Existence and Stability of Solitary Waves for NLS with Defects

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Existence and Stability of Solitary Waves for NLS with Defects

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Abstract

The main focus of this dissertation is to investigate the existence and stability of solitary waves to dispersive partial differential equations and, in particular to nonlinear Schrödinger equations with defects nonlinearity.

First we identify necessary and sufficient conditions for the existence of appropriately localized waves for the inhomogeneous semi-linear Schrödinger equation driven by the subLaplacian dispersion operators $(-\Delta)^s, 0 < s \leq 1$. We construct these waves and we establish sharp asymptotics, both at the singularity 0 and for large values. We show the non-degeneracy of these waves. Finally, we provide spectral and orbital stability classification, under slightly more restrictive assumptions.

Then we study the concentrated NLS on $\mathbf{R}^n$, with power non-linearities, driven by the fractional Laplacian, $(-\Delta)^s, s > \frac{n}{2}$. We construct the solitary waves explicitly, in an optimal range of the parameters, so that they belong to the natural energy space H^s. Next, we provide a complete classification of their spectral stability. Finally, we show that the waves are non-degenerate and consequently orbitally stable, whenever they are spectrally stable.

Incidentally, our construction shows that the soliton profiles for the concentrated NLS are in fact exact minimizers of the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$, which provides an alternative calculation and justification of the sharp constants in these inequalities.

Lastly, we consider the degenerate semi-linear Schrödinger and Korteweg-de Vries equations in one spatial dimension. We construct special solutions of the two models, namely standing wave solutions of NLS and traveling waves, which turn out to have compact support and are thus known as compactons. We show that the compactons are unique bell-shaped solutions of the corresponding PDE's and for appropriate variational problems as well. We provide a complete spectral characterization of such waves, for all values of p. Namely, we show that all waves are spectrally stable for $2 < p \leq 8$, while a single mode instability occurs for $p > 8$. This extends previous work of Germain, Harrop-Griffiths and Marzuola, [42], who have previously established orbital stability for some specific waves, in the range $p < 8$.

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Contents

Chapter 1

Introduction

1.1 Preliminaries

1.1.1 Function spaces, Fourier transform and basic operators

In order to fix the notations, we shall use the standard expressions for $\|\cdot\|_{L^p(\mathbf{R}^n)}, 1 \leq p \leq \infty$, or just $\|.\|_p$ as well as the following expression for the Fourier transform and its inverse

$$\hat{f}(\xi) = \int\limits_{\mathbf{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad f(x) = \int\limits_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

The operators $(-\Delta)^s, 0 < s < 1$ are defined in a classical way on the Schwartz class[1] $\mathcal{S}$ via $\widehat{(-\Delta)^s} f(\xi) = (2\pi|\xi|)^{2s} \hat{f}(\xi)$. Accordingly, the Sobolev spaces are taken

$$\|f\|_{\dot{H}^s} := \|(-\Delta)^{s/2} f\|_{L^2}, \|f\|_{H^s} = \|f\|_{\dot{H}^s} + \|f\|_{L^2}.$$

[1] and then by extension in any Banach space for which $\mathcal{S}$ is a dense subspace

More generally, the Sobolev spaces $W^{\alpha,p}, \alpha > 0, 1 < p < \infty$ are introduced as completions of the Schwartz family in the norms

$$\|f\|_{W^{\alpha,p}} := \|(-\Delta)^{s/2} f\|_{L^p} + \|f\|_{L^p}.$$

The use of weighted spaces is necessitated by the context, so we introduce

$$\|f\|_{\dot{L}^{q,-b}} = \left(\int_{\mathbf{R}^n} |x|^{-b} |f(x)|^q dx\right)^{1/q}.$$

The following commutator identity, see p. 1703, [36], will be of special interest

$$[(-\Delta)^s, x \cdot \nabla_x] = 2s(-\Delta)^s. \tag{1.1.1}$$

We will also need properties of the kernel of the operator $(I + (-\Delta)^s)^{-1}, s > 0$. We state a precise result next.

Lemma 1. *Let $0 < s < 1$. Then, the function $G_s(x) : \widehat{G_s}(\xi) = (1 + (4\pi^2|\xi|^2)^s)^{-1}$ satisfies.*

- *There is $C = C_{s,n}$, so that*

  $$G_s(x) \leq C_{s,n}|x|^{-n}$$

  when $|x| > 1$.

- *For $|x| \leq 1$, there is*

$$G_s(x) \sim \begin{cases} |x|^{2s-n} + O(1) & 2s < n \\ \ln(2/|x|) + o(x) & 2s = n \\ 1 + o(x) & 2s > n \end{cases}.$$

- *$G_s > 0$, $G_s \in L^1(\mathbf{R}^n)$.*

Regarding ∇G_s, we have the following bounds, in the regime $2s < n$

$$|\nabla G_s(x)| \leq C \begin{cases} |x|^{-n-1} & |x| > 1 \\ |x|^{2s-n-1} & |x| \leq 1 \end{cases} \tag{1.1.2}$$

Proof. First, take a partition of unity, so that there is a function φ, supported in $\{\xi : |\xi| < 1\}$ and $\zeta(\xi) := \varphi(\xi) - \varphi(2\xi)$, whence $\varphi(\xi) + \sum_{k=1}^{\infty} \zeta(2^{-k}\xi) = 1$. Let $|x| > 1$, say $|x| \sim 2^l, l \geq 0$. We have the partition of unity

$$1 = \varphi(2^l\xi) + (1 - \varphi(2^l\xi)) = \varphi(2^l\xi) + \sum_{k=1-l}^{\infty} \zeta(2^{-k}\xi)$$

whence

$$\begin{aligned} G_s(x) &= \int \frac{1}{1+(4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi} d\xi = \int \frac{1}{1+(4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi} \varphi(2^l\xi) d\xi + \\ &+ \sum_{k=1-l}^{\infty} \int \frac{1}{1+(4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi} \zeta(2^{-k}\xi) d\xi. \end{aligned}$$

In the first integral, we estimate the integrand by absolute value, whence we obtain the bound $C2^{-ln} \sim |x|^{-n}$. For a given x, we identify $j \in [1, n]$, so that $|x_j| \geq \frac{2^l}{n}$. Integrating by parts N times in the variable x_j (and $N > n+1$) and taking absolute values implies a bound

$$\sum_{k=1-l}^{\infty} \frac{1}{(2^k|x_j|)^N} 2^{kn} \lesssim 2^{-ln} \sim |x|^{-n}.$$

For $|x| < 1$, let us consider the case $2s < n$, as the others are similar and somewhat simpler. Say $|x| \sim 2^{-l}, l \geq 0$. We now use the partition of unity

$$1 = \varphi(2^{-l}\xi) + \sum_{k=l+1}^{\infty} \zeta(2^{-k}\xi)$$

Again, for the integral with $\varphi(2^{-l}\xi)$ we estimate by the absolute values

$$\left|\int \frac{1}{1+(4\pi^2|\xi|^2)^s} e^{-2\pi i x\cdot\xi}\varphi(2^{-l}\xi)d\xi\right| \leq C2^{l(n-2s)} \sim |x|^{2s-n},$$

while for the other integrals, we again integrate by parts N times in $|x_j| \geq \frac{2^{-l}}{n}$. The estimates are again

$$\sum_{k=l+1}^{\infty} \frac{1}{(2^k|x_j|)^N} 2^{k(n-2s)} \leq C2^{l(n-2s)} \sim |x|^{2s-n}.$$

For ∇G_s, the bounds (1.1.2) follow in an identical manner, once we recognize that taking derivatives results in an extra power of $|x|^{-1}$.

The statement $G_s > 0$ (and in fact G_s is bell-shaped), can be proved via the representation

$$\frac{1}{1+(4\pi^2|\xi|^2)^s} = \int_0^\infty e^{-t(1+(4\pi^2|\xi|^2)^s)}dt = \int_0^\infty e^{-t}e^{-t(4\pi^2|\xi|^2)^s)}dt$$

and the well-known fact that $\widehat{e^{-|\xi|^{2s}}}$ is a bell-shaped function, as long as $0 < s \leq 1$. Thus,

$$\|G_s\|_{L^1} = \int G_s(x)dx = \hat{G}_s(0) = 1.$$

□

1.1.2 Rearrangements

In this subsection, we discuss the techniques of rearrangements. Let A be a measurable set of finite volume in $\mathbb{R}^n$. Its symmetric rearrangement A^* is the open centered ball whose volume agrees with A, i.e. $A^* = \{x \in \mathbb{R}^n : |\omega_n||x|^n < Vol(A)\}$. For characteristic functions of measurable sets, define $(\chi_A)^* := \chi_{A^*}$

Definition 1. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a measurable function that vanishes at infinity, i.e. for all $t > 0$ we have $d_f(t) := |\{x : |f(x)| > t\}| < \infty$.*

We define the symmetric decreasing rearrangement f^* of f by symmetrizing its level set, namely $f^*(x) := \int_0^\infty \chi_{\{|f(x)|>t\}^*} dt$ and $d_{f^*}(t) = d_f(t)$. A function is called bell-shaped, if $f = f^*$. In particular, $f = f^* \geq 0$.

Recall the rearrangement inequality

$$\int_{\mathbf{R}^n} f(x)g(x)dx \leq \int_{\mathbf{R}^n} f^*(x)g^*(x)dx, \tag{1.1.3}$$

valid for all functions vanishing at infinity. In addition, if one of the functions, say f, is strictly decreasing, the equality is possible only if g is bell-shaped, i.e. $g = g^*$.

Next, we state the Polya-Szegö inequalities, which will be instrumental in our approach.

Lemma 2. *For $\beta \in (0, 1)$ and $f \in H^\beta(\mathbf{R}^n)$, its decreasing rearrangement $f^* \in H^\beta(\mathbf{R}^n)$ and*

$$\|(-\Delta)^{\frac{\beta}{2}} f\|_{L^2} \geq \|(-\Delta)^{\frac{\beta}{2}} f^*\|_{L^2}. \tag{1.1.4}$$

The full proof of this result is standard. It can be found, for example, in Appendix A, [33].

The next result is the Hardy's inequality.

Lemma 3. *Let $a < b$ and $f \in H^1(\mathbf{R})$, so that $f(a) = 0$. Then,*

$$\int_a^b \frac{|f(x)|^2}{|x-a|^2} dx \leq \int_a^b |f'(x)|^2 dx. \tag{1.1.5}$$

Remark: This is a slightly more general version of the classical statement

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^2} dx \leq \int_{-\infty}^{\infty} |f'(x)|^2 dx.$$

for function $f : f(0) = 0$. But it is clear that from the classical version, one obtains by an approximation argument that

$$\int_0^b \frac{|g(x)|^2}{|x|^2} dx \leq \int_0^b |g'(x)|^2 dx$$

for every $g : g(0) = 0,\ b > 0$. It is then clear that the formulation (1.1.5) reduces to this form, by taking $g : g(x) = f(x+a)$.

Our next proposition deals with a control of the weighted norms appearing in (2.2.2) in terms of a Sobolev embedding.

1.1.3 Weighted Sobolev inequality

Proposition 1. *For either one of the cases,*

- $n = 1, \sigma \in [\frac{1}{2}, 1),\ 0 < a < 1,\ 2 \leq q < \infty,$
- $n = 1, 0 < \sigma < \frac{1}{2},\ 0 < a < 2\sigma,\ 2 \leq q < 2 + \frac{4\sigma - 2a}{1 - 2\sigma},$
- $n \geq 2,\ 0 < \sigma < 1,\ 0 < a < 2\sigma,\ 2 \leq q < 2 + \frac{4\sigma - 2a}{n - 2\sigma},$

there exists C, depending on all parameters, so that

$$\left(\int_{\mathbf{R}^n} |x|^{-a} |\phi|^q dx \right)^{\frac{1}{q}} \leq C \|\phi\|_{H^\sigma(\mathbf{R}^n)}. \tag{1.1.6}$$

Remark: Note that the assumptions in Proposition 1 ensure that $a < n$. Also, for $q = 2$, there is the estimate

$$\left(\int_{\mathbf{R}^n} |x|^{-a} |\phi|^2 dx \right)^{\frac{1}{q}} \leq C_\epsilon \|\phi\|_{H^{\frac{a}{2}+\epsilon}(\mathbf{R}^n)}, \tag{1.1.7}$$

for every $\epsilon > 0$.

Proof. For the case $n \geq 2$, $\sigma > 0$, $0 < a < 2\sigma$, and $2 \leq q < 2 + \frac{4\sigma - 2a}{n - 2\sigma}$, we proceed as follows. By Sobolev embedding, we have, since $n\left(\frac{1}{2} - \frac{1}{q}\right) < \sigma$,

$$\left(\int_{|x|>1} |x|^{-a}|\phi|^q dx\right)^{\frac{1}{q}} \leq \left(\int_{|x|>1} |\phi|^q dx\right)^{\frac{1}{q}} \leq C\|\phi\|_{L^q} \leq C\|\phi\|_{H^\sigma}.$$

Next, for $|x| < 1$

$$\left(\int_{|x|<1} |x|^{-a}|\phi|^q dx\right)^{\frac{1}{q}} \leq C\left(\sum_{j=0}^{\infty} 2^{ja} \int_{|x|\tilde{}2^{-j}} |\phi|^q dx\right)^{\frac{1}{q}}$$

And by Hölder inequality we have for every $r \geq q$,

$$\int_{|x|\tilde{}2^{-j}} |\phi|^q \leq \left(\int |\phi|^r\right)^{\frac{q}{r}} (2^{-jn})^{(1-\frac{q}{r})}.$$

Thus

$$\left(\int_{|x|<1} |x|^{-a}|\phi|^q dx\right)^{\frac{1}{q}} \leq \left(\sum_{j=0}^{\infty} (2^{-jn})^{(1-\frac{q}{r})+ja}\|\phi\|^q_{L^r(|x|\tilde{}2^{-j})}\right)^{\frac{1}{q}}.$$

Select *any* $r \in (q, \infty)$, so that

$$a < n\left(1 - \frac{q}{r}\right),\ \ n\left(\frac{1}{2} - \frac{1}{r}\right) < \sigma$$

That is,

$$\frac{1}{2} - \frac{\sigma}{n} < \frac{1}{r} < \frac{1 - \frac{a}{n}}{q},$$

which is possible, due to the restriction $2 \leq q < 2 + \frac{4\sigma - 2a}{n - 2\sigma}$. We have

$$\left(\sum_{j=0}^{\infty} (2^{-jn})^{(1-\frac{q}{r})+ja}\|\phi\|^q_{L^r(|x|\tilde{}2^{-j})}\right)^{\frac{1}{q}} = \left(\sum_{j=0}^{\infty} (2^{j(a-n(1-\frac{q}{r}))}\|\phi\|^q_{L^r(|x|\tilde{}2^{-j})}\right)^{\frac{1}{q}}$$

$$\leq \quad C_r \sup_j \|\phi\|_{L^r(|x|\sim 2^{-j})} \leq C_r \|\phi\|_{L^r} \leq C_r \|\phi\|_{H^{n\left(\frac{1}{2}-\frac{1}{r}\right)}} \leq C_r \|\phi\|_{H^\sigma}.$$

where in the last step we have used the Sobolev embedding and $n\left(\frac{1}{2}-\frac{1}{r}\right) < \sigma$. The case $n = 1, \sigma \in (0, \frac{1}{2}), a < 2\sigma, 2 \leq q < 2 + \frac{4\sigma - 2a}{1-2\sigma}$ is done in an identical manner.

For the case $n = 1, \sigma \geq \frac{1}{2}, 2 \leq q < \infty$ is as follows. By Sobolev embedding $H^\sigma(\mathbf{R}) \hookrightarrow L^q(\mathbf{R})$, so

$$\left(\int_{|x|>1} |x|^{-b} |\phi|^q dx\right)^{\frac{1}{q}} \leq \left(\int_{|x|>1} |\phi|^q dx\right)^{\frac{1}{q}} \leq C\|\phi\|_{H^\sigma}.$$

The term $\left(\int_{|x|<1} |x|^{-b} |\phi|^q dx\right)^{\frac{1}{q}}$ is controlled in the same way as above, we omit the details.

$\square$

The Sobolev embedding will be of great importance, $\dot{W}^{s,p}(\mathbf{R}^n) \hookrightarrow L^q(\mathbf{R}^n)$, for $1 < p < q < \infty : s \geq n\left(\frac{1}{p}-\frac{1}{q}\right)$. Also, recall that for $s > \frac{n}{p}$, there is the embedding[2] $W^{s,p} \hookrightarrow C^{[s-\frac{n}{p}],\gamma}(\mathbf{R}^n) : 0 < \gamma < s - \frac{n}{p}$. As is well-known, the embedding $H^{\frac{n}{2}}(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$ fails, but sometimes an useful replacement estimate is the following for all $\delta \in (0, \frac{n}{2})$,

$$\|f\|_{L^\infty} \leq C_\delta(\|f\|_{\dot{H}^{\frac{n}{2}-\delta}} + \|f\|_{\dot{H}^{\frac{n}{2}+\delta}}). \tag{1.1.8}$$

The Sobolev embedding will be useful in the sequel, so we state it here - for each $1 < r < q < \infty$,

$$\|f\|_{L^q} \leq C_{r,q} \|f\|_{\dot{W}^{\frac{1}{r}-\frac{1}{q},r}}.$$

Next, we need the following endpoint Gagliardo-Nirenberg inequality

$$\|f\|_{L^q} \leq C_q \|f'\|_{L^2}^{\frac{1}{3}+\frac{2}{3q}} \|f\|_{L^1}^{\frac{2}{3}(1-\frac{1}{q})}, 1 \leq q < \infty. \tag{1.1.9}$$

[2]Here $\{x\} = x - [x]$, where $[x] = \max\{n : n \leq x\}$

1.1.4 The basics of the Hamiltonian index theory

We follow the setup described in [62], but the earlier versions of these results, [56, 83, 55, 68] provided much impetus in the developments of these methods.

Consider the Hamiltonian eigenvalue problem of the form

$$\mathcal{J}\mathcal{L}g = \lambda g, g \in H \tag{1.1.10}$$

where H is a Hilbert space, $\mathcal{J} : \mathcal{D}(\mathcal{J}) \subset H^* \to H$, and

$$\mathcal{L} : H \to H^*$$

We will give a brief introduction of the analysis of the number of unstable eigenvalues of (1.1.10). Assume that $\langle \mathcal{L}u, v\rangle$ is bounded symmetric bilinear form on $H \times H$, which gives rise to a self-adjoint operator $(\mathcal{L}, D(\mathcal{L}))$ Moreover, there exist a decomposition of H in $\mathcal{L}$ invariant subspaces

$$H = H_- \oplus ker\mathcal{L} \oplus H_+$$

where H_-, H_+ are the negative and positive subspaces respectively. More precisely, upon introducing the self-adjoint projections $P_- = \chi_{(-\infty,0)}(\mathcal{L}), P_+ = \chi_{(0,+\infty)}(\mathcal{L})$, we take $H_- = P_-[H], H_+ = P_+[H]$. Assume in addition that the Morse index of $\mathcal{L}$, that is $n(\mathcal{L}) = dim(H_-) < \infty$, while there exists $\delta > 0$, so that $\langle \mathcal{L}u, u\rangle \geq \delta\|u\|^2$, for all $u \in D(\mathcal{L}) \cap H_+$. For any $\lambda \in \sigma_{p.p.}(\mathcal{J}\mathcal{L})$, introduce the generalized eigenspace

$$E^{\lambda}_{gen} := \cup_{j=1}^{\infty} E^{\lambda}_j, E^{\lambda}_j = \{f \in H : (\mathcal{J}\mathcal{L} - \lambda I)^j f = 0\}$$

Assume also that the dual space H^* satisfies

$$\{\langle g \in H^* : \langle g, u\rangle = 0, \forall u \in H_- \oplus H_+\rangle\} \subset D(\mathcal{J}).$$

Further since, $Ker(\mathcal{L}) \subset E^0_{gen}$, decompose $E^0_{gen} = ker(\mathcal{L}) \oplus E_0$, where E_0 is finite dimensional with basis say $\{\psi_i\}_{i=1}^n \subset \mathcal{D}(\mathcal{L})$. Let D be the matrix with entries

$$D_{ij} = \langle \mathcal{L}\psi_i, \psi_j\rangle.$$

Next, we need to introduce the notion of negative Krein signature of a purely imaginary eigenvalue. More specifically, for $i\mu \in \sigma_{p.p.}(\mathcal{J}\mathcal{L}), \mu > 0$, consider $E_{i\mu} := Ker(\mathcal{J}\mathcal{L} - i\mu), P_{i\mu} : H \to E_{i\mu}$ and $k_i^{i\mu} = n(\mathcal{L}|_{E_{i\mu}}) = n(P_{i\mu}\mathcal{L}P_{i\mu})$. Finally, we define the total Krein signature

$$k_i^{\leq 0} = \sum_{\mu\neq 0: i\mu \in \sigma_{p.p.}(\mathcal{J}\mathcal{L})} k_i^{i\mu}.$$

In the most common case, when $i\mu$ is a simple eigenvalue, with say an eigenvector ψ_μ, $i\mu$ is of negative Krein signature exactly when the real quantity $\langle \mathcal{L}\psi_\mu, \psi_\mu\rangle < 0$. In such a case $k_i^{i\mu} = 1$.

By [62] see also [83][56] we have the following Hamiltonian-Krein index formula

$$k_{Ham.} := k_r + 2k_c + 2k_i^{\leq 0} = n(\mathcal{L}) - n(D) \tag{1.1.11}$$

where k_r is the number of real positive eigenvalues of $\mathcal{J}\mathcal{L}$ (counted with their multiplicities), k_c is the number of eigenvalues of $\mathcal{J}\mathcal{L}$ with positive real and imaginary part, and lastly $k_i^{\leq 0}$ is the total Krein signature introduced above. Note that $k_{Ham} = 0$ implies spectral stability for the model (1.1.10), as k_{Ham} counts all the instabilities, since $k_{Ham} \geq k_r + k_c$.

In the particular case $n(\mathcal{L}) = 1$, the formula (1.1.11) reduces the situation to two cases, namely $k_{Ham} = 1$, if $n(D) = 0$ and $k_{Ham} = 0$, if $n(D) = 1$. We have already discussed that $k_{Ham} = 0$ is a case of stability. If however $k_{Ham.} = 1$, by parity considerations, it follows that $k_r = 1$, while $k_c = k_i = 0$. In any event, this implies that the system has a real unstable growing mode. Thus, we have the following useful corollary of (1.1.11).

Corollary 1. *If $n(\mathcal{L}) = 1$, the the eigenvalue problem (1.1.10) is spectrally stable if $n(D) = 1$, and it has exactly one real unstable mode, if $n(D) = 0$.*

1.1.5 Ground states of dispersive PDEs with standard NLS as case study

In this section, the breakdown of the necessary steps needed to show the existence and stability of solitary waves for dispersive partial differential equations is given. In particular, the general Hamiltonian nonlinear fractional Schrödinger equation

$$iu_t + (-\Delta)^s u - F(|u|^2).u = 0, u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}, s > 0. \tag{1.1.12}$$

In the last chapter we consider degenerate nonlinear Schrödinger and KdV equation. The most familiar case is when $s = 1, F \equiv |u|^p u$. The Nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1} u = 0, \tag{1.1.13}$$

will be the prime example for this chapter. These models have many physical applications, the prominent ones are in nonlinear optics.

Another very important application is the study and existence and properties of the ground states, for the case of NLS is the standing wave solutions in the form $e^{i\omega}\phi_\omega$[3],$\phi_\omega > 0$. If we plug this solitary wave into (1.1.12) we have the elliptical profile equation

$$(-\Delta)^s\phi_\omega + \omega\phi_\omega - F(\phi_\omega)\phi_\omega = 0. \tag{1.1.14}$$

Or as in our prime example

$$-\Delta\phi_\omega + \omega\phi_\omega - \phi_\omega^p = 0. \tag{1.1.15}$$

From now on we will focus on the NLS equations, we will return to KdV in (4).

Next is to consider the linearized problems. To that end taking the ansatz $u = e^{i\omega t}(\phi_\omega + \vec{v}(t,x))$ and plugging it into the NLS equation (1.1.12) we obtain,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: \mathcal{J}\mathcal{L}\vec{v} \tag{1.1.16}$$

where we ignore any $O(v^2)$ term. Here $\vec{v} = \Re v + i\Im v := v_1 + iv_2$ and the following fractional Schrödinger operators are introduced

$$\mathcal{L}_- = (-\Delta)^s + \omega - F(\phi_\omega), \tag{1.1.17}$$

$$\mathcal{L}_+ = (-\Delta)^s + \omega - F'(\phi_\omega)\phi_\omega \tag{1.1.18}$$

with their domain H being a Hilbert space ($H^{2s}(\mathbb{R}^n)$ for continuous F).

For our specific example

[3]Here the subscript is to emphasize the dependency of ω not to be confuse with partial derivative with respect to ω

$$\mathcal{L}_- = -\Delta + \omega - \phi^{p-1}, \tag{1.1.19}$$

$$\mathcal{L}_+ = -\Delta + \omega - p\phi^{p-1} \tag{1.1.20}$$

with domain $(H^1(\mathbb{R}^n)$.

Clearly from the definition given in (1.1.16) we have

$$\mathcal{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathcal{L} := \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix},$$

and the assignment $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: e^{\lambda t}\vec{v}$, we obtain the following time-independent linearized eigenvalue problem

$$\mathcal{J}\mathcal{L}\vec{v} = \lambda\vec{v}. \tag{1.1.21}$$

Taking (1.1.14) into account, one immediate observation is that $\mathcal{L}_-\phi_\omega = 0$, also in the presence of transitional symmetry, taking the derivatives formally of (1.1.14) with respect to any $x_i, i = 1, \ldots, n$ we have $\mathcal{L}_+\partial_{x_i}\phi_\omega = 0$.

The spectral properties of the operator $\mathcal{L}_\pm$ play crucial role in the stability of the solitary waves. It is in general easy to see that zero is the bottom of the spectrum for $\mathcal{L}_-$ see theorem 3 of [80]. As for the kernel of $\mathcal{L}_+$, the conclusion is not as straight forward as $\mathcal{L}_-$. In the absence of symmetry $Ker[\mathcal{L}_+] = \{0\}$, note that proving this result is highly non trivial see (2.5). We give the definition of non-degenerate and weakly-non-degenerate as given in [80].

Definition 2. *We say that the wave ϕ_ω is non-degenerate if $Ker[\mathcal{L}_+] = span[\partial_i\phi_\omega], i = 1, \ldots, n]$. We say that ϕ_ω is weakly non-degenerate if $\phi_\omega \perp Ker[\mathcal{L}_+]$.*

Next, we give the formal notions of stability.

Definition 3. *We say that the wave ϕ_ω as a solution to the NLS problem (1.1.12) is spectrally stable if for (1.1.21) the set of solutions to (1.1.21) is empty.*

We say that the wave ϕ_ω is an orbitally stable solution of (1.1.21), if for any $\epsilon > 0 \exists \delta > 0$ such that for any initial data so that $\|u_0 - \phi_\omega\|_{H^1(\mathbb{R}^n)} < \delta$, then u satisfies

$$\sup_{t>0} \inf_{\theta \in [0,2\pi], y \in \mathbb{R}^n} \|u(t,.-y) - e^{i\theta}\phi_\omega\|_{H^1(\mathbb{R}^n)} < \epsilon.$$

Remark 1.1.1. *The notion of stability may depends on the operator or models as:*

- *The notion of spectral stability depends entirely on the spectrum of the operator $\mathcal{L}$.*
- *the notion of orbital stability depends on the number of symmetries in the model.*

Next we construct the waves and discuss their stability. Prior to construction of the waves, it is natural to ask for the range of parameters for which the solution of (1.1.14) exists. To that end we compute the Pohozaev identity.

First we start by taking the inner product of (1.1.14) with ϕ_ω and integrate by part assuming of course ϕ_ω has enough smoothness and decay properties we arrive at

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\phi_\omega| dx + \omega \int_{\mathbb{R}^n} \phi_\omega^2 dx = \int_{\mathbb{R}^n} H(\phi_\omega) dx$$

Next is to take the inner product of (1.1.14) with $x.\nabla_x \phi\omega = \sum_j^n x_j \partial_j \phi_\omega$ taking into account the commutation formula (1.1.1) we get

$$(s - \frac{n}{2}) \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\phi_\omega| dx + f_1(n,p,\alpha) \int_{\mathbb{R}^n} H(\phi_\omega) dx = \frac{n\omega}{2} \int_{\mathbb{R}^n} \phi_\omega^2 dx.$$

From this equations solving for $\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\phi_\omega| dx$ and $\int_{\mathbb{R}^n} H(\phi_\omega) dx$ we have the following relations

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\phi_\omega| dx = f(s,\omega,p,n,\alpha)\int_{\mathbb{R}^n}\phi_\omega^2 dx$$

and

$$\int_{\mathbb{R}^n} H(\phi_\omega) dx = g(s,\omega,p,n,\alpha)\int_{\mathbb{R}^n}\phi_\omega^2 dx$$

From here we can infer that if $H(\phi_\omega) > 0$ then the solutions of (1.1.14) exists if $f(s,\omega,p,n,\alpha) > 0, g(s,\omega,p,n,\alpha)$ which gives us the necessary conditions for the existence of such solutions.

Remark 1.1.2. *One can claim that the range above for the existence of the special solutions is the same as for the local well-posedness of* (1.1.12).

Applying this to our example, in particular (1.1.15) we have the following relations

$$\int_{\mathbb{R}^n} |\nabla\phi_\omega|^2 dx = \frac{2\omega(p+1)}{2n-(n-2)(p+1)}\int_{\mathbb{R}^n}\phi_\omega^2 dx$$

and

$$\int_{\mathbb{R}^n} \phi_\omega^{p+1} dx = \frac{\omega n(p+1)-2n}{2n-(n-2)(p+1)}\int_{\mathbb{R}^n}\phi_\omega^2 dx$$

So for existence of (1.1.15) we need to have $p \leq \frac{n+2}{n-2}, n > 2$

Variational Setup:

We construct the waves in (1.1.14) variationally. This can be accomplished in two ways.

The first is via the so called Weinstein functional, first introduced by Micheal Weinstein [49]. Consider the following functional

$$I_\omega[u] = \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u| dx + \omega \int_{\mathbb{R}^n} u^2 dx}{\left(\int_{\mathbb{R}^n} H(u) dx\right)^{\kappa(p)}}$$

Here $\kappa(p)$ is a number that makes the denominator a norm.

With appropriate assumptions and careful analysis, one can show that the unconstraint minimization problem $I_\omega[u] \to min$ has solution(sometimes bell-shaped, or even unique)in the appropriate space. Moreover, the miminizer satisfies (1.1.14) upto scaling.

Note that the waves constructed using this approach are not necessary normalized waves thus some of them are unstable.

Another approach is to construct the waves by imposing a constraint on the $L_2 - norm.$ To be more precise, one can consider the following constrained minimization problem.

$$\begin{cases} I[u] = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u| dx - \int_{\mathbb{R}^n} H(u) dx & \to min \\ \int_{\mathbb{R}^n} u^2 dx = \lambda, & \lambda > 0 \end{cases}$$

The question of the existence to the above minimization problem is hard to answer despite many recent progress. The difficulty of the variational problem and how to address it is not the focus of this work, but interested readers can check [80]. The point is that the normalized waves are stable. Thus the range of existence for the normalized waves is the same as the range of stability.

We will consider both approach for our example (1.1.15).

We start with the Weinstein functional

$$I_\omega[u] = \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx + \omega \int_{\mathbb{R}^n} u^2 dx}{\left(\int_{\mathbb{R}^n} u^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

Assume that $\omega > 0$, and $u \in H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), u \neq 0$, then $0 < \int_{\mathbb{R}^n} u^{p+1} dx < \infty$, thus $I_\omega[u]$ is well-defined. Next, take advantage of the scaling. To do that, for every $u \neq 0, I_\omega[u] > 0$, consider the non-negative number

$$m(\omega) := \inf_{u \in \mathcal{S}} I_\omega[u].$$

Note that if ϕ is a minimizer for $I_1[u] \to min$, then $\phi_\omega = \phi(\omega^{\frac{1}{2}} x)$ is a minimizer for $I_\omega[u] \to min$, and

$$m(\omega) = m(1)\omega^{\frac{n-2}{2(p+1)}(p - \frac{n+2}{n-2})}.$$

For the minimization to well-posed we need the estimate for some constant $C > 0$

$$\|u\|_{p+1} \leq C\|u\|_{H^1}.$$

The above estimate follows from Sobolev embedding inequality with $p \leq \frac{n+2}{n-2}$ which is the range we obtained from Pohozaev above, and also is the same for H^1 local well-posedness see [82, 17]. Thus we have

$$I_1[u] \geq \frac{1}{C}.$$

Hence the variational problem is well-posed.

Now, to construct normalized wave of (1.1.15), consider the constrained variational problem

$$\begin{cases} I[u] = \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^n} u^{p+1} dx & \to min \\ \int_{\mathbb{R}^n} u^2 dx = \lambda, & \lambda > 0 \end{cases}$$

For the above problem to be well-pose, we need the following estimate for some constant $C > 0$

$$\|u\|_{p+1} \leq C\|u\|_{\dot{H}^{\alpha}}.$$

Again this is Sobolev embedding inequality but this time with

$$1 < p < \frac{4}{n} + 1, \alpha = n(\frac{1}{2} - \frac{1}{p+1}).$$

Even though the above estimate is not enough to show well-posedness of the normalized waves, this can be accomplished using Hölder and Young's ϵ inequalities to show that

$$I[u] \geq -C_{\epsilon,\lambda}.$$

Remark 1.1.3. *We will see that the range for existence of normalized range is the range for which the waves are stable. One can easily claim that all normalized waves are stable.*

Another question one might ask is whether the only stable waves are the normalized ones.

The final step to construct these waves goes through minimizing sequences and notion of convergence. Since we are operating in functions spaces the most important topological property is compactness. Sometimes one can boil down the minimization problem to set of minimizers that are bell-shaped. Using the bell-shape properties one can get a point wise bound for the waves. This bound help to establish the so called the Kolmogorov-Relich-Riesz criteria for compactness in L_p. This together with the lower semi-continuity of the weak convergence in Hilbert space and the properties of the minimizing sequences helps to show convergence. For more details on concrete examples please see propositions(23)(3)(22). As for the analysis of the existence of waves of our prime example see [81].

Next, we discuss the Euler-Lagrange equation (1.1.15). Take a test function $h \in V_0^\infty(\mathbf{R}^n \setminus \{0\}$, and consider $u = \phi + \epsilon h$. Due to scaling let $\int \phi^{p+1} dx = 1$. Since ϕ is a minimizer we should have

$$I_\omega[\phi + \epsilon h] \geq m(1) = N(\phi).$$

Where $N(\phi) := \int |\nabla \phi|^2 + \int \phi^2$ and $D(\phi) := \int \phi^{p+1} dx$. Thus,

$$\begin{aligned} N(\phi + \epsilon h) &= \int |\nabla u(\phi + \epsilon h)|^2 + \int (\phi + \epsilon h)^2 \\ &= \int |\nabla u\phi + \epsilon |\nabla u h|^2 + \int (\phi^2 + 2\epsilon h\phi + \epsilon^2 h^2) \\ &= \int |\nabla u \phi|^2 + \int \phi^2 + 2\epsilon(\langle(\nabla u\phi, \nabla u h\rangle + \langle h, \phi\rangle) + O(\epsilon^2) \\ &= N(\phi) + 2\epsilon\langle(-\Delta + 1)\phi, h\rangle + O(\epsilon^2). \end{aligned}$$

Similarly,

$$D(\phi + \epsilon h) = \int (\phi + \epsilon h)^{p+1} dx = 1 + (p+1)\epsilon\langle \phi^p, h\rangle + O(\epsilon^2).$$

It follows that

$$\begin{aligned} I_1(\phi + \epsilon h) &= \frac{N(\phi + \epsilon h)}{D[\phi + \epsilon h]^{\frac{2}{p+1}}} = \frac{N(\phi) + 2\epsilon\langle(-\Delta + 1)\phi, h\rangle + O(\epsilon^2)}{1 + 2\epsilon\langle \phi^p, h\rangle + O(\epsilon^2)} \\ &= N[\phi] + 2\epsilon\langle(-\Delta + 1)\phi - N(\phi)\phi^p, h\rangle + O(\epsilon^2). \end{aligned}$$

As this holds for arbitrary function h and for all small ϵ, we have established upto scaling of $N(\phi)$ that ϕ solves (1.1.15) in a distributional sense.

Finally, fix h to be a real-valued function, $h \in C_0^\infty(\mathbf{R}^n)$. Starting again with the inequality

$$\frac{N(\phi+\epsilon h)}{D(\phi+\epsilon h)^{\frac{2}{p+1}}} \geq N(\phi),$$

but expanding to the second order ϵ^2 , we obtain

$$N[\phi]+\epsilon^2[\langle \mathcal{L}_+ h, h\rangle + N[\phi](p+3)(\langle \phi^p, h\rangle)^2] + O(\epsilon^3) \geq N[\phi],$$

after taking into account $\langle (-\Delta+1)\phi - N(\phi)\phi^p, h\rangle = 0$. After taking limits as $\epsilon \to 0$, we derive

$$\langle \mathcal{L}_+ h, h\rangle \geq -N[\phi](p+3)(\langle \phi^p, h\rangle)^2. \tag{1.1.22}$$

In particular, $\langle \mathcal{L}_+ h, h\rangle \geq 0$, if $\int \phi^p(x)h(x)dx = 0$.

This implies that $\mathcal{L}_+$ is positive on co-dimension one subspace $\{\phi^p\}^\perp$.

After constructing the waves, the natural next step is to study the stability of such waves. This heavily depends on the spectral properties of the operators $\mathcal{L}_\pm$. To this end we look closely into such properties.

Spectral Properties of $\mathcal{L}_\pm$

Before diving into the spectral properties for the operators we first start to investigating the self-adjointness and domain of the operators. The scope of this thesis is not the study of the general operators (1.1.17). For interested reader see [72], and for very specific examples see the next three chapters where we study specific cases of (1.1.17) for different nonlinearities.

Continuing with the NLS, the Friedrich's extensions of (1.1.19) are self-adjoint operators with domain $H^1(\mathbf{R}^n)$. Note that domain $H^1(\mathbf{R}^n)$ follows directly from Gagliardo Nirenberg Sobolev inequality. Next we introduce the quadratic forms $\mathcal{D}_\pm[g,g] := \langle \mathcal{L}_\pm g, g\rangle$

with form domain $H^1(\mathbf{R}^n)\times H^1(\mathbf{R}^n)$. Following the usual Friedrich's procedure, it suffices to show that $\mathcal{D}_\pm$ is bounded from below, which again follows from the Sobolev embedding.

- Another key property is the that the continuous spectrum of $\mathcal{L}_\pm$ is $[\omega,\infty)$. This follows from the Weyl's theorem since $\phi\to 0$ as $|x|\to\infty$.

- Next is to show that $\mathcal{L}_+$ has exactly one negative eigenvalue. To that end, since $\langle\mathcal{L}_+\phi,\phi\rangle = -(p-1)m(\omega)\int\phi^{p+1} < 0$, together with (1.1.22) we have $n(\mathcal{L}_+) = 1$.

- For $\mathcal{L}_-$, note that by inspection $\mathcal{L}_-\phi = 0$. And assume $-\alpha^2$ is the smallest eigenvalue of $\mathcal{L}_-$, then

$$-\alpha^2 = \inf_{\|u\|=1}\langle\mathcal{L}_-u,u\rangle.$$

On the other hand since $\mathcal{L}_-\phi = 0$, for any bell-shaped solution $\psi : \|\psi\| = 1$ of (1.1.15) we have

$$0 = \langle\mathcal{L}_-\phi,\psi\rangle = \langle\phi,\mathcal{L}_-\psi\rangle = \langle\phi,-\alpha^2\psi\rangle = -\alpha^2\langle\phi,\psi\rangle < 0$$

a contradiction, thus this established the Coercivity of $\mathcal{L}_-$, that is , $\mathcal{L}_-|_{\{\phi^\perp\}} \geq 0$.

The positivity of the second eigenfunction of $\mathcal{L}_+$ plays a key role in the degeneracy of the waves. For (1.1.17) see [36] while for (1.1.15) the positivity of the waves see [59].

Above we have established the basic Hamiltonian index theory. After analysis the spectral properties of $\mathcal{L}_\pm$ we can state the following corollary. Before we do that note that for the eigenvalue problem in the form (1.1.21) we have that $\mathcal{J}$ is invertible and anti-symmetric. We already see that $n(\mathcal{L}_+) = 1, n(\mathcal{L}_-) = 0$, thus $n(\mathcal{L}) = 1$.

Formally we can see that the eigenvalue zero of $\mathcal{L}_+$ is of multiplicity n, with $Ker[\mathcal{L}_+] = span\{\partial_1\phi_\omega,\ldots,\partial_n\phi_\omega\}$.

So,

$$Ker[\mathcal{L}] = \left\{ \begin{pmatrix} 0 \\ \phi_\omega \end{pmatrix}, \begin{pmatrix} \partial_1 \phi_\omega \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \partial_n \phi_\omega \\ 0 \end{pmatrix} \right\} := \{\varphi_0, \varphi_1, \dots, \varphi_n\}.$$

Clearly $\mathcal{J}^{-1} = -\mathcal{J} : Ker[\mathcal{L}] \to Ker[\mathcal{L}]^\perp$.

