$\alpha'(u, v, w, a_i, b_j) = ((\alpha\otimes\beta)\oslash(\alpha\otimes\beta)^{-A})(u, v, w, a_i, b_j)$

$= P(a_i|u, v)\ P(b_j|v, w, a_i) / \Sigma\{P(a_i|u, v)\ P(b_j|v, w, a_i) \mid a_i \in \Omega_A\}.$

The resulting BN is given on the right-hand side of Figure 4.

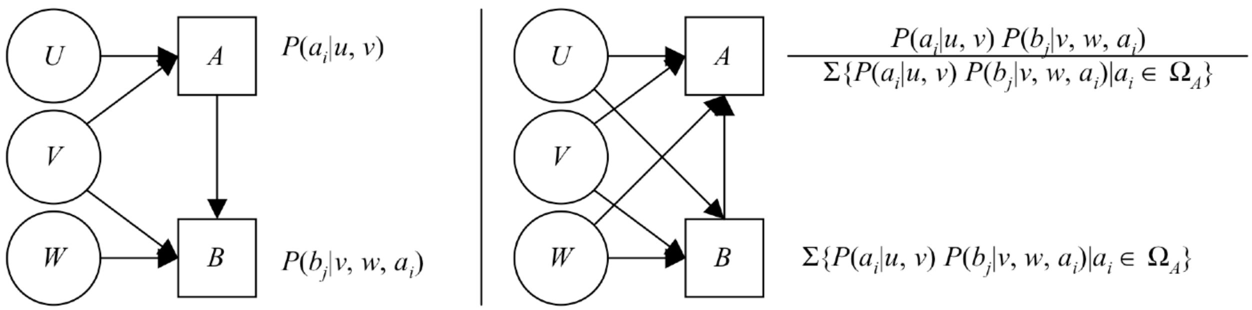

Figure 4. Arc reversal between two discrete nodes.

3.2 Two Continuous Variables

In this section, we describe arc reversals between two continuous variables. Consider the BN given on the left-hand side of Figure 5. In this BN, X has conditional PDF $f(x \mid u, v)$ and Y has conditional PDF $g(y \mid v, w, x)$. Let ξ and ψ denote the continuous potentials at X and Y, respectively, before arc reversal, and ξ' and ψ' after arc reversal. Then,

$$\xi(u, v, x) = f(x \mid u, v),$$

$$\psi(v, w, x, y) = g(y \mid v, w, x),$$

$$\psi'(u, v, w, y) = (\xi\otimes\psi)^{-X}(u, v, w, y) = \int f(x \mid u, v)\ g(y \mid v, w, x)\ \mathrm{d}x,$$

$$\xi'(u, v, w, x, y) = ((\xi\otimes\psi)\oslash(\xi\otimes\psi)^{-X})(u, v, w, x, y) = f(x \mid u, v)\ g(y \mid v, w, x) / (\int f(x \mid u, v)\ g(y \mid v, w, x)\ \mathrm{d}x).$$

The resulting BN is shown on the right-hand side of Figure 5.

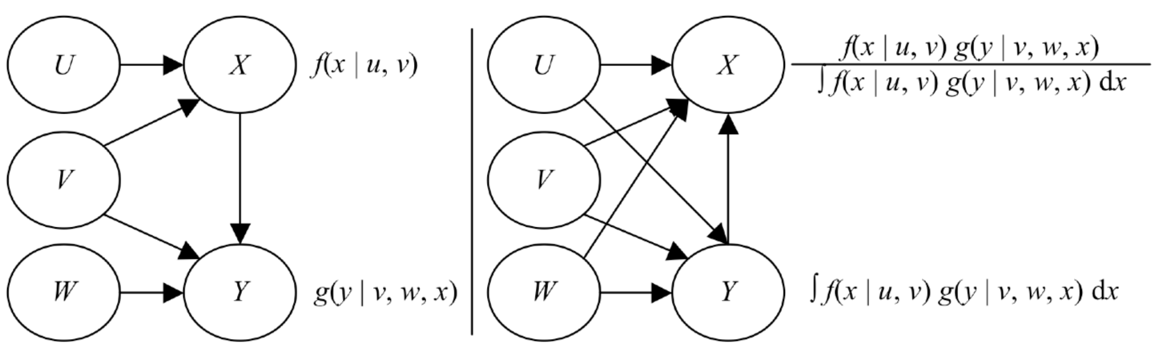

Figure 5. Arc reversal between two continuous nodes.

3.3 Continuous to Deterministic

As we have already discussed, the arc reversal between a continuous and a deterministic variable is slightly different from the arc reversal between two continuous variables since their joint PDF does not exist. After arc reversal, we transfer the density from the continuous node to the deterministic node, which results in the deterministic node being continuous and the continuous node having a Dirac potential.

Consider the situation shown in Figure 6. In this BN, X has continuous parents U and V, and Y has continuous parents V and W in addition to X. The density at X is f and the equation at Y is $Y = h(V, W, X)$. We assume h is invertible in X and differentiable on Ω_X. The potentials before and after arc reversals are as follows.

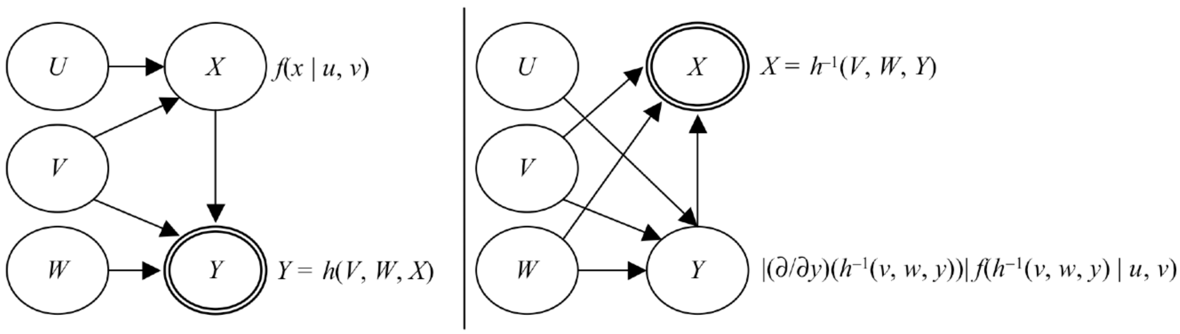

Figure 6. Arc reversal between a continuous and a deterministic variable.

