Some Problems in Stochastic Portfolio Theory

by

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Abstract

We consider some problems in the stochastic portfolio theory of equity markets. In the first part, we maximize the expected terminal value of a portfolio of equities. The optimal investment problem is then solved by the stochastic control approach. We next consider a portfolio optimization problem in a Lévy market with stochastic interest rates. Compared with Merton's model, there are two extra terms coming from the jump component of the stock price and the interest rate risk respectively in the optimal portfolio. The implication is that given the jump risk the investor should invest more in the equity when the stock price and the interest rate are positively correlated, and less when the two are negatively correlated. Our other observation is that given the interest rate risk and the same return as the pure diffusion case the investor should reduce her investment in the equity when the jump component presents in the stock price.

We consider relative arbitrage for an infinite market in the second part, and extend the relative arbitrage theory of equity markets to a market which consists of a countably infinite number of assets. One of our goals is to incorporate the bond market, because theoretically the zero coupon bond market is an infinite market. Our conclusion is that there exist relative arbitrage opportunities over arbitrary time horizons in this market under the condition that the capitalizations of the market follows the Pareto distribution. By doing so, we also improved the sufficient conditions of the relative arbitrage in the

equity market and provided partial answers to an open question proposed by Fernholz and Karatzas [9].

In the last part, we study a first-order model of equity markets. Here by first-order model we mean the growth rate and the volatility of the stock depend on the rank of the stock in the market. More precisely, we assume that the largest stock has zero growth rate and all the other stocks have positive growth rates; and the volatility of the stocks are the same and constant. Our purpose is to study the size effect of the equity market, which is often observed and means that the larger stocks have relatively smaller return and the small stocks have relatively larger return. We apply the stochastic portfolio theory to study the structure and the properties of this market, for example, the capital distribution, the portfolio performance and the diversity of the market.

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Chapter 1

Introduction

Modern portfolio theory or portfolio theory starts with Harry Markowitz and his paper "Portfolio Selection" [17] (1952). Because of his pioneering work, in 1990 he shared the Nobel Prize with Merton Miller and William Sharpe.

In his paper, Markowitz formalized the idea of diversification in mathematics and quantitatively showed why portfolio diversification works to reduce the risk of individual securities. He was the first to derive the concept of efficient portfolios, which are defined as those that have the smallest risk for a given level of expected return or the largest expected return for a given level of risk. Based on the mean-variance analysis, he proposed that an investor should select a portfolio from the efficient frontier, the set of efficient portfolios.

Tobin [31] (1958) extended Markowitz's work by adding a risk-free asset to the analysis. This made it possible to leverage portfolios on the efficient frontier and led to the super-efficient portfolio and the capital market line. By doing so, he also obtained the mutual fund separation theorem.

Following the line of efficient portfolios, Sharpe developed the renowned capital asset pricing model (CAPM) [29] (1964). According to this model, all investors should hold the market portfolio, which is actually Tobin's super-efficient portfolio, leveraged with positions in the risk-free asset. Most importantly, the model introduced the beta that related the required rate of return of risky assets and the market portfolio, which was assumed to be well-diversified and had only systematic risk. Roughly speaking, the CAPM implies that the higher risk should be accompanied by higher return. We just mention here that today these basic principles have been widely adopted by the financial community, especially the institutional investors.

Instead of Markowitz and Tobin's one-period static analysis, Samuelson [27] (1969) considered a multi-period portfolio section problem maximizing a utility function. The problem was solved by the stochastic dynamic programming method. A similar problem was also considered for a class of utility functions by Hakansson [14] (1970). But usually it is hard to obtain the explicit solutions for these problems.

The continuous-time case of portfolio selection problem was considered by Merton [19] [20] (1969, 1971). There the geometric Brownian motion was used to model the risky asset. The investor's goal was to choose optimal portfolio and consumption rules to maximize his utility. The problem was completely solved by the stochastic control approach. That is, the optimal portfolio and consumption rule were found explicitly.

Merton's model has an important impact on the development of mathematical finance, especially the field of portfolio optimization. Later on, all kinds of extensions have been made on this model, for instance, bankruptcy, drawdown constraints, stochastic volatility, transaction costs, stochastic interest rates, and the Brownian motion being replaced by other processes etc. For this contribution and his contribution to the options pricing, Merton was also one of the recipients of the Nobel Prize in 1997.

We remark here that other than the stochastic control approach, martingale methods are an alternative way to solve the utility maximization problem, see [15] and the references therein.

Stochastic portfolio theory is closely related to the portfolio selection problem but has something different. The theory was developed recently by R. Fernholz, a hedge fund manager. It proposes a new mathematical framework for constructing portfolios, analyzing the behavior of portfolios, and understanding the structure of equity markets. The main results are the relative arbitrage of the equity market and the use of the ranked process to analyze the equity market structure. Meanwhile, it also poses some interesting and significant problems in the areas of probability theory and stochastic analysis.

In this thesis, we shall consider some problems in the stochastic portfolio theory. Our main concern here is about the equity market, as the institutional investors do. We do not consider the consumption in the market as our purpose is to study the portfolio and the behavior of equity markets, so for most parts the risk free asset is not included.

Consider an equity market in which there are n stocks. The price of each stock satisfies the following stochastic differential equation

$$dX_i(t) = X_i(t)[b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dB_j(t)], \quad i = 1, 2, \cdots n. \tag{1.0.1}$$

A portfolio in the market is any measurable, adapted process $\pi(t) = (\pi_1(t), \cdots, \pi_n(t))$ such that $\sum_{i=1}^{n} \pi_i(t) = 1$, so the component $\pi_i(t)$ stands for the proportion or the weight of the ith stock in the portfolio π. The borrowing and the short-sale are not allowed in the market.

In particular, we define the market portfolio as the ratio of the market value of each stock to the total capital of the market. That is, $\mu(t) = (\mu_1(t), \cdots, \mu_n(t))$ is the market portfolio if the weights are given by

$$\mu_i(t) = X_i(t) / \sum_{i=1}^{n} X_i(t), \qquad i = 1, 2, \cdots n. \tag{1.0.2}$$

Given a portfolio π, let $Z_\pi(t)$ represent the value of this portfolio.

The first problem we consider is to maximize the following index function among all the portfolios

$$J(\pi) = \mathbb{E}[Z_\pi^p(T)] \tag{1.0.3}$$

where $0 < p < 1$ is a constant. The optimal investment problem is then solved by the stochastic control approach.

We next consider portfolio optimization in a Lévy market with stochastic interest rates. The problem is an extension to Merton's classic model in that we model the stock price by a Lévy process so that a jump can occur in the price of risky assets. We also replace the constant interest rate by stochastic interest rates which follow the Vasicek's term structure. By applying the dynamic programming method we solved the optimization problem. Some interesting implications and observations are obtained.

Our second problem is on the relative arbitrage in an infinite market. Initially we introduce the concept of diversity.

Let $\mu_{(1)}$ be the largest market weight, i.e. $\mu_{(1)}(t) = \max\,\{\mu_1(t), \cdots, \mu_n(t)\}$. We say the market is diverse if there exists $\delta \in (0,1)$ such that $\mu_{(1)}(t) < 1 - \delta, \quad t \geq 0$. The market is weakly diverse on $[0,T]$ if for some $\delta \in (0,1)$ such that

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)dt < 1 - \delta \quad a.s. \tag{1.0.4}$$

The concepts were originally used by Fernholz (1999). We assume that the market is nondegenerate. That is, there exists $\varepsilon > 0$ such that for all $t \geq 0$

$$x'\sigma(t)\sigma'(t)x \geq \varepsilon\|x\|^2, \quad x \in \mathbb{R}^n. \tag{1.0.5}$$

Then under this condition, it turns out there exist relative arbitrage opportunities over long terms if the market is diverse, see [8]. Formally, the portfolio π represents an arbitrage opportunity relative to μ over $[0,T]$ if with $Z^{\pi}(0) = Z^{\mu}(0) = z > 0$,

$$\mathbb{P}(Z^{\pi}(T) \geq Z^{\mu}(T)) = 1 \tag{1.0.6}$$

and

$$\mathbb{P}(Z^{\pi}(T) > Z^{\mu}(T)) > 0. \tag{1.0.7}$$

Recently, Fernholz and Karatzas et al. [10] have shown that relative arbitrage can exist over arbitrary time horizons even if the market is weakly diverse. In fact, the assumptions above can be weakened. The market need not be (weakly) diverse. The condition

$$\int_0^T \gamma_*^{\mu}(t)dt \geq \ln(n) + \zeta \quad a.s. \tag{1.0.8}$$

is sufficient to ensure relative arbitrage over time interval $[0,T]$, where $\gamma_*^{\mu}(t)$ is the excess growth rate of the market, n is the number of the stocks in the market and $\zeta > 0$ is a constant. More generally, the inequality above can be replaced by

$$\int_0^T \gamma_*^{\mu,p}(t)dt \geq \frac{n^{1-p}}{p}\ln(n) + \zeta \quad a.s. \tag{1.0.9}$$

where $0 < p < 1$ and $\gamma_*^{\mu,p}(t)$ is the generalized excess growth rate of the market,see Fernholz and Karatzas [9].

Then an open problem is, can we drop the term $\ln(n)$ in (1.0.8) or $\frac{n^{1-p}}{p}\ln(n)$ in (1.0.9) and therefore generalize the theory to the case which allows a countably infinite number of stocks in the market? Because the structure of infinite markets is somehow similar to that of finite markets in which stocks enter and exit. Furthermore, our another thought is to incorporate the bond market, because theoretically the zero coupon bond market is an infinite market.

Based on these considerations, in this thesis we assume that there are a countably infinite number of assets in the market and investigate the possible relative arbitrage opportunity in this market. As a matter of fact, our answer to the open problem is yes and we shall show that relative arbitrage can exist over arbitrary time horizons under the condition that the capitals of the market have a Pareto distribution. That is, the market weights have the form

$$\mu_i(t) = \frac{a(t)}{i^\alpha}, \quad \alpha > 1, \quad i = 1, 2, \cdots$$

where $a(t) > 0$ is a measurable, adapted process such that $\sum_{i=1}^{\infty} \frac{a(t)}{i^\alpha} = 1$.

Our assumption on the market portfolio above is based on the section 5.1 (pp93-96) in Fernholz [8]. There the capital distribution of markets is studied heuristically. It says that under some ideal conditions, the capital distribution curve is a straight line if the smallest stocks are replaced by constant rates. Here the capital distribution curve is the log-log plot of the market weights versus their respective ranks in descending order. This linear log-log weight distribution is called a Pareto distribution. There is some other evidence to support this assumption, see Fernholz, Karatzas et al. [1].

The last problem we consider is the size effect in equity markets. As we know, the size effect is often observed in equity markets in practice, so we wish to study the market models that reflect this phenomena.

The so called size or small-firm effect originally was studied by R. Banz [2]. According to the theory, if we divide the stocks in the market into different portfolios each year by firm size, i.e. the total value of outstanding equity, then the small-firm portfolios constantly have higher average annual returns. Of course, the small-firm portfolios tend to be riskier. But here we shall focus on the growth rate, though some conclusions can be extended to the case of time-dependent variances.

A first-order model is such a market model where the growth rate and the variance of stocks depend on the rank of the stocks in the market. Here by rank we mean the descending order of the stocks in terms of their capitalizations in the market. An atlas type of such models is studied in Fernholz [8] (see Example 5.3.3) and particularly in Banner, Fernholz et al. [1], where the same, constant variances are assigned to all the stocks; zero growth rate to all of the stocks but the smallest; and positive growth rate to the smallest.

In this thesis, we would like to study another type of first-order model. In fact, the setting of our model is somehow the reverse of the set-up in Banner, Fernholz et al. [1]. More precisely, we assign the same, constant variances to all the stocks; the positive growth rate to all the stocks but the largest; and zero growth rate to the largest.

We shall apply the stochastic portfolio theory, especially the concepts of portfolio generating functions and diversity to study the behaviors and the properties of the market. We show that the capital distribution of the market is asymptotically stable. Then we use the portfolio generating function to construct portfolios. It turns out that some of these function generated portfolios have a larger rate of growth than the market portfolio in the long term. We also consider the diversity of the market. A sufficient condition for ensuring the diversity of this market is given. We finally compare our market model with the atlas model.

Some simulations have been done to support our results.

Chapter 2

Portfolio Optimization

2.1 Introduction

The continuous time portfolio problem has a long history dating back to the pioneering work of Merton [19] [20]. It is concerned with finding the optimal investment strategy for individual investors. The investor tries to allocate her wealth into the risky and the risk free asset, and maximize her expected utility from consumption.

Many authors have considered the similar problems, i.e. finding the optimal portfolio and the optimal consumption which maximize the investor's utility from consumption, see [15] and the references therein. In particular, Korn & Kraft [16] considered a portfolio optimization problem in a savings & bond and/or a mixed savings & bond/stock market.

In this chapter, we formulate a portfolio optimization problem in an equity market consisting of n stocks. We consider only the investment in the market and maximize the expected terminal wealth, as those institutional investors do, with no consumption. The problem is then solved completely via the stochastic optimal control approach. That is, we can find the optimal percentage for each stock explicitly.

2.2 Formulation of The Problem

Consider an equity market in which there are n stocks. The price of each stock is described by the following stochastic differential equation,

$$dX_i(t) = X_i(t)[b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dB_j(t)], \quad i = 1, 2, \cdots, n, \tag{2.2.1}$$

where $X_i(t)$ is the price of the ith stock, $b_i(t)$ is the mean return of the ith stock, $B_j(t)$ is the jth component of an n dimensional Brownian motion, $\sigma_{ij}(t)$ is the volatility of the ith stock with respect to the jth source of uncertainty.

The Brownian motion B is defined on a given complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We denote by $(\mathscr{F}_t, t \geq 0)$ the natural filtration generated by B. i.e. $\mathscr{F}_t = \sigma(B(s), 0 \leq s \leq t)$.

Remark The stock price above is a log normal process in the sense that the equation (2.2.1) can be written as

$$d \ln X_i(t) = \gamma_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dB_j(t), \tag{2.2.2}$$

where $\gamma_i(t) = b_i(t) - \frac{1}{2} \sum_{j=1}^{n} \sigma_{ij}^2(t)$.

Now we define the portfolio in the market.

Definition 2.2.1. *A n dimensional measurable, adapted vector process* $\pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))$ *is called a portfolio if* $0 \leq \pi_i(t) \leq 1$ *for each* $i = 1, 2, \cdots, n$ *and* $\sum_{i=1}^{n} \pi_i(t) = 1$.

So here $\pi_i(t)$ stands for the weight of the ith stock in the portfolio π. Short sales or borrowings are not allowed in the market as $0 \leq \pi_i \leq 1$ for each i.

Then given a portfolio π, the value process of this portfolio, denoted by $Z_\pi(t)$, satisfies

$$
\begin{aligned}
\frac{dZ_\pi(t)}{Z_\pi(t)} &= \sum_{i=1}^{n} \pi_i(t)\ \frac{dX_i(t)}{X_i(t)} \\
&= \sum_{i=1}^{n} \pi_i(t) b_i(t) dt + \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t) dB_j(t).
\end{aligned}
\tag{2.2.3}
$$

Or in a matrix form the expression in (2.2.3) can be written as

$$
dZ_\pi(t) = Z_\pi(t)(\pi(t)' b(t)\ dt + \pi(t)'\sigma(t) dB(t)) \tag{2.2.4}
$$

where

$$
\pi(t) = \begin{pmatrix} \pi_1(t) \\ \vdots \\ \pi_n(t) \end{pmatrix}, \qquad b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}, \qquad B(t) = \begin{pmatrix} B_1(t) \\ \vdots \\ B_n(t) \end{pmatrix}
$$

are n dimensional column vectors and $\sigma(t) = (\sigma_{ij}(t))_{1 \le i,j \le n}$ is an $n \times n$ matrix.

The investor's goal is to maximize the following index function among all the portfolios

$$
J(\pi(\cdot)) = \mathbb{E}[Z_\pi^p(T)] \tag{2.2.5}
$$

where $0 < p < 1$ is a constant. In the following sections, we would like to find an optimal portfolio which maximizes the utility function above.

2.3 Stochastic Control Approach

In order to solve the problem formulated in the last section, we shall apply the stochastic control approach. Let $V(t,z)$ be the value function and we have the following result [1]

Theorem 2.3.1. *Assume that the optimal portfolio π exists. Then V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation*

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,z)+\sup_{\pi'\mathbf{1}=1}\{\mathscr{A}V(t,z)\}=0,\\&V(T,z)=z^p,\end{aligned}\tag{2.3.1}$$

where the operator

$$\mathscr{A}:=z\pi' b\frac{\partial}{\partial z}+\frac{1}{2}z^2\pi' A\pi\frac{\partial^2}{\partial z^2}$$

and $A=\sigma\sigma'$. $\mathbf{1}$ stands for the n dimensional vector whose components are all 1.

Let us consider the supremum term in (2.3.1). In view of the constraint of the portfolio, we construct the Lagrange function

$$l(\pi,\lambda)=z\pi' b\frac{\partial V}{\partial z}+\frac{1}{2}z^2\pi' A\pi\frac{\partial^2 V}{\partial z^2}-\lambda(\pi'\mathbf{1}-1)$$

where λ is the Lagrange multiplier factor. The first order condition for π is

$$z\frac{\partial V}{\partial z}b+z^2\frac{\partial^2 V}{\partial z^2}A\pi-\lambda\mathbf{1}=0.\tag{2.3.2}$$

Solving the equation for π gives

[1] For simplicity we ignore the functional dependence with respect to t

$$\pi = A^{-1}\left(\frac{\lambda}{z^2 V_{zz}}\mathbf{1} - \frac{V_z}{zV_{zz}}b\right), \tag{2.3.3}$$

where V_z, V_{zz} denote the first and the second derivative of V with respect to z respectively, and assume that A is invertible. Using the condition $\pi'\mathbf{1} = 1$ we obtain

$$\lambda = \frac{z^2 V_{zz} + zV_z \mathbf{1} A^{-1} b}{\mathbf{1}' A^{-1}\mathbf{1}}.$$

Substitute this into (2.3.3) and after some manipulations we have

$$\pi = \frac{1}{\mathbf{1}'A^{-1}\mathbf{1}}A^{-1}\mathbf{1} + \frac{V_z}{zV_{zz}}\left(\frac{\mathbf{1}'A^{-1}b}{\mathbf{1}'A^{-1}\mathbf{1}}A^{-1}\mathbf{1} - A^{-1}b\right). \tag{2.3.4}$$

Thus we obtained the candidate for the optimal portfolio π. However, the value function V in (2.3.4) is unknown. To determine the function, we assume that it has the form

$$V(t,z) = g(t)z^p, \tag{2.3.5}$$

where g satisfies $g(T) = 1$. Notice in this case $V_z = p\, g(t)z^{p-1}, V_{zz} = p(p-1)g(t)z^{p-2}$, so (2.3.4) becomes

$$\pi = \frac{1}{\mathbf{1}'A^{-1}\mathbf{1}}A^{-1}\mathbf{1} + \frac{1}{p-1}\left(\frac{\mathbf{1}'A^{-1}b}{\mathbf{1}'A^{-1}\mathbf{1}}A^{-1}\mathbf{1} - A^{-1}b\right). \tag{2.3.6}$$

Substitute (2.3.5) and (2.3.6) into (2.3.1), and we get

$$\frac{dg}{dt} + h(t)g(t) = 0, \tag{2.3.7}$$

where

$$h(t) = p\pi' b + \frac{1}{2} p(p-1)\pi' A\pi.$$

This ordinary differential equation can be solved as

$$g(t) = \exp(H(T) - H(t))$$

where $H(t)$ is the primitive of $h(t)$. i.e. $H'(t) = h(t)$.

From the expression above we get the value function

$$V(t,z) = z^p \exp(H(T) - H(t)). \tag{2.3.8}$$

In fact, by the verification theorem, the portfolio π given by (2.3.6) is also optimal. We summarize the analysis above as follows.

Theorem 2.3.2. *For the optimal investment problem in the equity market, the index function* (2.2.5) *is maximized by the optimal portfolio π given by* (2.3.6) *and the optimal value function is given by* (2.3.8).

2.4 Additional Risk-Free Asset

Suppose besides the equity, we have an additional risk free asset, say, a savings account, to invest. This asset, denoted by $S(t)$, evolves according to the equation

$$dS(t) = r(t)S(t)dt, \tag{2.4.1}$$

where $r(t)$ is the interest rate.

