Obstruction Theory in Algebra and Topology: A Homotopy Perspective

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Abstract

Based on the Homotopy theorem of Madhav V. Nori on smooth vector bundles and his Homotopy question on algebraic vector bundles [M2], we develop the theories of topological and algebraic obstructions as follows.

1. Let M be a *smooth manifold* of *dimension* d and $\mathscr{V}$ be a *smooth vector bundle* of *rank* n over M. We define an *obstruction set* $\pi_0\left(\mathcal{L}O(\mathscr{V})\right)$, to be called **Nori homotopy set**, and an *obstruction class* $\varepsilon(\mathscr{V}) \in \pi_0\left(\mathcal{L}O(\mathscr{V})\right)$. Then, if $2n \geq d+3$, we prove that

$$\varepsilon(\mathscr{V}) = neutral \Longleftrightarrow \mathscr{V} \cong \mathscr{V}_1 \oplus \mathbb{R}$$

2. Let A be an *essentially smooth ring* of *dimension* d over an infinite *perfect field* k, with $1/2 \in k$, and P be a *projective A-module* with $rank(P) = n$. We define a similar *obstruction set* as above $\pi_0\left(\mathcal{L}O(P)\right)$, and an *obstruction class* $\varepsilon(P) \in \pi_0\left(\mathcal{L}O(P)\right)$. Then, if $2n \geq d+3$, we prove that

$$\varepsilon(P) = neutral \Longleftrightarrow P \cong Q \oplus A$$

3. Further, for *real smooth affine schemes* $X = \operatorname{Spec}(A)$, we reconcile these two theories, as follows. Let M be the *manifold of real points* in X. Let P be a projective A-module, with $rank(P) = n$. Let $\mathscr{V}_{Top}(P)$ be the smooth vector bundle over M induced by P. Then, there is a natural map

$$\pi_0\left(\mathcal{L}O(P)\right) \longrightarrow \pi_0\left(\mathcal{L}O(\mathscr{V}^\star_{Top}(P))\right)$$

where $\mathscr{V}^\star_{Top}(P)$ denote the dual bundle.

Acknowledgements

"I spent most of a lifetime trying to be a mathematician-and what did I learn? What does it take to be one? I think I know the answer: you have to be born right, you must continually strive to become perfect, you must love mathematics more than anything else, you must work at it hard and without stop, and you must never give up", said Paul Halmos, one of the greatest mathematicians of twentieth century, in his 'automathography' called *I want to be a mathematician*. All he said above holds true in my case up to a varying degree (no doubt, some can have high divergence too) and this thesis is the culmination of that process spanning over the period of last 12 years.

Born right: Thanks to my parents, Banita and Gyanananda, for many things, from giving me the right education to enduring the innumerable number of goodbyes they say every time I leave for some other city, some other country. Thanks to my brother, Manas, for being there as a pillar of support whenever I have needed him. Last but not the least many thanks to my life partner, Archana, for willing to be my companinon in this journey through thick and thin (and making me write this for her).

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In the end, I would like to state that I am still trying to be a mathematician in whatever I am doing. And this thesis hopefully is the testament to my trial being on the right track.

Contents

Chapter 1

Introduction

Can one comb a hairy ball flat without creating any swirl? The question, which has a broader appeal in the mathematical context called *parallelizability* of the *spheres* in *Euclidean spaces*, translates to the question of whether the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ have two nonvanishing *vector fields* defined on it such that they span the *tangent spaces* at each point of the sphere. More generally, can one find n vector fields on the sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ such that they span the tangent spaces at each point of the concerned sphere? Those spheres for which the answer is affirmative are called *parallelizable spheres*. For $n = 2$, it has been answered negatively in an interesting theorem called *Hairy Ball theorem*. Moreover, this has been shown using the tools of K-theory that the spheres are parallelizable only for $n =$ 0, 1, 3 and 7. Now, let us look at a toned-down version of the above question in a relatively more general set up i.e. vector bundles. Let V be a vector bundle of rank n over a smooth manifold of dimension d. Does there exist a nonvanishing section of V defined on M (*so that it generates an one-dimensional vector subspace in the corresponding fiber space at each point of the manifold*)?

The question is interesting because if we indeed have a nonvanishing section defined on M, then it can be shown with some work that V has a trivial component, i.e. $V \cong V' \oplus \mathcal{L}$ where V' is a rank-$(n-1)$ vector bundle over M and $\mathcal{L} \cong M \times \mathbb{R}$. Historically, the question has been investigated in various implicit forms and one important result in this regard is that when $rank(V) > dim(M)$, there is always a nonvanishing section of V. So, the interest of study is when $rank(V) \leq dim(M)$. Another important obstruction for splitting off a nonzero direct summand is the *Euler character-*

istics which is defined for a vector bundle V over a smooth orientable manifold M. If it vanishes when $rank(V) = dim(M)$, then V has a nonvanishing section. However, tangent bundle over the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ does not have any nonvanishing section on it and more generally this is true for $\mathbb{S}^n$ for any even number n. So, there are a number of naturally occurring bundles for which there are no nonvanishing sections on them. This begs the natural question: what is the precise obstruction for such bundles in having nonvanishing sections (or in having *nonzero trivial summands*)?

We investigate this question in detail in this thesis. Of course, as mentioned above when rank of the vector bundle is equal to the dimension of the base manifold, *Euler characteristics* acts as an effective obstruction. But for the general case, the problem is wide open. Moreover, thanks to Swan this topological question is intricately related to the study of the algebraic structure of the *finitely generated projective modules*, especially over the ring of *smooth* functions over a manifold which also we investigate here thoroughly. Let X be a *smooth connected real manifold* and $C(X)$ be the ring of real valued smooth functions on X. Swan proved in [Sw] that the category of finitely generated projective modules over $C(X)$ is *equivalent* to the category of *vector bundles of finite rank* over X ($V \mapsto \Gamma(M, V)$, the set of *global sections* on V). Thus, if a vector bundle has a trivial summand, that will imply the associated projective module has a free component (i.e. $\Gamma(M, V) \cong Q \oplus A$). Especially, when the manifold X is contractible (e.g. $\mathbb{R}^n$), vector bundles over it are trivial. So, the associated projective modules are free. Restricting the attention to only *algebraic functions* (which are 'polynomial like' functions), Serre first posed the question in late 1960's whether finitely generated projective modules over $K[X_1, \ldots, X_n]$ are free where K is a field (known as *Serre's conjecture on projective modules*).

Since then, there has been considerable research in this direction which has culminated into a full fledged theory known as ***Algebraic Obstruction Theory***. Serre's conjecture was settled affirmatively by Quillen and Suslin independently in 1976. Then, Mohan Kumar and M.P. Murthy considered the following question: Suppose A is a *smooth affine algebra* over an *algebraically* *closed field* K, with Krull dimension d. Let P be a projective module over A of rank d. Then, does the vanishing of the top Chern class of P, denoted as $C^d(P)$, ensure that P has a free component, i.e. $P = Q \oplus A$ (where Q is some A module)? The intuition was that $C^d(P)$ acts as an obstruction to have a free component and thus acts as a barrier for P to be free. This turns out to be correct as Murthy [Mu] proved that $C^d(P)$ vanishes if and only if P has a free component. Later, Madhav Nori came up with two deep ideas: one involving the homotopy program around 1990's and another one with the definition of *Euler class groups* to detect the splitting for projective A-modules P, with $rank(P) = dim(A)$ around 1995's [MS]. Then, S. Bhatwadekar and R. Sridharan [BS2] developed the concept of *Euler Class Group* which also acts as an effective obstruction set in certain nice scenarios and there has been significant research done in this line of thought. However, despite getting enough attention of the research community, the program has not turned out to be as successful as expected, except when $rank(P) = dim(A)$.

We investigate here the idea of homotopy program as proposed by Nori and as it turns out, it has deep significance both in the algebraic as well as the topological context. This is now informally referred as **Nori's homotopy program**. He considered the following question:

Nori's question (topological version): Let V be a vector bundle of rank n over a smooth manifold M of dimension d. Let s_0 be a global section of V meeting the *zero section* of V transversally in a submanifold B_0 of M and let B be a submanifold of $M \times \mathbb{R}$ that meets $M \times 0$ transversally in B_0. Now, with some work one can show that s_0 induces an isomorphism from the *normal bundle* of B_0, denoted as $N(M, B_0)$, to V restricted to N_0, i.e. there is an induced map $[s_0] : N(M, B_0) \to V|_{B_0}$ which is an isomorphism. However, let us turn the table and ask: suppose we start with an isomorphism, $\phi : N(M \times \mathbb{R}, p^\star(V)|_N)$ (where $p : M \times \mathbb{R} \to M$ is the projection map and $p^\star(V)$ is the pullback bundle), which is compatible with s_0 i.e. $\phi|_{N_0 \times 0} = [s_0]$. Then, does there exist a global section s of $p^\star(V)$ which meets the zero section of $p^\star(V)$ transversally and $s|_{M \times 0} = s_0$ such that the induced isomorphism $[s]$ is same as the isomorphism we have begun with?

Nori answered this question affirmatively when $d \leq 2n - 3$ or $B = B_0 \times \mathbb{R}$ and posed the following question which is the algebraic counterpart of above question in the context of the projective modules [M2].

Nori's question (algebraic version): Let $X = spec(A)$ be a smooth affine variety of dimension d. Let P be a projective A-module of rank n and $\tilde{s} : P \to I$ be a surjective homomorphism from P on to an ideal I of A such that the zero set $V(I) = Y$ is a smooth affine subvariety of dimension $d - n$. Also assume that $Z := V(J)$ is a smooth affine subvariety of $X \times \mathbb{A}^1 = spec(A[T])$ such that Z intersects $X \times 0$ transversally in $Y \times 0$. Moreover, suppose that $\varphi : P[T] \to J/J^2$ is a surjective map which is compatible with $\tilde{s}$, i.e. $\varphi|_{T=0} = [\tilde{s}]$ the isomorphism induced by $\tilde{s} : P/IP \to I/I^2$. Then, does there exist a surjetcive map $\psi : P[T] \to J$ such that $\psi|_{T=0} = \tilde{s}$ and $\psi|_Z = \varphi$?

This question was settled affirmatively by Bhatwadekar and Keshari when $2n \geq d + 3$ [BK]. They used the theorem of Mandal and Verma which answred Nori's question when A is local [MV]. By this time, as it was clear that Euler class group as mentioned above is not detecting the splitting when $rank(P) < dim(A)$ Prof. Satya Mandal came up with the idea of **Nori Homotopy program** which was motivated from the above question of Nori and its answer by Bhatwadekar and Keshari. He proposed the definition of the ***Nori Homotopy obstruction set,*** $\pi_0(LO(P))$, where P is a projective module over A. And it is proven subsequently in [MM1] and [MM2] that this obstruction set, $\pi_0(LO(P))$, houses an **obstruction class**, $\varepsilon(P)$. We describe this in detail as follows.

Let A be a regular ring containing a field k with $\frac{1}{2} \in k$ (i.e. $char(k) \neq 2$). Let $dim(A) = d$ and J be an ideal of A and $w : A^n \twoheadrightarrow J/J^2$ be a surjective map. Denote the set of all such pairs (J, w) as $\mathcal{LO}(A, n)$. Two elements (J_0, w_0) and (J_1, w_1) in $\mathcal{LO}(A, n)$ are said to be *equivalent* if there is an ideal I in $A[T]$ and a surjective map $\phi : A[T]^n \twoheadrightarrow \frac{I}{I^2}$ such that $(I(0), \phi(0)) = (J_0, w_0)$ and $(I(1), \phi(1)) = (J_1, w_1)$. In a joint work with Prof. Mandal, we have proved with substantial effort that this is in fact an *equivalence relation* [MM2]. The set of equivalence classes is called the *Nori Homotopy set* and denoted as $\pi_0(\mathcal{LO}(A, n))$ [MM1].

The most crucial step in establishing the equivalence relation is proving an equivalent form of *Lindel's theorem* for *quadratic spaces*. Let A be an *essentially smooth* ring over a field K and $R = A[T_1, T_2, \ldots, T_n]$. Lindel proved that any finitely generated projective R-module P is *extended* from R i.e. $P \cong R \otimes_A \frac{P}{(T_1, T_2, \ldots, T_n)P}$, thus settled Bass-Quillen conjecture [L]. In this regard, Mandal proved in [MM1] that if (P, Φ) quadratic space over R such that $(P, \Phi) \otimes_A \frac{R}{(T_1, T_2, \ldots, T_n)R} \cong (A^n, q)$ *locally* where q is a rank r quadratic form, then (P, Φ) is extended from A. The definition of Nori homotopoy class for A^n is then naturally extended to the projective modules over A as follows: Let P be a A-projective module. Then $\mathcal{LO}(P)$ is the set of *local orientations* (J, w) where $w : P \twoheadrightarrow J/J^2$ and J is an ideal in A. Then, we establish similar equivalence relation on this set and define $\pi_0(\mathcal{LO}(P))$ in [MM2].

The main goal of defining Nori Homotopy set is of course to detect the obstruction to having free summand of the projective modules which we will describe now. We consider two distinguished elements $\epsilon(P) = (0, 0)$ and $e_1 = (A, 0)$ in $\mathcal{LO}(P)$. We call $[\epsilon(P)] \in \pi_0(\mathcal{LO}(P))$ the **Nori Homotopy Class of** P. The most important result we have proved in [MM2] is: suppose A is *essentially smooth* ring over an infinite field k with $\frac{1}{2} \in k$ such that $2.rank(P) \geq dim(A) + 3$. Then, we have $P \cong Q \oplus A \Leftrightarrow [\epsilon(P)] = [e_1]$ where Q is an A-module. That is, P has a nonzero free direct summand if and only if $\epsilon(P)$ and e_1 are *equivalent* in $\mathcal{LO}(P)$. So, this provides an effective obstruction for certain lower ranked projective modules (i.e. $2.rank(P) \geq dim(A) + 3$), nothing similar of which could be established in the earlier developed theory.

We also provide two important alternate descriptions of the obstruction set, $\pi_0(\mathcal{LO}(P))$. In the first one, we restrict our attention to those ideals which have *heights* precisely equal to the $rank(P)$. More formally, we define $\widetilde{\mathcal{L}O}(P)$ to be the set of *local orientations* (J, w) where $w : P \twoheadrightarrow J/J^2$ and J is an ideal in A and $height(J) = rank(P)$ or $J = A$. We establish similar equivalence relation as above in this set and define the set of equivalence classes, denoted as $\pi_0\left(\widetilde{\mathcal{L}O}(P)\right)$. Moreover, we prove that there is isomorphism/bijection between these two sets. Secondly, we consider the set, denoted as $\mathcal{L}O_{\mathfrak{s}}(P)$, of local orientations (J, w) where J is an ideal such that A/J is *smooth* or $J = A$. Again with substantial effort, we show that the homotopy relation on this set is an equivalence relation and thus, define $\pi_0\left(\mathcal{L}O_{\mathfrak{s}}(P)\right)$. Lastly, we proved that there is an ismorphism/bijection between this set with the earlier defined sets.

Moreover, it is of certain interest in the research community that whether there is any additional algebraic structure on the Nori Homotopy set. In this regard, we have shown that there is a natural *abelian monoid structure* on $\pi_0(\mathcal{LO}(P))$ [MM3] when 2.rank(P)$\geq dim(A) + 2$ (and this becomes a group in certain cases). This cumulatively provides us a fairly comprehensive theory on Algebraic Obstruction. However, similar results in topological context are yet to be proven and research in this regard is relatively at the nascent stage. So, we have explored very thoroughly the topological side involving vector bundles in this thesis.

Being motivated from the idea of *Nori homotopy set* mentioned above, we have come up with the idea of the obstruction set for a vector bundle in having a nonvanishing section which also we call **Nori Homotopy set** in topology. Let M be a smooth manifold of dimension d and $\mathcal{V}$ be a vector bundle of rank n over M. We consider the ***local $\mathcal{V}$-orientation*** as a pair (B, φ) where B is a submanifold of M with codim$(M, B) = n$ and $\varphi : N(M, B) \to \mathcal{V}|_B$ is an isomorphism. Let $\widetilde{\mathcal{L}O}(\mathcal{V})$ be the set of all such orientations which also includes the empty manifold ϕ by definition. We define a *chain homotopy* relation on this set by considering the vector bundle $p^\star(\mathcal{V})$ over $M \times \mathbb{R}$ where $p : M \times \mathbb{R} \to M$ is the projection map. This relation is an equivalence relation by construction. We denote the set of equivalence classes in $\widetilde{\mathcal{LO}}(\mathcal{V})$ as $\pi_0(\widetilde{\mathcal{LO}}(\mathcal{V}))$. This is the *Nori homotopy set* for a vector bundle $\mathcal{V}$. Similar to above algebraic case, we consider two distinguished elements in this set, denoted as e_0 and e_1, and prove that if these two elements belong to the same class (i.e. *chain-homotopic* to each other), then V will have a nonvanishing section when $d \leq 2n - 3$. Thus, this provides an effective description of the obstructions involved in vector bundles having a trivial summand.

The last crucial result we have proved here is in establishing relationship between the two said theories of obstruction based in Algebra and Topology. We consider the *real smooth affine scheme* $X = spec(A)$ and the corresponding associated *manifold* M *of real points* in X. Let P be a projective module over A and $\mathscr{V}_{Top}(P)$ be the induced smooth vector bundle over M. Then, we look at the Nori homotopy sets $\pi_0(\mathcal{L}O(P))$ defined for P and $\pi_0\left(\mathcal{L}O(\mathscr{V}^{\star}_{Top}(P))\right)$ defined for the *dual bundle* $\mathscr{V}^{\star}_{Top}(P)$. We establish a natural map between these two sets by considering an 'ideal theoretic' description of the *dual normal bundle* of a smooth manifold. Of course, existence of this map indicates that there is a deeper link between these two theories which require further investigation which we would like to do in future.

Chapter 2

Preliminaries

We are providing here some preliminaries of manifold theory.

2.1 Manifolds

Definition 2.1.1. *Let M be a Hausdorff topological space. Assume that for each point m in M, there is an open neighbourhood $U_m \subset M$ and an isomorphism, called* ***chart****, $\phi_m : U_m \to V_m$, where V_m is an open set in $\mathbb{R}^d$. Moreover, for any two points $m, n \in M$, let the composition maps $\phi_m \circ \phi_n^{-1} : V_m \cap V_n \to V_m \cap V_n$ is a smooth map in $\mathbb{R}^d$, whenever the map is defined. In this case, we say that M is a* ***smooth manifold****, with dimension d.*

Definition 2.1.2. *Let M_1 and M_2 be two smooth manifolds of dimension d_1 and d_2 respectively. Any map $s : M_1 \to M_2$ is called a* ***smooth map of manifolds*** *if $\phi_{2,s(m)} \circ s \circ \phi_{1,m}^{-1} : V_{1,m} \to V_{2,s(m)}$ is a smooth map in real sense from an open subset of $\mathbb{R}^{d_1}$ to an open subset of $\mathbb{R}^{d_2}$, whenever the map is defined.*

2.2 Vector Bundle

Definition 2.2.1. Let M be a smooth manifold as defined above. Consider two real smooth functions f, g defined on some open neighbourhoods of m. Define $f \sim g \Leftrightarrow f = g$ on some open neighbourhood of m. This relation is clearly an equivalence relation. We define the set of all equivalences classes as the ***germ*** of functions at m, denoted as $\widetilde{F_m}$. Now consider the class of functions in $\widetilde{F_m}$ which vanishes at m. Denote the set of such classes as F_m. Note that $\widetilde{F_m}$ is an algebra over $\mathbb{R}$ where the addition and multiplication are defined pointwise and F_m is an ideal of $\widetilde{F_m}$.

Definition 2.2.2. *Let v be a map $v : \widetilde{F_m} \to \mathbb{R}$. We call v, a linear derivation on $\widetilde{F_m}$ if: for any $f, g \in \widetilde{F_m}$ and $c \in \mathbb{R}$, $v(f + cg) = v(f) + cv(g)$ and $v(f.g) = v(f).g(m) + f(m).v(g)$ Such a v is also called a* ***tangent vector*** *v at the point m. Let T_mM denote the set of all tangent vectors at m, called as the* ***tangent space of m in M*** *. Note that T_mM has a natural vector space structure defined as follows: $(v + cw)(f) = v(f) + cw(f)$ for all $f \in \widetilde{F_m}$ and $c \in \mathbb{R}$.*

Now we have our first theorem:

Theorem 2.2.3. *[W, Lemma 1.16] T_mM is naturally isomorphic to $(F_m/F_m^2)^\star$ where '$\star$' denotes the dual of the vector space.*

Proof. Let $v \in T_mM$. Then, clearly v is a linear map on F_m. For any $f, g \in F_m$, $v(f.g) = v(f)g(m) + f(m)v(g) = 0$. So, v vanishes on F_m^2. Conversely, if $l \in (F_m/F_m^2)^\star$, then the corresponding linear derivation is defined as $v_l(f) := l(f - f(m))$ for any $f \in \widetilde{F_m}$. We can easily check, v_l is a linear derivation and these two assignments are inverse of each other thus establishing the isomorphism. ∎

Next, we have the following theorem:

Theorem 2.2.4. *[W, Theorem 1.17] $dim(F_m/F_m^2)^\star = dim(M)$.*

Proof. Note that locally M looks like $\mathbb{R}^d$. And F_m is only determined by the local behavior of the functions. So we can assume that $M = \mathbb{R}^d$. In $\mathbb{R}^d$, $\partial/\partial x_i$ for $i \in \{1, 2, \ldots, d\}$ generates the $dim(F_m/F_m^2)^\star$ by Taylor's expansion and are also linearly independent. Thus, the theorem follows.The proof is complete. ∎

Corollary 2.2.5. *From above two results, it follows that $dim(T_mM) = dim(M)$. In fact, we can conclude more that we have $\{\partial/\partial x_i\}_{i\in\{1,2,\ldots,d\}}$ as a basis of the tangent space of M at m.*

Definition 2.2.6. Let M and V be two topological spaces and $\pi : V \to M$ be a continuous surjection such that: [1] $\pi^{-1}(m)$ is a finite dimensional real vector space $\forall m \in M$ and [2] there is a neighbourhood $U_m \subset M$ and an isomorphism $\phi_m : U_m \times \mathbb{R}^{k_m} \to \pi^{-1}(U_m)$ for some $k_m \in \mathbb{N}$ such that $(\pi \circ \phi_m)(x, v) = x, \forall x \in U_m$ and the map $v \mapsto \phi_m(x, v)$ is a linear isomorphism between the vector space $\mathbb{R}^{k_m}$ and $\pi^{-1}(m), \forall m$. Then V is said to be a ***vector bundle*** over M. If M is connected, then k_m is independent of m and this number is said to be the **rank** of V. Moreover, if M is smooth and ϕ_m's are smooth, then we say that V is a smooth vector bundle. Any continuous map $s : M \to V$ is called **section** of the vector bundle V if $\pi \circ s(v) = v, \forall v \in V$. Moreover, if s is smooth, then we say that s is a *smooth section*.

Definition 2.2.7. *Let $\pi : V \to M$ and $\pi' : V' \to M'$ be two vector bundles. Then a smooth map $\phi : V \to V'$ is a **maps between vector bundles** if there is a smooth map $F : M \to N$ such that $\pi' \circ \phi = F \circ \pi$.*

The simplest example of vector bundle is *trivial bundle*: $M \times \mathbb{R}^k$. Note that if M is a smooth manifold and π and ϕ_m are smooth maps, then V has also a smooth manifold structure and locally looks like $\mathbb{R}^d \times \mathbb{R}^k$. If $s : M \to V$ is a section, then from the definition of manifold and vector bundle it follows that locally s looks like the map:

$$(x_1, x_2, \ldots, x_d) \mapsto \big(x_1, \ldots, x_d, f_1(x_1, .., x_d), f_2(x_1, .., x_d), \ldots, f_k(x_1, .., x_d)\big).$$

If s is smooth, then each f_i is smooth locally. For our purpose here in this article, we assume all maps are smooth and M is connected. One special *section*, worth mentioning here for our purpose, is the ***zero section*** of a vector bundle. In a vector bundle, there is a $0-vector$ in the vector space $\pi^{-1}(m)$ for each $m \in M$. The *zero setion* maps m to $0_m \in \pi^{-1}(m)$. Since locally this is same as $(x_1, x_2, \ldots, x_d) \mapsto \big(x_1, \ldots, x_d, 0, 0, \ldots, 0\big)$, the map is clearly smooth. Thus, it is a *smooth section* of the vector bundle.