Introducing the matrix $\mathcal{D}$ as

$$\mathcal{D}_{ij} := \langle \mathcal{L}^{-1}[\mathcal{J}^{-1}\varphi_i], \mathcal{J}^{-1}\varphi_j \rangle.$$

Then by the index counting theorem developed in [56], if $det(\mathcal{D}) \neq 0$, then

$$k_r + 2k_c + 2k_i^- = n(\mathcal{L}) - n(\mathcal{D}).$$

Note that Since $\partial_i \phi_\omega$ is odd in the i^{th} variable, while $\partial_j \phi_\omega$ is odd in the j^{th} variable we have

$$\mathcal{D}_{ij} = \langle \mathcal{L}_-^{-1} \partial_i \phi_\omega, \partial_j \phi_\omega \rangle = 0.$$

Also Since $\mathcal{L}_-^{-1}$ is positive on $Ker[\mathcal{L}_-]^\perp$ and $\partial_i \phi_\omega \perp Ker[\mathcal{L}_-]$ we then have

$$\mathcal{D}_{ii} = \langle \mathcal{L}_-^{-1} \partial_i \phi_\omega, \partial_i \phi_\omega \rangle > 0.$$

This reduces to $n(\mathcal{D}) = \langle \mathcal{L}^{-1}[\mathcal{J}^{-1}\varphi_0], \mathcal{J}^{-1}\varphi_0 \rangle = \langle \mathcal{L}_+^{-1} \phi_\omega, \phi_\omega \rangle$.

Continuing on our formal land, we can also compute formally $\mathcal{L}_+[\partial_\omega \phi_\omega] = -\phi_\omega$, which yields $\mathcal{L}_+^{-1} \phi_\omega = -\partial_\omega \phi_\omega$. So to compute the stability condition also known as Vakhitov–Kolokolov stability criterion $\langle \mathcal{L}_+^{-1} \phi_\omega, \phi_\omega \rangle$, the standard scaling argument is deployed.

First note that taking the argument above about $\mathcal{L}_+^{-1} \phi_\omega$ into account we have

$$\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle = \langle -\partial_\omega \phi_\omega, \phi_\omega\rangle = -\frac{1}{2}\partial_\omega \|\phi_\omega\|_2^2.$$

Note that $\phi_\omega := \omega^{\frac{1}{p-1}}\phi(\omega^{\frac{1}{2}}x)$ solve the profile equation (1.1.15). Thus

$$\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle = -\frac{1}{2}\partial_\omega \|\phi_\omega\|_2^2 = -\frac{1}{2}\partial_\omega \left[\omega^{\frac{2}{p-1}-\frac{n}{2}}\right] \|\phi\|_2^2.$$

Hence $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle < 0$ if and only if $1 < p < 1 + \frac{4}{n}$. Thus this is the range of p for which (1.1.21) is Spectrally stable. Note that this is the same range for orbital stability.

Chapter 2

Existence and stability of solitary waves for the inhomogeneous NLS

2.1 Introduction

The main object of consideration in this chapter will be the dynamics of the solutions to the Cauchy problem for the fractional inhomogeneous nonlinear Schrödinger equation[1] More precisely, we consider

$$\begin{cases} iu_t + (-\Delta)^s u - |x|^{-b}|u|^{p-1}u = 0, (t,x) \in \mathbf{R} \times \mathbf{R}^n, n \geq 1, \\ u(0,x) = u_0(x) \end{cases} \tag{2.1.1}$$

where we henceforth restrict ourselves to parameters (b,p,s), satisfying the following natural assumptions $b > 0, p > 1, s \in (0,1)$. This chapter has been published in [69]. Our goal in this article is the construction and the stability of solitary waves for (2.1.1). More specifically, the solitons are in the form of standing waves, that is special solutions in the

[1] see Section 1.1.1 for precise definitions of the fractional derivative operator

form $u(x,t) = e^{-i\omega t}\Phi_\omega(x), \Phi > 0$. These satisfy the profile equation[2]

$$(-\Delta)^s \Phi + \omega \Phi - |x|^{-b} \Phi^p = 0, x \in \mathbf{R}^n. \tag{2.1.2}$$

The nonlinear Schrödinger equation arises in various physical contexts such as nonlinear optics and plasma physics[82]. The equation with the inhomogeneous nonlinearity model the beam propagation in an inhomogeneous medium [10]. Fractional NLS also appears in many physical models like water models, quantum mechanics, Lévy stable process and the fractional Brownian motion[27]. Finally, the model (2.1.1), with $b > 0$ appears as an example, with a broken translational invariance, where special treatment is needed for the analysis of the associated eigenvalue problems.

We now turn to a review of the literature regarding the well-posedness results for (2.1.1).

2.1.1 The model - well-posedness results for the classical case $s = 1$

The classical model, $s = 1, b = 0, p > 1$ has been extensively studied in the literature, in terms of well-posedness of the Cauchy problem, long time behavior etc.. As these results are by now classical and well-known, we will not review them here, but we will rather refer the interested reader to the following sources [15, 44, 43, 61, 14, 18, 16, 30, 19, 20, 58, 9].

Recently the well-posedness of (2.1.1) appeared in the literature for the Laplacian case, i.e. $s = 1$. in fact, Farah [31] proved a Gagliardo-Nirenberg type estimate and use it to establish sufficient conditions for global existence and blow-up in $H^1(\mathbb{R}^n)$ for $\frac{4-2b}{n} < p < \frac{4-2b}{n-2}$ and $0 < b < \min(2, n)$, which was later extended by Dinh [26]. Moreover, Guzmán [46] showed that (2.1.1) is globally well-posed for the initial data in $H^s(\mathbb{R}^n)$ with $0 \leq s \leq 1$ using the contraction mapping principle based on the Strichartz estimates.

[2] The sense in which (2.1.2) holds is to be made precise later on, see Section 2.3 below.

In [39], the author showed the global well-posedness in $H^1(\mathbb{R}^n)$ of (2.1.1) with $s = 1$, using the assumption that if the initial data u_0 satisfies $\|u_0\|_{L^2} < \|\psi\|_{L^2}$, where ψ is the unique positive radial soliton of (2.1.2). Moreover they also showed strong instability of the standing waves.

In the paper [84], the author showed the global existence and blow up of solutions in $\mathbb{R}^2$, under various assumptions on the initial data. In addition, the paper [32] showed that if the initial datum $u_0 \in H^1(\mathbb{R}^3)$ satisfies that the momentum as well as the Hamiltonian of (2.1.1) with $s = 1, n = 3$ is dominated by same conserved quantities of (2.1.2) similarly, $\|\nabla u_0\|_{L^2}^{\frac{1+b}{2}} \|u_0\|_{L^2}^{\frac{1-b}{2}} < \|\nabla Q\|_{L^2}^{\frac{1+b}{2}} \|Q\|_{L^2}^{\frac{1-b}{2}}$ where Q satisfies (2.1.2), then the solution u to the Cauchy problem is global in $H^1(\mathbb{R}^3)$ for $0 < b < 1$, and asymptotically linear both forward and backward in time for u_0 radial and $0 < b < 1/2$. In [28], the author studied the decay properties of global solutions for the equation$(s = 1)$ when $1 < p < \frac{4-2b}{n-2}$ for $n \geq 3$ and using this they showed the energy scattering for the equation in the case $1 + \frac{4-2b}{n} < p < 1 + \frac{4-2b}{n-2}$. In [23], the authors have studied the global well-posedness for the defocusing inhomogeneous NLS, whose scaling critical index $s_c < 0$. In [12], the authors showed the L^2-norm concentration for the finite time blow-up solution for the focusing INLS. The same authors later in [11] investigated the blow-up and scattering criteria above the threshold for the same equation. Chen, [21] has considered the model (2.1.1), with non-linearity $|x|^b |u|^{p-1} u, b > 0$. He has identified essentially sharp conditions under which the solutions exist globally and others, under which the solutions blow up in finite time.

We now turn our attention to the issue of the existence of the solitary waves and their stability.

2.1.2 Solitary waves and stability in the classical case $s = 1$

The question for existence of solitary waves (2.1.2) and their stability was investigated in some specific instances of nonlinearity $g(x, |u|^2)u$ in the late 90's in [53]. Specifying to the case $V(x)|u|^{p-1}u$, and in particular to the case, $V = V(\epsilon|x|), 0 < \epsilon << 1$ was considered in [34], [63], see also the more recent work [51].

A general problem modeled by (2.1.1), was studied systematically for first time in the work of De Bouard-Fukuizumi, [10]. More precisely, they consider classical NLS (i.e. $s = 1$) with focussing nonlinearity $V(x)|u|^{p-1}u$, where $V \geq 0$,

$$V \in L_{loc.}^{\frac{2n}{n+2-(n-2)p}}(\mathbf{R}^n), \quad \lim_{x\to\infty} V(x)|x|^b = 1, \tag{2.1.3}$$

which of course contains the important case $V(x) = |x|^{-b}$, under the constraints $0 < b < 2, n \geq 3, 1 < p < 1 + \frac{4-2b}{n-2}$. In this work, they show the existence of non-negative solitary wave solutions under the same assumptions. Furthermore, they showed that there exists $\omega_* > 0$, so that the stability of the said solitary waves holds in the range $0 < b < 2, n \geq 3, 1 < p < 1 + \frac{4-2b}{n}$, when the spectral parameter $\omega \in (0, \omega_*)$. The key step in the stability proof is to show that the linear operator associated with the second variation of a Lyapunov functional[3], which is non-degenerate, for this they adapt a method of [54]. The work in a way supplements the earlier work [37], where the instability of the waves was shown in the range $p > 1 + \frac{4-2b}{n}, n \geq 3$, for small enough $\omega > 0$. Further, more general instability results have appeared in [64].

The authors in [40],[38] proved similar results (both for the stable and unstable waves, with frequency ω close to zero), but in the case of non-degeneracy of the linearized operator they employ the spherical of harmonics of the Laplacian.

[3]Although a key assumption, namely $b < 2$ has to be revised to $b < \frac{3}{2}$ in the case $n = 3$, more on this below

2.1.3 The fractional case $0 < s < 1$

The case of the fractional Schrödinger operator, that is $s \in (0,1)$, has also received considerable attention in recent years. Regarding the well-posedness for the standard power non-linearity, we mention the work of Dinh, [27] and the references therein. The paper [78] studied the well-posedness of (2.1.1) with $b < 0$. Unfortunately, we are not aware of any local and global well-posedness results for (2.1.1). It looks however that the work [22] seems to contain all necessary ingredients in terms of estimates and one has to proceeds as in [31]. We leave this line of investigation open to other researchers.

Regarding solitary waves for the fractional NLS, the real breakthrough came in the article [35], which deals with the case $b = 0, n = 1, s < 1$ about the existence of positive solution of (2.1.2) has been studied by the authors in [35]. Moreover, the non-degeneracy of the ground state is shown, which plays a very important role in orbital stability of such solutions. In a later work, [36] generalizes the above results in any dimension. More precisely, the uniqueness and non-degeneracy of the ground state solution for $(-\Delta)^s Q + Q - |Q|^{p-1}Q = 0$, with $Q \in H^s(\mathbb{R}^n)$ was established in $\mathbb{R}^n, n \geq 1, s \in (0,1)$ where $1 < p < 1 + \frac{4s}{n-2s}$ for $0 < 2s < n$ and $1 < p < \infty,\ 2s \geq n$.

Our goal is to investigate the existence of the waves Φ, given by (2.1.2), as well as their stability properties. Let us introduce the formally conserved quantities of 2.1.1:

- the L^2 norm

$$\mathcal{P}[u] = \int_{\mathbb{R}^n} |u(x)|^2 dx$$

- the Hamiltonian

$$\mathcal{H}[u] = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |x|^{-b} |u(x)|^{p+1} dx.$$

We will also make use of the total energy functional, defined as follows

$$E[u] := \mathcal{H}[u] + \frac{\omega}{2}\mathcal{P}[u].$$

In fact, a variant of the local well-posedness theory, presented in Theorem 4.6.6 in [16] for the case $s=1$, guarantees that for a data $u_0 \in H^s(\mathbf{R}^n)$, $1<p<1+\frac{4s-2b}{n-2s}$, there exists time $T_0 = T_0(\|u_0\|_{H^s})$, so that a strong solution $u(t,\cdot) \in H^s(\mathbf{R}^n)$ to (2.1.1) exists in $0<t<T_0$ and moreover $\mathcal{P}(u(t)) = \mathcal{P}(u_0), \mathcal{H}(u(t)) = \mathcal{H}(u_0)$.

Next, we discuss the linearization of (2.1.1) around the standing waves $e^{-i\omega t}\Phi_\omega$. We perform a standard linearization procedure, namely we take $u = e^{-i\omega t}[\Phi_\omega + v]$, plug it in (2.1.1) and ignoring the higher order terms $O(v^2)$, we arrive at the linearized system, which after $v = (\Re v, \Im v) =: (v_1, v_2)$ can be written as

$$\begin{pmatrix} \Re v \\ \Im v \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix} \begin{pmatrix} \Re v \\ \Im v \end{pmatrix}, \qquad (2.1.4)$$

where the following fractional Schrödinger operators are introduced

$$\begin{aligned} \mathcal{L}_+ &= (-\Delta)^s + \omega - p|x|^{-b}\Phi^{p-1}, \\ \mathcal{L}_- &= (-\Delta)^s + \omega - |x|^{-b}\Phi^{p-1}. \end{aligned}$$

Note that at this point, the properties of the potential $|x|^{-b}\Phi^{p-1}$ are not yet understood, but one has to definitely address the issue of its singularity at zero. This shall be a major concern going forward. We just mention that for the purposes of the stability considerations, it is convenient on using the standard domain $D(\mathcal{L}_\pm) = H^{2s}(\mathbf{R}^n)$, which will lead to some mild additional, perhaps unnecessary, restrictions on the parameters.

Upon the introduction of the operators

$$\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{L} := \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix},$$

and the assignment $\begin{pmatrix} \Re v \\ \Im v \end{pmatrix} \to e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: e^{\lambda t}\vec{v}$, we obtain the following time-independent linearized eigenvalue problem

$$\mathcal{J}\mathcal{L}\vec{v} = \lambda\vec{v}. \tag{2.1.5}$$

2.1.4 Main results

Before we formally state our results, we need a few rigorous definitions about the objects that we study. We employ the following standard definition of stability.

Definition 4. *We say that the wave $e^{-i\omega t}\Phi$ is spectrally stable, if the eigenvalue problem (2.1.5) has no non-trivial solutions $(\lambda, \vec{v})$, with $\Re\lambda > 0$. Otherwise, in the cases where there is $\lambda : \Re\lambda > 0$ and $\vec{v} \neq 0$, so that (2.1.5) is satisfied, we say that the wave $e^{-i\omega t}\Phi$ is spectrally unstable and λ is referred to as an unstable mode for (2.1.5) .*

We say that $e^{-i\omega t}\Phi$ is orbitally stable, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$, so that whenever $\|u_0 - \Phi\|_{H^s(\mathbf{R}^n)} < \delta$, then the following statements hold.

- *The solution u of (2.1.1), in appropriate sense, with initial data $u_0 \in H^s$ is globally in $H^s(\mathbf{R}^n)$, i.e. $u(t,\cdot) \in H^s(\mathbf{R}^n)$.*
- $$\sup_{t>0} \inf_{\theta\in\mathbf{R}} \|u(t,\cdot) - e^{i(\omega t+\theta)}\Phi(\cdot)\|_{H^s(\mathbf{R}^n)} < \epsilon.$$

Key Assumptions

Let Φ be a solution of (2.1.2). We assume that:

1. The solution map $g \to u_g$ has continuous dependence on initial data property in a neighborhood of Φ. That is, there exists $T_0 > 0$, so that for all $\epsilon > 0$, there exists $\delta > 0$, so that whenever $g : \|g - \Phi\|_{H^s} < \delta$, then $\sup_{0<t<T_0} \|u_g(t, \cdot) - e^{-i\omega t}\Phi_\omega\|_{H^s} < \epsilon$.

2. All initial data, sufficiently close to Φ_ω in H^s norm, generates a global in time solution u_g of (2.1.1). In addition, the L^2 norm and the Hamiltonian for these solutions are conserved. That is

$$\mathcal{P}[u_g(t)] = \mathcal{P}[g], \mathcal{H}[u_g(t)] = \mathcal{H}[g].$$

Remarks:

- The continuity dependence on initial data property stated above is a simple consequence of a standard local well-posedness result, in the spirit of Theorem 4.6.4, [16]. Since such result seems unavailable at the moment, we explicitly assume its veracity.

- There is also the notion of asymptotic stability for our waves. We do not formally introduce herein, as we do not have definite results in this direction. We conjecture it to be true, in all cases of spectral/orbital stability listed in our main theorems below.

Next, we introduce a subset in the parameters space (n, p, s, b), which will be helpful in the sequel

Definition 5. *We say that $(n, p, s, b) \in \mathcal{A}$, if the parameters are in the range below*

$$\mathcal{A} := \begin{cases} n = 1, \ \frac{1}{2} \leq s < 1, \ 0 < b < 1, \ 1 < p < \infty \\ n = 1, \ s \in (0, \frac{1}{2}), \ 0 < b < 2s, \ 1 < p < 1 + \frac{4s-2b}{1-2s} \\ n \geq 2, \ s \in (0, 1), \ 0 < b < 2s, \ 1 < p < 1 + \frac{4s-2b}{n-2s} \end{cases} .$$

This set will turn out to describe the necessary and sufficient conditions under which Φ_ω exists.

Our first theorem is a sufficiency result for the existence of the solitary waves Φ_ω.

Theorem 1. *(Existence results) Let $(n,p,s,b) \in \mathcal{A}$, $\omega > 0$. Then, there exits a bell-shaped function*[4] *$\Phi_\omega \in H^s(\mathbf{R}^n) \cap L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, so that the equation (2.1.2) is satisfied in a distributional sense. If (2.1.2) is also satisfied in the strong sense then*

$$\Phi_\omega = ((-\Delta)^s + \omega)^{-1}[|x|^{-b}\Phi_\omega^p]. \tag{2.1.6}$$

Finally, under the assumption $s \in (\frac{1}{2}, 1]$, we have that $\phi \in C^1(\mathbf{R}^n \setminus \{0\})$.

Remark: We have in fact much more precise description about the behavior of $\phi, \nabla\phi$ in Proposition 4 below.

Interestingly, we have the appropriate converse statement, which makes $\mathcal{A}$ the necessary and sufficient set of requirements for the solvability of (2.1.2).

Theorem 2. *Assume that a positive function $\psi \in H^s(\mathbf{R}^n) \cap L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ satisfies*

$$(-\Delta)^s\psi + \omega\psi = |x|^{-b}\psi^p$$

in a distributional sense. Then $(n,p,s,b) \in \mathcal{A}$ and $\omega > 0$.

Next, we are concerned with the stability of the waves constructed in Theorem 1.

Theorem 3. *Let $(n,p,s,b) \in \mathcal{A}$ and $\omega > 0$. In addition, assume that $2b < n$ and $s \in (\frac{1}{2}, 1]$. Let Φ_ω be the solution constructed in Theorem 1. Then,*

1. *the linearized operators $\mathcal{L}_\pm, D(\mathcal{L}_\pm) = H^{2s}(\mathbf{R}^n)$ are self-adjoint and $\Phi_\omega \in D(\mathcal{L}_+)$.*

[4] That is, a radial function, which is non-increasing in the radial variable

2. Φ_ω non-degenerate, in the sense that $Ker[\mathcal{L}_+] = \{0\}$.

For $1 < p < 1 + \frac{4s-2b}{n}$ the soliton $e^{-i\omega t}\Phi_\omega$ is spectrally and orbitally stable. In the complementary range,

$$1 + \frac{4s-2b}{n} < p < \begin{cases} \infty & n = 1 \\ 1 + \frac{4s-2b}{n-2s} & n \geq 2, \end{cases}$$

the soliton is spectrally unstable.

Remarks:

1. According to the necessity statements in Theorem 2, the results in Theorem 3 provide a full classification of the bell-shaped solutions that exists, in the cases $s \in (\frac{1}{2}, 1)$ and $2b < n$. Note that the constraint $2b < n$ is already contained in the necessity assumption for $n \geq 4$.

2. In the case $n = 3$, the constraint $b < \frac{3}{2}$ is slightly worse than the necessity assumptions, $b < 2$. This was the claim in [10], but one certainly faces some difficulties (specifically with $D(\mathcal{L}_+)$) in the range $b \in (\frac{3}{2}, 2)$.

3. Our results seem to be new even in the case $s = 1$, in low dimensions, $n = 1, 2$. The restrictions $b < \frac{1}{2}$ for $n = 1$ and $b < 1$ for $n = 2$ are more restrictive than the necessary assumptions $(n, p, s, b) \in \mathcal{A}$. It is interesting whether one can establish rigorously the stability situation for these parameters. As we discuss at length, the main issue is to make sense of the functional analytic framework, in particular the domains of the linearized operators $\mathcal{L}_\pm$.

4. The case $p = \frac{4s-2b}{n}$ is a bifurcation case, where one gets a crossing through zero of a pair of purely imaginary eigenvalues to a pair of stable/unstable real eigenvalues. This is also where the equation (2.1.1) enjoys an extra, so called pseudo-conformal

symmetry, hence the extra pair of eigenvalues at zero. Even though one has spectral stability for this case, one generally expects the corresponding waves to be spectrally unstable, as in the classical NLS, see the seminal paper [24] for details.

This chapter is planned as follows. In Section 2.2, we introduce the Pohozaev's identities, which in turn imply the necessary conditions for the existence of the waves, which is the content of Theorem 2. In Section 2.3, we present the variational construction of the waves along with some further properties of the profiles, such as boundedness, sharp asymptotics at zero and smoothness. In Section 2.4, we provide a self-adjoint realization of the linearized operators $\mathcal{L}_{\pm}$, followed by some preliminary coercivity properties. We also introduce the Frank-Lenzman-Silvestre Sturm oscillation theory for fractional Schrödinger operators as well as an adaptation of their method to our situation, which has to address singular potentials in the next section. In Section 2.5, we establish the non-degeneracy of the waves. This requires decomposition in spherical harmonics and careful analysis on the radial subspace by using the Frank-Lenzman-Silvestre theory developed in the previous section as well as an argument to rule out non-trivial elements in the first harmonic subspace. In Section 2.6, we provide a short introduction to the index counting theory, which provide an useful criteria for spectral stability. In Propositions 10 and 11, we show the coercivity of $\mathcal{L}_{\pm}$ on $\{\Phi\}^{\perp}$, which is an important ingredient of the orbital stability scheme. Finally, we show the orbital stability (whenever spectral stability holds) in Proposition 12.

2.2 Necessary conditions for the waves: proof of Theorem 1

The approach for the proof of Theorem 1 is to exploit the scaling and the associated Pohozaev's identities, which in due course will lead us to the set of constraints $\mathcal{A}$.

2.2.1 Pohozaev identities and consequences

Before we make assumptions on the smoothness and decay properties of the profiles ϕ, and in addition the sense in which (2.1.2) is satisfied, (2.1.2) remains a formal object. In order to further demystify the ranges in which one might expect reasonable solutions of (2.1.2), we provide the following Pohozaev type identities.

Lemma 4. *(Pohozaev identities) Assume that $0 < b < n$ and $\psi \in H^s(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$, with $\psi > 0$ satisfies*

$$(-\Delta)^s \psi + \omega\psi - |x|^{-b}\psi^p = 0 \tag{2.2.1}$$

in a distributional sense. Then,

$$\int_{\mathbf{R}^n} |x|^{-b}\psi^{p+1}dx = \frac{2ws(p+1)}{2(n-b)-(n-2s)(p+1)}\int_{\mathbf{R}^n}\psi^2 dx. \tag{2.2.2}$$

$$\int_{\mathbf{R}^n} |(-\Delta)^{s/2}\psi|^2 dx = \frac{w(n(p+1)-2(n-b))}{2(n-b)-(n-2s)(p+1)}\int_{\mathbf{R}^n}\psi^2 dx. \tag{2.2.3}$$

$$\omega\int_{\mathbf{R}^n}\psi(x)dx = \int_{\mathbf{R}^n}|x|^{-b}\psi^p dx. \tag{2.2.4}$$

Proof. A formal proof (i.e. one where we assume that ψ has enough smoothness and decay properties) is as follows. Take a dot product with ψ in (2.2.1) and integrating by part we get

$$\int |(-\Delta)^{s/2}\psi|^2 dx + \omega\int\psi^2(x)dx = \int |x|^{-b}\psi^{p+1}(x)dx.$$

If we take a dot product with $x\cdot\nabla_x\psi = \sum_{j=1}^n x_j\partial_j\psi$, taking into account the commutation formula (1.1.1) and various integration by parts calculations, we obtain another relation between $\int|(-\Delta)^{s/2}\psi|^2dx$ and $\int|x|^{-b}\psi^{p+1}(x)dx$, namely

$$(s-\frac{n}{2})\int|(-\Delta)^{s/2}\psi|^2dx + \frac{n-b}{p+1}\int|x|^{-b}\psi^{p+1}(x)dx = \frac{n\omega}{2}\int\psi^2(x)dx.$$

Solving the last two relations for $\int |(-\Delta)^{s/2}\psi|^2 dx, \int |x|^{-b}\psi^{p+1}(x)dx$, we obtain (2.2.2), (2.2.3). Integrating (2.2.1) yields (2.2.4).

For ψ, which is not necessarily smooth and decaying, one follows similar scheme. To establish (2.2.2), test the equation (2.2.1) by a sequence of Schwartz function ψ_N with $\lim_N \|\psi_N - \psi\|_{H^s(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)} = 0$ and then take limits. In order to show (2.2.3), test (2.2.1) by $x\cdot\nabla\psi_N$. Again taking into account the commutation relation $[(-\Delta)^s, x\cdot\nabla] = 2s(-\Delta)^s$ and taking limits as $\psi_N \to \psi$ establishes (2.2.3). The formula (2.2.4) is proved after testing (2.2.1) by a function $\chi(x/N), N >> 1$ (where χ is compactly supported and $\chi(x) = 1, |x| < 1$) and taking limits $N \to \infty$.

□

Implicit in the formulas (2.2.2), (2.2.3) displayed above is that the parameters need to satisfy certain conditions, so that ψ exists. We collect the necessary conditions in the following corollary.

Corollary 2. *Let $p > 1$, $n \geq 1, s \in (0,1), b > 0$. If ψ with properties listed in Lemma 4 exist, then $\omega > 0$ and the parameters must satisfy one of the following relations:*

- $n = 1$, $s \in [\frac{1}{2}, 1)$, $0 < b < 1$, $1 < p < \infty$.
- $n = 1$, $0 < s < \frac{1}{2}$, $b < 2s$,
$$1 < p < 1 + \frac{4s - 2b}{1 - 2s}.$$
- $n \geq 2$, $b < 2s$,
$$1 < p < 1 + \frac{4s - 2b}{n - 2s}. \tag{2.2.5}$$

Remark: Corollary 2 simply states that if ψ solves (2.2.1), then $(n, p, s, b) \in \mathcal{A}$.

Proof. The fact that $\omega > 0$ follows from (2.2.4). If $\psi(0) > 0$ and the integral on the left-hand side of (2.2.2) exists, it is non-singular at zero and hence $b < n$.

From the positivity of the left-hand sides of (2.2.2), (2.2.3) and $n(p+1)-2(n-b)=n(p-1)+2b>0$, it follows that $2(n-b)-(n-2s)(p+1)>0$. In particular, for $n=1$, the conditions are satisfied if $s\geq\frac{1}{2}, 1<p<\infty$ or $0<s<\frac{1}{2}$, but then $2s>b$, $1<p<1+2\frac{2s-b}{1-2s}$.

For $n\geq 2$, note that we always have $n-2s>0$, whence we come up with $b<2s$ and (2.2.5). □

Clearly, Corollary 2 establishes Theorem 2.

2.3 The Variational Construction and properties of the minimizers

We start with some elementary observations, which will identify conditions under which an important variational problem is well-posed.

2.3.1 Well-posedness of the variational problem

Consider the following functional

$$I_\omega[u]=\frac{\int_{\mathbb{R}^n}|(-\Delta)^{s/2}u|^2dx+\omega\int_{\mathbb{R}^n}u^2dx}{\left(\int_{\mathbb{R}^n}|x|^{-b}|u|^{p+1}\right)^{\frac{2}{p+1}}dx}.$$

We shall henceforth assume[5] that $b<n$, $\omega>0$ and $0<s<1$. So, for any function $u\in H^s(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n): u\neq 0$, we have that $0<\int_{\mathbb{R}^n}|x|^{-b}|u|^{p+1}dx<\infty$, so that the ratio $I_\omega[u]$ is well-defined. Since for $u\in\mathcal{S}$ For every $u\neq 0$, $I_\omega[u]>0$, we will consider the non-negative scalar function

$$m(\omega):=\inf_{u\in\mathcal{S}}I_\omega[u].$$

[5]and in fact, we shall pose some more restrictions later on

In the case when the parameters ensure that $m(\omega) > 0$, will be referred to well-posedness, versus the trivial case $m(\omega) = 0$ (which is certainly possible for certain parameter ranges) will be referred to as lack of well-posedness or ill-posedness. We have the following elementary lemma.

Lemma 5. *Assume that* $m(1) > 0$. *Then,*

$$m(\omega) = m(1)\omega^{\frac{(n-2s)}{2s(p+1)}[p-(1+\frac{4s-2b}{n-2s})]}. \tag{2.3.1}$$

In addition, if ϕ *is a minimizer for* $I_1[u] \to \min$*, i.e.* $m(1) = I_1(\phi)$*, then* $\phi_\omega(x) := \phi(\omega^{\frac{1}{2s}}x)$ *is a minimizer for* $I_\omega[u] \to \min$.

Proof. Take $\phi(x) = \psi(\lambda x)$ then

$$I_\omega[\phi] = \frac{\lambda^{-n+2s}\|(-\Delta)^{s/2}\psi\|^2 + \omega\lambda^{-n}\|\psi\|^2}{\lambda^{2(\frac{n-b}{p+1})}\left(\int_{\mathbf{R}^n}|x|^{-b}\psi^{p+1}\right)^{\frac{2}{p+1}}dx}.$$

Taking $\omega = \lambda^{2s}$ implies the formula

$$I_\omega[\phi] = \omega^{\frac{-n+2s-\frac{2(n-b)}{p+1}}{2s}} I_1(\psi),$$

whence the formula (2.3.1) follows by straightforward algebraic manipulations. □

Remarks:

- As was have discussed above, the well-posedness is equivalent to $m(1) > 0$. So far, we have not addressed this issue in a satisfactory manner. Lemma 5 just establishes that m is a specific power function, if the functional I_ω is bounded from a positive constant.

- Note however that under the standing assumptions $s > 0$, $p > 1$, the power of ω appearing in (2.3.1) is negative exactly when $(n,p,s,b) \in \mathcal{A}$.

2.3.2 Existence of minimizers

Our next goal is to obtain an existence result, which holds precisely when $(n,p,s,b) \in \mathcal{A}$. As is clear from Proposition 1, it suffices to consider the case $\omega = 1$.

Proposition 2. *Let $(n,p,s,b) \in \mathcal{A}$. Then the unconstrained minimization problem*

$$I_\omega[u] \to \min \tag{2.3.2}$$

has a bell-shaped solution $\phi \in H^s(\mathbf{R}^n) \cap L^{p+1,-b}$, in particular $m(\omega) > 0$.

If ϕ is a minimizer of (2.3.2), with $\|\phi\|_{L^{p+1,-b}} = 1$, then ϕ satisfies the Euler-Lagrange equation

$$(-\Delta)^s \phi + \omega\phi - m(\omega)|x|^{-b}\phi^p = 0 \tag{2.3.3}$$

in the following weak sense: for each $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, there is $\langle (-\Delta)^s\phi + \omega\phi - m(\omega)|x|^{-b}\phi^p, h\rangle = 0$. Finally, for the linearized operator,

$$\mathcal{L}_+ = (-\Delta)^s + \omega - pm(\omega)|x|^{-b}\phi^{p-1},$$

we have that for each real-valued $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\}) : \int |x|^{-b}\phi^p(x)h(x)dx = 0$, $\langle \mathcal{L}_+ h, h\rangle \geq 0$.

Remark:

- Proposition 2 does not claim the boundedness of the minimizer ϕ, i.e. the possibility that $\lim_{x\to 0}\phi(x) = \infty$ is left open.

- Related to the previous point, the Euler-Lagrange equation may have a significant singularity at zero, due to the presence of $|x|^{-b}$ and the possible singularity of ϕ at zero. We sidestep the issue for the moment, by testing (2.3.3) away from zero as $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$.

- The non-negativity property of $\mathcal{L}_+$ over the set $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\}), h \perp |x|^{-b}\phi^p$, normally would indicate that $\mathcal{L}_+$ has at most one negative eigenvalue. This would eventually turn out to be the case, see Proposition 5. Here, we are forced to restrict to a restricted set of test functions, namely $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, as we have not yet resolved the issue with the singularity of the potential $x \to |x|^{-b}\phi^p(x)$ at zero.

Proof. By the arguments in Lemma 5, it suffices to consider the case $\omega = 1$. By the assumption $(n,p,s,b) \in \mathcal{A}$, it follows from Proposition 1

$$\left(\int |x|^{-b}\phi^{p+1}\right)^{\frac{2}{p+1}} \leq C\|\phi\|_{H^s}^2.$$

Whence

$$\inf_{u \neq 0} I_1[u] \geq C^{-1}.$$

Thus, the variational problem (2.3.2) is well-posed or equivalently $m(1) > 0$.

We now need to show that (2.3.2) actually has a solution. To that end, observe that by the Polya-Szegö inequality (1.1.4), $\|(-\Delta)^{s/2}u\| \geq \|(-\Delta)^{s/2}u^*\|$. Also, $\|\phi^*\|_{L^2} = \|\phi\|_{L^2}$ and finally, by (1.1.3) and the fact that $|\cdot|^{-b}$ is bell-shaped and strictly decreasing,

$$\int_{\mathbf{R}^n} |x|^{-b}|\phi(x)|^{p+1}dx \leq \int_{\mathbf{R}^n} |x|^{-b}(|\phi(x)|^{p+1})^* dx = \int_{\mathbf{R}^n} |x|^{-b}(\phi^*(x))^{p+1}dx.$$

We conclude that $I_1[u] \geq I_1[u^*]$, which implies that we can reduce the set of possible minimizers to the set of bell-shaped functions, i.e. $\{u \in H^s(\mathbf{R}^n) \cap L^{p+1,b}(\mathbf{R}^n) : u = u^*\}$.

Next, by the dilation property of the functional $I_1(u) = I_1(au)$, we can without loss of generality further reduce to the set $\int_{\mathbf{R}^n} |x|^{-b} u^{p+1}(x)dx = 1$.

So, assume that ϕ_k is a minimizing sequence of bell-shaped functions, subject to the constraint $\int_{\mathbf{R}^n} |x|^{-b} \phi_k^{p+1}(x)dx = 1$. It follows that

$$\lim_k \|(-\Delta)^{s/2}\phi_k\|_{L^2}^2 + \|\phi_k\|_{L^2}^2 = m(1). \tag{2.3.4}$$

We will show that a subsequence of ϕ_k converges in the strong $H^{s/2}(\mathbf{R}^n)$ sense to a minimizer u, which we will show is the desired solution to the minimization problem (2.3.2). By weak compactness, we have that a subsequence of ϕ_k (which we will assume without loss of generality is ϕ_k itself) tends weakly in $H^{s/2}(\mathbf{R}^n)$ to a function ϕ, which is also trivially bell-shaped.

Since, for bell-shaped functions u we have the point-wise bound for each $x : |x| = R$,

$$|u(x)|^2 \leq |B_n|^{-1} R^{-n} \int_{|y|\leq R} |u(y)|^2 dy \leq |B_n|^{-1}|x|^{-n}\|u\|_{L^2}^2. \tag{2.3.5}$$

Based on this, we claim that (a subsequence of) ϕ_k converges to ϕ strongly in the topology of $L^{p+1,-b}$. This will follow from the Kolmogorov-Relich-Riesz criteria for compactness in L^p spaces from $\sup_k \|\phi_k\|_{H^{s/2}(\mathbf{R}^n)} < \infty$ (which is a corollary of (2.3.4)) and once we establish

$$\lim_N \sup_k \int_{|x|>N} |x|^{-b}|\phi_k(x)|^{p+1}dx = 0. \tag{2.3.6}$$

Indeed, (2.3.6) follows from the pointwise bounds for bell-shaped functions (2.3.5), since

$$\sup_k \int_{|x|>N} |x|^{-b}|\phi_k(x)|^{p+1}dx \leq C_n \sup_k \|\phi_k\|_{L^2}^{p+1} \int_{|x|>N} |x|^{-b-(p+1)\frac{n}{2}} dx$$

$$\leq C_n N^{-b-\frac{p-1}{2}n} \sup_k \|\phi_k\|_{L^2}^{p+1},$$

which clearly converges to zero as $N \to \infty$. Thus, up to a subsequence $\|\phi_k - \phi\|_{L^{p+1,-b}} \to 0$, whence $\int_{\mathbf{R}^n} |x|^{-b}\phi^{p+1}(x)dx = 1$. In particular, $I_1(\phi) = \|(-\Delta)^{s/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 \geq m(1)$.

Now, we have by the lower semicontinuity of the weak convergence in $H^{s/2}$ and (2.3.4) that

$$m(1) \leq \|(-\Delta)^{s/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 \leq \liminf_k \|(-\Delta)^{s/2}\phi_k\|_{L^2}^2 + \|\phi_k\|_{L^2}^2 = m(1).$$

It follows that $\lim_k \|(-\Delta)^{s/2}\phi_k\|_{L^2}^2 + \|\phi_k\|_{L^2}^2 = \|(-\Delta)^{s/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2$, whence by the uniform convexity of $\|\cdot\|_{L^2}$

$$\lim_k \|\phi_k - \phi\|_{H^{s/2}(\mathbf{R}^n)} = 0.$$

We conclude that $I_1[\phi] = m(1)$ and ϕ is a solution to (2.3.2).

Next, we discuss the Euler-Lagrange equation (2.3.3). Take a test function $h \in V_0^\infty(\mathbf{R}^n \setminus \{0\}$, that is h is supported in $\{x : |x| > \delta\}$ for some $\delta > 0$. Let also $0 < \epsilon << 1$ and consider $u = \phi + \epsilon h$. Recall $\int |x|^{-b}\phi^{p+1}dx = 1$. Since ϕ is a minimizer we should have

$$I_\omega[\phi + \epsilon h] \geq m(1) = N(\phi).$$

Where $N(\phi) := \int |(-\Delta)^{s/2}\phi|^2 + \int \phi^2$ and $D(\phi) := \int |x|^{-b}(\phi)^{p+1}dx$. Thus,

$$\begin{aligned} N(\phi+\epsilon h) &= \int |(-\Delta)^{s/2}(\phi+\epsilon h)|^2 + \int (\phi+\epsilon h)^2 \\ &= \int |(-\Delta)^{s/2}\phi + \epsilon(-\Delta)^{s/2}h|^2 + \int (\phi^2 + 2\epsilon h\phi + \epsilon^2 h^2) \\ &= \int |(-\Delta)^{s/2}\phi|^2 + \int \phi^2 + 2\epsilon(\langle(-\Delta)^{s/2}\phi, (-\Delta)^{s/2}h\rangle + \langle h, \phi\rangle) + O(\epsilon^2) \end{aligned}$$

$$= N(\phi) + 2\epsilon\langle((-\Delta)^s + 1)\phi, h\rangle + O(\epsilon^2).$$

Similarly,

$$D(\phi + \epsilon h) = \int |x|^{-b}(\phi + \epsilon h)^{p+1} dx = 1 + (p+1)\epsilon\langle|x|^{-b}\phi^p, h\rangle + O(\epsilon^2).$$

It follows that

$$\begin{aligned} I_1(\phi+\epsilon h) &= \frac{N(\phi+\epsilon h)}{D[\phi+\epsilon h]^{\frac{2}{p+1}}} = \frac{N(\phi)+2\epsilon\langle((-\Delta)^s+1)\phi,h\rangle+O(\epsilon^2)}{1+2\epsilon\langle|x|^{-b}\phi^p,h\rangle+O(\epsilon^2)} \\ &= N[\phi]+2\epsilon\langle((-\Delta)^s+1)\phi-|x|^{-b}N(\phi)\phi^p,h\rangle+O(\epsilon^2). \end{aligned}$$

As this holds for arbitrary function h and for all small ϵ, we have established that ϕ solves (2.3.3) in a distributional sense.

Finally, fix h to be a real-valued function, $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$. Starting again with the inequality

$$\frac{N(\phi+\epsilon h)}{D(\phi+\epsilon h)^{\frac{2}{p+1}}} \geq N(\phi),$$

but expanding to the second order[6] ϵ^2 , we obtain

$$N[\phi] + \epsilon^2[\langle\mathcal{L}_+ h, h\rangle + N[\phi](p+3)(\langle|\cdot|^{-b}\phi^p, h\rangle)^2] + O(\epsilon^3) \geq N[\phi],$$

after taking into account $\langle((-\Delta)^s+1)\phi - N(\phi)|x|^{-b}\phi^p, h\rangle = 0$. After taking limits as $\epsilon \to 0$, we derive

$$\langle\mathcal{L}_+ h, h\rangle \geq -N[\phi](p+3)(\langle|\cdot|^{-b}\phi^p, h\rangle)^2. \tag{2.3.7}$$

In particular, $\langle\mathcal{L}_+ h, h\rangle \geq 0$, if $\int |x|^{-b}\phi^p(x)h(x)dx = 0$. □

[6]Note that in the calculation above, the expansion in powers of ϵ is valid, since the fixed h that has its support away from zero

We shall now need to prove some further properties of the minimizers ϕ as well as some spectral results necessary for the sequel.