Let $\pi_0(t)$ be the percentage we invest in this risk free asset. We put the requirement $\sum_{i=1}^{n} \pi_i(t) + \pi_0(t) = 1$. Then the value of the portfolio, denoted by $Z(t)$, satisfies

$$
\begin{aligned}
\frac{dZ(t)}{Z(t)} &= \sum_{i=1}^{n} \pi_i(t)\, \frac{dX_i(t)}{X_i(t)} + \pi_0(t)\frac{dS(t)}{S(t)} \\
&= (\sum_{i=1}^{n} \pi_i(t) b_i(t) + r(t)\pi_0(t))dt + \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t) dB_j(t).
\end{aligned} \tag{2.4.2}
$$

Again in a matrix form (2.4.2) can be written as

$$
dZ(t) = Z(t)[(\pi(t)'b(t) + \pi_0(t)r(t))dt + \pi(t)'\sigma(t)dB(t)]. \tag{2.4.3}
$$

We still maximize the index function (2.2.5). In order to solve this problem, let $V(t,z)$ be the value function, and we have

Theorem 2.4.1. *Assume that the optimal portfolio π exists. Then it satisfies the HJB equation*

$$
\begin{aligned}
&\frac{\partial V}{\partial t}(t,z) + \sup_{\pi'\mathbf{1}+\pi_0=1} \{\mathscr{A}V(t,z)\} = 0, \\
&V(T,z) = z^p,
\end{aligned} \tag{2.4.4}
$$

where the operator

$$
\mathscr{A} := z(\pi' b + \pi_0 r)\frac{\partial}{\partial z} + \frac{1}{2} z^2 \pi' A \pi \frac{\partial^2}{\partial z^2}.
$$

Considering the supremum term in (2.4.4), we construct the Lagrange function

$$
l(\pi,\pi_0,\lambda) = z(\pi' b + r\pi_0)\frac{\partial V}{\partial z} + \frac{1}{2} z^2 \pi' A \pi \frac{\partial^2 V}{\partial z^2} - \lambda(\pi'\mathbf{1} - \pi_0 - 1).
$$

The first order conditions for π and π_0 are

$$
\begin{aligned}
& z\frac{\partial V}{\partial z}b + z^2\frac{\partial^2 V}{\partial z^2}A\pi - \lambda\mathbf{1} = 0 \\
& rz\frac{\partial V}{\partial z} - \lambda = 0.
\end{aligned}
$$

Solve these equations, and we obtain the candidates for the optimal portfolio

$$\pi = \frac{V_z}{zV_{zz}}A^{-1}(r\mathbf{1} - b) \tag{2.4.5}$$

and

$$\pi_0 = 1 - \pi'\mathbf{1}. \tag{2.4.6}$$

We next assume that the value function V has the same form as (2.3.5), so (2.4.5) can be written as

$$\pi = \frac{1}{p-1}A^{-1}(r\mathbf{1} - b). \tag{2.4.7}$$

Using (2.4.6), (2.4.7) and the equation (2.4.4), we can determine the value function

$$V(t,z) = z^p \exp(\tilde{H}(T) - \tilde{H}(t)), \tag{2.4.8}$$

where $\tilde{H}(t)$ satisfies $\tilde{H}'(t) = \tilde{h}(t) = p(\pi' b + \pi_0 r) + \frac{1}{2}\, p(p-1)\pi' A\pi$.

In fact, by the verification theorem, the necessary condition above is also sufficient. Therefore, we have the following

Theorem 2.4.2. *For the optimal investment problem in the market consisting of the equity and the risk free asset, the optimal portfolio is given by* (2.4.6) *and* (2.4.7)*. The value of the optimal portfolio is given as* (2.4.8).

The interesting observation here is that the investment in the equity and the risk-free market can be made separately. That is, we can just invest in the equity market and act like there is no risk-free asset. Then we allocate the remaining value in the risk-free asset.

2.5 Portfolio Optimization in a Lévy Market

2.5.1 Introduction

In Merton's classical model, a portfolio optimization problem is considered in a market consisting of two assets: a savings account and a stock. The interest rate for the savings account is taken to be a constant and the stock price is modeled by a geometric Brownian motion, which is a continuous process. However, in practice we know the interest rate fluctuates from time to time. Also, the stock price often jumps in view of some significant events.

In this section we consider a couple of extensions to Merton's model to characterize these properties of the market. More precisely, we assume that the interest rate is stochastic and follows Vasicek's term structure model. Furthermore, we model the stock price by a Lévy process, which is a natural extension of Brownian motion and has both continuous and discontinuous components in its decomposition. We feel it would be interesting to investigate this optimization problem and see how the investor's behaviors are affected under the random interests and the stock price with jumps. In fact, the results we have found are quite meaningful as we shall see later.

Some similar work has been done by other authors. For example, a bond/equity mixed portfolio optimization problem is considered in a pure diffusion market with stochastic interest rates in [16]; an optimal investment and consumption problem is considered in a jump diffusion market in [13]. Especially, we like to mention in [30] a similar optimal debt ratio is obtained in the debt management.

We arrange the section as the following. In the second part, we formulate the problem as a stochastic optimal control problem. In the third part, we solve the problem by the dynamic programming method. In the last part we compare our results with the solution in the pure diffusion case.

2.5.2 Formulation of The Problem

Suppose an investor invests her wealth $X(t)$ in a market consisting of two assets, a savings account and a stock. The stock price $P(t)$ satisfies the following equation

$$
\begin{aligned}
&dP(t) = P(t-)[b(t)dt + \sigma_1(t)dB_t^1 + \int_0^t \int_{\mathbb{R}} z\tilde{N}(dt,dz)], \\
&P(0-) = p > 0,
\end{aligned}
\tag{2.5.1}
$$

where $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is the compensated Poisson measure, $N(dt,dz)$ is the differential form of the Poisson measure, and $\nu(\cdot)$ is the Lévy measure. $(B_t^1, t \geq 0)$ is a standard Brownian motion on a filtered complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t, t \geq 0), \mathbb{P})$.

The equation (2.5.1) can be solved by Itô's formula for Lévy process and the solution is

$$
\begin{aligned}
P(t) = p\exp\{\int_0^t (b(s) - \frac{1}{2}\sigma_1^2(s))ds + \int_0^t \sigma_1(s)B_s^1 + t\int_{\mathbb{R}} [\ln(1+z) - z]\nu(dz) \\
+ \int_0^t \int_{\mathbb{R}} \ln(1+z)\tilde{N}(dt,dz)\}.
\end{aligned}
$$

We assume that the jump aptitude $z > -1$ so that the price is positive. Additionally, we assume $\int_{-1}^{\infty} z \vee z^2 \nu(dz) < \infty$ to ensure that $P(t)$ has finite first and second moments.

The money $M(t)$ in the savings account satisfies the following equation:

$$dM(t) = r(t)M(t)dt, \tag{2.5.2}$$

where $r(t)$ is the interest rate. As stated earlier, we assume that the term structure of the interest rate $r(t)$ follows

$$\begin{aligned} dr(t) &= \alpha(\theta - r(t))dt + \sigma_2 dB_t^2, \\ r(0) &= r_0 > 0, \end{aligned} \tag{2.5.3}$$

where α, θ, σ_2 are positive constants and $(B_t^2, t \geq 0)$ is another standard Brownian motion on the same filtered probability space and correlated with $(B_t^1, t \geq 0)$ by

$$\mathbb{E}[dB_t^1 \cdot dB_t^2] = \rho dt.$$

This is one of the earliest stochastic models of the term structure proposed by Vasicek in 1977. The feature of the model is the mean reversion property of the interest rate.

Suppose at time t the investor can invest the fraction $u(t)$ of her wealth into the stock and the rest into the savings account without transaction costs. We assume that $u(t) \in [0,1]$ is $\mathscr{F}_t$ measurable and self-financing. Then her wealth $X(t)$ evolves according to the equation

$$
\begin{aligned}
dX(t) = X(t)[r(t)+u(t)(b(t)-r(t))]dt + u(t)X(t)\sigma_1(t)dB_t^1 \\
+ u(t)X(t-)\int_0^t \int_{-1}^{\infty} z\tilde{N}(dt,dz),
\end{aligned} \tag{2.5.4}
$$

with the initial wealth $X(0-) = x_0 > 0$.

Remark The condition $u(t) \in [0,1]$ is sufficient to guarantee $X(t)$ is nonnegative for all $t \geq 0$, so the investor will not be in bankruptcy in the market.

The investor's goal is to find the optimal portfolio $u^*(t) \in [0,1]$ which maximizes her utility from the expected terminal wealth:

$$
J(u(\cdot); r_0, x_0) = \mathbb{E}[x(T)^{\gamma}] \tag{2.5.5}
$$

where $0 < \gamma < 1$.

In order to apply the stochastic optimal control approach, we varies the initial state from 0 to time $t \leq 0$ and let r, x be the corresponding interest rate and the wealth at the time respectively, $V(t, r, x)$ be the value function. In the following section, we shall find the optimal portfolio by the dynamic programming method.

2.5.3 Main Results

In this section, we shall solve the optimization problem formulated in the last section. First of all, We need a verification theorem in the Lévy market, for details see [24]. [2]

Theorem 2.5.1. *Suppose there exists a functional $W(t, r, x)$ such that*

$$
A^u W(t, r, x) \leq 0 \tag{2.5.6}
$$

[2] For simplicity from now on we ignore the functional dependence with respect to t

for all $t,x \geq 0, r \in \mathbb{R}$ *and* $u \in [0,1]$*, where*

$$\begin{aligned} A^u W(t,r,x) =& W_t + \alpha(\theta - r)W_r + x[r + u(b-r)]W_x \\ &+ \frac{1}{2}(\sigma_2^2\, W_{rr} + u^2x^2\sigma_1^2 W_{xx} + 2ux\rho\sigma_1\sigma_2 W_{xr}) \\ &+ \int_{-1}^{\infty}[W(t,r,x+uxz) - W(t,r,x) - uxzW_x]\ \nu(dz), \end{aligned} \tag{2.5.7}$$

and

$$W(T,r,x) = x^{\gamma}, \tag{2.5.8}$$

for all $r \in \mathbb{R}, x \geq 0$*, then*

$$W(t,r,x) \geq V(t,r,x)$$

for all $t,x \geq 0, r \in \mathbb{R}$ *and* $u \in [0,1]$.

Further, assume for each $t,x > 0$ *and* $r \in \mathbb{R}$ *there exists* $u_0 = u_0(t,r,x) \in [0,1]$ *such that*

$$A^{u_0} W(t,r,x) = 0. \tag{2.5.9}$$

Then $u_0(t)$ *is the optimal portfolio and* $W(t,r,x)$ *is the value function.*

Proof. By Dynkin's formula for a Lévy process and (2.5.6), we have

$$\begin{aligned} \mathbb{E}[W(T,r(T),X(T))] &= W(t,r,x) + \mathbb{E}[\int_t^T A^u W(s,r(s),X(s))ds] \\ &\leq W(t,r,x) \end{aligned}$$

for all $u(\cdot) \in [0,1]$.

However, by (2.5.8), $W(T, r(T), X(T)) = X(T)^{\gamma}$. Therefore,

$$\mathbb{E}[X(T)^{\gamma}] \leq W(t,r,x)$$

for all $u(\cdot) \in [0,1]$.

Take the supermum of both sides over $u(\cdot) \in [0,1]$ and it follows immediately

$$V(t,r,x) \leq W(t,r,x) \tag{2.5.10}$$

for all $t, x \geq 0$ and $r \in \mathbb{R}$.

Further, if (2.5.9) holds, then $W(t,r,x) = \mathbb{E}[X(T)^{\gamma}]$ for $u_0 = u_0(t,r,x) \in [0,1]$, so

$$W(t,r,x) \leq V(t,r,x). \tag{2.5.11}$$

Combine this inequality with (2.5.10) and we conclude that $W(t,r,x)$ is the value function and $u_0(t) := u_0(t, r(t), X(t-)) \in [0,1]$ is the optimal portfolio. $\square$

Now we state our main results.

Theorem 2.5.2. *For the portfolio problem in the Lévy market, the optimal portfolio u^* satisfies*

$$(b-r) + (\gamma-1)\sigma_1^2 u^* + \rho\sigma_1\sigma_2 a(t) + \int_{-1}^{\infty} z[(1+u^*z)^{\gamma-1} - 1]\ \nu(dz) = 0 \tag{2.5.12}$$

and the value function is given by

$$V(t,r,x) = x^{\gamma} g(t) \exp(a(t)r), \tag{2.5.13}$$

where $a(t)$ is defined by

$$a(t) = \frac{\gamma}{\alpha}(1 - \exp(\alpha(t-T)). \tag{2.5.14}$$

and

$$g(t) = \exp(\frac{1}{\gamma - 1}(H(T) - H(t))). \tag{2.5.15}$$

Here $H(t)$ is the primitive of $h(t)$, i.e. $H(t)' = h(t)$, and

$$\begin{aligned} h(t) = \alpha\theta a(t) + \gamma u(b-r) + \frac{1}{2}\sigma_2^2\, a(t)^2 + \frac{1}{2}\gamma(\gamma-1)\sigma_1^2(u^*)^2 + \\ \gamma\sigma_1\sigma_2 a(t)u^* + \int_{-1}^{\infty}[(1+u^*z)^\gamma - 1 - \gamma u^* z]\nu(dz) \end{aligned} \tag{2.5.16}$$

Moreover, we have the following upper bound for the optimal portfolio

$$u^* \leq \frac{b - r + \rho\sigma_1\sigma_2 a(t)}{(1-\gamma)\sigma_1^2 + \int_{-1}^{\infty} z[1-(1+z)^{\gamma-1}]\nu(dz)}. \tag{2.5.17}$$

Proof. Let $W(t,r,x) = f(t,r)x^\gamma$ with $f(T,r) = 1$. From (2.5.7), we get

$$\begin{aligned} f_t + \alpha(\theta - r)f_r + \gamma[r + u(b-r)]f + \frac{1}{2}\sigma_2^2 f_{rr} + \frac{1}{2}\gamma(\gamma-1)\sigma_1^2 u^2 f \\ + \gamma\rho\sigma_1\sigma_2 u f_r + f\int_{-1}^{\infty}[(1+uz)^\gamma - 1 - \gamma uz]\nu(dz) = 0. \end{aligned} \tag{2.5.18}$$

Let

$$f(t,r) = g(t)\exp(a(t)r)$$

with

$$g(T) = 1, \quad a(T) = 0. \tag{2.5.19}$$

Then (2.5.18) becomes

$$\begin{aligned} &g' + a'(t)rg + \alpha(\theta - r)a(t)g + \gamma[r + (b-r)]g + \frac{1}{2}\sigma_2^2 a(t)^2 g \\ &\quad + \frac{1}{2}\gamma(\gamma-1)\sigma_1^2 u^2 g + \gamma\rho\sigma_1\sigma_2 u a(t) g \\ &\quad + g\int_{-1}^{\infty}[(1+uz)^{\gamma} - 1 - \gamma uz]\nu(dz) = 0. \end{aligned} \tag{2.5.20}$$

By solving the first order necessary condition we obtain the optimal portfolio u^* satisfying the equation

$$(b-r) + (\gamma-1)\sigma_1^2 u^* + \rho\sigma_1\sigma_2 a(t) + \int_{-1}^{\infty} z[(1+u^*z)^{\gamma-1} - 1]\nu(dz) = 0. \tag{2.5.21}$$

Notice that if $u^* \in [0,1]$,

$$\int_{-1}^{\infty} z[(1+u^*z)^{\gamma-1} - 1]\nu(dz) \leq u^* \int_{-1}^{\infty} z[(1+z)^{\gamma-1} - 1]\nu(dz), \tag{2.5.22}$$

so we have the following upper bound for u^*:

$$u^* \leq \frac{b - r + \rho\sigma_1\sigma_2 a(t)}{(1-\gamma)\sigma_1^2 + \int_{-1}^{\infty} z[1-(1+z)^{\gamma-1}]\nu(dz)}. \tag{2.5.23}$$

To determine $a(t)$ and $g(t)$, we substitute u^* into (2.5.20), rearrange the terms and get that

$$
\begin{aligned}
g' + [a'(t) - \alpha a(t) + \gamma] rg + [\alpha\theta a(t) + \gamma\xi u + \frac{1}{2}\sigma_2^2 a(t)^2 \\
+ \frac{1}{2}\gamma(\gamma-1)\sigma_1^2 (u^*)^2 + \gamma\sigma_1\sigma_2 a(t) u^* \\
+ \int_{-1}^{\infty} [(1+u^*z)^\gamma - 1 - \gamma u^* z]\nu(dz)] g = 0.
\end{aligned} \tag{2.5.24}
$$

In the above equality, set

$$
a'(t) - \alpha a(t) + \gamma = 0.
$$

Solving the equation with the terminal value condition (2.5.19) gives

$$
a(t) = \frac{\gamma}{\alpha}(1 - \exp(\alpha(t-T)). \tag{2.5.25}
$$

Let

$$
\begin{aligned}
h(t) = \alpha\theta a(t) + \gamma\xi u + \frac{1}{2}\sigma_2^2 a(t)^2 + \frac{1}{2}\gamma(\gamma-1)\sigma_1^2 (u^*)^2 + \\
\gamma\sigma_1\sigma_2 a(t) u^* + \int_{-1}^{\infty} [(1+u^*z)^\gamma - 1 - \gamma u^* z]\nu(dz)
\end{aligned} \tag{2.5.26}
$$

and $H(t)$ be the primitive of $h(t)$, i.e. $H(t)' = h(t)$. Then from (2.5.19) and (2.5.24), we obtain

$$
g(t) = \exp(\frac{1}{\gamma-1}(H(T) - H(t))). \tag{2.5.27}
$$

It follows from the verification theorem that the above portfolio u^* is also optimal, and the value function is given by

$$V(t,r,x) = x^\gamma g(t)\exp(a(t)r).$$

Notice that if in (2.5.12) we use the approximation $(1+u^*z)^{\gamma-1} \approx 1+(\gamma-1)u^*z$, then the optimal portfolio u^* can be solved explicitly as

$$u^*(t) \approx \frac{b-r+\rho\sigma_1\sigma_2 a(t)}{(1-\gamma)\sigma_1^2+\int_{-1}^{\infty} z^2\, \nu(dz)}. \tag{2.5.28}$$

Comparing this solution with Merton's results, we see that there are two extra terms here. One is on the top and the other is on the bottom of the expression (2.5.28). The two terms come from two sources of risk respectively: the interest rate and the jump component of the stock price. It is not clear that the investor should invest more or less than the Merton line, since the optimal ration now is driven by the combined force of the two factors. But it is interesting to observe that given the risk from the jump component the investor should invest more in the equity if the interest rate is positively correlated to the stock price and invest less if the two are negatively correlated. This is actually reasonable because when the stock price increases the investor realizes an immediate gain from her investment in the equity. The positive correlation implies the interest rate increases too, so the gain can be invested at a higher than average rate of interest. Similarly when the stock price decreases, the investor makes an immediate loss which can be financed at a lower than average rate of interest because of the negative correlation. Another observation is that when the correlation of the interest rate and the stock price is zero or as time approaches T, the extra term on the top varnishes. This is also reasonable since in this case the interest rate risk is independent of or not important for the investment put in the equity.

2.5.4 Comparison with The Pure Diffusion Case

As we mentioned earlier, a bond/equity mixed portfolio optimization problem is considered in a pure diffusion market with stochastic interest rates in [16]. In this last section we are going to compare the results here and there. Although the context in that paper may be a bit different from our formulation, essentially we can use the same method to find the optimal portfolio in the pure diffusion case. We omit the calculations and summarize the results as the following.

Theorem 2.5.3. *For the portfolio optimization problem in the pure diffusion case, the optimal portfolio is given by*

$$\bar{u}(t) = \frac{b - r(t)}{(1-\gamma)\sigma_1^2} + \frac{\rho\sigma_2}{(1-\gamma)\sigma_1} a(t), \tag{2.5.29}$$

and the value function is given by

$$V(t,r,x) = x^\gamma \exp\{\frac{1}{\gamma - 1}(H(T) - H(t)) + \frac{\gamma}{\alpha}(1 - \exp(\alpha(t-T))r\}, \tag{2.5.30}$$

where

$$a(t) = \frac{\gamma}{\alpha}(1 - \exp(\alpha(t-T)). \tag{2.5.31}$$

and

$$g(t) = \exp(\frac{1}{\gamma - 1}(H(T) - H(t))). \tag{2.5.32}$$

Here $H(t)$ is the primitive of $h(t)$ and

$$h(t) = \alpha\theta(\gamma-1)a(t) + \frac{1}{2}(\gamma-1)\sigma_2^2 a(t)^2 - \frac{1}{2}\gamma(\lambda + \rho\sigma_2 a(t))^2.$$

Now compare this optimal portfolio $\bar{u}(t)$ with $u^*(t)$ in the Lévy market. Notice

$$\int_0^\infty z[1-(1+z)^{\gamma-1}]\nu(dz) \geq 0,$$

and it is easy to see that

$$u^*(t) \leq \bar{u}(t).$$

That is, in general the optimal ratio for the investment in the equity in the Lévy market is smaller than the corresponding one in the pure diffusion case with the same term structure of the interest rate. This implies that given the same interest rate risk and the return $b(t)$ when the stock price is modeled by a diffusion process, the investor should reduce her investment in the equity if the jump component presents in the stock price because the equity market is more risky. But the rule is not necessarily true if the investor has different returns in the two markets.