$$\xi(u, v, x) = f(x \mid u, v),$$

$$\psi(v, w, x, y) = \delta(y - h(v, w, x)),$$

$$\psi'(u, v, w, y) = (\xi \otimes \psi)^{-X}(u, v, w, y) = \int f(x \mid u, v)\, \delta(y - h(v, w, x))\, dx$$

$$= |(\partial/\partial y)(h^{-1}(v, w, y))|\, f(h^{-1}(v, w, y) \mid u, v), \text{ and}$$

$$\xi'(u, v, w, x, y) = ((\xi \otimes \psi) \oslash (\xi \otimes \psi)^{-X})(u, v, w, x, y)$$

$$= f(x \mid u, v)\, \delta(y - h(v, w, x)) \,/\, (|(\partial/\partial y)(h^{-1}(v, w, y))|\, f(h^{-1}(v, w, y) \mid u, v))$$

$$= \delta(x - h^{-1}(v, w, y)).$$

After we reverse the arc (X, Y), both X and Y inherit each other's parents, but X loses U as a parent. Also, Y has a density function and X has a deterministic conditional distribution. The determinism of the conditional for X after arc reversal is a consequence of the invertibility of the relationship at Y before arc reversal. The resulting BN is given on right-hand side of Figure 6. Also, some of the qualitative conclusions here, namely X loses U as a parent, Y has a density function, and X has a deterministic conditional distribution, are based on the assumption that U, V, W are continuous. If any of these are discrete, the conclusion can change, as we will demonstrate in Section 4.

As an example, consider the BN consisting of two continuous variables and a deterministic variable whose function is the sum of its two parents as shown in Figure 7. $X \sim f(x)$, $Y|x \sim g(y \mid x)$, and $Z = X + Y$. Let ξ, ψ, ζ denote the potentials associated with X, Y, and Z, respectively, before arc reversal, and ψ' and ζ' denote the revised potentials associated with Y and Z, respectively, after reversal of arc (Y, Z). Then,

$$\xi(x) = f(x),$$

$$\psi(x, y) = g(y \mid x),$$

$$\zeta(x, y, z) = \delta(z - x - y),$$

$$\zeta'(x, z) = (\psi \otimes \zeta)^{-Y}(x, z) = \int g(y \mid x)\, \delta(z - x - y)\, dy = \int g(y \mid x)\, \delta(y - (z - x))\, dy = g(z - x \mid x), \text{ and}$$

$$\psi'(x, y, z) = ((\psi \otimes \zeta) \oslash (\psi \otimes \zeta)^{-Y})(x, y, z) = g(y \mid x)\, \delta(z - x - y) \,/\, g(z - x \mid x) = \delta(y - (z - x)).$$

If we reverse the arc (X, Z) in the revised BN, we obtain the marginal distribution of Z,

$$\zeta''(z) = (\xi \otimes \zeta')^{-X}(z) = \int f(x)\, g(z - x \mid x)\, dx,$$

which is the convolution formula for Z. The revised potential at X,

$$\xi'(x, z) = ((\xi \otimes \zeta') \oslash (\xi \otimes \zeta')^{-X})(x, z) = f(x)\, g(z - x \mid x) / (\int f(x)\, g(z - x \mid x)\, dx),$$

represents the conditional distribution of X given z.

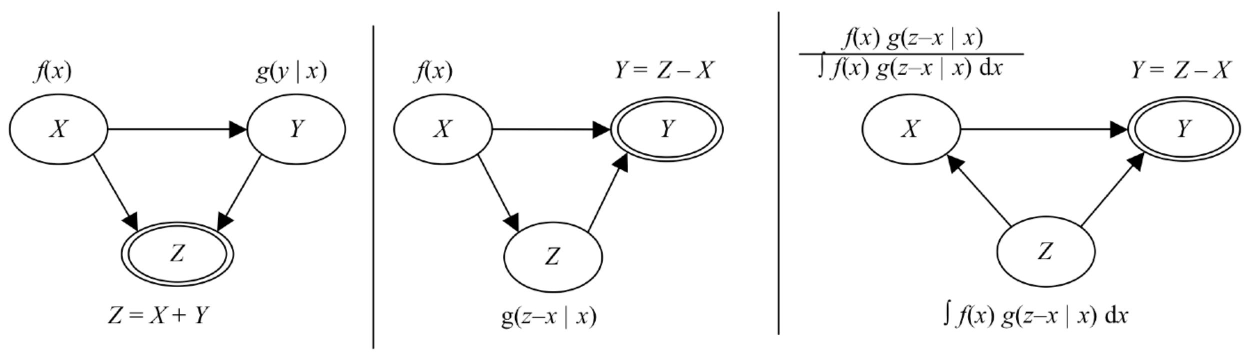

Figure 7. A continuous BN with a deterministic variable.

We have assumed that the function describing the deterministic variable is invertible and differentiable. Let us consider the case where the function is not invertible, but has known simple zeros, and is differentiable. For example, consider a BN with two continuous variables X and Y, where X has PDF $f(x)$ and Y is a deterministic function of X described by the function $Y = X^2$ as shown in Figure 8.

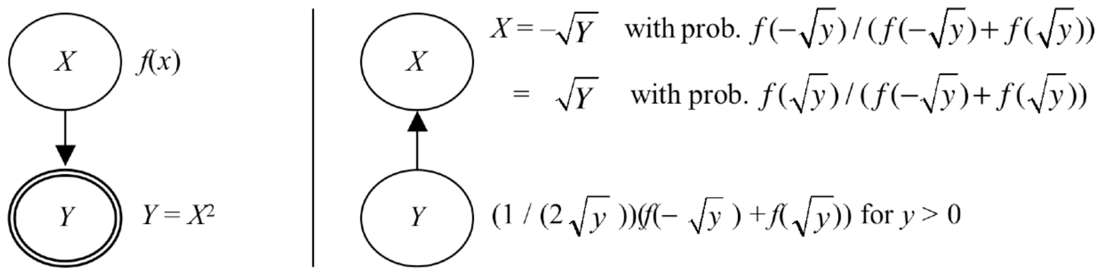

Figure 8. Arc reversal between a continuous node and a deterministic node with a non-invertible function.

This function is not invertible, but $y - x^2$ has two simple zeros at $x = \pm\sqrt{y}$. Suppose ξ and ψ denote the continuous potentials at *X* and *Y*, respectively, before arc reversal, and ξ′ and ψ′ after arc reversal. Then

$$\xi(x) = f(x),$$

$$\psi(x, y) = \delta(y - x^2) = \delta(x^2 - y) = (\delta(x + \sqrt{y}) + \delta(x - \sqrt{y}))/(2\sqrt{y})$$

$$\psi'(y) = (\xi\otimes\psi)^{-X}(y) = \int f(x)\, (\delta(x + \sqrt{y}) + \delta(x - \sqrt{y}))/(2\sqrt{y})\, dx$$

$$= (f(-\sqrt{y}) + f(\sqrt{y})) / (2\sqrt{y}), \text{ for all } y > 0.$$

$$\xi'(x, y) = f(x)\, (\delta(x + \sqrt{y}) + \delta(x - \sqrt{y})) / (f(-\sqrt{y}) + f(\sqrt{y}))$$

$$= (f(-\sqrt{y})\, \delta(x + \sqrt{y}) + f(\sqrt{y})\, \delta(x - \sqrt{y}) / (f(-\sqrt{y}) + f(\sqrt{y})), \text{ for all } y > 0.$$

Notice that the revised conditional for *X* is not deterministic if $f(-\sqrt{y}) > 0$ and $f(\sqrt{y}) > 0$, but it is a Dirac potential. The revised potential for *Y* is a density potential.