2.3.3 Boundedness of ϕ

In our next result, we use the already established (partial) coercivity of $\mathcal{L}_+$ on $\{|\cdot|^{-b}\phi^p\}^\perp \cap C_0^\infty(\mathbf{R}^n \setminus \{0\})$ in order to derive L^∞ bounds on ϕ. We believe that this is a new technique, which might be useful in the spectral analysis of other situations with singular potentials.

Once we show the boundedness of ϕ, we will go back to the claim about the coercivity of $\mathcal{L}_+$ on the full co-dimension one subspace $\{|\cdot|^{-b}\phi^p\}^\perp$.

Proposition 3. *Let $(n,s,p,b) \in \mathcal{A}$. Then, the minimizer ϕ constructed in Proposition 2 is a bounded function.*

Proof. Again, we assume $\omega = 1$, the other cases follow by rescaling.

We first show the boundedness of ϕ. Recall that since ϕ is a bell-shaped function, $\phi \in L^2(\mathbf{R}^n)$, we have that for every $x \neq 0$, $|\phi(x)| \leq C_n |x|^{-\frac{n}{2}} \|\phi\|_{L^2}$. This of course leaves the possibility that $\lim_{x\to 0}\phi(x) = \infty$, which we shall rule out for the remainder of the proof.

Our approach is by contradiction, that is assume that $\lim_{|x|\to 0}\phi(x) = \infty$. We now create a specifically designed test function $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\}) \cap \{|x|^{-b}\phi^p\}^\perp$. To this end, let χ be a radial positive C_0^∞ test function, supported in $\frac{1}{2} < |x| < 2$ and equal to 1 on $\frac{3}{4} < |x| < \frac{4}{3}$. Let $0 < \epsilon << 1$ and let

$$h(x) := \chi(x/\epsilon) - c_\epsilon \chi(x), \quad c_\epsilon = \frac{\int |x|^{-b}\phi^p(x)\chi(x/\epsilon)dx}{\int |x|^{-b}\phi^p(x)\chi(x)dx}.$$

Clearly, $h \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, where c_ϵ is designed so that $h \perp |\cdot|^{-b}\phi^p$. Note that the denominator of c_ϵ is bounded above and below by a constant independent on ϵ, so that

$$c_\epsilon \sim \int |x|^{-b}\phi^p(x)\chi(x/\epsilon)dx. \tag{2.3.8}$$

According to Proposition 2, we have that $\langle \mathcal{L}_+ h, h\rangle \geq 0$. As a consequence of this, after dropping some terms with favorable signs, we arrive at

$$\begin{aligned} c_\epsilon^2\langle(-\Delta)^s\chi,\chi\rangle - 2c_\epsilon\langle(-\Delta)^s\chi,\chi(\cdot/\epsilon)\rangle + \|(-\Delta)^{s/2}\chi(\cdot/\epsilon)\|^2 \\ \geq pm(1)\int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)dx. \end{aligned} \tag{2.3.9}$$

Let us estimate the terms on the left hand side of (2.3.9). Elementary estimates imply

$$\langle(-\Delta)^s\chi,\chi\rangle \leq C, \|(-\Delta)^{s/2}\chi(\cdot/\epsilon)\|^2 \leq C\epsilon^{n-2s}, c_\epsilon|\langle(-\Delta)^s\chi,\chi(\cdot/\epsilon)\rangle| \leq C\epsilon^n c_\epsilon.$$

The integral expression on the right hand side of (2.3.9) is essentially equivalent to c_ϵ, but not quite. In order to get the desired estimate, introduce the quantity $d_\epsilon := \int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)dx$, so that we now have proved the estimate

$$d_\epsilon \leq C(c_\epsilon^2 + \epsilon^{n-2s} + \epsilon^n c_\epsilon). \tag{2.3.10}$$

Furthermore, we have by Cauchy-Schwarz's inequality

$$c_\epsilon \leq C\int |x|^{-b}\phi^p(x)\chi(x/\epsilon)dx \leq C\left(\int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)dx\right)^{1/2}\left(\int_{|x|\sim\epsilon} |x|^{-b}\phi^p(x)dx\right)^{1/2}. \tag{2.3.11}$$

By our assumption, $\lim_{x\to 0}|\phi(x)| = \infty$, we have that for all small enough ϵ

$$\int_{|x|\sim\epsilon} |x|^{-b}\phi^p(x)dx \leq \frac{1}{\max_{x:|x|\sim\epsilon}\phi(x)}\int |x|^{-b}\phi^{p+1}(x)dx = \frac{1}{\max_{x:|x|\sim\epsilon}\phi(x)} = o(\epsilon).$$

Hence, we obtain that $c_\epsilon^2 = o(\epsilon)d_\epsilon$ and $\epsilon^n c_\epsilon \leq o(\epsilon)d_\epsilon + \epsilon^{2n}$. Substituting these estimates in (2.3.10) yields $d_\epsilon \leq Co(\epsilon)d_\epsilon + \epsilon^{n-2s}$, or after hiding $Co(\epsilon)d_\epsilon$ on the left-hand side, $d_\epsilon \leq 2\epsilon^{n-2s}$, for all small enough ϵ. This actually yields a very good point-wise estimate on ϕ. Indeed, recalling that ϕ is bell-shaped we estimate

$$c\epsilon^{n-b}\min_{x:|x|\sim\epsilon}\phi^p(x) \leq \int |x|^{-b}\phi^p(x)\chi^2(x/\epsilon)dx \leq C\epsilon^{n-2s},$$

whence for all $x \neq 0$,

$$\phi^p(x) \leq C|x|^{b-2s}. \tag{2.3.12}$$

This gives a contradiction and hence the required L^∞ bound, if $b \geq 2s$. Unfortunately, this covers only a small portion of the parameters space $\mathcal{A}$.

So, assume for the rest of the argument that $b < 2s$. In order to derive the L^∞ bounds for ϕ, in the case $b < 2s$, we shall need an additional bootstrap argument, based on the fact that ϕ is a (weak) solution of the Euler-Lagrange equation (2.3.3). To this end, we need to find a way to introduce $\tilde{\phi} := (1+(-\Delta)^s)^{-1}[|\cdot|^{-b}\phi^p]$. As of now, this is a *formal definition*, but it is clear that if we manage to define such an object in an appropriate way, this will be weak solution of (2.3.3). Since ϕ solves (2.3.3) in the weak sense described in Proposition 2, we will be eventually able to show that $\tilde{\phi} = \phi$ as L^q functions, for appropriate $q \in (2,\infty)$. To this end, we have the following claim.

Claim 1. *Assume* $(n,s,p,b) \in \mathcal{A}$ *and that a function* $f : \mathbf{R} \to \mathbf{R}$ *is bell-shaped and it satisfies* $f \in L^{p+1,-b}(\mathbf{R}^n)$ *and* $|f(x)| \leq C|x|^{\frac{b-2s}{p}}$. *Then,*

$$\tilde{z} = (1+(-\Delta)^s)^{-1}[|\cdot|^{-b} f^p] := G_s * [|\cdot|^{-b} f^p] \in \cap_{\frac{p+1}{p}<q} L^q(\mathbf{R}^n).$$

In particular $\tilde{z} \in L^2(\mathbf{R}^n)$.

Proof. (Claim 1) We consider the case $n > 2s$ only, as the case $n \leq 2s$ can arise only for $n = 1$, $s > \frac{1}{2}$, in which case the function G_s is bounded and the arguments are much simpler.

We split[7] $\tilde{z} = \tilde{z}_1 + \tilde{z}_2$

$$\tilde{z}_1 = G_s * [|\cdot|^{-b} f^p \chi_{|\cdot|<1}], \quad \tilde{z}_2 = G_s * [|\cdot|^{-b} f^p \chi_{|\cdot|\geq 1}].$$

Let us analyze $\tilde{z}_1$ first. We claim that due to the properties established in Lemma 1, we have that $\tilde{z}_1 \in \cap_{q<\infty} L^q(\mathbf{R}^n)$. Indeed, for $|x| < 2$, we can bound

$$|\tilde{z}_1(x)| \leq C|\cdot|^{2s-n}\chi_{|\cdot|<3} * |\cdot|^{-2s}\chi_{|\cdot|<1}.$$

Pick arbitrary $q_1, q_2 : 1 < q_1 < \frac{n}{n-2s}$, $1 < q_2 < \frac{n}{2s}$ and then $q \in (1,\infty) : \frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q}$. By Hardy-Littlewood-Sobolev inequality, we have

$$\|\tilde{z}_1\|_{L^q(|x|<2)} \leq C \||\cdot|^{2s-n}\chi_{|\cdot|<3}\|_{L^{q_1}(\mathbf{R}^n)} \||y|^{-2s}\chi_{|\cdot|<1}\|_{L^{q_2}(\mathbf{R}^n)} \leq C_q.$$

Clearly, in this way, we can generate any $q \in (1,\infty)$, by varying the choices q_1, q_2 in the specified intervals, so $\tilde{z}_1 \in \cap_{1<q<\infty} L^q(\mathbf{R}^n)$.

[7] here χ_I denotes the characteristic function of I

Regarding $\tilde{z}_2$, we split as follows

$$|\tilde{z}_2| \leq C[|\cdot|^{2s-n}\chi_{|\cdot|<1} * |\cdot|^{-b} f^p \chi_{|\cdot|\geq 1} + |\cdot|^{-n}\chi_{|\cdot|\geq 1} * |\cdot|^{-b} f^p \chi_{|\cdot|\geq 1}].$$

Clearly,

$$\||\cdot|^{2s-n}\chi_{|\cdot|<1} * |\cdot|^{-b} f^p \chi_{|\cdot|\geq 1}\|_{L^q} \leq C\||\cdot|^{2s-n}\chi_{|\cdot|<1}\|_{L^1}\||\cdot|^{-b} f^p \chi_{|\cdot|\geq 1}\|_{L^q} \leq C$$

as long as $\frac{p+1}{p} \leq q < \infty$, because

$$\||\cdot|^{-b} f^p \chi_{|\cdot|\geq 1}\|_{L^q}^q \leq \max_{|x|>1} |f^{qp-(p+1)}(x)| \int_{\mathbf{R}^n} |y|^{-b} f^{p+1}(y) dy \leq C.$$

Similarly, as long as $\frac{p+1}{p} < q < \infty$, we can find $\delta > 0$, so that $\frac{1}{1+\delta} + \frac{1}{q_\delta} = 1 + \frac{1}{q}$ and $q_\delta > \frac{p+1}{p}$. Then,

$$\||\cdot|^{-n}\chi_{|\cdot|\geq 1} * |\cdot|^{-b} f^p \chi_{|\cdot|\geq 1}\|_{L^q} \leq C\|\||\cdot|^{-n}\chi_{|\cdot|\geq 1}\|_{L^{1+\delta}}\||\cdot|^{-b} f^p \chi_{|\cdot|\geq 1}\|_{L^{q_\delta}} \leq C.$$

All in all, we have established $\tilde{z} \in \cap_{\frac{p+1}{p}<q<\infty} L^q(\mathbf{R}^n)$, as required. □

Now that we have established the claim and taking into account the properties of ϕ, which are already established, we can take $f = \phi$ in the Claim 1, whence we conclude that

$$\tilde{\phi} = (1 + (-\Delta)^s)^{-1}[|\cdot|^{-b}\phi^p]$$

is well-defined and element of $L^2(\mathbf{R}^n)$. Furthermore, for each integer k and each test function $f \in \mathcal{S}_k = \{f \in \mathcal{S} : supp\hat{f} \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}\}$, we have that

$$\langle \tilde{\phi}, (1 + (-\Delta)^s)^{-1} f\rangle = \langle |\cdot|^{-b}\phi^p, f\rangle = \langle \phi, (1 + (-\Delta)^s)^{-1} f\rangle,$$

where in the first equality we have used the definition of $\tilde{\phi}$, while in the second, we have used that ϕ is a weak solution of (2.3.3).

Since $(1+(-\Delta)^s)^{-1}$ is an isomorphism on each $\mathcal{S}_k$, it follows that $\langle\tilde{\phi}-\phi, f\rangle = 0$ for each $f \in \mathcal{S} : supp\hat{f} \subset \mathbf{R}^n \setminus \{0\}$. Since this is a dense set in $\mathcal{S}$ and hence in each $L^q, q \in [1,\infty)$, it follows that $\tilde{\phi} = \phi$ in the sense of $L^2(\mathbf{R}^n)$, that is

$$\phi = (1+(-\Delta)^s)^{-1}[|\cdot|^{-b}\phi^p] = G_s * [|\cdot|^{-b}\phi^p] \in L^2(\mathbf{R}^n). \tag{2.3.13}$$

According to the claim, the $L^2(\mathbf{R}^n)$ function on the right-hand side of (2.3.13) also belongs to $\cap_{\frac{p+1}{p}<q} L^q(\mathbf{R}^n)$. But then, since ϕ is bell-shaped and $\phi \in \cap_{\frac{p+1}{p}<q} L^q(\mathbf{R}^n)$, we have the point-wise bound

$$|x|^n|\phi(x)|^q \leq C\int_{|y|\sim|x|}|\phi(y)|^q dy \leq C_{q,n}\|\phi\|^q_{L^q(\mathbf{R}^n)}.$$

Whence $\phi(x) \leq C_q|x|^{-\frac{n}{q}}$. Recall that this is true for all $q < \infty$. That is, for each $\delta > 0$, there is C_δ, so that

$$\phi(x) \leq C_\delta|x|^{-\delta}. \tag{2.3.14}$$

This is almost, but not quite, that $\phi \in L^\infty(\mathbf{R}^n)$, which will yield the contradiction. On the other hand, we will show that (2.3.14) can be bootstrapped to $\phi \in L^\infty(\mathbf{R}^n)$, which will then be the desired contradiction.

By close inspection of the proof of Claim 1 (and under the assumptions in Claim 1) , we see that we can in fact place all but one piece in $L^\infty(\mathbf{R}^n)$. It thus remains to see why $|\cdot|^{2s-n}\chi_{|\cdot|<3} * |\cdot|^{-b}\phi^p\chi_{|\cdot|<1} \in L^\infty(\mathbf{R}^n)$. In view of the bound (2.3.14), we have for $\delta << 1$,

$$\begin{aligned}|\cdot|^{2s-n}\chi_{|\cdot|<3} * |\cdot|^{-b}\phi^p\chi_{|\cdot|<1}(x)| &\leq C\int\frac{\chi_{|x-y|<3}}{|x-y|^{n-2s}}\frac{\chi_{|y|<1}}{|y|^{b+\delta}}dy \\ &\leq C\||\cdot|^{2s-n}\chi_{|\cdot|<3}\|_{L^q}\|\chi_{|y|<1}|y|^{-b-\delta}\|_{L^r},\end{aligned}$$

where in the last step, we have applied the Hölder's inequality with $1 = \frac{1}{q} + \frac{1}{r}$, $q < \frac{n}{n-2s}$, $r(b+\delta) < n$. This last two conditions are possible to satisfy (i.e. such q, r exist) , for small δ, as long as $b < 2s$. This is another instance that this requirement is crucially used. In this way, we have reached contradiction with our assumption that ϕ is unbounded. Therefore, ϕ is $L^\infty(\mathbf{R}^n)$ function. □

2.3.4 Further properties of ϕ

We have the following proposition.

Proposition 4. *Let $(n, s, p, b) \in \mathcal{A}$. Then, $\phi \in L^1(\mathbf{R}^n)$, so by the bell-shapedness, in particular it satisfies the point-wise bound*

$$|\phi(x)| \leq C|x|^{-n}, |x| > 1. \tag{2.3.15}$$

If in addition, $s \in (\frac{1}{2}, 1)$, then

$$|\nabla \phi(x)| \leq C \begin{cases} |x|^{-n-1} & |x| > 1 \\ |x|^{2s-b-1} & |x| < 1 \end{cases}. \tag{2.3.16}$$

In particular, $\phi \in C^1(\mathbf{R}^n \setminus \{0\})$.

Remarks: As a corollary, we have

- $\phi \in \cap_{1 < q \leq \infty} L^q(\mathbf{R}^n)$.
- $|x||\nabla \phi(x)|$ is a bounded function, since $2s > b$. In fact, $|x||\nabla \phi| \in \cap_{1 < q \leq \infty} L^q(\mathbf{R}^n)$.

Proof. Even though $\phi \in L^1$ implies (2.3.15), it will be actually bootstrapped from it. So, we focus on the proof of (2.3.15). We already know that $|\phi(x)| \leq C|x|^{-n/2}, |x| > 1$. To obtain the higher decay rate, introduce the optimal decay rate,

$$\alpha := \sup\{s : |\phi(x)| \le A_s|x|^{-s}, |x| > 1\}.$$

Clearly $\alpha \ge \frac{n}{2}$. Assuming that $\alpha < n$ leads to a contradiction. Indeed, note the representation (2.3.13),

$$|\phi(x)| \le |G_s| * [|x|^{-b}\phi^p(x)|,$$

and the fact that G_s is integrable near zero. Moreover, there is the bound $|G_s(x)| \le C|x|^{-n}, |x| > 1$ and $|x|^{-n} * |x|^{-(b+p(\alpha-\epsilon))} \le C|x|^{-\min(n,b+p(\alpha-\epsilon))}$, for small enough ϵ, so that $b + p(\alpha - \epsilon) > \alpha$. But this implies a better decay rate than α. This contradicts our assumption $\alpha < n$, so it follows that $\alpha \ge n$. One can in fact see that $\alpha = n$, as this is the optimal decay rate for G_s.

The bound for $\|\phi\|_{L^1}$ follows easily now. We simply estimate

$$\|\phi\|_{L^1} \le \|G_s\|_{L^1}\||x|^{-b}\phi^p\|_{L^1} = \||x|^{-b}\phi^p\|_{L^1}.$$

But the function $|x|^{-b}\phi^p \sim |x|^{-b}, |x| < 1$, while $|x|^{-b}\phi^p \sim |x|^{-(b+np)}, |x| > 1$, so $|x|^{-b}\phi^p \in L^1(\mathbf{R}^n)$.

The bounds for $|\nabla\phi|$ for $|x| > 1$ follow as in the proof of (2.3.15), once we make sure that ∇G_s is integrable near zero, which since $|\nabla G_s(x)| \le C|x|^{2s-n-1}, |x| < 1$, requires that $s > \frac{1}{2}$. For the case $|\nabla\phi|, |x| < 1$, we again use the formula $\nabla\phi = \nabla G_s * [|\cdot|^{-b}\phi^p]$. One can see that for values $|x| < 1$,

$$|\nabla\phi(x)| \le C\int_{|y|<2} \frac{1}{|x-y|^{n+1-2s}}\frac{1}{|y|^b}dy + \text{bounded function}.$$

Integrating separately in the regions $|y| < \frac{|x|}{2}$ and $|y| \geq \frac{|x|}{2}$ yields the bound $|\nabla\phi(x)| \leq C|x|^{2s-b-1}$. $\square$

2.4 Preliminary spectral properties of $\mathcal{L}_\pm$

We start with the realization of $\mathcal{L}_\pm$ as a self-adjoint operator.

2.4.1 Self-adjointness of $\mathcal{L}_\pm$

The conclusion $\phi \in L^\infty(\mathbf{R}^n)$ is helpful in our study of $\mathcal{L}_+$ and $\mathcal{L}_-$. However, we still face difficulties, for example with regards to the self-adjointness, as the potential $|x|^{-b}\phi^{p-1}(x)$ is still singular at zero. The following non-trivial lemma resolves these issues.

Lemma 6. *Let $(n, s, p, b) \in \mathcal{A}$ and in addition $2b < n$. Then the Friedrich's extensions of $\mathcal{L}_\pm$ are self-adjoint operators with the natural domain $H^{2s}(\mathbf{R}^n)$.*

Proof. Before we proceed with the construction of the Friedriech's extension, let us show that the condition $n > 2b$ ensures that $\mathcal{L}_\pm(H^{2s}) \subset L^2(\mathbf{R}^n)$. This reduces to the estimate

$$\left(\int_{\mathbf{R}^n} |x|^{-2b}|h(x)|^2 dx\right)^{1/2} \leq C\|h\|_{H^{2s}(\mathbf{R}^n)},$$

which follows by (1.1.7), where $a = 2b$ and since $b < 2s$.

Next, introduce the quadratic forms $\mathcal{Q}_\pm[h, h] := \langle \mathcal{L}_\pm h, h\rangle$, with form domain $H^s(\mathbf{R}^n) \times H^s(\mathbf{R}^n)$. Via the usual Friedrich's procedure, it will suffice to show boundedness from below for $\mathcal{Q}_\pm$.

We proceed to bound $|\langle |x|^{-b}\phi^p, h\rangle|$. Clearly, the portion of the integral over $|x|>1$ is easy to control,

$$\int_{|x|>1} |x|^{-b}\phi^p(x)|h(x)|dx \leq C\|h\|_{L^2}\|\phi\|_{L^{2p}}^p \leq C\|h\|_{L^2}.$$

For the piece over $|x|\leq 1$, we have by Cauchy-Schwarz and Sobolev embedding, for any[8] $\sigma: 0<\sigma<s, 2b<n+2\sigma$

$$|\int_{|x|\leq 1} |x|^{-b}\phi^p(x)h(x)dx| \leq \|(-\Delta)^{\frac{\sigma}{2}}h\|_{L^2}\|(-\Delta)^{-\frac{\sigma}{2}}[|x|^{-b}\phi^p\chi_{|x|\leq 1}]\|_{L^2} \leq$$
$$\leq C\|(-\Delta)^{\frac{\sigma}{2}}h\|_{L^2}\||x|^{-b}\chi_{|x|\leq 1}\|_{L^{\frac{2n}{n+2\sigma}}} \leq C\|(-\Delta)^{\frac{\sigma}{2}}h\|_{L^2} \leq \kappa\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}+C_{\kappa,\sigma}\|h\|_{L^2}.$$

Next, for the integral $\int |x|^{-b}\phi^p h^2(x)dx$, we control it by applying Proposition 1, with $q=2$ and any $\sigma>\frac{b}{2}$,

$$\int |x|^{-b}\phi^p h^2(x)dx \leq C\|h\|_{H^\sigma}^2.$$

Choosing $\sigma<s$ as well, that is $\sigma\in(\frac{b}{2},s)$, we conclude that for each κ, there is C_κ, so that

$$\int |x|^{-b}\phi^p h^2(x)dx \leq \kappa\|h\|_{H^s}^2+C_\kappa\|h\|_{L^2}^2. \tag{2.4.1}$$

Combining the estimates for $\int |x|^{-b}\phi^p h dx$ and $\int |x|^{-b}\phi^p h^2(x)dx$, with (2.3.7), yields that there exists a sufficiently large C, so that for each $h\in H^s(\mathbf{R}^n)$, we have

$$\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}^2 - pm(\omega)\int |x|^{-b}\phi^p h^2(x)dx \geq -\kappa\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}^2 - C\|h\|_{L^2}^2.$$

Or

$$(1+\kappa)\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}^2 - pm(\omega)\int |x|^{-b}\phi^p h^2(x)dx \geq -C\|h\|_{L^2}^2. \tag{2.4.2}$$

[8] Clearly, one can select such $\sigma\in(0,s)$, as $b<n, b<2s$

So, again by (2.4.1) and (2.4.2),

$$(1+\kappa)\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}^2 - 2pm(\omega)\int |x|^{-b}\phi^p h^2(x)dx \geq -\kappa\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}^2 - C\|h\|_{L^2}^2,$$

whence for small enough κ,

$$2(\|(-\Delta)^{\frac{s}{2}}h\|_{L^2}^2 - pm(\omega)\int |x|^{-b}\phi^p h^2(x)dx) \geq -C\|h\|_{L^2}^2,$$

which is the desired boundedness from below for $\mathcal{L}_+$, once we divide by two and add $\omega\|h\|_{L^2}^2$. Since $\mathcal{L}_- \geq \mathcal{L}_+$, the boundedness from below (and hence the self-adjointness of the Friedrich's extension) for $\mathcal{L}_-$ follows.

□

Corollary 3. *Under the assumption* $2b < n$, $\phi \in H^{2s}(\mathbf{R}^n) = D(\mathcal{L}_\pm)$.

Proof. Since $\phi \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ is already clear, we just need to observe that $\phi = (1+(-\Delta)^s)^{-1}[|x|^{-b}\phi^p] \in \dot{H}^{2s}$. Indeed,

$$\|\phi\|_{\dot{H}^{2s}(\mathbf{R}^n)} = \|(-\Delta)^s(1+(-\Delta)^s)^{-1}[|x|^{-b}\phi^p]\|_{L^2} \leq C\||x|^{-b}\phi^p\|_{L^2},$$

which is finite, if $2b < n$ since $|x|^{-b}\phi^p \sim |x|^{-b}, |x| < 1$ and for $|x| > 1$, $|x|^{-b}\phi^p \leq \phi^p \in L^2(\mathbf{R}^n)$. □

Remark: The assumption $2b < n$ is automatic for $(n,p,s,b) \in \mathcal{A}$, if $n \geq 4$. In the case $n = 3$ however, this is not so and it amounts to the extra restriction $b < \frac{3}{2}$. In [10], the authors use the fact that $\phi \in D(\mathcal{L}_\pm)$, which is not justified in the full range $n = 3, b < 2$, but rather only in the range $b < \frac{3}{2}$. Their statement has to be modified accordingly in order to hold, at least based on the proof presented therein. Clearly, the restriction is even more severe in the lower dimensional cases $n = 1, 2$.

Now that we have properly realized $\mathcal{L}_\pm$ as self-adjoint operators, one can talk about their eigenvalues, coercivity properties etc. Our next result are in this direction.

2.4.2 Some basic coercivity properties of $\mathcal{L}_\pm$

Proposition 5. *Let $(n,s,p,b) \in \mathcal{A}$ and in addition $2b < n$. Then, the self-adjoint operators $\mathcal{L}_\pm$ enjoy the following properties:*

- *The continuous spectrum of $\mathcal{L}_\pm$ is $[\omega,\infty)$.*
- *$\mathcal{L}_+$ has exactly one negative eigenvalue.*
- *$\mathcal{L}_- \geq 0$, with $\mathcal{L}_-[\phi] = 0$ and moreover $\mathcal{L}_-|_{\{\phi\}^\perp\}} \geq 0$.*

Proof. Continuous spectrum for both operators consists of $[\omega,\infty)$ by Weyl's theorem. Clearly, since $\langle \mathcal{L}_+\phi,\phi\rangle = -(p-1)m(\omega)\int |x|^{-b}\phi^{p+1}dx < 0$, it follows that $\mathcal{L}_+$ has a negative eigenvalue. Then, the property $\langle \mathcal{L}_+h,h\rangle \geq 0, h \perp |\cdot|^{-b}\phi^p$, which was previously established only for $h \in C^\infty(\mathbf{R}^n \setminus \{0\})$, can now be extended to all $h \in \mathcal{S} : h \perp |\cdot|^{-b}\phi^p$, since $|\cdot|^{-b}\phi^p \in L^2(\mathbf{R}^n)$, due to the assumption $2b < n$ and the properties of ϕ. Thus, $n(\mathcal{L}_+) = 1$.

Regarding the claims for $\mathcal{L}_-$, assume that the lowest eigenvalue, say $-\sigma^2$ is a negative one. Then,

$$-\sigma^2 = \inf_{\|u\|=1} \langle \mathcal{L}_-u,u\rangle = \inf_{\|u\|=1} [\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 + \omega - m(\omega)\int_{\mathbf{R}^n} |x|^{-b}\phi^p|u|^2dx]$$

Similar to our considerations in the proof of Proposition 2, this variational problem has a bell-shaped solution, say $\psi : \|\psi\| = 1$, which satisfies $\mathcal{L}_-[\psi] = -\sigma^2\psi$. But on the other hand, by a direct inspection, $\mathcal{L}_-\phi = 0$, ϕ is bell-shaped as well. But then,

$$0 = \langle \mathcal{L}_-\phi,\psi\rangle = \langle \phi,\mathcal{L}_-\psi\rangle = -\sigma^2\langle \phi,\psi\rangle < 0,$$

a contradiction. Thus, $\mathcal{L}_-|_{\{\phi\}^\perp\}} \geq 0$.

□

Our next discussion will concern the Sturm-Liouville theory for fractional Schrödinger operators such as $\mathcal{L}_\pm$. We base our approach to a result due to Frank-Lenzmann-Silvester, [36].

2.4.3 Sturm oscillation theorem for the second eigenfunction of $\mathcal{L}_+$

Theorem 4. *(Frank-Lenzmann-Silvestre, Theorem 2.3, [36])*

Let $n \geq 1, s \in (0,1]$ and W satisfies

- $W = W(|x|)$ *and W is non-decreasing in $|x|$,*
- $W \in L^\infty(\mathbf{R}^n)$*,* $W \in C^\gamma, \gamma > \max(0, 1-2s)$*. That is*

$$|W(x) - W(y)| \leq C|x-y|^\gamma.$$

Then, assume that $H = (-\Delta)^s + W$ has two lowest radial eigenvalues E_0, E_1, so that $E_0 < E_1 < \inf \sigma_{ess}(H)$.

Then, the eigenvalue E_0 is simple and the corresponding eigenfunction is bell-shaped. Regarding E_1, the corresponding eigenfunction $\Psi_1 : \mathcal{H}\Psi_1 = E_1\Psi_1$ has exactly one change of sign. That is, there exists $r_0 \in (0,\infty)$, so that $\Psi_1(r) < 0, r \in (0, r_0)$ and $\Psi_1(r) > 0, r \in (r_0, \infty)$.

Remark: Note that the potentials involved in $\mathcal{L}_\pm$, while satisfying most of the requirements in Theorem 4, fail in a dramatic way the key boundedness requirement, as they are unbounded at zero. So, we shall need to employ an approximation argument to achieve the same result for $\mathcal{L}_+$.

Recall that according to Proposition 5, $\mathcal{L}_+$ has exactly one negative eigenvalue, $E_0 < 0$. The next *radial* eigenvalue E_1 (if there is one!) satisfies $E_1 \geq 0$.

Proposition 6. *(Sturm oscillation theorem for the second eigenfunction of $\mathcal{L}_+$) Let $(n, s, p, b) \in \mathcal{A}$ and in addition $2b < n$. Then, the smallest eigenvalue $E_0 < 0$ has a bell-shaped radial eigenfunction. Suppose that the operator $\mathcal{L}_+$ has a radial eigenvalue $E_1 < \omega$. Then, E_1 has a radial eigenfunction with exactly one change of sign.*

Remark: The condition $E_1 < \omega$ simply means that E_1 is not an embedded eigenvalue, as $\sigma_{ac}(\mathcal{L}_+) = [\omega, \infty)$.

Proof. Before we start with the proof, let us mention that as we discuss radial eigenfunctions, we restrict our considerations to the Hilbert space $L^2_{rad}(\mathbf{R}^n)$ for the purposes of this proof.

Recall $\mathcal{L}_+ = (-\Delta)^s + \omega - pm(\omega)|x|^{-b}\phi^{p-1}(x) =: (-\Delta)^s + \omega - W$. The statements regarding E_0 can be established directly, even for the unbounded potential W. Indeed, by the self-adjointness of $\mathcal{L}_+$ and the characterization of the lowest eigenvalue

$$E_0 = \min_{\|u\|_{L^2}=1} \langle \mathcal{L}_+ u, u\rangle = \omega + \min_{\|u\|_{L^2}=1} [\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2} - \int_{\mathbf{R}^n} W(x)|u|^2 dx].$$

By the Polya-Szegö inequality and since $W = W^*$, $\int_{\mathbf{R}^n} W(x)|u|^2 dx \leq \int_{\mathbf{R}^n} W(x)|u^*|^2 dx$, we conclude that the minimization problem $\min_{\|u\|_{L^2}=1} \langle \mathcal{L}_+ u, u\rangle$ has a bell-shaped solution $\Psi_0 : \|\Psi_0\|_{L^2} = 1$ and $\mathcal{L}_+\Psi_0 = E_0\Psi_0$. In particular, $\Psi_0 \in H^{2s}(\mathbf{R}^n)$. Moreover, E_0 is a simple eigenvalue, as the minimizers for $\min_{\|u\|_{L^2}=1} \langle \mathcal{L}_+ u, u\rangle$ need to be bell-shaped and as such, cannot be orthogonal to Ψ_0.

Next, we define an approximation of W, namely for every integer N, the bounded potentials,

$$W_N(r) = \begin{cases} W(r) & r > \frac{1}{N} \\ W(N^{-1}) & r \leq \frac{1}{N} \end{cases}$$

and the operators $\mathcal{L}_{+,N} := (-\Delta)^s + \omega - W_N$. Note that $\mathcal{L}_{+,N} \geq \mathcal{L}_+$, since $W_N \leq W$.

As $W_N = W_N^*$, they have, by the same arguments as above ground states $\Psi_{0,N} : \|\Psi_{0,N}\|_{L^2} = 1$, corresponding to the smallest eigenvalues $E_{0,N} \geq E_0$, so $\mathcal{L}_{+,N}\Psi_{0,N} = E_{0,N}\Psi_{0,N}$. We will show that $\lim_N E_{0,N} = E_0$. Indeed, we have that

$$E_0 \leq E_{0,N} = \min_{\|u\|_{L^2}=1} \langle \mathcal{L}_{+,N} u, u \rangle \leq \langle \mathcal{L}_{+,N}\Psi_0, \Psi_0 \rangle \leq E_0 + \int_{|x|<N^{-1}} W(|x|)\Psi_0^2(x)dx. \tag{2.4.3}$$

Since by (1.1.7), we have that

$$\left(\int_{|x|<1} |W(|x|)|\Psi_0^2(x)dx \right)^{1/2} \leq C \left(\int_{|x|<1} |x|^{-b}\Psi_0^2(x)dx \right)^{1/2} \leq C\|\Psi_0\|_{H^s(\mathbf{R}^n)}, \tag{2.4.4}$$

we conclude $\lim_{N\to\infty} \int_{|x|<N^{-1}} W(|x|)\Psi_0^2(x)dx = 0$, whence in combination with (2.4.3), we finally arrive at $\lim_N E_{0,N} = E_0$.

We now show that a subsequence of $\{\Psi_{0,N}\}$ converges strongly to Ψ_0. To that end, we need to show that $\{\Psi_{0,N}\}$ is pre-compact in the strong topology of $L^2(\mathbf{R}^n)$. Indeed, by (1.1.7), we have that, since $\frac{b}{2} < s$, there is C_s, so that

$$\int_{\mathbf{R}^n} W_N(|x|)\Psi_0^2 dx \leq C \int_{\mathbf{R}^n} |x|^{-b}\Psi_0^2 dx \leq C_s\|\Psi_0\|^2_{H^s(\mathbf{R}^n)}$$

Thus, by Gagliardo-Nirenberg's inequality

$$E_{0,N} = \langle \mathcal{L}_{+,N}\Psi_{0,N}, \Psi_{0,N} \rangle \geq \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|^2_{L^2} + \omega - C_s\|\Psi_0\|^2_{H^s(\mathbf{R}^n)}$$

$$\geq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2}^2 - C_{s,\omega},$$

whence $\sup_N \|\Psi_{0,N}\|_{H^s} < \infty$. Next, by the representation

$$\Psi_{0,N} = ((-\Delta)^s + \omega - E_{0,N})^{-1}[W_N \Psi_{0,N}],$$

$\|\Psi_{0,N}\|_{L^2} = 1$, and $\lim_N E_{0,N} = E_0 < 0$, we derive similar to the proof of (2.3.15), that there exists a constant $C = C_n$, but independent of N, so that $|\Psi_{0,N}(x)| \leq C_n |x|^{-n}$ for $|x| > 1$. This guarantees that $\lim_M \sup_N \int_{|x|>M} |\Psi_{0,N}(x)|^2 dx = 0$, which by Riesz-Relich-Kolmogorov criteria guarantees that $\{\Psi_{0,N}\}$ is pre-compact in $L^2(\mathbf{R}^n)$. That means that there is a subsequence $\Psi_{0,N_k} \to \Psi_0$. For simplicity of notations, we can assume without loss of generality that the sequence itself converges, i.e. $\lim_N \|\Psi_{0,N} - \Psi_0\|_{L^2} = 0$.

One can in fact show that (up to a further subsequence), $\lim_N \|\Psi_{0,N} - \Psi_0\|_{H^s} = 0$. Indeed, $\{\Psi_{0,N}\}$ being a bounded sequence in H^s has a weakly convergent subsequence (again assume that it is the sequence itself), which by uniqueness must be Ψ_0. Then, by lower semi-continuity of the L^2 norm with respect to weak convergence,

$$\liminf_N \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2} \geq \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}.$$

In addition, we claim that

$$\lim_N \int_{\mathbf{R}^n} W_N(|x|)\Psi_{0,N}^2(x)dx = \int_{\mathbf{R}^n} W(|x|)\Psi_0^2(x)dx. \tag{2.4.5}$$

Indeed, by (2.4.4), it suffices to show $\lim_N \left[\int_{\mathbf{R}^n} W_N(|x|)(\Psi_{0,N}^2(x) - \Psi_0^2(x))dx\right] = 0$. We have by Cauchy-Schwarz's that for every $\epsilon > 0$, there is C_ϵ

$$\left|\int_{\mathbf{R}^n} W_N(|x|)(\Psi_{0,N}^2(x) - \Psi_0^2(x))dx\right| \leq C \int_{\mathbf{R}^n} |x|^{-b}|\Psi_N(x) - \Psi_0(x)||\Psi_N(x) + \Psi_0(x)|dx$$

$$\leq \left(\int_{\mathbf{R}^n} |x|^{-b}|\Psi_N(x)+\Psi_0(x)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} |x|^{-b}|\Psi_N(x)-\Psi_0(x)|^2\right)^{\frac{1}{2}} \leq$$
$$\leq C_\epsilon(\|\Psi_N\|_{H^s}+\|\Psi_0\|_{H^s})\|\Psi_N-\Psi_0\|_{H^{\frac{b}{2}+\epsilon}}.$$

where we have used (1.1.7). Note that by Gagliardo-Nirenberg's, we have

$$\|\Psi_N-\Psi_0\|_{H^{\frac{b}{2}+\epsilon}} \leq C\|\Psi_N-\Psi_0\|_{H^s}^{\frac{b/2+\epsilon}{s}} \|\Psi_N-\Psi_0\|_{L^2}^{\frac{s-b/2-\epsilon}{s}},$$

which clearly converges to zero, as $N\to\infty$, as long as we select $0<\epsilon<s-b/2$.

Thus, having established (2.4.5) and $\liminf_N \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2} \geq \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}$, we conclude

$$\begin{aligned} E_0 &= \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}^2+\omega-\int_{\mathbf{R}^n} W(|x|)\Psi_0^2(x)dx \leq \\ &\leq \liminf_N[\|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2}^2+\omega-\int_{\mathbf{R}^n} W(|x|)\Psi_{0,N}^2(x)dx] = \liminf_N E_{0,N} = E_0. \end{aligned}$$

It follows that $\liminf_N \|(-\Delta)^{\frac{s}{2}}\Psi_{0,N}\|_{L^2} = \|(-\Delta)^{\frac{s}{2}}\Psi_0\|_{L^2}$, which implies that (up to a subsequence) $\lim_N \|\Psi_{0,N}-\Psi_0\|_{H^s}=0$.

We now turn to the second radial eigenfunction of $\mathcal{L}_+$. Let

$$h_1 \in D(\mathcal{L}_+) = H^{2s}(\mathbf{R}^n), \|h_1\|_{L^2}=1$$

is an eigenfunction corresponding[9] to E_1, so $\mathcal{L}_+h_1 = E_1h_1$. Clearly $h_1\perp\Psi_0$, whence $\lim_N\langle h_1,\Psi_{0,N}\rangle = 0$. By the Rayleigh characterization of the second smallest eigenvalue and since $\mathcal{L}_{+,N}\geq\mathcal{L}_+$, we have that $E_{1,N}\geq E_1$. Denote the corresponding *radial* eigenfunctions by $\Psi_{1,N}:\|\Psi_{1,N}\|_{L^2}=1$. Note that $-W_N$ satisfy the requirements of Theorem 4, with $\gamma=1$, as a bounded, piecewise defined function, whose components are Lipschitz.

[9]Even though the ultimate claim is that there is an eigenfunction Ψ_1, which has exactly one change of sign, we do not know that yet

Hence, due to Theorem 4, we may take those eigenfunctions $\Psi_{0,N}$ to have exactly one change of sign, say $r_N \in (0,\infty)$, say $\Psi_{0,N}|_{(0,r_N)} > 0, \Psi_{0,N}|_{(r_N,\infty)} < 0$.

Note

$$E_{1,N} = \inf_{\|u\|_{L^2}=1, u\perp\Psi_{0,N}} \langle \mathcal{L}_{+,N} u, u\rangle \leq \frac{\langle \mathcal{L}_{+,N}(h_1 - \langle h_1, \Psi_{0,N}\rangle \Psi_{0,N}), h_1 - \langle h_1, \Psi_{0,N}\rangle \Psi_{0,N}\rangle}{\|h_1 - \langle h_1, \Psi_{0,N}\rangle \Psi_{0,N}\|^2} =$$
$$= \langle \mathcal{L}_+ h_1, h_1\rangle + o(N^{-1}) = E_1 + o(N^{-1}).$$

It follows that $\lim_N E_{1,N} = E_1$. In particular, the assumption $E_1 < \omega$ guarantees that[10] $E_{1,N} < \omega$ for large enough N. Similar to the proofs for $\Psi_{0,N}$, (in particular note the representation $\Psi_{1,N} = ((-\Delta)^s + \omega - E_{1,N})^{-1}[W_N \Psi_{1,N}]$, which implies the bound $|\Psi_{1,N}(x)| \leq C|x|^{-n}$ for $|x| > 1$), the system $\{\Psi_{1,N}\}$ is pre-compact in $L^2(\mathbf{R}^n)$, so it has a convergent subsequence. Again, assume that it is the sequence itself. Denote its limit by $\Psi_1 : \lim_N \|\Psi_{1,N} - \Psi_1\|_{L^2} = 0$.

