THE WORLD ACCORDING TO GARP: NONPARAMETRIC TESTS OF WEAK SEPARABILITY AND ITS MONTE CARLO STUDIES

by

Seunghae Grace O

Submitted to the graduated degree program in Economics
and the Graduate Faculty of the University of Kansas
in partial fulfillment of the requirements
of the degree of Doctor of Philosophy.

Professor William Barnett
Chairperson
Professor Jianbo Zhang
Professor Shu Wu
Professor Shigeru Iwata
Professor Prakash Shenoy
Date Defended: May 6 2007

The Dissertation Committee for Seunghae Grace O certifies
that this is the approved version of the following dissertation:

THE WORLD ACCORDING TO GARP: NONPARAMETRIC TESTS OF WEAK SEPARABILITY AND ITS MONTE CARLO STUDIES

Committee:
Professor William Barnett
Chairperson
Professor Jianbo Zhang
Professor Shu Wu
Professor Shigeru Iwata
Professor Prakash Shenoy
Date Approved: May 6 2008

Table of Contents

List of Tables

List of Abbreviations

GARP	General Axiom of Revealed Preference
NONPAR	Varian 1982 Nonpar Software
NP	Varian 1982 Test using NONPAR
LP	Fleissig and Whitney 2003 Linear Programming Test
SW	Swofford and Whitney 1988 Joint Test
MP	Varian 1983 Minimal Perturbation Test
DP	de Peretti 2005 Test
Overall GARP	Utility Maximization Necessary Condition for Overall Data Set
Sub GARP	Utility Maximization Necessary Condition for Weakly Separable Subgroup Data Set
Power	Power of the Test (= 1 - Type II error)

Chapter 1

Weak Separability

The theoretical regularity conditions are generally viewed as maintained hypotheses. The simplifying separability restrictions are treated as null hypotheses. When it is accepted, weak separability (or blockwise weak separability) provides means of partitioning the economy into sectors, which can produce powerful parameter restrictions, hence substantially simplifying the estimation of large demand systems, without requiring homothetic assumption on preference, so that the utility system is capable of attaining the correct uncompensated own price elasticity and the income effect. Also, weak separability is consistent with decentralization in the decision making process, which makes two stage, or multi stage, optimization possible. Thus separability permits the use of aggregate data, in other words, the existence of consistence of aggregates.[1] However, homothetic preference assumption is very rarely necessary for any of above.

A group of goods is weakly separable from all other goods if the marginal rate of substitution between any pairs of goods in the group does not depend on the quantities consumed of any good that is not in the group. Weak separability implies that the demand for goods in the separable group depends only upon the prices of goods in the group, and secondly, the total group expenditure.

Following Strotz (1957) [36] and Gorman (1959) [27], Goldman and Uzawa (1964) [26] characterized the concept of weak separability by the form of the utiliy function and with the Slutsky terms of the corresponding demand function.

Let a set of all n commodities denoted by $N = \{1, ..., n\}$. Any grouping of the n commodities may be conveniently described by a partition of the set N into a class of mutually exclusive and exhaustive subsets, $\{N_1, ..., N_S\}$; namely, $N = \{N_1 \cup \ldots \cup N_S\}$, and $N_s \cap N_t = \emptyset$ for $1 \leq s, t \leq S$, $s \neq t$.

A commodity bundle, $x = (x_1, ..., x_n)$, is correspondingly partitioned into $(x^{(1)}, ..., x^{(S)})$, where, for each s, $1 \leq s \leq S$, the subvector $x^{(s)}$ is composed of x_i, for $i \in N_s$. Let $\{N_1, ..., N_S\}$ be the partition of the set N and $u(x)$ be a utility function for a preference relation.

The utility function $u(x)$ is called weakly separable with repect to a partition $\{N_1, ..., N_S\}$ if the marginal rate of substitution between two commodities i and j from N_s, $u_i(x)/u_j(x)$, is independent of the quantities of commodities outside of N_s. In other words,

$$\frac{\partial(u_i(x)/u_j(x))}{\partial x_k} = 0, \tag{1.1}$$

for all $i, j \in N_s$ and $k \notin N_s$. Thus the optimal quantities of these commodities depend only upon their prices and group expenditure.

Goldman and Uzawa (1964) [26] proved the theorem in which weak separability is characterized by the form of the utility function. A utility function $u(x)$ is weakly separable with respect to the partition $\{N_1, ..., N_S\}$, if and only if, $u(x)$ is of the form:

$$u(x) = \Phi(u^1(x^{(1)}), \ldots, u^S(x^S)),$$

where $\Phi(u^1, ..., u^S)$ is a function of S variables and, for each s, $u^s(x^{(s)})$ is a function of subvector $x^{(s)}$ alone.

Demand function $x_i = x_i(p, I)$, $i \in \{1, \ldots, n\}$ is derived by the maximization of the utility function $u(x)$, where $p = (p_1, ..., p_n)$ is the correponding price vector and I is the income. Then, the Slutsky term $K_{ij}(x)$ between commodities i and j is defined by

$$K_{ij}(x) = \frac{\partial x_i}{\partial p_i} + x_j \frac{\partial x_i}{\partial I}, \tag{1.2}$$

for $i, j = \{1, ..., n\}$.

A strictly concave utility function $u(x)$ is weakly separable with repect to a partition $\{N_1, ..., N_S\}$, if and only if, the Slutsky terms $K_{ij}(x)$ are of the form,

$$K_{ij}(x) = \kappa^{st}(x) \frac{\partial x_i}{\partial I} \frac{\partial x_j}{\partial I}, \tag{1.3}$$

with some functions $\kappa^{st}(x)$ defined for $s \neq t$ for all $i \in N_s$, $j \in N_t (s \neq t)$.

2

Chapter 2

Nonparametric Tests

2.1 Nonparametric Tests Description

Separability can be evaluated using either statistical tests of parameter restrictions on a parametric functional form or nonparametric tests of the appropriate necessary and sufficient conditions.

Barnett (1995) [6] proved that future uncertainty does not void contemporaneous conditional allocation under perfect certainty, when current prices are known with certainty, tastes are intertemporally separable, the right hand side of the contemporaneous expenditure allocation decision is realized (measured) expenditure allocated to the current period, and current period quantities demanded are realized (measured) quantities. The resulting allocation is conditional on the contemporaneous total expenditure realization, which does depend upon dynamics and future risk. But the contemporaneous conditional allocation is static and does not require perfect foresight or risk neutrality, so long as the assumptions of Barnett's result hold.

Thus the nonparametric tests in this dissertation condition upon realized current period total consumption expenditure, assume intertemporal separability, and assume that current period prices are known with certainty. When, however, current prices are random, Barnett's result does not apply. Then dynamics and future risk must be treated simultaneously with contemporaneous risk. No perfect certainty conditional current period allocation exists. For example, if current period prices include financial asset user cost prices, Barnett's result does not apply, since user cost prices depend upon current period interest rates that are not known until the end of the current period. In this case, nonparametric single period tests under perfect certainty cannot be applied. Instead separability can only be tested by parametric means using Euler equations, as shown and applied in Barnett and Hahm (1994) [9] and Barnett and Zhou (1994) [11].

The *revealed preference test* developed by Varian (1982) [41] is nonparametric, which shows some advantages. It can be used with any number of observations, unlike parametric test with the degree of freedom restriction. Nonparametric tests do not require certain functional forms and can avoid problems associated with model misspecification. Third, the nonparametric approach tests separability of direct utility functional form, where separability is based on quantity. On the contrary, the parametric approach is based on price, which does not test the separability over quantities unless the direct utility function is homothetic [3]. Additionally, the nonparametric test estimates global separability rather than local separability, which the parametric test does.

Varian (1982) [41] derived the necessary and sufficient conditions, based on a binary relation, for a data set to be constistent with neoclassical utility maximization and for weak separability. Varian also proved that a set of observed price and quantity data can be rationalized by a well-behaved uitility function if and only if the data set satisfies the Generalized Axiom of Revealed Preference (GARP), which was implemented with the software program NONPAR to test weak separability.

Varian's NONPAR procedure shows several deficiencies. The necessary and sufficient conditions for weak separability are tested sequentially rather than jointly. Second, it is assumed that there is no random measurement error in the data. Third, it is assumed that subgroup expenditure is fully adjusted to optimal levels in each period, which is equivalent to testing weak separability in a static model.

Barnett and Choi (1989) [7] found in Monte Carlo studies that the rejection rate of Varian's NONPAR test exceeded the nominal significance level in all cases, with the data set generated from the weak separable utility function.

