A Hopf Monoid On Set Families

Kevin Michael Marshall

Submitted to the graduate degree program in Department of Mathematics and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Jeremy Martin, Chairperson

Marge Bayer

Committee members

Hai Long Dao

José Samper

Eileen Nutting

Date defended: April 20, 2022

The Thesis Committee for Kevin Michael Marshall certifies
that this is the approved version of the following thesis :

A Hopf Monoid On Set Families

Jeremy Martin, Chairperson

Date approved: April 20, 2022

Abstract

We first introduce a Hopf monoid on set families called **SF**. Following that we will use the topological methods of Aguiar and Ardila [1] to find a cancellation-free formula for the Hopf submonoid of **SF** spanned by lattices of order ideals. We will then turn our attention to the Hopf submonoid spanned by simplicial complexes in which we derive an antipode formula for simplex skeletons.

We then turn our attention to the Hopf submonoid of **SF** spanned by chain gangs. The character group of this submonoid is related to formal power series. We proceed to show that the Hopf algebra of symmetric functions is a quotient of the Hopf algebra of chain gangs. Finally we conclude with suggestions for future research directions in the study of **SF**.

Acknowledgements

There are a large number of people I'd like to acknowledge for helping me get to this point; some fictitious and some not. I will leave it as an exercise to the reader to discern which are which. First and foremost I would like to thank my family for their support over the years. My parents have always been there to listen and give advice. My brother's visits to Lawrence have always been of benefit when I've needed a break from research. In addition I would like to thank my grandparents for letting me come visit any time I've needed a break. Over the years they've helped me in any number of situations and for that I can't thank them enough.

There are several members of the combinatorics community I would like to thank for their input over the years. First I would like to thank the participants of the KU Combinatorics Seminar for their questions and observations during talks I have given on my research. This includes Jeremy Martin, Marge Bayer, Mark Denker, Dania Morales, and many more. I would like to thank Geoff Critzer in particular for letting me come by his office and work through examples. I'd also like to thank Jacob White, Josè Samper, John Machacek, Jose Bastidas, and Alex McDonough for their questions and observations about the Hopf monoid on set families.

I have learned a lot about leadership during my time in graduate school. Joe Brennan showed me that even great leaders can go underappreciated. He along with the leadership advice from Colin Powell's books have made me into the confident teacher I am today. From the KU administration I would like to thank Douglas Girod and Barbara Bichelmeyer for their demonstration of superb leadership skills during these trying times. The COVID-19 pandemic has been a mixed bag of good and bad. With that in mind I must give some thanks to SARS-CoV-2 for freeing up my social life and giving me the time to do research.

In terms of the research process I cannot thank $C_8H_{10}N_4O_2$ and C_2H_6O enough for lubricating the machinery of the mind. That being said I must also thank the employees of Harbour Lights, On The Rocks, and McClain's for their services. Many great (and some not so great) ideas spawned over beer and coffee. Of course those ideas wouldn't exist without the brilliance of the degenerates in my life many of whom have gone on to better places. Namely Chad Duna, John Mangles, Lucian Grand, and Aaron Barrett. I am also indebted to Aaron for creating a workable thesis template out of that provided by Paul Johnson.

Over the years I have had to find various ways to deal with the stresses of being a graduate student. For that I'd like to start by thanking Robert Lopez, Kristen Anderson-Lopez, and Christophe Becke. Further I'd like to thank Kinoko Nasu and the animators at ufotable. I also cannot forget the inspiration gained from the ingenuity and tenacity shown by the members of the Future Laboratory and their work on the PhoneWave (Name TBA). Specifically I want to thank lab members 002 and 004 for giving me the emotional support to make it through the repetitive hellscape that is graduate school. Anytime I was faced with a problem that seemed untenable, Yuno's problem solving skills when faced with overwhelming adversity drove me to not give up. I also need to thank Boris and Persephone. The former for his years of support and life advice and the latter for her words of encouragement in the last semesters of graduate school. Finally to Ms. Koppleman who didn't think I would do well skipping from pre-algebra to geometry consider the following pages a counterexample to your statement.

"Keep the past, for all intents and purposes, where it is." – Rintaro Okabe

Contents

Introduction

Hopf algebras have their origins in algebraic topology and the theory of algebraic groups [2]. The term Hopf algebra was first coined in a paper by Borel [4] in 1953. By 1978 it was recognized that Hopf algebras could be used to answer problems in combinatorics [18]. Colloquially Hopf algebras give us a way to break apart and combine unlabeled objects. For our purposes we consider combinatorial objects such as simplicial complexes, posets, antimatroids, etc. For example there is a Hopf algebra on isomorphism classes of graphs. With this Hopf algebra Humpert and Martin were able to derive results about the Tutte polynomial [10].

There are times where working with labelled objects is of benefit. We can sacrifice some algebric structure in exchange for the ability to work with labelled objects. In such a case we can work with a Hopf monoid. In this transition from Hopf algebras to Hopf monoids we lose the ring structure that is provided by a Hopf algebra. The use of Hopf monoids in combinatorics is a relatively recent development compared to that of Hopf algebras. In particular the methods used by Aguiar and Ardila in their study of a Hopf monoid **GP** of generalized permutahedra [1] give a topological approach to computing a cancellation-free antipode for a Hopf monoid. As we will see this topological approach can be used in more Hopf monoids than the **GP**.

The first half of Chapter 1 gives an introduction into posets, the braid arrangement, and Hopf monoids. The braid arrangement is the pivotal tool used by Aguiar and Ardila in their study of **GP**. We can view the grouped terms of Takeuchi's formula as the Euler characteristics of objects on the braid arrangement. Thus the problem of finding a cancellation-free antipode formula turns into a topological problem. We give an outline of how this method is used by Aguiar and Ardila. The latter half of Chapter 1 gives a brief review of Hopf algebras, Fock functors, and symmetric functions which make an appearance in Chapter 6.

Chapter 2 covers the Hopf monoid on set families **SF** that is the object of study in subsequent chapters. One of the major pests we will encounter with **SF** is phantoms. We will see that they can be dealt with in a simple manner when evaluating the antipode. This alleviates most of the phantom problems we may face. In later chapters we make explicit where phantoms play a role so that the reader may avoid some of the pitfalls we encountered upon our journey.

Chapter 3 provides a survey of some Hopf submonoids of **SF**. Many of the Hopf submonoids considered are based on common combinatorial objects such as simplicial complexes and antimatroids. A brief review of each object is given. We will see that the Hopf submonoid spanned by simplicial complexes is the universal cocommutative Hopf submonoid of **SF**.

Chapter 4 walks through the process of finding a cancellation-free antipode formula for the Hopf submonoid of lattices of order ideals, **LOI**. We will make use of Birkhoff's theorem which gives a correspondence between a poset and its lattice of order ideals. This allows us to use the underlying poset structure. Understanding how the Hopf operations affect the underlying poset will give us the means to use the topological approach of Aguiar and Ardila for finding a cancellation-free antipode of **LOI**.

Chapter 5 moves away from **LOI** and focuses on the Hopf submonoid of simplicial complexes, **Simp**. We will start with some structural results about the inflators of a simplicial complex. In general it is not clear how to find a cancellation-free antipode formula for **Simp**. We will see that enumerative methods can be used to derive a cancellation-free antipode formula for independence complexes of uniform matroids.

Chapter 6 uses the results of Chapter 4 to analyze the character group of **LOI**. To motivate this we compute the character group of the Hopf submonoid **Bool** which is spanned by Boolean algebras, i.e., lattices of order ideals of antichains. We then turn our attention to the Hopf submonoid of chain gangs **CG**. The character group of **CG** contains a subgroup of the character group that is isomorphic to a multiplicative group of power series. This motivates us to turn our attention to the Hopf algebra of symmetric functions *CG*. We will see that symmetric functions are isomoprhic to a quotient Hopf algebra of CG.

Chapter 7 we consider future directions for study of the Hopf monoid on set families. The first direction involves a family of characters on **LOI** which relates to the Möbius function. This is followed by a short discussion about characters on **Simp**. The last half of the chapter concerns itself with some questions and observations about the Hopf submonoid **AMat**.

Chapter 1

Background

1.1 Posets

Definition 1.1. A *partially ordered set* (also called a *poset*) $(P, \leq_P)$ is a set P with a binary relation $\leq_P$ satisfying the following properties:

1. (*Reflexivity*) For all $x \in P$, $x \leq_P x$.

2. (*Antisymmetry*) If $x \leq_P y$ and $y \leq_P x$, then $x = y$.

3. (*Transitivity*) If $x \leq_P y$ and $y \leq_P z$, then $x \leq z$.

Often it is convenient to abuse notation and refer to the poset $(P, \leq_P)$ as the poset P. It is also convenient to write $\leq$ in place of $\leq_P$ when the context is clear. If $x < y$ and there is no z such that $x < z < y$, then we say that y *covers* x and write $x \lessdot y$. We can visualize the relations of a poset P by drawing its *Hasse diagram*. To do so we represent every element of P by a point with the following conditions:

1. If $x < y$, then the point x appears below the point y;

2. if $x \lessdot y$, then there is an edge between x and y.

For example, let Z be the "zigzag" poset with relations $1 < 3$, $2 < 3$, and $2 < 4$. Its Hasse diagram is shown in Figure 1.1.

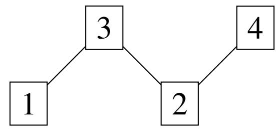


Figure 1.1: The Hasse diagram of the zigzag poset Z.

Definition 1.2. Q is a *subposet of* P if $Q \subseteq P$ as a set and if $x \leq_Q y$, then $x \leq_P y$. If $x \leq_Q y$ if and only $x \leq_P y$ for all $x, y \in Q$, then we say that Q is an *induced subposet of* P. We will use the notation $P|_Q$ to indicate the induced subposet Q of P. If Q does not inherit all relations from P we call Q a *weak subposet of* P.

As an example consider the zigzag poset Z from Figure 1.1 and take $Q = \{1,2,3\}$. Then $Z|_Q$ is the poset on $[3]$ with the relations $1 < 3$ and $2 < 3$. The poset on $[3]$ with the relation $2 < 3$ would be an example of a weak subposet of Z since the relation $1 < 3$ is not inherited.

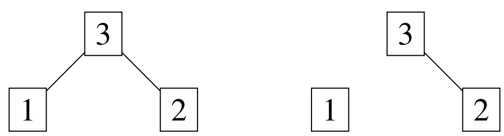


Figure 1.2: Induced and weak subposets of the zigzag poset.

An *order ideal* of a poset P is an induced subposet $Q \subseteq P$ with the property that if $x <_P y$ and $y \in Q$, then $x \in Q$. Similarly an *order filter* is a subposet $Q \subseteq P$ with the property that if $x <_P y$ and $x \in Q$, then $y \in Q$. We can think of an ideal (filter) as being "closed under going down (up)" in the poset. We use the notation $\lfloor X \rfloor$ to denote the order ideal generated by X, i.e,

$$\lfloor X \rfloor = \{y \in P : y \leq_P x \text{ for some } x \in X\}$$

and similarly use $\lceil X \rceil$ to denote the order filter generated by X.

We denote the set of order ideals of P by $J(P)$. There is a lattice structure on $J(P)$ where meet and join are respectively intersection and union. We refer to $J(P)$ as *the lattice of order ideals of* P. Since the union and intersection of ideals are ideals it follows that $J(P)$ is a distributive lattice.

Moreover, Birkhoff's theorem states that the finite distributive lattices are precisely the lattices $J(P)$ where P ranges over all finite posets[20, §3.4].

Birkhoff's theorem further states that $P \cong \mathrm{Irr}(J(P))$ where $\mathrm{Irr}(J(P))$ is the collection of *join-irreducible* elements of the lattice $J(P)$. For example the lattice of order ideals for the zigzag poset is shown in Figure 1.3. The shaded elements are the join-irreducible elements of the lattice.

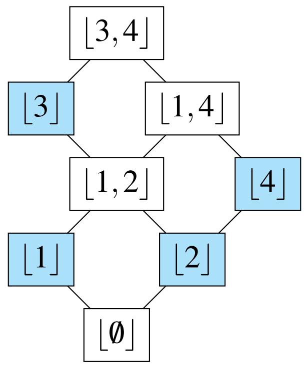


Figure 1.3: Lattice of order ideals of the zigzag poset.

Given two disjoint posets P and Q, the *disjoint union* $P+Q$ is the poset whose relations are exactly those inherited from P and Q. Thus the Hasse diagram of $P+Q$ is obtained by drawing the Hasse diagrams of P and Q side by side.

A *preposet* $(P, \leq_P)$ is a set P together with a binary relation $\leq$ that is reflexive and transitive, but not necessarily antisymmetric. For $x, y \in P$, write $x \equiv_P y$ if $x \leq_P y$ and $y \leq_P x$; then $\equiv_P$ is an equivalence relation, and $P/\equiv_P$ is a poset on the set of equivalence classes.

A preposet P is *linear*, or a *semiorder*, if every two elements are comparable. A *linear extension* of a preposet P is a linear preposet Q with the same underlying set and equivalence relation, such that every relation of P is a relation of Q. Equivalently, Q is a linear extension of $P/\equiv_P$ in the usual sense. We write $\mathscr{L}(P)$ for the set of linear extensions of P. For example, if P is the preposet on $[4]$ with relations $1 \equiv 2 \leq 3$ and $2 \leq 4$, then $\mathscr{L}(P) = \{12|3|4, 12|4|3\}$; see Figure 1.4.

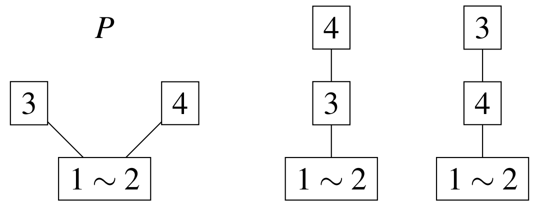


Figure 1.4: The preposet P and its linear extensions.

A *set composition* of $[n]$ is a partition of $[n]$ endowed with a linear order on the blocks of the partition. The notation $\mathrm{Comp}(n)$ denotes the set of all set compositions of n; we also write $\Phi \models [n]$ to mean $\Phi \in \mathrm{Comp}(n)$. Notice that a set composition is the same as a linear preposet. As such we use the convention that $\Phi = \Phi_1|\ldots|\Phi_k$ represents a set composition of $[n]$ with k blocks. Thus we can read off the linear order by going from left to right, i.e., $\Phi_1 < \cdots < \Phi_k$. We use the notation $x <_\Phi y$ when $x \in \Phi_i$ and $y \in \Phi_j$ with $i < j$.

1.2 The Braid Arrangement

The *braid arrangement* Br_n is constructed with the $\binom{n}{2}$ hyperplanes $x_i = x_j$ in $\mathbb{R}^n$. A face F of Br_n is a list of $\binom{n}{2}$ relations each indicating how x_i and x_j compare. These relations give rise to a set composition of $[n]$ where $i \sim_\Phi j$ if $x_i = x_j$ and $i <_\Phi j$ if $x_i < x_j$. This gives a bijection between set compositions of $[n]$ and the faces of Br_n. If $\Phi \models [n]$ has k blocks, then the resulting face has dimension k.

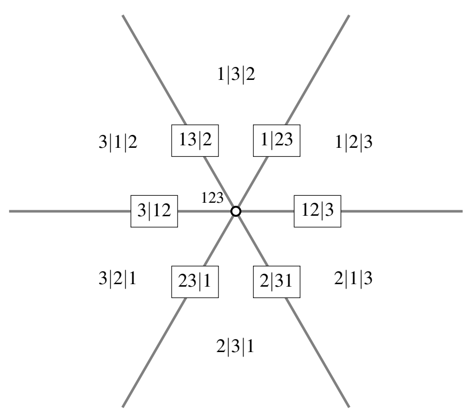


Figure 1.5: The braid arrangement Br_3.

Every preposet P on $[n]$ corresponds to a convex union $\|P\|$ of faces of the braid arrangement in $\mathbb{R}^n$, and this correspondence is a bijection. Moreover, the preposet linear extensions of P correspond to the maximal faces of $\|P\|$. For complete details on this "cone-preposet dictionary", see [16, §3].

By intersecting the braid fan with the unit sphere we obtain a triangulation of that sphere. The facets of the triangulation correspond to those regions of the braid fan for which $x_{\omega(1)} < \cdots < x_{\omega(n)}$ where $\omega \in \mathfrak{S}_n$. The lower dimensional faces are given by setting some of the inequalities to equality. For instance the faces in the triangulation of Br_3 corresponding to $x_1 < x_2 < x_3$ and $x_1 < x_3 < x_2$ share an edge corresponding to $x_1 < x_2 = x_3$. A preposet P corresponds to an open convex subfan of the braid fan given by taking $\|P\| \cap \mathbb{S}^{n-2}$ and intersecting the result with the hyperplane $\sum_{i=1}^{n} x_i = 0$.

A *polyhedron* is the intersection of a finite number of half-spaces. A bounded polyhedron is a *polytope*. A *face* of a polytope P is collection of points where a linear functional f is maximized in P. We refer to the dimension 0 faces as *vertices*, the dimension 1 faces as *edges*, and the codimension 1 faces as *facets*. In general we also include the polytope and the empty face as faces of the polytope. If we order the faces of a polytope by inclusion we obtain the *face lattice* of the polytope. In Figure 1.6 we see the square pyramid and its face lattice. In this example it happens that face lattice is dual to itself. This is not always the case.

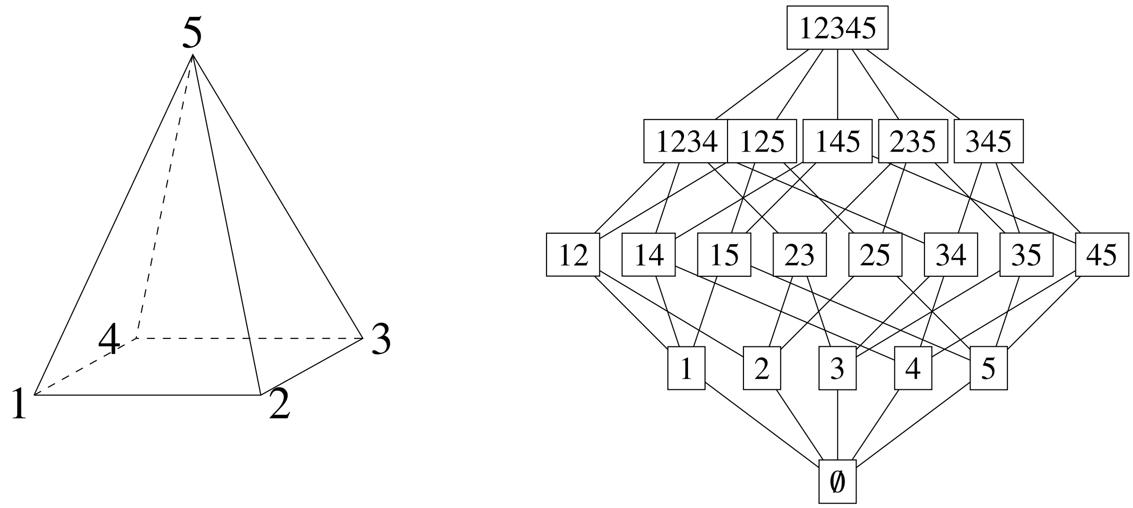


Figure 1.6: The square pyramid and its face lattice.

A subset S of $\mathbb{R}$ is a *cone* if for every $\lambda > 0$ and $x \in S$, then $\lambda x \in S$. It is not necessary for a cone to be convex. For example consider the cone consisting of the origin along with the first and third quadrants of the xy-plane.

A *polyhedral cone* is a polyhedron that is a cone. For example we can consider the cone $\mathscr{C}$ in $\mathbb{R}^2$ given by the set of points (x,y) such that $x \geq 0$ and $y \geq 0$. The faces of $\mathscr{C}$ are the origin, the positive x and y axis, as well as the first quadrant. Each of these faces is a cone.

A *fan* is a polyhedral complex of cones, i.e., the intersection of the closure of faces is a cone. If a fan $\mathscr{F}$ covers all of $\mathbb{R}^n$, then $\mathscr{F}$ is a *complete fan* in $\mathbb{R}^n$. We could extend $\mathscr{C}$ to a complete fan, call it $\mathscr{N}$, by adding in the negative x and y axes as well as the other three quadrants of $\mathbb{R}^2$.

Given a polyhedron P, the *normal cone* of a face F is the collection of linear functionals f such that $f|_P$ is maximized on F. The collection of normal cones for the faces of P is the *normal fan* of P. For example if P is the square whose vertices are $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, then the normal fan for P is $\mathscr{N}$ from above.

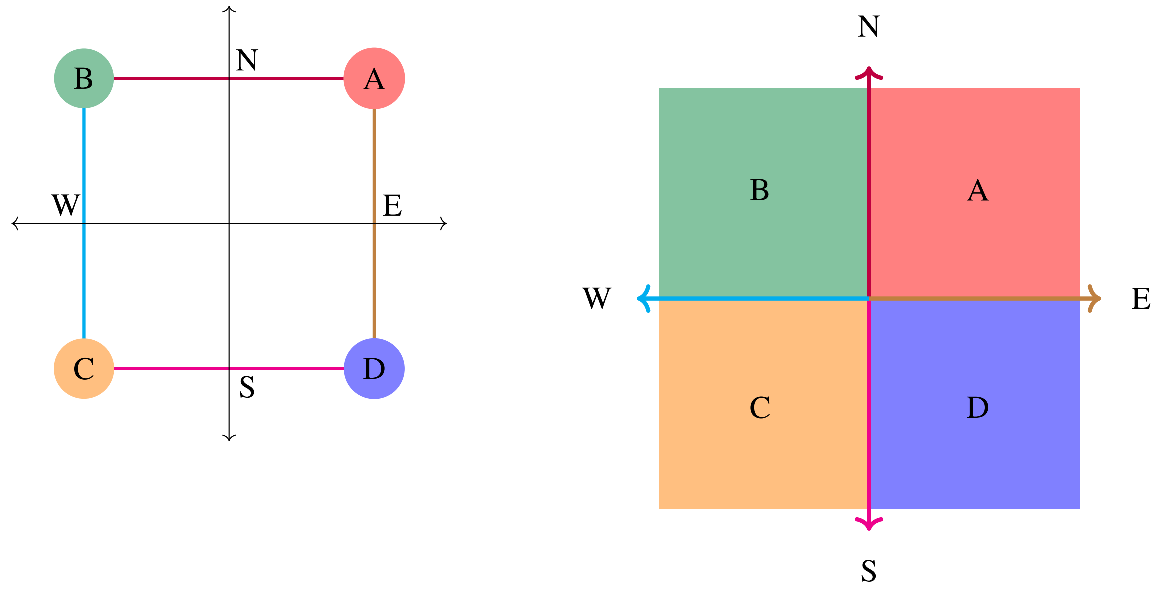


Figure 1.7: The square P and its associated normal fan.

1.3 Hopf Monoids

The presentation of Hopf monoids here follows [1, §2.2] closely. Before talking about Hopf monoids it is necessary to talk about set species.

Definition 1.3. A *set species* P consists of

1. a set $\mathrm{P}[I]$ for each finite set I, and

2. a function $\mathrm{P}[\sigma] : \mathrm{P}[I] \to \mathrm{P}[J]$ for each bijection $\sigma : I \to J$, satisfying the conditions $\mathrm{P}[\sigma \circ \tau] = \mathrm{P}[\sigma] \circ \mathrm{P}[\tau]$ and $\mathrm{P}[\mathrm{id}] = \mathrm{id}$.

A set species for which $\mathrm{P}[\emptyset]$ is a singleton is called *connected*.

We tend to consider I as a ground set for some combinatorial objects (such as graphs, posets, matroids, etc.). In this case $\mathrm{P}[I]$ corresponds to the set of all objects with ground set I. Additionally the map $\mathrm{P}[\sigma]$ associated with the bijection $\sigma : I \to J$ can be regarded as a renaming of elements in the ground set.

A set species is a functor from the category $\mathsf{FSet}^{\times}$ of finite sets with bijection to the category Set of sets with functions. The first condition of Definition 1.3 states how a finite set is mapped to a set, and the second condition states how a bijection are mapped to a function. By adding an algebraic structure to a set species, we strive to obtain a better understand of the species. This is where a *Hopf monoid in a set species* comes into play.

Definition 1.4. A *Hopf monoid in a set species* consists of a set species H such that for each finite set I and each decomposition $I = S \sqcup T$, there are product and coproduct maps

$$\mu_{S,T} : \mathrm{H}[S] \times \mathrm{H}[T] \to \mathrm{H}[I] \tag{Product}$$

$$\Delta_{S,T} : \mathrm{H}[I] \to \mathrm{H}[S] \times \mathrm{H}[T] \tag{Coproduct}$$

that satisfy the axioms of naturality, unitality, associativity, coassociativity, and compatibility, listed below.

Unless stated otherwise all Hopf monoids we consider are assumed to be connected. We can think of product as the merging of two objects whereas coproduct is the breaking up of a single object into two separate objects. Before describing the axioms it is necessary to introduce some notation that will be used throughout this document.

Suppose we have a decomposition $I = S \sqcup T$, and $x \in \mathrm{H}[S]$, $y \in \mathrm{H}[T]$, and $z \in \mathrm{H}[I]$. For the product of x and y we use $x \cdot y$, i.e., $\mu_{S,T}(x,y) = x \cdot y$. For the coproduct of z we write $\Delta_{S,T}(z) = (z|_S, z/_S)$ where $z|_S$ is the *restriction* of z to S and $z/_S$ is the *contraction* of S from z. The unique element of $\mathrm{H}[\emptyset]$ we call the *unit* of H and denote as 1.

Naturality. For each decomposition $I = S \sqcup T$ and bijection $\sigma : I \to J$ and any $x \in \mathrm{H}[S]$, $y \in \mathrm{H}[T]$, and $z \in \mathrm{H}[I]$, we require that the following conditions all hold.

$$\mathrm{H}[\sigma](x \cdot y) = \mathrm{H}[\sigma|_S](x) \cdot \mathrm{H}[\sigma|_T](y), \tag{1.1a}$$

$$\mathrm{H}[\sigma](z)|_S = \mathrm{H}[\sigma|_S](z|_S), \tag{1.1b}$$

$$\mathrm{H}[\sigma](z)/_S = \mathrm{H}[\sigma|_T](z/_S). \tag{1.1c}$$

If we think of I as labels for some combinatorial structure, naturality can be seen as ensuring that product and coproduct are preserved under relabeling. In most cases naturality should follow immediately from the definition of the monoid.

Unitality. For each I and $x \in \mathrm{H}[I]$, we require that the following conditions all hold.

$$x \cdot 1 = 1 \cdot x = x, \tag{1.2a}$$

$$x|_I = x/_\emptyset = x. \tag{1.2b}$$

Associativity. For each decomposition $I = R \sqcup S \sqcup T$, $x \in \mathrm{H}[R]$, $y \in \mathrm{H}[S]$, $z \in \mathrm{H}[T]$, we require

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{1.3}$$

Coassociativity. For each decomposition $I = R \sqcup S \sqcup T$ and $w \in \mathrm{H}[I]$, we require that

$$\Delta_{(R \sqcup S),T}(w) = \Delta_{R,(S \sqcup T)}(w) = \Delta_{R,S,T}(w). \tag{1.4}$$

Equivalently, the following conditions all hold.

$$(w|_{R\sqcup S})|_R = w|_R, \tag{1.5a}$$

$$(w|_{R\sqcup S})/_R = (w/_R)|_S, \tag{1.5b}$$

$$w/_{R\sqcup S} = (w/_R)/_S. \tag{1.5c}$$

By iterating this process we can take the product and coproduct with decompositions that have any number of parts. Thus if we have a set composition $S = S_1|S_2|\dots|S_k \models I$ then

$$\mu_S(x_1,\dots,x_k) = x_1 \cdots x_k, \tag{1.6a}$$

$$\Delta_S(y) = (y_1,\dots,y_k) \tag{1.6b}$$

where $x_i \in \mathrm{H}[S_i]$ and $y \in \mathrm{H}[I]$. Additionally,

$$y_i = (y|_{S_1\sqcup\cdots\sqcup S_i})/_{S_1\sqcup\cdots\sqcup S_{i-1}} \in H[S_i]. \tag{1.7}$$

Compatibility. Suppose we have decompositions $S \sqcup T = I = S' \sqcup T'$. If we consider the intersections $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, and $D = T \cap T'$ and $x \in \mathrm{H}[S]$, $y \in \mathrm{H}[T]$, then we require that the following conditions all hold:

$$(x \cdot y)|_{S'} = x|_A \cdot y|_C, \tag{1.8a}$$

$$(x \cdot y)/_{S'} = x/_A \cdot y/_C. \tag{1.8b}$$

The requirement of compatibility is for the breaking and merging of an object to give us the same result regardless of the order we compose our operations. We can visualize this in Figure 1.8.