Similar to the proof above for Ψ_0, we conclude that (after eventually taking a subsequence), $\lim_N \|\Psi_{1,N} - \Psi_1\|_{H^s} = 0$ and $\Psi_1 \perp \Psi_0$ is an eigenfunction for $\mathcal{L}_+$ corresponding to the eigenvalue E_1. It remains to show that Ψ_1 has exactly one sign change. To this end, consider the sequence $r_N \in (0,\infty)$ of sign changes for $\Psi_{1,N}$. There are three alternatives:

- $\{r_N\}$ converges to zero
- $\{r_N\}$ converges to $+\infty$
- $\{r_N\}$ has a subsequence, which converges to $r_0 \in (0,\infty)$.

We will show that the first two alternatives cannot really occur. Indeed, assume $r_N \to 0$. Then, pick a radial function $\zeta \in C_0^\infty(\mathbf{R}^n) : \zeta \geq 0$. We have

$$\langle \Psi_1, \zeta\rangle = \lim_N \langle \Psi_{1,N}, \zeta\rangle = \int_{|x|<r_N} \Psi_{1,N}\zeta(x)dx + \int_{|x|\geq r_N} \Psi_{1,N}\zeta(x)dx \leq 0.$$

[10]And in fact, we may claim that $\omega - E_{1,N} \geq \frac{\omega - E_1}{2}$.

Thus, we conclude that $\Psi_1 \leq 0$ a.e., which is then a contradiction with $\langle \Psi_1, \Psi_0 \rangle = 0$, as Ψ_0 is bell-shaped function. Similarly, the case $r_N \to \infty$ leads to the conclusion $\Psi_1 \geq 0$, which contradicts again $\Psi_1 \perp \Psi_0$.

Thus, the case $r_{N_k} \to r_0 > 0$ remains. For this subsequence, we clearly have that for each $\zeta : \zeta \in C_0^\infty(0, r_0), \zeta \geq 0$, we have $\langle \Psi_1, \zeta \rangle \geq 0$, while for $\zeta : \zeta \in C_0^\infty(r_0, \infty), \zeta \geq 0$, we have $\langle \Psi_1, \zeta \rangle \leq 0$. Equivalently, Ψ_0 changes sign exactly once, at $r_0 > 0$. □

2.5 The Non-degeneracy of Φ

In this section, we establish the non-degeneracy of the solutions of (2.1.2), obtained by means of rescaling of the constrained minimizers of (2.3.2). Let us outline the details of this construction. Start with a constrained minimizer ϕ_ω provided by Proposition 2. In particular, it satisfies (2.3.3), where recall $m(\omega)$ is in the form (2.3.1). Then, it suffices to take

$$\Phi_\omega(x) := m(\omega)^{\frac{1}{p-1}} \phi_\omega(x).$$

Clearly, with such a choice Φ_ω satisfies (2.1.2), which is bell-shaped and moreover enjoys all properties, as established for ϕ_ω in the Propositions 2, 3, 4. Note that $\mathcal{L}_\pm$ take the form

$$\mathcal{L}_+ = (-\Delta)^s + \omega - p|x|^{-b}\Phi_\omega^{p-1}, \mathcal{L}_- = (-\Delta)^s + \omega - |x|^{-b}\Phi_\omega^{p-1}.$$

The following result is the main conclusion of this section.

Proposition 7. *Assume $(n,p,s,b) \in \mathcal{A}$, and in addition $2b < n$ and $s \in (\frac{1}{2}, 1)$. Then,*

$$Ker[\mathcal{L}_+] = \{0\}.$$

We need to prepare the proof of Proposition 7 in several auxiliary results.

2.5.1 Differentiation with respect to parameters

We start this section with two *formal* calculations, which motivate our subsequent results.

Taking formal derivatives

Starting with the profile equation (2.1.2), we can *formally* take a derivative in any of the spatial variables, $\partial_{x_j}, j = 1, \ldots, n$. We obtain

$$\mathcal{L}_+[\partial_{x_j}\Phi] = -b\frac{x_j}{|x|^{b+2}}\Phi^p(x). \tag{2.5.1}$$

Let us emphasize again that (2.5.1) *is only a formal statement.* Indeed, such a formula is problematic at least in several ways - we need to have $\nabla\Phi \in D(\mathcal{L}_+) = H^{2s}$, the right-hand side of (2.5.1) is not in $L^2(\mathbf{R}^n)$, unless we assume $2(b+1) < n$ etc.

Similarly, by a simple scaling argument, the solution Φ_ω of (2.1.2) can be expressed through Φ_1, the solution for $\omega = 1$ as follows

$$\Phi_\omega(x) = w^{\frac{2s-b}{2s(p-1)}}\Phi_1(\omega^{\frac{1}{2s}}x) =: \omega^{\sigma_p}\Phi_1(\omega^{\frac{1}{2s}}x). \tag{2.5.2}$$

This highlights the dependence on the parameter ω in (2.1.2), which will be very useful in the sequel. More specifically, the *formal* differentiation in ω yields

$$\mathcal{L}_+[\partial_\omega\Phi_\omega] = -\Phi_\omega. \tag{2.5.3}$$

Again, the formula (2.5.3) *is only a formal statement.* In particular, note that since $\partial_\omega\Phi_\omega$ can be expressed as a linear combination of Φ_ω and $x \cdot \nabla\Phi_\omega$, we have the same issues with respect to the domain of $\mathcal{L}_+$. In both instances, that is (2.5.1) and (2.5.3), we heuristically expect them to hold in some sense. The required technical tools, which establish the corresponding rigorous statementys, are developed next.

A technical lemma

The following lemma shows that one can take weak derivatives with respect to the spatial variables x as well as the parameter ω.

Lemma 7. *Let $q, \nabla q \in L^2(\mathbf{R}^n)$. Then, for any $\psi \in \mathcal{S}$,*

$$\lim_{\delta\to 0}\langle\frac{q(x+\delta\mathbf{e}_j)-q(x)}{\delta},\psi\rangle = \langle\partial_{x_j}q,\psi\rangle, j=1,\dots,n, \tag{2.5.4}$$

Let now $q_\omega = f(\omega)q(g(\omega)x)$, where $f, g \in C^1(\mathbf{R}_+), g > 0$ and $q, x\cdot\nabla_x q \in L^2(\mathbf{R}^n)$. Then, for any $\psi \in \mathcal{S}$, we have

$$\lim_{\delta\to 0}\left\langle\frac{q_{\omega+\delta}-q_\omega}{\delta},\psi\right\rangle = \langle f'(\omega)q(g(\omega)\cdot)+f(\omega)g'(\omega)x\cdot\nabla_x q(g(\omega)\cdot),\psi\rangle. \tag{2.5.5}$$

Remark: Note that formally at least $\partial_\omega q = f'(\omega)q(g(\omega)\cdot)+f(\omega)g'(\omega)x\cdot\nabla_x q(g(\omega)\cdot)$, so the formula (2.5.5) is expected to be true.

Proof. We have by a simple change of variables

$$\lim_{\delta\to 0}\langle\frac{q(x+\delta\mathbf{e}_j)-q(x)}{\delta},\psi\rangle = \lim_{\delta\to 0}\langle q,\frac{\psi(\cdot-\delta\mathbf{e}_j)-\psi(\cdot)}{\delta}\rangle = -\langle q,\partial_j\psi\rangle = \langle\partial_j q,\psi\rangle,$$

where in the last step, we have used the Lebesgue's dominated convergence theorem integration by parts. This is justified since $\frac{\psi(\cdot-\delta\mathbf{e}_j)-\psi(\cdot)}{\delta} = -\partial_j\psi + O_{\|\cdot\|_{L^2}}(\delta)$ and $\nabla q \in L^2(\mathbf{R}^n)$. This establishes (2.5.4).

Regarding the proof of (2.5.5), by a change of variables and the Lebesgue's dominated convergence theorem

$$\lim_{\delta\to 0}\left\langle \frac{q_{\omega+\delta}-q_\omega}{\delta},\psi\right\rangle = \lim_{\delta\to 0}\int_{\mathbb{R}^n}\phi(y)\left(\frac{f(\omega+\delta)\psi(\frac{y}{g(\omega+\delta)})\frac{1}{g(\omega+\delta)^n}-f(\omega)\psi(\frac{y}{g(\omega)})\frac{1}{g(w)^n}}{\delta}\right)dy =$$
$$= \int_{\mathbb{R}^n} q(y)\partial_\omega\left[\frac{f(\omega)}{g(\omega)^n}\psi\left(\frac{y}{g(\omega)}\right)\right]dy = \left(\frac{f'(\omega)}{g^n(\omega)} - n\frac{f(\omega)g'(\omega)}{g^{n+1}(\omega)}\right)\int_{\mathbf{R}^n} q(y)\psi\left(\frac{y}{g(\omega)}\right)dy -$$
$$- \frac{f(\omega)g'(\omega)}{g^{n+2}(\omega)}\int_{\mathbf{R}^n} q(y)y\cdot\nabla_y\psi\left(\frac{y}{g(\omega)}\right)dy.$$

Clearly, the first term in (2.5.5) is accounted for as follows

$$\frac{f'(\omega)}{g^n(\omega)}\int_{\mathbf{R}^n} q(y)\psi\left(\frac{y}{g(\omega)}\right)dy = f'(\omega)\langle q(g(\omega)\cdot),\psi\rangle.$$

Next,

$$-n\frac{f(\omega)g'(\omega)}{g^{n+1}(\omega)}\int_{\mathbf{R}^n} q(y)\psi\left(\frac{y}{g(\omega)}\right)dy = -n\frac{f(\omega)g'(\omega)}{g(\omega)}\langle q(g(\omega)\cdot),\psi\rangle.$$

Finally, another change of variables and integration by parts (recall $q, x\cdot\nabla_x q \in L^2(\mathbf{R}^n)$ is assumed), yields

$$\int_{\mathbf{R}^n} q(y)y\cdot\nabla_y\psi\left(\frac{y}{g(\omega)}\right)dy = g^{n+1}(\omega)\int_{\mathbf{R}^n} q(g(\omega)x)x\cdot\nabla_x\psi(x)dx =$$
$$= -g^{n+1}(\omega)\int_{\mathbf{R}^n} div(xq(g(\omega)x))\psi(x)dx = -g^{n+1}(n\langle q(g(\omega)\cdot),\psi\rangle + g(\omega)\langle x\cdot\nabla_x q(g(\omega)\cdot),\psi\rangle).$$

Putting it all together yields the formula,

$$\lim_{\delta\to 0}\left\langle \frac{q_{\omega+\delta}-q_\omega}{\delta},\psi\right\rangle = f'(\omega)\langle q(g(\omega)\cdot),\psi\rangle + f(\omega)g'(\omega)\langle x\cdot\nabla_x q(g(\omega)\cdot),\psi\rangle$$

as required. □

Next, we have the following rigorous results which can be viewed as weaker versions of the formulas (2.5.1) and (2.5.3).

Rigorous versions of the formal differentiation formulas

Proposition 8. *Let* $(n,s,p,b) \in \mathcal{A}$, $s \in (\frac{1}{2},1), 2b < n$ *and* $\psi \in \mathcal{S}$. *Then, any solution* Φ_ω *of* (2.1.2)*, with the properties* $\Phi \in L^2 \cap L^\infty$ *and* $x \cdot \nabla\Phi \in L^2(\mathbf{R}^n)$ *satisfies*

$$\langle \partial_j \Phi_\omega, \mathcal{L}_+ \psi \rangle = -b \langle \frac{x_j}{|x|^{b+2}} \Phi^p, \psi \rangle,\ j = 1, \ldots, n \tag{2.5.6}$$

$$\langle \partial_\omega \Phi_\omega, \mathcal{L}_+ \psi \rangle = -\langle \Phi_\omega, \psi \rangle. \tag{2.5.7}$$

Remarks:

- Note that the expression $\langle \frac{x_j}{|x|^{b+2}} \Phi^p, \psi \rangle$ is well-defined, for smooth functions ψ, whenever $2(b+1) < n$. This is however not always satisfied under the assumptions in Proposition 8. The expression still makes sense, under the weaker assumptions herein, provided we interpret it in the form

$$\langle \frac{x_j}{|x|^{b+2}} \Phi^p, \psi \rangle = \int_{\mathbf{R}^n} \frac{x_j}{|x|^{b+2}} \Phi^p(x)(\psi(x) - \psi(0)) dx.$$

- The notation $\partial_\omega \Phi_\omega$ is used in (2.5.7) in the following sense

$$\partial_\omega \Phi_\omega = \sigma_p \omega^{\sigma_p - 1} \Phi_1(\omega^{\frac{1}{2s}} x) + \frac{\omega^{\sigma_p + \frac{1}{2s} - 1}}{2s} x \cdot \nabla_x \Phi_1(\omega^{\frac{1}{2s}} x). \tag{2.5.8}$$

  This is of course nothing but the formal derivative with respect to ω in (2.5.2). Note however that the expression on the right of (2.5.8) belongs to $L^2(\mathbf{R}^n)$, according to Proposition 4.

Proof. Our starting point is the formula (2.3.3). Applying it for x and $x+\delta\mathbf{e}_j$, taking the divided difference and then dot product with ψ yields

$$\langle((-\Delta)^s+\omega)[\frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta}],\psi\rangle = \langle\frac{|\cdot+\delta|^{-b}\Phi^p(\cdot+\delta\mathbf{e}_j)-|\cdot|^{-b}\Phi^p(\cdot)}{\delta},\psi\rangle. \quad (2.5.9)$$

Assume for the moment that ψ is so that $\hat{\psi}$ is supported in $\{\xi : |\xi| \geq \sigma > 0\}$. In this way, $\tilde{\psi} = ((-\Delta)^s+\omega)\psi \in \mathcal{S}$, since its Fourier transform, $(\omega+(2\pi|\cdot|)^{2s})\hat{\psi}$ is in Schwartz class[11].

So we have, by (2.5.4),

$$\langle((-\Delta)^s+\omega)[\frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta}],\psi\rangle = \langle\frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta},\tilde{\psi}\rangle \to \langle\partial_j\Phi,\tilde{\psi}\rangle.$$

It follows that

$$\lim_{\delta\to 0}\langle((-\Delta)^s+\omega)[\frac{\Phi(\cdot+\delta\mathbf{e}_j)-\Phi(\cdot)}{\delta}],\psi\rangle = \langle\partial_j\Phi,((-\Delta)^s+\omega)\psi\rangle.$$

This clearly can be extended from the set of Schwartz functions, which are Fourier supported away from zero to the whole set $\mathcal{S}$. Indeed, it suffices to observe that the set of Schwartz functions, which are Fourier supported away from zero is H^{2s} dense in $\mathcal{S}$.

For the right-hand side of (2.5.9), we could perform an identical argument, except that we do not have in general that $\partial_j|\cdot|^{-b}\Phi^p(\cdot) \in L^2(\mathbf{R}^n)$ (as we would need to require $2(b+1) < n$). Instead, we proceed with the direct proof. We have

$$\langle\frac{|\cdot+\delta|^{-b}\Phi^p(\cdot+\delta\mathbf{e}_j)-|\cdot|^{-b}\Phi^p(\cdot)}{\delta},\psi\rangle = \langle|\cdot|^{-b}\Phi^p(\cdot),\frac{\psi(\cdot-\delta\mathbf{e}_j)-\psi(\cdot)}{\delta}\rangle$$
$$\to -\langle|\cdot|^{-b}\Phi^p(\cdot),\partial_j\psi\rangle.$$

[11]Note that $|\xi|^{2s}\hat{\psi}(\xi)$ is not smooth at zero, unless $\hat{\psi}$ vanishes in a neighborhood of zero

If $\psi \in \mathcal{S}(\mathbf{R}^n \setminus \{0\})$, we can take integration by parts (as we avoid the singularity at zero), whence we arrive at

$$\lim_{\delta\to 0}\langle \frac{|\cdot+\delta|^{-b}\Phi^p(\cdot+\delta\mathbf{e}_j)-|\cdot|^{-b}\Phi^p(\cdot)}{\delta},\psi\rangle = \langle -b\frac{x_j}{|x|^{b+2}}\Phi^p + p|x|^{-b}\Phi^{p-1}\Phi',\psi\rangle.$$

Again, one may extend such a formula from $\psi \in \mathcal{S}(\mathbf{R}^n \setminus \{0\})$ to $\psi \in \mathcal{S}$. It follows that taking limits as $\delta \to 0$ in (2.5.9) results in (2.5.6).

For the proof of (2.5.7), we proceed in a similar fashion. More specifically, taking (2.1.2) at ω and then at $\omega+\delta$ and subtracting yields the relation

$$((-\Delta)^s+\omega)[\frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta}] - |x|^{-b}[\frac{\Phi^p_{\omega+\delta}-\Phi^p_\omega}{\delta}] = -\Phi_{\omega+\delta}.$$

Taking dot product with $\psi \in \mathcal{S}(\mathbf{R}^n \setminus \{0\})$ yields

$$\langle\frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta},((-\Delta)^s+\omega))\psi\rangle - \langle|x|^{-b}[\frac{\Phi^p_{\omega+\delta}-\Phi^p_\omega}{\delta}],\psi\rangle = -\langle\Phi_{\omega+\delta},\psi\rangle. \tag{2.5.10}$$

Clearly,

$$\langle\Phi_{\omega+\delta},\psi\rangle = \langle\Phi_\omega,\psi\rangle + \delta\langle\frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta},\psi\rangle \to \langle\Phi_\omega,\psi\rangle,$$

as the expression $\langle\frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta},\psi\rangle$ has a limit by (2.5.5), namely $\langle\frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta},\psi\rangle \to \langle\partial_\omega\Phi_\omega,\psi\rangle$.

Under the assumption $\psi \in \mathcal{S} : supp\hat{\psi} \subset \{\xi : |\xi| \geq \sigma > 0\}$, we introduce again $\tilde{\psi} = ((-\Delta)^s+\omega))\psi \in \mathcal{S}$. According to (2.5.2) and a simple change of variables

$$\lim_{\delta\to 0}\langle\frac{\Phi_{\omega+\delta}-\Phi_\omega}{\delta},((-\Delta)^s+\omega))\psi\rangle \quad = \quad \langle\partial_\omega\Phi_\omega,\tilde{\psi}\rangle = \langle\partial_\omega\Phi_\omega,((-\Delta)^s+\omega))\psi\rangle.$$

This is again extendable, as above to any $\psi \in \mathcal{S}$. Finally, by (2.5.5) and the formula[12] $\partial_\omega \Phi_\omega^p = p\Phi_\omega^{p-1}\partial_\omega \Phi_\omega$, we have[13]

$$\lim_{\delta\to 0}\langle |\cdot|^{-b}[\frac{\Phi^p_{\omega+\delta}-\Phi^p_\omega}{\delta}],\psi\rangle = \lim_{\delta\to 0}\langle \frac{\Phi^p_{\omega+\delta}-\Phi^p_\omega}{\delta}, |\cdot|^{-b}\psi\rangle = p\langle \partial_\omega \Phi_\omega, |\cdot|^{-b}\Phi_\omega^{p-1}\psi\rangle.$$

All in all, we obtain (2.5.7).

□

2.5.2 Conclusion of the non-degeneracy proof

In this section, we follow the arguments in [79]. We also assume that $n \geq 2$, as the one dimensional case $n = 1$ reduces to an easy argument, contained in the proof below.

We have from Proposition 5 that $\mathcal{L}_+$ has one simple negative eigenvalue and from the appendixA there is the decomposition of $\mathcal{L}_+$ in spherical harmonics as

$$\mathcal{L}_+ = \mathcal{L}_{+,0} \oplus \mathcal{L}_{+,\geq 1}.$$

The non-degeneracy of $\mathcal{L}_+$ follows from the following.

Proposition 9. *$\sigma_1(\mathcal{L}_{+,0}) > 0$ and there exists $\delta > 0$ so that $\mathcal{L}_{+,\geq 1} \geq \delta > 0$.*

Remark: We know that $\sigma_{ess.}(\mathcal{L}_+) = [\omega, \infty)$, whence the only remaining issue is the point spectrum.

Proof. We know that the smallest eigenvalue of $\mathcal{L}_+$, $E_0 < 0$ has a bell-shaped eigenfunction and hence, it is an eigenvalue of $\mathcal{L}_{+,0}$. The next *radial* eigenvalue E_1 cannot be negative since $n(\mathcal{L}_+) = 1$, thus $E_1 \geq 0$. If $E_1 > 0$, we will have shown $\sigma_1(\mathcal{L}_{+,0}) > 0$.

[12]This formula is of course correct formally, but in order to provide a rigorous justification, we need to took into account (2.5.2), and (2.5.8)

[13]noting that $|\cdot|^{-b}\psi \in L^2(\mathbf{R}^n)$ under the standing assumption $2b < n$

Assume, for a contradiction that $E_1 = 0$. Then by Proposition 6, there is an eigenfunction ψ_1 such that $\mathcal{L}_{+,0}\psi_1 = 0$, so that ψ_1 has exactly one change of sign. Without loss of generality, let $\psi_1(r) < 0, r \in (0, r_0)$ and $\psi_1(r) > 0$ for $r \in (r_0, \infty)$.

Next, we show now that $\Phi_\omega \perp Ker[\mathcal{L}_+]$. Indeed, for every $\psi \in ker[\mathcal{L}_+]$, we have that $\psi \in H^{2s}(\mathbf{R}^n)$. Thus, we can approximate by Schwartz functions $\psi_N \to \psi$ in $H^{2s}(\mathbf{R}^n)$ norm, whence $\lim_{N\to\infty} \|\mathcal{L}_+\psi_N - \mathcal{L}_+\psi\|_{L^2} = 0$. We have by (2.5.7) applied to ψ_N, that

$$0 = \langle \partial_\omega \Phi_\omega, \mathcal{L}_+\psi \rangle = \lim_{N\to\infty} \langle \partial_\omega \Phi_\omega, \mathcal{L}_+\psi_N \rangle = -\lim_{N\to\infty} \langle \Phi_\omega, \psi_N \rangle = -\langle \Phi_\omega, \psi \rangle.$$

It follows that $\Phi_\omega \perp Ker[\mathcal{L}_+]$. By a direct calculation we see that

$$\mathcal{L}_{+,0}\Phi = -|x|^{-b}(p-1)\Phi^p,$$

whence $|x|^{-b}\Phi^p \perp ker[\mathcal{L}_{+,0}]$. Note that since $2b < n$, $|x|^{-b}\Phi^p \in L^2(\mathbf{R}^n)$. Now consider

$$\varphi = c_0\Phi - r^{-b}\Phi^p = \Phi(c_0 - r^{-b}\Phi^{p-1}), c_0 := \frac{\Phi^{p-1}(r_0)}{r_0^b}.$$

Since Φ is bell-shaped, $\varphi(r) < 0, r \in (0, r_0)$ and $\varphi(r) > 0, r \in (r_0, \infty)$, but since $\varphi \perp ker[\mathcal{L}_{+,0}]$ we have $\langle \varphi, \psi_1 \rangle = 0$. On the other hand, $\varphi\psi_1 \geq 0$, and this is a contradiction. Hence $\sigma_1(\mathcal{L}_{+,0}) > 0$.

Finally we show that $\mathcal{L}_{+,\geq 1} > 0$. Note however that since $n(\mathcal{L}_+) = 1$ and $n(\mathcal{L}_{+,0}) = 1$, we have $\mathcal{L}_{+,\geq 1} \geq 0$. Hence, we just need to show that zero is not eigenvalue for $\mathcal{L}_{+,\geq 1}$.

Suppose, for a contradiction, that zero is an eigenvalue for $\mathcal{L}_{+,\geq 1}$. This implies that zero is an eigenvalue for $\mathcal{L}_{+,1}$. Indeed, otherwise zero is then eigenvalue for $\mathcal{L}_{+,\geq 2}$, say $\mathcal{L}_{+,\geq 2}\vartheta = 0$. Since $\mathcal{L}_{+,\geq 2} > \mathcal{L}_{+,1}$, it will follow that

$$\langle \mathcal{L}_{+,1}\vartheta, \vartheta \rangle < \langle \mathcal{L}_{+,\geq 2}\vartheta, \vartheta \rangle = 0.$$

Consequently, $\mathcal{L}_{+,1}$ has a negative eigenvalue, which is a contradiction, as we know $\mathcal{L}_{+,\geq 1} \geq 0$. Thus, we have reduced our contradiction argument to the case that $\mathcal{L}_{+,1}$ has an eigenvalue at zero, which we will need to refute now.

Since zero is now assumed to be an eigenvalue for $\mathcal{L}_{+,1}$ and $\mathcal{L}_{+,1} \geq 0$, it must be at the bottom of the spectrum. Its eigenfunctions are in the form $\psi_j = \psi(x)\frac{x_j}{|x|}, j = 1, \ldots, n$, where $\psi \in L^2_{rad}$. So, ψ is an eigenfunction at the bottom of the spectrum for the operator

$$\tilde{\mathcal{L}}_{+,1} = (-\partial_{rr} - \frac{n-1}{r}\partial_r + \frac{n-1}{r^2})^s + \omega - p|r|^{-b}\Phi^{p-1}(r),$$

acting on functions in L^2_{rad}. According to Lemma C.4, [36], $(-\Delta_l)^{\frac{s}{2}}, s \in (0,1)$ is positivity improving for each $l \geq 0$, i.e. for every $X_l \in \mathcal{X}_l$ and every $u \in \dot{H}^s_{rad}$,

$$\|(-\Delta_l)^{\frac{s}{2}}[uX_l]\|_{L^2_{rad}} \geq \|(-\Delta_l)^{\frac{s}{2}}|u|\|_{L^2_{rad}},$$

whence it is easy to see that $\langle\tilde{\mathcal{L}}_{+,1}u, u\rangle_{L^2_{rad}} \geq \langle\tilde{\mathcal{L}}_{+,1}|u|, |u|\rangle_{L^2_{rad}}$. Thus, we conclude that $\psi \geq 0$, since ψ is a solution of the constrained minimization problem

$$\begin{cases} \langle\tilde{\mathcal{L}}_{+,1}u, u\rangle_{L^2_{rad}} \to \min \\ \|u\|_{L^2_{rad}} = 1 \end{cases}.$$

We now apply formula (2.5.6) for a sequence of Schwartz functions Ψ_N approximating $\psi_1(x) = \psi(x)\frac{x_1}{|x|} \in Ker[\mathcal{L}_+]$ in the $H^{2s}(\mathbf{R}^n)$ norm. We have

$$\begin{aligned} 0 &= \langle\partial_{x_1}\Phi, \mathcal{L}_+\psi_1\rangle = \lim_{N\to\infty}\langle\partial_{x_1}\Phi, \mathcal{L}_+\Psi_N\rangle = -b\lim_{N\to\infty}\langle\frac{x_1}{|x|^{b+2}}\Phi^p, \Psi_N\rangle = \\ &= -b\langle\frac{x_1}{|x|^{b+2}}\Phi^p, \psi_1\rangle = -b\int_{\mathbf{R}^n}\frac{x_1^2}{|x|^{b+3}}\Phi^p(x)\psi(x)dx < 0. \end{aligned}$$

which is a contradiction. Note that the last integral, the singularity at zero is integrable, since $b+1<n$, as $b<\frac{n}{2}, n\geq 2$. This concludes the proof of the proposition as well as the non-degeneracy of Φ. $\square$

2.6 Spectral and orbital stability of the waves

2.6.1 Index counting theory for (2.1.5)

For the eigenvalue problem in the form (2.1.5), we have that $\mathcal{J}$ is invertible and anti-symmetric, $\mathcal{J}^{-1}=\mathcal{J}^*=-\mathcal{J}$ and $X=H^s(\mathbf{R}^n), X^*=H^{-s}(\mathbf{R}^n), n\geq 1$. Note that according to Proposition 5, we have that $n(\mathcal{L}_+)=1$, while $n(\mathcal{L}_-)=0$, whence $n(\mathcal{L})=n(\mathcal{L}_+)+n(\mathcal{L}_-)=1$. In addition,

$$Ker[\mathcal{L}]=span[\begin{pmatrix} ker[\mathcal{L}_+] \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ ker[\mathcal{L}_-] \end{pmatrix}]=span[\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}].$$

Thus, we have that $\mathcal{J}: Ker[\mathcal{L}]\to (Ker[\mathcal{L}])^{\perp}$. For the matrix $\mathcal{D}$, we need to solve Ψ : $\mathcal{J}\mathcal{L}\Psi=\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}$. So, $\Psi=\begin{pmatrix} \mathcal{L}_+^{-1}\Phi_\omega \\ 0 \end{pmatrix}$ and the matrix $\mathcal{D}$ is a scalar, with

$$\mathcal{D}=\langle \mathcal{L}\Psi,\Psi\rangle=\langle \mathcal{L}_+^{-1}\Phi_\omega,\Phi_\omega\rangle. \tag{2.6.1}$$

According to the formula (1.1.11), we conclude

$$k_r+2k_c+2k_0^{\leq 0}=1-n(\mathcal{D}).$$

Clearly, in our situation, it is always the case that $k_c = k_0^{\leq 0} = 0$, and $k_r = 1$ exactly when $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle > 0$ and $k_r = 0$, when $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle < 0$. We formulate our result in the following corollary.

Corollary 4. *For the eigenvalue problem* (2.1.5)*, spectral stability occurs exactly when* $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle < 0$ *and instability is when* $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle > 0$. *Moreover, the instability presents itself as a single, real unstable mode.*

Remarks:

- This is reminiscent of the standard Vakhitov-Kolokolov criteria for stability of waves in situations with a simple Morse index, i.e. Morse index equal to one.

- The case $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle = 0$ presents a transition from stability to instability, so a pair of eigenvalues crosses from being purely imaginary $\pm \imath\sigma$ symmetric with respect to the origin to being a pair of real ones $\pm\lambda$. In this case, the algebraic multiplicity of the zero eigenvalue for $\mathcal{J}\mathcal{L}$ is four, up from the algebraic multiplicity two in all other cases, corresponding to the modulational invariance still present in the system.

2.6.2 Coercivity of $\mathcal{L}_+$

In this section show the coercivity property of $\mathcal{L}_+$ on the space $\{\Phi_\omega\}^\perp$.

Proposition 10. *Let* $(n, s, p, b) \in \mathcal{A}$ *and* $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle < 0$. *Then, the operator* $\mathcal{L}_+$ *is coercive on* $\{\Phi_\omega\}^\perp \cap H^s$. *That is, there exists* $\delta > 0$, *so that for all*

$$\langle \mathcal{L}_+\Psi, \Psi\rangle \geq \delta \|\Psi\|_{H^s}^2, \ \ \forall \Psi \perp \Phi_\omega. \tag{2.6.2}$$

Proof. This is a version of a well-known lemma in the theory, see for example Lemma 6.7 and Lemma 6.9 in [67]. Recall that we have already showed $Ker[\mathcal{L}_+] = \{0\}$ and

$n(\mathcal{L}_+) = 1$. According to a result in [49] (see also Lemma 6.4, [67]), which state that under these conditions for $\mathcal{L}_+$

$$\alpha := \inf\{\langle \mathcal{L}_+ f, f\rangle : f \perp \Phi_\omega, \|f\|_{L^2} = 1\} \geq 0.$$

Consider the associated constrained minimization problem

$$\inf_{\|f\|=1, f\perp\Phi_\omega} \langle \mathcal{L}_+ f, f\rangle. \tag{2.6.3}$$

Take a minimizing sequence $f_k : \|f_k\| = 1, f_k \perp \Phi_\omega$, so that

$$\alpha = \lim_k \langle \mathcal{L}_+ f_k, f_k\rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} f_k\|^2 + \omega - p\int |x|^{-b}\Phi^{p-1}(x) f_k^2(x)dx].$$

By the properties

$$\|(-\Delta)^{\frac{s}{2}} f\| \geq \|(-\Delta)^{\frac{s}{2}} f^*\|, \quad \int |x|^{-b}\Phi^{p-1}(x) f^2(x)dx \leq \int |x|^{-b}\Phi^{p-1}(x)(f^*)^2(x)dx,$$

we can assume, without loss of generality that f_k are bell-shaped. Note that by (1.1.7) and the Gagliardo-Nirenberg's inequality

$$0 < \int |x|^{-b}\Phi^{p-1}(x) f_k^2(x)dx \leq C\|f_k\|^2_{H^{\frac{b}{2}+\epsilon}} \leq C\|f_k\|_{H^s}^{\frac{b/2+\epsilon}{s}} \|f_k\|_{L^2}^{\frac{s-b/2-\epsilon}{s}}.$$

Note that for $\epsilon = \frac{s-\frac{b}{2}}{2}$, by Young's inequality, we can derive the estimate (recall $\|f_k\|_{L^2} = 1$)

$$\langle \mathcal{L}_+ f_k, f_k\rangle \geq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} f_k\|^2 - C_{n,s,b}.$$

It follows that $\sup_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 < \infty$. By bell-shapedness of $f_k : \|f_k\|_{L^2} = 1$, we have the pointwise bound $|f_k(x)| \leq C|x|^{-n/2}$. This, along with $\sup_k \|f_k\|_{H^s} < \infty$, easily implies compactness in any $L^q(|x|>1), 2<q<\infty$. On the other hand, in the bounded domain $|x|<1$, there is compactness in $L^2(|x|<1)$. So, assume without loss of generality that f_k itself converges to f strongly in all $L^q(|x|>1), 2<q<\infty$ and in $L^2(|x|<1)$. In particular, f is bell-shaped, as f_k are bell-shaped. So, $f \neq 0$.

In addition to that, we can assume, without loss of generality a weak convergence in $H^s(\mathbf{R}^n)$, $f_k \rightharpoonup f$. Note that by the weak convergence,

$$f \perp \Phi_\omega, \ \ \liminf_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 \geq \|(-\Delta)^{\frac{s}{2}} f\|^2, \ \ \|f\|_{L^2} \leq \liminf \|f_k\|_{L^2} = 1.$$

Finally, by splitting in $|x|<1$ and $|x|>1$ and applying the different appropriate strong convergences in each (and uniform bounds in H^s), we obtain

$$\lim_k \int |x|^{-b} \Phi^{p-1}(x) f_k^2(x) dx = \lim_k \int |x|^{-b} \Phi^{p-1}(x) f^2(x) dx.$$

All in all, we obtain

$$\langle \mathcal{L}_+ f, f\rangle \leq \liminf \langle \mathcal{L}_+ f_k, f_k\rangle = \alpha. \tag{2.6.4}$$

We will now show that $\alpha>0$. Assume for a contradiction that $\alpha=0$. Since $f\neq 0$ (recall $f\perp \Phi_\omega$), we see from (2.6.4) that the function $g=\frac{f}{\|f\|}$ is a minimizer for (2.6.3). Writing the Euler-Lagrange equation for it implies

$$\mathcal{L}_+ g = \gamma g + c\Phi_\omega. \tag{2.6.5}$$

Taking dot product with g and taking into account $\langle \mathcal{L}_+ g, g\rangle = 0, g\perp \Phi_\omega$ implies that $\gamma=0$. This means that $g = c\mathcal{L}_+^{-1}\Phi_\omega$. But then,

$$0 = \langle \mathcal{L}_+ g, g\rangle = c^2 \langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle.$$

Since $\langle \mathcal{L}_+^{-1}\Phi_\omega, \Phi_\omega\rangle \neq 0$ by assumption, it follows $c = 0$. But then, since $Ker[\mathcal{L}_+] = \{0\}$, (2.6.5) implies that $g = 0$, which is a contradiction.

So, we have shown that $\alpha > 0$. In other words,

$$\langle \mathcal{L}_+\Psi, \Psi\rangle \geq \alpha\|\Psi\|^2, \ \ \forall \Psi \perp \Phi_\omega. \tag{2.6.6}$$

Note that (2.6.2) is however stronger than (2.6.6), as it involves $\|\cdot\|_{H^s}$ norms on the right-hand side. Nevertheless, we show that it is relatively straightforward to deduce it from (2.6.6). Indeed, assume for a contradiction in (2.6.2), that $g_k : \|g_k\|_{H^s} = 1, g_k \perp \Phi_\omega$, so that $\lim_k \langle \mathcal{L}_+ g_k, g_k\rangle = 0$.

Taking into account (2.6.6), this is only possible if $\lim_k \|g_k\|_{L^2} = 0$. So,

$$1 = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \|g_k\|_{L^2}^2] = \lim_k \|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2.$$

But then, we achieve a contradiction

$$0 = \lim_k \langle \mathcal{L}_+ g_k, g_k\rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \omega\|g_k\|^2 - p\int |x|^{-b}\Phi^{p-1}(x) g_k^2(x) dx] = 1,$$

since $\lim_k \int |x|^{-b}\Phi^{p-1}(x) g_k^2(x) dx = 0$, similar to some previous steps, as $\sup_k \|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2} < \infty$, $\|g_k\| \to 0$. A contradiction is reached, which completes the proof of Proposition 10.

□

Knowing that $\mathcal{L}_+|_{\{\Phi\}^\perp} \geq 0$ (and we have established something stronger in (2.6.2)), we can establishing the coercivity of $\mathcal{L}_-$.

2.6.3 Coercivity of $\mathcal{L}_-$

In Proposition 5, we have already established that $\mathcal{L}_-$ is non-negative on the subspace $\{\phi\}^\perp$. We need a stronger coercivity statement.

Proposition 11. *Let $(n,p,s,b) \in \mathcal{A}$. Then, there exists $\delta > 0$, so that*

$$\langle \mathcal{L}_- \Psi, \Psi \rangle \geq \delta \|\Psi\|_{H^s}^2, \forall \Psi \perp \Phi. \tag{2.6.7}$$

Proof. Recall that in Proposition 6, we have already seen that $\mathcal{L}_-|_{\{\Phi\}^\perp} \geq 0$. We will show first that

$$\inf_{\|u\|=1, u \perp \phi} \langle \mathcal{L}_- u, u \rangle > 0.$$

Assuming not, it follows that $\mathcal{L}_-$ has a second eigenfunction in its kernel, $\tilde{\Phi} \perp \Phi$. But then, since $\mathcal{L}_+ < \mathcal{L}_-$, we have $\langle \mathcal{L}_+ \tilde{\Phi}, \tilde{\Phi} \rangle < \langle \mathcal{L}_- \tilde{\Phi}, \tilde{\Phi} \rangle = 0$. Hence, $\mathcal{L}_+|_{\tilde{\Phi},\Phi\}^\perp} < 0$ and in particular, $\mathcal{L}_+$ has at least two negative eigenvalues, a contradiction. Thus, there exists $\delta > 0$, so that

$$\langle \mathcal{L}_- u, u \rangle \geq \delta \|u\|^2, u \perp \Phi. \tag{2.6.8}$$

We would like to upgrade, as before, the right-hand side to $\|u\|_{H^s}^2$. To that end, we assume for a contradiction, that there is a sequence $u_k : u_k \perp \Phi, \|u_k\|_{H^s} = 1$, while $\lim_k \langle \mathcal{L}_- u_k, u_k \rangle = 0$. From (2.6.8), it follows that $\lim_k \|u_k\| = 0$, so $\lim_k \|(-\Delta)^{\frac{s}{2}} u_k\| = 1$. Similar to the proof of Proposition 10 above this yields a contradiction as well, since

$$0 = \lim_k \langle \mathcal{L}_- u_k, u_k \rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2}^2 + \omega \|u_k\|^2 - \int |x|^{-b} \Phi^{p-1}(x) u_k^2(x) dx] = 1.$$

With this, (2.6.7) is established.

□

With Propositions 10 and 11 at hand, we are ready for the orbital stability result.

2.6.4 Orbital stability of Φ_ω

With the coercivity results in Proposition 10, one might argue that we have all the necessary ingredients for orbital stability, according to [45]. We are however missing one key piece of information, namely the map $\omega \to \Phi_\omega$ does not have the required C^1 smoothness. Therefore, we need a direct proof, which does not use the smoothness of this map.

Proposition 12. *Let the key assumptions (1), (2), (3) be satisfied and $\mathcal{L}_\pm|_{\{\Phi_\omega\}^\perp} \geq 0$, φ is non-degenerate, i.e $ker[\mathcal{L}_+] = \{0\}$, then $e^{-i\omega t}\Phi_\omega$ is orbitally stable solution of* (2.1.1).