Followed by Varian's sequential NONPAR test, Fleissig and Whitney (2003) [25] suggested another sequential test based on a linear programing using the superlative index numbers. Their test seeks to minimize the adjustments in absolute value term, needed for the superlative index to satisfy GARP. It was evaluated with the Cobb-Douglas setting with measurement errors, and their result shows that the data set violates the necessary and sufficient conditions for GARP in up to 20 percent of their trials with the observed quantity with measurement errors of 5 percent.

Swofford and Whitney (1994) [39] proposed a joint test of the necessary and sufficient conditions of weak separability, including the incomplete adjustment of category expenditure without prior restrictions on the nature of the adjustment.

With respect to the issue of being static, Varian (1985) [43] treated measurement error, calculating the minimal adjustment for the data set to be consistent with maximization of weakly separable utility maximization decision. de Peretti and Jones (2005) [17], Jones, Elger, Edgerton and Dutkowsky (2005) [30], Elger and Jones (2007) [22] incorporated measurement errors into nonparametric weak separability tests in consumer's decision problem and improved the efficiency of computation of Varian's test. Moreover, Jones, Elger, Edgerton and Dutkowsky (2005) [30] were able to run, on a standard PC [4], the full data set of Sworfford and Whitney's (1994) [39], which they needed to have two overlapping sub samples to solve minimization problems using a CRAY super computer.

This paper investigates the statistical properties of existing nonparmametric tests for weak separability with Monte Carlo experiments over various range of elasticities of substitution with measurement errors. It also illustrates the size of the power of existing nonparametric tests of weak separability to confirm our result. This

This dissertation is organized as follows; the rest of Chapter 2 introduces notations, definitions and necessary and sufficient conditions for weak separability. Chapter 3 contains the procedures in each tests. Chapter 4 presents data sets and Monte Carlo studies. Chapter 5 discusses the results and Chapter 6 concludes the paper. Notations, definitions and the settings for the experiments in this dissertation are following the existing literatures.

2.2 Notations and Definitions

Let $x_i = (x_{i,1}, ..., x_{i,K})'$, denote a $(K \times 1)$ vector of quantities, $p_i = (p_{i,1}, ..., p_{i,K})'$, the corresponding vector of prices for $i = \{1, \ldots, T\}$. Let also $x_i = (y_i, z_i)$ and $p_i = (r_i, v_i)$ denote partitions of the

quantity and price vectors into two groups, $(y_i = (y_{i,1}, ..., y_{i,m}), r_i = (r_{i,1}, ..., r_{i,m}))$ and
$(z_i = (z_{i,1}, ..., z_{i,K-m}), v_i = (v_{i,1}, ..., v_{i,K-m}))$.

Let $u = u(x) \equiv u(z, y)$ denote a utility function defined over all goods. The utility function is said to be weakly separable in block y if there exists a macrofunction, $\bar{u}$, and a sub-utility function or aggregator function[5], V, such that $u(x) \equiv u(z, y) = \bar{u}(z, V(y))$.

2.3 Necessary and Sufficient Conditions for Weak Separability

Varian (1982) [41] defined GARP within the Samuelson's (1947) [32] revealed preference theory, showing that if a data set satisfies GARP, it can be rationalized by a well-behaved utility function. The data set satisfies the GARP if there exists a sequence of each bundle satisfying the transitive closure of revealed preference, but not implying strictly direct revealed preference.

First, define the binary strict direct revealed preference relation P^0 by $x_i P^0 x_j$. It is said that x_i is strictly direct revealed preferred to x_j if $p_i \cdot x_i > p_i \cdot x_j$, $\forall i,j \in \{1, ..., T\}$. And the $(T \times T)$ P^0 matrix, whose element p^0_{ij} is defined as follows:

$$p^0_{ij} = \begin{cases} 1, & \text{if } p_i \cdot x_i > p_i \cdot x_j, \\ 0, & \quad otherwise. \end{cases} \tag{2.1}$$

Similarly, define the binary direct revealed preference relation R^0 by $x_i R^0 x_j$. It is said that x_i is direct revealed preferred to x_j if $p_i \cdot x_i \geq p_i \cdot x_j$, $\forall i,j \in \{1, ..., T\}$. And the $(T \times T)$ R^0 matrix, whose element r^0_{ij} is defined as follows:

$$r_{ij}^0 = \begin{cases} 1, & \text{if } p_i \cdot x_i \geq p_i \cdot x_j, \\ 0, & \quad otherwise. \end{cases} \tag{2.2}$$

The definition of binary revealed preference relation, R, is that $x_i R x_j$ if there exists a sequence between x_i and x_j such that $p_i \cdot x_i \geq p_i \cdot x_m, p_m \cdot x_m \geq p_m \cdot x_n, ..., p_p \cdot x_p \geq p_p \cdot x_j$ or $x_i R^0 x_m, x_m R^0 x_n, ..., x_p R^0 x_j$, where R is the transitive closure of R^0.

Using the above definitions, GARP is defined as follows:

Definition 2.3.1. Generalized Axiom of Revealed Preference (GARP)

The data satisfies the general axiom of revealed preference if $x_i R x_j$ implies not $x_j P^0 x_i$ ($r_{ij} = 1$ does not imply $p_{ij}^0 = 1$) or $x_i R x_j \Rightarrow p_j \cdot x_j \leq p_j \cdot x_i$, $\forall i, j \in \{1, \ldots, T\}$.

Using GARP, Varian (1982) [41] proved the following conditions, which refers to as Afriat's theorem.

Theorem 1. Afriat's theorem

The following conditions are equivalent.

i. There exists a nonsatiated, concave, monotonic and continuous utility function, which rationalizes the data.

ii. The data satisfies the generalized axiom of revealed preference (GARP).

iii. There exist numbers U_i, $\tau_i > 0$, such that

$$U_i \leq U_j + \tau_j p_j (x_i - x_j), \tag{2.3}$$

for $\forall\ i,\, j \in \{1, \ldots, T\}$.

Varian (1982) [41] provides a heuristic argument for Afriat's theorem to give more economic meaning to the Afriat inequalities.

Concavity of the utility function implies that

$$U(x_i) \leq U(x_j) + \Delta U(x_j)(x_i - x_j), \tag{2.4}$$

where the marginal utility of expenditure for the jth observation, τ_j, is greater than zero and utility maximization implies $\Delta U(x_j) = \tau_j p_j$. By substituting utility maximization condition to (2.4), we obtain (2.3). This shows that the Afriat conditions are a necessary condition for utility maximization assuming differentiability of the utility function.[6]

Varian (1983) [42] proved the necessary and sufficient nonparametric conditions for a data set to be consistent with weak separability. The necessary and sufficient conditions for weak separability are described by the following theorem.

Theorem 2. Varian's Separability Theorem

The following conditions are equivalent.

i. There exists a weakly separable, concave, monotonic, continuous and nonsatiated utility function, which rationalizes the data.

ii. There exist numbers U_i, V_j, τ_i, $\mu_i > 0$, such that, for $\forall\ i, j \in \{1, \ldots, T\}$

$$
\begin{aligned}
U_i &\leq U_j + \tau_j V_j(z_i - z_j) + \frac{\tau_j(V_i - V_j)}{\mu_j}, && (2.5)\\
V_i &\leq V_j + \mu_j r_j(y_i - y_j). && (2.6)
\end{aligned}
$$

iii. The data set (r_i, y_i) and $(\hat{p}_i, \hat{x}_i)$, where $\hat{p}_i = (v_i, ..., v_i, \frac{1}{\mu_i})$ and $\hat{x}_i = (z_i, ..., z_i, V_i)$ satisfy GARP, for some choices of (V_i, μ_i) satisfying the Afriat Theorem.

There exist numbers V_i, $\mu_i > 0$ satisfying (2.6) if and only if the observed data for the y block of goods, (r_i, y_i), satisfy GARP. (2.5) states that the overall price and quantity data for all goods, (p_i, x_i) also should satisfy GARP. If not, it cannot be rationalized by a utility function, or weak separablity. Condition (iii) states that there exist numbers V_i, $\mu_i > 0$, satisfying (2.6), such that the data set for all goods resulting from these replacement, $(\hat{p}_i, \hat{x}_i)$ satisfies GARP. It can be interpreted that weak separability implies that there exist group quantity and price indexes such that the quantities and prices for the separable block of goods are replaced by these group quantity and price indexes, then the data can still be rationalized by a well-behaved utility function. [7]

Chapter 3

Weak Separability Tests Description

3.1 Sequential Weak Separability Tests

3.1.1 Varian's NONPAR Test: NP Test

Varian (1982) [41] and Varian (1983) [42] constructed a software program, NONPAR, to test the hypothesis that a data set is consistent with the weakly separable utility maximization with using the following sequential procedure.