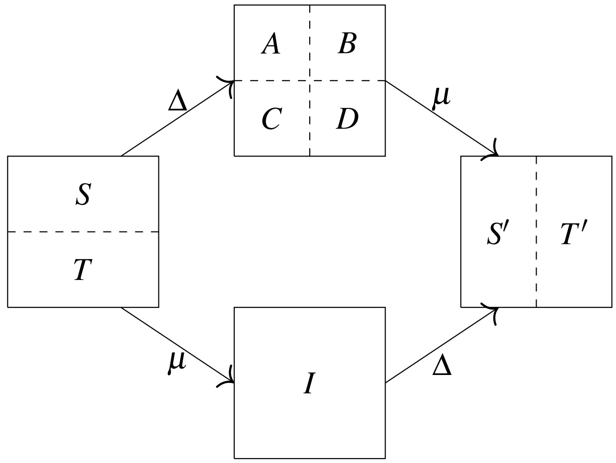


Figure 1.8: The decompositions of I

As with other algebraic structures we say that a Hopf monoid H is *commutative* if $xy = yx$ for all $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$ where $I = S \sqcup T$. Similarly, H is *cocommutative* if for all decompositions $I = S \sqcup T$ and $z \in \mathrm{H}[I]$, then $(z|_S, z/_S) = (z/_T, z|_T)$.

Example 1.5 (The Hopf monoid of matroids)**.** Let M be the species of matroid independence complexes on ground set E. For $E = S \sqcup T$, $A \in \mathrm{M}[S]$, and $B \in \mathrm{M}[T]$ we define

$$A \oplus B = \{I \cup J : I \in A, J \in B\} \qquad \text{(Direct Sum)}$$

$$A|_S = \{I \cap S : I \in A\}, \qquad \text{(Matroid Restriction)}$$

$$A/_S = \{I \cap T : I \cup \mathscr{B}_S \in A\} \qquad \text{(Matroid Contraction)}$$

where $\mathscr{B}_S$ is a facet of $A|_S$, i.e., a basis of the matroid whose independence complex is $A|_S$. Throughout this example we use $\mathscr{B}_X$ to denote a facet of $A|_X$. We claim that these operation give us a Hopf monoid where product is given by $A \oplus B$ and coproduct is given by $A|_S \otimes A/_S$.

The axioms of naturality, unitality, and associativity are straightforward to check so we proceed to check that coassociativity and compatibility both hold.

To check coassociativity we start with a matroid independence complex $A \in M[E]$ with $E = S \sqcup T \sqcup R$ and verify that (1.5a)-(1.5c) hold.

The first equation (1.5a) is easily verified

$$(A|_{R\sqcup S})|_R = \{I \subset R : I \in A|_{R\sqcup S}\} = \{I \subset R : I \in A\} = A|_R.$$

To verify the second equation (1.5b) we start by noting that

$$(A|_{R\sqcup S})/_R = \{I \subseteq S : I \cup \mathscr{B}_R \in A|_{R\sqcup S}\}.$$

Likewise since

$$A/_R = \{I \subseteq S \sqcup T : I \cup \mathscr{B}_R \in A\}$$

it follows that

$$(A/_R)|_S = \{I \subseteq S : I \cup \mathscr{B}_R \in A_{R\sqcup S}\}$$

as desired. Finally to verify (1.5c) note that $A/_R = \{I \subseteq S \sqcup T : I \cup B_r \in M\}$. Thus

$$\begin{aligned}(A/_R)/_S &= \{I \subseteq T : I \cup \mathscr{B}_S \in A/_R\}\\ &= \{I \subseteq T : I \cup \mathscr{B}_S \cup \mathscr{B}_R \in A\} = A/_{R\sqcup S}.\end{aligned}$$

Therefore coassociativity holds. To check compatibility we must show that equations (1.8a) and (1.8b) hold. Suppose that $S \sqcup T = I = S' \sqcup T'$ and consider the intersections $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, and $D = T \cap T'$. Further let $X \in \mathrm{M}[S]$ and $Y \in \mathrm{M}[T]$.

First we check (1.8a).

$$\begin{aligned}(X \cdot Y)|_{S'} &= \{(I_X \cup I_Y) \cap S' : I_X \in X, I_Y \in Y\}\\ &= \{(I_X \cap S') \cup (I_Y \cap S') : I_X \in X, I_Y \in Y\}\\ &= X|_{S'} \cdot Y_{S'} = X|_A \cdot Y|_C.\end{aligned}$$

Next we check (1.8b).

$$
\begin{aligned}
(X \cdot Y)/_{S'} &= \{I_X \cup I_Y \subset T' : (I_X \cup I_Y) \cup \mathscr{B}_{S'} \in X \cdot Y\} \\
&= \{I_X \subseteq B, I_Y \subseteq D : I_X \cup B_{S'} \in X, I_Y \cup \mathscr{B}_{S'} \in Y\} \\
&= \{I_X \subseteq B : I_X \cup \mathscr{B}_{S'} \in X\} \cdot \{I_Y \subseteq D : I_Y \cup \mathscr{B}_{S'} \in Y\} \\
&= X/_A \cdot Y/_C.
\end{aligned}
$$

Thus compatibility holds. Later we will describe a different Hopf monoid structure on the species M.

A *subspecies* of a species H is a species G such that $\mathrm{G}[I] \subset \mathrm{H}[I]$ for all I. A *Hopf submonoid* is a subspecies which inherits and is closed under the Hopf operations of the parent species. It is easily checked that intersection is well-defined on the level of Hopf monoids:

Proposition 1.6. *Suppose* H *is a Hopf monoid with Hopf submonoids* G_1 *and* G_2*. Then* $\mathrm{G}_1 \cap \mathrm{G}_2$ *is a Hopf submonoid of* H*, where* $(\mathrm{G}_1 \cap \mathrm{G}_2)[I] = \mathrm{G}_1[I] \cap \mathrm{G}_2[I]$.

If we replace the word "set" with the word "vector space" in Definition 1.3, we obtain the definition of a vector species. Fix a field $\Bbbk$.

Definition 1.7. A *vector species* **P** consists of

1. a vector space $\mathbf{P}[I]$, for each finite set I, and
2. a linear map $\mathbf{P}[\sigma] : \mathbf{P}[I] \to \mathbf{P}[J]$ for each bijection $\sigma : I \to J$, satisfying the conditions $\mathbf{P}[\sigma \circ \tau] = \mathbf{P}[\sigma] \circ \mathbf{P}[\tau]$ and $\mathbf{P}[\mathrm{id}] = \mathrm{id}$.

A vector species for which $\mathbf{P}[\emptyset] = \Bbbk$ is called *connected*.

Analogous to Definition 1.3, a vector species is a functor from the category $\mathsf{FSet}^\times$ of sets with bijections to the category Vec of vector spaces and linear maps. Specifically the second requirement in Definition 1.7 tells us that a vector species is a covariant functor. For our purposes we assume that all vector species are connected, unless stated otherwise.

Definition 1.8. Let P be a set species. The *linearization* of P is the vector species $\mathbf{P} = \Bbbk\mathrm{P}$ given by $\mathbf{P}[I] = \Bbbk\mathrm{P}[I]$. That is, $\mathbf{P}[I]$ is the vector space of formal $\Bbbk$-linear combinations of elements of $\mathrm{P}[I]$.

In a similar manner we can describe a Hopf monoid structure in vector species.

Definition 1.9. A *Hopf monoid in a vector species* consists of a vector species $\mathbf{H}$ such that for each finite set I and each decomposition $I = S \sqcup T$, there are operations called product and coproduct which are linear maps

$$\mu_{S,T} : \mathbf{H}[S] \otimes \mathbf{H}[T] \to \mathbf{H}[I], \tag{Product}$$

$$\Delta_{S,T} : \mathbf{H}[I] \to \mathbf{H}[S] \otimes \mathbf{H}[T] \tag{Coproduct}$$

that satisfy the axioms of naturality, unitality, associativity, coassociativity, and compatibility.

Definition 1.10. Let H be a Hopf monoid in set species. The *linearization* of H is the vector species $\mathbf{H} = \Bbbk\mathrm{H}$, equipped with a product and coproduct by extending those of H $\Bbbk$-linearly. A vector Hopf monoid of this form is called *linearized*.

Remark 1.11. There exist vector Hopf monoids which are not linearized. There are instances of non-linearized Hopf monoids on species which also admit a linearized Hopf monoid structures.

When talking about connected Hopf monoids in a vector species, the unit is the map $u : \Bbbk \to \mathbf{H}[\emptyset]$ which sends the multiplicative identity of $\Bbbk$ to the basis vector of $\mathbf{H}[\emptyset]$. Note that when $\mathbf{H}$ is the linearization of a set species there will be a canonical choice for the basis vector though in general this is not the case.

In addition to the unit which sends $\Bbbk$ to $\mathbf{H}[\emptyset]$ we also have the counit which is a linear map ε which sends the basis vector of $\mathbf{H}[\emptyset]$ to the multiplicative identity of $\Bbbk$ and each other element to zero.

Given a connected Hopf monoid $\mathbf{H}$ the *antipode* S is the solution to the commutative diagram shown in Figure 1.9.

$$\begin{tikzcd}
 & \mathbf{H}\otimes\mathbf{H} \arrow[rr, "id\otimes S"] & & \mathbf{H}\otimes\mathbf{H} \arrow[dr, "\mu"] & \\
\mathbf{H} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \Bbbk \arrow[rr, "u"] & & \mathbf{H} \\
 & \mathbf{H}\otimes\mathbf{H} \arrow[rr, "S\otimes id"'] & & \mathbf{H}\otimes\mathbf{H} \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Figure 1.9: The Antipode of $\mathbf{H}$

We can write out the antipode operation in the form of Takeuchi's formula

$$S_I(X) = \sum_{\Phi \models I} (-1)^{|\Phi|} \mu_\Phi(\Delta_\Phi(X)) \tag{1.9}$$

where $X \in \mathbf{H}[I]$. Although Takeuchi's formula gives an explicit way to compute the antipode, the number of terms in the sum grows rapidly as the set I increases in cardinality. Fortunately in many cases a lot of cancellation occurs between terms. The process of finding such a cancellation-free antipode formula tends to lead to fruitful combinatorial results. These results include things such as inversion of formal power series and determining the group of characters in a Hopf monoid.

1.4 The Hopf Monoid of Generalized Permutahedra

The *standard permutahedron* in $\mathbb{R}^n$ is the convex hull of the points $\{(w(1),\ldots,w(n)) : w \in \mathfrak{S}_n\}$. The normal fan of the standard permutahedron is in fact the braid fan. As an example consider the case where $n = 3$ in which the standard permutahedron is a hexagon. If we look down at the plane $x+y+z = 6$ which contains the hexagon we see that the normal fan is given by Figure 1.5. Given a vertex $v = (w(1),\ldots,w(n))$, the vertices adjacent to v are precisely those obtained from w by swapping $w(i)$ and $w(j)$ when $|w(i)-w(j)| = 1$. Further the edge is a parallel translate of

$e_i - e_j$ where $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.

A *generalized permutahedron* is a polyhedron P whose normal fan is a coarsening of the braid fan; that is, every normal cone of P is the union of faces of the braid fan [16]. Generalized permutahedra retain the property that every edge is parallel to $e_i - e_j$ for some i and j. This property fully characterizes generalized permutahedra. For example matroid polytopes are generalized permutahedra in which the edges correspond to basis exchange [7]. Readers interested in the theory of generalized permutahedra are refered to [16] and [17].

Many of the methods we use to study the Hopf monoid **SF** were first used by Aguiar and Ardila [1] to study the Hopf monoid **GP** of generalized permutahedra, defined as follows.

- As a vector species, $\mathbf{GP}[I]$ is spanned by generalized permutahedra in $\mathbb{R}^I$.

- Given $\mathfrak{p} \in \mathbf{GP}[S]$ and $\mathfrak{q} \in \mathbf{GP}[T]$, their product $\mu_{S,T}(\mathfrak{p},\mathfrak{q})$ is their Cartesian product $\mathfrak{p} \times \mathfrak{q} \in \mathbf{GP}[S \sqcup T]$.

- Given $\mathfrak{p} \in \mathbf{GP}[E]$ and $S \sqcup T = E$. We compute the coproduct $\Delta_{S,T}(\mathfrak{p})$ in the following way. We consider the face of $\mathfrak{p}$ that is maximized by the functional 1_S. It turns out that this face can be factored as $\mathfrak{p}|_S \times \mathfrak{p}/_S$ where $\mathfrak{p}|_S \subset \mathbb{R}^S$ and $\mathfrak{p}/_S \subset \mathbb{R}^T$. We define $\Delta_{S,T}(\mathfrak{p}) = \mathfrak{p}|_S \otimes \mathfrak{p}/_S$.

These operations satisfy the axioms of a Hopf monoid [1, §5]. For $\mathfrak{p} \in \mathbf{GP}[I]$, the antipode is given by

$$S_I(\mathfrak{p}) = (-1)^{|I|} \sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{\dim \mathfrak{q}} \mathfrak{q}$$

where $\mathfrak{q}$ ranges over the nonempty faces of $\mathfrak{p}$. The proof given in [1] starts by grouping the terms in Takeuchi's formula which gives us a sum over subfaces $\mathfrak{q}$ of $\mathfrak{p}$. The next step is to determine the values of the coefficients $a_{\mathfrak{q}}$. To do this we consider which cones of the braid fan are maximized by $\mathfrak{q}$. We end up with something that looks like Euler characteristic on a set of polyhedra $\mathscr{C}_{\mathfrak{q}}$. Since $\mathscr{C}_{\mathfrak{q}}$ is not a polyhedral complex we cannot interpret $a_{\mathfrak{q}}$ as an Euler characteristic. To remedy this inconvenience we use the fact that $\overline{\mathscr{C}}_{\mathfrak{q}}$ and $\overline{\mathscr{C}}_{\mathfrak{q}} - \mathscr{C}_{\mathfrak{q}}$ are polyhedral complexes where $\overline{\mathscr{C}}_{\mathfrak{q}}$ is the closure of $\mathscr{C}_{\mathfrak{q}}$. By intersecting these complexes with the unit sphere and the hyperplane $\sum_{i=1}^{n} x_i = 0$ we end up with two relative simplicial complexes. From there we can use the Euler characteristic to compute $a_{\mathfrak{q}}$ and arrive at a cancellation-free antipode formula for **GP**. We will see that this same method can be used for other Hopf monoids, not just **GP**.

1.5 Hopf Algebras & Fock Functors

A *bialgebra* B is a vector space over a field $\mathbb{k}$ with:

- associative $\mathbb{k}$-linear maps $\mu : B \otimes B \to B$,
- coassociative $\mathbb{k}$-linear maps $\Delta : B \to B \otimes B$,
- a unit $\eta : B \to \mathbb{k}$,
- and a counit $\varepsilon : \mathbb{k} \to B$

such that Δ and ε are algebra homomorphisms (equivalently μ and η are coalgebra homomorphisms). Given $x, y \in H$ we use the same convention as with Hopf monoids and use $x \cdot y$ in place of $\mu(x \otimes y)$. Similar to the operations of Hopf monoids, we can think of product as a way to combine objects in B and coproduct as a way to break apart objects in B. The result of taking the coproduct results in a summation over tensors. In the general case we often use *Sweedler notation* in which we do not concern ourselves with the specifics of the indices of summation. At first this may appear to be an odd convention to adopt, but it makes stating general facts about the coproduct very convenient. For example suppose $b \in B$, then we can represent the coproduct in Sweedler notation by

$$\Delta(b) = \sum b_1 \otimes b_2.$$

Further the property of coassociativity is simply written as

$$\Delta(b) = \sum b_1 \otimes \Delta(b_2) = \sum \Delta(b_1) \otimes b_2.$$

We can also use Sweedler notation to state what it means for ε to be a counit of B

$$b = \sum \varepsilon(b_1) \otimes b_2 = \sum b_1 \otimes \varepsilon(b_2). \tag{1.10}$$

Given a bialgebra B, a $\Bbbk$-vector subspace I of H is a *biideal* if I is an ideal and a coideal. Specifically

- for all $b \in B$, $bI \subseteq I$ and $Ib \subseteq I$,
- $\Delta(I) \subseteq I \otimes B + B \otimes I$,
- and $\varepsilon(I) = 0$.

A *Hopf algebra* H is a bialgebra (over a field $\Bbbk$) which admits an antipode $S : H \to H$. The antipode is the solution to the commutative diagram in Figure 1.10 where η is the unit and ε is the counit.

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "id\otimes S"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \Bbbk \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "S\otimes id"'] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Figure 1.10: The commutative diagram satisfied by S.

In Sweedler notation S is the map such that

$$\eta(\varepsilon(h)) = \sum h_1 \cdot S(h_2) = \sum S(h_1) \cdot h_2. \tag{1.11}$$

Suppose H is a Hopf algebra. A $\Bbbk$-vector subspace I of H is a *Hopf ideal* if I is a biideal and $S(I) \subseteq I$. The quotient H/I yields a Hopf algebra with structure inherited from H [13, §1.5].

A graded bialgebra $B = \oplus B_i$ over $\Bbbk$ is *connected* if $B_0 = \Bbbk$.

Lemma 1.12. *Suppose B is a graded and connected bialgebra. Fix $n > 0$ and suppose $b \in B_n$. Then*

$$\Delta(b) = b \otimes 1 + \sum b_1 \otimes b_2 + 1 \otimes b$$

where the Sweedler sum contains only elements of degree strictly between 0 and n.

Proof. Recall from (1.10) that $b = \sum b_1 \otimes \varepsilon(b_2)$. Thus there is a summand $b' \otimes b''$ of $\Delta(b)$ such that $b' \in B_n$. Since B is graded and connected this implies $b'' \in \Bbbk$. Since tensor product is a multilinear map we can group the terms of $B_n \otimes B_0$ into a single tensor of the form $b \otimes k$ for some $k \in \Bbbk$. Using properties of the counit we can show that $k = 1$. Similarly since $b = \sum \varepsilon(b_1) \otimes b_2$ it follows that $\Delta(b)$ has a summand of the form $1 \otimes b$. □

Proposition 1.13. *Suppose H is a graded and connected bialgebra. Then H admits a unique antipode S and H can be made into a Hopf algebra.*

Proof. We will use use (1.11) and Lemma 1.12 to recursively compute S. First note that $\eta(\varepsilon(1)) = 1 = 1 \otimes S(1) = S(1)$. Thus if $h \in H_0$, then $S(h) = h$. Assume we know how to compute S for the graded pieces up to but not including degree n and that $h \in H_n$. Then

$$\eta(\varepsilon(h)) = S(h) + \sum S(h_1) \otimes S(h_2) + S(1).$$

Using our induction assumption we can solve for $S(h)$ in terms of the lower degree pieces in the Sweedler sum. Lemma 1.12 guarantees that h_1 and h_2 will be in a lower graded piece of H than h. □

Given a Hopf monoid we can construct an associated Hopf algebra. This can be done via the *Fock functor*[1] denoted $\bar{\mathscr{K}}$. Before we can describe how $\bar{\mathscr{K}}$ turns a Hopf monoid $\mathbf{H}$ into a Hopf algebra H we need one additional definition. Two objects $X \in \mathbf{H}[I]$ and $Y \in \mathbf{H}[J]$ are *isomorphic* if

[1] There are actually four functors given the name Fock functor.

there is a bijection $\sigma : I \to J$ such that $\sigma X = Y$. In other words if we can relabel X to get Y we say that X and Y are isomorphic.

Suppose $|I| = n$ and consider $X \in \mathbf{H}[I]$. Then $\bar{\mathscr{K}}(X)$ sends X to the class of items that are isomorphic to X. We can think about this as removing the labels from X. We describe the graded Hopf algebra $H = \bar{\mathscr{K}}(\mathbf{H})$ as

$$H = \bigoplus_{n \geq 0} \text{span}\{\text{isomorphism classes of elements of } \mathbf{H}[I] \text{ for } |I| = n\}.$$

We denote the isomorphism class of $h \in \mathbb{H}[I]$ by $[h]$. To compute the product and coproduct in the Hopf algebra we can perform the product and coproduct by picking representatives from the monoid and performing the corresponding operations. Specifically if $h_1 \in \mathbf{H}[k_1]$, $h_2 \in \mathbf{H}[k_2]$, and $h \in \mathbf{H}[S \sqcup T]$, then

$$[h_1] \cdot [h_2] = [h_1 \cdot h_2^{k_1+}] \qquad \Delta([h]) = \sum_{[n]=S \sqcup T} [h|_S] \otimes [h/_S] \tag{1.12}$$

where $h_2^{k_1+} = \sigma^{+k_1}(h_2) \in \mathbf{H}[\{k_1+1, \ldots, k_1+k_2\}]$ is the result of applying the order-preserving bijection $\sigma^{+k_1} : [k_2] \to \{k_1+1, \ldots, k_1+k_2\}$ [1, §2.9].

1.6 Symmetric Functions

Suppose R is a commutative ring. The ring of formal power series on variables $x_1, x_2, \ldots$ with coefficients in R is denoted by $R[[x_1, x_2, \ldots]]$. A function $f \in R[[x_1, x_2 \ldots]]$ is a *symmetric function* if

$$f(x_1, x_2, \ldots) = f(x_{\omega(1)}, x_{\omega(2)}, \ldots)$$

for every $\omega \in \cup_{n=1}^{\infty} \mathfrak{S}_n = \mathfrak{S}_\infty$. For our purposes we are interested in the ring of symmetric functions over $\mathbb{C}$ which we denote by Λ. Note that $\Lambda = \oplus \Lambda_d$ where Λ_d is the collection of symmetric functions of degree d. There are several bases that are used when working with Λ. We will now review the bases that will be used in Chapter 6. For further information regarding the ring of symmetric functions we refer the reader to [12] and [19].

Suppose that $\alpha = (\alpha_1, \alpha_2, \dots)$ is a sequence such that all but finitely many entries are zero. We define $\mathbf{x}^\alpha = \prod_i x_i^\alpha$ to be the monomial with exponent vector α. Let $\lambda(\alpha) = (\lambda_1, \dots, \lambda_k)$ be the integer partition of $n = \sum \alpha_i$. We define the *monomial symmetric function* m_λ by

$$m_\lambda = \sum_{\alpha : \lambda(\alpha) = \lambda} \mathbf{x}^\alpha.$$

For example

$$m_{(2,2,1)} = x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2 + \dots.$$

If $\lambda = 1^k$, i.e., the partition with k 1's, then we obtain the k^{th} *elementary symmetric function* denoted e_k. That is to say

$$e_k = m_{1^k} = \sum_{1 \leq i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}.$$

For example

$$e_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_1 x_2 x_5 + \dots.$$

We set $e_0 = 1$. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we use the convention that $e_\lambda = e_{\lambda_1} \dots e_{\lambda_k}$.

If we only require the sequence i_k to be monotonically increasing as opposed to strictly increasing, then we arrive at the k^{th} *complete homogeneous symmetric function* denoted h_k. That is the function of the form

$$h_k = \sum_{1 \leq i_1 \leq \dots \leq i_k} x_{i_1} \dots x_{i_k}.$$

For example

$$h_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + \dots + x_1 x_2 + x_1 x_3 + x_1 x_4 + \dots.$$

Note that

$$h_k = \sum_{\lambda \vdash k} m_\lambda .$$

It is known [19, Section 7.6] that the generating functions for the elementary and complete homogeneous symmetric functions are given respectively by

$$E(t) = \sum e_k t^k = \prod_{i \geq 1} (1 + t x_i), \text{ and}$$
$$H(t) = \sum h_k t^k = \prod_{i \geq 1} \frac{1}{1 - t x_i}.$$

Noting that $E(t)H(-t) = 1$ we get a collection of relations between the e_ks and the h_ks given by

$$\sum_{k=0}^{n} (-1)^k e_k h_{n-k} = 0. \tag{1.13}$$

1.7 The Hopf Algebra of Symmetric Functions

As stated in Section 1.6 Λ is a graded ring. We can go further and endow Λ with a Hopf algebra structure. The coproduct of a symmetric function $F \in \Lambda$ is computed in the following way. For $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ we define $F(\mathbf{x}, \mathbf{y}) = F(x_1, y_1, x_2, y_2, \dots) \in \mathbb{C}[[\mathbf{x}, \mathbf{y}]]$. The power series $F(\mathbf{x}, \mathbf{y})$ is symmetric in both $\mathbf{x}$ and $\mathbf{y}$ and thus can be written in the form $\sum F_1(\mathbf{x}) F_2(\mathbf{y})$. We thus define

$$\Delta F = \sum F_1 \otimes F_2.$$

As an example let us compute Δh_2.

$$h_2(\mathbf{x},\mathbf{y}) = \sum x_i^2 + \sum_{i<j} x_i x_j + \sum x_i y_j + \sum y_i^2 + \sum_{i<j} y_i y_j$$
$$= h_2(\mathbf{x}) + h_1(\mathbf{x}) h_1(\mathbf{y}) + h_2(\mathbf{y}).$$

Thus

$$\Delta h_2 = h_2 \otimes h_0 + h_1 \otimes h_1 + h_0 \otimes h_2.$$

More generally

$$\Delta h_n = \sum_{i+j=n} h_i \otimes h_j \text{ and}$$
$$\Delta e_n = \sum_{i+j=n} e_i \otimes e_j.$$

To understand the antipode S of Λ we define the map $\omega : \Lambda \to \Lambda$ by $\omega(e_n) = h_n$ and extend ω algebraically since the e_ns form an algebra basis. In fact ω is an involutive automorphism. Since Λ is a connected and graded bialgebra we can use Proposition 1.13 to find the value of the antipode as applied to h_k. Using the relations from (1.13) we obtain

$$S(h_k) = (-1)^k e_k = (-1)^k \omega(h_k).$$

Chapter 2

A Hopf Monoid SF on Set Families

A *set family* is a pair $(\mathscr{F}, E)$, where E is a finite set and $\mathscr{F} \subseteq 2^E$. We say that $(\mathscr{F}, E)$ is *grounded* if $\emptyset \in \mathscr{F}$. Often we will simply refer to the set family $\mathscr{F}$ when the ground set E is clear from context. Let SF denote the set species of grounded set families. Note that SF$[\emptyset]$ contains a single object, namely $(\{\emptyset\}, \emptyset)$. Therefore SF is a connected set species (hence why we require set families to be grounded). Note that the notation SF is used in [1, Chapter 12] to denote the Hopf monoid of submodular functions. This Hopf monoid is unrelated to grounded set families. In our context SF will always refer to the set species of grounded set families.