Proof. Our proof proceeds by contradictions. More specifically, there is $\epsilon_0 > 0$ and a sequence of initial data $u_k : \lim_k \|u_k - \Phi\|_{H^s(\mathbb{R}^n)} = 0$, so that

$$\sup_{0 \leq t < \infty} \inf_{\theta \in \mathbf{R}} \|u_k(t, \cdot) - e^{i\theta}\Phi\|_{H^s} \geq \epsilon_0.$$

Recall that $E[u] = \mathcal{H}[u] + \frac{w}{2}\mathcal{P}[u]$. Introduce

$$\epsilon_k := |E[u_k(t)] - E[\Phi_\omega]]| + |\mathcal{P}[u_k(t)] - \mathcal{P}[\Phi_\omega]]|.$$

Since we have assumed the conservation laws, we have that ϵ_k is conserved and $\lim_k \epsilon_k = 0$ For all $\epsilon > 0$, define

$$t_k = \sup\{\tau : \sup_{0 < t < \tau} \|u_k(t) - \Phi\|_{H^s(\mathbb{R}^n)} < \epsilon\}.$$

Note that $t_k > 0$, by the local well-posedness assumption (1). If we let $u_k = v_n + iw_k$, then for $t \in (0, t_k)$, we have $\|w_k(t)\|_{H^s(\mathbb{R}^n)} \leq \|u_k(t) - \Phi\|_{H^s(\mathbb{R}^n)} < \epsilon$. Define the modulations parameter $\theta_k(t)$ so that $[w_k(t) - \sin(\theta_k(t))\Phi] \perp \Phi$, which is

$$\sin(\theta_k(t))\|\Phi\| = \langle w_k(t), \Phi\rangle. \tag{2.6.9}$$

Since $|\langle w_k(t), \Phi\rangle| \leq \epsilon \|\Phi\|_{L^2}$, there is an unique small solution $\theta_k(t)$ of 2.6.9, with $|\theta_k(t)| \leq \epsilon$. In addition, we have

$$\|u_k(t,\cdot) - e^{i\theta_k(t)}\varphi\|_{H^s} \leq \|u_k(t,\cdot) - \Phi\|_{H^s} + |e^{i\theta_k(t)} - 1|\|\Phi\|_{H^s} \leq C_0\epsilon,$$

where $C_0 = C_0(\|\Phi\|_{H^s})$ only. Let

$$T_k = \sup\{\tau : \sup_{0<t<\tau} \|u_k(t) - e^{i\theta_k(t)}\varphi(.)\|_{H^s(\mathbb{R}^n)} < 2C_0\epsilon\}.$$

Clearly $T_k > t_k > 0$ and to complete the proof it is enough to show that for all $\epsilon > 0$ and large k $T_k = \infty$, since we can choose $\epsilon_k : \epsilon_k << \epsilon_0$.

For $t \in (0, T_k)$, write

$$\psi_k(t,.) = u_k(t,\cdot) - e^{i\theta_k(t)}\Phi$$

and decompose into real and imaginary parts of ψ_k and then project on the vector $\begin{pmatrix} \Phi \\ 0 \end{pmatrix}$. This yields

$$\begin{pmatrix} v_n(t,\cdot) - \cos(\theta_k(t))\Phi \\ w_k(t,\cdot) - \sin(\theta_k(t))\Phi \end{pmatrix} = \mu_k(t)\begin{pmatrix} \Phi \\ 0 \end{pmatrix} + \begin{pmatrix} \eta_k(t,\cdot) \\ \zeta_k(t,\cdot) \end{pmatrix}, \quad \begin{pmatrix} \eta_k(t,\cdot) \\ \zeta_k(t,\cdot) \end{pmatrix} \perp \begin{pmatrix} \Phi \\ 0 \end{pmatrix}. \tag{2.6.10}$$

Note that this decomposition implies $\eta_k(t) \perp \Phi$, while $\zeta_k(t) = w_k(t,\cdot) - \sin(\theta_k(t))\Phi \perp \Phi$ by the choice of θ_k, see (2.6.9). Taking L^2 norms in (2.6.10) yields

$$|\mu_k(t)|^2\|\Phi\|_{L^2}^2 + \|\eta_k(t)\|_{L^2}^2 + \|\zeta_k(t)\|_{L^2}^2 = \|\psi_k(t)\|_{L^2}^2 \leq 4C_0^2\epsilon^2. \tag{2.6.11}$$

We now exploit the properties of the conserved quantities. We have

$$\mathcal{P}[u_k(t)] = \int_{\mathbf{R}^n} |e^{i\theta_k(t)}\Phi + \psi_k(t)|^2 dx = \mathcal{P}[\Phi] + \|\psi_k(t,\cdot)\|_{L^2}^2 +$$
$$2\int_{\mathbf{R}^n} \Phi(x)\Re[e^{i\theta_k(t)}\psi_k(t,x)]dx.$$

But

$$\int \Phi(x)\Re[e^{i\theta_k(t)}\psi_k(t,x)]dx = \int \Phi(x)[\cos(\theta_k)(v_n - \cos(\theta_k)\Phi) - \sin(\theta_k)(w_k - \sin(\theta_k)\Phi)]dx$$
$$= \mu_k(t)\cos(\theta_k(t))\|\phi\|^2,$$

due to $\eta_k \perp \Phi$ and $w_k - \sin(\theta_k)\Phi \perp \Phi$.

It follows that,

$$\mathcal{P}[u_k(t)] = \mathcal{P}[\Phi] + \|\psi_k(t,\cdot)\|_{L^2}^2 + 2\mu_k(t)\cos(\theta_k(t))\|\Phi\|^2,$$

whence by recalling that $\|\psi_k(t,\cdot)\|_{L^2} \leq 2C_0\epsilon$, in $t: 0 < t < T_k$

$$|\mu_k(t)| \leq \frac{|\mathcal{P}[u_k(t)] - \mathcal{P}[\phi]| + \|\psi_k(t,\cdot)\|_{L^2}^2}{2\cos(\theta_k(t)\|\Phi\|^2} \leq C(\epsilon_k + \|\psi_k(t,\cdot)\|_{L^2}^2) \leq C(\epsilon_k + \epsilon^2). \quad (2.6.12)$$

In the last estimate, recall that $|\theta_k(t)| \leq C_0\epsilon << 1$, whence $\cos(\theta_k(t)) \geq \frac{1}{2}$ and the denominator is harmless.

Next, we take advantage of an expansion for $E[u_k(t)] - E[\Phi]$. Indeed, for all sufficiently small ϵ, we have

$$E[u_k(t)] - E[\Phi] = E[e^{i\theta_k(t)}\Phi + \psi_k] - E[\Phi] = E[\Phi + e^{-i\theta_k(t)}\psi_k] - E[\Phi].$$

Generally, for small perturbations of the wave $\varrho_1 + i\varrho_2 \in H^s(\mathbf{R}^n)$ and by taking into account the specific form of the energy functional E, we have

$$E[\Phi + (\varrho_1 + i\varrho_2)] - E[\Phi] = \frac{1}{2}[\langle \mathcal{L}_+ \varrho_1, \varrho_1 \rangle + \langle \mathcal{L}_- \varrho_2, \varrho_2 \rangle] + Err[\varrho_1, \varrho_2], \tag{2.6.13}$$

where

$$|Err[\varrho_1, \varrho_2]| \leq C \int_{\mathbf{R}^n} |x|^{-b} \left| |\Phi + \varrho_1 + i\varrho_2|^{p+1} - \Phi^{p+1} - (p+1)\Phi^p \varrho_1 - \frac{p(p+1)}{2}\varrho_1^2 - \frac{p+1}{2}\varrho_2^2 \right| dx.$$

Observe that by elementary second order Taylor expansions of the function $z \to |z|^{p+1}$, there is the pointwise estimate

$$\left| |\Phi + \varrho_1 + i\varrho_2|^{p+1} - \Phi^{p+1} - (p+1)\Phi^p \varrho_1 - \frac{p(p+1)}{2}\varrho_1^2 - \frac{p+1}{2}\varrho_2^2 \right| \\ \leq C(\|\Phi\|_{L^\infty})(|\varrho_1| + |\varrho_2|)^{\min(p+1,3)},$$

whence, according to (1.1.6), we obtain the estimate

$$|Err[\varrho_1, \varrho_2]| \leq C \int_{\mathbf{R}^n} |x|^{-b}(|\varrho_1|^{\min(p+1,3)} + |\varrho_2|^{\min(p+1,3)})dx \\ \leq C(\|\varrho_1\|_{H^s}^{\min(p+1,3)} + \|\varrho_2\|_{H^s}^{\min(p+1,3)}).$$

Apply this expansion (2.6.13) to

$$\varrho_1 + i\varrho_2 = e^{-i\theta_k(t)}\psi_k = [\cos(\theta_k)(\mu_k\Phi + \eta_k) + \sin(\theta_k)\zeta_k] + i\,[\cos(\theta_k)\zeta_k - \sin(\theta_k)(\mu_k\Phi + \eta_k)]\,.$$

From (2.6.11), we see that $\|\varrho_1\|_{H^s} + \|\varrho_2\|_{H^s} \leq C\epsilon$, so we can bound the contribution of $|Err[\varrho_1, \varrho_2]|$ as follows

$$|Err[\varrho_1, \varrho_2]| \leq C\epsilon^{\min(p-1),1}(\|\varrho_1\|^2_{H^s} + \|\varrho_2\|^2_{H^s}). \tag{2.6.14}$$

Furthermore,

$$\langle \mathcal{L}_+\varrho_1, \varrho_1\rangle = \langle \mathcal{L}_-\eta_k, \eta_k\rangle - C(\epsilon^3 + \epsilon_k + \epsilon^2(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s}) + \epsilon(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s})^2)$$
$$\langle \mathcal{L}_-\varrho_2, \varrho_2\rangle \geq \langle \mathcal{L}_-\zeta_k, \zeta_k\rangle - C(\epsilon^3 + \epsilon_k + \epsilon^2(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s}) + \epsilon(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s})^2).$$

Due to the coercivity of $\mathcal{L}_-$ (see Proposition 11 and more specifically 2.6.7) and $\mathcal{L}_+$, which was established in Proposition 10, we have that for some $\kappa > 0$ and since $\eta_k, \zeta_k \perp \Phi$, we have

$$\epsilon_k \geq |E[u_k(t)] - E[\Phi]| \geq$$
$$\geq \kappa(\|\eta_k\|^2_{H^s} + \|\zeta_k\|^2_{H^s}) - C(\epsilon^3 + \epsilon_k + \epsilon^2(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s})$$
$$+\epsilon^{\min(p-1),1}(\|\eta_k\|_{H^s} + \|\zeta_k\|_{H^s})^2),$$

or in other words, after some algebraic manipulations and for sufficiently small ϵ (depending only on absolute constant),

$$\|\eta_k(t)\|^2_{H^s} + \|\zeta_k(t)\|^2_{H^s} \leq C(\epsilon^3 + \epsilon_k), \tag{2.6.15}$$

where C is a constant that depends on the parameters, but not on ϵ and n. We claim that this implies that $T^*_k = \infty$ for sufficiently small ϵ (depending on the parameters only) and then sufficiently large k, so that $\epsilon_k << \epsilon$. Indeed, assume that $T_k^* < \infty$. Then

$$2C_0\epsilon = \limsup_{t\to T_k^*-} \|\psi_k(t)\|_{H^s} \leq C(|\mu_k(t)| + \|\eta_k(t)\|_{H^s} + \|\zeta_k(t)\|_{H^s}) \leq C(\epsilon^{\frac{3}{2}} + \sqrt{\epsilon_k}).$$

This last inequality is a contradiction, if $\epsilon : C_0\epsilon \geq C\epsilon^{\frac{3}{2}}$ and then $C\sqrt{\epsilon_k} < C_0\epsilon$. Both of this can be arranged, so we obtain the required contradiction, which establishes Proposition 12. □

Chapter 3

On the standing waves of the Schrödinger equation with concentrated nonlinearity

3.1 Introduction

The (focusing) nonlinear Schrödinger equation, with generalized power non-linearity

$$iu_t + \Delta u + |u|^{2\sigma} u = 0, (t,x) \in \mathbf{R} \times \mathbf{R}^n \tag{3.1.1}$$

is a basic model in theoretical physics and applied mathematics. Example of such physical application is fractional quantum mechanics and Lévy path integrals [60]. Other application cal also be found in water waves theory and practical engineering applications . Equation (3.1.1) has been studied extensively in the last fifty years, in particular with regards to the well-posedness of the Cauchy problem and the stability of its solitary waves. The well-posedness theory is classical by now, [17] states that local well-posedness holds for any $\sigma > 0$, whenever the data $u_0 \in H^s(\mathbf{R}^n), s \geq 0$. The global well-posedness results rely upon the conservation law, which state that the following quantities, namely the mass

$M(u)$ and the energy $E(u)$

$$\begin{aligned} M(u) &= \int_{\mathbf{R}^n} |u(t,x)|^2 dx = const. \\ E(u) &= \frac{1}{2}\int_{\mathbf{R}^n} |\nabla u(t,x)|^2 dx - \frac{1}{2\sigma+2}\int_{\mathbf{R}^n} |u(t,x)|^{2\sigma+2} dx = const \end{aligned}$$

are conserved. As such, solutions with initial data $u_0 \in H^1(\mathbf{R}^n)$ yield global solutions, whenever the problem is L^2 sub-critical, i.e. $\sigma < \frac{2}{n}$, while for $\sigma \geq \frac{2}{n}$, some initial data gives rise to finite time blow-ups. Interestingly, the ground states for (3.1.1) are stable exactly in the L^2 sub-critical range $\sigma < \frac{2}{n}$, while they are unstable in the supercritical regime $\sigma > \frac{2}{n}$. In the L^2 critical case, $\sigma = \frac{2}{n}$, the equation (3.1.1) exhibits an additional symmetry, the so-called quasi-conformal invariance, which allows one to exhibit special self-similar type solutions, which show that blows up also occurs in the critical case.

In this work, published in [70] we analyze a related model, the focusing non-linear Schrödinger equation with concentrated non-linearity.

Now, the focusing NLS with concentrated non-linearity is the following

$$\begin{cases} iu_t = ((-\Delta)^s - |u|^{2\sigma}\delta_0)u, \quad (t,x) \in \mathbf{R}\times\mathbf{R}^n \\ u(0,x) = u_0(x) \end{cases} . \tag{3.1.2}$$

Our definition of a solution is as follows: a continuous in x function u is a weak solution of (3.1.2), if it satisfies

$$\begin{aligned} & i\left(\langle u(t,\cdot),\psi(t,\cdot)\rangle - \langle u_0,\psi(0,\cdot)\rangle - \int_0^t \langle u(\tau,\cdot),\psi_\tau(\tau,\cdot)\rangle d\tau\right) = \\ & = \int_0^t \langle(-\Delta)^{\frac{s}{2}} u(\tau,\cdot),(-\Delta)^{\frac{s}{2}}\psi(\tau,\cdot)\rangle d\tau - \int_0^t |u(\tau,0)|^{2\sigma} u(\tau,0)\psi(\tau,0) d\tau \end{aligned}$$

for all test functions ψ. For the case of the standard Laplacian, i.e. $s = 1$, the model (3.1.2) has been used to model resonant tunneling, [52], the dynamics of mixed states, [66], quantum turbulence, [8], the generation of weakly bounded states close to the instability, [85] among others.

The fractional Laplacian perturbed by a delta potential, together with their self-adjoint extensions and various applications, have been recently considered in [13]. In the case of one spatial dimension, $n = 1$ and $s > \frac{1}{2}$, the local well-posedness as well as the conservation of mass and energy

$$M(u) = \int_{\mathbf{R}^n} |u(t,x)|^2 dx = const. \tag{3.1.3}$$

$$E(u) = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \frac{1}{2\sigma+2}|u(t,0)|^{2\sigma+2} = const. \tag{3.1.4}$$

was recently established in [13]. Even though the results in [13] are stated for the one dimensional case only, it seems plausible that they can be extended in any dimension n and $s > \frac{n}{2}$ using similar techniques. It is important to note that since our interest is in continuous in x functions, the natural spaces for well-posedness, in the scale of the Sobolev spaces, should be $H^s(\mathbf{R}^n), s > \frac{n}{2}$. Another reason why this is, a more natural class of problems to consider, is that we would like waves which belong to the energy space $H^s(\mathbf{R}^n)$, as dictated by the conservation of $E(u)$. As we shall see below, the solitary waves belong to this space only for $s > \frac{n}{2}$.

It has to be noted however, that it is certainly possible (and it is in fact considerably more challenging, the furthest one is from the threshold $s = \frac{n}{2}$) to consider (3.1.2) in cases where $s < \frac{n}{2}$, and this has been addressed, at least in low dimensional situations, in the recent papers, [3, 4, 5, 6, 7]. Regarding analysis of blow up solutions for the concentrated NLS (although not necessarily in the case of interest $s > \frac{n}{2}$), this was carried out recently in [4].

Our main interest in the model (3.1.2) are its solitary waves and their stability. More specifically, we consider solutions in the form $u = e^{i\omega t}\phi$, ϕ real-valued, which naturally satisfy the profile equation. This is again understood in the weak sense described above

$$(-\Delta)^s\phi + \omega\phi - |\phi(0)|^{2\sigma}\phi(0)\delta_0 = 0. \tag{3.1.5}$$

We take the opportunity to note that in many cases considered herein, one cannot expect the positivity of ϕ, as in the classical case. This is why, we keep the absolute value in (3.1.5).

The concentration phenomena for fractional differential equation has some physical motivation. We encourage motivated reader to further explore the appendix of [50] and also [29]. Note that both papers deal with the fact that $s \in (0,1)$. We believe that our results can motivates further investigation of such structure for $s > 1$.

The question for the stability of these waves, when $s = 1$, has been considered in several contexts recently, see [2], [5], [6] for the three dimensional case $n = 3$ and [1], for $n = 2$. Again, some of these works consider cases mostly outside of the range of consideration herein $s > \frac{n}{2}$.

Before we address the construction of the solitons (that is, solutions of (3.1.5)), and since our situation is a bit non-standard, we would like to outline the framework for the stability of the waves.

3.1.1 Linearized problem for the concentrated NLS

As is customary, the spectral/linearized stability of the standing waves, i.e. the solutions of (3.1.5), guides us in the study of the actual non-linear dynamics, when one starts close to these solutions[1]. More specifically, if we linearize around the solitary waves and ignore

[1]And indeed in the understanding of the ranges of σ that give global existence viz. a viz blow up, as discussed above

quadratic and higher order contributions, we obtain a linear system, whose spectral information plays a part in the dynamics. To that end, we take $u = e^{i\omega t}(\phi + v)$ and plug it in (3.1.2), ignoring any $O(v^2)$ term, utilizing (3.1.5) and setting $(v_1, v_2) := (\Re v, \Im v) = v$, we obtain

$$\begin{pmatrix} \Re v \\ \Im v \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} \begin{pmatrix} \Re v \\ \Im v \end{pmatrix}, \tag{3.1.6}$$

where the following fractional Schrödinger operators are introduced

$$\begin{aligned} \mathcal{L}_+ &= (-\Delta)^s + \omega - (2\sigma+1)|\phi(0)|^{2\sigma}\delta_0, \\ \mathcal{L}_- &= (-\Delta)^s + \omega - |\phi(0)|^{2\sigma}\delta_0. \end{aligned}$$

This formulas are heuristic in the sense that the operators $\mathcal{L}_\pm$ are not yet properly defined, in terms of domains, etc. This is generally not an easy task,[2] nevertheless, will appropriately be define in later section, see Section 3.2.2. Introducing the operators

$$\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{L} := \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix},$$

and the assignment $\begin{pmatrix} \Re v \\ \Im v \end{pmatrix} \to e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: e^{\lambda t}\vec{v}$, we obtain the following time-independent linearized eigenvalue problem

$$\mathcal{J}\mathcal{L}\vec{v} = \lambda\vec{v}. \tag{3.1.7}$$

Since we are interested in stability of waves, it will be appropriate to give a standard definition of stability as follow.

[2] Although, as it turns out, we shall need to restrict to the case $s > \frac{n}{2}$, which will make such definitions in a sense canonical

Definition 6. *The wave* $e^{i\omega t}\phi$ *is said to be* **spectrally unstable** *if the eigenvalue problem* (3.1.7) *has a solution* $(\lambda, \vec{v})$ *with* $\Re\lambda > 0$ *and* $\vec{v} \neq 0, \vec{v} \in D(\mathcal{L})$*. Otherwise, i.e. , if* (3.1.7) *has no non-trivial solutions with* $\Re\lambda > 0$*, we say that the wave is* **spectrally stable**.

We say that $e^{i\omega t}\phi$ *is* **orbitally stable** *solution of* (3.1.2)*, if for every* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon)$*, so that whenever* $\|u_0 - \phi\|_{H^s(\mathbf{R}^n)} < \delta$*, then the following statements hold.*

- *The solution* u *of* (3.1.2)*, in appropriate sense, with initial data* $u_0 \in H^s$ *is global in* $H^s(\mathbf{R}^n)$*, i.e.* $u(t, \cdot) \in H^s(\mathbf{R}^n)$.

- $$\sup_{t>0} \inf_{\theta \in \mathbf{R}} \|u(t, \cdot) - e^{-i(\omega t+\theta)}\phi(\cdot)\|_{H^s(\mathbf{R}^n)} < \epsilon.$$

The connection between the two main notions of stability, namely spectral and orbital stability, has been explored extensively in the literature - see for example the excellent book [55]. Generally speaking, spectral stability is a prerequisite for orbital stability, and in many cases of interest and under some natural, but not necessarily easy to check conditions, see Section 5.2.2 in [55], spectral stability implies orbital stability. In the case under consideration, the Assumption 5.2.5 a) on p. 136, [55] does not apply. So, we provide a direct proof of orbital stability via contradiction argument, in the cases of spectral stability, by following the original idea by T.E. Benjamin.

We should also point out that the reverse connection, namely spectral instability implies orbital instability. Basic heuristics (or even some more formal arguments) may suggest that this must be indeed the case. However, in terms of rigorous results, see for example [57] which simply states that if there is a positive instability mode present, via a direct ODE Lyapunov method spectral instability implies orbital instability. As in the stability case, there is no satisfactory general result that would cover our examples, so we leave our rigorous conclusions at the level of spectral instability of the waves and we do not comment further on (the likely) orbital instability thereof.

3.1.2 Main results

Before we present our existence result for the singular elliptic problem (3.1.5), let us introduce a function $\mathcal{G}_s^\lambda$, which will be a basic building block in our analysis. Namely, for all $\lambda > 0$ and $s > 0$,

$$\widehat{\mathcal{G}_s^\lambda}(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \lambda}.$$

We first state a few results related to the existence of the waves ϕ_ω[3], under some conditions on the parameters s, ω, n, which turn out to be necessary as well. Then, we discuss the fact that these waves are also minimizers of a Sobolev embedding inequality and we present its exact constant.

Existence of the waves ϕ_ω

Theorem 5. *(Existence standing waves of the concentrated NLS) Let $\omega > 0, s > \frac{n}{2}$ and $\sigma > 0$. Then, the function ϕ, with*

$$\hat{\phi}_\omega(\xi) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega} d\xi \right)^{-(1+\frac{1}{2\sigma})} \frac{1}{(2\pi|\xi|)^{2s} + \omega}.$$

is a solution of (3.1.5)*. Alternatively,*

$$\phi_\omega(x) = \frac{\mathcal{G}_s^\omega(x)}{(\mathcal{G}_s^\omega(0))^{1+\frac{1}{2\sigma}}}.$$

[3]Here the subscript ω is to emphasis the ω dependency of ϕ. Whenever such dependency is deemed necessary ϕ will be written as ϕ_ω

Interestingly, the conditions for ω and s in Theorem 5 are necessary for the existence of solutions $\phi \in H^s(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ of (3.1.5).

Proposition 13. *Let $\phi \in H^s(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ be a weak solution of (3.1.5). Then, $\omega(s - \frac{n}{2}) > 0$.*

The proof of Proposition 13 proceeds via the Pohozaev's identities, see Section 3.2.1 below.

In the process of the variational construction of the waves ϕ_ω, we establish a non-surprising connection to the problem for the optimal constant in the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$. More specifically, we establish that $\mathcal{G}_s = \mathcal{G}_s^1$ (and consequently ϕ_1) are H^s functions that saturate the Sobolev embedding, with the optimal Sobolev constant

$$s 2^n \pi^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right) \sin\left(\frac{n\pi}{2s}\right) \|u\|_{L^\infty}^2 \leq \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \|u\|_{L^2}^2. \tag{3.1.8}$$

We formulate the result in the following proposition.

Proposition 14. *The function $\mathcal{G}_s$ is a solution to the Sobolev embedding minimization problem*

$$\inf_{u \in \mathcal{S}: u \neq 0} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 + \|u\|_{L^2}^2}{\|u\|_{L^\infty}^2} = s 2^n \pi^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right) \sin\left(\frac{n\pi}{2s}\right).$$

Next, we turn our attention towards the stability results. We first state spectral stability/instability result, followed by orbital stability statements.

Stability characterization of the waves ϕ_ω

Theorem 6. *Let $n \geq 1$, $s > \frac{n}{2}$ and $\omega > 0$. Then, the waves $e^{i\omega t}\phi_\omega$ are spectrally stable if and only if*

$$0 < \sigma < \frac{2s}{n} - 1.$$

That is, the waves are stable for all $0 < \sigma < \frac{2s}{n} - 1$ *and unstable, when* $\sigma > \frac{2s}{n} - 1$*. Moreover, the instability is due to a presence of a single and simple real mode in the eigenvalue problem* (3.1.7).

Finally, before we state our orbital stability results, we need to make some natural assumptions regarding the well-posedness of the Cauchy problem (3.1.2).

Clearly, the orbital stability is only expected to hold for the case $\sigma < \frac{2s}{n} - 1$, so we assume that henceforth. We make the following **key assumptions**:

1. The solution map $g \to u_g$ has **continuous dependence on initial data property in a neighborhood of** ϕ. That is, there exists $T_0 > 0$, so that for all $\epsilon > 0$, there exists $\delta > 0$, so that whenever $g : \|g - \phi\|_{H^s} < \delta$, then $\sup_{0<t<T_0} \|u_g(t, \cdot) - e^{-i\omega t}\phi_\omega\|_{H^s} < \epsilon$.

2. **All initial data, sufficiently close to** ϕ_ω **in** H^s **norm, generates a global in time solution** u_g of (3.1.2). In addition, the L^2 norm and the Hamiltonian for these solutions are conserved. That is

$$M[u_g(t)] = M[g], E[u_g(t)] = E[g].$$

First, let us mention that this exact result is already available in the one dimensional case $n = 1$, [13]. For dimensions higher than one, $n \geq 2$, we conjecture that this is also the case. That is, in parallel with the results for the standard semi-linear Schrödinger equation, we make the following conjecture - please refer to the definitions of the operator $\mathcal{L}_c$ and $D(\mathcal{L}_c)$ in (3.2.7) and (3.2.8) below.

Conjecture 1. *For* $s > \frac{n}{2}$, $u_0 \in D(\mathcal{L}_c)$*, there exists* $T > 0$ *such that* (3.1.2) *is locally well-posed and* (3.1.3) *are conserved*[4] *up to a possible blow-up time. In addition, if* $0 < \sigma <$

[4] For the case $n = 1, s > \frac{1}{2}$ is exactly the results in [13]. Thus the conjucture is for $n \geq 2$

$\frac{2s}{n} - 1$, *the solutions are global, whereas for* $\sigma \geq \frac{2s}{n} - 1$, *finite time blow-up is possible, for some initial data.*

We are now ready to state our orbital stability results.

Theorem 7. *Let* $n \geq 1$, $\omega > 0$, $s > \frac{n}{2}$, $0 < \sigma < \frac{2s}{n} - 1$. *In addition, assume continuous dependence on initial data and globality of the solutions close to* ϕ_ω, *as outlined above. Then, the solitons* $e^{i\omega t}\phi_\omega$ *is orbitally stable.*

We plan our this chapter as follows. In Section 1.1, we prove the Pohozaev's identities, which in turn imply the necessary conditions for existence of the waves. Then, we discuss a self-adjoint realization of the operators $(-\Delta)^s + \lambda - c\delta_0$ for $\lambda > 0, c > 0$.

In Section 3.3, we first provide a variational construction of the waves ϕ_ω. The special relation to the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n), s > \frac{n}{2}$ is highlighted. The precise results are stated in the explicit formulas in Proposition 14. Finally, in Section 3.3.4, we discuss the lower part of the spectrum for operators in the form $(-\Delta)^s + \lambda - \mu\delta_0$. In the particular case of the linearized operator $\mathcal{L}_+$, this yields the non-degeneracy of the waves, which in this case takes the form $Ker(\mathcal{L}_+) = \{0\}$, due to the broken translational symmetry.

In Section 3.4, we start with a short introduction to the instability index count theory in general, and then we apply it to the spectral stability of the waves ϕ_ω. We explicitly calculate the relevant Vakhitov-Kolokolov quantity $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega \rangle$, which provides the stability characterization of the waves described in Theorem 6. Finally, under the necessary and sufficient condition for spectral stability, $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega \rangle < 0$, we derive the coercivity of $\mathcal{L}_+$ on $\{\phi_\omega\}^\perp$, which is of course crucial in the proof of the orbital stability.

3.2 Preliminaries

3.2.1 Pohozaev's identities and consequences

We would like to address the question for existence of solutions for the profile equation (3.1.5). Eventually, we will write them down explicitly, but first, we need to identify some necessary conditions on the parameters, which turn out to be essentially sufficient as well. The approach here is classical (even though our problem is certainly not) - we build some Pohozaev's identities, which proceeds by establishing relations between various norms of the eventual solution ϕ, which are *a priori* assumed finite. As a consequence, we find that the parameters must meet certain constraints.

Proposition 15. *Let $\phi \in H^s(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ be a weak solution of (3.1.5). Then,*

$$\|\phi\|_{L^2}^2 = \frac{2s-n}{2s\omega}|\phi(0)|^{2\sigma+2} \tag{3.2.1}$$

$$\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2}^2 = \frac{n}{2s}|\phi(0)|^{2\sigma+2}. \tag{3.2.2}$$

Proof. Testing (3.1.5) with ϕ itself results in

$$\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - |\phi(0)|^{2\sigma+2} = 0. \tag{3.2.3}$$

Next, we test (3.1.5) against $x \cdot \nabla\Psi$, for a test function Ψ. We obtain, by taking into account the commutation relation $[(-\Delta)^s, x\cdot\nabla] = 2s(-\Delta)^s$,

$$\begin{aligned}
\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}[x\cdot\nabla\Psi]\rangle &= \langle\phi, x\cdot\nabla(-\Delta)^s\Psi\rangle + 2s\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}\Psi\rangle = \\
&= -\langle x\cdot\nabla\phi, (-\Delta)^s\Psi\rangle + (2s-n)\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}\Psi\rangle = \\
&= -\langle(-\Delta)^{\frac{s}{2}}[x\cdot\nabla\phi], (-\Delta)^{\frac{s}{2}}\Psi\rangle + (2s-n)\langle(-\Delta)^{\frac{s}{2}}\phi, (-\Delta)^{\frac{s}{2}}\Psi\rangle.
\end{aligned}$$

This implies

$$\langle(-\Delta)^{\frac{s}{2}}\phi,(-\Delta)^{\frac{s}{2}}[x\cdot\nabla\Psi]\rangle+\langle(-\Delta)^{\frac{s}{2}}[x\cdot\nabla\phi],(-\Delta)^{\frac{s}{2}}\Psi\rangle=(2s-n)\langle(-\Delta)^{\frac{s}{2}}\phi,(-\Delta)^{\frac{s}{2}}\Psi\rangle.$$

Note that the right-hand side of this expression makes sense for[5], $\Psi=\phi$ whence

$$\langle(-\Delta)^{\frac{s}{2}}\phi,(-\Delta)^{\frac{s}{2}}[x\cdot\nabla\Psi]\rangle=(s-\frac{n}{2})\|(-\Delta)^{\frac{s}{2}}\phi\|^2. \tag{3.2.4}$$

Also[6]

$$\langle\phi,x\cdot\nabla\Psi\rangle=-n\langle\phi,\Psi\rangle-\langle x\cdot\nabla\phi,\Psi\rangle,$$

which also makes sense for $\Psi=\phi$, whence

$$\langle\phi,x\cdot\nabla\Psi\rangle=-\frac{n}{2}\|\phi\|_{L^2}^2. \tag{3.2.5}$$

Finally, we claim that $\langle\delta_0,x\cdot\nabla\Psi\rangle=0$ for each test function Ψ. Indeed, Introduce a radial function $V:\mathbf{R}^n\to\mathbf{R}$, which is smooth and non-negative function, supported on $\mathbf{B}:=\{x\in\mathbf{R}^n:\|x\|<1\}$ and normalized so that $\int_{\mathbf{R}^n}V(x)dx=1$. It is well-known, that in a distribution sense, one can approximate $N^nV(Nx)\to\delta_0$. That is , $\lim_{N\to\infty}\langle N^nV(N\cdot),f\rangle=f(0)$. So,

$$\begin{aligned}\langle\delta_0,x\cdot\nabla\Psi\rangle &= \lim_{N\to\infty}N^n\sum_{j=1}^n\int_{\mathbf{R}^n}V(Nx)x_j\partial_j\Psi(x)dx=\\ &= \lim_{N\to\infty}\left[-nN^n\int_{\mathbf{R}^n}V(Nx)\Psi(x)dx-N^{n+1}\int_{\mathbf{R}^n}|x|V'(Nx)\Psi(x)dx\right]=0,\end{aligned}$$

[5] One can formally take limits of $\Psi_n:\|\Psi_n-\phi\|_{H^s}\to0$

[6] Note that $\phi\in H^1(\mathbf{R}^n)$ makes this well-defined

since

$$N^{n+1}\int_{\mathbf{R}^n}|x|V'(Nx)dx=\int_{\mathbf{R}^n}|y|V'(y)dy=|\mathbb{S}^{n-1}|\int_0^\infty V'(\rho)\rho^n d\rho$$
$$=-n\int_0^\infty V(\rho)\rho^{n-1}d\rho=-n.$$

Putting $\langle\delta_0, x\cdot\nabla\Psi\rangle=0$ together with (3.2.4), (3.2.5), implies

$$(s-\frac{n}{2})\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2}^2-\frac{\omega n}{2}\|\phi\|_{L^2}^2=0. \tag{3.2.6}$$

Solving the system of equations (3.2.3) and (3.2.6) results in the relations (3.2.1) and (3.2.2). □

An immediate corollary of these results follows from the positivity of the norms in both (3.2.1) and (3.2.2). This consequence is given by Proposition 13. Namely, either $\omega>0, s>\frac{n}{2}$ or $\omega<0, s<\frac{n}{2}$. Clearly, the case $\omega>0, s>\frac{n}{2}$ is a more physical situation to consider - after all, one has the embedding $H^s(\mathbf{R}^n)\hookrightarrow C(\mathbf{R}^n)$ and hence functions in the class $H^s(\mathbf{R}^n)$ are automatically continuous.

3.2.2 The self-adjoint operators $(-\Delta)^s+\lambda-c\delta_0$

In this section, we introduce the necessary self-adjoint extensions of the operators formally introduced as $(-\Delta)^s+\lambda-c\delta_0$. There has been quite a bit of recent work on the subject, see [1, 3, 2, 6, 13] among others. In these papers, the authors have introduced various (and sometimes all) self-adjoint extensions of such objects, under different assumptions on the parameters. As dictated by the results of Proposition 13, we work under the assumption $s>\frac{n}{2}$, which simplifies matters quite a bit, in the sense that the self-adjoint extension, which generates the standard quadratic form, is canonical.

More specifically, for given constants $\lambda > 0, c > 0$, we introduce the skew-symmetric quadratic form

$$\mathcal{Q}_c(f,g) = \langle \sqrt{((-\Delta)^s + \lambda)f}, \sqrt{((-\Delta)^s + \lambda)g} \rangle - cf(0)\bar{g}(0), f,g \in D(\mathcal{Q})$$

with domain $D(\mathcal{Q}) = H^s(\mathbf{R}^n)$. Note that as $D(\mathcal{Q}) \subset C(\mathbf{R}^n)$, the values $f(0), g(0)$ make sense. In addition, the form $\mathcal{Q}$ is bounded from below. This is a consequence of the Sobolev embedding $H^\alpha \hookrightarrow L^\infty(\mathbf{R}^n), \alpha > \frac{n}{2}$. Indeed, choose $\alpha : \frac{n}{2} < \alpha < s$ and estimate via the Sobolev and the Gagliardo-Nirenberg's inequalities

$$\begin{aligned}\mathcal{Q}_c(f,f) &\geq c_\lambda \|f\|_{H^s}^2 - k_\alpha \|f\|_{H^\alpha}^2 \geq c_\lambda \|f\|_{H^s}^2 - k_\alpha \left(\frac{c_\lambda}{2k_\alpha}\|f\|_{H^s}^2 + d_{\alpha,\lambda}\|f\|_{L^2}^2\right) \\ &\geq D_{\alpha,\lambda}\|f\|_{\dot{H}^s}^2 - M_{\alpha,\lambda}\|f\|_{L^2}^2.\end{aligned}$$

In addition, $\mathcal{Q}$ is closed form, as $\|f\|_{H^s}^2 \sim \mathcal{Q}(f,f) + M\|f\|^2$, for large enough M. According to the standard theory for quadratic forms, see Theorem VIII.15 in [72], there is an unique self-adjoint operator $\mathcal{L}_c$, so that

$$D(\mathcal{L}_c) \subset D(\mathcal{Q}),\ \ \mathcal{D}_c(f,g) = \langle \mathcal{L}_c f, g \rangle,\ \ \forall f \in D(\mathcal{L}_c), g \in D(\mathcal{Q}).$$

Identifying the exact form of $\mathcal{L}_c$ may not be an easy task, in general. In our case, this is not so hard, as the operator has been essentially constructed in previous works, see [13] for the one dimensional case. We follow their notations and approach. To this end, introduce the Green's function of the operator $(-\Delta)^s + \lambda$, namely the function $\mathcal{G}_s^\lambda$, so that

$$((-\Delta)^s + \lambda)\mathcal{G}_s^\lambda = \delta_0.$$

By taking the Fourier transform, we can write the following formula for $\mathcal{G}_s^\lambda$

$$\widehat{\mathcal{G}_s^\lambda}(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \lambda}.$$

Clearly, since $s > \frac{n}{2}$, $\mathcal{G}_s^\lambda \in H^s(\mathbf{R}^n) \subset C(\mathbf{R}^n)$. Introduce the domain of the operator $\mathcal{L}_c$ as

$$D(\mathcal{L}_c) = \{\psi \in H^s(\mathbf{R}^n) : \psi = g + c\psi(0)\mathcal{G}_s^\lambda, g \in H^{2s}(\mathbf{R}^n)\} \subset H^s(\mathbf{R}^n). \tag{3.2.7}$$

With this domain, its action is defined as

$$\mathcal{L}_c\psi := ((-\Delta)^s + \lambda)g. \tag{3.2.8}$$

Note that for $\psi \in D(\mathcal{L}_c)$ and $h \in H^s(\mathbf{R}^n) = D(\mathcal{Q})$, we have

$$\begin{aligned}\langle \mathcal{L}_c\psi, h\rangle &= \langle ((-\Delta)^s + \lambda)g, h\rangle \\ &= \langle \sqrt{(-\Delta)^s + \lambda}\psi, \sqrt{(-\Delta)^s + \lambda}h\rangle - c\psi(0)\langle ((-\Delta)^s + \lambda)\mathcal{G}_s^\lambda, h\rangle \\ &= \langle \sqrt{(-\Delta)^s + \lambda}\psi, \sqrt{(-\Delta)^s + \lambda}h\rangle - c\psi(0)\bar{h}(0) = \mathcal{Q}_c(\psi, h).\end{aligned}$$

Thus, $\mathcal{L}_c$ is a closed symmetric operator, with a quadratic form precisely $\mathcal{Q}$. Note that the role of the constant λ in the definition is to ensure that the function $\widehat{\mathcal{G}_s^\lambda}$ has no singularity at $\xi = 0$. We now need to show that $\mathcal{L}_c$ is precisely the unique self-adjoint operator with this property.

Lemma 8. *The closed symmetric operator $\mathcal{L}_c$, with domain given in* (3.2.7) *and whose action is defined in* (3.2.8)*, is self-adjoint.*

Proof. For technical reasons, let us first assume the condition

$$c\mathcal{G}_s^\lambda(0) \neq 1. \tag{3.2.9}$$

With that, we work on a different representation on $D(\mathcal{L}_c)$. More precisely, we would like to write ψ purely in terms of g. To this end, we evaluate the identity relating ψ and g at $x=0$. We obtain the equation for $\psi(0)$

$$\psi(0) = g(0) + c\psi(0)\mathcal{G}_s^\lambda(0).$$

This equation has a solution, under the condition (3.2.9),

$$\psi(0) = \frac{g(0)}{1-c\mathcal{G}_s^\lambda(0)}. \tag{3.2.10}$$

One can now write, for $c \neq \frac{1}{\mathcal{G}_s^\lambda(0)}$,

$$D(\mathcal{L}_c) = \{\psi \in L^2(\mathbf{R}^n) : \psi = g + c\mathcal{G}_s^\lambda \frac{g(0)}{1-c\mathcal{G}_s^\lambda(0)}, g \in H^{2s}(\mathbf{R}^n)\},$$

which describes $D(\mathcal{L}_c)$ purely in terms of an arbitrary function $g \in H^{2s}(\mathbf{R}^n)$.