Chapter 3

Relative Arbitrage

3.1 Introduction

The concept of market diversity was first introduced by R. Fernholz [7]. Roughly speaking, it says that no single company is allowed to dominate the entire market in terms of relative capitalization. Assume that the market is nondegenerate. That is, if there exists a $\delta > 0$ such that

$$x\sigma(t)\sigma(t)x' \geq \delta \|x\|^2, \quad \forall x \in \mathbb{R}^n, \tag{3.1.1}$$

where the $n \times n$ matrix-valued process $\sigma(t) = (\sigma_{ij}(t))_{1 \leq i,j \leq n}$ is the volatility of the market. The interesting implication under these reasonable assumptions is that there exist relative arbitrage opportunities in the market over long term time horizons, see R. Fernholz [8]. In fact, recently R. Fernholz and I. Karatzas et al. [9] have shown that relative arbitrage can exist over arbitrary time horizons even if the market is weakly diverse. It turns out the assumptions above can be weakened. The market need not be (weakly) diverse. The condition

$$\int_0^T \gamma_*^{\mu}(t)dt \geq \ln(n) + \zeta, \quad a.s. \tag{3.1.2}$$

is sufficient to ensure relative arbitrage over the time interval $[0,T]$, where $\gamma_*^{\mu}(t)$ is the excess growth rate of the market, n is the number of the stocks in the market and $\zeta > 0$ is a constant. More generally, the inequality (3.1.2) can be replaced by the following

$$\int_0^T \gamma_*^{\mu,p}(t)dt \geq \frac{n^{1-p}}{p}\ln(n) + \zeta, \quad a.s. \quad 0 < p < 1, \tag{3.1.3}$$

where $\gamma_*^{\mu,p}(t)$ is the generalized excess growth rate the market,see R. Fernholz and I. Karatzas [9].

Then an open problem is, can we drop the term $\ln(n)$ in (3.1.2) or $\frac{n^{1-p}}{p}\ln(n)$ in (3.1.3) and thereby generalize the theory to the case which allows countably infinite stocks in the market? The structure of infinite markets is somehow similar to that of finite markets in which stocks enter and exit, as mentioned in R. Fernholz [8]. Besides, our another idea is to incorporate the bond market since theoretically the zero coupon bond market is an infinite market. Based on these considerations, we assume that there are a countably infinite number of assets in the market and investigate the possible relative arbitrage opportunity. As a matter of fact, our answer to the question is positive and we shall show that relative arbitrage can exist over arbitrary time horizons provided the capitals of the market have a Pareto distribution.

This chapter is organized as the following. In section 2, we introduce the concepts of market portfolios and relative arbitrage. Section 3 is a result on functionally generated portfolios. Section 4 is our main results, and in section 5 we test the assumptions of the results in section 4. The final section 6 is our conclusions.

3.2 Market Portfolios

Consider an equity market in which there are a countably infinite number of stocks. We assume that each company has only one share outstanding since the shares of a company can be infinitely divisible. The price of stocks is modeled by a geometric Brownian motion. That is,

$$dX_i(t) = X_i(t)[b_i(t)dt + \sum_{j=1}^{\infty} \sigma_{ij}(t)dB_j(t)], \quad i = 1, 2, \cdots, \tag{3.2.1}$$

where $X_i(t)$ is the price of the ith stock, $b_i(t)$ is the mean return of the ith stock, $B_j(t)$ is the jth component of an infinite dimensional Brownian motion, $\sigma_{ij}(t)$ is the volatility of the ith stock with respect to the jth source of uncertainty.

The Brownian motion B is defined on a given complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We denote by $(\mathscr{F}_t, t \geq 0)$ the natural filtration generated by B. i.e. $\mathscr{F}_t = \sigma(W(s), 0 \leq s \leq t), \quad t \geq 0$.

Remark The equality (3.2.1) can be written as

$$d\ln X_i(t) = \gamma_i(t)dt + \sum_{j=1}^{\infty} \sigma_{ij}(t)dB_j(t), \tag{3.2.2}$$

where $\gamma_i(t) = b_i(t) - \frac{1}{2}\sum_{j=1}^{\infty} \sigma_{ij}^2(t)$ is called the growth rate of the ith stock.

Definition 3.2.1. *A nonnegative vector* $\pi(t) = (\pi_1(t), \pi_2(t), \cdots)$ *is called a portfolio process if it is measurable, adapted and satisfies* $0 \leq \pi_i(t) \leq 1, i = 1, 2, \cdots, \sum_{i=1}^{\infty} \pi_i(t) = 1$.

So here $\pi_i(t)$ represents the weight of the ith stock in the portfolio π. Short sales or borrowings are not allowed in the market since $0 \leq \pi_i(t) \leq 1$ for each $i \geq 1$. Then with this definition, the value process of the portfolio π, denoted by $Z^\pi(t)$, satisfies

$$\begin{aligned} \frac{dZ^\pi(t)}{Z^\pi(t)} &= \sum_{i=1}^{\infty} \pi_i(t)\ \frac{dX_i(t)}{X_i(t)} \\ &= \sum_{i=1}^{\infty} \pi_i(t) b_i(t) dt + \sum_{j=1}^{\infty} \sigma_j^\pi(t) dB_j(t), \end{aligned} \tag{3.2.3}$$

where $\sigma_j^\pi(t) = \sum_{i=1}^{\infty} \pi_i(t)\sigma_{ij}(t)$. That is, the portfolio π is self-financing.

Remark The equality (3.2.3) can be written as

$$d \ln Z^\pi(t) = \gamma^\pi(t) dt + \sum_{j=1}^{\infty} \sigma_j^\pi(t) dB_j(t), \tag{3.2.4}$$

where

$$\gamma^\pi(t) = \sum_{i=1}^{\infty} \pi_i(t)\gamma_i(t) + \gamma_*^\pi(t), \tag{3.2.5}$$

and

$$\gamma_*^\pi(t) = \frac{1}{2}(\sum_{i=1}^{\infty} \pi_i(t)\xi_{ii}(t) - \sum_{i,j=1}^{\infty} \pi_i(t)\xi_{ij}(t)\pi_j(t)) \tag{3.2.6}$$

are called the growth rate and the excess growth rate of the portfolio π, respectively, where $\xi(t) = \sigma(t)\sigma'(t)$.

From (3.2.4), (3.2.5) and (3.2.2), we can express the log of value processes in terms of the log price of stocks and the excess growth rate of portfolios, that is,

$$d \ln Z^\pi(t) = \sum_{i=1}^{\infty} \pi_i(t) d \ln X_i(t) + \gamma_*^\pi(t) dt. \tag{3.2.7}$$

Definition 3.2.2. *The portfolio μ with weights*

$$\mu_i(t) := \frac{X_i(t)}{\sum_{i=1}^{\infty} X_i(t)} \quad t \geq 0,$$

for $i = 1, 2, \cdots$ is called the market portfolio if $\sum_{i=1}^{\infty} X_i(t) < \infty$ a.s.

The market portfolio plays a fundamental role in the stochastic portfolio theory. One reason is that if one invests according to the portfolio he will achieve "the whole market" by a constant multiplier difference. In fact, it is not hard to verify that

$$\frac{dZ^{\mu}(t)}{Z^{\mu}(t)} = \frac{dZ(t)}{Z(t)},$$

where $Z(t) = \sum_{i=1}^{\infty} X_i(t)$, so $Z^{\mu}(t) = kZ(t)$, k is a constant.

Finally, we introduce the concept of relative arbitrage.

Definition 3.2.3. *We say that the portfolio π represents an arbitrage opportunity relative to η over $[0, T]$ if with $Z^{\pi}(0) = Z^{\eta}(0) = z > 0$,*

$$\mathbb{P}(Z^{\pi}(T) \geq Z^{\eta}(T)) = 1 \tag{3.2.8}$$

and

$$\mathbb{P}(Z^{\pi}(T) > Z^{\eta}(T)) > 0. \tag{3.2.9}$$

Here are some comments about the definition. First, if we treat the log of our value processes as the usual wealth process, it recovers the usual arbitrage definition. Second, relative arbitrage can be treated as a criterion to judge the performance of portfolios in the following sense. If the portfolio π is an arbitrage opportunity relative to η, then the portfolio π will have a better performance than η in the market.

3.3 Functionally Generated Portfolios

We introduce the functionally generated portfolios in this section. This concept plays a central role in the stochastic portfolio theory. Define $\Delta = \{x \in l^2(\mathbb{R}) : 0 < x_i < 1, i = 1, 2, \cdots, \sum_{i=1}^{\infty} x_i = 1\}$.

Definition 3.3.1. *Let S be a positive continuous function defined on Δ and π be a portfolio. We say that S generates π if there exists a measurable, bounded variation process θ such that*

$$\ln \frac{Z^{\pi}(t)}{Z^{\mu}(t)} = \ln S(\mu(t)) + \theta(t), \quad a.s. \quad t \in [0, T]. \tag{3.3.1}$$

If S generates the portfolio π, then we call S the portfolio generating function.

Remark We often use the differential form of the equation (3.3.1). That is,

$$d \ln \frac{Z^{\pi}(t)}{Z^{\mu}(t)} = d \ln S(\mu(t)) + d\theta(t), \quad a.s. \quad t \in [0, T]. \tag{3.3.2}$$

The definition provides us with a powerful tool to generate portfolios. In fact, we can generate portfolios by any C^2 function, as stated in the following theorem. Note that the notation D_i stands for the partial derivative with respect to the ith variable, and D_{ij} is the second derivative with respect to the ith and the jth variable.

Theorem 3.3.1. *Let S be a positive C^2 function defined on a neighborhood of Δ such that for each i, $x_i D_i \ln S(x)$ is bounded on Δ. Then S generates the portfolio π with weights*

$$\pi_i(t) = (D_i \ln S(\mu(t)) + 1 - \sum_{j=1}^{\infty} \mu_j(t) D_j \ln S(\mu(t)))\mu_i(t), \quad i = 1, 2, \cdots, \tag{3.3.3}$$

and the bounded variation process θ satisfies

$$d\theta(t) = \frac{-1}{2S(\mu(t))} \sum_{i,j=1}^{\infty} D_{ij}S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt, \tag{3.3.4}$$

where $\tau_{ij}(t)$ is the cross variation process of $\ln\mu_i(t)$ and $\ln\mu_j(t)$, i.e.

$$\langle \ln\mu_i(t), \ln\mu_j(t)\rangle_t = \tau_{ij}(t), \quad i,j = 1,2,\cdots. \tag{3.3.5}$$

As some readers may have seen, the theorem above is nothing but the extension of Theorem 3.1.5 (pp46-48) in R. Fernholz [8]. In order to prove the theorem, we need the following lemma.

Lemma 3.3.1. *Let $\tau(t) = (\tau_{ij}(t))_{i,j\geq 1}$ be defined by* (3.3.5)*. Then the null space of $\tau(t)$ is spanned by $\mu(t)$. In particular, $\tau(t)\mu(t)' = 0 \quad a.s.$*

Proof. From (3.3.5), we have

$$\begin{aligned}\tau_{ij}(t) &= \langle \ln\mu_i(t), \ln\mu_j(t)\rangle_t = \langle \ln\frac{X_i(t)}{Z^\mu(t)}, \ln\frac{X_j(t)}{Z^\mu(t)}\rangle_t \\ &= \langle \ln X_i(t) - \ln Z^\mu(t), \ln X_j(t) - \ln Z^\mu(t)\rangle_t \\ &= \langle \ln X_i(t), \ln X_j(t)\rangle_t - \langle \ln X_i(t), \ln Z^\mu(t)\rangle_t \\ &\quad - \langle \ln X_j(t), Z^\mu(t)\rangle_t + \langle \ln Z^\mu(t), \ln Z^\mu(t)\rangle_t.\end{aligned} \tag{3.3.6}$$

Recall $\xi_{ij}(t) = \sum_{k=1}^{\infty} \sigma_{ik}(t)\sigma_{jk}(t)$. Let

$$\xi_{i\mu}(t) = \sum_{k=1}^{\infty} \mu_k(t)\xi_{ik}(t)$$

and

$$\xi_{\mu\mu}(t) = \sum_{i,j=1}^{\infty} \mu_i(t)\xi_{ij}(t)\mu_j(t).$$

Then (3.3.6) can be written as

$$\tau_{ij}(t) = \xi_{ij}(t) - \xi_{i\mu}(t) - \xi_{j\mu}(t) + \xi_{\mu\mu}(t). \tag{3.3.7}$$

By (3.3.7), for any $0 \neq x = (x_1, x_2, \cdots) \in l^2(\mathbb{R})$, we get

$$x\tau(t)x' = x\xi(t)x' - 2x\xi(t)\mu(t)'\sum_{i=1}^{\infty} x_i + \mu(t)\xi(t)\mu(t)'(\sum_{i=1}^{\infty} x_i)^2. \tag{3.3.8}$$

Replacing x by $\mu(t)$ and noticing $\sum_{i=1}^{\infty}\mu_i(t) = 1$, from (3.3.8) it yields

$$\mu(t)\tau(t)\mu(t)' = 0. \tag{3.3.9}$$

However, since $\tau(t)$ is positive definite, we conclude that $\mu(t)$ spans the null space of $\tau(t)$. In particular, $\tau(t)\mu(t)' = 0 \quad a.s.$ □

Corollary 3.3.1. *Let π be a portfolio. Then*

$$\gamma_*^{\pi}(t) = \frac{1}{2}(\sum_{i=1}^{\infty}\pi_i(t)\tau_{ii}(t) - \sum_{i,j=1}^{\infty}\pi_i(t)\tau_{ij}(t)\pi_j(t)). \quad a.s. \tag{3.3.10}$$

Proof. From (3.3.7), we have

$$\sum_{i=1}^{\infty}\pi_i(t)\tau_{ii}(t) = \sum_{i=1}^{\infty}\pi_i(t)\xi_{ii}(t) - 2\sum_{i=1}^{\infty}\pi_i(t)\xi_{i\mu}(t) + \xi_{\mu\mu}(t)$$

and

$$\sum_{i,j=1}^{\infty}\pi_i(t)\tau_{ij}(t)\pi_j(t) = \sum_{i,j=1}^{\infty}\pi_i(t)\xi_{ij}(t)\pi_j(t) - \sum_{i=1}^{\infty}\pi_i(t)\xi_{i\mu}(t) - \sum_{j=1}^{\infty}\pi_j(t)\tau_{j\mu}(t) + \xi_{\mu\mu}(t).$$

Then (3.3.10) follows immediately from the above two equalities. □

When the portfolio π is the market portfolio, due to Lemma 4.3.1, (3.3.10) has a simpler form

$$\gamma_*^{\mu}(t) = \frac{1}{2}\sum_{i=1}^{\infty}\mu_i(t)\tau_{ii}(t). \quad a.s. \tag{3.3.11}$$

we are now ready to show the theorem.

Proof of Theorem 4.3.1. It is easy to see that the $\pi(t)$ defined by (3.3.3) is a portfolio, so to show the theorem we only need to verify (3.3.2).

Apply the Itô formula to $\mu_i(t) = \exp(\ln\mu_i(t))$, and we have

$$\begin{aligned} d\mu_i(t) &= \mu_i(t)d\ln\mu_i(t) + \frac{1}{2}\mu_i(t)d\langle\mu_i(t),\mu_i(t)\rangle_t \\ &= \mu_i(t)d\ln\mu_i(t) + \frac{1}{2}\mu_i(t)\tau_{ii}(t)dt. \end{aligned} \tag{3.3.12}$$

From (3.3.12), it follows

$$\begin{aligned} d\langle\mu_i(t),\mu_j(t)\rangle_t &= \mu_i(t)\mu_j(t)d\langle\ln\mu_i(t),\ln\mu_j(t)\rangle_t \\ &= \mu_i(t)\mu_j(t)\tau_{ij}(t)dt. \end{aligned} \tag{3.3.13}$$

Then again by the Itô formula, we have

$$
\begin{aligned}
d\ln S(\mu(t)) &= \sum_{i=1}^{\infty} D_i \ln S(\mu(t)) d\mu_i(t) + \frac{1}{2}\sum_{i,j=1}^{\infty} D_{ij}\ln S(\mu(t)) d\langle \mu_i(t), \mu_j(t)\rangle_t \\
&= \sum_{i=1}^{\infty} D_i \ln S(\mu(t)) d\mu_i(t) + \frac{1}{2}\sum_{i,j=1}^{\infty} D_{ij}\ln S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt.
\end{aligned}
\tag{3.3.14}
$$

But notice that

$$
\begin{aligned}
D_{ij}\ln S(\mu(t)) &= \frac{S(\mu(t))D_{ij}S(\mu(t)) - D_iS(\mu(t))D_jS(\mu(t))}{S^2(\mu(t))} \\
&= \frac{D_{ij}S(\mu(t))}{S(\mu(t))} - D_i \ln S(\mu(t)) D_j \ln S(\mu(t)).
\end{aligned}
\tag{3.3.15}
$$

So (3.3.14) turns out to be

$$
\begin{aligned}
d\ln S(\mu(t)) = &\sum_{i=1}^{\infty} D_i \ln S(\mu(t)) d\mu_i(t) + \frac{1}{2S(\mu(t))}\sum_{i,j=1}^{\infty} D_{ij}S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt \\
&- \frac{1}{2}\sum_{i,j=1}^{\infty} D_i \ln S(\mu(t)) D_j \ln S(\mu(t))\mu_i(t), \mu_j(t)\tau_{ij}(t)dt.
\end{aligned}
\tag{3.3.16}
$$

On the other hand, by (4.2.7), (4.3.12) and (4.3.10) we have

$$
\begin{aligned}
d\ln \frac{Z^{\pi}(t)}{Z^{\mu}(t)} &= \sum_{i=1}^{\infty} \pi_i(t) d\ln \mu_i(t) + \gamma_*^{\pi}(t)dt \\
&= \sum_{i=1}^{\infty} \frac{\pi_i(t)}{\mu_i(t)} d\mu_i(t) - \frac{1}{2}\sum_{i=1}^{\infty} \pi_i(t)\tau_{ii}(t)dt + \gamma_*^{\pi}(t)dt \\
&= \sum_{i=1}^{\infty} \frac{\pi_i(t)}{\mu_i(t)} d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^{\infty} \pi_i(t)\pi_j(t)\tau_{ij}(t)dt.
\end{aligned}
\tag{3.3.17}
$$

Now compare (4.3.16) and (4.3.17), and we let $\phi(t) = 1 - \sum_{j=1}^{\infty} \mu_j(t) D_j \ln S(\mu(t))$, $\pi_i(t) = (D_i \ln S(\mu(t)) + \phi(t))\mu_i(t), i = 1, 2, \cdots$. With this $\pi(t)$,

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{\pi_i(t)}{\mu_i(t)} d\mu_i(t) &= \sum_{i=1}^{\infty} (D_i \ln S(\mu(t)) + \phi(t)) d\mu_i(t) \\ &= \sum_{i=1}^{\infty} D_i \ln S(\mu(t)) d\mu_i(t) + \phi(t) \sum_{i=1}^{\infty} d\mu_i(t) \\ &= \sum_{i=1}^{\infty} D_i \ln S(\mu(t)) d\mu_i(t),\end{aligned} \tag{3.3.18}$$

since $\sum_{i=1}^{\infty} d\mu_i(t) = 0$.

Meanwhile, by Lemma 4.3.1 we have

$$\begin{aligned}\sum_{i,j=1}^{\infty} \pi_i(t)\pi_j(t)\tau_{ij}(t)dt &= \sum_{i,j=1}^{\infty} (D_i \ln S(\mu(t)) + \phi(t))(D_j \ln S(\mu(t)) \\ &\qquad + \phi(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt \\ &= \sum_{i,j=1}^{\infty} D_i \ln S(\mu(t)) D_j \ln S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt \\ &\quad + 2\phi(t) \sum_{i,j=1}^{\infty} D_i \ln S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt \\ &\quad + \phi^2(t) \sum_{i,j=1}^{\infty} \mu_i(t)\mu_j(t)\tau_{ij}(t)dt \\ &= \sum_{i,j=1}^{\infty} D_i \ln S(\mu(t)) D_j \ln S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt.\end{aligned} \tag{3.3.19}$$

Therefore, define $\theta(t)$ as

$$d\theta(t) = \frac{-1}{2S(\mu(t))} \sum_{i,j=1}^{\infty} D_{ij} S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt,$$

and we have shown

$$d \ln \frac{Z^{\pi}(t)}{Z^{\mu}(t)} = d \ln S(\mu(t)) + d\theta(t) \quad a.s. \qquad \square$$

3.4 Relative Arbitrage

In this section we show some relative arbitrage opportunities in the market by portfolio generating functions. As a matter of fact, we assume that the market weights have the form

$$\mu_i(t) = \frac{a(t)}{i^{\alpha}}, \quad \alpha > 1, \ \ i = 1, 2, \cdots \tag{3.4.1}$$

where $a(t) > 0$ is a jointly measurable and adapted process such that $\sum_{i=1}^{\infty} \frac{a(t)}{i^{\alpha}} = 1$.