If the deterministic function is such that it's zeros are not so easily determined or it is not differentiable, then we would not be able to write a closed form expression for the distributions of *X* and *Y* after arc reversal.

3.4 Deterministic to Continuous

In this subsection, we describe arc reversal between a deterministic and a continuous variable. Consider a BN as shown on the left-hand side of Figure 9. *X* is a deterministic variable associated with a function, $X = h(U, V)$, and *Y* is a continuous variable and the conditional distribution of $Y|(v, w, x)$ is distributed as $g(y\,|v, w, x)$. Suppose we wish to reverse the arc (*X*, *Y*). Since there is no density potential at *X*, Shenoy [2006] suggests to first reverse arc (*U*, *X*) or (*V*, *X*) (resulting in a density potential at *X*), and then reverse arc (*X*, *Y*) using the rules for arc reversal between two continuous nodes. However, here we show that it is possible to reverse an arc between a deterministic node and a continuous node directly without having to reverse other arcs.

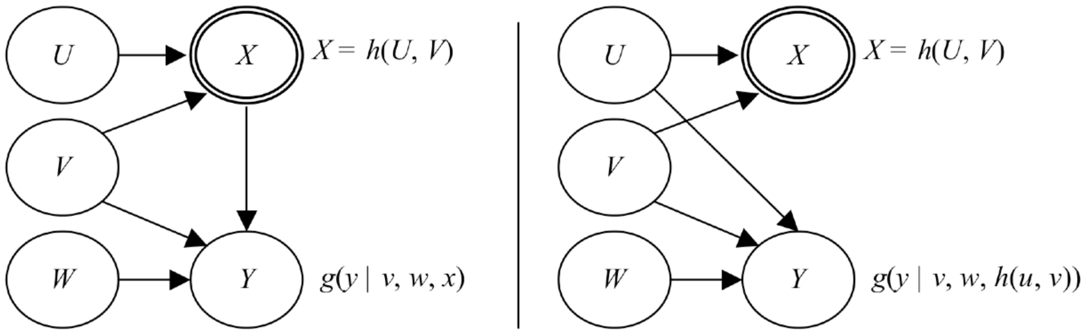

Figure 9. Arc reversal between a deterministic and a continuous node.

Consider again the BN given on left-hand side of Figure 9. Suppose we wish to reverse the arc (X, Y). Let ξ and ψ denote the continuous potentials at X and Y, respectively, before arc reversal, and ξ' and ψ' after arc reversal. Then,

$$\xi(u, v, x) = \delta(x - h(u, v)),$$

$$\psi(v, w, x, y) = g(y \mid v, w, x),$$

$$\psi'(u, v, w, y) = ((\xi \otimes \psi)^{-X})(u, v, w, y) = \int \delta(x - h(u, v))\, g(y \mid v, w, x)\, \mathrm{d}x = g(y \mid v, w, h(u, v)), \text{ and}$$

$$\xi'(u, v, w, x, y) = (\xi \otimes \psi) \oslash ((\xi \otimes \psi)^{-X})(u, v, w, x, y) = \delta(x - h(u, v))\, g(y \mid v, w, x) / g(y \mid v, w, h(u, v))$$

$$= \delta(x - h(u, v)).$$

Notice that ξ' does not depend on either W or Y. Thus, after arc reversal, there is no arc from Y to X, i.e., the arc being reversed disappears, and X does not inherit an arc from W. The resulting BN is shown on the right-hand side of Figure 9.

3.5 Deterministic to Deterministic

In this subsection, we describe arc reversal between two deterministic variables. Consider the BN on the left-hand side of Figure 10. X is a deterministic function of its parents $\{U, V\}$, and Y is also a deterministic function of its parents $\{X, V, W\}$. Suppose we wish to reverse the arc (X, Y). Let ξ and ψ denote the potentials associated with X and Y, respectively, before arc reversal, and ξ' and ψ' after arc reversal. Then,

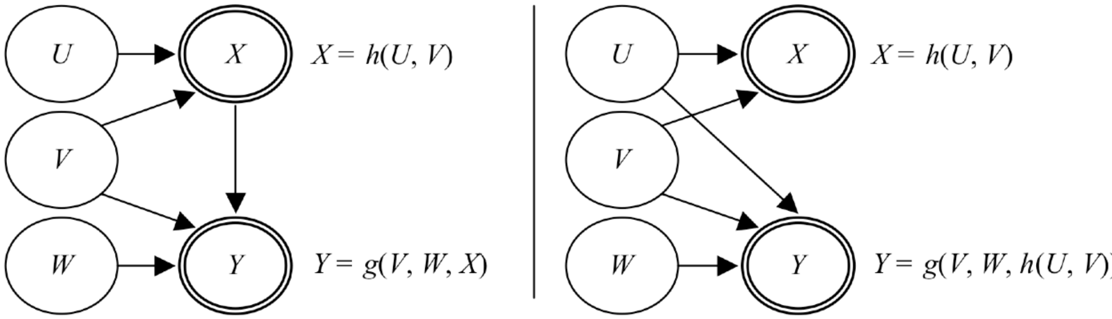

Figure 10. Arc reversal between two deterministic nodes.

$$\xi(u, v, x) = \delta(x - h(u, v)),$$

$$\psi(v, w, x, y) = \delta(y - g(v, w, x)),$$

$$\psi'(u, v, w, y) = (\xi \otimes \psi)^{-X}(u, v, w, y) = \int \delta(x - h(u, v))\, \delta(y - g(v, w, x))\, dx$$

$$= \delta(y - g(v, w, h(u, v))), \text{ and}$$

$$\xi'(u, v, w, x, y) = ((\xi \otimes \psi) \oslash (\xi \otimes \psi)^{-X})(u, v, w, x, y)$$

$$= \delta(x - h(u, v))\, \delta(y - g(v, w, x)) / \delta(y - g(v, w, h(u, v))) = \delta(x - h(u, v)).$$

N otice that ξ' does not depend on either Y or W. The arc being reversed disappears, and X does not inherit a parent of Y.

