Operator Theory on Univariate Analytic Function Spaces

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Abstract

The goal of this thesis is to study certain aspects of Toeplitz, Hankel, and composition operators, as well as the associated operator algebras, on various Hardy and Bergman spaces of univariate analytic functions. The work belongs to the general area of function theoretic operator theory, with added flavors of uniform function algebras and C*-algebras. It is hoped that the results obtained reveal new and perhaps interesting connections between properties of the symbol functions and those of the induced operators. It is also hoped that some of the methods and techniques developed in this research could help solve other problems in the theory of Toeplitz and Hankel operators beyond the scope of this thesis.

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Contents

Introduction

This thesis investigates certain classes of bounded linear operators defined by functions, referred to as symbols, on Banach and Hilbert spaces or their isometrically isomorphic copies of analytic functions in a complex variable. The central theme is to reveal connections between various operator theoretic properties and those of the symbols. The results obtained are mainly driven by operator theoretic questions, for the effective treatment of which operator algebraic framework and tools are often tapped on the basis of specific relations among the generating operators. Thus the development of this work is in the spirit of viewing operator theory and operator algebras as two inextricably linked subjects. Nonetheless, the chief interest here is not in general structural questions on operator algebras but rather in their relevant features that produce concrete results for the operators involved. In this regard, Chapter 3 and 4 see widespread application of C*-algebraic and, perhaps to a lesser extent, commutative Banach algebraic methods to Toeplitz, Hankel, and composition operators on the classical Hardy space over the unit circle.

For a majority of problems in operator theory on the Hardy space, the largest possible symbol class consists of the essentially bounded measurable complex functions on the circle. These functions are more profitably viewed as continuous functions on three distinct regions of the maximal ideal space of the algebra of bounded analytic functions: the embedded disc on which the functions take the form of bounded harmonic functions; the non-trivial Gleason parts other than the disc where the composed functions by the Hoffman maps arise again as harmonic functions; and the Shilov boundary, partitioned in fibers and containing the support sets of representing measures, over which the functions live in restrictions of uniform algebras. This point of view makes a variety of analytical, algebraic, and topological techniques applicable.

The C*-algebra of quasi-continuous functions on the circle consists of functions that together with their complex conjugates are sums of bounded analytic functions and continuous functions. By a well-known characterization, they are precisely the bounded functions of vanishing mean oscillation and thus closed under various real-analysis-style operations on the circle. Moreover, the effect of these operations on the asymptotic behavior of their harmonic extensions in the disc is often well understood. Through a natural isomorphism, this commutative C*-algebra or a certain C*-subalgebra plays an important role in Allan-Douglas localization as the center of the Toeplitz-Hankel C*-subalgebra of the Calkin algebra. This in turn necessitates the study of restriction algebras on the corresponding fibers of the maximal ideal space of the bounded functions. It is via these connections that Douglas algebras enter in the study of Toeplitz and Hankel operators. Besides, Douglas algebras realized on various compact spaces are independently studied in the literature as an important class of uniform algebras, with [117] being one of the earliest surveys on this subject.

Now we turn to the actual content of the thesis after these general remarks.

The first chapter of the thesis serves several purposes. It reviews basic concepts and definitions to be used in later chapters, records fundamental theorems in the selected fields, and sets up notation and preliminary facts in preparation for further development. As such, almost all the results included in this chapter are known to various degrees. Some of them are found in standard reference books, for which we shall not provide explicit citation. Instead, these books are listed as follows: [115] on real and complex analysis, [121] on function theory on the circle, [78, 116, 35] on complex Banach and Hilbert spaces and their operators, [111, 17, 78] on Banach algebras, [60, 93, 126, 61, 5] on uniform algebras, [46, 49, 141] on C*-algebras, [79, 62] on Douglas algebras, [79, 55] on Hardy and [157] on Bergman spaces, [49, 18, 157, 98] on Toeplitz and Hankel operators, and [42, 98] on composition operators. Others appear in the journal literature, and citation is given wherever possible. Occasionally, some proofs are still provided if they seem to be new and interesting, or motivate and connect with ideas appearing in later developments, or are not readily available in the literature.

The next three chapters treat various aspects of operator theory on analytic function spaces. Chapter 2 concerns analytic multiplication operators on Hardy and weighted Bergman spaces over planar regions, containing results on Banach spaces obtained by complex analysis and duality, and results on Hilbert spaces based on unitary equivalence or similarity. The results of this chapter appeared in [145]. Chapter 3 includes two applications of Allan-Douglas localization for Toeplitz and Hankel operators on the Hardy space using C*-algebraic and Douglas-algebraic methods. This chapter has appeared as [144]. Chapter 4 concerns Hankel operators with piecewise quasi-continuous symbols, Toeplitz-composition C*-algebras with quasi-continuous symbols and a subgroup of conformal automorphisms of the disc, and commutative Toeplitz-composition subalgebras of the Calkin algebra generated by certain piecewise quasi-continuous symbols and a linear fractional non-automorphism. The latter is sourced from and improves on a submitted manuscript [146]. A synopsis of the main results of each chapter is to constitute the rest of this Introduction, while a more detailed overview with surveys of literature and discussions of motivation will appear in each beginning section.

Chapter 2 studies some aspects of commutant theory and functional calculus for analytic multiplication operators on Hardy and weighted Bergman spaces over bounded planar regions. Multiplication operators defined by univalent functions are shown to commute only with multiplication operators. This result is generalized to a tuple of operators, and a sufficient condition is given for irreducibility of that induced by finite Blaschke products. Operators defined by fairly general ancestral functions are shown to commute with no nonzero compact operators, and these include the ones by monomial functions over annuli for which we also characterize the commutants. Norm and sequential weak closures of the analytic functional calculus algebra generated by a multiplication operator are characterized and essential spectral mapping properties obtained. Generalizing to a larger class of weighted Bergman spaces the similarity for multiplication operators defined by finite Blaschke products, the commutant classification of these operators is obtained and seen strictly finer than the similarity classification. It also follows that, when the degree is greater than one, these operators commute with no nonzero compact operators and yet are reducible in the Banach space sense, and a characterization is obtained for the commutant of a tuple of multiplication operators to equal that of a given finite Blaschke product.

Motivated by results in uniform algebras, a distance localization formula in C*-algebras is established in Chapter 3 under the framework of the Allan-Douglas localization principle, and is used to derive a locality result for products of Hankel operators as compact perturbations of Hankel operators. Using localization and certain quasi-continuous functions, it is proved that the essential spectrum of the commutator of a Toeplitz and a Hankel operator is antipodal symmetric under a mild condition on the Hankel symbol function. Under the same condition the essential spectrum of a Hankel operator also exhibits this symmetry. Conjugates of interpolating Blaschke products and characteristic functions are constructed that satisfy the condition, while examples show the condition is only sufficient.

Chapter 4 is focused on some special symbol classes for Toeplitz, Hankel, and composition operators on the Hardy space. The essential spectrum of commutators of Hankel operators with arbitrary piecewise quasi-continuous symbols is shown contractible, antipodal symmetric, connected and having a connected complement. Therefore, it follows from the Brown-Douglas-Fillmore theory that such commutators are compact perturbations of normal operators and unitarily equivalent modulo compact operators to their additive inverses, and that the associated C*-algebra extensions split. Using a relation between fibers and support sets, compact commutators and self-commutators are characterized in terms of symbol behavior on support sets and Douglas algebras generated by the symbols. Next, the C*-algebra generated by quasi-continuous Toeplitz operators and a subgroup of automorphic composition operators is shown to extend the compact operators by a crossed-product C*-algebra, from which there follows a characterization of Fredholm operators generated by quasi-continuous Toeplitz operators and a single rational rotation. Lastly, composition operators defined by linear fractional non-automorphims of the disc fixing a boundary point are considered. The maximal ideal space as well as the Shilov boundary of certain associated commutative Calkin subalgebras are completely identified, which yields explicit formulas for the essential spectrum and essential norm. Fredholm index determinations are also obtained.

The last chapter gathers a number of open problems for further investigation. These problems either naturally grow from, or closely relate to, the results already obtained and are motivated by important questions and results found in the literature. Except for a few side results, most of the material in this chapter is exploratory and non-definitive. It is hoped that solutions of, or advances on, these problems may generate more insights in this field of research.

Chapter 1

Preliminaries in Functional Analysis and Operator Theory

1.1 Analytic functions of a complex variable and compact planar sets

First recall the classical theorems of Montel and Vitali as follows.

Theorem 1.1.1 (Montel). *Let Ω be an open subset of $\mathbb{C}$. If f_n is a sequence of analytic functions on Ω that are uniformly bounded on compact subsets of Ω, then there is a subsequence f_{n_k} that converges to an analytic function uniformly on compact subsets of Ω.*

Theorem 1.1.2 (Vitali). *Let Ω be an open connected subset of $\mathbb{C}$. Let f_n be a sequence of analytic functions on Ω that are uniformly bounded on compact subsets of Ω. If $\lim_{n\to\infty} f_n$ exists at every point of a subset $S \subset \Omega$ having an accumulation point in Ω, then f_n converges to an analytic function uniformly on compact subsets of Ω.*

Let ξ be a *univalent* analytic function on an open subset $\Omega \subset \mathbb{C}$. Then ξ is a homeomorphism between the open subsets Ω and $\xi(\Omega)$ with an analytic inverse ξ^{-1} satisfying $((\xi^{-1})' \circ \xi)\xi' = 1$. Since the Jacobian of ξ equals $|\xi'|^2$, the change of variables formulas by ξ and ξ^{-1} take the following form. For f measurable on $\xi(\Omega)$,

$$\int_\Omega (f \circ \xi)|\xi'|^2 da = \int_{\xi(\Omega)} f da,$$
$$\int_{\xi(\Omega)} f|(\xi^{-1})'|^2 da = \int_\Omega (f \circ \xi) da.$$

Here da is the Lebesgue area measure. We shall use change of variables to prove

Proposition 1.1.3. *Let f be analytic on an open subset $\Omega \subset \mathbb{C}$. Let $A \subset \Omega$ be a Borel set with $a(A) = 0$. Then $f(A)$ is a Borel set with $a(f(A)) = 0$.*

Proof. Let $\Omega = \bigsqcup_n \Omega_n$ be the decomposition in connected components. Then $A = \bigsqcup_n (A \bigcap \Omega_n)$ and $f(A) = \bigcup_n f(A \bigcap \Omega_n)$. By working with each Ω_n and $A \bigcap \Omega_n$ instead of Ω and A, we can assume Ω is connected and f is non-constant. Then, $f(A \bigcap Z(f'))$ is at most countable for $Z(f')$ is so. Hence it suffices to show $f(B)$ is a Borel set with $a(f(B)) = 0$, $B := A \setminus Z(f')$.

To this end, notice first that at every $\lambda \in B$, f is univalent with bounded f' on some open disc Δ_λ, $\lambda \in \Delta_\lambda \subset \Omega$. Because $\mathbb{C}$ is a Lindelöf space,

$$B \subset \bigcup_{\lambda \in B} \Delta_\lambda = \bigcup_k \Delta_k \Rightarrow f(B) = \bigcup_k f(B \bigcap \Delta_k).$$

Since $f|\Delta_k$ is a homeomorphism and $B \bigcap \Delta_k$ is a Borel set, $f(B \bigcap \Delta_k)$ and $f(B)$ are Borel. Next, using the change of variables under $f|\Delta_k$, one has

$$\begin{aligned} a(f(B \bigcap \Delta_k)) &= \int_{f(\Delta_k)} 1_{f(B \bigcap \Delta_k)} da = \int_{\Delta_k} (1_{f(B \bigcap \Delta_k)} \circ f)|f'|^2 da \\ &= \int_{\Delta_k} 1_{B \bigcap \Delta_k} |f'|^2 da = \int_{B \bigcap \Delta_k} |f'|^2 da = 0, \end{aligned}$$

the last step due to $a(B \bigcap \Delta_k) \le a(A) = 0$. Therefore, $a(f(B)) = 0$. $\square$

Definition 1.1.4. For a nonempty compact subset K of an open subset Ω of $\mathbb{C}$, an envelope Γ of K in Ω is a finite disjoint union of closed simple Jordan curves in $\Omega \setminus K$ whose aggregate winding number equals 1 relative to the points of K, and 0 relative to the points of $\mathbb{C} \setminus \Omega$.

The hexagon lemma ensures the existence of envelopes (indeed furnishes polygonal ones). If f is analytic in Ω and Γ is an envelope of K in Ω, then the Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in K.$$

Definition 1.1.5. The polynomial-convex hull hull(K) of a nonempty compact set $K \subset \mathbb{C}$ is

$$\text{hull}(K) := \{z \in \mathbb{C} : |p(z)| \leq \|p|K\|_\infty \text{ for every polynomial } p\}.$$

Theorem 1.1.6. *The complement of* hull(K) *equals the unbounded component of that of K.*

Let $C(K)$ be the space of continuous complex functions on K equipped with the sup-norm, $P(K)$ the norm closure of polynomials on K, $R(K)$ the closure of the collection $H(K)$ of functions on K admitting an analytic extension in an open neighborhood of K, and $A(K)$ the collection of functions continuous on K and analytic in the topological interior K^o. The inclusions $P(K) \subset R(K) \subset A(K) \subset C(K)$ are obvious. Note that $R(K)$ is also the closure of rational functions on K, due to Runge's theorem, although the latter is a proper subset of $H(K)$.

Theorem 1.1.7 (Runge)**.** *Every function analytic in an open neighborhood of K can be uniformly approximated on K by rational functions with poles in distinct bounded components of $\mathbb{C} \setminus K$.*

It is well known that $P(K) = R(K)$ if and only if K has connected complement (i.e. $K =$ hull(K)), and that $A(K) = C(K)$ if and only if K has empty interior. Besides these, only sufficient conditions are available for the other equalities. For example,

Theorem 1.1.8 (Mergelyan)**.** *If the diameters of the components of $\mathbb{C} \setminus K$ are bounded away from zero, then $R(K) = A(K)$.*

Lastly, we recall the basic result on (complex) Banach-space valued analytic functions, as an application of the uniform boundedness principle.

Theorem 1.1.9. *Let X be a Banach space with dual space X^*, and Ω be an open subset of $\mathbb{C}$. Then a function $f : \Omega \to X$ is norm analytic if and only if $x^*(f)$ is analytic for every $x^* \in X^*$, a function $g : \Omega \to X^*$ is dual-norm analytic if and only if $g(x)$ is analytic for every $x \in X$, and a function $T : \Omega \to \mathcal{L}(Y, X)$, Y another Banach space, is operator-norm analytic if and only if $x^*(Ty)$ is analytic for every $x^* \in X^*$ and $y \in Y$.*

1.2 Bounded analytic and harmonic functions in the disc

Let f be a complex valued function smooth in the open unit disc $\mathbb{D} \hookrightarrow \mathbb{R}^2$. Define two differential operators $\partial, \overline{\partial}$ acting on f as follows

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad z = x + iy$$

for which the following identities are well known

$$\overline{\partial f} = \overline{\partial}\bar{f}, \qquad \Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\overline{\partial}\partial = 4\partial\overline{\partial}.$$

Under these notations, f is analytic if it satisfies the Cauchy-Riemann equation $\overline{\partial} f = 0$ (and in this case the complex derivative $f' = \partial f$), while harmonic if it satisfies the Laplace equation $\Delta f = 0$. In particular, ∂f of a harmonic function f is analytic with $(\partial f)' = \partial^2 f$.

Let $h^\infty(\mathbb{D})$ be the Banach space of bounded harmonic functions in $\mathbb{D}$, equipped with the sup-norm, and $H^\infty(\mathbb{D})$ the Banach subalgebra of bounded analytic functions in $\mathbb{D}$. Each function in $h^\infty(\mathbb{D})$ has a non-tangential limit almost everywhere on $\partial\mathbb{D}$, and this induces a *-linear isometry from $h^\infty(\mathbb{D})$ onto L^∞, the C*-algebra of essentially bounded measurable functions on $\partial\mathbb{D}$. The restriction of this linear isometry to the subalgebra $H^\infty(\mathbb{D})$ is multiplicative whose range is denoted by H^∞, a weak-star closed subalgebra of L^∞. The inverse of the isometry between $h^\infty(\mathbb{D})$ and L^∞ is given explicitly by the Poisson integral

$$\hat{f}(z) = \int_{\partial\mathbb{D}} f P_z d\theta, \quad f \in L^\infty, \; z \in \mathbb{D},$$

where $d\theta$ is the normalized linear Lebesgue measure on $\partial\mathbb{D}$ and P_z is the Poisson kernel corresponding to $z \in \mathbb{D}$

$$P_z(\lambda) = \Re\frac{\lambda + z}{\lambda - z} = \frac{1 - |z|^2}{|\lambda - z|^2}, \quad \lambda \in \partial\mathbb{D}.$$

Evidently, $z \mapsto P_z(\lambda)$ is a harmonic function in the disc with range precisely $(0, \infty)$ while $\lambda \mapsto P_z(\lambda)$ is continuous on the circle. In such a pairing, $\hat{f} \in h^\infty(\mathbb{D})$ is called the harmonic extension of the boundary function $f \in L^\infty$. For $f \in H^\infty$, the harmonic and analytic extensions via the Poisson and respectively Cauchy integrals coincide, that is,

$$\hat{f}(z) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f(\lambda)}{\lambda - z} d\lambda, \quad f \in H^\infty, \; z \in \mathbb{D}.$$

By the solution to the classical Dirichlet boundary value problem, the harmonic extensions of $C := C(\partial\mathbb{D}) \subset L^\infty$ consist exactly of the functions in $h^\infty(\mathbb{D}) \bigcap C(\overline{\mathbb{D}})$, and those of $H^\infty \bigcap C$ are exactly the functions in the disc algebra $H^\infty(\mathbb{D}) \bigcap C(\overline{\mathbb{D}})$.

Strictly positive harmonic functions in $\mathbb{D}$ that are not necessarily bounded admit a more general Poisson integral representation via finite positive measures not necessarily having a Radon-Nikodym derivative relative to $d\theta$. The measure $d\mu$ in the following classical result exists as a weak-star cluster point (indeed the limit) of the net of measures $d\mu_r := f_r d\theta \in C^*$, $0 < r \uparrow 1$, where $\|\mu_r\| = \int_{\partial\mathbb{D}} |f_r| d\theta = \int_{\partial\mathbb{D}} f_r d\theta = f(0)$ for the radial functions $f_r(\lambda) := f(r\lambda)$, $\lambda \in \partial\mathbb{D}$.

Theorem 1.2.1 (Herglotz). *If f is a strictly positive harmonic function in $\mathbb{D}$, then there exists a unique finite positive measure $d\mu$ on $\partial\mathbb{D}$ such that*

$$f(z) = \int_{\partial\mathbb{D}} P_z d\mu, \quad z \in \mathbb{D}.$$

Since the closed balls in $L^\infty \cong (L^1)^*$ are weak-star compact while L^1 is separable, these balls are sequentially compact in the metrizable relative weak-star topology. Therefore, any uniformly bounded sequence of functions in the weak-star closed H^∞ possesses a subsequence converging in the weak-star topology to a limit function in H^∞. Alternately, this follows from Montel's theorem and a characterization for weak-star convergent sequences.

Proposition 1.2.2. *A sequence $\{h_n\}_n$ converges to h in the (relative) weak-star topology in H^∞ if and only if $\sup_n \|h_n\|_\infty < \infty$ and $\hat{h}_n(z) \to \hat{h}(z)$ at every $z \in \mathbb{D}$.*

A sequence $\{z_n\}_n$ in $\mathbb{D}$ is called a Blaschke sequence if $\sum_n (1 - |z_n|) < \infty$ (or equivalently $\prod_n |z_n| > 0$ excluding a finite number of zero entries). By a classical result of F. Riesz, the zero sets of nonzero functions in the Hardy classes $H^p(\mathbb{D})$, $p \in (0, \infty]$, are Blaschke sequences. The converse is also true. In fact, a Blaschke product b is a function in $H^\infty(\mathbb{D})$ defined by

$$b(z) = \prod_n \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \qquad z \in \mathbb{D}$$

whose zero sequence $\{z_n\}_n$ in $\mathbb{D}$ (counting multiplicities) is any given Blaschke sequence. Here the convention is that for $z_n = 0$, the corresponding factor is taken to be z. Therefore, any nonzero function in $H^p(\mathbb{D})$ factors through a Blaschke product corresponding to its zero set. Consequently, closed ideals in $H^\infty(\mathbb{D})$ of the form $\{h \in H^\infty(\mathbb{D}) : h|\Omega \equiv 0\}$, $\Omega \subset \mathbb{D}$, are either zero or principal ideals generated by Blaschke products.

A Blaschke sequence $\{z_n\}_n$ is further called an *interpolating* sequence and the corresponding Blaschke product b an interpolating Blaschke product if

$$\delta(b) := \inf_n \prod_{k:k\neq n} \rho(z_n, z_k) = \inf_n (1 - |z_n|^2)|b'(z_n)| > 0$$

where ρ is the pseudo-hyperbolic distance on $\mathbb{D}$ and $\delta(b)$ is the *uniform separation constant* of b. A finite Blaschke product with simple zeros is interpolating. The sequence is called a *sparse* sequence and the corresponding Blaschke product b a sparse Blaschke product if

$$\lim_{n\to\infty} \prod_{k:k\neq n} \rho(z_n, z_k) = \lim_{n\to\infty} (1 - |z_n|^2)|b'(z_n)| = 1.$$

A sparse Blaschke product is a finite Blaschke product times an interpolating one, that is, an *almost interpolating* Blaschke product, but not vice versa. Finite products of interpolating Blaschke products are called Carleson-Newman Blaschke products, and these obviously include the almost interpolating ones. The following deep result of Garnett and Nicolau strengthens a theorem of D. E. Marshall replacing Blaschke products by the much more restricted class of interpolating ones.

Theorem 1.2.3 (Garnett-Nicolau)**.** *The Banach space H^∞ is generated by interpolating Blaschke products.*

A function in $H^\infty(\mathbb{D})$ is called *inner* if its boundary function on $\partial\mathbb{D}$ is unimodular (almost everywhere). A function in $H^\infty(\mathbb{D})$ is called *outer* if it does not belong to any principal ideal in $H^\infty(\mathbb{D})$ generated by a non-constant inner function. Blaschke products are inner functions with zeros in $\mathbb{D}$. On the other hand, an inner function is called *singular* if it is nonvanishing in $\mathbb{D}$. It follows from the Herglotz theorem that singular inner functions s correspond to finite positive Borel measures $d\mu \perp d\theta$ on $\partial\mathbb{D}$ via

$$s(z) = e^{it} \exp\left(-\int_{\partial\mathbb{D}} \frac{\lambda+z}{\lambda-z} d\mu(\lambda)\right), \quad z \in \mathbb{D}.$$

A classical result of Frostman implies, among other things, that inner functions u can be uniformly approximated by Blaschke products on $\mathbb{D}$, that their ranges $u(\mathbb{D})$ exhaust $\mathbb{D}$ except for subsets of zero (logarithmic) capacity, and that their Nevanlinna counting functions satisfy

$$N_u(z) = \ln\left|\frac{1-\bar{u}(0)z}{z-u(0)}\right|$$

for all $z \in \mathbb{D}$ except for subsets of zero capacity.

Theorem 1.2.4 (Frostman)**.** *Let $u \in H^\infty$ be an inner function. Then the set of $\alpha \in \mathbb{D}$ for which the Frostman shift $(u-\alpha)/(1-\bar{\alpha}u)$ is not a Blaschke product is of zero capacity.*

Relatively closed subsets of $\mathbb{D}$ with zero capacity are precisely the sets of omitted values of inner functions.

Proposition 1.2.5. *A subset $F \subset \mathbb{D}$ is relatively closed with zero capacity if and only if $F = \mathbb{D} \setminus u(\mathbb{D})$ for an inner function u.*

Proof. Sufficiency is an immediate consequence of Frostman's theorem. To show necessity, any analytic covering map from $\mathbb{D}$ onto $\mathbb{D} \setminus F$ for a relatively closed subset $F \subset \mathbb{D}$ with zero capacity is known to be an inner function. $\square$

Let h be a nonzero function in $H^\infty(\mathbb{D})$ and $r \in (0,1)$. Then $\int_{\lambda\in\partial\mathbb{D}} |\ln|h(r\lambda)||d\theta < \infty$ considering the multiplicity of zeros lying on the compact $r\partial\mathbb{D} \subset \mathbb{D}$. By a classical result of

Szegö, $\int_{\partial\mathbb{D}} |\ln|h||d\theta < \infty$ for its boundary function as well. By subharmonicity, $\int_{\lambda\in\partial\mathbb{D}} \ln|h(r\lambda)|d\theta$ increases in $r \in (0,1)$. The following integral criterion is convenient. It shows that each of these three classes of functions is closed under products.

Theorem 1.2.6. *A nonzero function $h \in H^\infty(\mathbb{D})$ is a Blaschke product if and only if*

$$\lim_{r\to 1^-} \int_{\lambda\in\partial\mathbb{D}} \ln|h(r\lambda)|d\theta = \int_{\partial\mathbb{D}} \ln|h|d\theta,$$

a singular inner function if and only if

$$\lim_{r\to 1^-} \int_{\lambda\in\partial\mathbb{D}} \ln|h(r\lambda)|d\theta < \int_{\partial\mathbb{D}} \ln|h|d\theta,$$

and an outer function if and only if

$$\int_{\partial\mathbb{D}} \ln|h|d\theta = \ln|h(0)|.$$

The following unique factorization result is pivotal. In particular, it reveals a rigid divisibility structure in H^∞.

Theorem 1.2.7. *Every nonzero function $h \in H^\infty(\mathbb{D})$ admits a unique (modulo unimodular constants) factorization $h = bsF$ in a Blaschke product b, a singular inner function s, and an outer function F.*

The question of when singular inner factors are absent from the factorization is answered by a result of W. Rudin [113] (cf. [36, p. 11]). It generalizes Frostman's theorem because, for h inner, the inner factor of $h - h(\zeta)$ is just the Frostman shift induced by $h(\zeta) \in \mathbb{D}$.

Theorem 1.2.8 (Rudin)**.** *For every nonconstant $h \in H^\infty(\mathbb{D})$, the set of $\zeta \in \mathbb{D}$ for which the inner factor of $h - h(\zeta)$ is not a Blaschke product with all simple zeros is of zero capacity.*

Since inner functions are analytic self maps of $\mathbb{D}$, it is natural to compose analytic functions on $\mathbb{D}$ by inner functions. Such compositions preserve inner and outer functions.

Theorem 1.2.9. *If u_1, u_2 are inner and F is outer, then $u_2 \circ u_1$ is inner and $F \circ u_1$ is outer.*

The distribution $\theta \circ u^{-1}$ on $\partial\mathbb{D}$ of a non-constant inner function u equals $P_{u(0)} d\theta$, a probability measure boundedly equivalent to $d\theta$ because Poisson kernels are nonvanishing continuous function on $\partial\mathbb{D}$. It immediately follows that the essential range of u on $\partial\mathbb{D}$, that is, the support of $\theta \circ u^{-1}$, consists of the entire $\partial\mathbb{D}$ (there holds a much deeper result due to Sarason). Next we give a measure theoretic proof of a Poisson kernel identity. The boundary function of any conformal automorphism γ of $\mathbb{D}$ is a homeomorphism of $\partial\mathbb{D}$ and we shall not distinguish the two.

Proposition 1.2.10. *For $\gamma \in \mathrm{Aut}(\mathbb{D})$ and $z \in \mathbb{D}$, one has $(P_z \circ \gamma^{-1}) P_{\gamma(0)} \equiv P_{\gamma(z)}$ on $\partial\mathbb{D}$.*

Proof. Choose $\gamma_z \in \mathrm{Aut}(\mathbb{D})$ with $\gamma_z(0) = z$. Then $\gamma \circ \gamma_z \in \mathrm{Aut}(\mathbb{D})$ with $(\gamma \circ \gamma_z)(0) = \gamma(z)$. For any Borel subset $A \subset \partial\mathbb{D}$, one deduces

$$\begin{aligned}
\int_A P_{\gamma(z)} d\theta = \theta((\gamma \circ \gamma_z)^{-1}(A)) = \theta(\gamma_z^{-1}(\gamma^{-1}(A))) &= \int_{\gamma^{-1}(A)} P_z d\theta \\
= \int_{\gamma^{-1}(A)} P_z \circ \gamma^{-1} \circ \gamma \; d\theta &= \int_{\partial\mathbb{D}} ((P_z \circ \gamma^{-1}) 1_A) \circ \gamma \; d\theta \\
= \int_{\partial\mathbb{D}} (P_z \circ \gamma^{-1}) 1_A P_{\gamma(0)} d\theta &= \int_A (P_z \circ \gamma^{-1}) P_{\gamma(0)} d\theta.
\end{aligned}$$

Therefore, $(P_z \circ \gamma^{-1}) P_{\gamma(0)} = P_{\gamma(z)}$ $d\theta$-a.e., and one concludes the proof by continuity. □

We close this section with a well-known result. In particular, it ensures uniqueness of the measure in the Herglotz theorem.

Proposition 1.2.11. *The linear span of $\{P_z : z \in \mathbb{D}\}$ is dense in C on the circle.*

Proof. By the Hahn-Banach theorem and the Riesz respresentation of the dual space of C, it suffices to show that if a complex Borel measure μ on $\partial\mathbb{D}$ satisfies $\int_{\partial\mathbb{D}} P_z d\mu = 0, \forall z \in \mathbb{D}$, then $\int_{\partial\mathbb{D}} f d\mu = 0$ for every $f \in C$. Splitting μ into its real and imaginary parts and noting P_z is real, we can and do assume that μ is a finite real measure. Then the condition becomes

$$\Re \int_{\partial\mathbb{D}} \frac{\lambda + z}{\lambda - z} d\mu(\lambda) = 0, \; \forall z \in \mathbb{D}.$$

It follows that the analytic function $z \in \mathbb{D} \mapsto \int_{\partial\mathbb{D}} \frac{\lambda+z}{\lambda-z} d\mu(\lambda)$ is constant. Therefore,

$$\int_{\partial\mathbb{D}} \frac{\lambda+z}{\lambda-z} d\mu(\lambda) \equiv \mu(\partial\mathbb{D}) = 0.$$

Subtraction gives $2z \int_{\partial\mathbb{D}} \frac{d\mu(\lambda)}{\lambda-z} \equiv 0$, thus $\int_{\partial\mathbb{D}} \frac{d\mu(\lambda)}{\lambda-z} \equiv 0$, and differentiation on z further gives

$$\int_{\partial\mathbb{D}} \frac{d\mu(\lambda)}{(\lambda-z)^n} \equiv 0, \ \forall n \in \mathbb{N}.$$

Setting $z = 0$, this gives $\int_{\partial\mathbb{D}} \bar{\lambda}^n d\mu(\lambda) = 0$. Taking complex conjugates, $\int_{\partial\mathbb{D}} \lambda^k d\mu(\lambda) = 0$ for all $k \in \mathbb{Z}$. Now it follows from Fejér's uniform approximation theorem that $\int_{\partial\mathbb{D}} f d\mu = 0$ for every $f \in C$, as required. □

1.3 Commutative Banach algebras and maximal ideals

Let A be a commutative unital Banach algebra. The (proper) maximal ideals of A correspond to the (nonzero) multiplicative linear functionals on A as their kernels. The set of these functionals is denoted by $M(A)$ and is called the maximal ideal space (a.k.a. spectrum, character space, carrier space, structure space, etc) of A. Equipped with the relative weak-star topology, $M(A) \subset A^*$ is a compact Hausdorff space. For every $a \in A$, the Gelfand transform $\hat{a} \in C(M(A))$ is defined by $\hat{a}(m) := m(a), \ \forall m \in M(A)$. (It is customary to write a in lieu of $\hat{a}$ when confusion is not an issue.) The basic facts are that $\sigma(a; A) = a(M(A))$, and that the Gelfand transform is a contractive algebra homomorphism from A into $C(M(A))$.

Definition 1.3.1. The radical $R(A)$ of a commutative unital Banach algebra A is the intersection of all maximal ideals of A. Equivalently, $R(A) := \{a \in A : \sigma(a) = \{0\}\}$ is the kernel of the Gelfand transform.

By the spectral radius formula $\rho(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$, the radical consists exactly of the quasi-nilpotent elements of A. It also follows that $R(A)$ contains no nonzero differences of idempotents in A. An algebra is *semi-simple* if its radical is trivial. Subalgebras of semi-simple algebras are obviously semi-simple. The following observation is sometimes useful.

Proposition 1.3.2. *The subset $M(A) \subset A^*$ is linearly independent.*

Proof. Let m_k, $k = 1, ..., n$, be distinct functionals in $M(A)$ such that $\sum_{k=1}^n \beta_k m_k = 0$ for complex scalars β_k. It remains to show each $\beta_k = 0$.

There exists $a \in A$ such that $m_k(a)$, $k = 1, ..., n$, are distinct numbers. For, the closed subsets

$$A_{j,k} := \{a \in A : m_j(a) = m_k(a)\}, \qquad j \neq k$$

all have empty interior in A, so that the finite union $\bigcup_{j\neq k} A_{j,k}$ is of first category. By the Baire category theorem there exists $a \in A \setminus \bigcup_{j\neq k} A_{j,k}$, and such a gives distinct $\{m_k(a)\}$.

Now one has a homogeneous system of n linear equations of n unknowns $\{\beta_k\}$

$$\sum_{k=1}^{n} (m_k(a))^j \beta_k = 0, \qquad j = 0, 1, ..., n-1$$

whose coefficients form a Vandermonde matrix with generators $\{m_k(a)\}$. Thus, each $\beta_k = 0$. $\square$

Next we recall some deep connections between A and its maximal ideal space $M(A)$ obtained by methods of several complex variables. If $p \in A$ is a nontrivial idempotent, then $p \in C(M(A))$ is a nontrivial characteristic function which renders $M(A)$ disconnected. Proved by reduction to finitely generated subalgebras and then uniform approximation in several complex variables (the Oka-Weil theorem), the Shilov idempotent theorem supplies the converse.

Theorem 1.3.3 (G. Shilov)**.** *For a commutative unital Banach algebra A, $M(A)$ is disconnected if and only if A contains nontrivial idempotents.*

The multiplicative group A^{-1} of invertible elements of A is open in A. Its principal connected component (i.e. the one containing the unit) is a subgroup denoted by A_0^{-1}, $A_0^{-1} = e^A$. The quotient group A^{-1}/A_0^{-1} consists of the components of A^{-1} as the cosets and is called the *abstract index group* of A. The proof of the following theorem uses the multivariate analytic functional calculus and Oka's solution of the Cousin problem for polynomial polyhedra. Note that by general facts in algebraic topology, the quotient group $C(M(A))^{-1}/C(M(A))_0^{-1}$ is isomorphic to the first cohomotopy group of the compact Hausdorff space $M(A)$, and to its first Čech cohomology group with integer coefficients.

Theorem 1.3.4 (Arens-Royden). $A^{-1}/A_0^{-1} \cong C(M(A))^{-1}/C(M(A))_0^{-1}$.

Besides these analytical and topological aspects of the maximal ideal space, there is an algebraic aspect and the three closely interact with each other.

Definition 1.3.5. Let A be a commutative unital Banach algebra. The hull of an ideal of A is the collection of the maximal ideals containing the ideal; The kernel of a collection of maximal ideals is the intersection of the maximal ideals in the collection.

The zero set $Z(a)$ of $a \in A$ is defined as $Z(a) := \{m \in M(A) : a(m) = 0\}$. The hull of an ideal J of A is then exactly the zero set $Z(J) := \bigcap_{a \in J} Z(a)$. Evidently, J is contained in the kernel of its hull $Z(J)$. The relatively rare case of an ideal which equals the kernel of its hull, that is, a *radical* ideal, is of importance in certain problems. Let $a_1, ..., a_n \in A$. The closed ideal in A generated by $a_1, ..., a_n$ is proper if and only if these elements live in a common maximal ideal. Therefore, the following statements are equivalent:

(i) The Bézout equation $\sum_{k=1}^{n} a_k b_k = 1$ has a solution $(b_1, ..., b_n)$ in A;

(ii) The Bézout equation $\sum_{k=1}^{n} \hat{a}_k f_k = 1$ has a solution $(f_1, ..., f_n)$ in $C(M(A))$;

(iii) $\sum_{k=1}^{n} |\hat{a}_k(m)| \geq \epsilon > 0$ for all m in a dense subset of $M(A)$;

(iv) $\bigcap_{k=1}^{n} Z(a_k) = \emptyset$;

(v) $(0, ..., 0) \notin \sigma(a_1, ..., a_n)$

where the *joint* spectrum $\sigma(a_1, ..., a_n) := \{(a_1(m), ..., a_n(m)) : m \in M(A)\}$ is a compact subset of $\mathbb{C}^n$. It is therefore of interest to study zero sets.

G. Shilov showed that the intersection of the closed subsets of $M(A)$ on which every function $\hat{a} \in C(M(A))$, $a \in A$, attains its maximum modulus retains this property. That is, there exists a (unique) smallest such closed subset of $M(A)$.

Definition 1.3.6. The Shilov boundary ∂A of a commutative unital Banach algebra A is the smallest closed subset of $M(A)$ on which every function $\hat{a} \in C(M(A))$, $a \in A$, attains its maximum modulus.

Remark 1.3.7. If G is an open neighborhood in $M(A)$ of $m \in \partial A$, then there exists $a \in A$ such that $\|\hat{a}|(M(A)\backslash G)\|_\infty < \|\hat{a}\|_\infty$. For otherwise the closed subset $M(A)\backslash G$ would contain $\partial A \ni m$, creating a contradiction.

The Shilov boundary figures in H. Rossi's celebrated local maximum modulus principle which states that Gelfand transforms of commutative Banach algebra elements behave like analytic functions *off* the Shilov boundary. However, it was first introduced, among other things, to play a role in the following extension result on maximal ideals. We shall prove a somewhat stronger version of Shilov's original theorem (which stated $r(M(A)) \supset \partial B$, cf. [78, Theorem 4.15.6]).

Theorem 1.3.8. *Let B be a closed subalgebra of a commutative unital Banach algebra A, and let $r : M(A) \to M(B)$ be the restriction map. Then, $r(\partial A) \supset \partial B$.*

Proof. Since $r(\partial A)$ is a closed subset of $M(B)$, it suffices to show that every $\hat{b} \in C(M(B))$, $b \in B$, attains on $r(\partial A)$ its maximum modulus $\rho(b; B)$. But this follows from the relation $\tilde{b} = \hat{b} \circ r$ where the Gelfand transform $\tilde{b} \in C(M(A))$ of $b \in B \subset A$ attains on ∂A its maximum modulus $\rho(b; A) = \lim_{n\to\infty} \|b^n\|^{1/n} = \rho(b; B)$. This completes the proof. □

The algebraic interpretation is that every maximal ideal in the Shilov boundary of B is the trace in B (i.e. intersection with B) of a maximal ideal in the Shilov boundary of A. This result finds an application in Subsection 4.6.2.

Definition 1.3.9. Let B be a closed subalgebra of a commutative unital Banach algebra A, and let $r : M(A) \to M(B)$ be the restriction map. The fiber $M_m(A)$ of $M(A)$ over $m \in M(B)$ is defined to be the pre-image $r^{-1}(m)$.

Remark 1.3.10. A fiber may be empty unless it is over the Shilov boundary. In case the subalgebra B is a C*-algebra, we have $\partial B = M(B)$ so every fiber $M_m(A)$ is non-empty.

We close this section with some observations on spectra relative to subalgebras. Let U be a non-commutative unital Banach algebra. The closed subalgebra $A(a)$ singly generated by $a \in U$ is commutative. It is well known that

$$\sigma(a, A(a)) = \mathrm{hull}(\sigma(a, U)).$$

Next let B be a commutative closed subalgebra of U. In general, $\sigma(b, U) \subsetneq \sigma(b, B) = b(M(B))$ for $b \in B$, and it is in general the smaller spectrum relative to U that is of interest. Taking A to be either the double commutant algebra of B or a maximal commutative subalgebra of U containing B, one has $\sigma(b, U) = \sigma(b, A) = \tilde{b}(M(A)) = \hat{b}(r(M(A))) \supset \hat{b}(\partial B)$ for $b \in B \subset A$ with Gelfand transform $\tilde{b}$ on $M(A)$. Combining the two inclusions gives

$$b(\partial B) \subset \sigma(b, U) \subset b(M(B)), \qquad \forall b \in B. \tag{1.3.1}$$

This relation can sometimes be useful if the Shilov boundary ∂B is relatively large in $M(B)$.

1.4 Regular measures on compact Hausdorff spaces

Let X be a compact Hausdorff space equipped with its Borel σ-algebra $\mathcal{F}(X)$ generated by the open subsets of X.

Definition 1.4.1. A probability measure μ defined on $\mathcal{F}(X)$ is called regular if for every $G \subset X$ open and $\epsilon > 0$, there exists a compact subset $K \subset G$ such that $\mu(K) > \mu(G) - \epsilon$. A finite complex measure ν defined on $\mathcal{F}(X)$ is regular if its normalized total-variation measure $|\nu|/||\nu||$ is a regular probability measure.

Regular measures on compact spaces X are important because they are identified with bounded linear functionals on the space $C(X)$ of continuous complex functions on X.

Theorem 1.4.2 (F. Riesz-Markov-Kakutani)**.** *The dual space $C(X)^*$ is isometrically isomorphic via integration to the space of regular finite complex measures on X.*

In what follows, we focus only on probability measures because certain closed subsets of them represent multiplicative linear functionals on subalgebras of $C(X)$, although the development can be easily adapted for finite complex measures.

Definition 1.4.3. The support set $\text{supp}\mu$ of a regular probability measure μ on X is defined as

$$\text{supp}\mu := \{x \in X : \ \mu(G) > 0 \text{ for all open neighborhood } G \text{ of } x \text{ in } X\}.$$

Support sets of regular probability measures admit a useful characterization as follows.

Proposition 1.4.4. *The support set of μ is the smallest closed subset of X with full μ-measure.*

Remark 1.4.5. The nontrivial fact that $\mu(X \setminus \mathrm{supp}\mu) = 0$ is due to $\mu(K) = 0$ for every $K \subset X \setminus \mathrm{supp}\mu$ compact and then to regularity.

Under the isometric isomorphism, the set of regular probability measures is precisely the weak-star compact, convex set $\mathcal{S}$ of *states* on $C(X)$, $\mathcal{S} := \{\phi \in C(X)^* : \|\phi\| = \phi(1) = 1\}$. Moreover, the point masses on X are precisely the *extreme* points of $\mathcal{S}$, in the sense of the Krein-Milman theorem, called the extreme states. For, point masses are extreme by a consideration of support sets. Conversely, suppose that $\mathrm{supp}\mu$ of $\mu \in \mathcal{S}$ contains two distinct points $x_1, x_2 \in X$. Choose by separation a closed subset $F \subset X$ containing x_1 in its interior but not containing x_2. Then $t = \mu(F) \in (0, 1)$. Define $\mu_1(\Omega) := \mu(\Omega \bigcap F)$, $\Omega \in \mathcal{F}(X)$, and put

$$\mu_2 = \mu - \mu_1, \qquad \nu_1 = \mu_1/t, \qquad \nu_2 = \mu_2/(1-t).$$

Then, $\mu = \mu_1 + \mu_2 = t\nu_1 + (1-t)\nu_2$ with $\nu_1 \neq \nu_2 \in \mathcal{S}$ indicates that μ is not extreme. (Note that $\nu_1 \perp \nu_2$ with $\mathrm{supp}\nu_1 \subset F$, while $\mathrm{supp}\nu_2$ may not be contained in $X \setminus F$, so that the two support sets may intersect.) Therefore, $\mathcal{S}$ equals the weak-star closure of the convex hull of the point masses, which can also be proved by direct constructions.

We next recall some immediate properties about regularity and support sets.

Proposition 1.4.6. *Let $f : X \to Y$ be a continuous map between compact spaces X, Y. Let μ be a regular probability measure on X with* $\mathrm{supp}\mu$. *Then, the distribution $\mu \circ f^{-1}$ is a regular probability measure on Y with* $\mathrm{supp}(\mu \circ f^{-1}) = f(\mathrm{supp}\mu)$.

Proposition 1.4.7. *Let $F \subset X$ be a closed subset of a compact space X. Let μ be a regular probability measure on F with* $\mathrm{supp}\mu$. *Then, the canonical extension ν on X defined by $\nu(\Omega) := \mu(\Omega \bigcap F)$, $\Omega \in \mathcal{F}(X)$, is a regular probability measure on X with* $\mathrm{supp}\nu = \mathrm{supp}\mu$.

1.5 Uniform algebras and representing measures

Uniform algebras constitute an important class of semi-simple commutative unital Banach algebras. In fact the sup-norm closure of the Gelfand transform of any commutative unital Banach algebra is a uniform algebra, and the two algebras obviously share the same Shilov boundary. In the classical setting of approximation theory, the algebras $P(K), R(K), A(K)$ are all uniform algebras on compact subsets K of $\mathbb{C}$. In their broadest scope, however, abstract uniform algebras naturally arise from the celebrated Stone-Weierstrass theorem and universally model *non-self-adjoint* closed subalgebras of commutative unital C*-algebras that separate multiplicative linear functionals (i.e. the extreme states).

Theorem 1.5.1 (M. H. Stone-Weierstrass)**.** *The algebra of continuous complex functions on a compact Hausdorff space coincides with any uniformly closed, point-separating, self-adjoint, unital subalgebra.*

Definition 1.5.2. For X a compact Hausdorff space and $C(X)$ the algebra of continuous complex functions on X, a uniform algebra A on X is a uniformly closed unital subalgebra of $C(X)$ that separates the points of X.

Remark 1.5.3. The point separating requirement is equivalent to that the topology on X coincide with the one generated by the functions in A, and it ensures that the canonical embedding $X \hookrightarrow M(A)$ via point evaluations is homeomorphic. This embedding of X in $M(A)$ contains the Shilov boundary ∂A. The three realizations of A on X, $M(A)$, ∂A, are isometrically isomorphic.

Definition 1.5.4. A uniform algebra $A \subset C(X)$ on X is called antisymmetric if its maximal C*-subalgebra $Q_A := A \bigcap \overline{A}$ is trivial.

Definition 1.5.5. A uniform algebra $A \subset C(X)$ on X is called natural if $M(A) = X$.

Natural uniform algebras are simply Gelfand transforms of uniform algebras.

Definition 1.5.6. For $A \subset C(X)$ a uniform algebra on X, a regular probability measure μ_m on

X is called a representing measure for $m \in M(A)$ if

$$\int_X f d\mu_m = m(f), \ \forall f \in A.$$

Equivalently, a representing measure is a Hahn-Banach (i.e. norm-preserving) extension in $C(X)^*$ of $m \in M(A) \subset A^*$.

Thus, each $m \in M(A)$ possesses a representing measure which may or may not be unique.

Definition 1.5.7. A representing measure μ_m for $m \in M(A)$ of a uniform algebra A on X is called a Jensen measure if

$$\ln |m(f)| = \ln \left| \int_X f d\mu_m \right| \leq \int_X \ln |f| d\mu_m, \ \forall f \in A.$$

Evidently, the inequality above becomes an equality for all the invertible elements $f \in A^{-1}$. That is, Jensen measures satisfy the Arens-Singer property (but not vice versa). The universal existence of Jensen measures is given by

Theorem 1.5.8 (E. Bishop)**.** *Every $m \in M(A)$ of a uniform algebra A on X possesses a Jensen representing measure.*

We next discuss minimal support sets of $m \in M(A)$ and their relations to the support sets $\text{supp}\mu_m$ of representing measures μ_m for $m \in M(A)$.

Definition 1.5.9. A support set of $m \in M(A)$ of a uniform algebra A on X is a closed subset S of X which satisfies $|m(f)| \leq \|f|S\|_\infty, \ \forall f \in A$.

For a representing measure μ_m for $m \in M(A)$,

$$|m(f)| = \left| \int_X f d\mu_m \right| = \left| \int_{\text{supp}\mu_m} f d\mu_m \right| \leq \int_{\text{supp}\mu_m} |f| d\mu_m \leq \|f|\text{supp}\mu_m\|_\infty, \ \forall f \in A,$$

so that $\text{supp}\mu_m$ is a support set of $m \in M(A)$. The collection of support sets of a fixed $m \in M(A)$ is partially ordered by set inclusion, and a standard application of Zorn's lemma yields the existence of minimal support sets of $m \in M(A)$. Not all support sets of representing measures are minimal support sets. But every minimal support set must be the support set of a representing measure. We shall prove the following result.

Proposition 1.5.10. *For every minimal support set S of $m \in M(A)$ of a uniform algebra A on X, there exists a representing measure μ_m for m with* $\mathrm{supp}\mu_m = S$.

Proof. The key observation is that every support set S of $m \in M(A)$ contains $\mathrm{supp}\mu_m$ of some representing measure μ_m. For, it follows from definition that there exists a contractive multiplicative linear functional m' on the restriction algebra $A|S \subset C(S)$ satisfying

$$m = m' \circ r$$

where r is the restriction map from A onto $A|S$. Then, m' admits an extension in $M(\overline{A|S})$ of the uniform algebra $\overline{A|S}$ on the compact space S, which is in turn represented by a regular probability measure ν on S. So we have for every $f \in A$ that

$$m(f) = m'(f|S) = \int_S (f|S) d\nu = \int_X f d\mu,$$

where μ is the canonical extension on X of ν, a regular probability measure with $\mathrm{supp}\mu = \mathrm{supp}\nu \subset S$. This together with the integral representation shows that μ is a representing measure for m, so that $\mathrm{supp}\mu \subset S$ is a support set of m. By minimality of S, we assert $S = \mathrm{supp}\mu$ as required. □

The importance of minimal support sets is manifested in the following result.

Theorem 1.5.11 (K. Hoffman)**.** *Let A be a uniform algebra on X, and S a minimal support set of $m \in M(A)$. If $f \in A$ vanishes identically on a nonempty open subset of S, then $m(f) = 0$.*

Remark 1.5.12. Representing measures which have the Jensen property, or whose support sets are minimal support sets, are apparently useful. It is then a favorable situation when m has a *unique* representing measure μ_m, for in this case μ_m has the Jensen property and $\mathrm{supp}\mu_m$ is the minimal support set of m. If, moreover, *every* $m \in M(A)$ has a unique representing measure μ_m on X, then the map $m \in M(A) \mapsto \mu_m \in C(X)^*$ is a weak-star homeomorphic embedding.

In general, not all $m \in M(A)$ have unique representing measures, not even all $x \in X \hookrightarrow M(A)$. The Choquet boundary $\mathrm{Ch}(A)$ consists of the points $x \in X \hookrightarrow M(A)$ having unique representing measures (the point masses at x). One has $\overline{\mathrm{Ch}(A)} = \partial A \subset X$. The Choquet boundary can be characterized as certain extreme points and has a non-commutative analog in Arveson's boundary representations for operator algebras [7].

We recall two more notions, *logmodular* algebras and *regular* algebras. Note that $C(X;\mathbb{R})$ is the set of continuous real functions on a compact space X.

Definition 1.5.13. A uniform algebra A on X is called logmodular if the set $\{\ln|f| : f \in A^{-1}\}$ is uniformly dense in $C(X;\mathbb{R})$; It is called strongly logmodular if the two sets are equal.

A uniform algebra A on X is a Dirichlet algebra if the set $\{\Re f : f \in A\} = \{\ln|f| : f \in e^A = A_0^{-1}\}$ is uniformly dense in $C(X;\mathbb{R})$. Logmodular algebras generalize Dirichlet algebras while keeping most, if not all, important properties [80] of the latter.

Theorem 1.5.14 (K. Hoffman)**.** *For a logmodular algebra A on X,* $\mathrm{Ch}(A) = \partial A = X$ *and every $m \in M(A)$ has a unique representing measure on X.*

Definition 1.5.15. A uniform algebra A on X is called regular if for every closed subset F of X and a point $x \in X \setminus F$, there is a function $f \in A$ such that $f \equiv 0$ on F and $f(x) \neq 0$.

Remark 1.5.16*.* There is a property weaker than regularity but stronger than point separation. A uniform algebra A on X is called *separating* if for every closed subset F of X and a point $x \in X \setminus F$, there is a function $f \in A$ such that $f(x) \notin f(F)$. An important example [128] is that H^∞ on $M(H^\infty)$ is non-regular but is separating.

A regular algebra A on X certainly satisfies $\partial A = X$. More importantly, *natural* regular algebras have the Shilov property as follows. Note that this property is a weaker substitute for the radicality of every closed ideal of the algebra.

Theorem 1.5.17 (Shilov)**.** *Let A be a regular algebra on $M(A)$, and let J be a closed ideal of A. If $f \in A$ vanishes identically on an open neighborhood of $Z(J)$ in $M(A)$, then $f \in J$.*

Remark 1.5.18. Neither logmodularity nor regularity carries over to isometrically isomorphic realizations of uniform algebras on different compact spaces. For instance, H^∞ is a logmodular algebra on $M(L^\infty)$ but not on $M(H^\infty)$, and $H^\infty + C$ is a regular algebra on $M(L^\infty)$ [8] but not on $M(H^\infty + C)$ (so that Shilov's theorem above does not apply). Nonetheless, there are natural non-regular uniform algebras which still have the Shilov property. Gorkin and Mortini [65] showed that $H^\infty + C$ is such an example.

1.6 Restrictions of uniform algebras

Very often one need consider restrictions of a uniform algebra on closed subsets of the compact Hausdorff space on which the algebra is defined.

Definition 1.6.1. Given a uniform algebra A on a compact space X, a nonempty closed subset $F \subset X$ is called a peak set for A if there exists a function $f \in A$ such that $f \equiv 1$ on F while $|f| < 1$ off F. A weak peak set for A is a nonempty intersection of peak sets for A.