Definition 2.2.8. We define the ***tangent bundle*** of a smooth manifold M, denoted as TM, $TM = \bigsqcup_m T_m M$, where $T_m M$ is the tangent space of M at m as defined above. This has a natural vector bundle structure with $\pi : TM \to M$ defined as $v \mapsto m$ where $v \in T_m M$.

Then, from Corollary 2.2.5, it follows that the rank of TM is equal to the dimension of M.

Now consider two manifolds M and V with their corresponding tangent spaces TM and TV. Let $s : M \to V$ be a smooth map. Then we have an induced map in the tangent bundles, $ds : TM \to TV$, defined as follows: If $v \in M$, then $ds(v)$ is a tangent vector at $s(m)$, which acts on any smooth function f defined on some neighbourhood of $s(m)$, $ds(v)(f) = v(f \circ s)$. Note that ds is a linear map between vector spaces $T_m M \to T_{s(m)} V$ at each $m \in M$, denoted as ds_m.

Definition 2.2.9. *Let $\phi : N \to M$ be a smooth map of manifolds. If $d\phi_n$ is non-singular (i.e. injective) for each $n \in N$, then ϕ is said to be an **immersion of manifolds**. And if ϕ is injective and $d\phi_n$ is non-singular for each n, then N is said to be a **submanifold** of M.* A standard example of *submanifold* is when $B \subset M$ and the charts of B and M are compatible via the inclusion map*:* $B \hookrightarrow M$. *Moreover, if ϕ is a homeomorphism between N and $\phi(N) (\subset M)$, then ϕ is called **embedding**.*

Definition 2.2.10. *Let $i : B \hookrightarrow M$ be a submanifold. Define $N_b(M, B) := T_b M / T_b B$.* (Notice that $T_b B$ can be naturally thought of a vector subspace of $T_b M$ through the induced map di). *There is a natural vector bundle structure on the set $N(M, B) := \bigsqcup_{b \in B} N_b(M, B)$. This vector bundle is called the **normal bundle** of B in M.*

A simple example of normal bundle is: Consider, $B = \mathbb{R}^d \times \bar{0} \hookrightarrow \mathbb{R}^{d+k} = M$. Then $T_b B = \langle \{\partial/\partial x_i\}_{i \in \{1,\ldots,d\}} \rangle \cong \mathbb{R}^d$. And $T_b M = \langle \{\partial/\partial x_i\}_{i \in \{1,\ldots,d+k\}} \rangle \cong \mathbb{R}^{d+k}$.Then we have $N_b(B, M) = \langle \{\partial/\partial x_i\}_{i \in \{d+1,\ldots,d+k\}} \rangle \cong \mathbb{R}^k$.

Definition 2.2.11. *Let $f : N \to M$ be a map of manifolds. Let $\pi : V \to M$ be a vector bundle. Then we can construct a vector bundle on N canonically as follows: Define a set $f^*V := \{(n, v) \in N \times V | f(n) = \pi(v)\}$ and equip it with the subspace topology of $M \times V$ and the projection map $\pi' : f^\star V \to N$ defined as $\pi'(n, v) = n$. This establishes a vector bundle structure on $f^\star V$ induced from that of V. This is called the **pullback bundle of V by f**.*

Definition 2.2.12. Let V be a vector bundle on a manifold M. Let s and t be two *sections* of V and $B := \{m \in M | s(m) = t(m)\}$. We say that s and t meet ***transversally along B*** if $ds_m(T_mM) + dt_m(T_mM) = T_{s(m)}V$, *for all* $m \in B$.

Now, we are going prove a theorem which is important with regard to the Nori's homotopy question. Before that we are going to mention one specific *pullback bundle*: Let N be a submanifold of M with $i : N \hookrightarrow M$. Let V be a vector bundle over M. Then, $i^\star V$, the pullback bundle over N, is denoted $V|_N$.

2.3 Isomorphism Theorem

Theorem 2.3.1. *Let s be a section of a vector bundle V over M. Assume that s meets the zero section $\tilde{0}$ of V transversally over a submanifold B of M. Then we have an induced isomorphism $[s]$ of vector bundles:*

$$[s] : N(M, B) \to V|_B,$$

defined by $[s](b, v) = (b, ds_b(v))$ where $v \in N_b(M, B) \subseteq T_b(M)$.

Proof. For simplicity, we assume $V = M \times \mathbb{R}^n$. Now, locally we know $s(x) = \big(x, f_1(x), \ldots, f_n(x)\big)$, where $f_i(x) : M \to \mathbb{R}$ is a smooth map for each i. Now the transversality in the hypothesis implies $ds(T_bM) + d\tilde{0}(T_bM) = T_{(b,\bar{0})}(M \times \mathbb{R}^n)$. Then,

$$\big(Id_b, Df_1, \ldots, Df_n\big)(T_bM) + \big(Id_b, 0, \ldots, 0\big)(T_bM) = T_bM \oplus T_0\mathbb{R}^n.$$

Here, Id_b denotes the map induced on the tangent spaces by $b \mapsto (b, 0)$ and Df_i are derivatives of maps. Let $L = \big(Df_1, \ldots, Df_n\big) : T_bM \to T_0\mathbb{R}^n$. Then we have,

$$(Id, L)(T_bM) + (Id, 0)(T_bM) = T_bM \oplus T_0\mathbb{R}^n.$$

That implies,

$$T_bM + L(T_bM) = T_bM \oplus T_0\mathbb{R}^n.$$

Hence, we must have $L(T_bM) = T_0\mathbb{R}^n$ for all $b \in B$. Note that $T_b(M) = T_bB \oplus N_b(M, B)$ and since $(f_1, \ldots, f_n)(b) = (0, \ldots, 0)$ for all $b \in B$, $L(T_bB) = 0$. Thus, $L(N_b(M, B)) = T_0\mathbb{R}^n \Rightarrow (Id, L)(T_bB, N_b(B, M)) = T_{(b,0)}(B \times \mathbb{R}^n)$. This implies that we have an induced isomorphism of vector bundles

$$[s] : N(M, B) \to B \times \mathbb{R}^n (= V|_B)$$

defined by $(b, v) \mapsto (b, (Df_1(b), \ldots, Df_n(b))(v))$. The general case follows by extending this local isomorhism to the global one. ■

Chapter 3

Nori's Theorem

We will consider a fundamental natural question in this chapter, initially posed by Madhav Nori.

3.1 Nori's Homotopy Question

Question 3.1.1. *Let B_0 be a submaniold of a smooth manifold M and V be a vector bundle over M. Let s_0 be a global section of V that meets the zero section of V transversally in B_0. Consider, $\widetilde{M} := M \times \mathbb{R}$ and B be a submanifold of $\widetilde{M}$ such that it meets $M \times \{0\}$ transversally along B_0. Let $p : \widetilde{M} \to M$ be the projection map sending $(m, t) \mapsto m$. Let p^*V be the pull-back bundle over $\widetilde{M}$(see 2.2.11). Assume that we have an isomorphism $\phi : N(\widetilde{M}, B) \to p^*V|_B$ such that $\phi|_{B_0} = [s_0]$, where $[s_0]$ is the isomorphism as defined in 2.3.1. Then, the question is whether we can find a global section s of p^*V that meets zero section of p^*V transversally along B, so that $[s] = \phi$ and $s|_{M \times \{0\}} = s_0$.*

Nori provided an answer to the above question which we will discuss below. However, before that some more preliminaries have to be explained as follows.

3.2 Tubular Neighbourhood

Definition 3.2.1. Let $B \subset M$ be a submanifold. A **tubular neighbourhood** of B is a pair (f, T) where $T = (p, E, B)$ is a vector bundle over B and $f : E \to M$ is an embedding such that:

1. The following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "0_B"] \arrow[dr, hook] & E \arrow[d, "f"] \\ & M, \end{tikzcd}$$

   where 0_B is the zero section of B in E.

2. $f(E)$ is an open neighbourhood of B in M.

Now, we are going to show why such a neighbourhood exists always. But before that let's state some lemmas.

Lemma 3.2.2. *Let M and N be two smooth manifolds. Let $f : M \to N$ be a smooth proper map and an immersion. Assume that f is injective on a closed subset K of M. Then, there exists an open subset U of M such that f is injective on U and $K \subset U$.*

Proof. First, embedd N in $\mathbb{R}^m$ for some m (we can do so using *Whitney Embedding Theorem*). Fix a point n_0 in N. Let g be a function defined as: $g(n) = d(n, n_0)$ for all $n \in N$ (where d is the Eucledean metric in $\mathbb{R}^m$). Then, we can express $f(K)$ as union of following compact sets: $f(K) = \bigcup_k K_k$ where $K_k = g^{-1}([k, k+1]) \bigcap f(K)$. Since f is proper, $f^{-1}(K_k)$ is compact in M. Let $J_k := K \bigcap f^{-1}(K_k)$. Then, J_k is also compact for each k. Moreover, f is injective on each $J_{k-1} \bigcup J_k \bigcup J_{k+1}$(since this is a subset of K). Then from previous part, we have neighbourhood U_k of $J_{k-1} \bigcup J_k \bigcup J_{k+1}$ such that f is injective on U_k for each k. So, in this case f is in fact a diffeomorphism between U_k and $f(U_k)$. Since K_k is closed, we can define $d(K_k, A)$ for any closed set A as:

$$d(K_k, A) := inf\{\|x - a\| : x \in K, a \in A\}$$

Note that $d(K_k, \bigcup_{j>k+1} K_j) = d_k > 0$ from the construction. Now, choose a decreasing sequence $\{\epsilon_k\}$ such that $\epsilon_k < d_k/2$ for each k and $\epsilon_k \to 0$. We claim that $B_{\epsilon_k}(K_k) \bigcap B_{\epsilon_l}(K_l) = \phi$ for

$|k-l| > 1$ where $B_\epsilon(A)$ denotes the union of balls of radius ϵ around all points in A. This is because if x lies in the intersection, then there are points $y_l \in K_l$ and $y_k \in K_k$ such that $d(x, y_k) + d(x, y_l) < \epsilon_k + \epsilon_l < 2\epsilon_k < d_k$(assuming wlog $k < l$ and thus, $\epsilon_k > \epsilon_l$). But that implies $d(y_k, y_l) < d_k$ which is a contradiction. Now, consider:

$$V_k := U_{k-1} \bigcap U_k \bigcap U_{k+1} \bigcap f^{-1}(B_{\epsilon_k}(K_k))$$

Note that V_k is an open neighbourhood of J_k and f is injective on V_k. Let $W_k = f(V_k)$. Let $V := \bigcup_k V_k$ and $W = \bigcup_k W_k$. Then, we claim that V is the required open neighbourhood of K on which f is injective. Suppose not. Let x, y be two distinct points in V such that $f(x) = f(y)$. Note that if $x \in V_k$ and $y \in V_l$ for $|k-l| > 1$, then it is a contradiction as $f(x) = f(y) \in B_{\epsilon_k}(K_k) \bigcap B_{\epsilon_l}(K_l) = \phi$ for $|k-l| > 1$. So, $\{x, y\} \subseteq V_{k-1} \bigcup V_k \bigcup V_{k+1}$. But, by construction $V_{k-1} \bigcup V_k \bigcup V_{k+1} \subseteq U_k$ and f is injetcive on U_k. So, contradiction. ∎

3.2.1 Existence of tubular neighbourhood

Lemma 3.2.3. *Let $B \subset M$ be a submanifold, $T = (p, E, B)$ be a vector bundle and $U \subseteq E$ be a neighbourhood of zero section. Suppose, $f : U \to M$ is an embedding such that $f \circ 0_B = id|_B$ and $f(U)$ is open in M. Then, there is a smooth map $g : E \to M$ such that (g, T) is a tubular neighbourhood of B and $g = f$ in a neighbourhood of B.*

Proof. (See [Hirsch, pp. 109, Section 4.5]) Let us first fix a orthogonal structure on E. Then we choose a map $\tau : M \to \mathbb{R}_+$ such that if $y \in E_x$ and $|y| \leq \tau(x)$, then $y \in f(U)$. Let $\lambda : [0, \infty) \to [0, 1)$ be a diffeomorphism equal to the identity near 0. Define an embedding: $h : E \to E$ by $h(y) = \tau(p(y))\lambda(|y|)|y|^{-1}y$. Then $h(E) \subset U$ and $h =$ identity near M. Now define g to be $g = f.h$. This completes the proof. ∎

Theorem 3.2.4. *Let B be a submanifold of M. Then, B has a tubular neighbourhood in M.*

Proof. First, assume that $M = \mathbb{R}^n$. Then, TM can be identified with $M \times \mathbb{R}^n$. Let $E =$

$N(B, M)$(see 2.2.10). Note that $E = \{(b, y) \in M \times \mathbb{R}^n : y \in N_b(B, M)\}$. If the dimension of B as a manifold is k, the dimension of $T_b(B)$ is k as a vector space $\forall b \in B$. So, the dimension of $N_b(B, M)$ is $n - k$. Thus, as a manifold the dimension of E is n $(= rank(E) + dim(B))$. In fact, there is a natural isomorphism between E and $B \times \mathbb{R}^{n-k}$ as vector bundles. Let's denote the zero-section of E i.e. $B \times \{0\}$ in $N(B, M)$ as 0_E. Note that the tangent space $T_{(b,0)}E$ at each $b \in B$ has a natural splitting: $T_{(b,0)}E = T_bB \oplus TN_b(B, M) \cong T_bB \oplus N_b(B, M)$. Now, consider the map

$$f : E \to \mathbb{R}^n$$

$$f((b, y)) = b + y$$

We have $f(b, 0) = b$ at all points on the zero-section 0_E of E(i.e. $f \circ 0_B = id|_B$). So, it is injective on the closed set 0_E of E. Now consider the map: $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\tilde{f}((x, y)) = x+y$. Note that f is just the restriction of $\tilde{f}$ to $E = N(B, M) \subseteq \mathbb{R}^n \times \mathbb{R}^n$. Also note that the differential matrix $d\tilde{f}$ is precisely (I_n, I_n), where I_n is $(n \times n)$-identity matrix and this has rank n at all points in the domain. Also we have $df_{(b,0)}$ is an identity map on T_bB and on $N_b(B, M)$. Thus, it is an immersion at all points of E. Taking union of all such neighbourhoods, we will get a neighbourhood of the zero section. We note that $f \circ 0_B = id|_B$. So, now Lemma 3.2.2 ensures that $f|_U$ is an embedding of some open neighbourhood $U \subset E$ of B. So f is an immersion in some neighbourhood of the zero-section. Now, by lemma 3.2.3, it follows that B has a tubular neighbourhood in M.

Now, we will tackle the general case. We assume that $M \subseteq \mathbb{R}^n$. Assume that M is embedded in some $\mathbb{R}^n$ (by Whitney Embedding Theorem). Then, by previous analysis we have a tubular neighbourhood $W \subseteq \mathbb{R}^n$ of M. Let $r : W \to M$ be a retraction. Let $E = N(B, M)$ be the normal bundle of B. For each $b \in B$, let $U_b = \{(b, y) \in N_b(B, M) : b+y \in W\}$. Consider $U = \bigcup_{b \in B} U_b$. Then, U is open and the map $r|_U : U \to M$ gives the required tubular neighbourhood. ∎

Note that from our understanding of the proof of above theorem, it is enough for us to consider the *normal bundle* of a *submanifold* as the required tubular neighbourhood of the *submanifold*. We will now define *closed tubular neighbourhood* of a manifold which conceptually is very close to the notion of *tubular neighbourhood*.

Definition 3.2.5. *Let B be a submanifold of a manifold M, V a vector bundle over M and (f, T) be a tubular neighbourhood of B where $T = (p, E, B)$. Then, a* ***closed tubular neighbourhood of radius*** *$\epsilon > 0$, denoted as $D_\epsilon(T)$, is the disk bundle:*

$$D_\epsilon(T) = \big\{x \in E : |x| < \epsilon\big\}$$

where the norm is taken with respect to some orthogonal structure defined on E.

It is clear from the definition that if a *tubular neighbourhood* exists, then so does a *closed tubular neighbourhood* for any $\epsilon > 0$.

Now, we will go back to the question of Nori, 3.1.1. From the hypothesis of the question, we have a map, which is an isomorphism, ϕ: $N(\widetilde{M}, B) \to p^*(V)|_B$. Now from the proof of the existence of tubular neighbourhood i.e. Theorem 3.2.4, we know that normal bundle of a submanifold can be considered as a *tubular neighbourhood* of the submanifold. We will consider $N(\widetilde{M}, B)$ as a *tubular neighbourhood* of B in $M \times \mathbb{R}$ with an embedding $\psi : N(\widetilde{M}, B) \to M$. We will denote the *image* of ψ in M as $\widetilde{W}$. Thus, we can consider the *pullback-bundle* $p^\star(V)|_{\widetilde{W}}$. Then, the isomorphism map ϕ induces a section, $s' : \widetilde{W} \to p^\star(V)|_{\widetilde{W}}$. This can be seen locally as follows. Note that locally $M \times \mathbb{R} \cong \mathbb{R}^k \times \mathbb{R}^{d-k+1}$, $B \cong \mathbb{R}^k$ and $p^\star(V) \cong \mathbb{R}^{d+1} \times \mathbb{R}^n$. From the isomorphism ϕ, we have the following:

$$\phi : N(\widetilde{M}, B) \to p^\star(V)|_B$$

$$\mathbb{R}^k \times \mathbb{R}^{d-k+1} \to \mathbb{R}^k \times \mathbb{R}^n$$

$$(\tilde{x}, \tilde{y}) \mapsto (\tilde{x}, f(\tilde{x}, \tilde{y}))$$

Now we define the section s' induced by ϕ locally as follows:

$$s : M \times \mathbb{R} \to p^{\star}(V)$$

$$(\tilde{x}, \tilde{y}) \mapsto (\tilde{x}, \tilde{y}, f(\tilde{x}, \tilde{y}))$$

We can check the compatibility condition easily. Thus we have a section s' over $\widetilde{W}$. Let W be a *closed tubular neighbourhood* of B associated with $\widetilde{W}$) for some fixed ϵ. Then, restricting s' to W we have a section s' on W.

Now, notice that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "0_B"] \arrow[dr, "0_B"'] & N(\widetilde{M}, B) \arrow[d, "\phi"] \\ & p^{*}(V)\big|_B \end{tikzcd}$$

where 0_B is the zero section of B in respective bundles. So, the *section* s' vanishes on B in W (i.e. taking elements of B in W to the 0-vectors in $p^{\star}(V)\big|_B$) as ϕ vanishes on B in $N(\widetilde{M}, B)$. Notice that this is related to our requirement in Nori's question (3.1.1) that the global *section* s, provided of course if such a section exists, should precisely meet the zero section transversally on B.

Now, we claim that the induced map of section s' as defined in Theorem 2.3.1, denoted as $[s'] : N(W, B) \to p^{\star}(V)|_B$ is same as the map ϕ. This is clear if we check the map locally. Thus, we are able to extend the section s_0 over B_0 to a section s' over W with the satisfaction of required conditions in Nori's question(3.1.1). Stating formally what we have achieved so far:

Lemma 3.2.6. *With the notations as above, there is a closed tubular neighbourhood W of B that* *intersects $M \times 0$ in a closed tubular neighbourhood of B_0 and a section s' of $p^\star(V)|_W$ that vanishes precisely on B so that (i) the induced isomorphism $[s'] = \phi$ and (ii) $s'|_{M\times 0 \cap W} = s_0|_{M\times 0\cap W}$*

Now, we will extend this section s' to the whole space $M \times \mathbb{R}$. This will be done by an application of an important theorem related to such an extension which will be stated below. This requires a number of definitions and structures which is a part of *Topological Obstruction Theory*. The central question considered in this field of study is: Let $\mathcal{B}$ be a vector bundle over an n-simplicial space X. Let L be a subcomplex of X. Suppose there is a continuous map $s : L \to \mathcal{B}$. Then, can s be extended to a section over the whole space X? If this is not possible, what is the precise obstruction for such an extension? The answer to these questions lies in the cohomology groups of the pair (X, L) and an element associated with the given map s called *obstruction cocycle* in these groups. In simple words, if the obstruction cocycle vanishes in an i-th cohomology group, then extension to i-simplex is possible. However, before elaborating this further in a rigorous way, we will start with a few definitions.

We have presented below all the definitions and results that are required for proving the Theorem of obstruction 3.3.13 all of which have been represented from the book *The topology of fibre bundles* by Neeman Steenrod [Sn]. The purpose here is to give the definitions of all the terms that are being used and outline the ideas leading to the proof of Theorem 3.3.13.

3.3 Coordinate bundle

Definition 3.3.1. *A topological transformation group is a triple (X, G, π), where $\pi : G \times X \to X$ defined as $\pi(g, x) = g.x$ is a continuous action of a topological group G on the topological space X. We call G to be effective if $g.x = x$ for all x implies $g = e$, the identity elemnt in G.*

Definition 3.3.2. *A coordinate bundle $\mathcal{B}$ is a collection of following information:*

1. *A space B called the bundle space,*

2. *a space X called the base space,*

3. *a surjective map $p : B \to X$ called projection,*

4. *a space Y called fibre,*

5. *an effective topological transformation group G of Y called the 'group' of bundle,*

6. *an open cover $\{V_j\}_J$ of X called coordinate neighbourhoods,*

7. *a set of homeomorphisms, $\{\phi_j\}_J$ where $\phi_j : V_j \times Y \to p^{-1}(V_j)$ for all $j \in J$, called coordinate functions*

such that they satisfy the following properties:

1. *$p\phi_j(x, y) = x$, for all $x \in V_j$ and $y \in Y$*

2. *The maps $\phi_{j,x} : Y \to p^{-1}(x)$ defined by setting $\phi_{j,x}(y) := \phi_j(x, y)$ satisfies: for each $i, j \in J$ and $x \in V_i \cap V_j$, the homeomorphisms defined by, $\phi_{j,x}^{-1}\phi_{i,x} : Y \to Y$ coincides with a group action of an element of G,*

3. *for each pair $i, j \in J$, the map $g_{ji} : V_i \cap V_j \to G$, defined by $g_{ij}(x) = \phi_{j,x}^{-1}\phi_{i,x}$, is continuous.*

Definition 3.3.3. *A space Y is called q-simple if it is pathwise-connected and for each $y_0 \in Y$ the fundamental group $\pi_1(Y, y_0)$ acts trivially on the q-th homotopy group $\pi_q(Y, y_0)$.*

The action is defined in certain natural way which we will see later. However, for us it is enough to know that if the fundamnetal group vanishes i.e. $\pi_1(Y, y_0) = 0$, then Y is q-simple for all q.

Definition 3.3.4. *A bundle of coefficients is a coordinate bundle in which the fibre Y is an abelian group and the group of the bundle G is totally disconnected and acts as automorphisms of Y*

One specific example of *bundle of coefficients* is '*associated bundle*' associated with a given coordinate bundle, defined as follows and the structure of coefficients bundle on this has been proved below.