3.6 Continuous to Discrete

In this section, we will describe arc reversal between a continuous and a discrete node. Consider the BN as shown in Figure 11. X is a continuous node with conditional PDF $f(x \mid u, v)$, and A is a discrete node with conditional masses $P(a_i \mid v, w, x)$ for each $a_i \in \Omega_A$. Let ξ and α denote the density and discrete potentials associated with X and A, respectively, before arc reversal, and ξ' and α' after arc reversal. Then

$$\xi(u, v, x) = f(x \mid u, v),$$

$$\alpha(v, w, x, a_i) = P(a_i \mid v, w, x),$$

$$\alpha'(u, v, w, a_i) = (\xi \otimes \alpha)^{-X}(u, v, w, a_i) = \int f(x \mid u, v)\, P(a_i \mid v, w, x)\, dx, \text{ and}$$

$$\xi'(u, v, w, x, a_i) = ((\xi \otimes \alpha) \oslash (\xi \otimes \alpha)^{-X})(u, v, w, x, a_i)$$

$$= f(x \mid u, v)\, P(a_i \mid v, w, x) / (\int f(x \mid u, v)\, P(a_i \mid v, w, x)\, dx).$$

The BN on the RHS of Figure 11 depicts the results after arc reversal.

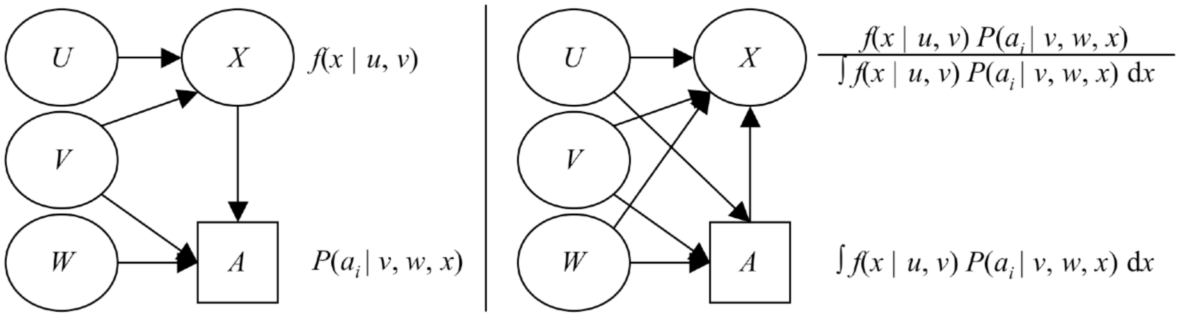

Figure 11. Arc reversal between a continuous and a discrete node.

For a concrete example, consider the simpler hybrid BN shown on the LHS of Figure 12. X is a continuous variable, distributed as $N(0, 1)$. A is a discrete variable with two states $\{a_1, a_2\}$. The conditional probability mass functions of A are as follows: $P(a_1 \mid x) = 1/(1 + e^{-2x})$ and $P(a_2 \mid x) = e^{-2x}/(1 + e^{-2x})$. Let α and ξ denote the potentials associated with A and X, respectively, before arc reversal, and α' and ξ' after arc reversal. Then,

$\alpha(a_1, x) = 1/(1 + e^{-2x})$,

$\alpha(a_2, x) = e^{-2x}/(1 + e^{-2x})$,

$\xi(x) = \varphi_{0,1}(x)$, where $\varphi_{0,1}(x)$ is the PDF of the standard normal distribution,

$\alpha'(a_1) = (\alpha\otimes\xi)^{-X}(a_1) = \int (1/(1 + e^{-2x}))\ \varphi_{0,1}(x)\ \mathrm{d}x = 0.5$,

$\alpha'(a_2) = (\alpha\otimes\xi)^{-X}(a_2) = \int (e^{-2x}/(1 + e^{-2x})\ \varphi_{0,1}(x)\ \mathrm{d}x = 0.5$,

$\xi'(a_1, x) = ((\alpha\otimes\xi)\oslash(\alpha\otimes\xi)^{-X})(a_1, x) = (1/(1 + e^{-2x}))\ \varphi_{0,1}(x))/0.5 = (2/(1 + e^{-2x}))\ \varphi_{0,1}(x)$,

$\xi'(a_2, x) = (\alpha\otimes\xi)\oslash(\alpha\otimes\xi)^{-X}(a_2, x) = (e^{-2x}/(1 + e^{-2x}))\ \varphi_{0,1}(x))/0.5 = (2e^{-2x}/(1 + e^{-2x}))\ \varphi_{0,1}(x)$,

The resulting BN after the arc reversal is given on the RHS of Figure 12.

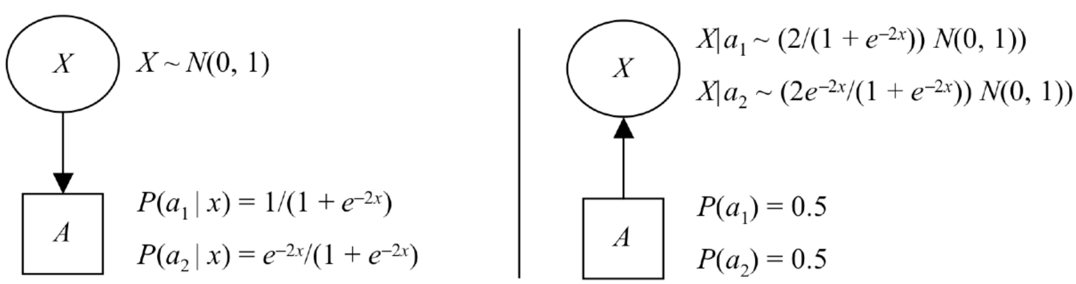

Figure 12. Arc reversal between a continuous and a discrete node.

3.7 Deterministic to Discrete

In this subsection, we describe reversal of an arc between a deterministic and a discrete variable. Consider the hybrid BN shown on the left-hand side of Figure 13. Let ξ and α denote the potentials at X and A, respectively, before arc reversal, and let ξ' and α' denote the potentials after arc reversal. Then,

$$\xi(u, v, x) = \delta(x - h(u, v)),$$

$$\alpha(v, w, x, a_i) = P(a_i \mid v, w, x),$$

$$\alpha'(u, v, w, a_i) = \int \delta(x - h(u, v))\, P(a_i \mid v, w, x)\, dx = P(a_i \mid v, w, h(u, v)), \text{ and}$$

$$\xi'(u, v, w, x, a_i) = \delta(x - h(u, v))\, P(a_i \mid v, w, x)/P(a_i \mid v, w, h(u, v)) = \delta(x - h(u, v)).$$