Our assumption on the market portfolio above is based on the section 5.1 (pp93-96) in R. Fernholz [8]. There the capital distribution of markets is studied heuristically. It says that under some ideal conditions, the capital distribution curve is a straight line if the smallest stocks are replaced by constant rates. Here the capital distribution curve is the log-log plot of the market weights versus their respective ranks in descending order. This linear log-log weight distribution is called a Pareto distribution.

Now let us see some relative arbitrage opportunities in the market.

Proposition 3.4.1. *Suppose* $0 < a(t) < \exp(\frac{\alpha A - C}{C})$ *is strictly increasing on* $[0, \infty)$*, where*

$$A := \sum_{i=1}^{\infty} \frac{\ln i}{i^{\alpha}}, \qquad C := \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}}. \tag{3.4.2}$$

Then the portfolio π *with weights*

$$\pi_i(t) = -\frac{\mu_i(t)\ln\mu_i(t)}{S(\mu(t))}, \quad i = 1, 2, \cdots \tag{3.4.3}$$

represents an arbitrage opportunity relative to the market portfolio on $[0, T]$*, where* $S(x)$ *is defined on* Δ *by*

$$S(x) = -\sum_{i=1}^{\infty} x_i \ln x_i. \tag{3.4.4}$$

In particular,

$$\mathbb{P}(Z^{\pi}(T) > Z^{\mu}(T)) = 1.$$

Proof. We first verify $0 < S(\mu(t)) < \infty$ with $S(x)$ defined by (4.4.3). In fact, by (4.4.1) we have

$$\begin{aligned} S(\mu(t)) &= -\sum_{i=1}^{\infty} \frac{a(t)}{i^{\alpha}} \ln \frac{a(t)}{i^{\alpha}} \\ &= \alpha a(t) A - a(t) \ln a(t) C, \end{aligned} \tag{3.4.5}$$

where A and C are defined by (4.4.2). By the Cauchy integral test, the series A and C are convergent since $\alpha > 1$, so $0 < S(\mu(t)) < \infty$.

Next we show that $S(\mu(t))$ is strictly increasing on $[0, \infty)$ with respect to t. To this end, we differentiate the both sides of (4.4.5) and obtain

$$\begin{aligned} S'(\mu(t)) &= \alpha a'(t) A - a'(t) \ln a(t) C - C a'(t) \\ &= (\alpha A - C) a'(t) - C a'(t) \ln a(t) \\ &> 0, \end{aligned} \tag{3.4.6}$$

since $(\alpha A - C) = \sum_{i=1}^{\infty} \frac{\alpha \ln i - 1}{i^{\alpha}} > 0$, $a'(t) > 0$ and $\ln a(t) < 0$. That is, $S(\mu(t))$ is strictly increasing on $[0, \infty)$ with respect to t. Then for any $T > 0$, we have

$$S(\mu(T)) > S(\mu(0)). \quad a.s. \tag{3.4.7}$$

Now choose $S(x)$ as our portfolio generating function. By Theorem 4.3.1, S generates the portfolio π in (4.4.3) and satisfies

$$d\ln\frac{Z^{\pi}(t)}{Z^{\mu}(t)} = d\ln S(\mu(t)) + d\theta(t), \quad a.s. \tag{3.4.8}$$

where

$$d\theta(t) = \frac{1}{2S(\mu(t))}\sum_{i=1}^{\infty}\mu_i(t)\tau_{ii}(t)dt = \frac{\gamma_*^{\mu}(t)}{S(\mu(t))}. \tag{3.4.9}$$

Thus, with $Z^{\pi}(0) = Z^{\mu} = z > 0$, by (4.4.8) and (4.4.9) we have

$$\ln\frac{Z^{\pi}(T)}{Z^{\mu}(T)} = \ln\frac{S(\mu(T))}{S(\mu(0))} + \int_0^T \frac{\gamma_*^{\mu}(t)}{S(\mu(t))}dt. \quad a.s. \tag{3.4.10}$$

However, by (4.4.7) the first term on the right hand side above is positive, and the second term is nonnegative, so $\ln\frac{Z^{\pi}(T)}{Z^{\mu}(T)} > 0. \quad a.s.$ That is, $Z^{\pi}(T) > Z^{\mu}(T). \quad a.s.$ $\square$

Corollary 3.4.1. *Let A and C are defined as (4.4.2). Suppose $0 < a(t) < \exp(\frac{\alpha A - C}{C})$ is increasing on $[0, \infty)$, $\inf_{t \geq 0} a(t) > 0$ and there exists $\zeta > 0$ such that*

$$\int_0^T \gamma_*^{\mu}(t)dt \geq \zeta > 0, \tag{3.4.11}$$

Then we have the same conclusion in Proposition 4.4.1.

Proof. We only need to notice in this case everything remains the same in the proof of proposition 4.4.1 but (4.4.10) becomes

$$
\begin{aligned}
\ln \frac{Z^{\pi}(T)}{Z^{\mu}(T)} &= \ln \frac{S(\mu(T))}{S(\mu(0))} + \int_0^T \frac{\gamma_*^{\mu}(t)}{S(\mu(t))} dt \\
&\geq \frac{\int_0^T \gamma_*^{\mu}(t) dt}{\sup_{0 \leq t \leq T} S(\mu(t))} \\
&\geq \frac{\zeta}{\sup_{0 \leq t \leq T} S(\mu(t))} > 0, \quad a.s.
\end{aligned}
\tag{3.4.12}
$$

so the same conclusion in Proposition 4.4.1 follows. □

Before showing another relative arbitrage opportunity in the market, we introduce the following quantity. For $0 < p < 1$, define

$$
\gamma_*^{\mu,p} = \frac{1}{2} \sum_{i=1}^{\infty} \mu_i^p(t) \tau_{ii}(t). \tag{3.4.13}
$$

Notice when $p = 1$, $\gamma_*^{\mu,1}(t) = \gamma_*^{\mu}(t)$. In this sense, the quantity is treated as the generalization of $\gamma_*^{\mu}(t)$.

Proposition 3.4.2. *Suppose $a(t) > 0$ is strictly increasing on $[0, \infty)$ and $p\alpha > 1$, then the portfolio π with weights*

$$
\pi_i(t) = \frac{p\mu_i^p(t)}{S(\mu(t))} + (1-p)\mu_i(t), \quad i = 1, 2, \cdots \tag{3.4.14}
$$

represents an arbitrage opportunity relative to the market portfolio on $[0, T]$, where $S(x)$ is defined on Δ by

$$
S(x) = \sum_{i=1}^{\infty} x_i^p. \tag{3.4.15}
$$

In particular,

$$
\mathbb{P}(Z^{\pi}(T) > Z^{\mu}(T)) = 1.
$$

Proof. First of all, with the $S(x)$ defined by (4.4.15) we claim $0 < S(\mu(t)) < \infty$. In fact, in this case it is easy to see $0 < S(\mu(t)) = \sum_{i=1}^{\infty} \frac{a^p(t)}{i^{p\alpha}} < \infty$ since $p\alpha > 1$. $S(\mu(t))$ is also strictly increasing for $t \geq 0$ since

$$S'(\mu(t)) = pa(t)^{p-1}a'(t)\sum_{i=1}^{\infty}\frac{1}{i^{p\alpha}} > 0. \tag{3.4.16}$$

From (4.4.16) we conclude that for any $T > 0$,

$$S(\mu(T)) > S(\mu(0)). \quad a.s. \tag{3.4.17}$$

Now choose $S(x)$ as our portfolio generating function. Again by Theorem 4.3.1 S generates the portfolio π in (4.4.14) and the value of the portfolio π satisfies

$$d\ln\frac{Z^{\pi}(t)}{Z^{\mu}(t)} = d\ln S(\mu(t)) + d\theta(t), \quad a.s. \tag{3.4.18}$$

where

$$d\theta(t) = \frac{p(1-p)}{2S(\mu(t))}\sum_{i=1}^{\infty}\mu_i^p(t)\tau_{ii}(t)dt = p(1-p)\frac{\gamma_*^{\mu,p}(t)}{S(\mu(t))}dt. \tag{3.4.19}$$

Thus, with $Z^{\pi}(0) = Z^{\mu}(0) = z > 0$, by (4.4.18) and (4.4.19) we have

$$\ln\frac{Z^{\pi}(T)}{Z^{\mu}(T)} = \ln\frac{S(\mu(T))}{S(\mu(0))} + p(1-p)\int_0^T\frac{\gamma_*^{\mu,p}(t)}{S(\mu(t))}dt. \quad a.s. \tag{3.4.20}$$

However, by (4.4.17) the first term on the right hand side above is positive, and the second term is nonnegative, so $\ln\frac{Z^{\pi}(T)}{Z^{\mu}(T)} > 0. \quad a.s.$ That is, $Z^{\pi}(T) > Z^{\mu}(T) \quad a.s.$ $\square$

Similar to Corollary 4.4.1, we have the following

Corollary 3.4.2. *Suppose $a(t) > 0$ is increasing on $[0, \infty)$ and there exists $\zeta > 0$ such that*

$$\int_0^T \gamma_*^{\mu,p}(t)dt \geq \zeta > 0, \tag{3.4.21}$$

Then we have the same conclusion as in Proposition 4.4.2.

Proof. We only need to notice in this case (4.4.20) becomes

$$\begin{aligned}
\ln \frac{Z^\pi(T)}{Z^\mu(T)} &= \ln \frac{S(\mu(T))}{S(\mu(0))} + p(1-p) \int_0^T \frac{\gamma_*^{\mu,p}(t)}{S(\mu(t))} dt \\
&\geq \frac{p(1-p)}{\sup_{0\leq t\leq T} S(\mu(t))} \int_0^T \gamma_*^{\mu,p}(t)dt \\
&\geq \frac{p(1-p)\zeta}{\sup_{0\leq t\leq T} S(\mu(t))} > 0, \quad a.s.
\end{aligned} \tag{3.4.22}$$

so we have the same conclusion as in Proposition 4.4.2. □

We finally show the last relative arbitrage opportunity in this section. For $0 < p < 1$, define the diversity function $D(x)$ on Δ by

$$D(x) = (\sum_{i=1}^{\infty} x_i^p)^{\frac{1}{p}}. \tag{3.4.23}$$

We now verify the following proposition.

Proposition 3.4.3. *Suppose $a(t) > 0$ is strictly increasing on $[0, \infty)$ and $p\alpha > 1$, then the portfolio π with weights*

$$\pi_i(t) = \frac{\mu_i^p(t)}{D^p(\mu(t))}, \quad i = 1, 2, \cdots \tag{3.4.24}$$

represents an arbitrage opportunity relative to the market portfolio on $[0, T]$. In particular,

$$\mathbb{P}(Z^\pi(T) > Z^\mu(T)) = 1.$$

Proof. Firstly, it is easy to verify that $0 < D(\mu(t)) < \infty$ since $p\alpha > 1$. Secondly, $D(\mu(t))$ is strictly increasing with respect to t as $a(t)$ is. Apply Theorem 4.3.1 to the function (3.4.23), and we have

$$d\ln\frac{Z^{\pi}(t)}{Z^{\mu}(t)} = d\ln D(\mu(t)) + d\theta(t), \quad a.s. \tag{3.4.25}$$

where π is given by (3.4.24) and $d\theta(t) = (1-p)\gamma_*^{\pi}(t)dt$.

With $Z^{\pi}(0) = Z^{\mu}(0) = z > 0$, (3.4.25) can be written as

$$\ln\frac{Z^{\pi}(T)}{Z^{\mu}(T)} = \ln\frac{D(\mu(T))}{D(\mu(0))} + (1-p)\int_0^T \gamma_*^{\mu}(t)dt \quad a.s. \tag{3.4.26}$$

The term on the right hand side above is positive, so we get $\ln\frac{Z^{\pi}(T)}{Z^{\mu}(T)} > 0$ $a.s.$ or $Z^{\pi}(T) > Z^{\mu}$ $a.s.$ The completes the proof. □

3.5 Linear Capital Distribution Curve

In the last section, we assume that the market weights have the form

$$\mu_i(t) = \frac{a(t)}{i^{\alpha}}, \quad \alpha > 1, i = 1, 2, \cdots \tag{3.5.1}$$

where $a(t) > 0$ is a measurable and adapted process such that $\sum_{i=1}^{\infty}\frac{a(t)}{i^{\alpha}} = 1$. We call this linear log-log plot of market weights a Pareto distribution. In this section we like to test the assumption. In fact, in the following we take $a(t)$ as a constant.

We first simulate the average capital distribution curve of US equity markets from the year 1990 to 1999, see Figure 3.5.1.

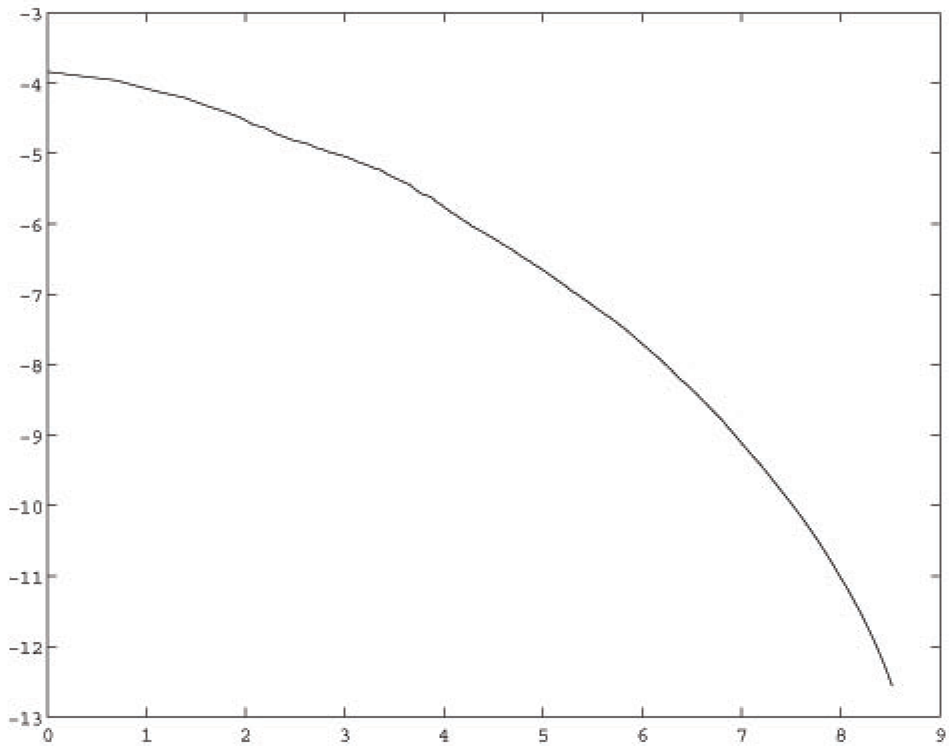


Figure 3.5.1: The capital distribution curve of US equity markets 1990-1999

The curve is a log-log plot of the market values of the largest 5000 stocks versus their ranks in the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX), and the NASDAQ Stock Market. The data are from the monthly database of the Center for Research in Securities Prices (CRSP) at the University of Chicago after removing all REIT's, all closed-end funds and ADR's. Clearly the curve is not strictly a straight line.

Now we would like to find a linear form of $\mu_i(t)$. To do so, we use the least squares method to estimate the parameters a and α in the form $\ln(a) - \alpha \ln(i)$. We have done this piecewisely, see the following figures.

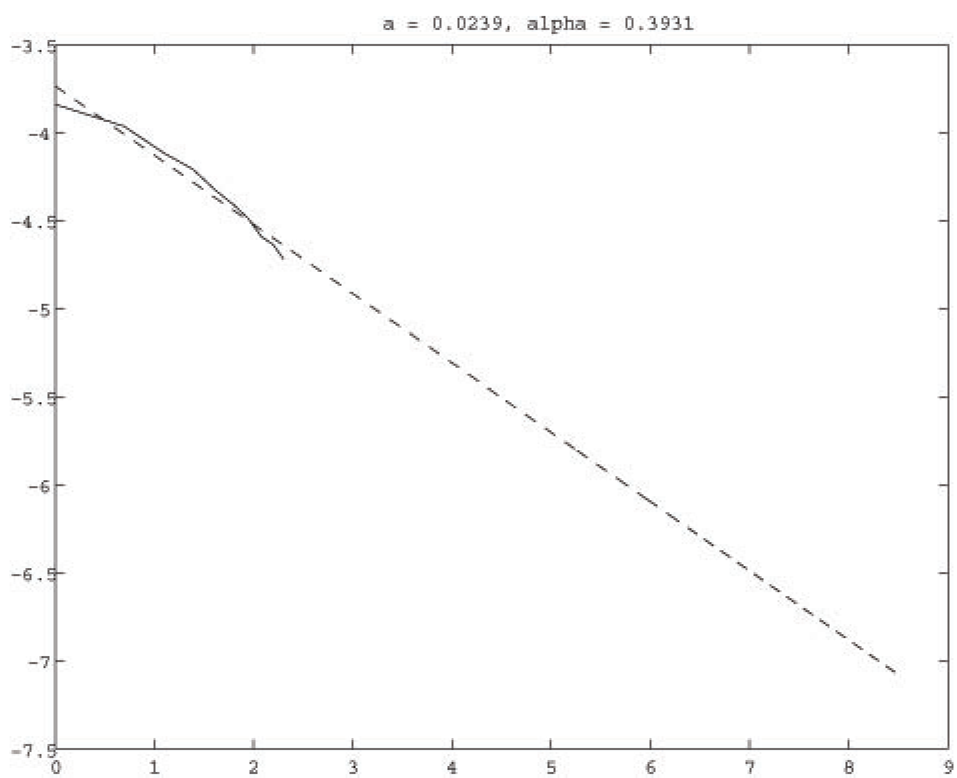


Figure 3.5.2: Linear approximation of the capital distribution curve $i = 1 : 10$

From these figures, we see that generally the linear form is piecewisely a good approximation to the average capital distribution curve of US equity markets from the

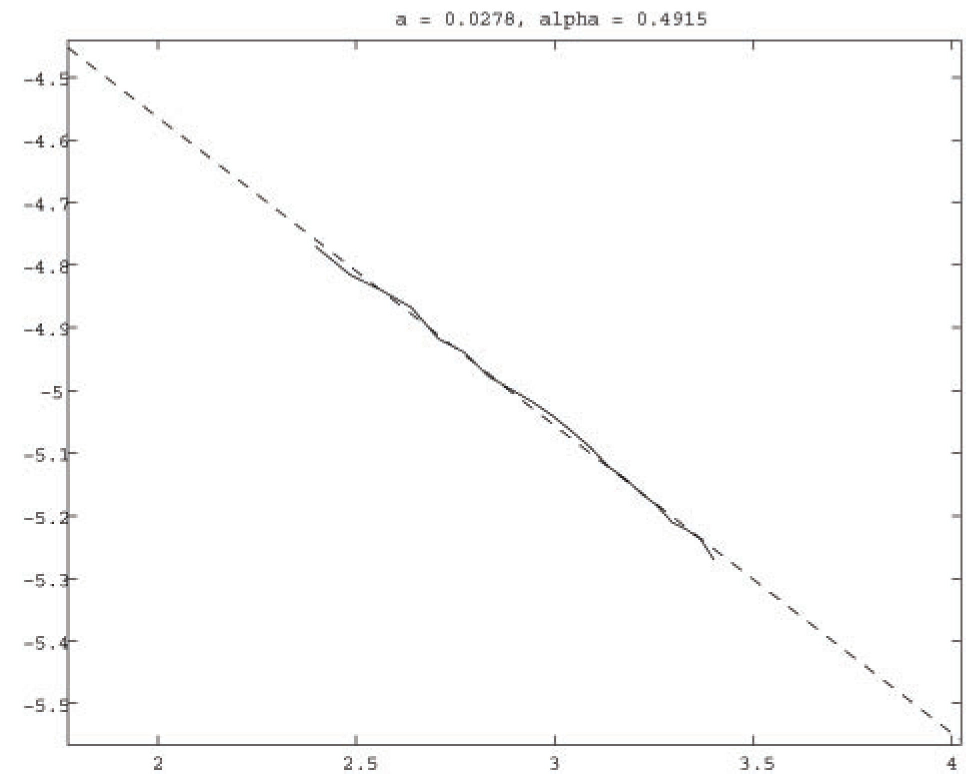


Figure 3.5.3: Linear approximation of the capital distribution curve $i = 11 : 30$