Definition 3.3.5. *Let $\mathcal{B}$ be a bundle $\{B, p, X, Y, G\}$. Let $\pi = \pi_q(Y)$, $\Pi = \bigsqcup_x \pi_q(Y_x)$, and the map be $\rho : \Pi \to X$ taking $\pi_q(Y_x)$ to x. Note that each $g \in G$ induces an automorphism of $\pi(Y)$. Let H be the subgroup of G which acts as identity on $\pi(Y)$. Let $\Gamma = G/H$. Consider the space $\{\Pi, \rho, X, \pi(Y), \Gamma\}$. This bundle is the required associated bundle of $\mathcal{B}$, denoted as $\mathcal{B}(\pi_q)$.*

Theorem 3.3.6. *If X is locally path-connected and Y is q-simple, then there is a coordinate bundle structure on $\{\Pi, \rho, X, \pi_q, \Gamma\}$ as defined above.*

Proof. Note that for each isomprhism $\phi_{j,x}$ between Y and Y_x that comes with the bundle structure in $\mathcal{B}$, we have an isomorphism of fundamental groups, denoted as $\phi^*_{j,x} : \pi \to \pi_x$. Now, for each coordinate neighbourhood V_j, define $\psi_j : V_j \times \pi \to \rho^{-1}(V_j)$ by $\psi_j(x, \alpha) = \phi^*_{j,x}(\alpha)$. These will be the required coordinate functions once we define topology on on Π. It is easy to see that ψ_j's are injective. The bundle structure requires the hypothesis of X being locally path-connected. Associate π and Γ with discrete topology and lastly note that the topology on Π should be such that the injective maps ψ_j's should be homeomorphisms. For this, given $\alpha \in \Pi$, we choose a j such that $\rho(\alpha) \in V_j$ and the neighbourhoods of α are the images of neighbourhoods of $(\rho(\alpha), \rho_j(\alpha))$ under ψ_j. ■

Next, we will explain the concept of **Cohomology groups based on a bundle of coefficients**. This is similar to standard singular cohomology groups we have seen introductory graduate level course in Cohomology theory with a modification because of the bundle structure: Let K be a finite simplex and $X = |K|$ be the associated space of K. Let $\mathcal{B}$ be the bundle of coefficients $\{B, p, X, Y, G\}$ over the space X. For each q-cell σ of K, we choose a reference point x_σ in σ and denote π_σ the fibre of $\mathcal{B}$ over x_σ (called the *coefficient group of* σ).

A q-*cochain* of K with coefficients in $\mathcal{B}$ is a function c to each oriented q-cell σ an element

$c(\sigma)$ of π_σ and satisfies $c(-\sigma) = -c(\sigma)$ where $-\sigma$ denotes the opposite orientation of σ. The q-cochain c is said to be zero on a subcomplex L if $c(\sigma) = 0$ for each $\sigma \in L$. The abelian groups generated by such q- cochains are denoted by $C^q(K, L; \mathcal{B})$. Similarl to the singular cohomology groups, we can define the coboundary maps: $\delta : C^q(K, L; \mathcal{B}) \to C^{q+1}(K, L; \mathcal{B})$ and check that $\delta^2 = 0$. thsi defines a chain complex and we can consider the homology groups of this chain complex (i.e. the cokernel modulo the kernel) denoted as $H^q(K, L; \mathcal{B})$. With some work, one can show that these groups are independent of the choices of base points. It is notable here that when the bundle of coefficients $\mathcal{B}$ is a trivial bundle, $H^q(K, L; \mathcal{B})$ reduces to the ordinary cohomology groups $H^q(K, L; \pi)$.

Now we will briefly describe the functorial property of this cohomology groups. Let $\mathcal{B}$ and $\mathcal{B}'$ be two associated bundles over K and K' and let $h : \mathcal{B} \to \mathcal{B}'$ be a bundle map such that the induced map $\bar{h} : (K, L) \to (K', L')$ 'preserves' the cell-structure (more rigorously defined as $\bar{h}$ having a *solid carrier* $\{E_\sigma\}$). Then there is induced map of the corresponding cohomology groups: $h^* : H^q(K, L; \mathcal{B}) \to H^q(K', L'; \mathcal{B}')$. And this map is also independent of any choice of carrier we have considered for preserving the cell-structure. More generally, it is true that if $h : \mathcal{B} \to \mathcal{B}'$ and $h' : \mathcal{B}' \to \mathcal{B}''$, then $(h'h)^* = h^* h'^*$ and the identity map $id : \mathcal{B} \to \mathcal{B}$ induces the identity map of $H^q(K, L; \mathcal{B})$. Thus, we get that $H^q(K, L; \mathcal{B})$ is a topological invariant. Moreover, it is invariant under homotopy, i.e. if $h_0, h_1 : \mathcal{B} \to \mathcal{B}'$ are two maps which are homotopic to each other, then $h_0^* = h_1'^*$.

Next, we consider the *Coboundary operator*. Let $\mathcal{B}$ be a bundle of coefficinets over (K, L) and let i and j be the inclusion maps:

$$L \xrightarrow{i} K \xrightarrow{j} (K, L).$$

Similar to the singular cohomology cases, this induces a map on the cohomology groups and we can get a map: $\delta : H^q(L; \mathcal{B}) \to H^{q+1}(K, L; \mathcal{B})$, for $q = 0, 1, \ldots$. Thus we have a long exact sequence of cohomology groups:

$$\cdots \to H^{q-1}(L;\mathcal{B}) \xrightarrow{\delta} H^q(K,L;\mathcal{B}) \xrightarrow{j^*} H^q(K;\mathcal{B}) \xrightarrow{i^*} H^q(L;\mathcal{B}) \to \cdots$$

Obstruction cycle: Let K be a finite complex, L be one of its subcomplex and $\mathcal{B}$ be a bundle over K. Suppose there is a *section* $f : L \to \mathcal{B}|_L$. We want to extend this section to a global section over K. First we will start with a definition:

Definition 3.3.7. *A space Y is said to be q-connected ($q \geq 0$) if it is path connected and the homotopy groups $\pi_i(Y) = 0$ for $i = 1, \ldots, q$.*

Then we have the following theorem:

Theorem 3.3.8. *If f is a section on the subcomplex L of K to a bundle $\mathcal{B}$, then f can be extended to a section over $L \cup K_0$ where K_0 is the zero-complex of K. Moreover, if Y is q-connected, then f can be extended to a section over $L \cup K^{q+1}$.*

Note that this extension has nothing to do with the bundle structure on $\mathcal{B}$. It is true for any topological space.

Proof. Note that if L does not contain all zero-dimensional simplices in K_0, then we can extend this to $L \cup K_0$ by assigning arbitrary points in the corresponding fibres of $\mathcal{B}$. Now, suppose f is defined on $L \cup K^q$ and we consider the problem of extending this to $L \cup K^{q+1}$. Let σ be any $(q+1)$-cell of K not in L. The problem is to define the map f on σ. Since σ is contractible, $\mathcal{B}|_\sigma$ is a product bundle and thus there is an isomorphism $\phi_\sigma : \mathcal{B}|_\sigma \cong \sigma \times Y$ and a map $p_\sigma : B_\sigma \to Y$ such that $\phi_\sigma(p(b), p_\sigma(b)) = b$ for all $b \in B_\sigma$ where B_σ is an open ball in σ. Note that since f is defined on $L \cup K^q$, it is also defined on the *boundary* of σ, denoted as $\dot{\sigma}$. Let $f_{\dot{\sigma}} := p_\sigma[f|_{\dot{\sigma}}]$. Then since $\pi_q(Y) = 0$ by hypothesis, this map extends to a map f_σ such that $f_\sigma|_{\dot{\sigma}} = f_{\dot{\sigma}}$. Thus, f extends as required. ∎

Now, if $\pi_q(Y) \neq 0$, then we will come across the obstruction for f be extended to a global section. Now we will compute this obstruction more precisely. Assume that $\pi_q(Y) \neq 0$. For any $(q+1)$-cell σ, we have a map $f|_{\dot{\sigma}} : \dot{\sigma} \to Y$ whose extendibility over σ is equivalent to the extendibility of f. So, we focus on extendibility of $f_{\dot{\sigma}}$ over σ. It is obvious to see that $f|_{\dot{\sigma}}$ can be extended to a map over σ if and only if it can be extended to a map of σ in to B_σ. Let Y_x be a fiber over a point $x \in \sigma$. Then the inclusion map of Y_x in B_σ induces an isomorphism $\pi_q(Y_x) \cong \pi_q(B_\sigma)$. Therefore, $f|_{\dot{\sigma}}$ is homotopic to a map $f|_{\dot{\sigma}} : \dot{\sigma} \to Y_x$. We define $c(f, \sigma)$ to be the element of $\pi_q(Y_\sigma)$ associated with $f|_{\dot{\sigma}}$. From the discussion above, it is obvious to see the following proposition.

Proposition 3.3.9. *A section f over $L \cup K^q$ is extendable over $L \cup K^{q+1}$ if and only if $c(f, \sigma) = 0$ for each $(q+1)$-cell σ.*

Thus, we have the following definition of obstruction.

Definition 3.3.10. *Assume the notations in the preceding paragraph. Let $c(f)$ be a function of $(q+1)$-cells, given by $c(f)(\sigma) = c(f, \sigma)$. We call $c(f)$ the **obstruction cocycle** of f.*

Recall from the theorem 3.3.2 that when Y is q-simple, the groups $\pi_q(Y_x)$ form a bundle $\mathcal{B}(\pi_q)$ of coefficients over K. Since $c(f)$ is afunction on the set of all $(q+1)$-cells in K, this can be seen as a $(q+1)$-cochain of K *mod* L with coefficients in $\mathcal{B}(\pi_q)$. Then, we have the following theorem from [Sn, Theorem 32.4] which shows that $c(f)$ is in fact a cocycle.

Theorem 3.3.11. *With the assumptions and notations above, the obstruction cochain $c(f)$ is a cocycle.*

Next, we have the following theorem [Sn, Theorem 34.2] which shows that $c(f)$ is precisely the obstruction we are looking for.

Theorem 3.3.12. *Let f be a section of $\mathcal{B}|_{L \cup K^{q-1}}$ and let f be extendable over $L \cup K^q$. Then the set $c(f')$ of $(q+1)$-dimensional obstruction cocycles of all such extensions f' of f form a single cohomology class $\bar{c}(f) \in H^{q+1}(K, L; \mathcal{B}(\pi_q))$ and f is exendable on $L \cup K^{q+1}$ if and only if $\bar{c}(f) = 0$.*

The following result, which we need for our subsequent work, follows immediately from the above theorem.

Theorem 3.3.13. *[Sn, Cor. 34.4] If $\mathcal{B}$ is a bundle over (K, L) and for each $q = 0, 1, \ldots, dim(K \setminus L)$, Y is $(q-1)$-simple and $H^q(K, L; \mathcal{B}(\pi_{q-1})) = 0$, then any section f of $\mathcal{B}|_L$ can be extended to a full-section of $\mathcal{B}$.*

3.4 Nori's homotopy theorem

From 3.2.6, we have a section s' on W which is non-vanishing on $W \cup M \times 0 \setminus Int(W)$ (as it vanishes only on B). We want to extend this as a non-vanishing section to $M \times \mathbb{R} \setminus Int(W)$. Now we will apply 3.3.13 so that such an extension will be possible. In this case, $(K, L) = (M \times \mathbb{R} \setminus Int(W), W \cup M \times 0 \setminus Int(W))$. We will prove that the cohomology groups $H^i(M \times \mathbb{R} \setminus Int(W), W \cup M \times 0 \setminus Int(W); \mathcal{L})$ vanishes for all $i \geq n$ and all *local systems* $\mathcal{L}$ (definition below) on $M \times \mathbb{R}$.

Definition 3.4.1. *Local system: A local system (of abelian groups) on $X$is a locally constant sheaf (of abelian groups) on X.*

First we will state some of the results we will be using from basic Algebraic Topology course as follows.

Theorem 3.4.2. *Let $A, B \subset X$ be two subspaces of X such that $Int(A) \cup Int(B) = X$, then we have a natural isomorphism: $H^n(X, A) \cong H^n(B, A \cap B; G)$ induced by the inclusion $(B, A \cap B) \hookrightarrow (X, A)$. Similarly, if we have a sequence of subspaces $Z \subset A \subset X$ such that $\bar{Z} \subset Int(A)$. Then the inclusion $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $i^* : H^n(X, A; G) \to H^n(X \setminus Z, A \setminus Z; G)$ for all n.*

Theorem 3.4.3. *Let $B \subset A \subset X$ a sequence of subspaces in X. We have a long exact sequence* *of cohomology groups as follows:*

$$\cdots \to H^n(X, A; G) \to H^n(X, B; G) \to H^n(A, B; G) \to H^{n+1}(X, A; G) \to \cdots$$

Now we have all the essential ingredients to prove Nori's answer to the question 3.1.1. The result is as follows:

Theorem 3.4.4. *Assume the hypothesis in Question 3.1.1. Let $dim(M) = d$ and $rank(P) = n$. Then such a global section exists if one of the following conditions hold:*

1. $2n \geq d + 3$
2. $B = B_0 \times \mathbb{R}$

Proof. First note that since W is a closed set, $In\bar{t}W \subset W = \bar{W}$. Thus, we can apply 3.4.2 and get:

$$H^i(M \times \mathbb{R} \setminus Int(W), W \cup M \times 0 \setminus Int(W); \mathcal{L}) \cong H^i(M \times \mathbb{R} \setminus Int(W), W \cup M \times 0 \setminus Int(W); \mathcal{L})$$

Now consider the triple $(M \times \mathbb{R}, W \cup M \times 0, M \times 0)$. By 3.4.3, we have the long exact sequence of cohomologies:

However, as $M \times \mathbb{R}$ is homotopic to $M \times 0$ $H^i(M \times \mathbb{R}, M \times 0) = 0$. Thus,

$$H^{i+1}(M \times \mathbb{R}, W \cup M \times 0; \mathcal{L}) \cong H^i(W \cup M \times 0, M \times 0; \mathcal{L}), \forall i$$

But applying 3.4.2 excision again, we have:

$$H^i(W \cup M \times 0, M \times 0; \mathcal{L}) \cong H^i(W, W \cap M \times 0; \mathcal{L})$$

Since W is homotopic to B and $W \cap M \times 0$ is homotopic to B_0, we have the final isomorphism:

$$H^i(W, W \cap M \times 0; \mathcal{L}) \cong H^i(B, B_0 : \mathcal{L})$$

However, these groups vanish for all $i \geq n$ if

1. $dim(B) \leq n - 2$. That implies $dim(M) \leq 2n - 3$ as $1 + dim(M) = n + dim(B)$. Or

2. $B = B_0 \times \mathbb{R}$ (by homotopy).

■

Chapter 4

Obstruction Theory for Vector Bundles

We consider now the broad question:

Question 4.0.1. *Let V be a vector bundle of rank n over a smooth manifold M of dimension d. Then, under what conditions does V split i.e. there is a vector bundle V' such that $V \cong V' \oplus \mathbb{R}$?*

Note that there are some general result in this regard. We will begin with a few definitions.

Definition 4.0.2. *Let X be a topological space and $\{\phi_\beta\}_{\beta \in I}$ be a set of maps $X \to [0,1]$ where I is some indexing set. We say $\{\phi_\beta\}_{\beta \in I}$ is a partition of unity of X if for each point $x \in X$:*

1. *There is a neighbourhood of x in which all but finitely many ϕ_β's are zero, and*

2. $\sum_\beta \phi_\beta(x) = 1$

Let $\{U_\alpha\}$ be an open cover of X. Then, we say a partition of unity of X is ***subordinate*** to the open cover if for all β, $supp(\phi_\beta) \subseteq U_\alpha$ for some α (where $supp(\phi_\beta) := \overline{\phi^{-1}(\mathbb{R} \setminus 0)}$).

Definition 4.0.3. *A Hausdorff space M is said to paracompact if for each open cover $\{U_\alpha\}$ of M there is a partition of unity $\{\phi_\beta\}$ subordinate to the cover.*

The above definition of paracompactness is equivalent to another definition often found in the literature. That is, X is Hausdorff and has *locally finite* open refinement i.e. if $\{U_\alpha\}$ is a given open cover of X and then there is an open cover $\{V_\beta\}$ such that for each β $V_\beta \subseteq U_\alpha$ for some α and for each x in X, there is a neighbourhood of x which intersects only finitely many of the V_β's.

Note that most of the spaces we normally come across in Topology are paracompact. For example, *CW-spaces, metric spaces, compact Hausdorff spaces* are paracompact. Since smooth manifolds admit CW-complex structures, they are too paracompact. Next, we have the following lemma.

Lemma 4.0.4. *An inner product exists on a vector bundle (V, M, π) if M is paracompact.*

Proof. Let $\{U_\alpha\}$ be a chat of (V, M) and $\{\phi_\beta\}$ be the associated partition of unity. By definition, locally on each U_α, $V \cong U_\alpha \times \mathbb{R}^n$ and there is an inner product denoted as $\langle , \rangle_\alpha$ on this. Then define $\langle v, w\rangle := \sum_\beta \phi_\beta(\pi(v))\langle v, w\rangle_{\alpha(\beta)}$ where $supp(\phi_\beta) \subseteq U_{\alpha(\beta)}$. ■

Then we have the following theorem:

Theorem 4.0.5. *Let (V, M, π) be a vector bundle. If there is a novanishing section s on M, then V splits off a rank-one trivial subbundle (i.e. $V \cong V' \oplus (M \times \mathbb{R})$ for some vector bundle (V', M, π')).*

Proof. Let W be the subbundle such that each of its fiber W_m is generated by the vector $s(m)$ for all $m \in M$. Note that W is trivial bundle of rank 1 on M. This can be seen by showing the map: $\psi : M \times \mathbb{R} \to W$ which takes $(m, r) \mapsto (m, r.s(x))$ an isomorphism of vector bundles. Since by 4.0.4, there is an inner product on V, we can define $W^\perp := \{v \in V : \langle v, w\rangle = 0, \forall w \in W\}$. It is easy to see that $W^\perp$ is a subbundle and $V \cong W \oplus W^\perp$. ■

Thus, when we have a nonvanishing global section $s : M \to V$, then V splits. Especially, $n > d$ such a section exists and thus the vector bundle splits. So, we will consider only the vector bundles of rank $n \leq d$. In general, it is not true that vector bundles always split off trivial subbundles. For example, the tangent bundles over sphere $\mathbb{S}^n$ do not split except when $n = 1, 3$ and 7. So, our goal is to study the obstruction to such splitting. In this regard, Nori's result motivates us to define an obstruction class as we will see soon. However, let's start with a useful and interesting theorem first.

Theorem 4.0.6. *M is a smooth manifold, and (V, M, π) be a smooth vector bundle, with $rank(V) = n \leq \dim M = d$. Let $s_0, s_1 : M \to E$ be two sections such that they meet the zero section transver-*

sally at B_0 and B_1 respectively. Consider the pull back

$$\begin{tikzcd} p^\star(V) \arrow[r, dashed] \arrow[d, dashed] & V \arrow[d, "\pi"] \\ M \times \mathbb{R} \arrow[r, "p"'] & M \end{tikzcd}$$

Then, there is a section $s : M \times \mathbb{R} \longrightarrow p^\star(V)$ such that

1. $s|_{T=0} = s_0$ *and* $s|_{T=1} = s_1$.

2. s *meets the zero section transversally.*

Proof. We start with $s' = (1 - T)s_0 + Ts_1$. Then, clearly F meets the zero section transversally on $(M \times 0) \cup (M \times 1)$. We want to show that there is a section s such that

1. s is transverse to the zero-section of $p^\star(V)$.

2. $s(x) = s'(x)$ for all $x \in (M \times 0) \cup (M \times 1)$.

We will use the following lemma proven by Nori [M2]:

Lemma 4.0.7. *Let s' be a section of a vector bundle V of finite rank over M and A be a closed manifold of M. Suppose $s'|_A$ meets the zero section transversally. Then there is section s of V such that s is transverse to the zero section of V and $s(x) = s'(x), \forall x \in A$.*

Proof. It is known that there is a finite set of global sections $s_1, \ldots, s_m$ of V that generates the fiber V_x over each $x, \forall x \in M$ [Husemoller]. Next, we know that there is a smooth function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = A$. [Hirsch, Chapter 2.2 Exc-1] Now consider the set of global sections

$$u_k := f \cdot s_k : M \longrightarrow V \qquad k = 1, 2, \ldots, m$$

Then, the $\{u_k\}$'s have the properties:

1. all the u_k vanish on A

2. the $\{u_k\}$ generate V_x for every $x \in M \setminus A$.

Let $\phi : M \times \mathbb{R}^m \to V$ defined by

$$\varphi(x, t_1, \ldots, t_m) = s(x) + \sum t_i u_i(x)$$

Then, by (1) and (2) φ is transverse to the zero section of V. Then, $S := \phi^{-1}$(0-section) is a submanifold of $M \times \mathbb{R}^m$ [Hirsch, Chapter-1 Theorem 3.3]. Thus we have the following diagram:

$$\begin{tikzcd}
 & M \times t \arrow[d] \arrow[dr, bend left] & \\
S \arrow[r, hook] \arrow[dr, bend right] & M \times \mathbb{R}^m \arrow[r] \arrow[d] & V \\
 & \mathbb{R}^m &
\end{tikzcd}$$

Now we can apply Sard's theorem [Hirsch, Chapter-3 Theorem 2.7] to the projection $S \to \mathbb{R}^m$ to get a global section s' of V satisfying

1. s' is transverse to the zero section of V.

2. $s'(x) = s(x)$ for all $\forall x \in A$. ■

Now from Lemma 4.0.7 Theorem 4.0.6 easily follows where $A = (M \times 0) \cup (M \times 1)$. ■

4.1 Homotopy Obstruction Set for Vector Bundles

Definition 4.1.1. Let M be a smooth manifold, with $\dim M = d$ and $\mathscr{V}$ be vector bundle with $rank(\mathscr{V}) = n$. By a **local $\mathscr{V}$-orientation**, we mean a pair (B, φ) where $B \subseteq M$ is a submanifold of M, with $co\dim(M, B) = n$ and $\varphi : N(M, B) \xrightarrow{\sim} \mathscr{V}|_B$ is an isomorphism. The empty submanifold of M would be denoted by ϕ.

Denote

$$\begin{cases} \mathcal{LO}^n(\mathscr{V}) = \{(B,\varphi) : ((B,\varphi)) \text{ is a local } \mathscr{V} \text{ orientation}\} \\ \text{Let } \widetilde{\mathcal{LO}}(\mathscr{V}) = \mathcal{LO}^n(\mathscr{V}) \cup \{\phi\} \end{cases} \tag{4.1}$$

For $(B_0,\varphi_0),(B_1,\varphi_1) \in \mathcal{LO}(\mathscr{V})$, we say that (B_0,ϕ_0) is *homotopic* to (B_1,φ_1) and write $(B_0,\varphi_0) \sim_H (B_1,\varphi_1)$, if there is an $(B,\varphi) \in \mathcal{LO}(p^\star(\mathscr{V}))$ such that $(B,\varphi)|_{T=0} = (B_0,\varphi_0)$ and $(B,\varphi)|_{T=1} = (B_1,\varphi_1)$. Similarly, $(B_0,\varphi_0) \sim_H \{\phi\}$ if there is $(B,\varphi) \in \mathcal{LO}(p^\star(\mathscr{V}))$ such that $(B,\varphi)|_{T=0} = (B_0,\varphi_0)$ and $B \cap (M \times 1) = \phi$. The relation $\sim_H$ generates a chain equivalence relation $\sim$ on $\mathcal{LO}(\mathscr{V})$, which we call the *chain homotopy relation*. Then, the (Nori) **Homotopy obstruction** set $\pi_0\left(\widetilde{\mathcal{LO}}(\mathscr{V})\right) := \frac{\widetilde{\mathcal{LO}}(\mathscr{V})}{\sim}$ is defined to be the set of equivalence classes of elements in $\widetilde{\mathcal{LO}}(\mathscr{V})$.

We will define the obstruction element $\varepsilon(V)$ as follows:

1. Take a section $s_0 : M \longrightarrow V$ such that $B_0 = (s_0 = 0)$ has a codimension n.

   This induces an isomorphism

$$\overline{s_0} : N(M, B_0) \xrightarrow{\sim} V|_{B_0}$$

2. Define the obstruction class

$$\mathbf{e}_0 := \varepsilon(V) = [(B_0, \overline{s_0})] \in \pi_0(\widetilde{\mathcal{LO}}(V)))$$

   This is well-defined. Because, let $s_1 : M \longrightarrow V$ be another such section, with $B_1 = (s_1 = 0)$ of codimension n. Let $F = (1-T)s_0 + Ts_1$. Then, by Theorem 4.0.6 these two sections are homotopic, thus belong to the same class.