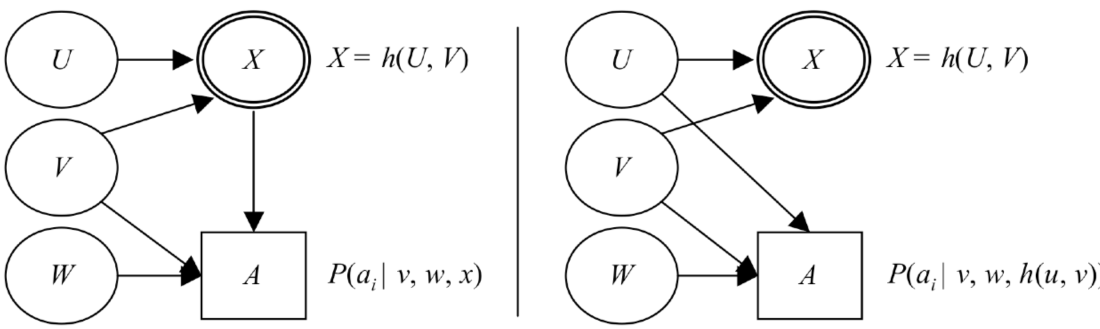

Figure 13. Arc reversal between a deterministic and a discrete variable.

Notice that ξ' depends on neither A nor W. The illustration of an arc reversal between a deterministic and discrete node with parents is given in Figure 13.

For a concrete example, consider the BN given on the LHS of Figure 14. The continuous variable $V \sim U[0, 2]$, deterministic variable $X = V^2$, and discrete variable A with two states $\{a_1, a_2\}$ has the conditional distribution $P(a_1 \mid v, x) = 1$ if $v \le x$, and $P(a_1 \mid v, x) = 0$ if $v > x$. Let ξ and α denote the potentials associated with X and A, respectively, before arc reversal, and ξ' and α' after arc reversal. Then,

$$\xi(v, x) = \delta(x - v^2)$$

$$\alpha(a_1, v, x) = P(a_1 \mid v, x) \begin{array}{l} = 1 \text{ if } v \le x \\ = 0 \text{ if } v > x, \end{array}$$

$$\alpha(a_2, v, x) = P(a_2 \mid v, x) \begin{array}{l} = 0 \text{ if } v \le x \\ = 1 \text{ if } v > x, \end{array}$$

$$\alpha'(a_1, v) = \int \delta(x - v^2)\, \alpha(a_1, v, x)\, dx = \alpha(a_1, v, v^2) = P(a_1 \mid v) \begin{array}{l} = 1 \text{ if } v \le v^2, \text{ and} \\ = 0 \text{ if } v > v^2, \end{array}$$

$$\alpha'(a_2, v) = \int \delta(x - v^2)\, \alpha(a_2, v, x)\, dx = \alpha(a_2, v, v^2) = P(a_2 \mid v) \begin{array}{l} = 0 \text{ if } v \le v^2, \text{ and} \\ = 1 \text{ if } v > v^2, \end{array}$$

$$\xi'(a_1, v, x) = \delta(x - v^2)\ \alpha(a_1, v, x)/\alpha(a_1, v, v^2) = \delta(x - v^2)$$
$$\xi'(a_2, v, x) = \delta(x - v^2)\ \alpha(a_2, v, x)/\alpha(a_2, v, v^2) = \delta(x - v^2)$$

The situation after arc-reversal is shown in the RHS of Figure 14.

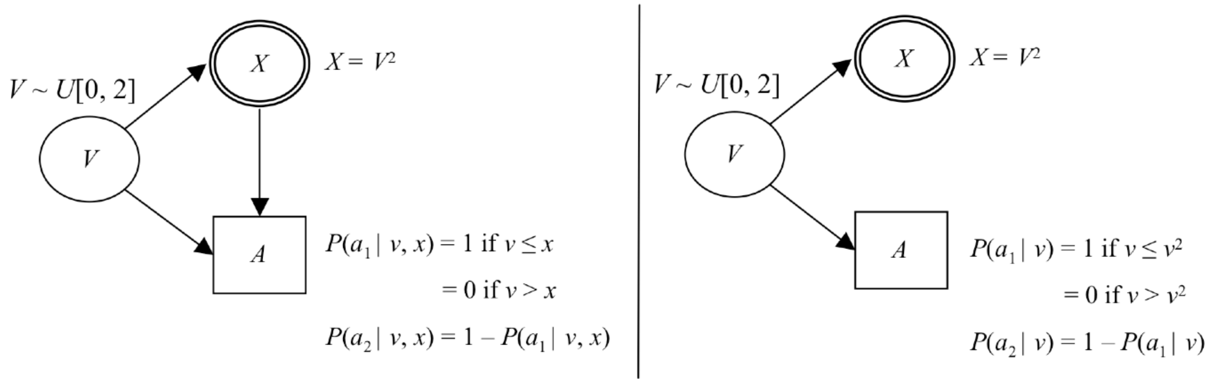

Figure 14. An example of arc-reversal between a deterministic and a discrete variable.

3.8 Discrete to Continuous

In this subsection, we describe reversal of an arc from a discrete to a continuous variable. Consider the hybrid BN shown on the LHS of Figure 15. Let α and ξ denote the potentials associated with A and X, respectively, before arc reversal, and α' and ξ' after arc reversal. Then,

$$\alpha(u, v, a_i) = P(a_i \mid u, v),$$
$$\xi(v, w, x, a_i) = f_i(x \mid v, w),$$
$$\xi'(u, v, w, x) = (\alpha\otimes\xi)^{-A}(u, v, w, x) = \Sigma\{P(a_i \mid u, v)\ f_i(x \mid v, w) \mid a_i \in \Omega_A\},$$
$$\alpha'(u, v, w, x, a_i) = ((\alpha\otimes\xi)\oslash(\alpha\otimes\xi)^{-A})(u, v, w, x, a_i)$$
$$= P(a_i \mid u, v)\ f_i(x \mid v, w)/\Sigma\{P(a_i \mid u, v)\ f_i(x \mid v, w) \mid a_i \in \Omega_A\}.$$

The density at X after arc reversal is a mixture density.

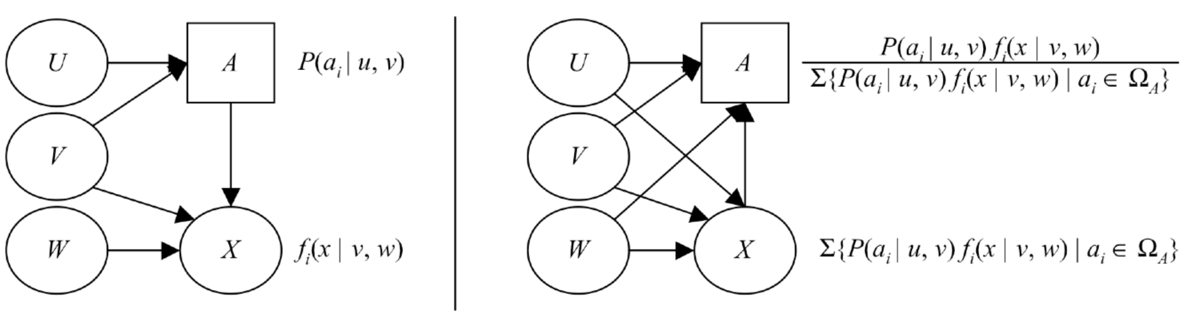

Figure 15. Arc reversal between a discrete and a continuous variable.