In order to show that $\mathcal{L}_c = \mathcal{L}_c^*$, it suffices to show that it has a real number in its resolvent set, see Corollary on p. 137, [71]. To this end, let $M >> 1$, and we will show that $-M-\lambda \in \rho(\mathcal{L}_c)$. Let $f \in L^2(\mathbf{R}^n)$ is arbitrary and consider

$$(\mathcal{L}_c + M - \lambda)\psi = f. \tag{3.2.11}$$

This is of course equivalent to the equation $((-\Delta)^s + M)g = f$, where

$$\psi = g + c\mathcal{G}_s^\lambda \frac{g(0)}{1-c\mathcal{G}_s^\lambda(0)}.$$

which has the unique solution $g = ((-\Delta)^s + M)^{-1} f \in H^{2s}(\mathbf{R}^n)$. Thus, we can uniquely solve (3.2.11) as follows

$$\psi = g + c\mathcal{G}_s^\lambda \frac{g(0)}{1 - c\mathcal{G}_s^\lambda(0)}, \quad g = ((-\Delta)^s + M)^{-1} f \in H^{2s}(\mathbf{R}^n).$$

In terms of estimates $\|g\|_{H^{2s}} \le C_M \|f\|_{L^2}$ and consequently

$$\|\psi\|_{L^2} \le \|g\|_{L^2} + C|g(0)| \le \|g\|_{H^s} \le C_M \|f\|_{L^2}.$$

This shows that all $\mathcal{L}_c{}'s$, with c satisfying (3.2.9) are self-adjoint. What about c, which fails (3.2.9)? In this case

$$1 = c\mathcal{G}_s^\lambda(0) = c\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \lambda} d\xi$$

It follows that for every $\tilde{\lambda} \neq \lambda$, say $\tilde{\lambda} > \lambda$, we have that $c\mathcal{G}_s^{\tilde{\lambda}}(0) \neq 1$. Thus, following the scheme described in the previous arguments, the operator $\mathcal{L}_c^{\tilde{\lambda}}$, formally defined through $(-\Delta)^s + \tilde{\lambda} - c\delta_0$ is self-adjoint. This means that

$$\mathcal{L}_c = \mathcal{L}_c^\lambda = \mathcal{L}_c^{\tilde{\lambda}} + (\lambda - \tilde{\lambda}) Id,$$

is self-adjoint as well. □

Remark 1. *In particular, we have the following important formula*[7] *for the action of* $\mathcal{Q}_c$ *on functions* $\psi \in H^s$*, with* $\psi(0) = 0$*,*

$$\mathcal{Q}_c(\psi, \psi) = \|(-\Delta)^{\frac{s}{2}} \psi\|_{L^2}^2 + \lambda \|\psi\|_{L^2}^2. \tag{3.2.12}$$

[7] This is important because all our calculations for the linearized operators will be at the level of the quadratic forms

3.3 Variational construction of the waves ϕ_ω and spectral consequences

We first construct, in a variational manner, some approximate solutions to the elliptic profile problem (3.1.5). This will turn out to be important in our subsequent considerations.

3.3.1 Variational constructions

Let $\omega, \sigma > 0$. For a radial function $V : \mathbf{R}^n \to \mathbf{R}$ as before[8] and $N >> 1$, consider the functional

$$I_{\omega,N}[u] = \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx + \omega \int_{\mathbb{R}^n} u^2 dx}{\left(\int_{\mathbb{R}^n} N^n V(Nx)|u|^{2\sigma+2} dx\right)^{\frac{1}{\sigma+1}}},$$

and the corresponding unconstrained variational problem $I_{\omega,N}[u] \to \min$. Clearly, $I_{\omega,N}[u] > 0$, so its optimal value is well-defined

$$m_N(\omega) := \inf_{u \in \mathcal{S}, u \neq 0} I_{\omega,N}[u].$$

Proposition 16. *Let $s > \frac{n}{2}$. Then the unconstrained minimization problem*

$$I_{\omega,N}[u] \to \min \tag{3.3.1}$$

has a real-valued solution $\phi_N \in H^s(\mathbf{R}^n) \cap L^\infty$, in particular $m_N(\omega) > 0$. Moreover, ϕ_N may be chosen to satisfy

$$N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2} dx = 1.$$

[8] That is, V is non-negative, radial, smooth and supported on the unit ball $\mathbf{B} \subset \mathbf{R}^n$, with $\int_{\mathbf{B}} V(x) dx = 1$

Finally, ϕ_N satisfies the Euler-Lagrange equation

$$(-\Delta)^s\phi_N + \omega\phi_N - m_N(\omega)N^nV(Nx)|\phi_N|^{2\sigma}\phi_N = 0 \tag{3.3.2}$$

in distributional sense.

Proof. Since $\|V\|_{L^1} = 1$, we have for $u \in H^s(\mathbf{R}^n) \subset L^\infty$,

$$\left(N^n\int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2}dx\right)^{\frac{1}{\sigma+1}} \leq \|u\|^2_{L^\infty(\mathbf{R}^n)} \leq C\|u\|^2_{H^s(\mathbf{R}^n)}, \tag{3.3.3}$$

whence (3.3.1) is a well-posed variational problem and $m_N(\omega) > 0$. Next, due to dilation properties of the functional $I_{\omega,N}$, we can assume that the infimum is taken only over functions with the normalization property

$$N^n\int_{\mathbb{R}^n} V(Nx)|u(x)|^{2\sigma+2}dx = 1.$$

Let u_k be a minimizing sequence such that $\int_{\mathbb{R}^n} N^nV(Nx)|u_k|^{2\sigma+2}dx = 1$ and hence

$$\lim_k(\|(-\Delta)^{\frac{s}{2}}u_k\|^2_{L^2} + \omega\|u_k\|^2_{L^2}) = m_N(\omega).$$

By weak compactness, we can select a weakly convergent subsequence (which we assume is just $\{u_k\}$), $u_k \rightharpoonup u$. By the lower semi-continuity of the norms, with respect to weak convergence

$$\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2} + \omega\|u\|^2_{L^2} \leq \liminf_k(\|(-\Delta)^{\frac{s}{2}}u_k\|^2_{L^2} + \omega\|u_k\|^2_{L^2}) = m_N(\omega). \tag{3.3.4}$$

We now show that $\{u_k\}$ is pre-compact in $C(\mathbf{B})$. Indeed, since $s > \frac{n}{2}$, we have by the Sobolev embedding that

$$\|u_k\|_{C^\gamma(\mathbf{R}^n)} \leq C\|u_k\|_{H^s}, \tag{3.3.5}$$

for $0 < \gamma < \{s - \frac{n}{2}\}$. Consequently, u_k are uniformly Hölder-continuous, hence equicontinuous as elements of $C(\mathbf{B})$. Also, $\{u_k\}$ is a totally bounded by (3.3.5). By Arzelà-Ascoli, we have that $\{u_k\}_{k=1}^\infty$ is pre-compact in $C(\mathbf{B})$, i.e , for a subsequence, which we again assume it is just u_k, we have that $u_k \rightrightarrows_{\mathbf{B}} u$. It is now clear that

$$1 = \lim_k N^n \int_{\mathbf{R}^n} V(Nx)|u_k(x)|^{2\sigma+2}dx = N^n \int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2}dx. \tag{3.3.6}$$

Thus, by (3.3.4)and (3.3.6), we conclude that $I_{\omega,N}[u] \leq m_N(\omega)$. This, by the definition of $m_N(\omega)$ means that $I_{\omega,N}[u] = m_N(\omega)$. In particular,

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 = m_N(\omega),$$

so u actually solves the minimization problem (3.3.1). This is the solution ϕ_N that we were interested in.

Next we show that the minimizer satisfies the Euler Lagrange equation. To that end take an arbitrary test function h and let $\epsilon > 0$ consider $u = \phi_N + \epsilon h$, and recall that

$$\int N^n V(Nx)|\phi_N|^{2\sigma+2}dx = 1.$$

Since ϕ_N is a minimizer we have that $I_{\omega,N}[u] \geq m_N(\omega)$. Expanding in powers of ϵ, we obtain

$$\int |(-\Delta)^{s/2}(\phi_N + \epsilon h)|^2 dx + \omega \int (\phi_N + \epsilon h)dx = m_N(\omega) + 2\epsilon\langle((-\Delta)^s + \omega)\phi_N, h\rangle + O(\epsilon^2).$$

Similarly,

$$\begin{aligned}
&\int V(Nx)|\phi_N+\epsilon h|^{2\sigma+2}dx\\
&=\int V(Nx)|\phi_N|^{2\sigma+2}dx+(2\sigma+2)\epsilon\int V(Nx)|\phi_N|^{2\sigma}\phi_N h+O(\epsilon^2)\\
&=1+(2\sigma+2)\epsilon\int V(Nx)|\phi_N|^{2\sigma}\phi_N h+O(\epsilon^2).
\end{aligned}$$

Thus, after rationalizing the denominator we arrived at

$$\begin{aligned}
I_{\omega,N}&=\frac{m_N(\omega)+2\epsilon\langle((-\Delta)^s+\omega)\phi_N,h\rangle+O(\epsilon^2)}{1+2\epsilon\int N^nV(Nx)|\phi_N|^{2\sigma}\phi_N hdx+O(\epsilon^2)}\\
&=m_N(\omega)+2\epsilon\langle((-\Delta)^s+\omega)\phi_N-m_N(\omega)N^nV(Nx)|\phi_N|^{2\sigma}\phi_N,h\rangle+O(\epsilon^2).
\end{aligned}$$

Since this hold for any arbitrary test function h and any $\epsilon>0$ we have that ϕ_N solves (3.3.2). □

Next, we have the following technical result.

Lemma 9. *There exists constants $C_1(\omega),C_2(\omega)$, but independent on N, so that*

$$C_1(\omega)\leq m_N(\omega)\leq C_2(\omega).$$

Furthermore, the sequence $\{\phi_N\}_{N=1}^{\infty}$, is a pre-compact in every set of the form $C(K)$, where K is a compact subset of $\mathbf{R}^n$.

Proof. The lower bound, with a constant independent on N follows from (3.3.3). The upper bound follows by testing against a concrete function like $u_0(x)=e^{-|x|^2}$. Since $\frac{1}{3}<u_0(x)\leq 1$, on the support of $V(Nx), N\geq 1$, we have that

$$m_N(\omega)\leq I_{\omega,N}[u_0]\leq 9\left(\|(-\Delta)^{\frac{s}{2}}u_0\|_{L^2}^2+\omega\|u_0\|_{L^2}^2\right)=:C_2(\omega).$$

Next, since ϕ_N satisfy $N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2}dx = 1$, we have that

$$I_{\omega,N}[\phi_N] = \|(-\Delta)^{\frac{s}{2}}\phi_N\|_{L^2}^2 + \|\phi_N\|_{L^2}^2 = m_N(\omega).$$

Thus, by Sobolev embedding

$$\|\phi_N\|_{C^\gamma(\mathbf{R}^n)} \leq C\|\phi_N\|_{H^s} \leq C(\omega)m_N(\omega) \leq C_3(\omega).$$

for $0 < \gamma < \min\{1, s - \frac{n}{2}\}$. It follows that for each compact $K \subset \mathbf{R}^n$, $\{\phi_N\}$ is pre-compact in $C(K)$ by Arzela-Ascolli's theorem. □

Clearly, Lemma 9 allows us to take a convergent (sub) sequence as $N \to \infty$. We wish to learn what the limit is expected to be. It turns out that it is nothing but the minimizer for the Sobolev inequality $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$. We justify that in the next section.

3.3.2 Relation to the minimizers for the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$

For $s > \frac{n}{2}, \omega > 0$, we study up the functional

$$J_\omega[u] = \frac{\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2}{\|u\|_{L^\infty}^2}$$

and the corresponding minimization problem $J_\omega[u] \to \min$. Finally, denote

$$c^2(\omega) := \inf_{u \in \mathcal{S}: u \neq 0} J_\omega[u].$$

The described optimization problem has a clear analytical interpretation, namely that c is the exact constant in the Sobolev embedding estimate

$$c(\omega)\|u\|_{L^\infty} \le |||u|||_{H^s} := \sqrt{\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2}.$$

We know from the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$ that c is well-defined and we can alternatively introduce it as follows $c(\omega) = \sup\{C > 0 : C\|u\|_{L^\infty} \le |||u|||_{H^s}, \forall u \in \mathcal{S}\}$.

Another useful observation is that one can assume, without loss of generality, that in the infimum procedure described above, $\|u\|_{L^\infty}$ is replaced by $|u(0)|$. That is ,

$$c^2(\omega) = \inf_{u \in H^s : u(0) \neq 0} \frac{\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2}{|u(0)|^2}.$$

Lemma 10. *Let $s > \frac{n}{2}, \omega > 0$ and $\gamma < \min(1, s - \frac{n}{2})$. Then, there exists $C = C(s, \omega, \gamma)$, so that*

$$c^2(\omega) \le m_N(\omega) \le c^2(\omega) + CN^{-\gamma} \tag{3.3.7}$$

Proof. By (3.3.3), we see that for every $N \ge 1$, $I_{\omega,N} \ge J_\omega$, whence $m_N(\omega) \ge c^2(\omega)$.

For the opposite inequality, observe first that since $m_N(\omega) \le C_2(\omega)$, we can take

$$m_N(\omega) = \inf_{u \in \mathcal{S} : u \neq 0} I_{\omega,N}[u] = \inf_{N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2}dx = 1;\ \ |||u|||_{H^s} \le 10C_2} I_{\omega,N}[u].$$

So, let $u \in H^s : N^n \int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2}dx = 1;\ \ |||u|||_{H^s} \le 10C_2$. Recall that for every $q > 1$, there is C_q, so that for $a > 0, b > 0$ $|a^q - b^q| \le C_q|a - b|(a^{q-1} + b^{q-1})$. As a consequence, and by Sobolev embedding and together with the definition $\|u\|_{C^\gamma} := \sup_{x \neq 0} \frac{|u(x) - u(0)|}{|x|^\gamma}$ we have,

$$\left||u(x)|^{2\sigma+2} - |u(0)|^{2\sigma+2}\right| \le C_\sigma|u(x) - u(0)|\|u\|_{L^\infty}^{2\sigma+1} \le C_{\gamma,\sigma}|x|^\gamma\|u\|_{C^\gamma(\mathbf{R}^n)}^{2\sigma+1}$$

$$\leq C_{\gamma,\sigma}|x|^{\gamma}\|u\|_{H^s}^{2\sigma+1},$$

and since $\|\!|u|\!\|_{H^s} \leq 10C_2$, we conclude

$$\big||u(x)|^{2\sigma+2} - |u(0)|^{2\sigma+2}\big| \leq C_{\gamma,\sigma,\omega}|x|^{\gamma}. \tag{3.3.8}$$

It follows that

$$\begin{aligned}
\big||u(0)|^{2\sigma+2} - 1\big| &= \left||u(0)^{2\sigma+2} - N^n \int_{\mathbf{R}^n} V(Nx)|u(x)|^{2\sigma+2}dx\right| = \\
&= N^n \left|\int_{\mathbf{R}^n} V(Nx)[|u(x)|^{2\sigma+2} - |u(0)|^{2\sigma+2}dx\right| \\
&\leq C_{\gamma,\sigma,\omega}N^n \int_{\mathbf{R}^n} V(Nx)|x|^{\gamma}dx \\
&\leq C_{\gamma,\sigma,\omega}N^{-\gamma} \int_{\mathbf{R}^n} V(y)|y|^{\gamma}dy \leq C_{\gamma,\sigma,\omega}N^{-\gamma},
\end{aligned}$$

so $|u(0)| \leq 1 + C_{\gamma,\sigma,\omega}N^{-\gamma}$. It follows that

$$\begin{aligned}
m_N(\omega) &= \inf_{N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2}dx=1;\ \|\!|u|\!\|_{H^s}\leq 10C_2} \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 \leq \\
&\leq (1 + C_{\gamma,\sigma,\omega}N^{-\gamma}) \inf_{\|\!|u|\!\|_{H^s}\leq 10C_2, u(0)\neq 0} \frac{\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2}{|u(0)|^2} \\
&\leq c^2 + C_{\gamma,\sigma,\omega}N^{-\gamma}.
\end{aligned}$$

□

We now take limit as $N \to \infty$. In view of our discussion so far, it is not surprising that this yields the minimizers for the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow L^{\infty}(\mathbf{R}^n)$. In turn, this allows us to present an explicit formula for the solutions of (3.1.5) and to interpret them as minimizers of the Sobolev embedding problem.

3.3.3 Description of the solutions for the profile equation (3.1.5)

Lemma 11. *Let* $s > \frac{n}{2}, \omega > 0$. *Then, for every constant* $C \neq 0$, *the function*

$$\hat{\phi}(\xi) = \frac{C}{(2\pi|\xi|)^{2s} + \omega}, \tag{3.3.9}$$

is a minimizer of the problem $\min_{u \in H^s} J_\omega[u]$. *In particular, the optimal Sobolev constant is given by the formula*

$$c^2(\omega) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega} d\xi \right)^{-1}.$$

Proof. From Lemma 10, it follows that $\lim_N m_N(\omega) = c^2(\omega)$. In addition, as we have pointed out, maximizers can be taken, with the property $\|\phi_N\|_{H^s} \leq C(\omega)$. As $H^s(\mathbf{R}^n)$ embeds in $C^\gamma(\mathbf{R}^n), 0 < \gamma < s - \frac{n}{2}$ and this is compact embedding on bounded domains, we can select

$$\phi_N : N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2} dx = 1,$$

so that ϕ_N is uniformly convergent, on the compact subsets of $\mathbf{R}^n$ to $\phi \in H^s(\mathbf{R}^n)$.

We will show that $\phi(0) = 1$ and ϕ is in the form (3.3.9). We have, for each $N \geq 1$,

$$\begin{aligned} \left|1 - |\phi(0)|^{2\sigma+2}\right| &\leq N^n \int_{\mathbf{R}^n} V(Nx) \left||\phi_N(x)|^{2\sigma+2} - |\phi(0)|^{2\sigma+2}\right| dx \\ &\leq C_\sigma(\|\phi_N\|_{L^\infty}^{2\sigma+1} + |\phi(0)|^{2\sigma+1}) N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x) - \phi(0)| dx. \end{aligned}$$

But $\|\phi_N\|_{L^\infty} \leq \|\phi_N\|_{H^s} < C(\omega)$, while

$$|\phi_N(x) - \phi(0)| \leq |\phi_N(x) - \phi_N(0)| + |\phi_N(0) - \phi(0)| \leq C_\gamma |x|^\gamma + |\phi_N(0) - \phi(0)|.$$

Plugging this back in our estimate for $|1-|\phi(0)|^{2\sigma+2}|$, we obtain, for each $0<\gamma<s-\frac{n}{2}$,

$$|1-|\phi(0)|^{2\sigma+2}| \leq C|\phi_N(0)-\phi(0)|+CN^n\int_{\mathbf{R}^n}V(Nx)|x|^\gamma dx \leq C|\phi_N(0)-\phi(0)|+CN^{-\gamma}.$$

Clearly, the expression on the right goes to zero as $N\to\infty$, as $\phi_N \rightrightarrows_{\mathbf{B}} \phi$. By adjusting the sign of ϕ_N, if necessary, this implies that we can take $\phi(0)=\lim_N \phi_N(0)=1$.

Next, ϕ_N satisfies the Euler-Lagrange equation (3.3.2). Test this equation with ψ. We obtain

$$\langle\phi_N,((-\Delta)^s+\omega)\psi\rangle = m_N(\omega)N^n\int_{\mathbf{R}^n}V(Nx)|\phi_N|^{2\sigma}\phi_N(x)\psi(x)dx. \tag{3.3.10}$$

Taking limits in N then yields, after taking into account $\phi(0)=1$,

$$\langle\phi,((-\Delta)^s+\omega)\psi\rangle = c^2(\omega)\psi(0). \tag{3.3.11}$$

In other words, ϕ satisfies the equation

$$((-\Delta)^s+\omega)\phi-c^2\delta_0=0. \tag{3.3.12}$$

in a distributional sense.

By taking ψ in (3.3.10) to be an appropriate approximation of the function $\mathcal{G}_s^\omega(\cdot+x)$, we conclude that

$$\phi(x)=const.\mathcal{G}_s^\omega(x)$$

which is of course the same as (3.3.9). Additionally, by testing (3.3.12) by ϕ itself, we obtain

$$\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2}^2+\omega\|\phi\|_{L^2}^2=c^2\phi(0)^2=c^2.$$

This shows that ϕ is a minimizer for $\min_{u\in H^s} J_\omega[u]$ and so any function in the form (3.3.9) is one as well. Also,

$$c^2(\omega) = \frac{\|(-\Delta)^{\frac{s}{2}}\mathcal{G}_s^\omega\|_{L^2}^2 + \omega\|\mathcal{G}_s^\omega\|_{L^2}^2}{(\mathcal{G}_s^\omega(0))^2} = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-1}. \tag{3.3.13}$$

□

We now state a result that describes the solutions of (3.1.5).

Lemma 12. *The non-trivial solutions to* (3.1.5)*, with $\phi(0) > 0$ are given by*

$$\hat{\phi}(\xi) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-(1+\frac{1}{2\sigma})} \frac{1}{(2\pi|\xi|)^{2s}+\omega}. \tag{3.3.14}$$

Proof. We can proceed as in the proof of Lemma 11 to see that

$$\hat{\phi}(\xi) = |\phi(0)|^{2\sigma}\phi(0)\frac{1}{(2\pi|\xi|)^{2s}+\omega}.$$

In order to determine $\phi(0)$, we apply the inverse Fourier transform to obtain an equation for it as follows

$$\phi(0) = \int_{\mathbf{R}^n} \hat{\phi}(\xi) d\xi = |\phi(0)|^{2\sigma}\phi(0)\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi.$$

It follows that

$$|\phi(0)|^{2\sigma} = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-1},$$

which proves the claim. □

3.3.4 The spectrum of $(-\Delta)^s + \omega - \mu\delta_0$

In this section, we develop some tools to study the bottom of the spectrum of the operators $(-\Delta)^s + \omega - \mu\delta_0$, depending on the value of μ. More specifically, we have the following result.

Proposition 17. *Let $s > \frac{n}{2}, \omega > 0$ and $L_\mu = (-\Delta)^s + \omega - \mu\delta_0$ be the self-adjoint operator introduced in Lemma 8. Then,*

- *If $\mu > c^2(\omega)$, the operator L_μ has one simple negative eigenvalue, $-\lambda_{\omega,\mu} < 0$, with eigenfunction $\Psi_0 : \widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s}+\omega+\lambda_{\omega,\mu}}$. For the rest of the spectrum*

$$\sigma(L_\mu) \setminus \{-\lambda_{\omega,\mu}\} \subset [\omega, \infty).$$

  In particular, $L_\mu|_{\{\Psi_0\}^\perp} \geq \omega$.

- *If $\mu = c^2(\omega)$, $L_\mu \geq 0$, 0 is a simple eigenvalue, with an eigenfunction Ψ_0 defined as above. For the rest of the spectrum, there is $\sigma(L_\mu) \setminus \{0\} \subset [\omega, \infty)$. In particular, $L_\mu|_{\{\Psi_0\}^\perp} \geq \omega$.*

- *If $\mu < c^2(\omega)$, there is a simple eigenvalue $\lambda_{\omega,\mu} \in (0, \omega)$, with eigenfunction $\Psi_0 : \widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s}+\omega-\lambda_{\omega,\mu}}$ and $\sigma(L_\mu) \setminus \{\lambda_{\omega,\mu}\} \subset [\omega, \infty)$. In particular, $L_\mu|_{\{\Psi_0\}^\perp} \geq \lambda_{\omega,\mu} > 0$.*

Proof. Assume first $\mu > c^2$. We would like to formally analyze the eigenvalue problem associated with the lowest eigenvalue of L_μ. So, we are looking for $f \neq 0, f \in D(L_\mu)$, so that $L_\mu f = -\lambda f$ for some $\lambda > 0$. This is the equation

$$((-\Delta)^s + \omega + \lambda)f = \mu f(0)\delta_0. \tag{3.3.15}$$

Arguing as in the proof of Lemma 11, by taking Fourier transform etc., we find that all possible solutions are in the form

$$\hat{f}(\xi) = \frac{\mu f(0)}{(2\pi|\xi|)^{2s} + \omega + \lambda}.$$

Clearly, $f \in D(L_\mu)$ and we need to see that there exists $\lambda > 0$, so that it solves (3.3.15). To this end, we have

$$f(0) = \int_{\mathbf{R}^n} \hat{f}(\xi) d\xi = \mu f(0) \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda} d\xi.$$

As we seek non-trivial solutions f (and hence $f(0) \neq 0$), this amounts to finding λ, so that for the given ω, we have

$$\mu \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda} d\xi = 1. \tag{3.3.16}$$

We claim that under the condition $\mu > c^2$, there is exactly one solution $\lambda = \lambda_{\omega,\mu} \in (0, \infty)$. Indeed, consider the continuous and decreasing function

$$h(\lambda) := \mu \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda} d\xi - 1.$$

Computing its limits at the ends of the interval

$$\lim_{\lambda \to 0+} h(\lambda) = \mu \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega} d\xi - 1 = \frac{\mu}{c^2} - 1 > 0, \lim_{\lambda \to +\infty} h(\lambda) = -1,$$

implies that there is an unique eigenvalue $\lambda_{\omega,\mu} > 0$. Moreover, the corresponding eigenfunction is, up to a multiplicative constant

$$\widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda_{\omega,\mu}}.$$

We now prove the statement about the rest of the spectrum. Consider the spectral decomposition of the self-adjoint operator L_μ. Assume for a contradiction that for any $\delta > 0$, we have that $\sigma(L_\mu) \cap (-\lambda_{\omega,\mu} + \delta, \omega - \delta) \neq \emptyset$. Let $\Psi \in Image(\mathbb{P}_{(-\lambda_{\omega,\mu}+\delta,\omega-\delta)})$ (i.e. $\Psi = \mathbb{P}_{(-\lambda_{\omega,\mu}+\delta,\omega-\delta)}\Psi$) and then normalize it, that is $\|\Psi\|_{L^2} = 1$. As

$$\Psi_0(0) = \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega + \lambda_{\omega,\mu}} d\xi > 0,$$

consider the well-defined element of $D(L_\mu)$,

$$\tilde{\Psi} := \Psi - \frac{\Psi(0)}{\Psi_0(0)} \Psi_0.$$

Note that $\tilde{\Psi}(0) = 0$, so according to (3.2.12), we have,

$$\langle L_\mu \tilde{\Psi}, \tilde{\Psi} \rangle = \|(-\Delta)^{\frac{s}{2}} \tilde{\Psi}\|_{L^2}^2 + \omega \|\tilde{\Psi}\|_{L^2}^2 \geq \omega \|\tilde{\Psi}\|_{L^2}^2 \geq \omega \|\Psi\|_{L^2}^2 = \omega.$$

where we have used that $\Psi \perp \Psi_0$, and hence $\|\tilde{\Psi}\|_{L^2}^2 = \|\Psi\|_{L^2}^2 + \frac{\Psi^2(0)}{\Psi_0^2(0)} \|\Psi_0\|_{L^2}^2 \geq \|\Psi\|_{L^2}^2 = 1$.

On the other hand, again by $\Psi \perp \Psi_0, L_\mu \Psi \perp \Psi_0$, and the properties of the spectral projections,

$$\langle L_\mu \tilde{\Psi}, \tilde{\Psi} \rangle = \langle L_\mu \Psi, \Psi \rangle + \frac{\Psi^2(0)}{\Psi_0^2(0)} \langle L_\mu \Psi_0, \Psi_0 \rangle \leq (\omega - \delta) - \lambda_{\omega,\mu} \frac{\Psi^2(0)}{\Psi_0^2(0)} \leq \omega - \delta.$$

Clearly, the two estimates that we have obtained for $\langle L_\mu \tilde{\Psi}, \tilde{\Psi} \rangle$ are contradictory, which is due to the assumption $\sigma(L_\mu) \cap (-\lambda_{\omega,\mu}, \omega - \delta) \neq \emptyset$. Thus, $\sigma(L_\mu) \cap (-\lambda_{\omega,\mu}, \omega) = \emptyset$ or $\sigma(L_\mu) \setminus \{-\lambda_{\omega,\mu}\} \subset [\omega, \infty)$, which was the claim.

The proof for $\mu = c^2$ goes along similar lines. Indeed, for any test function $\Psi \in H^s$, we have

$$\langle L_\mu \Psi, \Psi \rangle = \|(-\Delta)^{\frac{s}{2}} \Psi\|_{L^2}^2 + \omega \|\Psi\|_{L^2}^2 - c_s^2 |\Psi(0)|^2 \geq 0,$$

by the definition of $c^2 = \inf J_\omega[\Psi]$. Hence, $L_\mu \geq 0$. Furthermore, by direct inspection $L_\mu[\mathcal{G}_\omega^s] = 0$, whence 0 is an eigenvalue (and it would have to be at the bottom of the spectrum). Finally, $\sigma(L_\mu) \setminus \{0\} \subset [\omega, \infty)$ is shown in the exact same way as in the case $\mu > c^2$.

For the case $\mu < c^2$, we can similarly identify an unique $\lambda_{\omega,\mu} \in (0, \omega)$, so that

$$\mu \int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s} + \omega - \lambda} d\xi = 1.$$

This $\lambda_{\omega,\mu} > 0$ is an eigenvalue for L_μ, with eigenfunction, $\Psi_0 : \widehat{\Psi}_0(\xi) = \frac{1}{(2\pi|\xi|)^{2s}+\omega-\lambda}$. Moreover, $\sigma(L_\mu) \setminus \{\lambda_{\omega,\mu}\} \subset [\omega, \infty)$ is proved in the same fashion as above. $\square$

Remark 2. *Note that the operator $\mathcal{L}_\pm$ have the form*

$$\begin{aligned}
\mathcal{L}_- &= (-\Delta)^s + \omega - |\phi(0)|^{2\sigma}\delta_0 = (-\Delta)^s + \omega - c^2(\omega)\delta_0 \\
\mathcal{L}_+ &= (-\Delta)^s + \omega - (2\sigma+1)c^2(\omega)\delta_0.
\end{aligned}$$

As a direct consequence of the results of Proposition 17 and Remark 2, we have the following corollary.

Corollary 5. *Let $s > \frac{n}{2}$, $\omega > 0$, $\sigma > 0$. Then,*

- *$\mathcal{L}_- \geq 0$, 0 is a simple eigenvalue, with eigenfunction $\mathcal{G}_s^\omega$ and*

$$\sigma(\mathcal{L}_-) \setminus \{0\} \subset [\omega, \infty).$$

Also, $\mathcal{L}_-|_{\{\mathcal{G}_s^\omega\}^\perp} \geq \omega$.

- *$\mathcal{L}_+$ has a simple negative eigenvalue, with an eigenfunction Ψ_0. Also,*

$$\mathcal{L}_+|_{\{\Psi_0\}^\perp} \geq \omega > 0.$$

3.4 Stability of the waves

In this section, we identify the regions of stability for the waves.

3.4.1 Instability index count for (3.1.6)

In our specific case, we need to apply the instability index counting theory to the eigenvalue problem (3.1.6). Recall that $\mathcal{J}^* = -\mathcal{J} = \mathcal{J}^{-1}$, while $\mathcal{L} = \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix}$, whence

$$n(\mathcal{L}) = n(\mathcal{L}_+) + n(\mathcal{L}_-) = 1 + 0 = 1,$$

due to the results of Corollary 5. Also, again by the description in Corollary 5,

$$Ker(\mathcal{L}) = \begin{pmatrix} Ker(\mathcal{L}_-) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Ker(\mathcal{L}_+) \end{pmatrix} = span \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix}.$$

It follows that Corollary 1 is applicable to the eigenvalue problem (3.1.6), and in fact the spectral stability of it is equivalent to the condition

$$\langle \mathcal{L}_+^{-1} \phi_\omega, \phi_\omega \rangle < 0. \tag{3.4.1}$$

Since, $\phi_\omega = c\mathcal{G}_s^\omega$, it suffice s to compute $\langle \mathcal{L}_+^{-1} \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle$. We accomplish this in the following proposition.

Proposition 18. *Let $n \geq 1$, $\omega > 0$, $\sigma > 0$ and $s > \frac{n}{2}$. Then,*

$$sgn\langle \mathcal{L}_+^{-1} \phi_\omega, \phi_\omega \rangle = sgn\langle \mathcal{L}_+^{-1} \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle = sgn\left(\sigma - \frac{2s-n}{n}\right).$$

In particular, the waves ϕ_ω are spectrally stable if and only if

$$0 < \sigma < \frac{2s}{n} - 1.$$

Proof. We first need to find $\mathcal{L}_+^{-1}\mathcal{G}_s^\omega$. That is, we need to solve $\mathcal{L}_+\psi = \mathcal{G}_s^\omega$. Based on the formula (3.2.8) however, we need to solve

$$\mathcal{G}_s^\omega = \mathcal{L}_+\psi = ((-\Delta)^s + \omega)g$$

whence, we can actually find g pretty easily by taking Fourier transform. Namely,

$$((2\pi|\xi|)^{2s} + \omega)\hat{g}(\xi) = \widehat{\mathcal{G}_s^\omega}(\xi) = \frac{1}{(2\pi|\xi|)^{2s} + \omega}.$$

It follows that

$$\hat{g}(\xi) = \frac{1}{((2\pi|\xi|)^{2s} + \omega)^2},$$

or equivalently $g = \mathcal{G}_s^\omega * \mathcal{G}_s^\omega$. We can now proceed to find ψ from (3.2.10). Namely, taking into account that $\mathcal{L}_+ = (-\Delta)^s + \omega - (2\sigma+1)c^2$, we compute

$$\psi = g + (2\sigma+1)c^2 \frac{g(0)}{1 - (2\sigma+1)c^2\mathcal{G}_s^\omega(0)} \mathcal{G}_s^\omega.$$

Note however that $g(0) = \mathcal{G}_s^\omega * \mathcal{G}_s^\omega(0) = \|\mathcal{G}_s^\omega\|_{L^2}^2$. Also, according to (3.3.13), $c_s^2\mathcal{G}_s^\omega(0) = 1$, so

$$\psi = \mathcal{G}_s^\omega * \mathcal{G}_s^\omega - \frac{2\sigma+1}{2\sigma} \frac{\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi}{\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi} \mathcal{G}_\omega^s.$$

So ,

$$\langle \mathcal{L}_+^{-1}\mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle = \langle \psi, \mathcal{G}_s^\omega \rangle = \langle \mathcal{G}_s^\omega * \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle - \frac{2\sigma+1}{2\sigma} \frac{\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi}{\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi} \langle \mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle =$$

$$= \int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^3} d\xi - \frac{2\sigma+1}{2\sigma} \frac{\left(\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi\right)^2}{\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi}.$$

So, it remains to compute

$$\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^j} d\xi, j=1,2,3.$$

which we have done in the Appendix, see Proposition 29. More specifically, substituting the formulas (B.0.1), (B.0.2), (B.0.3) in the expression for $\langle \mathcal{L}_+^{-1}\mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle$, we obtain

$$\begin{aligned} \langle \mathcal{L}_+^{-1}\mathcal{G}_s^\omega, \mathcal{G}_s^\omega \rangle &= \frac{\pi|\mathbf{S}^{n-1}|\omega^{\frac{n}{2s}-3}}{4s(2\pi)^n \sin(\frac{n\pi}{2s})} \left(\left(1-\frac{n}{2s}\right)\left(2-\frac{n}{2s}\right) - \frac{2\sigma+1}{\sigma}\left(1-\frac{n}{2s}\right)^2 \right) = \\ &= \frac{n\pi|\mathbf{S}^{n-1}|\omega^{\frac{n}{2s}-3}}{8s^2\sigma(2\pi)^n \sin(\frac{n\pi}{2s})} \left(1-\frac{n}{2s}\right)\left(\sigma - \frac{2s-n}{n}\right). \end{aligned}$$

Note that, as $s > \frac{n}{2}$, only the last term in the expression changes sign over the parameter space. We have this established Proposition 18 in full. □

Having the above spectral properties of the operator $\mathcal{L}_\pm$, we have one last step before arriving at the orbital stability of the wave. More specifically, we need to argue the coerciveness of $\mathcal{L}_\pm$ on the space $H^s(\mathbf{R}^n)$. To that end we have the following proposition

.

Proposition 19. *Let $s > \frac{n}{2}, \omega > 0$, $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega \rangle < 0$. Then, the operator $\mathcal{L}_+$ is coercive on $\{\phi_\omega\}^\perp$. That is, there exists $\delta > 0$, so that for all*

$$\langle \mathcal{L}_+\Psi, \Psi \rangle \geq \delta \|\Psi\|_{H^s}^2, \ \ \forall \Psi \perp \phi_\omega. \tag{3.4.2}$$

Proof. This is a version of a well-known lemma in the theory, see for example Lemma 6.7 and Lemma 6.9 in [67]. Recall that we have already showed $Ker[\mathcal{L}_+] = \{0\}$ and

$n(\mathcal{L}_+) = 1$. According[9] to Lemma 6.4, [67] under these conditions for $\mathcal{L}_+$ we have that for any $g \perp \phi_\omega$,

$$\langle \mathcal{L}_+ g, g \rangle \geq 0. \tag{3.4.3}$$

Consider the associated constrained minimization problem

$$\inf_{\|f\|=1, f \perp \phi_\omega} \langle \mathcal{L}_+ f, f \rangle \tag{3.4.4}$$

and set

$$\alpha := \inf\{\langle \mathcal{L}_+ f, f \rangle : f \perp \phi_\omega, \|f\|_{L^2} = 1\} \geq 0.$$

We will show that $\alpha > 0$. Assume for a contradiction that $\alpha = 0$.

Take a minimizing sequence $f_k : \|f_k\| = 1, f_k \perp \phi_\omega$, so that

$$\alpha = \lim_k \langle \mathcal{L}_+ f_k, f_k \rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} f_k\|^2 + \omega - (2\sigma+1)c^2 |f_k(0)|^2].$$

However, by Sobolev embedding and the Gagliardo-Nirenberg's inequalities (recall $\|f_k\|_{L^2} = 1$), we have that for all $\beta : \frac{n}{2} < \beta < s$ and for all $\epsilon > 0$,

$$|f(0)| \leq \|f\|_{L^\infty} \leq C_\beta(\|f\|_{\dot{H}^\beta} + C\|f\|_{L^2} \leq C_\beta \|f\|_{\dot{H}^s}^{\frac{\beta}{s}} \|f\|_{L^2}^{1-\frac{\beta}{s}} + C\|f\|_{L^2} \leq \epsilon \|f\|_{\dot{H}^s} + C_\epsilon \|f\|_{L^2}.$$

Applying this estimate, we obtain a lower bound for $\langle \mathcal{L}_+ f_k, f_k \rangle$ (recall $\|f_k\|_{L^2} = 1$), as follows

$$\langle \mathcal{L}_+ f_k, f_k \rangle \geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} f_k\|^2 - C.$$

Since, $\alpha = \lim_k \langle \mathcal{L}_+ f_k, f_k \rangle$, this implies that $\sup_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 < \infty$. This means that we can select a subsequence of $\{f_k\}$ (denoted by the same), so that f_k converges weakly to

[9] And this is already explicit in a much earlier work by Weinstein

$f \in H^s(\mathbf{R}^n)$. In addition, by the Sobolev embedding $H^s(\mathbf{R}^n) \hookrightarrow C^\gamma(\mathbf{R}^n), \gamma < s - \frac{n}{2}$, we can, as we have done previously, without loss of generality assume that $f_n \rightrightarrows f$ on the compact subsets of $\mathbf{R}^n$. In particular, $\lim_k f_k(0) = f(0)$. Note that by the weak convergence, $\langle f, \phi_\omega \rangle = \lim_k \langle f_k, \phi_\omega \rangle = 0$, so $f \perp \phi_\omega$ and

$$\liminf_k \|(-\Delta)^{\frac{s}{2}} f_k\|^2 \geq \|(-\Delta)^{\frac{s}{2}} f\|^2, \quad \|f\|_{L^2} \leq \liminf \|f_k\|_{L^2} = 1. \tag{3.4.5}$$

It follows that

$$\langle \mathcal{L}_+ f, f \rangle \leq \liminf_k \langle \mathcal{L}_+ f_k, f_k \rangle = 0. \tag{3.4.6}$$

But by (3.4.3), and since $f \perp \phi_\omega$, we have that $\langle \mathcal{L}_+ f, f \rangle \geq 0$. It follows that $0 = \langle \mathcal{L}_+ f, f \rangle = \lim_k \langle \mathcal{L}_+ f_k, f_k \rangle$. But this means that all inequalities in (3.4.5) and (3.4.6) are equalities and in particular

$$\lim_k \|(-\Delta)^{\frac{s}{2}} f_k\|_{L^2} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2},$$

$$\lim_k \|f_k\|_{L^2} = \|f\|_{L^2}.$$

These last identities, in addition to the H^s weak convergence f_k to f, implies strong convergence, that is $\lim_k \|f_k - f\|_{H^s} = 0$. In particular, $\|f\|_{L^2} = \lim_k \|f_k\|_{L^2} = 1$. In other words, f is a minimizer for the constrained minimization problem (3.4.4). Write the Euler-Lagrange equation for f

$$\mathcal{L}_+ f = df + c\phi_\omega. \tag{3.4.7}$$

Taking dot product with f and taking into account $\langle \mathcal{L}_+ f, f \rangle = 0$, $f \neq 0$ and $f \perp \phi_\omega$ implies that $d = 0$. This means that $f = c\mathcal{L}_+^{-1} \phi_\omega$. But then,

$$0 = \langle \mathcal{L}_+ f, f \rangle = c^2 \langle \mathcal{L}_+^{-1} \phi_\omega, \phi_\omega \rangle.$$

Since $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle \neq 0$, it follows $c = 0$. But then, since $Ker[\mathcal{L}_+] = \{0\}$, (3.4.7) implies that $f = 0$, which is a contradiction. Thus, we have shown that $\alpha > 0$. As a consequence,

$$\langle \mathcal{L}_+\Psi, \Psi\rangle \geq \alpha\|\Psi\|_{L^2}^2, \ \ \forall\Psi \perp \phi_\omega. \tag{3.4.8}$$

Note that (3.4.2) is however stronger than (3.4.8), as it involves $\|\cdot\|_{H^s}$ norms on the right-hand side. Nevertheless, we show that it is relatively straightforward to deduce it from (3.4.8). Indeed, assume for a contradiction in (3.4.2), that $g_k : \|g_k\|_{H^s} = 1, g_k \perp \phi_\omega$, so that $\lim_k\langle \mathcal{L}_+ g_k, g_k\rangle = 0$.