Nonempty intersections of finitely (indeed countably) many peak sets are peak sets, and peak sets meet the Shilov boundary ∂A. Therefore, by compactness of X, weak peak sets meet the Shilov boundary. The significance of weak peak sets also lies in the following result.

Theorem 1.6.2. *For a uniform algebra A on X and a weak peak set F for A, the restriction algebra $A|F$ is uniformly closed in $C(F)$. That is, $A|F$ is a uniform algebra on F.*

Definition 1.6.3. Given a uniform algebra A on a compact space X, a closed subset $\Omega \subset X$ is called an antisymmetric set for A if the restriction algebra $A|\Omega$ is antisymmetric.

Zorn's lemma implies that X admits a partition in *maximal* antisymmetric sets for A, which are necessarily closed. The maximal antisymmetric decomposition is a useful tool for the central question of membership in, and more generally distance to, uniform algebras.

Theorem 1.6.4 (E. Bishop-Glicksberg)**.** *For a uniform algebra A on X and a function $f \in C(X)$, the sup-norm distances satisfy*

$$d(f, A) = \max\{d(f|\Omega, A|\Omega) : \Omega \text{ a maximal antisymmetric set for } A\}.$$

Moreover, each such Ω is a weak peak set for A, so that $A|\Omega$ is uniformly closed in $C(\Omega)$.

Remark 1.6.5. (i) The theorem in its original and more general form is stated for the distances to a closed ideal of A and without the point-separating condition on A (so that A can be a proper self-adjoint subalgebra of $C(X)$). (ii) Every maximal antisymmetric set for A is contained in a fiber $X_y = M_y(C(X))$ of X over $y \in M(Q_A)$, $Q_A = A \bigcap \overline{A}$ the largest C*-subalgebra contained in A. (In case A is self-adjoint, the two partitions of X are actually identical [18, 1.27(c)].) Consequently, the maximum of the localized distances over the maximal antisymmetric sets Ω can be replaced by that over the fibers X_y.

The *essential set* $\mathcal{E}(A)$ of an algebra $A \subset C(X)$ is the closed union of the non-singleton maximal antisymmetric sets for A. It is characterized as the smallest closed subset of X such that every $f \in C(X)$ vanishing on the closed subset lies in A. That is, it determines the largest ideal of $C(X)$ contained in A.

Proposition 1.6.6. *Let A be a uniform algebra on X. If $f \in C(X)$ vanishes on $\mathcal{E}(A)$, then $f \in A$. Conversely, if a closed subset $F \subset X$ is such that every $f \in C(X)$ vanishing on F lies in A, then $F \supset \mathcal{E}(A)$.*

Proof. The first part trivially follows from the Bishop-Glicksberg theorem. For the converse, fix such a closed subset $F \subset X$ and suppose $F \not\supset \mathcal{E}(A)$. Then there exists a non-singleton maximal antisymmetric set Ω and $x \in \Omega$ such that $x \notin F$. Let $y \neq x \in \Omega$, and choose a $[0, 1]$-valued $f \in C(X)$ with $f \equiv 0$ on $F \bigcup \{y\}$ while $f(x) = 1$. So, $f \in A$ while $f|\Omega$ is not constant. This is a contradiction since Ω is an antisymmetric set. □

Remark 1.6.7. Since the algebra $H^\infty + C$ on $M(L^\infty)$ contains no nontrivial ideals of L^∞ (proved later), one has $\mathcal{E}(H^\infty + C) = M(L^\infty)$. Equivalently, the union of all singleton maximal antisymmetric sets for $H^\infty + C$ has empty interior. On the other hand, every nonempty clopen subset of a fiber $M_\lambda(L^\infty)$ over $\lambda \in \partial\mathbb{D}$ is known [63] to contain a singleton maximal antisymmetric set.

It is important to understand the maximal ideal space of restriction algebras. The basic observation is that for a uniform algebra A on X and a closed subset $F \subset X$, the adjoint r^* of the dense-range restriction map $r : A \to \overline{A|F}$ is a homeomorphism from $M(\overline{A|F})$ into $M(A)$. Then, the image of r^* in $M(A)$ is a crucial object to identify, for $f|F$ is invertible in $\overline{A|F}$ if and only if $\hat{f}$ of $f \in A$ is nonvanishing on this image. More generally, the Bézout equation $\sum_{k=1}^{n}(f_k|F)g_k = 1$ has a solution $\{g_k\}$ in $\overline{A|F}$ if and only if $\hat{f}_k$, $k = 1, ..., n$, do not have a common zero on this image. The identification can be achieved via a number of notions related to support sets.

Definition 1.6.8. For a uniform algebra A on X and a closed subset $F \subset X$, define the A-convex hull of F in X to be $\operatorname{hull}_A(F) := \{x \in X : |f(x)| \le \|f|F\|_\infty, \forall f \in A\}$. We say F is A-convex if $\operatorname{hull}_A(F) = F$.

Thus the polynomial-convex hull of a compact planar set K is just the (polynomial algebra) $P\{|z| \le \|z|K\|_\infty\}$-convex hull of K in $\{|z| \le \|z|K\|_\infty\}$. The A-convex hull of F in X is simply the point evaluation functionals in $X \hookrightarrow M(A)$ having F as a support set. Weak peak sets and in particular maximal antisymmetric sets for A are A-convex.

Lemma 1.6.9. *If Ω is a weak peak set for a uniform algebra A on X, then Ω is A-convex.*

Proof. Fix an arbitrary $x \in X \setminus \Omega$, and we shall find $f \in A$ vanishing on Ω but not at x. This would demonstrate $x \notin \operatorname{hull}_A(\Omega)$ and conclude the proof. To this end, $x \in X \setminus \Omega$ implies $x \in X \setminus F$ for a peak set F containing Ω. If $f \in A$ is such that $f \equiv 1$ on F while $|f| < 1$ off F, then $f - 1 \in A$ vanishes on Ω but not at x, as required. $\square$

The identification of maximal ideal spaces of restriction algebras now follows from the definitions together with some standard argument used earlier.

Theorem 1.6.10. *Let A be a uniform algebra on X, and F be a closed subset of X. Let*

$r : A \to \overline{A|F}$ *be the restriction map. Then*

$$\begin{aligned} r^*(M(\overline{A|F})) =& \{m \in M(A) : F \text{ is a support set for } m\} \\ =& \{m \in M(A) : F \supset \operatorname{supp}\mu_m \text{ of a representing measure } \mu_m\} \\ =& \operatorname{hull}_{\hat{A}}(F \hookrightarrow M(A)). \end{aligned}$$

Remark 1.6.11. It follows that the sets on the right are closed subsets of $M(A)$.

We end with an observation on the preceding results. Let A be a (natural) uniform algebra on $X = M(A)$, and F be a closed subset of $M(A)$. Then the uniform algebra $\overline{A|F}$ on F is naturally realized as $\overline{A|\operatorname{hull}_A(F)}$ on $\operatorname{hull}_A(F)$ with its Shilov boundary contained in F. In particular, if F is A-convex, then $M(\overline{A|F}) = F$. If in addition F is a maximal antisymmetric set for A, then $M(A|F) = F$ is connected by the Shilov idempotent theorem since the antisymmetric algebra $A|F$ contains no nontrivial idempotents. We record this useful fact (cf. [13, 140, 152] where $X = M(H^\infty)$) in view of the Bishop-Glicksberg theorem.

Proposition 1.6.12. *Maximal antisymmetric sets for natural uniform algebras are connected.*

1.7 Elements of the theory of C*-algebras

A classical theorem of Gelfand and Naimark states that commutative unital C*-algebras are simply the algebras $C(X)$ of continuous complex functions on compact spaces X.

Theorem 1.7.1 (Gelfand-Naimark)**.** *Every commutative unital C*-algebra $\mathcal{U}$ is isometrically *-isomorphic to $C(M(\mathcal{U}))$ via its Gelfand transform.*

Let X be a Hausdorff space on which the C*-algebra $C_b(X)$ of bounded continuous complex functions is point-separating. Then the continuous canonical embedding $X \hookrightarrow M(C_b(X))$ is *dense*, by Urysohn's lemma and the preceding theorem, under which every $f \in C_b(X)$ extends to its Gelfand transform $\hat{f} \in C(M(C_b(X)))$. $M(C_b(X))$ is the Stone-Čech compactification of X. When X is discrete, $C_b(X) = l^\infty(X)$ and the compactification is totally disconnected. Now, recall that a sequence $\{z_n\}_n$ in $\mathbb{D}$ is called interpolating if $\inf_n \prod_{k:k\neq n} \rho(z_n, z_k) > 0$. A classical result of L. Carleson [24] identifies the interpolating sequences as precisely those for which the contractive algebra homomorphism

$$\Phi : h \in H^\infty(\mathbb{D}) \mapsto (h(z_n))_n \in l^\infty(\{z_n\}_n)$$

is *surjective.* In this case, the relative topology on $\{z_n\}_n$ is necessarily discrete, and the adjoint map Φ^* is a homeomorphism from the compactification $M(l^\infty(\{z_n\}_n))$ of $\{z_n\}_n$ into $M(H^\infty(\mathbb{D}))$. Since Φ^* maps the embedding in $M(l^\infty(\{z_n\}_n))$ of z_n to its embedding in $M(H^\infty(\mathbb{D}))$, the closure $\overline{\{z_n\}_n}$ in $M(H^\infty)$ is precisely the homeomorphic image under Φ^* of its compactification. Let $b \in H^\infty$ be the Blaschke product associated with the sequence and observe $\ker \Phi = bH^\infty$ by the inner-outer factorization. Then the range of Φ^* consists exactly of those $m \in M(H^\infty)$ such that $\ker m \supset \ker \Phi = bH^\infty$, that is, $m(b) = 0$. Writing $Z(b)$ for the zero set of $\hat{b}$ on $M(H^\infty)$, we have just deduced $Z(b) = \overline{\{z_n\}_n}$ is a totally disconnected subspace of the connected space $M(H^\infty)$, an important fact due to K. Hoffman (cf. [79]).

The ideal structure of $C(X)$ on a compact space X, and therefore that of any commutative unital C*-algebra, is completely characterized by closed subsets of X or the maximal ideal space. In particular, all closed ideals of commutative unital C*-algebras are radical.

Theorem 1.7.2 (M. H. Stone)**.** *Let X be a compact space. Then, $M(C(X)) = X$. In addition, there is a one-to-one order-reversing correspondence between the closed ideals of $C(X)$ and the closed subsets of X via the hull-kernel relation.*

Remark 1.7.3. In particular, for $f \in C(X)$ with zero set $Z(f) \subset X$, the closed principal ideal $\overline{fC(X)}$ of $C(X)$ equals the ideal $\mathcal{I}_{Z(f)} := \{g \in C(X) : g|Z(f) \equiv 0\}$. This is seen as an approximate factorization result in $C(X)$. In addition, for a collection $\{X_\alpha\}_\alpha$ of closed subsets of X, one has $\bigvee_\alpha \mathcal{I}_{X_\alpha} = \mathcal{I}_{\bigcap_\alpha X_\alpha}$, $\bigwedge_\alpha \mathcal{I}_{X_\alpha} = \mathcal{I}_{\overline{\bigcup_\alpha X_\alpha}}$.

Let $\mathcal{A}$ be a C*-subalgebra of a commutative unital C*-algebra $\mathcal{U}$, and let $r : M(\mathcal{U}) \to M(\mathcal{A})$ be the continuous, surjective restriction map. Membership of $u \in \mathcal{U}$ in $\mathcal{A}$ and relations between closed ideals of $\mathcal{U}$ and those of $\mathcal{A}$ are captured in the following results, their proofs depending respectively on the two theorems above.

Proposition 1.7.4. *Let $u \in \mathcal{U}$ with Gelfand transform $\hat{u}$ on $M(\mathcal{U})$. Then, $u \in \mathcal{A}$ if and only if $\hat{u}$ is constant on every fiber $M_\alpha(\mathcal{U})$, $\alpha \in M(\mathcal{A})$.*

Proposition 1.7.5. *If $\mathcal{I}$ is a closed ideal of $\mathcal{U}$ with hull $Z(\mathcal{I}) \subset M(\mathcal{U})$, then the trace $\mathcal{I} \bigcap \mathcal{A}$ in $\mathcal{A}$ is a closed ideal of $\mathcal{A}$ with hull $r(Z(\mathcal{I})) \subset M(\mathcal{A})$.*

Proposition 1.7.6. *If $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ with hull $Z(\mathcal{I}) \subset M(\mathcal{A})$, then the hull in $M(\mathcal{U})$ of the closed ideal of $\mathcal{U}$ generated by $\mathcal{I}$ is $r^{-1}(Z(\mathcal{I}))$.*

Using these basic results, we shall work through an important example. Let Ω be a bounded open subset of $\mathbb{C}$ with compact closure $\overline{\Omega}$ and boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$. Identifying $C(\overline{\Omega}) = \mathcal{A}\{z, \bar{z}\}$ with its restriction on Ω, we let A be a C*-subalgebra of $C_b(\Omega)$ containing the C*-subalgebra $C(\overline{\Omega})$. The special case of A being various C*-subalgebras of $\mathcal{A}(h^\infty(\mathbb{D})) \subset C_b(\mathbb{D})$ is often found in the theory of Toeplitz operators on the Hardy and Bergman spaces over the disc.

Since $A \ni z$ the identity function, the canonical embedding $\Omega \hookrightarrow M(A)$ is homeomorphic and dense. Also, $M(A) \setminus \Omega$ is closed, more precisely

$$M(A) \setminus \Omega = \{m \in M(A) : \hat{z}(m) \in \partial\Omega\} = \bigcap \{\overline{\Omega \setminus K} : K \subset \Omega \text{ compact}\}$$

where the closure is taken in $M(A)$. For, it is clear that

$$M(A) \setminus \Omega \supset \{m \in M(A) : \hat{z}(m) \in \partial\Omega\} \supset \bigcap \{\overline{\Omega \setminus K} : K \subset \Omega \text{ compact}\}.$$

Fix $m \in M(A) \setminus \Omega$ and $K \subset \Omega$ compact. The homeomorphic embedding of K in $M(A)$ is compact, so that $M(A) \setminus K$ is an open neighborhood of m in $M(A)$. By the density of $\Omega \hookrightarrow M(A)$, m is in the closure of $\Omega \bigcap (M(A) \setminus K) = \Omega \setminus K$. That is, $M(A) \setminus \Omega \subset \bigcap \{\overline{\Omega \setminus K} : K \subset \Omega \text{ compact}\}$ proving the equalities.

Next define for $f \in A$ the *cluster set at* $\partial\Omega$ to be

$$f(\partial\Omega) := \bigcap \{\overline{f(\Omega \setminus K)} : K \subset \Omega \text{ compact}\},$$

and we will show

$$f(\partial\Omega) = \hat{f}(M(A) \setminus \Omega).$$

Fix $m \in M(A)\setminus\Omega$ and $K \subset \Omega$ compact. Let z_β be a net in Ω with $z_\beta \to m$ in $M(A)$. Then there exists β_0 such that $z_\beta \in \Omega \setminus K$ for all $\beta \geq \beta_0$. Since $f(z_\beta) \to \hat{f}(m)$ for $\beta \geq \beta_0$, $\hat{f}(m) \in \overline{f(\Omega \setminus K)}$ gives one inclusion. To see the other, let $\zeta \in \overline{f(\Omega \setminus K)}$ for every $K \subset \Omega$. Choose $z_K \in \Omega\setminus K$ such that $|\zeta - f(z_K)| < d(K, \partial\Omega)$. The net $\{z_K : K \subset \Omega \text{ compact}\}$ in $M(A)$ has a cluster point m, and the cluster point $\hat{z}(m)$ of the net $\{z_K : K \subset \Omega \text{ compact}\}$ in $\mathbb{C}$ lies in $\partial\Omega$. That is, $m \in M(A)\setminus\Omega$ by an earlier result. It also follows from $|\zeta - f(z_K)| < d(K, \partial\Omega)$ that the net $f(z_K) \to \zeta$, so that its cluster point $\hat{f}(m) = \zeta$. The required inclusion is established.

Using the cluster set representation define

$$A_0 := \{f \in A : f(\partial\Omega) = \{0\}\} = \{f \in A : \hat{f}|(M(A) \setminus \Omega) \equiv 0\},$$

a closed ideal of A, and note the fact

$$A_0 = \{f \in C(\overline{\Omega}) : f(\partial\Omega) = \{0\}\} \subset C(\overline{\Omega}).$$

Let $r : M(A) \to M(C(\overline{\Omega})) = \overline{\Omega}$ be the restriction map. One has $r(m_1) \neq r(m_2)$ whenever $m_1 \neq m_2 \in M(A)$ and $m_1 \in \Omega$. For, let $f \in A$ with $\hat{f}(m_1) = 1$ while $\hat{f}(m_2) = 0$ and $\hat{f}(M(A) \setminus \Omega) \equiv 0$. Then $f \in A_0 \subset C(\overline{\Omega})$ so that $\hat{f}(m) = f(r(m))$ for all $m \in M(A)$, giving in particular $r(m_1) \neq r(m_2)$. Then it follows that the fiber $M_z(A) = \{z\}$ for every $z \in \Omega \subset \overline{\Omega}$, and that $r^{-1}(\partial\Omega) = M(A) \setminus \Omega$. The singleton fibers imply an approximate factorization result in A: for $g \in C(\overline{\Omega})$ with zero set $Z(g) \subset \Omega$, the closed ideal $\{f \in A : f(Z(g)) \equiv 0\}$ equals the principal ideal $\overline{gA}$, because the hull of the latter is precisely $r^{-1}(Z(g)) = Z(g)$. In particular, taking $g(z) = z - \zeta$ for $\zeta \in \Omega$, the maximal ideal of A corresponding to ζ is principal.

If A is a non-self-adjoint Banach subalgebra of $C_b(\Omega)$ containing $C(\overline{\Omega})$, then one still has the homeomorphic embedding $\Omega \hookrightarrow M(A)$, and the fiber $M_z(A) = \{z\}$ over every $z \in \Omega \subset \overline{\Omega}$ (by the same proof of [140, Lemma, p. 368] applied to Ω instead of $\mathbb{D}$). Therefore, the embedding

$\Omega = r^{-1}(\Omega)$ is open (although in general not dense) in $M(A)$ with $M(A) \setminus \Omega = r^{-1}(\partial\Omega)$.

Non-commutative unital C*-algebras are identified with operator algebras on Hilbert spaces by another classical theorem of Gelfand and Naimark.

Theorem 1.7.7 (Gelfand-Naimark)**.** *Every unital C*-algebra is (isometrically) *-isomorphic to a C*-subalgebra of bounded linear operators on a Hilbert space.*

Such an isomorphism is just one of a large family of *faithful representations* of the algebra, and can indeed be taken to be *irreducible* on the underlying Hilbert space giving a sense of "minimality" among the representations. Using an arbitrary representation, many general C*-algebraic questions can be answered as if on a Hilbert space. On the other hand, one can try to explicitly construct convenient representations to study concrete problems.

Some of the most fundamental properties of unital C*-algebras are summarized as follows.

(i) C*-subalgebras of unital C*-algebras possess spectral permanence.

(ii) *-homomorphisms from one unital C*-algebra into another are contractive and have closed ranges. In particular, *-monomorphisms from one unital C*-algebra into another are necessarily isometric.

(iii) Closed bideals of unital C*-algebras are self-adjoint.

(iv) Quotient algebras of unital C*-algebras over closed bideals are C*-algebras with the quotient norms.

An element x of a unital C*-algebra is normal if the commutator $[x, x^*] := xx^* - x^*x = 0$, self-adjoint if $x^* = x$, and unitary if $xx^* = x^*x = 1$. A self-adjoint element is precisely a normal element with real spectrum. A self-adjoint element with nonnegative (real) spectrum is called positive. Typical positive elements are of the form y^*y, $(y^*y)^{1/2}$, for y in the algebra.

Normal elements generate commutative C*-subalgebras, admit the continuous functional calculus, and possess an important property as follows.

Theorem 1.7.8 (Fuglede-Putnam)**.** *Let x_1, x_2 be normal elements of a C*-algebra $\mathcal{U}$. If $y \in \mathcal{U}$ and $x_1 y = y x_2$, then $x_1^* y = y x_2^*$.*

Another important result is that the unitary elements of a C*-algebra linearly span the algebra (without taking closure).

Theorem 1.7.9 (Russo-Dye-Palmer-Harris)**.** *The convex hull of the unitary elements of a unital C*-algebra contains the open unit ball in the C*-algebra.*

Remark 1.7.10. It follows from this theorem and the bounded functional calculus for unitary operators that a von Neumann algebra is the closed span of its member projections.

Next, in view of the development in Sections 4.4 and 4.5, we briefly review the basic constructions of crossed products of C*-algebras by *discrete* groups. Given a C*-dynamical system (A, G, α) determined by a homomorphism $\alpha : s \in G \mapsto \alpha_s \in \mathrm{Aut}(A)$ from a discrete group G into the automorphism group $\mathrm{Aut}(A)$ of a C*-algebra A, the *algebraic* crossed product $A \rtimes_{\mathrm{alg}} G$ is a *-algebra consisting of finitely supported functions from G to A with addition and scalar multiplication defined point-wise, and multiplication and involution defined by

$$(f \star g)(s) := \sum_{t \in G} f(t)\alpha_t(g(t^{-1}s))$$

$$f^*(s) := \alpha_s(f(s^{-1})^*)$$

for $f, g \in A \rtimes_{\mathrm{alg}} G$, $s \in G$. Then, C*-norms are induced by *-representations of $A \rtimes_{\mathrm{alg}} G$ which in turn are synthesized from covariant representations of (A, G, α). A covariant representation (ρ, U) of the given system consists of a representation ρ of A and a unitary representation U of G on a common Hilbert space H, such that the automorphism pairs $(\alpha_s, U_s \cdot U_s^*)$, $\forall s \in G$, are all covariant in the sense of commuting diagrams intertwined by ρ. Namely, the following operator equalities on H are satisfied

$$\rho(\alpha_s(a)) = U_s \rho(a) U_s^*, \quad \forall a \in A, s \in G.$$

Every covariant representation (ρ, U) on H defines a *-representation $\rho \rtimes U$ of $A \rtimes_{\text{alg}} G$ as

$$(\rho \rtimes U)f := \sum_{s\in G} \rho(f(s))U_s \in \mathcal{L}(H), \quad f \in A \rtimes_{\text{alg}} G$$

with range $(\rho \rtimes U)(A \rtimes_{\text{alg}} G)$ equal to the (pre-closure) *-algebra generated by $\rho(A)$ and $U(G)$ in $\mathcal{L}(H)$. In particular, if τ is a *faithful* representation of A on a Hilbert space H, then

$$\rho : a \in A \mapsto \bigoplus_{t\in G} \tau(\alpha_{t^{-1}}(a)) \in \mathcal{L}\left(\bigoplus_G H\right)$$

$$U_s\left(\bigoplus_{t\in G} h_t\right) = \bigoplus_{t\in G} h_{s^{-1}t}, \quad s \in G,\ h_t \in H,\ \sum_{t\in G} \|h_t\|^2 < \infty$$

constitute a covariant representation on $\bigoplus_G H \cong l^2(G) \bigotimes H$ called the (left) *regular* covariant representation. It has the key property that the corresponding $\rho \rtimes U$, called the (left) regular *-representation of $A \rtimes_{\text{alg}} G$, is *injective*. Now define respectively the *universal* and *reduced* C*-norms for $f \in A \rtimes_{\text{alg}} G$ by

$$\|f\|_u := \sup\{\|(\rho \rtimes U)f\| : (\rho, U) \text{ is a covariant representation of the system}\},$$

$$\|f\|_r := \|(\rho \rtimes U)f\|, \text{ where } (\rho, U) \text{ is the regular covariant representation.}$$

The completions of the *-algebra $A \rtimes_{\text{alg}} G$ relative to the two C*-norms are called the full and reduced crossed products, respectively, and are written $A \rtimes G$ when confusion is not an issue. By construction, for every covariant representation (ρ, U), the *contractive* *-representation $\rho \rtimes U$ relative to the universal C*-norm extends to a *-homomorphism from the full crossed product *onto* the C*-algebra generated by $\rho(A)$ and $U(G)$ on the underlying Hilbert space. When (ρ, U) is the regular covariant representation, this C*-algebra is by construction isomorphic to the reduced crossed product, so there also exists a *-homomorphism from the full onto the reduced crossed product extending the identity map of the dense *-subalgebra $A \rtimes_{\text{alg}} G$. Note that when the C*-algebra A is commutative and unital, the automorphic action α of the group G on A corresponds, via the adjoint map, to a homeomorphic action on the compact Hausdorff space

$M(A)$. The crossed products in this case are also called *transformation group* C*-algebras.

Lastly, we discuss the commutator ideal $\mathcal{I}_c = \mathcal{I}_c(\mathcal{U})$ of a unital C*-algebra $\mathcal{U}$, which is defined to be the closed bideal of $\mathcal{U}$ generated by all the commutators $[a, b]$, $a, b \in \mathcal{U}$. Besides the fact that it is the smallest closed bideal over which the quotient algebra is commutative, the commutator ideal also has a useful characterization by multiplicative linear functionals. In particular, it follows that $\mathcal{I}_c = \mathcal{U}$ if and only if the only multiplicative linear functional on $\mathcal{U}$ is the trivial one.

Theorem 1.7.11 (Arveson [7]). *Let $\mathcal{U}$ be a unital C*-algebra. Then its commutator ideal*

$$\mathcal{I}_c = \bigcap \{\ker m : \ m \text{ is a multiplicative linear functional on } \mathcal{U}\}.$$

Proof. For every such functional m on $\mathcal{U}$, which must satisfy $\|m\| = 0, 1$, one has $m([a, b]) = m(a)m(b) - m(b)m(a) = 0$ which implies that the closed bideal $\ker m \supset \mathcal{I}_c$. Thus we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "m"] \arrow[d, "q"'] & & \mathbb{C} \\
\mathcal{U}/\mathcal{I}_c \arrow[urr, "m'"'] & &
\end{tikzcd}$$

for a multiplicative linear functional m' on the commutative unital C*-algebra $\mathcal{U}/\mathcal{I}_c$, so that $\ker m = q^{-1}(\ker m')$. Conversely, every $\xi \in M(\mathcal{U}/\mathcal{I}_c) \bigcup \{0\}$ arises in this manner as m' from such functionals m on $\mathcal{U}$. Therefore, the intersection of the kernels equals

$$q^{-1}\left(\bigcap \left\{\ker \xi : \ \xi \in M(\mathcal{U}/\mathcal{I}_c) \bigcup \{0\}\right\}\right) = q^{-1}(0) = \mathcal{I}_c,$$

which completes the proof. $\square$

The C*-algebra generated by analytic Toeplitz operators on the Hardy space [49] as well as on the Bergman space [99] properly contains its commutator ideal, while the full Toeplitz algebra on the Bergman space coincides with its commutator ideal [131, 92].

1.8 The algebra QC and the space VMO

Let the unit circle $\partial\mathbb{D}$ be equipped with the normalized Lebesgue measure $d\theta$. Let $L^\infty \cong C(M(L^\infty))$ be the commutative unital C*-algebra of essentially bounded measurable complex functions on $\partial\mathbb{D}$, and $H^\infty \subset L^\infty$ the closed subalgebra of boundary functions of bounded analytic functions in $H^\infty(\mathbb{D})$. Write $C := C(\partial\mathbb{D})$ and $QC := (H^\infty + C) \bigcap \overline{H^\infty + C}$ for the C*-subalgebras of L^∞ consisting of continuous and quasi-continuous functions, respectively. Let us recall the definition of and basic facts about the spaces BMO and VMO on $\partial\mathbb{D}$.

Definition 1.8.1. The mean and mean oscillation of $f \in L^1(\partial\mathbb{D})$ over a subarc $I \subset \partial\mathbb{D}$ are

$$f_I := \frac{1}{\theta(I)} \int_I f d\theta, \qquad f_I^{MO} := \frac{1}{\theta(I)} \int_I |f - f_I| d\theta.$$

Define the sets of functions of bounded and, respectively, vanishing, mean oscillation as

$$\begin{aligned} BMO &:= \{f \in L^1(\partial\mathbb{D}) : \|f\|_{BMO} := \sup_I f_I^{MO} < \infty\}, \\ VMO &:= \{f \in L^1(\partial\mathbb{D}) : f_I^{MO} \to 0 \text{ as } \theta(I) \to 0\} \subset BMO. \end{aligned}$$

Sarason [120] introduced the convenient tool of integral gaps for VMO functions.

Definition 1.8.2. The integral gap of $f \in L^1(\partial\mathbb{D})$ at $\lambda \in \partial\mathbb{D}$ is defined as

$$\gamma_\lambda(f) := \limsup_{\delta \downarrow 0} |f_{(\lambda,\lambda+\delta)} - f_{(\lambda-\delta,\lambda)}| \in [0, \infty].$$

Our first lemma extends Sarason's original statement about vanishing integral gaps.

Lemma 1.8.3. *If $f \in VMO$ and $s, t > 0$, then uniformly in $\lambda \in \partial\mathbb{D}$*

$$\lim_{\delta \downarrow 0} |f_{(\lambda,\lambda+t\delta)} - f_{(\lambda-s\delta,\lambda)}| = 0.$$

Proof. Set $r = s/(s+t) \in (0, 1)$ and $r' = r \wedge (1-r)$. Given any $\epsilon > 0$, we have for all sufficiently small $\delta > 0$ and uniformly in $\lambda \in \partial\mathbb{D}$ that

$$f^{MO}_{(\lambda - s\delta, \lambda + t\delta)} = \frac{1}{(s+t)\delta} \int_{\lambda - s\delta}^{\lambda + t\delta} |f - f_{(\lambda - s\delta, \lambda + t\delta)}| d\theta < r'\epsilon.$$

Splitting the integral in two, it follows without loss of generality that

$$\frac{1}{t\delta} \int_{\lambda}^{\lambda + t\delta} |f - f_{(\lambda - s\delta, \lambda + t\delta)}| d\theta < r'\epsilon.$$

Then we can deduce

$$r'\epsilon > \left| \frac{1}{t\delta} \int_{\lambda}^{\lambda + t\delta} (f - f_{(\lambda - s\delta, \lambda + t\delta)}) d\theta \right| = \left| \frac{1}{t\delta} \int_{\lambda}^{\lambda + t\delta} f d\theta - f_{(\lambda - s\delta, \lambda + t\delta)} \right| = |f_{(\lambda, \lambda + t\delta)} - f_{(\lambda - s\delta, \lambda + t\delta)}|.$$

In view of $f_{(\lambda - s\delta, \lambda + t\delta)} = r f_{(\lambda - s\delta, \lambda)} + (1 - r) f_{(\lambda, \lambda + t\delta)}$, we have

$$r'\epsilon > r |f_{(\lambda, \lambda + t\delta)} - f_{(\lambda - s\delta, \lambda)}| \geq r' |f_{(\lambda, \lambda + t\delta)} - f_{(\lambda - s\delta, \lambda)}|,$$

so $|f_{(\lambda, \lambda + t\delta)} - f_{(\lambda - s\delta, \lambda)}| < \epsilon$ as required. □

Lemma 1.8.4 (Sarason)**.** *If $f \in VMO$ piecewise on $\partial\mathbb{D}$ and $\gamma_{\lambda_n}(f) = 0$ at every end point λ_n of the arcs, then $f \in VMO$ on $\partial\mathbb{D}$.*

Being able to draw upon standard techniques from real analysis on the circle, one's treatment of QC functions becomes very flexible, due to the following characterization of Sarason [119] obtained on the basis of fundamental results of Fefferman and Stein [57].

Theorem 1.8.5 (Sarason)**.** $QC = VMO \bigcap L^\infty$.

It can be directly verified that VMO is a closed linear subspace of BMO in the semi-norm $\|\cdot\|_{BMO}$ and that $BMO \supset L^\infty$. It is also known ([121]) that the $\|\cdot\|_{BMO}$-closure of C lies between QC and VMO. The following result states that certain series of QC functions, converging pointwise and absolutely in the BMO-norm rather than in the L^∞-norm, are still in QC.

Lemma 1.8.6. *If the partial sums $s_n = \sum_{k=1}^{n} f_k$ of QC functions f_k satisfy $\sup_n \|s_n\|_\infty < \infty$, $\sum_{k=1}^{\infty} \|f_k\|_{BMO} < \infty$, and if $s = \sum_{k=1}^{\infty} f_k$ exists a.e., then $s \in QC$.*

Proof. Evidently, $s \in L^\infty$. it remains to show $\|s - s_n\|_{BMO} \to 0$, so that $s \in VMO$ by the preceding remark. But this limit follows from the estimate

$$\|s - s_n\|_{BMO} \le \sum_{k=n+1}^{\infty} \|f_k\|_{BMO}$$

by the bounded convergence theorem for integrals, and from $\sum_{k=1}^{\infty} \|f_k\|_{BMO} < \infty$. □

For any subarc $I \subset \partial\mathbb{D}$, consider the averaging functional $m_I \in QC^*$ given by $m_I(f) = f_I$, $f \in QC$. Let I_ω be a net of subarcs such that $\lim_\omega \theta(I_\omega) = 0$. Then the net m_{I_ω} in the weak-star compact unit ball of QC^* has a subnet converging to a cluster point $m \in QC^*$. It turns out that m is multiplicative on QC.

Proposition 1.8.7 (Sarason)**.** *If* $\lim_\omega \theta(I_\omega) = 0$, *and if* $\lim_\omega m_{I_\omega} = m$ *in the weak-star topology of* QC^*, *then* $m \in M(QC)$.

Proof. Fix arbitrary $f, g \in QC$. We have by weak-star convergence that

$$m(fg) = \lim_\omega \frac{1}{\theta(I_\omega)} \int_{I_\omega} fg d\theta \quad \text{while} \quad m(f)m(g) = \lim_\omega f_{I_\omega} g_{I_\omega}.$$

Since $f, g \in VMO$ and $\lim_\omega \theta(I_\omega) = 0$, we have $\lim_\omega f_{I_\omega}^{MO} = \lim_\omega g_{I_\omega}^{MO} = 0$. Therefore,

$$\begin{aligned} \lim_\omega \left| \frac{1}{\theta(I_\omega)} \int_{I_\omega} fg d\theta - f_{I_\omega} g_{I_\omega} \right| &\le \lim_\omega \frac{1}{\theta(I_\omega)} \int_{I_\omega} |fg - f_{I_\omega} g_{I_\omega}| d\theta \\ &\le \lim_\omega \left(f_{I_\omega}^{MO} \|g\|_\infty + g_{I_\omega}^{MO} \|f\|_\infty \right) = 0. \end{aligned}$$

This estimate shows $m(fg) = m(f)m(g)$, that is, $m \in M(QC)$. □

The next proposition gives a certain type of constructions of QC functions from given ones.

Proposition 1.8.8. *For* $f \in QC$ *and* $t_0 = 0 < t_1 < ... < t_n < 1 = t_{n+1}$, *define*

$$f_{t_1,\dots,t_n} = \prod_{k=0}^{n} f_k, \qquad f_k(e^{i2\pi t}) = f(e^{i2\pi(t_k + (t_{k+1} - t_k)t)}), \qquad 0 < t < 1.$$

Then $f_{t_1,\dots,t_n} \in QC$.

Proof. Writing $\lambda_k := e^{i2\pi t_k}$, each f_k is the dilation to $\partial\mathbb{D}\setminus\{1\}$ of the rotation by $\bar{\lambda}_k$ of the subarc restriction $f|(\lambda_k, \lambda_{k+1})$. Therefore, $\|f_k\|_\infty \le \|f\|_\infty$ and f_k is of VMO on $\partial\mathbb{D}\setminus\{1\}$ because f is so on $\partial\mathbb{D}$. Fix an arbitrary $\epsilon > 0$. It follows that for every subarc I of $\partial\mathbb{D}\setminus\{1\}$ with sufficiently small $\theta(I) > 0$,

$$\frac{1}{\theta(I)}\int_I \left| f_k - \frac{1}{\theta(I)}\int_I f_k d\theta \right| d\theta < \frac{\epsilon}{2(n+1)\|f\|_\infty^n}, \quad k = 0, 1, ..., n,$$

so that

$$\begin{aligned}
&\left| \frac{1}{\theta(I)}\int_I f_{t_1,\ldots,t_n} d\theta - \prod_{k=0}^{n}\left(\frac{1}{\theta(I)}\int_I f_k d\theta\right)\right| \\
=&\left| \frac{1}{\theta(I)}\int_I \left[\prod_{k=0}^{n} f_k - \prod_{k=0}^{n}\left(\frac{1}{\theta(I)}\int_I f_k d\theta\right)\right] d\theta \right| \\
\le&\frac{1}{\theta(I)}\int_I \left| \prod_{k=0}^{n} f_k - \prod_{k=0}^{n}\left(\frac{1}{\theta(I)}\int_I f_k d\theta\right)\right| d\theta \\
\le&\sum_{k=0}^{n} \frac{\|f\|_\infty^n}{\theta(I)}\int_I \left| f_k - \frac{1}{\theta(I)}\int_I f_k d\theta \right| d\theta < \frac{\epsilon}{2}
\end{aligned} \tag{1.8.1}$$

where the second inequality is obtained by splitting the difference of products in the integrand into a telescopic sum. Then, the deduction proceeds as

$$\begin{aligned}
&\frac{1}{\theta(I)}\int_I \left| f_{t_1,\ldots,t_n} - \frac{1}{\theta(I)}\int_I f_{t_1,\ldots,t_n} d\theta \right| d\theta \\
<&\frac{\epsilon}{2} + \frac{1}{\theta(I)}\int_I \left| \prod_{k=0}^{n} f_k - \prod_{k=0}^{n}\left(\frac{1}{\theta(I)}\int_I f_k d\theta\right)\right| d\theta \\
<&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

where the first inequality is due to (1.8.1) and the second to the same argument used therein. That is, we have just shown that $f_{t_1,\ldots,t_n}$ is of VMO on $\partial\mathbb{D}\setminus\{1\}$. Since evidently $\|f_{t_1,\ldots,t_n}\|_\infty \le \|f\|_\infty^{n+1} < \infty$, it remains only to show the integral gap $\gamma_1(f_{t_1,\ldots,t_n}) = 0$.

To this end, again fix an $\epsilon > 0$. It follows from (1.8.1) that for all sufficiently small $\delta > 0$,

$$\begin{aligned}
\frac{\epsilon}{3} &> \left| \frac{1}{\delta} \int_1^{1+\delta} f_{t_1,\dots,t_n} d\theta - \prod_{k=0}^{n} \left(\frac{1}{\delta} \int_1^{1+\delta} f_k d\theta \right) \right| \\
&= \left| \frac{1}{\delta} \int_1^{1+\delta} f_{t_1,\dots,t_n} d\theta - \prod_{k=0}^{n} \left(\frac{1}{(t_{k+1}-t_k)\delta} \int_{\lambda_k}^{\lambda_k+(t_{k+1}-t_k)\delta} f d\theta \right) \right|, \qquad (1.8.2) \\
\frac{\epsilon}{3} &> \left| \frac{1}{\delta} \int_{1-\delta}^{1} f_{t_1,\dots,t_n} d\theta - \prod_{k=0}^{n} \left(\frac{1}{\delta} \int_{1-\delta}^{1} f_k d\theta \right) \right| \\
&= \left| \frac{1}{\delta} \int_{1-\delta}^{1} f_{t_1,\dots,t_n} d\theta - \prod_{k=0}^{n} \left(\frac{1}{(t_{k+1}-t_k)\delta} \int_{\lambda_{k+1}-(t_{k+1}-t_k)\delta}^{\lambda_{k+1}} f d\theta \right) \right|. \qquad (1.8.3)
\end{aligned}$$

Next, Lemma 1.8.3 with $s = t_{k+1} - t_k$, $t = t_{k+2} - t_{k+1}$ asserts that for all sufficiently small $\delta > 0$,

$$\left| \frac{1}{(t_{k+2}-t_{k+1})\delta} \int_{\lambda_{k+1}}^{\lambda_{k+1}+(t_{k+2}-t_{k+1})\delta} f d\theta - \frac{1}{(t_{k+1}-t_k)\delta} \int_{\lambda_{k+1}-(t_{k+1}-t_k)\delta}^{\lambda_{k+1}} f d\theta \right| < \frac{\epsilon}{3(n+1)\|f\|_\infty^n}$$

for $k = 0, \dots, n-1$ and

$$\left| \frac{1}{t_1\delta} \int_1^{1+t_1\delta} f d\theta - \frac{1}{(1-t_n)\delta} \int_{1-(1-t_n)\delta}^{1} f d\theta \right| < \frac{\epsilon}{3(n+1)\|f\|_\infty^n},$$

so that

$$\left| \prod_{k=0}^{n} \left(\frac{1}{(t_{k+1}-t_k)\delta} \int_{\lambda_k}^{\lambda_k+(t_{k+1}-t_k)\delta} f d\theta \right) - \prod_{k=0}^{n} \left(\frac{1}{(t_{k+1}-t_k)\delta} \int_{\lambda_{k+1}-(t_{k+1}-t_k)\delta}^{\lambda_{k+1}} f d\theta \right) \right| < \frac{\epsilon}{3}.$$

Now it follows from this estimate and (1.8.2), (1.8.3) that for all sufficiently small $\delta > 0$,

$$\left| \frac{1}{\delta} \int_1^{1+\delta} f_{t_1,\dots,t_n} d\theta - \frac{1}{\delta} \int_{1-\delta}^{1} f_{t_1,\dots,t_n} d\theta \right| < \epsilon.$$

That is, $\gamma_1(f_{t_1,\dots,t_n}) = 0$ which completes the proof. $\square$

We also derive an integral characterization for invertible functions in QC. Its proof will use a property of $M(QC)$ which has a generalization to $M(L^\infty)$, to be proved first with an idea from the proof of [120, Lemma 7]. There may be independent interest in this property of $M(L^\infty)$, for

Goldstine's theorem implies that the canonical embedding of the closed unit ball of L^1 is weak-star dense in that of the bidual $(L^1)^{**} \cong (L^\infty)^*$. Let $m_I \in (L^\infty)^*$ be the averaging functional in $(L^\infty)^*$, which is also the canonical embedding in $(L^1)^{**}$ of the normalized indicator function in L^1, of a subarc $I \subset \partial\mathbb{D}$, $\theta(I) > 0$.

Proposition 1.8.9. *For every $\xi \in M(L^\infty)$, there exists a collection $\{I_\omega : \omega \in \mathscr{W}\}$ of subarcs indexed by a directed set $\mathscr{W}$ such that the net $\{m_{I_\omega} : \omega \in \mathscr{W}\}$ converges to ξ in the weak-star topology of $(L^\infty)^*$ while $\{\theta(I_\omega) : \omega \in \mathscr{W}\}$ converges to 0.*

Proof. Let $\mathscr{W}$ be the collection of all pairs (F, ϵ), where F is a (nonempty) finite subset of L^∞ and $\epsilon > 0$. Define a partial order " $\leq$ " on $\mathscr{W}$ by

$$(F_1, \epsilon_1)\text{``}\leq\text{''}(F_2, \epsilon_2) \iff F_1 \subset F_2 \text{ and } \epsilon_1 \geq \epsilon_2.$$

Since $(F_{1,2}, \epsilon_{1,2})\text{``}\leq\text{''}(F_1 \bigcup F_2, \epsilon_1 \wedge \epsilon_2)$, one has $(\mathscr{W}, \text{``}\leq\text{''})$ as a directed set.

Next we construct a subarc $I := I_\omega$ for every $\omega := (F, \epsilon) \in \mathscr{W}$ satisfying

$$\theta(I) < \epsilon \text{ and } |m_I(f) - \xi(f)| < \epsilon, \forall f \in F. \tag{1.8.4}$$

To this end define on $\partial\mathbb{D}$ the function

$$g = \sum_{f \in F} |f - \xi(f)| \in L^\infty.$$

Since $\xi(g) = \sum_{f \in F} |\xi(f) - \xi(f)| = 0$ for $\xi \in M(L^\infty)$, g is not invertible in L^∞. Therefore, the Borel subset $B := \{\lambda \in \partial\mathbb{D} : g(\lambda) < \epsilon/2\}$ has positive measure $\theta(B) > 0$. The Lebesgue density theorem gives $\lambda_0 \in \partial\mathbb{D}$ such that

$$\lim_{\delta \downarrow 0} \frac{\theta((\lambda_0 - \delta, \lambda_0 + \delta) \bigcap B)}{2\delta} = 1.$$

Thus one can choose $\delta_0 \in (0, \epsilon/2)$ and put $I = (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$ such that $\theta(I) = 2\delta_0 < \epsilon$, $\theta(I \setminus B)/(2\delta_0) < \epsilon/(2\|f\|_\infty)$. Using the partition $I = (I \bigcap B) \bigsqcup (I \setminus B)$, routine integral estimates then give $m_I(g) < \epsilon$. In particular,

$$|m_I(f) - \xi(f)| = |m_I(f - \xi(f))| \leq m_I(|f - \xi(f)|) \leq m_I(g) < \epsilon, \qquad \forall f \in F.$$

So the subarc I fulfills (1.8.4).

To see $\lim_{\omega \in (\mathscr{W}, \text{``}\leq\text{''})} m_{I_\omega} = \xi$ in $(L^\infty)^*$, note that every weak-star neighborhood of ξ contains a basic neighborhood

$$G(F, \epsilon) := \{x^* \in (L^\infty)^* : |x^*(f) - \xi(f)| < \epsilon, \ \forall f \in F\}$$

for a finite subset F of L^∞ and $\epsilon > 0$. By (1.8.4), $m_{I_\omega} \in G(F, \epsilon)$ whenever $(F, \epsilon)\text{``} \leq \text{''}\omega$ in $\mathscr{W}$. This establishes the weak-star convergence of the net of functionals. The convergence $\lim_{\omega \in (\mathscr{W}, \text{``}\leq\text{''})} \theta(I_\omega) = 0$ is obvious by construction. The proof is complete. □

Proposition 1.8.10. *If $f \in QC$, then $f \in QC^{-1}$ if and only if there exist $\epsilon, \delta > 0$ such that*

$$\left| \frac{1}{\theta(I)} \int_I f d\theta \right| \geq \epsilon$$

for every subarc I of $\partial\mathbb{D}$ with $0 < \theta(I) < \delta$.

Proof. Related to the development in [120, p. 822], we claim

$$M(QC) = \{\lim_{\omega \in \mathscr{W}} (m_{I_\omega}|QC) : \lim_{\omega \in \mathscr{W}} \theta(I_\omega) = 0\} \tag{1.8.5}$$

where the nets of averaging functionals restricted to QC converge in the weak-star topology of QC^*. For, if $\lim_{\omega \in \mathscr{W}} (m_{I_\omega}|QC) = m$ in QC^* and $\lim_{\omega \in \mathscr{W}} \theta(I_\omega) = 0$, then $m \in M(QC)$ by Proposition 1.8.7. To show the other inclusion, the preceding proposition states

$$M(L^\infty) \subset \{\lim_{\omega \in \mathscr{W}} m_{I_\omega} : \lim_{\omega \in \mathscr{W}} \theta(I_\omega) = 0\},$$

which under the weak-star continuous restriction map $r : (L^\infty)^* \to QC^*$ yields

$$M(QC) = r(M(L^\infty)) \subset \{\lim_{\omega \in \mathscr{W}} (m_{I_\omega}|QC) : \lim_{\omega \in \mathscr{W}} \theta(I_\omega) = 0\}$$

and asserts equality in (1.8.5).

Hence for $f \in QC$, $f \in QC^{-1}$ if and only if $\min_{y \in M(QC)} |f(y)| > 0$, which in turn amounts to the said integral condition due to (1.8.5) and existence of weak-star convergent subnets of averaging functionals all with unit norm in QC^*. $\square$

In connection with the determination of the essential commutant of all the Toeplitz and Hankel operators on the Hardy space using Davidson's determination [45] of that of the Toeplitz operators, S. Power [104] suggested proving the following result based on the characterization $QC = VMO \bigcap L^\infty$. We shall first supply such a proof.

Proposition 1.8.11. *$H^\infty + C$ contains no non-trivial ideals of L^∞.*

First proof. It suffices to show that if $f \in L^\infty$ is such that $fL^\infty \subset H^\infty + C$, then $f = 0$ a.e. Suppose on the contrary $|f| > 0$ on a subset of $\partial\mathbb{D}$ of positive measure. Then for some $\epsilon > 0$, $V := \{|f| \geq \epsilon\}$ has positive measure, and the Lebesgue density theorem gives a $\lambda_0 \in V$ with

$$\lim_{\delta \to 0} \frac{\theta((\lambda_0 - \delta, \lambda_0 + \delta) \bigcap V)}{2\delta} = 1.$$

Thus there exists $\delta_0 > 0$ such that $\theta((\lambda_0 - \delta, \lambda_0 + \delta) \bigcap V) > \delta$ for every $\delta \in (0, \delta_0)$. This yields a sequence $\delta_n \downarrow 0$ in $(0, \delta_0)$ satisfying, without loss of generality,

$$\theta((\lambda_0, \lambda_0 + \delta_n) \bigcap V) > \delta_n/2.$$

Let $\phi := (|f|/f)1_{\{f \neq 0\}}$, $\phi \in L^\infty$ and $f\phi = |f|$. By hypothesis,

$$|f|1_{(\lambda_0, \lambda_0+\delta_0)} = f\phi 1_{(\lambda_0, \lambda_0+\delta_0)} \in fL^\infty \subset H^\infty + C,$$

so the real function $\psi := |f|1_{(\lambda_0, \lambda_0+\delta_0)} \in QC \subset VMO$ has a zero integral gap at $\lambda_0 \in \partial\mathbb{D}$. In particular, $\psi_{(\lambda_0,\lambda_0+\delta_n)} = \psi_{(\lambda_0,\lambda_0+\delta_n)} - \psi_{(\lambda_0-\delta_n,\lambda_0)} \to 0$. On the other hand,

$$\psi_{(\lambda_0,\lambda_0+\delta_n)} \geq \frac{1}{\delta_n}\int_{(\lambda_0,\lambda_0+\delta_n)\bigcap V} |f|d\theta \geq \frac{1}{\delta_n}\cdot\epsilon\cdot\frac{\delta_n}{2} = \frac{\epsilon}{2} > 0,\ \forall n.$$

This is absurd, and the proof is done. □

Next, we give a more abstract and shorter proof independent of the results above. Instead, it exploits the ideal structure of the commutative C*-algebra L^∞ and relies on a theorem of K. Hoffman stating that the maximal ideal space $M(H^\infty + C)$ is connected (cf. [79]).

Second proof. Suppose $H^\infty + C$ contains a non-trivial ideal J of $L^\infty \cong C(M(L^\infty))$, and let $Z(\overline{J})$ be the hull in $M(L^\infty)$ of the closed ideal $\overline{J}$. Since $Z(\overline{J})$ is a proper closed subset of the totally disconnected $M(L^\infty)$, there exists a non-trivial characteristic function $1_V \in L^\infty$ which vanishes identically on $Z(\overline{J})$. Then, $1_V \in \overline{J} \subset H^\infty + C$ due to the radicality of $\overline{J}$. This generates a contradiction since $M(H^\infty + C)$ is connected. The proof is complete. □

Remark 1.8.12. Hoffman's connectedness result for $M(H^\infty + C)$ could also be given an alternate proof using $QC = VMO \bigcap L^\infty$. By the Shilov idempotent theorem, we need to show $H^\infty + C$, equivalently QC, does not contain the characteristic function of any Borel subset of $\partial\mathbb{D}$ whose measure is neither zero nor full. This can be proved by showing such characteristic functions have nonvanishing integral gaps at some points and thus can not be in VMO.

Let $C_{\bar{z}} \in \mathcal{L}(L^2)$ be the composition operator by the complex conjugate $z \mapsto \bar{z}$ on $\partial\mathbb{D}$ and write $\tilde{f} := C_{\bar{z}}f = f \circ \bar{z}$ for $f \in L^2$. Noting that $C_{\bar{z}}|QC \in \mathcal{L}(QC)$, we write $\bar{y} := y \circ C_{\bar{z}}|QC \in M(QC)$ for $y \in M(QC)$. Define $QC_s = \{f \in QC : \tilde{f} = f\}$, a C*-subalgebra of QC. To apply the Allan-Douglas localization principle for Toeplitz and Hankel operators, it is useful to determine the fibers of $M(QC)$ over $M(QC_s)$ as follows.

Proposition 1.8.13 (S. Power [105])**.** *For $x \in M(QC_s)$, the fiber $M_x(QC) = \{y, \bar{y}\}$.*

The key to the proof of Power's result consists in showing the existence of two QC_s functions $f_\pm$ associated with any QC function f, each of which coincides with f on half of the circle. To achieve this, set $\partial\mathbb{D}_+ = \partial\mathbb{D}\bigcap\{\Im z \geq 0\}$ and define

$$f_+ = \begin{cases} f, & \text{on } \partial\mathbb{D}_+ \\ \tilde{f}, & \text{on } \partial\mathbb{D}\setminus\partial\mathbb{D}_+ \end{cases}, \qquad f_- = \begin{cases} \tilde{f}, & \text{on } \partial\mathbb{D}_+ \\ f, & \text{on } \partial\mathbb{D}\setminus\partial\mathbb{D}_+ \end{cases}.$$

Since clearly $\gamma_{\pm 1}(f_\pm) = 0$, it follows from construction and the preceding results that $f_\pm \in QC_s$ with $f_+ = f$ on $\partial\mathbb{D}_+$ and $f_- = f$ on $\partial\mathbb{D}\setminus\partial\mathbb{D}_+$, fulfilling the requirement.

We shall have more occasions to apply some of these results in the construction of QC functions pivotal to various operator theoretic problems. See Sections 3.4, 3.5, 4.5, and [147].