Step 1. Check $\{(r_i, y_i),\ i = 1, \ldots, T\}$ and $\{(p_i, x_i),\ i = 1, \ldots, T\}\}$ for GARP.

Step 2. If neither condition from Step 1 is violated, construct a particular pair of Afriat indexes, $\hat{V}_i$, $\hat{\mu}_i$ for $i \in \{1, \ldots, T\}$, satisfying (2.5) and (2.6) in Varian's Theorem (ii). If Afriat index does not exist satisfying conditions, then the test rejects weak separability hypothesis.

Step 3. Check $\{(\hat{p}_i, \hat{x}_i)\} = \left\{(v_i, 1/\hat{\mu}_i), (z_i, \hat{V}_i)\ i = 1, ..., T\right\}$ for GARP.

If any of the three conditions are not satisfied, then weak separability hypothesis is rejected by the sequential test.

The Varian's test, NP test, however, shows bias towards rejecting weak separability, see Barnett and Choi (1989) [7]. What NP test states is that there exist some choices of Afriat indexes satisfying (2.5) and (2.6). However, the choice of Afriat indexes is not unique. By changing the scales, Afriat indexes can be found. Also, NP test is static, implying that a stochastic error in a data set can violate one of conditions of GARP, which leads the rejection of weak separability hypothesis, even if the true data set is consistent with maximization problem of a weakly separable utility function.

3.1.2 Fleissig and Whitney's Test: LP Test

Fleissig and Whitney (2003) [25] proposed another method using a linear programing *(LP test)* to construct Afriat indexes using superlative index, which can provide a second-order approximation to the true quantity aggregate, see Diewert (1976) [21]. They seek to minimize the adjustment, in absolute value term, needed for superlative indexes. Tornqvist index, the discrete time approximation to the Divisia index, was used to satisfy the Afriat inequalities. From Varian's separability theorem (iii), they label $\frac{1}{\mu_i}$ as the group price index and V_i as the group quantity index for the separable y goods.

Let the superlative index number QV_i be an initial estimate for V_i in (2.3) from Afriat theorem. If the estimates give lower bound in the μ_i that are all positive, the superlative index solves the Afriat indexes.

If the superlative index number fails, a solution can be obtained with a small adjustment to QV_i by adding a positive number, $Q_i^p \geq 0$ and a negative number, $-Q_i^n \leq 0$. Then, equation (2.3) becomes,

$$QV_i + Q_i^p - Q_i^n \leq QV_j + Q_j^p - Q_j^n + \mu_j r_j (y_i - y_j) \tag{3.1}$$

for $\forall\ i, j \in \{1, \ldots, T\}$.

Let the deviations μ_i, μ_i^p and μ_i^n, be from the QV_i/Exp_{iy}, where Exp_{iy} is the expenditure on y separable goods in period i. Using group quantity index and group price index, the deviation μ_i from the QV_i/Exp_{iy} can be derived by adding up the positive and negative deviations, $\mu_i^p \geq 0$, $-\mu_i^n \leq 0$ respectively.

Linear program model for testing weak separability for good y becomes

$$\min F_{LP} = \sum_{i=1}^{T} (Q_i^p + Q_i^n + \mu_i^p + \mu_i^n) \tag{3.2}$$

subject to

$$QV_i + Q_i^p - Q_i^n \leq QV_j + Q_j^p - Q_j^n + \mu_j r_j(y_i - y_j)$$

$$\mu_i = QV_i/Exp_{iy} + \mu_i^p - \mu_i^n$$

$$\mu_i \geq \epsilon_{\mu_i}$$

$$QV_i + Q_i^p - Q_i^n \geq \epsilon_{QV_i}$$

$$Q_i^p,\ Q_i^n,\ \mu_i^p,\ \mu_i^n\ \geq 0, \tag{3.3}$$

with small number ϵ_{QV_i} and ϵ_{μ_i}, for $\forall\ i,\ j \in \{1, \ldots, T\}$.

LP test was evaluated with the Cobb-Douglas setting with measurement errors. The data set violates GARP in up to 20% of Fleissig and Whitney's (2003) [25] trials with the observed quantities with measurment errors of 5%.

3.2 Two Extensions to Varian's (1985) Test

3.2.1 Varian's 1985 Test: MP Test

Varian (1985) [43] introduced a test of optimizing behavior of firms, based on the weak axiom of cost minimization (WACM), within an explicit stochastic framework that can account for measurment error. Jones et al. [30], Jones et al. [17] and Elger et al. [22] extended Varian's (1985) [43] test to consumer's utility optimization based on GARP. If a dataset violates GARP, Varian(1985) [43] proposed computing the minimal perturbation of observed quntity data needed to render it consistent with GARP. Varian's suggested statistical minimal perturabation method can be used to determine whether or not such revealed preference violations can be attributed to measurement error. Elger et al. [22] generalized his method with the necessary and sufficient conditions for the maximization of weakly separable utility.

Assume that the true unobserved data set is $\{(x_i^*, p_i) : i = 1, ..., T\}$ and it is related to the observed one, $\{(x_i, p_i) : i = 1, ..., T\}$, by a multiplicative error term, $x_{i,k}^* = x_{i,k}(1 + \varepsilon_{i,k})$, where the measurement error, $\varepsilon_{i,k}$, is distributed as $f(\cdot)$; $f(\cdot)$ is assumed to have zero mean and $i.i.d$ with the variance σ^2 for at time $i \in \{1, \ldots, T\}$, and the number of good $k \in \{1, \ldots, K\}$. If the observed data set violates GARP, then each adjustment procedure calculates its perturbation of the observed quantity data that can make the data be consistent with GARP.

Varian's (1985) [43] procedure is minimizing the objective function, F_{MP}, over perturbed quantity, ζ_i, for $\forall\ i \in \{1, \ldots, T\}$. Let ζ_i^y be the m elements of ζ_i, which correspond to the perturbed quantity of y separable group and ζ_i^z be the $(K - m)$ remaining elements of ζ_i, which correspond to the perturbed quantity of the z goods. Then the problem is:

$$\min F_{MP}(\zeta_1, ..., \zeta_T) = \sum_{i=1}^{T}\sum_{k=1}^{K}(\frac{\zeta_{i,k}}{x_{i,k}} - 1)^2 \tag{3.4}$$

subject to

$$V_i \leq V_j + \mu_j r_j(\zeta_i^y - \zeta_j^y) \tag{3.5}$$

$$U_i \leq U_j + \tau_j v_j(\zeta_i^z - \zeta_j^z) + \frac{\tau_j}{\mu_j}(V_i - V_j) \tag{3.6}$$

$$V_i \geq 0,\ \mu_i \geq 0, \tag{3.7}$$

for $\forall\ i, j \in \{1, ..., T\}$. The solution for ζ_i arguments is called the minimal perturbation, and the minimal perturbation of the quantity data is denoted by $\hat{\zeta}_i$ for $i \in \{1, ..., T\}$. Then $\hat{\zeta}_i = (\hat{\zeta_{i,1}}, \ldots, \hat{\zeta_{i,K}})$ the $(K \times 1)$ vector of adjusted quantity in period $i \in \{1, ..., T\}$, the minimally perturbed quantity data is consistent with the necessary and sufficient condition for weak separability given the observed price data.

Let $\hat{\varepsilon_{i,k}}$ be the computed error of good k in period i defined by $\hat{\varepsilon}_{i,k} = \frac{\hat{\zeta_{i,k}}}{x_{i,k}} - 1$. With the assumption over ε, $\frac{F(x_1^*, ..., x_T^*)}{\sigma^2} = \frac{\sum_{i=1}^{T}\sum_{k=1}^{K}(\varepsilon_{i,k}^2)}{\sigma^2}$ is distributed as χ^2_{TK}. If $\frac{F(x_1^*, ..., x_T^*)}{\sigma^2} > C_\alpha$, then the weak separability hypothesis would be rejected, when C_α is the critical value for a χ^2_{TK} at the α significance level. However, the test statistics cannot be computed because the true data is unobserved, implying σ^2 is unknown.

To resolve this problem, Varian (1985) [43] suggested that the weak separability hypothesis is rejected if $\bar{\sigma^2} \equiv \frac{F(\hat{\zeta}_1,\ldots,\hat{\zeta}_T)}{C_\alpha} > \sigma^2$. In Varian's (1985) [43], "if $\bar{\sigma}$ is much smaller than our prior opinions concerning the precision with which these data have been measured, we may well want to accept the maximization hypothesis". Also it is stated that "The bound statistic, $\bar{\sigma}$, is a transparent way of reporting empirical results, since it can be easily compared to one's own subjective prior regarding the standard deviation of measurement errors in the data".