Taking into account (3.4.8), this is only possible if $\lim_k \|g_k\|_{L^2} = 0$. So,

$$1 = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \|g_k\|_{L^2}^2] = \lim_k \|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2.$$

Note that by (1.1.8), we have that for all $0 < \delta < s - \frac{n}{2}$, we have that

$$\begin{aligned} |g_k(0)| &\leq \|g_k\|_{L^\infty} \leq C(\|g_k\|_{\dot{H}^{\frac{n}{2}+\delta}} + \|g_k\|_{\dot{H}^{\frac{n}{2}-\delta}}) \\ &\leq C(\|g_k\|_{\dot{H}^s}^{\frac{\frac{n}{2}+\delta}{s}} \|g_k\|_{L^2}^{1-\frac{\frac{n}{2}+\delta}{s}} + \|g_k\|_{\dot{H}^s}^{\frac{\frac{n}{2}-\delta}{s}} \|g_k\|_{L^2}^{1-\frac{\frac{n}{2}-\delta}{s}}, \end{aligned}$$

whence $\lim_k \|g_k(0)| = 0$. But then, we achieve a contradiction, since

$$0 = \lim_k \langle \mathcal{L}_+ g_k, g_k\rangle = \lim_k [\|(-\Delta)^{\frac{s}{2}} g_k\|_{L^2}^2 + \omega\|g_k\|_{L^2}^2 - (2\sigma+1)c_s^2|g_k(0)|^2] = 1.$$

□

3.4.2 Orbital stability

In this section, we prove that the spectrally stable solutions are in fact orbitally stable. There is, in general, a straightforward way to obtain orbital stability, based on spectral stability, see for example Theorem 5.2.11, [55]. While this is the case in general, we are dealing with non-standard linearized operators and their domains. In particular, the Assumption 5.2.5 a) on p. 136, [55] does not apply. Thus, we need to consider a direct proof, based on the Benjamin's approach.

As was established already, the case $0 < \sigma < \frac{2s}{n} - 1$ represents the spectrally stable waves, which we now analyze for orbital stability.

Proposition 20. *Let* $\omega > 0$*,* $n \geq 1, s > \frac{n}{2}$*,* $0 < \sigma < \frac{2s}{n} - 1$ *and the key assumptions (1), (2) are satisfied. Then* $e^{i\omega t}\phi_\omega$ *is orbitally stable solution of* (3.1.2).

Proof. Let us outline first what the consequences of our assumptions are. By Proposition 18, we have that $\langle \mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle < 0$, which by Proposition 19 means that the coercivity estimate (3.4.2) holds. By Corollary 5, $Ker(\mathcal{L}_+) = \{0\}$, that is the wave ϕ_ω is non-degenerate.

We now concentrate on the orbital sdtability. Our proof is by a contradiction argument. That is, there is $\epsilon_0 > 0$ and a sequence of initial data $u_k : \lim_k \|u_k - \phi\|_{H^s(\mathbb{R}^n)} = 0$, so that

$$\sup_{0 \leq t < \infty} \inf_{\theta \in \mathbf{R}} \|u_k(t,\cdot) - e^{-i\theta}\phi\|_{H^s} \geq \epsilon_0. \tag{3.4.9}$$

Using the conserved quantities (3.1.3) and (3.1.4) we define new conserved quantity

$$\mathcal{E}[u] := E[u] + \frac{\omega}{2}M[u],$$

$$\epsilon_k := |\mathcal{E}[u_k(t)] - \mathcal{E}[\phi_\omega]]| + |M[u_k(t)] - M[\phi_\omega]]|,$$

and for all $\epsilon > 0$,

$$t_k := \sup\{\tau : \sup_{0 < t < \tau} \|u_k(t) - \phi\|_{H^s(\mathbb{R}^n)} < \epsilon\}.$$

Note that ϵ_k is conserved and $\lim_k \epsilon_k = 0$ and by the assumption that we have local well-posedness $t_k > 0$.

Consider $t \in (0,t_k)$ and let $u_k = v_k + iw_k$ and $\|w_k(t)\|_{H^s(\mathbb{R}^n)} \leq 2\|u_k - \phi\|_{H^s(\mathbb{R}^n)} < \epsilon$. This leads to the definition of the modulation parameter $\theta_k(t)$ such that $w_k + \sin\theta_k(t)\phi \perp \phi$, that is ,

$$-\sin(\theta_k(t))\|\phi\|_{L^2} = \langle w_k(t), \phi\rangle. \tag{3.4.10}$$

By Cauchy-Schwarz we have $|\langle w_k(t),\phi\rangle| \leq \epsilon\|\phi\|_{L^2}$ and this means there is an unique small solution $\theta_k(t)$ of (3.4.10), with $|\theta_k(t)| \leq \epsilon$. Also

$$\|u_k(t,\cdot) - e^{-i\theta_k(t)}\phi\|_{H^s} \leq \|u_k(t,\cdot) - \phi\|_{H^s} + |e^{-i\theta_k(t)} - 1|\|\phi\|_{H^s} \leq C(\|\phi\|_{H^s})\epsilon.$$

Now define

$$T_k := \sup\{\tau : \sup_{0<t<\tau} \|u_k(t,\cdot) - e^{-i\theta_k(t)}\varphi(\cdot)\|_{H^s(\mathbb{R}^n)} < 2C\epsilon\}.$$

Clearly $0 < t_k < T_k$. From this we see that to get contradiction of (3.4.9) it is enough to show that for all $\epsilon > 0$ and large $k, T_k = \infty$. To that end let $t \in (0,T_k)$ write

$$\psi_k = u_k - e^{-i\theta_k(t)}\phi = v_k + iw_k - e^{-i\theta_k(t)}\phi,$$

and decompose into real and imaginary part of ψ_k and projecting on $\begin{pmatrix}\phi\\0\end{pmatrix}$ with

$$\begin{pmatrix}\eta_k(t,\cdot)\\ \zeta_k(t,\cdot)\end{pmatrix} \perp \begin{pmatrix}\phi\\ 0\end{pmatrix}$$

yield

$$\begin{pmatrix} v_k(t,\cdot)-\cos(\theta_k(t))\phi \\ w_k(t,\cdot)+\sin(\theta_k(t))\phi \end{pmatrix} = \mu_k(t)\begin{pmatrix} \phi \\ 0 \end{pmatrix} + \begin{pmatrix} \eta_k(t,\cdot) \\ \zeta_k(t,\cdot) \end{pmatrix}. \tag{3.4.11}$$

By the choice of θ_k we have $\zeta_k \perp \phi$, and from the above decomposition we also have $\eta_k \perp \phi$. So taking the L^2 norm of (3.4.11) we have

$$|\mu_k(t)|^2\|\phi\|_{L^2}^2 + \|\eta_k(t)\|_{L^2}^2 + \|\zeta_k(t)\|_{L^2}^2 = \|\psi_k(t)\|_{L^2}^2 \le 4C^2\epsilon^2. \tag{3.4.12}$$

Next we take advantage of the two conserved quantities, to that end we consider the mass

$$\begin{aligned} M[u_k(t)] &= \int_{\mathbf{R}^n} |e^{-i\theta_k(t)}\phi + \psi_k(t)|^2 dx \\ &= M[\phi] + \|\psi_k(t,\cdot)\|_{L^2}^2 + 2\int_{\mathbf{R}^n} \phi(x)\Re[e^{-i\theta_k(t)}\psi_k(t,x)]dx \\ &= M[\phi] + \|\psi_k(t,\cdot)\|_{L^2}^2 + 2\mu_k(t)\cos(\theta_k(t))\|\phi\|_{L^2}^2. \end{aligned}$$

Here we use the fact that $w_k + \sin\theta_k(t)\phi \perp \phi$ and $\eta_k \perp \phi$. Solving for $\mu_k(t)$ and since $|\theta_k|$ is very small and $\|\psi_k(t,\cdot)\|_{L^2} \le 2C\epsilon$, in $t: 0 < t < T_k$ we have

$$|\mu_k(t)| \le \frac{|M[u_k(t)] - M[\phi]| + \|\psi_k(t,\cdot)\|_{L^2}^2}{2\cos(\theta_k(t)\|\phi\|_{L^2}^2} \le C(\epsilon_k + \|\psi_k(t,\cdot)\|_{L^2}^2) \le C(\epsilon_k + \epsilon^2). \tag{3.4.13}$$

Now we will expand $\mathcal{E}[u_k(t)] - \mathcal{E}[\phi]$ but first for any small perturbations of the wave $\alpha_1 + i\alpha_2 \in H^s(\mathbf{R}^n)$ and using (3.1.5) we have

$$E[\phi + (\alpha_1 + i\alpha_2)] - E[\phi] = \frac{1}{2}[\langle \mathcal{L}_+\alpha_1, \alpha_1\rangle + \langle \mathcal{L}_-\alpha_2, \alpha_2\rangle] + Err[\alpha_1, \alpha_2], \tag{3.4.14}$$

where

$$|Err[\alpha_1,\alpha_2]| \le C|((\phi(0)+\alpha_1(0))^2+\alpha_2^2(0))^{\sigma+1}-\phi(0)^{2\sigma+2}$$
$$-(2\sigma+2)\phi(0)^{2\sigma+1}\alpha_1(0)-\frac{(2\sigma+2)(2\sigma+1)}{2}\phi^{2\sigma}(0)\alpha_1^2(0)-(2\sigma+2)\phi^{2\sigma}(0)\alpha_2^2(0)|$$
$$\le C(\|\phi\|_{L^\infty})(|\alpha_1(0)|+|\alpha_2(0)|)^{\min(2\sigma+2,3)}.$$

Note that

$$e^{i\theta_k(t)}\psi_k=[\cos(\theta_k)(\mu_k\phi+\eta_k)-\sin(\theta_k)\zeta_k]+i\,[\cos(\theta_k)\zeta_k+\sin(\theta_k)(\mu_k\phi+\eta_k)]\,.$$

Now apply the expansion (3.4.14) with

$$\alpha_1=\cos(\theta_k)(\mu_k\phi+\eta_k)-\sin(\theta_k)\zeta_k,\alpha_2=\cos(\theta_k)\zeta_k+\sin(\theta_k)(\mu_k\phi+\eta_k)$$

together with (3.4.12), we see that $\|\alpha_1\|_{H^s}+\|\alpha_2\|_{H^s}\le C\epsilon$. So , we can bound the contribution of $|Err[\alpha_1,\alpha_2]|$ as follows

$$|Err[\alpha_1,\alpha_2]|\le C\epsilon^{\min(2\sigma,1)}(\|\alpha_1\|_{H^s}^2+\|\alpha_2\|_{H^s}^2). \tag{3.4.15}$$

By the Sobolev embeddings, $\mathcal{L}_-\phi=0$ and $\mathcal{L}_+=\mathcal{L}_--2\sigma|\phi(0)|^{2\sigma}\delta$ together with (3.4.12) and (3.4.13) we have

$$\langle\mathcal{L}_+\alpha_1,\alpha_1\rangle\ge\langle\mathcal{L}_+\eta_k,\eta_k\rangle-C(\epsilon^3+\epsilon_k+\epsilon^2(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})+\epsilon(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})^2)$$
$$\langle\mathcal{L}_-\alpha_2,\alpha_2\rangle\ge\langle\mathcal{L}_-\zeta_k,\zeta_k\rangle-C(\epsilon^3+\epsilon_k+\epsilon^2(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})+\epsilon(\|\eta_k\|_{H^s}+\|\zeta_k\|_{H^s})^2).$$

Taking advantage of the coercivity of $\mathcal{L}_-$ and $\mathcal{L}_+$, which was established in Proposition 17, we have that for some $\kappa > 0$ and since $\eta_k, \zeta_k \perp \phi$ together with some algebraic manipulations yield

$$\|\eta_k(t)\|_{H^s}^2 + \|\zeta_k(t)\|_{H^s}^2 \leq C(\epsilon^3 + \epsilon_k). \tag{3.4.16}$$

Here C is independent of ϵ and k. This implies that $T_k^* = \infty$, since if we assume that $T_k^* < \infty$, then

$$2C_0\epsilon = \limsup_{t\to T_k^*-} \|\psi_k(t)\|_{H^s} \leq C(|\mu_k(t)| + \|\eta_k(t)\|_{H^s} + \|\zeta_k(t)\|_{H^s}) \leq C(\epsilon^{\frac{3}{2}} + \sqrt{\epsilon_k}). \tag{3.4.17}$$

which is a contradiction, if ϵ is so that $C_0\epsilon > C\epsilon^{\frac{3}{2}}$ and then k is so large, and hence ϵ_k is so small, that $C_0\epsilon > C\sqrt{\epsilon_k}$, which certainly contradicts (3.4.17). Hence the wave is orbitally stable.

□

Chapter 4

On the stability of the compacton waves for the degenerate KdV and NLS models

4.1 Introduction

In this chapter, we shall be interested in some aspects of the dynamics of dispersive equations, driven by degenerate dispersion. In order to fix ideas, we settle on the one dimensional case on the line, and recall the standard dispersive models. More specifically, for $p > 2$, consider the (generalized) Korteweg de Vries equation

$$u_t + u_{xxx} + \partial_x(|u|^{p-2}u) = 0, \quad u : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \tag{4.1.1}$$

and the non-linear Schrödinger equation

$$iu_t + u_{xx} + |u|^{p-2}u = 0, \quad u : \mathbf{R} \times \mathbf{R} \to \mathbb{C}. \tag{4.1.2}$$

The Cauchy problem for these models is well-understood, even in low regularity setting. In particular, and for classical solutions, the solutions conserve energy

$$E[u(t)] = \frac{1}{2}\int_{\mathbf{R}} |u'|^2 dx - \frac{1}{p}\int_{\mathbf{R}} |u|^p dx = E[u_0]$$

and mass

$$M[u(t)] = \int_{\mathbf{R}} |u|^2 dx = M[u_0].$$

The solitary waves of these models, namely the standing waves $e^{i\omega t}\phi$ of (4.1.2) and the traveling waves $\phi(x-\omega t)$ of (4.1.1), have received ample attention in the literature. It turns out that they are unique (i.e. for each $\omega > 0, p > 2$, there is an unique function ϕ with this property). Their stability is also a classical fact by now (see for example [18], but also [14, 16, 51]) - namely these waves are spectrally stable for all $\omega > 0$, when $2 < p < 6$, while they become spectrally unstable for $p > 6$, again for all values of $\omega > 0$.

In this chapter published in [47], we investigate the degenerate KdV and NLS. More specifically, for $p > 2$, the degenerate KdV is given by

$$u_t + \partial_x(u\partial_x(u\partial_x u) + |u|^{p-2}u) = 0, \quad u : \mathbf{R}\times\mathbf{R} \to \mathbf{R}. \tag{4.1.3}$$

The degenerate KdV model was introduced in [73, 74], with the main interest in the compacton traveling waves. A more general Hamiltonian version of this problem, was proposed in [25, 86] and subsequently in [65]. For more details on Compactons, their structure, emergence, and other properties see [75, 76, 77]

The degenerate NLS takes the form

$$iu_t + \bar{u}\partial_x(u\partial_x u) + |u|^{p-2}u = 0, \quad u : \mathbf{R}\times\mathbf{R} \to \mathbb{C}. \tag{4.1.4}$$

The well-posedness of the Cauchy problem for the degenerate KdV equation (4.1.3) has been studied in [41] and quite recently the well-posedness for degenerate NLS (4.1.4) was studied in [48]. In particular, these models conserve the mass $M[u] = \int_{\mathbf{R}} |u|^2$ and the Hamiltonian, which in this case takes the form

$$\mathcal{H}[u(t)] = \frac{1}{2}\int_{\mathbf{R}} |u\partial_x u|^2 - \frac{1}{p}\int_{\mathbf{R}} |u|^p dx = \mathcal{H}[u_0].$$

The non-linear dynamics of an equilibrium solution of (4.1.3) and (4.1.4) heavily depends on the spectral/linear stability of the equilibrium. To that end, note that the degenerate KdV, (4.1.3) can be written in the Hamiltonian form

$$\partial_t u = \mathcal{J}\frac{\partial \mathcal{H}}{\partial u}$$

with $L^2(\mathbb{R})$ skew-symmetric operator $\mathcal{J} = \partial_x : H^1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$, while the degenerate NLS, in the form

$$\partial_t z = \mathcal{J}\frac{\partial \mathcal{H}}{\partial z}$$

where $z = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Of particular interest will be the solitary wave solutions of (4.1.3) and (4.1.4) respectively. That is, consider solutions of the type $u(t,x) = \phi(x-\omega t)$ in (4.1.3) and $u(t,x) = e^{i\omega t}\phi(x)$ in (4.1.4), for a real-valued[1] function ϕ, we are lead to the same profile equation for ϕ, namely

$$-\phi\partial_x(\phi\partial_x\phi) + \omega\phi - |\phi|^{p-2}\phi = 0. \tag{4.1.5}$$

[1] In fact, we will construct non-negative solutions $\phi > 0$.

The existence of such waves, with appropriate properties, such as smoothness, decay at infinity etc. has been considered in the literature, [42]. These results are relatively straightforward, we present a version, which suffice for our purposes, see Proposition 21 below.

If we restrict our considerations to bell-shaped and decaying at ∞ solutions, one can say much more. This is the subject of our next first existence and uniqueness result, Proposition 21.

Proposition 21. *(Existence and uniqueness of bell-shaped compactons)*

Let $\omega > 0, p > 2$. Then, there exists a compactly supported bell-shaped solution ϕ, which satisfies the profile equation (4.1.5) and consequently,

$$\phi' = -\sqrt{\omega - \frac{2}{p}\phi^{p-2}}, 0 < x < L. \tag{4.1.6}$$

Letting $L : supp\varphi = [-L, L]$ and $\phi_0 := \phi(0)$, we have the formulas

$$L = L(\omega, p) = \frac{p^{\frac{1}{p-2}}}{2^{\frac{p-1}{p-2}}}\omega^{\frac{4-p}{2(p-2)}} \int_0^1 \frac{1}{\sqrt{z - z^{\frac{p}{2}}}}dz; \quad \phi_0 = \phi_0(\omega, p) = \left(\frac{p\omega}{2}\right)^{\frac{1}{p-2}}. \tag{4.1.7}$$

Conversely, suppose that ϕ is a bell-shaped solution ϕ, which vanishes at ∞. Then, ϕ is necessarily of the form described in (4.1.6). In particular, it is compactly supported and unique. More precisely, for each $\omega > 0, p > 2$, there exists an unique bell-shaped element $\phi = \phi_{\omega,p}$, with the properties described above.

As alluded to above, our main interest is in the stability of the waves ϕ. In order to introduce the relevant notions, we need to derive the corresponding linearized dynamics. We develop the necessary background material in the next section.

4.1.1 Linearizations about the solitary waves and spectral stability

Take the ansatz $u = \phi(x-\omega t) + e^{\lambda t} v(x-\omega t)$ and plug it in the degenerate KdV model (4.1.3). Ignoring high order terms $O(v^2)$, we arrive at the following linearized system

$$\mathcal{J}\mathcal{L}_+ v = \lambda v. \tag{4.1.8}$$

where $\mathcal{J} = \partial_x$, while

$$\mathcal{L}_+ f = -\phi\partial_x^2(\phi f) + (\omega - (\phi\phi')' - (p-1)\phi^{p-2})f = -\phi\partial_x^2(\phi f) - (p-2)\phi^{p-2} f, \tag{4.1.9}$$

where we have used the relation $\omega - (\phi\phi')' = \phi^{p-1}$, which follows readily from (4.1.5). Note that the relation that we just used only holds on the support of ϕ (i.e. on the interval $[-L, L]$). This is the relevant form of the eigenvalue problem in the context of the degenerate KdV model (4.1.1).

For the NLS equation (4.1.4), we linearize around the standing wave $e^{i\omega t}\phi_\omega$, that is we take $u = e^{i\omega t}(\Phi_\omega + e^{\lambda t} v)$ and plug it into (4.1.4). Taking (4.1.5) into account and ignoring the higher order terms $O(|v|^2)$, setting the real and imaginary part of $v = v_1 + iv_2$ as $(\Re v, \Im v) := (v_1, v_2)$ we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix} \begin{pmatrix} \Re v \\ \Im v \end{pmatrix} = \lambda \begin{pmatrix} \Re v \\ \Im v \end{pmatrix} \tag{4.1.10}$$

where $\mathcal{L}_+$ is as above and

$$\mathcal{L}_- f = -\phi\partial_x^2(\phi f) + (\omega + (\phi\phi')' - \phi^{p-2})f = -\phi\partial_x^2(\phi f) + 2(\omega - \phi^{p-2})f, \tag{4.1.11}$$

again by (4.1.5). A concise form of (4.1.10) is given by

$$\mathcal{J}\mathcal{L}\vec{v} = \lambda\vec{v} \tag{4.1.12}$$

where we have introduced

$$\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{L} := \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}.$$

The standard notion of spectral stability is as follows.

Definition 7. *We say that the solution $\phi(x-\omega t)$ of the generalized KdV problem* (4.1.3) *is called spectrally stable, if the eigenvalue problem* (4.1.8) *does not have non-trivial solutions $(\lambda, v) : \Re\lambda > 0, v \neq 0, \vec{v} \in D(\mathcal{J}\mathcal{L}_+)$.*

Similarly, the solution $e^{i\omega t}\phi$ of the degenerate NLS equation (4.1.4) *is called spectrally stable, if the eigenvalue problem* (4.1.12) *does not have non-trivial solutions $(\lambda, \vec{v}) : \Re\lambda > 0, \vec{v} \neq 0, v \in D(\mathcal{J}\mathcal{L})$.*

Note that in this definition, we completely sidestep the important issue for local/global well-posedness of the corresponding Cauchy problems. The local aspect of the theory is discussed in the recent paper [41], but the global well-posedness theory (which is more relevant as far as stability is concerned), seems lacking at the moment.

We now state our main results.

4.1.2 Main results

We start with an existence result for the waves ϕ, which must satisfy the ordinary differential equation (4.1.5). In our construction of the waves, we take advantage of a variational construction, so we introduce a few of our main players. More specifically, consider the following constrained minimization problem

$$\begin{cases} N[u] = \int_{\mathbf{R}} |u\partial_x u|^2 + \omega \int |u|^2 \to \min \\ \text{subject to } \int |u|^p dx = 1. \end{cases} \tag{4.1.13}$$

Equivalently, the same solutions are achieved via the minimization of the so-called Weinstein functional,

$$J_\omega[u] := \frac{\int_{\mathbf{R}} |u\partial_x u|^2 + \omega \int |u|^2}{\|u\|_{L^p}^2}.$$

Note that formally, $J_\omega[u]$ is unconstrained, but one can see that without loss of generality, one may consider only $u : \|u\|_{L^p} = 1$, which is of course (4.1.13).

The following is the main result of this paper.

Theorem 8. *Let $p > 2$ and $\omega > 0$. Then, there exists an unique bell-shaped solution ϕ of the profile equation* (4.1.5)*, which is a compacton (i.e. it has a compact support). Their half-period L and their amplitude are given by* (4.1.7)*. Moreover, such solutions are the unique constrained minimizers of the variational problem* (4.1.13)*.*

By construction, $\phi(x - \omega t)$ is a traveling wave solution of the degenerate KDV (4.1.3)*, while $e^{i\omega t}\phi$ is a standing wave solution of the degenerate NLS,* (4.1.4)*.*

Regarding spectral stability, $\phi(x - \omega t)$ and $e^{i\omega t}\phi(x)$ are spectrally stable solutions (of (4.1.3) *and* (4.1.4) *respectively) if and only if $2 < p \leq 8$.*

Remark: For the values $2 < p < 8$, the solution ϕ may be generated as a normalized wave. That is, as the minimizer of the following constrained variational problems

$$\begin{cases} N[u] = \frac{1}{4} \int_{\mathbf{R}} |u\partial_x u|^2 - \frac{1}{p} \int |u|^p \to \min \\ \text{subject to } \int |u|^2 dx = \lambda. \end{cases}$$

Note that this constrained variational problem also has unique solution, for each $\lambda > 0, p > 2$. Then, one has a special one-to-one correspondence $\omega = \omega(\lambda) : \mathbf{R}_+ \to \mathbf{R}_+$, which is an increasing function.

4.2 Existence of the compacton waves

We start with the proof of Proposition 21.

4.2.1 Proof of Proposition 21

We solve the ODE (4.1.5). As long as $\phi \neq 0$, say on an interval $[-L, L]$, we may divide by ϕ, which then leads us to the ODE

$$-\frac{1}{2}\partial_x^2(\phi^2) + \omega - |\phi|^{p-2} = 0. \tag{4.2.1}$$

Denoting $Q := \phi^2 \geq 0$, this is equivalent to

$$-\frac{1}{2}Q'' + \omega - Q^{\frac{p}{2}-1} = 0. \tag{4.2.2}$$

Multiplying the equation by Q' and integrating on the interval $[-L, L]$, and imposing that $Q'(-L) = Q'(L) = Q(L) = Q(-L) = 0$, we reduce the order of the ODE, namely to

$$(Q')^2 = 4\omega Q - \frac{8}{p}Q^{\frac{p}{2}}. \tag{4.2.3}$$

If we further require that Q is decreasing on $(0, L)$, i.e. $Q'(r) < 0, r > 0$, we can finally setup the ODE

$$Q'(r) = -2\sqrt{\omega Q(r) - \frac{2}{p}Q^{\frac{p}{2}}(r)}, 0 < r < L. \tag{4.2.4}$$

Note that this immediately implies (4.1.6). Clearly, (4.2.4) has an unique solution of the required form, namely with $Q(0) = Q_0 = \left(\frac{p\omega}{2}\right)^{\frac{2}{p-2}}, Q'(0) = 0$, if we select

$$L = \int_0^{Q_0} \frac{1}{2\sqrt{\omega Q - \frac{2}{p} Q^{\frac{p}{2}}}} dQ = \frac{p^{\frac{1}{p-2}}}{2^{\frac{p-1}{p-2}}} \omega^{\frac{4-p}{2(p-2)}} \int_0^1 \frac{1}{\sqrt{z - z^{\frac{p}{2}}}} dz. \tag{4.2.5}$$

Clearly, such solution is bell-shaped as it satisfies (4.2.4).

Conversely, assume that ϕ is a bell-shaped solution of (4.1.5), which vanishes at $\pm\infty$. Then, $Q := \phi^2$ will satisfy (4.2.2) as argued above, and it will also vanish at $\pm\infty$. We observe now that the support of Q may not be infinite, since then, it must be that

$$+\infty = \int_0^{Q(0)} \frac{dQ}{\sqrt{\omega Q - \frac{2}{p} Q^{\frac{p}{2}}}},$$

which is clearly false as the integral is convergent, due to the mild singularity at both 0 and $Q(0)$. Thus, Q (and subsequently ϕ) is supported on a finite interval $[-L, L]$. Then, it becomes clear that since $Q'(L) = 0$, it is the case that $Q(0) = Q_0 = \left(\frac{p\omega}{2}\right)^{\frac{2}{p-2}}$ and the solution is unique from (4.2.4). In particular, L is given by the formula displayed in (4.2.5).

4.2.2 An alternative variational problem

Looking at the form of the functional $N[u]$, it is pretty standard to replace $u = \sqrt{v}$, especially since we are looking for positive solutions of (4.1.5). Specifically, we shall consider

$$\begin{cases} N_0[v] = N[\sqrt{v}] = \frac{1}{4} \int_{\mathbf{R}} |v'|^2 + \omega \int |v| \to \min \\ \text{subject to } \int |v|^{p/2} dx = 1. \end{cases} \tag{4.2.6}$$

Clearly, $N_0[v] \geq 0$, so we introduce

$$m(\omega) := \inf_{v \in \mathcal{S}} J_\omega(u) = \inf_{\int |u|^p dx = 1} N[u] = \inf_{\int |v|^{p/2} dx = 1} N_0[v].$$

We now show that (4.2.6) has a solution.

Proposition 22. *Let $p > 2, \omega > 0$. Then, the constrained minimization problem (4.2.6) has a bell-shaped solution $\varphi = \varphi^{\#}$.*

Proof. By the Szegö inequality, $\int_{\mathbf{R}} |v'|^2 \geq \int_{\mathbf{R}} |\partial_x v^{\#}|^2$, while $\int |v| = \int |v^{\#}|, \int |v|^{p/2} = \int |v^{\#}|^{p/2}$, whence it is clear that it suffices to restrict the problem (4.2.6)to bell-shaped entries v. Take a minimizing sequence, $v_n : \|v_n\|_{L^{p/2}} = 1$, $v_n \in H^1 \cap L^1$,

$$\lim_n \left(\frac{1}{4} \int_{\mathbf{R}} |v_n'|^2 + \omega \int v_n \right) = m(\omega).$$

It follows that $\sup_n \|v_n'\|_{L^2} < 4m(\omega)$, $\sup_n \int v_n \leq m(\omega)$. By the bell-shapedness, we conclude the point-wise decay $|v_n(x)| \leq \frac{C}{1+|x|}$. Thus, by Kolmogorov-Rellich criteria, the set $\{v_n : \sup_n \|v_n'\|_{L^2} < \infty, |v_n(x)| \leq \frac{C}{1+|x|}\}$ is pre-compact in $L^{p/2}(\mathbf{R})$. Without loss of generality, we can assume that $\lim_n \|v_n - \varphi\|_{L^{p/2}} = 0$ and $v_n' \to \varphi'$ weakly. Similarly, fixing a compact subset $K \subset \mathbf{R}$, the set $\{v_n : \sup_n \|v_n'\|_{L^2} < \infty, \|v_n\|_{L^{p/2}} = 1\}$ is a pre-compact subset in any $C^\alpha(K), \alpha < \frac{1}{2}$. In particular, we can without loss of generality assume in addition that v_n converges uniformly to φ on the compact subsets of $\mathbf{R}$. By Fatou's lemma, $\liminf_n \int v_n \geq \int \varphi$. Finally, by the lower semi-continuity of the L^2 norm, with respect to the weak topology, $\liminf_n \int (\partial_x v_n)^2 \geq \int (\varphi')^2$. Putting it all together, we have that $\int \varphi^{p/2} = 1$, while

$$N_0[\varphi] \leq \liminf_n N_0[v_n] = m(\omega).$$

It follows that $N_0[\varphi] = m(\omega)$, whence φ is indeed a constrained minimizer of (4.2.6). In addition, φ is clearly bell-shaped (as limit of bell-shaped functions).

□

We will eventually establish that the minimizers φ of (4.2.6) are unique (up to translations) and they also have compact support, whence the reference to "compactons". We would like the reader to keep this in mind, as this is somewhat non-standard situation, which develops herein. For this, we need to derive the Euler-Lagrange relation for (4.2.6).

Proposition 23. *The solution φ of (4.2.6) is compactly supported function, which satisfies the Euler-Lagrange equation*

$$-\frac{1}{2}\varphi'' + \omega - c(\omega,p)\varphi^{\frac{p}{2}-1} = 0, -L < x < L \tag{4.2.7}$$

where $c = c(\omega,p)$ are explicit. In fact, there is the formula

$$c(\omega,p) = p^{\frac{3}{p+1}}\omega^{\frac{p+4}{2(p+1)}}\left(2\int_0^1 \sqrt{z - z^{\frac{p}{2}}}dz + \int_0^1 \frac{z}{\sqrt{z - z^{\frac{p}{2}}}}dz\right)^{\frac{p-2}{p+1}}. \tag{4.2.8}$$

Finally, there are the following formulas for the behavior of φ at $\pm L$

$$\varphi(x) = \omega(L-x)^2 + O((L-x)^3), \;\; \varphi(x) = \omega(x+L)^2 + O((x+L)^3). \tag{4.2.9}$$

Before we proceed with the proof of Proposition 23, we would like to make some important remarks.

1. Similar to (4.2.8), one may compute various quantities involving norms of φ, as well as the half-period L etc. This is despite the lack of explicit formulas for the function φ.

2. It is pretty clear that once c is given by a formula like (4.2.8), the uniqueness results from Proposition 21 apply here as well. In particular, the variational problem (4.2.6) has an unique bell-shaped solution φ.

Proof. (Proposition 23)

Let $\epsilon : |\epsilon| << 1$ and consider a test function h, so that it vanishes outside the support of φ. Let us look at a perturbation $\varphi + \epsilon h$. Note that due to the constraint in (4.2.6) and the support property of h, namely $supp\ h \subset supp\ \varphi$, we have the formula

$$\|\varphi + \epsilon h\|_{L^{\frac{p}{2}}} = 1 + \epsilon \langle \varphi^{\frac{p}{2}-1}, h\rangle + O(\epsilon^2).$$

From the minimization property of $\varphi > 0$, we must have that the scalar function

$$\begin{aligned} g(\epsilon) :&= N_0\left[\frac{\varphi+\epsilon h}{\|\varphi+\epsilon h\|_{L^{\frac{p}{2}}}}\right] = \frac{1}{4}\int_{\mathbb{R}}\left(\frac{\varphi'+\epsilon h'}{\|\varphi+\epsilon h\|_{L^{\frac{p}{2}}}}\right)^2 dx + \omega\int_{\mathbf{R}}\frac{(\varphi+\epsilon h)}{\|\varphi+\epsilon h\|_{L^{\frac{p}{2}}}}dx = \\ &= \frac{1}{4}\int_{\mathbf{R}}(\varphi'+\epsilon h')^2 dx(1-2\epsilon\langle\varphi^{\frac{p}{2}-1},h\rangle) + \omega\int_{\mathbf{R}}(\varphi+\epsilon h)dx(1-\epsilon\langle\varphi^{\frac{p}{2}-1},h\rangle) + O(\epsilon^2) \\ &= g(0) + \epsilon\left(\frac{1}{2}\langle-\varphi'',h\rangle - \langle\varphi^{\frac{p}{2}-1},h\rangle(\frac{1}{2}\int_{\mathbf{R}}(\varphi')^2dx + \omega\int\varphi) + \omega\langle h,1\rangle\right) + O(\epsilon^2). \end{aligned}$$

□

achieves its minimum at zero. It follows that $g'(0) = 0$, whence for all test functions h

$$\langle -\frac{1}{2}\varphi'' + \omega - c\varphi^{\frac{p}{2}-1}, h\rangle = 0,$$

where we have denoted $c = c(\omega, p) := \frac{1}{2}\int(\varphi')^2 + \omega\int\varphi$. It follows that on the support of φ, the Euler-Lagrange equation (4.2.7) holds true. As φ is bell-shaped, this must be an interval of the form $[-L, L]$ for some $L > 0$ or $\mathbf{R}$.

We show now that such minimizer φ must be compactly supported. Indeed, suppose that $supp\ \varphi$ is **R**. Multiplying (4.2.7) by φ' and integrating once (and taking into account that φ, φ' vanish at $\pm\infty$ and $\varphi'(x) < 0, x > 0$, due to bell-shapedness), we obtain the relation

$$\varphi'(x) = -2\sqrt{\omega\varphi(x) - \frac{2c}{p}\varphi^{\frac{p}{2}}(x)}, x > 0. \tag{4.2.10}$$

Now, clearly as the function φ achieves its maximum at zero, we have that φ_0 is a zero of the function $z \to \omega - \frac{2c}{p} z^{\frac{p}{2}-1}$, so $\varphi_0 = \varphi(0)$ satisfies

$$\varphi_0^{\frac{p}{2}-1} = \frac{p\omega}{2c(\omega,p)}. \tag{4.2.11}$$

If the relation (4.2.10) holds for a function φ supported on **R**, it must be that

$$\infty = \int_0^{\varphi_0} \frac{d\varphi}{2\sqrt{\omega\varphi - \frac{2c}{p}\varphi^{\frac{p}{2}}}}.$$

This is however clearly false, as the integral in question is convergent, due to the mild singularities at $0, \varphi_0$. So, $supp\ \varphi = [-L, L]$, and in fact, using the relation (4.2.11) yields

$$L = \int_0^{\varphi_0} \frac{d\varphi}{2\sqrt{\omega\varphi - \frac{2c(\omega)}{p}\varphi^{\frac{p}{2}}}} = \frac{\sqrt{\varphi_0}}{2\sqrt{\omega}} \int_0^1 \frac{dz}{\sqrt{z - z^{p/2}}}.$$

We now compute explicitly $c(\omega, p)$. We have

$$\begin{aligned} c(\omega,p) &= \frac{1}{2}\int (\varphi')^2 + \omega \int \varphi = \int_0^L (\varphi')^2 + 2\omega \int_0^L \varphi = \\ &= 2\int_0^{\varphi_0} \sqrt{\omega\varphi - \frac{2c}{p}\varphi^{\frac{p}{2}}}\, d\varphi + \omega \int_0^{\varphi_0} \frac{\varphi}{\sqrt{\omega\varphi - \frac{2c}{p}\varphi^{\frac{p}{2}}}} d\varphi = \\ &= \varphi_0^{\frac{3}{2}}\sqrt{\omega}\left(2\int_0^1 \sqrt{z - z^{p/2}}\, dz + \int_0^1 \frac{z}{\sqrt{z - z^{p/2}}} dz\right). \end{aligned}$$

Together with (4.2.11), this yields a system of two relations for the unknowns $\varphi_0(\omega, p), c(\omega, p)$, which results in the formula (4.2.8).

Finally, we discuss the behavior of φ in a proximity of the endpoints $\pm L$. From bell-shapedness and (4.2.7), we obtain that $\varphi(\pm L) = \varphi'(\pm L) = 0$, whereas $\varphi''(\pm L) = 2\omega > 0$. Thus, (4.2.9) holds true.

4.2.3 Spectral theory for the linearized operator $\mathcal{L}_+$

Clearly, by the equivalence between the constrained minimization problems (4.1.13) and (4.2.6), we have that (4.1.13) has a solution $\Phi := \sqrt{\varphi}$, which is also bell-shaped. We establish further properties of Φ.

Proposition 24. *The solution Φ of (4.1.13), satisfies the Euler-Lagrange equation*

$$-\Phi\partial_x(\Phi\Phi') + \omega\Phi - c(\omega)\Phi^{p-1} = 0, \quad -L < x < L. \tag{4.2.12}$$

In addition, consider the symmetric operator

$$\mathcal{L}_+ f = -\Phi\partial_x^2(\Phi f) + \omega f - (\Phi\Phi')' f - (p-1)c(\omega)\Phi^{p-2} f, \tag{4.2.13}$$

with a base Hilbert space $L^2[-L, L]$. Any self-adjoint extension of $\mathcal{L}_+$ (also denoted by $\mathcal{L}_+$), has the property $\mathcal{L}_+|_{\{\Phi^{p-1}\}^\perp} \geq 0$. In particular, (any self-adjoint extension of) $\mathcal{L}_+$ has at most one negative eigenvalue.

Remark: Note that by the relation (4.2.12), we have that $-(\Phi\Phi')' + \omega = c(\omega)\Phi^{p-2}$, whence we can rewrite the linearized operator and the corresponding quadratic form as follows

$$\mathcal{L}_+ f = -\Phi\partial_x^2(\Phi f) - (p-2)c(\omega)\Phi^{p-2} f \tag{4.2.14}$$

$$q(u,v) = \langle \partial_x(\Phi u), \partial_x(\Phi v)\rangle - (p-2)c(\omega,p)\langle \Phi^{p-2}u, v\rangle. \tag{4.2.15}$$

Proof. For $\epsilon : |\epsilon| << 1$ and a test function $h : supp\ h \subset (-L, L)$, consider a perturbation $\Phi + \epsilon h$. According to the minimization property of Φ, we must have that the scalar function

$$\begin{aligned} f(\epsilon) &:= N\left[\frac{\Phi+\epsilon h}{\|\Phi+\epsilon h\|_{L^p}}\right] \\ &= \frac{1}{4\|\Phi+\epsilon h\|_{L^p}^4}\int (2\Phi\Phi' + 2\epsilon(\Phi h)' + \epsilon^2\partial_x(h^2))^2 + \frac{\omega}{\|\Phi+\epsilon h\|_{L^p}^2}\int (\Phi+\epsilon h)^2. \end{aligned}$$

has a minimum at $\epsilon = 0$. As a consequences of the minimization property, $f'(0) = 0$, while $f''(0) \geq 0$. It is however easier to work with expansions in powers of ϵ, instead of differentiating directly in ϵ. We have

$$\|\Phi+\epsilon h\|_{L^p}^p = 1 + p\epsilon\langle \Phi^{p-1}, h\rangle + \frac{p(p-1)}{2}\epsilon^2\langle \Phi^{p-2}, h^2\rangle + O(\epsilon^3).$$

To this end, let us do the first order expansion.

$$\begin{aligned} f(\epsilon) &= \int (\Phi\Phi' + \epsilon(\Phi h)')^2(1 - 4\epsilon\langle \Phi^{p-1}, h\rangle) + \omega\int (\Phi^2 + 2\epsilon\Phi h)(1 - 2\epsilon\langle \Phi^{p-1}, h\rangle) + O(\epsilon^2) = \\ &= f(0) + 2\epsilon\left(-\int \Phi(\Phi\Phi')h + \omega\langle \Phi, h\rangle - \langle \Phi^{p-1}, h\rangle\left(2\int (\Phi\Phi')^2 + \omega\int \Phi^2\right)\right) + O(\epsilon^2). \end{aligned}$$

Hence we have the Euler-Lagrange equation (4.2.12) in weak sense, with

$$c(\omega) = 2\int (\Phi\Phi')^2 + \omega\int \Phi^2,$$

which is the same coefficient that we have encountered in Proposition 23.