3. Next we will consider another distinguished element in $\pi_0(\widetilde{\mathcal{LO}}(V))$. Let $e_1 := [\phi]$ be the class associated with the empty-manifold. Then we have the following important theorem that we are looking for in the beginning of the section.

Theorem 4.1.2. *Let M be a smooth manifold of dimension d and V be a vector bundle of rank n on M. Let $d \leq 2n-3$. Then consider $\epsilon(V)$ as defined above. If $\epsilon(V) = e_1$, then V splits.*

Proof. Let $(B_0, \overline{s_0})$ be such that $\epsilon(V) = [(B_0, \overline{s_0})]$ (as defined above). Since $[(B_0, \overline{s_0})] = e_1$ by hypothesis, we have $(B_0, \varphi_0) \sim_H (B_1, \varphi_1) \sim_H \cdots \sim_H (B_k, \varphi_k) \sim_H \{\phi\}$. By Theorem 3.4.4 when $d \leq 2n-3$, there are sections $\widetilde{s_i} : M \times \mathbb{R}$ such that $\overline{\widetilde{s_i}} : N(M \times R, B_i) \xrightarrow{\sim} p^{\star}(V)|_{B_i}$ where $\widetilde{s_i}|_{M \times 0} = s_i$ and $B_i = \{x \in M | s_i(x) = 0\}$, for all $i \leq k$. Since $B_k \sim_H \{\phi\}$, by definition $B_k|_{M \times 1} = \phi$ and thus, $s_k|_{M \times 1}$ is a non-vanishing section on $M \times 1 \cong M$. Thus, V splits by theorem 4.0.5. The proof is complete. ■

Moreover, we can say a bit more.

Corollary 4.1.3. *For any isomorphism $\phi_0 : N(M, B_0) \to V|_{B_0}$, such that (B_0, ϕ_0) is homotopic to $\epsilon(V) = [(B_0, \overline{s_0})]$, then that isomorphism is induced from a section $s : M \to V$.*

Proof. Similar to above. ■

Chapter 5

Obstruction Theory in Algebra

Similar to the question of *triviality* and obstructions to it for vector bundles, which we have dealt in the previous chapters, there is also a quest for investigating the obstructions to having *direct summands* for the projective modules, the algebraic counterpart of vector bundles. We will start with the definition of projective modules:

Definition 5.0.1. *An A-module P is said to be projective if there is an A-module Q such that $P \oplus Q$ is free.*

There are two more equivalent formulations of the definition of the projective modules which are used interchangeably in the literature. However, for our purpose the above definition would suffice. Moreover, we will always assume A is Noetherian and P is finitely generated. Now the question of interest here is:

Question 5.0.2. *What is the obstruction for P to split off a direct summand i.e. $P \cong Q \oplus A$ for some A-module Q?*

The desire to define an obstruction class, for P to split off a free direct summand, is age old and might have been considered too bold. Though there has been significant research on this starting from Serre's question on projective modules from mid-twentieth century, it is quite far from culminating satisfying results so far. We will develop here an invariant called **Nori homotopy class** for a projective module which acts as an important obstruction element.

5.1 Foundation of Homotopy Obstructions

In this section, we establish some notations and for a projective module P over a noetherian ring A, give several descriptions of the *homotopy sets*.

Notations 5.1.1. Throughout, A will denote a commutative Noetherian ring with $\dim A = d$ and k will denote a field. Often, but not always, we will assume $\frac{1}{2} \in A$ and/or $k \subseteq A$. For A-modules M, N, we denote $M[T] := M \otimes A[T]$ and $M^* = Hom(M, A)$. For $f \in Hom(M, N)$, denote $f[T] := f \otimes 1 \in Hom(M[T], N[T])$.

For ideals $I \subseteq A$, homomorphisms $f : M \longrightarrow \frac{I}{I^2}$ would be identified with the induced maps $\frac{M}{IM} \longrightarrow \frac{I}{I^2}$. For surjective homomorphisms $\omega_1 : M \twoheadrightarrow \frac{I_1}{I_1^2}$, $\omega_2 : M \twoheadrightarrow \frac{I_2}{I_2^2}$, where I_1, I_2 are two ideals, with $I_1 + I_2 = A$, $\omega_1 \star \omega_2 : M \twoheadrightarrow \frac{I_1 I_2}{(I_1 I_2)^2}$ will denote the unique surjective map induced by ω_1, ω_2.

For a projective A-module P, $\mathbb{Q}(P) = (\mathbb{Q}(P), q)$ will denote the quadratic space $\mathbb{H}(P) \perp A$, where $\mathbb{H}(P) = P^* \oplus P$ is the hyperbolic space. So, $P^* \oplus P \oplus A$ is the underlying projective module of $\mathbb{Q}(P)$ and, for $(f, p, s) \in P^* \oplus P \oplus A$, $q(f, p, s) = f(p) + s^2$.

Definition 5.1.2. Let A be a noetherian commutative ring, $X = \mathrm{Spec}\,(A)$ and P be a projective A-module. By a **local P-orientation**, we mean a pair (I, ω) where I is an ideal of A and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism, which is identified with the surjective homomorphism $\frac{P}{IP} \twoheadrightarrow \frac{I}{I^2}$, induced by ω. A local P-orientation will simply be referred to as a **local orientation**, when P is understood.

Denote

$$\begin{cases} \mathcal{LO}(P) = \{(I, \omega) : (I, \omega) \text{ is a local } P \text{ orientation}\} \\ \mathcal{Q}(P) = \{(f, s) \in P^* \oplus A : s(1 - s) \in f(P)\} \\ \widetilde{\mathcal{Q}}(P) = \{(f, p, s) \in P^* \oplus P \oplus A : f(p) + s(s - 1) = 0\} \\ \widetilde{\mathcal{Q}}'(P) = \{(f, p, z) \in P^* \oplus P \oplus A : f(p) + z^2 = 1\} \end{cases} \tag{5.1}$$

There is a commutative diagram of set theoretic maps, denoted as follows:

$$\begin{tikzcd} \widetilde{\mathcal{Q}}(P) \arrow[r, two heads, "\nu"] \arrow[d, "\eta"'] & \mathcal{Q}(P) \arrow[dl, "\eta'"] \\ \mathcal{LO}(P) & \end{tikzcd} \qquad \text{where, for } (f,p,s) \in \widetilde{\mathcal{Q}}(P), \quad \nu(f,p,s) = (f,s) \tag{5.2}$$

and $\eta'(f,s) = \eta(f,p,s) = (I,\omega)$, where $I = f(P) + As$ and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is the homomorphism induced by f. These maps η, η', ν are surjective. If and when $1/2 \in A$ (*which we often assume*), there is also a bijection

$$\kappa : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}'(P) \quad \text{sending} \quad (f,p,s) \mapsto (2f, 2p, 2s-1) \tag{5.3}$$

Definition 5.1.3. *Let X, Y and Z are three objects in a category with morphisms $f : Z \to X$ and $g : Z \to Y$. Consider a tripple (P, i_1, i_2) where P is an object with two morphisms i_1 and i_2 from X and Y respectively such that the following diagram commutes:*

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \\ Y \arrow[r, "i_2"'] & P \end{tikzcd} \tag{5.4}$$

*and such that (P, i_1, i_2) is universal with respect to above diagram. Then, (P, i_1, i_2) is defined to be the **pushout** or **push forward** of (f,g).*

Definition 5.1.4. *Assume the notations of 5.1.2. The **Nori Homotopy obstruction set** $\pi_0(\mathcal{LO}(P))$ defined by the pushout diagrams, in $\underline{Sets}$, as follows:*

$$\begin{tikzcd} \mathcal{LO}(P[T]) \arrow[r, "T=0"] \arrow[d, "T=1"'] & \mathcal{LO}(P) \arrow[d] \\ \mathcal{LO}(P) \arrow[r] & \pi_0(\mathcal{LO}(P)) \end{tikzcd} \qquad \text{in} \quad \underline{Sets}. \tag{5.5}$$

Indeed, $\pi_0(\mathcal{LO}(P))$ was the Homotopy obstruction explicitly envisioned by Nori (see [M2]).

Similarly, $\pi_0(\mathcal{Q}(P))$, $\pi_0(\widetilde{\mathcal{Q}}(P))$ and $\pi_0(\widetilde{\mathcal{Q}}'(P))$ are defined.

We record, the following basic lemma:

Lemma 5.1.5. *Assume $1/2 \in A$. Then, the bijection κ, induces an isomorphism*

$$\overline{\kappa} : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$$

Further, the maps η, ν, η' (in diagram 5.2) induce set theoretic maps, as denoted in the commutative diagram of maps of pre-sheaves:

$$\begin{tikzcd} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \arrow[r, two heads, "\overline{\nu}"] \arrow[d, "\overline{\eta}"'] & \pi_0\left(\mathcal{Q}(P)\right) \arrow[dl, "\overline{\eta}'"] \\ \pi_0\left(\mathcal{L}O(P)\right) & \end{tikzcd}$$

Proof. It follows from definition of pushout. ■

We proceed to prove that, the above is a commutative triangle of bijections:

$$\begin{tikzcd} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \arrow[r, two heads, "\overline{\nu}", "\sim"'] \arrow[d, "\overline{\eta}"', "\wr"] & \pi_0\left(\mathcal{Q}(P)\right) \arrow[dl, "\sim"', "\overline{\eta}'"] \\ \pi_0\left(\mathcal{L}O(P)\right) & \end{tikzcd} \tag{5.6}$$

We fix notations, for $(f, p, s) \in \widetilde{\mathcal{Q}}(P)$, its equivalence class in $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ will be denoted by $[(f, p, s)]$ and similar notations will be used for $(f, s) \in \mathcal{Q}(P)$ and $(I, \omega) \in \mathcal{L}O(P)$. Note, given $(I, \omega) \in \mathcal{L}\mathcal{O}(P)$, ω lifts to a homomorphism f, as follows:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "\omega"'] & I \arrow[d, two heads] \\ & \frac{I}{I^2} \end{tikzcd} \tag{5.7}$$

By Nakayama's lemma there is an element $s \in I$ such that $(1 - s)I \subseteq f(P)$. Consequently,

$(f, s) \in \mathcal{Q}(P)$ and $I = (f(P), s)$. This association would not be unique. Such a pair $(f, s) \in \mathcal{Q}(P)$ will be referred to as a lift of (I, ω) in $\mathcal{Q}(P)$. Now define the map:

$$\chi : \mathcal{LO}(P) \longrightarrow \pi_0\left(\mathcal{Q}(P)\right) \quad \text{by} \quad \chi\left(I, \omega\right) = [(f, s)] \in \pi_0\left(\mathcal{Q}(P)\right) \tag{5.8}$$

where $(f, s) \in \mathcal{Q}(P)$ is any lift of (I, ω) in $\mathcal{Q}(P)$, (*as in diagram 5.7*) and $[(f, s)]$ is its equivalence class. In several lemmas, we establish that χ is well defined.

Lemma 5.1.6. *Let $(I, \omega_I) \in \mathcal{LO}(P)$ and $(f, s) \in \mathcal{Q}(P)$ be a lift, as in diagram (5.7). Further, assume that $t(1-t) \in f(P)$, with $I = (f(P), s) = (f(P), t)$. Then*

$$[(f, s)] = [(f, t)] \in \pi_0\left(\mathcal{Q}(P)\right).$$

Proof. First note, $(1-s)I \subseteq f(P)$ and $(1-t)I \subseteq f(P)$. Write $I[T] = IA[T]$. So,

$$I[T] = f(P)A[T] + sA[T] = f(P)A[T] + tA[T].$$

Let $S(T) = t + T(s-t)$. Clearly, $S(T) \in I[T]$. Further, we claim

$$(1 - S(T))I[T] \subseteq f(P)A[T].$$

We have $(1 - S(T))I[T] = (1 - S(T))\left(f(P)A[T] + sA[T]\right)$. So, we only need to prove that $(1 - S(T))s \in f(P)A[T]$. But

$$(1 - S(T))s = (1-t)s - T(s-t)s = (1-t)s - T[(s-1)s + (1-t)s] \in f(P)A[T]$$

So, the claim is established. Therefore, $(1 - S(T))S(T) \in f(P)A[T]$. Denote $f[T] := f \otimes 1_{A[T]}$. Then, $f[T] : P[T] \twoheadrightarrow f(P)A[T]$ is a surjection. Clearly, $(f[T], S(T)) \in \mathcal{Q}(P[T])$. Now, $(f[T], S(T))_{T=0} = (f, t)$ and $(f[T], S(T))_{T=1} = (f, s)$. ■

Lemma 5.1.7. *Suppose* $(I,\omega) \in \mathcal{LO}(P)$ *and* f, g *be two lifts of* ω *as follows:*

$$\begin{tikzcd} P \arrow[r, two heads, "f"] \arrow[dr, "\omega"'] & f(P) \arrow[d, two heads] \\ & \frac{I}{I^2} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[r, two heads, "g"] \arrow[dr, "\omega"'] & g(P) \arrow[d, two heads] \\ & \frac{I}{I^2} \end{tikzcd}$$

$$\ni \quad I = (f(P), s) = (g(P), t) \text{ and } s(1-s) \in f(P), \ t(1-t) \in g(P).$$

Then

$$[(f,s)] = [(g,t)] \in \pi_0\left(\mathcal{Q}(P)\right)$$

Proof. Note, $(g-f)(P) \subseteq I^2$. Let $F = f[T] + T(g[T] - f[T]) \in P[T]^*$. It is obvious that

$$I[T] = F(P[T]) + I[T]^2$$

For completeness, we give a proof.

$$\forall\, x \in I, x = (1-s)x + sx = f(p) + sx \quad \text{where} \quad p \in P, \ sx \in I^2$$

So,

$$(\text{modulo } I[T]^2) \quad x \equiv f(p) \equiv F[T](p).$$

So,

$$\exists \quad S(T) \in I[T] \ \ni (1 - S(T))I[T] \subseteq F[T](P[T])$$

So, $(F[T], S(T)) \in \mathcal{Q}(P[T])$. Therefore,

$$[(f, S(0))] = [(F(0), S(0))] = [(F(1), S(1))] = [(g, S(1))]$$

Now, the proof is complete by (5.1.6). ■

Theorem 5.1.8. *Let* $(I,\omega) \in \mathcal{LO}(P)$. *Then,* $\chi(I,\omega)$ *as defined in equation (5.8), is well defined.*

Proof. Follows from Lemma 5.1.7. ■

Now, we prove that $\overline{\nu}$ is a bijection, as follows.

Theorem 5.1.9. *The map, as defined in 5.1.5,*

$$\overline{\nu} : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \twoheadrightarrow \pi_0\left(\mathcal{Q}(P)\right)$$

is a bijection.

Proof. Define a map $\Psi_0 : \mathcal{Q}(P) \to \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ as follows: Given $(f,s) \in \mathcal{Q}(P)$, $\exists\ p \in P \ni f(p) = s(1-s)$. Define

$$\Psi_0(f,s) := [(f,p,s)] \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

We show that this association is a well defined map. To show this, suppose there is another $q \in P$ such that $f(q) = s(1-s)$. Note $f(p-q) = 0$. So, $f[T](p + T(q-p)) = f(p) + Tf(q-p) = f(p) + 0 = s(1-s)$. Therefore,

$$H(T) := (f[T], p + T(q-p), s) \in \widetilde{\mathcal{Q}}(P[T])$$

and, hence

$$H(0) = (f,p,s) \sim H(1) = (f,q,s).$$

This establishes that Ψ_0 is well defined. Now, we show that Ψ_0 is homotopy invariant. To see this, suppose $H(T) = (F, S(T)) \in \mathcal{Q}(P[T])$. Then, $S(T)(1-S(T)) = F(p(T))$, for some $p(T) \in P[T]$. Write $\widetilde{H} = (F, p(T), S(T)) \in \widetilde{\mathcal{Q}}(P[T])$. So,

$$\Psi_0(F(0), s(0)) = [\widetilde{H}(0)] = [\widetilde{H}(1)] = \Psi_0(F(1), S(1))$$

This establishes that Ψ_0 factors through a map

$$\Psi : \pi_0\left(\mathcal{Q}(P)\right) \to \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

It is easy to check that $\overline{\nu}$ and Ψ are inverse of each other. The proof is complete. ■

Lemma 5.1.10. *The map $\chi : \mathcal{L}O(P) \longrightarrow \pi_0\left(\mathcal{Q}(P)\right)$ (see (5.8)) induces a well defined map $\overline{\chi} : \pi_0\left(\mathcal{L}O(P)\right) \longrightarrow \pi_0\left(\mathcal{Q}(P)\right)$, which is the inverse of the map $\overline{\eta}' : \pi_0\left(\mathcal{Q}(P)\right) \longrightarrow \pi_0\left(\mathcal{L}O(P)\right)$.*

Consequently, all the maps $\overline{\eta}$, $\overline{\eta}'$, $\overline{\nu}$ in diagram 5.6, are bijections.

Proof. The latter statement follows from the first one. Given a homotopy $H(T) \in \mathcal{L}O(P[T])$, it lifts to a homotopy $\widetilde{H}(T) = (F(T), S(T)) \in \mathcal{Q}(P[T])$. So, $\chi(H(0)) = [(F(0), S(0))] = [(F(1), S(1))] = \chi(H(1))$. So, χ is homotopy invariant, hence $\overline{\chi}$ is well defined. It is easy to see that this induced map is the inverse of $\overline{\eta}'$. The proof is complete. ■

Corollary 5.1.11. *Recall the notation*

$$Q_{2n}(A) = \left\{(x_1, \ldots, x_n; y_1, \ldots, y_n; z) \in A^{2n+1} : \sum x_i y_i = z(1-z)\right\}.$$

If $P = A^n = \oplus Ae_i$ is free, then $Q_{2n}(A) \cong \widetilde{\mathcal{Q}}(P)$ is a bijection. This bijection induces a bijection $\pi_0\left(Q_{2n}(A)\right) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$.

Before we proceed, we introduce the following notions.

Notations 5.1.12. Suppose A is a commutative noetherian ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. Denote $\zeta = \overline{\nu}^{-1}\chi : \mathcal{L}O(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ and $\zeta_0 : \widetilde{\mathcal{Q}}(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. So, we have a commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{Q}}(P) \arrow[d, "\eta"'] \arrow[dr, "\zeta_0"] & \\
\mathcal{L}O(P) \arrow[r, "\zeta"'] & \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)
\end{tikzcd}$$

5.2 Homotopy Equivalence

In this section, we prove the following key homotopy theorem.

Theorem 5.2.1. *Let A be a regular ring over a field k, with $1/2 \in k$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$* (see 5.1.1). *Recall $\widetilde{\mathcal{Q}}'(P) \subseteq \mathbb{Q}(P) = P^* \oplus P \oplus A$. Suppose $H(T) \in \widetilde{\mathcal{Q}}'(P[T])$. Then, there is an orthogonal transformation $\sigma(T) \in O(\mathbb{Q}(P[T]), q)$, such that*

$$H(T) = \sigma(T)(H(0)) \qquad \text{and} \qquad \sigma(0) = 1.$$

Proof. Let $H(T) = (f(T), p(T), s(T)) \in \widetilde{\mathcal{Q}}'(P[T])$ be a homotopy, as above. So, $H(0) \in \widetilde{\mathcal{Q}}'(P)$. Then,

$$A[T]H(T) \cong A[T]H(0) \cong (A[T], q_0) \qquad \text{are isometric,}$$

where q_0 is the trivial quadratic space of rank one. The bilinear inner product in $\mathbb{Q}(P)$ will be denoted $\langle -, - \rangle$. We have the following split exact sequences of quadratic spaces:

$$0 \longrightarrow K \longrightarrow \mathbb{Q}(P[T]) \xrightarrow{\langle H(T), - \rangle} A[T] \longrightarrow 0$$

$$0 \longrightarrow K_0 \longrightarrow \mathbb{Q}(P) \xrightarrow[\langle H(0), - \rangle]{} A \longrightarrow 0$$

Therefore, $K = (A[T]H(T))^{\perp}$, $K_0 = (AH(0))^{\perp}$ are orthogonal complements. Write $\overline{K} := K \otimes \frac{A[T]}{(T)}$. Note, for $\wp \in \operatorname{Spec}(A)$, $\mathbb{Q}(P)_{\wp} \cong (A, q_{2n+1})$, where $q_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z^2$. So, $\overline{K}_{\wp} \cong (K_0)_{\wp}$ are isometric. It is standard (see [MM1, Lemma 4.1]), that $(K_0)_{\wp} = (A_{\wp}H(0))^{\perp} \cong (A, q_{2n})_{\wp}$ where $q_{2n} = \sum_{i=1}^{n} X_i Y_i$. In other words, $\overline{K}$ is locally trivial. By the Quadratic version [MM1, Theorem 3.5] of Lindel's theorem [L], there is an isometry $\tau : K \xrightarrow{\sim} \overline{K} \otimes A[T]$. Further, it follows $\overline{K} = (AH(0))^{\perp} \cong K_0$. Therefore, there is an isometry $\sigma_0 : \overline{K} \xrightarrow{\sim} K_0$, which extends to an isometry $\sigma_0 \otimes 1 : \overline{K} \otimes A[T] \xrightarrow{\sim} K_0 \otimes A[T]$. Then, $\sigma_1 := (\sigma_0 \otimes 1)\tau : K \xrightarrow{\sim} K_0 \otimes A[T]$ is an isometry. Finally, note

$$(A[T]H(T), q_{|A[T]H(T)}) \cong (A[T], q_0) \cong (A[T]H(0), q_{|A[T]H(0)}).$$

Now, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\sigma_1"'] & \mathbb{Q}(P[T]) \arrow[r, "\langle H(T),-\rangle"] \arrow[d, dashed, "\sigma(T)"] & A[T] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_0 \otimes A[T] \arrow[r] & \mathbb{Q}(P[T]) \arrow[r, "\langle H(0),-\rangle"'] & A[T] \arrow[r] & 0 \end{tikzcd} \tag{5.9}$$

of quadratic spaces. In this diagram, the horizontal lines are split exact sequences of quadratic spaces. Hence, there is an isometry $\sigma(T) \in O\left(\mathbb{Q}(P[T]), q\right)$, such that the diagram commutes. That means, for all $\mathbf{v} \in \mathbb{Q}(P[T])$, we have $\langle H(T), \mathbf{v}\rangle = \langle H(0), \sigma(T)\mathbf{v}\rangle$. Replacing $\sigma(T)$ by $\sigma(T)^{-1}$, we have $\sigma(T)H(0) = H(T)$. So, we have $\sigma(0)H(0) = H(0)$. Again, by replacing $\sigma(T)$ by $\sigma(T)\sigma(0)^{-1}$, we have $\sigma(0) = 1$. The proof is complete. ∎

The following Corollary would be of some importance for our future discussions.

Corollary 5.2.2. *Let A be a regular ring over a field k, with $1/2 \in k$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$. Let $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{Q}}'(P)$ be such that $[\mathbf{u}] = [\mathbf{v}] \in \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$. Then, there is a homotopy $H(T) \in \widetilde{\mathcal{Q}}'(P[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$. Equivalently, for $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{Q}}(P)$ if $\zeta_0(\mathbf{u}) = \zeta_0(\mathbf{v}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, then there is a homotopy $H(T) \in \widetilde{\mathcal{Q}}(P[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$.*

Proof. Suppose $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{Q}}(P)$ such that $[\mathbf{u}] = [\mathbf{v}] \in \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$. Then, there is a sequence of homotopies $H_1(T), \ldots, H_m(T) \in \widetilde{\mathcal{Q}}'(P[T])$ such that $\mathbf{u} =: \mathbf{u}_0 := H_1(0)$, $\mathbf{u}_m := H_m(1) = \mathbf{v}$ and $\forall\ i = 1, \ldots, m-1$, we have $\mathbf{u}_i := H_i(1) = H_{i+1}(0)$. By Theorem 5.2.1, for $i = 1, \ldots, m$ there are orthogonal matrices $\sigma_i(T) \in O(\mathbb{Q}(P[T]), q)$ such that $\sigma_i(0) = 1$ and $H_i(T) = \sigma_i(T)H_i(0) = \sigma_i(T)\mathbf{u}_{i-1}$. Therefore, $\mathbf{u}_i = H_i(1) = \sigma_i(1)\mathbf{u}_{i-1}$.