1.9 Douglas algebras, Gleason parts, and Hoffman maps

A *Douglas algebra* B is any closed subalgebra of L^∞ containing H^∞ on the unit circle $\partial\mathbb{D}$. Since $H^\infty\overline{H^\infty}$ is dense in L^∞, a result proved by R. G. Douglas and W. Rudin in 1969, B separates the points of $M(L^\infty)$ and is thus a uniform algebra on $M(L^\infty)$. Perhaps the first important property of Douglas algebras is (strong) logmodularity on $M(L^\infty)$, due to $|(L^\infty)^{-1}| = |(H^\infty)^{-1}|$. This is especially fortunate since the totally disconnected space $M(L^\infty)$ does not carry any Dirichlet algebra other than L^∞ [79]. It then follows from the general theory of logmodular uniform algebras that the Shilov boundary of $M(B)$ is $M(L^\infty)$, the restriction map homeomorphically embeds $M(B)$ in $M(H^\infty)$ as a closed subset, and every $m \in M(H^\infty)$ possesses a unique representing measure μ_m with support set $S_m := \operatorname{supp}\mu_m \subset M(L^\infty)$. For $z \in \mathbb{D} \hookrightarrow M(H^\infty)$, μ_z is given by

$$\int_{M(L^\infty)} f d\mu_z = \int_{\partial\mathbb{D}} f P_z d\theta, \quad f \in L^\infty.$$

Since the map $m \in M(H^\infty) \mapsto \mu_m \in C(M(L^\infty))^*$ is weak-star continuous, one has the isometric *-linear extension

$$f \in L^\infty \cong C(M(L^\infty)) \mapsto \hat{f} \in C(M(H^\infty)), \qquad \hat{f}(m) := \int_{M(L^\infty)} f d\mu_m,\ m \in M(H^\infty),$$

and the restrictions $\hat{f}|\mathbb{D}$ on the dense [25] subset $\mathbb{D} \hookrightarrow M(H^\infty)$, which are just the harmonic extensions, fill the space $h^\infty(\mathbb{D})$. By the Stone-Weierstrass theorem on $M(H^\infty)$,

$$C(M(H^\infty)) \cong \mathcal{A}(h^\infty(\mathbb{D})) = \mathcal{A}(H^\infty(\mathbb{D}) \bigcup \overline{H^\infty(\mathbb{D})})$$

via restriction to $\mathbb{D} \hookrightarrow M(H^\infty)$. The restriction to $M(L^\infty) \subset M(H^\infty)$ of $g \in C(M(H^\infty))$ is the Gelfand transform of the boundary limit function in L^∞ of the restriction $g|\mathbb{D} \in \mathcal{A}(h^\infty(\mathbb{D}))$. In addition, for a Douglas algebra B, the map $f \in B \mapsto \hat{f}|M(B) \in C(M(B))$ is simply the Gelfand transform of B. We shall use a common symbol to denote an L^∞ function, its bounded harmonic extension in $\mathbb{D}$, and its continuous extension on $M(H^\infty)$.

The Chang-Marshall theorem is a powerful result on the structure of Douglas algebras. An immediate consequence is that $M(B_1) \subset M(B_2) \Rightarrow B_1 \supset B_2$ for Douglas algebras B_1, B_2. It also allows for determination of membership of $m \in M(H^\infty)$ in $M(B)$ by its support S_m.

Theorem 1.9.1 (Chang-Marshall)**.** *Every Douglas algebra is generated by H^∞ and the complex conjugates of interpolating Blaschke products invertible in the algebra.*

Corollary 1.9.2. *Let B be a Douglas algebra. Then,*

$$M(B) = \{m \in M(H^\infty) : u|S_m = \text{const}, \forall u \in B^{-1} \text{ an inner function}\}.$$

In particular, if $m \in M(H^\infty)$ with $S_m \subset S_{m'}$ for some $m' \in M(B)$, then $m \in M(B)$.

Corollary 1.9.3. *Let B be a Douglas algebra, $f \in B$, $m \in M(B)$. Then, $f|S_m \in H^\infty|S_m$.*

It is well known that S_m is a singleton if and only if $m \in M(L^\infty)$, and that $S_m = M(L^\infty)$ if and only if $m \in \mathbb{D} \hookrightarrow M(H^\infty)$. The following result covers what lies between these extremities. Note that when $B = H^\infty + C$, the smallest Douglas algebra other than H^∞, we have $Q_B := B \bigcap \overline{B} = QC$ and $M(H^\infty + C) = M(H^\infty) \setminus \mathbb{D}$.

Proposition 1.9.4. *Let B be a Douglas algebra, $m \in M(H^\infty)$ and $y \in M(Q_B)$. Then, $m \in M_y(B)$ if and only if $S_m \subset M_y(L^\infty)$.*

Proof. Suppose $m \in M_y(B)$. Then S_m is an antisymmetric set for B and is therefore contained in the fiber $M_{y'}(L^\infty)$ over some $y' \in M(Q_B)$. For every $q \in Q_B$, we have

$$q(y) = q(m) = \int_{S_m} q d\mu_m = q(y').$$

That is, $y = y' \in M(Q_B)$ and $S_m \subset M_y(L^\infty)$.

Next suppose $S_m \subset M_y(L^\infty)$. For $u \in B^{-1}$ an inner function, $u \in Q_B$ is constant on $M_y(L^\infty) \supset S_m$. It then follows from the corollary to the Chang-Marshall theorem that $m \in M(B)$. Now for every $q \in Q_B$ we have

$$q(m) = \int_{S_m} q d\mu_m = q(y),$$

which amounts to $m \in M_y(B)$. The proof is complete. $\square$

It was mentioned in Section 1.2 that for a nonconstant inner function $u \in H^\infty$, the essential range $\sigma(u, L^\infty) = u(M(L^\infty)) = \partial\mathbb{D}$. Under $M(H^\infty + C) = M(H^\infty) \setminus \mathbb{D}$, one can use the fact that $u(M(H^\infty) \setminus \mathbb{D})$ equals the *cluster set* $u(\partial\mathbb{D})$ of u at $\partial\mathbb{D}$ to determine $\sigma(u, H^\infty + C)$. In particular, the only inner functions invertible in $H^\infty + C$ are the finite Blaschke products.

Theorem 1.9.5. *If $u \in H^\infty$ is a nonconstant inner function, then*

$$\sigma(u, H^\infty + C) = u(M(H^\infty) \setminus \mathbb{D}) = \partial\mathbb{D} \text{ or } \overline{\mathbb{D}}$$

where the former holds if and only if u is a finite Blaschke product.

Proof. One need only prove that, if there exists $z_0 \in \mathbb{D}$ such that $z_0 \notin u(\partial\mathbb{D})$, then u is a finite Blaschke product. To see this, $z_0 \notin u(\partial\mathbb{D})$ implies that the inner factor of $u - z_0$ is a finite Blaschke product. In view of the factorization

$$u - z_0 = \frac{u - z_0}{1 - \bar{z}_0 u}(1 - \bar{z}_0 u)$$

where the first factor is inner and the second invertible, the Frostman shift $(u - z_0)/(1 - \bar{z}_0 u) = b$

a finite Blaschke product. So, u is a Frostman shift of b, thus itself a finite Blaschke product. $\square$

Now we look at two ways in which Douglas algebras arise naturally, for which the maximal ideal spaces are explicitly identified as subsets of $M(H^\infty)$. First, let $H^\infty[g_\alpha : \alpha \in \Lambda]$ be the Douglas algebra generated by $\{g_\alpha\}_{\alpha\in\Lambda}$ in L^∞ over H^∞. The Chang-Marshall theorem gives (with minor changes to the proof of [67, Lemma 1.5])

Theorem 1.9.6.

$$M(H^\infty[g_\alpha : \alpha \in \Lambda]) = \{m \in M(H^\infty) : g_\alpha|S_m \in H^\infty|S_m, \forall \alpha \in \Lambda\}.$$

Next, for F a compact subset of $M(L^\infty)$ such that $H^\infty|F$ is *closed* in the sup-norm on F, define the extension of H^∞ over F as

$$H^\infty_F := \{f \in L^\infty : f|F \in H^\infty|F\} = H^\infty + \mathcal{I}_F$$

where $\mathcal{I}_F := \{f \in L^\infty : f|F \equiv 0\}$. It is easy to see that H^∞_F is a Douglas algebra.

Theorem 1.9.7.

$$M(H^\infty_F) = M(L^\infty)\bigcup\{m \in M(H^\infty) : S_m \subset F\}.$$

Proof. Clearly, $M(L^\infty) \subset M(H^\infty_F)$. Let $m \in M(H^\infty)$ with $S_m \subset F$, and consider its representing measure $\mu_m \in C(M(L^\infty))^*$. If $f_1, f_2 \in H^\infty_F$, then $f_1|F = h_1|F$, $f_2|F = h_2|F$ for some $h_1, h_2 \in H^\infty$. So one has

$$\int_{M(L^\infty)} f_1 f_2 d\mu_m = \int_{S_m} h_1 h_2 d\mu_m = \int_{S_m} h_1 d\mu_m \int_{S_m} h_2 d\mu_m = \int_{S_m} f_1 d\mu_m \int_{S_m} f_2 d\mu_m.$$

That is, μ_m is multiplicative on H^∞_F and so $m \in M(H^\infty_F)$, proving one inclusion.

For the other inclusion, suppose $m \in M(H^\infty_F) \subset M(H^\infty)$ but $m \notin M(L^\infty)$, and we need only prove $S_m \subset F$. Assume not. Then there exist two distinct points $\xi, \xi' \in S_m$ with $\xi \notin F$. Let E be a clopen neighborhood of ξ in the totally disconnected $M(L^\infty)$ with $E\bigcap(F\bigcup\{\xi'\}) = \emptyset$. Since both $\mu_m(E)$ and $\mu_m(M(L^\infty) \setminus E)$ are strictly positive and $F \subset M(L^\infty) \setminus E$ by construction, the continuous function on $M(L^\infty)$

$$f := \mu_m(M(L^\infty) \setminus E)1_E - \mu_m(E)1_{M(L^\infty)\setminus E}$$

is in $(H^\infty_F)^{-1}$. However, $m(f) = \int_{M(L^\infty)} f d\mu_m = 0$ generates the desired contradiction. $\square$

For a uniform algebra A, the pseudo-hyperbolic distance ρ on $M(A)$ is defined by $\rho(m, n) = \sup\{|a(m)| : a(n) = 0, \|a\| \leq 1, a \in A\}$. One has

$$\rho(m, n) < 1 \Leftrightarrow \|m - n\| < 2$$

and these define an equivalence relation $m \sim n$ on $M(A)$. The resulting equivalence classes form a partition of $M(A)$ into the so-called *Gleason parts* of $M(A)$. A part is *trivial* if it is a singleton. As is common to all logmodular algebras, the nontrivial Gleason parts of $M(H^\infty)$ are analytic discs [80]. That is, there exist bijective continuous (but not necessarily homeomorphic) maps L, called *Hoffman maps*, from $\mathbb{D}$ to the nontrivial parts such that $f \circ L \in H^\infty(\mathbb{D})$ for every $f \in H^\infty$. In fact, the Hoffman map $L_m : \mathbb{D} \to P_m$ onto a nontrivial part $P_m \ni m$ with $L_m(0) = m$ is explicitly defined as the point-wise limit $L_m = \lim_\alpha L_{z_\alpha}$ in $M(H^\infty)$ for any net $z_\alpha \in \mathbb{D} \hookrightarrow M(H^\infty)$ converging to m [81]. Here,

$$L_{z_\alpha}(z) := \frac{z + z_\alpha}{1 + \bar{z}_\alpha z}, \quad z \in \mathbb{D}.$$

Consequently, for $f \in H^\infty$, $f \circ L_m = \lim_\alpha f \circ L_{z_\alpha}$ is the point-wise limit of uniformly bounded analytic functions. Therefore, the convergence of the derivatives $(f \circ L_{z_\alpha})^{(k)}$ to $(f \circ L_m)^{(k)}$, for any order $k \geq 0$, is uniform on compact subsets of $\mathbb{D}$. More generally, for $f \in L^\infty \cong h^\infty(\mathbb{D})$, the point-wise convergence $f \circ L_m = \lim_\alpha f \circ L_{z_\alpha}$ of uniformly bounded harmonic functions is uniform on compact subsets, giving $f \circ L_m \in h^\infty(\mathbb{D})$ a bounded harmonic function. Finally, for $f \in \mathcal{A}(h^\infty(\mathbb{D}))$, one still has $f \circ L_m = \lim_\alpha f \circ L_{z_\alpha}$ uniformly on compact subsets of $\mathbb{D}$, by passing to finite sums of finite products of $h^\infty(\mathbb{D})$ functions and then to the limit in sup-norm.

Each L_m admits a unique, surjective, continuous extension $L^*_m : M(H^\infty) \to \overline{P_m}$ which is indeed the adjoint map to the algebra homomorphism $f \in H^\infty \mapsto f \circ L_m \in H^\infty(\mathbb{D})$ [22]. Therefore, in view of the dense embedding $\mathbb{D} \hookrightarrow M(H^\infty)$, the Gelfand transform of $f \circ L_m \in H^\infty(\mathbb{D})$ for $f \in H^\infty$ is exactly $f \circ L_m^*$, and the unique continuous extension of $f \circ L_m \in h^\infty(\mathbb{D})$ for $f \in L^\infty$ is also $f \circ L_m^*$.

On the other hand, the trivial points of the connected $M(H^\infty)$ form a closed subset *properly* containing the totally disconnected $M(L^\infty) = \partial H^\infty$. Actually the set of trivial points is itself totally disconnected [129]. Although no interpolating Blaschke products can vanish identically on a nontrivial part, other Blaschke products can. The following result is known (e.g. [23, 130]).

Proposition 1.9.8. *If P is a nontrivial part of $M(H^\infty)$ disjoint from $\mathbb{D}$, then $\overline{P} \bigcap M(L^\infty) = \emptyset$.*

Proof. Let L be a Hoffman map taking $\mathbb{D}$ onto P. By a result of Hoffman [81], for every $n > 1$ the nontrivial point $L(1/n) \in \overline{\{\zeta_{n,k}\}_k} \setminus \mathbb{D}$ for an interpolating sequence $\{\zeta_{n,k}\}_k$ in $\mathbb{D}$. Dropping finitely many points, one can assume $\sum_k (1 - |\zeta_{n,k}|) < 1/2^n$, $\forall n$, so that $\sum_{n,k}(1 - |\zeta_{n,k}|) < \infty$. If b is the infinite Blaschke product determined by the zeros $\{\zeta_{n,k}\}_{n,k}$, then $b(L(1/n)) = 0$ for every n, and the analytic function $b \circ L$ must vanish identically in $\mathbb{D}$. That is, b vanishes on P and $\overline{P}$. Since b is unimodular on $M(L^\infty)$, the conclusion follows at once. $\square$

In particular, since the map $L^* : M(H^\infty) \to \overline{P}$ takes parts to parts [22, 23], the trivial point $L^*(\xi) \in \overline{P}$ for $\xi \in M(L^\infty)$ is not in $M(L^\infty)$, asserting the proper inclusion of $M(L^\infty)$ in the set of trivial points of $M(H^\infty)$.

The Hoffman maps are an indispensable tool in studying the topological structure of $M(H^\infty)$, the ideal structure of Douglas algebras, and uniform algebras on $M(H^\infty)$ in connection with harmonic Toeplitz operators on the Bergman space [13, 153, 127].

1.10 Hardy spaces and Bergman spaces

Throughout this section, fix $p \in [1, \infty)$ and let $d\theta$, da be respectively the normalized linear Lebesgue measure on $\partial\mathbb{D}$ and the normalized area measure on $\mathbb{D}$. For any analytic function $f : \mathbb{D} \to \mathbb{C}$, the function $|f|^p = e^{p \ln |f|}$ is subharmonic, so that the integral $\int_{\lambda \in \partial\mathbb{D}} |f(r\lambda)|^p d\theta$ is increasing in $r \in (0,1)$ and that

$$\sup_{0<r<1}\left(\int_{\lambda\in\partial\mathbb{D}}|f(r\lambda)|^p d\theta\right)^{1/p}=\lim_{r\to1^-}\left(\int_{\lambda\in\partial\mathbb{D}}|f(r\lambda)|^p d\theta\right)^{1/p}$$

while the quantity may be infinite. Naturally then, the Hardy space $H^p(\mathbb{D})$ consists of the analytic functions on $\mathbb{D}$, with point-wise linear operations, for which the quantity above is finite and defines the norm. For $f \in H^p(\mathbb{D})$, Fatou's classical result states that the radial limits $\lim_{r\to1^{-1}} f(r\lambda) =: f(\lambda)$ exist for almost all $\lambda \in \partial\mathbb{D}$, and a theorem of Hardy and Littlewood asserts the radial maximal function is p-th integrable on $\partial\mathbb{D}$, that is, for some constant A_p

$$\int_{\lambda\in\partial\mathbb{D}}\sup_{0<r<1}|f(r\lambda)|^p d\theta \le A_p^p \sup_{0<r<1}\int_{\lambda\in\partial\mathbb{D}}|f(r\lambda)|^p d\theta<\infty.$$

Therefore, the boundary function $f \in L^p := L^p(\partial\mathbb{D})$ satisfies

$$\|f\|_p=\left(\int_{\partial\mathbb{D}}|f|^p d\theta\right)^{1/p}=\lim_{r\to1^-}\left(\int_{\lambda\in\partial\mathbb{D}}|f(r\lambda)|^p d\theta\right)^{1/p}.$$

Thus the radial limit induces a linear isometry from $H^p(\mathbb{D})$ onto a subspace H^p of L^p which can be shown to be the closure in L^p of the polynomials in z. In particular, the Banach space $H^p(\mathbb{D}) \cong H^p$ for $1 < p < \infty$ is reflexive and separable.

Szegö's classical result characterizes moduli of H^p functions in L^p.

Theorem 1.10.1 (Szegö)**.** *If $0 \neq h \in H^p$, then $\int_{\partial\mathbb{D}}|\ln|h||d\theta < \infty$. Conversely, if $f \in L^p$ satisfies $\int_{\partial\mathbb{D}}|\ln|f||d\theta < \infty$, then $|f| = |h|$ θ-a.e. for some $0 \neq h \in H^p$.*

For all $\alpha \in \mathbb{D}$, the evaluation functionals $v_\alpha f = f(\alpha)$, $f \in H^p(\mathbb{D})$, are in the dual space $H^p(\mathbb{D})^*$ having the following convergence properties.

Proposition 1.10.2. *As $|\alpha| \to 1^-$, $\|v_\alpha\| \to \infty$ and $v_\alpha/\|v_\alpha\| \to 0$ in the weak-star topology.*

Proof. Suppose the first convergence is false. Then $\|v_{\alpha_n}\| \le M$ for some $M \le \infty$ and $|\alpha_n| \to 1^-$. Passing to a subsequence if needed, we assume $\alpha_n \to \lambda_0$, $|\lambda_0| = 1$. Fix an integer $m > p$ and note that the nonvanishing analytic function $z \mapsto (z-\lambda_0)^{-1}$ on $\mathbb{D}$ has an analytic m-th root f

with $|f(z)| = |z - \lambda_0|^{-1/m}$. The elementary estimate $|r\lambda - \lambda_0| > |r\lambda - r\lambda_0|$ for $r \in (0, 1)$ and $|\lambda| = 1$ yields the upper bound

$$\int_{\lambda\in\partial\mathbb{D}} |f(r\lambda)|^p d\theta \le r^{-p/m} \int_{\lambda\in\partial\mathbb{D}} |\lambda - \lambda_0|^{-p/m} d\theta.$$

Since the integral on the right is finite due to $m > p$, it follows that $f \in H^p(\mathbb{D})$. Thus, $|f(\alpha_n)| \le M\|f\|$ while $|f(\alpha_n)| = |\alpha_n - \lambda_0|^{-1/m} \to \infty$, and we arrive at a contradiction.

Now for every polynomial p one has $p(\alpha)/\|v_\alpha\| \to 0$ as $|\alpha| \to 1^-$, because p is bounded on $\mathbb{D}$ while $\|v_\alpha\| \to \infty$. This extends to all $f \in H^p(\mathbb{D})$ by density and $\|v_\alpha/\|v_\alpha\|\| = 1$. That is, one has the second convergence in the weak-star topology. □

In particular, the space $H^2(\mathbb{D}) \cong H^2$ is a *reproducing kernel* Hilbert space with kernel $K_\alpha(z) = 1/(1 - \bar{\alpha}z)$ corresponding to the evaluation functional at $\alpha \in \mathbb{D}$, where the *normalized* kernel $k_\alpha(z) = K_\alpha(z)/\|K_\alpha\| = \sqrt{1 - |\alpha|^2}/(1-\bar{\alpha}z)$ converges weakly to 0 as $|\alpha| \to 1^-$. It is of interest to note that the kernel functions are invertible elements in the disc algebra $P(\overline{\mathbb{D}})$. It is also obvious that $\{z^n : n \ge 0\}$ is an orthonormal basis.

Next we turn to the Bergman space $A^p(\mathbb{D})$ consisting of the analytic functions $f : \mathbb{D} \to \mathbb{C}$, with point-wise linear operations, such that

$$\left(\int_{\mathbb{D}} |f|^p da\right)^{1/p} = \left(\int_{r=0}^{r=1} \int_{\lambda\in\partial\mathbb{D}} |f(r\lambda)|^p d\theta \; rdr\right)^{1/p}$$

is finite and defines the norm. One can identify $A^p(\mathbb{D})$ as a closed subspace of $L^p(\mathbb{D})$. Again, the Banach space $A^p(\mathbb{D})$ is reflexive, if $1 < p < \infty$, and the polynomials in z are dense. The point evaluation functionals on $A^p(\mathbb{D})$ are bounded and satisfy the same convergence properties as on $H^p(\mathbb{D})$. Note that $H^\infty(\mathbb{D}) \subsetneq H^p(\mathbb{D}) \subsetneq A^p(\mathbb{D})$ and that functions in $A^p(\mathbb{D})$ may not possess boundary limit functions.

The space $A^2(\mathbb{D})$ is a reproducing kernel Hilbert space with kernel $K_\alpha(z) = 1/(1 - \bar{\alpha}z)^2$ at $\alpha \in \mathbb{D}$, and the normalized kernel $k_\alpha(z) = K_\alpha(z)/\|K_\alpha\| = (1 - |\alpha|^2)/(1 - \bar{\alpha}z)^2$ converges weakly to 0 as $|\alpha| \to 1^-$. The kernel functions are again invertible elements in the disc algebra $P(\overline{\mathbb{D}})$. It is also clear that $\{\sqrt{n+1}z^n : n \geq 0\}$ is an orthonormal basis.

1.11 Toeplitz, Hankel, and composition operators

We start with multiplication and composition operators on the Banach space $X = H^p(\mathbb{D})$ or $A^p(\mathbb{D})$. Every $h \in H^\infty(\mathbb{D})$ defines a bounded operator $T_h \in \mathcal{L}(X)$ by multiplication, $T_h f = hf$, $f \in X$. Conversely, a function $g : \mathbb{D} \to \mathbb{C}$ is called a *multiplier* on X if $gf \in X$ whenever $f \in X$. Obviously such $g \in X$. Less obvious is that g must be bounded.

Proposition 1.11.1. *Every multiplier on X is in $H^\infty(\mathbb{D})$.*

Proof. By definition, every multiplier g on X defines by multiplication a linear map M_g from X into X. Observe that M_g has a closed graph in $X \times X$. For, if $f_n \to f$ and $gf_n \to h$ in X, then applying the point evaluations $v_z \in X^*$ gives

$$f_n(z) \to f(z), \quad g(z)f_n(z) \to h(z), \quad z \in \mathbb{D}.$$

It follows that $h = gf = M_g f$ as desired. By the closed graph theorem then, $M_g \in \mathcal{L}(X)$ and $M_g^* \in \mathcal{L}(X^*)$. Since $M_g^* v_z = g(z)v_z$ and $v_z \neq 0$ (for $v_z(1) = 1$), taking norms in X^* gives $|g(z)| \leq \|M_g^*\|$, $z \in \mathbb{D}$. That is, g is bounded and, of course, analytic. $\square$

Recall that an isometry $T \in \mathcal{L}(X)$ is called *pure* if $\bigcap_{n=1}^\infty T^n X = \{0\}$. Purity in the following result would be immediate by a multiplicity consideration if u had a zero in $\mathbb{D}$.

Lemma 1.11.2. *For any non-constant inner function u, T_u on $H^p(\mathbb{D})$ is a pure isometry.*

Proof. Identifying $H^p(\mathbb{D})$ and H^p, we see T_u is an isometry. To show purity, let $f \in \bigcap_{n=1}^\infty T_u^n H^p$. Then $f = u^n f_n$, $f_n \in H^p$, $\|f\| = \|f_n\|$, $\forall n$. On the circle $2^{-1}\partial\mathbb{D}$, we have $M := \max_{2^{-1}\partial\mathbb{D}} |u| < 1$

and the estimates

$$\begin{aligned}\int_{\lambda\in\partial\mathbb{D}} |f(\lambda/2)|^p d\theta &= \int_{\lambda\in\partial\mathbb{D}} |u(\lambda/2)|^{np}|f_n(\lambda/2)|^p d\theta \\ &\leq M^{np}\int_{\lambda\in\partial\mathbb{D}} |f_n(\lambda/2)|^p d\theta \\ &\leq M^{np}\|f_n\|^p = M^{np}\|f\|^p.\end{aligned}$$

Sending $n\to\infty$, $\int_{\lambda\in\partial\mathbb{D}}|f(\lambda/2)|^p d\theta = 0$ implies $f\equiv 0$ on $2^{-1}\partial\mathbb{D}$ and hence on $\mathbb{D}$. □

Remark 1.11.3. For an inner function u, the orthogonal complement to the closed subspace uH^2 in H^2 is called a model space. Operator theory on model spaces is out of the scope of this thesis.

Relative to the orthonormal basis $\{z^n : n \geq 0\}$ of the Hilbert space $H^2(\mathbb{D})$, T_z is the (unilateral) shift whose self-commutator $[T_z^*, T_z] = I - T_zT_z^*$ is the rank-one projection $1\otimes 1$. On the other hand, relative to the orthonormal basis $\{\sqrt{n+1}z^n : n\geq 0\}$ of $A^2(\mathbb{D})$, T_z is a *weighted* shift for which $I - T_z^*T_z$, $I - T_zT_z^*$ are respectively represented by the diagonal matrices

$$\operatorname{diag}[1-(n+1)/(n+2)]_{n\geq 0}, \qquad \operatorname{diag}[1-n/(n+1)]_{n\geq 0}.$$

Since the entries tend to 0, both operators are compact. So, in either case, T_z is essentially unitary with essential spectrum $\sigma_e(T_z) = \partial\mathbb{D}$ and essential norm $\|T_z\|_e = \|T_z\| = 1$. It also follows that compact operators can be characterized in terms of operator norm involving T_z^n.

Proposition 1.11.4. *Let $X = H^2(\mathbb{D})$ or $A^2(\mathbb{D})$. Then $T\in\mathcal{L}(X)$ is compact if and only if $\|TT_z^n\|\to 0$ as $n\to\infty$.*

Proof. To prove the sufficient part, write

$$T = TT_z^nT_z^{n*} + T(I - T_z^nT_z^{n*}).$$

The first term $\|TT_z^nT_z^{n*}\| \leq \|TT_z^n\|\to 0$ by hypothesis, while the second term is compact due to $I - T_zT_z^*$ thus $I - T_z^nT_z^{n*}$ being compact. So, T is also compact.

Conversely, suppose T is compact but $\|TT_z^{n_k}\| > \epsilon > 0$ for a subsequence. Choose, for each k, $f_k \in X$ with $\|f_k\| = 1$ and $\|TT_z^{n_k} f_k\| > \epsilon$. Since the sequence $\{T_z^{n_k} f_k\}_k$ lie in the closed unit ball of the separable Hilbert space X, it has a subsequence, still indexed by k for notational simplicity, converging to $f \in X$ weakly. Applying the kernels $K_\alpha \in X$, $\alpha \in \mathbb{D}$, one obtains $\alpha^{n_k} f_k(\alpha) \to f(\alpha)$, while $|f_k(\alpha)| \leq \|K_\alpha\|$ implies $\alpha^{n_k} f_k(\alpha) \to 0$. Thus $T_z^{n_k} f_k \to f = 0$ weakly, and $TT_z^{n_k} f_k \to 0$ in norm since T is compact. This contradiction against $\|TT_z^{n_k} f_k\| > \epsilon > 0$ completes the proof. □

Every non-constant analytic map $\phi : \mathbb{D} \to \mathbb{D}$ defines a bounded (cf. [42]) injective operator $C_\phi \in \mathcal{L}(X)$ by composition, $C_\phi f = f \circ \phi$, $f \in X$. One immediately has the defining property $C_\phi^* v_z = v_{\phi(z)}$ for adjoints of composition operators, and the intertwining property $C_\phi T_g = T_{g \circ \phi} C_\phi$ between composition and multiplication operators. By the open mapping theorem, C_ϕ is a Banach-space isomorphism if and only if its range is closed. Composition operators on $H^p(\mathbb{D})$ by inner functions are isomorphisms. For the general case, see [28, 158] for characterizations.

Proposition 1.11.5. *For any non-constant inner function u, C_u on $H^p(\mathbb{D})$ is an isomorphism.*

Proof. Identifying $H^p(\mathbb{D})$ and H^p, $C_u f = f \circ u$ for the boundary functions f, u on $\partial\mathbb{D}$. Since

$$\int_{\partial\mathbb{D}} |f \circ u|^p d\theta = \int_{\partial\mathbb{D}} |f|^p P_{u(0)} d\theta$$

while the Poisson kernel $P_{u(0)}$ is bounded away from 0, C_u is bounded below. □

Next we introduce the Toeplitz and Hankel operators on the Hilbert spaces H^2 and $A^2(\mathbb{D})$ respectively. For $f \in L^\infty$, $M_f \in \mathcal{L}(L^2)$ denotes the multiplication operator by f on L^2 whose compression to the Hardy space H^2, $T_f := PM_f|H^2$, is the Toeplitz operator with symbol f. Let $C_{\bar{z}} \in \mathcal{L}(L^2)$ be the composition operator by the complex conjugate $z \mapsto \bar{z}$ on $\partial\mathbb{D}$, and define the Hankel operator H_f with symbol $f \in L^\infty$ to be the compression of $C_{\bar{z}} M_z M_f \in \mathcal{L}(L^2)$ to H^2

$$H_f := PC_{\bar{z}} M_z M_f |H^2 = C_{\bar{z}} M_z (I - P) M_f |H^2.$$

Note that the above definition of H_f differs from the other familiar definition using $(I-P)M_f|H^2 \in$

$\mathcal{L}(H^2, L^2 \ominus H^2)$, only by the unitary factor $U := C_{\bar{z}} M_z$ mapping $L^2 \ominus H^2$ onto H^2. Under the canonical orthonormal basis of H^2, the matrix forms of general Toeplitz and Hankel operators display distinctive patterns: Toeplitz operators have constant entries on the diagonals while Hankel operators have constant entries on the cross diagonals. Conversely, by classical results of Brown-Halmos and Nehari respectively, all infinite matrices displaying these patterns *and* inducing bounded operators on H^2 are given by L^∞ symbols. The following well-known formulas for the norm and essential norm are useful

$$\|T_f\| = \|T_f\|_e = \|f\|_\infty, \quad \|H_f\| = d(f, H^\infty), \quad \|H_f\|_e = d(f, H^\infty + C).$$

Since H^∞ is weak-star closed in L^∞, it follows from Alaoglu's theorem that the distance $d(f, H^\infty)$ for $f \in L^\infty$ is attained. Although no Douglas algebras other than H^∞ and L^∞ are weak-star closed in L^∞, $H^\infty + C$ is among a class of Douglas algebras B for which $d(f, B)$ is always attained. See [10, 148, 95].

When $f \in H^\infty$, the analytic Toeplitz operator $T_f \in \mathcal{L}(H^2)$ is subnormal with the minimal normal extension $M_f \in \mathcal{L}(L^2)$, and is unitarily equivalent to the multiplication operator by $f \in H^\infty(\mathbb{D})$ on $H^2(\mathbb{D})$. Similarly, composition operators on $H^2(\mathbb{D})$ have their unitary equivalents on H^2.

Fix an analytic self-map ϕ of $\mathbb{D}$. For every $\lambda \in \partial\mathbb{D}$, the strictly positive harmonic function

$$z \in \mathbb{D} \mapsto P_{\phi(z)}(\lambda) = \Re \frac{\lambda + \phi(z)}{\lambda - \phi(z)}$$

admits, by the Herglotz theorem, an integral representation

$$P_{\phi(z)}(\lambda) = \int_{\partial\mathbb{D}} P_z d\mu_\lambda, \quad z \in \mathbb{D}$$

for a unique finite positive measure μ_λ on $\partial\mathbb{D}$ called the Aleksandrov-Clark measure [31, 3, 29] of ϕ at $\lambda \in \partial\mathbb{D}$. Let μ_λ^s be its singular part (relative to the linear Lebesgue measure) with support set $\text{supp}\mu_\lambda^s$, and define

$$E(\phi) = \overline{\bigcup_{\lambda\in\partial\mathbb{D}} \text{supp}\mu_\lambda^s}.$$

When the boundary function satisfies $|\phi| < 1$ a.e. on $\partial\mathbb{D}$, $E(\phi)$ is known ([90], Theorem 3.1) to play a key role in determining the essential norm of products of Toeplitz operators and C_ϕ.

On the other hand, every $f \in L^\infty(\mathbb{D})$ defines a multiplication operator $M_f \in \mathcal{L}(L^2(\mathbb{D}))$, a Toeplitz operator $T_f \in \mathcal{L}(A^2(\mathbb{D}))$, and a Hankel operator $H_f \in \mathcal{L}(A^2(\mathbb{D}), L^2(\mathbb{D}) \ominus A^2(\mathbb{D}))$

$$M_f g = fg,\ g \in L^2(\mathbb{D}); \quad T_f = PM_f|A^2(\mathbb{D}); \quad H_f = (I-P)M_f|A^2(\mathbb{D})$$

where P is the orthogonal projection from $L^2(\mathbb{D})$ onto the Bergman space $A^2(\mathbb{D})$. When $f \in H^\infty(\mathbb{D})$ or $f \in h^\infty(\mathbb{D})$, one has the analytic Toeplitz operators simply as multiplications, or the extensively studied harmonic Toeplitz operators, respectively. Although the matrix forms of these operators on the Bergman space lack the patterns present on the Hardy space, Toeplitz operators are restrictions on the invariant subspace $A^2(\mathbb{D}) \subset L^2(\mathbb{D})$ of integral operators

$$\begin{aligned} T_f g(z) &= \langle Pfg, K_z\rangle = \langle fg, K_z\rangle \\ &= \int_{\mathbb{D}} f(\zeta)g(\zeta)\overline{K_z(\zeta)}da(\zeta) = \int_{\mathbb{D}} \frac{f(\zeta)g(\zeta)}{(1-z\bar{\zeta})^2}da(\zeta), \quad g \in A^2(\mathbb{D}),\ f \in L^\infty(\mathbb{D}) \end{aligned}$$

where the integral kernel

$$(z,\zeta) \in (\mathbb{D}, da) \times (\mathbb{D}, da) \mapsto \frac{f(\zeta)}{(1-z\bar{\zeta})^2}$$

has singularities on $\partial\mathbb{D} \times \partial\mathbb{D}$ with

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{1}{|1-z\bar{\zeta}|^4}da(z)da(\zeta) = \infty.$$

1.12 C*-subalgebras of the Calkin algebra with nontrivial centers

Various properties of Hilbert space operators can be effectively studied by embedding the operators in certain C*-algebras. In case these algebras have a reasonably sizable center, the Allan-Douglas localization theorem gives an isomorphism of the algebra into the direct sum of the quotient algebras over the ideals generated by the maximal ideals of any central C*-subalgebra. Although many naturally arising algebras of concrete operators have only a trivial center, mainly because they contain all the compact operators, their projections in the Calkin algebra often have nontrivial centers. Hence the localization method is applicable to these C*-subalgebras of the Calkin algebra to obtain certain *essential* properties of the operators under study. Moreover, employing the full center in the localization framework results in a *larger* number of nonetheless structurally *simpler* local (i.e. quotient) algebras than using central subalgebras does, which in general yields finer results. It is therefore desirable to identify the center itself whenever possible.

We exclusively focus in this section on operators in $\mathcal{L} = \mathcal{L}(H^2)$ of the Hardy space, where $\pi : T \in \mathcal{L} \mapsto [T] \in \mathcal{L}/\mathcal{K}$ denotes the quotient map onto the Calkin algebra on H^2. For a subset $S \subset L^\infty$, $\mathcal{T}(S)$ denotes the Toeplitz C*-subalgebra of $\mathcal{L}$ generated by the Toeplitz operators with symbol in S, $\mathcal{TH}(S)$ the Toeplitz-Hankel C*-algebra generated by $\mathcal{T}(S)$ and the Hankel operators with symbol in S, and $\mathcal{TC}(S, \Gamma)$ the Toeplitz-composition C*-algebra generated by $\mathcal{T}(S)$ and the composition operators of a collection Γ of analytic self-maps of $\mathbb{D}$. Note that if $S \supset C$, then these operator algebras contain $\mathcal{K}$ and thus have trivial centers. The rest of this section discusses the center of several Toeplitz-Hankel and Toeplitz-composition C*-subalgebras of the Calkin algebra appearing in later chapters.

It is well-known that the centers of the Calkin C*-subalgebras $\mathcal{T}(L^\infty)/\mathcal{K}$ and $\mathcal{TH}(L^\infty)/\mathcal{K}$ are respectively $\mathcal{T}(QC)/\mathcal{K}$ and $\mathcal{T}(QC_s)/\mathcal{K}$, where QC_s is the C*-subalgebra of symmetric QC functions. In either case, the isomorphism $f \in QC \mapsto [T_f] \in \mathcal{T}(QC)/\mathcal{K}$ or its restriction to QC_s allows one to naturally identify the maximal ideals of the center with those of QC or QC_s. Certainly, $\mathcal{T}(QC_s)/\mathcal{K}$ is also a central subalgebra of the non-commutative $\mathcal{TH}(PQC)/\mathcal{K}$, although it is not known whether its center is indeed larger.

Next consider $\mathcal{TC}(L^\infty, \gamma)/\mathcal{K}$ for a conformal automorphism γ of $\mathbb{D}$. The boundary function of γ is a homeomorphism of $\partial\mathbb{D}$ which induces a composition on QC. The QC functions fixed by this composition form a C*-subalgebra $QC_\gamma := \{f \in QC : f \circ \gamma = f\}$, and it is the center of $\mathcal{TC}(L^\infty, \gamma)/\mathcal{K}$. More precisely,

Proposition 1.12.1. *The center of $\mathcal{TC}(L^\infty, \gamma)/\mathcal{K}$ is naturally isomorphic to QC_γ.*

Proof. Since the commutant of $\mathcal{T}(L^\infty)/\mathcal{K}$ in the Calkin algebra is $\mathcal{T}(QC)/\mathcal{K}$ ([45]), the commutant as well as the center of $\mathcal{TC}(L^\infty, \gamma)/\mathcal{K}$ is naturally isomorphic to the C*-algebra of QC functions f satisfying $[T_f][C_\gamma] = [C_\gamma][T_f]$, hence also satisfying $[T_f][C_\gamma^*] = [C_\gamma^*][T_f]$ because $[T_f]$ is normal. However, it is known that for QC functions f and automorphisms γ

$$[C_\gamma][T_f] = [T_{f\circ\gamma}][C_\gamma].$$

Therefore, the condition becomes $[T_f][C_\gamma] = [T_{f\circ\gamma}][C_\gamma]$. Since C_γ is invertible, this in turn amounts to $f \circ \gamma = f$, that is, $f \in QC_\gamma$. $\square$

Although it is not clear whether QC_γ is nontrivial for a general automorphism γ, it is actually isomorphic to QC if γ is a *rational* rotation. See Lemma 4.5.3.

Let $PQC \subset L^\infty$ be the C*-subalgebra of *piecewise quasicontinuous* functions on $\partial\mathbb{D}$ generated by PC and QC. We now consider Toeplitz-composition C*-algebras with Toeplitz symbols in subsets of PQC and a composition symbol ϕ being a non-automorphism, linear fractional self-map of $\mathbb{D}$ with $\|\phi\|_\infty = 1$. Such ϕ must satisfy $\phi(\zeta) = \eta$ for a *unique* pair $\zeta, \eta \in \partial\mathbb{D}$, and the *Krein adjoint* (cf. [41]) ψ of ϕ is of the same type with $\psi(\eta) = \zeta$. If $\zeta = \eta$ *and* $\phi'(\zeta) = 1$, ϕ is *parabolic* and takes the explicit form

$$\phi(z) = \frac{(2-\alpha)z + \alpha\zeta}{-\alpha\bar{\zeta}z + (2+\alpha)}$$

where α, $\Re\alpha > 0$, is called the *translation number* of ϕ. Otherwise, ϕ is *non-parabolic.* In either case, $|\phi| < 1$ everywhere on $\partial\mathbb{D}$ except ζ, and the set $E(\phi)$ associated with the Aleksandrov-Clark measures of ϕ is the singleton $\{\zeta\}$. Thus, the following lemma is a special case of Corollary 2.2

in [88]. In particular, C_ϕ is not compact.

Lemma 1.12.2. *Let ϕ be a non-automorphism, linear fractional self-map of $\mathbb{D}$ satisfying $\phi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Then for every $f \in L^\infty$ continuous at ζ, $T_f C_\phi$ is compact if and only if $f(\zeta) = 0$.*

Recall that a function $f \in L^\infty$ is said to be continuous at $\zeta \in \partial\mathbb{D}$ with $f(\zeta) = c$ if $\lim_{\epsilon\to 0} \|f|(\zeta - \epsilon, \zeta + \epsilon) - c\|_\infty = 0$, or equivalently $f|M_\zeta(L^\infty) \equiv c$, for some constant c. It follows from the preceding lemma that for every $f \in L^\infty$ continuous at both ζ and η,

$$[T_f][C_\phi] = f(\zeta)[C_\phi], \qquad [T_{\bar{f}}][C_\psi] = \overline{f(\eta)}[C_\psi].$$

Taking adjoints of the latter equality yields $[C_\psi^*][T_f] = f(\eta)[C_\psi^*]$. Since ϕ is the Krein adjoint of ψ, $[C_\psi^*] = [C_\phi]/s$ where $s = |1/\phi'(\zeta)| > 0$ ([88], Theorem 3.1, 3.6). Therefore, one has as in [88]

$$\begin{aligned}
[T_f][C_\phi] &= f(\zeta)[C_\phi], \qquad [C_\phi][T_f] = f(\eta)[C_\phi];\\
[T_f][C_\psi] &= f(\eta)[C_\psi], \qquad [C_\psi][T_f] = f(\zeta)[C_\psi].
\end{aligned}$$

Let $PQC(\zeta) = \{f \in PQC : f \text{ is continuous at } \zeta\}$. That is, the C*-subalgebra $PQC(\zeta)$ consists of the functions $f \in PQC$ of which the Gelfand transforms are constant on the fiber $M_\zeta(PQC)$.

Let ϕ be parabolic. Since C_ϕ is essentially normal [20] and $\mathcal{T}(PQC)/\mathcal{K}$ is commutative [120], it follows from the equalities above that $\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K}$ is commutative.

On the other hand, there are two different cases for a non-parabolic ϕ. That is, $\zeta \neq \eta$ in $\phi(\zeta) = \eta$, or $\phi(\zeta) = \zeta = \eta$ but $\phi'(\zeta) \neq 1$. For the first case, define $PQC(\zeta, \eta) := PQC(\zeta) \bigcap PQC(\eta)$, $PQC_{\zeta,\eta} := \{f \in PQC(\zeta, \eta) : f(\zeta) = f(\eta)\}$. By [20], neither $\mathcal{TC}(PQC(\zeta, \eta), \phi)/\mathcal{K}$ in the first case nor $\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K}$ in the second case is commutative. Their centers are not known, so it is of interest to identify large enough central C*-subalgebras.

In the first case, reasoning as above yields that $[T_f]$ for every $f \in PQC_{\zeta,\eta}$ lies in the center of $\mathcal{TC}(PQC(\zeta, \eta), \phi)/\mathcal{K}$. Moreover, since $[C_\phi^2] = [C_{\phi\circ\phi}] = 0$ due to $E(\phi \circ \phi) = \emptyset$, one directly verifies that $[C_\phi C_\phi^* + C_\phi^* C_\phi] = s[C_\phi C_\psi + C_\psi C_\phi]$ also lies in the center. Therefore, the relatively large Calkin C*-subalgebra generated by $\{[C_\phi C_\phi^* + C_\phi^* C_\phi], [T_f] : f \in PQC_{\zeta,\eta}\}$ is central.

For the second case, one no longer has $[C_\phi^2] = 0$ since ϕ fixes $\zeta \in \partial\mathbb{D}$, and it seems impossible to introduce an element generated by $[C_\phi]$ in the center of $\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K}$. It is not known if there exists a central C*-subalgebra larger than $\mathcal{T}(PQC(\zeta))/\mathcal{K}$, which hinders an effective application of the localization method in this case. However, essential spectra and Fredholm indices of a class of operators will be determined via a different approach in Subsection 4.6.2.

Chapter 2

Analytic Multiplication Operators over Planar Regions

2.1 Overview

A planar region is understood to be a nonempty connected open subset of the complex plane $\mathbb{C}$. Let Ω always denote in this chapter a bounded planar region without assuming finite connectivity nor boundary regularity, and let $\mathbb{D}$ denote the open unit disc. Fix a power index p, $p \in [1, \infty)$ unless otherwise specified.

Let w be a nonnegative function in $L^1(\Omega, da)$, da being the Lebesgue area measure on Ω, such that on every $K \subset \Omega$ compact,

$$\int_K w^{-s} da < \infty \tag{2.1.1}$$

for some $s = s(K) \in (0, \infty)$. For example, condition (2.1.1) holds for moduli of nonvanishing continuous functions or nonzero analytic functions on Ω. The collection $\mathcal{W}(\Omega)$ of all such functions w form a convex cone. The space $A^p(\Omega, wda)$ of analytic functions h on Ω satisfying

$$\|h\| := \left(\int_\Omega |h|^p w da \right)^{1/p} < \infty \tag{2.1.2}$$

is referred to as the weighted Bergman space with weight $w \in \mathcal{W}(\Omega)$. Some special cases are the weighted Bergman spaces $A^p_r(\mathbb{D})$ with standard Bergman weights $w_r(z) = (1+r)(1-|z|^2)^r$, $r > -1$, and the spaces $A^2(\mathbb{D}, |q^2|da)$ for polynomials q arising in the (one-variable) essential normality problem of cyclic Bergman submodules ([53], Sec. 4.1).

Following [112] (also [55, Section 10.5]), the Hardy space $H^p(\Omega, \omega)$ for a fixed reference point

$\omega \in \Omega$ consists of analytic functions h on Ω for which the subharmonic function $|h|^p$ admits a harmonic majorant on Ω. Equipped with the norm

$$\|h\| := u(\omega)^{1/p} \tag{2.1.3}$$

where u is the least harmonic majorant for $|h|^p$, $H^p(\Omega, \omega)$ is a Banach space, and a Hilbert space if $p = 2$.

Let $H^\infty(\Omega)$ be the Banach algebra of bounded analytic functions on Ω equipped with the sup-norm $\|.\|_\infty$. Consider the Banach space (Lemma 2.2.1)

$$X = A^p(\Omega, wda) \text{ or } H^p(\Omega, \omega).$$

It is clear that $H^\infty(\Omega) \subset X$, and that every $f \in H^\infty(\Omega)$ defines a multiplication operator $T_f \in \mathcal{L}(X)$, $T_f h = fh$, $h \in X$. (By Proposition 1.11.1 and Lemma 2.2.1, $H^\infty(\Omega)$ is indeed the multiplier algebra of X.) These operators over general regions were studied in [9] on $A^p(\Omega, da)$, [34] on $H^p(\Omega, \omega)$, [12] on $A^2(\Omega, da)$, and so on.

After establishing some basic lemmas in Section 2.2, a characterization of the commutant $\{T_f\}'$ is generalized in Section 2.3 from $H^2(\mathbb{D})$ [36] to X, from which the commutant inclusion result of [36] follows. This characterization and an explicit description of the annihilator $(T_f X)^\perp$ serve as the starting point in a duality approach to the commutant problems treated in this paper. For any univalent symbol ξ, $\{T_\xi\}'$ is shown to consist only of multiplication operators, with a partial converse. This result is then generalized to multiple operators, which is of interest in view of Theorems 2.6.5 and 2.7.10. Let b be a finite Blaschke product with degree n. Despite the unitary equivalence $T_b \cong \bigoplus_n T_z$ on the Hardy space $H^2(\mathbb{D})$ [47], the similarity $T_b \sim \bigoplus_n T_z$ on the weighted Bergman space $A^2_r(\mathbb{D})$ [85], and the Riemann surface representations [36, 71, 52] for operators in the commutant $\{T_b\}'$, there are no explicit global characterizations of $\{T_b\}'$ (except for $n = 2$ on $H^2(\mathbb{D})$ [135]). On the other hand, deep results on the reducing subspaces for T_b on the Bergman space $A^2(\mathbb{D})$ are obtained in [154, 74, 71, 52, 51]. We give a sufficient condition in terms of boundary behavior for irreducibility of a k-tuple $\{T_{b_1}, ..., T_{b_k}\}$ on X, and leave more results about $\{T_b\}'$ on $A^2(\mathbb{D}, wda)$ to Section 2.7. X-ancestral symbol functions are taken up following C. C. Cowen's original concept [36]. It is proved that T_f commutes with no nonzero compact operators if the cluster set of the X-ancestral f does not exhaust its range.

Reducing subspaces of T_{z^n} on $A^2_r(\mathbb{D})$ are obtained in [155, 125]. Over an annulus $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$, $H^2(R)$ and $A^2(R)$ have the orthonormal bases $\{c_k z^k : k \in \mathbb{Z}\}$ for which T_{z^n} is a weighted n-step bilateral shift. The structure allows for characterizations of $\{T_{z^n}\}'$ and its reducing subspaces [50]. These results motivate the treatment in Section 2.4 of $\{T_{z^n}\}'$ on the Banach space X over R where a matching direct-sum decomposition is not available. We characterize $\{T_{z^n}\}'$ and show it contains a nontrivial idempotent thus making T_{z^n} reducible, while T_{z^n} does not commute with nonzero compact operators. This leads to the result that $\{T_{z^{n_1}}, ..., T_{z^{n_k}}\}$ is reducible if and only if $\gcd\{n_1, ..., n_k\} > 1$. Consequently, if S denotes the restriction to a certain invariant subspace of the analytic Toeplitz operator $T_{\pi_\omega} \in \mathcal{L}(H^p(\partial\mathbb{D}))$ of the covering map π_ω from $\mathbb{D}$ onto R, then the k-tuple of iterates $\{S^{n_1}, ..., S^{n_k}\}$ is reducible if and only if $\gcd\{n_1, ..., n_k\} > 1$.

The norm and sequential weak closures of the algebra of operators obtained from T_f by the analytic functional calculus are characterized in Section 2.5, with mapping properties obtained for the essential spectrum and Browder's essential spectrum of operators in the norm closure. On unweighted Bergman and Hardy spaces Axler [9] and Conway [34] actually identify the essential spectrum. However, such results are not available in the literature for weighted spaces nor for Browder's essential spectrum.

Section 2.6 concerns commutant problems on the Hilbert space $H^2(\mathbb{D})$. It has been an important problem to find symbol conditions under which the commutant of analytic Toeplitz operators equals that of the operator defined by *some* inner function. Only sufficient conditions of varied strength are known ([14, 134, 136, 137, 36]), and under these conditions the inner function is a finite Blaschke product. Instead, we obtain a sufficient and necessary condition for the commutant of a family of analytic Toeplitz operators on $H^2(\mathbb{D})$ to equal that of T_ϕ for a *given* inner function ϕ.

The last section treats the corresponding problems on the Hilbert space $A^2(\mathbb{D}, wda)$ for w in a subclass $\mathcal{W}_d(\mathbb{D}) \subset \mathcal{W}(\mathbb{D})$. After generalizing from $A^2_r(\mathbb{D})$ to $A^2(\mathbb{D}, wda)$ the similarity $T_b \sim \bigoplus_n T_z$ [85, 71] for a finite Blaschke product b, we obtain various results on $\{T_b\}'$ using the approach in Section 2.6. The added generality beyond the standard Bergman weights seems justified by these results.

For a bounded linear operator $T \in \mathcal{L}(E)$ on a Banach space E with dual space E^*, $\sigma(T)$ denotes the spectrum and $\rho(T)$ its radius, $\sigma_e(T)$ the essential spectrum, and $T^* \in \mathcal{L}(E^*)$ the adjoint operator. For a subset $\mathcal{O} \subset \mathcal{L}(E)$ of operators, the commutant is $\mathcal{O}' := \{T \in \mathcal{L}(E) : TS = ST, \forall S \in \mathcal{O}\}$, and the essential commutant is $\mathcal{O}'_e := \{T \in \mathcal{L}(E) : TS - ST \in \mathcal{K}(E), \forall S \in \mathcal{O}\}$ where $\mathcal{K}(E)$ is the ideal of compact operators on E. For a subset $W \subset E$, $W^\perp \subset E^*$ denotes its annihilator.

2.2 Preliminary lemmas

We first need a key estimate which in particular implies that $A^p(\Omega, wda)$ is a Banach space, and that the point evaluation functionals over compact subsets of Ω are uniformly bounded in the dual space $A^p(\Omega, wda)^*$.

