus X$. We have seen that if $y \in \beta X \setminus X$ is uniform then it is a non-normality point of $\beta X \setminus X$. Suppose that $y \in \beta X \setminus X$ is not uniform. That is, there exists $Z \in y$ such that $w(Z) < w(X)$. Let $Z \in y$ be such that $w(Z)$ is minimum. Then, there is a cover of Z consisting of sets $\mathrm{cl}\, B$ from a subcollection, $\mathscr{Z}$, of $\mathscr{B}_0$ of size $w(Z)$. Let $Y = \bigcup\{\mathrm{cl}\, B : B \in \mathscr{Z}\}$. Since $\mathscr{B}_0$ is locally finite, Y is closed. Each $B \in \mathscr{Z}$ has no isolated points, so Y has no isolated points. Also, $y \in \mathrm{cl}_{\beta X} Y$. Since X is normal and Y is closed, Y is C^*-embedded in X. Therefore, $\beta Y = \mathrm{cl}_{\beta X} Y$ and $y|_Y$ is uniform on Y. So, by the theorem, y is a non-normality point of the set $\mathrm{cl}_{\beta X} Y \setminus Y$ and hence is a non-normality point of $\beta X \setminus X$.

$\square$

Chapter 4

Open questions

4.1 Coarser connected topologies

We know that a metric space has a coarser connected Hausdorff topology if and only if it is not compact [11]. We also know that if a metric space has weight $\geq \mathfrak{c}$ then it has a coarser connected metric topology if and only if it is not compact 2.3.8. There are still open questions about coarser connected metrizable topologies of spaces with smaller weight.

Question 4.1.1. Which non-compact metric spaces have coarser connected metrizable topologies?

Druzhinina [4] asks the following question.

Question 4.1.2. Let X be a dense G_δ-subset of a connected metrizable space. Does X have a coarser connected metrizable topology?

Fleissner, Porter and Roitman, in [7] and [8], investigated coarser connected topologies on ordinal spaces. They characterized all ordinal spaces that have a coarser connected Hausdorff topology. An ordinal δ has a 'minimal decomposition' of the form $\alpha + \beta$ where $\alpha \leq 2^{|\beta|}$ if and only if δ has a coarser connected Hausdorff topology.

From [21], no ordinal has a coarser connected regular topology. Urysohn is a separation property stronger than Hausdorff and weaker than regular. If an ordinal has a coarser connected Urysohn topology, then it has cofinality $\aleph_0$ [8]. The following is an open problem:

Question 4.1.3. Which ordinals of countable cofinality have coarser connected Urysohn topologies?

4.2 Non-normality points

The special π-base for the metric space X was important in the proof of Theorem 3.3.4. In particular, since X was locally compact, every member, B, of the π-base had weight $\aleph_0$. It seems similar techniques can be applied to a metric space that is not necessarily locally compact, but has a homogeneous π-base. We mean by homogeneous that each member of the π-base has the same weight or even if every pair of comparable members of the π-base have the same weight. More generally, we would like to know if the following is true.

Question 4.2.1. Under GCH, is every $y \in \beta X \setminus X$ a non-normality point of $\beta X \setminus X$, for a metric space X without isolated points?

The following are technical questions which would help to answer the above general question.

Question 4.2.2. Can there be a cardinal κ and an ultrafilter p on $D(\kappa)$ such that $\mathrm{cf}(\omega^\kappa/p)$ is uncountable and weakly compact?

For more information about the above questions see the following paper by Jin and Shelah [13].

Question 4.2.3. Let X be a metric space, or more generally, any completely regular space. Let y be a z-ultrafilter on X. Is there $Z_0 \in y$ such that y relative to Z_0 is a remote point? In other words, is there $Z_0 \in y$ such that for all $Z \in y$, $\mathrm{int}_{Z_0}(Z \cap Z_0) \neq \emptyset$?

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