Write $H(T) = \sigma_m(T)\cdots\sigma_1(T)\mathbf{u}_0$. Then, $H(T) \in \widetilde{\mathcal{Q}}'(P[T])$ and $H(0) = \mathbf{u}_0$ and $H(1) = \mathbf{u}_m$.

This establishes first part of the statement on $\pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$. The latter assertion on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ follows from the former, by the bijective correspondences $\widetilde{\mathcal{Q}}'(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P)$ and $\widetilde{\mathcal{Q}}'(P[T]) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P[T])$. This completes the proof. ■

Remark 5.2.3. Another way to state (5.2.2) would be that the homotopy relation on $\widetilde{\mathcal{Q}}(P)$ is actually an equivalence relation.

In a slightly more formal language, the above is summarized as follows.

Theorem 5.2.4. *Let A be a regular ring over a field k, with $1/2 \in k$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$. For, $\sigma(T) \in O\left(\mathbb{Q}(P[T]), q\right)$ and $\mathbf{u} \in \widetilde{\mathcal{Q}}'(P)$, define the* (left) *action $\sigma(T)\mathbf{u} := \sigma(1)\mathbf{u} \in \widetilde{\mathcal{Q}}'(P)$. Denote $O\left(\mathbb{Q}(P[T]), q, T\right) = \{\sigma(T) \in O\left(\mathbb{Q}(P[T]), q\right) : \sigma(0) = 1\}$. Then, the map*

$$\frac{\widetilde{\mathcal{Q}}'(P)}{O\left(\mathbb{Q}(P), q, T\right)} \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right) \quad \text{is a bijection.}$$

Proof. Similar to the proof of (5.2.2). ■

5.3 Homotopy Triviality and Lifting

In this section, under further smoothness conditions, we establish that for $(I, \omega_I) \in \mathcal{L}O(P)$, the triviality of $\zeta(I, \omega_I)$ implies that ω_I lifts to a surjective map $P \twoheadrightarrow I$. We start this section with the following notations and definitions.

Definition 5.3.1. Suppose A is a commutative noetherian ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. There are two distinguished points in $\widetilde{\mathcal{Q}}(P)$, namely:

$$\mathbf{0} := (0, 0, 0) \in \widetilde{\mathcal{Q}}(P), \quad \mathbf{1} := (0, 0, 1) \in \widetilde{\mathcal{Q}}(P)$$

$$\text{We denote} \quad \mathbf{e}_0 = \zeta_0(\mathbf{0}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right), \quad \text{and} \quad \mathbf{e}_1 = \zeta_0(\mathbf{1}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

Use the same notations $\mathbf{e}_0,\ \mathbf{e}_1 \in \pi_0\left(\mathcal{L}O(P)\right) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, to denote their respective images. Define the obstruction class

$$\varepsilon(P) := \mathbf{e}_0 \in \pi_0\left(\mathcal{L}O(P)\right) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

In the light of (4.0.6), $\varepsilon(P)$ will be referred to as **(Nori) Homotopy Class** of P, which may sometimes be shortened. Note, for any $f \in P^*$ and $p \in P$, $\varepsilon(P) := \mathbf{e}_0 = \zeta_0(f, 0, 0) = \zeta_0(0, p, 0) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$.

We record the following obvious observation.

Lemma 5.3.2. *Suppose A is a commutative noetherian ring with $\dim A = d$ and P is a projective A-module. Let $p \in P$ and $f \in P^*$ be such that $f(p) = 1$ (i. e. $P \cong Q \oplus A$). Let*

$$\mathbf{0} = (0, 0, 0),\ \mathbf{u} = (f, 0, 0),\ \mathbf{1} = (0, 0, 1) \in \widetilde{\mathcal{Q}}(P).$$

Then, $\zeta_0(\mathbf{0}) = \zeta_0(\mathbf{u}) = \zeta_0(\mathbf{1}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. In other words,

$$\varepsilon(P) = \mathbf{e}_0 = \mathbf{e}_1.$$

Proof. The first equality is obvious and was mentioned above (5.3.1). To prove the second equality, write $H(T) = ((1-T)f, Tp, T)$. Then, $(1-T)f(Tp) = T(1-T)$. So, $H(T) \in \widetilde{\mathcal{Q}}(P[T])$. We have $H(0) = \mathbf{u}$ and $H(1) = (0, p, 1)$.

Now write $G(T) = (0, (1-T)p, 1)) \in \widetilde{\mathcal{Q}}(P[T])$. Then, $G(0) = (0, p, 1)$ and $G(1) = (0, 0, 1)$. The proof is complete. ∎

The following is the main result in this section.

Theorem 5.3.3. *Suppose A is an essentially smooth ring over an infinite perfect field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$, with $2n \geq d+3$. Suppose $(I, \omega_I) \in \mathcal{LO}(P)$, with $height(I) \geq n$. Then, ω_I lifts to a surjective map $P \twoheadrightarrow I$ if and only if $\varepsilon(P) = \zeta(I, \omega_I)$.*

Proof. Suppose ω_I lifts to a surjective map $f : P \twoheadrightarrow I$. Write $H(T) = (f(T), 0, 0) \in \widetilde{\mathcal{Q}}(P[T])$. Then, $\zeta(I, \omega_I) = \zeta_0(H(1)) = \zeta_0(H(0)) = \zeta_0(\mathbf{0}) = \varepsilon(P)$.

Conversely, suppose $\zeta(I, \omega_I) = \zeta_0(\mathbf{0})$. For notational convenience, fix $f_0 \in P^*$, and let $\mathbf{v}_0 = (f_0, 0, 0) \in \widetilde{\mathcal{Q}}(P)$. Then, $\zeta(I, \omega_I) = \zeta_0(\mathbf{0}) = \zeta_0(\mathbf{v}_0)$. There is an element $\mathbf{u} = (f_1, p_1, s_1) \in \widetilde{\mathcal{Q}}(P)$ such that $\eta(\mathbf{u}) = (I, \omega_I)$. By Moving Lemma argument 5.4.5 (below), we can assume that $height(f_0(P)) \geq n$ and $height(f_1(P)) \geq n$. We have, $\zeta_0(\mathbf{u}) = \zeta_0(\mathbf{v}_0)$. By (5.2.2), there is a homotopy $H(T) = (f(T), p(T), S(T)) \in \widetilde{\mathcal{Q}}(P[T])$ such that $H(0) = \mathbf{v}_0$ and $H(1) = \mathbf{u}$. Write $\eta(H(T)) = (J, \Omega)$. We would apply [BK, Theorem 4.13], for which we would need $height(J) \geq n$. So, we modify $H(T)$, as follows. Denote $Z(T) = 1 - S(T)$. Write $\mathscr{P} = \{\wp \in \mathrm{Spec}\,(A[T]) : height(\wp) \leq n-1, T(1-T)Z(T) \notin \wp\}$. For $\wp \in \mathscr{P}$, let $\delta(\wp)$ be the maximum of the length of chains in $\mathscr{P}$, ending at $\wp$. Then $\delta : \mathscr{P} \longrightarrow \mathbb{N}$ is a generalized dimension functions (*consult* [M1, pp. 36-37]). Note, $\forall\, \wp \in \mathscr{P}$, we have $\delta(\wp) \leq n-1$. Now, $(f(T), T(1-T)Z(T)^2) \in P[T]^* \oplus A[T]$ is basic on $\mathscr{P}$. So, there is an element $g(T) \in P[T]^*$ such that $F(T) = f(T) + T(1-T)Z(T)^2 g(T)$ is basic on $\mathscr{P}$. It follows, $F(0) = f(0)$ and $F(1) = f(1)$. We have $Z(T)(1 - Z(T)) = (1 - S(T))S(T) =$

$$f(T)(p(T)) = F(T)(p(T)) - T(1-T)Z(T)^2 g(T)(p(T))$$

Write $\mathcal{J} = (f(T)(P[T]), Z(T))$. Then $\mathcal{J} = (F(T)(P[T]), Z(T))$. Write $M = \frac{\mathcal{J}}{F(T)(P[T])}$. Let $p_1, \ldots, p_m$ be a set of generators of P. So, $\mathcal{J}$ is generated by $f(T)(p_1), \ldots, f(T)(p_m), Z(T)$. Use "overline" to denote the images in M and repeat the proof of Nakayama's Lemma, as follows:

$$\begin{pmatrix} \overline{f(T)(p_1)} \\ \overline{f(T)(p_2)} \\ \cdots \\ \overline{f(T)(p_m)} \\ \overline{Z(T)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -T(1-T)Z(T)g(T)(p_1) \\ 0 & 0 & \cdots & 0 & -T(1-T)Z(T)g(T)(p_2) \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & -T(1-T)Z(T)g(T)(p_m) \\ 0 & 0 & \cdots & 0 & Z(T)-T(1-T)Z(T)g(T)(p(T)) \end{pmatrix} \begin{pmatrix} \overline{f(T)(p_1)} \\ \overline{f(T)(p_2)} \\ \cdots \\ \overline{f(T)(p_m)} \\ \overline{Z(T)} \end{pmatrix}$$

So,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & T(1-T)Z(T)g(T)(p_1) \\ 0 & 1 & \cdots & 0 & T(1-T)Z(T)g(T)(p_2) \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 1 & T(1-T)Z(T)g(T)(p_m) \\ 0 & 0 & \cdots & 0 & 1-Z(T)+T(1-T)Z(T)g(T)(p(T)) \end{pmatrix} \begin{pmatrix} \overline{f(T)(p_1)} \\ \overline{f(T)(p_2)} \\ \cdots \\ \overline{f(T)(p_m)} \\ \overline{Z(T)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

With $Z'(T) = Z(T) - T(1-T)Z(T)g(T)(p(T))$, the determinant of this matrix is $1 - Z'(T)$. It follows, $(1 - Z'(T))\mathcal{J} \subseteq F(T)(P[T])$. So,
$(1 - Z'(T))Z'(T) = F(T)(q(T))$ for some $q(T) \in P[T]$. Note, $Z'(0) = Z(0)$ and $Z'(1) = Z(1)$. Therefore, $(F(T), q(T), Z'(T)) \in \widetilde{\mathcal{Q}}(P[T])$. Also, with $S'(T) = 1 - Z'(T)$, $(F(T), q(T), S'(T)) \in \widetilde{\mathcal{Q}}(P[T])$. We have,
$S'(T)(1 - S'(T)) = (1 - Z'(T))Z'(T) = F(T)(q(T))$
$S'(0) = 1 - Z'(0) = 1 - Z(0) = S(0) = 0$ and
$S'(1) = 1 - Z'(1) = 1 - Z(1) = S(1)$.
Write $\mathcal{H}(T) = (F(T), q(T), S'(T))$ and $\eta(\mathcal{H}(T)) = (J', \Omega')$. It is clear $\mathcal{H}(0) = (f_0, q(0), 0)$, $\mathcal{H}(1) = (f_1, q(1), S(1))$. So, $\eta(\mathcal{H}(0)) = \eta(\mathbf{v}_0)$ and $\eta(\mathcal{H}(1)) = \eta(\mathbf{u}) = (I, \omega_I)$.

We have $J' = (F(T)(P[T]), S'(T))$. We claim that $height(J') \geq n$. To see this, let $J' \subseteq \wp \in \operatorname{Spec}(A[T])$. If $T \in \wp$, then $I_0 := f_0(P) \subseteq \wp$ and hence $height(\wp) \geq n$. Likewise, if $1 - T \in \wp$, then $I_1 := f_1(P) \subseteq \wp$ and hence $height(\wp) \geq n$. So, we assume $T(1-T) \notin \wp$. If $Z(T) \in \wp$, then $\mathcal{J} = (F(T)(P[T]), Z'(T)) = (F(T)(P[T]), Z(T)) \subseteq \wp$, which is impossible because $S'(T) \in \wp$. So, $T(1-T)Z(T) \notin \wp$. Since F is basic on $\mathscr{P}$, $height(\wp) \geq n$. This establishes the claim.So, $\mathcal{H}(T) = (F(T), q(T), S'(T)) \in \widetilde{\mathcal{Q}}(P[T])$ is such that $\eta(\mathcal{H}(0)) = (I_0, \omega_{I_0})$, $\eta(\mathcal{H}(1)) = (I, \omega_I)$ and with $\eta(\mathcal{H}(T)) = (J', \Omega')$, we have $height(J') \geq n$. If $T \in \wp \in Ass\left(\frac{A[T]}{J'}\right)$ then $(J'(0), T) = (I_0, T) \subseteq \wp$. Then, $height(\wp) \geq n+1$. This is impossible because $A[T]$ is regular (Cohen-Macaulay) and J' is local complete intersection ideal. Hence,

$$\begin{tikzcd} \frac{A[T]}{TJ'} \arrow[r, "T=0"] \arrow[d] & A \arrow[d] \\ \frac{A[T]}{J'} \arrow[r] & \frac{A}{J'(0)} \end{tikzcd}$$

is a patching diagram (see (5.3.4) below). So, the map $\Omega' : P[T] \twoheadrightarrow \frac{J'}{(J')^2}$ and $f_0 : P \twoheadrightarrow I_0$ combines to give a surjective maps $\phi : P[T] \twoheadrightarrow \frac{J'}{T(J')^2}$. Now, by [BK, Theorem 4.13], there is a surjective homomorphism $\varphi : P[T] \twoheadrightarrow J'$ such that $\varphi(0) = f_0$ and $\varphi \otimes \frac{A[T]}{J'} = \Omega'$. Now, it follows that $\varphi(1)$ is a lift of ω_I. This completes the proof. ∎

We used the following lemma above which needs a proof. The standard references for Patching diagrams are [Mi, O]. We will be specific in the following statement, because the literature does not seem complete regarding definitions of Patching diagrams of modules that are not projective.

Lemma 5.3.4. *Let R be a noetherian commutative ring and $A = R[T]$. Let I be a locally complete intersection ideal of A with $height(I) = r$. Assume $T : \frac{A}{I} \hookrightarrow \frac{A}{I}$ is injective (i. e. $T \notin \wp \in Ass\left(\frac{A}{I}\right)$). Then,*

$$\begin{tikzcd} \frac{I}{TI^2} \arrow[r] \arrow[d] & \frac{I}{TI} \arrow[d] \\ \frac{I}{I^2} \arrow[r] & \frac{I}{I^2+TI} \end{tikzcd}$$

is a Patching diagram, in the sense that it is a Cartesian square. Further,

1. $\frac{I}{TI} \xrightarrow{\sim} I(0)$.

2. $\frac{I}{I^2+TI} \xrightarrow{\sim} \frac{I(0)}{I(0)^2}$.

Proof. The patching diagram follows, because $I^2 \cap (TI) = TI^2$.

To see this, first we have $TI^2 \subseteq I^2 \cap (TI)$. Suppose $f \in I^2 \cap (TI)$. Then, $f = Tg$ with $g \in I$. Now, consider the map

$$T : \frac{I}{I^2} \longrightarrow \frac{I}{I^2}$$

Since $\frac{I}{I^2}$ is projective $\frac{A}{I}$-module and $T : \frac{A}{I} \hookrightarrow \frac{A}{I}$ is injective, T is also injective on $\frac{I}{I^2}$. So, $g \in I^2$. So, $f = Tg \in TI^2$.

Now, we prove $\frac{I}{TI} \xrightarrow{\sim} I(0)$. Obviously, the map is surjective. Suppose $f(T) \in I$ and $f(0) = 0$.Then, $f = Tg$. Since T is non zero divisor on $\frac{A}{I}$, $g \in I$. So, $f \in TI$.

Finally, we prove $\frac{I}{I^2+TI} \xrightarrow{\sim} \frac{I(0)}{I(0)^2}$. Again, the map is on to. Suppose $f(T) \in I$ and $f(0) \in I(0)^2$. Then, $f(0) = \sum f_i(0)g_i(0)$. Then, $f - \sum f_i g_i \in (T) \cap I = TI$ (by the above, if we like). So, $f \in I^2 + TI$. ∎

The following is a converse of Lemma 5.3.2.

Theorem 5.3.5. *Suppose A is an essentially smooth ring over an infinite perfect field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$. Assume $2n \geq d+3$. Then,*

$$\varepsilon(P) = \mathbf{e}_1 \iff P \cong Q \oplus A$$

for some projective A-module Q.

Proof. Suppose $P \cong Q \oplus A$. Then, by (5.3.2), $\varepsilon(P) = \mathbf{e}_0 = \mathbf{e}_1$. Conversely, suppose $\varepsilon(P) = \mathbf{e}_0 = \mathbf{e}_1$. Fix $f_0 \in P^*$ such that $height(f_0(P)) = n$. Then, $\zeta_0(f_0, 0, 0) = \mathbf{e}_0 = \mathbf{e}_1$. Then, it follow from Theorem 5.3.3 that $\eta(0, 0, 1)$ lifts to a surjective map $P \twoheadrightarrow A$. This completes the proof. ∎

This provides a comprehensive answer to the Question-5.0.2 we have started with. Moreover, there is also certain interest in the research community about whether there is any additional algebraic structure on the obstruction set, $\pi_0(\mathcal{LO}(P))$. We would like to explore this in detail in the following sections.

5.4 The Involution

In this section, we introduce an involution map $\Gamma : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. This can be thought of as a substitute to additive inverse map, without any regard to existence of an addition.

Definition 5.4.1. *Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$. For $(f, p, s) \in \widetilde{\mathcal{Q}}(P)$, define $\Gamma(f, p, s) = (f, p, 1 - s)$. This association, $\mathbf{v} \mapsto \Gamma(\mathbf{v})$, establishes a bijective correspondence*

$$\Gamma : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P), \quad \text{such that} \quad \Gamma^2 = 1.$$

We would say that Γ is an involution on $\widetilde{\mathcal{Q}}(P)$, which will be a key instrument in the subsequent discussions. (This notation Γ will be among the standard notations throughout this article.)

We record the following obvious lemma.

Lemma 5.4.2. *Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$ and $\Gamma : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P)$ is the involution. Let $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$ and denote $\eta(\mathbf{v}) = (I, \omega_I)$ and $\eta(\Gamma(\mathbf{v})) = (J, \omega_J)$. Then,*

1. *$I \cap J = f(P)$.*
2. *For $H(T) \in \widetilde{\mathcal{Q}}(P[T])$, we have $\Gamma(H(T))_{T=t} = \Gamma(H(t))$.*
3. *Therefore, $\forall\, \mathbf{v}, \mathbf{w} \in \widetilde{\mathcal{Q}}(P) \qquad \zeta_0(\mathbf{v}) = \zeta_0(\mathbf{w}) \Longleftrightarrow \zeta_0(\Gamma(\mathbf{v})) = \zeta_0(\Gamma(\mathbf{w}))$.*

In deed, Γ factors through an involution on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, as follows.

Corollary 5.4.3. *Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$. Then, the involution $\Gamma : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P)$ induces a bijective map $\widetilde{\Gamma}$: $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, such that $\widetilde{\Gamma}^2 = 1$ and $\zeta_0\Gamma = \widetilde{\Gamma}\zeta_0$. We say $\widetilde{\Gamma}$ is an involution.* (The notation $\widetilde{\Gamma}$ will also be among our standard notations throughout this article.)

Proof. First, consider the map $\zeta_0\Gamma : \widetilde{\mathcal{Q}}(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. For, $H(T) \in \widetilde{\mathcal{Q}}(P[T])$, we have $\zeta_0\Gamma(H(0)) = \zeta_0\Gamma(H(1))$. Therefore, $\zeta_0\Gamma$ is homotopy invariant. Hence, it induces a well defined map $\widetilde{\Gamma} : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. Clearly, $\widetilde{\Gamma}^2 = 1$ and $\widetilde{\Gamma}$ is a bijection. The proof is complete.■

The following is a way to compute the involution.

Corollary 5.4.4. *Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$. Suppose $(I, \omega) \in \mathcal{LO}(P)$. For any $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$ with $\eta(\mathbf{v}) = (I, \omega)$, write $\eta(\Gamma(\mathbf{v})) = (J, \omega_J)$. Then,*

$$\widetilde{\Gamma}(\zeta(I, \omega)) = \zeta(J, \omega_J) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

Proof. Obvious. ■

Next we prove two versions of *Moving Lemma Argument* which is useful not only for our purpose but also in other computational aspects in commutative algebra.

Lemma 5.4.5 (Moving Lemma)**.** *Suppose A is a commutative noetherian ring with $\dim A = d$ and P is a projective A-module with $rank(P) = n$. Assume $2n \geq d + 1$. Let $K \subseteq A$ be an ideal with $height(K) \geq n$ and $(I, \omega_I) \in \mathcal{LO}(P)$. Then, there is an element $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$ such that $\eta(\mathbf{v}) = (I, \omega_I)$. Further, with $J = f(P) + A(1 - s)$, we have $height(J) \geq n$ and $J + K = A$.*

Proof. Let $f_0 : P \twoheadrightarrow I$ be any lift of ω_I. Then, $I = f_0(P) + I^2$. By Nakayama's Lemma, there is an element $t \in I$, such that $(1 - t)I \subseteq f_0(P)$. Therefore, $t(1 - t) = f_0(p_0)$ for some $p_0 \in P$.

(*Readers are referred to [M1] regarding generalities on Basic Element Theory and generalized dimension functions.*) Write

$$\mathscr{P} = \{\wp \in \mathrm{Spec}\,(A) : t \notin \wp, \text{ and either } K \subseteq \wp \text{ or } height(\wp) \leq n-1\}$$

There is a generalized dimension function (see [M1]) $\delta : \mathscr{P} \longrightarrow \mathbb{N}$, such that $\delta(\wp) \leq n-1 \; \forall \; \wp \in \mathscr{P}$. Now $(f_0, t^2) \in P^* \oplus A$ is basic on $\mathscr{P}$. So, there is an element $g \in P^*$ such that $f := f_0 + t^2 g$ is basic on $\mathscr{P}$. It follows, $f(P) + At = f_0(P) + At = I$ and $I = f(P) + I^2$. By Nakayama's Lemma, there is an element $s \in I$, such that $(1-s)I \subseteq f(P)$ and hence $f(p) = s(1-s)$, for some $p \in P$, Hence, $I = (f(P), s)$. Now, write $J = f(P) + A(1-s)$. For $J \subseteq \wp \in \mathrm{Spec}\,(A)$, $s \notin \wp$ and hence $t \notin \wp$. Since, f is basic on $\mathscr{P}$, $height(\wp) \geq n$. This establishes, $height(J) \geq n$.

Now suppose $J + K \subseteq \wp \in \mathrm{Spec}\,(A)$. By the same argument above, $t \notin \wp$. Hence, $\wp \in \mathscr{P}$. This is impossible, because f is basic on $\mathscr{P}$. So, $J + K = A$. Now, $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$, satisfies the requirement. ∎

The following is another version of the Moving Lemma 5.4.5.

Lemma 5.4.6 (Moving Representation)**.** *Suppose A is a commutative noetherian ring, with* $\dim A = d$*. Let P be a projective A-module, with $rank(P) = n$ and $2n \geq d+1$. Let $x \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ and let $K \subseteq A$ be an ideal with $height(K) \geq n$. Then, there is a local P-orientation $(J, \omega_J) \in \mathcal{LO}(P)$ such that $x = \zeta(J, \omega_J)$, $height(J) \geq n$ and $J + K = A$.*

Proof. Let $x = \zeta(I, \omega_I)$. First, $\eta(\mathbf{u}) = (I, \omega_I)$ for some $\mathbf{u} \in \widetilde{\mathcal{Q}}(P)$. Denote $(I_0, \omega_{I_0}) := \eta(\Gamma(\mathbf{u}))$. Then, $\tilde{\Gamma}(x) = \zeta(I_0, \omega_{I_0})$.