For a concrete example, consider the BN given on the LHS of Figure 16. The discrete variable A has two states $\{a_1, a_2\}$ with $P(a_1) = 0.5$ and $P(a_2) = 0.5$. X is a continuous variable whose conditional distributions are $X|a_1 \sim N(0, 1)$ and $X|a_2 \sim N(2, 1)$. Let α and ξ denote the potentials associated with A and X, respectively, before arc reversal, and α' and ξ' after arc reversal. Then,

$$\alpha(a_1) = 0.5,$$

$$\alpha(a_2) = 0.5,$$

$$\xi(a_1, x) = \varphi_{0,1}(x),$$

$$\xi(a_2, x) = \varphi_{2,1}(x),$$

$$\xi'(x) = (\alpha\otimes\xi)^{-A}(x) = 0.5\ \varphi_{0,1}(x) + 0.5\ \varphi_{2,1}(x),$$

$$\alpha'(a_1, x) = ((\alpha\otimes\xi)\oslash(\alpha\otimes\xi)^{-A})(a_1, x) = (0.5\ \varphi_{0,1}(x))/(0.5\ \varphi_{0,1}(x) + 0.5\ \varphi_{2,1}(x)),$$

$$\alpha'(a_2, x) = ((\alpha\otimes\xi)\oslash(\alpha\otimes\xi)^{-A})(a_2, x) = (0.5\ \varphi_{2,1}(x))/(0.5\ \varphi_{0,1}(x) + 0.5\ \varphi_{2,1}(x)),$$

The resulting BN after the arc reversal is given on the RHS of Figure 16.

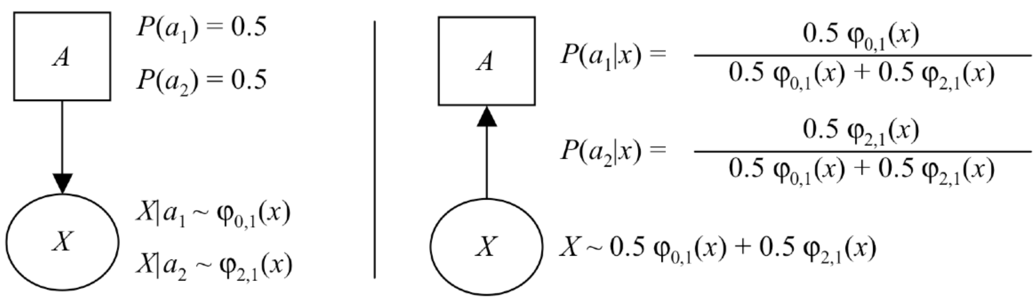

Figure 16. An example of an arc reversal between a discrete and a continuous variable.

3.9 Discrete to Deterministic

In this subsection, we describe reversal of an arc between a discrete and a deterministic variable. Consider the hybrid BN as shown on the left-hand side of Figure 17. Suppose that $\Omega_A = \{a_1, \ldots, a_k\}$. Let α and ξ denote the potentials associated with A and X, respectively, before arc reversal, and α' and ξ' after arc reversal. Then,

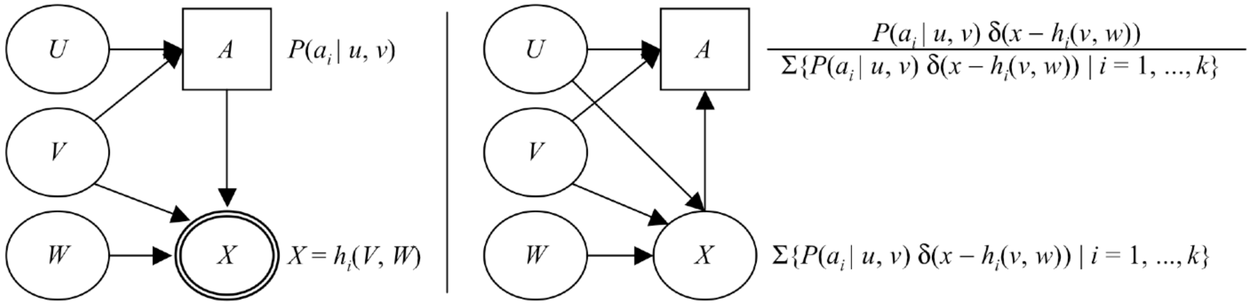

Figure 17. Arc reversal between a discrete and a deterministic variable.

$$\alpha(u, v, a_i) = P(a_i \mid u, v),$$
$$\xi(v, w, x, a_i,) = \delta(x - h_i(v, w)),$$
$$\xi'(u, v, w, x) = (\alpha \otimes \xi)^{-A}(u, v, w, x) = \Sigma\{P(a_i \mid u, v)\ \delta(x - h_i(v, w)) \mid i = 1, \ldots, k\},$$
$$\alpha'(u, v, w, x, a_i) = P(a_i \mid u, v)\ \delta(x - h_i(v, w)) \,/\, \Sigma\{P(a_i \mid u, v)\ \delta(x - h_i(v, w)) \mid i = 1, \ldots, k\}.$$

The situation after arc reversal is shown on the right-hand side of Figure 17. Notice that after arc reversal, X has a weighted sum of Dirac delta functions. Since the variances of the conditional for X after arc reversal may not be zeros, X may not be deterministic after arc reversal.

For a concrete example, consider the simpler hybrid BN shown on the LHS of Figure 18. V has the uniform distribution on (0, 1). A has two states $\{a_1, a_2\}$ with $P(a_1 \mid v) = 1$ if $0 < v \leq 0.5$, and = 0 otherwise, and $P(a_2 \mid v) = 1 - P(a_1 \mid v)$. X is deterministic with equations $X = V$ if $A = a_1$, and $X = -V$ if $A = a_2$. After arc reversal, the conditional distributions at A and X are as shown in the RHS of Figure 17 (these are special cases of the general formulae given in Figure 16). Let ϖ denote the density potential at V. Then $\varpi(v) = 1$ if $0 < v < 1$. We can find the marginal of X from the BN on the RHS of Figure 17 by reversing arc (V, X) as follows.

$$(\varpi \otimes \xi')^{-V}(x) = \int \varpi(v)\ \mathrm{P}(a_1 \mid v)\ \delta(x - v)\ \mathrm{d}v + \int \varpi(v)\ \mathrm{P}(a_2 \mid v)\ \delta(x + v)\ \mathrm{d}v = 1 \text{ if } 0 < x \leq 0.5 \text{ or } -1 < x < -0.5.$$

Thus, the marginal distribution of X is uniform on the interval $(-1, -0.5) \cup (0, 0.5)$.