The *join* $\mathscr{F}_1 * \mathscr{F}_2$ of two set families $\mathscr{F}_1$ and $\mathscr{F}_2$ is the set family

$$\mathscr{F}_1 * \mathscr{F}_2 = \{X \cup Y : X \in \mathscr{F}_1, Y \in \mathscr{F}_2\}.$$

Let $(\mathscr{F}, E)$ be a set family and $A \subseteq E$. We define the *restriction* $\mathscr{F}|_A$ and the *contraction* $\mathscr{F}/_A$ to be the set families

$$\mathscr{F}|_A = \{F \cap A | F \in \mathscr{F}\},$$
$$\mathscr{F}/_A = \{F \in \mathscr{F} | F \cap A = \emptyset\}.$$

Proposition 2.1. *The set species* SF *admits the structure of a commutative Hopf monoid, with product*

$$(\mathscr{F}_1, E_1) \cdot (\mathscr{F}_2, E_2) = (\mathscr{F}_1 * \mathscr{F}_2, E_1 \cup E_2),$$

coproduct

$$(\Delta_{A,\bar{A}}(\mathscr{F},E) = ((\mathscr{F}|_A,A),(\mathscr{F}/_A,E\setminus A)),$$

and unit $(\{\emptyset\},\emptyset)$*. As a consequence of linearization it follows that the vector species* **SF** *also admits a commutative Hopf monoid structure.*

Proof. Unitality and Naturality: Unitality follows immediately from the definitions of unit, join, restriction, and contraction. *Commutativity and associativity:* Associativity of product comes directly from associativity of union. Commutativity also follows from commutativity of union. *Coassociativity:* Suppose that $(\mathscr{F},E)$ is a set family and that $E = S\sqcup T\sqcup R$. Recall we need to show that (1.5a)-(1.5c) hold. Indeed,

$$\begin{aligned}
(\mathscr{F}|_{RS})|_R &= \{(F\cap(R\cup S))\cap R : F\in\mathscr{F}\}\\
&= \{((F\cap R)\cup(F\cap S))\cap R : F\in\mathscr{F}\}\\
&= \{((F\cap R)\cap R)\cup((F\cap S)\cap R) : F\in\mathscr{F}\}\\
&= \{F\cap R : F\in\mathscr{F}\}\\
&= \mathscr{F}|_R,
\end{aligned}$$

$$\begin{aligned}
(\mathscr{F}|_{RS})/_R &= \{X\in\mathscr{F}|_{RS} : X\cap R=\emptyset\}\\
&= \{F\cap(R\cup S) : F\in\mathscr{F}, F\cap(R\cup S)\cap R=\emptyset\}\\
&= \{(F\cap R)\cup(F\cap S) : F\in\mathscr{F}, F\cap R=\emptyset\}\\
&= \{F\cap S : F\in\mathscr{F}, F\cap R=\emptyset\}\\
&= (F/_R)|_S,
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{F}/_{RS} &= \{F \in \mathscr{F} : F \cap (R \cup S) = \emptyset\} \\
&= \{F \in \mathscr{F} : (F \cap R) \cup (F \cap S) = \emptyset\} \\
&= \{F \in \mathscr{F} : (F \cap R) = \emptyset, (F \cap S) = \emptyset\} \\
&= \mathscr{F}/_R/_S.
\end{aligned}$$

Therefore coassociativity holds.

Compatibility: Recall that for compatibility we need to show that (1.8a) and (1.8b) hold when $S \sqcup T = E = S' \sqcup T'$ with pairwise intersections $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, and $D = T \cap T'$.

We check that (1.8a) and (1.8b) hold.

$$\begin{aligned}
(\mathscr{F} \cdot \mathscr{F}')|_{S'} &= \{(X \cup Y) \cap S' : X \in \mathscr{F}, Y \in \mathscr{F}'\} \\
&= \{(X \cap S') \cup (Y \cap S') : X \in \mathscr{F}, Y \in \mathscr{F}'\} \\
&= \{X' \cup Y' : X' \in \mathscr{F}|_{S'}, Y' \in \mathscr{F}'|_{S'}\} = \mathscr{F}|_A \cdot \mathscr{F}'|_C.
\end{aligned}$$

$$\begin{aligned}
(\mathscr{F} \cdot \mathscr{F}')/_{S'} &= \{X \cup Y : X \in \mathscr{F}, Y \in \mathscr{F}', (X \cup Y) \cap S' = \emptyset\} \\
&= \{X \cup Y : X \in \mathscr{F}, Y \in \mathscr{F}', (X \cap S') \cup (Y \cap S') = \emptyset\} \\
&= \{X \cup Y : X \in \mathscr{F}, Y \in \mathscr{F}', (X \cap S') = \emptyset, (Y \cap S') = \emptyset\} = \mathscr{F}/_A \cdot \mathscr{F}'/_C.
\end{aligned}$$

□

We will use the symbol $*$ for the linear extension of the join operation to **SF**.

Throughout this document we will make use of the higher product and coproduct operations

μ_Φ and Δ_Φ. In particular suppose $\mathscr{F}$ is a grounded set family on E and a set composition $\Phi = \Phi_1|\ldots|\Phi_m \models E$. Then

$$\Delta_\Phi(\mathscr{F}) = (\mathscr{F}_1, \ldots, \mathscr{F}_m) \tag{2.1}$$

where

$$\mathscr{F}_i = \{A \cap \Phi_i : A \in \mathscr{F} \text{ and } A \cap \Phi_j = \emptyset \;\; \forall j < i\} \tag{2.2}$$

Remark 2.2. Given a grounded set family $(\mathscr{F}, E)$ we do not require that every element of E appear in a member of $\mathscr{F}$. An element of E that does not appear in any member of $\mathscr{F}$ will be referred to as a *phantom*.

Example 2.3. Consider the topology $\tau = \{\emptyset, 1, 12\}$ on ground set $I = \{1,2\} = R \sqcup S$ where $R = \{1\}$ and $S = \{2\}$. Then $\mu_{R,S}(\Delta_{R,S}(\tau)) = \mu_{R,S}((\{\emptyset,1\},R),(\{\emptyset\},S)) = (\{\emptyset,1\},I)$. We see that 2 is a phantom of the set family $\mu_{R,S}(\Delta_{R,S}(\tau))$.

Phantoms pose a menace when working with the monoid **SF**. Specifically, they have an effect when computing the antipode for a set family. As an example consider $\mathscr{F} = \{\emptyset, 1\}$. Then

$$S(\mathscr{F}, [1]) = -(\mathscr{F}, [1]), \text{ but}$$
$$S(\mathscr{F}, [2]) = (\mathscr{F}, [2]).$$

Thus phantoms cannot be completely ignored. Fortunately we can get a handle on the phantom menace when working with the antipode. Given a finite set E and an element $x \notin E$ we define the *conjuration map* $\gamma : E \to E \cup \{x\}$ to be the linear map such that

$$\gamma(\mathscr{F}, E) = (\mathscr{F}, E \cup \{x\})$$

where $\mathscr{F} \in \mathbf{SF}[E]$. Since set families with ground set E form a basis for the space $\mathbf{SF}[E]$ this is sufficient information to determine how γ acts on an arbitrary element of $\mathbf{SF}[E]$. With the conjuration map in place we can now handle phantoms in the anitpode.

Proposition 2.4. *Suppose $\mathscr{F} \in \mathbf{SF}[E]$ and that $x \notin E$. Then*

$$S(\gamma(\mathscr{F},E)) = -\gamma(S(\mathscr{F},E)).$$

Proof. Recall Takeuchi's formula:

$$\begin{aligned} S(\gamma(\mathscr{F},E)) &= \sum_{\Phi \models E\cup\{x\}} (-1)^{|\Phi|}\mu_\Phi(\Delta_\Phi(\mathscr{F})) \\ &= \sum_{\substack{\Phi \models E\cup\{x\} \\ \{x\}\in\Phi}} (-1)^{|\Phi|}\mu_\Phi(\Delta_\Phi(\mathscr{F})) + \sum_{\substack{\Phi \models E\cup\{x\} \\ \{x\}\notin\Phi}} (-1)^{|\Phi|}\mu_\Phi(\Delta_\Phi(\mathscr{F})) \end{aligned} \tag{2.3}$$

Suppose $\Phi \models E \cup \{x\}$ and consider the set composition $\Psi \models E$ that arises by erasing x from Φ. If x is a singleton block, then $|\Psi| = |\Phi| - 1$. In addition there are $|\Psi| + 1$ positions where x could be inserted to get a set composition of $E \cup \{x\}$.

In a similar manner, if x were not a singleton block, then $|\Psi| = |\Phi|$. Given Ψ, there are $|\Psi|$ blocks where x could be inserted to get a set composition of $E \cup \{x\}$. Therefore Equation (2.3) can be rewritten as

$$\begin{aligned} &\gamma\left(\sum_{\Psi\models E}(|\Psi|+1)(-1)^{|\Psi|+1}\mu_\Psi(\Delta_\Psi(\mathscr{F})) + \sum_{\Psi\models E}|\Psi|(-1)^{|\Psi|}\mu_\Psi(\Delta_\Psi(\mathscr{F}))\right) \\ &= -\gamma\left(\sum_{\Psi\models E}(-1)^{|\Psi|}\mu_\Psi(\Delta_\Psi(\mathscr{F}))\right) \\ &= -\gamma(S(\mathscr{F},E)) \end{aligned}$$

as desired. $\square$

Chapter 3

Hopf Submonoids of SF

There are many collections of grounded set families such as simplicial complexes that are commonplace in the study of combinatorics. We will see that many of these collections give rise to Hopf submonoids of **SF**.

Remark 3.1. Suppose **X** is a subspecies of **SF** and that $(\mathscr{F},E) \in \mathbf{X}[E]$. We need to keep in mind that $(\mathscr{F},E)$ may have phantoms in which case $\bigcup_{F\in\mathscr{F}} F$ is a proper subset of E. For example we shall see that simplicial complexes form a Hopf submonoid called **Simp** in which $\mathbf{Simp}[E]$ is spanned by simplical complexes whose vertex set is a subset of E.

3.1 Accessible Set Systems (Acc)

Definition 3.2. A set family $(\mathscr{F},E)$ is called *accessible* if for each non-empty $X \in \mathscr{F}$ there is an element $x \in X$ such that $X - \{x\} \in \mathscr{F}$.

It is immediate from this definition that non-empty accessible set systems are grounded.

Proposition 3.3. *If $\mathscr{F}$ and $\mathscr{G}$ are accessible set families, then $\mathscr{F} * \mathscr{G}$ is a an accessible set family.*

Proof. Suppose there is a non-empty set $X = F \cup G \in \mathscr{F} * \mathscr{G}$ where $F \in \mathscr{F}$ and $G \in \mathscr{G}$. Then without loss of generality $F \neq \emptyset$ and therefore using the accessibility of $\mathscr{F}$ it follows that there is an $x \in S$ such that $F - \{x\} \in \mathscr{F}$. Hence $X - \{x\} \in \mathscr{F} * \mathscr{G}$ and $\mathscr{F} * \mathscr{G}$ is accessible. $\square$

Proposition 3.4. *If $(\mathscr{F},E)$ is an accessible set family and $A \subseteq E$, then $\mathscr{F}|_A$ and $\mathscr{F}/_A$ are accessible set families.*

Proof. Suppose that $X \in \mathscr{F}|_A$ with $X \neq \emptyset$. Then $X = X' \cap A$ for some non-empty $X' \in \mathscr{F}$. It follows that there is an $x \in E$ such that $Y' = X' - \{x\} \in \mathscr{F}$. Either $x \in A$ or $x \notin A$. If $x \in A$, then

$$Y' \cap A = (X' - \{x\}) \cap A = X - \{x\} \in \mathscr{F}|_A.$$

If on the other hand $x \notin A$, then

$$Y' \cap A = (X' - \{x\}) \cap A = X' \cap A \neq \emptyset.$$

In this case we can apply accessibility to Y' since $Y' \neq \emptyset$ and repeat the above argument until we obtain an element $z \in A$ in which case

$$(Y' - \{z\}) \cap A = (X' - \{z\}) \cap A = X - \{z\} \in \mathscr{F}|_A.$$

We have therefore verified that $\mathscr{F}|_A$ is an accessible set system.

If $X \in \mathscr{F}/_A$, then $X \cap A = \emptyset$. Further if X is non-empty, then there is an $x \in E$ such that $X - \{x\} \in \mathscr{F}$. Since $X - \{x\} \subseteq X$ and $X \cap A = \emptyset$, then it follows that $(X - \{x\}) \cap A = \emptyset$ implying $X - \{x\} \in \mathscr{F}/_A$. Therefore $\mathscr{F}/_A$ is an also an accessible set system. $\square$

Corollary 3.5. *The subspecies* **Acc** *spanned by accessible set families is a Hopf submonoid of* **SF**.

3.2 Set Systems Closed Under Intersection (Int)

Definition 3.6. A set family $(\mathscr{F}, E)$ is *closed under intersection* if for every $X, Y \in \mathscr{F}$ it follows that $X \cap Y \in \mathscr{F}$.

Proposition 3.7. *If $\mathscr{F}$ and $\mathscr{G}$ are set families (with disjoint ground sets) closed under intersection, then $\mathscr{F} * \mathscr{G}$ is a set family closed under intersection.*

Proof. If $X, y \in \mathscr{F} * \mathscr{G}$, then there exist $F_1, F_2 \in \mathscr{F}$ and $G_1, G_2 \in \mathscr{G}$ such that $X = F_1 \cup G_1$ and $Y = F_2 \cup G_2$. It follows that

$$
\begin{aligned}
X \cap Y &= (F_1 \cup G_1) \cap (F_2 \cup G_2) \\
&= (F_1 \cap F_2) \cup (G_1 \cap F_2) \cup (F_1 \cap G_2) \cup (G_1 \cap G_2) \\
&= (F_1 \cap F_2) \cup (G_1 \cap G_2) \in \mathscr{F} * \mathscr{G}.
\end{aligned}
$$

□

Proposition 3.8. *If $(\mathscr{F}, E)$ is a set family closed under intersection and $A \subseteq E$, then $\mathscr{F}|_A$ and $\mathscr{F}/_A$ are also set families closed under intersection.*

Proof. Suppose $X, Y \in \mathscr{F}|_A$. Then $X = X' \cap A$ and $Y = Y' \cap A$ for some $X', Y' \in \mathscr{F}$. Therefore

$$X \cap Y = (X' \cap A) \cap (Y' \cap A) = (X' \cap Y') \cap A \in \mathscr{F}|_A.$$

Next suppose $X, Y \in \mathscr{F}/_A \subseteq \mathscr{F}$. Then $X \cap A = \emptyset$ and likewise $Y \cap A = \emptyset$. Therefore

$$(X \cap Y) \cap A = (X \cap A) \cap (Y \cap A) = \emptyset.$$

Therefore $X \cap Y \in \mathscr{F}/_A$. □

Corollary 3.9. *The subspecies* **Int** *spanned by grounded set families closed under intersection is a Hopf submonoid of* **SF**.

3.3 Set Systems Closed Under Union (Union)

Definition 3.10. A set system $(\mathscr{F}, E)$ is *closed under union* if for every $X, Y \in \mathscr{F}$ it follows that $X \cup Y \in \mathscr{F}$.

Proposition 3.11. *If $\mathscr{F}$ and $\mathscr{G}$ are set families closed under union, then $\mathscr{F} * \mathscr{G}$ is a set family closed under union.*

Proof. If $X, Y \in \mathscr{F} * \mathscr{G}$, then there exist $F_1, F_2 \in \mathscr{F}$ and $G_1, G_2 \in \mathscr{G}$ such that $X = F_1 \cup G_1$ and $Y = F_2 \cup G_2$. Considering the union

$$X \cup Y = (F_1 \cup G_1) \cup (F_2 \cup G_2) = (F_1 \cup F_2) \cup (G_1 \cup G_2).$$

Therefore $\mathscr{F} * \mathscr{G}$ is closed under union. □

Proposition 3.12. *If $(\mathscr{F}, E)$ is a set family closed under union and $A \subseteq E$, then $\mathscr{F}|_A$ and $\mathscr{F}/_A$ are also set families closed under union.*

Proof. Suppose that $X, Y \in \mathscr{F}|_A$. Then $X = X' \cap A$ and $Y = Y' \cap A$ for some $X', Y' \in \mathscr{F}$. Therefore

$$X \cup Y = (X' \cap A) \cup (Y' \cap A) = (X' \cup Y') \cap A \in \mathscr{F}|_A.$$

Next suppose $X, Y \in \mathscr{F}/_A \subseteq \mathscr{F}$. Then $X \cup Y \in \mathscr{F}$ and

$$(X \cup Y) \cap A = (X \cap A) \cup (Y \cap A) = \emptyset$$

which implies $X \cup Y \in \mathscr{F}/_A$. □

Corollary 3.13. *The subspecies* **Union** *spanned by grounded set families closed under union is a Hopf submonoid of* **SF**.

Recall that a set family $(E, \mathscr{F})$ is a *topological space* if

1. $\emptyset, E \in \mathscr{F}$,

2. given a collection $\{F_\alpha\} \subseteq \mathscr{F}$, then $\cup F_\alpha \in \mathscr{F}$,

3. and given a finite collection $\{F_\alpha\} \subseteq \mathscr{F}$, then $\cap F_\alpha \in \mathscr{F}$.

The elements of $\mathscr{F}$ are called the *open sets* of the space. If E is a finite set and $(\mathscr{F}, E)$ is a topological space, then we call $(\mathscr{F}, E)$ a *finite topological space*. Further finite topological spaces on a set E are equivalent to the sublattices of the boolean lattice of E. As such a finite topological space is a set family that is closed under intersection and union. Using the fact that the intersection of two Hopf submonoids is a Hopf submonoid it follows that topological spaces span a Hopf submonoid of **SF**.

Corollary 3.14. *The subspecies* **Top** *of finite topological spaces is a Hopf submonoid of* **SF**. *Further* **Top** *is a Hopf submonoid of* **Int** *and* **Union**.

Due to phantoms we need to be aware that **Top**$[E]$ contains more than just finite topological spaces on E, but instead contains finite topological spaces on subsets of E. For example consider $E = \{x, y\}$ and $\mathscr{F} = \{E, \emptyset\}$. Then $(\mathscr{F}, E) \in$ **Top**$[E]$. Taking $S = \{x\}$ and $T = \{y\}$ we obtain

$$(\mathscr{F}, E)|_S = (S, \{S, \emptyset\}) \text{ and}$$
$$(\mathscr{F}, E)/_S = (T, \{\emptyset\}).$$

Notice that the restriction gives a topological space on S and yet the contraction fails to give a topological space on T.

3.4 Simplicial Complexes (Simp)

Definition 3.15. A *simplicial complex* is a set family $(\mathscr{F}, E)$ such that if $X \in \mathscr{F}$ and $Y \subseteq X$, then $Y \in \mathscr{F}$.

This definition of simplicial complexes allows for the possibility of phantom vertices. By adding the requirement that $\{x\} \in \mathscr{F}$ for each $x \in E$ as some sources require we would still be able to form a perfectly valid Hopf submonoid of simplicial complexes. We will consider the former definition to maintain consistency with the other Hopf submonoids that are discussed.

Proposition 3.16. *The subspecies* **Simp** *spanned by simplicial complexes is a Hopf submonoid of* **SF**.

Proof. We first show that if $\mathscr{F}$ and $\mathscr{G}$ are simplicial complexes, then so is $\mathscr{F} * \mathscr{G}$. We know that $\emptyset \in \mathscr{F} * \mathscr{G}$, thus it remains to show that $X \in \mathscr{F} * \mathscr{G}$ implies $Y \in \mathscr{F} * \mathscr{G}$ for all $Y \subseteq X$. We know that $X = F \cup G$ for some $F \in \mathscr{F}$ and $G \in \mathscr{G}$. Since $Y \subseteq X$, it follows that $Y = F' \cup G'$ for some $F' \subseteq F$ and $G' \subseteq G$. Hence $Y \in \mathscr{F} * \mathscr{G}$. Thus simplicial complexes are closed under taking join.

Next suppose we have a simplicial complex $\mathscr{F}$ with ground set E and $A \subseteq E$. To show that simplicial complexes are closed under restriction and contraction we make use of the following fact.

$$\begin{aligned}\mathscr{F}/_A &= \{F \in \mathscr{F} : F \cap A = \emptyset\}\\ &= \{F \in \mathscr{F} : F \subseteq \bar{A}\}\\ &= \{F \cap \bar{A} : F \in \mathscr{F}\} = \mathscr{F}|_{\bar{A}}.\end{aligned}$$

Thus it suffices to show that $\mathscr{F}/_A$ is a simplicial complex. Suppose $X \in \mathscr{F}/_A$ and $Y \subseteq X$. Then $Y \subseteq X \subseteq A$ and $Y \in \mathscr{F}$ in which case $Y \in \mathscr{F}/_A$ as desired. Therefore simplicial complexes form a Hopf submonoid of **SF**. □

A consequence of the above proof is that **Simp** is cocommutative.

Proposition 3.17. **Simp** *is the universal cocommutative Hopf submonoid of* **SF**.

Proof. We have already seen that cocommutativity holds in **Simp**. Suppose that **Z** is a cocommutative Hopf submonoid of **SF** and that $\mathscr{F} \in \mathbf{Z}[E]$. Further suppose that $X \in \mathscr{F}$ and $Y \subseteq X$.

Then $Y \in \mathscr{F}|_Y = \mathscr{F}/_{\bar{Y}}$. Hence $Y \in \mathscr{F}$ and as a result $\mathscr{F}$ is a simplicial complex. Therefore $\mathbf{Z} \subseteq \mathbf{Simp}$. □

3.5 Matroids (Mat)

There is also a Hopf submonoid of **SF** whose underlying species is spanned by independence complexes of matroids. The Hopf structure of this Hopf submonoid is different from the more familiar Hopf monoid of matroids defined in §1.5.

Proposition 3.18. *The subspecies* **Mat** *spanned by matroid independence complexes is a cocommutative Hopf submonoid of* **SF**.

Proof. Suppose $(\mathscr{I}, E)$ is a matroid independence system and $A \subseteq E$. Since **Mat** is a subspecies of **Simp** it suffices to show that the restriction and join operations produce matroids. The operation of restriction corresponds to matroid restriction and join corresponds to taking the direct sum of matroids [15, Prop. 4.2.12]. □

Remark 3.19. It is worth noting that **Mat** is a cocommutative Hopf monoid unlike the non-cocommutative Hopf monoid of matroids given by Aguiar and Ardila in [1]. Also note that the loops of a matroid are the phantoms of the matroid independence complex.

3.6 Boolean Lattices (Bool)

If $(\mathscr{I}, E)$ is a matroid, then $x \in E$ is a *loop* if x appears in every basis. Similarly $x \in E$ is a *coloop* if x appears in no basis. A *Boolean lattice* is a set family of the form $B = (2^F, E)$ where $F \subseteq E$. The set family B is the independence complex for the matroid that has $|F|$ coloops and $|E| - |F|$ loops.

Proposition 3.20. *The submonoid* **Bool** *spanned by Boolean lattices (considered as set families) is a Hopf submonoid of* **SF**.

Proof. Since Boolean lattices are the matroid independence complexes of simplices, **Bool** $\subseteq$ **Simp** and hence it is only require to show that Boolean lattices are closed under taking joins and restrictions. To that end consider the Boolean lattices $(2^I, E_1)$ and $(2^J, E_2)$ such that $I \subseteq E_1$ and $J \subseteq E_2$ where $E_1 \cap E_2 = \emptyset$. Then

$$\begin{aligned} 2^I * 2^J &= \{A \cup B : A \subseteq I, B \subseteq J\} \\ &= \{C : C \subseteq I \cup J\} \\ &= 2^{I \cup J}. \end{aligned}$$

Similarly if $A \subseteq E_1$

$$\begin{aligned} 2^I|_A &= \{B \cap A : B \subset I\} \\ &= \{C : C \subset A \cap I\} \\ &= 2^{I \cap A}. \end{aligned}$$

Therefore **Bool** is a Hopf submonoid of **SF**. □

3.7 Antimatroids (AMat)

Definition 3.21. An *antimatroid* is an accessible set family $(\mathscr{F}, E)$ that is closed under taking unions.

Definition 3.21 is the most convenient characterization of antimatroids for showing that they form a Hopf submonoid of **SF**, but that definition does not satisfy the desire to know what relation antimatroids have to matroids. This requires a bit of a cryptomorphismological journey.

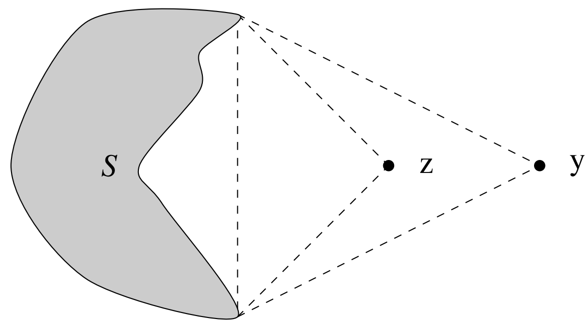


Figure 3.1: A visualization of the anti-exchange axiom.

Given an antimatroid $(\mathscr{F}, E)$ consider the set system

$$(\mathscr{G}, E) = \{E \setminus F : F \in \mathscr{F}\}.$$

The set family $\mathscr{G}$ is is called a *convex geometry* and the sets in $\mathscr{G}$ are referred to as *convex sets*. Since the convex sets are complements of sets of the antimatroid it follows that convex sets are closed under intersection. Further given a convex set $C \neq E$ there is an $x \in E$ such that $C \cup \{x\}$ is a convex set. We can equivalently define a convex geometry in terms of a closure operator.

Definition 3.22. A *closure operator* is a map $\mathrm{cl} : 2^E \to 2^E$ such that for $S \in 2^E$ and $T \subseteq S$ the following are satisfied:

- $S \subseteq \mathrm{cl}(S)$,
- $\mathrm{cl}(T) \subseteq \mathrm{cl}(S)$,
- $\mathrm{cl}(\mathrm{cl}(S)) = \mathrm{cl}(S)$.

The resulting collection of closed sets from a closure operator is closed under intersection, but a set family of closed sets in general lacks the second property required for a convex geometry. For that we require the *anti-exchange axiom*. The anti-exchange axiom states that if $y, z \in E \setminus \mathrm{cl}(S)$ with $y \neq z$ and $z \in \mathrm{cl}(S \cup \{y\})$, then $y \notin \mathrm{cl}(S \cup \{z\})$.

Given a closure operator that satisfies the anti-exchange axiom and a closed set S, then we can place a partial order on the elements of E not in $\mathrm{cl}(S)$ for which $x \leq y$ if $x \in \mathrm{cl}(S \cup \{y\})$. If x is a minimal element in this poset, then $S \cup \{x\}$ is closed. In other words, when $S \neq E$ there is an x such that $S \cup \{x\}$ is closed. Thus closed sets of a closure operator satisfying the anti-exchange axiom are convex sets and form a convex geometry. Hence the complements of these closed sets form an antimatroid.

The name antimatroid comes from the anti-exchange axiom acting in an analogous way to a similar axiom for the rank function definition of matroids. Another reason for the name antimatroid is that antimatroids are greedoids that satisfy the interval property without lower bounds whereas matroids are greedoids that satisfy the interval property without upper bounds.

Definition 3.23. The set family $(\mathscr{F}, E)$ is a *greedoid* if it satisfies the following conditions:

1. If $X \in \mathscr{F}$ is non-empty, then there exists $x \in X$ such that $X \setminus \{x\} \in \mathscr{F}$.

2. If $X, Y \in \mathscr{F}$ with $|X| > |Y|$, then there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathscr{F}$.