Lemma 2.2.1. *Fix $w \in \mathcal{W}(\Omega)$. Then for every $K \subset \Omega$ compact, there exists a finite constant C_K such that*

$$\sup_K |h| \leq C_K \|h\|, \quad \forall h \in A^p(\Omega, wda). \tag{2.2.1}$$

Proof. Choose $\epsilon = \epsilon(K) > 0$ such that $\overline{K^\epsilon} \subset \Omega$, and let $s = s(\overline{K^\epsilon}) \in (0, \infty)$ be the corresponding exponent as in (2.1.1) for the compact $\overline{K^\epsilon}$. Set

$$r = \frac{ps}{1+s} \in (0, \infty).$$

Then $p/r = 1 + s^{-1}$ with conjugate index $(p/r)' = 1 + s$. Note that

$$(r/p)(p/r)' = s.$$

Also, $w \in (0, \infty)$ $a.e.$ on $\overline{K^\epsilon}$ so that $w^{r/p} w^{-r/p} = 1$ $a.e.$ Using polar coordinates to integrate the subharmonic function $|h|^r$ on $\overline{\mathbb{D}}(z, \epsilon)$, $z \in K$, which satisfies

$$|h(z)|^r \leq \int_{\lambda \in \partial \mathbb{D}} |h(z + \delta\lambda)|^r d\theta, \; 0 \leq \delta \leq \epsilon,$$

and by Hölder's inequality with the conjugate indices p/r and $(p/r)'$, one deduces

$$\begin{aligned}
&\sup_K |h|^r \leq \frac{1}{\pi\epsilon^2} \int_{\overline{K^\epsilon}} |h|^r da = \frac{1}{\pi\epsilon^2} \int_{\overline{K^\epsilon}} |h|^r w^{r/p} w^{-r/p} da \\
&\leq \frac{1}{\pi\epsilon^2} \left(\int_{\overline{K^\epsilon}} |h|^p w da \right)^{r/p} \left(\int_{\overline{K^\epsilon}} w^{-(r/p)(p/r)'} da \right)^{[(p/r)']^{-1}} \\
&= \frac{1}{\pi\epsilon^2} \left(\int_{\overline{K^\epsilon}} |h|^p w da \right)^{r/p} \left(\int_{\overline{K^\epsilon}} w^{-s} da \right)^{r/(sp)} \\
&\leq \frac{1}{\pi\epsilon^2} \|h\|^r \left[\left(\int_{\overline{K^\epsilon}} w^{-s} da \right)^{1/(sp)} \right]^r,
\end{aligned}$$

which yields (2.2.1) by letting

$$C_K = (\pi\epsilon^2)^{-1/r} \left(\int_{\overline{K^\epsilon}} w^{-s} da \right)^{1/(sp)} < \infty.$$

This completes the proof for any $h \in A^p(\Omega, wda)$. □

Actually, we have shown more than (2.2.1), in that the weighted L^p norm can be replaced by the integral over a certain subset of Ω containing K. This observation together with the maximum modulus principle will be used next to show $w1_{\Omega \setminus K} da$ for any compact $K \subset \Omega$ is a reverse Carleson measure for the space $A^p(\Omega, wda)$, a fact known [94] for $A^p_r(\mathbb{D})$.

Lemma 2.2.2. *Fix $w \in \mathcal{W}(\Omega)$. Then for every $K \subset \Omega$ compact, there exists a finite constant D_K such that for every analytic function h in Ω,*

$$\int_\Omega |h|^p w da \leq D_K \int_{\Omega \setminus K} |h|^p w da. \tag{2.2.2}$$

Proof. Let Γ be an envelope of K in Ω, and choose $\epsilon > 0$ such that $\overline{\Gamma^\epsilon} \subset \Omega \setminus K$. Then it follows from the previous proof that for every analytic function h in Ω,

$$\sup_\Gamma |h| \le C_\Gamma \left(\int_{\overline{\Gamma^\epsilon}} |h|^p w da \right)^{1/p} \le C_\Gamma \left(\int_{\Omega \setminus K} |h|^p w da \right)^{1/p}.$$

With the maximum modulus principle, this implies

$$\int_K |h|^p w da \le (\sup_\Gamma |h|)^p \int_K w da \le (C_\Gamma)^p \int_{\Omega \setminus K} |h|^p w da \int_\Omega w da,$$

from which (2.2.2) follows by choosing $D_K = 1 + (C_\Gamma)^p \int_\Omega w da < \infty$. $\square$

The next lemma collects some basic properties of the point evaluation functionals v_α, $\alpha \in \Omega$, on X and those of X itself.

Lemma 2.2.3. *Let v_α be the point evaluation functional on X at $\alpha \in \Omega$. Then,*

(i) *The dual Banach space valued function $\alpha \in \Omega \mapsto v_\alpha \in (X^*, \|.\|)$ is analytic, and its n-th order derivative $v_\alpha^{(n)} \in X^*$ equals the n-th order derivative functional at α. That is, $v_\alpha^{(n)} h = h^{(n)}(\alpha)$ for $h \in X$.*

(ii) *The subset $\{v_\alpha, v_\alpha^{(n)} : \alpha \in \Omega, n \in \mathbb{N}\} \subset X^*$ is linearly independent.*

(iii) *For $\alpha \in \Omega$ and a set $\Theta \subset \Omega$ having an accumulation point in Ω, the linear spans $\mathrm{sp}\{v_\alpha, v_\alpha^{(n)} : n \in \mathbb{N}\}$, $\mathrm{sp}\{v_\beta : \beta \in \Theta\}$ are weak-star dense.*

(iv) *For $p \in (1, \infty)$, X is reflexive and separable.*

Proof. (i). By Lemma 2.2.1 and respectively [112] Lemma 2.3, $v_\alpha \in X^*$. For each $h \in X$, the function $\alpha \in \Omega \mapsto v_\alpha h = h(\alpha) \in \mathbb{C}$ is analytic. So, by completeness of X and a classical result [78, Theorem 3.10.1], the function $\alpha \mapsto v_\alpha$ is norm-analytic. Consider an arbitrary $h \in X$, and we have

$$v'_\alpha h = (\lim_{\beta \to 0} (v_{\alpha+\beta} - v_\alpha)/\beta) h = \lim_{\beta \to 0} (h(\alpha + \beta) - h(\alpha))/\beta = h'(\alpha).$$

The proof will be complete after an induction on the order of differentiation.

(ii). Let α_j, $1 \le j \le k$, be distinct points in Ω such that

$$\sum_{j=1}^{k}\left(\lambda_j v_{\alpha_j} + \sum_{n=1}^{n_j}\lambda_{j,n} v_{\alpha_j}^{(n)}\right) = 0 \tag{2.2.3}$$

for scalars $\lambda_j, \lambda_{j,n}$. It suffices to show all these scalars vanish. Fix an arbitrary l, $1 \le l \le k$. For each m, $0 \le m \le n_l$, define the polynomial

$$p_m(z) = (z-\alpha_l)^m \prod_{j \ne l}(z-\alpha_j)^{n_j+1}.$$

Then $p_m \in H^\infty(\Omega) \subset X$. Applying (2.2.3) on p_m yields

$$\lambda_l p_m(\alpha_l) + \sum_{n=1}^{n_l}\lambda_{l,n} p_m^{(n)}(\alpha_l) = 0.$$

Letting m decrease from n_l to 0, we iteratively deduce that $\lambda_{l,n}$, $n_l \ge n \ge 1$, and λ_l all vanish. This completes the proof.

(iii). Weak-star density of the spans is ensured by the Hahn-Banach theorem and the identity theorem implying that

$$\bigcap_{n\in\mathbb{N}} \ker v_\alpha^{(n)} \bigcap \ker v_\alpha = \bigcap_{\beta\in\Theta} \ker v_\beta = \{0\}.$$

(iv). X is uniformly convex due to it being isometrically isomorphic to closed subspaces of certain L^p-spaces, $p \in (1,\infty)$, over Ω or $\partial\mathbb{D}$. Reflexivity follows from uniform convexity by the Milman-Pettis theorem. So, the weak-star and weak topologies on X^* coincide, and the span $\mathrm{sp}\{v_\alpha, v_\alpha^{(n)} : n \in \mathbb{N}\}$ in (iii) is dense in the weak and norm topologies. That is, X^* and X are both separable. □

Remark 2.2.4. Norm-continuity of the function $\alpha \mapsto v_\alpha$ for $X = H^p(\Omega,\omega)$ was shown in [34] Prop. 1.5. Boundedness of the n-th order derivative functionals on X can also be proved directly by the Cauchy integral formula with Lemma 2.2.1 and respectively [112] Lemma 2.3.

Denote by $Z(f)$ the zero set of an analytic function $f \not\equiv 0$ in Ω and by $m_f(\alpha) \in \mathbb{N}$ the multiplicity of a zero $\alpha \in Z(f)$. The *cluster set* $f(\partial\Omega)$ of $f \in H^\infty(\Omega)$ *at the boundary* $\partial\Omega$ is defined as

$$f(\partial\Omega) := \bigcap_{\epsilon>0} \overline{f((\partial\Omega)^\epsilon \cap \Omega)} = \bigcap\{\overline{f(\Omega \setminus K)} : K \subset \Omega \text{ compact}\}. \tag{2.2.4}$$

Identifying Ω with its canonical embedding in $M(H^\infty(\Omega))$, one has the inclusions

$$\overline{f(\Omega)} \setminus f(\Omega) \subset f(\partial\Omega) \subset \hat{f}(M(H^\infty(\Omega)) \setminus \Omega) \tag{2.2.5}$$

where the latter is indeed an equality if $\Omega \hookrightarrow M(H^\infty(\Omega))$ is dense (i.e. the corona theorem holds for $H^\infty(\Omega)$). If f has a continuous extension on $\overline{\Omega}$, then $f(\partial\Omega)$ coincides with the image of $\partial\Omega$ under the extension. We are interested in the case $0 \notin f(\partial\Omega)$.

Lemma 2.2.5. *Let $f \in H^\infty(\Omega)$ with $0 \notin f(\partial\Omega)$. Then $Z(f)$ is finite and*

$$T_f X = \bigcap\{\mathrm{ker} v_\alpha, \mathrm{ker} v_\alpha^{(n)} : \alpha \in Z(f), 1 \leq n \leq m_f(\alpha) - 1\}, \tag{2.2.6}$$

$$(T_f X)^\perp = \mathrm{sp}\{v_\alpha, v_\alpha^{(n)} : \alpha \in Z(f), 1 \leq n \leq m_f(\alpha) - 1\}, \tag{2.2.7}$$

$$T_f \text{ is bounded below.} \tag{2.2.8}$$

Proof. By (2.2.4), $0 \notin f(\partial\Omega)$ implies that $0 \notin \overline{f(\Omega \setminus K)}$ for some $K \subset \Omega$ compact. In particular, $Z(f) \subset K$ must be finite. Let

$$\epsilon := \inf\{|f(z)| : z \in \Omega \setminus K\} > 0$$

and write N for the intersection of the kernels in (2.2.6). It suffices to establish $N \subset T_f X$. For then $T_f X = N$, hence (2.2.7) holds due to finiteness of the set of the functionals, and (2.2.8) holds by the open mapping theorem and injectivity of T_f. To that end fix an $h \in N$. That is, $h \in X$ has a zero at every $\alpha \in Z(f)$ and

$$m_h(\alpha) \geq m_f(\alpha).$$

It follows that there exists an analytic function g on Ω such that $h = fg = T_f g$. It remains to show $g \in X$.

Suppose $X = A^p(\Omega, wda)$. Since

$$\int_K |g|^p wda \le \left(\max_K |g|\right)^p \int_\Omega wda < \infty \text{ and}$$

$$\int_{\Omega\setminus K} |g|^p wda \le \int_{\Omega\setminus K} \frac{|h|^p}{\epsilon^p} wda \le \frac{1}{\epsilon^p}\int_\Omega |h|^p wda < \infty,$$

one has $g \in A^p(\Omega, wda)$. Next suppose $X = H^p(\Omega, \omega)$. If u is a harmonic majorant for $|h|^p$ on Ω, then $u_1 := u/\epsilon^p$ is a harmonic majorant for $|g|^p$ so that $g \in H^p(\Omega, \omega)$. For, clearly

$$|g|^p = |h|^p/|f|^p \le u/\epsilon^p = u_1$$

on $\Omega \setminus K$. On the other hand let Γ be an envelope of K in Ω. Since Γ is contained in $\Omega \setminus K$ on which the subharmonic function $|g|^p \le u_1$, one as well has $|g|^p \le u_1$ in the enclosure of Γ which contains K. That is $|g|^p \le u_1$ on Ω. The proof is complete. □

Remark 2.2.6. Since $\lambda \notin f(\partial\Omega) \Leftrightarrow 0 \notin (f-\lambda)(\partial\Omega)$ and $\ker T_{f-\lambda} = \{0\}$ for $f - \lambda \not\equiv 0$, Lemma 2.2.5 implies $\sigma_e(T_f) \subset f(\partial\Omega)$. In fact, Axler [9] on unweighted $A^p(\Omega, da)$ and Conway [34] on $H^p(\Omega,\omega)$ have identified $\sigma_e(T_f)$ as the cluster set of f at certain subsets of $\partial\Omega$. It also follows from Lemma 2.2.5 and 2.2.3(ii)-(iv) that if $f(\alpha_0) \notin f(\partial\Omega)$ for some $\alpha_0 \in \Omega$, then the adjoint T_f^* on the separable Hilbert space X (when $p = 2$) is in the Cowen-Douglas class $\mathcal{B}_n(\mathbb{D}(\bar{f}(\alpha_0), \delta))$ [43], with index $n = \sum\{m_{f-f(\alpha_0)}(\alpha) : \alpha \in Z(f - f(\alpha_0))\}$ and some $\delta > 0$ small, so that the von Neumann algebra $\{T_f, T_f^*\}'$ is finite dimensional [52, 51]. Lastly, for $\Omega = \mathbb{D}$, $0 \notin f(\partial\mathbb{D}) = \hat{f}(M(H^\infty)\setminus\mathbb{D}) = \sigma(f, H^\infty + C)$ if and only if f has its inner factor a finite Blaschke product and outer factor an invertible function.

We close this section with a lemma on finite Blaschke products

$$b(z) = \prod_{k=1}^{n} \frac{\zeta_k - z}{1 - \bar{\zeta}_k z}, \ \zeta_k \in \mathbb{D}, \quad n =: \deg(b) \ge 1, \tag{2.2.9}$$

defined on $\Omega = \mathbb{D}$ and extended to rational functions with poles off $\overline{\mathbb{D}}$. Finite Blaschke products are precisely those functions in the disc algebra with unimodular boundary values. Recall a key fact (cf. [71], p. 359) that the derivatives b' are nonvanishing on $\partial\mathbb{D}$.

Lemma 2.2.7. *Let $b_1, ..., b_k$ be finite Blaschke products on $\mathbb{D}$ that separate points on $\partial\mathbb{D}$. Then there exists $t \in (0,1)$ such that*

$$\bigcap_{j=1}^{k} Z(b_j - b_j(\alpha)) = \{\alpha\}, \ t < |\alpha| < 1.$$

Proof. Suppose not. Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{D}$, such that $|\alpha_n| \to 1$ and

$$b_j(\beta_n) = b_j(\alpha_n) \text{ while } \beta_n \neq \alpha_n \tag{2.2.10}$$

for all j and n. After passing to subsequences, we may take that

$$\alpha_n \to \alpha_0 \text{ and } \beta_n \to \beta_0 \text{ in } \overline{\mathbb{D}}.$$

By continuity of b_j, one has by (2.2.10) that

$$b_j(\beta_0) = b_j(\alpha_0), \ \forall j. \tag{2.2.11}$$

Since clearly $|\alpha_0| = 1$, one has in particular $|b_1(\beta_0)| = |b_1(\alpha_0)| = 1$ so that $|\beta_0| = 1$. Now we claim $\alpha_0 \neq \beta_0$, which would create a contradiction between (2.2.11) and the separation hypothesis on $\partial\mathbb{D}$, and hence complete the proof.

Suppose otherwise $\alpha_0 = \beta_0$, that is, $\alpha_n \to \alpha_0$ and $\beta_n \to \alpha_0$. Since $b_1'(\alpha_0) \neq 0$ at $\alpha_0 \in \partial\mathbb{D}$, b_1 is locally univalent at α_0. Therefore $b_1(\beta_n) \neq b_1(\alpha_n)$ for n large, due to $\beta_n \neq \alpha_n$. This contradiction against (2.2.10) asserts the claim. □

2.3 Commutants over general regions

A *vector-valued functional Banach space* F is a Banach space of V-valued, V a normed linear space, functions on a set Z with point-wise linear operations, such that point evaluations on

F to V are bounded. A subset $Y \subset Z$ is called a set of uniqueness for F if $f \in F$ vanishing on Y implies $f \equiv 0$. The Fundamental Lemma in [36] (p.3) characterizes the commutant of a multiplication operator on $H^2(\mathbb{D})$ and underlies most results in this direction, e.g. the explicit representation of the commutants of multiplication operators by covering maps ([36], Thm. 7) or semiautomorphic functions ([39], Thm. 2.1). This simple lemma is valid on vector-valued functional Banach spaces with essentially the same proof.

Proposition 2.3.1. *Let F be a V-valued functional Banach space on Z with point evaluations v_α, $\alpha \in Z$. Let $g : Z \to \mathbb{C}$ be such that the multiplication operator $T_g f = gf$ is in $\mathcal{L}(F)$. For $\alpha \in Z$ and $x^* \in V^*$ put $w_{\alpha,x^*} := x^* \circ v_\alpha \in F^*$. Then for $S \in \mathcal{L}(F)$ the following are equivalent*

(i) *$ST_g = T_g S$.*

(ii) *$S^* w_{\alpha,x^*}$ annihilates $(g - g(\alpha))F$ for every $\alpha \in Z$ and $x^* \in V^*$.*

(iii) *$S^* w_{\alpha,x^*}$ annihilates $(g - g(\alpha))F$ for every α in a set Y of uniqueness for F and $x^* \in V^*$.*

Proof. (i)$\Rightarrow$(ii). $ST_g = T_g S$ implies $S^*(T_g - g(\alpha)I)^* = (T_g - g(\alpha)I)^* S^*$, so that $\ker(T_g - g(\alpha)I)^*$ is invariant for S^*, $\forall \alpha \in Z$. Since

$$w_{\alpha,x^*} \in \ker(T_g - g(\alpha)I)^*, \tag{2.3.1}$$

$S^* w_{\alpha,x^*} \in \ker(T_g - g(\alpha)I)^*$ annihilates $\operatorname{ran}(T_g - g(\alpha)I) = (g - g(\alpha))F$.

(iii)$\Rightarrow$(i). For every $f \in F$, $\alpha \in Y$, $x^* \in V^*$,

$$0 = (S^* w_{\alpha,x^*})(gf - g(\alpha)f) = x^*(S(gf)(\alpha) - g(\alpha)(Sf)(\alpha)) = x^*((ST_g f)(\alpha) - (T_g S f)(\alpha)).$$

By the Hahn-Banach theorem on the normed linear space V, $ST_g f - T_g S f \in F$ vanishes on Y. So, $ST_g f = T_g S f$ and $ST_g = T_g S$. $\square$

For multiplication operators by functions in $H^\infty(\Omega)$ on vector-valued functional Banach spaces on Ω, commutant inclusion under analytic composition follows from the above proposition exactly as the corollaries in [36, pp. 3-4]. More precisely, we have

Corollary 2.3.2. *Let F be a vector-valued functional Banach space on Ω. Suppose for every $h \in H^\infty(\Omega)$ the multiplication operator $T_h f = hf$ is in $\mathcal{L}(F)$. Then for $h \in H^\infty(\Omega)$ nonconstant and $g \in H^\infty(h(\Omega))$, one has $\{T_h\}' \subset \{T_{g\circ h}\}'$. If $g \in H^\infty(h(\Omega))$ is univalent, then $\{T_h\}' = \{T_{g\circ h}\}'$.*

Proof. The argument for the first part using Proposition 2.3.1 rests on the observation that $(g \circ h - (g \circ h)(\alpha))F \subset (h - h(\alpha))F$, due to a factorization ([36], p.4) and the hypothesis that $T_h \in \mathcal{L}(F)$ for every $h \in H^\infty(\Omega)$. For the second part, one has $h = g^{-1} \circ (g \circ h)$, where the inverse map $g^{-1} \in H^\infty((g \circ h)(\Omega))$. Thus, $\{T_{g\circ h}\}' \subset \{T_h\}'$ while $\{T_h\}' \subset \{T_{g\circ h}\}'$, both by the first part. $\square$

Remark 2.3.3. The first conclusion of Corollary 2.3.2 is equivalent to

$$\{T_h\}'' \supset \{T_{g\circ h} : g \in H^\infty(h(\Omega))\}.$$

We note in passing that the representation $h \mapsto T_h$ of the algebra $H^\infty(\Omega)$ on F is necessarily continuous by the closed graph theorem applied to the complete spaces $H^\infty(\Omega)$ and $\mathcal{L}(F)$. Also, in view of the second part of Corollary 2.3.2, one may ask if $\{T_h\}' = \{T_{h\circ\psi}\}'$ for $h \in H^\infty(\Omega)$ and automorphisms $\psi \in \mathrm{Aut}(\Omega)$. However, neither inclusion is true in general.

Example 2.3.4. Let $h(z) = z^2 \in H^\infty(\mathbb{D})$ and $\psi(z) = (\alpha - z)/(1 - \bar{\alpha}z) \in \mathrm{Aut}(\mathbb{D})$ where $0 \neq \alpha \in \mathbb{D}$. Let $X = H^p(\mathbb{D})$ or $A^p_r(\mathbb{D})$, and $C_{-1} \in \mathcal{L}(X)$ be the composition operator defined by $(C_{-1}f)(z) = f(-z)$. Evidently, $C_{-1} \in \{T_{z^2}\}'$. Suppose $C_{-1} \in \{T_{\psi^2}\}'$ as well. Then, $T_{\psi^2}C_{-1} = C_{-1}T_{\psi^2} = T_{\psi^2(-z)}C_{-1}$ gives $\psi^2(z) \equiv \psi^2(-z)$, while the latter is false by letting $z = \alpha$. Thus, $\{T_h\}' \not\subset \{T_{h\circ\psi}\}'$. Taking $g = h \circ \psi$, $\phi = \psi^{-1}$, this also means $\{T_{g\circ\phi}\}' \not\subset \{T_g\}'$.

It is clear that the preceding results are valid for T_h, $h \in H^\infty(\Omega)$, on X. Proposition 2.3.1 and Lemma 2.2.5 serve as the starting point in the duality approach to commutant problems.

It was proved in [2, Lemma 1.8] that for Ω finitely connected with regular boundary, the commutant $\{T_z\}'$ on $H^2(\Omega, \omega)$ consists only of multiplication operators, which also holds on $A^2(\Omega, da)$. We prove this result without any assumption on Ω.

Theorem 2.3.5. $\{T_z\}' = \{T_f : f \in H^\infty(\Omega)\}$ *on X. More generally,* $\{T_\xi\}' = \{T_f : f \in H^\infty(\Omega)\}$ *for every univalent function $\xi \in H^\infty(\Omega)$.*

Proof. Fix $S \in \{T_z\}'$. Since $z(\partial\Omega) = \partial\Omega \not\ni \alpha$ for $\alpha \in \Omega$, it follows from Lemma 2.2.5 that $((z-\alpha)X)^\perp = \mathbb{C}v_\alpha$, so that Proposition 2.3.1 yields

$$S^* v_\alpha = f(\alpha) v_\alpha \tag{2.3.2}$$

at every $\alpha \in \Omega$ for some scalar function $f : \Omega \to \mathbb{C}$. Hence

$$|f(\alpha)| \|v_\alpha\| \leq \|S^*\| \|v_\alpha\|.$$

Because $v_\alpha \neq 0$, f is bounded by $\|S^*\| = \|S\| < \infty$. Set $g := S1 \in X$. Applying (2.3.2) on the function $\phi \equiv 1$ gives $g(\alpha) = f(\alpha)$ over Ω, so $f = g \in X$ is analytic on Ω. That is, $f \in H^\infty(\Omega)$ and the multiplication operator T_f is well-defined on X. Now apply (2.3.2) on an arbitrary $\phi \in X$ to obtain

$$(S\phi)(\alpha) = v_\alpha(S\phi) = (S^* v_\alpha)\phi = f(\alpha) v_\alpha \phi = f(\alpha)\phi(\alpha) = (T_f \phi)(\alpha).$$

Since this holds for every $\alpha \in \Omega$, $S = T_f$ which proves the nontrivial half of $\{T_z\}' = \{T_f : f \in H^\infty(\Omega)\}$.

If $\xi \in H^\infty(\Omega)$ is univalent, then $\{T_\xi\}' = \{T_{\xi \circ z}\}' = \{T_z\}'$ by the second part of Corollary 2.3.2, and the conclusion still holds. □

Corollary 2.3.6. *If $\xi \in H^\infty(\Omega)$ is univalent, then T_ξ is irreducible and $\{T_\xi\}'' = \{T_f : f \in H^\infty(\Omega)\}$ is a maximal commutative, weakly closed subalgebra of $\mathcal{L}(X)$.*

The univalence of $\xi \in H^\infty(\Omega)$ can be relaxed to ξ being univalent on $\xi^{-1}(G)$ for a non-empty open set $G \subset \xi(\Omega)$. See [156]. Theorem 2.3.5 has a generalization to multiple operators.

Theorem 2.3.7. *Let $f_1, ..., f_k \in H^\infty(\Omega)$. If there exists a compact $K \subset \Omega$ such that at every $\alpha \in \Omega \setminus K$,*

(i) $f_j(\alpha) \notin f_j(\partial\Omega)$, $j = 1, ..., k$,

(ii) $\bigcap_{j=1}^k Z(f_j - f_j(\alpha)) = \{\alpha\}$,

(iii) $\alpha \notin \bigcap_{j=1}^k Z(f_j')$,

then $\{T_{f_1}, ..., T_{f_k}\}' = \{T_g : g \in H^\infty(\Omega)\}$ on X. In this case the k-tuple $\{T_{f_1}, ..., T_{f_k}\}$ is irreducible.

Proof. Fix $S \in \{T_{f_1}, ..., T_{f_k}\}'$. At every $\alpha \in \Omega \setminus K$, we have by condition (i), Lemma 2.2.5, and Proposition 2.3.1 that

$$S^* v_\alpha \in \bigcap_{j=1}^{k} \mathrm{sp}\{v_\beta, v_\beta^{(n)} : \beta \in Z(f_j - f_j(\alpha)), 1 \le n \le m_{f_j - f_j(\alpha)}(\beta) - 1\}.$$

Using linear independence of the functionals (Lemma 2.2.3(ii)), an induction on k asserts that

$$\begin{aligned} &\bigcap_{j=1}^{k} \mathrm{sp}\{v_\beta, v_\beta^{(n)} : \beta \in Z(f_j - f_j(\alpha)), 1 \le n \le m_{f_j - f_j(\alpha)}(\beta) - 1\} \\ &= \mathrm{sp}\{v_\beta, v_\beta^{(n)} : \beta \in \bigcap_{j=1}^{k} Z(f_j - f_j(\alpha)), 1 \le n \le \wedge_{j=1}^{k} m_{f_j - f_j(\alpha)}(\beta) - 1\}. \end{aligned}$$

These expressions together with conditions (ii) and (iii) yield

$$S^* v_\alpha = f(\alpha) v_\alpha, \ \forall \alpha \in \Omega \setminus K \tag{2.3.3}$$

for some scalar function $f : \Omega \setminus K \to \mathbb{C}$.

Taking norms in X^*, it follows from (2.3.3) that $f(\alpha)$ is bounded on $\Omega \setminus K$. Set $g := S1 \in X$. Applying (2.3.3) on the function $\phi \equiv 1$ gives

$$g(\alpha) = f(\alpha), \ \forall \alpha \in \Omega \setminus K,$$

so in particular g is bounded on $\Omega \setminus K$. Since the analytic function g is bounded on the compact $K \subset \Omega$, we have $g \in H^\infty(\Omega)$ and the multiplication operator T_g is in $\mathcal{L}(X)$. Now apply (2.3.3) on any $\phi \in X$ to get

$$(S\phi)(\alpha) = v_\alpha(S\phi) = (S^* v_\alpha)\phi = g(\alpha) v_\alpha \phi = g(\alpha)\phi(\alpha) = (T_g \phi)(\alpha)$$

over $\Omega \setminus K$ which is a nonempty open subset of Ω. So, the two analytic functions $S\phi = T_g \phi$. That is, $S = T_g$. The rest follows by standard argument. □

As an application of this theorem, we invoke Lemma 2.2.7 to obtain a simple sufficient condition for irreducibility of several multiplication operators by finite Blaschke products on the disc $\mathbb{D}$.

Corollary 2.3.8. *Let $b_1, ..., b_k$ be finite Blaschke products on $\mathbb{D}$ that separate points on $\partial\mathbb{D}$. Then $\{T_{b_1}, ..., T_{b_k}\}' = \{T_g : g \in H^\infty(\mathbb{D})\}$ on X over $\mathbb{D}$. In particular, the k-tuple $\{T_{b_1}, ..., T_{b_k}\}$ is irreducible.*

Proof. By Lemma 2.2.7 let $t \in (0,1)$ be such that $\bigcap_{j=1}^k Z(b_j - b_j(\alpha)) = \{\alpha\}$ for all $t < |\alpha| < 1$. Put

$$K := \{z : |z| \leq t\} \bigcup Z(b_1')$$

where $Z(b_1')$ in $\mathbb{D}$ is finite. It is clear that K is a compact subset of $\mathbb{D}$ for which conditions (ii) and (iii) in Theorem 2.3.7 both hold. Also, condition (i) holds for every $\alpha \in \mathbb{D}$, for $b_j(\mathbb{D}) = \mathbb{D}$ while the cluster set $b_j(\partial\mathbb{D})$ equals the range of b_j on $\partial\mathbb{D}$, which is $\partial\mathbb{D}$. Therefore Theorem 2.3.7 asserts the conclusion. $\square$

There readily exist examples of finite Blaschke products $b_1, ..., b_k$ separating the points on $\partial\mathbb{D}$. By the Stone-Weierstrass theorem, the point-separating condition is equivalent to that the uniformly closed unital subalgebra of $C(\partial\mathbb{D})$ generated by $b_1, ..., b_k$ and $1/b_1 = \bar{b}_1, ..., 1/b_k = \bar{b}_k$ contains the function z.

By Proposition 2.3.1, for every $f \in H^\infty(\Omega), \alpha \in \Omega$, the weak-star closure of the linear set $\{S^* v_\alpha : S \in \{T_f\}'\}$ is contained in the annihilator in X^* of the range $(f - f(\alpha))X$. Following [36], $f \in H^\infty(\Omega)$ is called X-*ancestral* if the closure equals the annihilator for *every* $\alpha \in \Omega$. A key property of ancestral symbol functions over general regions lies in an extension of [36, Theorem 1] and its corollary, which can be proved in the same way using Proposition 2.3.1 instead. We state the result but omit the proof.

Theorem 2.3.9. *If $f \in H^\infty(\Omega)$ is nonconstant and X-ancestral, and $\{T_f\}' \subset \{T_h\}'$ on X for some $h \in H^\infty(\Omega)$, then $h = g \circ f$ for some $g \in H^\infty(f(\Omega))$. In particular, if $\{T_f\}' = \{T_h\}'$ for f and h ancestral, then $h = g \circ f$ for some univalent $g \in H^\infty(f(\Omega))$.*

Now we have a partial converse to Theorem 2.3.5. Note that the class of functions $f \in H^\infty(\Omega)$ for which $\{T_f\}' = \{T_z\}'$, that is, T_f is totally abelian, is in general much larger than the univalent functions. For instance, see [44] for meromorphic functions f acting on $A^2(\mathbb{D})$ and $H^2(\mathbb{D})$ with $\{T_f\}' = \{T_z\}'$.

Corollary 2.3.10. *Let $\xi \in H^\infty(\Omega)$ be nonconstant. Then ξ is univalent if and only if $\{T_\xi\}' = \{T_z\}'$ and ξ is X-ancestral.*

Proof. Any univalent $\xi \in H^\infty(\Omega)$ satisfies $\xi(\alpha) \notin \xi(\partial\Omega)$, $\forall \alpha \in \Omega$. For, $\xi(\mathbb{D}(\alpha, \epsilon_1))$ contains $\mathbb{D}(\xi(\alpha), \epsilon_2)$ for some small $\epsilon_1, \epsilon_2 > 0$, which implies $\xi(\Omega \setminus \overline{\mathbb{D}}(\alpha, \epsilon_1))$ is bounded away from $\xi(\alpha)$ by ϵ_2 due to injectivity of ξ. So, $(T_{\xi - \xi(\alpha)} X)^\perp = \mathbb{C} v_\alpha$ by Lemma 2.2.5 and ξ is ancestral. The rest follows by Theorem 2.3.9. □

Next we establish a relation between ancestral functions and the double commutants of their multiplication operators, part of which was noticed in the classical context of $H^2(\mathbb{D})$ in [36, 39].

Theorem 2.3.11. *Let $f \in H^\infty(\Omega)$ be nonconstant. If f is X-ancestral, then $\{T_f\}'' = \{T_{g \circ f} : g \in H^\infty(f(\Omega))\}$ on X. Conversely, if $\{T_f\}'' = \{T_{g \circ f} : g \in H^\infty(f(\Omega))\}$ and $\{T_f\}' = \{T_h\}'$ for some X-ancestral h, then f is X-ancestral.*

Proof. Observe first that $\{T_f\}'' = \{T_{g\circ f} : g \in H^\infty(f(\Omega))\}$ is equivalent to $\{T_f\}'' \subset \{T_{g \circ f} : g \in H^\infty(f(\Omega))\}$, for the other inclusion always holds (Remark 2.3.3). Secondly, $\{T_f\}' \ni T_z \Rightarrow \{T_f\}'' \subset \{T_z\}'$ while $\{T_z\}' = \{T_h : h \in H^\infty(\Omega)\}$ by Theorem 2.3.5.

Now suppose f is ancestral and $S \in \{T_f\}''$. From the second observation above, $S = T_h$ for some $h \in H^\infty(\Omega)$. Then $\{T_f\}' \subset \{T_h\}'$, and Theorem 2.3.9 asserts $h = g \circ f$ for some $g \in H^\infty(f(\Omega))$. This proves the first part of the theorem in view of the first observation above.

Conversely, $\{T_f\}' = \{T_h\}'$ implies $T_h \in \{T_h\}'' = \{T_f\}''$. So, $T_h = T_{g\circ f}$, $h = g \circ f$, and by a factorization ([36], p. 4),

$$(h - h(\alpha))X \subset (f - f(\alpha))X \tag{2.3.4}$$

for any $\alpha \in \Omega$. Therefore, $\{S^* v_\alpha : S \in \{T_f\}'\} = \{S^* v_\alpha : S \in \{T_h\}'\}$ is weak-star dense in the annihilator of $(h - h(\alpha))X$, a fortiori in that of $(f - f(\alpha))X$ by (2.3.4). That is, f is ancestral. □

Over the disc $\mathbb{D}$, the example ϕ ([39], p. 178) of a semi-automorphic function was shown to be not $H^2(\mathbb{D})$-ancestral although $\{T_\phi\}'' = \{T_{g\circ\phi} : g \in H^\infty(\phi(\mathbb{D}))\}$ on $H^2(\mathbb{D})$. So, $\{T_f\}'' = \{T_{g\circ f} : g \in H^\infty(f(\Omega))\}$ alone does not imply f ancestral.

The rest of this section concerns the existence of nonzero compact operators in the commutant $\{T_f\}'$ on X, which was investigated by Cowen [36, 38, 39] in the classical context of $X = H^2(\mathbb{D})$. It was shown $\{T_f\}'$ on $H^2(\mathbb{D})$ contains no nonzero compact operators if the nonconstant $f \in H^\infty(\mathbb{D})$ either is $H^2(\mathbb{D})$-ancestral ([36], p. 27), or has its $\{T_f\}'$ lift to $\{M_f\}'$ of the minimal normal extension M_f on $L^2(\partial\mathbb{D})$ [38, Theorem 1]. On the other hand, $\{T_\phi\}'$ does contain a nonzero compact operator [39, Theorem 1.1] for the example ϕ of a semi-automorphic function. We adapt the argument of [36], p. 27, to prove the following nonexistence result on Hardy and weighted Bergman spaces over general regions. Note that Lemma A in [36], p. 26, on quasi-nilpotence of compact operators in the commutant remains true in this generality.

Theorem 2.3.12. *If $f \in H^\infty(\Omega)$ is a nonconstant X-ancestral function and $f(\alpha_0) \notin f(\partial\Omega)$ for some $\alpha_0 \in \Omega$, then $\{T_f\}'$ on X contains no nonzero compact operators.*

Proof. Since the cluster set $f(\partial\Omega)$ is compact, it follows from the hypothesis that $f(\alpha) \notin f(\partial\Omega)$ for all α in an open neighborhood U_0 of α_0 in Ω. Put

$$U_{00} := U_0 \setminus f^{-1}(f(Z(f'))). \tag{2.3.5}$$

Since f is nonconstant, both $Z(f')$ and the pre-image under f of any point are countable, so that $f^{-1}(f(Z(f')))$ is countable. Therefore, U_{00} must be uncountable.

Let $K \in \{T_f\}'$ be compact. We shall show $K^* v_\alpha = 0$ for all $\alpha \in U_{00}$. Then it would follow that

$$\mathrm{sp}\{v_\alpha : \alpha \in U_{00}\} \subset \ker K^*.$$

Since the uncountable subset U_{00} has an accumulation point in Ω, the span on the left is weak-star dense (Lemma 2.2.3(iii)) while $\ker K^*$ is weak-star closed. Therefore, $X^* = \ker K^*$ and $K = 0$ as desired.

To that end fix $\alpha \in U_{00}$. Noting that (2.3.5) ensures that $Z(f-f(\alpha))$ consists of simple zeros only, Lemma 2.2.5 applies to $f-f(\alpha)$ to give

$$((f-f(\alpha))X)^{\perp} = \mathrm{sp}\{v_{\alpha_k} : 1 \leq k \leq l\} \tag{2.3.6}$$

where $\{\alpha_k : 1 \leq k \leq l\} = f^{-1}(f(\alpha))$ consists of distinct points.

Suppose $K^*v_\alpha \neq 0$. By Proposition 2.3.1, $\{S^*v_\alpha : S \in \{T_f\}'\} \subset ((f-f(\alpha))X)^{\perp}$. So in view of (2.3.6)

$$K^*v_\alpha = \lambda v_{\alpha_j} + x$$

for some j, $1 \leq j \leq l$, $x \in \mathrm{sp}\{v_{\alpha_k} : 1 \leq k \leq l, k \neq j\}$, and $\lambda \neq 0$. Put

$$p(z) := \prod_{1 \leq k \leq l, k \neq j} (z - \alpha_k),$$

$$\tilde{K} := (\lambda p(\alpha_j))^{-1} K T_p \in \{T_f\}'.$$

One verifies that

$$\tilde{K}^* v_\alpha = (\lambda p(\alpha_j))^{-1} T_p^* (\lambda v_{\alpha_j} + x) = (\lambda p(\alpha_j))^{-1} (\lambda p(\alpha_j) v_{\alpha_j} + 0) = v_{\alpha_j},$$

where the second equality is due to $T_p^* v_\beta = p(\beta) v_\beta, \forall \beta \in \Omega$, while $p(\alpha_k) = 0$ for $k \neq j$.

Note that $f(\alpha_j) = f(\alpha)$. So one has by Proposition 2.3.1 and (2.3.6) that

$$\{S^* v_{\alpha_j} : S \in \{T_f\}'\} \subset ((f - f(\alpha_j))X)^{\perp} = ((f-f(\alpha))X)^{\perp} = \mathrm{sp}\{v_{\alpha_k} : 1 \leq k \leq l\}.$$

Since f is ancestral, the weak-star closure of the finite-dimensional linear set $\{S^* v_{\alpha_j} : S \in \{T_f\}'\}$, which is itself, equals $((f - f(\alpha_j))X)^{\perp}$. So the above gives

$$\{S^* v_{\alpha_j} : S \in \{T_f\}'\} = \mathrm{sp}\{v_{\alpha_k} : 1 \leq k \leq l\}.$$

Thus $v_\alpha = S^* v_{\alpha_j}$ for some $S \in \{T_f\}'$, noting $\alpha \in \{\alpha_k : 1 \leq k \leq l\}$. Because $\tilde{K}S \in \{T_f\}'$ is compact, it must be quasi-nilpotent. However, $(\tilde{K}S)^* v_\alpha = v_\alpha$ indicates $1 \in \sigma((\tilde{K}S)^*) = \sigma(\tilde{K}S)$. This contradiction shows $K^* v_\alpha = 0, \forall \alpha \in U_{00}$, and concludes the proof. □

Remark 2.3.13. The condition that $f(\alpha_0) \notin f(\partial\Omega)$ for some $\alpha_0 \in \Omega$ (also see Remark 2.2.6) is not automatically satisfied for nonconstant $f \in H^\infty(\Omega)$ ([137, Example 1] where f is in the disc algebra). On the other hand, suppose $\partial\Omega$ has area measure zero and f is analytic in a neighborhood of $\overline{\Omega}$. Then using local univalence one sees that $f(\partial\Omega)$, as the image of $\partial\Omega$ under f, has area measure zero (Proposition 1.1.3) while $f(\Omega)$ is open, so that the condition is satisfied.

2.4 The commutant of T_{z^n} over centered annuli

This section concerns the monomial multiplication operators $T_{z^n} = T_z^n$, $n > 1$, on $X = H^p(R, \omega)$ or $A^p(R, wda)$ over an annulus $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$. Note that $H^p(R, \omega)$ isometrically embeds in $L^p(\partial R, d\mu_\omega)$ [2], where $d\mu_\omega$ is the harmonic measure on ∂R for $\omega \in R$ which is different but equivalent to the linear Lebesgue measure. Put

$$\lambda := e^{i2\pi/n}$$

and define the composition operators C_k, $k = 1, ..., n$, by the rotations $(C_k f)(z) = f(\lambda^k z)$, $f \in X$. By [112, Lemma 2.3], composition operators on $H^p(\Omega, \omega)$ by analytic self maps of a general region Ω are bounded. A subclass of weight functions w ensures that C_k is bounded on $A^p(R, wda)$.

Definition 2.4.1. For an annulus $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$, $\mathcal{W}_a(R)$ consists of the functions $w \in \mathcal{W}(R)$ such that, for some $\epsilon_w > 0$ and $B_w \in (1, \infty)$,

$$B_w^{-1} w(\beta) \le w(\alpha) \le B_w w(\beta) \tag{2.4.1}$$

whenever $|\alpha| = |\beta| \in (r_1, r_1 + \epsilon_w) \bigcup (r_2 - \epsilon_w, r_2)$.

Lemma 2.4.2. *If $w \in \mathcal{W}_a(R)$, then $C_k \in \mathcal{L}(A^p(R, wda))$.*

Proof. Let $K = \{z : r_1 + \epsilon_w \le |z| \le r_2 - \epsilon_w\}$. For $f \in A^p(R, wda)$, Lemma 2.2.2, rotation invariance of the area measure, and (2.4.1) give the following estimates for the analytic function

$C_k f$ in R

$$\|C_k f\|^p \le D_K \int_{R\setminus K} |f(\lambda^k z)|^p w(z) da(z) = D_K \int_{R\setminus K} |f(z)|^p w(\lambda^{-k} z) da(z)$$
$$\le D_K B_w \int_{R\setminus K} |f(z)|^p w(z) da(z) \le D_K B_w \|f\|^p.$$

That is, $\|C_k\| \le (D_K B_w)^{1/p}$. $\square$

Theorem 2.4.3. *Let $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$ and $w \in \mathcal{W}_a(R)$ in case $X = A^p(R, wda)$.*

$$\{T_{z^n}\}' = \left\{ \sum_{k=1}^{n} T_{\psi_k} C_k : \ \psi_k \in H^\infty(R) \right\}.$$

Proof. Let $S \in \{T_{z^n}\}' = \{T_{z^{-n}}\}' = \{T^n_{z^{-1}}\}'$. By [37, Lemma] there exist $Y_k \in \mathcal{L}(X)$, $k = 1, ..., n$, such that

$$T_{z^{-1}} Y_k = \lambda^k Y_k T_{z^{-1}}, \tag{2.4.2}$$

$$S = n^{-1} T_{z^{n-1}} \sum_{k=1}^{n} Y_k \tag{2.4.3}$$

noting $(T^{n-1}_{z^{-1}})^{-1} = T_{z^{n-1}}$. It follows from (2.4.2) that

$$Y_k C_{n-k} T_{z^{-1}} = \lambda^{k-n} Y_k T_{z^{-1}} C_{n-k} = \lambda^{k-n} \lambda^{-k} T_{z^{-1}} Y_k C_{n-k} = T_{z^{-1}} Y_k C_{n-k}.$$

Therefore, $Y_k C_{n-k} \in \{T_{z^{-1}}\}' = \{T_z\}'$ implies by Theorem 2.3.5 that $Y_k C_{n-k} = T_{f_k}$ thus

$$Y_k = T_{f_k} C_k$$

for some $f_k \in H^\infty(R)$. Together with (2.4.3), one then arrives at

$$S = n^{-1} T_{z^{n-1}} \sum_{k=1}^{n} T_{f_k} C_k = \sum_{k=1}^{n} T_{\psi_k} C_k$$

where $\psi_k(z) := n^{-1} z^{n-1} f_k(z)$ are functions in $H^\infty(R)$. On the other hand, operators of this form obviously lie in $\{T_{z^n}\}'$. The proof is complete. $\square$

Despite the lack of a basis in X compatible with T_{z^n}, it does possess a reducing subspace.

Theorem 2.4.4. *Let $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$ and $w \in \mathcal{W}_a(R)$ in case $X = A^p(R, wda)$. Then T_{z^n} on X is reducible for each $n > 1$.*

Proof. Note $\lambda := e^{i2\pi/n} \neq 1$ for $n > 1$. For each $1 \leq m \leq n$, put

$$s_m = \sum_{k=1}^{m-1} \lambda^{-k}, \quad t_m = \sum_{k=m}^{n} \lambda^{-k},$$

so that $s_m + t_m = 0$ (by convention $s_1 = 0$). Consider the operator

$$P := \sum_{k=1}^{n} T_{f_k} C_k \in \{T_{z^n}\}', \text{ where } f_k(z) := \lambda^{-k}(z + n^{-1}). \tag{2.4.4}$$

One has $P \neq 0, I$, for $Pz = nz^2 + z \not\equiv 0, z$. It remains to verify $P^2 = P$. Now,

$$\begin{aligned} P^2 &= \sum_{j,k=1}^{n} T_{f_j} C_j T_{f_k} C_k = \sum_{j,k=1}^{n} T_{f_j} T_{f_k(\lambda^j z)} C_{j+k} \\ &= \sum_{m=1}^{n} \left(\sum_{j+k=m,n+m} T_{f_j(z) f_k(\lambda^j z)} \right) C_m. \end{aligned}$$

In view of (2.4.4), it clearly suffices to show

$$\sum_{j+k=m,n+m} f_j(z) f_k(\lambda^j z) = f_m(z), \ \forall m = 1, ..., n. \tag{2.4.5}$$

Elementary computations yield

$$\begin{aligned} f_j(z) f_k(\lambda^j z) &= \lambda^{-k}(z^2 + n^{-1} z) + \lambda^{-j-k} n^{-1}(z + n^{-1}), \\ \sum_{j+k=m,n+m} \lambda^{-k} &= s_m + t_m = 0, \\ \sum_{j+k=m,n+m} \lambda^{-j-k} &= (m - 1 + n - m + 1)\lambda^{-m} = n\lambda^{-m}. \end{aligned}$$

These equalities assert (2.4.5) and conclude the proof. □

The following result is known [47, Corollary 7] for $X = H^2(\mathbb{D})$.

Theorem 2.4.5. *Let* $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$*. Let* $\gcd\{n_1, ..., n_k\} = n$*. Then* $\{T_{z^{n_1}}, ..., T_{z^{n_k}}\}' = \{T_{z^n}\}'$ *on* X *over* R*.*

Proof. Write $\lambda_j := e^{i2\pi/n_j}$, $j = 1, ..., k$, and $\lambda := e^{i2\pi/n}$. Let $S \in \{T_{z^{n_1}}, ..., T_{z^{n_k}}\}'$. Since z^{n_j} is continuous on $\overline{R}$, the cluster set $z^{n_j}(\partial R) = \{z : |z| = r_1^{n_j} \text{ or } r_2^{n_j}\}$. Thus for every $\alpha \in R$, $\alpha^{n_j} \notin z^{n_j}(\partial R)$ and of course $Z(z^{n_j} - \alpha^{n_j}) = \{\lambda_j^m \alpha : 1 \le m \le n_j\}$ with all simple zeros. Lemma 2.2.5 applies to the function $z^{n_j} - \alpha^{n_j}$ to yield

$$((z^{n_j} - \alpha^{n_j})X)^\perp = \mathrm{sp}\{v_{\lambda_j^m \alpha} : 1 \le m \le n_j\}. \tag{2.4.6}$$

It follows from Proposition 2.3.1 and (2.4.6) that for every $\alpha \in R$

$$S^* v_\alpha \in \bigcap_{j=1}^{k} \mathrm{sp}\{v_{\lambda_j^m \alpha} : \ 1 \le m \le n_j\}.$$

By linear independence, an induction on k gives

$$\bigcap_{j=1}^{k} \mathrm{sp}\{v_{\lambda_j^m \alpha} : \ 1 \le m \le n_j\} = \mathrm{sp}\{v_{\lambda^m \alpha} : \ 1 \le m \le n\}.$$

Thus, $S \in \{T_{z^n}\}'$ by Proposition 2.3.1. The other direction is trivial. □

Remark 2.4.6. Evidently, Theorem 2.4.4 is valid with $\mathbb{D}$ in lieu of R. Theorem 2.4.5 is also valid over $\mathbb{D}$, with the same proof using all $\alpha \in \mathbb{D} \setminus \{0\}$.

Lemma 2.4.7. *Let* $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$ *and* $w \in \mathcal{W}_a(R)$ *in case* $X = A^p(R, w da)$*. Then the function* z^n *is* X*-ancestral.*

Proof. Fix $\alpha \in R$ and we have $((z^n - \alpha^n)X)^\perp = \mathrm{sp}\{v_{\lambda^k \alpha} : 1 \le k \le n\}$ by (2.4.6). Since the composition operators $C_k \in \{T_{z^n}\}'$ satisfy $C_k^* v_\alpha = v_{\lambda^k \alpha}$, the function z^n is X-ancestral by definition. □

We have an immediate consequence to Lemma 2.4.7 and Theorem 2.3.12.

Corollary 2.4.8. *Let $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$ and $w \in \mathcal{W}_a(R)$ in case $X = A^p(R, wda)$. Then T_{z^n} does not commute with nonzero compact operators.*

The results obtained so far have implications for iterates of the restriction of the analytic Toeplitz operator $T_{\pi_\omega} \in \mathcal{L}(H^p(\partial\mathbb{D}))$ on a certain invariant subspace, for covering maps $\pi = \pi_\omega$ from $\mathbb{D}$. We recall the notations and facts needed now and later in the proof of Theorem 2.5.5. An analytic function π from $\mathbb{D}$ onto an open connected planar set Ω is called a covering map [138] if every $z \in \Omega$ has a connected neighborhood $\Delta \subset \Omega$ such that π conformally maps each component of $\pi^{-1}(\Delta)$ onto Δ, in which case the group G of deck transformations of π consists of all disc automorphisms $L \in \mathrm{Aut}(\mathbb{D})$ such that $\pi = \pi \circ L$. The class of covering maps is obviously closed under left and right compositions by conformal mappings. It also follows from the Koebe uniformization theorem that two covering maps π_1, π_2 from $\mathbb{D}$ onto Ω must satisfy $\pi_1 = \pi_2 \circ L$ for some $L \in \mathrm{Aut}(\mathbb{D})$ (eg. [72, p. 1234]). Following [112], composition by π_ω, $\pi_\omega(0) = \omega \in \Omega$, is a linear isometry from $H^p(\Omega, \omega)$ onto the closed subspace $H^p(\mathbb{D}, G) \subset H^p(\mathbb{D})$ consisting of G-automorphic functions in $H^p(\mathbb{D})$, which in turn is isometric via the radial-limit map to the closed subspace $H^p(\partial\mathbb{D}, G) \subset H^p(\partial\mathbb{D})$ consisting of those $H^p(\partial\mathbb{D})$ functions measurable in the σ-subalgebra of $d\theta$-essentially G-invariant Borel subsets of $\partial\mathbb{D}$ [59].

For the Hilbert space case, T_{π_ω} on $H^2(\partial\mathbb{D})$ was shown in [1, 38] to be reducible by spectral projections of a certain unitary operator, the product of an invertible analytic Toeplitz operator and the composition operator by a nontrivial $L \in G$. The question arises as to the reducibility of iterates of the restriction of T_{π_ω} on the invariant subspace $H^p(\partial\mathbb{D}, G)$. To this we have

Proposition 2.4.9. *Let $\omega \in \Omega$ and π_ω, $\pi_\omega(0) = \omega$, be a covering map of $\mathbb{D}$ onto Ω with the group G. Write $S := T_{\pi_\omega}|H^p(\partial\mathbb{D}, G)$. Then*

(i) *S is irreducible.*

(ii) *For $\Omega = \{z : 0 < r_1 < |z| < r_2 < \infty\}$, the k-tuple of iterates*

$$\{S^{n_1}, ..., S^{n_k}\}$$

is reducible if and only if $\gcd\{n_1, ..., n_k\} > 1$.