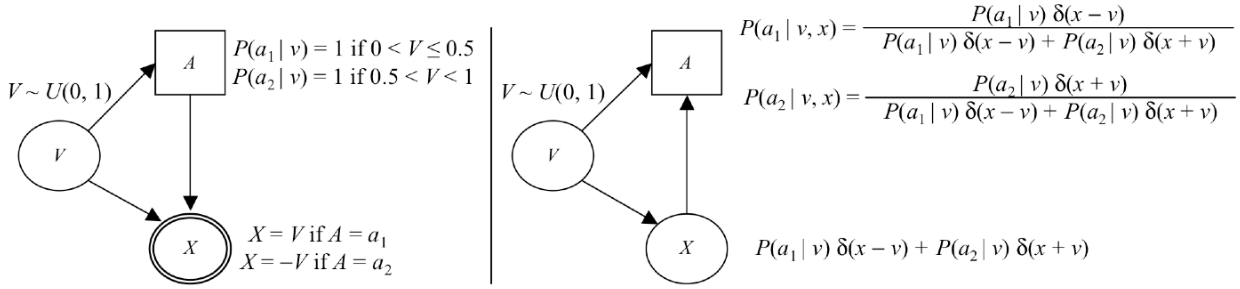

Figure 18. An example of arc reversal between a discrete and deterministic variable.

4 Partially Deterministic Distributions

In this section, we describe a new kind of conditional distribution called partially deterministic. Partially deterministic distributions arise in the process of arc reversals in hybrid BNs.

The conditional distributions associated with a deterministic variable have zero variances. If some of the conditional distributions have zero variances and some have positive variances, we say that the distribution is *partially deterministic*.

We get such distributions during the process of the arc reversals between a continuous node and a deterministic node with discrete and continuous parents. Consider the BN shown on the left-hand side of Figure 19. Let ξ and ζ denote the continuous potentials at X and Z, respectively, before arc reversal, and ξ′ and ζ′ after arc reversal. Then,

$$\xi(x) = f(x),$$

$$\zeta(x, y, z, a_1) = \delta(z - x) = \delta(x - z),$$

$$\zeta(x, y, z, a_2) = \delta(z - y) = \delta(y - z),$$

$$\zeta'(y, z, a_1) = (\xi\otimes\zeta)^{-X}(y, z, a_1) = \int f(x)\ \delta(x - z)\ \mathrm{d}x = f(z),$$

$$\zeta'(y, z, a_2) = (\xi\otimes\zeta)^{-X}(y, z, a_2) = \delta(y - z) \int f(x)\ \mathrm{d}x = \delta(y - z),$$

$$\xi'(x, y, z, a_1) = (\xi\otimes\zeta)\oslash(\xi\otimes\zeta)^{-X}(x, y, z, a_1) = f(x)\ \delta(x - z) / f(z) = f(z)\ \delta(x - z) / f(z) = \delta(z - x),$$

$$\xi'(x, y, z, a_2) = (\xi\otimes\zeta)\oslash(\xi\otimes\zeta)^{-X}(x, y, z, a_2) = f(x)\ \delta(y - z) / \delta(y - z) = f(x).$$

Thus, after arc reversal, both X and Z have partially deterministic distributions.

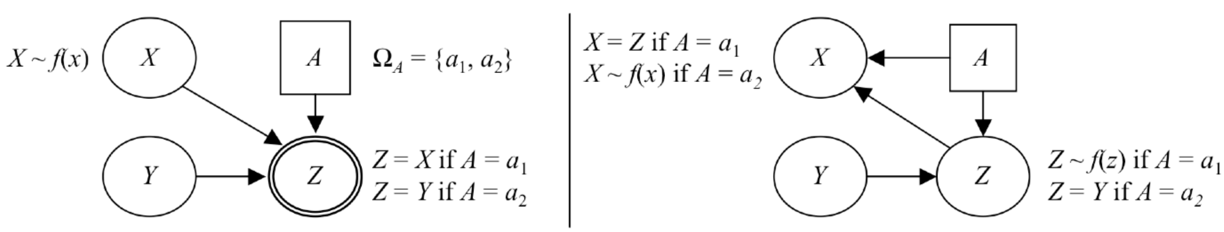

Figure 19. Arc reversal leading to partially deterministic distributions.

The significance of partially deterministic distributions is as follows. If we have a Bayesian network with all continuous variables such that each continuous variable is associated with a density potential, then we can propagate the density potentials similar to discrete potentials in a discrete Bayesian network. The only difference is that we use integration for marginalizing continuous variables (instead of summation for discrete variables). This assumes that the joint potential obtained by combining all density potentials represents the joint density for all variables in the Bayesian network. However, if even a single variable has a deterministic or a partially deterministic conditional, then the joint potential (obtained by combining all conditionals associated with the variables) no longer represents the joint density as the joint density does not exist. Thus, one cannot assume that in a Bayesian network with no deterministic variables (as in the case of RHS of Figure 19), that the joint density exists for all continuous variables in the network. It is clear from the Bayesian network in the LHS of Figure 19, that the joint density for $\{X, Y, Z\}$ (conditioned on the states of A) does not exist. And since the joint distributions of the two Bayesian networks are the same, the joint density for $\{X, Y, Z\}$ does not exist also for the Bayesian network in the RHS of Figure 19.

5 Conclusions and Summary

We have described arc reversals in hybrid BNs with deterministic variables between all possible kinds of pairs of variables. In some cases, there is no closed form for the distributions after arc reversals. For example, if a deterministic variable has a function that is not differentiable, then we cannot describe the distributions after arc reversal in closed form. We do believe, however, that the framework described in Section 2 is sufficient to describe arc reversals in those cases where there is a closed form for the revised distributions. Also, we have described a new kind of conditional distribution called partially deterministic that can arise after arc reversals.

The arc-reversal theory facilitates the task of approximating general BNs with mixture of Gaussians BNs. Also, the arc-reversal theory is potentially useful in solving hybrid influence diagrams, i.e., influence diagrams with discrete, continuous, and deterministic chance variables. We conjecture that Olmsted's arc-reversal algorithm for solving discrete influence diagrams would apply to hybrid influence diagrams also. The arc-reversal theory described here would make this possible. Of course, this is a topic that needs further investigation.