Now, we apply Moving Lemma 5.4.5, to (I_0, ω_{I_0}) and K. There is $\mathbf{v} \in \widetilde{\mathcal{Q}}(P)$, such that $\eta(\mathbf{v}) = (I_0, \omega_{I_0})$, and with $\eta(\Gamma(\mathbf{v})) = (J, \omega_J)$, we have $height(J) \geq n$ and $J + K = A$. Now, $x = \tilde{\Gamma}(\tilde{\Gamma}(x)) = \tilde{\Gamma}(\zeta(I_0, \omega_{I_0})) = \zeta(J, \omega_J)$. The proof is complete. ∎

5.5 The Monoid Structure on $\pi_0(\mathcal{L}O(P))$

In this section, we define and establish a natural monoid structure on the homotopy obstruction set $\pi_0(\mathcal{L}O(P)) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, when $2rank(P) \geq \dim A + 2$ and A is a regular ring over a field k, with $1/2 \in k$. We start with the following basic ingredient of the group structure.

Definition 5.5.1. *Let A be a commutative noetherian ring, with $\dim A = d$, and P be a projective A-module, with $rank(P) = n \geq 2$. Let $(I, \omega_I), (J, \omega_J) \in \mathcal{L}O(P)$ be such that $I + J = A$. Let $\omega := \omega_I \star \omega_J : P \twoheadrightarrow \frac{IJ}{(IJ)^2}$ be the unique surjective map induced by ω_I, ω_J. We define a pseudo-sum*

$$(I, \omega_I)\hat{+}(J, \omega_J) := (IJ, \omega) \in \pi_0(\mathcal{L}O(P)).$$

Note, pseudo-sum commutes.

In the rest of this section, we establish that the pseudo sum respects homotopy, when $2n \geq d+2$, and A is a regular ring over a field k, with $1/2 \in k$. Consequently, this leads to a addition operation on $\pi_0(\mathcal{L}O(P))$. The following is the key lemma.

.

Lemma 5.5.2. *Let A be a commutative noetherian ring and P be a projective A-module, with $\dim A = d$, $rank(P) = n$, and $2n \geq d + 2$. Consider a homotopy*

$$H(T) = (f(T), p(T), Z(T)) \in \widetilde{\mathcal{Q}}(P[T]).$$

Write $\eta(H(0)) = (K_0, \omega_{K_0})$ and $\eta(H(1)) = (K_1, \omega_{K_1})$. Further suppose $(J, \omega_J) \in \mathcal{L}O(P)$ such that $K_0 + J = K_1 + J = A$ and $height(J) \geq n$. Then, there is a homotopy $\mathcal{H}(T) \in \widetilde{\mathcal{Q}}(P[T])$ such that $\eta(\mathcal{H}(0)) = (K_0J, \omega_{K_0J})$ and $\eta(\mathcal{H}(1)) = (K_1J, \omega_{K_1J})$, where, for $i = 0, 1$ $\omega_{K_iJ} := \omega_{K_i} \star \omega_J : P \twoheadrightarrow \frac{K_iJ}{(K_iJ)^2}$. Consequently,

$$(K_0, \omega_{K_0})\hat{+}(J, \omega_J) = (K_1, \omega_{K_1})\hat{+}(J, \omega_J) \in \pi_0(\mathcal{L}O(P)).$$

Proof. We will write $f = f(T)$, $p = p(T)$ and $Z = Z(T)$. Denote $Y = 1 - Z$ and $\eta(\Gamma(H(T)) = (\mathbb{J}, \omega_{\mathbb{J}})$. Then, $\mathbb{J} = (f(P[T]), Y)$. Write

$$\mathscr{P} = \{\wp \in \mathrm{Spec}\,(A[T]) : YT(1-T) \notin \wp, J \subseteq \wp\}.$$

There is a generalized dimension function $\delta : \mathscr{P} \longrightarrow \mathbb{N}$ such that $\forall\, \wp \in \mathscr{P}$, $\delta(\wp) \leq \dim\left(\frac{A[T]}{JA[T]}\right) \leq d+1-height(J) \leq d+1-n \leq n-1$. Further, $(f, Y^2T(1-T))$ is a basic element in $P[T]^* \oplus A[T]$, on $\mathscr{P}$. Therefore, there is an element $\lambda := \lambda(T) \in P[T]^*$ such that

$$f' = f + Y^2T(1-T)\lambda \text{ is basic on } \mathscr{P}. \quad \text{So,} \quad f'(0) = f(0),\ f'(1) = f(1).$$

We have $\mathbb{J} = (f(P[T]), Y) = (f'(P[T]), Y)$. Further,

$$Z(1-Z) = Y(1-Y) = f(p) = f'(p) - Y^2T(1-T)\lambda(p).$$

So,

$$Y = f'(p) - Y^2T(1-T)\lambda(p) + Y^2$$

Write $M = \frac{\mathbb{J}}{f'(P[T])}$. Let $p_1, \ldots, p_m$ be a set of generators of P. Use "overline" to indicate images in M. We intend to repeat the proof of Nakayama's Lemma and we have

$$\begin{pmatrix} \overline{f(p_1)} \\ \overline{f(p_2)} \\ \cdots \\ \overline{f(p_m)} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\lambda(p_1)YT(1-T) \\ 0 & 0 & \cdots & 0 & -\lambda(p_2)YT(1-T) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -\lambda(p_m)YT(1-T) \\ 0 & 0 & 0 & 0 & Y - \lambda(p)YT(1-T) \end{pmatrix} \begin{pmatrix} \overline{f(p_1)} \\ \overline{f(p_2)} \\ \cdots \\ \overline{f(p_m)} \\ \overline{Y} \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda(p_1)YT(1-T) \\ 0 & 1 & \cdots & 0 & \lambda(p_2)YT(1-T) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \lambda(p_m)YT(1-T) \\ 0 & 0 & 0 & 0 & 1-Y+\lambda(p)YT(1-T) \end{pmatrix} \begin{pmatrix} \overline{f(p_1)} \\ \overline{f(p_2)} \\ \cdots \\ \overline{f(p_m)} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

Multiplying by the adjoint matrix and computing the determinant, with $Y' = Y - \lambda(p)YT(1-T)$, we have

$$(1-Y')\mathbb{J} \subseteq f'(P[T]).$$

We have $Y'(0) = Y(0) = 1 - Z(0)$, $Y'(1) = Y(1) = 1 - Z(1)$. Further,

$$Y'(1-Y') = f'(p') \qquad \text{for some} \quad p' \in P[T].$$

$$\text{Therefore} \qquad H'(T) = (f', p', Y') \in \widetilde{\mathcal{Q}}(P[T]).$$

We have

$$\mathbb{J} = (f(P[T]), Y) = (f'(P[T]), Y) = (f'(P[T]), Y').$$

In fact, $\eta(H'(T)) = (\mathbb{J}, \omega_{\mathbb{J}})$ and write $\eta(\Gamma(H'(T))) = (\mathbb{I}, \omega_{\mathbb{I}})$. Claim

$$\mathbb{I} + JA[T] = A[T]. \qquad i.e. \qquad (f'(P[T]), 1-Y') + JA[T] = A[T].$$

To see this, let

$$\mathbb{I} + JA[T] \subseteq \wp \in \operatorname{Spec}(A[T])$$

1. If $Y \in \wp$ then $\mathbb{J} = (f'(P[T]), Y) = (f'(P[T]), Y') \subseteq \wp$. So, $Y' \in \wp$, which is impossible, since $1 - Y' \in \wp$. So, $\wp \in D(Y)$.

2. Since f' is unimodular of $\mathscr{P}$ and $\wp \in D(Y)$, we must have $T(1-T) \in \wp$.

3. Now, $T \in \wp$ implies,

$$\mathbb{I}(0) + J = (f'(0)(P), 1 - Y'(0)) + J = (f(0)(P), 1 - Y(0)) + J = K_0 + J = A \subseteq \wp$$

   which is impossible.

4. Likewise, $1 - T \in \wp$ implies,

$$\mathbb{I}(1) + J = (f'(1)(P), 1 - Y'(1)) + J = (f(0)(P), 1 - Y(1)) + J = K_1 + J = A \subseteq \wp.$$

   This is also impossible.

This establishes the claim. Recall, $\omega_{\mathbb{I}} : P[T] \twoheadrightarrow \frac{\mathbb{I}}{\mathbb{I}^2}$ is induced by f'. Extend $\omega_J : A^n \twoheadrightarrow \frac{J}{J^2}$ to a surjective map $\omega_{JA[T]} : A[T]^n \twoheadrightarrow \frac{JA[T]}{J^2A[T]}$. Let

$$\Omega := \omega_{\mathbb{I}} \star \omega_{JA[T]} : P[T] \twoheadrightarrow \frac{J\mathbb{I}}{J^2\mathbb{I}^2} \quad \text{be induced by } \omega_{\mathbb{I}}, \text{ and } \omega_{JA[T]}.$$

Now, there is a homotopy$\mathcal{H}(T) \in \widetilde{\mathcal{Q}}(P[T])$, such that $\eta(\mathcal{H}(T)) = (\mathbb{I}JA[T], \Omega)$. Specializing at $T = 0$ and $T = 1$, we have

$$\eta(\mathcal{H}(0)) = (K_0J, \omega_{K_0J}), \qquad \eta(\mathcal{H}(1)) = (K_1J, \omega_{K_1J}).$$

The proof is complete. ■

Now, we define addition on $\pi_0(\mathcal{L}O(P))$.

Definition 5.5.3. *Let A be a regular ring, containing a field k, with $1/2 \in k$, with $\dim A = d$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $2n \geq d + 2$. Let $x, y \in \pi_0(\mathcal{L}O(P))$. By Moving Lemma 5.4.6, we can write $x = [(I, \omega_I)]$, $y = [(J, \omega_J)]$, for some $(I, \omega_I), (J, \omega_J) \in$*

$\mathcal{LO}(P)$, with $height(IJ) \geq n$, and $I + J = A$. Define

$$x + y := (I, \omega_I)\hat{+}(J, \omega_J) \in \pi_0\left(\mathcal{LO}(P)\right) \qquad \text{as defined in } (5.5.1).$$

We establish that $x + y$ is well defined (5.5.4).

Proposition 5.5.4. *Under the setup and notations, as in (5.5.3), $x + y$ is well defined.*

Proof. Let $x = [(I_1, \omega_{I_1})], y = [(J_1, \omega_{J_1})] \in \pi_0\left(\mathcal{LO}(P)\right)$, be another pair of choices, as in (5.5.3). That means, $height(I_1 J_1) \geq n$, $I_1 + J_1 = A$. We prove

$$(I, \omega_I)\hat{+}(J, \omega_J) = (I_1, \omega_{I_1})\hat{+}(J_1, \omega_{J_1}).$$

By Moving Lemma 5.4.5, there is $(K, \omega_K) \in \mathcal{LO}(P)$ such that $x = [(K, \omega_K)]$, $hieight(K) \geq n$ and $K + I_1 \cap J_1 = A$.

We have $\mathbf{u}, \mathbf{u}_1 \in \widetilde{\mathcal{Q}}(P)$ such that $\eta(\mathbf{u}) = (I, \omega_I)$, and $\eta(\mathbf{u}_1) = (K, \omega_K)$. Since $x = [(I, \omega_I)] = [(K, \omega_K)] \in \pi_0\left(\mathcal{LO}(P)\right)$, it follows $\mathbf{u}$, $\mathbf{u}_1$ are equivalent in $\widetilde{\mathcal{Q}}(P)$. By (5.2.2), there is a homotopy $H(T) \in \widetilde{\mathcal{Q}}(P[T])$ such that $H(0) = \mathbf{u}$, and $H(1) = \mathbf{u}_1$. It follows from Lemma 5.5.2,

$$(I, \omega_I)\hat{+}(J, \omega_J) = (K, \omega_K)\hat{+}(J, \omega_J) = (J, \omega_J)\hat{+}(K, \omega_K)$$

Likewise, the above is

$$= (J_1, \omega_{J_1})\hat{+}(K, \omega_K) = (K, \omega_K)\hat{+}(J_1, \omega_{J_1}) = (I_1, \omega_{I_1})\hat{+}(J_1, \omega_{J_1})$$

The proof is complete. ■

The final statement on the binary structure on $\pi_0\left(\mathcal{LO}(P)\right) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, is as follows.

Theorem 5.5.5. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$. Assume $2n \geq d + 2$.* (Subsequently, we use the notations in $\pi_0(\mathcal{L}O(P))$ and $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ interchangeably.) *Then, the addition operation on* $\pi_0(\mathcal{L}O(P))$*, defined in (5.5.3) has the following properties.*

1. *The addition in* $\pi_0(\mathcal{L}O(P))$ *is commutative and associative. Further, the image* $\mathbf{e}_1 := [(A,0)] \in \pi_0(\mathcal{L}O(P))$*, of* $(0,0,1) \in \widetilde{\mathcal{Q}}(P)$*, acts as the additive identity in* $\pi_0(\mathcal{L}O(P))$*. In other words,* $\pi_0(\mathcal{L}O(P))$ *has a structure of an abelian monoid.*

2. *Let* $\mathbf{e}_0 := [(0,0)] \in \pi_0(\mathcal{L}O(P))$ *be the image of* $(0,0,0) \in \widetilde{\mathcal{Q}}(P)$*. Then,* $x + \widetilde{\Gamma}(x) = \mathbf{e}_0$*,* $\forall\, x \in \pi_0(\mathcal{L}O(P))$*, where* $\widetilde{\Gamma}$ *is the involution map.*

3. *If* $\mathbf{e}_0 = \mathbf{e}_1 \in \pi_0(\mathcal{L}O(P))$*, then* $\pi_0(\mathcal{L}O(P))$ *is an abelian group, under this addition.* (Recall (5.3.5), $\mathbf{e}_0 = \mathbf{e}_1$ if and only if $P \cong Q \oplus A$.)

Proof. Given $x, y, z \in \pi_0(\mathcal{L}O(P))$, by the Moving Lemma 5.4.6, we can write

$$x = [(K, \omega_K)],\ y = [(I, \omega_I)],\ z = [(J, \omega_J)] \quad \ni\ K + I = K + J = I + J = A$$

and $height(K) \geq n$, $height(I) \geq n$, $height(J) \geq n$. By definition (5.5.3),

$$(x + y) + z = ((K, \omega_K)\hat{+}(I, \omega_I))\hat{+}(J, \omega_J) = x + (y + z).$$

$$\text{and} \qquad x + y = (K, \omega_K)\hat{+}(I, \omega_I) = (I, \omega_I)\hat{+}(K, \omega_K) = y + x.$$

So, the associativity and commutativity hold. It is obvious that, for all $x \in \pi_0(\mathcal{L}O(P))$, we have $x + \mathbf{e}_1 = x$. So, $\mathbf{e}_1$ acts as the additive identity. This establishes (1).

Let $x = [(K, \omega_K)] \in \pi_0(\mathcal{L}O(P))$, with $height(K) \geq n$. There is $\mathbf{u} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$, with $\eta(\mathbf{u}) = (K, \omega_K)$. Write $\eta(\Gamma(\mathbf{u})) = (I_1, \omega_{I_1})$. We can assume $height(I_1) \geq n$. It follows.

$$x + \widetilde{\Gamma}(x) = \zeta_0(f, 0, 0) = \mathbf{e}_0. \qquad \text{This establishes (2).}$$

If $\mathbf{e}_0 = \mathbf{e}_1$, it follows from (2) that, $\pi_0(\mathcal{L}O(P))$ has a group structure. This establishes (3).

This completes the proof. ∎

Remark 5.5.6. Use the notation as in (5.5.5). When $\mathbf{e}_0 \neq \mathbf{e_1}$, the results in (5.5.5) describe a situation similar to the construction of Witt group, from the monoid of isometry classes of quadratic spaces.

For $x, y \in \pi_0\left(\mathcal{L}O(P)\right)$ define $x \sim y$ if $x + n\mathbf{e}_0 = y + m\mathbf{e}_0$, for integers $m, n \geq 0$. This is easily checked to be an equivalence relation. Let $\mathbb{E}\left(\pi_0\left(\mathcal{L}O(P)\right)\right)$ be the set of all equivalence classes. Then, $\mathbb{E}\left(\pi_0\left(\mathcal{L}O(P)\right)\right)$ has a structure of an abelian group, induced by the additive structure on $\pi_0\left(\mathcal{L}O(P)\right)$. The natural map

$$\ell : \pi_0\left(\mathcal{L}O(P)\right) \twoheadrightarrow \mathbb{E}\left(\pi_0\left(\mathcal{L}O(P)\right)\right)$$

is a surjective homomorphism of monoids. The identity element of $\mathbb{E}\left(\pi_0\left(\mathcal{L}O(P)\right)\right)$ is $\ell(\mathbf{e}_0) = \ell(\mathbf{e}_1)$. For $x \in \pi_0\left(\mathcal{L}O(P)\right)$, the additive inverse of $\ell(x)$ is $\ell(\widetilde{\Gamma}(x))$.

Clearly, if $\mathbf{e}_0 = \mathbf{e_1}$, then $\mathbb{E}\left(\pi_0\left(\mathcal{L}O(P)\right)\right) = \pi_0\left(\mathcal{L}O(P)\right)$.

We can say more when $rank(P) = dim(A)$. That is, $\pi_0\left(\mathcal{L}O(P)\right)$ has natural abelian group structure even if it does not split when there is an isomorphism of top determinants of P and another projective module $P' \oplus A$. We omit the proof here and refer interested reader to [MM3] for more details.

Chapter 6

Relation between Obstructions in Algebra and Topology

The goal of this chapter is to relate the algebraic obstructions $\pi_0(\mathcal{L}O(P))$ and the topological obstructions $\pi_0(\mathcal{L}O(\mathscr{V}_{Top}(P)))$, where P is a finitely generated projective module over the smooth real affine ring A and $\mathscr{V}_{Top}(P)$ is the smooth vector bundle over the manifold M of *real points* in $X = \mathrm{Spec}\,(A)$.

First we will give an alternate description of the *Nori homotopy set* in Algebra, $\pi_0(\mathcal{L}O(P))$, defined for a projective module P over a ring A (5.1.4).

6.1 Alternate Description of the Obstructions, $\pi_0(\mathcal{L}O(P))$

Let A be a noetherian commutative ring, with $\dim A = d$ and P be a projective A-module with $rank(P) = n$. Let us recall the essential components in the construction of $\pi_0(\mathcal{L}O(P))$. By a **local P-orientation**, we mean a pair (I, ω) where I is an ideal of A and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism. We will use the same notation ω for the map $\frac{P}{IP} \twoheadrightarrow \frac{I}{I^2}$, induced by ω.

Denote

$$\begin{cases} \mathcal{L}O(P) = \{(I, \omega) : (I, \omega) \text{ is a local } P \text{ orientation}\} \\ \widetilde{\mathcal{L}O}(P) = \{(I, \omega) \in \mathcal{L}O(P) : height(I) = n \text{ or } I = A\} \end{cases} \tag{6.1}$$

We would like to define $\pi_0\left(\widetilde{\mathcal{L}O}(P)\right)$ similar to the way $\pi_0(\mathcal{L}O(P))$ (5.1.4) is defined. However, note that the substitutions $T = 0, 1$ would not yield any map from $\widetilde{\mathcal{L}O}(P[T])$ to $\widetilde{\mathcal{L}O}(P)$. Nevertheless, the definition of $\pi_0(\mathcal{L}O(P))$ by push forward diagram (5.5), is only an alternate way of saying the following:

1. For $(I_0,\omega_0),(I_1,\omega_1) \in \mathcal{L}O(P)$, we write $(I_0,\omega_0) \sim (I_1,\omega_1)$, if there exists

$$(I,\omega) \in \mathcal{L}O(P[T]) \quad \ni \quad \begin{cases} (I(0),\omega(0)) = (I_0,\omega_0) \\ (I(1),\omega(1)) = (I_1,\omega_1) \end{cases}$$

   The homotopy relation $\sim$ generates an equivalence relation on $\mathcal{L}O(P)$, which we denote by $\approx$.

2. The above definition (5.5) means, $\pi_0\left(\mathcal{L}O(P)\right)$ is the set of all equivalence classes in $\mathcal{L}O(P)$.

Definition 6.1.1. *With notations as above, if we restrict the relation $\approx$ to $\widetilde{\mathcal{L}O}(P) \subseteq \mathcal{L}O(P)$, we get an equivalence relation on $\widetilde{\mathcal{L}O}(P)$. That is, two elements $(I_0,\omega_0),(I_1,\omega_1) \in \widetilde{\mathcal{L}O}(P)$ are 'related' if and only if $(I_0,\omega_0) \approx (I_1,\omega_1)$ in $\mathcal{L}O(P)$. Define $\boldsymbol{\pi_0}\left(\widetilde{\boldsymbol{\mathcal{L}O}}\boldsymbol{(P)}\right)$ to be the set of all equivalence classes in $\widetilde{\mathcal{L}O}(P)$.*

Note that there is another natural way to define chain equivalence relation in $\widetilde{\mathcal{L}O}(P)$: let $(I_0,\omega_0),(I_1,\omega_1) \in \widetilde{\mathcal{L}O}(P)$ be two elements. Then, $(I_0,\omega_0) \sim (I_1,\omega_1)$ if there is $\mathcal{H}(T) = (I,\omega) \in \widetilde{\mathcal{L}O}(P[T])$ such that $\mathcal{H}(0) = (I_0,\omega_0)$, $\mathcal{H}(1) = (I_1,\omega_1)$. We will show now that these two equivalence relations induce the same equivalence classes on $\widetilde{\mathcal{L}O}(P)$.

Lemma 6.1.2. Let $(I_0,\omega_0),(I_1,\omega_1) \in \widetilde{\mathcal{L}O}(P)$ be such that $[(I_0,\omega_0)] = [(I_1,\omega_1)]$. Then there is $\mathcal{H}(T) = (I,\omega) \in \widetilde{\mathcal{L}O}(P[T])$ such that $\mathcal{H}(0) = (I_0,\omega_0)$, $\mathcal{H}(1) = (I_1,\omega_1)$.

Proof. First, there is $H(T) = (I,\omega) \in \mathcal{L}O(P[T])$ such that $H(0) = (I_0,\omega_0)$, $H(1) = (I_1,\omega_1)$. Let $\varphi : P[T] \twoheadrightarrow J = I \cap K \subseteq I$ that lifts ω, with $height(K) \geq n$, and $I + K = A[T]$. Write $1 = s(T) + t(T) \in I^2 + K^2$. Clearly, $t(T)I \subseteq I \cap K = J$ and $s(T)K \subseteq I \cap K =: J$. Let

$$\mathscr{P} = \{\wp \in \operatorname{Spec}(A) : height(\wp) \leq n-1, K \not\subseteq \wp, T(1-T) \notin \wp\}$$

I claim, $(\varphi, T(1-T)t(T))$ is unimodular in $\mathscr{P}$. To, see this, suppose

$$(I \cap K, T(1-T)t(T)) \subseteq \wp \in \mathscr{P} \Longrightarrow A = (I, t(T)) \subseteq \wp$$

So, there is $\lambda \in P[T]^*$, such that $\Phi = \varphi + T(1-T)t(T)\lambda$ is basic on $\mathscr{P}$. It follows,

$$K = \Phi(P[T]) + K^2 \quad \text{Clearly,} \quad RHS \subseteq K$$

Suppose, $x \in K$. Then,

$$x = xs + xt = \varphi(p) + xt = \Phi(p) - T(1-T)t(T)\lambda(p) + xt \in RHS$$

So,

$$\Phi(P[T]) = \tilde{I} \cap K, \quad \text{with,} \quad \tilde{I} + K = A[T], \quad height(\tilde{I}_{T(1-T)}) \geq n$$

Since, $\Phi_{|T=0} = \varphi_{|T=0}$, we have $\tilde{I}(0) = I(0) = I_0$. Similarly, $\tilde{I}(1) = I(1) = I_1$. Since, $height(I_0) \geq n$ and $height(I_1) \geq n$, it follows $height(\tilde{I}) \geq n$. Consider,

$$\mathcal{H}(T) = (\tilde{I}, \tilde{\omega}) \in \widetilde{\mathcal{L}O}(P[T]) \quad \text{where} \quad \begin{tikzcd} P[T] \arrow[r, two heads] \arrow[dr, bend right, "\tilde{\omega}"'] & \tilde{I} \cap K \arrow[d] \\ & \frac{\tilde{I}}{\tilde{I}^2} \end{tikzcd}$$

Again, since $\Phi_{|T=0} = \varphi_{|T=0}$, we have $\mathcal{H}(0) = (I_0, \omega_0)$, and $\mathcal{H}(1) = (I_1, \omega_1)$. The proof is complete. ∎

It follows that there is natural map which is simply the inclusion:

$$\varphi : \pi_0\left(\widetilde{\mathcal{L}O}(P)\right) \longrightarrow \pi_0\left(\mathcal{L}O(P)\right)$$

Next, we would like to show that the map φ is in fact an isomorphism which leads us to the following new description of $\pi_0\left(\mathcal{L}O(P)\right)$.