Proof. From the preceding discussion, the Banach space isometry of $H^p(\Omega, \omega)$ onto $H^p(\partial\mathbb{D}, G)$ is seen to induce a similarity between the multiplication operators T_f on $H^p(\Omega, \omega)$, $f \in H^\infty(\Omega)$, and restrictions of the analytic Toeplitz operators $T_{f\circ\pi_\omega}$ on the invariant subspace $H^p(\partial\mathbb{D}, G)$. Here $f \circ \pi_\omega$ refers to the radial limit in $H^\infty(\partial\mathbb{D})$. Take $f(z) = z$, and Corollary 2.3.6 gives (i) under the similarity. Take each $f_j(z) = z^{n_j}$, and (ii) follows from Theorems 2.4.5, 2.4.4, and Corollary 2.3.6. □

Since covering maps are $H^2(\mathbb{D})$-ancestral ([36], Corollary on p. 22), T_{π_ω} on $H^2(\partial\mathbb{D})$ does not commute with nonzero compact operators ([36], p. 27). Using ideas as above, we obtain the following result on the restriction $T_{\pi_\omega}|H^p(\partial\mathbb{D}, G)$.

Proposition 2.4.10. *Let* $\omega \in \Omega$ *and* π_ω, $\pi_\omega(0) = \omega$, *be a covering map of* $\mathbb{D}$ *onto* Ω *with the group* G. *Write* $S := T_{\pi_\omega}|H^p(\partial\mathbb{D}, G)$. *Then*

(i) S *does not commute with nonzero compact operators.*

(ii) *For* $\Omega = \{z : 0 < r_1 < |z| < r_2 < \infty\}$ *and* $n \in \mathbb{N}$, *the iterate* S^n *does not commute with nonzero compact operators.*

Proof. Under the intertwining Banach space isometry between $H^p(\Omega, \omega)$ and $H^p(\partial\mathbb{D}, G)$, (i) follows from Theorem 2.3.5. To see this, suppose T_h, $h \in H^\infty(\Omega)$, is a compact operator on $H^p(\Omega, \omega)$. Then the spectrum $\sigma(T_h) = \overline{h(\Omega)}$ (see the proof of Theorem 2.5.3(i)) is connected, while every nonzero point of which, if there is any, is isolated. This forces $h \equiv 0$. Similarly, (ii) follows from Corollary 2.4.8. □

Now let π_ω be a covering map from $\mathbb{D}$ onto the annulus R with $\pi_\omega(0) = \omega$ and group G. Consider the closed subspace of the Hardy space $H^p(\mathbb{D})$

$$H^p(\mathbb{D}, G) = \{f \in H^p(\mathbb{D}) : f \circ L = f, \forall L \in G\},$$

and the closed subalgebra of $H^\infty(\mathbb{D})$

$$H^\infty(\mathbb{D}, G) = \{g \in H^\infty(\mathbb{D}) : g \circ L = g, \forall L \in G\}.$$

$H^p(\mathbb{D}, G)$ is an invariant subspace for the multiplication operators $T_g \in \mathcal{L}(H^p(\mathbb{D})), \forall g \in H^\infty(\mathbb{D}, G)$ including π_ω. For each $k = 1, ..., n-1$, the rotation $\rho_k : z \mapsto \lambda^k z$ is a conformal automorphism of R, so

$$\rho_k \circ \pi_\omega = \pi_\omega \circ L_k \tag{2.4.7}$$

for some $L_k \in \mathrm{Aut}(\mathbb{D})$. One has the following simple observation.

Lemma 2.4.11. *The composition operator C_{L_k} on $H^p(\mathbb{D}, G)$ defined by L_k is in $\mathcal{L}(H^p(\mathbb{D}, G))$.*

Proof. Since the composition operator C_{L_k} on $H^p(\mathbb{D})$ is in $\mathcal{L}(H^p(\mathbb{D}))$, it suffices to show invariance of the subspace $H^p(\mathbb{D}, G)$. For each $L \in G$, (2.4.7) gives

$$\pi_\omega \circ L_k \circ L = \rho_k \circ \pi_\omega \circ L = \rho_k \circ \pi_\omega = \pi_\omega \circ L_k,$$

so that $L_k \circ L \circ L_k^{-1} \in G$. Therefore, one has for $f \in H^p(\mathbb{D}, G)$ that

$$f \circ L_k \circ L = f \circ L_k \circ L \circ L_k^{-1} \circ L_k = f \circ L_k.$$

That is, $f \circ L_k \in H^p(\mathbb{D}, G)$, proving invariance under C_{L_k}. $\square$

Remark 2.4.12. The L_k satisfying (2.4.7) is not unique. Indeed, another $\tilde{L_k}$ satisfies (2.4.7) if and only if $\tilde{L_k} = L \circ L_k$ for some $L \in G$. However, the composition operator C_{L_k} on $H^p(\mathbb{D}, G)$ is uniquely defined, for $f \circ \tilde{L_k} = f \circ L \circ L_k = f \circ L_k$, $\forall f \in H^p(\mathbb{D}, G)$. Also, covering maps onto annuli have explicit forms, so that the automorphisms L_k could be written out.

The development above and Theorem 2.4.3 give the following

Theorem 2.4.13. *Let π_ω be a covering map from $\mathbb{D}$ onto R with $\pi_\omega(0) = \omega \in R$, the group G,* *and the automorphisms L_k. Then on $H^p(\mathbb{D}, G)$ the commutant*

$$\{T^n_{\pi_\omega} | H^p(\mathbb{D}, G)\}' = \left\{ \sum_{k=0}^{n-1} (T_{g_k} | H^p(\mathbb{D}, G)) C_{L_k} : \; g_k \in H^\infty(\mathbb{D}, G) \right\}, \tag{2.4.8}$$

and each operator in the commutant has a unique representation of this form.

2.5 Extensions of the analytic functional calculus

For nonconstant $f \in H^\infty(\Omega)$, it is seen from previous discussions that $\{T_f\}' \subset \{T_{g \circ f}\}'$, $g \in H^\infty(f(\Omega))$. By extending the analytic functional calculus for T_f, further connections between T_f and $T_{g\circ f}$ will be obtained for g in certain subalgebras of $H^\infty(f(\Omega))$. For $K \subset \mathbb{C}$ nonempty and compact, let $H(K)$ denote the restriction to K of the algebra of analytic functions in neighborhoods of K, and set $R(K) := \overline{H(K)}$ in the sup-norm $\|\cdot\|_\infty$ over K.

Let $(A, \|\cdot\|)$ be a unital Banach algebra. For $a \in A$, $\sigma(a)$ denotes the spectrum, $\rho(a)$ the spectral radius, $\mathcal{F}(a) := \{f(a) : f \in H(\sigma(a))\} \subset A$ the subalgebra obtained via the analytic functional calculus, and $\overline{\mathcal{F}(a)}$ the norm closure in A. The map

$$\tau : f(a) \in \mathcal{F}(a) \mapsto f|\sigma(a) \in H(\sigma(a))$$

is a well-defined contractive homomorphism and extends by continuity to

$$\overline{\tau} : x \in \overline{\mathcal{F}(a)} \mapsto \overline{\tau}(x) \in \overline{H(\sigma(a))} = R(\sigma(a))$$

where $\overline{\tau}(x)$ is the *spectral function* [54] of x. It is well known that $\hat{a}$ on $M(\overline{\mathcal{F}(a)})$ is a homeomorphism onto $\sigma(a)$ and that

$$\overline{\tau}(x) = \hat{x} \circ \hat{a}^{-1} \tag{2.5.1}$$

in terms of the Gelfand transform $\hat{x}$ on $M(\overline{\mathcal{F}(a)})$. In particular, and due to spectral permanence in the commutative subalgebra $\overline{\mathcal{F}(a)}$ of A,

$$\sigma(x) = \hat{x}(M(\overline{\mathcal{F}(a)})) = (\hat{x} \circ \hat{a}^{-1})(\sigma(a)) = [\overline{\tau}(x)](\sigma(a)). \tag{2.5.2}$$

Moreover, under any bounded unital homomorphism θ of A into another Banach algebra B, it is easily seen that $\theta(x) \in \overline{\mathcal{F}(\theta(a))}$ with spectral function

$$\overline{\tau}(\theta(x)) = \overline{\tau}(x)|\sigma(\theta(a)). \tag{2.5.3}$$

This observation together with (2.5.2) gives

$$\sigma(\theta(x)) = [\overline{\tau}(\theta(x))](\sigma(\theta(a))) = [\overline{\tau}(x)](\sigma(\theta(a))). \tag{2.5.4}$$

For $A = H^\infty(\Omega)$ the analytic functional calculus assumes a natural form, and one can characterize $\overline{\mathcal{F}(a)}$ and identify the spectral functions.

Lemma 2.5.1. *For every nonconstant $f \in H^\infty(\Omega)$, one has*

(i) $g(f) = g \circ f$, $\forall g \in H(\overline{f(\Omega)})$.

(ii) $\overline{\mathcal{F}(f)} = \{g \circ f : g \in R(\overline{f(\Omega)})\}$, *and for such g the spectral function of $g \circ f \in \overline{\mathcal{F}(f)}$ is g.*

Proof. Obviously $\sigma(f) = \overline{f(\Omega)}$. For $g \in H(\overline{f(\Omega)}) = H(\sigma(f))$ and Γ an envelope of $\sigma(f)$ in the domain of g, the analytic functional calculus for $f \in H^\infty(\Omega)$ gives

$$g(f) = \frac{1}{2\pi i} \oint_\Gamma g(\zeta)(\zeta - f)^{-1} d\zeta$$

with norm-convergence in $H^\infty(\Omega)$. Applying the point evaluation functionals in $M(H^\infty(\Omega))$, one derives for $\omega \in \Omega$ that

$$(g(f))(\omega) = \frac{1}{2\pi i} \oint_\Gamma g(\zeta)(\zeta - f(\omega))^{-1} d\zeta = g(f(\omega)) = (g \circ f)(\omega)$$

which proves (i).

(ii). The homomorphism $\tau : g(f) \in \mathcal{F}(f) \mapsto g|\sigma(f) \in H(\sigma(f))$ is a surjective isometry, hence its extension $\overline{\tau}$ is an isometry from $\overline{\mathcal{F}(f)}$ onto $R(\sigma(f)) = R(\overline{f(\Omega)})$ such that

$$\overline{\tau}^{-1} = \overline{\tau^{-1}}.$$

The homomorphism $\kappa : g \in R(\overline{f(\Omega)}) \mapsto g \circ f \in H^\infty(\Omega)$ is an isometry. Also it follows from (i) that

$$\tau^{-1} = \kappa|H(\overline{f(\Omega)})$$

so that $\overline{\tau^{-1}} = \kappa$. Therefore, $\overline{\tau}^{-1} = \kappa$ and

$$\overline{\mathcal{F}(f)} = \overline{\tau}^{-1}(R(\overline{f(\Omega)})) = \kappa(R(\overline{f(\Omega)})) = \{g \circ f : g \in R(\overline{f(\Omega)})\}.$$

In addition, for $g \in R(\overline{f(\Omega)})$, $g \circ f = \kappa(g) = \overline{\tau}^{-1}(g)$ so $\overline{\tau}(g \circ f) = g$. That is, the spectral function of $g \circ f$ is g. This completes the proof. $\square$

Next we extend the mapping theorem ([68], Thm. 4; [101], Thm. 1) for the Browder essential spectrum from $\mathcal{F}(T)$ to $\overline{\mathcal{F}(T)}$. For $T \in \mathcal{L}(E)$ on an infinite dimensional Banach space E, let $\alpha(T), \delta(T), n(T), d(T)$ be respectively the ascent, descent, nullity, and defect of T. Define

$$\Phi := \{T \in \mathcal{L}(E) : n(T) = d(T) < \infty, \alpha(T) = \delta(T) < \infty\},$$

and the Browder essential spectrum of $T \in \mathcal{L}(E)$ is (written $\sigma_8(T)$ in [68])

$$\sigma_{be}(T) := \{\zeta \in \mathbb{C} : \zeta I - T \notin \Phi\}.$$

Proposition 2.5.2. *Let $T \in \mathcal{L}(E)$ and $S \in \overline{\mathcal{F}(T)}$ with spectral function f. Then*

$$\sigma_{be}(S) = f(\sigma_{be}(T)).$$

Proof. Let $\mathcal{U}$ be a maximal commutative subalgebra of $\mathcal{L}(E)$ containing T and therefore containing $\overline{\mathcal{F}(T)} \ni S$. Let

$$\theta : \mathcal{U} \to \mathcal{U}/(\mathcal{U} \bigcap \mathcal{K}(E))$$

be the quotient map modulo the ideal $\mathcal{U} \bigcap \mathcal{K}(E)$ of $\mathcal{U}$. By [68] Lemma 3,

$$\sigma_{be}(T) = \sigma(\theta(T)), \quad \sigma_{be}(S) = \sigma(\theta(S)),. \tag{2.5.5}$$

Consider the homomorphism $\overline{\tau}$ on $\overline{\mathcal{F}(T)}$ *relative to* $\mathcal{U}$ and note that $\overline{\tau}(S) = f$ *regardless of* $\mathcal{U}$. It follows from (2.5.4) for $A = \mathcal{U}$ and $B = \mathcal{U}/(\mathcal{U} \bigcap \mathcal{K}(E))$ that

$$\sigma(\theta(S)) = f(\sigma(\theta(T))). \tag{2.5.6}$$

The proof is completed by combining (2.5.5) and (2.5.6). □

For a multiplication operator T_f on X over Ω, $f \in H^\infty(\Omega)$, the basic analytic functional calculus states $T_{g\circ f} = T_{g(f)} = g(T_f)$ for $g \in H(\overline{f(\Omega)})$, due to Lemma 2.5.1(i) and the homomorphism $f \in H^\infty(\Omega) \mapsto T_f \in \mathcal{L}(X)$. The following theorem characterizes $\overline{\mathcal{F}(T_f)}$, and extends essential commutant inclusion and essential spectral mapping to multiplication operators in $\overline{\mathcal{F}(T_f)}$.

Theorem 2.5.3. *For nonconstant $f \in H^\infty(\Omega)$, one has on the space X that*

(i) $\overline{\mathcal{F}(T_f)} = \{T_{g\circ f} : g \in R(\overline{f(\Omega)})\}$, *and for such g the spectral function of $T_{g\circ f} \in \overline{\mathcal{F}(T_f)}$ is g.*

(ii) *For every $g \in R(\overline{f(\Omega)})$, $\{T_f\}'_e \subset \{T_{g\circ f}\}'_e$, $\sigma_e(T_{g\circ f}) = g(\sigma_e(T_f))$, and $\sigma_{be}(T_{g\circ f}) = g(\sigma_{be}(T_f))$.*

Proof. (i). In view of the homomorphism $f \in H^\infty(\Omega) \mapsto T_f \in \mathcal{L}(X)$, one has $\sigma(T_f) \subset \sigma(f)$. On the other hand, for every $\alpha \in \Omega$, $1 \notin (f(\alpha)I - T_f)X$ implies $f(\Omega) \subset \sigma(T_f)$ and hence $\sigma(f) = \overline{f(\Omega)} \subset \sigma(T_f)$. That is, $\sigma(T_f) = \sigma(f)$. Since clearly $\|T_f\| \le \|f\|_\infty$, one further has

$$\|f\|_\infty \ge \|T_f\| \ge \rho(T_f) = \rho(f) = \|f\|_\infty. \tag{2.5.7}$$

That is, the map $f \in H^\infty(\Omega) \mapsto T_f \in \mathcal{L}(X)$ is an isometric isomorphism which preserves spectra. Then the characterization of $\overline{\mathcal{F}(T_f)}$ follows from Lemma 2.5.1(ii), with $g \in R(\overline{f(\Omega)})$ being precisely the spectral function of $T_{g\circ f} \in \overline{\mathcal{F}(T_f)}$.

(ii). $\{T_f\}'_e \subset \{T_{g\circ f}\}'_e$ follows from (i) and $\mathcal{K}(X)$ being norm closed. Consider the quotient map $\theta : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$ onto the Calkin algebra. We have by (i) and (2.5.4) the equalities

$$\sigma_e(T_{g\circ f}) = \sigma(\theta(T_{g\circ f})) = g(\sigma(\theta(T_f))) = g(\sigma_e(T_f)).$$

The remaining statement involving the Browder essential spectrum σ_{be} follows from Proposition 2.5.2. The proof is complete. $\square$

Remark 2.5.4. If f is such that the diameters of the components of $\mathbb{C} \setminus \overline{f(\Omega)}$ are bounded away from zero and that the interior of the closure $\overline{f(\Omega)}^o = f(\Omega)$, then a result of Mergelyan on rational approximations (cf. [60], Thm. II.10.4) asserts $R(\overline{f(\Omega)}) = H^\infty(f(\Omega)) \bigcap C(\overline{f(\Omega)})$, so that Theorem 2.5.3(ii) applies to $T_{g\circ f}$ for analytic functions g on $f(\Omega)$ with a continuous extension to $\overline{f(\Omega)}$.

The *sequential* weak and strong closures of $\mathcal{F}(T_f)$ are respectively

$$\overrightarrow{\mathcal{F}(T_f)}^w := \{T \in \mathcal{L}(X) : T = \text{w-}\lim_{n\to\infty} S_n, S_n \in \mathcal{F}(T_f)\},$$

$$\overrightarrow{\mathcal{F}(T_f)}^s := \{T \in \mathcal{L}(X) : T = \text{s-}\lim_{n\to\infty} S_n, S_n \in \mathcal{F}(T_f)\}.$$

Although the weak and strong closures of $\mathcal{F}(T_f)$ coincide, it is not clear whether the corresponding sequential closures do. The next result characterizes $\overrightarrow{\mathcal{F}(T_f)}^w$ on X and shows $\overrightarrow{\mathcal{F}(T_f)}^w = \overrightarrow{\mathcal{F}(T_f)}^s$ on $A^p(\Omega, wda)$. On a related note, for finitely connected Ω with regular boundary the weak closure of $\mathcal{F}(T_z)$ on $H^2(\Omega, \omega)$ was shown in the proof of [2], Prop. 1.9, to be $\{T_g : g \in H^\infty(\Omega)\}$.

Theorem 2.5.5. *For $f \in H^\infty(\Omega)$ nonconstant, $\overrightarrow{\mathcal{F}(T_f)}^w$ on X consists of all $T_{g\circ f}$ where g is the point-wise limit on $f(\Omega)$ of a sequence $g_n \in H(\overline{f(\Omega)})$ satisfying $\sup_n \|g_n|f(\Omega)\|_\infty < \infty$. Moreover, if $X = A^p(\Omega, wda)$, then one has $\overrightarrow{\mathcal{F}(T_f)}^w = \overrightarrow{\mathcal{F}(T_f)}^s$.*

Proof. Let g be the point-wise limit on $f(\Omega)$ of a sequence $g_n \in H(\overline{f(\Omega)})$ with $\sup_n \|g_n|f(\Omega)\|_\infty < \infty$. Clearly the convergence is uniform on compact subsets and $g \in H^\infty(f(\Omega))$, so that $g \circ f \in H^\infty(\Omega)$ and $T_{g\circ f}$ is well-defined on X. We have $T_{g_n\circ f} = g_n(T_f) \in \mathcal{F}(T_f)$ for $g_n \in H(\overline{f(\Omega)})$.

If $X = A^p(\Omega, wda)$, then $T_{g_n \circ f} \to T_{g \circ f}$ strongly, for

$$\int_\Omega |(g_n \circ f)h - (g \circ f)h|^p wda = \int_\Omega |(g_n - g) \circ f|^p |h|^p wda \to 0 \tag{2.5.8}$$

by bounded convergence with respect to the finite measure $|h|^p wda$ for each $h \in A^p(\Omega, wda)$. Thus in this case $T_{g \circ f} \in \overrightarrow{\mathcal{F}(T_f)}^s$.

Next consider $X = H^p(\Omega, \omega)$. Let π_ω, $\pi_\omega(0) = \omega$, be a covering map of $\mathbb{D}$ onto Ω with the group G. Fix $h \in H^p(\partial\mathbb{D}, G)$ and $x^* \in H^p(\partial\mathbb{D}, G)^*$. The functional x^* on $H^p(\partial\mathbb{D}, G) \subset L^p(\partial\mathbb{D})$ extends to a functional on $L^p(\partial\mathbb{D})$ which is in turn given by some $u \in L^q(\partial\mathbb{D})$, $1/p + 1/q = 1$, noting $p \neq \infty$. Since the uniformly bounded sequence $g_n \circ f \circ \pi_\omega \in H^\infty(\mathbb{D})$ converges point-wise on $\mathbb{D}$ to $g \circ f \circ \pi_\omega \in H^\infty(\mathbb{D})$, we have by a well-known fact that, upon passing to radial limits in $H^\infty(\partial\mathbb{D})$,

$$g_n \circ f \circ \pi_\omega \to g \circ f \circ \pi_\omega \tag{2.5.9}$$

in the $L^1(\partial\mathbb{D})$-topology on $H^\infty(\partial\mathbb{D})$. Therefore,

$$x^*(T_{g_n \circ f \circ \pi_\omega} h) = \int_{\partial\mathbb{D}} (g_n \circ f \circ \pi_\omega) hu d\theta \to \int_{\partial\mathbb{D}} (g \circ f \circ \pi_\omega) hu d\theta = x^*(T_{g \circ f \circ \pi_\omega} h)$$

due to $hu \in L^1(\partial\mathbb{D})$. That is,

$$T_{g_n \circ f \circ \pi_\omega} | H^p(\partial\mathbb{D}, G) \to T_{g \circ f \circ \pi_\omega} | H^p(\partial\mathbb{D}, G)$$

weakly in $\mathcal{L}(H^p(\partial\mathbb{D}, G))$. In view of the similarity as in the proof of Proposition 2.4.9, $T_{g_n \circ f} \to T_{g \circ f}$ weakly in $\mathcal{L}(H^p(\Omega, \omega))$, so that $T_{g \circ f} \in \overrightarrow{\mathcal{F}(T_f)}^w$.

Now suppose $S = \text{w-}\lim h_n(T_f)$ on X for a sequence $h_n \in H(\sigma(T_f))$. Recall that $\sigma(T_f) = \sigma(f) = \overline{f(\Omega)}$ and $h_n(T_f) = T_{h_n \circ f}$. It follows from the uniform boundedness principle that

$$\sup_n \|T_{h_n \circ f}\| = \sup_n \|h_n(T_f)\| < \infty.$$

Since $\|T_{h_n \circ f}\| = \|h_n \circ f\|_\infty = \|h_n|f(\Omega)\|_\infty$, one has

$$\sup_n \|h_n|f(\Omega)\|_\infty < \infty.$$

By Montel's theorem, the uniformly bounded sequence $h_n|f(\Omega)$ in $H^\infty(f(\Omega))$ has a pointwise convergent subsequence

$$h_{n_k}|f(\Omega) \to h \in H^\infty(f(\Omega)).$$

It then follows from the first part of the proof that $h_{n_k}(T_f) \to T_{h\circ f}$ weakly, strongly if $X = A^p(\Omega, wda)$. Now by uniqueness of limits,

$$S = T_{h\circ f}$$

is of the desired form associated with the subsequence $\{h_{n_k}\}_k$ and its limit h.

Finally noting $\overrightarrow{\mathcal{F}(T_f)}^w \supset \overrightarrow{\mathcal{F}(T_f)}^s$, the proof for both assertions of the theorem is complete. $\square$

Corollary 2.5.6. *Let $f \in H^\infty(\Omega)$. If $V \subset X$ is an invariant subspace for T_f and $G_n \not\subset \sigma(T_f|V)$ for every bounded component G_n of $\mathbb{C}\backslash\overline{f(\Omega)}$, then V is invariant for all $T_{g\circ f}$, where g is the pointwise limit on $f(\Omega)$ of a sequence $g_n \in H(\overline{f(\Omega)})$ satisfying $\sup_n \|g_n|f(\Omega)\|_\infty < \infty$. In particular, if $\mathbb{C}\setminus\overline{f(\Omega)}$ is connected, then every invariant subspace of X for T_f is invariant for all such $T_{g\circ f}$.*

Proof. Noting $\overline{f(\Omega)} = \sigma(T_f)$, there exists by hypothesis λ_n in every bounded component G_n of the resolvent set of T_f such that $\lambda_n I - T_f|V$ is an automorphism of V. Thus $(\lambda_n I - T_f)V = V$ and $(\lambda_n I - T_f)^{-1}V = V$ because $\lambda_n I - T_f$ is an automorphism of X. Since $\mathcal{F}(T_f)$ lies in the norm closure of the algebra $\mathcal{A}$ generated by $I, T_f, \{(\lambda_n I - T_f)^{-1}\}_n$ ([17] p. 24, Theorem 11) for each of which V is invariant, the closed linear subspace V is invariant for the weak closure of $\mathcal{A}$ which contains $\overrightarrow{\mathcal{F}(T_f)}^w$. Since $T_{g\circ f} \in \overrightarrow{\mathcal{F}(T_f)}^w$ by Theorem 2.5.5, V is invariant for all such operators, as desired. $\square$

Remark 2.5.7. We discuss two scenarios in which the preceding condition on g is readily satisfied. Suppose $g \in H^\infty(f(\Omega))$ extends to a function $\tilde{g} \in H^\infty(\Lambda)$ on some open set $\Lambda \supseteq f(\Omega)$. If for the first scenario the geometry of the set Λ allows for the construction of a sequence $\psi_n : \Lambda_n \to \Lambda$ of analytic functions on open sets $\Lambda_n \supset \overline{\Lambda}$ with $\psi_n(z) \to z, \forall z \in \Lambda$, then one can define $g_n := \tilde{g} \circ \psi_n$. This happens if for instance Λ is starlike without centrifugal cuts. The sequence $\{g_n\}_n$ so defined verifies the condition on g in the theorem and corollary, for g_n is analytic on $\Lambda_n \supset \overline{f(\Omega)}$, $\|g_n|f(\Omega)\|_\infty \leq \|g_n\|_\infty \leq \|\tilde{g}\|_\infty$, and $g_n(z) = \tilde{g}(\psi_n(z)) \to \tilde{g}(z) = g(z)$ at every $z \in f(\Omega)$. Therefore, $g_n(T_f) \to T_{g\circ f}$ weakly/strongly. The special case of $\Omega = \Lambda = \mathbb{D}$ and an inner function f will be used to prove Propositions 2.6.2 and 2.7.7. In the second scenario, Λ is an annulus with center z_0 and radii $0 < r_1 < r_2 < \infty$. Considering the Laurent series of $\tilde{g}$, one has $\tilde{g} = g_1 + g_2$ where g_1 is *bounded* and analytic outside the inner circle and g_2 is likewise inside the outer circle [115, p. 229, Exercise 25(d)]. Define the dilates $g_{1,n}(z) := g_1(z_0 + (n+1)(z - z_0)/n)$ outside smaller concentric inner circles, $g_{2,n}(z) := g_2(z_0 + n(z - z_0)/(n+1))$ inside larger outer ones, and put $g_n := g_{1,n} + g_{2,n}$. Then it is similarly verified that the sequence $\{g_n\}_n$ is analytic and uniformly bounded on neighborhoods of $\overline{\Lambda}$ and converges pointwise in Λ to $\tilde{g}$, so that g again satisfies the condition. Note that in general $H^\infty(\Lambda) \not\subset R(\overline{f(\Omega)})$ for $\overline{f(\Omega)} \not\subset \Lambda$, so such $T_{g\circ f}$ is not necessarily in $\overline{\mathcal{F}(T_f)} = \{T_{g\circ f} : g \in R(\overline{f(\Omega)})\}$ (Theorem 2.5.3).

2.6 Commutants on $H^2(\mathbb{D})$

In this section $H^2 := H^2(\mathbb{D}, 0) = H^2(\mathbb{D})$ and $H^\infty := H^\infty(\mathbb{D})$ are the classical Hardy spaces over the disc. We consider the commutant of a collection of multiplication operators by H^∞-composites of a common inner function on the Hilbert space $X = H^2$.

The method employed in this section is based on unitary equivalence of operators. If $\phi \in H^\infty$ is a nonconstant inner function, then $T_\phi \in \mathcal{L}(X)$ is a pure isometry with defect $d := \dim(X \ominus \phi X)$, $1 \leq d \leq \aleph_0$, so that a spatial isometry from X onto $\bigoplus_d X$ intertwines T_ϕ and $\bigoplus_d T_z$ (cf. [47]). In what follows, unitary equivalence of operators on X and on $\bigoplus_d X$ is always relative to this fixed spatial isometry once an inner function is given. The following simple lemma is included for completeness.

Lemma 2.6.1. *Let H be a Hilbert space. The map $S \in \mathcal{L}(H) \mapsto \bigoplus_d S \in \mathcal{L}(\bigoplus_d H)$ is an isometric $*$-isomorphism. In addition, it is sequentially continuous with respect to the weak* *operator topologies.*

Proof. The first part is directly verified. To prove the second let $S_n \to S$ weakly in $\mathcal{L}(H)$. Then $\sup_n \|S_n\| < \infty$ and $\langle S_n g, h\rangle \to \langle Sg, h\rangle, \forall g, h \in H$. For $\bigoplus_{k=1}^d g_k$, $\bigoplus_{k=1}^d h_k \in \bigoplus_d H$, one has as $n \to \infty$ that

$$\left\langle (\bigoplus_d S_n)(\bigoplus_{k=1}^d g_k), \bigoplus_{k=1}^d h_k \right\rangle = \sum_{k=1}^d \langle S_n g_k, h_k\rangle$$
$$\to \sum_{k=1}^d \langle S g_k, h_k\rangle = \left\langle (\bigoplus_d S)(\bigoplus_{k=1}^d g_k), \bigoplus_{k=1}^d h_k \right\rangle,$$

which follows from the Lebesgue dominated convergence theorem since

$$|\langle S_n g_k, h_k\rangle| \le (\sup_n \|S_n\|)\|g_k\|\|h_k\|, \text{ and}$$
$$\sum_{k=1}^d (\sup_n \|S_n\|)\|g_k\|\|h_k\| \le (\sup_n \|S_n\|) \left(\sum_k \|g_k\|^2\right)^{1/2} \left(\sum_k \|h_k\|^2\right)^{1/2} < \infty.$$

That is, $\bigoplus_d S_n \to \bigoplus_d S$ weakly in $\mathcal{L}(\bigoplus_d H)$. □

Proposition 2.6.2. *Let $g \in H^\infty$ and let $\phi \in H^\infty$ be a nonconstant inner function. Then $T_{g\circ\phi}$ is unitarily equivalent to $\bigoplus_d T_g$.*

Proof. Consider the dilates $g_n(z) := g(nz/(n+1))$, $z \in (1+1/n)\mathbb{D}$. Then Remark 2.5.7 for $\Lambda = \mathbb{D}$ and the inner function ϕ asserts that in the weak operator topology

$$g_n(T_\phi) \to T_{g\circ\phi}. \tag{2.6.1}$$

Similarly,

$$g_n(T_z) \to T_{g\circ z} = T_g,$$

which in turn yields

$$\bigoplus_d g_n(T_z) \to \bigoplus_d T_g \tag{2.6.2}$$

in the weak operator topology by the second part of Lemma 2.6.1. Since T_ϕ is unitarily equivalent to $\bigoplus_d T_z$, so is $g_n(T_\phi)$ to $g_n(\bigoplus_d T_z)$ which in turn equals $\bigoplus_d g_n(T_z)$ by the first part of Lemma 2.6.1. Passing to the limit, it follows from (2.6.1) and (2.6.2) that $T_{g\circ\phi}$ is equivalent to $\bigoplus_d T_g$, completing the proof. $\square$

Remark 2.6.3. The above result may be known. The proof is motivated by that of Lemma 1 in [14] and generalizes it to any $g \in H^\infty \supsetneq H(\overline{\mathbb{D}})$, and will also be used to derive Proposition 2.7.7 from Theorem 2.7.6.

For a Hilbert space H, let $\mathcal{P}, \mathcal{Q}$ be the set of (orthogonal) projections on H and $\bigoplus_d H$, respectively. The first part of the following lemma is essential to the approach of this and next sections.

Lemma 2.6.4. *Let H be a Hilbert space and $S_\alpha, S_\beta \in \mathcal{L}(H)$, $\alpha \in \Lambda'$, $\beta \in \Lambda''$. Then one has*

$$\bigcap_\alpha \{\bigoplus_d S_\alpha\}' = \bigcap_\beta \{\bigoplus_d S_\beta\}' \;\;\Leftrightarrow\;\; \bigcap_\alpha \{S_\alpha\}' = \bigcap_\beta \{S_\beta\}',\;\; \textit{and}$$
$$\bigcap_\alpha \{\bigoplus_d S_\alpha\}' \bigcap \mathcal{Q} = \bigcap_\beta \{\bigoplus_d S_\beta\}' \bigcap \mathcal{Q} \;\;\Rightarrow\;\; \bigcap_\alpha \{S_\alpha\}' \bigcap \mathcal{P} = \bigcap_\beta \{S_\beta\}' \bigcap \mathcal{P}.$$

Proof. The basic fact is that, for an operator block matrix $[A_{kl}]_{1\le k,l\le d} \in \mathcal{L}(\bigoplus_d H)$ with entries $A_{kl} \in \mathcal{L}(H)$ and an $S \in \mathcal{L}(H)$,

$$[A_{kl}] \in \{\bigoplus_d S\}' \Leftrightarrow A_{kl} \in \{S\}', \forall k, l.$$

Suppose $\bigcap_\alpha \{S_\alpha\}' \subseteq \bigcap_\beta \{S_\beta\}'$. Then

$$\begin{aligned}[A_{kl}] \in \bigcap_\alpha \{\bigoplus_d S_\alpha\}' &\Leftrightarrow A_{kl} \in \bigcap_\alpha \{S_\alpha\}' \\ \Rightarrow A_{kl} \in \bigcap_\beta \{S_\beta\}' &\Leftrightarrow [A_{kl}] \in \bigcap_\beta \{\bigoplus_d S_\beta\}'.\end{aligned}$$

On the other hand suppose $\bigcap_\alpha \{\bigoplus_d S_\alpha\}' \subseteq \bigcap_\beta \{\bigoplus_d S_\beta\}'$. Then

$$\begin{aligned} A \in \bigcap_\alpha \{S_\alpha\}' &\Leftrightarrow \bigoplus_d A \in \bigcap_\alpha \{\bigoplus_d S_\alpha\}' \\ &\Rightarrow \bigoplus_d A \in \bigcap_\beta \{\bigoplus_d S_\beta\}' \Leftrightarrow A \in \bigcap_\beta \{S_\beta\}'. \end{aligned}$$

Finally suppose $\bigcap_\alpha \{\bigoplus_d S_\alpha\}' \bigcap \mathcal{Q} \subseteq \bigcap_\beta \{\bigoplus_d S_\beta\}' \bigcap \mathcal{Q}$. Then

$$\begin{aligned} A \in \bigcap_\alpha \{S_\alpha\}' \bigcap \mathcal{P} &\Leftrightarrow \bigoplus_d A \in \bigcap_\alpha \{\bigoplus_d S_\alpha\}' \bigcap \mathcal{Q} \\ &\Rightarrow \bigoplus_d A \in \bigcap_\beta \{\bigoplus_d S_\beta\}' \bigcap \mathcal{Q} \Leftrightarrow A \in \bigcap_\beta \{S_\beta\}' \bigcap \mathcal{P}. \end{aligned}$$

The implications of the commutant inclusions are therefore established, and those of the commutant equalities follow by symmetry. □

The main theorem of this section gives a sufficient and necessary condition under which the commutant of a family of multiplication operators equals that of the multiplication operator by a given inner function.

Theorem 2.6.5. *Let $\{f_\alpha\}_\alpha$ be a collection in H^∞ and $\phi \in H^\infty$ a nonconstant inner function. Then $\bigcap_\alpha \{T_{f_\alpha}\}' = \{T_\phi\}'$ if and only if there exist functions $g_\alpha \in H^\infty$ such that $f_\alpha = g_\alpha \circ \phi$ and $\bigcap_\alpha \{T_{g_\alpha}\}' = \{T_z\}'$.*

Proof. Let $\{g_\alpha\}_\alpha$ be a collection in H^∞. In view of Proposition 2.6.2, we deduce

$$\bigcap_\alpha \{T_{g_\alpha \circ \phi}\}' = \{T_\phi\}' \quad \Leftrightarrow \quad \bigcap_\alpha \{\bigoplus_d T_{g_\alpha}\}' = \{\bigoplus_d T_z\}' \tag{2.6.3}$$

$$\Leftrightarrow \quad \bigcap_\alpha \{T_{g_\alpha}\}' = \{T_z\}', \tag{2.6.4}$$

the second equivalence due to Lemma 2.6.4.

So, the sufficiency part follows immediately. Now assume $\bigcap_\alpha \{T_{f_\alpha}\}' = \{T_\phi\}'$. Since inner functions are H^2-ancestral ([36], Thm. 2), $\{T_\phi\}' \subseteq \{T_{f_\alpha}\}'$ implies that $f_\alpha = g_\alpha \circ \phi$ for some function g_α bounded and analytic on $\phi(\mathbb{D})$ [36, Thm. 1]. The inner function ϕ has its range $\phi(\mathbb{D})$

an open subset of $\mathbb{D}$ with $\mathbb{D} \setminus \phi(\mathbb{D})$ of zero capacity, so each g_α analytically continues from $\phi(\mathbb{D})$ to $\mathbb{D}$ and hence by the monodromy theorem admits an extension in H^∞ (cf. [32]). An appeal to the previous paragraph then proves the necessity part. $\square$

In connection with the commonly asked questions in [47] about the commutant of a multiplication operator on H^2, we have

Proposition 2.6.6. *If $g \in H^\infty$ is such that $\{T_g\}' = \{T_z\}'$, then for any nonconstant inner function ϕ one has*

(i) *$\{T_{g\circ\phi}\}'$ lifts isometrically to $\{M_{g\circ\phi}\}'$ of the minimal normal extension $M_{g\circ\phi}$ on $L^2(\partial\mathbb{D})$.*

(ii) *$\{T_{g\circ\phi}\}'$ contains no nonzero compact operators.*

(iii) *$\{T_{g\circ\phi}\}' \subset \{T_\chi\}' \bigcap \{T_F\}'$ for the inner-outer factorization $g\circ\phi = \chi F$.*

Proof. By Theorem 2.6.5, $\{T_{g\circ\phi}\}' = \{T_\phi\}'$. Because $\{T_\phi\}'$ lifts isometrically [38], so does $\{T_{g\circ\phi}\}'$ ([38], Corollary on p. 2), which proves (i). Since inner functions are H^2-ancestral, $\{T_{g\circ\phi}\}' = \{T_\phi\}'$ contains no nonzero compact operators ([36], p. 27), giving (ii).

To prove (iii) write $g = \chi_1 F_1$ for the inner-outer factorization, so that

$$g \circ \phi = (\chi_1 \circ \phi)(F_1 \circ \phi) = \chi F$$

implies $\chi_1 \circ \phi = \lambda\chi$, $F_1 \circ \phi = \lambda^{-1}F$ for some unimodular constant λ. Hence, $\{T_{g\circ\phi}\}' = \{T_\phi\}' \subset \{T_{\chi_1\circ\phi}\}' \bigcap \{T_{F_1\circ\phi}\}' = \{T_\chi\}' \bigcap \{T_F\}'$. $\square$

2.7 Similarity and commutants on $A^2(\mathbb{D}, wda)$

We shall work with a reasonably general subclass of weight functions on $\mathbb{D}$.

Definition 2.7.1. $\mathcal{W}_d(\mathbb{D})$ consists of the functions $w \in \mathcal{W}(\mathbb{D})$ such that, for any $M \in [1, \infty)$, there exist $\epsilon_w > 0$ and $B_w \in (1, \infty)$ depending on M and satisfying

$$B_w^{-1} w(\beta) \le w(\alpha) \le B_w w(\beta) \tag{2.7.1}$$

whenever $|\alpha|, |\beta| \in (1 - \epsilon_w, 1)$ and $M^{-1}(1 - |\beta|) \le 1 - |\alpha| \le M(1 - |\beta|)$.

Remark 2.7.2. The standard Bergman weights $w_r(z) = (1+r)(1-|z|^2)^r$, $r > -1$, are certainly in the class $\mathcal{W}_d(\mathbb{D})$.

We first establish two lemmas for a general power index p. The first uses essentially the argument in [53, Sect. 4.1]. The second, for non-radial weights, does not follow from the existing results found in [42, Sect. 3.1]. Instead, it is inspired by the main features of the proofs in [85, 71].

Lemma 2.7.3. *If* $1 < p < \infty$ *and* $w \in \mathcal{W}_d(\mathbb{D})$*, then the set* $\mathscr{P}$ *of polynomials is dense in* $A^p(\mathbb{D}, wda)$*.*

Proof. Fix an arbitrary $f \in A^p(\mathbb{D}, wda)$. For $r \in (0,1)$, write $f_r(z) = f(rz)$ for the dilate of f defined and analytic in $r^{-1}\mathbb{D} \supset \overline{\mathbb{D}}$. In view of its Taylor series, f_r is uniformly approximable on $\overline{\mathbb{D}}$ by polynomials, *a fortiori* $f_r \in \overline{\mathscr{P}}$ in the weighted L^p norm. We claim $\sup_r \|f_r\| < \infty$.

Notice first the subharmonic function $|f|^p$ satisfies

$$\int_{\lambda\in\partial\mathbb{D}} |f_r(t\lambda)|^p d\theta \le \int_{\lambda\in\partial\mathbb{D}} |f(t\lambda)|^p d\theta, \quad r, t \in (0,1).$$

With $M = 1$, there exist $\epsilon_w > 0$ and $B_w \in (1, \infty)$ such that (2.7.1) holds whenever $|\alpha| = |\beta| \in (1-\epsilon_w, 1)$. Let $K = (1-\epsilon_w)\overline{\mathbb{D}}$. Then, invoking Lemma 2.2.2 and polar coordinates, we deduce for any $r \in (0,1)$ that

$$\begin{aligned}
\|f_r\|^p &\le D_K \int_{\mathbb{D}\setminus K} |f_r|^p w da \\
&= D_K \int_{1-\epsilon_w}^1 \left(\int_{\lambda\in\partial\mathbb{D}} |f_r(t\lambda)|^p w(t\lambda) d\theta \right) tdt \\
&\le D_K B_w \int_{1-\epsilon_w}^1 \left(\int_{\lambda\in\partial\mathbb{D}} |f(t\lambda)|^p d\theta \right) w(t) tdt \\
&\le D_K B_w^2 \int_{1-\epsilon_w}^1 \left(\int_{\lambda\in\partial\mathbb{D}} |f(t\lambda)|^p w(t\lambda) d\theta \right) tdt \\
&= D_K B_w^2 \int_{\mathbb{D}\setminus K} |f|^p w da < \infty.
\end{aligned}$$

Now by reflexivity and separability (Lemma 2.2.3(iv)), the bounded sequence $\{f_{n/(n+1)}\}$ has a subsequence converging weakly in $A^p(\mathbb{D}, wda)$ to some g, $g \in \overline{\mathscr{P}}$ where the norm and weak closures coincide. Since the point evaluations are all in $A^p(\mathbb{D}, wda)^*$, one has $g(z) = f(z), \forall z \in \mathbb{D}$, by passing to the limit. That is, $\overline{\mathscr{P}} = A^p(\mathbb{D}, wda)$ as required. $\square$

Lemma 2.7.4. *For b a finite Blaschke product and $w \in \mathcal{W}_d(\mathbb{D})$, the composition operator $C_b f = f \circ b$ is in $\mathcal{L}(A^p(\mathbb{D}, wda))$, $p \in [1, \infty)$.*

Proof. Notice that for some finite constant $M_b > 1$ depending on b ([85], p. 2966),

$$M_b^{-1}(1 - |z|) \leq 1 - |b(z)| \leq M_b(1 - |z|), \quad z \in \mathbb{D}. \tag{2.7.2}$$

Then there exist $\epsilon_w > 0$ and $B_w \in (1, \infty)$, depending only on M_b, such that (2.7.1) holds whenever $|\alpha|, |\beta| \in (1 - \epsilon_w, 1)$ and $M_b^{-1}(1 - |\beta|) \leq 1 - |\alpha| \leq M_b(1 - |\beta|)$. Since $b(\partial\mathbb{D}) = \partial\mathbb{D}$ and b' is nonvanishing on $\partial\mathbb{D}$, there exists $\delta_w \in (0, \epsilon_w)$ such that $|b(z)| \in (1 - \epsilon_w, 1)$ whenever $|z| \in (1 - \delta_w, 1)$, and that $0 < A_1 \leq |b'(z)| \leq A_2 < \infty$ for $|z| \in [1 - \delta_w, 1 + \delta_w]$. The former condition and (2.7.2) yield from (2.7.1) that

$$B_w^{-1} w(z) \leq w(b(z)) \leq B_w w(z), \quad |z| \in (1 - \delta_w, 1). \tag{2.7.3}$$

The latter condition and compactness give a finite cover $\bigcup_j \Delta_j \supset \{z : 1 - 2^{-1}\delta_w \leq |z| \leq 1\}$ with open discs $\Delta_j \subset \{z : 1 - \delta_w < |z| < 1 + \delta_w\}$, $j = 1, ..., J$, on each of which b is univalent with inverse map b^{-1}.

Let $K = (1 - 2^{-1}\delta_w)\overline{\mathbb{D}}$. Consider an arbitrary $f \in A^p(\mathbb{D}, wda)$ and the analytic function $f \circ b$ on $\mathbb{D}$. Using Lemma 2.2.2, a change of variables by the univalent b on each $\Delta_j \bigcap \mathbb{D}$, and the left half of (2.7.3), one has the estimates

$$\begin{aligned}
\|f\circ b\|^p &\le D_K\int_{\mathbb{D}\setminus K}|f\circ b|^p w da \le D_K\sum_j\int_{\Delta_j\bigcap\mathbb{D}}|f\circ b|^p w da\\
&= D_K\sum_j\int_{b(\Delta_j\bigcap\mathbb{D})}\frac{|f|^p(w\circ b^{-1})}{|b'\circ b^{-1}|^2}da\\
&\le \frac{D_K B_w}{A_1^2}\sum_j\int_{b(\Delta_j\bigcap\mathbb{D})}|f|^p w da\\
&\le \frac{D_K B_w J}{A_1^2}\int_{\mathbb{D}}|f|^p w da = \frac{D_K B_w J}{A_1^2}\|f\|^p.
\end{aligned}$$

This concludes the proof. □

Although not needed here, $C_b \in \mathcal{L}(A^p(\mathbb{D}, wda))$ is also bounded below, with a similar proof.

For the rest of the section, fix $w \in \mathcal{W}_d(\mathbb{D})$, $X = A^2(\mathbb{D}, wda)$, and write $H^\infty = H^\infty(\mathbb{D})$. We follow the approach used in [71] to derive similarity for T_b. The core step is a representation as in [71, Theorem 2.1] for X induced by $C_b \in \mathcal{L}(X)$.

Proposition 2.7.5. *If b is a Blaschke product with* $\deg b = n$*, then there are $T_k \in \mathcal{L}(X)$, $k = 1, ..., n$, satisfying the operator interpolation equation*

$$I = \sum_{k=1}^{n} T_z^{k-1} C_b T_k \tag{2.7.4}$$

on X. Moreover, if $f, f_1, ..., f_n \in X$ satisfy

$$f = \sum_{k=1}^{n} T_z^{k-1} C_b f_k,$$

then $f_k = T_k f$, $k = 1, ..., n$.

Proof. The second part follows from the first and the fact ([71], pp. 359-360) that if $f_1, ..., f_n \in X$ satisfy $\sum_{k=1}^n T_z^{k-1} C_b f_k = 0$, then $f_1 = ... = f_n = 0$.

To prove the first part, W. Rudin's representation [114, Chapter 7] states that for every polynomial $p \in \mathscr{P}$, there are unique rational functions ψ_k, $k = 1, ..., n$, with poles off $\overline{\mathbb{D}}$ that satisfy

$$p = \sum_{k=1}^{n} T_z^{k-1} C_b \psi_k. \tag{2.7.5}$$

This allows each T_k to be well defined as a linear map on the dense (Lemma 2.7.3) linear subspace $\mathscr{P} \subset X$ by $T_k p = \psi_k \in X$ according to (2.7.5). We shall show the existence of a finite constant C such that

$$\|\psi_k\| \leq C\|p\|, \quad p \in \mathscr{P},\ k = 1, ..., n. \tag{2.7.6}$$

Recycling the notations and constructions from the proof of Lemma 2.7.4, let $\epsilon := (2M_b)^{-1}\delta_w$. Then for any $|z| \in [1-\epsilon, 1]$, there exist an open disc Δ_z, $z \in \Delta_z \subset \{1 - 2\epsilon < |z|\}$, and n local inverse maps $\beta_1 := \beta_{1,z}, ..., \beta_n := \beta_{n,z}$ of b defined on Δ_z which satisfy $\overline{\beta_j(\Delta_z)} \bigcap \overline{\beta_k(\Delta_z)} = \emptyset$ for $j \neq k$. The latter ensures that the modulus of the n by n Vandermonde determinant with generators $\{\beta_k\}$ is bounded away from zero on Δ_z.

Consider a finite subcover $\bigcup_{l=1}^{N} \Delta_l \supset \{1 - \epsilon \leq |z| \leq 1\}$. From (2.7.5) the argument used in [71], before (2.7), leads to the upper bound

$$|\psi_k|^2 \leq C' \sum_{j=1}^{n} |p \circ \beta_j|^2, \quad k = 1, ..., n$$

on each Δ_l, $l = 1, ..., N$, for a finite constant C'. Note that $b^{-1}(\Delta_l \bigcap \mathbb{D}) \subset \{1 - \delta_w < |z| < 1\}$ by the left half of (2.7.2), and this implies

$$w \leq B_w w \circ \beta_k, \quad k = 1, ..., n$$

on $\Delta_l \bigcap \mathbb{D}$ by the right half of (2.7.3). Then, using Lemma 2.2.2 with $K = (1-\epsilon)\overline{\mathbb{D}}$ we derive for each k the following estimates

$$\begin{aligned}\|\psi_k\|^2 &\le D_K \int_{\mathbb{D}\setminus K} |\psi_k|^2 w da \le D_K \sum_l \int_{\Delta_l \bigcap \mathbb{D}} |\psi_k|^2 w da \\ &\le D_K C' \sum_l \sum_{j=1}^n \int_{\Delta_l \bigcap \mathbb{D}} |p \circ \beta_j|^2 w da \\ &\le D_K C' B_w \sum_l \sum_{j=1}^n \int_{\Delta_l \bigcap \mathbb{D}} |p \circ \beta_j|^2 (w \circ \beta_j) da \\ &= D_K C' B_w \sum_l \sum_{j=1}^n \int_{\beta_j(\Delta_l \bigcap \mathbb{D})} |p|^2 w |b'|^2 da \\ &\le D_K C' B_w A_2^2 N n \|p\|^2.\end{aligned}$$

This establishes (2.7.6) with $C = (D_K C' B_w A_2^2 N n)^{1/2}$.

Then the bounded, densely defined linear maps T_k extend to $T_k \in \mathcal{L}(X)$. Since $C_b \in \mathcal{L}(X)$ by Lemma 2.7.4, both sides of (2.7.4) are in $\mathcal{L}(X)$ and agree on the dense $\mathscr{P}$ by construction of T_k. Thus, (2.7.4) holds on the whole space X. $\square$

Exactly as [71, Corollary 2.4], similarity now follows from Proposition 2.7.5 via the spatial isomorphism $\Psi := (T_1, ..., T_n) : X \to \bigoplus_n X$.

Theorem 2.7.6. *If b is a Blaschke product with $\deg b = n$, then $T_b \in \mathcal{L}(X) \sim \bigoplus_n T_z \in \mathcal{L}(\bigoplus_n X)$.*

In what follows, similarity of operators on X and on $\bigoplus_n X$ is always relative to the spatial isomorphism Ψ corresponding to a given finite Blaschke product. The next result generalizes Theorem 2.7.6 using extensions of the analytic functional calculus. The proof is similar to that of Proposition 2.6.2, using Remark 2.5.7, Lemma 2.6.1 and Theorem 2.7.6, which is omitted. The special case of $g \in H(\overline{\mathbb{D}})$ on $A_r^2(\mathbb{D})$ was used to prove the main result in [85], pp. 2980-2981.

Proposition 2.7.7. *Let $g \in H^\infty$ and let b be a Blaschke product with $\deg b = n$. Then $T_{g \circ b} \sim \bigoplus_n T_g$.*

Using similarity we are ready to obtain several interesting results on multiplication operators by finite Blaschke products. Among these, the similarity classification (i) was obtained in [85, 71] when $X = A^2_r(\mathbb{D})$.

Theorem 2.7.8. *Let b, b_1, b_2 be finite Blaschke products. Then*

(i) $T_{b_1} \sim T_{b_2}$ *if and only if* $\deg b_1 = \deg b_2$.

(ii) *If $f \in H^\infty$ and $\{T_b\}' \subset \{T_f\}'$, then $f = g \circ b$ for some $g \in H^\infty$.*

(iii) *$\{T_{b_1}\}' \subset \{T_{b_2}\}'$ if and only if $b_2 = B \circ b_1$ for some finite Blaschke product B. In particular, if $\deg b_1 \nmid \deg b_2$, then $\{T_{b_1}\}' \not\subset \{T_{b_2}\}'$.*

(iv) *$\{T_{b_1}\}' = \{T_{b_2}\}'$ if and only if $b_2 = L \circ b_1$ for some $L \in \mathrm{Aut}(\mathbb{D})$.*

(v) *$\{T_b\}'$ contains no nonzero compact operators.*

Proof. (i) follows from Theorem 2.7.6 and invariance of the Fredholm index under similarity, noting $\mathrm{ind} \bigoplus_n T_z = n(\mathrm{ind} T_z) = -n$ by Lemma 2.2.5.