3.2.2 de Peretti's Test: DP Test

de Peretti's (2005) [16] test is based on minimizing the Varian's objective function (3.4) iteratively with different constraints using the transitive closure matrix from the definition of GARP. Using de Peretti's test, a necessary condition for utility maximization and weak separability can be tested.

The method is to minimize sum of square of errors (3.8) over ζ_i subject to (3.9) and (3.10) in ζ_i.

$$F_{DP}(\zeta_1, ..., \zeta_T) = \sum_{i=1}^{T} \sum_{k=1}^{K} (\frac{\zeta_{i,k}}{x_{i,k}} - 1)^2 \tag{3.8}$$

subject to

$$\zeta_i^y \ R \ y_j \ \textit{implies not} \ y_j \ P^0 \ \zeta_i^y \tag{3.9}$$

$$\zeta_i \ R \ x_j \ \textit{implies not} \ x_j \ P^0 \ \zeta_i \tag{3.10}$$

for $\forall i, \, j \in \{1, ..., T\}$.

The constraints can be attained by implementing the followings:

$$\zeta_i^y \cdot r_i = y_i \cdot r_i \tag{3.11}$$

$$\zeta_i^y \cdot r_j \geq y_j \cdot r_j \; for \; \forall j \; such \; that \; y_i \; R \; y_j \tag{3.12}$$

$$\zeta_i \cdot p_i = x_i \cdot p_i \tag{3.13}$$

$$\zeta_i \cdot p_j \geq x_j \cdot p_j \; for \; \forall j \; such \; that \; x_i \; R \; x_j \tag{3.14}$$

Procedures in DP test eliminate GARP violations by adjusting bundles along the observed budget lines, however, in Varian's MP test the minimal perturbation is not constrained in terms of expenditure and all bundles are adjusted simultaneously. Therefore, MP test eliminates GARP violations, in part, by adjusting total expenditure. Additionally Varian's Minimal Perturbation test is based on the magnitude of the errors, meanwhile DP test focuses on testing the distribution of errors.

The test procedure follows by testing the computed errors for independence and identical distribution. Let the two sets of residuals s^1 and s^2. Let $S^1_{\hat{\varepsilon}}$ be a $(T \times K)$ matrix whose element at the ith row and jth column is given by $\hat{\zeta}_{ij}/x_{ij} - 1$, and let s^1 be a $(Tk \times 1)$ vector defined as $s^1 = vec(S^1_{\hat{\varepsilon}})$. Let $s^2 = vec(S^2_{\hat{\varepsilon}})$. $S^2_{\hat{\varepsilon}}$ is a $(r \times k)$ matrix whose element at the ith row and jth column is given by $\hat{\zeta}_{ij}/x_{ij} - 1$ if and only if $\hat{\zeta}_{ij} - x_{ij} \neq 0$. r is the number of bundles altered to enable the data consistent with GARP.

Given s^1 or s^2, following Spanos (1999) [35], the test to check whether residuals are $i.i.d.$ is attained by testing restrictions from the estimating two auxiliary regressions followed by.

de Peretti (2005) [16] suggested

$$s^a_t = c_1 + \alpha \cdot trend + \sum_{j=1}^{T_1} \gamma_j s^a_{t-j}, \;\; a = 1 \; or \; 2 \tag{3.15}$$

for first order dependence and trend heterogeniety. The joint significance of coefficients α and γ_j, $j \in \{1, \ldots, T_1\}$ are tested.

For second-order dependence and trend heterogeniety,

$$(s^a_t)^2 = c_2 + \delta \cdot trend + \sum_{j=1}^{T_2} \sum_{k=1}^{T_2} d\beta_{jk} s^a_{t-j} s^a_{t-k}, \;\; a = 1, \;\; or \, 2 \tag{3.16}$$

, where

$$d = \begin{cases} 1, & \text{if } k \geq j, \\ 0, & otherwise. \end{cases} \tag{3.17}$$

The joint significance of coefficients δ and β_{jk}, for $j, k \in \{1, ..., T_2\}$ are tested.

de Peretti (2005) [16] suggests the test procedure as following.

Let P_1 and P_2 be the probabilities associated with F test and Wald test, repectively, for the two auxiliary regressions 3.15 and 3.16. At a significant level α, the decision rule is:

$H_0 : min(P_1, P_2) \geq \alpha$, that is, violations are caused by stochastic elements as measurement error; utility maximization hypothesis is not rejected.

$H_1 : min(P_1, P_2) < \alpha$, that is, violations are not caused by stochastic elements; utility maximization hypothesis is rejected. In other words, data set is not generated by the maximization behavior, or there exists one or several "disruptions" in a data set.

3.3 Joint Test of Weak Separability

3.3.1 Swofford and Whitney's Test: SW Test

Swofford and Whitney (1994) [39] derived necessary and sufficient conditions for the joint test of weak separability. The test also allows incomplete adjustment, which has been pointed by many researchers, see Serletis (1991)[33] [8], of group expenditures and puts no prior restrictions on the nature of the adjustment.

The test procedure is based on minimizing the objective function,

$$F_{SW} = \sum_{i=1}^{T} (\tau_i - \mu_i \phi_i)^2 \tag{3.18}$$

subject to

$$U_i \leq U_j + \tau_j v_j (z_i - z_j) + \phi_j (V_i - V_j) \tag{3.19}$$

$$V_i \leq V_j + \mu_j r_j (y_i - y_j) \tag{3.20}$$

$$U_i, \;\; V_i, \;\; \tau_i, \; \mu_i, \; \phi_i > 0 \tag{3.21}$$

for $\forall \; i, \; j \in \{1, \ldots, T\}$.[9]

If a feasible solution is found, such that the objective function is minimized to zero, then ϕ_i becomes τ_i/μ_i for $\forall \; i$, indicating that weak separability is said to hold with complete adjustment. If SW minimization problem has no solution, weak separability with incomplete adjustment is rejected.

Swofford and Whitney defined a new variable θ_i, such that $\phi_i =$

$(\tau_i + \theta_i)/\mu_i$. Then the minimization of F_{SW} is to find a feasible solution that $\theta_i = 0$ for all $\forall\ i$, since the objective function becomes

$$F_{SW} = \sum(\theta_i)^2. \tag{3.22}$$

Swofford and Whitney showed that if an agent maximizes a weakly separable utility function subject to a standard budget constraint and an additional expenditure constraint on the group y, then θ_i can be interpreted as a shadow price associated with the additional constraint. Additionally, τ_i is a shadow price associated with the standard budget constraint. Therefore, θ_i/τ_i represents the ratio of the shadow price of the expenditure constraint on the group y to the marginal utility of income at each observation.

The necessary and sufficient conditions for weak separability are satisfied if there is a feasible solution to the constraints, such that the shadow price of the additional expenditure constraint, θ_i is zero for $\forall$ i. If there is a feasible solution to the constraints, but θ_i is nonzero for some i, then there is incomplete adjustment of expenditure on group y

for some observations.

It is natural to take the average value of $|\theta_i/\tau_i|$ over $\forall\ i$ as a measure of incomplete adjustment of expenditure in group y.

Chapter 4

Monte Carlo Experiment

4.1 WS-Branch Utility System

The true underlying utility function for the Monte Carlo simulation is the WS-branch utility tree, see Barnett (1977) [2]. The WS-branch utility tree is the only blockwise weakly separable utility function and can be shown to be a flexible form when there are no more than two goods in each block and no more than total of two blocks. It is homothetic in supernumerary quantities, but not homothetic in the elementary quantities. Hence, a homothetic utility function can be converted into a nonhomothetic utility function by translating the quantities into supernumerary quantities.

The generalized quadratic mean of order ρ with two groups, q_1 and q_2, is

$$U = U(q_1, q_2) = A(B_{11}q_1^{2\rho} + 2B_{12}q_1^{\rho}q_2^{\rho} + B_{22}q_2^{2\rho})^{1/2\rho} \ , \tag{4.1}$$

where $q_1 = q_1(x_1, x_2)$ and $q_2 = q_2(x_3)$ and with restrictions on parameters, $\rho < 1/2$, $B_{ij} > 0$ for $i, j = 1, 2, B_{ij} = B_{ji}$, for $i \neq j$ and $\sum_i \sum_j B_{ij} = 1$.