Definition 3.24. Suppose that $(\mathscr{F}, E)$ is a greedoid and that $F, G, H \in \mathscr{F}$ with $F \subseteq G \subseteq H$. $(\mathscr{F}, E)$ is said to be a greedoid satisfying the *interval property* if for any $x \in E \setminus H$ such that $F \cup \{x\}, H \cup \{x\} \in \mathscr{F}$, then it is the case that $G \cup \{x\} \in \mathscr{F}$.

Definition 3.25. Suppose that $(\mathscr{F}, E)$ is a greedoid and that $G, H \in \mathscr{F}$ with $G \subseteq H$. $(\mathscr{F}, E)$ is said to be a greedoid satisfying the *interval property without lower bounds* if for any $x \in E \setminus H$ such that $H \cup \{x\} \in \mathscr{F}$, then it is the case that $G \cup \{x\} \in \mathscr{F}$.

Definition 3.26. Suppose that $(\mathscr{F}, E)$ is a greedoid and that $F, G, H \in \mathscr{F}$ with $F \subseteq G$. $(\mathscr{F}, E)$ is said to be a greedoid satisfying the *interval property without upper bounds* if for any $x \in E \setminus G$ such that $F \cup \{x\} \in \mathscr{F}$, then it is the case that $G \cup \{x\} \in \mathscr{F}$.

From this we see that matroids and antimatroids share some kind of not quite duality. A survey paper by Dietrich covers some of the parallel results between both objects[6]. For further information on antimatroids and greedoids [3] and [11] are recommended.

Proposition 3.27. *The subpecies* **AMat** *spanned by antimatroids is a Hopf submonoid of* **SF**.

Proof. Since $\mathbf{AMat} = \mathbf{Acc} \cap \mathbf{Union}$ it follows that **AMat** is a Hopf submonoid of **SF**. □

We might ask whether or not the subspecies spanned by convex geometries is a Hopf submonoid of **SF**. Consider $\mathscr{F} = \{\emptyset, 1, 2, 3, 12, 23, 123\} \in \mathbf{SF}[[3]]$. Then the contraction $\mathscr{F}/_2$ is not a convex geometry. Hence it follows that the subspecies spanned by convex geometries does not form a Hopf submonoid of **SF**.

Remark 3.28. It should be noted that White has a Hopf monoid on antimatroids [21, §6.5]. The Hopf monoid **AMat** shares the same product operation, but differs in the coproduct. Notably the tensor factors of the coproduct in White's monoid will yield antimatroid minors whereas this is not always the case in **AMat**.

3.8 Lattices of Order Ideals (LOI)

Finally we consider the subspecies **LOI** of **SF** spanned by the lattices of order ideals of posets, i.e., set families of the form $(J(P), E)$ where P is a poset with $P \subseteq E$. We will show that **LOI** is a Hopf submonoid of **SF**. In following chapters **LOI** will be a primary focus for our work.

Lemma 3.29. *If $A \subseteq P$, then $J(P|_A) = J(P)|_A$, i.e.,*

$$\{I : I \text{ is an order ideal of } P|_A\} = \{I \cap A : I \in J(P)\}$$

Proof. Suppose that $I = \langle x_1, \ldots, x_k \rangle \in J(P|_A)$. Then consider $\tilde{I} = \langle x_1, \ldots, x_k \rangle \in J(P)$. It follows that $\tilde{I} \cap A = I \in J(P)|_A$.

Conversely if $I \in J(P)|_A$, then $I = \tilde{I} \cap A$ for some $\tilde{I} \in J(P)$. If $p \in I$ and $q \in A$ with $q <_P p$, then it follows that $q \in \tilde{I}$ and thus that $q \in I$. thus I is an order ideal of $P|_A$. □

Lemma 3.30. *Suppose P and Q are disjoint posets. Then $J(P+Q) = J(P) * J(Q)$, where $+$ is disjoint union of posets.*

Proof. Suppose that P and Q are disjoint posets and consider $I \in J(P+Q)$. If $I \notin J(P) * J(Q)$, then without loss of generality there exist $x \in P$ and $y \in Q$ such that $x, y \in I$ and $x <_{P+Q} y$. This cannot happen due to the definition of $P+Q$. Thus $I = M \sqcup N$ where $M \in J(P)$ and $N \in J(Q)$ and hence $I \in J(P) * J(Q)$.

Conversely suppose $I \in J(P) * J(Q)$. Then $I = M \sqcup N$ where $M \in J(P)$ and $N \in J(Q)$. If $x \in I$, then without loss of generality $x \in M$. Therefore if $y \leq x$, then $y \in M$ and hence $y \in I$. Thus $I \in J(P+Q)$. $\square$

Proposition 3.31. *The subspecies* **LOI** *is a Hopf submonoid of* **SF**.

Proof. Let P, Q be posets, so that $(J(P), E)$ and $(J(Q), F)$ are set families on ground sets E and F respectively. Thus $P \subseteq E$ and $Q \subseteq F$ as sets. First, $\mu_{P,Q}((J(P), E) \otimes (J(Q), F)) = (J(P) * J(Q), E \cup F) = (J(P+Q), E \cup F)$. Second, let $E = A \sqcup B$ (as sets). Then

$$\begin{aligned} \Delta_{A,B}(J(P), E) &= (J(P)|_A, A) \otimes (J(P)/_A, B) \\ &= (\{I \cap A : \ I \in J(P)\}, A) \otimes (\{I \in J(P) : I \subseteq B\}, B) \\ &= (\{I \cap A : \ I \in J(P)\}, A) \otimes (\{I \in J(P) : I \cap A = \emptyset\}, B) \end{aligned}$$

By Lemma 3.29, the left tensor factor is $(J(P|_A), A)$. For the right term, consider P restricted to the set

$$\tilde{B} = \{x \in B : x \not> y \ \ \forall y \in A\}.$$

Note that $\tilde{B}$ is the complement of the order filter generated by A. Suppose that $\tilde{I} \in J(P|_{\tilde{B}})$ and consider the ideal $I = \langle \tilde{I} \rangle_P$. Suppose that there is some $x \in I \cap (B \setminus \tilde{B})$; then (since $x \notin \tilde{B}$) there is some $y \in A$ such that $y < x$, but (since $x \in I$) we have $y \in I$. But then $I \cap A \neq \emptyset$, which contradicts the definition of $\tilde{B}$. Thus it must be that $I \cap (B \setminus \tilde{B}) = \emptyset$, i.e., $\tilde{I} = I$. It follows that

$$J(P|_{\tilde{B}}) \subseteq \{I \in J(P) : I \cap A = \emptyset\}. \tag{3.1}$$

Conversely if I is an order ideal of P that doesn't intersect A, then any element $x \in I$ must be in $\tilde{B}$. If not, then $\langle x \rangle \cap A \neq \emptyset$, but $\langle x \rangle \subset I$ a contradiction since $I \cap A = \emptyset$. Thus equality holds in (3.1), and therefore the right term of the coproduct is also a lattice of order ideals. Hence

$$\Delta_{A,B}(J(P)) = (J(P|_A), A) \otimes (J(P|_{\tilde{B}}), B)$$

and **LOI** is a Hopf submonoid of **SF**. □

Corollary 3.32. **LOI** *is a Hopf submonoid of* **AMat**.

Proof. Suppose P is a poset. If $I, I' \in J(P)$, then $I \cup I' \in J(P)$. Further if $I \in J(P)$ where $I \neq \emptyset$, then a generator $x \in I$ can be removed and $I \setminus \{x\} \in J(P)$. Thus by Definition 3.21, $J(P)$ is an antimatroid. □

Suppose $A \subseteq E$. Then A is called *free* if $\mathscr{F}|_A = 2^A$. A *circuit* is a minimal non-free set. That is to say that $A \subseteq E$ is a circuit if $\mathscr{F}|_A \neq 2^A$, but $\mathscr{F}|_B = 2^B$ for every proper subset $B \subsetneq A$. Given a circuit $C \subseteq E$ we say that $a \in C$ is a *root* of C if $F|_C = 2^C \setminus \{\{a\}\}$. Every circuit has a unique root[3, §8.7.C]. Given a circuit C with root a we denote the *rooted circuit* by the pair (C, a). An antimatroid is determined by its collection of rooted circuits. In the language of antimatroids the set family $(J(P), P)$ is known as the *poset antimatroid* of P. Using circuits we can determine exactly when an antimatroid is a poset antimatroid.

Proposition 3.33 (Refer to [11, Corollary 3.10])**.** *An antimatroid is a poset antimatroid if and only if all of its circuits have cardinality 2.*

3.9 The Hierarchy of Hopf Submonoids

We end this chapter by ordering the Hopf submonoids of **SF** by inclusion. Figure 3.2 illustrates the relationships shared by the various Hopf submonoids mentioned in this chapter. Set families gain additional structure as we descend down the hierarchy of Hopf submonoids. It should be noted that there are many more Hopf submonoids than those listed in this chapter. For example we know that **Int**$\cap$**Acc** is a Hopf submonoid of **SF** that properly contains **Simp** and **LOI**.

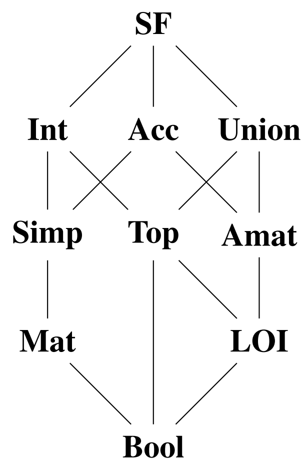


Figure 3.2: Hopf submonoids of **SF**

Chapter 4

A Cancellation-Free Antipode Formula for LOI

In this section, we derive a cancellation-free formula for the antipode of **LOI**, using the topological approach of [1]. Before starting down this road it is vital that we keep in mind phantoms. Specifically, an element $(J(P),E) \in \mathbf{SF}[E]$ consists of a poset P whose underlying set is a *subset* of E. The elements of E that don't appear in P are phantoms. Proposition 2.4 tells us that we can first derive a cancellation-free formula for phantomless elements of **LOI** and then account for phantoms afterwards.

Definition 4.1. Let P be a poset and $(J(P),E) \in \mathbf{SF}[E]$, and $\Phi \models E$. An element $x \in E$ is *betrayed* (with respect to P and Φ) if there exists $y \in E$ such that $y <_P x$ and $y <_\Phi x$. The set of betrayed elements in Φ_i will be denoted $B(\Phi_i)$, and we put $B(\Phi) = \bigcup_i B(\Phi_i)$. Evidently $B(\Phi) \cap \mathrm{Min}(P) = \emptyset$, where $\mathrm{Min}(P)$ denotes the set of minimal elements of P.

The reason for saying that y "betrays" x is that y is below x in the order given by P, yet jumps in front of x in Φ.

Proposition 4.2. *Suppose P is a poset and suppose that $\Phi = \Phi_1|\ldots|\Phi_m \models P$. Then*

$$\Delta_\Phi(J(P),P) = \bigotimes_{i=1}^{m} (J(K_i),\Phi_i)$$

where K_i is the restriction of P to $\Phi_i \setminus B(\Phi_i)$.

Here and subsequently, the notation $(J(P),P)$ specifies that the underlying set for the set family $J(P)$ is just the underlying set of the poset P (rather than a superset of it). That is, there are no phantoms.

Proof. By iterating coproduct, we see that the i^{th} tensor factor is the set family

$$(\mathscr{F}_i, \Phi_i) = \{I \cap \Phi_i : I \in J(P) \text{ and } I \cap (\Phi_1 \cup \cdots \cup \Phi_{i-1}) = \emptyset\}.$$

Suppose that $I \in \mathscr{F}_i$; then $I = \tilde{I} \cap \Phi_i$ for some $\tilde{I} \in J(P)$ and $\tilde{I} \cap (\Phi_1 \cup \cdots \cup \Phi_{i-1}) = \emptyset$. Therefore $x \in I$ implies $y \not<_P x$ for $y \in \Phi_1 \cup \cdots \cup \Phi_{i-1}$, for otherwise $\tilde{I} \cap (\Phi_1 \cup \cdots \cup \Phi_{i-1}) \neq \emptyset$ since $y \in \tilde{I} \cap (\Phi_1 \cup \cdots \cup \Phi_{i-1})$. Thus $x \in K_i$ and consequently, $I \in J(K_i)$.

Conversely suppose that $I = \langle x_1, \ldots, x_m \rangle \in J(K_i)$. Let $\tilde{I} = \langle x_1, \ldots, x_m \rangle \in J(P)$; then $I = \tilde{I} \cap \Phi_i$ by Lemma 3.29. Since the generators x_j all belong to K_i, it follows that $\tilde{I} \cap (\Phi_1 \cup \cdots \cup \Phi_{i-1}) = \emptyset$. Thus $I \in \mathscr{F}_i$. □

Applying μ_Φ to the formula of Proposition 4.2, we obtain the following equation in $\mathbf{SF}[P]$:

$$\mu_\Phi(\Delta_\Phi(J(P), P)) = (J(K_1), \Phi_1) * \cdots * (J(K_m), \Phi_m) = (J(K_1 + \cdots + K_m), P). \tag{4.1}$$

(Here the operator $*$ indicates the join of the $(J(K_i), \Phi_i)$ as set families, which is equivalent to the product of the $J(K_i)$ as lattices.) Equation (4.1) asserts that every term in the antipode has the form $(J(Q), P)$, where $Q = K_1 + \cdots + K_m$ and $K_i = P|_{\Phi_i \setminus B(\Phi_i)}$ for some $\Phi \models P$. In particular, each (Hasse) component of Q is contained in some block of Φ. Note that the betrayed elements have become phantoms.

Definition 4.3. Let P be a poset. A *fracturing* Q of P is a disjoint sum of induced subposets of P. We require only that $Q \subseteq P$ as sets, not that $Q = P$. The *support system* of $(J(Q), E)$ with respect to $(J(P), E)$ is

$$\mathrm{Supp}(Q) = \mathrm{Supp}_P^E(Q) = \{\Phi \models E : \ \mu_\Phi(\Delta_\Phi(J(P), E)) = (J(Q), E)\}.$$

The previous discussion implies that $\mathrm{Comp}(E)$ is the disjoint union of the sets $\mathrm{Supp}_P^E(Q)$, as Q ranges over all fracturings of P. The fracturing Q is called *good* if $\mathrm{Supp}_P^E(Q) \neq \emptyset$; we write $\mathrm{Good}(P)$ for the set of all good fracturings of P. Observe that $\mathrm{Good}(P)$ depends only on P, not on E.

Proposition 4.2 implies that

$$\Phi \in \mathrm{Supp}_P^E(Q) \quad \Longrightarrow \quad B(\Phi) = P \setminus Q \text{ (as sets).} \tag{4.2}$$

In particular, every good fracturing must contain $\mathrm{Min}(P)$ as a subset.

Example 4.4. Let P be the poset on $E = \{1,2,3\}$ with relations $1 < 3, 2 < 3$. The antichain Q on $\{1,2\}$ is a good fracturing of P, with $\mathrm{Supp}_P^E(Q) = \{1|3|2,\ 1|23,\ 1|2|3,\ 12|3,\ 2|1|3,\ 2|31,\ 2|3|1\}$. The corresponding subfan of the braid arrangement is shaded in Figure 4.1. This example illustrates that $\|\mathrm{Supp}_P^E(Q)\|$ need not be a convex fan.

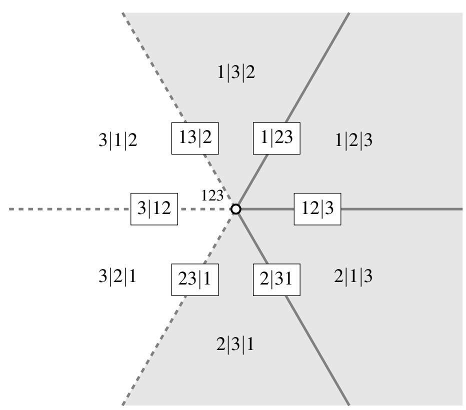


Figure 4.1: A (non-convex) example of $\|\mathrm{Supp}_E^P(Q)\|$.

Observe that

$$
\begin{aligned}
S(J(P),E) &= \sum_{\Phi \models E} (-1)^{|\Phi|} \mu_\Phi \Delta_\Phi(J(P),E) \\
&= (-1)^{|E \setminus P|} \sum_{\Phi \models P} (-1)^{|\Phi|} \mu_\Phi \Delta_\Phi(J(P),P) && \text{(by Prop. 2.4)} \\
&= (-1)^{|E \setminus P|} \sum_{Q \in \mathrm{Good}(P)} (J(Q),P) \underbrace{\left(\sum_{\Phi \in \mathrm{Supp}_P^P(Q)} (-1)^{|\Phi|} \right)}_{c_Q}. && (4.3)
\end{aligned}
$$

In particular, the coefficient c_Q can always be computed in terms of set compositions of (the underlying set of) P, rather than some superset including phantoms. As in [1, §7], we will compute the coefficients c_Q by regarding them as the "relative Euler characteristics" of subfans of the braid fan. The geometry is more complicated than the situation of [1], since these fans are not always convex.

Proposition 4.5. *Let Q be a good fracturing of P. Then $\Phi \in \mathrm{Supp}_P^P(Q)$ if and only if the following conditions all hold:*

$$\forall x, y \in Q: \; y <_Q x \implies x \sim_\Phi y; \tag{4.4}$$

$$\forall x \in Q: \; \forall y \in Q: \; y <_P x \text{ and } y \not<_Q x \implies x <_\Phi y; \tag{4.5}$$

$$\forall x \in Q: \; \forall y \in P \setminus Q: \; y <_P x \implies x <_\Phi y; \tag{4.6}$$

$$\forall y \in P \setminus Q: \; \exists x \in P: \; x <_P y \text{ and } x <_\Phi y. \tag{4.7}$$

Proof. Suppose $\Phi \in \mathrm{Supp}_P^P(Q)$. By Proposition 4.2, each component of Q is contained in some block of Φ, implying (4.4). By (4.2), $B(\Phi) \supseteq P \setminus Q$, which is equivalent to (4.7); and $B(\Phi) \subseteq P \setminus Q$, which implies (4.5) and (4.6).

Conversely, suppose that $\Phi \models P$ satisfies (4.4)–(4.7). Let Q' be the good fracturing of P such that $\Phi \in \mathrm{Supp}(Q')$. First, we claim that $B(\Phi) = P \setminus Q$. The inclusion $B(\Phi) \supseteq P \setminus Q$ is just (4.7).

For the reverse inclusion, if $x \in B(\Phi) \cap Q$, then there exists $y \in P$ such that $y <_P x$ and $y <_\Phi x$. If $y \in P \setminus Q$ then (4.6) fails, while if $y \in Q$ then (4.5) implies $y <_Q x$, but then $x \sim_\Phi y$ by (4.4), a contradiction, so the claim is proved. In particular, $Q' = Q$ as sets. By (4.4), every relation in Q is a relation in Q'; conversely, if $y <_{Q'} x$, then $y <_P x$ and $y \sim_\Phi x$, so (4.5) implies $y <_Q x$. Therefore, $Q = Q'$. □

As mentioned earlier, our goal is to calculate the "Euler characteristic" of the fan $\|\operatorname{Supp}_P^P(Q)\|$. By condition (4.4), $\|\operatorname{Supp}_P^P(Q)\|$ is contained in the subspace $V_Q \subset \mathbb{R}^{|P|}$ defined by equalities $x_i = x_j$ whenever i, j belong to the same component of Q; the dimension of this subspace is $u+k$, where u is the number of components of Q and $k = |P \setminus Q|$. Observe that condition (4.7) gives rise to a disjunction of linear inequalities rather than a conjunction, which is why $\|\operatorname{Supp}_P^P(Q)\|$ need not be convex (q.v. Example 4.4). Accordingly, our next step is to express $\|\operatorname{Supp}_P^P(Q)\|$ as a union of convex fans.

Definition 4.6. Let Q be a fracturing of P. A *betrayal function* is a map $\beta : P \setminus Q \to P$ such that $\beta(b) <_P b$ for every $b \in P \setminus Q$. Observe that Q has a betrayal function if and only if $Q \supseteq \operatorname{Min}(P)$. Let

$$\operatorname{Supp}_\beta^P(Q) = \{\Phi \in \operatorname{Supp}_P^P(Q) : \ \beta(b) <_\Phi b \ \ \forall b \in P \setminus Q\}.$$

Applying Proposition 4.5, we see that $\operatorname{Supp}_\beta^P(Q)$ consists of set compositions $\Phi \in \operatorname{Supp}_P^P(Q)$ satisfying (4.4), (4.5), (4.6), and

$$\forall b \in P \setminus Q : \ \ \beta(b) <_\Phi b. \tag{4.8}$$

Observe that $\|\operatorname{Supp}_\beta^P(Q)\|$ is a *convex* subfan of the braid arrangement for every β: for each $b \in P \setminus Q$, the disjunction (4.7) has been replaced by a single inequality. Moreover, $\operatorname{Supp}_P^P(Q) = \bigcup_\beta \operatorname{Supp}_\beta^P(Q)$, though in general this is not a disjoint union.

Proposition 4.7. *Let Q be a good fracturing of P and β a betrayal function for Q. Then $\|\operatorname{Supp}_\beta^P(Q)\|$ is homeomorphic to $\mathbb{R}^{u+k}$, where u is the number of components of Q and $k = |P \setminus Q|$.*

Proof. The affine hull of $\|\operatorname{Supp}^P_\beta(Q)\|$ is defined by the linear equalities (4.4), hence has one degree of freedom for each component of Q and each element of $P \setminus Q$. The inequalities given by (4.5), (4.6), and (4.8) define $\|\operatorname{Supp}^P_\beta(Q)\|$ as a convex open subset of its affine hull. The conclusion follows by [8]. $\square$

Example 4.8. Recall the poset P and good fracturing Q of Example 4.4. In Figure 4.2, each face of $\operatorname{Supp}^P_P(Q)$ is colored green, blue, or red, depending on whether $3 \in P$ is betrayed by only by 1, only by 2, or by both 1 and 2. There are two betrayal functions $\beta_1, \beta_2 : \{3\} \to \{1,2\}$, given by $\beta_i(3) = i$. Thus $\|\operatorname{Supp}^P_{\beta_1}(Q)\|$ is the subfan consisting of the green and red faces, and $\|\operatorname{Supp}^P_{\beta_2}(Q)\|$ consists of the blue and red faces. Observe that both subfans are convex.

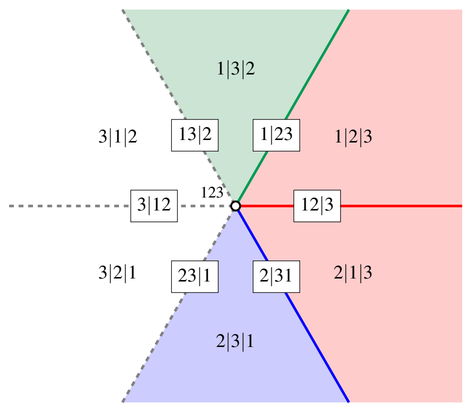


Figure 4.2: An example of $\|\operatorname{Supp}^P_\beta(Q)\|$.

In order to understand the support of $S(J(P),P)$, we need to know which fracturings are good. The following definition and proposition give a usable criterion for goodness, together with a way of constructing explicit elements of $\operatorname{Supp}^P_P(Q)$ for a fracturing Q.

Definition 4.9. Let Q be a fracturing of P with (Hasse) components $Q_1, \dots, Q_u$. The *conflict digraph* $\operatorname{Con}_P(Q)$ of Q is the digraph (directed graph) with vertices $\{Q_1, \dots, Q_u\}$ and edges

$$\{Q_i \to Q_j : \ i \neq j \text{ and there exist } x \in Q_i \text{ and } y \in Q_j \text{ such that } y <_P x\}.$$

Example 4.10. Consider the poset $P = \{1 < 2 < 3, 4 < 5 < 6, 1 < 4, 2 < 5, 3 < 6\}$ and the fracturing whose components are the induced subposets $Q_1 = \{4\}$, $Q_2 = \{1 < 2 < 5\}$, $Q_3 = \{3 < 6\}$. Then $\mathrm{Con}_P(Q)$ is the digraph with vertices Q_i and edges $Q_2 \to Q_2$, $Q_2 \to Q_1$, $Q_3 \to Q_2$, and $Q_3 \to Q_1$.

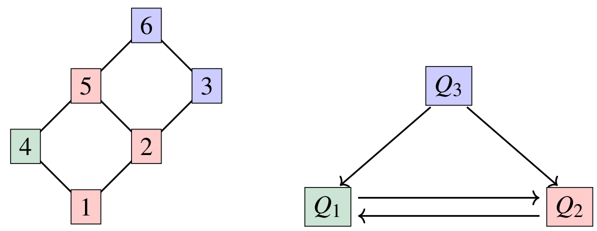


Figure 4.3: P and $\mathrm{Con}_P(Q)$.

Observe that Q cannot be a good fracturing. For any set composition $\Phi \models [6]$ satisfying (4.4), either 4 betrays 5, or 1 betrays 4, or $\{1,2,4,5\}$ is contained in some block of Φ, in which case $\mu_\Phi(\Delta_\Phi(P))$ includes the relations $1 < 4 < 5$. This obstruction to goodness is captured by the antiparallel edges between Q_1 and Q_2 in $\mathrm{Con}_P(Q)$. In fact, cycles in $\mathrm{Con}(Q)$ form obstructions to goodness, as we now explain.

Definition 4.11. Let P be a poset. An *acyclic fracturing* Q of P is a fracturing of P such that the conflict digraph $\mathrm{Con}_P(Q)$ (see Definition 4.9) is acyclic; we write $\mathrm{Acyc}(P)$ for the set of all acyclic fracturings of P.

Proposition 4.12. *Let Q be a fracturing of P with components $Q_1, \ldots, Q_u$. Then Q is a good fracturing if and only if $Q \supseteq \mathrm{Min}(P)$ and Q is an acyclic fracturing.*

Proof. ($\Longrightarrow$) Suppose that Q is a good fracturing, and let $\Phi \in \mathrm{Supp}_P^E(Q)$. We have observed in Definition 4.3 that $Q \supset \mathrm{Min}(P)$. Now, suppose that $\mathrm{Con}_P(Q)$ contains a cycle, which we may take to be $Q_1 \to \cdots \to Q_s \to Q_1$. Then there are elements $x_1, \ldots, x_s, y_1, \ldots, y_s$ of $P \setminus Q$, with $x_j, y_j \in Q_j$ and $x_j <_P y_{j+1}$ for all j (taking indices mod s). For each j, since x_j does not betray y_{j+1}, it follows that $Q_j \subseteq \Phi_{r_j}$ and $Q_{j+1} \subseteq \Phi_{r_{j+1}}$, where $r_{j+1} \leq r_j \leq m$. But then $r_1 \geq r_2 \geq \cdots \geq r_m \geq r_1$, so all the r_i are equal (say to r) and $Q_1 \cup \cdots \cup Q_s \subset \Phi_r$. By Proposition 4.2, $x_j <_Q y_{j+1}$ for all j, but then the Q_i are in fact identical. So the cycle is a self-loop, which is prohibited in the construction of $\mathrm{Con}_P(Q)$. We conclude that $\mathrm{Con}_P(Q)$ is acyclic and thus Q is an acyclic fracturing.