The proof of (ii) uses techniques from the proof of Lemma 3.3 in [85]. Write $\bigoplus_n T_z = \Psi T_b \Psi^{-1}$ for the similarity. Then the hypothesis $\{T_f\}' \supset \{T_b\}'$ implies

$$\{\Psi T_f \Psi^{-1}\}' \supset \{\bigoplus_n T_z\}' \tag{2.7.7}$$

while Theorem 2.3.5 gives the matrix form

$$\{\bigoplus_n T_z\}' = \{[T_{h_{ij}}]_{1 \le i,j \le n} : h_{ij} \in H^\infty\}. \tag{2.7.8}$$

First choose an arbitrary j from $\{1, ..., n\}$ and set $h_{jj} = 1$ with all other entries zero in (2.7.8). A consideration in view of (2.7.7) and with each j asserts

$$\Psi T_f \Psi^{-1} = \bigoplus_n T_j, \quad T_j \in \mathcal{L}(X).$$

Next $\bigoplus_n T_z \in \{\bigoplus_n T_j\}'$ implies each $T_j = T_{g_j}$, $g_j \in H^\infty$, by Theorem 2.3.5 again. Then choose the functions h_{ij} in (2.7.8) to be either identically 1 or 0 using exactly the patterns in [85], p.

2975, and we have $g_1 = ... = g_n$ so that

$$\Psi T_f \Psi^{-1} = \bigoplus_n T_g, \quad g \in H^\infty.$$

On the other hand, Proposition 2.7.7 states for such $g \in H^\infty$ that

$$\Psi T_{g\circ b} \Psi^{-1} = \bigoplus_n T_g.$$

Comparing the last two equalities, we have $T_f = T_{g\circ b}$ and $f = g \circ b$.

Now it follows from (ii) that if $\{T_{b_1}\}' \subset \{T_{b_2}\}'$, then $b_2 = g \circ b_1$ for some $g \in H^\infty$. Write $g = BsF$ where B is a Blaschke product, s a singular inner function, and F an outer function. Then,

$$b_2 = g \circ b_1 = ((Bs) \circ b_1)(F \circ b_1)$$

is the inner-outer factorization for b_2, which renders F constant. So we take

$$b_2 = (B \circ b_1)(s \circ b_1). \tag{2.7.9}$$

It is seen from (2.7.9) that B can not have infinitely many zeros in $\mathbb{D}$ because $b_1(\mathbb{D}) = \mathbb{D}$, so B is a finite product. Since $s \circ b_1$ is singular, (2.7.9) implies s is constant. This proves (iii) since the other direction is trivial by Corollary 2.3.2, and since $b_2 = B \circ b_1$ implies $\deg(b_2) = \deg(B)\deg(b_1)$. (iv) follows at once.

To prove (v), suppose $[T_{h_{ij}}]_{1\le i,j\le n}$, $h_{ij} \in H^\infty$, is compact on $\bigoplus_n X$. Then each $T_{h_{ij}}$ is compact on X which forces $T_{h_{ij}} = 0$ (by a spectral consideration as before). In view of (2.7.8) then, $\{\bigoplus_n T_z\}'$, and $\{T_b\}'$ as well, contains no nonzero compact operators. □

If $\deg b > 1$, then $\{\bigoplus_{\deg b} T_z\}'$ contains nontrivial projections, thus $\{T_b\}'$ contains nontrivial idempotents. Note that for $X = A^2(\mathbb{D})$, $\{T_b\}'$ actually contains projections (cf. [155]), which is not known for general spaces X.

Remark 2.7.9. Since $\mathrm{Aut}(\mathbb{D})$ consists exactly of the rotations of the Möbius maps, which preserve the degree when acting on Blaschke products but *not* vice versa, one sees from (i), (iv) that the commutant equivalence classes of finite Blaschke multiplication operators are *strictly finer* than the similarity equivalence classes. Also, if b were $A^2(\mathbb{D}, wda)$-ancestral, (ii)-(v) would have followed from Theorems 2.3.9 and 2.3.12. Although inner functions are H^2-ancestral [36, Theorem 2], the argument therein does not work for non-isometries like T_b on Bergman spaces.

The next two results correspond to results in Section 2.6.

Theorem 2.7.10. *Let $\{f_\alpha\}_\alpha$ be a collection in H^∞ and b a finite Blaschke product. Then $\bigcap_\alpha \{T_{f_\alpha}\}' = \{T_b\}'$ if and only if there exist functions $g_\alpha \in H^\infty$ such that $f_\alpha = g_\alpha \circ b$ and $\bigcap_\alpha \{T_{g_\alpha}\}' = \{T_z\}'$.*

Proof. In view of Proposition 2.7.7 and Lemma 2.6.4, we have

$$\begin{aligned}\bigcap_\alpha \{T_{g_\alpha \circ b}\}' = \{T_b\}' \quad &\Leftrightarrow \quad \bigcap_\alpha \{\bigoplus_n T_{g_\alpha}\}' = \{\bigoplus_n T_z\}' \\ &\Leftrightarrow \quad \bigcap_\alpha \{T_{g_\alpha}\}' = \{T_z\}'.\end{aligned}$$

This gives the sufficiency part. For the necessity part, $\{T_b\}' \subset \{T_{f_\alpha}\}'$ for each index α asserts by Theorem 2.7.8(ii) that $f_\alpha = g_\alpha \circ b$ for some $g_\alpha \in H^\infty$, and the implications above complete the proof. □

Proposition 2.7.11. *If $g \in H^\infty$ is such that $\{T_g\}' = \{T_z\}'$, then for any finite Blaschke product b one has*

(i) *$\{T_{g\circ b}\}'$ contains no nonzero compact operators.*

(ii) *$\{T_{g\circ b}\}' \subset \{T_\chi\}' \bigcap \{T_F\}'$ for the inner-outer factorization $g \circ b = \chi F$.*

Proof. By Theorem 2.7.10, $\{T_{g\circ b}\}' = \{T_b\}'$. Then, (i) follows from Theorem 2.7.8(v), and the proof of (ii) is identical to that of Proposition 2.6.6(iii). □

Jiang and Zheng [85] obtained a similarity classification for a class of multiplication operators on $A^2_r(\mathbb{D})$: *For $f, g \in H(\overline{\mathbb{D}})$, $T_f \sim T_g$ on $A^2_r(\mathbb{D})$ if and only if $f = h \circ b_1$ and $g = h \circ b_2$ for finite Blaschke products $b_1, b_2, \deg b_1 = \deg b_2$ and $h \in H(\overline{\mathbb{D}})$.* We shall obtain a commutant classification for a larger class on X.

We first need a result on multiplication by outer functions. The proof relies on an argument used by Cowen ([38], pp. 4-5) for reciprocals of outer functions.

Lemma 2.7.12. *Let $p \in [1, \infty)$ and $w \in \mathcal{W}(\mathbb{D})$. Let $f \in H^\infty$ be an outer function. Then, the range of $T_f \in \mathcal{L}(A^p(\mathbb{D}, wda))$ is dense.*

Proof. We rephrase Cowen's argument to fit in our setting. Evidently, we can assume $\|f\|_\infty = 1$. Identifying the algebra H^∞ with $H^\infty(\partial\mathbb{D}) \subset L^\infty(\partial\mathbb{D})$, the truncated functions $|f|^{-1} \wedge n$, $n \in \mathbb{N}$, on $\partial\mathbb{D}$ are invertible functions in $L^\infty(\partial\mathbb{D})$. So there exist invertible H^∞ functions h_n with $|h_n| = |f|^{-1} \wedge n$ on $\partial\mathbb{D}$. Note that for each n, $\|h_n f\|_\infty \leq 1$ and $-\ln|f| \geq -\ln|h_n f| \geq 0$ on $\partial\mathbb{D}$. Since

$$\int_{\partial\mathbb{D}} -\ln|f| d\theta < \infty,$$

the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty} \int_{\partial\mathbb{D}} -\ln|h_n f| d\theta = 0,$$

while the outer functions $h_n f$ satisfy

$$\int_{\partial\mathbb{D}} -\ln|h_n f| d\theta = -\ln|h_n(0) f(0)|.$$

Thus $\lim_{n\to\infty} |h_n(0) f(0)| = 1$. The uniformly bounded sequence $\{h_n f\}$ in H^∞ has a convergent subsequence $h_{n_k} f \to h \in H^\infty$ pointwise in $\mathbb{D}$. Since $\|h\|_\infty \leq 1$ and $|h(0)| = 1$, $h \equiv \lambda$ a unimodular constant.

Fix $g \in A^p(\mathbb{D}, wda)$. Since $h_{n_k} f \to \lambda$ pointwise in $\mathbb{D}$ and $\|h_{n_k} f - \lambda\|_\infty \leq 2$,

$$\int_{\mathbb{D}} |h_{n_k} f - \lambda|^p |g|^p w da \to 0$$

by bounded convergence relative to the finite measure $|g|^p w da$. That is, $h_{n_k} f g \to \lambda g$ in $A^p(\mathbb{D}, wda)$. Thus λg and g are in the closure of the range of T_f. □

Next, like [85, Theorem 3.1] and [44, Theorem 1.1], we need the space-X version of a result of Cowen ([36], Corollary on p. 19) and Thomson [136, 137] originally proved on the Hardy space H^2. This line of results [73, Chapter 3] remains valid on weighted Bergman spaces X, with identical proofs, based on two facts: First, the reproducing kernels $K_z \in X$ are co-analytic in $z \in \mathbb{D}$. For, the inner product $\langle K_{\bar{z}}, f\rangle = \overline{\langle f, K_{\bar{z}}\rangle} = \bar{f}(\bar{z})$ is analytic in z for every $f \in X$, that is, $z \in \mathbb{D} \mapsto K_{\bar{z}} \in X$ is norm-analytic. Secondly, Lemma 2.7.12 ensures $\ker T_f^* = \ker T_\phi^*$ on X for the inner factor ϕ of $f \in H^\infty$. Therefore, we state without proof

Theorem 2.7.13 (Cowen-Thomson). *Let $f \in H^\infty$. If the inner factor of $f - f(\alpha)$ is a finite Blaschke product for an $\alpha \in \mathbb{D}$, then there is a finite Blaschke product b such that $\{T_f\}' = \{T_b\}'$ on X.*

Denote by $\mathcal{CT}$ [44] the Cowen-Thomson class of H^∞ functions f as above. The product of a finite Blaschke product and an outer function is in $\mathcal{CT}$. Also if $f(\mathbb{D}) \not\subset f(\partial\mathbb{D})$, equivalently $\sigma(f, H^\infty) \supsetneqq \sigma(f, H^\infty + C)$, then $f \in \mathcal{CT}$ (Remark 2.2.6), and this subsumes the case of nonconstant $f \in H(\overline{\mathbb{D}})$ (Remark 2.3.13). On the other hand it is seen that, using the factorization in the proof of Theorem 1.9.5, the only inner functions in $\mathcal{CT}$ are the finite Blaschke products. Note in passing that the Cowen-Thomson theorem together with Theorem 2.7.8(v) eliminates the ancestral condition and relaxes the other in Theorem 2.3.12 for the special case of $X = A^2(\mathbb{D}, w da)$, $w \in \mathcal{W}_d(\mathbb{D})$. Now we obtain a commutant classification for $\{T_f : f \in \mathcal{CT}\}$.

Theorem 2.7.14. *For $f, g \in \mathcal{CT}$, $\{T_f\}' = \{T_g\}'$ on X if and only if $f = h_1 \circ b$ and $g = h_2 \circ b$ for a finite Blaschke product b and $h_i \in H^\infty$ with $\{T_{h_i}\}' = \{T_z\}'$, $i = 1, 2$.*

Proof. Suppose $\{T_f\}' = \{T_g\}'$ on X. Then Theorem 2.7.13 gives $\{T_f\}' = \{T_g\}' = \{T_b\}'$ for a finite Blaschke product b. By Theorem 2.7.10, this implies $f = h_1 \circ b$ and $g = h_2 \circ b$ for $h_i \in H^\infty$ with $\{T_{h_i}\}' = \{T_z\}'$, $i = 1, 2$, as desired.

The other direction follows from Theorem 2.7.10 which gives $\{T_f\}' = \{T_b\}' = \{T_g\}'$. □

In contrast to finite Blaschke products, $\{T_f\}' = \{T_g\}'$ does not imply $T_f \sim T_g$ despite certain analogy between the commutant and similarity classifications. For, simply take $g = f + \alpha$, α a nonzero constant, and note $\sigma(T_f) \neq \sigma(T_f) + \alpha$. Then, $\{T_f\}' = \{T_g\}'$ while $T_f \not\sim T_g$.

Chapter 3

Allan-Douglas Localization for Toeplitz and Hankel Operators

3.1 Overview

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. L^2 and L^∞ are the Lebesgue spaces of square integrable and essentially bounded measurable complex functions, respectively, on the unit circle $\partial\mathbb{D}$ equipped with the normalized Lebesgue measure $d\theta$. $H^2 \subset L^2$ denotes the Hardy subspace, and $H^\infty \subset L^\infty$ the subalgebra of boundary functions of bounded analytic functions in $\mathbb{D}$. Let C, PC, and $QC := (H^\infty + C) \bigcap \overline{H^\infty + C} = VMO \bigcap L^\infty$ be the C*-subalgebras of L^∞ consisting respectively of continuous, piecewise continuous, and quasicontinuous functions on $\partial\mathbb{D}$. Here VMO is the set of functions of vanishing mean oscillation on $\partial\mathbb{D}$. The C*-algebra generated by PC and QC in L^∞ is denoted by PQC.

For $f \in L^\infty$, $M_f \in \mathcal{L}(L^2)$ denotes the multiplication operator by f on L^2 whose compression to H^2, $T_f := PM_f|H^2$, is the Toeplitz operator with symbol f. Let $C_{\bar{z}} \in \mathcal{L}(L^2)$ be the composition operator by the complex conjugate $z \mapsto \bar{z}$ on $\partial\mathbb{D}$ and write $\tilde{f} := C_{\bar{z}} f = f \circ \bar{z}$ for $f \in L^2$. Define the Hankel operator H_f with symbol $f \in L^\infty$ to be the compression of $C_{\bar{z}} M_z M_f \in \mathcal{L}(L^2)$ to H^2

$$H_f := PC_{\bar{z}} M_z M_f|H^2 = C_{\bar{z}} M_z (I - P) M_f|H^2.$$

One has $H_f^* = H_{\tilde{\bar{f}}}$ and the following crucial identities (cf. [18])

$$T_{fg} = T_f T_g + H_{\tilde{f}} H_g, \tag{3.1.1}$$

$$H_{fg} = H_f T_g + T_{\tilde{f}} H_g. \tag{3.1.2}$$

For a subset $S \subset L^\infty$, $\mathcal{T}(S)$ and $\mathcal{TH}(S)$ are the norm-closed subalgebras of $\mathcal{L}(H^2)$ generated by Toeplitz and, respectively, Toeplitz and Hankel operators with symbols in S, and write $\mathcal{H}_S := \{H_f : f \in S\}$. Identifying L^∞ functions with their Gelfand transform on $M(L^\infty)$, let $S|F$ be the restriction of S to a compact subset $F \subset M(L^\infty)$. $\mathcal{K}(H^2)$ denotes the ideal of compact operators on H^2 and $\pi : \mathcal{L}(H^2) \to \mathcal{L}(H^2)/\mathcal{K}(H^2)$ the quotient map onto the Calkin algebra. The maximal ideal space $M(A)$ of a commutative unital Banach algebra A is endowed with the Gelfand topology, and the fiber $M_x(A)$ consists of extensions in $M(A)$ of $x \in M(B)$ for a subalgebra B.

Noting [18] 2.79(b), a result of Sarason [120] (p. 825) states that for $S \subset L^\infty$ a module over C with $M(C) = \partial\mathbb{D}$, the sup-norm distances satisfy

$$d(f, S) = \max\{d(f|M_\lambda(L^\infty), S|M_\lambda(L^\infty)) : \lambda \in \partial\mathbb{D}\}, \quad f \in L^\infty. \tag{3.1.3}$$

Sarason indicated the partition-of-unity argument ([118], p. 12) to prove formula (3.1.3), an idea attributed to [123]. Based on the Allan-Douglas localization principle, we establish in Section 3.2 a distance localization formula for unital C*-algebras with nontrivial centers. The proof uses intrinsic similarities between the two cases.

While originally used by Douglas to derive various essential properties of Toeplitz operators on H^2 [49, 48], the localization principle has had its applicability extended considerably [99, 104, 105, 106, 124, 88]. Also see [18], p. 43. In view of these applications in the Calkin algebra, we apply the distance formula to show locality of the problem of when the product of two Hankel operators is a compact perturbation of a Hankel operator. While the important special case of complex conjugates of two inner functions is solved in [27], the general case is still open. Note that QC is invariant under the composition operator $C_{\bar{z}}$, and $M(QC)$ is invariant for the adjoint

$C_{\bar{z}}^*$. Writing $\bar{y} := C_{\bar{z}}^* y = y \circ C_{\bar{z}} \in M(QC)$ for $y \in M(QC)$, the main result in Section 3.3 is

Theorem 3.1.1. *Let $f, g \in L^\infty$. Then $H_f H_g \in \mathcal{H}_{L^\infty} + \mathcal{K}(H^2)$ if and only if for every $y \in M(QC)$ and $\epsilon > 0$, there exist $\phi(y,\epsilon), \psi(y,\epsilon) \in L^\infty$ satisfying*

$$\|(f - \phi(y,\epsilon))|M_y(L^\infty)\|_\infty < \epsilon, \tag{3.1.4}$$

$$\|(g - \psi(y,\epsilon))|M_y(L^\infty)\|_\infty < \epsilon, \tag{3.1.5}$$

$$H_{\phi(y,\epsilon)} H_{\psi(\bar{y},\epsilon)} \in \mathcal{H}_{L^\infty} + \mathcal{K}(H^2). \tag{3.1.6}$$

Power ([104], Theorem 2.4) showed that the essential spectrum $\sigma_e(H_g) = \sigma(\pi H_g)$ of a Hankel operator is antipodal symmetric if $g \in L^\infty$ satisfies

$$g|M_\lambda(L^\infty) \in H^\infty|M_\lambda(L^\infty), \quad \lambda = 1, -1. \tag{3.1.7}$$

Note that $H_g = S_{zg}$ in Power's notation, which does not matter here since z is constant on the fibers. The commutators $[T_f, H_g] := T_f H_g - H_g T_f$ occur naturally in the theory. For instance, commuting Toeplitz and Hankel operators with L^∞ symbols were characterized in [96] and the essentially commuting ones characterized in [75]. Using ideas from [104] and constructing certain QC functions, we apply the localization principle to establish antipodal symmetry of $\sigma_e[T_f, H_g]$ for arbitrary $f \in L^\infty$ while $g \in L^\infty$ satisfies a condition much weaker than (3.1.7). Let $\lambda \in \partial\mathbb{D}$. For $\delta > 0$, let $m_{\lambda,\delta}$ in the closed unit ball of the dual space QC^* be the averaging functional over the subarc $(\lambda - \delta, \lambda + \delta)$ of $\partial\mathbb{D}$, and define [120] in the weak-star topology of QC^*

$$M_\lambda^0(QC) := \overline{\{m_{\lambda,\delta} : \delta > 0\}} \bigcap M_\lambda(QC) = \{\lim_\omega m_{\lambda,\delta_\omega} : \lim_\omega \delta_\omega = 0\}. \tag{3.1.8}$$

The main result in Section 3.4 is

Theorem 3.1.2. *Let $f \in L^\infty$. Let $g \in L^\infty$ satisfy*

$$g|M_y(L^\infty) \in H^\infty|M_y(L^\infty), \quad \forall y \in M_1^0(QC) \bigsqcup M_{-1}^0(QC). \tag{3.1.9}$$

Then, $\sigma_e[T_f, H_g] = -\sigma_e[T_f, H_g]$.

We restate Power's theorem under the weaker condition (3.1.9), a result of independent interest. In particular, it yields a result parallel to the theorem above.

Theorem 3.1.3. *If* $g \in L^\infty$ *satisfies condition* (3.1.9)*, then* $\sigma_e(H_g) = -\sigma_e(H_g)$.

In Section 3.5, the difference in strength between conditions (3.1.7) and (3.1.9) will be made clear using a simple result on support sets which will also yield an equivalent form of (3.1.9). In addition, we characterize (3.1.9) in terms of QC functions for conjugates of interpolating Blaschke products, from which examples are constructed of those products with conjugates satisfying (3.1.9) but not (3.1.7). These ideas also produce characteristic functions satisfying (3.1.9) but not (3.1.7). On the other hand, the converse (excluding the trivial case of $f = 0$) to Theorem 3.1.2 fails. We do not know whether the converse, or certain partial converses, to Theorem 3.1.3 is true. We are also led to ask if there exist $f, g \in L^\infty$ such that $\sigma_e[T_f, H_g]$ is *not* antipodal symmetric?

3.2 Localization of distances in unital C*-algebras with centers

The Allan-Douglas localization principle is related to central decompositions of (unital) C*-algebras and to sheaf theory. This paper uses the following version.

Theorem 3.2.1 ([49, 48, 18])**.** *Let* $\mathscr{A}$ *be a C*-subalgebra contained in the center of a C*-algebra* $\mathscr{U}$*. For every maximal ideal* $\alpha \in M(\mathscr{A})$*, let* $\mathscr{I}_\alpha$ *be the closed bideal of* $\mathscr{U}$ *generated by* α*, and let* $\Phi_\alpha : \mathscr{U} \to \mathscr{U}/\mathscr{I}_\alpha$ *be the quotient map onto the quotient* (*local*) *C*-algebra. Then*

(i) *The map* $\bigoplus\{\Phi_\alpha : \alpha \in M(\mathscr{A})\} : \mathscr{U} \to \bigoplus\{\mathscr{U}/\mathscr{I}_\alpha : \alpha \in M(\mathscr{A})\}$ *is an isometric* $*$*-isomorphism.*

(ii) *For* $u \in \mathscr{U}$*, the map* $\alpha \in M(\mathscr{A}) \mapsto \|\Phi_\alpha(u)\|$ *is upper semicontinuous.*

(iii) *For* $u \in \mathscr{U}$*,* $\sigma(u) = \bigcup\{\sigma(\Phi_\alpha(u)) : \alpha \in M(\mathscr{A})\}$*.*

The main result of this section states that the distance in a C*-algebra from an element to a submodule over a central C*-subalgebra equals the maximum of the distances in the local C*-algebras induced by the subalgebra.

Theorem 3.2.2. *Let $\mathscr{A}$ be a C^*-subalgebra contained in the center of a C^*-algebra $\mathscr{U}$, and $\mathscr{M} \subset \mathscr{U}$ be a submodule over $\mathscr{A}$. For every maximal ideal $\alpha \in M(\mathscr{A})$, let $\mathscr{I}_\alpha$ be the closed bideal of $\mathscr{U}$ generated by α, and let $\Phi_\alpha : \mathscr{U} \to \mathscr{U}/\mathscr{I}_\alpha$ be the quotient map. Then for $u \in \mathscr{U}$,*

$$d(u, \mathscr{M}) = \max\{d(\Phi_\alpha(u), \Phi_\alpha(\mathscr{M})) : \alpha \in M(\mathscr{A})\}.$$

Proof. First note by Theorem 3.2.1(ii) that the map

$$\alpha \in M(\mathscr{A}) \mapsto d(\Phi_\alpha(u), \Phi_\alpha(\mathscr{M})) = \inf\{\|\Phi_\alpha(u - v)\| : \ v \in \mathscr{M}\}$$

remains upper semicontinuous, so that its supremum over the compact space $M(\mathscr{A})$ is indeed a maximum.

Denote the maximum by M and fix an arbitrary $\epsilon > 0$. At every $\alpha \in M(\mathscr{A})$,

$$d(\Phi_\alpha(u), \Phi_\alpha(\mathscr{M})) < M + \epsilon \Rightarrow \|\Phi_\alpha(u - v_\alpha)\| = \|\Phi_\alpha(u) - \Phi_\alpha(v_\alpha)\| < M + \epsilon$$

for some $v_\alpha \in \mathscr{M}$. Then, for $u - v_\alpha \in \mathscr{U}$, upper semicontinuity of the map $\beta \in M(\mathscr{A}) \mapsto \|\Phi_\beta(u - v_\alpha)\|$ gives an open neighborhood G_α of α with

$$\|\Phi_\beta(u) - \Phi_\beta(v_\alpha)\| = \|\Phi_\beta(u - v_\alpha)\| < M + \epsilon, \quad \beta \in G_\alpha. \tag{3.2.1}$$

Let $\{G_{\alpha_1}, ..., G_{\alpha_n}\}$ be a finite subcover of the open cover $\{G_\alpha : \alpha \in M(\mathscr{A})\}$ of the compact space $M(\mathscr{A})$, and let $\{\psi_1, ..., \psi_m\}$ be a partition of unity on $M(\mathscr{A})$ subordinate to the subcover. That is, each ψ_k, $k = 1, ..., m$, is a continuous function on $M(\mathscr{A})$ taking values in $[0, 1]$ and supported in some $G_{\alpha_{n(k)}}$, $1 \leq n(k) \leq n$, such that $\sum_{k=1}^m \psi_k \equiv 1$. Identifying via the Gelfand transform each ψ_k with an element of $\mathscr{A}$, we observe that

$$v^\epsilon := \sum_{k=1}^m v_{\alpha_{n(k)}} \psi_k \in \mathscr{M} \tag{3.2.2}$$

since $\mathscr{M}$ is a module over $\mathscr{A}$.

Let $\beta \in M(\mathscr{A})$. Since $\psi_k - \psi_k(\beta)e \in \mathscr{I}_\beta$, one has

$$\Phi_\beta(\psi_k) = \psi_k(\beta)\Phi_\beta(e).$$

Hence, the image in the local C*-algebra of $v^\epsilon \in \mathscr{M}$ defined in (3.2.2) is

$$\Phi_\beta(v^\epsilon) = \sum_{k=1}^{m} \Phi_\beta(v_{\alpha_{n(k)}})\Phi_\beta(\psi_k) = \sum_{k=1}^{m} \psi_k(\beta)\Phi_\beta(v_{\alpha_{n(k)}}). \tag{3.2.3}$$

If $k \in \{1, ..., m\}$ is such that $G_{\alpha_{n(k)}} \ni \beta$, then $\|\Phi_\beta(u) - \Phi_\beta(v_{\alpha_{n(k)}})\| < M + \epsilon$ by (3.2.1). If otherwise $G_{\alpha_{n(k)}} \not\ni \beta$, then $\psi_k(\beta) = 0$ since ψ_k is supported in $G_{\alpha_{n(k)}}$. Therefore, using (3.2.3) and $\sum_{k=1}^m \psi_k(\beta) = 1$ we have the estimate

$$\begin{aligned} \|\Phi_\beta(u) - \Phi_\beta(v^\epsilon)\| &\leq \sum_{k=1}^{m} \psi_k(\beta)\|\Phi_\beta(u) - \Phi_\beta(v_{\alpha_{n(k)}})\| \\ &\leq \sum_{k=1}^{m} \psi_k(\beta)(M + \epsilon) = M + \epsilon. \end{aligned}$$

We have shown that $\|\Phi_\beta(u - v^\epsilon)\| = \|\Phi_\beta(u) - \Phi_\beta(v^\epsilon)\| \leq M + \epsilon$ for every $\beta \in M(\mathscr{A})$, so that $\|u - v^\epsilon\| \leq M + \epsilon$ by applying Theorem 3.2.1(i) to $u - v^\epsilon \in \mathscr{U}$. Because $v^\epsilon \in \mathscr{M}$, $d(u, \mathscr{M}) \leq \inf_{\epsilon > 0} \|u - v^\epsilon\| \leq M$. The other direction is trivial due to $\|\Phi_\alpha\| \leq 1$, $\forall \alpha \in M(\mathscr{A})$. This ends the proof. □

Remark 3.2.3. In view of (3.2.2), it suffices to assume $\mathscr{M} + \mathscr{M} \subset \mathscr{M}$ and $\mathscr{M}\mathscr{A}^+ \subset \mathscr{M}$, where $\mathscr{A}^+$ is the set of positive elements of $\mathscr{A}$.

If the C*-algebra $\mathscr{U}$ is commutative, then the localization principle in Theorem 3.2.1 relative to a C*-subalgebra $\mathscr{A}$ reduces to the case in which $\mathscr{I}_\alpha$ for $\alpha \in M(\mathscr{A})$ is the ideal of the elements of $\mathscr{U}$ vanishing on the fiber $M_\alpha(\mathscr{U})$, and $\Phi_\alpha : \mathscr{U} \to \mathscr{U}/\mathscr{I}_\alpha$ is identical via an isometric isomorphism to the restriction of $\mathscr{U} \cong C(M(\mathscr{U}))$ to $M_\alpha(\mathscr{U})$. With these observations we have the following immediate corollary.

Corollary 3.2.4. *Let $\mathscr{A}$ be a C*-subalgebra of a commutative C*-algebra $\mathscr{U}$, and $\mathscr{M} \subset \mathscr{U}$ be a* *module over $\mathscr{A}$. Then*

$$d(u, \mathscr{M}) = \max\{d(u|M_\alpha(\mathscr{U}), \mathscr{M}|M_\alpha(\mathscr{U})) : \alpha \in M(\mathscr{A})\}, \quad u \in \mathscr{U}.$$

Evidently, Corollary 3.2.4 contains Sarason's result (3.1.3) and Shilov's result that [123]

$$d(f, A) = \max\{d(f|X_y, A|X_y) : y \in M(Q_A)\}, \quad f \in C(X) \tag{3.2.4}$$

for a uniform algebra $A \subset C(X)$ on a compact Hausdorff space X, $Q_A := A \bigcap \overline{A}$ and $X_y := M_y(C(X))$. For the larger class of $\mathscr{M}$ as in Remark 3.2.3, it parallels the Bishop-Glicksberg theorem for closed ideals of uniform algebras. The proof of the latter is based on different ideas, and uses the Krein-Milman theorem and the fact that the extreme annihilating measures for the ideal are supported in maximal sets of antisymmetry for the algebra.

3.3 Compact perturbations of Hankel operators

Consider the C*-subalgebras $C_s := \{f \in C : \tilde{f} = f\}$ and $QC_s := \{f \in QC : \tilde{f} = f\}$ of L^∞. The Allan-Douglas localization principle was adapted to the C*-algebra $\pi\mathcal{TH}(L^\infty)$ relative to its central subalgebra $\pi\mathcal{T}(C_s) \cong C_s$ [104], and to $\pi\mathcal{TH}(PQC)$ relative to $\pi\mathcal{T}(QC_s) \cong QC_s$ [105, 124]. Since Power [104] and M. Hoffman [82] showed that the commutant of $\pi\mathcal{TH}(L^\infty)$ in the Calkin algebra is $\pi\mathcal{T}(QC_s)$, the center of $\pi\mathcal{TH}(L^\infty)$ is precisely $\pi\mathcal{T}(QC_s)$. Therefore, one immediately has the following version of the localization principle for $\pi\mathcal{TH}(L^\infty)$ relative to its center $\pi\mathcal{T}(QC_s) \cong QC_s$, where the identification is via $f \mapsto \pi T_f$. (Here the ideals $\mathscr{I}_x$ are the Glimm ideals of $\pi\mathcal{TH}(L^\infty)$.) Theorem 3.3.1(ii) states that restrictions of symbol functions on fibers dominate the *local* Toeplitz and Hankel operators.

Theorem 3.3.1. *For every $x \in M(QC_s)$, let $\mathscr{I}_x$ be the closed bideal of $\pi\mathcal{TH}(L^\infty)$ generated by the maximal ideal of $\pi\mathcal{T}(QC_s)$ corresponding to x, and let $\Phi_x : \pi\mathcal{TH}(L^\infty) \to \pi\mathcal{TH}(L^\infty)/\mathscr{I}_x$ be the quotient map onto the local C*-algebra. Then*

(i) *For $T \in \mathcal{TH}(L^\infty)$, $\sigma_e(T) = \bigcup\{\sigma(\Phi_x(\pi T)) : x \in M(QC_s)\}$.*

(ii) *For $f \in L^\infty$ and $x \in M(QC_s)$, one has*

$$\|\Phi_x(\pi T_f)\| \leq \|f|M_x(L^\infty)\|_\infty, \quad \|\Phi_x(\pi H_f)\| \leq \|f|M_x(L^\infty)\|_\infty.$$

Proof. (i) is trivially due to $\sigma(\pi T) = \sigma(\pi T, \pi\mathcal{TH}(L^\infty))$ and Theorem 3.2.1(iii).

For (ii), we use the standard argument by Douglas in [49, Prop. 7.50]. Let $M := \|f|M_x(L^\infty)\|_\infty$ and fix an arbitrary $\epsilon > 0$. Since the fiber

$$M_x(L^\infty) \subset U_\epsilon := \{\xi \in M(L^\infty) : |f(\xi)| < M + \epsilon\},$$

U_ϵ an open subset of $M(L^\infty)$, there exists an open neighborhood G_ϵ of x in $M(QC_s)$ such that

$$\bigsqcup\{M_y(L^\infty) : y \in G_\epsilon\} \subset U_\epsilon. \tag{3.3.1}$$

Choose $\psi \in QC_s$ with $\psi(M(QC_s)) \subset [0,1]$, $\psi(x) = 0$, and $\psi \equiv 1$ on $M(QC_s) \setminus G_\epsilon$. By (3.3.1), $f(1-\psi)$ vanishes on $M(L^\infty) \setminus U_\epsilon$ while $\|f(1-\psi)|U_\epsilon\|_\infty \leq M + \epsilon$, so

$$\|f(1-\psi)\|_\infty \leq M + \epsilon.$$

By (3.1.1) and Hartman's theorem,

$$\Phi_x(\pi T_f) = \Phi_x(\pi T_{f\psi}) + \Phi_x(\pi T_{f(1-\psi)}) = \Phi_x(\pi T_f)\Phi_x(\pi T_\psi) + \Phi_x(\pi T_{f(1-\psi)})$$

where $\Phi_x(\pi T_\psi) = 0$ for $\psi(x) = 0 \Rightarrow \pi T_\psi \in \mathscr{I}_x$, and $\|\Phi_x(\pi T_{f(1-\psi)})\| \leq \|f(1-\psi)\|_\infty \leq M + \epsilon$. Thus we have $\|\Phi_x(\pi T_f)\| \leq M$ by sending $\epsilon \downarrow 0$.

Similarly, by (3.1.2) and Hartman's theorem,

$$\Phi_x(\pi H_f) = \Phi_x(\pi H_f)\Phi_x(\pi T_\psi) + \Phi_x(\pi H_{f(1-\psi)}) = \Phi_x(\pi H_{f(1-\psi)})$$

where $\|\Phi_x(\pi H_{f(1-\psi)})\| \leq \|f(1-\psi)\|_\infty \leq M + \epsilon$. Thus we have $\|\Phi_x(\pi H_f)\| \leq M$ as well. The proof is complete. $\square$

For $x \in M(QC_s)$, Power [105] pointed out that the fiber $M_x(QC) = \{y, \bar{y}\}$, $\bar{\bar{y}} = y$, and that $y = \bar{y}$ if and only if

$$y \in M_1^0(QC) \bigsqcup M_{-1}^0(QC) =: K^0. \tag{3.3.2}$$

We are ready to prove Theorem 3.1.1 by Theorem 3.2.2 and (ii) of Theorem 3.3.1.

Proof of Theorem 3.1.1. Necessity is trivial by setting $\phi(y,\epsilon) = f, \psi(y,\epsilon) = g$ for all y, ϵ. For sufficiency, fix an arbitrary $x \in M(QC_s)$ with $M_x(QC) = \{y, \bar{y}\}$ and consider two cases.

First suppose $y = \bar{y}$. Then one has $M_x(L^\infty) = M_y(L^\infty)$, so that (3.1.4), (3.1.5), and Theorem 3.3.1(ii) give

$$\|\Phi_x(\pi H_f) - \Phi_x(\pi H_{\phi(y,\epsilon)})\| < \epsilon, \quad \|\Phi_x(\pi H_g) - \Phi_x(\pi H_{\psi(y,\epsilon)})\| < \epsilon.$$

A standard estimate then gives

$$\|\Phi_x(\pi H_f \pi H_g) - \Phi_x(\pi H_{\phi(y,\epsilon)} \pi H_{\psi(y,\epsilon)})\| \le \epsilon(\|f\|_\infty + \|g\|_\infty + \epsilon),$$

while (3.1.6) and $y = \bar{y}$ imply $\Phi_x(\pi H_{\phi(y,\epsilon)} \pi H_{\psi(y,\epsilon)}) \in \Phi_x(\pi \mathcal{H}_{L^\infty})$. Therefore, after sending $\epsilon \downarrow 0$,

$$d(\Phi_x(\pi H_f \pi H_g), \Phi_x(\pi \mathcal{H}_{L^\infty})) = 0. \tag{3.3.3}$$

Next suppose $y \neq \bar{y}$, and one has

$$M_x(L^\infty) = M_y(L^\infty) \bigsqcup M_{\bar{y}}(L^\infty). \tag{3.3.4}$$

For $\phi_1 = \phi(y, \epsilon), \psi_1 = \psi(y, \epsilon), \phi_2 = \phi(\bar{y}, \epsilon), \psi_2 = \psi(\bar{y}, \epsilon)$, the hypotheses read

$$\|(f - \phi_1)|M_y(L^\infty)\|_\infty < \epsilon, \quad \|(f - \phi_2)|M_{\bar{y}}(L^\infty)\|_\infty < \epsilon,$$
$$\|(g - \psi_1)|M_y(L^\infty)\|_\infty < \epsilon, \quad \|(g - \psi_2)|M_{\bar{y}}(L^\infty)\|_\infty < \epsilon,$$
$$H_{\phi_1} H_{\psi_2} \in \mathcal{H}_{L^\infty} + \mathcal{K}(H^2), \quad H_{\phi_2} H_{\psi_1} \in \mathcal{H}_{L^\infty} + \mathcal{K}(H^2).$$

Choose $\eta \in QC$ such that $\eta(y) = 1,\ \eta(\bar{y}) = 0$, and set

$$j := \eta\phi_1 + (1 - \eta)\phi_2,$$
$$k := \eta\psi_1 + (1 - \eta)\psi_2.$$

Then one has by (3.3.4) and the construction that

$$\|(f - j)|M_x(L^\infty)\|_\infty < \epsilon, \quad \|(g - k)|M_x(L^\infty)\|_\infty < \epsilon.$$

As above, it follows that

$$\|\Phi_x(\pi H_f \pi H_g) - \Phi_x(\pi H_j \pi H_k)\| < \epsilon(\|f\|_\infty + \|g\|_\infty + \epsilon) \tag{3.3.5}$$

where

$$\begin{aligned} \pi H_j \pi H_k &= \pi H_{\eta\phi_1} \pi H_{(1-\eta)\psi_2} + \pi H_{(1-\eta)\phi_2} \pi H_{\eta\psi_1} \\ &\quad + \pi H_{\eta\phi_1} \pi H_{\eta\psi_1} + \pi H_{(1-\eta)\phi_2} \pi H_{(1-\eta)\psi_2}. \end{aligned} \tag{3.3.6}$$

For the first two terms of (3.3.6), since $\pi H_{\phi_1} \pi H_{\psi_2} = \pi H_h$ for some $h \in L^\infty$, repeated applications of (3.1.2) and Hartman's theorem yield

$$\begin{aligned} \pi H_{\eta\phi_1} \pi H_{(1-\eta)\psi_2} &= \pi T_{\tilde{\eta}} \pi H_{\phi_1} \pi H_{\psi_2} \pi T_{1-\eta} = \pi T_{\tilde{\eta}} \pi H_h \pi T_{1-\eta} \\ &= \pi H_h \pi T_\eta \pi T_{1-\eta} = \pi H_h \pi T_{\eta(1-\eta)}. \end{aligned}$$

By construction $\eta(1-\eta)|M_x(L^\infty) \equiv 0$, so that

$$\Phi_x(\pi H_{\eta\phi_1}\pi H_{(1-\eta)\psi_2}) = \Phi_x(\pi H_h)\Phi_x(\pi T_{\eta(1-\eta)}) = 0$$

due to Theorem 3.3.1(ii). Similarly,

$$\Phi_x(\pi H_{(1-\eta)\phi_2}\pi H_{\eta\psi_1}) = 0.$$

For the remaining two terms of (3.3.6), a different arrangement gives

$$\pi H_{\eta\phi_1}\pi H_{\eta\psi_1} = \pi H_{\phi_1}\pi T_\eta \pi T_{\tilde{\eta}}\pi H_{\psi_1} = \pi H_{\phi_1}\pi T_{\eta\tilde{\eta}}\pi H_{\psi_1},$$

while $\eta\tilde{\eta}|M_x(L^\infty) \equiv 0$ implies $\Phi_x(\pi T_{\eta\tilde{\eta}}) = 0$. Therefore

$$\Phi_x(\pi H_{\eta\phi_1}\pi H_{\eta\psi_1}) = 0.$$

Observing $(1-\eta)(1-\tilde{\eta})|M_x(L^\infty) \equiv 0$ as well, one similarly has

$$\Phi_x(\pi H_{(1-\eta)\phi_2}\pi H_{(1-\eta)\psi_2}) = 0.$$

In view of these assertions and (3.3.6),

$$\Phi_x(\pi H_j \pi H_k) = 0. \tag{3.3.7}$$

Sending $\epsilon \downarrow 0$ in (3.3.5) then gives $\Phi_x(\pi H_f \pi H_g) = 0$. In particular, (3.3.3) holds.

Now set in Theorem 3.2.2 $u = \pi H_f \pi H_g \in \pi\mathcal{T}\mathcal{H}(L^\infty) = \mathscr{U}$, $\mathscr{A} = \pi\mathcal{T}(QC_s)$ and $\mathscr{M} = \pi\mathcal{H}_{L^\infty}$, and one directly verifies $\mathscr{M} + \mathscr{M} \subset \mathscr{M}$, $\mathscr{M}\mathscr{A} \subset \mathscr{M}$. The latter inclusion is due to

$$\pi H_h \pi T_\eta = \pi H_{h\eta}, \quad h \in L^\infty,\ \eta \in QC_s.$$

Therefore, from (3.3.3) for every $x \in M(QC_s)$, we arrive at

$$d(\pi H_f \pi H_g, \pi \mathcal{H}_{L^\infty}) = 0. \tag{3.3.8}$$

By the well-known essential norm formula (cf. [18]) stating that

$$\|\pi H_h\| = d(h, H^\infty + C), \;\; h \in L^\infty,$$

one has the well-defined linear isometry

$$h + (H^\infty + C) \in L^\infty/(H^\infty + C) \mapsto \pi H_h \in \mathcal{L}(H^2)/\mathcal{K}(H^2),$$

whose range $\pi\mathcal{H}_{L^\infty}$ must be closed in the Calkin algebra due to completeness of the quotient space $L^\infty/(H^\infty + C)$. (This argument is inspired by [117], p. 290). Hence (3.3.8) yields $\pi H_f \pi H_g \in \pi\mathcal{H}_{L^\infty}$, that is, $H_f H_g \in \mathcal{H}_{L^\infty} + \mathcal{K}(H^2)$ as desired. □

Remark 3.3.2. Theorem 3.3.1(ii) also yields a necessary condition for a Hankel operator to be a compact perturbation of a product of two Hankel operators, and hence in $\mathcal{T}(L^\infty)$ (see [27, Sect. 3] for construction of such a noncompact Hankel operator). *If for some $h, f, g \in L^\infty$ and $K \in \mathcal{K}(H^2)$*

$$H_h = H_f H_g + K,$$

then H_h for every $y \in M(QC) \setminus K^0$ lies in the compact perturbation of the closed bideal $\mathcal{J}_y$ of $\mathcal{TH}(L^\infty)$ generated by $\{T_\xi : \xi \in QC_s, \xi(y) = 0\}$. Indeed, with $y \neq \bar{y}$, we choose $\eta \in QC$ with $\eta(y) = 1, \eta(\bar{y}) = 0$, and write

$$\pi H_h = \pi H_f \pi H_g = (\pi H_{\eta f} + \pi H_{(1-\eta)f})(\pi H_{\eta g} + \pi H_{(1-\eta)g})$$

to get $\Phi_x(\pi H_h) = 0$ as in (3.3.7) for $x := y|QC_s$. That is,

$$\pi H_h \in \mathscr{I}_x = \pi \mathcal{J}_y$$

and $H_h \in \mathcal{J}_y + \mathcal{K}(H^2)$ as desired. Note $\mathcal{K}(H^2) \not\subset \mathcal{J}_y$.

Theorem 3.1.1 gives equivalent local conditions in terms of uniform approximation of L^∞ symbol functions on the fibers $M_y(L^\infty)$ over $y \in M(QC)$. Theorem 1 in [27] provides sufficient conditions per support set for the product $H_{\tilde{\theta}_1} H_{\bar{\theta}_2}$, θ_1, θ_2 inner functions, and therefore the product $H_{f\tilde{\theta}_1} H_{g\bar{\theta}_2}$ for $f, g \in H^\infty + C$, to be in $\mathcal{H}_{L^\infty} + \mathcal{K}$. It is hoped that these together with uniform approximation results on fibers of $M(L^\infty)$ could be used to approach the compact Hankel perturbation problem of [27], and that one should have better approximation on the finer fibers over $y \in M(QC)$ than those over $\lambda \in \partial\mathbb{D}$ which only reflect the behavior of symbol functions in shrinking neighborhoods of λ on $\partial\mathbb{D}$.

3.4 Antipodal symmetry of essential spectra

Let $r : M(QC) \to M(QC_s)$ be the restriction map and write

$$K_0 := r(K^0)$$

for K^0 defined in (3.3.2). The proof of Theorems 3.1.2 and 3.1.3 requires the construction of QC functions to go with Theorem 3.3.1.

Lemma 3.4.1. *For any open neighborhood U of K_0 in $M(QC_s)$, there exist functions $q_0, q_1, q_2 \in QC$ such that $\sum_{k=0}^2 q_k^2 = 1$, $\tilde{q}_1 q_1 = \tilde{q}_2 q_2 = 0$, and $\tilde{q}_0 q_0 \psi = 0$ for every $\psi \in L^\infty$ vanishing on $\bigsqcup\{M_x(L^\infty) : x \in U\}$.*

Proof. Fix such U. The subset $K_0 \subset M(QC_s)$ is compact since K^0 is compact in $M(QC)$. Urysohn's lemma applies to the compact Hausdorff space $M(QC_s)$ to give some $p_0 \in QC_s$ satisfying $p_0(M(QC_s)) \subset [0,1]$, $p_0 = 0$ on $M(QC_s) \setminus U$, and $p_0 = 1$ on K_0. Set $p = 1 - p_0 \in QC_s$. Then $p(M(QC_s)) \subset [0,1]$, $p = 1$ on $M(QC_s) \setminus U$, and $p = 0$ on K_0.

Next define $p_1 \in L^\infty$ to be p on the upper half of $\partial\mathbb{D}$ and identically 0 on the lower half. Let $p_2 = p - p_1$. That is, p_2 equals 0 on the upper half and p on the lower half. Obviously,

$$\tilde{p}_1 p_1 = \tilde{p}_2 p_2 = 0. \tag{3.4.1}$$

Now $p \in VMO$ implies, by construction, that p_1 is of VMO over the upper and lower half of $\partial\mathbb{D}$ respectively. Also, $\tilde{p} = p$ implies

$$\int_{\pm 1}^{\pm 1+\delta} p d\theta = \int_{\pm 1-\delta}^{\pm 1} p d\theta, \quad \delta > 0. \tag{3.4.2}$$

Moreover, $p = 0$ on K_0 is equivalent to $p = 0$ on K^0. It follows from the latter and (3.1.8) that

$$\frac{1}{2\delta} \int_{\pm 1-\delta}^{\pm 1+\delta} p d\theta \to 0 \text{ as } \delta \downarrow 0. \tag{3.4.3}$$

Combining (3.4.2) and (3.4.3) gives

$$\frac{1}{\delta} \int_{\pm 1}^{\pm 1+\delta} p d\theta = \frac{1}{\delta} \int_{\pm 1-\delta}^{\pm 1} p d\theta \to 0 \text{ as } \delta \downarrow 0.$$

Therefore, by construction again,

$$\frac{1}{\delta} \int_{1}^{1+\delta} p_1 d\theta - \frac{1}{\delta} \int_{1-\delta}^{1} p_1 d\theta = \frac{1}{\delta} \int_{1}^{1+\delta} p d\theta \to 0,$$
$$\frac{1}{\delta} \int_{-1}^{-1+\delta} p_1 d\theta - \frac{1}{\delta} \int_{-1-\delta}^{-1} p_1 d\theta = -\frac{1}{\delta} \int_{-1-\delta}^{-1} p d\theta \to 0$$

as $\delta \downarrow 0$. That is, the integral gaps

$$\gamma_{\pm 1}(p_1) := \limsup_{\delta \downarrow 0} \left| \frac{1}{\delta} \int_{\pm 1}^{\pm 1+\delta} p_1 d\theta - \frac{1}{\delta} \int_{\pm 1-\delta}^{\pm 1} p_1 d\theta \right| = 0.$$

It then follows from Lemma 2 in [120] that $p_1 \in VMO$. So we assert $p_1, p_2 \in QC$.

Note that $p(M(QC_s)) \subset [0,1] \Leftrightarrow p(M(L^\infty)) \subset [0,1] \Leftrightarrow p(M_\lambda(L^\infty)) \subset [0,1], \forall \lambda \in \partial\mathbb{D}$. By the construction of p_1 from p and the well-known result (cf. [18, 2.79(a)]) on equality of the local essential range at $\lambda \in \partial\mathbb{D}$ and the range on $M_\lambda(L^\infty)$ of an L^∞ function, one deduces $p_1(M_\lambda(L^\infty)) \subset [0,1], \forall \lambda \in \partial\mathbb{D}$, so that $p_1(M(QC)) \subset [0,1]$. The same holds for p_2. Therefore, there exist $[0,1]$-valued continuous functions on $M(QC_s)$ and respectively $M(QC)$, whose squares equal p_0 on $M(QC_s)$ and respectively p_1, p_2 on $M(QC)$, thus giving $q_0 \in QC_s$ and $q_1, q_2 \in QC$

such that

$$\sum_{k=0}^{2} q_k^2 = \sum_{k=0}^{2} p_k = 1.$$

It remains to show such q_0, q_1, q_2 satisfy the other two conditions. For $k = 1, 2$ one has by (3.4.1) that

$$(\tilde{q_k} q_k)^2 = \tilde{q_k^2} q_k^2 = \tilde{p_k} p_k = 0 \Rightarrow \tilde{q_k} q_k = 0.$$

Suppose $\psi \in L^\infty$ vanishes on $\bigsqcup\{M_x(L^\infty) : x \in U\}$. Since $q_0 \in QC_s$, one has $\tilde{q_0} q_0 \psi = q_0^2 \psi = p_0 \psi$ while $p_0 = 0$ on $\bigsqcup\{M_x(L^\infty) : x \in M(QC_s) \setminus U\}$. So $p_0 \psi = 0$ on $M(L^\infty)$ and $\tilde{q_0} q_0 \psi = 0$. The proof is complete. $\square$

For a unital C*-algebra A and an element $a \in A$, consider the closed span of even and respectively odd products of a and a^* as follows

$$S_e(a) := \overline{\text{sp}} \left\{ \prod_{k=1}^{2m} a_k : a_k = a \text{ or } a^*, m \geq 0 \right\},$$
$$S_o(a) := \overline{\text{sp}} \left\{ \prod_{k=1}^{2n+1} b_k : b_k = a \text{ or } a^*, n \geq 0 \right\}.$$

Note $S_o(a)$ is a self-adjoint bimodule over the C*-algebra $S_e(a)$ with $S_o(a) S_o(a) \subset S_e(a)$. The following observation is needed in order to apply the results in [103] on odd-even decompositions of singly generated C*-algebras.

Lemma 3.4.2. *$S_e([T_f, H_g]) \subset \mathcal{T}(L^\infty)$ for $f, g \in L^\infty$.*

Proof. Since $[T_f, H_g]^* = [H_g^*, T_f^*] = [H_{\tilde{\bar{g}}}, T_{\bar{f}}]$, an even product of $[T_f, H_g]$ and $[T_f, H_g]^*$ expands to a sum of products of Toeplitz and Hankel operators each having an even number of Hankel factors. Invoking the identities (3.1.1) and (3.1.2), an induction on the (even) number of Hankel factors, and on the gap between the first two Hankel factors, shows that such products of Toeplitz and Hankel operators lie in $\mathcal{T}(L^\infty)$ (also see [75], p. 133). This completes the proof. $\square$

The proof of Theorem 3.1.2 is modeled after Power's proof of [104, Theorem 2.4] using instead the more refined localization in Theorem 3.3.1, and relies on the two lemmas above.

Proof of Theorem 3.1.2. It follows from the discussion preceding the proof of Theorem 3.1.1 that for every $x \in K_0$, $M_x(QC) = \{y\}$ so that $M_x(L^\infty) = M_y(L^\infty)$ where y is the unique element of K^0 with $r(y) = x$. Hence (3.1.9) becomes

$$g|M_x(L^\infty) \in H^\infty|M_x(L^\infty), \quad \forall x \in K_0,$$

under the assumption of which the proof will be carried out.