The subutility function, q_1 and q_2 are

$$q_1(x_1, x_2) = (A_{11}y_1^{2\delta} + 2A_{12}y_1^{\delta}y_2^{\delta} + A_{22}y_2^{2\delta})^{1/2\delta}, \tag{4.2}$$

$$q_2(x_3) = x_3 - a_3 \ , \tag{4.3}$$

where $\delta < 1/2$, $A_{ij} > 0$ for $i, j = 1, 2$, $A_{ij} = A_{ji}$, for $i \neq j$ and $\sum_i \sum_j A_{ij} = 1$. The supernumerary quantities, y_i, are $y_i = x_i - a_i$, for $i = 1, 2, 3$, where a_i is "committed quantities". The parameter restrictions are to guarantee that the utility function satisfies the regularity conditions, i.e. quasi-concavity and monotonicity. For a parameter A, $A > 0$, can be normalized to one. The WS-branch utility function is blockwise weakly separable, with x_1 and x_2 representing the first group (block) and x_3 representing the second group (block). The B_{12} and A_{12} are called the interaction coefficients. If the interaction coefficients are all zero, then both the subutility function and macro function are CES utility function. When both interaction coefficients are zero, $A_{12} = B_{12} = 0$, and $\delta \to 0$ and $\rho \to 0$, then they become Cobb-Douglas.

Barnett and Choi (1989) [7] showed that the elasticity of substitution between goods from two macro (aggregator) functions is given by

$$\sigma(q_1, q_2) = \frac{1}{(1 - \rho + \Theta)} \ , \tag{4.4}$$

where

$$\Theta = \frac{-\rho(B_{11}B_{22} - B_{12}^2)}{(B_{11}Q^{-\rho} + B_{12})(B_{12} + B_{22}Q^{\rho})} \ , \tag{4.5}$$

where $Q = \frac{q_1}{q_2}$.

If $a_1 = a_2 = 0$ [10], then the substitution elasticity between goods in the first group is given by

$$\sigma(x_1, x_2) = \frac{1}{(1 - \delta + \Phi)} \,, \tag{4.6}$$

where

$$\Phi = \frac{-\delta(A_{11}A_{22} - A_{12}^2)}{(A_{11}X^{-\delta} + A_{12})(A_{12} + A_{22}X^{\delta})}, \tag{4.7}$$

where $X = \frac{x_1}{x_2}$.

Given that homotheticity is imposed on the WS-branch utility tree, $\sigma(x_1, q_2) = \sigma(x_2, q_2)$, implying that only substitution elasticities between groups, $\sigma_{13} = \sigma(x_1, q_2)$, and between x_1 and x_2, $\sigma_{12} = \sigma(x_1, x_2)$, need to be specified in the Monte Carlo experiments.[11]

4.2 Calibration

The model is calibrated with the values of substitution elasticities for the subutility function, $\sigma_{12} = \sigma(x_1, x_2)$, and for the macro function, $\sigma_{13} = \sigma(x_1, q_2)$, drawn from $\{0.1, 0.6, 1.0, 3.0, 5.0\}$.

It is assumed that $A_{11} = A_{22} = 1/2$, $A_{12} = A_{21} = 0$, $B_{11} = 2/3$, $B_{22} = 1/3$ and $B_{12} = B_{21} = 0$. Under these assumptions, the values of δ and ρ can be calculated by the followings:

$$\sigma_{12} = \frac{1}{1 - 2\delta} \tag{4.8}$$

$$\sigma_{13} = \frac{1}{1 - 2\rho}. \tag{4.9}$$

4.3 Data Generation Process

Barnett and Choi (1989) [7] constructed a Monte Carlo experiment, constructing 60 pre-selected observations of quantities for three goods and total expenditure. They used real data from Barnett (1981) (Appendix D) [4] and solved for the prices by using the inverse WS-branch model. These quantities are normalized to equal 20 each at 31st observation and expenditure is normalized to equal 60 at the 31st observation. Then, for each set of quantity and price values, white noise was added, in a manner that preserved the total expenditure stream.

In this study, the data set is generated from the WS branch tree with generalized quadratic utility function defined by Barnett (1977) [2]. The generated data set represents the demand for three goods with our aggregation setting. Unlike Barnett and Choi (1989)[7], the data set is generated according to the following five steps:

Step 1. Generate a set of sub group quantities, x_1, x_2, and x_3, and total expenditure, m, with random walk plus drift VAR(1) model.

Step 2. Derive first order conditions of subutility, (4.2) and (4.3), with sub group expenditure. Solve for the sub group price system p_1 and p_2. With preselected elasticity of subsititution values and parameter values, calculate the macro (aggregator) quantity, q_1.

Step 3. Using Fisher reversal test, calculate aggregate price index, P_1, from the first order conditions of macro utility maximization problem. With calculated q_1, q_2, P_1, and preselected parmameters, calculate the aggregate price for the second group, P_2.

Step 4. Add white noise the measurement errors, 1%, 10% and 20% each, into the quantities generated from the previous step in a manner of preserving the total expenditure. In addition, following Barnett and Choi [7], the simulated quantities and expenditure data are normalized to equal 20 and 60, repectively at the 31st observation.

Step 5. All Monte Carlo simulation is repeated over a sample size T=60 and with 1000 replications.

Chapter 5

Results

5.1 Type I Error

For all Monte Carlo experiments, the null hypothesis H_0 is that both true data and adjusted data are consistent with maximization of weakly separable utility. For NONPAR test, when Afriat indexes satisfy GARP with the overall data and with sub group data and sufficient condition, (iii) in Varian's theorem, then the null is not rejected. If there is any negative Afriat numbers, we reject the null.[12] For Varian's Minimal perturbation test (MP test), the null is rejected if $\bar{\sigma^2} \equiv \frac{F(\hat{\zeta_1},\ldots,\hat{\zeta_T})}{C_\alpha} > \sigma^2$. In de Peretti's iterative test (DP test), the null is rejected according to the the test procedure of checking probability associated with two auxiliary regressions 3.15 and 3.16 whether residuals are *i.i.d.*. In LP test and SW test, when the problem does not have feasible solutions, hence there exist no Afriat indexes satisfying necessary and sufficient conditions, then we reject the null.

SW is attractive since weak separability is tested jointly allowing the incomplete adjustment. The burden of the claculation is challenging to actually solve, with $T(T-1)$ nonlinear inequality constraints and $T+T(T-1)$ linear constriants with $4T$ sign restrictions. Consequently, SW test has not being widely used in empirical work. For Varian's 1985 Minimal purturbation test is quite burdensome as well. The problem has $T(K+2)$ variables with $T(T-1)$ nonlinear inequalities and $2T$ sign restrictions over $T(K+2)$ variables. In practice, when NONPAR test and LP test reject weak separability, then SW test and Varian's Minimal purturbation test are applied to the data [13].

5.1.1 NONPAR test, LP test and SW test

σ_{12}	σ_{13}	Sub GARP	Overall GARP	NONPAR	LP	SW
0.1	0.1	0	0	82.8	0	0
0.6	0.6	0	0	82.8	0	0
1	1	0	0	88.6	0	0
3	3	0	0	65.8	0	0
5	5	0	0	55.5	0	0

Table 5.1: Violation rate of NONPAR test and LP test with 1% error: Same Elasticity of Substitution

The data sets, generated without error and with 1% error, satisfy GARP with overall data and with subgroup weakly separable data. The violation ratio of the Afriat inequalities for the NONPAR and LP tests is shown in Table 5.1, Table 5.2, and Table 5.3. The violation rate of overall GARP is the ratio of violation of GARP with overall data.

The violation rate of sub GARP, similarly, is the ratio of violation of GARP with subgroup weakly separable data, which has not violated the overall GARP.

The violation rates of nonstochastic Vairan's NONPAR test are high in all assumed elasticiy of substitution values, especially it is notable under Cobb- Douglas setting, 88.6% (See Table 5.1). Fleissig and Whitney's LP test shows powerful results when the data is generated without error and with 1% error in all assumed coefficient values in CES and Cobb- Douglas settings. More precisely, with the data with 1% error, LP test successfully found Afriat indexes satisfying Varian's theorem. When there exists one single violation in Afriat inequalities, the test rejects the null, H_0, the data is consistent with maximization of weakly separable utility function.