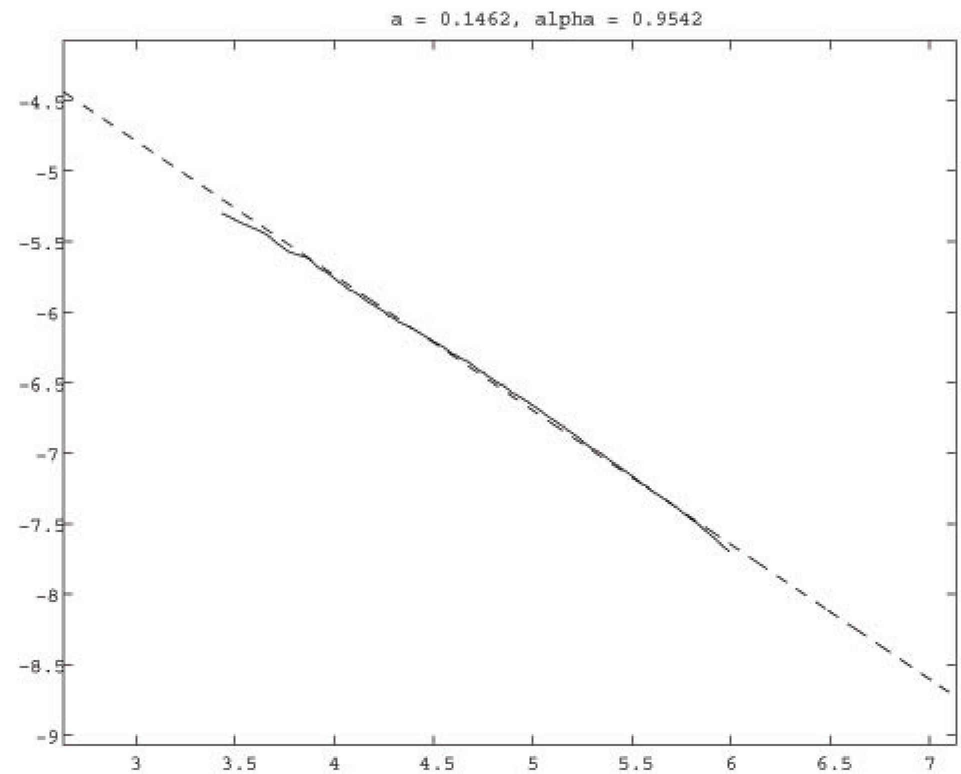


Figure 3.5.4: Linear approximation of the capital distribution curve $i = 31 : 400$

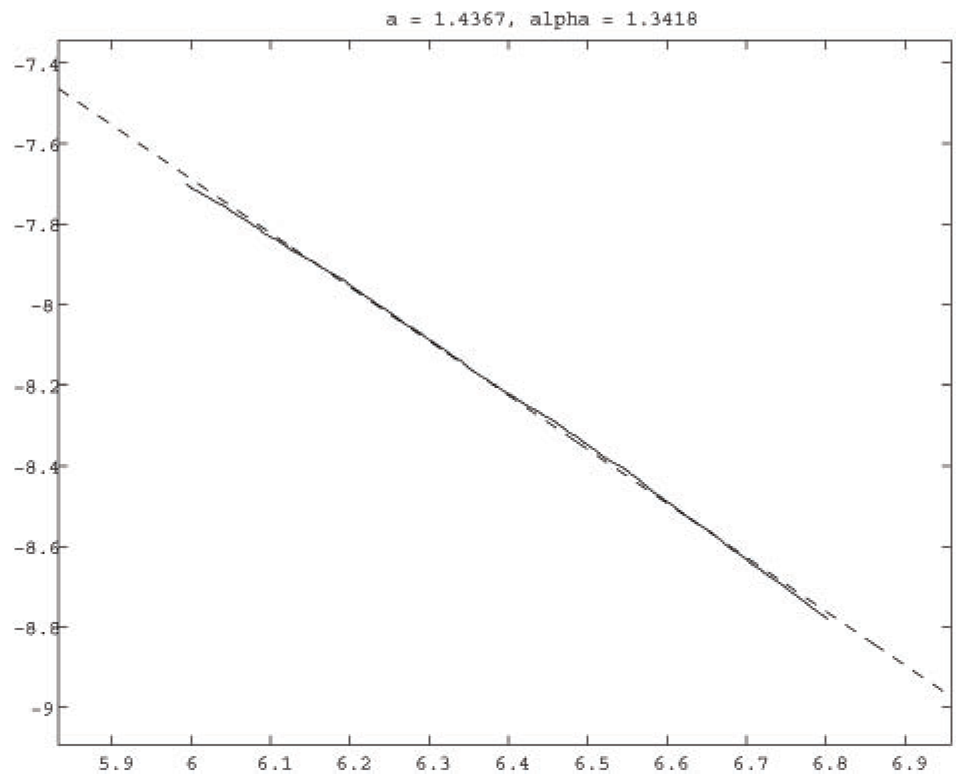


Figure 3.5.5: Linear approximation of the capital distribution curve $i = 401 : 900$

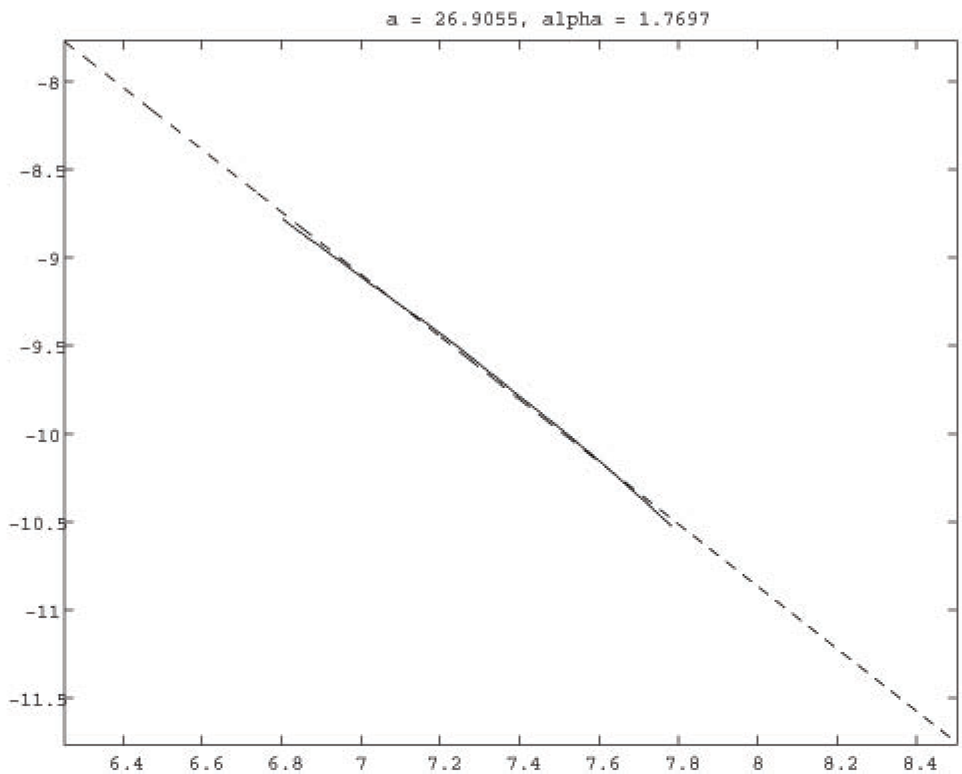


Figure 3.5.6: Linear approximation of the capital distribution curve $i = 901 : 2400$

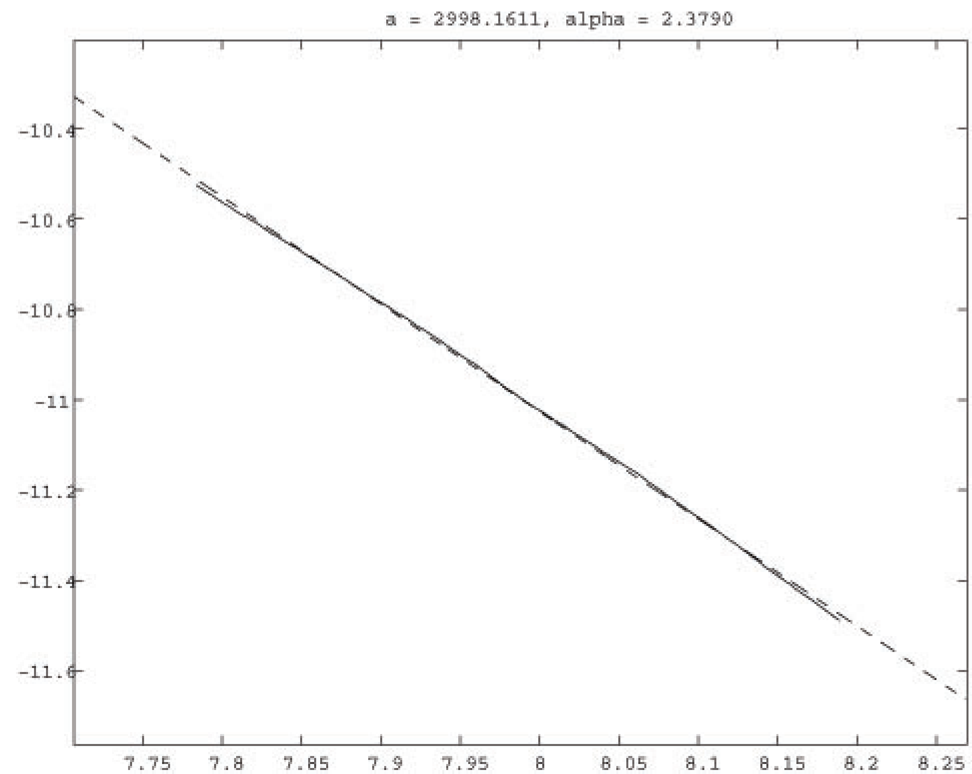


Figure 3.5.7: Linear approximation of the capital distribution curve $i = 2401 : 3600$

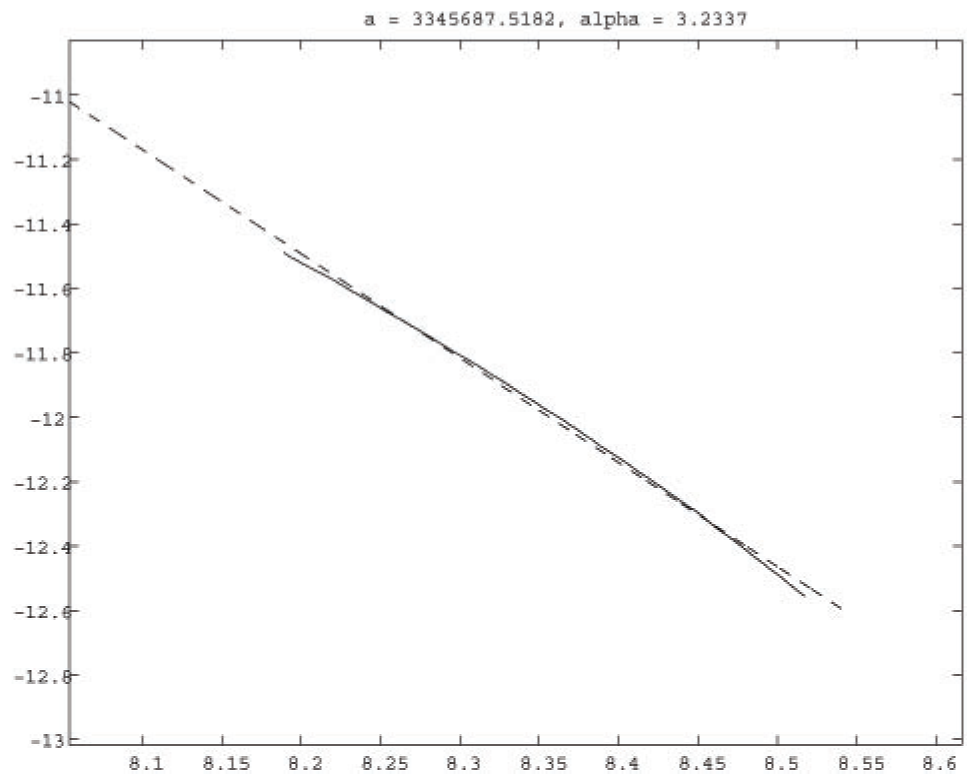


Figure 3.5.8: Linear approximation of the capital distribution curve $i = 3601 : 5000$

year 1990 - 1999. In other words, our assumption $\mu_i(t) = \frac{a(t)}{i^\alpha}$ stands well if we treat $a(t)$ and α as parameters. The trouble seems to be at the beginning for the largest 10 or 30 stocks, where we have α values less than 1. However this is not a serious issue at all in an infinite market or if the number of the stocks in the market is sufficiently large.

In fact, we also tried this linear approximation to the capital distribution curve of US equity markets for each year 1990 - 1999. It agrees very well in each case. The following table gives the average a and α values for each year.

	1990	1991	1992	1993	1994
a	9.2667e+08	1.3352e+09	7.3105e+07	5.3577e+04	6.4465e+04
α	1.7987	1.7853	1.6003	1.4120	1.3927

Table 3.5.1: Average a and α values for the year 1990 -1994.

	1995	1996	1997	1998	1999
a	3.8081e+04	3.9061e+03	3.3326e+03	2.1751e+04	1.9957e+05
α	1.3686	1.3060	1.3561	1.5045	1.5657

Table 3.5.2: Average a and α values for the year 1995 -1999.

The tables show that α values are stable at about 1.5 for the ten years. However, a values behave very irregularly, as we also see in those figures. But one thing we notice is that a values could be increasing with respect to time t, as shown in the tables for the year 1990-1991, 1993-1994 and 1997-1999. This verifies the assumption of Proposition 4.1 - 4.3, where we require that $a(t)$ is increasing on $[0,\infty)$. We are not saying that then arbitrary opportunities exist in those particular years. We just mean that it is possible $a(t)$ is an increasing function.

3.6 Conclusions

We have seen some relative arbitrage opportunities in an infinite market in the last section. Now let us go back to the question at the beginning. Our answer is positive under the condition that the capitals of the market have a Pareto distribution and the market generating function is strictly increasing. In fact, in this case relative arbitrage opportunities exist in the market over arbitrary time horizons. Also, it seems easier to find relative arbitrage in infinite markets than in finite markets. All of this verifies our intuition in reality.

Chapter 4

A First Order Model

4.1 Introduction

A first-order model is a market model in which the growth rate and the variance of stocks depend on the rank of the stocks in the market. Here by rank we mean the descending order of the stocks in terms of their capitalizations in the market. An atlas type of such models is studied in Fernholz [8] (see Example 5.3.3) and particularly in Banner, Fernholz et al. [1], where the same, constant variances are assigned to all the stocks; zero growth rate to all the stocks but the smallest; and positive growth rate to the smallest.

In this chapter, we would like to study another type of first-order model. In fact, the setting of our model is somehow the reverse of the set-up in Banner, Fernholz et al. [1]. More precisely, we assign the same, constant variances to all the stocks; the positive growth rate to all the stocks but the largest; and zero growth rate to the largest.

Our purpose is to study the size effect in the equity market. The so called size or small-firm effect originally studied by R. Banz [2]. According to the theory, if we divide the stocks in the market into different portfolios each year by firm size, i.e. the total value of outstanding equity, then the small-firm portfolios constantly have higher average annual returns. Of course, the small-firm portfolios tend to be riskier. But in this thesis, we want to focus on this growth rate and ignore the variances of stocks, though some conclusions can be extended to the case of time-dependent variances. Anyway, we assume that all the stocks in the market face the same, constant variance; the largest stocks or the large-firm portfolios have zero growth rate; and the small stocks or the small-firm portfolios have relatively higher growth rate.

Although the structure of the model is simple, we shall see that the model has some nice properties. Some coincide with the observation in reality.

The rest of the chapter is organized as the following. In section 2, we formulate the model strictly in mathematics and introduce the notation used through the chapter. In section 3, we show that the capital distribution of the market is asymptotically stable. In section 4, we investigate possible investment opportunities in the market. Some portfolios and their performance in the market are analyzed. In section 5, we discuss the diversity of the market. A sufficient condition for market diversity is given. In section 6, we compare our market model with the atlas model. In section 7, the simulation on the capital distribution is performed. Finally, section 8 is our conclusion and some comments.

4.2 Formulation of the Model

Consider an equity market in which there are n stocks. The price of each stock follows a geometric Brownian motion. That is,

$$d\ln X_i(t) = \gamma_i(t)dt + \sigma_i(t)dB_i(t), \quad t \geq 0, \quad i = 1, 2, \cdots, n, \tag{4.2.1}$$

where $X_i(t)$ is the price of the ith stock; $\gamma_i(t)$ is the growth rate of the ith stock; $\sigma_i(t)$ is the variance of the ith stock; and $(B_1(t), B_2(t), \cdots, B_n(t))$ is an n-dimensional standard Brownian motion. We assume that each company has only one share outstanding in the market since the shares of a company can be infinitely divisible. Then $X_i(t)$ also stands for the total capitalization of the ith company in the market.

As we observe in practice, the growth rate and the variance of a stock are usually related to the rank of the stock in the market, so we would like to reflect the observation in our model. To make the point clear, we assign the positive growth rate to all but the largest stock; zero growth rate to the largest; the same, constant variances to all the stocks.

Let $X_{(k)}(t), k = 1, 2, \cdots, n$, be the inverse order statistics. That is, $X_{(k)}(t)$ satisfies

$$X_{(n)}(t) \leq X_{(n-1)}(t) \leq \cdots \leq X_{(k+1)}(t) \leq X_{(k)}(t) \leq X_{(k-1)}(t) \leq X_{(1)}(t).$$

We denote by $(p_t(1), p_t(2), \cdots, p_t(n))$ a permutation of $(1, 2, \cdots, n)$ and define

$$X_{p_t(k)}(t) = X_{(k)}(t), \tag{4.2.2}$$

$$p_t(k) < p_t(k+1) \quad if \quad X_{(k)}(t) = X_{(k+1)}(t). \tag{4.2.3}$$

That is, $p_t(k)$ is the index of the kth largest stock in the market and the ties are resolved by turning to the smaller index. With these notations, we write $\gamma_i(t), \sigma_i(t)$ as the following form

$$\gamma_i(t) = \begin{cases} 0, & X_i(t) = X_{p_t(1)}(t), \\ g, & otherwise, \end{cases} \qquad \sigma_i(t) = \sigma, \tag{4.2.4}$$

where g and σ are positive constants.

It can be shown that the equation (4.2.1) with the condition (4.2.4) has a weak solution, which is unique in the sense of probability law. Interested readers can refer to Banner, Fernholz et al. [1] or Bass & Pardoux [3] for more discussion on this type of stochastic differential equations.

Definition 4.2.1. *A nonnegative vector $\pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))$ is called a portfolio process if it is measurable, adapted and satisfies $\Sigma_{i=1}^n \pi_i(t) = 1$.*

So again here $\pi_i(t)$ represents the weight of the ith stock in the portfolio π. Short sales and borrowings are not allowed in the market since $0 \leq \pi_i(t) \leq 1$ for each i. Then with the definition, the value process $Z^\pi(t)$ of the portfolio π satisfies

$$\begin{aligned} \frac{dZ^\pi(t)}{Z^\pi(t)} &= \Sigma_{i=1}^n \pi_i(t) \, \frac{dX_i(t)}{X_i(t)} \\ &= \Sigma_{i=1}^n \pi_i(t) b_i(t) dt + \Sigma_{j=1}^n \pi_i(t) \sigma_i(t) dB_i(t), \end{aligned} \tag{4.2.5}$$

where $b_i(t) = \gamma_i(t) + \frac{1}{2}\sigma_i^2(t)$. That is, the portfolio is self-financing.

The equality (4.2.5) can also be written as

$$d \ln Z^\pi(t) = \gamma^\pi(t) dt + \Sigma_{i=1}^n \pi_i(t) \sigma_i(t) dB_i(t), \tag{4.2.6}$$

where

$$\gamma^{\pi}(t) = \Sigma_{i=1}^{n}\pi_i(t)\gamma_i(t) + \gamma_*^{\pi}(t), \tag{4.2.7}$$

and

$$\gamma_*^{\pi}(t) = \frac{1}{2}\Sigma_{i=1}^{n}\pi_i(t)(1-\pi_i(t))\sigma_i^2(t) \tag{4.2.8}$$

are called the growth rate and the excess growth rate of the portfolio π, respectively.

From (4.2.6), (4.2.7) and (4.2.1), we can express the log of value processes in terms of the log price of stocks and the excess growth rate of portfolios. That is,

$$d\ln Z^{\pi}(t) = \Sigma_{i=1}^{n}\pi_i(t)d\ln X_i(t) + \gamma_*^{\pi}(t)dt. \tag{4.2.9}$$

Definition 4.2.2. *The portfolio μ with weights*

$$\mu_i(t) := \frac{X_i(t)}{Z(t)} \quad i = 1, 2, \cdots, n,$$

where $Z(t) = \Sigma_{i=1}^{n}X_i(t)$, is called the market portfolio.