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Appendix: Properties of Dirac Delta Functions

In this appendix, we describe some basic properties of Dirac delta functions [Dirac 1927, Dirac 1958, Hoskins 1979, Kanwal 1998, Saichev and Woyczynski 1997, Khuri 2004]. We attempt to justify most of the properties. These justifications should not be viewed as formal mathematical proofs, but rather as examples of the use of Dirac delta functions that lead to correct conclusions.

(i) (*Sampling*) If $f(x)$ is any function, $f(x)\,\delta(x) = f(0)\,\delta(x)$. Thus, if $f(x)$ is continuous in the neighborhood of 0, then $\int f(x)\,\delta(x)\,\mathrm{d}x = f(0)\int \delta(x)\,\mathrm{d}x = f(0)$. The range of integration need not be from $-\infty$ to ∞, but can cover any domain containing 0.

(ii) (*Change of Origin*) If $f(x)$ is any function which is continuous in the neighborhood of a, then $\int f(x)\,\delta(x-a)\,\mathrm{d}x = f(a)$.

(iii) $\int \delta(x - h(u, v))\,\delta(y - g(v, w, x))\,\mathrm{d}x = \delta(y - g(v, w, h(u, v)))$. This follows from property (ii) of Dirac delta functions.

(iv) (*Rescaling*) If $g(x)$ has real (non-complex) zeros at $a_1, \ldots, a_n$, and is differentiable at these points, and $g'(a_i) \neq 0$ for $i = 1, \ldots, n$, then $\delta(g(x)) = \Sigma_i\, \delta(x - a_i)/|g'(a_i)|$. In particular, if $g(x)$ has only one real zero at a_0, and $g'(a_0) \neq 0$, then $\delta(g(x)) = \delta(x - a_0)/|g'(a_0)|$.

(v) $\delta(ax) = \delta(x)/|a|$ if $a \neq 0$. $\delta(-x) = \delta(x)$, i.e., δ is symmetric about 0.

(vi) Suppose $Y = g(X)$, where g is invertible and differentiable on Ω_X. Then $\delta(y) = \delta(g(x)) = \delta(x - a_0)\,/\,|g'(a_0)|$, where $a_0 = g^{-1}(0)$. Also, $\delta(y - g(x)) = \delta(g(x) - y) =$
$\delta(x - g^{-1}(y))\,/\,|(\mathrm{d}/\mathrm{d}x)(g(g^{-1}(y)))| = \delta(x - g^{-1}(y))\,/\,|\mathrm{d}y/\mathrm{d}x| = \delta(x - g^{-1}(y))\,|\mathrm{d}x/\mathrm{d}y|$
$= \delta(x - g^{-1}(y))\,|(\mathrm{d}/\mathrm{d}y)(g^{-1}(y))|$.

(vii) Consider the Heaviside function $H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x \geq 0$. Then, $\delta(x)$ can be regarded as the "generalized" derivative of $H(x)$ with respect to x, i.e., $(\mathrm{d}/\mathrm{d}x)H(x) = \delta(x)$. $H(x)$ can be regarded as the limit of certain differentiable functions (such as, e.g., the cumulative distribution functions (CDF) of the Gaussian random variable with mean 0 and variance σ^2 in the limit as $\sigma \to 0$). Then, the generalized derivative of $H(x)$ is the limit of the derivative of these functions.

(viii) Suppose continuous variable X has probability density function (PDF) $f_X(x)$ and $Y = g(X)$. Then Y has PDF $f_Y(y) = \int f_X(x)\, \delta(y - g(x))\, dx$. The function g does not have to be invertible. To show the validity of this formula, let $F_Y(y)$ denote the cumulative distribution function of Y. Then, $F_Y(y) = P(g(X) \leq y) = \int f_X(x)\, H(y - g(x))\, dx$, where $H(\cdot)$ is the Heaviside function defined in (vii). Then, $f_Y(y) = (d/dy)(F_Y(y)) = \int f_X(x)\, (d/dy)(H(y - g(x)))\, dx =$ $\int f_X(x)\, \delta(y - g(x))\, dx$.

(ix) Suppose continuous variable X has pdf $f_X(x)$ and $Y = g(X)$, where g is invertible and differentiable on Ω_X. Then the pdf of Y is $f_Y(y) = \int f_X(x)\, \delta(y - g(x))\, dx =$ $|(d/dy)(g^{-1}(y))| \int f_X(x)\, \delta(x - g^{-1}(y))\, dx = |(d/dy)(g^{-1}(y))|\, f_X(g^{-1}(y))$. Also, $f_X(x)\, \delta(y - g(x)) / (|(d/dy)(g^{-1}(y))|\, f_X(g^{-1}(y))) = \delta(x - g^{-1}(y))$. This is because if we consider the left-hand side as a function of x, say $\phi(x)$, it is equal to 0 if $x \neq g^{-1}(y)$, and $\int \phi(x)\, dx = 1$. Therefore, by definition, $\phi(x) = \delta(x - g^{-1}(y))$. Finally, $f_X(x)\, \delta(y - g(x)) =$ $(|(d/dy)(g^{-1}(y))|\, f_X(g^{-1}(y)))\, \delta(x - g^{-1}(y))$.

(x) The definition of δ can be extended to $\mathbb{R}^n$, the n-dimensional Euclidean space. Thus, if $\boldsymbol{x} \in \mathbb{R}^n$, $\delta(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \neq \boldsymbol{0}$, and $\int \ldots \int \delta(\boldsymbol{x})\, d\boldsymbol{x} = 1$, where $d\boldsymbol{x} = dx_1 \ldots dx_n$. Thus, e.g., $\int \ldots \int f(\boldsymbol{x})\, \delta(\boldsymbol{x} - \boldsymbol{x}_0)\, d\boldsymbol{x} = f(\boldsymbol{x}_0)$.

(xi) Suppose $X_1, \ldots, X_n$ are continuous variables with joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$. Then, the deterministic variable $Y = g(X_1, \ldots, X_n)$ has PDF $f_Y(y) = \int \ldots \int f_{\boldsymbol{X}}(\boldsymbol{x})\, \delta(y - g(\boldsymbol{x}))\, d\boldsymbol{x}$. The function g does not have to be invertible.

(xii) Suppose $X_1, \ldots, X_n$ are continuous variables with joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$. Then the joint PDF of deterministic variables $Y = g(X_1, \ldots, X_n)$ and $Z = h(X_1, \ldots, X_n)$ is given by $f_{Y,Z}(y, z) = \int \ldots \int f_{\boldsymbol{X}}(\boldsymbol{x})\, \delta(y - g(\boldsymbol{x}))\, \delta(z - h(\boldsymbol{x}))\, d\boldsymbol{x}$. The functions g and h do not have to be invertible.