First let $x \in K_0$. It follows from assumption that $g|M_x(L^\infty) = h|M_x(L^\infty)$ for some $h \in H^\infty$, so that Theorem 3.3.1(ii) gives $\Phi_x(\pi H_g) = \Phi_x(\pi H_h) = 0 \Rightarrow \Phi_x(\pi[T_f, H_g]) = 0$. In particular,

$$\sigma(\Phi_x(\pi[T_f, H_g])) = -\sigma(\Phi_x(\pi[T_f, H_g])). \tag{3.4.4}$$

Next consider the case $x \in M(QC_s) \setminus K_0$. Choose an open neighborhood U in $M(QC_s)$ of the compact subset K_0 such that $x \notin \overline{U}$. Let $\phi \in QC_s$ with $\phi(x) = 1$ and $\phi = 0$ on $\overline{U}$. It follows that $\phi f = f$ and $\phi g = g$ on $M_x(L^\infty)$ while $\phi f = \phi g = 0$ on $\bigsqcup\{M_x(L^\infty) : x \in U\}$. Write $a := \phi f$ and $b := \phi g$. Then $a = b = 0$ on $\bigsqcup\{M_x(L^\infty) : x \in U\}$ while by Theorem 3.3.1(ii)

$$a = f \text{ and } b = g \text{ on } M_x(L^\infty) \Rightarrow \Phi_x(\pi[T_a, H_b]) = \Phi_x(\pi[T_f, H_g]). \tag{3.4.5}$$

In addition, a, b vanishing on $\bigsqcup\{M_x(L^\infty) : x \in U\}$ implies so for $\tilde{a}, \tilde{b}$ as well. For, given $\xi \in \bigsqcup\{M_x(L^\infty) : x \in U\}$, certainly $\xi \circ C_{\bar{z}} \in M(L^\infty)$. Since $(\xi \circ C_{\bar{z}})|QC_s = \xi|QC_s$, one has $\xi \circ C_{\bar{z}} \in \bigsqcup\{M_x(L^\infty) : x \in U\}$ as ξ does, and the conclusion ensues.

Now let $q_0, q_1, q_2 \in QC$ be the functions constructed in Lemma 3.4.1 that possess the said properties relative to U. Define a bounded linear operator $\Psi \in \mathcal{L}(\pi\mathcal{L}(H^2))$ by

$$\Psi(S) = \sum_{k=0}^{2} (\pi T_{q_k}) S (\pi T_{q_k}), \quad S \in \pi\mathcal{L}(H^2).$$

Since the commutant of $\pi\mathcal{T}(L^\infty)$ equals $\pi\mathcal{T}(QC)$ and $q \mapsto \pi T_q$ is multiplicative on QC, one has for $S \in \pi\mathcal{T}(L^\infty)$ that

$$\Psi(S) = \sum_{k=0}^{2} S\pi T_{q_k}\pi T_{q_k} = S\pi \sum_{k=0}^{2} T_{q_k^2} = S.$$

So, $\Psi = I$ on $\pi\mathcal{T}(L^\infty) \supset S_e(\pi[T_a, H_b])$ due to Lemma 3.4.2. On the other hand, $\Psi = 0$ on $S_o(\pi[T_a, H_b])$. For, each odd product $T \in \pi\mathcal{L}(H^2)$ of $\pi[T_a, H_b]$ and $\pi[T_a, H_b]^*$ equals either $S\pi[T_a, H_b]$ or $S\pi[T_a, H_b]^* = S\pi[H_{\tilde{\bar{b}}}, T_{\bar{a}}]$ for some even product $S \in \pi\mathcal{T}(L^\infty)$, and therefore by (3.1.2) assumes the form

$$T = S_1\pi H_a + S_2\pi H_b + S_3\pi H_{ab}, \text{ or } T = S_4\pi H_{\bar{a}} + S_5\pi H_{\tilde{\bar{b}}} + S_6\pi H_{\bar{a}\tilde{\bar{b}}}, \tag{3.4.6}$$

for some $S_j \in \pi\mathcal{T}(L^\infty), j = 1, ..., 6$. Let $R := S_j\pi H_c$ be a generic term in (3.4.6) where $c \in L^\infty$ vanishes on $\bigsqcup\{M_x(L^\infty) : x \in U\}$ as shown earlier. Repeated applications of (3.1.2) yield

$$\Psi(R) = S_j \sum_{k=0}^{2} \pi T_{q_k}\pi H_c \pi T_{q_k} = S_j \sum_{k=0}^{2} \pi T_{q_k}\pi H_{cq_k} = -S_j \sum_{k=0}^{2} \pi H_{\tilde{q}_k c q_k} = 0,$$

where we have exploited the implications $q_k \in QC \Rightarrow \tilde{q}_k \in QC \Rightarrow \pi H_{q_k} = \pi H_{\tilde{q}_k} = 0$ and the properties $\tilde{q}_1 q_1 = \tilde{q}_2 q_2 = \tilde{q}_0 q_0 c = 0$ due to c vanishing on $\bigsqcup\{M_x(L^\infty) : x \in U\}$. By linearity and continuity of Ψ, one concludes that $\Psi = 0$ on $S_o(\pi[T_a, H_b])$ as desired.

We shall apply throughout this paragraph the results in [103] (cf. [104], Theorem 1.2). Let A be the C*-subalgebra generated by $\pi[T_a, H_b]$ in $\pi\mathcal{TH}(L^\infty)$. One has the direct-sum decomposition $A = S_e(\pi[T_a, H_b]) \bigoplus S_o(\pi[T_a, H_b])$ induced by the idempotent $\Psi \in \mathcal{L}(A)$ with range $S_e(\pi[T_a, H_b])$ and kernel $S_o(\pi[T_a, H_b])$. Since the bideal $\mathscr{I}_x$ in Theorem 3.3.1 is invariant for Ψ, there is a unique idempotent $\Psi_x \in \mathcal{L}(\Phi_x(A))$ satisfying the intertwining property

$$\Psi_x\Phi_x = \Phi_x\Psi \text{ on } A, \tag{3.4.7}$$

where $\Phi_x(A)$ equals the C*-subalgebra generated by $\Phi_x(\pi[T_a, H_b])$ in the local algebra, and both

$\mathrm{ran}\Psi_x$ and $\ker \Psi_x$ are closed. So it follows from (3.4.7) that

$$\begin{aligned}\mathrm{ran}\Psi_x &= \Phi_x(S_e(\pi[T_a,H_b])) = \overline{\Phi_x(S_e(\pi[T_a,H_b]))} = S_e(\Phi_x(\pi[T_a,H_b])),\\ \ker \Psi_x &\supset \overline{\Phi_x(S_o(\pi[T_a,H_b]))} = S_o(\Phi_x(\pi[T_a,H_b])).\end{aligned}$$

Therefore, one has the direct-sum decomposition

$$\Phi_x(A) = S_e(\Phi_x(\pi[T_a,H_b]))\bigoplus S_o(\Phi_x(\pi[T_a,H_b]))$$

which asserts antipodal symmetry of $\sigma(\Phi_x(\pi[T_a,H_b]))$. In view of (3.4.5), then, (3.4.4) stills holds in this case.

Finally, antipodal symmetry of $\sigma_e[T_f,H_g]$ follows from (3.4.4) for all $x\in M(QC_s)$ and Theorem 3.3.1(i) for $[T_f,H_g]\in\mathcal{TH}(L^\infty)$. $\square$

The proof of Theorem 3.1.3 requires only minor changes and is omitted.

Remark 3.4.3. For $g\in L^\infty$ satisfying (3.1.9) and $f\in L^\infty$, the proofs for $[T_f,H_g]$ and H_g actually establish the direct-sum decompositions in the Calkin algebra

$$A(\pi[T_f,H_g]) = S_e(\pi[T_f,H_g])\bigoplus S_o(\pi[T_f,H_g]), \tag{3.4.8}$$

$$A(\pi H_g) = S_e(\pi H_g)\bigoplus S_o(\pi H_g). \tag{3.4.9}$$

For, it is seen from the proof above that

$$S_e(\Phi_x(\pi[T_f,H_g]))\bigcap S_o(\Phi_x(\pi[T_f,H_g])) = \{0\}, \quad \forall x\in M(QC_s).$$

If $S\in S_e(\pi[T_f,H_g])\bigcap S_o(\pi[T_f,H_g])$, then for every $x\in M(QC_s)$

$$\Phi_x(S)\in S_e(\Phi_x(\pi[T_f,H_g]))\bigcap S_o(\Phi_x(\pi[T_f,H_g])) = \{0\},$$

from which Theorem 3.2.1(i) gives $S=0$. Thus (3.4.8) and similarly (3.4.9) hold.

Theorem 3.1.3 has an interesting corollary which would have been available only for continuous Toeplitz symbols under its original condition. Note that if in addition $g \in PQC$ is assumed in the following result, then $\sigma_e[T_f, H_g]$ is actually a union of antipodal symmetric line segments, by the essential spectrum formula [105, Theorem 5] for Hankel operators with PQC symbols.

Corollary 3.4.4. *If $f \in QC$ and $g \in L^\infty$, then $\sigma_e[T_f, H_g] = -\sigma_e[T_f, H_g]$.*

Proof. First it follows from (3.1.2), invariance of QC under $C_{\bar{z}}$, and Hartman's theorem that for $f \in QC$ and $g \in L^\infty$,

$$\pi[T_f, H_g] = \pi(T_f H_g) - \pi(H_g T_f) = \pi H_{\tilde{f}g} - \pi H_{fg} = \pi H_{(\tilde{f}-f)g}. \tag{3.4.10}$$

Then recall for each $y \in K^0$, $y = \bar{y}$, hence

$$\begin{aligned}(\tilde{f} - f)g|M_y(L^\infty) &= ((\tilde{f} - f)|M_y(L^\infty))(g|M_y(L^\infty)) = (\tilde{f}(y) - f(y))g|M_y(L^\infty) \\ &= (f(\bar{y}) - f(y))g|M_y(L^\infty) \equiv 0 \in H^\infty|M_y(L^\infty).\end{aligned}$$

The proof is complete by (3.4.10) and Theorem 3.1.3. □

For $\lambda \in \partial\mathbb{D}$ and $y \in M(QC)$, the fibers $F := M_\lambda(L^\infty), M_y(L^\infty)$ possess the Clancey-Gosselin property ([30]; cf. [18, 4.60(b)]). That is, for any inner function u, $\bar{u}|F \in H^\infty|F \Leftrightarrow u|F = \text{const}$. Hence inner functions u discontinuous at $\lambda = 1$ or -1 on $\partial\mathbb{D}$ do not satisfy (3.1.7) for $g = \bar{u}$, albeit $u|M_y(L^\infty)$ over each $y \in M_\lambda(QC)$ may or may not be constant. See Example 3.5.5 in the next section for a class of interpolating Blaschke products b discontinuous at λ while $b|M_y(L^\infty) = \text{const}$ over each $y \in M_\lambda^0(QC)$.

We close this section with an application to Hankel symbols $g = \bar{u}$ or $\tilde{u}$ for quasi-inner functions u, that is, unimodular functions u in $H^\infty + C$. Let

$$Z(u) := \{m \in M(H^\infty + C) : \ u(m) = 0\}$$

be the zero set of u in $M(H^\infty + C)$. For a subset Ω of $M(H^\infty + C)$, write $\Omega|QC := \{m|QC :$

$m \in \Omega\} \subset M(QC)$. For $m \in M(H^\infty)$, let S_m be the compact support set of its representing measure μ_m on $M(L^\infty)$, S_m a singleton if and only if $m \in M(L^\infty)$, and $S_m = M(L^\infty)$ if and only if $m \in \mathbb{D} \hookrightarrow M(H^\infty)$. If $m \in M(H^\infty + C)$, then $S_m \subset M_{m|QC}(L^\infty)$ and $u|M_{m|QC}(L^\infty) \in H^\infty|M_{m|QC}(L^\infty)$. Finally, $\mathcal{G}$ denotes the set of nontrivial points of $M(H^\infty + C)$, that is, those points whose Gleason parts are analytic discs. The following characterization of (3.1.9) for quasi-inner functions uses the proof of [64, Theorem 3.3].

Theorem 3.4.5. *Let u be a quasi-inner function. Then the following are equivalent.*

(i) $\bar{u}|M_y(L^\infty) \in H^\infty|M_y(L^\infty)$, $\forall y \in K^0$.

(ii) $Z(u)|QC \bigcap K^0 = \emptyset$.

(iii) $(Z(u) \bigcap \mathcal{G})|QC \bigcap K^0 = \emptyset$.

In this case and for $f \in L^\infty$,

$$\sigma_e[T_f, H_{\bar{u}}] = -\sigma_e[T_f, H_{\bar{u}}], \quad \sigma_e(H_{\bar{u}}) = -\sigma_e(H_{\bar{u}}); \tag{3.4.11}$$

$$\sigma_e[T_f, H_{\tilde{u}}] = -\sigma_e[T_f, H_{\tilde{u}}], \quad \sigma_e(H_{\tilde{u}}) = -\sigma_e(H_{\tilde{u}}). \tag{3.4.12}$$

Proof. It is trivial that (i)⇒(ii)⇒(iii).

Assume (iii) and fix an arbitrary $y \in K^0$. Suppose $u|M_y(L^\infty)$ is not invertible in $H^\infty|M_y(L^\infty)$. Then $u(\xi) = 0$ for some $\xi \in M(H^\infty + C)$ with $S_\xi \subset M_y(L^\infty)$, S_ξ not a singleton for $|u| \equiv 1$ on $M(L^\infty)$. If $\xi \in \mathcal{G}$, then we would have $y = \xi|QC \in (Z(u) \bigcap \mathcal{G})|QC$ contradicting (iii). By Corollary 3.2 in [64], the trivial point $\xi \in M(H^\infty + C) \setminus M(L^\infty)$ then lies in the closure in $M(H^\infty + C)$ of the subset $\{x \in \mathcal{G} : S_x \subset S_\xi\}$. Thus for each $n \in \mathbb{N}$, there exists $x_n \in \mathcal{G}$ with $S_{x_n} \subset S_\xi$ and $|u(x_n)| < 1/n$. By Theorem 1.2 in [64], the sequence $\{x_n\}_n$ in $\mathcal{G}$ has a cluster point $x \in \mathcal{G}$. We have $S_x \subset S_\xi$ due to each $S_{x_n} \subset S_\xi$, and $u(x) = 0$ for $u(x_n) \to 0$. But $y = x|QC \in (Z(u) \bigcap \mathcal{G})|QC$ is again a contradiction to (iii). Therefore, $u|M_y(L^\infty)$ is invertible in $H^\infty|M_y(L^\infty)$, giving $\bar{u}|M_y(L^\infty) \in H^\infty|M_y(L^\infty)$. That is, (i) holds.

Now Theorems 3.1.2 and 3.1.3 give (3.4.11). Taking adjoints $[T_{\bar{f}}, H_{\bar{u}}]^* = -[T_f, H_{\tilde{u}}]$, $H^*_{\bar{u}} = H_{\tilde{u}}$, (3.4.12) follows from (3.4.11). ∎

3.5 Investigation of conditions and construction of examples

Recall that a Douglas algebra is by definition a closed subalgebra of L^∞ containing H^∞. Let $H^\infty[g]$ be the Douglas algebra generated by $g \in L^\infty$ over H^∞. For F a compact subset of $M(L^\infty)$ such that $H^\infty|F$ is *closed* in the sup-norm, define the Douglas algebra

$$H_F^\infty := \{f \in L^\infty : f|F \in H^\infty|F\}$$

whose closure in L^∞ is due to that of $H^\infty|F$. The maximal ideal space of any Douglas algebra is identified with its restriction in $M(H^\infty)$.

Lemma 3.5.1. *For $g \in L^\infty$ and F as above, $g|F \in H^\infty|F$ if and only if $g|S_m \in H^\infty|S_m$ for all support sets $S_m \subset F$.*

Proof. Suppose $g|S_m \in H^\infty|S_m$ for all $S_m \subset F$. It is known (Theorem 1.9.7) that for such F

$$M(H_F^\infty) = M(L^\infty) \bigcup \{m \in M(H^\infty) : S_m \subset F\}, \tag{3.5.1}$$

while Lemma 1.5 in [67] states that

$$M(H^\infty[g]) = \{m \in M(H^\infty) : g|S_m \in H^\infty|S_m\}.$$

It follows from these two equalities and the hypothesis that

$$M(H_F^\infty) \subset M(H^\infty[g]),$$

so that $H^\infty[g] \subset H_F^\infty$ by the Chang-Marshall theorem. That is, $g \in H_F^\infty$, and $g|F \in H^\infty|F$ by definition of H_F^∞. The converse is trivial. □

Corollary 3.5.2. *For $g \in L^\infty$ and $\lambda \in \partial\mathbb{D}$, the following are equivalent*

(i) $g|M_\lambda(L^\infty) \in H^\infty|M_\lambda(L^\infty)$,

(ii) $g|M_y(L^\infty) \in H^\infty|M_y(L^\infty),\ \forall y \in M_\lambda(QC)$.

Proof. It suffices to show (ii)⇒(i). But this follows from the above lemma, since $H^\infty|M_\lambda(L^\infty)$ is closed due to $M_\lambda(L^\infty)$ being a peak set for H^∞ [93], and the fact $\{S_m : S_m \subset M_\lambda(L^\infty)\} = \bigsqcup_{y \in M_\lambda(QC)} \{S_m : S_m \subset M_y(L^\infty)\}$. □

From this it is clear for general L^∞ symbols that condition (3.1.9) is much weaker than (3.1.7), because $M^0_\lambda(QC)$ is just a small subset of $M_\lambda(QC)$ ([120], p. 823). Next, we use Lemma 3.5.1 to derive an equivalent form of (3.1.9). Define the compact subset of $M(L^\infty)$

$$F(K^0) := \bigsqcup\{M_y(L^\infty) : y \in K^0\}$$

as the pre-image of K^0 under the restriction map from $M(L^\infty)$ onto $M(QC)$.

Proposition 3.5.3. *Condition* (3.1.9) *is equivalent to* $g|F(K^0) \in H^\infty|F(K^0)$.

Proof. First we show $(H^\infty+C)|F(K^0) \subset H^\infty|F(K^0)$, so that $(H^\infty+C)|F(K^0) = H^\infty|F(K^0)$. Let $h \in H^\infty$ and $c \in C$. Since $F(K^0)$ partitions in $F(K^0) \bigcap M_1(L^\infty) = F_1$ and $F(K^0) \bigcap M_{-1}(L^\infty) = F_{-1}$, $c|F(K^0)$ assumes the constant $c(1)$ on F_1 and $c(-1)$ on F_{-1}. For the function $\phi \in H^\infty \bigcap C$ given by

$$\phi(z) = 2^{-1}[(c(1) - c(-1))z + c(1) + c(-1)],$$

we have $\phi(1) = c(1)$, $\phi(-1) = c(-1)$, and $(h + c)|F(K^0) = (h + \phi)|F(K^0) \in H^\infty|F(K^0)$ as desired.

Next we show the restriction $(H^\infty + C)|F(K^0)$ is closed. Write

$$K^0 = \bigcap\{G : \ G \text{ an open neighborhood of } K^0 \text{ in } M(QC)\}. \tag{3.5.2}$$

For every such G, choose $f_G \in QC$ such that f_G on $M(QC)$ ranges in $[0, 1]$, $f_G \equiv 1$ on K^0 while $f_G \equiv 0$ on $M(QC) \setminus G$. Consider the compact set $F_G := \{y \in M(QC) : f_G(y) = 1\}$. Evidently, f_G *peaks* (cf. [93]) on F_G, so $\bigcap_G F_G$ is a weak peak set for QC on $M(QC)$. By construction, $K^0 \subset F_G \subset G$, so that $K^0 \subset \bigcap_G F_G \subset \bigcap_G G = K^0$ in view of (3.5.2). It follows that the pre-image $F(K^0)$ in $M(L^\infty)$ is a weak peak set for QC, *a fortiori* for $H^\infty + C$, on $M(L^\infty)$. Hence, the restriction algebra $(H^\infty + C)|F(K^0)$ is closed by a well-known result on uniform algebras (cf. [93]).

Now that $H^\infty|F(K^0)$ is shown closed, the nontrivial direction of the proposition follows from Lemma 3.5.1 and the fact that $\{S_m : S_m \subset F(K^0)\} = \bigsqcup_{y \in K^0}\{S_m : S_m \subset M_y(L^\infty)\}$. This completes the proof. $\square$

For their inherent importance, we shall construct examples of interpolating Blaschke products b with $g = \bar{b}$ satisfying (3.1.9), thus showing antipodal symmetry (3.4.11), (3.4.12) for such $u = b$, while condition (3.1.7) fails. To do so, we first characterize condition (3.1.9) for $g = \bar{b}$ in terms of existence of certain QC functions. For $f \in L^\infty$ and $z \in \mathbb{D}$, $f(z)$ denotes the harmonic extension. Since (3.5.3) is equivalent to $\lim_{t \to \pm 1} \phi(t) = 0$ for $\phi \in QC$ ([120], Lemma 5), the characterization requires that if $z_n \to \pm 1$, the convergence be adequately tangential as measured by boundary behavior of harmonic extensions of QC functions.

Proposition 3.5.4. *Let b be an infinite interpolating Blaschke product with zero sequence $\{z_n\}_n$ in $\mathbb{D}$. Then condition* (3.1.9) *holds for $g = \bar{b}$ if and only if $\exists \phi \in QC$,*

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{\pm 1 - \delta}^{\pm 1 + \delta} \phi \, d\theta = 0 \tag{3.5.3}$$

$$\text{and} \qquad \lim_{n \to \infty} \phi(z_n) = 1. \tag{3.5.4}$$

Proof. In view of Theorem 3.4.5 (i)$\Leftrightarrow$(ii), it suffices to prove the equivalence with

$$Z(b)|QC \bigcap K^0 = \emptyset \tag{3.5.5}$$

in lieu of (3.1.9) for $g = \bar{b}$. Since $Z(b)|QC$ and K^0 are compact subsets of $M(QC)$, (3.5.5) is in turn equivalent to the existence of $\psi \in QC$ such that

$$\psi \equiv 0 \quad \text{on} \quad K^0, \tag{3.5.6}$$

$$\psi \equiv 1 \quad \text{on} \quad Z(b)|QC. \tag{3.5.7}$$

Fix an arbitrary function $f \in QC$. It follows from (3.1.8) that f satisfies (3.5.3) if and only if it does (3.5.6). We claim that f satisfies (3.5.4) if and only if it does (3.5.7). For, in this case $Z(b) = \overline{\{z_n\}_n} \setminus \mathbb{D}$ [79] consists exactly of the cluster points of the sequence $\{z_n\}_n$ in $M(H^\infty)$. If (3.5.4) holds for f, and a subnet $z_{n_\beta} \to m$ in $M(H^\infty)$, then $\mu_{z_{n_\beta}} \to \mu_m$ in the weak-star topology of $C(M(L^\infty))^*$, and the implication

$$f(z_{n_\beta}) = \int_{M(L^\infty)} f d\mu_{z_{n_\beta}} \to \int_{M(L^\infty)} f d\mu_m = f(m|QC) = 1$$

yields (3.5.7) for f. Conversely, any subnet $\{z_{n_\omega}\}_\omega$ has a cluster point $\xi \in Z(b)$. Passing to another subnet still denoted by $\{z_{n_\omega}\}_\omega$ for convenience, we deduce

$$z_{n_\omega} \to \xi \;\Rightarrow\; f(z_{n_\omega}) \to f(\xi|QC) = 1$$

assuming (3.5.7) for f. That is, every subnet of $\{f(z_n)\}_n$ has a subnet converging to 1, hence we get (3.5.4) for f. The proof is complete. $\square$

The following construction of QC functions is enabled by [70, Lemma 1] which states that for any pair of concentric subarcs $J \subset I$ of $\partial\mathbb{D}$, there exists a $[0,1]$-valued function $f \in C$ with $f \equiv 1$ on J, $f \equiv 0$ off I, and $\|f\|_{BMO} \leq M/\ln(\theta(I)/\theta(J))$. Here $M \in (0,\infty)$ is a universal constant and $\| \ \|_{BMO}$ is the maximum-mean-oscillation norm. We note in passing that related constructions using this result are also found in [15, 142], but our focus is condition (3.5.3), (3.5.4).

Example 3.5.5. Given three infinite sequences $\{r_n\}_n, \{t_n\}_n, \{p_n\}_n$ of *positive* numbers under the following set of conditions

$$r_1 > r_2 > \cdots \to 0, \tag{3.5.8}$$

$$t_1 \geq t_2 \geq \cdots \to 0, \tag{3.5.9}$$

$$1/2 > r_1 + t_1/2, \tag{3.5.10}$$

$$r_n - r_{n+1} \geq t_n, \tag{3.5.11}$$

$$\inf\{p_{n+1} - p_n : \ n \in \mathbb{N}\} > 0, \tag{3.5.12}$$

$$\sum_n 1/p_n < \infty, \tag{3.5.13}$$

write $s_n := t_n e^{-p_n}$ and choose $j_n = \pm 1, n \in \mathbb{N}$, at will. Let

$$z_n = (1 - s_n)e^{i2\pi j_n r_n} \in \mathbb{D}.$$

The sequence $\{z_n\}_n$ is exponential, hence interpolating (cf. [62]), for

$$(1 - |z_{n+1}|)/(1 - |z_n|) = s_{n+1}/s_n \leq e^{-p_{n+1}}/e^{-p_n} \leq e^{-\inf_n(p_{n+1}-p_n)} < 1$$

by (3.5.9), (3.5.12). Let b be the interpolating Blaschke product with zeros $\{z_n\}_n$. Let $J_n \subset I_n$ be the concentric subarcs centered at $z_n/|z_n|$ with $\theta(J_n) = s_n, \theta(I_n) = t_n$, and let f_n be a $[0,1]$-valued continuous function with $f_n \equiv 1$ on J_n, $f_n \equiv 0$ off I_n, and $\|f_n\|_{BMO} \leq M/\ln(\theta(I_n)/\theta(J_n)) = M/p_n$. By (3.5.9), (3.5.11), the support subarcs $\{I_n\}_n$ are pairwise disjoint. Now define

$$\phi_+ = \sum_{j_n=1} f_n, \quad \phi_- = \sum_{j_n=-1} f_n, \quad \phi = \phi_+ + \phi_-$$

pointwise. Evidently, $\|\phi_\pm\|_\infty \leq 1$. Since the convergence of the continuous partial sums to $\phi_\pm$ is in the BMO norm due to $\sum_n \|f_n\|_{BMO} < \infty$ by (3.5.13), and since $VMO \supset C$ is closed in the BMO norm, $\phi_\pm, \phi \in VMO \bigcap L^\infty = QC$. The function ϕ_+ vanishes on a subarc $(1-\epsilon, 1)$ for some $\epsilon > 0$ so that the integral gap $\gamma_1(\phi_+) = 0$ for $\phi_+ \in VMO$ implies

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{1-\delta}^{1+\delta} \phi_+ d\theta = \lim_{\delta \to 0} \frac{1}{2\delta} \int_{1}^{1+\delta} \phi_+ d\theta = 0,$$

and similarly for ϕ_-. This verifies (3.5.3) for ϕ, ϕ vanishing in a neighborhood of $\lambda = -1$ due to (3.5.10). Since $1 - |z_n| = s_n = \theta(J_n) \to 0$, J_n centered at $z_n/|z_n|$ and θ normalized, $\phi \in QC$ satisfies ([120], Lemma 5)

$$\phi(z_n) - \frac{1}{\theta(J_n)} \int_{J_n} \phi d\theta \to 0,$$

while by construction

$$\int_{J_n} \phi d\theta = \int_{J_n} \phi_+ d\theta + \int_{J_n} \phi_- d\theta = \int_{J_n} f_n d\theta = \theta(J_n).$$

These verify (3.5.4). Therefore, Proposition 3.5.4 asserts condition (3.1.9) for $g = \bar{b}$. On the other hand, $s_n \to 0$ and $j_n r_n \to 0$ give $z_n \to 1$. Thus b is discontinuous at 1 on $\partial\mathbb{D}$ and condition (3.1.7) fails for $g = \bar{b}$. An easy choice for the sequences is

$$r_n = (n+2)^{-1}, \quad t_n = (n+3)^{-2}, \quad p_n = p^n, \quad p > 1.$$

Finally, given $j_n = \pm 1$, $\{r_n\}_n$ and $\{p_n\}_n$ satisfying (3.5.8), (3.5.12), (3.5.13), one can then inductively define $\{t_n\}_n$ satisfying (3.5.9), (3.5.11), and $\{z_n\}_n$ satisfying

$$\rho(z_n, z_k) := |z_n - z_k|/|1 - \bar{z}_k z_n| \geq e^{-1/2^n}, \quad n > k.$$

Therefore, $\{z_n\}_n$ can in fact be taken to be a sparse sequence, that is,

$$\lim_{n\to\infty} \prod_{k\neq n} \rho(z_n, z_k) = 1.$$

Let $\{z_n\}_n$ be a sparse sequence constructed in the preceding example. We have shown that

$$(\overline{\{z_n\}_n} \setminus \mathbb{D})|QC \bigcap M_1^0(QC) = \emptyset, \tag{3.5.14}$$

while $z_n \to 1$ implies

$$\overline{\{z_n\}_n} \setminus \mathbb{D} \subset M_1(H^\infty + C) \Rightarrow (\overline{\{z_n\}_n} \setminus \mathbb{D})|QC \subset M_1(QC). \tag{3.5.15}$$

Consider the injective [83, Lemma 5] thus homeomorphic embedding $m \in \overline{\{z_n\}_n} \setminus \mathbb{D} \mapsto m|QC \in M_1(QC) \setminus M_1^0(QC)$ by (3.5.14), (3.5.15). Since the Stone-Čech compactification $\overline{\{z_n\}_n}$ of the discrete space $\{z_n\}_n$ is uncountable and totally disconnected, and so is $\overline{\{z_n\}_n} \setminus \mathbb{D} = \overline{\{z_n\}_n} \setminus \{z_n\}_n$, there are uncountably many totally disconnected $\{y = m|QC\} \subset M_1(QC) \setminus M_1^0(QC)$ such that each $M_y(L^\infty) \supset S_m$ for a sparse point $m \in \mathcal{G}$. Meanwhile, for every $y \in M_1^0(QC)$, a refinement of the argument in [120], p. 826, and [64, Corollary 3.2] give an $m \in \mathcal{G}$ with $S_m \subset M_y(L^\infty)$. We have therefore the following result on support sets of nontrivial points, part of which will yield in the next example characteristic functions satisfying (3.1.9) but not (3.1.7). Note that by a result of K. Hoffman (cf. [79]), $M_\lambda(H^\infty + C)$ for $\lambda \in \partial\mathbb{D}$ is connected and so is its continuous image $M_\lambda(QC)$. Also note that for $y \in M(QC)$, $M_y(L^\infty)$ is not a singleton if and only if it contains S_m for some $m \in M(H^\infty + C) \setminus M(L^\infty)$ ([84], p. 183), and that such m can indeed be taken in $\mathcal{G}$. However, it is an open problem if all fibers of $M(L^\infty)$ over $M(QC)$ are non-singleton.

Proposition 3.5.6. *Let $\lambda \in \partial\mathbb{D}$. Then there exist uncountably many totally disconnected $y \in M_\lambda(QC) \setminus M_\lambda^0(QC)$ such that each $M_y(L^\infty) \supset S_m$ for a sparse point $m \in \mathcal{G}$, while for every $y \in M_\lambda^0(QC)$ there exists $m \in \mathcal{G}$ such that $M_y(L^\infty) \supset S_m$.*

Example 3.5.7. Choose any $y \in M_1(QC) \setminus M_1^0(QC)$ such that $M_y(L^\infty)$ contains at least two distinct points, say ξ_1, ξ_2. Let $E \not\ni \xi_2$ be a clopen subset of $M(L^\infty)$ containing the compact $F(K^0) \bigsqcup \{\xi_1\}$. The characteristic function $1_E \in C(M(L^\infty))$ equals 1 identically on each fiber of $M(L^\infty)$ over K^0, so that $g := 1_V \mapsto 1_E$, V the corresponding Borel subset of $\partial\mathbb{D}$, satisfies (3.1.9). But $1_E(\xi_1) = 1$, $1_E(\xi_2) = 0$ imply $1_E|M_y(L^\infty) \notin H^\infty|M_y(L^\infty)$ for 1_E is not constant on the fiber $M_y(L^\infty)$ [63, Theorem 2.8]. Thus, $1_E|M_1(L^\infty) \notin H^\infty|M_1(L^\infty)$ and $g = 1_V$ does not satisfy (3.1.7). Moreover, a result of Marshall states (cf. [62]) $H^\infty[1_E] = H^\infty[\bar{u}]$ for some inner function u, while a result of Younis [150] asserts $H^\infty[\bar{u}] = H^\infty[\bar{b}]$ for some interpolating Blaschke product b. Then $H^\infty[1_E] = H^\infty[\bar{b}]$ implies $\bar{b}$ like 1_E satisfies (3.1.9) but not (3.1.7), giving indirect constructions of such products.

Remark 3.5.8. Incidentally, the Blaschke products b with zero sequence $\{z_n\}_n \subset \mathbb{D}$ of Example 3.5.5 satisfy $[H_{\bar{b}}, T_z] \in \mathcal{K}(H^2)$, due to $(z - \bar{z})\bar{b} \in C$, while $[H_{\bar{b}}, T_f] \notin \mathcal{K}(H^2)$ for a family of QC functions f. See [15] and [27, Sect. 4] for background and related constructions. To prove the latter, take any $m \in \overline{\{z_n\}_n} \setminus \mathbb{D}$. Then

$$y := m|QC \in M_1(QC) \setminus M_1^0(QC)$$

implies $y \neq \bar{y}$. Let $f \in QC$ with $f(y) \neq f(\bar{y})$. Since $\bar{b}|S_m \notin H^\infty|S_m$ due to $b(m) = 0$, and $(f - \tilde{f})|S_m \equiv \text{const} = f(y) - f(\bar{y}) \neq 0$ due to $S_m \subset M_y(L^\infty)$, we have $(f - \tilde{f})\bar{b}|S_m \notin H^\infty|S_m$. So, $(f - \tilde{f})\bar{b} \notin H^\infty + C$, and $[H_{\bar{b}}, T_f] \notin \mathcal{K}(H^2)$ in view of (3.4.10).

Lastly, the necessity of condition (3.1.9) to Theorem 3.1.2 will be examined. Although the following example uses PQC functions for the Hankel symbol, functions of other special types are readily seen to exist, although not necessarily easy to explicitly construct, that fail condition (3.1.9). In fact for each $y \in K^0$, $M_y(L^\infty)$ contains a non-singleton S_m, so that there exists an inner function (even an infinite Blaschke product) u with $|m(u)| < 1$. Since the corresponding $m \in M(H^\infty)$ induces an element in $M(H^\infty|M_y(L^\infty))$, the unimodular $u|M_y(L^\infty)$ is not invertible in $H^\infty|M_y(L^\infty)$. That is, $\bar{u}|M_y(L^\infty) \notin H^\infty|M_y(L^\infty)$ failing (3.1.9) for $g = \bar{u}$. However, one evidently does not know the exact form of such u.

Example 3.5.9. Since $M_y(PQC)$ over $y \in M_{\pm 1}(QC) \setminus K^0$ is a singleton ([120], Lemma 13 and surrounding remarks), $M_y(L^\infty)$ is a fiber over a point of $M(PQC)$. Hence it follows from Corollary 3.5.2 that a function $g \in PQC$ satisfies (3.1.9) if and only if it satisfies (3.1.7). By the local version of Lemma 10 in [120], closure of the fiber algebra $H^\infty|M_\lambda(L^\infty)$, and [18, 2.79(b)], the latter in turn amounts to the vanishing of the integral gaps $\gamma_{\pm 1}(g)$. Since in particular any $g \in PQC$ with a jump discontinuity $g(\lambda+) \neq g(\lambda-)$ at either $\lambda = 1$ or -1 has $\gamma_\lambda(g) = |g(\lambda+) - g(\lambda-)| \neq 0$, one has a large class of explicit counter-examples, in view of Corollary 3.4.4, to show that the converse to Theorem 3.1.2 does not hold.

To conclude this section, we note that although there are Hankel operators with non-symmetric essential spectra, for instance $\sigma_e(H_g) = [0, \pi]$ if H_g is represented by Hilbert's matrix, the author has not succeeded finding a non-symmetric essential spectrum of the commutator $[T_f, H_g]$ for any $f, g \in L^\infty$.

Chapter 4

Toeplitz, Hankel, and Composition Operators of Special Symbol Classes

4.1 Overview

Let $C_{\bar{z}}$ be the composition by the complex conjugate $z \in \partial\mathbb{D} \mapsto \bar{z} \in \partial\mathbb{D}$ and write $\tilde{f} := C_{\bar{z}} f = f \circ \bar{z}$, $f^* := \bar{\tilde{f}} = \tilde{\bar{f}}$, for any function f on $\partial\mathbb{D}$. Recall that PQC is the C*-algebra generated by PC and QC in L^∞ over the circle $\partial\mathbb{D}$. For a subset $S \subset L^\infty$, $\mathcal{T}(S)$ and $\mathcal{TH}(S)$ denote the norm-closed subalgebras of $\mathcal{L}(H^2)$ generated by the Toeplitz and, respectively, the Toeplitz and Hankel operators with symbol class S. Also write $\mathcal{H}_S := \{H_f : f \in S\}$. Let $\mathcal{F}(H^2)$ be the set of finite-rank operators on H^2, $\mathcal{K}(H^2)$ the ideal of compact operators, and $\pi : \mathcal{L}(H^2) \to \mathcal{L}(H^2)/\mathcal{K}(H^2)$ the quotient map onto the Calkin algebra.

In [69, Theorem 10] several sufficient and necessary conditions, (2) through (5), were derived for the compactness of a finite sum $\sum_{k=1}^{n} H_{f_k} H_{g_k}$ of products of Hankel operators with L^∞ symbols f_k and g_k. The key to the proof is a distribution function inequality involving harmonic analysis, which asserts compactness of the sum from condition (2). The main interest, however, lies in conditions (4) and (5), in terms of membership of certain linear combinations of the symbol functions in the restriction algebra of H^∞ on each support set in $M(L^\infty)$, and in terms of certain Douglas algebras generated by the symbols, respectively. When $n = 1$, the matrices A and R in those conditions are simply scalars and thus explicit, resulting in explicit characterizations in [154] of compact semicommutators $T_{fg} - T_f T_g = H_{\tilde{f}} H_g$ of Toeplitz operators, which adds to and provides an alternative proof of the Axler-Chang-Sarason-Volberg theorem [11, 139]. When

$n = 2$ and for the commutator $[T_f, T_g] = T_f T_g - T_g T_f = H_{\tilde{f}} H_{-g} + H_{\tilde{g}} H_f$, a peculiar cancellation mechanism allows for explicit characterizations of essentially commuting Toeplitz operators [67, Theorem 0.6, 0.8]. Also see [66, Theorem 20] for the case of $H_{\tilde{f}} H_g + H_{\tilde{g}} H_f$. However, for a general sum $H_{f_1} H_{g_1} + H_{f_2} H_{g_2}$ of two products, the lack of knowledge about A and R leaves such results out of reach.

Hankel operators with PQC symbols extend those with PC symbols which continue to be studied in the recent literature, e.g. [108, 56]. Motivated by the considerations above, and after obtaining distance formulas from a PQC function to restriction algebras of H^∞, we treat in Section 4.3 commutators $[H_f, H_g]$ with $f, g \in PQC$. By (3.1.1) and since PQC is invariant under $C_{\bar{z}}$, $\pi[H_f, H_g]$ lies in the commutative C*-subalgebra $\pi\mathcal{T}(PQC)$ of the Calkin algebra, so that the essential norm $\|[H_f, H_g]\|_e = \|\pi[H_f, H_g]\|$ and essential spectrum $\sigma_e[H_f, H_g] = \sigma(\pi[H_f, H_g])$ can be obtained by Sarason's description [120] of the Gelfand transform of $\pi\mathcal{T}(PQC)$. In particular, every such commutator is normal plus compact, and the compact commutators are characterized in explicit terms just like [66, 67, 75]. As a special case, essentially normal Hankel operators with PQC symbols are characterized. These results have certain contact with the Brown-Douglas-Fillmore theory.

Recall that for a subset $S \subset L^\infty$ and a collection Γ of analytic self-maps of $\mathbb{D}$, $\mathcal{TC}(S, \Gamma)$ denotes the C*-subalgebra of $\mathcal{L}(H^2)$ generated by the Toeplitz operators with symbol in S and the composition operators with symbol in Γ. Such Toeplitz-composition C*-algebras on H^2 are first studied in [87, 88] with the focus on continuous Toeplitz operators and linear fractional composition operators. The latter fall into two classes, automorphisms or otherwise, with differing structures of the algebras. The cases of piecewise continuous Toeplitz operators and composition operators defined by finite Blaschke products are respectively considered in [122, 76].

For Γ a non-elementary discrete subgroup of automorphisms of $\mathbb{D}$, it is shown in [87] that the Calkin C*-subalgebra $\mathcal{C}(\Gamma)/\mathcal{K} = \mathcal{TC}(C, \Gamma)/\mathcal{K}$ is isomorphic to the crossed product $C \rtimes \Gamma$ determined by the homeomorphic action of Γ on $\partial\mathbb{D}$. Extending [87, Theorem 3.1], Section 4.4 shows that the larger algebra $\mathcal{TC}(QC, \Gamma)/\mathcal{K}$ is isomorphic to the crossed product $QC \rtimes \Gamma$ for the action of Γ on QC by composition, after deriving some properties of the associated homeomorphic group action on $M(QC)$. Then considering in Section 4.5 the special case of Γ being the finite cyclic group generated by a rational rotation γ of the disc, the Fredholm operators in $\mathcal{TC}(QC, \gamma)$ are characterized, which generalizes part of the corresponding result in [86] for continuous symbols. Although certain evidence seems to suggest that the index formula should extend as well, we have not yet found a proof.

Now let ϕ be a non-automorphism, linear fractional self-map of $\mathbb{D}$ with $\phi(\zeta) = \eta$ for a pair $\zeta, \eta \in \partial\mathbb{D}$ (that is, C_ϕ is not compact). Then the results in [90] on compact Toeplitz-weighted composition operators and the explicit formula [41] for the adjoint C_ϕ^* imply that continuous Toeplitz operators interact with C_ϕ like constant multiples, and that C_ϕ^* equals a constant multiple of a closely related composition, both modulo $\mathcal{K}(H^2)$. Letting $T \in \mathcal{L}(H^2) \mapsto [T] \in \mathcal{L}(H^2)/\mathcal{K}(H^2)$ be the quotient map onto the Calkin algebra, these key properties were first found and used in [88] to construct an isomorphism of the non-commutative C*-algebra $[\mathcal{TC}(C, \phi)]$, $\phi(\zeta) = \eta \neq \zeta$, based on Allan-Douglas localization [49, 18], by which essential spectra of operators in $\mathcal{TC}(C, \phi)$ were explicitly computed. See [89, 110] for further developments along these lines. On the other hand, for ϕ fixing ζ, isomorphisms of $[\mathcal{TC}(C, \phi)]$ were established in [109] depending on parabolicity of ϕ. In particular, the isomorphism [109, Corollary 6.6] of the non-commutative $[\mathcal{TC}(C, \phi)]$ for a non-parabolic ϕ involves a crossed product of continuous functions by a cyclic group action of infinite order, under which essential spectra remain elusive.

While piecewise continuous Toeplitz symbols were considered in [122], quasicontinuous symbols have fundamentally different behavior. In fact, $PC \bigcap QC = C$ by Sarason's characterization [119] of QC indicates they extend C in different directions. Section 4.6 investigates Toeplitz-composition algebras generated by certain PQC symbols and a ϕ fixing a boundary point. For the parabolic case, we consider the commutative Calkin C*-subalgebra of the generators and obtain the complete fiber structure of its maximal ideal space. This is achieved in part by leveraging Sarason's work [120], and in part by a direct-sum decomposition using partial knowledge of the maximal ideals which then uncovers the complete structure. For the non-parabolic case, however, we consider a commutative non-self-adjoint Calkin subalgebra instead. The maximal ideal space is similarly described, and the fairly large yet proper Shilov boundary is identified on the basis of the Toeplitz C*-subalgebra and an abundance of singleton fibers. Essential spectra and Fredholm indices are then obtained as a result of these identifications.

There are several reasons for the choice of the commutative non-self-adjoint algebra in lieu of the larger non-commutative C*-algebra for the non-parabolic case. The crossed-product approach as in [109] seems intractable due to the elevated complexity of the PQC functions. The localization approach used in [88] derives its effectiveness from the availability of a large central C*-subalgebra containing a crucial element generated by C_ϕ and C_ϕ^* due to $[C_\phi]^2 = 0$. For ϕ fixing a boundary point, this critical feature disappears and the localization approach seems in doubt. On the flip side, one does lose the ability to bring C_ϕ^* in the combination for spectral analysis, in contrast to the C*-algebraic approaches.

We end this section by recalling additional notations. The spectrum of an element a of a unital Banach algebra U is denoted by $\sigma(a)$, or more specifically $\sigma(a, U)$ when subalgebras are in the context, and the spectral radius by $\rho(a)$. The maximal ideal space $M(A)$ of a commutative unital Banach algebra A is equipped with the Gelfand topology, and the fiber $M_x(A)$ consists of the extensions in $M(A)$ of $x \in M(B)$ for a subalgebra B. $C(\Omega)$ stands for the commutative C*-algebra of continuous complex functions on a compact Hausdorff space Ω, while C is reserved for that on $\partial\mathbb{D}$. The homeomorphism group of Ω is written $\mathrm{Homeo}(\Omega)$. The automorphism group of a C*-algebra A is written $\mathrm{Aut}(A)$, while the group of conformal automorphisms of $\mathbb{D}$ is denoted by $\mathrm{Aut}(\mathbb{D})$.

4.2 The distance from a PQC function to restrictions of H^∞

For reference throughout this and the next section, we first recall the structure of the fibers $M_\lambda(QC)$ over $\lambda \in \partial\mathbb{D} \cong M(C)$, and $M_y(PQC)$ over $y \in M(QC)$. The closed ideal $\{f \in QC : f(\lambda+) = 0\}$ of QC corresponds to a unique compact subset $M_\lambda^+(QC)$ of $M(QC)$, in that the ideal consists of all QC functions vanishing on $M_\lambda^+(QC)$. One similarly determines $M_\lambda^-(QC)$. For $\delta > 0$, let $m_{\lambda,\delta}$ in the closed unit ball of the dual space QC^* be the averaging functional over the subarc $(\lambda - \delta, \lambda + \delta)$ of $\partial\mathbb{D}$ centered at λ, and define

$$M_\lambda^0(QC) := \overline{\{m_{\lambda,\delta} : \delta > 0\}} \bigcap M(QC) = \{\lim_\omega m_{\lambda,\delta_\omega} : \lim_\omega \delta_\omega = 0\} \tag{4.2.1}$$

in the weak-star topology in QC^*. Sarason proved that [120, Lemma 8]

$$M_\lambda^+(QC) \bigcup M_\lambda^-(QC) = M_\lambda(QC), \quad M_\lambda^+(QC) \bigcap M_\lambda^-(QC) = M_\lambda^0(QC),$$

and that ([120], Lemma 13 and surrounding remarks) $M_y(PQC)$ over $y \in M_\lambda^\pm(QC) \setminus M_\lambda^0(QC)$ consists of a single functional $y\pm$, whose action on $f \in PC$ gives $f(\lambda\pm)$, while $M_y(PQC)$ over $y \in M_\lambda^0(QC)$ consists of two distinct functionals $y+, y-$, again determined by $f(y\pm) = f(\lambda\pm), \forall f \in PC$. Following [105], the *jump* of $g \in PQC$ over $y \in M_\lambda^0(QC)$ is

$$g_y := 2^{-1}(g(y+) - g(y-)).$$

Identifying L^∞ functions with their Gelfand transform on $M(L^\infty)$, let $H^\infty|F$ be the restriction algebra of H^∞ to a compact subset $F \subset M(L^\infty)$. For $m \in M(H^\infty + C)$, let μ_m be its unique representing measure on $M(L^\infty)$. The support set of μ_m is denoted by S_m, the smallest compact subset of $M(L^\infty)$ satisfying $\mu_m(S_m) = 1$. For $y \in M_\lambda^0(QC)$, write

$$S_{m,y+} := S_m \bigcap M_{y+}(L^\infty), \quad S_{m,y-} := S_m \bigcap M_{y-}(L^\infty).$$

Using a key idea from Sarason's proof of [120, Lemma 10], we have

Lemma 4.2.1. *If $S_{m,y+}, S_{m,y-} \neq \emptyset$, then one has in the sup-norm that*

$$d(g|S_m, H^\infty|S_m) = |g_y|, \ \forall g \in PQC.$$

Proof. Since such S_m is a set of antisymmetry for $H^\infty + C$, it is contained in a unique fiber of $M(L^\infty)$ over $M(QC)$. It is clear then $S_m \subset M_y(L^\infty)$ by hypothesis, which together with $M_y(PQC) = \{y+, y-\}$ gives the nontrivial partition $S_m = S_{m,y+} \bigsqcup S_{m,y-}$ where $g \in PQC$

assumes $g(y+)$ on $S_{m,y+}$, $g(y-)$ on $S_{m,y-}$. Considering the constant function $(g(y+)+g(y-))/2$ in H^∞, one has that the distance is at most $2^{-1}|g(y+)-g(y-)| = |g_y|$.

To prove the reverse direction, suppose $\|g|S_m - h|S_m\| = d < |g_y|$ for some $h \in H^\infty$. Then, one has $h(S_{m,y+}) \subset \overline{\mathbb{D}}(g(y+), d)$, $h(S_{m,y-}) \subset \overline{\mathbb{D}}(g(y-), d)$, while $\overline{\mathbb{D}}(g(y+), d) \bigcap \overline{\mathbb{D}}(g(y-), d) = \emptyset$. It follows from Runge's theorem applied to the compact $\overline{\mathbb{D}}(g(y+), d) \bigsqcup \overline{\mathbb{D}}(g(y-), d)$ in $\mathbb{C}$ that the real non-constant indicator function $1_{S_{m,y+}}$ on S_m can be uniformly approximated by polynomials in $h|S_m$. Since $H^\infty|S_m$ is closed in the sup-norm, this implies $1_{S_{m,y+}} \in H^\infty|S_m$, a contradiction to the fact that S_m is a set of antisymmetry for H^∞. Therefore, $\|g|S_m - h|S_m\| \geq |g_y|, \forall h \in H^\infty$, which proves the lower bound for the distance. $\square$

For $f \in L^\infty$ and $z \in \mathbb{D}$, let $f(z)$ be the harmonic extension of f at z via the Poisson kernel P_z. Given $y \in M_\lambda^0(QC)$, by showing the existence of an S_m satisfying the lemma's hypothesis, we obtain the following distance formulas. Such S_m for every $y \in M_\lambda^0(QC)$ is obtained by refining an argument used by Sarason ([120], p. 826) which yields such S_m only for some $y \in M_\lambda^0(QC)$.

Theorem 4.2.2. *For $\lambda \in \partial\mathbb{D}$, $y \in M_\lambda^0(QC)$, and $g \in PQC$, one has in the sup-norms*

$$d(g|M_y(L^\infty), H^\infty|M_y(L^\infty)) = |g_y| \tag{4.2.2}$$

$$= \max\{d(g|S_m, H^\infty|S_m) : S_m \subset M_y(L^\infty)\}, \tag{4.2.3}$$

where the maximum is attained at any S_m satisfying $S_{m,y+}, S_{m,y-} \neq \emptyset$.

In particular, the following are equivalent:

(i) $g_y = 0$,

(ii) $g|M_y(L^\infty) \in H^\infty|M_y(L^\infty)$,

(iii) $g|S_m \in H^\infty|S_m$ *for every* $S_m \subset M_y(L^\infty)$.

Proof. Since $y \in M_\lambda^0(QC)$, a net $\{t_\omega\}_\omega \subset (0,1)$ exists ([120], p. 822) such that $\lim_\omega t_\omega = 1$ and

$$\lim_\omega f(t_\omega \lambda) = y(f), \quad \forall f \in QC. \tag{4.2.4}$$

Meanwhile, the net $\{t_\omega \lambda\}_\omega \subset \mathbb{D} \hookrightarrow M(H^\infty)$ has a cluster point in $M(H^\infty)$. Passing to a subnet if needed, we have $\lim_\omega t_\omega\lambda = m$ in $M(H^\infty)$ for some $m \in M(H^\infty) \setminus \mathbb{D} = M(H^\infty + C)$, due to $\lim_\omega t_\omega\lambda = \lambda \in \partial\mathbb{D}$ in $\mathbb{C}$. Then we have weak-star convergence in the dual space $C(M(L^\infty))^*$, that is

$$\lim_\omega f(t_\omega\lambda) = \lim_\omega \int_{\partial\mathbb{D}} f P_{t_\omega\lambda} d\theta = \int_{S_m} f d\mu_m, \quad \forall f \in L^\infty. \tag{4.2.5}$$

Combining (4.2.4) and (4.2.5) yields

$$y(f) = \int_{S_m} f d\mu_m = m(f), \quad \forall f \in QC \subset H^\infty + C.$$

That is, $m \in M_y(H^\infty + C)$, which asserts $S_m \subset M_y(L^\infty)$ by a standard fact (cf. [18]). Define $\chi \in PC$ to be 1 on the half circle originating counter clockwise from $\lambda \in \partial\mathbb{D}$, and 0 elsewhere. One has

$$\chi(y+) = \chi(\lambda+) = 1, \quad \chi(y-) = \chi(\lambda-) = 0,$$

and $\chi(t_\omega\lambda) \equiv 1/2$ by reflection symmetry of the Poisson kernel $P_{t_\omega\lambda}$ on $\partial\mathbb{D}$ about the diameter $[-\lambda, \lambda]$. Then (4.2.5) applies to χ and yields

$$\begin{aligned} 1/2 = \int_{S_m} \chi d\mu_m &= \int_{S_{m,y+}} \chi d\mu_m + \int_{S_{m,y-}} \chi d\mu_m \\ &= \chi(y+)\mu_m(S_{m,y+}) + \chi(y-)\mu_m(S_{m,y-}) = \mu_m(S_{m,y+}). \end{aligned}$$

In particular, $S_{m,y+}, S_{m,y-} \neq \emptyset$. Since $d(g|M_y(L^\infty), H^\infty|M_y(L^\infty)) \leq |g_y|$ by invoking the constant function $(g(y+) + g(y-))/2$ again, Lemma 4.2.1 gives

$$\begin{aligned} & d(g|M_y(L^\infty), H^\infty|M_y(L^\infty)) \\ \geq & \sup\{d(g|S_m, H^\infty|S_m) : S_m \subset M_y(L^\infty)\} \\ \geq & |g_y| \\ \geq & d(g|M_y(L^\infty), H^\infty|M_y(L^\infty)). \end{aligned}$$

So, we have equality across, and the supremum is actually a maximum which is attained at any S_m

satisfying $S_{m,y+}, S_{m,y-} \neq \emptyset$. Lastly, (i)$\Leftrightarrow$(ii) follows from (4.2.2) and the fact that $H^\infty|M_y(L^\infty)$ is closed, and (i)$\Leftrightarrow$(iii) from (4.2.3) and $H^\infty|S_m$ being closed. $\square$

Remark 4.2.3. Let $\mathcal{G}$ be the set of *nontrivial* points of $M(H^\infty + C)$, that is, those points whose Gleason part consists of more than one point [82]. It is known that $\mathcal{G} \subsetneq M(H^\infty + C) \setminus M(L^\infty)$. If $y \in M^0_\lambda(QC)$, then there indeed exists $\eta \in \mathcal{G}$ such that $S_{\eta,y+}, S_{\eta,y-} \neq \emptyset$. For, let $m \in M_y(H^\infty + C)$ and $\chi \in PC$ be as in the proof above. Then in particular $m \notin M(L^\infty)$. By [64, Corollary 3.2], $m_\alpha \to m$ in $M(H^\infty)$ for a net $\{m_\alpha\}_\alpha$ in $\mathcal{G}$ with $S_{m_\alpha} \subset S_m \subset M_y(L^\infty)$, so that weak-star convergence $\mu_{m_\alpha} \to \mu_m$ of the representing measures gives

$$\begin{aligned}\int_{S_{m_\alpha}} \chi d\mu_{m_\alpha} = \int_{S_{m_\alpha,y+}} \chi d\mu_{m_\alpha} + \int_{S_{m_\alpha,y-}} \chi d\mu_{m_\alpha} = \mu_{m_\alpha}(S_{m_\alpha,y+}) \\ \to \int_{S_m} \chi d\mu_m = 1/2.\end{aligned}$$

Thus, $\mu_{m_\alpha}(S_{m_\alpha,y+}) \in (0,1)$ for some m_α, and $\eta = m_\alpha \in \mathcal{G}$ gives $S_{\eta,y+}, S_{\eta,y-} \neq \emptyset$.