The market portfolio plays a fundamental role in the stochastic portfolio theory. One reason is that if one invests according to the portfolio he will achieve "the whole market" by a constant multiplier difference. In fact, it is not hard to verify that

$$\frac{dZ^{\mu}(t)}{Z^{\mu}(t)} = \frac{dZ(t)}{Z(t)},$$

so $Z^{\mu}(t) = kZ(t)$, where $k = \frac{Z^{\mu}(0)}{Z(0)}$. In view of the importance of the portfolio, we shall revisit it in section 4.

Before we move to the next section, let us see a property of the market. From Proposition 2.3 in Banner, Fernholz et al. [1], we have the following equality

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T I_{\{0\}}(X_i(t)-X_{p_t(k)}(t))dt=\frac{1}{n}. \tag{4.2.10}$$

In other words, each stock asymptotically spends $\frac{1}{n}$ of the time serving as the kth rank. In particular,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T I_{\{0\}}(X_i(t)-X_{p_t(1)}(t))dt=\frac{1}{n}. \tag{4.2.11}$$

By (4.2.11) and the strong law of large number of Brownian motion, it follows from (4.2.1) that

$$\begin{aligned}
\lim_{T\to\infty}\frac{\ln X_i(T)}{T} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_i(t)dt \\
&= g-g\lim_{T\to\infty}\frac{1}{T}\int_0^T I_{\{0\}}(X_i(t)-X_{p_t(1)}(t))dt \\
&= (1-\frac{1}{n})g, \qquad i=1,2,\cdots,n.
\end{aligned} \tag{4.2.12}$$

(4.2.12) states that asymptotically the growth rate of each stock is $(1-\frac{1}{n})g$. Then by Proposition 2.1.2 in Fernholz [8], we conclude that the market is coherent. That is, none of the stocks in the market declines too quickly.

By the same proposition, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma^{\mu}(t)dt=(1-\frac{1}{n})g, \tag{4.2.13}$$

so asymptotically the growth rate of the market is also $(1-\frac{1}{n})g$. This is the case since all but the largest stock have a positive growth rate g; and the largest stock has zero growth rate. Certainly the zero growth rate of the largest stock has an impact on the market. Asymptotically each stock is being the largest about $\frac{1}{n}$ of the time, so in the long run the market growth rate is reduced by $\frac{1}{n}\,g$.

Remark The right hand side of (4.2.13) becomes g when n goes to infinity. In other words, asymptotically the growth rate of an infinite market is g. Clearly in this case the impact of the largest is neutralized since there are a large quantity of stocks in the market.

4.3 Capital Distribution of the Market

In this section we show that the capital distribution of the market exhibits some sort of stability property. Initially we introduce the concept of local time for a semimartingale.

Definition 4.3.1. *Let X be a continuous semimartingale. Then the local time at 0 for X is the process Λ_X defined by*

$$\Lambda_X(t) = \frac{1}{2}(|X(t)| - |X(0)| - \int_0^t sgn(X(s))dX(s)), \tag{4.3.1}$$

where $sgn(x) = 2\,I_{(0,\infty)}(x) - 1$. We also denote by $\langle X \rangle_t$ the quadratic variation process for X.

The following definitions are taken from Fernholz [8].

Definition 4.3.2. *The market is coherent if for $i = 1, 2, \cdots, n$,*

$$\lim_{t\to\infty} \frac{\ln \mu_i(t)}{t} = 0, \qquad a.s. \tag{4.3.2}$$

Definition 4.3.3. *The market is asymptotically stable if it is coherent and for $k = 1, 2, \cdots, n$,*

$$\lim_{t\to\infty}\frac{1}{t}\Lambda_{\ln\mu_{(k)}-\ln\mu_{(k+1)}}(t)=\lambda_{k,k+1}, \qquad a.s. \tag{4.3.3}$$

$$\lim_{t\to\infty}\frac{1}{t}\langle\ln\mu_{(k)}-\ln\mu_{(k+1)}\rangle_{(t)}=\sigma^2_{k,k+1}, \qquad a.s. \tag{4.3.4}$$

where $\lambda_{k,k+1},\sigma^2_{k,k+1}$ are positive constants.

Now we are ready to show the following

Proposition 4.3.1. *The capital distribution of the market is asymptotically stable.*

Proof. By Proposition 4.1.11 in Fernholz [8], we have the following dynamics for the ranked processes

$$\begin{aligned} d\ln X_{(1)}(t) &= \Sigma_{i=1}^n I_{\{0\}}(X_i(t)-X_{p_t(1)}(t))dX_i(t)+\frac{1}{2}d\Lambda_{\ln X_{(1)}-\ln X_{(2)}}(t) \\ &= \sigma dB_1(t)+\frac{1}{2}d\Lambda_{\ln X_{(1)}-\ln X_{(2)}}(t); \end{aligned} \tag{4.3.5}$$

$$\begin{aligned} d\ln X_{(k)}(t) &= \Sigma_{i=1}^n I_{\{0\}}(X_i(t)-X_{p_t(k)}(t))dX_i(t) \\ &\qquad +\frac{1}{2}d\Lambda_{\ln X_{(k)}-\ln X_{(k+1)}}(t)-\frac{1}{2}d\Lambda_{\ln X_{(k-1)}-\ln X_{(k)}}(t) \\ &= gdt+\sigma dB_k(t)+\frac{1}{2}d\Lambda_{\ln X_{(k)}-\ln X_{(k+1)}}(t) \\ &\quad -\frac{1}{2}d\Lambda_{\ln X_{(k-1)}-\ln X_{(k)}}(t), \quad k=2,\cdots,n-1; \end{aligned} \tag{4.3.6}$$

$$\begin{aligned} d\ln X_{(n)}(t) &= \Sigma_{i=1}^n I_{\{0\}}(X_i(t)-X_{p_t(n)}(t))dX_i(t)-\frac{1}{2}d\Lambda_{\ln X_{(n-1)}-\ln X_{(n)}}(t) \\ &= gdt+\sigma dB_n(t)-\frac{1}{2}d\Lambda_{\ln X_{(n-1)}-\ln X_{(n)}}(t). \end{aligned} \tag{4.3.7}$$

On the other hand, from (4.2.12) it follows that

$$\lim_{T\to\infty}\frac{\ln X_{(k)}(T)}{T}=(1-\frac{1}{n})g, \qquad k=1,2,\cdots,n. \tag{4.3.8}$$

Using (4.3.8) and the strong law of large numbers of Brownian motion, by (4.3.5), (4.3.6) and (4.3.7) we obtain the following estimates for ${\lambda_{k,k+1}}'s$,

$$\lambda_{1,2}=\lim_{t\to\infty}\frac{\Lambda_{\ln X_{(1)}-\ln X_{(2)}}(t)}{t}=2(1-\frac{1}{n})g; \tag{4.3.9}$$

$$\begin{aligned}\lambda_{k,k+1}-\lambda_{k-1,k}&=\lim_{t\to\infty}\frac{\Lambda_{\ln X_{(k)}-\ln X_{(k+1)}}(t)}{t}-\lim_{t\to\infty}\frac{\Lambda_{\ln X_{(k)}-\ln X_{(k+1)}}(t)}{t}\\&=2(1-\frac{1}{n})g-2g\\&=-\frac{2}{n}g;\end{aligned} \tag{4.3.10}$$

$$\lambda_{n-1,n}=\lim_{t\to\infty}\frac{\Lambda_{\ln X_{(n-1)}-\ln X_{(n)}}(t)}{t}=2g-2(1-\frac{1}{n})g=\frac{2}{n}g. \tag{4.3.11}$$

By (4.3.9), (4.3.10) and (4.3.11), we induce

$$\lambda_{k,k+1}=2(1-\frac{k}{n})g, \quad k=1,2,\cdots,n-1. \tag{4.3.12}$$

We next calculate the value of the parameters ${\sigma^2_{k,k+1}}'s$. In fact, it is not hard to verify that for $k=1,2,\cdots,n-1$,

$$\begin{aligned}\sigma^2_{k,k+1}&=\lim_{t\to\infty}\frac{1}{t}\langle\ln\mu_k-\ln\mu_{k+1}\rangle_t\\&=\lim_{t\to\infty}\frac{1}{t}\langle\ln X_{(k)}-\ln X_{(k+1)}\rangle_t\\&=2\,\sigma^2.\end{aligned} \tag{4.3.13}$$

Recall that we have shown the market is coherent. Therefore, by Definition 3.3, the market is asymptotically stable. $\square$

Now let us make some observations from the proposition.

First. Define $g_k(t) = \gamma_{p_t(k)}(t) - \gamma^{\mu}(t)$ and $G_k = \lim_{T\to\infty} \int_0^T g_k(t)dt$ for $k = 1, 2, \cdots, n$. That is, G_k represents the asymptotic relative growth rate of $X_{(k)}(t)$. Then by Proposition 5.3.2 in Fernholz [8], we get from (4.3.12)

$$G_k = \frac{1}{2}(\lambda_{k-1,k} - \lambda_{k,k+1}) = \frac{2}{n}g, \qquad k = 1, 2, \cdots, n. \tag{4.3.14}$$

In other worlds, asymptotically the relative growth rate for the ranked stocks is $\frac{2}{n}g$.

Second. The capital distribution curve refers to the log-log plot of the market weights versus their respective ranks in descending order. Since the capital distribution of the market is equivalent to the size distribution of the firms, it has been studied extensively. Now from (5.3.13) in Fernholz [8] we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\ln \mu_{(k)}(t) - \ln \mu_{(k+1)}(t)}{\ln(k) - \ln(k+1)} dt \approx -\frac{k\sigma_{k,k+1}^2}{2\lambda_{k,k+1}} = -\frac{k\sigma^2}{2(1-\frac{k}{n})g}. \tag{4.3.15}$$

That is, the asymptotic log-log slope between $\mu_{(k)}(t)$ and $\mu_{(k+1)}(t)$ is $-\frac{k\sigma^2}{2(1-\frac{k}{n})g}$. We shall simulate the capital distribution curve of the market and see how precisely the result is related to the simulation.

4.4 The Portfolios and Their Performance

In this section we shall study the investment in the market. In other words, we analyze some portfolios and their performance in the market. The first one we see here is the market portfolio, which was defined in section 2.

Example 4.1 (The Market portfolio) We have shown earlier that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma^{\mu}(t)dt = (1-\frac{1}{n})g. \tag{4.4.1}$$

Notice that in this case the expression (4.2.6) has the form

$$d\ln Z^{\mu}(t) = \gamma^{\mu}(t)dt + \sigma\Sigma_{i=1}^{n}\mu_i(t)dB_i(t). \tag{4.4.2}$$

From (4.4.1), (4.4.2), and by the strong law of large numbers we obtain

$$\lim_{T\to\infty}\frac{\ln Z^{\mu}(T)}{T} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma^{\mu}(t)dt = (1-\frac{1}{n})g. \tag{4.4.3}$$

That is, in the long run the log of the total market capitalization is in proportion to time t. Moreover, by (4.2.12) and (4.4.3) it is easy to see that

$$\lim_{t\to\infty}\frac{\ln\mu_i(t)}{t} = 0, \qquad i = 1,2,\cdots,n. \tag{4.4.4}$$

The equality (4.4.4) recovers the definition of market coherence. □

Example 4.2 (The equally-weighted portfolio) We next consider a portfolio with the same, constant weights. That is, the investor holds each stock equally in the market.

Let

$$\pi_i(t) = \frac{1}{n}, \qquad i = 1,2,\cdots,n. \tag{4.4.5}$$

Then from (4.2.6) we have

$$d\ln Z^{\pi}(t) = (1-\frac{1}{n})(g+\frac{1}{2}\sigma^2)dt + \frac{\sigma}{n}\,\Sigma_{i=1}^{n} dB_i(t). \tag{4.4.6}$$

By the strong law of large numbers of Brownian motion, it follows that

$$\lim_{T\to\infty}\frac{\ln Z^{\pi}(T)}{T} = (1-\frac{1}{n})(g+\frac{1}{2}\sigma^2). \tag{4.4.7}$$

In other words, asymptotically the growth rate of the portfolio is $(1-\frac{1}{n})(g+\frac{1}{2}\sigma^2)$. Compared to the asymptotic growth rate of the market portfolio, the difference of the two is $\frac{1}{2}(1-\frac{1}{n})\sigma^2$. Certainly this type of the portfolios is desirable in practice, since it can "beat" the market. □

We now consider some rank-dependent portfolios. By their names, the weights of these portfolios are determined by the rank of the stocks in the market. The following theorem is used frequently through the rest of the section. It is a modified version of Theorem 4.2.1 in Fernholz [8]. Notice that in this case the condition of pathwise mutually nondegeneracy is automatically satisfied by the very structure of our model. Before stating the theorem, we introduce the concept of functionally generating portfolios. Define $\Delta = \{x \in \mathbb{R}^n : 0 \le x_i \le 1, i = 1,2,\cdots,n.\ \Sigma_{i=1}^{n} x_i = 1\}$.

Definition 4.4.1. *Let S be a positive continuous function defined on Δ and π be a portfolio. We say that S generates π if there exists a measurable, bounded variation process θ such that*

$$\ln\frac{Z^{\pi}(t)}{Z^{\mu}(t)} = \ln\mu(t) + \theta(t), \quad a.s. \quad t\in[0,T]. \tag{4.4.8}$$

If S generates the portfolio π, then we call S the portfolio generating function.

Remark We often use the differential form of the equation (4.4.8). That is,

$$d\ln\frac{Z^{\pi}(t)}{Z^{\mu}(t)} = d\ln\mu(t) + d\theta(t), \quad a.s. \quad t\in[0,T]. \tag{4.4.9}$$

Theorem 4.4.1. *Let* $\boldsymbol{S}$ *be a function defined on a neighborhood* U *of* Δ*. Suppose that there exists a positive* C^2 *function* S *defined on* U *such that for* $x\in U$*,*

$$\boldsymbol{S}(x_1,x_2,\cdots,x_n) = S(x_{(1)},x_{(2)},\cdots,x_{(n)}),$$

and for each i*,* $x_i D_i \ln S(x)$ *is bounded on* Δ*. Then* S *generates the portfolio* π *with weights*

$$\pi_{(k)}(t) = (D_k \ln S(\mu_{(\cdot)}(t)) + 1 - \Sigma_{j=1}^{n}\mu_{(j)}(t)D_j\ln S(\mu_{(\cdot)}(t)))\mu_{(k)}(t), \quad k=1,2,\cdots,n, \tag{4.4.10}$$

and the bounded variation process θ *satisfies*

$$\begin{aligned} d\theta(t) = &\frac{-1}{2\boldsymbol{S}(\mu(t))}\Sigma_{i,j=1}^{n} D_{ij}S(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)dt \\ &+\frac{1}{2}\Sigma_{k=1}^{n-1}(\pi_{(k+1)}(t)-\pi_{(k)}(t))d\Lambda_{\ln\mu_{(k)}-\ln\mu_{(k+1)}}(t), \end{aligned} \tag{4.4.11}$$

where $\tau_{ij}(t)$ *is the cross variation process of* $\ln\mu_{(i)}(t)$ *and* $\ln\mu_{(j)}(t)$*, i.e.*

$$\langle \ln\mu_{(i)}(t), \ln\mu_{(j)}(t)\rangle_t = \tau_{(ij)}(t), \quad i,j=1,2,\cdots,n. \tag{4.4.12}$$

Example 4.3 (The largest stock) Consider the function $S(x) = x_{(1)}$, where $x\in\Delta$ and $x_{(1)}$ is the largest component of x. By Theorem 4.1 above, the function generates the portfolio π with weights

$$\pi_{(1)}(t) = 1; \quad \pi_{(k)}(t) = 0, \qquad k = 2,3,\cdots,n. \tag{4.4.13}$$

That is, the investor holds only the largest stock in the market. The value process Z^π of the portfolio satisfies

$$d\ln\frac{Z^\pi(t)}{Z^\mu(t)} = d\ln\mu_{(1)}(t) - \frac{1}{2}d\Lambda_{\ln\mu_{(1)}-\ln\mu_{(2)}}(t), \qquad t \geq 0. \tag{4.4.14}$$

Recall that $\lim_{t\to\infty}\frac{\ln\mu_i(t)}{t} = 0, \quad i = 1,2,\cdots,n$, so

$$\lim_{t\to\infty}\frac{\ln\mu_{(1)}(t)}{t} = 0. \tag{4.4.15}$$

Then by (4.4.15), (4.4.3) and (4.3.9), we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln Z^\pi(t)}{t} &= \lim_{t\to\infty}\frac{\ln Z^\mu(t)}{t} - \frac{1}{2}\lim_{t\to\infty}\frac{\Lambda_{\ln\mu_{(1)}-\ln\mu_{(2)}}(t)}{t}\\ &= (1-\frac{1}{n})g - \frac{1}{2}\cdot 2(1-\frac{1}{n})g\\ &= 0.\end{aligned} \tag{4.4.16}$$

The example shows that in the long term the growth rate of the portfolio will be slower than time t. Compared to the growth rate of the market portfolio, obviously the portfolio is not a good choice for the investor. □

Example 4.4 (Hold the non-largest stocks) Now we consider a portfolio which is contrary to the one in Example 4.3, that is, we suppose that the investor holds all the stocks but the largest. Let $S(x) = x_{(2)} + x_{(3)} + \cdots + x_{(n)}$, where $x \in \Delta$ and $x_{(k)}, k = 2,3,\cdots,n$ are the components of x in descending order. The function generates the portfolio π with weights

$$\pi_{(1)}(t) = 0; \quad \pi_{(k)}(t) = \mu_{(k)}(t)/\Sigma_{i=2}^{n}\mu_{(i)}(t), \qquad k = 2, 3, \cdots, n. \tag{4.4.17}$$

The value process of the portfolio satisfies

$$\begin{aligned} d\ln\frac{Z^{\pi}(t)}{Z^{\mu}(t)} &= d\ln S(\mu(t)) + \frac{1}{2}\,\Sigma_{k=2}^{n-1}(\pi_{(k+1)}(t) - \pi_{(k)}(t))d\Lambda_{\ln\mu_{(k)} - \ln\mu_{(k+1)}}(t) \\ &= d\ln(1 - \mu_{(1)}(t)) + \frac{1}{2}\,\pi_{(2)}(t)d\Lambda_{\ln\mu_{(1)} - \ln\mu_{(2)}}(t), \end{aligned} \tag{4.4.18}$$

where we used the property of the local time of semimartingales

$$I_{\{0\}}(X(t))d\Lambda_X(t) = d\Lambda_X(t). \tag{4.4.19}$$

The first term on the right hand side in (4.4.18) is negative and the second term is positive, so the performance of the portfolio is determined by the difference of the two terms. If the largest market $\mu_{(1)}(t)$ stays stable over time, then the portfolio will be better off than the market portfolio in the long run since the second term on the right hand side in (4.4.18) is increasing. □

Example 4.5 (The diversity-weighted portfolio) For $0 < p < 1$, define the diversity function $D(x)$ by

$$D(x) = (\Sigma_{i=1}^{n} x_{(i)}^{p})^{\frac{1}{p}}. \tag{4.4.20}$$

where $x \in \Delta$ and $x_{(i)}, \quad i = 1, 2, \cdots, n$, is the components of x in descending order. The portfolio π generated by this function is called the diversity-weighted portfolio, which has weights

$$\pi_{(k)}(t) = \mu_{(k)}^{p}(t)/D^{p}(\mu(t)), \qquad k = 1, 2, \cdots, n. \tag{4.4.21}$$

The value process Z^{π} of the portfolio satisfies

$$\begin{aligned} d\ln\frac{Z^{\pi}(t)}{Z^{\mu}(t)} &= d\ln S(\mu(t)) + (1-p)\gamma_{*}^{\pi}(t)dt \\ &\quad + \frac{1}{2}\Sigma_{k=1}^{n-1}(\pi_{k+1}(t) - \pi_{k}(t))d\Lambda_{\ln\mu_{(k)}-\ln\mu_{(k+1)}}(t) \\ &= d\ln S(\mu(t)) + (1-p)\gamma_{*}^{\pi}(t)dt, \end{aligned} \tag{4.4.22}$$

where the local time part is dropped because of the property (4.4.19). In fact, notice the generating function $D(x)$ is symmetric, so is the portfolio generated by $D(x)$. Therefore, in this case the value process of the rank-dependent portfolios is the same as that of the rank-independent portfolios. i.e. the portfolios generated by the function

$$D(x) = (\Sigma_{i=1}^{n} x_i^{p})^{\frac{1}{p}}, \qquad x \in \Delta. \tag{4.4.23}$$

The same thing is true for the portfolio generated by the entropy function $S(x) = -\Sigma_{i=1}^{n} x_i \ln x_i, \quad x \in \Delta$, since the function S is also symmetric.