($\Longleftarrow$) Suppose that $Q \supset \mathrm{Min}(P)$ and that Q is an acyclic fracturing, i.e., $\mathrm{Con}_P(Q)$ is acyclic. As observed in Definition 4.6, the first assumption implies that Q admits a betrayal function β. Acyclicity of $\mathrm{Con}_P(Q)$ implies that there is a well-defined preposet $O_\beta(Q)$ on $[n]$ with equivalence classes $\{Q_1,\ldots,Q_u\} \cup (P \setminus Q)$ and relations

$$Q_j < b \quad \text{whenever } \beta(b) \in Q_j; \tag{4.9a}$$

$$Q_i < Q_j \quad \text{whenever } Q_i \to Q_j \text{ is an edge in } \mathrm{Con}_P(Q). \tag{4.9b}$$

Let $\Phi \in \mathscr{L}(O_\beta(Q))$. That is, Φ is a linear preposet with the same equivalence classes that contains the relations (4.9a) and (4.9b). The conditions (4.9a) imply that $P \setminus Q \subseteq B(\Phi)$, while the conditions (4.9b) imply the reverse inclusion. Moreover, the posets K_i of Proposition 4.2 are precisely the components of Q. Therefore $\mu_\Phi(\Delta_\Phi(P)) = Q$, i.e., $\Phi \in \mathrm{Supp}_P^P(Q)$, and now (4.9a) implies that in fact $\Phi \in \mathrm{Supp}_\beta^P(Q)$. $\square$

Remark 4.13. The components Q' of a good fracturing Q have a notable property: if $x, y \in Q'$, $z \in Q$, and $x \leq_P z \leq_P y$, then $z \in Q'$. (In poset terminology, each Q' is interval-closed as a subposet of $(Q, \leq_P)$.) Indeed, let β be a betrayal function and let $\Phi \in \mathscr{L}(O_\beta(Q))$. If $z \notin B(\Phi)$. Let Φ_i be the block of Φ containing z and Φ_j be the block of Φ containing x and y. It is not the case that $i < j$ (when z betrays y) or that $i > j$ (when x betrays z), so $i = j$, and it follows that $z \in Q_j$.

Proposition 4.14. *Let P be a poset, let $Q \in \mathrm{Good}(P)$, and let $\mathscr{B}$ be the collection of all betrayal functions β such that $\mathrm{Supp}_\beta^P(Q) \neq \emptyset$. Then $\bigcap_{\beta \in \mathscr{B}} \mathrm{Supp}_\beta^P(Q) \neq \emptyset$.*

Proof. Let D be the digraph $(V_1 \cup V_2, E_1 \cup E_2 \cup E_3)$, where $V_1 = \{Q_1,\ldots,Q_u\}$ and $V_2 = P \setminus Q$, and

$$E_1 = \{Q_i \to Q_j :\ i \neq j \text{ and there exist } x \in Q_i \text{ and } y \in Q_j \text{ such that } y <_P x\},$$

$$E_2 = \{b \to b' :\ b, b' \in P \setminus Q \text{ and } b <_P b'\},$$

$$E_3 = \{Q_j \to b :\ b \in P \setminus Q \text{ and there exists } x \in Q_j \text{ such that } x <_P b\}.$$

It suffices to show that D is acyclic, for then every linear extension of D belongs to $\bigcap_{\beta\in\mathscr{B}} \mathrm{Supp}^P_\beta(Q)$. Indeed, the subdigraph (V_1,E_1) is just the conflict digraph $\mathrm{Con}_P(Q)$, which is acyclic by Proposition 4.12, and the subdigraph (V_2,E_2), is the transitive closure of the poset $P\setminus Q$. Moreover, every edge in E_3 points from V_1 to V_2. Thus D is acyclic as desired. □

Theorem 4.15. *Let P be a finite poset, so that $J(P)\in \mathbf{LOI}[P]$. Then the antipode of $J(P)$ in* **LOI** *is given by the following cancellation-free and grouping-free formula:*

$$S(J(P),P) = \sum_{Q\in\mathrm{Good}(P)} (-1)^{c(Q)+|P\setminus Q|}(J(Q),P)$$

where $c(Q)$ is the number of components of Q.

Proof. Recall from (4.3) that

$$S(J(P),P) \;=\; \sum_{Q\in\mathrm{Good}(P)} c_Q(J(Q),P)$$

where

$$c_Q = \sum_{\Phi\in\mathrm{Supp}^P_P(Q)} (-1)^{|\Phi|}.$$

Fix a good fracturing Q, and abbreviate $u=c(Q)$ and $k=|P\setminus Q|$. Recall that $\mathrm{Supp}^P_P(Q) = \bigcup_{\beta\in\mathscr{B}} \mathrm{Supp}^P_\beta(Q)$, where $\mathscr{B}$ is the set of all betrayal functions for Q. For $\mathscr{A}\subseteq\mathscr{B}$, define

$$Y_{\mathscr{A}} = Y_{\mathscr{A},\mathscr{Q}} = \bigcap_{\beta\in\mathscr{A}} \mathrm{Supp}^P_\beta(Q).$$

By inclusion-exclusion, we have

$$c_Q = \sum_{\emptyset\neq\mathscr{A}\subseteq\mathscr{B}} (-1)^{|\mathscr{A}|+1} \sum_{\Phi\in Y_{\mathscr{A}}} (-1)^{|\Phi|}. \tag{4.10}$$

As in the proof of [1, Thm. 7.1], the inner sum can be interpreted as the reduced Euler characteristic of $\|Y_{\mathscr{A}}\|$ as a relative polyhedral complex (or equivalently of $\|Y_{\mathscr{A}}\| \cap \mathbb{S}^{|P|-2}$ as a relative simplicial complex; see §1.2). Since each $\|\operatorname{Supp}_{\beta}^{P}(Q)\|$ is open, convex, and homeomorphic to $\mathbb{R}^{u+k}$ (by Proposition 4.7), so is their intersection $\|Y_{\mathscr{A}}\|$. Thus $\|Y_{\mathscr{A}}\| \cap \mathbb{S}^{|P|-2}$ is homeomorphic to an open ball of dimension $u+k-2$, and thus

$$\sum_{\Phi \in Y_{\mathscr{A}}} (-1)^{|\Phi|} = \tilde{\chi}(\overline{Y_{\mathscr{A}}}) - \tilde{\chi}(\partial Y_{\mathscr{A}}) = \tilde{\chi}(\mathbb{B}^{u+k}) - \tilde{\chi}(\mathbb{S}^{u+k-1}) = 0 - (-1)^{u+k-1} = (-1)^{u+k}.$$

Substituting into (4.10), we obtain

$$c_Q = \sum_{\emptyset \neq \mathscr{A} \subseteq \mathscr{B}} (-1)^{|\mathscr{A}|+1+u+k} = (-1)^{u+k+1} \sum_{\emptyset \neq \mathscr{A} \subseteq \mathscr{B}} (-1)^{|\mathscr{A}|} = (-1)^{u+k}$$

which establishes the desired formula for $S(J(P),P)$. □

Should the ground set contain phantoms we can use Proposition 2.4 to adjust the antipode accordingly.

Corollary 4.16. *Suppose $(J(P),E)$ has k phantoms. Then*

$$S((J(P),E)) = (-1)^k \sum_{Q \in \mathrm{Good}(P)} (-1)^{c(Q)+|P \setminus Q|} (J(Q),E) \tag{4.11}$$

We will see that in general the antipode formula can be difficult to work with although for some poset operations like dual and direct sum we will be able to develop formulae. The first tool we obtain from the cancellation-free antipode formula is a formula for the dual poset P^* in terms of P.

Proposition 4.17. *Suppose P is a poset and that P^* is the dual of P. Then*

$$S(J(P^*),P) = \sum_{\substack{Q \in \mathrm{Acyc}(P) \\ \mathrm{Max}(P) \subseteq Q}} (-1)^{c(Q)+|P \setminus Q|} (J(Q),P).$$

Proof. By Proposition 4.12 we see that Q^* is a good fracturing of P^* if and only if $Q^* \in \mathrm{Acyc}(P^*)$

and $\mathrm{Min}(P^*) \subseteq Q^*$. By taking the dual of each component of Q^* we obtain a fracturing Q of P. Q is an acyclic fracturing (of P) since dualizing each component will reverse the edges of the conflict graph of Q^*. It also follows that $\mathrm{Min}(P^*) \subseteq Q$ and since $\mathrm{Min}(P^*) = \mathrm{Max}(P)$ we can write $\mathrm{Max}(P) \subseteq Q$. Since $c(Q) = c(Q^*)$ and $|P \setminus Q| = |P^* \setminus Q^*|$ we can rewrite the antipode formula from Theorem 4.15 for $(J(P^*), P)$ as

$$S(J(P^*), P) = \sum_{\substack{Q \in \mathrm{Acyc}(P) \\ \mathrm{Max}(P) \subseteq Q}} (-1)^{c(Q)+|P \setminus Q|} (J(Q), P).$$

$\square$

Remark 4.18. The duality map $\varphi : \mathbf{LOI} \to \mathbf{LOI}$ given by $\varphi((J(P), P)) = (J(P^*), P)$ is not a Hopf automorphism or antiautomorphism. It respects the Hopf product (join) and restriction:

$$\begin{aligned}
\varphi((J(P_1), P_1) * (J(P_2), P_2)) &= \varphi((J(P_1 + P_2), P_1 \sqcup P_2)) \\
&= J((P_1 + P_2)^*, P_1 \sqcup P_2) = (J(P_1^* + P_2^*), P_1 \sqcup P_2) \\
&= (J(P_1^*), P_1) * (J(P_2^*), P_2) = \varphi((J(P_1), P)) * \varphi((J(P_2), P_2)).
\end{aligned}$$

and

$$\begin{aligned}
\varphi((J(P)|_A, A)) &= \varphi((J(P|_A), A)) \\
&= (J(P^*|_A), A) = (J(P^*)|_A, A) = \varphi(J(P), P)|_A.
\end{aligned}$$

On the other hand, it does not respect contraction. Consider the zigzag poset $Z = \{1 < 3, 2 < 3, 2 < 4\}$. If $A = \{2\}$, then $\varphi(J(Z)/_A) = J(\{1\})$ whereas $\varphi(J(Z))/_A = J(\{3 < 1, 4\})$.

Suppose C_n is the naturally ordered chain on $[n]$, so that $J(C_n)$ is the chain $\emptyset \subset [1] \subset [2] \subset \cdots \subset [n]$. Observe that $\mathrm{Min}(C_n) = \{1\}$, so the good fracturings are disjoint sums of chains $Q = Q_1 + \cdots + Q_u$ such that $1 \in Q_1$ and the blocks of Q give a natural set composition Ψ of a subset

$V \subseteq [n]$. In other words $\Psi = \Psi_1|\dots|\Psi_u \models V$ is such that if $i < j$ with $x \in \Psi_i$ and $y \in \Psi_j$, then $x < y$. As a result we can write out a cancellation-free antipode formula for C_n.

Proposition 4.19. *Suppose* $(J(C_n),[m]) \in \mathbf{LOI}[[m]]$. *Then,*

$$S(J(C_n),[m]) = (-1)^{m-n} \sum_{\substack{V\subseteq[n]:\\ 1\in V}} (-1)^{n-|V|} \sum_{\substack{\Psi=\Psi_1|\cdots|\Psi_u\models V\\ \textit{natural}}} (-1)^u (J(C_{\Psi_1}),\Psi_1) * \cdots * (J(C_{\Psi_u},\Psi_u),[n])$$

Example 4.20. Suppose $P = A_n$ is the antichain with elements $[n]$. Then $\mathrm{Min}(A_n) = [n]$, so the only good fracturing of A_n is A_n itself; the components are its singleton subsets. We have $u = n$ and $k = 0$, so

$$S(J(A_n),E) = (-1)^{n+p}(J(A_n),E)$$

where p is the number of phantom elements of $(J(A_n),E)$,i.e., $|E \setminus [n]|$. Moreover, $J(A_n)$ is just the Boolean algebra with atoms $[n]$, so this formula describes the antipode in the Hopf submonoid **Bool**.

Example 4.21. For more complex posets, the family of good fracturings can have complex structure, making it hard to write down a more concrete formula than that of Theorem 4.15. For instance, consider the *complete ranked poset*: a poset P with ground set $E = E_1 \cup \cdots \cup E_v$ such that every element in E_k is less than every element in E_{k+1}. (Equivalently, P is an ordinal sum of antichains.) The posets C_n from Example 4.19 are complete ranked posets with $E_k = \{k\}$.

Let us describe a good fracturing $Q = Q_1 + \cdots + Q_u$ of a complete ranked poset P. Observe that $\mathrm{Min}(P) = E_1$, thus $E_1 \subseteq Q$. Moreover, the requirement that $\mathrm{Con}_P(Q)$ be acyclic naturally induces a total ordering (with ties allowed) on the components of Q, as we now show. Define

$$a_i = \min\{k : \; Q_i \cap E_k \neq \emptyset\}, \qquad b_i = \max\{k : \; Q_i \cap E_k \neq \emptyset\}.$$

Evidently $a_i \leq b_i$ for all i. Moreover, it cannot be the case that both $a_i > b_j$ and $a_j > b_i$, or else Q_i

and Q_j would form a 2-cycle in the conflict digraph $\mathrm{Con}(Q)$. Therefore, either

$$a_i \leq b_i \leq a_j \leq b_j \qquad \text{or} \qquad a_j \leq b_j \leq a_i \leq b_i.$$

(Conversely, every fracturing Q that admits such a relation on its components is acyclic.) Therefore, we have a complete transitive relation on the pairs (a_i, b_i), which we may as well assume is the natural ordering.

Pictorially, consider a $u \times n$ rectangular grid in which the box in position (i, j) is shaded in if and only if $Q_i \cap E_j \neq \emptyset$. The shading pattern is a *ribbon*: no square is strictly southwest of any other. Columns other than the first are allowed to be empty, but rows must be nonempty. (See Figure 4.4 for an example.) Each ribbon diagram gives rise to many good fracturings, determined by filling each box in the kth column of the ribbon with one or more elements of E_k (using no element more than once, and using all elements of E_1). The antipode of P can thus be expressed as a sum over such fillings.

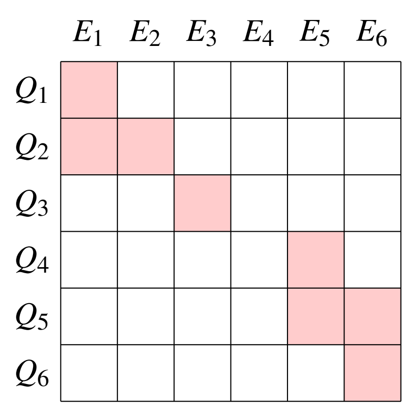


Figure 4.4: Distributing elements of P into a good fracturing.

It appears to be very difficult to give a self-contained description of good fracturings of posets that are not complete ranked.

Example 4.22. For a non-complete-ranked-poset the situation is harder to work out. The smallest example is the zigzag poset P (see Figure 1.1).

If we look at the 15 good fracturings of P (Figure 4.5) we see that the situation is a bit messier than Example 4.21. Since 3 has two potential betrayers and 4 has only a single potential betrayer intuition would suggest that 3 should be betrayed more often. But looking at the fracturings that appear, 3 is betrayed in only 3 cases whereas 4 is betrayed in 5 cases. If we look on the level of ordered set partitions our intuition is rewarded. For 41 of the 75 ordered set compositions 3 is betrayed whereas 4 is betrayed in only 31 of the 75 ordered set compositions.

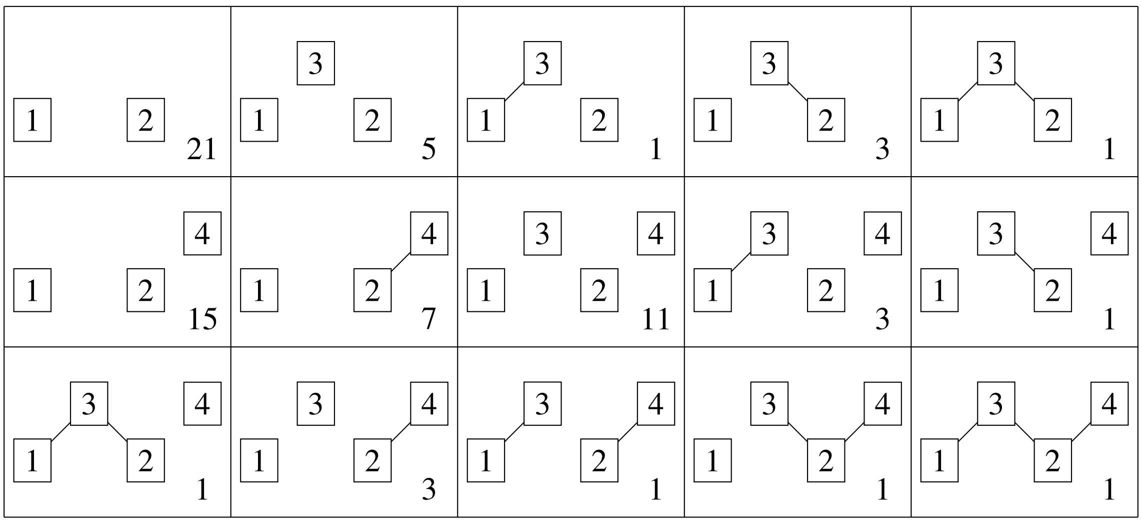


Figure 4.5: The good fracturings of P. The cardinality of $\mathrm{Supp}_P^P(Q)$ is given in the bottom right corner of each box.

4.1 Ordinal sums

As we saw in Example 4.22, working out the good fracturings of a generic poset can be messy. To that end it will be of service to have ways to deal with some of the common structures that appear in posets. Recall that the *ordinal sum* $P_{\mathrm{lo}} \oplus P_{\mathrm{hi}}$ of two posets P_{lo} and P_{hi} is constructed from $P_{\mathrm{lo}} + P_{\mathrm{hi}}$ by adding the relations $x < y$ for all $x \in P_{\mathrm{lo}}$ and $y \in P_{\mathrm{hi}}$. It turns out that we can classify a good fracturing Q of the ordinal sum into one of two cases. Before doing so we have to introduce a new type of fracturing. Further we will assume unless otherwise stated that we are working with $(J(P), P)$ since we can use Corollary 4.16 to adapt any results in the case where phantoms exist.

Definition 4.23. Let Q be a good fracturing of $P = P_{\text{lo}} \oplus P_{\text{hi}}$. First, we say that Q is **pure** if $Q = Q_{\text{lo}} + Q_{\text{hi}}$, where Q_{lo} is a good fracturing of P_{lo} and Q_{hi} is an acyclic fracturing of P_{hi}. (That is, every component of Q is a subposet either of P_{lo} or of P_{hi}.) Second, we say that Q is **mixed** if it has a component H such that $H \cap P_{\text{lo}} \neq \emptyset$ and $H \cap P_{\text{hi}} \neq \emptyset$. In this case H is a **hybrid component**. Note that Q can have at most one hybrid component, since any two such would form a 2-cycle in the conflict digraph, so we may write $Q = Q_{\text{lo}} + Q_{\text{hi}} + H$, where Q_{lo} (resp., Q_{hi}) is the subposet consisting of components contained in P_{lo} (resp., Q_{hi}).

Evidently every good fracturing is either pure or mixed, so

$$\begin{aligned} S(J(P),P) &= \sum_{Q \in \text{Good}(P)} (-1)^{c(Q)+|P \setminus Q|} (J(Q),P) \\ &= \sum_{\text{pure } Q} (-1)^{c(Q)+|P \setminus Q|} (J(Q),P) + \sum_{\text{mixed } Q} (-1)^{uc(Q)+|P \setminus Q|} (J(Q),P) \end{aligned}$$

where as before $c(Q)$ is the number of components of Q.

Proposition 4.24 (Classification of pure fracturings)**.** *Let $P = P_{\text{lo}} \oplus P_{\text{hi}}$ where $P_{\text{lo}} \neq \emptyset$. Then*

$$\sum_{\textit{pure } Q} (-1)^{c(Q)+|P \setminus Q|} (J(Q),P) = S(J(P_{\text{lo}}),P_{\text{lo}}) * \sum_{Q_{\text{hi}} \in \text{Acyc}(P_{\text{hi}})} (-1)^{c(Q_{\text{hi}})+|P_{\text{hi}} \setminus Q_{\text{hi}}|} (J(Q_{\text{hi}}),P_{\text{hi}}). \tag{4.12}$$

Proof. If Q is pure, then $Q = Q_{\text{lo}} + Q_{\text{hi}}$ where Q_{lo} is a good fracturing of P_{lo} and Q_{hi} is an acyclic fracturing of P_{hi}. The conflict digraph $\text{Con}_P(Q)$ is formed from the acyclic digraph $\text{Con}_{P_{\text{lo}}}(Q_{\text{lo}}) + \text{Con}_{P_{\text{hi}}}(Q_{\text{hi}})$ by adding edges from every component of Q_{hi} to every component of Q_{lo}, but not vice versa (see Figure 4.6); in particular it too is acyclic. Moreover, $\text{Min}(P) = \text{Min}(P_{\text{lo}})$, so Q is in fact a good fracturing of P. Therefore

$$
\begin{aligned}
&\sum_{\text{pure } Q} (-1)^{c(Q)+|P\setminus Q|}(J(Q),P) \\
&\qquad = \sum_{Q_{\text{lo}}\in \text{Good}(P_{\text{lo}})} \sum_{Q_{\text{hi}}\in \text{Acyc}(P_{\text{hi}})} (-1)^{c(Q_{\text{lo}}+Q_{\text{hi}})+|(P_{\text{lo}}\sqcup P_{\text{hi}})\setminus(Q_{\text{lo}}\sqcup Q_{\text{hi}})|}(J(Q_{\text{lo}}+Q_{\text{hi}}),P_{\text{lo}}\sqcup P_{\text{hi}}) \\
&\qquad = \sum_{Q_{\text{hi}}\in \text{Acyc}(P_{\text{hi}})} \sum_{Q_{\text{lo}}\in \text{Good}(P_{\text{lo}})} (-1)^{c(Q_{\text{lo}})+c(Q_{\text{hi}})+|P_{\text{lo}}\setminus Q_{\text{lo}}|+|P_{\text{hi}}\setminus Q_{\text{hi}}|}(J(Q_{\text{lo}}),P_{\text{lo}})*(J(Q_{\text{hi}}),P_{\text{hi}}) \\
&\qquad = \sum_{Q_{\text{hi}}\in \text{Acyc}(P_{\text{hi}})} (-1)^{c(Q_{\text{hi}})+P_{\text{hi}}\setminus Q_{\text{hi}}} \sum_{Q_{\text{lo}}\in \text{Good}(P_{\text{lo}})} (-1)^{c(Q_{\text{lo}})+|P_{\text{lo}}\setminus Q_{\text{lo}}|}(J(Q_{\text{lo}}),P_{\text{lo}})*(J(Q_{\text{hi}}),P_{\text{hi}}) \\
&\qquad = S(J(P_{\text{lo}}),P_{\text{lo}}) * \sum_{Q_{\text{hi}}\in \text{Acyc}(P_{\text{hi}})} (-1)^{c(Q_{\text{hi}})+|P_{\text{hi}}\setminus Q_{\text{hi}}|}(J(Q_{\text{hi}}),P_{\text{hi}}). \qquad \square
\end{aligned}
$$

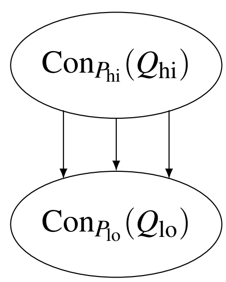


Figure 4.6: The conflict graph of a pure fracturing.

Proposition 4.25 (Classification of mixed fracturings)**.** *Suppose $P = P_{\text{lo}} \oplus P_{\text{hi}}$ is a poset. Further suppose $Q = Q_{\text{lo}} + Q_{\text{hi}} + H$ is a mixed fracturing of P, as in Definition 4.23. Then Q is a good fracturing of P if and only if*

1. *the induced subposet $Q|_{Q\cap P_{\text{lo}}}$ is a good fracturing of P_{lo};*

2. *the induced subposet $Q|_{Q\cap P_{\text{hi}}}$ is an acyclic fracturing of P_{hi};*

3. *$H \cap P_{\text{lo}}$ is a order filter in $Q \cap P_{\text{lo}}$;*

4. *$H \cap P_{\text{hi}}$ is a order ideal in $Q \cap P_{\text{hi}}$.*

Proof. First note that $\mathrm{Min}(Q) = \mathrm{Min}(P_{\mathrm{lo}}) \subseteq P_{\mathrm{lo}} \cap Q$. To show that $Q \cap P_{\mathrm{lo}}$ and $Q \cap P_{\mathrm{hi}}$ are acyclic, observe that we obtain $\mathrm{Con}_{P_{\mathrm{lo}}}(Q|_{Q\cap P_{\mathrm{lo}}})$ from $\mathrm{Con}_P(Q)$ by removing all components whose restriction to P_{lo} is empty. Thus $\mathrm{Con}_{P_{\mathrm{lo}}}(Q|_{Q\cap P_{\mathrm{lo}}})$ is isomorphic to a subgraph of $\mathrm{Con}(Q)$ and thus $\mathrm{Con}_{P_{\mathrm{lo}}}(Q|_{Q\cap P_{\mathrm{lo}}})$ is acyclic. Likewise, $\mathrm{Con}_{P_{\mathrm{lo}}}(Q|_{Q\cap P_{\mathrm{hi}}})$ is isomorphic to a subgraph of $\mathrm{Con}_P(Q)$ and thus $\mathrm{Con}_{P_{\mathrm{lo}}}(Q|_{Q\cap P_{\mathrm{hi}}})$ is acyclic. Hence $Q|_{Q\cap P_{\mathrm{lo}}}$ is a good fracturing and $Q|_{Q\cap P_{\mathrm{hi}}}$ is an acyclic fracturing of P_{hi}.

To show that $H \cap P_{\mathrm{lo}}$ is an order filter of $Q \cap P_{\mathrm{lo}}$, suppose that $a, b \in P_{\mathrm{lo}}$ such that $a < b$, $a \in H$ and b belongs to some component Y of Q. Then $\mathrm{Con}_P(Q)$ has an edge $Y \to H$. Since $H \cap P_{\mathrm{hi}} \neq \emptyset$ we also know there is an edge $H \to Y$. Since $\mathrm{Con}_P(Q)$ is acyclic, it must be the case that $H = Y$ and thus $b \in H \cap P_{\mathrm{lo}}$. Hence $X \cap P_{\mathrm{lo}}$ is an order filter of $Q \cap P_{\mathrm{lo}}$. A similar argument shows that $H \cap P_{\mathrm{hi}}$ is an order ideal of $Q \cap P_{\mathrm{hi}}$. Thus we have verified that conditions (1)–(4) are necessary for Q to be a good fracturing of P.

Conversely, suppose Q is a fracturing of P with a unique hybrid component H and that conditions (1)–(4) hold. Since $\mathrm{Min}(P) = \mathrm{Min}(P_{\mathrm{lo}}) \subseteq Q|_{Q\cap P_{\mathrm{lo}}} \subset Q$, we need only show that $\mathrm{Con}_P(Q)$ is acyclic.

Consider the induced subdigraphs $G_{\mathrm{lo}} = \mathrm{Con}_{P_{\mathrm{lo}}}(Q_{\mathrm{lo}})$ and $G_{\mathrm{hi}} = \mathrm{Con}_{P_{\mathrm{hi}}}(Q_{\mathrm{hi}})$. As subdigraphs of an acyclic digraph, G_{lo} and G_{hi} are acyclic. Furthermore, we claim that $\mathrm{Con}_P(Q)$ consists of the disjoint union $G_{\mathrm{lo}} + G_{\mathrm{hi}}$ together with the vertex H and the edges

$$\{B_{\mathrm{hi}} \to B_{\mathrm{lo}},\ H \to B_{\mathrm{lo}},\ B_{\mathrm{hi}} \to H \ \mid\ B_{\mathrm{lo}} \in G_{\mathrm{lo}},\ B_{\mathrm{hi}} \in G_{\mathrm{hi}}\}.$$

Indeed, the graph thus constructed is a subgraph of $\mathrm{Con}(Q)$ since every edge comes from a relation between two components of Q. Since every element of P_{hi} is greater than every element of P_{lo}, there are no edges of the form $B_{\mathrm{lo}} \to B_{\mathrm{hi}}$. There are no edges of the form $B_{\mathrm{lo}} \to H$ by assumption (3), and no edges of the form $H \to B_{\mathrm{hi}}$ by assumption (4). Hence the constructed graph must be $\mathrm{Con}_P(Q)$. Thus Q is a good fracturing of P. $\square$

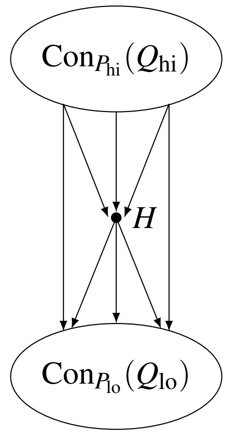


Figure 4.7: The conflict graph of a mixed fracturing.