Under Cobb Douglas setting, data set with 10% error shows that overall data violates GARP with 35.2% and with 27.11% weakly sepa-

σ_{12}	σ_{13}	Sub GARP	Overall GARP	NONPAR	LP	SW
0.6	0.1	0	0	65	0	0
1	0.1	0	0	51.5	0	0
1	0.6	0	0	71.6	0	0
3	0.1	0	0	54.3	0	0
3	0.6	0	0	56.4	0	0
3	1	0	0	61.3	0	0
5	0.1	0	0	19.2	0	0
5	0.6	0	0	22.9	0	0
5	1	0	0	44.2	0	0
5	3	0	0	55.6	0	0

Table 5.2: Violation rate of NONPAR test and LP test with 1% error: Macro elasticity of substitution is greater than within group elasticity of substitution

σ_{12}	σ_{13}	Sub GARP	Overall GARP	NONPAR	LP	SW
0.1	0.6	0	0	96	0	0
0.1	1	0	0	88.4	0	0
0.1	3	0	0	94.2	0	0
0.1	5	0	0	97.2	0	0
0.6	1	0	0	65	0	0
0.6	3	0	0	83.3	0	0
0.6	5	0	0	92.4	0	0
1	3	0	0	84.9	0	0
1	5	0	0	85.7	0	0
3	5	0	0	66.6	0	0

Table 5.3: Violation rate of NONPAR test and LP test with 1% error: Macro elasticity of substitution is smaller than within group elasticity of substitution

rable group data violates GARP. The violation rates of NONPAR test in all parameter settings record very high, over 90% with measurement error, reconfirming the fact that Varian's nonstochastic test is heavily biased towards rejection in all settings, which was found by Barnett and Choi (1989) [7] in their Monte Carlo experiment.

σ_{12}	σ_{13}	Sub GARP	Overall GARP	LP
0.1	0.1	30.7	26.9	92.3
0.6	0.6	18.5	14.8	53.8
1	1	27.11	35.2	13.3
3	3	29.5	18.7	27.7
5	5	6.3	13.2	63.6

Table 5.4: Violation rate of LP test with 10% error: Same Elasticity

LP test, however, shows lower rate of violation percentage shares in all assumed values of substitution than ones from NONPAR test. More precisely, when σ_{13}, the substitution elasticity between groups, is low, the violation rate of LP test is very high relardless of the magnitude of the elasticity of substition between subgroup goods, good 1 and good 2. However, under the Cobb Douglas setting, with $\sigma_{13} = 1$, $\sigma_{12} = 1$, the violation ratio is as low as 13.3%, which can be naturally explained that LP test recognizes the structure as weak separable under the additive separability under the Cobb Douglas setting. The violation rates of overall GARP and sub GARP found the same pattern. Swofford and Whiteney's test successfully recognizes the maximization of weak separable utility under the setting with 1% of error.

σ_{12}	σ_{13}	Sub GARP	Overall GARP	LP
0.6	0.1	24.9	25.1	99.1
1	0.1	45.3	36.2	71.4
1	0.6	20.2	30.3	53.8
3	0.1	0.8	15.3	98.3
3	0.6	1	12.5	11.1
3	1	0.2	3.9	2.1
5	0.1	2.9	16.1	90.1
5	0.6	0.01	17.6	63.6
5	1	0.05	4.1	6.2
5	3	0.1	20.4	42.8

Table 5.5: Violation rate of LP test with 10% error: Macro elasticity of substitution is greater than within group elasticity of substitution

σ_{12}	σ_{13}	Sub GARP	Overall GARP	LP
0.1	0.6	27.7	30.9	22.2
0.1	1	16.1	1.3	2.7
0.1	3	40.3	24.9	57.1
0.1	5	41.5	18.9	91.6
0.6	1	7.4	0.2	4.3
0.6	3	23.3	12.6	61.5
0.6	5	9.8	4.8	63.6
1	3	43.8	37.6	81.8
1	5	34.6	40	86.6
3	5	30.5	17.2	57.1

Table 5.6: Violation rate of LP test with 10% error: Macro elasticity of substitution is smaller than within group elasticity of substitution

5.1.2 MP test and DP test

In this section, the data replication under the Cobb Douglas setting with 1%, 10% and 20% measurement errors, is used to evaluate the tests. Table 5.7 and Table 5.8 show the minimal perturbations of quantities needed to be consistent with the data consistent with weak separability conditions under Varian's (1985) [43] MP test. When the error is 1%, the size of perterbation is as small as 0.00172 and as the magnitude of error gets larger, the size needed to be adjusted to satisfy weak separability conditions gets larger. Table 5.7 reports the bound statistic, $100 \times \bar{\sigma}$, where $\bar{\sigma^2} = \hat{F}/C_\alpha$, which measures the unknown standard deviation of measurement error, where one would have to reject weak separability.

One will not reject the maximization hypothesis unless the true standard deviation of measurement error in the data quantity replica-

Error	Objective[14]	$100 \times \bar{\sigma}$
1%	0.00172	0.3643
10%	0.0381	1.7148
20%	3.241	15.816

Table 5.7: Minimal Perturbation Test: Utility Maximization

Error	Objective[15]	$100 \times \bar{\sigma}$
1%	0.00082	0.2571
10%	0.019	1.2378
20%	6.3	22.5402

Table 5.8: Minimal Perturbation Test: Weak Separability

tions is less than 0.36%, 1.71% and 15.81% in each experiment accordingly. The data set is generated with 1%, 10% and 20% of measurement errors respectively, implying the true standard deviation of measurement error in the data replications is greater than the bound statistics, so that the utility maximization hypothesis, or overall GARP, cannot be rejected. For weakly separable group, the value of bound statistic is 0.25%, which is less than the true standard deviation of measurement errors, 1%, the weak separability should be accepted as well. Surprisingly, minimal perturbation test shows powerful results with very volitile measurement errors with the standard deviation 10% in overall and sub group. The bound statistic with 10% error in utility maximization hypothesis is 1.71%, and for weakly separable group is 1.23%, in which one cannot reject the weak separability hypothesis.

Error	Objective	Std. Error	Adjustment [16]	s_1 [17]	s_2[18]
1%	0.000998	0.00673	4.1	0	0
10%	0.00623	0.00897	13.9	65.9	72.7
20%	0.02109	0.0295	35.1	85.6	91.9

Table 5.9: de Peretti's DP test: Utility Maximization

Error	Objective	Std. Error	Adjustment [19]	s_1[20]	s_2[21]
1%	0.0012	0.0089	5.6	0	0
10%	0.00832	0.0165	14.2	88.7	91.6
20%	0.04213	0.049	17.8	13.4	94.1

Table 5.10: de Peretti's DP test: Necessary Condition for Weak Separability

Table 5.9 and Table 5.10 report the results of de Peretti's test with the iterative method. This test investgates the two necessary conditions of weak separability, Utility maximization with the overall data and utility maximization of weakly separable group data. Utility maximization with the overall data results are shown in Table 5.9. It turns out that DP test works well under the small size of measurement error. With the minimized adjustment 0.000998, quantities satisfy utility maximization conditions, equivalently saying GARP, with 1% error. de Peretti test explains that expenditure has to be held to satisfy DP test's constraints, therefore, when the quantities of some goods is adjusted downward, and it is necessary for other goods quantities will be upward within the expenditure, as a result, the adjustment process is along the budget line.

The violation rate is the ratio of number of violation (with overall data and weakly separable data, respectively) to the number of replication. The violation rate of utility maximization, or GARP with overall data, is 65.9% and the violation rate of a necessary condition of weakly separability, or GARP with weakly separable group, is 88.7% with 10% error in testing residuals for the first order independence and trend heterogeniety. A result of testing a necessary condition of weak separability recordts as high as 94.1% of violation rate. It should be pointed out that the average number of adjustment of DP test is very small, 4.1, with 1% error, which enables the calculation very efficient. In practice, DP test is easy to solve and much faster to conduct than Varian's minimal perturbation test, even when the size of the data gets larger or with the large replication, the calculation burden does not increase overwhelmingly . When it is believed that the data has stochastic errors or very few violation, de Peretti's iterative method can be applied before MP test. [22]

5.2 Power of the tests

The power of nonparametric tests for weak separability is ideally tested by calculating the rejection rate of groups generated to be inconsistent with weak separability. To evaluate the power, the data is generated violating weak separability. The utility structure is given by $\tilde{u} = u(x_2, V(x_1, x_3))$, implying x_3 is not weakly separable from x_1, repectively.

The power of the test is the probability that rejects the null that the data sets are separable when the null hypothesis, $H_0 : u = u(x_2, V(x_1, x_3))$ is false.

The Monte Carlo studies for the power is the following:

Step 1. Take the data which is consistent with GARP.

Step 2. Take a test of the subgroup quantities, x_1 and x_3, for the necessary conditions for GARP.

Step 3. Weak separability is evaluated with the data set which has passed Step 1 and Step 2.