The performance of the diversity-weighted portfolio relative to the market portfolio is determined by the two terms on the right hand side of (4.4.22). However, the case is more complicated. We stop the discussion and leave the problem to the interested readers.

4.5 On the Market Diversity

The concept of market diversity was first introduced by R. Fernholz [7]. Roughly speaking, if the market is diverse, then no single company is allowed to dominate the entire market in terms of relative capitalization. Here is the formal definition.

Definition 4.5.1. *The market is diverse if there exists* $\varepsilon \in (0,1)$ *such that*

$$\mu_{(1)}(t) < 1-\varepsilon, \quad t \geq 0. \tag{4.5.1}$$

The market is weakly diverse if for some $\varepsilon \in (0,1)$

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)dt < 1-\varepsilon, \quad a.s. \tag{4.5.2}$$

The interesting thing is that relative arbitrage can exist in a (weakly) diverse market over long time horizons. For more discussion about the two concepts see Fernholz [8], Fernholz, Karatzas et al. [10] etc.

Regarding our market model, since the drift and the diffusion coefficients are bounded, the usual equivalent martingale measure holds. Therefore, the market is not (weakly) diverse in this case. However, in view of the special structure of the model, we have a simple sufficient condition to ensure the market diversity.

For our purpose, we cite the following lemma from Fernholz, Karatzas etal. [10].

Lemma 4.5.1. *Let* $\delta \in (0, 1-\mu_{(1)}(0))$*. Suppose that on the event* $\{\frac{1}{2} \leq \mu_{(1)}(t) < 1-\delta\}$ *we have*

$$\gamma_{(k)}(t) \geq 0 \geq \gamma_{(1)}(t), \quad k = 2,\cdots,n, \tag{4.5.3}$$

$$\min_{2\le k\le n}\gamma_{(k)}(t)-\gamma_{(1)}(t)+\frac{\varepsilon}{2}\ge\frac{M}{\delta Q(t)}, \tag{4.5.4}$$

where $Q(t):=\ln\frac{1-\delta}{\mu_{(1)}(t)}$, ε *and* M *are respectively the lower and the upper bound of the volatility of the market. Then on any finite time horizon* $[0,T]$ *the market is diverse and* $\int_0^T Q^{-2}(t)dt<\infty$ *holds a.s.*

Notice that in our market model, $\gamma_{(1)}(t)=0$, $\gamma_{(k)}(t)=g$ for $k=2,3,\cdots,n$ and the volatility of the market is constantly σ^2. Therefore, by the lemma above, immediately we have the following

Theorem 4.5.1. *Let* δ *and* $Q(t)$ *be defined as in Lemma 5.1. Suppose that on the event* $\{\frac{1}{2}\le\mu_{(1)}(t)<1-\delta\}$ *we have*

$$(\frac{1}{2}+\frac{g}{\sigma^2})Q(t)\ge\frac{1}{\delta}. \tag{4.5.5}$$

Then on any finite time horizon $[0,T]$ *the market is diverse and* $\int_0^T Q^{-2}(t)dt<\infty$. *a.s.*

4.6 Comparison with The Atlas Model

As we mentioned earlier, Fernholz [8] and Banner, Fernholz [1] have studied an atlas type of the first-order model, where the same, constant variances are assigned to all stocks; zero growth rate to all the stocks but the smallest; and positive growth rate to the smallest, the " atlas" stock. According to these assumptions, $\gamma_i(t)$ and $\sigma_i(t)$ in their model are written as the following form

$$\gamma_i(t) = \begin{cases} ng, & X_i(t) = X_{p_t(n)}(t), \\ 0, & otherwise, \end{cases} \qquad \sigma_i(t) = \sigma, \tag{4.6.1}$$

where g and σ are positive constants.

Since both models belong to the first model, we would like to make a brief comparison to see similarities and differences between the two. The first conclusion we draw is that the market of the both models is coherent. Some differences of the two models are listed in the following table.

	Our Market Model	Atlas Model
Asymptotic average market growth rate	$(1-\frac{1}{n})g$	g
Log-log slope between $\mu_{(k)}$ and $\mu_{(k+1)}$	$-\frac{k\sigma^2}{2(1-\frac{k}{n})g}$	$-\frac{\sigma^2}{2g}$
Growth rate of equally-weighted portfolio	$(1-\frac{1}{n})(g+\frac{1}{2}\sigma^2)$	$g+\frac{n-1}{2n}\sigma^2$

Table 4.6.1: Comparison with The Atlas Model

Remark 1. Note that the log-log slope $-\frac{\sigma^2}{2g}$ between $\mu_{(k)}$ and $\mu_{(k+1)}$ of the atlas model is a constant, so asymptotically the capital distribution curve in that case is a straight line. The linear capital distribution is called the Pareto distribution. In other words, asymptotically the capital distribution of the atlas model follows the Pareto distribution.

2. Interestingly the equally-weighted portfolio has a better performance in the both models. Moreover, both growth rates of the portfolio tend to $g+\frac{1}{2}\sigma^2$ as $n \to \infty$.

3. On the market diversity, a sufficient condition is given in our market model; while in the atlas model, it is shown heuristically that

$$\mathbb{P}(\frac{1}{T}\int_0^T \mu_{(1)}(t)dt < 1-\varepsilon) \approx 1. \tag{4.6.2}$$

That is, the market is ”almost” weakly diverse.

4.7 Simulations

4.7.1 Test The Model Assumption

In this section we would like to test the assumptions of our market model. That is, the largest stock has zero growth rate, and all the other stocks have positive growth rates, and the variances of the stocks are the same, constant. More precisely, we study the 30 stocks under the Dow index for the time period 2006. The selection is for simplicity.

Figure 1 shows the distribution of the largest and the smallest stocks according to their market values in the time period. From the graph we see that except for a few days Exxon Mobil (XOM) takes the first place, while General Electric (GM) is the last all the time, so without loss of generality, we assume that XOM is the largest stock in the market. The graph also suggests that the rank among these 30 stocks are relatively stable because of their large capital values. Roughly speaking, we have the following order according to the market values of the stocks.

Figure 2 shows the average daily growth rate and the growth rate in the year 2006 of the 30 stocks with 1 corresponding to the largest, 2, the second largest and so on. Recall that the growth rate is defined as $\gamma = \mu - \frac{1}{2}\sigma^2$, where μ is the expected return and σ is the volatility. From the chart we can see that although it is not that obvious, generally the smaller stocks have larger growth rates, especially the second half whose market values are below 100 billion. But realize that we are considering the largest 30 stocks in the market. Also, as we know, in 2006 the oil industry made good profit, so if we exclude XOM from the list, the second place GE actually grows fairly slow. On the other hand, the smallest stock GM had a largest growth rate even though the auto industry was not so good in 2006. That is a very interesting observation.

Figure 3 shows that volatility of the 30 stocks. They are fairly stable at 15% - 18 % in the time period. Once again we think this is because these are large stocks. The volatility will be increasing if the firm's capital value is small.

Our last observation is about the performance of the largest stock. We calculate that the return of XOM is 31.06% in 2006 while the second largest GE is 5.20 % compared with Dow index 14.90%. Again be ware of the oil industry's good profit, GE's return was much lower than the general performance of the market.

Finally, to improve the results above we think a longer time period is needed. More importantly, we must have a much larger universe of the stocks and get a better sampling. In those cases, the distribution of the rank among the stocks over time will change more frequently. We believe that our market model will fit better then. On the other hand, as well-believed we do think the firm size plays an important role in the equity market. But our empirical study is very rough here, and our market model is also rather simple. However, it is another attempt to modeling the size phenomena of the equity market. We have not seen much work in the subject.

4.7.2 Test The Capital Distribution

In this section we simulate the capital distribution curve in the US equity markets and compare it with the capital distribution of our market model. Recall that the capital distribution curve refers to the log-log plot of the market weights versus their respective ranks in descending order.

We generate the capital distribution curves for the year 1990 - 1999 using the monthly database from the Center for Research in Securities Prices (CRSP) at the Uni-

1	2	3
EXXON MOBIL (XOM)	GEN ELECTRIC (GE)	MICROSOFT (MSFT)
4	5	6
CITIGROUP (C)	PROCTER GAMBLE (PG)	WAL MART (WMT)
7	8	9
PFIZER (PFE)	JOHNSON AND JOHNS (JNJ)	AMER INTL GROUP (AIG)
10	11	12
ALTRIA GROUP (MO)	JP MORGAN CHASE (JPM)	INTL BUSINESS MACH (IBM)
13	14	15
INTEL (INTC)	AT&T (T)	VERIZON (VZ)
16	17	18
COCA COLA (KO)	HEWLETT PACKARD (HPQ)	HOME DEPOT (HD)
19	20	21
MERCK (MRK)	AMER EXPRESS (AXP)	BOEING (BA)
22	23	24
UNITED TECH (UTX)	3M (MMM)	WALT DISNEY (DIS)
25	26	27
CATERPILLAR (CAT)	MCDONALDS (MCD)	DU PONT (DD)
28	29	30
HONEYWELL (HON)	ALCOA (AA)	GEN MOTORS (GM)

Table 4.7.2: Order of the stocks

versity of Chicago. The database, called the CRSP Universe, consists of the stocks traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX) and the NASDAQ Stock Market. Since the smallest stocks exhibits some irregular behaviors in the log-log plot, we only consider the largest 5121 stocks in the market after the removal of all REITs, all closed-end funds and those ADRs.

For the capital distribution of our market model, we make use of (4.3.15), which says that asymptotically the log-log slope between $\mu_{(k)}(t)$ and $\mu_{(k+1)}(t)$ is $-\frac{k\alpha}{2(1-\frac{k}{n})}$, where $\alpha = \frac{\sigma^2}{g}$. Taking $k = 1, 5, 10, 20, 40, 80, \cdots, 5120$, and assuming that $\mu_{(k)}(t)'s$ are constants, by these slopes we use the piecewise cubic Hermite interpolation method to estimate the capital distribution curve of our market model. We then compare the curves with those resulted from the real data, treating α as a parameter.

Here are our results. The x-axis is the log of the ranks and the y-axis is the log of the market weights. The solid line is the observed capital distribution curve and the broken line is the estimated curve.

From Figure 4.7.4 - 4.7.13, we see that the estimated capital distribution curve of our market model is compatible with the observed curve in the US equity markets in the year 1990 - 1999. In particular, the estimation gets better when α is smaller. Recall $\alpha = \frac{\sigma^2}{g}$, where σ and g are the variance and the growth rate of the stocks, so the model applies better to the case in which the market has a larger growth rate and a smaller volatility. In addition, our estimation seems robust over the 10-year period.

4.8 Conclusions and Comments

As we see, the first-order model can be quite useful in practice. In particular, it proposes a new method on the portfolio analysis in equity markets. In the real world most cases the investment is made more or less related to the size of the firms, so the model and the method should be important for practitioners.

We comment here that the structure of our model is rather simple. But the conclusions can be easily extended to the following cases:

1. The variances $\sigma_i(t)$ is a bounded process and $\gamma_i(t)$ takes more than two values. i.e.

$$\gamma_i(t) = \begin{cases} 0, & X_i(t) = X_{p_t(1)}(t), \\ g_m, & X_{p_t(2)}(t) \le X_i \le X_{p_t(i_m)}(t), \\ \cdots & \cdots \\ g_n, & X_{p_t(i_m+1)}(t) \le X_i \le X_{p_t(n)}(t), \end{cases} \tag{4.8.1}$$

where g_m, g_n are constants in an ascending order.

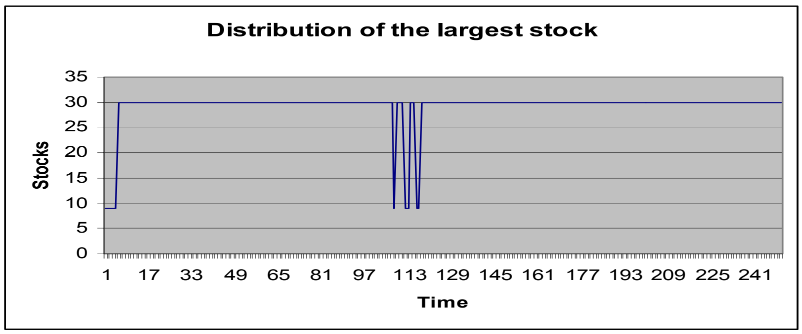
2. $\gamma_i(t)$ and $\sigma_i(t)$ have the following form

$$\gamma_i(t) = \gamma + \Sigma_{k=1}^n g_k I_{\{X_{p_t(k)}(t)\}}(X_i(t)), \qquad \sigma_i(t) = \Sigma_{k=1}^n \sigma_k I_{\{X_{p_t(k)}(t)\}}(X_i(t)), \tag{4.8.2}$$

where $\gamma > 0, g_k, \sigma_k$ are constants such that $\gamma + g_1 = 0$ and for $k = 1, 2, \cdots, n$,

$$g_{k+1} \ge g_k, \qquad \sigma_k > 0. \tag{4.8.3}$$

Certainly some more complicated models would be desirable for research.



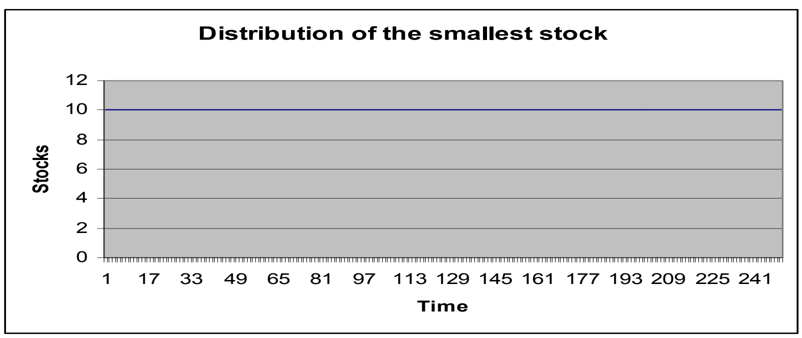
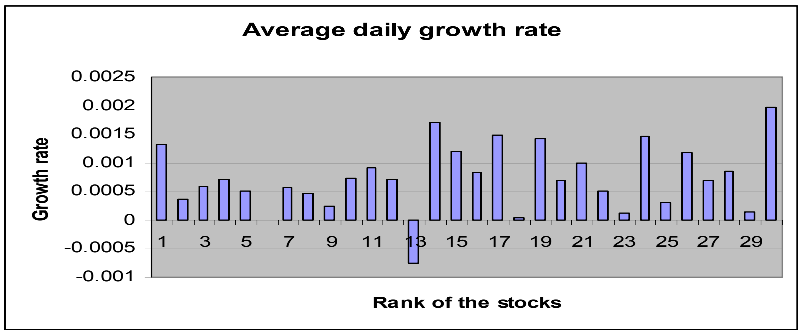


Figure 4.7.1: Distribution of the largest and the smallest stocks over the time



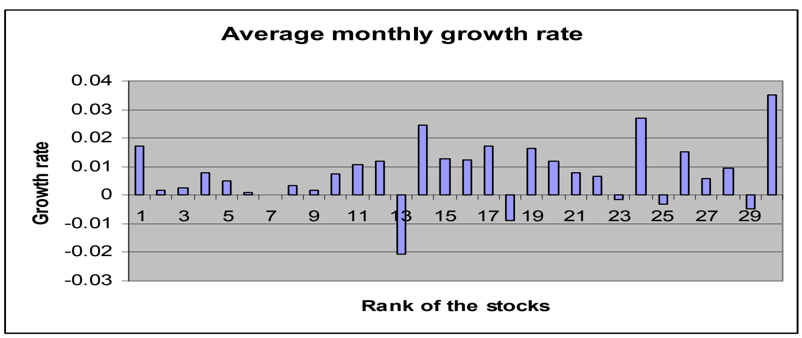
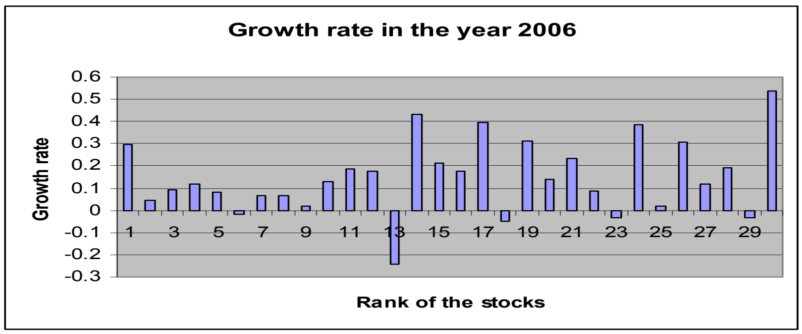