Proposition 6.1.3. Let A be a regular ring with $\dim A = d$, containing a filed k with $1/2 \in k$. Let P be a projective A-module with $rank(P) = n$. Then, the natural map

$$\pi_0\left(\widetilde{\mathcal{L}O}(P)\right) \xrightarrow{\sim} \pi_0\left(\mathcal{L}O(P)\right)$$

is an isomorphism.

Proof. Follows from (6.1.2). ∎

Remark 6.1.4. We record the following, with notations as in (5.1.2):

1. Assume A is a Cohen Macaulay ring. Then, $\widetilde{\mathcal{L}O}(P)$ is in bijection with the set

$$\left\{(I,\omega) \in \mathcal{L}O(P) : height(I) = n,\ \omega : \frac{P}{IP} \xrightarrow{\sim} \frac{I}{I^2} \text{ is an isomorphism}\right\} \cup \{(A,0)\}$$

Our next goal is to provide yet another description of $\pi_0\left(\mathcal{L}O(P)\right)$, by considering P-orientations (I,ω), with $\frac{A}{I}$ *smooth*. We will start with some definitions and relevant results as follows.

6.2 Smoothness

6.2.1 Perfect fields

First, we recall the following definition.

Definition 6.2.1. Let k be a field. We say k is *perfect* if every field extension of k is *separable* over k.

Lemma 6.2.2. *A field k is perfect if and only if it is a field of characteristic 0 or a field of characteristic $p > 0$ such that every element has a p^{th} root.*

Lemma 6.2.3. Let k be a field. Then there is a purely inseparable extension $k \hookrightarrow K$, such that K is perfect. This K is unique, to be called the **perfect closure**, to be denoted by k_p.

Proof. If $ch(k) = 0$ then $K = k$. If $ch(k) = p > 0$, then $K = \bigcup k^{\frac{1}{p^n}}$. ■

Theorem 6.2.4. Let k be a perfect field. Any reduced k algebra is geometrically reduced over k. Let R, S be k-algebras. Assume both R and S are reduced. Then the k-algebra $R \otimes_k S$ is reduced.

Proof. See [H, pp. 93 Ex. 3.15].

6.2.2 Regualrity in $ch(k) = 0$

Theorem 6.2.5. *[K, pp. 118, Theorem 7.2]* Let k be a filed and $ch(k) = 0$. Let A be an affine algebra over k and $\wp \in \mathrm{Spec}\,(A)$. Then the following are equivalent:

1. $A_\wp$ is a regular local ring.

2. $\Omega_{A_\wp/k}$ is free $A_\wp$-module.

In this case,

$$rank(\Omega_{A_\wp/k}) = \dim_\wp A := height(\wp) + \dim V(\wp) = height(\wp) + Tr.\deg(\kappa(\wp))/k$$

6.2.3 Jacobian criterion for regularity

First, we give the Jacobian criterion for regularity.

Definition 6.2.6. *[K, pp. 123, Jacobian] Let k be a perfect field. Let*

$$A = \frac{k[X_1, \ldots, X_n]}{(F_1, \ldots, F_m)} \quad \text{be an affine algebra.}$$

Let $\wp \in \operatorname{Spec}(A)$. Define the Jacobian matrix

$$J(\wp) = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \cdots & \frac{\partial F_2}{\partial X_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial X_1} & \frac{\partial F_m}{\partial X_2} & \cdots & \frac{\partial F_m}{\partial X_n} \end{pmatrix} \in \mathbb{M}_{m\times n}(\kappa(\wp))$$

The following is the Jacobian criterion.

Theorem 6.2.7. *[K, pp. 123, Theorem 7.8]* Use the notations as in (6.2.6). In particular k is **perfect**. Assume A is *equidimensional* (i.e. $dim(A/\wp) := Trdeg((k(\wp)/k)$ is independedent of $\wp$, where $\wp$ is any minimal prime ideal in A). Then,

1. $rank(J(\wp)) \leq n - \dim A$.

2. We have

$$A_\wp \quad \text{is regular} \Longleftrightarrow rank(J(\wp)) = n - \dim A$$

6.2.4 Geometric regularity

We introduce the following notation from [K, pp. 296].

Definition 6.2.8. Let A be a ring and $\wp \in \operatorname{Spec}(A)$. Then, the dimension of A at $\wp$ is defined to be the supremum of the lengths of all chains in $\operatorname{Spec}(A)$, passing through $\wp$. So,

$$\dim_\wp A = \dim A_\wp + \dim \frac{A}{\wp}$$

If A is affine algebra over a field k, then

$$\dim_{\wp} A = \dim A_{\wp} + \dim \frac{A}{\wp} = \dim A_{\wp} + Tr \deg (\kappa(\wp)/k) \tag{6.2}$$

In fact, the last term makes sense when A is **essentially finite** over k. So, for any **essentially finite** algebra B over k and $\wp \in \operatorname{Spec}(B)$, define

$$\dim_{\wp}^{Tr} B = \dim B_{\wp} + Tr \deg (\kappa(\wp)/k) \tag{6.3}$$

Proposition 6.2.9. *[K, pp. 127, Proposition 7.12]* Let A be an **affine algebra**, over k. Let $\wp \in \operatorname{Spec}(A)$, such that $A_{\wp}$ is regular. Then, for any separable extension of fields $k \hookrightarrow K$, $K \otimes_k A_{\wp}$ is regular.

More generally, it follows from this that if A is of essentially finite type over k, then the **same holds.**

Definition 6.2.10. *[K, pp. 128]* Let $(R, \mathfrak{m})$ be noetherian local, containing a field $k \subseteq R$. We say $(R, \mathfrak{m})$ is **geometrically regular over** k, if $K \otimes_k R$ is regular for all finite field extension $k \hookrightarrow K$.

Corollary 6.2.11. Let k is perfect and A is essentially of finite type over k. Let $\wp \in \operatorname{Spec}(A)$ such that $A_{\wp}$ is regular. Then, then $A_{\wp}$ is geometrically regular over k.

Proof. Follows from (6.2.9). ∎

Theorem 6.2.12. *[K, pp. 129, Theorem 7.14]* Let k be a field (not necessarily perfect) and A be an **affine algebra** over k. Let $\wp \in \operatorname{Spec}(A)$. Then, the following are equivalent:

1. $A_{\wp}$ is geometrically regular.

2. $\Omega_{A_{\wp}/k}$ is free of

$$\mathbf{rank}\left(\Omega_{A_{\wp}/k}\right) = \dim_{\wp} A := \dim A_{\wp} + \dim V(\wp) = height(\wp) + Tr \deg (\kappa(\wp)/k)$$

3. $\mu_\wp(\Omega_{A/k}) \leq \dim_\wp A$

Two comments:

1. If A is **essentially finite** over k, then the same works if we replace $\dim_\wp A$ by $\dim_\wp^{Tr} A$.

2. If k is **perfect**, we can drop the word "geometrically" by (6.2.11).

6.2.5 Smoothness

Definition 6.2.13. *[K, pp. 138]* Suppose k is a field. Let A be an essentially finitely type ring over k. For $\wp \in \operatorname{Spec}(A)$, we say A is *smooth* at $\wp$ over k, if $A_\wp$ is *geometrically regular* over k.

Corollary 6.2.14. Let k be **perfect field**. Let A be an essentially finitely type ring over k. For $\wp \in \operatorname{Spec}(A)$,

$$A_\wp \quad \text{is regular} \quad \Longleftrightarrow \quad A_\wp \quad \text{is smoothover } k.$$

Proof. Follows from (6.2.11). ■

Theorem 6.2.15. *[K, pp. 139, Theorem 8.1]* Let k be a field, not necessarily perfect. $A = \left(\frac{k[X_1,X_2,\ldots,X_n]}{I}\right)_S = \left(\frac{P}{I}\right)_S$ be essentially finite type algebra. Let $\wp \in \operatorname{Spec}(A)$. Then, $A_\wp = \left(\frac{P}{I}\right)_{\mathcal{N}}$, where $\mathcal{N} \in \operatorname{Spec}(P)$ is the inverse image of $\wp$. Then, the following are equivalent:

1. A/k is smooth at $\wp$.

2.
$$\mu_\wp(\Omega_{A/k}) \leq \dim_\wp^{Tr} A_\wp := height(\wp) + Tr.\deg(\kappa(\wp))$$

3. $\Omega_{A_\wp/k}$ is free, of

$$\textbf{rank}\left(\Omega_{A_\wp/k}\right) = \dim_\wp^{Tr} A_\wp := height(\wp) + Tr.\deg(\kappa(\wp))$$

4. The sequence

$$\begin{array}{ccccccccc} & & & & A_\wp^n & & & & \\ & & & & \| & & & & \\ 0 & \longrightarrow & \left(\frac{I}{I^2}\right)_{\mathcal{N}} & \longrightarrow & \left(\frac{\Omega_{P/k}}{I\Omega_{P/k}}\right)_{\mathcal{N}} & \longrightarrow & \Omega_{A_\wp/k} & \longrightarrow & 0 \end{array}$$

   is split exact.

5. If $ch(k) = 0$, then above are equivalent to $\Omega_{A_\wp/k}$ is free.

We summarize the results when k is perfect.

Proposition 6.2.16. Suppose k is a perfect field and A is a algebra over k, **essentially of finite type**. Let $\wp \in \operatorname{Spec}(A)$. Then, the following are equivalent.

1. A/k is smooth at $\wp$.

2. $$\mu_\wp(\Omega_{A/k}) \leq \dim_\wp^{Tr} A_\wp := height(\wp) + Tr.\deg(\kappa(\wp))$$

3. $\Omega_{A_\wp/k}$ is free of

$$\textbf{rank}\left(\Omega_{A_\wp/k}\right) = \dim_\wp^{Tr} A_\wp := height(\wp) + Tr.\deg(\kappa(\wp))$$

4. $A_\wp$ is regular.

5. The rank (borrowing notations from (6.2.7)):

$$rank(J(\wp)) = n - \dim A = height(F_1, \ldots, F_m)$$

Proof. By (6.2.15), we have $(1) \iff (2) \iff (3)$.

Then, by (6.2.14) $(1) \iff (4)$.

Finally, $(4) \iff (6.2.7)$. The proof is complete. ■

6.2.6 Smooth P-orientations

Theorem 6.2.17. Suppose k is a perfect field. Let A be a smooth affine algebra over k. Let $I \subseteq A$ be an ideal. Then, $\frac{A}{I}$ is smooth if and only if the sequence

$$0 \longrightarrow \frac{I}{I^2} \longrightarrow \frac{\Omega_{A/k}}{I\Omega_{A/k}} \xrightarrow{\varphi} \Omega_{\frac{A}{I}/k} \longrightarrow 0 \quad \text{is split exact.}$$

Proof. Suppose $\frac{A}{I}$ is smooth over k. Fix a maximal ideal $\mathfrak{m} \in V(I)$. Since $A_\mathfrak{m}$ and $\left(\frac{A}{I}\right)_\mathfrak{m}$ are regular, by [Mh, Thm. 36, pp.121] $I_\mathfrak{m}$ is complete intersection. Let $\dim(A_\mathfrak{m}) = d(\mathfrak{m})$ and $\dim\left(\frac{A}{I}\right)_\mathfrak{m} = d(\mathfrak{m}) - r(\mathfrak{m})$. Then, $I_\mathfrak{m}$ is complete intersection of height $r(\mathfrak{m})$.

By (6.2.16),

$$rank\left(\Omega_{\frac{A}{I}/k}\right)_\mathfrak{m} = \dim\left(\frac{A}{I}\right)_\mathfrak{m} = d(\mathfrak{m}) - r(\mathfrak{m}) = rank\left(\Omega_{A_\mathfrak{m}/k}\right) - r(\mathfrak{m})$$

So, $rank\left(\ker\left(\varphi_\mathfrak{m}\right)\right) = r(\mathfrak{m}) = rank\left(\frac{I}{I^2}\right)_\mathfrak{m}$. Since the map $\left(\frac{I}{I^2}\right)_\mathfrak{m} \twoheadrightarrow \ker(\varphi_\mathfrak{m})$ is a surjective map of free modules of same rank, the sequence is exact on the left.

Conversely, suppose the sequence is split exact. Since $A_\mathfrak{m}$ is smooth, the middle term is free of rank $\dim A_\mathfrak{m} =: d(\mathfrak{m})$, and hence other two terms are free. Let $height(I_\mathfrak{m}) = h$. Then,

$$h \leq rank\left(\frac{I}{I^2}\right)_\mathfrak{m} \Longrightarrow rank\left(\Omega_{\frac{A}{I}/k}\right)_\mathfrak{m} = d(\mathfrak{m}) - rank\left(\frac{I}{I^2}\right)_\mathfrak{m} \leq d(\mathfrak{m}) - h = \dim\left(\frac{A}{I}\right)_\mathfrak{m}$$

By (6.2.16(2)), $\left(\frac{A}{I}\right)_\mathfrak{m}$ is smooth. This completes the proof. ∎

We will now quote an important theorem, referred as Bertini's theorem [Mu, pp. 413], which establishes the crucial link between *height* of an ideal and smoothness. We will state the version in which we assume when $Ch(k) = 0$, more generally if k is perfect (which essentially simplifies *geometrically reduced* to *reduced*). We would mainly be interested in the case of smooth real affine schemes and we would assume that the ground field is perfect, in which case the statement of the theorem is a bit simpler.

Proposition 6.2.18. Let X be a reduced scheme over an infinite perfect field k, with locally closed embedding $X \hookrightarrow \mathbb{P}^N = \text{Proj}\,(k[T_0, T_1, \ldots, T_N])$. Let P be a locally free sheaf over X, with $rank(P) = n$. Assume

1. P is globally generated by finitely generated k-vector space $V \subseteq \Gamma(X, P)$.

2. $(s, a) \in \Gamma\left(X, P(1) \oplus \mathcal{O}_X(1)\right)$ be unimodular.

3. Set
$$W = \sum_{i=0}^{N} t_i \otimes V = Im\left(\Gamma\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)\right)\right) \otimes_k V \longrightarrow \Gamma\left(X, P(1)\right)$$

Then, there is a nonempty open set $U \subseteq W \subseteq \Gamma\left(X, P(1)\right)$ and $y \in U$ such that

1. $Z(s + ay)$ is a reduced subscheme of X, of pure codimension n (or empty).

2. For all irreducible component X_i, the components of $Z(s+ay) \cap X_i$ have dimension $\dim X_i - n$.

3. $Z(s + ay)$ is smooth.

For clarity, we write an affine version of (6.2.18).

Proposition 6.2.19. Let $A = k[t_1, t_2, \ldots, t_N] = \frac{k[T_1, T_2, \ldots, T_N]}{\mathscr{I}}$ be a reduced affine ring, over an infinite perfect field k and $X = \text{Spec}\,(A)$. (*Note* $X = D(T_0) = \text{Spec}\,(k[t_1, t_2, \ldots, t_N])$, *is embedded as, with* $t_i = \frac{T_i}{T_0}$, $\text{Spec}\,(k[t_1, \ldots, t_N]) \longrightarrow \text{Proj}\,(k[T_0, T_1, \ldots, T_N])$ and $\mathcal{O}(1)|_X = A$

Let P be a projective A-module with $rank(P) = n$. Let $P^* = \sum_{i=1}^{m} A\varphi_i$ be a projective

A-module and let $V = \sum_{i=1}^{m} k\varphi_i \subseteq P^*$. Let

$$W = \sum_{j=1}^{N} t_j V = \sum_{j=1}^{N} \sum_{i=1}^{m} k t_j \varphi_i \subseteq P^*$$

Let X_i denote the irreducible components of X. Let $(\varphi, a) \in P^* \oplus A$ be unimodular. Then, there is a nonempty open subset $U \subseteq W$, such that $\forall y \in U$, with $\Phi = \varphi + ay$, and $\mathscr{I} = \Phi(P)$, we have

1. $\mathscr{I} = A$ or $\frac{A}{\mathscr{I}}$ is reduced of pure $height(\mathscr{I}) = n$.

2. $\dim X_i \cap V(\mathscr{I}) = \dim X_i - n$ for all components X_i of X.

3. $\frac{A}{\mathscr{I}}$ is smooth.

In particular, we have a choice of $y = \sum a_{ij} t_j \varphi_i$, with $a_{ij} \in k$ and $\varphi_i \in P^*$.

The following is a projective module version of [Mu, Cor. 2.4].

Corollary 6.2.20. Let A be reduced affine ring, as in (6.2.19), over a perfect field k and $X = \operatorname{Spec}(A)$. For simplicity, assume with $\dim X = \dim X_i = d$, for all irreducible components components where X_i of X. For simplicity, we assume A is smooth.

Let $J \subseteq A$ be an ideal and P be a projective A-module with $rank(P) = n \geq 1 + \dim V(J)$. Let $\omega : P \twoheadrightarrow \frac{I}{I^2}$ be a surjective map, where I is an ideal with $I + J = A$. Then, there is a lift $\varphi : P \twoheadrightarrow I \cap K$, a surjective lift of ω, such that

1. $I + K = J + K = A$, with $height(K) \geq n$ (i.e. $height(K) = n$ or $K = A$).

2. $\frac{A}{K}$ is smooth.

Proof. Let $\psi : P \twoheadrightarrow I_0 \subseteq I$ be any surjective lift of ω, We write $\overline{\psi} = \omega$. Then, $(1-s)I \subseteq I_0$ for some $s \in I$, and $I = (I_0, s)$. Since, $(I_0, s) + J = A$, (ψ, s^2) is basic on $V(J)$. So, by basic element theory, $\psi + s^2\gamma$ is basic on $V(J)$. Replacing ψ by $\psi + s^2\gamma$, we can assume $I_0 + J = A$. Consider

$$\mathfrak{X} = X - V(JI^2), \quad \mathscr{E} = P^*_{|\mathfrak{X}}, \quad P^* = \sum_{i=1}^{m} A\lambda_i, \qquad JI^2 = \sum_{j=1}^{m} Aa_i$$

Let

$$V = \sum_{i=1}^{m}\sum_{j=1}^{m} k\lambda_i a_j \tag{6.4}$$

Note V generates $\mathscr{E} = P^*_{|\mathfrak{X}}$, globally.

Then, with $a = 1$, we have (ψ, a) is basic on $\mathfrak{X}$. By (6.2.18), there is $y = \sum t_r\mu_r$, with $\mu_r \in \sum t_l V$ such that $Z(\psi + ay)$ has smooth locus on all points of $\mathfrak{X}$. This means, with $\Phi = \psi + ay$ and $\mathcal{I} = \Phi(P)$, the ring $\left(\frac{A}{\mathcal{I}}\right)_\wp$ is smooth, at all points in $\wp \in \mathfrak{X}$.

1. Since $a_i \in I^2$, as in (6.4), Φ remains a lift of $\overline{\psi}\omega$.
2. As usual, by Nakayama lemma,

$$\Phi(P) = I \cap K \quad \text{with} \quad I + K = A$$

3. Further, since $a_j \in J$, we have

$$A = I_0 + J = \psi(P) + J = \Phi(P) + J \qquad \Longrightarrow \qquad K + J = A$$

4. Now suppose, $K \subseteq \wp \in \operatorname{Spec}(A)$. Then, $I^2 J \not\subseteq \wp$. So, $\wp \in \mathfrak{X}$, and $\left(\frac{A}{K}\right)_\wp$ is smooth.

The proof is complete. ■

Corollary 6.2.21. The hypothesis of (6.2.20) can be relaxed and the conclusion works for any reduced ring A, essentially of finite type over an infinite perfect field k. This means $A = S^{-1}B$, where $B = k[x_1, \ldots, x_r]$ is fintely generated.

Now we are ready to state the main result in this section.

Theorem 6.2.22. Suppose k is a perfect field. Let A be an essentially smooth ring over k, with $1/2 \in k$, and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$. Let

$$\mathcal{L}O_{\mathfrak{s}}(P) = \left\{ (I, \omega) \in \widetilde{\mathcal{L}O}(P) : \frac{A}{I} \text{ is smooth and height(I) = n} \right\} \bigcup \{(A, 0)\}$$

Define $\pi_0\left(\mathcal{L}O_{\mathfrak{s}}(P)\right)$ by using homotopies in $\mathcal{L}O_{\mathfrak{s}}(P[T])$. Then,

1. There is a natural map

$$\Theta : \pi_0\left(\mathcal{L}O_{\mathfrak{s}}(P)\right) \twoheadrightarrow \pi_0\left(\widetilde{\mathcal{L}O}(P)\right) \qquad \text{sending} \quad [(I, \omega)] \mapsto [(I, \omega)] \tag{6.5}$$

2. The map Θ is surjective.

3. The map Θ is a bijection.

Proof. The proof of (1) is obvious. Let $x = [(I, \omega)] \in \pi_0\left(\widetilde{\mathcal{L}O}(P)\right)$, and $\varphi : P \twoheadrightarrow I \cap K$, as in (6.2.20), (with $J = A$). Let $\omega_K : P \twoheadrightarrow \frac{K}{K^2}$ be the P-orientation, induced by φ. Then $[(K, \omega_K)] = \Gamma(x) \in \pi_0\left(\widetilde{\mathcal{L}O}(P)\right)$ is the involution, with $\frac{A}{K}$ smooth. By one more application of the same argument, it follows $x = \Gamma^2(x) = [(K_1, \omega_{K_1})]$, with $\frac{A}{K_1}$ smooth. So, the map Θ is surjective. This settles (2). To prove (3), we will follow the proof of Lemma 6.1.2.

Suppose $(I_0, \omega_0), (I_1, \omega_1) \in \mathcal{L}O_{\mathfrak{s}}(P)$ such that $\Theta\left([(I_0, \omega_0)]\right) = \Theta\left([(I_1, \omega_1)]\right)$. Recall, homotopy is an equivalence relation. By (6.1.2), there is $\mathcal{H}(T) = (I, \omega) \in \widetilde{\mathcal{L}O}(P[T])$ such that $\mathcal{H}(0) = (I_0, \omega_0)$, $\mathcal{H}(1) = (I_1, \omega_1)$. By (6.2.20), with $J = A[T]$, there there is a lift $\varphi : P \twoheadrightarrow I \cap K$, a surjective lift of ω, such that

1. $I + K = A[T]$, with $height(K) \geq n$.

2. $\frac{A[T]}{K}$ is smooth.

Write $1 = s(T) + t(T) \in I^2 + K^2$. We write $J = I \cap K$. So, $t(T)I \subseteq I \cap K = J$ and $s(T)K \subseteq I \cap K = J$.

Write $X[T] = \text{Spec}\,(A[T])$. Consider

$$\mathfrak{X} = X[T] - V(T(1-T)K^2), \quad \mathscr{E} = P^*_{|\mathfrak{X}}, \quad P[T]^* = \sum_{i=1}^{m} A[T]\lambda_i, \qquad K^2 = \sum_{j=1}^{m} A[T]a_j$$

Let

$$V = \sum_{i=1}^{m}\sum_{j=1}^{m} k\lambda_i A[T]a_j$$

Given $\wp \in \mathfrak{X}$, $K^2 \nsubseteq \wp$. So, a_i is unit in $A[T]_\wp$ for some i. So, V generates $\mathscr{E} = P[T]^*_{|\mathfrak{X}}$, globally .