σ_{12}	σ_{13}	No error	1% error	10% error
0.6	0.6	0.518	0.757	1
1	1	0.674	0.680	1
3	3	0.080	0.205	1
5	5	0.080	0.211	1

Table 5.11: Power of LP test: Same Elasticity

σ_{12}	σ_{13}	No error	1% error	10% error
1	0.1	1	1	1
1	0.6	0.953	0.978	1
3	0.1	1	1	1
3	0.6	0.906	0.975	1
3	1	0.777	0.745	1
5	0.1	1	1	1
5	0.6	0.97	1	1
5	1	0.754	0.850	1
5	3	0.12	0.224	1

Table 5.12: Power of LP test: Macro elasticity of substitution is greater than within group elasticity of substitution

σ_{12}	σ_{13}	No error	1% error	10% error
0.1	1	0.854	0.956	1
0.1	3	0.060	- [23]	-
0.1	5	0.089	-	-
0.6	1	0.740	0.801	1
0.6	3	0.030	0.137	1
0.6	5	0.083	0.108	1
1	3	0.525	0.727	1
1	5	0.479	0.560	1
3	5	0.035	0.197	1

Table 5.13: Power of LP test: Macro elasticity of substitution is smaller than within group elasticity of substitution

For Fleissing and Whitney's LP test, Type II error is the number of times for the test failing to reject the false null divided by the number of data satisfying the necessary condition.

Table 5.11, Table 5.12 and Table 5.13 show the results of power test experiment of LP test with data with no error, 1% error and 10% error repectively. Under Cobb Douglas setting, LP Test correctly rejects the data without error with 67.4% and with 68% for 1% error. When the elasticity of substitution between x_1 and x_2, σ_{12}, is much lower than the substitution elasticity between the group, σ_{13}, the results records the size of power as low as 3% with $\sigma_{12} = 0.6$, $\sigma_{13} = 3$ without errors. In this particular setting, true data itself does not reveal strongly weak separability structure, rather it has high substitutibility between a separable group quantities and x_3. Hence, LP test does not often recognize the weak separability utility tree and does not rejects the false null hypothesis. The other case that draws our attention is the case with both σ_{12} and σ_{13} are set high. For the data generated with no error and 1% error, the false null is rejected 8% and 21.1% each with $\sigma_{12} = \sigma_{13} = 5$. Again, it is suspicious that the true structure correctly grasp the structure if x_1, x_2, which are weakly separable to x_3 under the high substitutibility between groups. For the data set with 10% errors are rejected with a probablity of 1, regardless of the assumed values of elasticity of substitution. Naturally, when the true substitution setting delivers clear weak separability structure, the power of LP test is closer to 1.

The results also reports several settings without feasible solution to LP minimization, in particular with the elasticity of substitution with 0.1, instead of recording the power to 1. For the date which does not satisfy the necessay conditon of weak separability, or GARP of the group x_1 and x_3, LP test rejects the false null as well.

Error	No Error	1%	10%	20%
objective	0.051	0.074	0.4814	8.677
$100 \times \bar{\sigma}$	2.0305	2.4447	16.2312	26.4531
Power[24]	0.799	0.832	0.992	1

Table 5.14: Power of Varian's Minimal Perturbation Test

The power of the Varian's minimal perturbation is calculated under the Cobb Douglas setting. With 1% error in the data set, the MP test correctly rejects the false null with a probability of 0.832. With the 20% of errors, MP test also rejects the false null 100%. The results shows that MP test records higher power than LP test regardless the magnitude of errors. Also, the result presents the minimized objective function in average and bound statistics. For all the error sizes, the false null is rejected.

Table 5.15 presents the type II errors associated with the testing procedure of DP test, in other words, testing *i.i.d.* of calculated residuals. For dePeretti's iterative method, the Cobb Douglas setting is assumed.

The Type II error is the number of times when test statistic is less than the value of significance level, $\alpha = 5\%$, divided by the number of data, satisfying the necessary condition of utility maximization or weak separability, respectively. The probability of accepting the false null is as small as 3% for s_2 under 10% error, 10.97% without any error assumed.

The nature of DP test is testing the normality of calculated residual, it should be carefully interpreted, especially when it is compared to other tests. Additionally, the test is testing necessary conditions of weak separability, hence, one can employ alternative way of calculating the power of the test. In alternative way, the power of the test would be calculated with random behavior data, which is not utility maximization nor weakly separable. Then it can be tested for the normality of the calculated rediduals. In this case, the rejection may be caused by that the data is not weakly separable, or that the data is not utility maximization behavior.

Error	No error	1%	10%
Objective	9.976	11.092	24.532
(std err)	(5.367)	(6.109)	(8.0295)
Type II Error (s_1)[25]	0.2134	0.1883	0.0491
Type II Error (s_2)	0.1097	0.1271	0.0336
Power (s_1)[26]	0.7866	0.8117	0.9509
Power (s_2)	0.8903	0.8729	0.9664

Table 5.15: Power of dePeretti's Iterative Method

Chapter 6

Conclusions

Barnett and Choi (1989) [7] studied Monte Carlo simulation based on WS-branch utility tree model focusing on parametric weak separability test on four flexible functional form models. They also examined Varian's NONPAR test, which failed to recognize the weak separability with no random disturbances, such as measurement errors, in the data.

In this study, current available nonparametric weak separability tests are investgated with generated data with and without random error, such as measurement errors. In particular, it is investigated the problem of nonparametric weak separability, that not all nonparametric weak separability tests correctly recognize weak separability. The data set is generated over a wide range of elasticity of substitution with different assumptions on the magnitude of error.

Monte Carlo studies confirm that Varian's NONPAR test is heavily biased toward rejecting weak separability even without any errors in the generated data set. Fleissig and Whitney's LP test captures the weak separability structures in utility function, where nonstochastic test is not able to capture and rejects the weak separability structure. LP test also considers errors on the test, and it correctly recognizes the weak separability under the small size of disturbances.

Varian's minimal perturbation MP test shows very powerful results with the volatile environment. De Peretti's iterative method tests necessary conditions for utility maximization and weak separability. It is fast and easy to practice with small number of adjustment. With an assumption of small stochastic errors in the data set, DP test can be applied easily.

Jones, Dutkowsky, and Elger (2005) [29] designed the sweep program to calculating adjusted data to be consistent with weak separability, based on Varian's minimal perturbation test. Their procedure is with LP test by testing firstly GARP with overall data and then using MP test, it tests GARP with weakly separable group data sequentially by adjusting data set, to be consistent with the maximization of weak separable utility function. Lastly, to make the results sure, they test the adjusted data set with SW test.

For future research it can be condidered the idea of de Peretti's test, which takes account the nature of the errors in the data. In extension, future works can be focused on the significance of violations of the utility maximization hypothesis rather than focusing on Afriat indexes testing weak separability. In deed, Barnett and de Peretti (2008) [8] suggests another nonparametric weak separability test based on the property of weak separability utility function: the marginal substitution within group is independent of the commodity in outside of the group. With this test, one may finally answer to Barnett and Choi (1989) [7]

Chapter Appendix

Notes

[1]See Deaton and Muellbauer (1980) [18] for a thorough discussion of weak separability, Gorman form and hierarchical decision making.

[2]For a detailed discussion of functional separability and its theoretical implications, see Shepard (1970) [34], Berndt and Christensen (1973) [12], Berndt and Christensen (1974) [13], Denny and Fuss (1977) [19], Blackorby, Primont, and Russell (1977) [14], and Denny and Pinto (1978)[20].

[3]See Barnett and Choi (1989) [7]

[4]They ran the test in FORTRAN using the commercial solver FF-

SQP.

[5] It is true under the assumption of homotheticity.

[6] See Varian (1982) [41] pp. 969-970 for the corresponding sufficient conditions and the formal proof of Afriat theorem.

[7] See Barnett(1987)[5] for the discussion of the homotheticity in aggregation theory.

[8] "Due to habit persistence, adjustment costs, the formation of expectations, or a combination of reasons, consumers are thought to take time to adjust some group expenditures." Swofford and Whitney (1994) [39]

[9] The utility maximization problem becomes $\max U(y, V(z))$ subject to $r'y + v'z = Y$, Y = total expenditure. The following Afriat inequalities are again derived from the basic properties of concave functions, $U(y_i, V_i) \leq U(y_j, V_j) + \mathbf{D}U(y_j)(y_i - y_j) + (\partial U_j / \partial V_j)(V_i - V_j))$ and $V_i \leq \mathbf{D}V(z_j)(z_i - z_j)$. Utility maximization implies $\mathbf{D}U(y) = (\tau + \theta)r$,

$\mathbf{D}U(z) = (\partial U/\partial V)(\partial V/\partial z) = \tau v$, and $\mathbf{D}V(z) = \partial V/\partial z = \mu r$, where θ can be viewed as a measure of the degree to which z is adjusted to be consistent with a weakly separable utility maximization problem. By substituting first order conditions, we get $\partial U/\partial V = (\tau+\theta)/\mu$. With the substituted first order conditions and Afriat inequalities, we get Swoffrod and Whiteney's minimization problem, min $\sum \theta^2 = \sum (\tau - \mu\phi)^2$.