As a consequence of Proposition 4.25,

$$\sum_{\text{mixed } Q} (-1)^{c(Q)+|P\setminus Q|}(J(Q),P)$$
$$= \sum_{H\in \mathrm{Hyb}(P_{\mathrm{lo}},P_{\mathrm{hi}})} \sum_{\substack{Q_{\mathrm{lo}}\in \mathrm{Good}(P_{\mathrm{lo}}\setminus \lceil H\rceil)\\ Q_{\mathrm{hi}}\in \mathrm{Acyc}(P_{\mathrm{hi}}\setminus \lfloor H\rfloor)}} (-1)^{c(Q_{\mathrm{lo}})+c(Q_{\mathrm{hi}})+1+|P|-|Q_{\mathrm{lo}}|-|Q_{\mathrm{hi}}|-|H|}(J(Q_{\mathrm{lo}}\oplus H\oplus Q_{\mathrm{hi}}),P). \tag{4.13}$$

where $\mathrm{Hyb}(P_{\mathrm{lo}},P_{\mathrm{hi}})$ denotes the set of induced subposets H of P that intersect both P_{lo} and P_{hi} (thus, potential hybrid components of a mixed fracturing).

Combining (4.12) and (4.13) yields the following cancellation-free formula for the antipode of the ordinal sum of two posets:

Theorem 4.26. *Suppose $P = P_{\mathrm{lo}}\oplus P_{\mathrm{hi}}$. Then*

$$S(J(P),P) = S(J(P_{\mathrm{lo}}),P_{\mathrm{lo}}) * \sum_{Q_{\mathrm{hi}}\in \mathrm{Acyc}(P_{\mathrm{hi}})} (-1)^{c(Q_{\mathrm{hi}})+|P_{\mathrm{hi}}\setminus Q_{\mathrm{hi}}|}(J(Q_{\mathrm{hi}}),P_{\mathrm{hi}})$$
$$+ \sum_{H\in \mathrm{Hyb}(P_{\mathrm{lo}},P_{\mathrm{hi}})} \sum_{\substack{Q_{\mathrm{lo}}\in \mathrm{Good}(P_{\mathrm{lo}}\setminus \lceil H\rceil)\\ Q_{\mathrm{hi}}\in \mathrm{Acyc}(P_{\mathrm{hi}}\setminus \lfloor H\rfloor)}} (-1)^{c(Q_{\mathrm{lo}})+c(Q_{\mathrm{hi}})+1+|P|-|Q_{\mathrm{lo}}|-|Q_{\mathrm{hi}}|-|H|}(J(Q_{\mathrm{lo}}\oplus H\oplus Q_{\mathrm{hi}}),P).$$

When P is an antichain, setting $P_{\mathrm{hi}} = P$ and $P_{\mathrm{lo}} = \emptyset$ recovers Example 4.20. Moreover, if P_{lo} is a complete ranked poset and P_{hi} is an antichain, then $P = P_{\text{lo}} \oplus P_{\text{hi}}$ is a complete ranked poset, and $\text{Acyc}(P_{\text{hi}})$ is just the power set of P_{hi}. Giving a formula for complete ranked posets:

Corollary 4.27. *Suppose P is a complete ranked poset. Then $P = P_{\text{lo}} \oplus P_{\text{hi}}$ where P_{lo} is a complete ranked poset and P_{hi} is an antichain. Furthermore,*

$$S(J(P),P) = S(J(P_{\text{lo}}),P_{\text{lo}}) * \sum_{Q_{\text{hi}} \subseteq P_{\text{hi}}} (-1)^{|Q_{\text{hi}}|}(2^{Q_{\text{hi}}},P_{\text{hi}})$$
$$+ \sum_{H \in \text{Hyb}(P_{\text{lo}},P_{\text{hi}})} \sum_{Q_{\text{lo}} \in \text{Good}(P_{\text{lo}} \setminus \lceil H \rceil)} \sum_{Q_{\text{hi}} \subseteq P_{\text{hi}} \setminus \lfloor H \rfloor} (-1)^{c(Q_{\text{lo}})+1+|(P_{\text{lo}} \setminus H) \setminus Q_{\text{lo}}|+|P_{\text{hi}} \setminus H|}(J(Q_{\text{lo}} \oplus H \oplus Q_{\text{hi}}),P).$$

Note that this formula is tautological when $P_{\text{lo}} = P$.

Corollary 4.27 can be applied recursively until P is an antichain, when the antipode is easily evaluated as in Example 4.20. By comparison to the discussion of complete ranked posets in Example 4.21, the recursive step here corresponds to removing the rightmost column from the diagram in Figure 4.4. Hybrid components can arise from rows with more than one shaded square; more specifically, an element of $\text{Hyb}(P_{\text{lo}},P_{\text{hi}})$ arises from a row with shaded squares in columns corresponding to both of P_{lo} and P_{hi}. For example, if we put $P_{\text{hi}} = E_v$ as in Corollary 4.27, then the hybrid component in the ribbon in Figure 4.4 is Q_5.

Example 4.28. The chain C_n is an ordered sum of n 1-element posets, so we can use the foregoing discussion to analyze its antipode. We can regard C_n either as $[1,n-1] \oplus \{n\}$ or as $\{1\} \oplus [2,n]$, where intervals are equipped with the natural ordering.

First, let $P_{\text{lo}} = [1,n-1]$ and $P_{\text{hi}} = \{n\}$. Then the elements of $\text{Hyb}(P_{\text{lo}},P_{\text{hi}})$ are of the form $A \oplus \{n\}$ where $A \subseteq [1,n-1]$. Using the equation in Corollary 4.27 we obtain

$$
\begin{aligned}
S(J(C_n),[n]) &= S(J(C_{n-1}),[n-1]) * \sum_{Q_{\text{hi}} \subseteq \{n\}} (-1)^{|Q_{\text{hi}}|}(2^{Q_{\text{hi}}}, P_{\text{hi}}) \\
&\quad + \sum_{A \subseteq C_{n-1}} \sum_{Q_{\text{lo}} \in \text{Good}(C_{n-1} \setminus \lceil H \rceil)} (-1)^{c(Q_{\text{lo}})+1+|(C_{n-1} \setminus A) \setminus Q_{\text{lo}}|} (J(Q_{\text{lo}} \oplus A \oplus \{n\}), Q_{\text{lo}} \sqcup A \sqcup \{n\}) \\
&= S(J(C_{n-1}),[n-1]) * ((\{\emptyset\},\{n\}) - (\{\emptyset,\{n\}\},\{n\})) \\
&\quad + \sum_{A \subseteq C_{n-1}} (-1)^{n-|A|} \sum_{Q_{\text{lo}} \in \text{Good}(C_{n-1} \setminus \lceil A \rceil)} (-1)^{c(Q_{\text{lo}})-|Q_{\text{lo}}|} (J(Q_{\text{lo}} \oplus A \oplus \{n\}), Q_{\text{lo}} \sqcup A \sqcup \{n\}).
\end{aligned}
$$

The first sum includes those good fracturings for which n is either a single component or is omitted. The second (double) sum includes those good fracturings for which n belongs to a component of size at least 2.

Next, let $P_{\text{lo}} = \{1\}$ and $P_{\text{hi}} = [2,n]$. Then the elements of $\text{Hyb}(P_{\text{lo}}, P_{\text{hi}})$ are of the form $\{1\} \oplus A$ where $A \subseteq [2,n]$. Once again, using the equation in Theorem 4.26, we obtain

$$
\begin{aligned}
S(J(C_n)) &= (-1)^1(2^{\{1\}},\{1\}) * \sum_{Q_{\text{hi}} \in \text{Acyc}([2,n])} (-1)^{c(Q_{\text{hi}})+|P_{\text{hi}} \setminus Q_{\text{hi}}|} (J(Q_{\text{hi}}), P_{\text{hi}}) \\
&\quad + \sum_{A \subseteq [2,n]} \sum_{Q_{\text{hi}} \in \text{Acyc}([2,n] \setminus \lfloor A \rfloor)} (-1)^{1+c(Q_{\text{hi}})+|([2,n] \setminus A) \setminus Q_{\text{hi}}|} (J(\{1\} \oplus A \oplus Q_{\text{hi}}), \{1\} \sqcup A \sqcup Q_{\text{hi}}) \\
&= -(\{\emptyset,\{1\}\},\{1\}) * \sum_{Q_{\text{hi}} \in \text{Acyc}([2,n])} (-1)^{c(Q_{\text{hi}})+|P_{\text{hi}} \setminus Q_{\text{hi}}} J(Q_{\text{hi}}) \\
&\quad + \sum_{A \subseteq [2,n]} (-1)^{n-|A|} \sum_{Q_{\text{hi}} \in \text{Acyc}([2,n] \setminus \lfloor A \rfloor)} (-1)^{c(Q_{\text{hi}})+|Q_{\text{hi}}|} (J(\{1\} \oplus A \oplus Q_{\text{hi}}), \{1\} \sqcup A \sqcup Q_{\text{hi}}).
\end{aligned}
$$

The first includes those good fracturings for which 1 is a single component. The second (double) sum includes those good fracturings for which 1 belongs to a component of size at least 2.

Chapter 5

The Hopf submonoid Simp

In Section 3.4 we showed that the species of simplicial complexes forms a Hopf submonoid of **SF** called **Simp**. Since **Simp** is cocommutative contraction by a set A becomes restriction to the complement of A. As a result we do not have to worry about additional phantoms being formed when we take products and coproducts of members of **Simp**. Thus we shall assume that we are working with phantomless complexes since we can account for phantoms using Proposition 2.4. Even with this bane lifted from our minds we shall see that finding a cancellation-free antipode formula for **Simp** remains a non-trivial task.

5.1 The Structure of Inflations and Inflators

In order to analyze the antipode we first make some initial observations about product and coproduct in **Simp**.

Proposition 5.1. *Let A be a simplicial complex with vertices $[n]$ and $\Phi = \Phi_1|\ldots|\Phi_k \models [n]$. Then:*

1. *A is a subcomplex of $\mu_\Phi(\Delta_\Phi(A))$.*

2. *Suppose $\delta \subset \Phi_i$ for some i. Then $\delta \in A$ if and only if $\delta \in \mu_\Phi(\Delta_\Phi(A))$.*

3. *σ is a minimal nonface of $\mu_\Phi(\Delta_\Phi(A))$ if and only if (i) σ is a minimal nonface of A and (ii) $\sigma \subseteq \Phi_i$ for some i.*

Proof. Composing the product and coproduct

$$\begin{aligned}
\mu_\Phi(\Delta_\Phi(A)) &= \mu(A|_{\Phi_1} \otimes \cdots \otimes A|_{\Phi_k}) \\
&= A|_{\Phi_1} * \cdots * A|_{\Phi_k} \\
&= \langle B = B_1 \cup \cdots \cup B_k : B_i \text{ is a facet of } A|_{\Phi_i} \rangle \\
&= \{F = F_1 \cup \cdots \cup F_k : F_i \text{ is a face of } A|_{\Phi_i}\}.
\end{aligned}$$

The first two assertions are an immediate result of this observation. The final assertion follows because the minimal nonfaces of a join are precisely the minimal nonfaces of the join factors. □

Since coproduct is a restriction to the respective blocks of Φ and join is commutative, we can focus on set partitions as opposed to set compositions when dealing with the antipode of **Simp**. Specifically we can rewrite Takeuchi's formula as

$$S_{[n]}(X) = \sum_{\Phi \models n} (-1)^{|\Phi|} \mu_\Phi(\Delta_\Phi(X)) = \sum_{\Phi \in \Pi_n} (-1)^{|\Phi|} (|\Phi|)!\, \mu_\Phi(\Delta_\Phi(X)). \tag{5.1}$$

The factorial on the right hand side of (5.1) accounts for the fact that the order of blocks in a set partition is irrelevant. For example $1|23$ and $23|1$ are different set compositions, but represent the same set partition. As a reminder the collection Π_n of set partitions has a lattice structure. In particular we will make use of the meet operation of the partition lattice. Given two set partitions $\Phi, \Psi \in \Pi_n$ their meet $\Phi \wedge \Psi$ is the coarsest common refinement of Φ and Ψ,

$$\Phi \wedge \Psi = \{\Phi_i \cap \Psi_j : \Phi_i \in \Phi, \Psi_j \in \Psi, \Phi_i \cap \Psi_j \neq \emptyset\}.$$

Definition 5.2. Suppose A and B are simplicial complexes on vertex set $[n]$. The *support system* of B with respect to A is

$$\mathrm{Supp}_A(B) = \{\Phi \in \Pi_n : \ B = \mu_\Phi(\Delta_\Phi(A))\}.$$

If $\mathrm{Supp}_A(B) \neq \emptyset$ then we say that B is an *inflation* of A. We denote the set of inflations of A by $\mathrm{Inf}(A)$ Further if $\Phi \in \mathrm{Supp}_A(B)$, then we say that Φ *inflates* A into B.

The notation $\mathrm{Supp}_A(B)$ bears resemblance to the notation $\mathrm{Supp}_P(Q)$ used in Chapter 4. Grouping (5.1) further we have

$$\begin{aligned} S_{[n]}(X) &= \sum_{\Phi \in \Pi_n} (-1)^{|\Phi|}(|\Phi|)!\ \mu_\Phi(\Delta_\Phi(X)) \\ &= \sum_{B \in \mathrm{Inf}(A)} B \underbrace{\left(\sum_{\Phi \in \mathrm{Supp}_A(B)} (-1)^{|\Phi|}(|\Phi|)! \right)}_{c_B}. \end{aligned} \tag{5.2}$$

Thus if we can understand which complexes B are inflations of A as well as the elements of $\mathrm{Supp}_B(A)$, then we can compute c_B in (5.2). The remainder of this section is devoted to understanding the combinatorics of $\mathrm{Inf}(A)$ and $\mathrm{Supp}_B(A)$. We shall start our journey by understanding the structure of $\mathrm{Supp}_B(A)$ as it pertains to $\Pi_{|A|}$.

Proposition 5.3. *Suppose that A and B are simplicial complex on vertex set $[n]$. Then $\mathrm{Supp}_A(B)$ is interval closed in Π_n.*

Proof. Suppose Φ, Ψ, and Ω are integer partitions such that $\Phi \in \mathrm{Supp}_A(B)$, $\Psi \in \mathrm{Supp}_A(B)$, and $\Phi \leq \Omega \leq \Psi$ when ordered by reverse refinement. Observe that

$$B = \mu_\Phi(\Delta_\Phi(A)) \supseteq \mu_\Omega(\Delta_\Omega(A)) \supseteq \mu_\Psi(\Delta_\Psi(A)) = B.$$

Thus $\mu_\Omega(\Delta_\Omega(A)) = B$. □

Proposition 5.4. *Suppose that A and B are simplicial complexes on vertex set $[n]$. Further suppose that Φ and Ψ are set partitions of $[n]$ such that $\Phi \in \mathrm{Supp}_A(B)$ and $\Psi \in \mathrm{Supp}_A(B)$. Then $\Phi \wedge \Psi \in \mathrm{Supp}_A(B)$.*

Proof. The hypothesis $\mu_\Phi(\Delta_\Phi(A)) = \mu_\Psi(\Delta_\Psi(A))$ is equivalent to

$$A_{\Phi_1} * \cdots * A_{\Phi_k} = A_{\Psi_1} * \cdots * A_{\Psi_\ell}$$

i.e.,

$$\{\sigma_1 \cup \cdots \cup \sigma_k : \ \sigma_i \in A, \ \sigma_i \subseteq \Phi_i\} = \{\tau_1 \cup \cdots \cup \tau_\ell : \ \tau_i \in A, \ \tau_i \subseteq \Psi_i\}.$$

If $\Omega = \Phi \wedge \Psi$, then both $\mu_\Phi(\Delta_\Phi(A))$ and $\mu_\Psi(\Delta_\Psi(A))$ are subcomplexes of $\mu_\Omega(\Delta_\Omega(A))$. Suppose we have a collection of faces $\{\omega_{ij} \in A_{\Phi_i \cap \Psi_j}\}$ where i ranges over $[k]$ and j ranges over $[\ell]$. We need to show that $\omega_{11} \cup \cdots \cup \omega_{1\ell} \in A_{\Phi_1}$ (and equivalently for the other $k+\ell-1$ possibilities). Note that $\omega_{1s} \in A_{\Phi_1}$ and $\omega_{1s} \in A_{\Psi_s}$ for $s \in [\ell]$. Therefore

$$\omega_{11} \cup \cdots \cup \omega_{1\ell} \in A_{\Psi_1} * \cdots * A_{\Psi_\ell} = A_{\Phi_1} * \cdots * A_{\Phi_k}.$$

Further $\omega_{11} \cup \cdots \cup \omega_{1\ell} \subset \Phi_1$, so in fact $\omega_{11} \cup \cdots \cup \omega_{1\ell} \in A_{\Phi_1}$. □

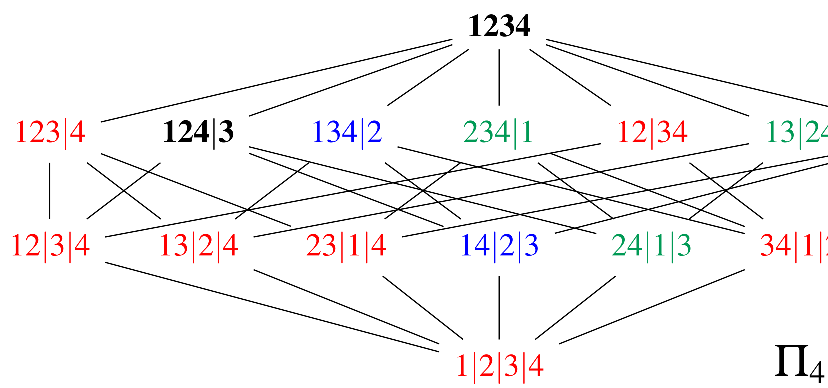
Corollary 5.5. *Suppose A and B are simplicial complexes such that $B \in \mathrm{Inf}(A)$. Then B has a unique finest decomposition as a (non-trivial) join of subcomplexes of A.*

We call the unique finest decomposition of B the *canonical join representation* of B.

Definition 5.6. If Φ is the maximally refined inflator of B, then we call Φ the *fundamental inflator* of B, written $\Phi = \mathrm{FI}_A(B)$. Further we denote the collection of fundamental inflators by

$$\mathrm{Fund}(A) = \{\Phi \in \Pi_n : \Phi = \mathrm{FI}_A(B) \text{ for some } B\}.$$

Example 5.7. Consider the simplicial complex $A = \langle 123, 34 \rangle$. If we look at the lattice of set partitions we can see canonical join representations for each of the complexes that appear in the antipode. In the diagram we use boldface to denote A, red to denote the full simplex, blue to denote $\langle 123, 234\rangle$, and green to denote $\langle 123, 134\rangle$. The respective fundamental inflators are $124|3$, $1|2|3|4$, $14|2|3$, and $24|1|3$.



We can go through and compute the antipode of A as well.

$$\begin{aligned} S(A) &= (2!-1!)A + (2\cdot 2! - 4\cdot 3! + 4!)\langle 1234\rangle + (2\cdot 2! - 3!)\langle 123, 234\rangle + (2\cdot 2! - 3!)\langle 123, 134\rangle \\ &= A + 4\langle 1234\rangle - 2\langle 123, 234\rangle - 2\langle 123, 134\rangle \end{aligned}$$

Example 5.8. Let A be the complete 0-dimensional complex on $[n]$, i.e., $A = \langle \{i\} : i \in [n]\rangle$. Note that this is the independence complex of the uniform rank-1 matroid. Suppose $\Phi \in \Pi_n$. Then $B_\Phi = \mu_\Phi(\Delta_\Phi(A))$ is the join of the 0-dimensional complexes on the blocks of Φ and

$$S(A) = \sum_{\Phi \in \Pi_n} (-1)^{|\Phi|} |\Phi|! B_\Phi.$$

Remark 5.9. Observe that every support system is isomorphic to an order ideal in a partition lattice. Specifically, $\mathrm{Supp}_A(B)$ is an order ideal in the interval $[\mathrm{FI}_A(B), \hat{1}] \subseteq \Pi_n$, which is isomorphic to the partition lattice $\Pi_{|\mathrm{FI}_A(B)|}$.

Proposition 5.10. *Suppose A and B are simplicial complexes such that $B \in \mathrm{Inf}(A)$. Then $\Psi = \Psi_1 | \ldots | \Psi_k \in \mathrm{Supp}_A(B)$ is the fundamental inflator of B if and only if $B = A_{\Psi_1} * \cdots * A_{\Psi_k}$ is the canonical join decomposition of B.*

Proof. Suppose $\Psi = \Psi_1|\dots|\Psi_k = \mathrm{FI}_A(B)$ and that the canonical join decomposition of B is $A_{\Phi_1} * \cdots * A_{\Phi_\ell}$ for $\Phi = \Phi_1|\dots|\Phi_\ell \in \Pi_n$. Our goal is to show that $\Phi = \Psi$.

Since $B = A_{\Psi_1} * \cdots * A_{\Psi_k}$ is a join decomposition of B it follows that Ψ coarsens Φ (since Φ is the canonical join decomposition).

To show that Φ coarsens Ψ we will begin with the fact that the join decomposition of Ψ is coarsened by the join decomposition of Φ.

$$\begin{aligned}
A_{\Phi_1} * \cdots * A_{\Phi_\ell} &\supseteq A_{\Psi_1} * \cdots * A_{\Psi_k} \\
&= B && \text{(since } \Psi \text{ inflates } A \text{ into } B\text{)} \\
&= B_{\Phi_1} * \cdots * B_{\Phi_\ell} && \text{(since } \Phi \in \mathrm{Supp}_A(B)\text{)} \\
&\supseteq A_{\Phi_1} * \cdots * A_{\Phi_\ell} && \text{(since } B \supseteq A\text{)}
\end{aligned}$$

so equality holds throughout. In particular Φ inflates A into B, so Φ coarsens Ψ. Hence $\Phi = \Psi$. □

Corollary 5.11. *Suppose A and B are simplicial complexes with vertex set $[n]$. Then the following are equivalent:*

1. $B \in \mathrm{Inf}(A)$;
2. $\mu_\Phi(\Delta_\Phi(A)) = B$ *for some join-decomposition* Φ *of* B;
3. $\mu_\Phi(\Delta_\Phi(A)) = B$ *for* $\Phi = \mathrm{FI}_A(B)$.

Proof. (1) $\implies$ (3) follows from Proposition 5.10. (3) $\implies$ (2) and (2) $\implies$ (1) are immediate. □

Proposition 5.12. *Suppose A and B are simplicial complexes such that $B \in \mathrm{Inf}(A)$. Then the non-singleton blocks of* $\mathrm{FI}_A(B)$ *must be non-faces of A.*

Proof. Suppose $\Phi = \mathrm{FI}_A(B)$ and that Φ_i is a non-singleton block of Φ. Then Proposition 5.10 says that A_{Φ_i} cannot be factored further as a non-trivial join of two smaller complexes (otherwise Φ would not be fundamental). Since any simplex can be decomposed into its vertices it's not possible for A_{Φ_i} to be a simplex and hence it cannot be a face of A. $\square$

At this point we have a lot of structural information about $\mathrm{Supp}_A(B)$. On the other hand it is not clear how to find information about which complexes B are inflations of A. Our best bet is to narrow our focus to structures that have a lot of symmetry such as simplex skeletons. We shall shortly see that in such a case it is possible to exploit symmetry to find the fundamental inflators (equivalently the inflations) of the complex A.

5.2 A Cancellation-Free Antipode Formula for the Join Closure of Simplex Skeletons

Given a set a vertices V, the *simplex skeleton* $\mathrm{sk}(m,V)$ is the complex whose faces are the subsets of V whose cardinality is at most m, i.e.,

$$\mathrm{sk}(m,V) = \{A \subseteq V : |A| \leq m\}.$$

For simplicity when $V = [n]$ we write $\mathrm{sk}(m,n)$. Note that $\mathrm{sk}(m,n)$ is a pure complex of dimension $m-1$. Suppose $X \subseteq [n]$. Then

$$\mathrm{sk}(m,n)|_X = \mathrm{sk}(\min(|X|,m),X). \tag{5.3}$$

Lemma 5.13. $\mathrm{sk}(m,n)$ *is join indecomposible if and only if* $m < n$.

Proof. If $m \geq n$, then $\mathrm{sk}(m,n)$ is a simplex and hence the join of vertices. On the other hand suppose $m < n$. Suppose that $\mathrm{sk}(m,n) = \mathscr{F}_1 * \cdots * \mathscr{F}_s$ where $\mathscr{F}_i$ has non-empty vertex set V_i.

Setting $d_i = \min(|V_i|, m)$ it follows from (5.3) that

$$\mathscr{F}_i = \mathrm{sk}(m,n)|_{V_i} = \mathrm{sk}(d_i, V_i).$$

Pick a facet σ_i from $\mathscr{F})_i$. Then $\sigma = \sigma_1 \cup \cdots \cup \sigma_s \in \mathrm{sk}(m,n)$. However note that

$$|\sigma| = \sum d_i > m.$$

Thus $\sigma \notin \mathrm{sk}(m,n)$ even though $\sigma \in \mathscr{F}_1 * \cdots * \mathscr{F}_s$. Therefore $\mathrm{sk}(m,n)$ must be join indecomposible. $\square$

Proposition 5.14. *$\Phi \in \Pi_n$ is a fundamental inflator of $\mathrm{sk}(m,n)$ if and only if for each block Φ_i either*

$$|\Phi_i| = 1, \tag{5.4a}$$

$$\text{or } |\Phi_i| > m. \tag{5.4b}$$

Proof. Suppose each block of $\Phi \in \Pi_n$ satisfies (5.4a) or (5.4b). Further suppose that Φ inflates $\mathrm{sk}(m,n)$ into B. Since any non-singleton block restricts to a simplex skeleton (that is not a simplex), Lemma 5.13 implies that no block of Φ can be further refined to get another partition that inflates $\mathrm{sk}(m,n)$ into B. Thus it follows that $\Phi = \mathrm{FI}_{\mathrm{sk}(m,n)}(B)$.

For the converse suppose that $\Phi \in \Pi_n$ and that Φ inflates $\mathrm{sk}(m,n)$ into B. Suppose that $1 < |\Phi_i| \leq m+1$ for some block Φ_i. Then (5.3) tells us that $\mathrm{sk}(m,n)|_{\Phi_i}$ is a simplex. Hence Φ can be further refined by breaking Φ_i into singletons to obtain a more refined partition that inflates $\mathrm{sk}(m,n)$ into B and thus $\Phi \neq \mathrm{FI}_{\mathrm{sk}(m,n)}(B)$. $\square$

We use $p_{a,b}(M)$ to denote the number of ways to partition the set M into b subsets each of whose cardinalities is at most a.