Figure 4.7.2: Distribution of the largest and the smallest stocks over the time



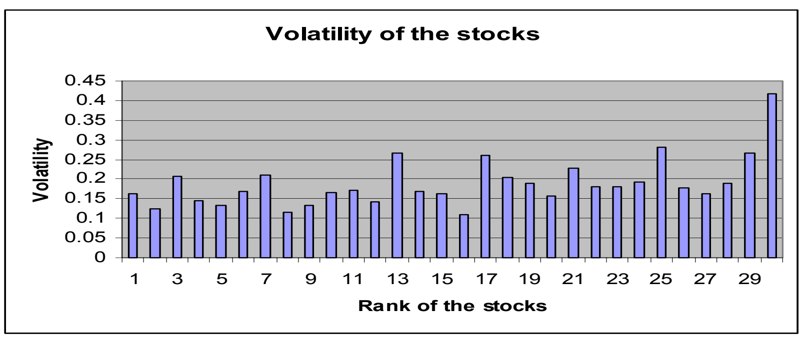


Figure 4.7.3: Distribution of the largest and the smallest stocks over the time

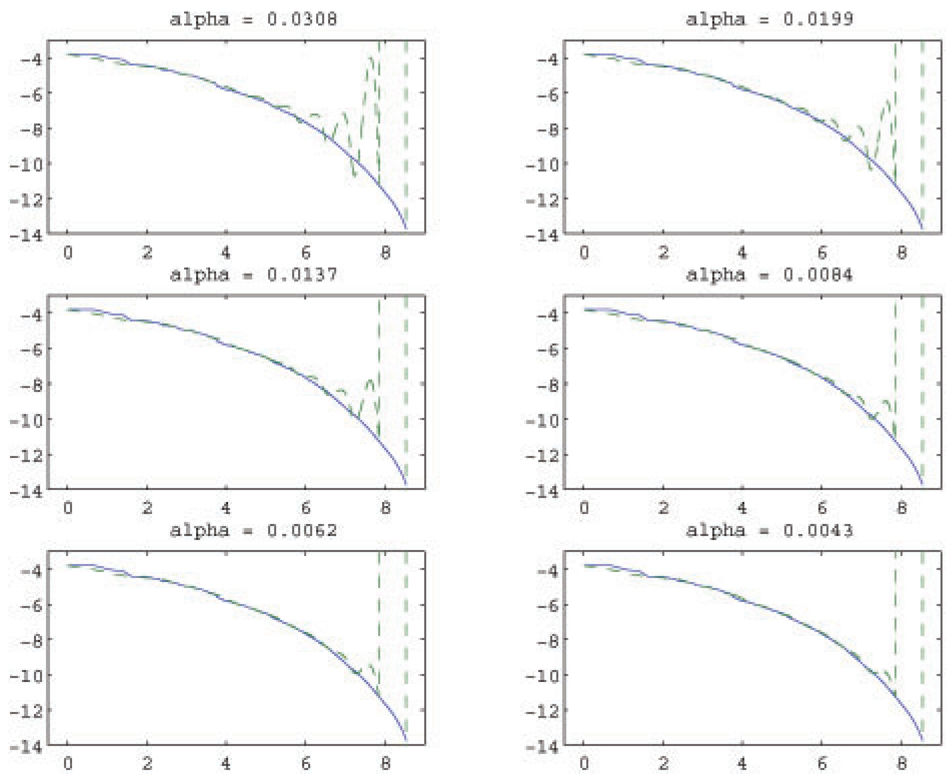


Figure 4.7.4: Estimated and observed capital distribution curve for 1990

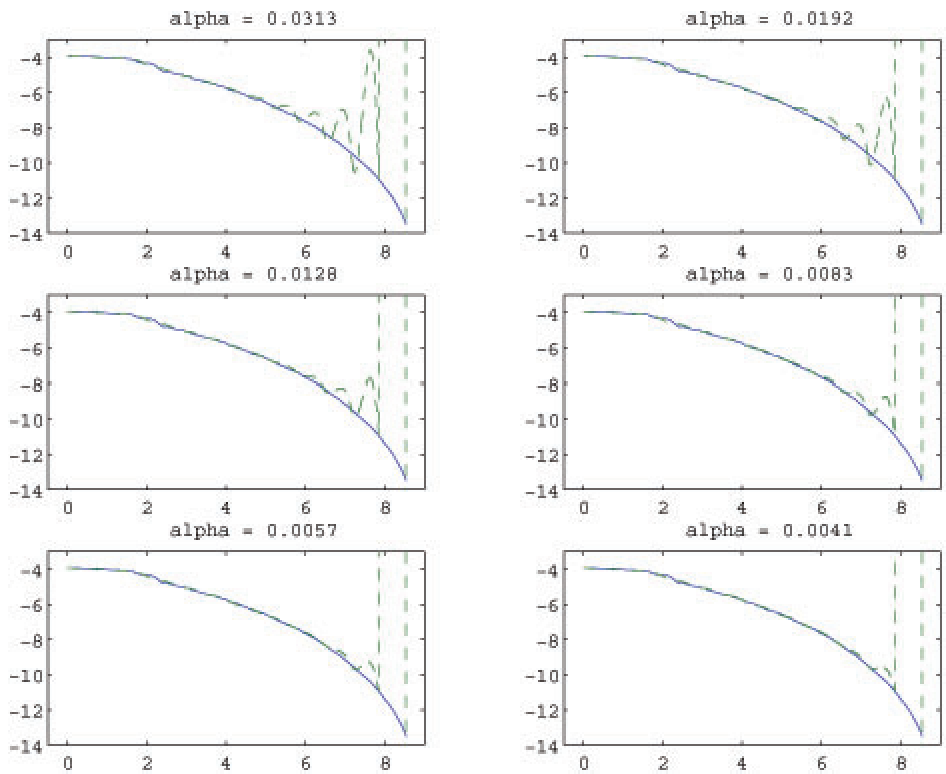


Figure 4.7.5: Estimated and observed capital distribution curve for 1991

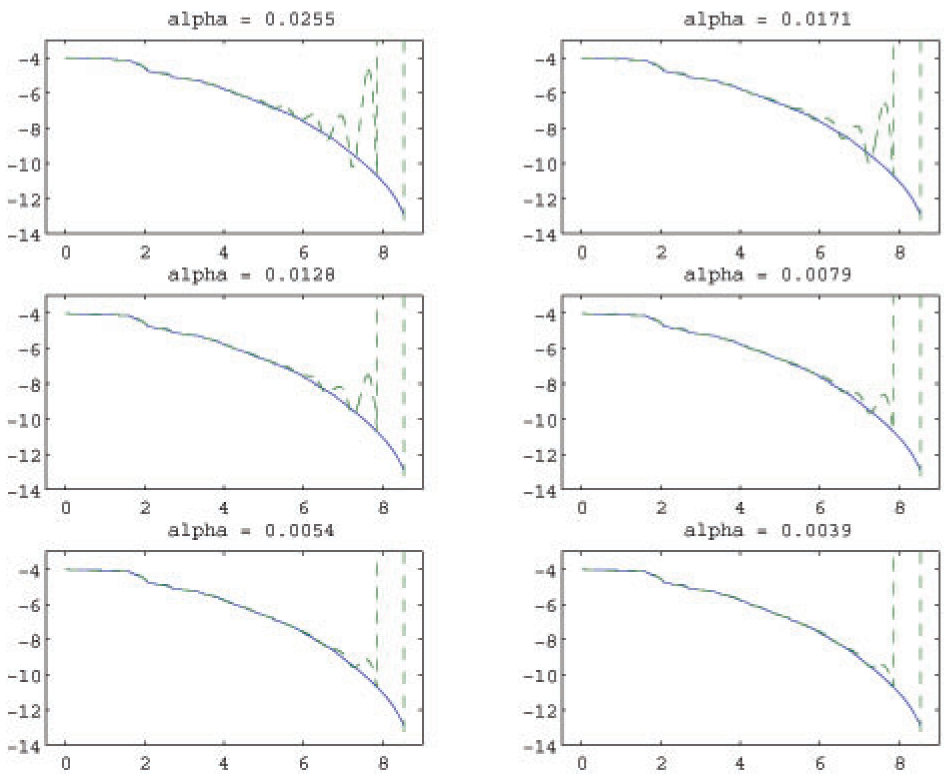


Figure 4.7.6: Estimated and observed capital distribution curve for 1992

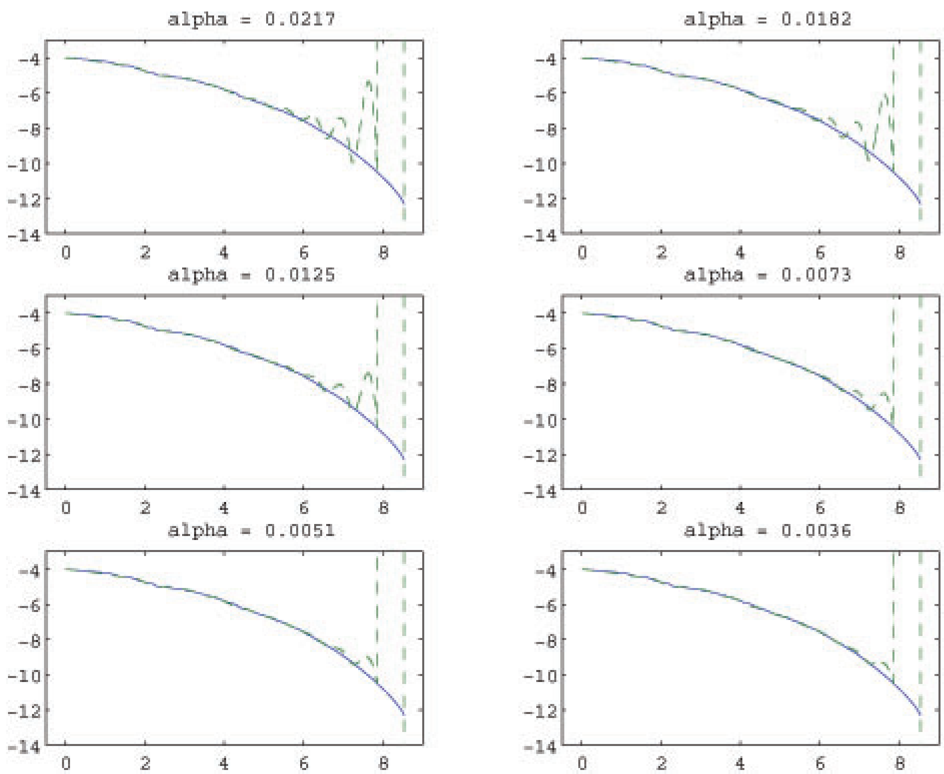


Figure 4.7.7: Estimated and observed capital distribution curve for 1993

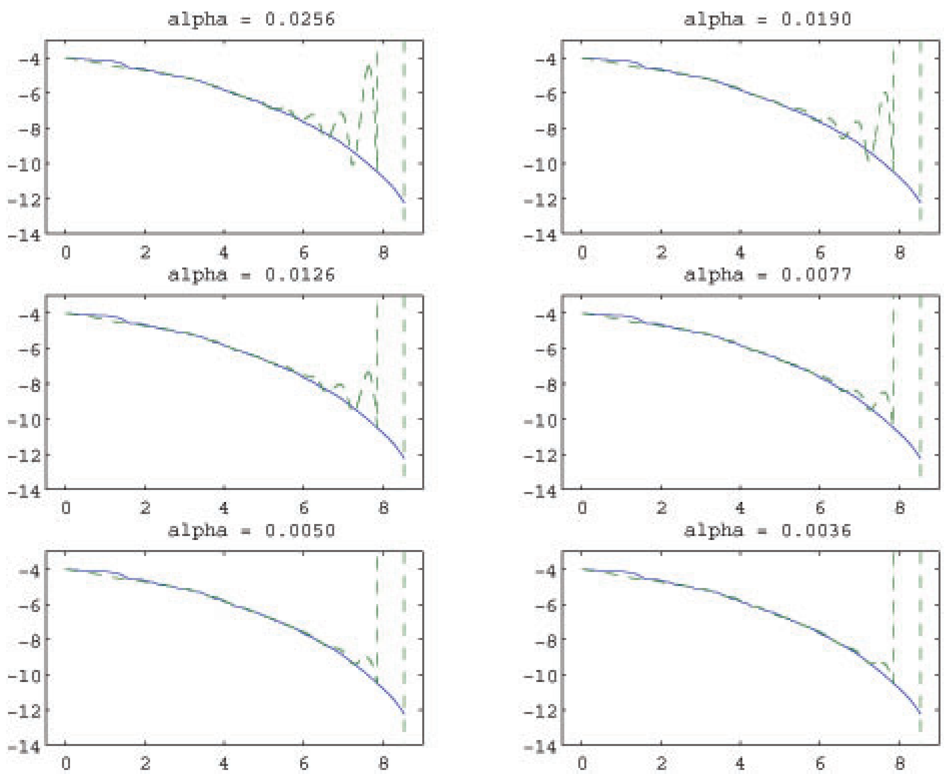


Figure 4.7.8: Estimated and observed capital distribution curve for 1994

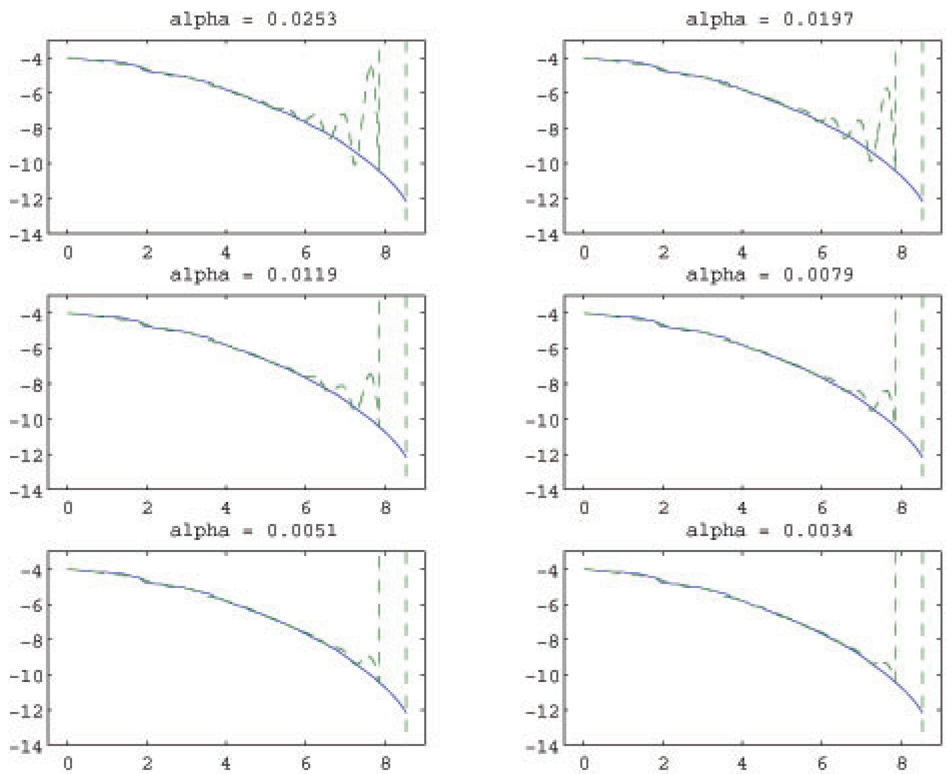


Figure 4.7.9: Estimated and observed capital distribution curve for 1995

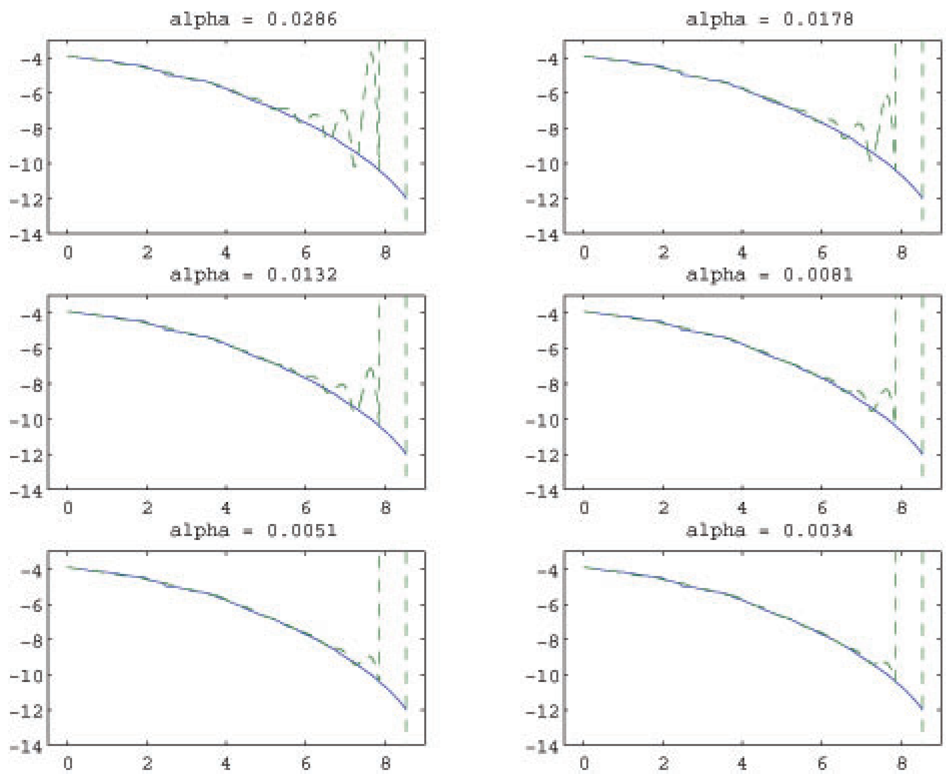


Figure 4.7.10: Estimated and observed capital distribution curve for 1996

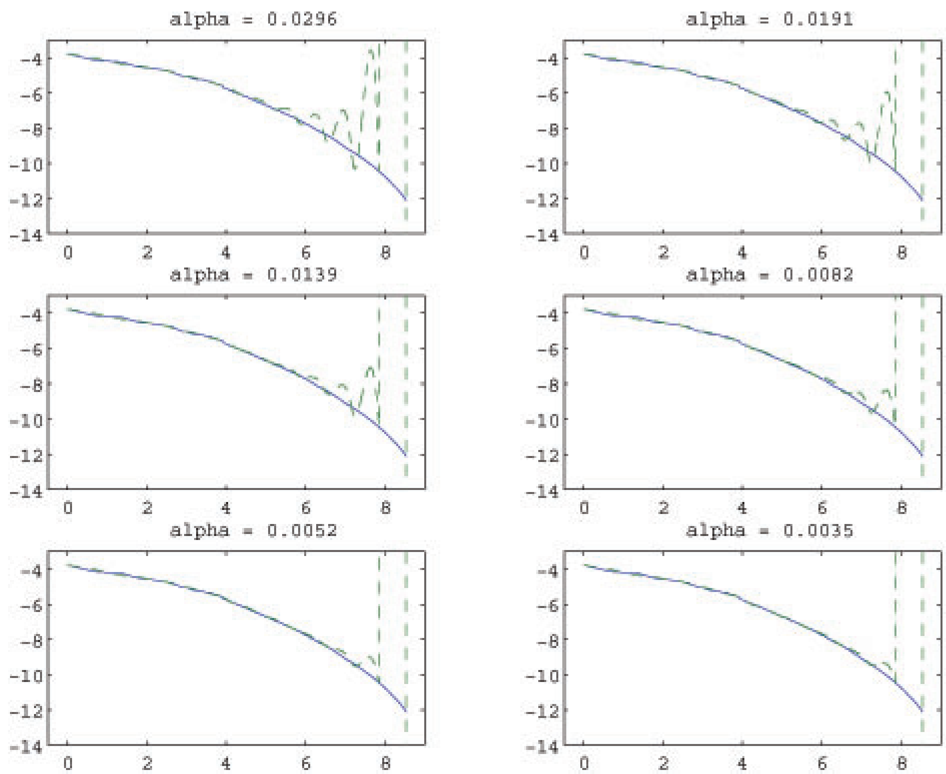


Figure 4.7.11: Estimated and observed capital distribution curve for 1997

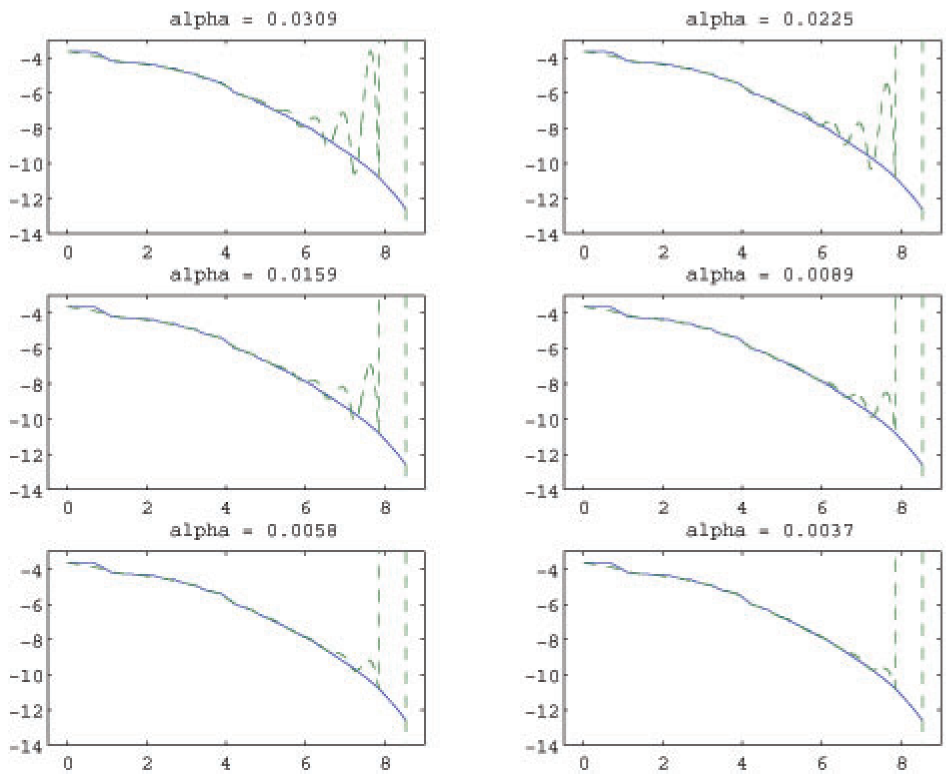


Figure 4.7.12: Estimated and observed capital distribution curve for 1998

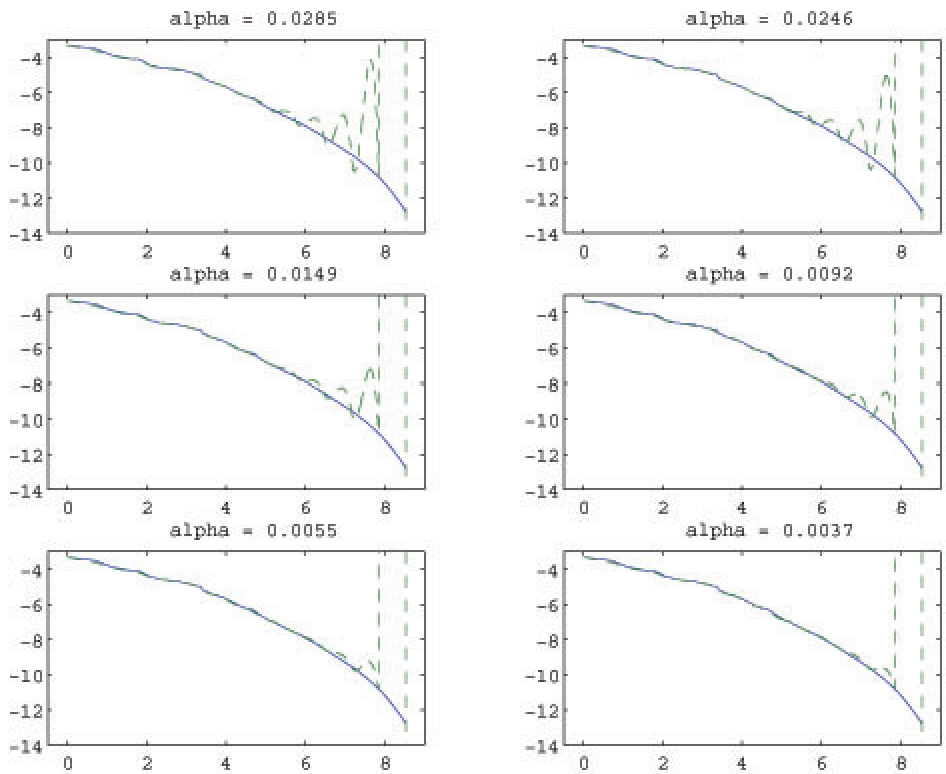


Figure 4.7.13: Estimated and observed capital distribution curve for 1999

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