[10]Otherwise, Theorem 2.2, Barnett and Choi (pp. 366) [7], is applied.

[11]See Theorem 2.1. Barnett and Choi (1989 pp. 365) [7].

[12]Whenever there is a negative Afriat number, we can change scales and then test.

[13]See Jones, Barry E. and Elger, Thomas and Edgerton, D. L. [29] and D.H. Dutkowsky [30]

[14] Objective in average. Objective= $\frac{1}{R}\sum_R \hat{F}$, where R = # of replications

[15]Objective in average. Objective= $\frac{1}{R}\sum_R \hat{F}$, where $R = \#$ of replications

[16]Number of adjustment in average.

[17]Violation rate from (s_1) in average.

[18]Violation rate from (s_2) in average.

[19]Number of adjustment in average.

[20]Violation rate from (s_1) in average.

[21]Violation rate from (s_2) in average.

[22]Empirical data shows that in MP test, the lower bound statistic, estimated standard deviation of error, is as large as 0.132%, Elger and Jones (2007) [22] calculated bound statistic of MP test with US monetary assets from US from 1993 to 2001.

[23]- indicates that there is at least one occasion with infeasible solu-

tion.

[24]Power = Number of Rejection of false null / Number of data consistent with a necessary condition of weak separabililty

[25]Type II Error = Number of times acceptance of a necessary condition of weak separability / Number of data consistent with the utility maximization.

[26]Power = 1- Type II error

Bibliography

[1] S. N. Afriat, *The construction of a utility function from expenditure data*, International Economic Review, 8 (1967), pp. 67–77.

[2] W. A. Barnett, *Recursive subaggregation and a generalized hypocycloidal demand model*, Econometrica, 45 (1977), pp. 1117–1136.

[3] ——, *The user cost of money*, Economic Letters, 1 (1978), pp. 145–149.

[4] ——, *Consumer Demand and Labor Supply: Goods, Monetary Assets, and Time*, North-Holland Publishing Company, 1981.

[5] ——, *The microeconomic theory of monetary aggregation. Barnett W. and Singleton K. ed, New approaches to monetary economics: proceedings of the second international symposium in economic theory and econometrics*, Cambridge University Press, 1987.

[6] ——, *Exact Aggregation Under Risk*, The Theory of Monetary Aggregation, W. A. Barnett and A. Serletis (eds.), Amsterdam:North-Holland, 1995.

[7] W. A. Barnett and S. Choi, *A monte carlo study of tests of blockwise weak separability*, Journal of Business and Economics Statistics, 7 (1989), pp. 363–377.

[8] W. A. Barnett and P. de Peretti, *A necessary and sufficient stochastic semi-nonparametric test for weak separability*, Working paper, (2008).

[9] W. A. Barnett and J. H. Hahm, *Financial-Firm PRoduction of Monetary Services: A Generalized Symmetric Barnett Variable-Profit-Function Approach*, Functional Structure Inference,W. A. Barnett and A. Serletis (eds.), Amsterdam:North-Holland, 1994.

[10] W. A. Barnett and O. Seck, *Rotterdam vs. almost ideal model will the best demand specificaion please stand up?* working paper,

2006.

[11] W. A. Barnett and G. Zhou, *Financial-Firms' Production and Supply-Side Monetary Aggregation Under Dynamic Uncertainty*, Functional Structure Inference,W. A. Barnett and A. Serletis (eds.), Amsterdam:North-Holland, 1994.

[12] E. R. Berndt and L. R. Christensen, *The interal structure of functional relationships: Separability, substitution, and aggregation*, Review of Economic Studies, 40 (1973), pp. 403–410.

[13] ——, *Testing for the existence of a consistent aggregate index of labor inputs*, American Economic Review, 64 (1974), pp. 391–404.

[14] C. Blackorby, D. Primont, and R. R. Russell, *On testing separability restrictions with flexible functional forms*, Journal of Econometrics, 5 (1977), pp. 195–209.

[15] ——, *Duality, Separability, and Functional Structure*, Amsterdam: North-Holland, 1978.

[16] P. de Peretti, *Testing the significance of the departures from utility maximization*, Macroeconomic Dynamic, 9 (2005), pp. 372–397.

[17] P. de Peretti and B. E. Jones, *A comparison of two methods for testing the utility maximization hypothesis when quantity data are mea- sured with error,*, Macroeconomic Dynamics, 9 (2005), pp. 612–629.

[18] A. Deaton and J. Muellbauer, *An almost ideal demand system*, The American Economic Review, 70 (1980), pp. 312–326.

[19] M. Denny and M. Fuss, *The use of approximation analysis to test for separability and the existence of consistent aggregates*, American Economic Review, 67 (1977), pp. 404–418.

[20] M. Denny and C. Pinto, *An Aggregate Model with Multi-Product Technologies*, In Production Economics: A Dual Approach

to Theory and Applications, Vol. 2, M. Fuss and D. McFadden (eds.), Amsterdam:North-Holland, 1978.

[21] W. E. Diewert, *Exact and superlative index numbers*, The Journal of Political Economy,, (1976), pp. 481–507.

[22] T. Elger and B. E. Jones, *Can rejection of weak separability be attributed to random measurement errors in the data?*, Economics Letters, forthcoming (2007).

[23] D. Fisher and A. R. Fleissig, *Monetary aggregation and the demand for assets*, Journal of Money, Credit and Banking, 29 (1997), pp. 458–475.

[24] A. R. Fleissig and J. Swofford, *A dynamic asymptotically ideal model of money demand*, Journal of Monetary Economics, 37 (1996), pp. 371–380.

[25] A. R. Fleissig and G. A. Whitney, *A new pc-based test for varian's weak separability conditions*, Journal of Business and Eco-

nomic Statistics, (2003), pp. 133–144.

[26] S. M. Goldman and H. Uzawa, *A note on separability in demand analysis*, Econometrica, 32 (1964), pp. 387–398.

[27] W. M. Gorman, *Separable utility and aggregation*, Econometrica, 27 (1959), pp. 469–481.

[28] P. Hjerstrand, *A monte carlo study of the necessary and sufficient conditions for weak separability*. Lund University working paper, 2003.

[29] B. E. Jones, D. Dutkowsky, and T. Elger, *Sweep programs and optimal monetary aggregation*, Journal of Banking and Finance, 29 (2005), pp. 483–508.

[30] B. E. Jones, T. Elger, D. L. Edgerton, and D. H. Dutkowsky, *Toward a unified approach to testing for weak separability*, Economics Bulletin, 3 (2005), pp. 1–7.

[31] N. M. Jones, B. E. and D. L. Edgerton, *Nonparametric Tests of the Necessary and Sufficient Conditions for Separability*, Functional Structure Inference,W. A. Barnett and A. Serletis (eds.), Amsterdam:North-Holland, 2007.

[32] P. A. Samuelson, *Foundations of Economic Analysis*, Harvard University Press, 1947.

[33] A. Serletis, *The demand for divisia money in the united states: A dynamic flexible demand system*, Journal of Money, Credit and Banking, 23 (1991), pp. 35–52.

[34] R. W. Shephard, *Theory of Cost and Production Functions*, Princeton, NJ: Princeton University Press., 1970.

[35] A. Spanos, *Probability Theory and Statistical Inference*, Cambridge University Press, 1999.

[36] R. H. Strotz, *The empirical implications of a utility tree*, Econometrica, 25 (1957), pp. 269–280.

[37] J. L. Swofford and G. A. Whitney, *Nonparametric tests of utility maximization and weak separability for consumption, leisure and money*, Review of Economics and Statistics, (1987), pp. 458–464.

[38] ——, *A comparison of nonparametric tests of weak separability for annual and quarterly data on consumption, leisure and money*, Journal of Business and Economics Statistics, 6 (1988), pp. 241–246.

[39] ——, *A revealed preference test for weakly separable utility maximization with incomplete adjustment*, Journal of Econometrics, 60 (1994), pp. 235–249.

[40] L. Tornqvist, *The bank of finland's consumption price index*, ank of Finland Monthly Bulletin, 10 (1936), pp. 1–8.

[41] H. R. Varian, *The nonparametric approach to demand analysis*, Econometrica, 50 (1982), pp. 945–973.

[42] ——, *Nonparametric test of consumer behavior*, Review of Economic Studies, (1983), pp. 99–110.

[43] ——, *Nonparametric demand analysis*. unpublished mimeo, 1985.