Theorem 5.15. *Suppose* $\Gamma = \mathrm{sk}(m,n)$*. Further suppose that* $\Phi = \Phi_1|\dots|\Phi_k \in \mathrm{Fund}(\Gamma)$ *has exactly* v *non-singleton blocks, say* $\Phi_{k-v+1}, \Phi_{k-v+2}, \dots, \Phi_k$*. Then*

$$S(\Gamma,[n]) = \sum_{\Phi \in \mathrm{Fund}(\Gamma)} \left(\sum_{w=0}^{k-v} (v+w)!(-1)^{v+w} p_{m+1,w}(\Phi_{k-v+1} \cup \dots \cup \Phi_k) \right) (\Gamma|_{\Phi_1} * \dots * \Gamma|_{\Phi_k}, [n]) \tag{5.5}$$

Proof. Previously in (5.2) we showed that we can group Takeuchi's formula to get a sum over inflations of Γ. Since each inflation is in one-to-one correspondence with fundamental inflators we can equivalently sum over the fundamental inflators of Γ

$$S(\Gamma,[n]) = \sum_{\Phi \in \mathrm{Fund}(\Gamma)} \underbrace{\left(\sum_{\Phi \in \mathrm{Supp}_\Gamma(\mu_\Phi \Delta_\Phi(\Gamma))} (-1)^{|\Phi|}(|\Phi|)! \right)}_{c_\Phi} \mu_\Phi \Delta_\Phi(\Gamma,[n]) \tag{5.6}$$

Fixing $\Phi \in \mathrm{Fund}(\Gamma)$ we now show that

$$c_\Phi = \sum_{w=0}^{k-v} (v+w)!(-1)^{v+w} p_{m+1,w}(\Phi_{k-v+1} \cup \dots \cup \Phi_k). \tag{5.7}$$

Suppose that $\Phi = \mathrm{FI}_\Gamma(H)$. As a consequence of Proposition 5.14, the only way to obtain another inflator Θ that inflates Γ into H is to merge the singleton blocks of Γ. We must take care not to merge more than $m+1$ singleton blocks together otherwise Θ will be a fundamental inflator (different from Φ) as a result of Proposition 5.14. In summary, we need to know how many ways we can partition the set $\Phi_{k-v+1} \cup \Phi_{k-v+2} \cup \dots \cup \Phi_k$ into subsets of cardinality at most $m+1$ with w blocks. This can be done $p_{m+1,w}(\Phi_{k-v+1} \cup \dots \cup \Phi_k)$ ways.

The final step is to iterate over w. If we merge the $k-v$ singletons into w blocks, then the result is a partition of $[n]$ into $v+w$ blocks. Therefore we multiply by $(v+w)!$ to account for the number of ways we can rearrange the blocks. Finally we need to include a factor of $(-1)^{v+w}$ as required by Takeuchi's formula. □

Equation (5.5) is not quite cancellation-free as the inner sum to compute the coefficient c_Φ is prone to have positive and negative terms which may cancel with one another. This further exemplifies the challenges faced when working with **Simp**.

Chapter 6

Characters

Definition 6.1. Suppose $\mathbf{H}$ is a connected Hopf monoid in vector species. A *character* ζ on $\mathbf{H}$ is a collection of linear maps

$$\zeta_I : \mathbf{H}[I] \to \mathbb{k}$$

for each finite set I subject to the conditions:

1. *Naturality*: For each bijection $\sigma : I \to J$ and $x \in \mathbf{H}[I]$, we have $\zeta_J(\mathbf{H}[\sigma](x)) = \zeta_I(x)$.

2. *Multiplicativity*: For each $I = S \sqcup T$, $x \in \mathbf{H}[S]$, and $y \in \mathbf{H}[T]$, we have $\zeta_I(x \cdot y) = \zeta_S(x)\zeta_T(y)$.

3. *Unitality*: $\zeta_\emptyset(1) = 1$.

Note that $\mathbf{H}$ must be connected in order for unitality to make sense. Recall from Section 1.3 that we assume $\mathbb{k}$ to be a field of characteristic 0. The *convolution product* between two characters χ and ϕ is the operation defined by

$$(\chi * \phi)_I(x) = \sum_{I=S\sqcup T} \chi_S(x|_S)\phi_T(x/_S). \tag{6.1}$$

It turns out that the convolution product gives us a character of $\mathbf{H}$. In order to show that $(\chi * \phi)_I$ is a character we need to check that it is multiplicative. We will use compatibility of the Hopf monoid to do this. Take $I = S \sqcup T = I = S' \sqcup T'$, as well as the intersections $A = S \cap S'$, $B = S \cap T'$, $C = T \cap S'$, and $D = T \cap T'$ (see Figure 1.8). Suppose $x \in \mathbf{H}[S]$, $y \in \mathbf{H}[T]$.

Then

$$
\begin{aligned}
(\chi * \phi)_I(x\cdot y) &= \sum_{I=S'\sqcup T'} \chi_{S'}((x\cdot y)|_{S'})\phi_{T'}((x\cdot y)/_{S'}) \\
&= \sum_{I=S'\sqcup T'} \chi_{S'}(x|_A \cdot y|_C)\phi_{T'}(x/_A \cdot y/_C) \\
&= \sum_{I=S'\sqcup T'} \chi_{S'}(x|_A)\chi_{S'}(y|_C)\phi_{T'}(x/_A)\phi_{S'}(y/_C) \\
&= \sum_{\substack{S=A\sqcup B\\ T=C\sqcup D}} \chi_A(x|_A)\chi_C(y|_C)\phi_B(x/_A)\phi_B(y/_C) = (\chi*\phi)_S(x)(\chi*\phi)_T(y).
\end{aligned}
$$

It is easy to check that the character ε defined by $\varepsilon_I = 0$ if $I \neq \emptyset$ and $\varepsilon_\emptyset(1) = 1$ acts as an identity for characters. Further if S is the antipode of $\mathbf{H}$, then $\phi \circ S = \phi^{-1}$.

Proposition 6.2. *Suppose* $\mathbf{H}$ *is a Hopf monoid and let* $\mathbb{X}(\mathbf{H})$ *be the set of characters of* $\mathbf{H}$. *Then* $\mathbb{X}(\mathbf{H})$ *forms a group under convolution product.*

6.1 Characters on Bool

Recall from Section 3.6 that the Hopf submonoid **Bool** is spanned as a vector species by set families of the form $(2^J, I)$ where $J \subseteq I$. Due to naturality and multiplicativity, every character of $\mathbb{X}(\mathbf{Bool})$ is determined by its value on the set families $(\{\emptyset,\{1\}\},[1])$ and $(\{\emptyset\},[1])$, i.e., the singleton set family with no phantoms and the set family consisting of a single phantom. Further each set family of **Bool** up to relabeling is determined by the number of non-phantoms in the family and the size of the ground set of the family. Thus every character of $\mathbb{X}(\mathbf{Bool})$ can be written in the form $\chi_{x,y}(2^J, I) = x^p y^{q-p}$ where $p = |J|$ and $q = |I|$. For brevity we define $\chi_{x,y}(p,q) := \chi_{x,y}(2^{[p]},[q])$. Note that in particular

$$
\begin{aligned}
&\chi_{x,y}(1,1) = x, \text{ and} \\
&\chi_{x,y}(0,1) = y.
\end{aligned}
$$

We can compute the convolution product of $\chi_{x,y}(p,q) = x^p y^{q-p}$ and $\chi_{r,s}(p,q) = r^p s^{q-p}$:

$$\begin{aligned}
(\chi_{x,y} * \chi_{r,s})(p,q) &= \sum_{[q]=S\sqcup T} \chi_{x,y}(|[p]\cap S|,|S|)\chi_{r,s}(|[p]\cap T|,|T|) \\
&= \sum_{m=0}^{q-p}\sum_{n=0}^{p} \binom{q-p}{m}\binom{p}{n} \chi_{x,y}(n,m+n)\chi_{r,s}(p-n,q-m-n) \qquad (6.2) \\
&= \sum_{m=0}^{q-p}\sum_{n=0}^{p} \binom{q-p}{m}\binom{p}{n} x^n y^m r^{p-n} s^{q-p-m} \\
&= \left(\sum_{n=0}^{p}\binom{p}{n} x^n r^{p-n}\right)\left(\sum_{m=0}^{q-p}\binom{q-p}{m} y^m s^{q-p-m}\right) \\
&= (x+r)^p (y+s)^{q-p}.
\end{aligned}$$

As a result we have an isomorphism $\phi : \mathbb{X}(\mathbf{Bool}) \to \mathbb{C}^2$ given by $\phi(\chi_{x,y}) = (x,y)$.

As an application we can obtain a combinatorial identity that is elementary though not obvious. Now consider taking convolution powers of $\chi_{x,y}(p,q)$. The above calculation shows us that

$$(\chi_{x,y} * \chi_{x,y})(p,q) = (2x)^p(2y)^{q-p} = 2^q x^p y^{q-p}. \qquad (6.3)$$

On the other hand setting $r = x$ and $s = y$ and interchanging the sums of (6.2) we get

$$\begin{aligned}
(\chi_{x,y} * \chi_{x,y})(p,q) &= \sum_{n=0}^{p}\sum_{m=n}^{q} \binom{q-p}{m-n}\binom{p}{n} \chi_{x,y}(n,m)\chi_{x,y}(p-n,q-m) \\
&= \sum_{n=0}^{p}\sum_{m=n}^{q-p+n} \binom{q-p}{m-n}\binom{p}{n} x^p y^{q-p} \\
&= \sum_{n=0}^{p}\sum_{k=0}^{q-p} \binom{q-p}{k}\binom{p}{n} x^p y^{q-p} \\
&= x^p y^{q-p} \sum_{n=0}^{p}\sum_{k=0}^{q-p} \binom{q-p}{k}\binom{p}{n}.
\end{aligned}$$

Combining the result with (6.3) we obtain the combinatorial identity

$$2^q = \sum_{n=0}^{p} \sum_{k=0}^{q-p} \binom{q-p}{k} \binom{p}{n} \tag{6.4}$$

for $p \leq q$.

Now let us consider how to interpret (6.4) combinatorially. Since the left hand side of this equation is 2^q we would expect that the double sum on the right somehow counts the number of subsets of $[q]$. Since each factor inside the double sum is independent of either n or k and thus we can reorganize the sums

$$\begin{aligned} 2^q &= \sum_{n=0}^{p} \sum_{k=0}^{q-p} \binom{q-p}{k} \binom{p}{n} \\ &= \sum_{n=0}^{p} \binom{p}{n} \sum_{k=0}^{q-p} \binom{q-p}{k}. \end{aligned}$$

We can interpret this product of sums as follows. In order to create a subset of $[q]$ we can first split $[q]$ into two disjoint subsets. One subset will have size p and the other will have size $q-p$. Next we pick a subset from each disjoint set. By taking the union of the chosen subsets we have now formed a subset of $[q]$. Further every subset of $[q]$ can be formed in this way.

Let us turn our attention to a slightly broader Hopf submonoid of **LOI**.

6.2 The Hopf submonoid of chain gangs

Recall that the Boolean lattices are the lattices of order ideals of antichains. We can think of an antichain as a disjoint union of chains where each chain has a single element. A poset A is a *chain gang* if A is the disjoint union of chains. The species **CG** is the subspecies of **LOI** generated by lattice of order ideals of chain gangs (with the possibility of phantoms).

Proposition 6.3. **CG** *is a Hopf submonoid of* **LOI**.

Proof. Let E and F be disjoint finite sets. Given two chain gangs $A \subset E$ and $B \subset F$ we know from Proposition 3.30 that $(J(A),E) * (J(B),F) = (J(A+B),E \sqcup F)$. Since the disjoint union of two chain gangs is a chain gang it follows that **CG** is closed under join. Recall from Section 3.8 that taking the restriction or contraction of $(J(A),E)$ will result in the lattice of order ideals of an induced subposet of A. Since an induced subposet of a chain gang is a chain gang it follows that **CG** is closed under the operations of restriction and contraction. Thus **CG** is a Hopf submonoid of **LOI**. $\square$

Since every chain gang is the disjoint union of chains it follows that lattices of order ideals of chains form a basis for the Hopf monoid **CG**. Therefore it will be beneficial to introduce notation for working with chains. We will use the following notation throughout the rest of this chapter

C_X is the chain on $X \subset [n]$ whose order is inherited from $\mathbb{N}$,

$C_X^Y = (J(C_X),Y)$ where $X \subseteq Y$,

C_n is the chain on $[n]$ with $1 < 2 < \cdots < n$,

$C_m^n = (J(C_m),[n])$, and

$C(A;F)$ is the lattice of order ideals of a chain gang A with phantom set F.

In particular C_0^n is the set family $(\{\emptyset\},[n])$, i.e., the set family consisting of n phantoms.

6.3 Characters of chain gangs

Since lattices of order ideals of chains form a basis for **CG**, naturality and multiplicativity imply that a character on **CG** is determined by its values on C_0^1 and C_n^n for all $n \geq 1$. This is akin to our analysis of $\mathbb{X}(\mathbf{Bool})$ in Section 6.1. Specifically, suppose that $C(A;F) \in \mathbf{CG}$, where A has a_n

chains of length n and $f = |F|$. Then every character in $\mathbb{X}(\mathbf{CG})$ can be written as

$$\zeta_{u,\vec{t}}(C(A;F)) = u^f t_1^{a_1} t_2^{a_2} t_3^{a_3} \dots$$

where $u, t_1, t_2, \dots \in \mathbb{C}$ and $\vec{t} = (t_1, t_2, t_3, \dots)$. Note that

$$\zeta_{u,\vec{t}}(C_n^n) = t_n, \text{ and}$$
$$\zeta_{u,\vec{t}}(C_0^1) = u.$$

Proposition 6.4. *Suppose* $\zeta_{u,\vec{t}}, \zeta_{v,\vec{s}} \in \mathbb{X}(\mathbf{CG})$. *Then*

$$(\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_m^n) = (u+v)^{n-m} \left(s_m + \sum_{q=1}^{m} s_{q-1} \sum_{b=0}^{m-q} \binom{m-q}{b} t_{b+1} v^{m-q-b} \right) \tag{6.5}$$

Proof. Suppose that $\zeta_{u,\vec{t}}, \zeta_{v,\vec{s}} \in \mathbb{X}(\mathbf{CG})$. Multiplicativity tells us that

$$(\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_m^n) = \left((\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_0^1)\right)^{n-m} \left((\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_m^m)\right)$$

Thus we can evaluate $(\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_0^1)$ and $(\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_m^m)$ individually before multiplying them together in order to get the desired result. For the first computation we have

$$(\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_0^1) = \sum_{A \in \{\emptyset, \{1\}\}} \zeta_{u,\vec{t}}(C_0^1|_A) \zeta_{v,\vec{s}}(C_0^1/_A) = u + v. \tag{6.6}$$

The second computation gives

$$\begin{aligned}
(\zeta_{u,\vec{t}} * \zeta_{v,\vec{s}})(C_m^m) &= \sum_{A\subseteq[m]} \zeta_{u,\vec{t}}(C_m^m|_A)\zeta_{v,\vec{s}}(C_m^m/_A) \\
&= \sum_{A\subseteq[m]} \zeta_{u,\vec{t}}(C_{|A|}^{|A|})\zeta_{v,\vec{s}}(C_{\min(A)-1}^{m-|A|}) \\
&= \sum_{A\subseteq[m]} t_{|A|}v^{m-|A|-\min(A)+1}s_{\min(A)-1} \\
&= s_m + \sum_{q=1}^{m} \sum_{\substack{B\subseteq[q+1,m] \\ A=B\cup\{q\}}} t_{|B|+1}v^{m-q-|B|}s_{q-1} \\
&= s_m + \sum_{q=1}^{m} s_{q-1} \sum_{b=0}^{m-q} \binom{m-q}{b} t_{b+1}v^{m-q-b}. \qquad (6.7)
\end{aligned}$$

Multiplying (6.6) and (6.7) gives the desired result. $\square$

Observe that when $u = v = 0$ then the right hand side of (6.5) will vanish if $m < n$. In other words if we input a set family with phantoms, then the resulting character will return 0. Further $\zeta_{0,\vec{t}} * \zeta_{0,\vec{s}} = \zeta_{0,\vec{r}}$ where $r_n = (\zeta_{0,\vec{t}} * \zeta_{0,\vec{s}})(C_n^n)$. Therefore:

Corollary 6.5. *The set $\mathbb{EX}(\mathbf{CG})$ of characters of the form $\zeta_{0,\vec{t}}$ is a subgroup of $\mathbb{X}(\mathbf{CG})$. We call $\mathbb{EX}(\mathbf{CG})$ the* exorcism subgroup *of $\mathbb{X}(\mathbf{CG})$.*

Aguiar and Ardila showed that the character group of permutahedra in **GP** is isomorphic to the multiplicative group of exponential formal power series [1, Thm. 9.2]. The exorcism subgroup of $\mathbb{X}(\mathbf{CG})$ gives us an analogous result for **CG**.

Definition 6.6. A power series $\sum t_n x^n$ is *unital* if $t_0 = 1$.

Theorem 6.7. *The subgroup $\mathbb{EX}(\mathbf{CG})$ is isomorphic to the multiplicative group U of unital power series.*

Proof. Let $\zeta_{0,\vec{t}}, \zeta_{0,\vec{s}} \in \mathbb{EX}(\mathbf{CG})$. In the expression for $(\zeta_{0,\vec{t}} * \zeta_{0,\vec{s}})(C_m^m)$ given by (6.5), the inner sum yields zero except when $m-q-b=0$. Thus

$$
\begin{aligned}
(\zeta_{0,\vec{t}} * \zeta_{0,\vec{s}})(C_m^m) &= s_m + \sum_{q=1}^{m} s_{q-1} t_{m-q-1} \\
&= \sum_{q=0}^{m} s_q t_{m-q}
\end{aligned}
$$

where $s_0 = t_0 = 1$. In other words, $\zeta_{0,\vec{t}} * \zeta_{0,\vec{s}} = \zeta_{0,\vec{r}}$ where $r_m = \sum_{q=0}^{m} s_q t_{m-q}$. Observe that if $f(x) = \sum s_n x^n$ and $g(x) = \sum t_n x^n$, then

$$
f(x)g(x) = \sum_{m=0}^{\infty} \sum_{q=0}^{m} s_q t_{m-q} x^m = \sum_{m=0}^{\infty} r_m x^m.
$$

Therefore the map $\phi : \mathbb{EX}(\mathbf{CG}) \to U$ defined by

$$
\phi(\zeta_{0,\vec{t}}) = 1 + \sum_{m=1}^{\infty} t_m x^m
$$

is an isomorphism. □

6.4 Inverting Formal Power Series

Given a character $\zeta_{0,\vec{s}} \in \mathbb{EX}(\mathbf{CG})$ we know that $\zeta_{0,\vec{s}}^{-1} = \zeta_{0,\vec{s}} \circ S$ where S is the antipode. Further Theorem 6.7 implies that inverting $\zeta_{0,\vec{s}}$ should be akin to inverting power series. To that end we shall first review how one might invert a power series without the aid of the antipode.

6.4.1 Inverting by Elementary Means

An elementary method to finding the multiplicative inverse of a formal power series uses systems of equations. Suppose $f(x) = \sum_n t_n x^n$. Computing $1/f(x)$ is equivalent to finding s_n for $n \geq 0$ such

that
$$\left(\sum_{n=0}^{\infty} s_n x^n\right)\left(\sum_{n=0}^{\infty} t_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} s_{n-k}t_k\right)x^n = 1.$$

Thus we can set up a system of $n+1$ equations in order to solve for s_k where $0 \leq k \leq n$

$$\begin{aligned}
s_n t_0 + s_{n-1}t_1 + \cdots + s_0 t_n &= 0\\
s_{n-1}t_0 + s_{n-1}t_1 + \cdots + s_0 t_{n-1} &= 0\\
&\vdots\\
s_1 t_0 + s_0 t_1 &= 0\\
s_0 t_0 &= 1.
\end{aligned}$$

Writing the system in terms of matrices

$$\begin{bmatrix} t_0 & t_1 & \dots & t_{n-1} & t_n \\ 0 & t_0 & \dots & t_{n-2} & t_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & t_0 & t_1 \\ 0 & 0 & \dots & 0 & t_0 \end{bmatrix}\begin{bmatrix} s_n \\ s_{n-1} \\ \vdots \\ s_1 \\ s_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

If we assume that $t_0 = 1$ as in Theorem 6.7, then the determinant of the matrix on the left hand side is 1. Applying Cramer's Rule to solve for s_n we obtain

$$s_n = \begin{vmatrix} 0 & t_1 & \dots & t_{n-1} & t_n \\ 0 & 1 & \dots & t_{n-2} & t_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & t_1 \\ 1 & 0 & \dots & 0 & 1 \end{vmatrix} = (-1)^n \det \underbrace{\begin{bmatrix} t_1 & \dots & t_{n-1} & t_n \\ 1 & \dots & t_{n-2} & t_{n-1} \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 1 & t_1 \end{bmatrix}}_{A}. \tag{6.8}$$

The second equality is obtained by expanding along the first column. Thus we can compute s_n as a determinant of a matrix.

Alternatively, we can approach the problem using geometric series and integer compositions. Let

$$f(x) = 1 + \sum_{k=1}^{\infty} t_k x^k = 1 + g(x).$$

Then

$$\begin{aligned} \frac{1}{f(x)} = \frac{1}{1+g(x)} &= \frac{1}{1-(-g(x))} \\ &= 1 + \sum_{r=1}^{\infty} (-g(x))^r \\ &= 1 + \sum_{r=1}^{\infty} \left(-\sum_{k=1}^{\infty} t_k x^k \right)^r \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{\alpha=(\alpha_1,\alpha_2,\dots,\alpha_r)\models n} (-1)^r \prod_{i=1}^{r} t_{\alpha_i} \right) x^n. \end{aligned} \tag{6.9}$$

We explain why (6.9) and (6.8) are equivalent. Let $a_{i,j} = t_{j-i+1}$ denote the entries of the matrix A defined in (6.8). Let $\alpha = (\alpha_1, \dots \alpha_k)$ be an integer composition of of n. We claim that there is exactly one permutation σ such that

$$\prod_{\ell=1}^{n} a_{\ell,\sigma_\ell} = \prod_{i=1}^{k} t_{\alpha_i}, \tag{6.10}$$

namely $\sigma = (\sigma_1, \ldots, \sigma_n)$, where

$$\sigma_i = \begin{cases} \alpha_1 & \text{if } i = 1; \\ \alpha_1 + \cdots + \alpha_j & \text{if } i = \alpha_1 + \cdots + \alpha_{j-1} + 1; \\ i - 1 & \text{otherwise.} \end{cases} \tag{6.11}$$

One can check that (6.10) is satisfied by σ. In addition, $\operatorname{sgn}(\sigma) = \prod(-1)^{\alpha_i - 1} = (-1)^{n-k}$. As an example consider $n = 6$ and $\alpha = (2,1,1,2)$. We want find the σ associated with the term $t_2 t_1 t_1 t_2$. By (6.11) $\sigma = (213465)$. The elements a_{i,σ_i} have been highlighted in Figure 6.4.1.

$$\begin{bmatrix} t_1 & \color{red}{t_2} & t_3 & t_4 & t_5 & t_6 \\ \color{red}{1} & t_1 & t_2 & t_3 & t_4 & t_5 \\ 0 & 1 & \color{red}{t_1} & t_2 & t_3 & t_4 \\ 0 & 0 & 1 & \color{red}{t_1} & t_2 & t_3 \\ 0 & 0 & 0 & 1 & t_1 & \color{red}{t_2} \\ 0 & 0 & 0 & 0 & \color{red}{1} & t_1 \end{bmatrix}$$

Thus far we have shown that every term from (6.12) appears a term in the determinant (6.8). Now we want to show that these are the only non-zero terms in (6.8). If we we expand out the determinant we end up with

$$\begin{aligned} \det(a_{i,j}) &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{\ell=1}^{n} a_{\ell,\sigma_\ell} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{\ell=1}^{n} t_{\sigma_i - i + 1}. \end{aligned}$$

Notice that when $\sigma_i - i + 1 < 0$ that σ contributes nil to the determinant. Thus the only non-zero contributions come from $\sigma \in \mathfrak{S}_n$ in which $\sigma_i \geq i - 1$ for all $i \in [n]$.

Suppose we try to construct a $\sigma \in \mathfrak{S}_n$ that gives a nonzero summand in the determinant. For

σ_1 we have our choice of t_j where $j \in [n]$. If $\sigma_1 = 1$, then we reduce down to an $(n-1) \times (n-1)$ matrix of the same form as the original $n \times n$ matrix. We then have our choice of $\sigma_2 = m$ where $2 \leq m \leq n$. On the other hand if $\sigma_1 > 1$ then we are forced to pick $\sigma_2 = 1$ as picking $\sigma_m = 1$ for $m > 2$ would give $a_{m,\sigma_m} = 0$. By a similar logic we must pick $\sigma_i = i-1$ for $2 \leq i < \sigma_1$. At this point we've now reduced down to selecting the remaining terms of σ from an $(n-\sigma_1) \times (n-\sigma_1)$ matrix.

The resulting σ thus takes the form of (6.11). We can obtain the integer composition α by first letting $i_1 < \cdots < i_k$ be the sequence of indices such that $\sigma_{i_j} > i_j - 1$. If we set $\alpha_j = \sigma_{i_j} - \sigma_{i_{j-1}}$ (with $\alpha_1 = \sigma_1$), then we have recovered the integer partition we seek. Succinctly what we have shown is that if $\sigma \in \mathfrak{S}_n$, then $\sigma_i \geq i-1$ for all i if and only if σ satisfies (6.11) for some integer partition α of n.

Thus we have shown that the non-zero terms in the determinant expansion (6.8) are precisely the terms from (6.9).

6.4.2 Inverting with Characters

Utilizing the antipode formula from Proposition 4.19 we can obtain the n^{th} coefficient of the inverse power series by computing $\zeta_{\vec{t}}^{-1}(C_n^n)$. Doing this we obtain a result equivalent to (6.9).

Proposition 6.8. *Suppose $f(x) = \sum_{n=0}^{\infty} t_n x^n$ is a power series such that $t_0 = 1$. Then*

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} s_n x^n$$

where

$$s_n = \sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\models n} (-1)^k \prod_{i=1}^{k} t_{\alpha_i}. \tag{6.12}$$

Proof. Recall from Proposition 4.19 the antipode formula

$$S(C_n^n) = \sum_{\substack{V\subseteq[n]:\\ 1\in V}} (-1)^{n-|V|} \sum_{\substack{\Psi=\Psi_1|\cdots|\Psi_u\models V\\ \text{natural}}} (-1)^u (C_{\Psi_1}^{\Psi_1} * \cdots * C_{\Psi_u}^{\Psi_u}, [n])$$

where as before $C_{\Psi_i}^{\Psi_i} = (J(C_{\Psi_i}), \Psi_i)$. We can compute the values of $\zeta_{0,\vec{t}}^{-1}$ as follows:

$$\begin{aligned}
\zeta_{0,\vec{t}}^{-1}(C_n^n) = \zeta_{0,\vec{t}}(S(C_n^n)) &= \sum_{\substack{V\subseteq[n]:\\ 1\in V}} (-1)^{n-|V|} \sum_{\substack{\Psi=\Psi_1|\cdots|\Psi_u\models V\\ \text{natural}}} (-1)^u \zeta_{0,\vec{t}} \left(C_{\Psi_1}^{\Psi_1} * \cdots * C_{\Psi_u}^{\Psi_u}, [n] \right) \\
&= \sum_{\substack{\Psi=\Psi_1|\cdots|\Psi_u\models [n]\\ \text{natural}}} (-1)^u \prod_{i=1}^{u} t_{|\Psi_i|} \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_k)\\ \alpha_i>0\\ \sum\alpha_j=n}} (-1)^k \prod_{i=1}^{k} t_{\alpha_i}.
\end{aligned}$$

□

6.5 Applications to Symmetric Functions

Chain gangs are intimately intertwined with symmetric functions. First we can use (6.12) to unravel the relations between the bases of h_n's and e_n's discussed in Section 1.6. In addition we shall see that the Hopf algebra of symmetric functions from Section 1.7 is a quotient of the Hopf algebra of chain gangs.