Next, we deal with the second order condition, namely $f''(0) \geq 0$. To simplify matters, take $h \perp \Phi^{p-1}$, that is $\langle \Phi^{p-1}, h \rangle = 0$. Looking at the next order, that is the terms containing ϵ^2. We obtain

$$\frac{f''(0)}{2} = -(p-1)\langle \Phi^{p-2}, h^2 \rangle \left(2\int (\Phi\Phi')^2 + \omega \int \Phi^2 \right)$$
$$+\langle (\Phi h)', (\Phi h)' \rangle + \int \Phi\Phi' \partial_x(h^2) + \omega \int h^2.$$

It is now clear, after integration by parts, that one can write the previous identity in the form

$$\frac{f''(0)}{2} = \langle \mathcal{L}_+ h, h \rangle.$$

As this is valid for all $h : \langle \Phi^{p-1}, h \rangle = 0$ and $f''(0) \geq 0$, we conclude that $\mathcal{L}_+|_{\{\Phi^{p-1}\}^\perp} \geq 0$. □

From Proposition 24 and the minimax formula for the eigenvalues of a self-adjoint operator, we conclude that $n(\mathcal{L}_+) \leq 1$. Our next task is to characterize the rest of the spectrum of $\mathcal{L}_+$. We will need the relation between Φ_x^2 and $\Phi\Phi_{xx}$ with Φ^{p-2}. Multiplying (4.2.12) by Φ' and integrating, taking into account that $\Phi(\pm L) = 0$, we obtain

$$(\Phi')^2 = \omega - \frac{2c(\omega)}{p}\Phi^{p-2}. \tag{4.2.16}$$

Another very important relation, which is obtained from (4.2.12) and (4.2.16) is

$$\Phi\Phi'' = \frac{2-p}{p}c(\omega)\Phi^{p-2}. \tag{4.2.17}$$

We now state a technical result, which connects the degenerate operator $\mathcal{L}_+$, posed on domain contained in $L^2[-L,L]$ to a standard Schrödinger operator, defined herein

$$L_+ := -\partial_t^2 + \frac{\omega}{4} - c(\omega)\frac{2p^2-5p+3}{2p}\Phi^{p-2}(x(t)). \tag{4.2.18}$$

To this end, we borrow an important idea from [42]. Namely, a change of variables is introduced, which transform the degenerate operator $\mathcal{L}_+$, acting on $D(\mathcal{L}_+) \subset L^2[-L,L]$ into a standard Schrödinger operator L_+, with exponentially decaying potential, acting on a subspace of $L^2(\mathbf{R})$.

Lemma 13. *The transformation*

$$t(x) = \int_0^x \frac{dy}{\Phi(y)} : (-L,L) \to \mathbf{R} \tag{4.2.19}$$

is a one-to-one mapping from $(-L,L)$ to $\mathbf{R}$. The inverse function $x : \mathbf{R} \to (-L,L)$ satisfies

$$\lim_{t\to\pm\infty} x(t) = \pm L$$

and moreover, it approaches the limits with exponential rates. More specifically, for every $\epsilon > 0$, there exists C_ϵ, so that

$$L - C_\epsilon e^{(-\sqrt{\omega}+\epsilon)t} < x(t) < L, \;\; t > 0 \tag{4.2.20}$$

$$-L < x(t) < -L + C_\epsilon e^{(\sqrt{\omega}-\epsilon)t}, \;\; t < 0. \tag{4.2.21}$$

For the Schrödinger operator L_+, defined in (4.2.18), there is the relation

$$\mathcal{L}_+ f = \frac{L_+(\sqrt{\Phi} f)}{\sqrt{\Phi}}. \tag{4.2.22}$$

Remark: Note that due to the asymptotics (4.2.20) and (4.2.21), and (4.2.9), the potential $W(t) = c(\omega)\frac{2p^2-5p+3}{2p}\Phi^{p-2}(x(t))$ obeys the exponential decay estimate

$$0 < W(t) \leq C_\epsilon e^{-((p-2)\sqrt{\omega}-\epsilon)|t|}, t \in \mathbf{R}$$

for every $\epsilon > 0$ and some C_ϵ.

Proof. By (4.2.9), we have that $|\Phi(y)| \sim |L-y|, y \sim L$ and $|\Phi(y)| \sim |L+y|, y \sim -L$, we have that the integral in (4.2.19) converges to $-\infty$ as $x \to -L$, while it approaches ∞, as $x \to L$. That is, the transformation (4.2.19) provides a one-to-one homeomorphic map $t : (-L, L) \to (-\infty, \infty)$. Its inverse, which will be frequently used, is denoted $x(t)$. We need the asymptotic behavior of $x(t)$, which we analyze next. A simple L'Hospital calculation shows that

$$\lim_{x \to L-} \frac{\ln(L-x)}{t(x)} = -\lim_{x \to L-} \frac{\Phi(x)}{L-x} = -\sqrt{\omega},$$

where we have used the asymptotic (4.2.9) and similar at $x = -L$. In short, we obtain that for every $\epsilon > 0$, we have for every $\epsilon > 0$, (4.2.20) and (4.2.21) hold true.

For the formula (4.2.22), note first that have the relation $\partial_t = \Phi\partial_x$. We compute $\Phi^{\frac{1}{2}}\mathcal{L}_+[f\Phi^{-\frac{1}{2}}]$. By direct calculations,

$$\begin{aligned} -\Phi^{\frac{3}{2}}\partial_x^2[\sqrt{\Phi}f] &= -\Phi^2\partial_x^2 f - \Phi\Phi_x\partial_x f - \frac{2\Phi''\Phi - (\Phi')^2}{4}f = \\ &= -\Phi^2\partial_x^2 f - \Phi\Phi_x\partial_x f + \frac{\omega}{4}f - \frac{c(\omega)(3-p)}{2p}\Phi^{p-2}f, \end{aligned}$$

where we have used the relations (4.2.16) and (4.2.17). It follows that

$$\Phi^{\frac{1}{2}}\mathcal{L}_+[f\Phi^{-\frac{1}{2}}] = -\Phi^2\partial_x^2 f - \Phi\Phi_x\partial_x f + \frac{\omega}{4}f - c(\omega)\Phi^{p-2}\frac{2p^2-5p+3}{2p}f.$$

On the other hand, note that

$$-\partial_t^2 f = -\partial_t(\Phi\partial_x f) = -\Phi(\partial_x(\Phi\partial_x f)) = -\Phi^2\partial_x^2 f - \Phi\Phi_x\partial_x f.$$

It follows that $L_+ f = \Phi^{\frac{1}{2}}\mathcal{L}_+[f\Phi^{-\frac{1}{2}}]$, or alternatively, (4.2.22). $\square$

Proposition 25. *Let $\mathcal{L}_+$ be any self-adjoint extension of the symmetric operator introduced in* (4.2.13)*. Then, $\mathcal{L}_+$ has exactly one negative eigenvalue, say $-\zeta^2$, a simple eigenvalue at zero, with $Ker(\mathcal{L}_+) = span[\Phi']$, while the rest of the spectrum is away from zero. That is, there exists $\delta > 0$, so that*

$$spec(\mathcal{L}_+) \setminus \{-\zeta^2, 0\} \subset [\delta, +\infty). \tag{4.2.23}$$

Proof. Direct inspection shows that

$$\begin{aligned} \mathcal{L}_+\Phi' &= -\Phi\partial_x^2(\Phi\Phi') + \omega\Phi' - (\Phi\Phi')'\Phi' - (p-1)c(\omega)\Phi^{p-2}\Phi' = \\ &= -\Phi\partial_x^2(\Phi\Phi') - (\Phi\Phi')'\Phi' + (\omega\Phi - c(\omega)\Phi^{p-1})' = \\ &= -\Phi\partial_x^2(\Phi\Phi') - (\Phi\Phi')'\Phi' + (\Phi\partial_x(\Phi\Phi'))' = 0, \end{aligned}$$

whence $\Phi' \in Ker(\mathcal{L}_+)$. This is of course a consequence of the translational invariance of the profile equation (4.2.12). Also by a direct inspection, and using the equation (4.2.12), we conclude

$$\mathcal{L}_+\Phi = -2\omega\Phi - (p-4)c(\omega)\Phi^{p-1}.$$

Taking dot product with Φ, we obtain,

$$\langle \mathcal{L}_+\Phi, \Phi \rangle = -2\omega \int \Phi^2 - (p-4)c(\omega). \tag{4.2.24}$$

We will now show that such quantity is negative, for all $p > 2$, which would establish that $\mathcal{L}_+$ has exactly one negative eigenvalue. One could use the arguments of Proposition 23 to evaluate the quantity $\langle \mathcal{L}_+ \Phi, \Phi \rangle$ explicitly in terms of p, ω. Instead, we provide an easier roundabout argument, which shows $\langle \mathcal{L}_+ \Phi, \Phi \rangle < 0$. To this end, recall that we have already established the relation

$$c(\omega) = \frac{1}{2}\int (\varphi')^2 + \omega \int \varphi = 2\int (\Phi\Phi')^2 + \omega \int \Phi^2.$$

We now obtain another identity based on taking dot product of (4.2.12) with $x\Phi'$. Due to $\int \Phi^p dx = 1$, we have

$$c(\omega) = \frac{p\omega}{2}\int \Phi^2 - \frac{p}{2}\int (\Phi\Phi')^2.$$

Combining this with (4.2.12), we establish the relationship

$$\int (\Phi\Phi')^2 = \frac{p-2}{p+4}\omega \int \Phi^2 dx,$$

which in turn implies

$$c(\omega) = \frac{3p\omega}{p+4}\int \Phi^2.$$

Substituting this in (4.2.24) results in the formula

$$\langle \mathcal{L}_+ \Phi, \Phi \rangle = -\omega \frac{(3p^2 - 10p + 8)}{p+4}\int \Phi^2 dx < 0,$$

for $p > 2, \omega > 0$. Thus, $n(\mathcal{L}_+) = 1$.

Next, we establish that the eigenvalue at zero is simple and there is a gap between the zero and the non-negative portion of the spectrum. Recall that we work with arbitrary self-adjoint extension of $\mathcal{L}_+ : D(\mathcal{L}_+) \subset L^2[-L, L]$, given by the quadratic form q. Since

$q(u,u) \geq 0$ for all $u \perp \Phi^{p-1}$, it follows that $\mathcal{L}_+$ has only one simple negative eigenvalue, say $-\zeta^2$, and $\sigma(\mathcal{L}_+) \setminus \{-\zeta^2\} \subset [0,\infty)$. For any Ω, a relatively open subset in $[0,\infty)$, denote the spectral projection $P_\Omega := \chi_\Omega(\mathcal{L}_+)$. Supposing for a contradiction that (4.2.23) fails, we select a sequence f_n, with $P_{[0,\frac{1}{n})} f_n = f_n, \|f_n\|_{L^2} = 1, f_n \perp \Phi'$. It follows that

$$\langle \mathcal{L}_+ f_n, f_n \rangle = q(f_n, f_n) = \|(\Phi f_n)'\|^2_{L^2[-L,L]} - (p-2)c(\omega) \int_{-L}^{L} \Phi^{p-2} f_n^2 dx \to 0. \quad (4.2.25)$$

Denote $g_n := \Phi f_n$. Clearly, $\|g_n\|_{L^2} \leq \|\Phi\|_{L^\infty}$, while

$$\limsup_n \|g_n'\|_{L^2} = \limsup_n [q(f_n, f_n) + (p-2)c(\omega) \int_{-L}^{L} \Phi^{p-2} f_n^2] \leq C_p \|\Phi\|^{p-2}_{L^\infty},$$

whence $\sup_n \|g_n\|_{H^1} < \infty$.

From the various convergences and the weak compactness of bounded sets in $L^2[-L,L]$, and the compactness of the embedding $H^1[-L,L]$ into $L^2[-L,L]$, it follows that there exist a subsequence (denoted the same), so that the weak convergences $f_n \rightharpoonup_{L^2} U$, $g_n \rightharpoonup_{H^1} g$ hold, as well as the strong convergence $\lim_n \|g_n - g\|_{L^2[-L,L]} = 0$. In particular, one can see that for every $\delta > 0$, $\lim_n \|g_n - g\|_{L^2[-L+\delta, L-\delta]} = 0$. Moreover, since Φ does not vanish on $[-L+\delta, L-\delta]$, we have that $g = \Phi U$ and $\lim_n \|f_n - U\|_{L^2[-L+\delta,L-\delta]} = 0$, for every $\delta > 0$. This is now enough to conclude that

$$\lim_n \int_{-L}^{L} \Phi^{p-2} f_n^2 dx = \int_{-L}^{L} \Phi^{p-2} U^2 dx \quad (4.2.26)$$

Indeed, since $\Phi(x) \leq C(L+x), -L < x$ and $\Phi(x) \leq C(L-x), x <$, we obtain

$$\left|\int_{[-L,-L+\delta)\cup(L-\delta,L)} \Phi^{p-2} f_n^2 dx\right| \leq C\delta^{p-2} \|f_n\|^2_{L^2} = C\delta^{p-2},$$

and similarly for the integrals $|\int_{[-L,-L+\delta)\cup(L-\delta,L)}\Phi^{p-2}U^2dx|$. On the other hand, on the interval $[-L+\delta, L-\delta]$, we use the convergence $\lim_n \|f_n-U\|_{L^2[-L+\delta,L-\delta]}=0$, to estimate

$$|\int_{-L+\delta}^{L-\delta}\Phi^{p-2}f_n^2dx-\int_{-L+\delta}^{L-\delta}\Phi^{p-2}U^2dx|\leq\|\Phi\|_{L^\infty}^{p-2}\|f_n-U\|_{L^2[-L+\delta,L-\delta]}(\|f_n\|_{L^2}+\|U\|_{L^2}).$$

Putting all this together ensures (4.2.26).

We now claim that U is not identically zero. Assume for a contradiction that $U=0$. In view of (4.2.25) and (4.2.26), this implies that $\lim_n\|g_n'\|_{L^2}=0$. Since $g_n\in H^1[-L,L]$, it follows that it is uniformly continuous function, whence the limit $\lim_{x\to L-}g_n(x)=\lim_{x\to L-}\Phi(x)f_n(x)=:c_n$ exists. Then, $c_n=0$, since otherwise $|f_n(x)|\geq\frac{|c_n|}{2\Phi(x)}\geq\frac{C_n}{(L-x)}$ for all $x\in(L-\delta,L)$ and some $C_n>0$. But then,

$$1=\int_{-L}^{L}f_n^2(x)dx\geq\int_{L-\delta}^{L}f_n^2(x)dx\geq C_n^2\int_{L-\delta}^{L}\frac{1}{(L-x)^2}dx=\infty,$$

a contradiction. It follows that $g_n(L)=0$. Similarly, $g_n(-L)=0$. Now, we can estimate

$$\int_{-L}^{0}|f_n(x)|^2dx=\int_{-L}^{0}\frac{|g_n(x)|^2}{\Phi^2(x)}dx\leq C\int_{-L}^{0}\frac{|g_n(x)|^2}{(L+x)^2}dx\leq C\int_{-L}^{0}|g_n'(x)|^2dx,$$

where we have first observed that $\Phi(x)>C(L+x)$ on $x\in(-L,0)$, and in the last step, we have applied the Hardy's inequality (see (1.1.5)), as $g_n(-L)=0$. Similarly,

$$\int_{0}^{L}|f_n(x)|^2dx\leq C\int_{0}^{L}|g_n'(x)|^2dx.$$

Combining the last two inequalities, we obtain

$$1=\int_{-L}^{0}|f_n(x)|^2dx+\int_{0}^{L}|f_n(x)|^2dx\leq C\int_{-L}^{L}|g_n'(x)|^2dx,$$

which is contradictory as $\lim_n \|g_n'\|_{L^2} = 0$. This proves that U is not identically zero.

As $f_n = P_{[0,1/n)} f_n$ and $f_n \rightharpoonup U$, it is clear that U is an eigenfunction for $\mathcal{L}_+$. This can also be seen as a consequence of (4.2.26), the inequality $\liminf_n \|g_n'\|_{L^2} \geq \|g\|_{L^2}$ (which is the lower semi-continuity of L^2 norm, with respect to weak convergence) and (4.2.25).

In any case, we conclude that $\mathcal{L}_+ U = 0$. Recall that $f_n \perp \Phi'$, whence their weak limit $f_n \rightharpoonup U$ also satisfies $U \perp \Phi'$. According to (4.2.22) however, this implies that

$$L_+(\sqrt{\Phi} U) = 0,$$

at least in a distributional sense[2], against the compactly supported test functions. Standard elliptic theory, together with the decay properties of $\sqrt{\Phi} U$ proves that $\sqrt{\Phi} U$ is indeed an L^2 eigenfunction for L_+. In addition, due to $\mathcal{L}_+[\Phi'] = 0$ and again (4.2.22), we also have $L_+[\sqrt{\Phi}\Phi'] = 0$ as well. According to the standard Sturm-Liouville theory for Schrödinger operators acting on **R**, with exponentially decaying potentials, each eigenvalue of L_+ is simple. In our case however, we have two candidates for eigenfunctions corresponding to the zero eigenvalue, namely $\sqrt{\Phi} U, \sqrt{\Phi}\Phi'$. So, it must be that $\sqrt{\Phi} U = const. \sqrt{\Phi}\Phi'$, or $U = const. \Phi'$. This is in turn contradictory as $U \perp \Phi'$ and U is not identically zero. This concludes the proof of Proposition 25. □

4.2.4 Spectral theory for the operator $\mathcal{L}_-$

In order to analyze the Schrödinger eigenvalue problem (4.1.10), we shall need also basic properties of the spectrum of $\mathcal{L}_-$. In the classical theory, such operator is non-negative, with a simple eigenvalue at zero. The same results holds here as well.

[2]This is due to the presence of the exponentially growing in the spatial variable factor $\Phi^{-1/2}$ in the formula (4.2.22).

We start however with a formula for $\mathcal{L}_-$ in the spirit of (4.2.22). Namely, for the Schrödinger operator

$$L_- = -\partial_t^2 + \frac{9\omega}{4} - \frac{3(p+1)}{2p} c(\omega)\Phi^{p-2}$$

defined on $L^2(\mathbf{R})$, there is the relation

$$\sqrt{\Phi}\mathcal{L}_- f = L_-(\sqrt{\Phi} f). \tag{4.2.27}$$

Proposition 26. *Let $\mathcal{L}_-$ be any self-adjoint extension of the symmetric operator*

$$\mathcal{L}_- f = -\Phi\partial_x^2[\Phi f] + 2(\omega - c(\omega)\Phi^{p-2}) f,$$

defined through the quadratic form $q(u,v) = \langle \Phi u, \Phi v\rangle + 2\langle(\omega - c(\omega)\Phi^{p-2})u, v\rangle$ for $u,v \in C_0^\infty(-L,L)$. Then, $\mathcal{L}_- \geq 0$, where zero is a simple eigenvalue, with $Ker[\mathcal{L}_-] = span[\Phi]$.

Proof. A direct inspection shows that

$$\mathcal{L}_-[\Phi] = -\Phi\partial_x^2[\Phi^2] + 2(\omega - c(\omega)\Phi^{p-2})\Phi = 0,$$

due to the profile equation (4.2.12).

Next, we show that there is the point-wise domination $\mathcal{L}_- \geq \mathcal{L}_+$, as in the classical case. Indeed, we have

$$\begin{aligned} \mathcal{L}_- - \mathcal{L}_+ &= \frac{9\omega}{4} - \frac{3(p+1)}{2p} c(\omega)\Phi^{p-2} - \left(\frac{\omega}{4} - \frac{2p^2 - 5p + 3}{2p} c(\omega)\Phi^{p-2}\right) = \\ &= 2\omega - (4-p)c(\omega)\Phi^{p-2} \end{aligned}$$

If $p \geq 4$, this is clearly a positive quantity. In the case $2 < p < 4$, taking into account that $\Phi \leq \Phi(0) = \left(\frac{p\omega}{2c(\omega)}\right)^{\frac{1}{p-2}}$, we conclude again

$$\mathcal{L}_- - \mathcal{L}_+ \geq 2\omega - \frac{(4-p)p\omega}{2} = \frac{\omega}{2}(p-2)^2 > 0$$

As it was already shown in Proposition 24, that $\mathcal{L}_+|_{\{\Phi^{p-1}\}^\perp} \geq 0$, it follows that $\mathcal{L}_-|_{\{\Phi^{p-1}\}^\perp} \geq 0$. In addition, such an inequality guarantees that $\mathcal{L}_-$ may have at most a single negative eigenvalue at the bottom of its spectrum. We proceed to rule this out. Recalling the relationship (4.2.27), it follows that if $\mathcal{L}_-$ has a negative eigenvalue, then the operator L_- has negative eigenvalue as well. Thus, it remains to rule out negative eigenvalues for L_-.

Recall that since $\mathcal{L}_-[\Phi] = 0$, by (4.2.27), it follows that $L_-[\Phi^{\frac{3}{2}}] = 0$. Thus, the Schrödinger operator L_- has an eigenvalue at zero, with corresponding positive eigenfunction $\Phi^{\frac{3}{2}}$. By Sturm-Liouville's theorem, this means that zero is at the bottom of the spectrum for L_-. We have thus ruled out negative eigenvalues for L_-, whence $\mathcal{L}_- \geq 0$.

Finally, suppose that Ψ is an eigenfunction for $\mathcal{L}_-$, corresponding to the zero eigenvalue. That is, $\mathcal{L}_-[\Psi] = 0$. By (4.2.27), $L_-[\Psi\sqrt{\Phi}] = 0$. As we have just seen, zero is at the bottom of the spectrum for L_- and it is hence a simple eigenvalue, with a corresponding eigenfunction $\Phi^{\frac{3}{2}}$. It follows that $\Psi\sqrt{\Phi} = const.\Phi^{\frac{3}{2}}$ or $\Psi = const.\Phi$. Thus $Ker[\mathcal{L}_-] = span[\Phi]$ and the proof of Proposition 26 is complete.

□

4.3 Spectral stability of the compacton waves

Our next task is to study the stability of the waves ϕ, which satisfy (4.1.5). Before we address these issues, for both the degenerate NLS and KdV cases, we would like to comment on the precise relation between the solutions φ to the variational problem (4.2.6) and the waves ϕ.

Proposition 27. *The profile equation* (4.1.5) *has unique bell-shaped solution* $\phi : [-L, L] \to$ **R** *Moreover, this solution is related to the unique solution* Φ *of the variational problem* (4.1.13) *via a formula*

$$\Phi_\omega(x) = c_{p,\omega}\omega^{-\alpha}\phi(\omega^\alpha x), \alpha = \frac{p+4}{2(p+1)(p-2)}. \tag{4.3.1}$$

Finally,

$$\Phi_\omega(x) = \omega^{\frac{1}{2(p+1)}}\Phi_1(\omega^{\frac{p}{2p+2}}x), \tag{4.3.2}$$

where Φ_1 *is the unique minimizer of* (4.1.13) *with* $\omega = 1$.

Proof. We have already discussed the uniqueness of φ, and consequently of $\Phi = \sqrt{\varphi}$, see the remarks after Proposition 23. It remains to note that due to the relation (4.2.8), we have that $c(\omega, p) = const(p)\omega^{\frac{p+4}{2(p+1)}}$, and so plugging in the relation (4.3.1) in the Euler-Lagrange equation (4.1.13) yields (4.1.5), for appropriately chosen constant $c_{\omega,p}$. Note that this constant $c_{\omega,p}$ can be derived explicitly based on (4.2.8), but we will not do so herein.

The formula for Φ_ω in terms of Φ_1 is due to an elementary scaling transformation, which transforms the variational problem (4.1.13) for general $\omega > 0$, into the one for $\omega = 1$. □

Based on the results of Proposition 27, we can claim the following properties of the operators $\mathcal{L}_\pm$, based on the corresponding $\mathcal{L}_\pm$.

Proposition 28. *Any self-adjoint extensions of the symmetric operators* $\mathcal{L}_\pm$ *satisfy the following properties*

- $\mathcal{L}_- \geq 0$, *with a simple eigenvalue at zero, given by* Φ, *i.e.*

$$Ker(\mathcal{L}_-) = span[\phi], \mathcal{L}_-|_{\{\phi\}^\perp} > 0.$$

- *The operator $\mathcal{L}_+$ has exactly one negative eigenvalue, the second smallest eigenvalue is zero, which is also simple. In fact, $n(\mathcal{L}_+) = 1$, while $Ker(\mathcal{L}_+) = span[\phi']$.*

We are now ready to proceed with the analysis of the spectral stability of the compacton waves ϕ. We start with the degenerate NLS case.

4.3.1 Spectral stability of the degenerate NLS compactons

According to the instability index theory developed in Section 1.1.4, we start with $Ker(\mathcal{L})$. Clearly,

$$Ker(\mathcal{J}\mathcal{L}) = Ker(\mathcal{L}) = span[\begin{pmatrix} Ker(\mathcal{L}_+) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Ker(\mathcal{L}_-) \end{pmatrix}] = span[\begin{pmatrix} \phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi \end{pmatrix}].$$

Looking for adjoints, we solve $\mathcal{J}\mathcal{L}\vec{f} = \begin{pmatrix} \phi' \\ 0 \end{pmatrix}$, which yields $\vec{f} = -\begin{pmatrix} 0 \\ \mathcal{L}_-^{-1}\phi' \end{pmatrix}$. Same as in the classical cases, further adjoints are impossible, behind $\vec{f}$. Indeed, assuming $\mathcal{J}\mathcal{L}\vec{g} = \vec{f}$, we need to solve $\mathcal{L}_+ g_1 = -\mathcal{L}_-^{-1}[\phi']$. This is however a contradiction by Fredholm theory, since

$$0 = \langle \mathcal{L}_+ g_1, \phi' \rangle = -\langle \mathcal{L}_-^{-1}[\phi'], \phi' \rangle < 0,$$

due to the fact that $\mathcal{L}_-^{-1}|_{\{\phi\}^\perp} > 0$.

On the other hand, solving $\mathcal{J}\mathcal{L}\vec{f} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$, produces an adjoint $\vec{f} = \begin{pmatrix} \mathcal{L}_+^{-1}\phi \\ 0 \end{pmatrix}$. Looking for further adjoints involves the quation $\mathcal{J}\mathcal{L}\vec{g} = \begin{pmatrix} \mathcal{L}_+^{-1}\phi \\ 0 \end{pmatrix}$, which results in $\mathcal{L}_- g_2 = -\mathcal{L}_+^{-1}\phi$. By Fredholm theory, this requires a solvability condition $\langle \mathcal{L}_+^{-1}\phi, \phi \rangle = 0$.

Thus, we may conclude that as long as $\langle \mathcal{L}_+^{-1}\phi, \phi\rangle \neq 0$,

$$gKer(\mathcal{J}\mathcal{L}) \ominus Ker(\mathcal{L}) = span[\begin{pmatrix} \mathcal{L}_+^{-1}\phi \\ 0 \end{pmatrix}].$$

According to the instability index theory, the matrix D is one dimensional and the stability is determined by the sign of $\langle \mathcal{L}_+^{-1}\phi, \phi\rangle$.

It turns out that this is a quantity easy to compute. In fact, from the results of Proposition 27, we can see that the mapping $\omega \to \phi_\omega$ is a C^1 mapping from the positive line into the set of functions. This allows us to take a Frechet derivative with respect to ω in the profile equation (4.1.5), just as in the classical case. In short, we obtain

$$\mathcal{L}_+[\partial_\omega \phi_\omega] = -\phi,$$

whence

$$\langle \mathcal{L}_+^{-1}\phi, \phi\rangle = -\langle \partial_\omega, \phi_\omega, \phi_\omega\rangle = -\frac{1}{2}\partial_\omega \int \phi^2 dx.$$

4.3.2 Spectral stability for the degenerate KdV waves

We use, again, the procedure, outlined in Section 1.1.4. As established in Proposition 28, the eigenvalue problem (4.1.8) clearly has a one dimensional kernel, namely $Ker(\mathcal{L}_+) = span[\phi']$. We now proceed to find the generalized kernel of $\partial_x \mathcal{L}_+$. So, we need a $\psi \in D(\partial_x \mathcal{L}_+) \subset L^2(\mathbf{R})$, so that $\partial_x \mathcal{L}_+ \psi = \phi'$. It follows that

$$\mathcal{L}_+\psi = \phi + c, x \in \mathbf{R}. \tag{4.3.3}$$

Taking into account the specific form of the operator $\mathcal{L}_+$, in particular (4.1.9), we conclude that for $|x|>L$, one must have

$$\mathcal{L}_+\psi(x)=\omega\psi(x)=c, |x|>L,$$

since $supp\phi\subset(-L,L)$. This is of course impossible, as $\psi\in L^2(\mathbf{R})$, unless $c=0$. Thus, (4.3.3) becomes $\mathcal{L}_+\psi=\phi$.

Noting that $\phi\perp Ker(\mathcal{L}_+)$, so that $\mathcal{L}_+^{-1}\phi$ exists (uniquely in the subspace $Ker(\mathcal{L}_+)^\perp$), it follows that $\psi\in\mathcal{L}_+^{-1}\phi+Ker(\mathcal{L}_+)$, so we may select

$$\psi=\mathcal{L}_+^{-1}\phi,$$

and this is the unique element generating the subspace $gKer(\partial_x\mathcal{L}_+)\ominus Ker(\mathcal{L}_+)$. It follows that the stability of the traveling wave $\phi(x-\omega t)$, once again is equivalent to the condition $\langle\mathcal{L}_+^{-1}\phi,\phi\rangle<0$.

4.3.3 Computation of $\langle\mathcal{L}_+^{-1}\phi,\phi\rangle$

Since $\phi:(0,\infty)\to(0,\phi_0)$ is a bijection and based on the representation (4.1.6), we have

$$\begin{aligned}\int_{-\infty}^{\infty}\phi^2dx &= 2\int_0^\infty\phi^2dx=-2\int_0^\infty\frac{\phi^2}{\phi'}dx=2\int_0^{\phi_0}\frac{\phi^2}{\sqrt{\omega-\frac{2}{p}\phi^{p-2}}}d\phi=\\ &= 2\phi_0^3\int_0^1\frac{z^2}{\sqrt{\omega-\frac{2}{p}\phi_0^{p-2}z^{p-2}}}dz=2\left(\frac{p}{2}\right)^{\frac{3}{p-2}}\omega^{\frac{3}{p-2}-\frac{1}{2}}\int_0^1\frac{z^2}{\sqrt{1-z^{p-2}}}dz,\end{aligned}$$

so that

$$D=-\frac{1}{2}\partial_\omega\int_{-\infty}^{\infty}\phi^2dx=-\left(\frac{p}{2}\right)^{\frac{3}{p-2}}\frac{(8-p)}{2(p-2)}\omega^{\frac{3}{p-2}-\frac{3}{2}}\int_0^1\frac{z^2}{\sqrt{1-z^{p-2}}}dz,$$

This calculation yields the necessary and sufficient condition[3] for stability $2 < p \leq 8$.

[3]The point $p = 8$ is the threshold for stability, similar to the case $p = 5$ for the standard NLS. At this value, if we use p as a bifurcation parameter, going from $p < 8$ to $p > 8$, there is a crossing of a pair of purely imaginary eigenvalues through zero to a real pair of eigenvalues, one positive and one negative

Appendix A

Spherical harmonics and fractional Schrödinger operators

In this section, we give the final preparatory material before we establish the non-degeneracy in section 2.5, in the case $n \geq 2$. The approach is to decompose the fractional Schrödinger operator $\mathcal{L}_+ = (-\Delta)^s + \omega - p|x|^{-b}\Phi^{p-1}$, with a base space $L^2(\mathbf{R}^n)$ onto simpler, essentially one dimensional subspaces of the spherical harmonics (SH for short). This is convenient due to the radiality of the potential $W := p|x|^{-b}\Phi^{p-1}$, which allows for such decompositions to be invariant. In addition, the objects of interest are confined to the radial subspace and at most to the next SH subspace, which allows us to use Proposition 6. Similar approach was taken in the recent paper [79]. We continue now with the specifics.

The Laplacian on $\mathbb{R}^n$ is given in the spherical coordinates by

$$\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \frac{\Delta_{\mathbf{S}^{n-1}}}{r^2},$$

where $\Delta_{\mathbf{S}^{n-1}}$ is the self-adjoint Laplace-Beltrami operator on the sphere. Its action may be uniquely described as

$$\Delta_{\mathbf{S}^{n-1}} P[\vec{x}/r] = r^2 \Delta[P[\vec{x}/r]],$$

for each polynomial of n variables P. There are many useful properties of $\Delta_{\mathbf{S}^{n-1}}$, we will just concentrate the discussion on those that are directly relevant to our argument. In particular, its spectrum is explicitly given by

$$\sigma(-\Delta_{\mathbf{S}^{n-1}}) = \{l(l+n-2), l = 0, 1, ...\}.$$

In fact, there are the finite dimensional eigenspaces $\mathcal{X}_l \subset L^2(\mathbf{S}^{n-1})$, corresponding to the eigenvalue $l(l+n-2)$, which give rise to the orthogonal decomposition $L^2(\mathbf{S}^{n-1}) = \oplus_{l=0}^{\infty}\mathcal{X}_l$. It is worth noting that $\mathcal{X}_0 = span[1]$, whereas $\mathcal{X}_1 = span\{\frac{x_j}{r}, j = 1, 2, ..., n\}$. Denote $\mathcal{X}_{\geq 1} := \oplus_{l=1}^{\infty}\mathcal{X}_l$, so that $L^2(\mathbb{R}^n) = L^2_{rad}(r^{n-1}dr) \oplus L^2(r^{n-1}dr, \mathcal{X}_{\geq 1})$. We henceforth use the notation L^2_{rad} as a shorthand for $L^2_{rad}(r^{n-1}dr)$. Note that if we restrict $-\Delta$ to L^2_{rad} , we have

$$-\Delta|_{L^2_{rad}} = -\partial_{rr} - \frac{n-1}{r}\partial_r,$$

while

$$-\Delta|_{L^2(r^{n-1}dr, \mathcal{X}_{\geq 1})} \geq -\partial_{rr} - \frac{n-1}{r}\partial_r + \frac{n-1}{r^2}.$$

For every Banach space $X \hookrightarrow L^2(\mathbb{R}^n)$, we denote its radial subspace $X_{rad} := X \cap L^2_{rad}$.

Now consider a fractional Schrödinger operator $\mathcal{H} = (-\Delta)^s + W$, where W is radial. $\mathcal{H}$ acts invariantly on $L^2(r^{n-1}dr, \mathcal{X}_l)$ for each l. Upon introducing $\mathcal{H}_l = \mathcal{H}|_{L^2(r^{n-1}dr, \mathcal{X}_l)}$, we have the decomposition

$$\mathcal{H} = \oplus_{l=0}^{\infty}\mathcal{H}_l : \oplus_{l=0}^{\infty}L^2(r^{n-1}dr, \mathcal{X}_l) \to \oplus_{l=0}^{\infty}L^2(r^{n-1}dr, \mathcal{X}_l).$$

We also make use of the notation $\mathcal{H}_{\geq 1} := \oplus_{l=1}^{\infty}\mathcal{H}_l$ for $\mathcal{H}$ restricted to $\oplus_{l=1}^{\infty}L^2(r^{n-1}dr, \mathcal{X}_l)$. Clearly $D(\mathcal{H}_l) = D(\mathcal{H}) \cap L^2(r^{n-1}dr, \mathcal{X}_l)$ and $\sigma(\mathcal{H}) = \bigcup_{l=0}^{\infty}\sigma(\mathcal{H}_l)$ and

$$\mathcal{H}_0 < \mathcal{H}_1 < \mathcal{H}_2 < \dots$$

We shall also use the notation $\sigma_0(\mathcal{H}_l)$ for the bottom eigenvalue, $\sigma_1(\mathcal{H}_l)$ for the second smallest eigenvalue and so on.

Appendix B

The integrals $\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^j} d\xi$

Herein, we compute the integrals that arise in the calculation of the Vakhitov-Kolokolov index in Proposition 18.

Proposition 29. *For $\omega > 0$, we have*

$$\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi = \frac{\pi|\mathbf{S}^{n-1}|}{2s(2\pi)^n} \frac{\omega^{\frac{n}{2s}-1}}{\sin(\frac{n\pi}{2s})} \tag{B.0.1}$$

$$\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^2} d\xi = \frac{\pi|\mathbf{S}^{n-1}|}{2s(2\pi)^n} \left(1-\frac{n}{2s}\right) \frac{\omega^{\frac{n}{2s}-2}}{\sin(\frac{n\pi}{2s})} \tag{B.0.2}$$

$$\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^3} d\xi = \frac{\pi|\mathbf{S}^{n-1}|}{4s(2\pi)^n} \left(1-\frac{n}{2s}\right)\left(2-\frac{n}{2s}\right) \frac{\omega^{\frac{n}{2s}-3}}{\sin(\frac{n\pi}{2s})}. \tag{B.0.3}$$

Proof. We easily pass to integrals in the radial variable as follows

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{1}{((2\pi|\xi|)^{2s}+\omega)^j} d\xi &= |\mathbf{S}^{n-1}| \int_0^\infty \frac{\rho^{n-1}}{((2\pi\rho)^{2s}+\omega)^j} d\rho = |\mathbf{S}^{n-1}| \frac{\omega^{\frac{n}{2s}-j}}{2s(2\pi)^n} \int_0^\infty \frac{\rho^{\frac{n}{2s}-1}}{(\rho+1)^j} d\rho = \\ &= |\mathbf{S}^{n-1}| \frac{\omega^{\frac{n}{2s}-j}}{2s(2\pi)^n} \int_{-\infty}^\infty \frac{e^{t\frac{n}{2s}}}{(e^t+1)^j} dt.\end{aligned}$$

So, with $a := \frac{n}{2s} \in (0,1)$, matters are clearly reduced to computing the integrals

$$\int_{-\infty}^\infty \frac{e^{ta}}{(e^t+1)^j} dt,$$

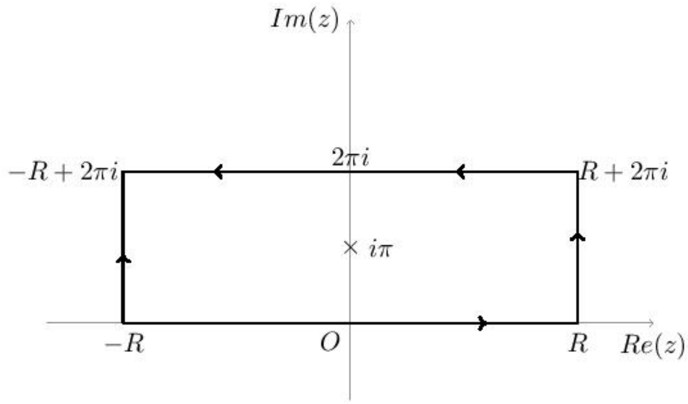

Figure B.1: Contour of integration

for $a \in (0,1)$, $j = 1,2,3$. In order to compute this integral, we use the residue theorem formula

$$\int_{\gamma_R} \frac{e^{az}}{(e^z+1)^j} dz = 2\pi i Res\left(\frac{e^{az}}{(e^z+1)^j}, \pi i\right).$$

where $R >> 1$, and $\gamma_R = \gamma_R^1 \cup \gamma_R^2 \cup \gamma_R^3 \cup \gamma_R^4$, and the curves $\gamma_r^m, m = 1,2,3,4$ are given, together with their orientation as follows,

$$\gamma_R^1 = \{x \in (-R,R)\}, \gamma_R^2 = \{R + ih, h \in [0,2\pi]\},$$
$$\gamma_R^3 = \{x + 2\pi i, x \in (R,-R)\}, \gamma_R^4 = \{-R + ih, h \in [2\pi, 0]\}.$$

Clearly,

$$\int_{\gamma_R^1} \frac{e^{az}}{(e^z+1)^j} dz + \int_{\gamma_R^3} \frac{e^{az}}{(e^z+1)^j} dz = (1 - e^{2\pi a i}) \int_{-R}^{R} \frac{e^{ta}}{(e^t+1)^j} dt,$$

while for $R >> 1$,

$$\left| \int_{\gamma_R^2} \frac{e^{az}}{(e^z+1)^j} dz \right| \le C \frac{e^{Ra}}{(e^R-1)^j}, \quad \left| \int_{\gamma_R^4} \frac{e^{az}}{(e^z+1)^j} dz \right| \le C e^{-aR}.$$

It follows that

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{az}}{(e^z+1)^j}dz=(1-e^{2\pi ai})\int_{-\infty}^{\infty}\frac{e^{ta}}{(e^t+1)^j}dt.$$

It remains to compute the residues associated with this complex integration. This is a straightforward calculation, the results of which are below

$$Res\left(\frac{e^{az}}{e^z+1},\pi i\right) = -e^{ia\pi} \tag{B.0.4}$$

$$Res\left(\frac{e^{az}}{(e^z+1)^2},\pi i\right) = -(1-a)e^{ia\pi} \tag{B.0.5}$$

$$Res\left(\frac{e^{az}}{(e^z+1)^3},\pi i\right) = -\frac{1}{2}(2-a)(1-a)e^{ia\pi}. \tag{B.0.6}$$

The formulas (B.0.1), (B.0.2), (B.0.3) follow by substituting these expressions in the residue formulas and taking $R\to\infty$. □

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