Since $T(1-T)$ acts as an unit in $\mathfrak{X}$, we have, $(\varphi, T(1-T))$ is basic on $\mathfrak{X}$. So, there is

$$\Phi = \varphi + T(1-T)\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i a_j \quad \text{with } a_j \in K^2$$

such that $Z(\Phi)$ is smooth on $\mathfrak{X}$, with $co\dim(Z(\Phi) = n$ on $\mathfrak{X}$. In other words, if $\mathscr{I} = \Phi(P[T])$, then $\frac{A[T]}{\mathscr{I}}$ is smooth on $\mathfrak{X}$, with $height(\mathscr{I}_\wp) = n$ for all $\wp \in \mathfrak{X}$ (or $Z(\Phi)$ is empty). Claim,

$$K = \Phi(P[T]) + K^2 \qquad \text{Clearly,} \quad RHS \subseteq K$$

Let $x \in K$. Then

$$x = xs(T) + xt(T) = \varphi(p) + xt(T) = \Phi(p) - \left(T(1-T)\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i(p)a_j\right) + xt(T)$$

which is in $\Phi(P[T]) + K^2$. So, the equality is established. So,

$$\Phi(P[T]) = \tilde{I} \cap K \quad \text{with} \quad \tilde{I} + K = A[T]$$

Let $\tilde{I} \subseteq \wp$ and $T(1-T) \notin \wp$. Then, $\wp \in \mathfrak{X}$, and hence $\left(\frac{A[T]}{\mathscr{I}}\right)_\wp = \left(\frac{A[T]}{\tilde{I}}\right)_\wp$ is smooth.

Since, $\Phi_{|T=0} = \varphi_{|T=0}$, we have

$$I(0) \cap K(0) = \varphi(P[T])(0) = \Phi(P[T])(0) = \tilde{I}(0) \cap K(0)$$

This decomposition is unique (if $K(0) = J + K(0)^2$ and $(1-t)K(0) \subseteq J$ for some $t \in K(0)$ and $I(0) = \tilde{I}(0) = (J, 1-t)$). So, $\tilde{I}(0) = I(0) = I_0$. Similarly, $\tilde{I}(1) = I(1) = I_1$. Consider,

$$\mathcal{H}(T) = (\tilde{I}, \tilde{\omega}) \in \widetilde{\mathcal{L}O}(P[T]) \quad \text{where} \quad \begin{tikzcd} P[T] \arrow[r, two heads] \arrow[dr, bend right, "\tilde{\omega}"'] & \tilde{I} \cap K \arrow[d] \\ & \frac{\tilde{I}}{\tilde{I}^2} \end{tikzcd}$$

Again, since $\Phi_{|T=0} = \varphi_{|T=0}$, we have $\mathcal{H}(0) = (I_0, \omega_0)$, and $\mathcal{H}(1) = (I_1, \omega_1)$.

It remains to show that $\frac{A[T]}{\tilde{I}}$ is smooth at primes $\wp \in V(T(1-T))$. First, we establish that $height(\tilde{I}) = n$ (unless $\tilde{I} = A[T]$). Let $\tilde{I} \subseteq \wp \in \operatorname{Spec}(A[T])$). If $T(1-T) \notin \wp$, then it follows from above $height(\wp) \geq n$. Now assume, $T \in \wp$. So, $\wp = (\wp_0, T)$ and $I_0 \subseteq \wp_0$. Since $height(I_0) \geq n$, it follows $height(\wp) \geq n+1$. By same reasoning, for $(\tilde{I}, 1-T) \subseteq \wp \in \operatorname{Spec}(A[T])$, $height(\wp) \geq n+1$). This establishes that $height(\tilde{I}) = n$ (unless $\tilde{I} = A[T]$).

Since T and $T-1$ are interchangeable, we only need to prove, for prime ideals, $\wp \in V(\tilde{I})$ with $T \in \wp$, $\frac{A[T]}{\tilde{I}}$ is smooth at $\wp$. This is same as proving $\frac{A[T]}{\tilde{I} \cap K}$ is smooth at $\wp$. This formulation would make applicability of the the Jacobian criterion (6.2.7), as given in [K, pp. 123].

We write $\wp = (\wp_0, T)$, with $\wp_0 \in \operatorname{Spec}(A)$. By hypothesis $\frac{A}{I_0}$ is smooth at $\wp_0$, we might as well say that $\frac{A}{I_0 \cap K_0}$ is smooth at $\wp_0$. Write

$$A = \frac{k[X_1, X_2, \ldots, X_N]}{(F_1, \ldots, F_m)} \quad P = \sum_{i=1}^{r} Ap_i$$

Write

$$\begin{cases} \varphi(p_i) = \overline{g_i(X_1, \dots, X_N, T)} \\ \Phi(p_i) = \overline{g_i + T(1-T) \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i(p_i) a_j} \quad \text{with } a_j \in K^2 \\ = \overline{g_i + T(1-T) H_i(X_1, \dots, X_N, T)} \quad \text{with } H_i = \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i(p_i) a_j \\ \text{Write } G_i(X_1, \dots, X_N, T) = g_i + T(1-T) H_i(X_1, \dots, X_N, T) \end{cases}$$

So,

$$(g_1, \dots, g_r) = I \cap K \quad (G_1, \dots, G_r) = \tilde{I} \cap K$$

We have

$$J(g(X,0), \wp_0) = \left(\begin{array}{cccc} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial X_1} & \frac{\partial F_m}{\partial X_2} & \cdots & \frac{\partial F_m}{\partial X_N} \\ \hline \frac{\partial g_1(X,0)}{\partial X_1} & \frac{\partial g_1(X,0)}{\partial X_2} & \cdots & \frac{\partial g_1(X,0)}{\partial X_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_r(X,0)}{\partial X_1} & \frac{\partial g_r(X,0)}{\partial X_2} & \cdots & \frac{\partial g_r(X,0)}{\partial X_N} \end{array} \right) \in \kappa(\wp_0) = \frac{A_{\wp_0}}{\wp_0 A_{\wp_0}} \tag{6.6}$$

By Jacobian criterion (6.2.7)

$$rank(J(g(X,0), \wp_0) = N - \dim(V(I_0)) = N - (d-n) \tag{6.7}$$

And

$$J(G, \wp) = \left(\begin{array}{cccc|c} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_N} & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{\partial F_m}{\partial X_1} & \frac{\partial F_m}{\partial X_2} & \cdots & \frac{\partial F_m}{\partial X_N} & 0 \\ \hline \frac{\partial G_1(X,T)}{\partial X_1} & \frac{\partial G_1(X,T)}{\partial X_2} & \cdots & \frac{\partial G_1(X,T)}{\partial X_N} & \frac{G_1(X,T)}{\partial T} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial G_r(X,T)}{\partial X_1} & \frac{\partial G_r(X,T)}{\partial X_2} & \cdots & \frac{\partial G_r(X,T)}{\partial X_N} & \frac{\partial G_r(X,T)}{\partial T} \end{array} \right)$$

$$
= \left(\begin{array}{ccc|c}
\frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} & 0 \\
\cdots & \cdots & \cdots & 0 \\
\frac{\partial F_m}{\partial X_1} & \cdots & \frac{\partial F_m}{\partial X_N} & 0 \\
\hline
\frac{\partial g_1(X,T)}{\partial X_1} + T(1-T)\frac{\partial H_1(X,T)}{\partial X_1} & \cdots & \frac{\partial g_1(X,T)}{\partial X_N} + T(1-T)\frac{\partial H_1(X,T)}{\partial X_N} & \frac{G_1(X,T)}{\partial T} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial g_r(X,T)}{\partial X_1} + T(1-T)\frac{\partial H_r(X,T)}{\partial X_1} & \cdots & \frac{\partial G_r(X,T)}{\partial X_N} + T(1-T)\frac{\partial H_r(X,T)}{\partial X_N} & \frac{\partial G_r(X,T)}{\partial T}
\end{array}\right)
$$

The image of $J(G, \wp)$ in $\kappa(\wp) = \kappa(\wp_0)$ is same as the image of

$$
= \left(\begin{array}{ccc|c}
\frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} & 0 \\
\cdots & \cdots & \cdots & 0 \\
\frac{\partial F_m}{\partial X_1} & \cdots & \frac{\partial F_m}{\partial X_N} & 0 \\
\hline
\frac{\partial g_1(X,0)}{\partial X_1} & \cdots & \frac{\partial g_1(X,0)}{\partial X_N} & \frac{G_1(X,T)}{\partial T} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial g_r(X,0)}{\partial X_1} & \cdots & \frac{\partial g_r(X,0)}{\partial X_N} & \frac{\partial G_r(X,T)}{\partial T}
\end{array}\right)
$$

The left side of the vertical line coincides with (6.6), and its rank by (6.7) is $= N - (d - n)$. So,

$$rank(J(G, \wp)) \geq N - (d - n) = (N + 1) - (d + 1 - n) = N + 1 - \dim V(\tilde{I} \cap K)$$

By Jacobian criterion (6.2.7(1)),we have

$$rank(J(G, \wp) = N - (d - n) = (N + 1) - (d + 1 - n) = N + 1 - \dim V(\tilde{I} \cap K)$$

By Jacobian criterion (6.2.7(2)) $\frac{A[T]}{\tilde{I} \cap K}$ is smooth at $\wp = (\wp_0, T)$. The proof is complete. ■

6.3 Real affine schemes

In this section, we discuss real affine schemes. First, we establish the general setup and some notations.

Notations 6.3.1. In this section $A = \mathbb{R}[x_1, \ldots, x_r] = \frac{\mathbb{R}[X_1, \ldots, X_r]}{\mathscr{I}}$ would denote a smooth real affine ring, with $\dim A = d$. Let $X = \mathrm{Spec}\,(A)$ be the corresponding affine scheme. We assume all components X_i of X have $\dim X_i = d$.

1. $\Omega_{A/\mathbb{R}}$ would denote the module of (Kähler) differentials. Note, $\Omega_{A/\mathbb{R}}$ is a projective A-module of rank d (6.2.5, 6.2.14, 6.2.15, 6.2.16).

2. Let $M := M(A)$ denote the manifold of real points in X and $T(M)$ denote the tangent bundle on M. Note, M can be empty or may have several connected components.

3. For real points $x = (a_1, a_2, \ldots, a_r) \in M$, $\mathfrak{m}_x = (X_1 - a_1, X_2 - a_2, \ldots, X_r - a_r) \in \mathrm{Spec}\,(A)$ would denote corresponding maximal ideal. Also, let $\kappa(x) = \frac{A}{\mathfrak{m}_x}$.

4. The vector space duals $Hom(-, \mathbb{R})$ would be denoted by *. We continue to denote $\mathcal{M}^\star = Hom(M, A)$ for any A-module $\mathcal{M}$. To be consistent, subsequently, for a vector bundle $\mathscr{V}$ on M, its dual would be denoted by $\mathscr{V}^\star$.

The following connection between algebraic cotangent bundles and topological cotangent bundles is worth recording.

Lemma 6.3.2. Use the notations as in (6.3.1). Suppose $x = (a_1, a_2, \ldots, a_r) \in M$. Then, the fibers of the cotangent and tangent bundles are

$$\begin{cases} T(M)_x^\star = \Omega_{A/\mathbb{R}} \otimes \kappa(x) = \Omega_{A/\mathbb{R}} \otimes \frac{A}{\mathfrak{m}_x}, \\ T(M)_x \cong T(M)_x^{\star\star} = \left(\Omega_{A/\mathbb{R}} \otimes \kappa(x)\right)^\star = Hom_A(\Omega_{A/\mathbb{R}}, A) \otimes \kappa(x) \end{cases}$$

Proof. We only prove $T(M)^\star_x = \Omega_{A/\mathbb{R}} \otimes \kappa(x)$. Since, both A and $\frac{A}{I}$ are smooth, the sequence

$$0 \longrightarrow \frac{\mathscr{I}}{\mathscr{I}^2} \longrightarrow \Omega_{\mathbb{R}[X_1,\ldots,X_r]/\mathbb{R}} \longrightarrow \Omega_{A/\mathbb{R}} \longrightarrow 0$$

is split exact. Therefore,

$$0 \longrightarrow \frac{\mathscr{I}}{\mathscr{I}^2} \otimes \kappa(x) \longrightarrow \Omega_{\mathbb{R}[X_1,\ldots,X_r]/\mathbb{R}} \otimes \kappa(x) \longrightarrow \Omega_{A/\mathbb{R}} \otimes \kappa(x) \longrightarrow 0$$

is also split exact. There are natural vertical maps (see 6.9), and a commutative diagram, as follows

$$\begin{tikzcd} 0 \arrow[r] & \frac{\mathscr{I}}{\mathscr{I}^2} \otimes \kappa(x) \arrow[r] \arrow[d, dashed] & \Omega_{\mathbb{R}[X_1,\ldots,X_r]/\mathbb{R}} \otimes \kappa(x) \arrow[r, "\lambda"] \arrow[d, "\wr"'] & \Omega_{A/\mathbb{R}} \otimes \kappa(x) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N(\mathbb{R}^r, M)^\star_x \arrow[r] & T(\mathbb{R}^r)^\star_x \arrow[r, "\beta"'] & T(M)^\star_x \arrow[r] & 0 \end{tikzcd} \tag{6.8}$$

1. First note that $T(M)^\star_x = \frac{\mathbf{I}(M)_x}{\mathbf{I}(M)^2_x}$, where $I(M)_x$ denote the ideal of germs of smooth functions $f(x)$ near x, with $f(x) = 0$. This gives a derivation

   $$A \longrightarrow T(M)^\star \qquad \text{sending} \qquad f \mapsto f - f(x)$$

   By universal property, we obtain the vertical natural maps

   $$\begin{tikzcd} A \arrow[r] \arrow[dr] & \Omega_{A/\mathbb{R}} \arrow[d, dashed] \\ & T(M)^\star_x \end{tikzcd} \tag{6.9}$$

2. Take a set of generators, $\mathscr{I} = (F_1, F_2, \ldots, F_m)$. Then,

   $$\begin{cases} \ker(\lambda) = \sum \kappa(x) dF_k \\ \ker(\beta) = \sum \kappa(x) \delta F_k \end{cases} \qquad \text{where} \quad \delta F_k = \sum_i \frac{\partial F_k}{\partial X_i}(x) dX_i \tag{6.10}$$

   The first equality follows because the first line in (6.8) is right exact (in fact exact). To see the latter part, write $V = \sum \kappa(x)\delta F_k \subseteq T(\mathbb{R}^r)^\star_x$. Then, with

$$J(F) = \begin{pmatrix} \frac{\partial F_1}{\partial X_1}(x) & \frac{\partial F_1}{\partial X_2}(x) & \cdots & \frac{\partial F_1}{\partial X_r}(x) \\ \frac{\partial F_2}{\partial X_1}(x) & \frac{\partial F_2}{\partial X_2}(x) & \cdots & \frac{\partial F_2}{\partial X_r}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial X_1}(x) & \frac{\partial F_m}{\partial X_2}(x) & \cdots & \frac{\partial F_m}{\partial X_r}(x) \end{pmatrix}$$

The surjective map,

$$T(\mathbb{R}^r)^\star = \bigoplus_{i=1}^r \kappa(x)dX_i \xrightarrow{J(F)} V$$

sends

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_r \end{pmatrix} \mapsto \begin{pmatrix} z_1 & z_2 & \cdots & z_r \end{pmatrix} J(F) \begin{pmatrix} dX_1 \\ dX_2 \\ \cdots \\ dX_r \end{pmatrix}$$

So, $\dim V = rank(J(F)) = \dim \ker(\beta)$. Since $V \subseteq \ker(\beta)$ we have $V = \ker(\beta)$.

3. Since, $V = \ker(\beta)$, if follows from (6.10) that the dotted vertical map (6.8) is surjective, and hence an isomorphism.

4. Now, since the first two vertical maps in (6.8) are isomorphisms, so is the 3^{rd}-vertical map.

The proof is complete. ∎

The following connection between algebraic conormal bundles and topological conormal bundles is of our particular interest.

Proposition 6.3.3. Let $X = \mathrm{Spec}\,(A)$ be a smooth affine space, over $\mathbb{R}$, with $\dim X = d$. Assume all components $X_i \subseteq X$, $\dim X_i = d$. Let $Y = \mathrm{Spec}\left(\frac{A}{I}\right) \subseteq X$ be a closed subscheme, with Y smooth, over $\mathbb{R}$, pure codimension n or pure $\dim Y = d - n$. Let M be the set of real points in X and $B \subseteq M$ be the set of real points in Y. Then,

1. Then, M is a smooth manifold, with $\dim M = d$ (or $M = \phi$).

2. Further, $B \subseteq M$ is a submanifold (or $B = \phi$).

3. The conormal bundle is given by

$$N(M,B)^\vee = \coprod_{x\in B} \frac{I}{I^2} \otimes \frac{A}{\mathfrak{m}_x} = \coprod_{x\in B} \frac{I}{I^2 + \mathfrak{m}_x I} = \coprod_{x\in B} \frac{I}{\mathfrak{m}_x I} \quad \text{where } \mathfrak{m}_x \text{ is the idealof } x. \tag{6.11}$$

Proof. First, (1) follows from Implicit function theorem. By same reasoning, B is also a manifold. The sequence

$$0 \longrightarrow \frac{I}{I^2} \longrightarrow \Omega_{A/\mathbb{R}} \longrightarrow \Omega_{\frac{A}{I}/\mathbb{R}} \longrightarrow 0 \tag{6.12}$$

is a split exact sequence. For $x \in B$, and $\kappa(x) = \frac{A}{\mathfrak{m}_x} \cong \mathbb{R}$, the sequence leads to the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \frac{I}{I^2} \otimes \kappa(x) \arrow[r] & \Omega_{A/\mathbb{R}} \otimes \kappa(x) \arrow[r] \arrow[d, "\chi_M"', "\wr"] & \Omega_{\frac{A}{I}/\mathbb{R}} \otimes \kappa(x) \arrow[r] \arrow[d, "\wr"', "\chi_B"] & 0 \\ 0 \arrow[r] & N(M,B)^\star_x \arrow[r] & T(M)^\star_x \arrow[r, "q"'] & T(B)^\star_x \arrow[r] & 0 \end{tikzcd} \tag{6.13}$$

The natural maps χ_M and χ_B are isomorphisms, by (6.3.2). Therefore, the map q is surjective. This settles (2) that $B \subseteq M$ is a submanifold. Now, χ_M induces an isomorphism $\frac{I}{I^2} \otimes \kappa(x) \xrightarrow{\sim} N(M,B)_x$. So, the conormal bundle is as in (6.11). This settles (3). The proof is complete. ∎

6.3.1 Algebraic to Topological obstructions

Before we proceed, we recall the following construction.

Definition 6.3.4. Let $A = \mathbb{R}[x_1, \ldots, x_r] = \frac{\mathbb{R}[X_1,\ldots,X_r]}{\mathscr{I}}$ be a smooth real affine ring, with $\dim A = d$, and $X = \mathrm{Spec}\,(A)$. be the corresponding affine scheme. Assume , for all components $X_i \subseteq X$, $\dim X_i = d$. Let P be a projective A-module, with $rank(P) = n \leq d$. Let $\mathcal{S}ym(P) = \oplus_{i=0}^\infty S_i P$ be the the symmetric algebra. Then,

1. $\mathscr{V}_a(P) = \mathrm{Spec}\left(\mathcal{S}ym(P)\right)$ is defined to be the algebraic vector bundle, corresponding to P. It comes with the natural structure map $\mathscr{V}_a(P) \longrightarrow X$, induced by the map $A \longrightarrow \mathcal{S}ym(P)$. Note, the module of sections of $\mathscr{V}_a(P) \longrightarrow X$ is $P^\star := Hom(P, A)$.

2. Let $M := M(X)$ be the manifold of real points in X. Also, let $\mathscr{V}_{\mathbb{R}}(P)$ be the real points in $\mathscr{V}_a(P)$. Then, $\mathscr{V}_{\mathbb{R}}(P) \longrightarrow M$ is defined to be the real vector bundle, over M, corresponding to P. In fact,
$$\mathscr{V}_{\mathbb{R}}(P) = \coprod_{x \in M} \frac{P}{\mathfrak{m}_x P} = \coprod_{x \in M} P \otimes \kappa(x)$$

Now we define a natural map, from the algebraic obstructions to the topological obstructions.

Theorem 6.3.5. Use all the notations from (6.3.4). Then, there is a natural map

$$\pi_0\left(\mathcal{L}O(P)\right) \longrightarrow \pi_0\left(\mathcal{L}O(\mathscr{V}_{\mathbb{R}}^\star(P))\right)$$

Proof. By (6.2.22), $\pi_0\left(\mathcal{L}O(P)\right) \cong \pi_0\left(\mathcal{L}O_{\mathfrak{s}}(P)\right)$. So, we define natural a map

$$\pi_0\left(\mathcal{L}O_{\mathfrak{s}}(P)\right) \longrightarrow \pi_0\left(\mathcal{L}O(\mathscr{V}_{\mathbb{R}}^\star(P))\right)$$

Let $(I, \omega) \in \mathcal{L}O_{\mathfrak{s}}(P)$. So, $\omega : \frac{P}{IP} \xrightarrow{\sim} \frac{I}{I^2}$ is an isomorphism. This induces, isomorphism of the algebraic vector bundles:

$$\begin{tikzcd}
\mathscr{V}_a\left(\frac{P}{IP}\right) \arrow[rr, "\omega_a", "\sim"'] \arrow[dr] & & \mathscr{V}_a\left(\frac{I}{I^2}\right) \arrow[dl] \\
& \mathrm{Spec}\left(\frac{A}{I}\right) &
\end{tikzcd}$$

Let $B(I) \subseteq M$ denote the set of all real points on $\mathrm{Spec}\left(\frac{A}{I}\right)$. Restricting on the real points, the above diagram gives the isomorphisms of real vector bundles:

$$\begin{tikzcd}
\mathscr{V}_t(P)|_{B(I)} \arrow[r, equal] & \mathscr{V}_t\left(\frac{P}{IP}\right) \arrow[rr, "\omega_t", "\sim"'] \arrow[dr] & & \mathscr{V}_t\left(\frac{I}{I^2}\right) \arrow[r, "\sim"'] \arrow[dl] & N(M, B(I))^\star \\
& & B(I) & &
\end{tikzcd}$$

Here, the left hand equality $\mathscr{V}_t\left(\frac{P}{IP}\right) = \mathscr{V}_t(P)|_{B(I)}$ is obvious. The right hand isomorphism $\mathscr{V}_t\left(\frac{I}{I^2}\right) \xrightarrow{\sim} N(M, B(I))^\star$, is obtained by (6.3.3). Dualising, we obtain an isomorphism

$$\omega_t^\star : N(M, N) \xrightarrow{\sim} \mathscr{V}_t^\star(P)|_{B(I)}$$

This defines a map

$$\varphi : \mathcal{L}O_{\mathfrak{s}}(P) \longrightarrow \pi_0\left(\mathcal{L}O(\mathscr{V}_{\mathbb{R}}^\star(P))\right) \qquad \text{sending} \quad (I, \omega) \mapsto [(B(I), \omega_t^\star)]$$

In case, $B(I) = \phi$ is empty, interpret $[(B(I), \omega_t^\star)] := 0$.

Now suppose (I_0, ω_0) and $(I_1, \omega_1) \in \mathcal{L}O_s(P)$ such that $[(I_0, \omega_0)] = [(I_1, \omega_1)] \in \pi_0\left(\mathcal{L}O_{\mathfrak{s}}(P)\right)$. Then, by the proof of (6.2.22), there is $(I, \omega) \in \mathcal{L}O_s(P[T])$ such that $(I, \omega)|_{T=0} = (I_0, \omega_0)$ and $(I, \omega)|_{T=1} = (I_1, \omega_1)$. It follows

$$((B(I), \omega_t^\star)|_{T=1} = (B(I_1), (\omega_1)_t^\star), \quad \text{and} \quad ((B(I), \omega_t^\star)|_{T=0} = (B(I_0)(\omega_0)_t^\star)$$

Therefore, φ factors through a map Φ, as follows:

$$\begin{tikzcd}
\mathcal{L}O_{\mathfrak{s}}(P) \arrow[d] \arrow[dr, bend left, "\varphi"] & \\
\pi_0\left(\mathcal{L}O_{\mathfrak{s}}(P)\right) \arrow[r, dashed, "\Phi"'] & \pi_0\left(\mathcal{L}O(\mathscr{V}_{\mathbb{R}}^\star(P))\right)
\end{tikzcd}$$

The proof is complete. ∎

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