6.5.1 Applications to h_n and e_n

Recall from Section 1.6 that $E(t)H(-t) = 1$ where

$$E(t) = \sum e_k t^k = \prod_{i \geq 1} (1 + t x_i), \text{ and}$$
$$H(t) = \sum h_k t^k = \prod_{i \geq 1} \frac{1}{1 - t x_i}.$$

We can also use (6.8) and write

$$e_n = \det(h_{i-j+1}), \tag{6.13}$$
$$h_n = \det(e_{i-j+1}). \tag{6.14}$$

If we note that h_n is the Schur function s_λ where $\lambda \vdash n$ has a single part and e_n is the Schur function $s_{\lambda'}$ where $\lambda' \vdash n$ has n parts, then equations (6.13) and (6.14) can be obtained by traditional means with [19, Theorem 7.16.1].

Using the inversion formula (6.12) we obtain formulae to compute the h_n's and e_n's in terms of one another

$$h_n = (-1)^n \sum_{\alpha = (\alpha_1, \ldots, \alpha_k) \models n} (-1)^k \prod_{i=1}^{k} e_{\alpha_i} \tag{6.15}$$
$$e_n = (-1)^n \sum_{\alpha = (\alpha_1, \ldots, \alpha_k) \models n} (-1)^k \prod_{i=1}^{k} h_{\alpha_i}. \tag{6.16}$$

6.5.2 The Hopf Algebra of Chain Gangs

We shall now take some time and look at the Hopf algebra $CG = \bar{\mathscr{K}}(\mathbf{CG})$.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash [n]$ and p be a non-negative integer. Define $C(\lambda, p)$ to be the unlabelled chain gang with chains of lengths λ_i and p phantoms. The degree k piece of CG has vector space basis $\{C(\lambda, p) : |\lambda| + p = k\}$. The operations of product and coproduct are inherited from the Hopf monoid as described in (1.12). In other words we pick a representative from each isomorphism class and perform the Hopf operations on the level of the monoid. Specifically, suppose $C(\lambda,p)$ and $C(\nu,q)$ are chain gangs in CG. The product will be

$$C(\lambda,p)\cdot C(\nu,q) = C(\lambda\cup\nu, p+q) \tag{6.17}$$

where $\lambda\cup\nu$ is the multiset union of λ and ν, sorted in decreasing order.

Since coproduct is a morphism of algebras it suffices to compute $\Delta(C((),1))$ and $\Delta(C((n),0))$. The former is simply

$$\Delta(C((),1)) = 1\otimes C((),1) + C((),1)\otimes 1. \tag{6.18}$$

The latter is given by

$$\begin{aligned}
\Delta(C((n),0)) &= \sum_{S\subseteq[n]} [C_n^n|_S]\otimes[C_n^n/_S] \\
&= \sum_{S\subseteq[n]} [C_{|S|}^{|S|}]\otimes[C_{\min(S)-1}^{n-|S|}] \\
&= \sum_{S\subseteq[n]} C((|S|),0)\otimes C((\min(S)-1), n-|S|-\min(S)+1) \\
&= 1\otimes C((n),0) + \sum_{q=1}^{n}\sum_{b=0}^{n-q}\binom{n-q}{b} C((b+1),0)\otimes C((q-1), n-q-b).
\end{aligned} \tag{6.19}$$

Note that $q=\min(S)$ and $b+1=|S|$. Given $\lambda=(\lambda_1,\ldots,\lambda_\ell)\vdash n$ and a non-negative integer p we can compute the coproduct of $C(\lambda,p)$ as

$$\Delta(C(\lambda,p)) = \Delta(C((\lambda_1),0)*\cdots*\Delta(C((\lambda_\ell),0))*(\Delta((),1))^p \tag{6.20}$$

where $C((\lambda_i),0)$ and $C((),1)$ can be computed using (6.18) and (6.19).

Recall that in the monoid **CG** the operation of contraction may introduce phantoms. Specifically if we expand (6.20) with (6.19) we see that phantoms will appear if λ has at least one part of size at least 2.

Proposition 6.9. *Let $\widehat{CG}$ be the vector subspace of CG spanned by $\{C(\lambda,0)\}$ where λ runs over all partitions. Then $\widehat{CG}$ is a subalgebra of CG isomorphic to Λ, but not a subcoalgebra of CG.*

Proof. Consider the map $f : \Lambda \to \widehat{CG}$ defined by $f(e_\lambda) = C(\lambda,0)$ and extended linearly to all of Λ. Note this includes $f(e_0) = C(0,0) = 1$. Then

$$\begin{aligned} f(e_\lambda e_\nu) &= f(e_{\lambda\cup\nu}) \\ &= C(\lambda\cup\nu,0) = C(\lambda,0)\cdot C(\nu,0) = f(e_\lambda)\cdot f(e_\nu) \end{aligned}$$

as desired. On the other hand $\widehat{CG}$ is not closed under coproduct. For example

$$\Delta(C((2),0)) = C((2),0) + C((1),0)\otimes C((1),0) + C((1),0)\otimes C((),1) + 1\otimes C((2),0). \quad (6.21)$$

Note that the term in red has a phantom. □

Theorem 6.10. *Let $\mathfrak{F}$ be the vector space spanned by $\{C(\lambda,p) :\ p>0\}$. Then $\mathfrak{F}$ is a Hopf ideal and $CG/\mathfrak{F} \cong \Lambda$.*

Proof. Suppose $C(\lambda,p) \in \mathfrak{F}$ and $C(\nu,q) \in CG$. Then $C(\lambda\cup\nu, p+q) \in \mathfrak{F}$. Thus $\mathfrak{F}$ is an ideal of CG. Considering the summands of (6.20) along with what we know about restriction and contraction it follows that $[C(A;F)|_S]$ or $[C(A;F)/_S]$ will have at least one phantom. Therefore $\Delta\mathfrak{F} \subseteq \mathfrak{F}\otimes CG + CG\otimes\mathfrak{F}$ and $\mathfrak{F}$ is a coideal. Since $CG/\mathfrak{F}$ is a commutative bialgebra it follows from Theorem 1 of [14] that $\mathfrak{F}$ is a Hopf ideal. Given $X \in CG$ we will use the convention $\overline{X}$ for the image of X in $CG/\mathfrak{F}$.

Taking the quotient $CG/\mathfrak{F}$ has the effect of killing off any terms of CG that contain phantoms. Thus the chain gangs $\overline{C(\lambda,0)}$ where λ ranges over all integer partitions, form a graded basis for the quotient. Consider the map $f : \Lambda \to CG/\mathfrak{F}$ defined by $f(e_\lambda) = \overline{C(\lambda,0)}$ and extended linearly. Showing that product is preserved under f follows the same steps from Proposition 6.9.

Suppose $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Then (6.17) and the fact that coproduct is an algebra morphism gives

$$\Delta(\overline{C(\lambda, 0)}) = \Delta(\overline{C((\lambda_1), 0)} * \cdots * \Delta(\overline{C((\lambda_\ell), 0)}).$$

Taking the quotient has the effect of removing any terms of (6.19) that contain phantoms. The only terms that survive are those that occur when $b = n - q$. Specifically for $C((\lambda_i), 0)$,

$$\Delta(\overline{C((\lambda_i), 0)}) = \sum_{q=0}^{\lambda_i} \overline{C((q), 0)} \otimes \overline{C((\lambda_i - q), 0)}. \tag{6.22}$$

Thus

$$\begin{aligned}
\Delta(f(e_{\lambda_i})) = \Delta(\overline{C((\lambda_i), 0)}) &= \sum_{q=0}^{\lambda_i} \overline{C((q), 0)} \otimes \overline{C((\lambda_i - q), 0)} \\
&= \sum_{q=0}^{\lambda_i} f(e_q) \otimes f(e_{\lambda_i - q}) \\
&= f\left(\sum_{q=0}^{\lambda_i} e_q \otimes e_{\lambda_i - q}\right) \\
&= f(\Delta(e_{\lambda_i}))
\end{aligned}$$

and as such

$$\Delta(f(e_\lambda)) = f(\Delta(e_\lambda)).$$

□

6.6 Geometric Characters on Chain Gangs

At this point we have seen that the character group $\mathbb{X}(\mathbf{CG})$ contains a subgroup isomorphic to formal power series whose degree zero term is unity under multiplication. We might ask if there are any other characters of note in $\mathbb{X}(\mathbf{CG})$.

Definition 6.11. A character $\zeta_{u,\vec{t}}$ is a *geometric character* if $t_n = r^n$ for a scalar r. We denote such a character by $\gamma_{u,r}$.

Proposition 6.12. *Fix $u, v, x, y \in \mathbb{C}$ and consider the geometric characters $\gamma_{u,x}, \gamma_{v,y} \in \mathbb{X}(\mathbf{CG})$. Then*

$$(\gamma_{u,x} * \gamma_{v,y})(C_m^n) = (u+v)^{n-m}\left(y^m + x\left(\frac{(x+v)^m - y^m}{x+v-y}\right)\right). \tag{6.23}$$

Proof. Substituting in the appropriate variables into (6.5) we obtain

$$\begin{aligned}
(\gamma_{u,x} * \gamma_{v,y})(C_m^n) &= (u+v)^{n-m}\left(y^m + \sum_{q=1}^{m} y^{q-1} \sum_{b=0}^{m-q} \binom{m-q}{b} x^{b+1} v^{m-q-b}\right) \\
&= (u+v)^{n-m}\left(y^m + \frac{x}{y} \sum_{q=1}^{m} y^{q} \sum_{b=0}^{m-q} \binom{m-q}{b} x^{b} v^{m-q-b}\right) \\
&= (u+v)^{n-m}\left(y^m + \frac{x}{y} \sum_{q=1}^{m} y^{q} (x+v)^{m-q}\right) \\
&= (u+v)^{n-m}\left(y^m + x(x+v)^{m-1}\left(\frac{1-\left(\frac{y}{x+v}\right)^m}{1-\frac{y}{x+v}}\right)\right) \\
&= (u+v)^{n-m}\left(y^m + x\left(\frac{(x+v)^m - y^m}{x+v-y}\right)\right)
\end{aligned}$$

as desired. □

Considering the right hand side of (6.23) we see that geometric characters fail to form a subgroup of $\mathbb{X}(\mathbf{CG})$. By restricting our attention to specific families of geometric characters we see that not all hope is lost.

Corollary 6.13. *The geometric characters of the form $\gamma_{u,u}$ give a subgroup of $\mathbb{X}(\mathbf{CG})$ isomorphic to $\mathbb{C}$.*

Proof. Given $\gamma_{u,u}$ and $\gamma_{v,v}$ it follows

$$\begin{aligned}
(\gamma_{u,u} * \gamma_{v,v})(C_m^n) &= (u+v)^{n-m}\left(v^m + u\left(\frac{(u+v)^m - v^m}{u+v-v}\right)\right) \\
&= (u+v)^{n-m}\left(v^m + (u+v)^m - v^m\right) \\
&= (u+v)^n.
\end{aligned}$$

It follows that $\phi(\gamma_{u,u}) = u$ gives the desired isomorphism. □

Consequently we see that $\gamma_{u,u}^{-1} = \gamma_{-u,-u}$. We can compare this to inverting $\gamma_{u,u}$ using the antipode.

Proposition 6.14. *Given the chain C_m^n*

$$1 = \sum_{Q \in \mathrm{Good}(C_m^n)} (-1)^{c(Q)+|Q|} \tag{6.24}$$

where $c(Q)$ is the number of components of Q.

Proof. Using the fact that $\gamma_{-u,-u} = \gamma_{u,u}^{-1} = \zeta \circ S$ we end up with

$$\begin{aligned}
\gamma_{u,u}^{-1}(C_m^n) &= \zeta(S(C_m^n)) \\
(-u)^n &= \sum_{Q \in \mathrm{Good}(C_m^n)} (-1)^{c(Q)+n-|Q|} \gamma_{u,u}(J(Q),[n]) \\
&= \sum_{Q \in \mathrm{Good}(C_m^n)} (-1)^{c(Q)-|Q|}(-1)^n u^n \\
&= (-u)^n \sum_{Q \in \mathrm{Good}(C_m^n)} (-1)^{c(Q)-|Q|}.
\end{aligned}$$

Dividing both sides by $(-u)^p$ yields the desired identity. □

The above results could be viewed as looking at the characters $\gamma_{u,ux}$ where $x = 1$. It is natural then to ask what happens if we fix $u = 1$ and vary x.

Proposition 6.15. *Suppose $\gamma_{1,x}$, $\gamma_{1,y}$ are geometric characters. Then*

$$\gamma_{1,x} * \gamma_{1,y} = \gamma_{1,y-1} * \gamma_{1,x+1}.$$

Proof. By setting $H_n(x,y) = \sum_{k=0}^{n} x^k y^{n-k}$, then we can simplify (6.23) as

$$\begin{aligned}
(\gamma_{u,x} * \gamma_{v,y})(C_m^n) &= (u+v)^{n-m}\left(y^m + x\left(\frac{(x+v)^m - y^m}{x+v-y}\right)\right) \\
&= (u+v)^{n-m}(y^m + xH_{m-1}(x+v,y)) \\
&= (u+v)^{n-m}\Big(y^m + (x+v)H_{m-1}(x+v,y) - vH_{m-1}(x+v,y)\Big) \\
&= (u+v)^{n-m}\Big(H_m(x+v,y) - vH_{m-1}(x+v,y)\Big) \\
&= (u+v)^{n-m}\Big(H_m(x+v,y) - vH_{m-1}(x+v,y)\Big).
\end{aligned}$$

If we set $u = v = 1$, then

$$\begin{aligned}
(\gamma_{1,x} * \gamma_{1,y})(C_m^n) &= (2u)^{n-m}\Big(H_m(x+1,y) - H_{m-1}(x+1,y)\Big) \\
&= (2u)^{n-m}\Big(H_m(y,x+1) - H_{m-1}(y,x+1)\Big) \\
&= (\gamma_{1,y-1} * \gamma_{1,x+1})(C_m^n).
\end{aligned}$$

□

Chapter 7

Open Questions and Future Directions

At this point we have barely scratched the surface of **SF**. Our focus has been on **LOI** and **Simp** which lie at the base of Figure 3.2. Even at that altitude there are many unanswered questions. We will finish by discussing some of the possible avenues for future research.

7.1 The Antichain Detection Character on LOI

Consider the following parametrized family of characters

$$\alpha_{x,y}(J(P),E) = \begin{cases} x^{|P|}y^{f} & \text{if } P \text{ is an antichain,} \\ 0 & \text{otherwise.} \end{cases}$$

We use f to denote the number of phantoms, i.e., $|E \setminus P|$. We can compute the inverse of α. In the following computation we will use $p = |P|$ and $m = |\mathrm{Min}(P)|$.

$$
\begin{aligned}
\alpha_{x,y}^{-1}(J(P),E) &= \sum_{\substack{Q\in\mathrm{Good}(P)\\ Q\text{ antichain}}} (-1)^{c(Q)+|P\setminus Q|+f} x^{|Q|} y^{f+|P\setminus Q|} \\
&= (-1)^{|P|+f} y^{|P|+f} \sum_{\substack{Q\in\mathrm{Good}(P)\\ Q\text{ antichain}}} \left(\frac{x}{y}\right)^{|Q|} \\
&= (-y)^{p+f} \sum_{k=0}^{p-m} \binom{p-m}{k} \left(\frac{x}{y}\right)^{m+k} \\
&= (-y)^{p+f} \left(\frac{x}{y}\right)^{m} \sum_{k=0}^{p-m} \binom{p-m}{k} \left(\frac{x}{y}\right)^{k} \\
&= (-y)^{p+f} \left(\frac{x}{y}\right)^{m} \left(1+\frac{x}{y}\right)^{p-m} \\
&= (-y)^{p+f} x^m y^{-p} (x+y)^{p-m}.
\end{aligned}
$$

The character $\mu = \alpha_{-1,1}$ is the Möbius function of $J(P)$[20, §3.9]. In this special case $x+y=0$ which at first glance makes it seem like $\mu^{-1}=0$. An exception arises when $p-m=0$, i.e., when P is an antichain. In this case there is only one good fracturing (namely P) and we get a value of $(-1)^f$. Therefore

$$
\alpha_{-1,1}^{-1}(J(P)) = \begin{cases} (-1)^f & \text{if } P \text{ is an antichain,} \\ 0 & \text{otherwise.} \end{cases}
$$

Note that $\mu^{-1}(J(P))$ is just $\alpha_{1,-1}(J(P))$. If $\beta = \alpha_{2,-1}$ we end up with $\beta^{-1}(J(P)) = 2^m$ We could also consider $\beta = \alpha_{1,1}$ in which case

$$
\beta^{-1}(J(P)) = (-1)^{f+p} 2^{p-m}.
$$

Can we understand the k^{th} convolution power of $\alpha_{x,y}$? For example

$$\alpha_{x,y}^2(J(P)) = \sum_{E=S\sqcup T} \alpha_{x,y}(J(P)|_S)\alpha_{x,y}(J(P)/_S).$$

This question seems to require a balancing act between S and T. We need enough elements in S to make sure that $J(P)/_S$ is an antichain, on the other hand we still need $J(P)|_S$ to be an antichain.

7.2 Characters of Simp

In a conversation with Mark Denker it was suggested that working $\mathbb{X}(\mathbf{Simp})$ might yield tangible results. This lies in the observation that **Simp** is cocommutative and thus 6.1 can be written as

$$(\chi * \phi)_I(x) = \sum_{I=S\sqcup T} \chi_S(x|_S)\phi_T(x|_T).$$

Thus in the convolution product we only need to know how characters are affected by restriction. This has its benefits. In Section 6.3 we had to keep track of phantoms formed by contraction. This is not the case in $\mathbb{X}(\mathbf{Simp})$ since restriction does not introduce new phantoms.

For a quick example of the types of characters that appear in $\mathbb{X}(\mathbf{Simp})$ consider

$$\alpha(\Gamma, E) = \begin{cases} 1, \text{ if } \Gamma \text{ is dimension } 0 \\ 0, \text{ otherwise.} \end{cases}$$

Then $\alpha^k(\Gamma, E)$ is akin to properly coloring the 1-skeleton with exactly k colors.

At this time we only have a "nice" antipode formula for independence complexes of uniform matroids. Thus our ability to invert characters in $\mathbb{X}(\mathbf{Simp})$ is currently limited.

7.3 A Question About AMat

Castillo, Martin, and Samper defined a Hopf monoid **OMat** on ordered matroids[5]. Further Gillespie has a paper [9] in which Gillespie constructs an antimatroid from an ordered matroid. José Samper asked if Gillespie's construction might yield a Hopf morphism from **OMat** to **AMat**. We have not had a chance to delve into this question. Even if the answer is negative, it could be that Gillespie's construction yields a Hopf morphism from **OMat** to White's monoid mentioned in Remark 3.28.

7.4 Generalizing from LOI to AMat

For now we will conclude with a short discussion about generalizing the cancellation-free formula for the antipode in **LOI** to the larger Hopf submonoid **AMat**. This seems like a fitting conclusion as it was antimatroids that first sparked the inspiration for **SF**.

Let us begin by first reviewing some terminology from Section 3.7. An *antimatroid* is a set family $(\mathscr{F}, E)$ such that:

- if $X, Y \in \mathscr{F}$, then $X \cup Y \in \mathscr{F}$, and
- if $X \in \mathscr{F}$ and $X \neq \emptyset$, then there exists $x \in X$ such that $X \setminus \{x\} \in \mathscr{F}$.

If $X \in \mathscr{F}$ then we say that X is a *feasible set*.

Suppose $X \subseteq E$. Then X is called *free* if $\mathscr{F}|_X = 2^X$. A *circuit* is a minimal non-free set. That is to say that $X \subseteq E$ is a circuit if $\mathscr{F}|_X \neq 2^X$, but $\mathscr{F}|_Y = 2^Y$ for every proper subset $Y \subsetneq X$. Given a circuit $C \subseteq E$ we say that $r \in C$ is a *root* of C if $F|_C = 2^C \setminus \{\{r\}\}$. Every circuit has a unique root [3, §8.7.C]. Given a circuit C with root r we denote the *rooted circuit* by the pair (C, r). An antimatroid is determined by its collection of rooted circuits. Given the rooted circuits of an antimatroid we can precisely determine when a set is feasible.

Proposition 7.1 (Refer to [3, Proposition 8.7.11]). *Let $(\mathscr{F},E)$ be an antimatroid and $A \subseteq E$. then $A \in \mathscr{F}$ if and only if $C \cap A \neq \{r\}$, for every rooted circuit (C,r).*

Our first step when determining a cancellation-free antipode formula for **LOI** or **Simp** was to group like terms of Takeuchi's formula. We will adapt our previous notation accordingly. Let $A = (\mathscr{F},E) \in$ **AMat**. Given $B \in$ **AMat**$[E]$ define the *support system* of B with respect to A as

$$\mathrm{Supp}_A(B) = \{\Phi \models E : B = \mu_\Phi(\Delta_\Phi(A))\}.$$

We want to know when $\mathrm{Supp}_A(B) \neq \emptyset$. In Chapter 4 this was done using betrayal. Recall that if P is a poset, $(J(P),E) \in$ **LOI**, and $\Phi \models E$, then $x \in P$ is betrayed by $y \in P$ if $y <_P x$ and $y <_\Phi x$. Further recall from Proposition 3.33 that every circuit of the antimatroid $(J(P),E)$ takes the form $(\{a,b\},b)$. Specifically $(\{a,b\},b)$ is a circuit of $(J(P),E)$ if and only if $a <_P b$. Thus it seems reasonable that betrayal could be adapted to antimatroids as a whole by looking at the circuits of an antimatroid. Let us start with an example.

Example 7.2. Consider the set family $A = (2^{[3]} \setminus \{\{3\}\}, [3])$. Observe that A is a member of **AMat** but not a member of **LOI**. Further the only circuit of A is $([3],3)$. Using Takeuchi's formula

$$S(A) = -A + (\{\emptyset,1,2,12,23,123\},[3]) + (\{\emptyset,1,2,12,13,123\},[3]) - (2^{[2]},[3]) - (2^{[3]},[3]) \quad (7.1)$$

with

$$\begin{aligned} \mathrm{Supp}_A(A) &= \{123\}, \\ \mathrm{Supp}_A(\{\emptyset,1,2,12,23,123\}) &= \{1|23\}, \\ \mathrm{Supp}_A(\{\emptyset,1,2,12,13,123\}) &= \{2|13\}, \\ \mathrm{Supp}_A(2^{[2]}) &= \{12|3, 1|2|3, 2|1|3\}, \\ \mathrm{Supp}_A(2^{[3]}) &= \{13|2, 23|1, 3|12, 1|3|2, 2|3|1, 3|1|2, 3|2|1\}. \end{aligned}$$

Note that $\mathrm{Supp}_A(2^{[2]})$ is the only term of (7.1) for which "betrayal" occurs. Furthermore the elements $\mathrm{Supp}_A(2^{[2]})$ are those for which the root of $([3],3)$ occurs in a block after all of the non-root elements of the circuit. We posit that this is the correct generalization of betrayal to antimatroids.

Definition 7.3. Suppose $(\mathscr{F},E) \in \mathbf{AMat}$. We say that $r \in E$ is *betrayed* with respect to $\Phi = (\Phi_1,\ldots,\Phi_k) \models E$ if there exists a rooted circuit (C,r) of $(\mathscr{F},E)$ such that

$$r \in \Phi_i \text{ and} \tag{7.2}$$

$$C \setminus \{\{r\}\} \subseteq \Phi_1 \cup \cdots \cup \Phi_{i-1}. \tag{7.3}$$

Akin to the definition of betrayal for **LOI** we say that C betrays a.

Proposition 7.4. *Suppose $A = (\mathscr{F},E) \in \mathbf{AMat}$, that $r \in \bigcup_{F \in \mathscr{F}} F$, and that $\Phi = (\Phi_1,\ldots,\Phi_k) \models E$. If r is betrayed with respect to Φ, then r is a phantom of $\mu_\Phi(\Delta_\Phi(A))$.*

Proof. Suppose that r is betrayed by C with respect to Φ. Coassociativity of **AMat** implies that the i^{th} factor of $\Delta_\Phi(A)$ is

$$(A|_{\Phi_i})/_{\Phi_1\cup\cdots\cup\Phi_{i-1}} = (\{F \cap \Phi_i : F \in \mathscr{F} \text{ and } F \cap (\Phi_1 \cup \cdots \cup \Phi_{i-1}) = \emptyset\}, \Phi_i). \tag{7.4}$$

Suppose that $r \in F$ such that $F \cap (\Phi_1 \cup \cdots \cup \Phi_{i-1}) = \emptyset$. Then $F \cap C = \emptyset$ and by Proposition 7.1 it follows that $F \notin \mathscr{F}$. Thus r appears in no member of $(A|_{\Phi_i})/_{\Phi_1\cup\cdots\cup\Phi_{i-1}}$. Hence r must be a phantom of $\mu_\Phi(\Delta_\Phi(A))$. ☐

Conjecture 7.5. *Suppose $A = (\mathscr{F},E) \in \mathbf{AMat}$, that $r \in \bigcup_{F \in \mathscr{F}} F$, and that $\Phi = (\Phi_1,\ldots,\Phi_k) \models E$. If r is a phantom of $\mu_\Phi(\Delta_\Phi(A))$, then r is betrayed with respect to Φ.*

References

[1] Marcelo Aguiar and Federico Ardila. Hopf monoids and generalized permutahedra. Preprint, arXiv:1709.07504, 2017.

[2] Nicolás Andruskiewitsch and Walter Ferrer Santos. The beginnings of the theory of Hopf algebras. *Acta Appl. Math.*, 108(1):3–17, 2009.

[3] Anders Björner and Günter M. Ziegler. Introduction to greedoids. In *Matroid applications*, volume 40 of *Encyclopedia Math. Appl.*, pages 284–357. Cambridge Univ. Press, Cambridge, 1992.

[4] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953.

[5] Federico Castillo, Jeremy L. Martin, and José A. Samper. Hopf monoids of ordered simplicial complexes. Preprint, arXiv:2011.14955, 2020.

[6] Brenda L. Dietrich. Matroids and antimatroids—a survey. *Discrete Math.*, 78(3):223–237, 1989.

[7] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. in Math.*, 63(3):301–316, 1987.

[8] Stefan Geschke. Convex open subsets of $\mathbb{R}^n$ are homeomorphic to n-dimensional open balls. math.uni-hamburg.de/home/geschke/papers/ConvexOpen.pdf, retrieved July 2, 2021, 2012.

[9] Bryan R. Gillespie. Convexity in ordered matroids and the generalized external order. *Electron. J. Combin.*, 27(3):Paper No. 3.41, 20, 2020.

[10] Brandon Humpert and Jeremy L. Martin. The incidence Hopf algebra of graphs. *SIAM J. Discrete Math.*, 26(2):555–570, 2012.

[11] Bernhard Korte, László Lovász, and Rainer Schrader. *Greedoids*, volume 4 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 1991.

[12] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition.

[13] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[14] Warren D. Nichols. Quotients of Hopf algebras. *Comm. Algebra*, 6(17):1789–1800, 1978.

[15] James Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[16] Alex Postnikov, Victor Reiner, and Lauren Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.

[17] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.

[18] Gian-Carlo Rota. Hopf algebra methods in combinatorics. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 363–365. CNRS, Paris, 1978.

[19] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[20] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[21] Jacob White. Coloring complexes and combinatorial Hopf monoids. preprint, arXiv:1611.04079, 2020.