Analogous to [120, Lemma 10], a global distance formula from a PQC function to $H^\infty + C$ in L^∞ follows, in terms of the size of the jumps of the PQC function.

Proposition 4.2.4. *If $g \in PQC$, then*

$$d(g, H^\infty + C) = \max\{|g_y| : y \in M^0_\lambda(QC), \lambda \in \partial\mathbb{D}\}. \tag{4.2.6}$$

Proof. Since each maximal set of antisymmetry for $H^\infty + C$ on $M(L^\infty)$ is contained in a fiber $M_y(L^\infty)$ over $y \in M(QC)$, and $(H^\infty + C)|M_y(L^\infty) = H^\infty|M_y(L^\infty)$, the Bishop-Glicksberg antisymmetry theorem gives

$$d(g, H^\infty + C) = \max\{d(g|M_y(L^\infty), H^\infty|M_y(L^\infty)) : y \in M(QC)\}.$$

On the other hand, $M_y(PQC)$ over each $y \in M_\lambda(QC) \setminus M^0_\lambda(QC)$, $\lambda \in \partial\mathbb{D}$, is a singleton, so that $g \in PQC$ is constant on such $M_y(L^\infty)$ and that $d(g|M_y(L^\infty), H^\infty|M_y(L^\infty)) = 0$. Thus the above expression together with (4.2.2) yields (4.2.6) at once. $\square$

It follows from [120, Lemmas 9, 10] that the distance from a PQC function to $H^\infty + C$ equals the distance to the proper subset QC. We give an alternate proof of this fact avoiding some of the rather involved function theoretic constructions on the circle.

Proposition 4.2.5 (Sarason [120])**.** *If $g \in PQC$, then $d(g, H^\infty + C) = d(g, QC)$.*

Proof. The Bishop-Glicksberg theorem applies to the C*-subalgebra QC on $M(PQC)$ and gives

$$d(g, QC) = \max\{d(g|M_y(PQC), QC|M_y(PQC)) : y \in M(QC)\}.$$

One has $d(g|M_y(PQC), QC|M_y(PQC)) = 0$ for the singleton $M_y(PQC)$ over each $y \in M_\lambda(QC) \setminus M_\lambda^0(QC)$, $\lambda \in \partial\mathbb{D}$. On $M_y(PQC)$ over $y \in M_\lambda^0(QC)$, g assumes two values $g(y+)$, $g(y-)$, while $QC|M_y(PQC)$ consists only of constants. Since for $c \in \mathbb{C}$,

$$2(|g(y+) - c| \vee |g(y-) - c|) \geq |g(y+) - c| + |g(y-) - c| \geq |g(y+) - g(y-)| = 2|g_y|$$

and equality is attainable, $d(g|M_y(PQC), QC|M_y(PQC)) = |g_y|$ for such y. Therefore,

$$d(g, QC) = \max\{|g_y| : y \in M_\lambda^0(QC), \lambda \in \partial\mathbb{D}\},$$

which together with Proposition 4.2.4 completes the proof. □

Remark 4.2.6. For $f \in L^1$, Sarason [120] defined its *integral gap* at $\lambda \in \partial\mathbb{D}$ as

$$\gamma_\lambda(f) := \limsup_{\delta \downarrow 0} \left| \frac{1}{\delta} \int_\lambda^{\lambda+\delta} f d\theta - \frac{1}{\delta} \int_{\lambda-\delta}^{\lambda} f d\theta \right|.$$

In the proof of [120, Lemma 10], Sarason also showed, in view of [18, 2.79(b)], that on the coarser fibers $M_\lambda(L^\infty)$ over $\lambda \in \partial\mathbb{D}$

$$d(g|M_\lambda(L^\infty), H^\infty|M_\lambda(L^\infty)) = 2^{-1}\gamma_\lambda(g) \tag{4.2.7}$$

$$= \max\{d(g|S_m, H^\infty|S_m) : S_m \subset M_\lambda(L^\infty)\} \tag{4.2.8}$$

for $g \in PQC$. Since $H^\infty|M_\lambda(L^\infty)$ is closed, the following are in fact equivalent:

(i) $\gamma_\lambda(g) = 0$,

(ii) $g|M_\lambda(L^\infty) \in H^\infty|M_\lambda(L^\infty)$,

(iii) $g|S_m \in H^\infty|S_m$ for every $S_m \subset M_\lambda(L^\infty)$.

Therefore, Theorem 4.2.2 and Proposition 4.2.4 constitute a complete analog on the finer fibers over $M(QC)$ to Sarason's result, which will be used to analyze Hankel operators in the next section. This analogy also links $\gamma_\lambda(g)$ and $|g_y|$ as follows.

Corollary 4.2.7. *If $\lambda \in \partial\mathbb{D}$ and $g \in PQC$, then*

$$\gamma_\lambda(g) = 2\max\{|g_y| : y \in M_\lambda^0(QC)\}. \tag{4.2.9}$$

Proof. One has $\{S_m : S_m \subset M_\lambda(L^\infty)\} = \bigsqcup_{y\in M_\lambda(QC)}\{S_m : S_m \subset M_y(L^\infty)\}$, and g is constant on S_m for $y \in M_\lambda(QC) \setminus M_\lambda^0(QC)$ and $S_m \subset M_y(L^\infty)$. The conclusion then follows from (4.2.8) and (4.2.3). $\square$

4.3 Commutators of Hankel operators with PQC symbols

This section concerns the commutator $[H_f, H_g] = H_fH_g - H_gH_f$ for $f, g \in PQC$. The key starting point is Sarason's determination ([120, Lemma 14, 15, 16]; cf. [105, Theorem 4], [18, 4.87]) of the Gelfand transform of πT_f on the fibers $M_y(\pi\mathcal{T}(PQC))$ over $y \in M(QC) \cong M(\pi\mathcal{T}(QC))$.

Theorem 4.3.1 (Sarason)**.** *Let $\lambda \in \partial\mathbb{D}$. If $y \in M_\lambda^\pm(QC) \setminus M_\lambda^0(QC)$, then $M_y(\pi\mathcal{T}(PQC))$ consists of a single functional assuming the value $f(y\pm)$ at each πT_f, $f \in PQC$; If $y \in M_\lambda^0(QC)$, then $M_y(\pi\mathcal{T}(PQC))$ is homeomorphic to $[0,1]$ via the *-isomorphism between $C(M_y(\pi\mathcal{T}(PQC)))$ and $C[0,1]$ determined by*

$$\pi T_f|M_y(\pi\mathcal{T}(PQC)) \mapsto (t \in [0,1] \mapsto tf(y+) + (1-t)f(y-)),\ \ f \in PQC.$$

The algebra PC is clearly invariant under the action of $C_{\bar{z}}$. So are VMO, L^∞, and hence QC. Consequently, PQC as generated by PC and QC is also invariant under $C_{\bar{z}}$. Also, $M(QC) \subset QC^*$

is invariant for the adjoint $C_{\bar{z}}^* \in \mathcal{L}(QC^*)$, and both $C_{\bar{z}}^*$ and $C_{\bar{z}}^*|M(QC)$ are homeomorphisms in the weak-star topologies. The same holds for PQC. Write $\bar{y} := C_{\bar{z}}^* y = y \circ C_{\bar{z}} \in M(QC)$ for $y \in M(QC)$ and use the same notation $\overline{y\pm} \in M(PQC)$ for $y\pm \in M(PQC)$. The map $C_{\bar{z}}^*$ has the following properties, part of which was indicated and used without proof in [105].

Lemma 4.3.2 (cf. [105])**.** *Let $\lambda \in \partial\mathbb{D}$ and $y \in M(QC)$. Then,*

(i) $\bar{y} \in M_{\bar{\lambda}}^{\mp}(QC) \setminus M_{\bar{\lambda}}^0(QC)$ *if* $y \in M_\lambda^{\pm}(QC) \setminus M_\lambda^0(QC)$;

(ii) $\bar{y} \in M_{\bar{\lambda}}^0(QC)$ *if* $y \in M_\lambda^0(QC)$;

(iii) $\bar{y} = y$ *if and only if* $y \in M_1^0(QC) \bigsqcup M_{-1}^0(QC)$;

(iv) $\overline{y\pm} = \bar{y}\mp$, $\overline{y\pm} \neq y\pm$;

(v) $\bar{\bar{y}} = y$, $\overline{\overline{y\pm}} = y\pm$.

Proof. Since the map $C_{\bar{z}}$ takes the ideal $\{f \in QC : f(\bar{\lambda}+) = 0\}$ to the ideal $\{f \in QC : f(\lambda-) = 0\}$, and $\{f \in QC : f(\bar{\lambda}-) = 0\}$ to $\{f \in QC : f(\lambda+) = 0\}$, one has by definition that $\bar{y} \in M_{\bar{\lambda}}^{\mp}(QC)$ if $y \in M_\lambda^{\pm}(QC)$, from which (i), (ii), and the necessity part of (iii) follow in view of $M_\lambda^+(QC) \bigcap M_\lambda^-(QC) = M_\lambda^0(QC)$. The sufficiency part of (iii) follows from (4.2.1), weak-star continuity, and the fact that $m_{\pm 1,\delta}$, $\delta > 0$, are all fixed by $C_{\bar{z}}^*$. Finally, (iv) is due to $\tilde{f}(\lambda\pm) = f(\bar{\lambda}\mp)$ for $f \in PC$, and (v) to $C_{\bar{z}}^2 = I$. □

It immediately follows from definitions and Lemma 4.3.2 (iv) that, for $\alpha \in \mathbb{C}$, $f, g \in PQC$, $\lambda \in \partial\mathbb{D}$ and $y \in M_\lambda^0(QC)$,

$$(\alpha f + g)_y = \alpha(f_y) + g_y, \tag{4.3.1}$$

$$\tilde{f}_y = -f_{\bar{y}}, \tag{4.3.2}$$

$$f_y^* = -\overline{f_{\bar{y}}}. \tag{4.3.3}$$

The essential spectrum and essential norm of $[H_f, H_g]$ are obtained as follows, after an elementary identity is recorded:

$$[ta\alpha + (1-t)b\beta] - [ta + (1-t)b][t\alpha + (1-t)\beta] = t(1-t)(a-b)(\alpha - \beta) \tag{4.3.4}$$

where $a, b, \alpha, \beta \in \mathbb{C},\ t \in [0,1]$. The line segment joining α, β is written $[\alpha, \beta]$.

Theorem 4.3.3. *For $f, g \in PQC$, one has*

$$\sigma_e[H_f, H_g] = \bigcup\{[-(f_y g_{\bar{y}} - f_{\bar{y}} g_y),\ f_y g_{\bar{y}} - f_{\bar{y}} g_y] :\ y \in M^0_{e^{i\theta}}(QC), \theta \in (0,\pi)\},$$
$$\|[H_f, H_g]\|_e = \max\{|f_y g_{\bar{y}} - f_{\bar{y}} g_y| :\ y \in M^0_{e^{i\theta}}(QC), \theta \in (0,\pi)\}.$$

Proof. We have by (3.1.1) that

$$\begin{aligned}\pi[H_f, H_g] &= (\pi T_{\tilde{f}g} - \pi T_{\tilde{f}} \pi T_g) - (\pi T_{\tilde{g}f} - \pi T_{\tilde{g}} \pi T_f)\\ &=: S_1 - S_2 =: S \in \pi\mathcal{T}(PQC). \end{aligned} \tag{4.3.5}$$

Thus, it follows from elements of the Gelfand theory that

$$\begin{aligned}\sigma_e[H_f, H_g] &= \sigma(S) = S(M(\pi\mathcal{T}(PQC)))\\ &= \bigcup\{S(M_y(\pi\mathcal{T}(PQC))) :\ y \in M_\lambda(QC), \lambda \in \partial\mathbb{D}\},\\ \|[H_f, H_g]\|_e &= \|S\| = |S(M(\pi\mathcal{T}(PQC)))|\\ &= \max\{|S(M_y(\pi\mathcal{T}(PQC)))| :\ y \in M_\lambda(QC), \lambda \in \partial\mathbb{D}\},\end{aligned}$$

where $|\Omega| := \max\{|\zeta| : \zeta \in \Omega\}$ is the radius of any compact set Ω of complex numbers. Now apply Theorem 4.3.1 for the Gelfand transform of each term of S on the fibers $M_y(\pi\mathcal{T}(PQC))$. There are two cases for every $\lambda \in \partial\mathbb{D}$, as follows.

If $y \in M^\pm_\lambda(QC) \setminus M^0_\lambda(QC)$, one has $S(M_y(\pi\mathcal{T}(PQC))) = \{0\}$, for the fiber $M_y(\pi\mathcal{T}(PQC))$ is a singleton whose action on S gives the value

$$[(\tilde{f}g)(y\pm) - \tilde{f}(y\pm)g(y\pm)] - [(\tilde{g}f)(y\pm) - \tilde{g}(y\pm)f(y\pm)] = 0.$$

If $y \in M^0_\lambda(QC)$, then $S(M_y(\pi\mathcal{T}(PQC)))$ equals the range of the function

$$t \in [0,1] \mapsto \phi^y(t) - \psi^y(t)$$

with ϕ^y, ψ^y corresponding respectively to S_1, S_2 in (4.3.5). Using (4.3.4),

$$\begin{aligned}\phi^y(t) &= [t\tilde{f}(y+)g(y+) + (1-t)\tilde{f}(y-)g(y-)] \\ &\quad - [t\tilde{f}(y+) + (1-t)\tilde{f}(y-)][tg(y+) + (1-t)g(y-)] \\ &= 4t(1-t)\tilde{f}_y g_y\end{aligned}$$

and similarly $\psi^y(t) = 4t(1-t)\tilde{g}_y f_y$. So, by invoking (4.3.2),

$$\phi^y(t) - \psi^y(t) = 4t(1-t)(\tilde{f}_y g_y - \tilde{g}_y f_y) = 4t(1-t)(f_y g_{\bar{y}} - f_{\bar{y}} g_y).$$

It then becomes clear that in this case

$$\begin{aligned}S(M_y(\pi\mathcal{T}(PQC))) &= [0,\ f_y g_{\bar{y}} - f_{\bar{y}} g_y], \\ |S(M_y(\pi\mathcal{T}(PQC)))| &= |f_y g_{\bar{y}} - f_{\bar{y}} g_y|.\end{aligned} \tag{4.3.6}$$

Some consolidations using Lemma 4.3.2 are available for the second case. For $\lambda = \pm 1$, one has $\bar{y} = y$, so $S(M_y(\pi\mathcal{T}(PQC))) = \{0\}$. For $\lambda = e^{i\theta}, \theta \in (0, \pi)$, one has $\bar{y} \neq y$ and $y \in M_\lambda^0(QC) \mapsto \bar{y} \in M_{\bar{\lambda}}^0(QC)$ is a bijection with

$$f_{\bar{y}} g_{\bar{\bar{y}}} - f_{\bar{\bar{y}}} g_{\bar{y}} = -(f_y g_{\bar{y}} - f_{\bar{y}} g_y). \tag{4.3.7}$$

Thus, by (4.3.6) and (4.3.7), the range of S on the union of the two distinct fibers over y and $\bar{y}$ is exactly the line segment

$$[-(f_y g_{\bar{y}} - f_{\bar{y}} g_y),\ f_y g_{\bar{y}} - f_{\bar{y}} g_y]$$

centered at the origin. The proof is concluded by combining these cases. $\square$

It is seen from Theorem 4.3.3 that $\sigma_e[H_f, H_g]$ is antipodal symmetric, a property shared by other classes of operators involving Hankel operators [144]. Evidently, it is also connected. Moreover, $\sigma_e[H_f, H_g]$ is a star domain about 0, so that its complement in $\mathbb{C}$ is path connected, forcing the Fredholm index

$$\operatorname{ind}(\lambda I - [H_f, H_g]) \equiv 0, \quad \lambda \notin \sigma_e[H_f, H_g].$$

Therefore the Brown-Douglas-Fillmore theorem (cf. [49, Chapter 9]) implies that the essentially normal $[H_f, H_g]$ is a compact perturbation of a normal operator, and that two such commutators are unitarily equivalent modulo $\mathcal{K}$ if and only if they have identical essential spectra given explicitly as above. In particular, $[H_f, H_g]$ and $[H_g, H_f]$ are unitarily equivalent modulo $\mathcal{K}$. For $f, g \in PC$, $\sigma_e[H_f, H_g]$ reduces to a countable union of line segments thus has zero area measure, analogous to the case of Hankel operators with PC symbols. In contrast, based on the homeomorphism [120, p. 823]

$$M_\lambda^0(QC) \cong M(SO[0,1)) \setminus [0,1)$$

where $SO[0,1)$ denotes the C*-algebra of bounded continuous functions on $[0,1)$ of slow oscillation near 1, it is shown in [105] that any compact star domain $\Omega \subset \mathbb{C}$ about the origin is the essential spectrum of a Hankel operator with PQC symbol. This is also true for commutators assuming Ω antipodal symmetric in addition. For, if $p \in PC, q \in QC$ and $y \in M_\lambda^0(QC)$, then

$$(pq)_y = 2^{-1}(p(\lambda+) - p(\lambda-))q(y), \qquad (pq)_{\bar{y}} = 2^{-1}(p(\bar{\lambda}+) - p(\bar{\lambda}-))q(\bar{y}).$$

So for $p \in PC$ continuous except a jump discontinuity at the imaginary unit $\lambda = i$ with $p(i+) - p(i-) = 2$, and $q \in QC$, the essential spectrum formula gives

$$\sigma_e[H_{pq}, H_{\tilde{p}}] = \bigcup \{[-q(y), q(y)] : y \in M_i^0(QC)\}. \tag{4.3.8}$$

As in [105, p. 52] choose $\phi \in SO[0,1)$ such that the range $\phi(M(SO[0,1)) \setminus [0,1)) = \Omega$, which is the cluster set of ϕ at 1. Then the homeomorphism induces a continuous function on $M_i^0(QC)$ from the continuous function $\phi|(M(SO[0,1)) \setminus [0,1))$, which extends to a continuous function on

$M(QC)$. This gives a function $q \in QC$ with $q(M_i^0(QC)) = \Omega$. It follows from (4.3.8) and the geometric assumptions about Ω that $\sigma_e[H_{pq}, H_{\tilde{p}}] = \Omega$ for such p, q. We summarize these findings.

Proposition 4.3.4. *For $f, g \in PQC$, one has*

(i) *$[H_f, H_g]$ is a compact perturbation of a normal operator.*

(ii) *$[H_f, H_g]$ and $[H_g, H_f]$ are unitarily equivalent modulo $\mathcal{K}$.*

(iii) *$\sigma_e[H_f, H_g]$ is an antipodal symmetric star domain about the origin.*

Conversely, given a compact antipodal symmetric star domain $\Omega \subset \mathbb{C}$ about the origin, there exist $f, g \in PQC$ such that $\sigma_e[H_f, H_g] = \Omega$.

Denote by $H^\infty[f_1, ..., f_n]$ the Douglas algebra generated by $f_1, ..., f_n \in L^\infty$ over H^∞. For an arbitrary collection $\{B_\alpha\}_{\alpha \in \Lambda}$ of Douglas algebras, Younis and Zheng [151, Theorem 3] proved that for some constant $J > 1$

$$d(f, \bigcap_{\alpha \in \Lambda} B_\alpha) \leq J \sup_{\alpha \in \Lambda} d(f, B_\alpha), \quad \forall f \in L^\infty.$$

It follows that $\bigcap_{\alpha \in \Lambda} B_\alpha \subset H^\infty + C$ if and only if for some constant $L > 0$

$$d(f, H^\infty + C) \leq L \sup_{\alpha \in \Lambda} d(f, B_\alpha), \quad \forall f \in L^\infty.$$

Noting this fact and applying results from Section 4.2, we set the essential norm zero to characterize the compact commutators of Hankel operators with PQC symbols, in forms similar to [66, 67, 75, 149].

Theorem 4.3.5. *For $f, g \in PQC$, the following are equivalent*

(i) *$[H_f, H_g]$ is compact.*

(ii) *$f_y g_{\bar{y}} = f_{\bar{y}} g_y, \ \forall y \in M_\lambda^0(QC), \ \lambda \in \partial\mathbb{D}$.*

(iii) *For every $m \in M(H^\infty + C)$, there are $a_m, b_m \in \mathbb{C}$ with $|a_m| + |b_m| > 0$ such that both $(a_m f + b_m g)|S_m$ and $(a_m f + b_m g)^*|S_m$ are in $H^\infty|S_m$.*

(iv) *For every $m \in \mathcal{G}$,* (iii) *holds.*

(v)

$$\bigcap_{|a|+|b|>0} H^\infty[af+bg,(af+bg)^*] \subset H^\infty + C.$$

(vi) *For some constant* $L > 0$

$$d(\phi, H^\infty + C) \le L \sup_{|a|+|b|>0} d(\phi, H^\infty[af+bg,(af+bg)^*]), \quad \forall \phi \in L^\infty.$$

Proof. Theorem 4.3.3 gives (i)⇔(ii), while (iii)⇔(v) will be proved in Proposition 4.3.7 for any $f, g \in L^\infty$. Also, (v)⇔(vi) as above. It remains to show (ii)⇒(iii) and (iv)⇒(ii).

Assume (ii) holds. Every $m \in M(H^\infty + C)$ has its support $S_m \subset M_y(L^\infty)$ for a unique $y \in M_\lambda(QC)$ which falls in two cases. If $y \in M_\lambda(QC) \setminus M_\lambda^0(QC)$, then $M_y(PQC)$ is a singleton, so that any PQC function is constant on $M_y(L^\infty) \supset S_m$ giving in particular (iii). Next let $y \in M_\lambda^0(QC)$. If either $S_{m,y+}$ or $S_{m,y-} = \emptyset$, then either $S_m \subset M_{y-}(L^\infty)$ or $S_m \subset M_{y+}(L^\infty)$, and we have the same situation as before. It remains to consider $S_{m,y+}, S_{m,y-} \neq \emptyset$, in which case Lemma 4.2.1 gives that, for $a, b \in \mathbb{C}$,

$$\begin{aligned} &(af+bg)|S_m \text{ and } (af+bg)^*|S_m \in H^\infty|S_m \\ \Leftrightarrow &\begin{cases} f_y a + g_y b = 0 \\ f_{\bar{y}} a + g_{\bar{y}} b = 0 \end{cases} \end{aligned} \tag{4.3.9}$$

by (4.3.1) and (4.3.3). Since the system (4.3.9) of two homogeneous linear equations in the two unknowns a, b has a nonzero solution (a_m, b_m) if and only if the determinant

$$\det \begin{bmatrix} f_y & g_y \\ f_{\bar{y}} & g_{\bar{y}} \end{bmatrix} = f_y g_{\bar{y}} - f_{\bar{y}} g_y = 0,$$

(iii) is seen to follow from $f_y g_{\bar{y}} = f_{\bar{y}} g_y$.

Next assume (iv) holds and let $y \in M_\lambda^0(QC), \lambda \in \partial\mathbb{D}$. There exists by Remark 4.2.3 $m \in \mathcal{G}$ with $S_{m,y+}, S_{m,y-} \neq \emptyset$. One is again in the situation surrounding (4.3.9), and the assumption that (4.3.9) has a nonzero solution (a_m, b_m) implies $f_y g_{\bar{y}} = f_{\bar{y}} g_y$, proving (ii). $\square$

Essentially normal Toeplitz operators with L^∞ symbols are characterized in [67, Corollary 3.4] in terms of symbol behavior on support sets. Such characterizations for essentially normal Hankel operators are not available in the literature. We address a special case of this problem for PQC symbols. Since $PC \bigcap (H^\infty + C) = C$ and $PQC \bigcap (H^\infty + C) = QC$ (by Proposition 4.2.5), there are plenty of non-compact essentially normal Hankel operators with symbols in PC and $PQC \setminus PC$.

Corollary 4.3.6. *For $f \in PQC$, the following are equivalent*

(i) *H_f is essentially normal.*

(ii) *H_f is a compact perturbation of a normal operator.*

(iii) $|f_y| = |f_{\bar{y}}|, \ \forall y \in M_\lambda^0(QC), \ \lambda \in \partial\mathbb{D}$.

(iv) *For every $m \in M(H^\infty + C)$, either both $f|S_m$ and $f^*|S_m$ are in $H^\infty|S_m$, or $(\lambda_m f + f^*)|S_m$ is in $H^\infty|S_m$ for some $\lambda_m \in \partial\mathbb{D}$.*

(v) *For every $m \in \mathcal{G}$,* (iv) *holds.*

(vi)

$$H^\infty[f, f^*] \bigcap \bigcap_{|\lambda|=1} H^\infty[\lambda f + f^*] \subset H^\infty + C.$$

(vii) *For some constant $L > 0$*

$$d(\phi, H^\infty + C) \leq L(d(\phi, H^\infty[f, f^*]) \vee \sup_{|\lambda|=1} d(\phi, H^\infty[\lambda f + f^*])), \quad \forall \phi \in L^\infty.$$

Proof. Set $g = f^*$, and Theorem 4.3.5(i)(ii) become (i)(iii) here, respectively, noting $H_f^* = H_{f^*}$ and using (4.3.3).

Since $\operatorname{ind}(\lambda I - H_f) \equiv 0, \forall \lambda \notin \sigma_e(H_f)$ [106, p. 425], the BDF theorem as before gives (i)$\Leftrightarrow$(ii).

See Proposition 4.3.7 for (iv)$\Leftrightarrow$(vi). It remains to show that Theorem 4.3.5(iii) with $g = f^*$ is equivalent to (iv) here, since the situation with the equivalence of Theorem 4.3.5(iv) and (v) here is identical.

First assume Theorem 4.3.5(iii) with $g = f^*$ holds. That is, for every $m \in M(H^\infty + C)$, there exist $a_m, b_m \in \mathbb{C}$, $|a_m| + |b_m| > 0$, such that

$$(a_m f + b_m f^*)|S_m \in H^\infty|S_m, \tag{4.3.10}$$

$$(\bar{a}_m f^* + \bar{b}_m f)|S_m \in H^\infty|S_m. \tag{4.3.11}$$

Suppose at least one of $f|S_m$ and $f^*|S_m$ is not in $H^\infty|S_m$. If $f|S_m \notin H^\infty|S_m$ and $a_m = 0$, then $\bar{b}_m \neq 0$ and (4.3.11) creates a contradiction; If $f|S_m \notin H^\infty|S_m$ and $b_m = 0$, then $a_m \neq 0$ and (4.3.10) is absurd. Hence, if $f|S_m \notin H^\infty|S_m$, then $a_m b_m \neq 0$, which further implies $f^*|S_m \notin H^\infty|S_m$ by (4.3.10) or (4.3.11). One has the same situation with f and f^* switched. That is, either way we have $f|S_m \notin H^\infty|S_m$, $f^*|S_m \notin H^\infty|S_m$, and $a_m b_m \neq 0$, so that (4.3.10) and (4.3.11) become

$$(\lambda_m f + f^*)|S_m \in H^\infty|S_m \tag{4.3.12}$$

$$(\bar{\lambda}_m f^* + f)|S_m \in H^\infty|S_m \tag{4.3.13}$$

where $\lambda_m := a_m/b_m$. Multiply (4.3.12) by $\bar{\lambda}_m$ then subtract by (4.3.13) to get

$$(|\lambda_m|^2 - 1) f|S_m \in H^\infty|S_m.$$

Since $f|S_m \notin H^\infty|S_m$, we have $|\lambda_m| = 1$ as desired, and (iv) follows.

Next assume (iv) holds. Then we have Theorem 4.3.5(iii), $g = f^*$, with either $(a_m, b_m) = (1, 0)$ or $(\lambda_m, 1)$, noting in the latter case

$$(\lambda_m f + f^*)^* = \bar{\lambda}_m f^* + f = \bar{\lambda}_m(\lambda_m f + f^*).$$

The proof is complete. $\square$

Let $k_z \in H^2$ be the normalized reproducing kernel at $z \in \mathbb{D}$,

$$k_z(e^{i\theta}) = \frac{\sqrt{1 - |z|^2}}{1 - \bar{z}e^{i\theta}}.$$

For $f \in L^\infty$ and $m \in M(H^\infty + C)$, Lemma 1.5, 2.5 and 2.6 in [67] establish

$$m \in M(H^\infty[f]) \Leftrightarrow f|S_m \in H^\infty|S_m \Leftrightarrow \lim_{z \to m} \|H_f k_z\|_2 = 0 \tag{4.3.14}$$

where the limit is taken relative to every net of $z \in \mathbb{D} \hookrightarrow M(H^\infty)$ converging to m in $M(H^\infty)$. Using this key relation and following mostly the proofs of [67, Lemma 1.1] and [75, Lemma 16], we next show (iii)$\Leftrightarrow$(v) in Theorem 4.3.5, 4.3.6 for general symbols. Recall that the maximal ideal space of any Douglas algebra is identified with its restriction in $M(H^\infty)$.

Proposition 4.3.7. *For $f, g \in L^\infty$,* Theorem 4.3.5(iii)$\Leftrightarrow$(v) *and* Corollary 4.3.6(iv)$\Leftrightarrow$(vi).

Proof. Suppose Theorem 4.3.5(v) holds. Then we have

$$\begin{aligned} M(H^\infty + C) \subset & M\left(\bigcap_{|a|+|b|>0} H^\infty[af + bg, (af + bg)^*]\right) \\ = & \overline{\bigcup_{|a|+|b|>0} M(H^\infty[af + bg, \bar{a}f^* + \bar{b}g^*])} \\ = & M(H^\infty[f, f^*]) \bigcup M(H^\infty[g, g^*]) \bigcup \\ & \overline{\bigcup_{\alpha \neq 0} M(H^\infty[f + \alpha g, f^* + \bar{\alpha}g^*])} \end{aligned} \tag{4.3.15}$$

with the substitution $\alpha := b/a$. If $m \in M(H^\infty + C)$ is in $M(H^\infty[f, f^*])$ or $M(H^\infty[g, g^*])$, then [67, Corollary 1.6] puts $f|S_m$ and $f^*|S_m$ in $H^\infty|S_m$, or $g|S_m$ and $g^*|S_m$ in $H^\infty|S_m$. That is, we have (iii) with $(a_m, b_m) = (1, 0)$ or $(0, 1)$. By (4.3.15) it remains to consider

$$\begin{aligned} m \in & \overline{\bigcup_{\alpha \neq 0} M(H^\infty[f + \alpha g, f^* + \bar{\alpha}g^*])} \\ = & \overline{\bigcup_{0<|\alpha|\leq 1} M(H^\infty[f + \alpha g, f^* + \bar{\alpha}g^*])} \bigcup \overline{\bigcup_{0<|\beta|\leq 1} M(H^\infty[\beta f + g, \bar{\beta}f^* + g^*])} \end{aligned}$$

where $\beta := 1/\alpha$. It suffices to only consider m in the first closure. To this end let $m_\omega \to m$ in $M(H^\infty)$ for a net $m_\omega \in M(H^\infty[f + \alpha_\omega g, f^* + \bar{\alpha}_\omega g^*])$, which, by [67, Corollary 1.6] and (4.3.14), implies that for each ω

$$\lim_{z\to m_\omega} \|H_{f+\alpha_\omega g}k_z\|_2 = 0, \quad \lim_{z\to m_\omega} \|H_{f^*+\bar{\alpha}_\omega g^*}k_z\|_2 = 0.$$

Passing to a subnet of $\alpha_\omega \in \overline{\mathbb{D}}$ if necessary, we assume $\alpha_\omega \to \alpha_m$ and hence $\bar{\alpha}_\omega \to \bar{\alpha}_m$ in $\overline{\mathbb{D}}$. From this point on, the argument in [75], pp. 136-137, shows

$$\lim_{z\to m} \|H_{f+\alpha_m g}k_z\|_2 = 0, \quad \lim_{z\to m} \|H_{f^*+\bar{\alpha}_m g^*}k_z\|_2 = 0.$$

That is, both $f + \alpha_m g|S_m$ and $f^* + \bar{\alpha}_m g^*|S_m$ are in $H^\infty|S_m$, and Theorem 4.3.5(iii) holds with $(a_m, b_m) = (1, \alpha_m)$. The implication Corollary 4.3.6(vi)⇒(iv) is proved in the same way, if not easier. We omit the details.

On the other hand, (4.3.14) and the Chang-Marshall theorem (cf. [62]) give the converses. This ends the proof. □

The determination of essential spectra in Theorem 4.3.3 and characterization of essentially normal Hankel operators in Corollary 4.3.6 are of interest from a C*- algebra extension perspective. Let $T = [H_f, H_g]$ or H_h, where $f, g \in PQC$, and $h \in PQC$ is such that H_h is essentially normal. Then the essentially normal operator T induces via the continuous functional calculus for πT a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}^*\{T\} + \mathcal{K} \longrightarrow C(\sigma_e(T)) \longrightarrow 0. \tag{4.3.16}$$

This extension of $\mathcal{K}$ by $C(\sigma_e(T))$ *splits*, that is, there exists a $*$-monomorphism from $C(\sigma_e(T))$ into $\mathcal{L}(H^2)$ mapping the identity function z to $T + K$ for a compact operator K. This is because $\sigma_e(T) \subset \mathbb{C}$ is a star domain about 0 in either case of T (see [105, Theorem 5] in case $T = H_h = S_{zh}$), so that it is contractible to 0. Therefore, the group $\mathrm{Ext}(\sigma_e(T))$ of equivalence classes of extensions is trivial ([21]; cf. [49]), and the particular extension (4.3.16) associated with T must correspond to the zero element of $\mathrm{Ext}(\sigma_e(T))$. That is, the extension splits.

4.4 Toeplitz-composition C*-algebras and crossed products

Let $\beta : QC \to C(M(QC))$ be the Gelfand transform. For every $\gamma \in \mathrm{Aut}(\mathbb{D}) \hookrightarrow \mathrm{Homeo}(\partial\mathbb{D})$, the distribution of γ on $\partial\mathbb{D}$ under the linear measure is mutually absolutely continuous, and the algebra QC on $\partial\mathbb{D}$ is easily seen invariant under the well-defined composition operator on L^∞ by γ. Writing $\Phi_\gamma \in \mathrm{Aut}(QC)$ for the composition $\Phi_\gamma f = f \circ \gamma$, $\forall f \in QC$, the automorphism $\beta\Phi_\gamma\beta^{-1}$ of $C(M(QC))$ equals the composition $\Psi_{\underline{\gamma}}\hat{f} = \hat{f} \circ \underline{\gamma}$, $\forall \hat{f} \in C(M(QC))$, defined by $\underline{\gamma} = \Phi_\gamma^* \in \mathrm{Homeo}(M(QC))$. That is, we have the commutative diagram

$$\begin{tikzcd} QC \arrow[r, "\Phi_\gamma"] \arrow[d, "\beta"'] & QC \arrow[d, "\beta"] \\ C(M(QC)) \arrow[r, "\Psi_{\underline{\gamma}}"] & C(M(QC)) \end{tikzcd} \tag{4.4.1}$$

A basic observation is that $\underline{\gamma}$ maps fibers to fibers. Note in passing that further partitions of the fibers of $M(QC)$ over $\partial\mathbb{D}$ are obtained in [120], which interact with $\underline{\gamma}$ in a manner more refined but not needed for the discussions here.

Lemma 4.4.1. *If $r : M(QC) \to M(C) = \partial\mathbb{D}$ is the restriction map, then $\underline{\gamma}(M_\lambda(QC)) = M_{\gamma(\lambda)}(QC)$ for $\lambda \in \partial\mathbb{D}$. Equivalently, one has the commutative diagram*

$$\begin{tikzcd} M(QC) \arrow[r, "\underline{\gamma}"] \arrow[d, "r"'] & M(QC) \arrow[d, "r"] \\ \partial\mathbb{D} \arrow[r, "\gamma"] & \partial\mathbb{D} \end{tikzcd}$$

Proof. Since $\underline{\gamma} = \Phi_\gamma^*$, the diagram amounts to $r(y\Phi_\gamma) = \gamma(r(y))$, $\forall y \in M(QC)$. To see the latter, let $\phi(z) = z$ be the coordinate function in C and we deduce

$$r(y\Phi_\gamma) = (r(y\Phi_\gamma))(\phi) = (y\Phi_\gamma)(\phi) = y(\gamma) = \gamma(r(y))$$

as desired. □

Later proofs will use only a special case of the next result when $A = QC$.

Proposition 4.4.2. *Let A be a C^*-subalgebra of L^∞ containing C and suppose $\Lambda \subset \partial\mathbb{D}$ has zero θ-measure. Then the union of fibers $\bigsqcup_{\lambda\in\Lambda} M_\lambda(A)$ has empty interior in $M(A)$.*

Proof. In view of the continuous restriction map r from $M(L^\infty)$ onto $M(A)$, the conclusion follows from a stronger result, that is, the union of fibers

$$\bigsqcup_{\lambda\in\Lambda} M_\lambda(L^\infty) = r^{-1}(\bigsqcup_{\lambda\in\Lambda} M_\lambda(A))$$

has empty interior in $M(L^\infty)$. Suppose the stronger result is false. Then, since $M(L^\infty)$ is totally disconnected, $\bigsqcup_{\lambda\in\Lambda} M_\lambda(L^\infty)$ contains a nonempty basic clopen set $F \subset M(L^\infty)$. The characteristic function $1_F \in C(M(L^\infty))$ corresponds via the Gelfand transform to $1_E \in L^\infty$ for a Borel subset E of $\partial\mathbb{D}$ with strictly positive measure. By the Lebesgue density theorem, the set of density points of E has positive measure as well. However, since 1_F vanishes identically on the fiber $M_\lambda(L^\infty)$ over every $\lambda \in \partial\mathbb{D}\setminus\Lambda$, the local essential range of 1_E is $\{0\}$ (cf. [18, 2.79]) at every $\lambda \in \partial\mathbb{D}\setminus\Lambda$. That is,

$$\theta((\lambda-\epsilon, \lambda+\epsilon)\bigcap E) = 0$$

for sufficiently small $\epsilon > 0$. Such λ is certainly not a density point of E. So the density points are all contained in the zero-measure set Λ. This contradiction completes the proof. □

For a subgroup Γ of $\mathrm{Aut}(\mathbb{D})$, the map $\gamma \mapsto \underline{\gamma} = \Phi_\gamma^*$ is a group isomorphism from Γ onto $\underline{\Gamma}$, both of which are endowed with the discrete topology. Consider the sub-action of $\underline{\Gamma} \subset \mathrm{Homeo}(M(QC))$ on the compact Hausdorff space $M(QC)$. Recall that a homeomorphic action $g \cdot \omega$ of a discrete group G on a compact space Ω is said to be *topologically free* (cf. [6]) if the set of fixed points of the action of any non-identity element has empty interior in Ω. The action is said to be (topologically) *amenable* ([102], Definition 2.1; also [87], p. 3175) if there exists a sequence of functions $\xi_n : G\times\Omega \to [0,1]$ satisfying $\xi_n(g,\cdot) \in C(\Omega)$ for all $n \in \mathbb{N}$ and $g \in G$, $\sum_{g\in G} \xi_n(g,\cdot) \equiv 1$ on Ω, and that for every $g \in G$

$$\sup_{\omega\in\Omega}\left(\sum_{g'\in G} |\xi_n(g^{-1}g', \omega) - \xi_n(g', g\cdot\omega)|\right) \to 0 \text{ as } n \to 0. \tag{4.4.2}$$

Both properties are well-known for the action of Γ on $\partial\mathbb{D}$, from which we will derive them for the corresponding action of $\underline{\Gamma}$ on $M(QC)$, essentially by reduction via Lemma 4.4.1.

Lemma 4.4.3. *Let* Γ *be a discrete subgroup of* $\mathrm{Aut}(\mathbb{D})$. *Then the homeomorphic action on* $M(QC)$ *of the discrete group* $\underline{\Gamma}$ *is topologically free and amenable.*

Proof. The action on $\partial\mathbb{D}$ of every non-identity element of Γ has at most two fixed points $\lambda_1, \lambda_2 \in \partial\mathbb{D}$. Therefore, by Lemma 4.4.1, the set of fixed points of the action on $M(QC)$ of every non-identity element of $\underline{\Gamma}$ is a subset of $\bigsqcup_{k=1,2} M_{\lambda_k}(QC)$. By Proposition 4.4.2 then, it has empty interior in $M(QC)$. That is, the action of $\underline{\Gamma}$ on $M(QC)$ is topologically free.

Amenability of the action follows from a rather trivial fact ([102], p. 1565) involving continuous maps relative to which actions are covariant, where in this case the covariance is supplied exactly by the commutative diagram in Lemma 4.4.1. For, if a sequence of functions $\xi_n : \Gamma \times \partial\mathbb{D} \to [0,1]$ satisfies $\xi_n(\gamma, \cdot) \in C$, $\sum_{\gamma\in\Gamma} \xi_n(\gamma, \cdot) \equiv 1$ on $\partial\mathbb{D}$, and that for every $\gamma \in \Gamma$ condition (4.4.2) holds for $G = \Gamma$ and $\Omega = \partial\mathbb{D}$, then define $\eta_n : \underline{\Gamma} \times M(QC) \to [0,1]$ by

$$\eta_n(\underline{\gamma}, y) = \xi_n(\gamma, r(y))$$

where $r : M(QC) \to \partial\mathbb{D}$ is the restriction map. Obviously, $\eta_n(\underline{\gamma}, \cdot) \in C(M(QC))$ and $\sum_{\underline{\gamma}\in\underline{\Gamma}} \eta_n(\underline{\gamma}, \cdot) \equiv 1$ on $M(QC)$. To verify condition (4.4.2) for η_n, simply note

$$\begin{aligned}
&\sup_{y\in M(QC)} \left(\sum_{\underline{\gamma}'\in\underline{\Gamma}} |\eta_n(\underline{\gamma}^{-1}\underline{\gamma}', y) - \eta_n(\underline{\gamma}', \underline{\gamma}(y))| \right) \\
&= \sup_{y\in M(QC)} \left(\sum_{\gamma'\in\Gamma} |\xi_n(\gamma^{-1}\gamma', r(y)) - \xi_n(\gamma', \gamma(r(y)))| \right) \\
&= \sup_{\lambda\in\partial\mathbb{D}} \left(\sum_{\gamma'\in\Gamma} |\xi_n(\gamma^{-1}\gamma', \lambda) - \xi_n(\gamma', \gamma(\lambda))| \right) \to 0 \text{ as } n \to 0
\end{aligned}$$

for every $\underline{\gamma} \in \underline{\Gamma}$, where Lemma 4.4.1 gave the first equality. By definition then, amenability of the action of $\underline{\Gamma}$ on $M(QC)$ follows from that of Γ on $\partial\mathbb{D}$. $\square$

Now consider the C*-dynamical system (QC, Γ) with the group isomorphism $\gamma \in \Gamma \mapsto$

$\Phi_{\gamma^{-1}} \in \mathrm{Aut}(QC)$, and the system $(C(M(QC)), \underline{\Gamma})$ with the isomorphism $\underline{\gamma} \in \underline{\Gamma} \mapsto \Psi_{\underline{\gamma}^{-1}} \in \mathrm{Aut}(C(M(QC)))$. They determine the crossed products [141] $QC \rtimes \Gamma$ and $C(M(QC)) \rtimes \underline{\Gamma}$, respectively. For every representation ρ of $C(M(QC))$ on a Hilbert space H and a unitary representation U of $\underline{\Gamma}$ on H such that

$$\rho\Psi_{\underline{\gamma}^{-1}}\beta f = U_{\underline{\gamma}}(\rho\beta f)U_{\underline{\gamma}}^*, \quad \forall f \in QC,\ \gamma \in \Gamma,$$

the representation $\rho\beta$ of QC and the unitary representation $\gamma \mapsto U_{\underline{\gamma}}$ of Γ on the same space satisfy, by (4.4.1),

$$\rho\beta\Phi_{\gamma^{-1}} f = U_{\underline{\gamma}}(\rho\beta f)U_{\underline{\gamma}}^*.$$

These relations for every $\gamma \in \Gamma$ can be summarized in the following commutative diagram

$$\begin{tikzcd}
C(M(QC)) \arrow[r, "\Psi_{\underline{\gamma}^{-1}}"] \arrow[d, "\rho"] & C(M(QC)) \arrow[d, "\rho"'] \\
\mathcal{L}(H) \arrow[r, "U_{\underline{\gamma}}(\cdot)U_{\underline{\gamma}}^*"] & \mathcal{L}(H) \\
QC \arrow[u, "\rho\beta"'] \arrow[r, "\Phi_{\gamma^{-1}}"] \arrow[uu, bend left, "\beta"] & QC \arrow[u, "\rho\beta"] \arrow[uu, bend right, "\beta"']
\end{tikzcd}$$

That is, every covariant representation for the system $(C(M(QC)), \underline{\Gamma})$ corresponds to one for (QC, Γ), and vice versa, via the isomorphism between the two systems. Thus, the crossed products are *-isomorphic, and the full and reduced versions [141] in either case coincide since the homeomorphic action of $\underline{\Gamma}$ on $M(QC)$ is amenable by Lemma 4.4.3. The extension result below is the main theorem of this section and the basis for the next. The proof follows that of [87, Theorem 3.1], using instead the properties of the action of $\underline{\Gamma}$ on $M(QC)$. Note that, obviously, we don't need Γ to be non-elementary as in [87], for $\mathcal{K} \subset \mathcal{T}(QC)$.

Theorem 4.4.4. *Let Γ be a discrete subgroup of* $\mathrm{Aut}(\mathbb{D})$*. Then one has the following short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{TC}(QC, \Gamma) \longrightarrow QC \rtimes \Gamma \longrightarrow 0.$$

Proof. In view of the preceding paragraph, it suffices to find a *-isomorphism

$$C(M(QC)) \rtimes \underline{\Gamma} \cong \mathcal{TC}(QC, \Gamma)/\mathcal{K}. \tag{4.4.3}$$

Consider the faithful representation of $C(M(QC))$ (cf. [18])

$$\rho : \beta f \in C(M(QC)) \mapsto f \in QC \mapsto [T_f]$$

and the unitary representation of $\underline{\Gamma}$ ([87], p. 3179)

$$U : \underline{\gamma} \in \underline{\Gamma} \mapsto \gamma \in \Gamma \mapsto [V_{\gamma^{-1}}],$$

both in the Calkin algebra. Here $V_\gamma = (C_\gamma C_\gamma^*)^{-1/2} C_\gamma$ is the unitary operator associated with the polar decomposition of the invertible C_γ, for any $\gamma \in \mathrm{Aut}(\mathbb{D})$. By the deductions leading up to (3.5) in [87] and noting $C_\gamma T_h = T_{h\circ\gamma} C_\gamma$ for $h \in H^\infty$, one has

$$[V_\gamma][T_f][V_\gamma]^* = [T_{f\circ\gamma}] \tag{4.4.4}$$

for all f in $H^\infty \bigcup C$ and thus in $QC \subset H^\infty + C$. Therefore, using (4.4.1) for the first and (4.4.4) for the third equality, one has for all $f \in QC$ and $\underline{\gamma} \in \underline{\Gamma}$ that

$$\rho \Psi_{\underline{\gamma}^{-1}}(\beta f) = \rho\beta\Phi_{\gamma^{-1}} f = [T_{f\circ\gamma^{-1}}] = [V_{\gamma^{-1}}][T_f][V_{\gamma^{-1}}]^* = U_{\underline{\gamma}}(\rho(\beta f))U_{\underline{\gamma}}^*.$$

That is, (ρ, U) is a covariant representation in the Calkin algebra for the C*-dynamical system consisting of $C(M(QC))$, $\underline{\Gamma}$, and the isomorphism $\underline{\gamma} \in \underline{\Gamma} \mapsto \Psi_{\underline{\gamma}^{-1}} \in \mathrm{Aut}(C(M(QC)))$. For the same reason as in [87], this representation generates the C*-subalgebra $\mathcal{TC}(QC, \Gamma)/\mathcal{K}$ in the Calkin algebra.

Therefore, by universality of $C(M(QC)) \rtimes \underline{\Gamma}$, there exists a surjective *-homomorphism $\tau : C(M(QC)) \rtimes \underline{\Gamma} \to \mathcal{TC}(QC, \Gamma)/\mathcal{K}$ extending the *-representation $\rho \rtimes U$ of the algebraic crossed product, and thus satisfying

$$\tau(\beta f) = \rho(\beta f) = [T_f], \ \forall f \in QC \tag{4.4.5}$$

under the natural embedding $C(M(QC)) \hookrightarrow C(M(QC)) \rtimes \underline{\Gamma}$, and

$$\tau(1_{\underline{\gamma}}) = U(\underline{\gamma}) = [V_{\gamma^{-1}}], \ \forall \gamma \in \Gamma \tag{4.4.6}$$

for the point-indicator functions $1_{\underline{\gamma}}$ on $\underline{\Gamma}$.

Consider the ideal $\mathcal{J} = \ker \tau$. Since $[T_f] = 0$ if and only if $f = 0$, one has $\mathcal{J} \bigcap C(M(QC)) = \{0\}$. Now because the action of $\underline{\Gamma}$ on $M(QC)$ is both topologically free and amenable, $\mathcal{J} = \{0\}$ as in [87]. That is, τ implements the desired *-isomorphism (4.4.3). □

Corollary 4.4.5. *Let* Γ *be a subgroup of* $\mathrm{Aut}(\mathbb{D})$. *Then* $\mathcal{TC}(QC,\Gamma)/\mathcal{K}$ *is the closed linear span of the set* $\{[T_f][C_\gamma] : f \in QC, \gamma \in \Gamma\}$.

Proof. By definition, $C(M(QC)) \rtimes \underline{\Gamma}$ is the closed span of the set $\{(\beta f) \star 1_{\underline{\gamma}} : f \in QC, \gamma \in \Gamma\}$. In view of the *-isomorphism τ satisfying (4.4.5) and (4.4.6), $\mathcal{TC}(QC,\Gamma)/\mathcal{K}$ is the closed span of $[T_f][V_\gamma]$, $f \in QC, \gamma \in \Gamma$. Since $[V_\gamma] = [T_{1/|g|}][C_\gamma]$ for some $g \in C^{-1}$ [87, Lemma 3.2], $[T_f][V_\gamma] = [T_{f/|g|}][C_\gamma]$ where $f/|g| \in QC$, which completes the proof. □

It is of interest to characterize compact sums of L^∞-weighted composition operators. See [19, 90, 88] for various special cases of this problem involving linear fractional non-automorphisms as the composing functions. On the other hand, the problem for sums of QC-weighted automorphic compositions has a trivial answer as a direct consequence of Theorem 4.4.4.

Corollary 4.4.6. *Let* $\gamma_1, ..., \gamma_n \in \mathrm{Aut}(\mathbb{D})$ *be distinct and let* $f_1, ..., f_n \in QC$. *Then* $\sum_{k=1}^n T_{f_k} C_{\gamma_k}$ *is compact only if* $f_1 = ... = f_n = 0$.

Proof. Suppose the sum is a compact operator. Since each $[C_{\gamma_k}] = [T_{g_k}][V_{\gamma_k}]$ for some $g_k \in C^{-1}$ as above, one has $\sum_{k=1}^n [T_{f_k g_k}][V_{\gamma_k}] = 0$. Take Γ to be the subgroup of $\mathrm{Aut}(\mathbb{D})$ generated by

$\gamma_1, ..., \gamma_n$. In view of the *-isomorphism τ again, one has in the algebraic crossed product

$$\sum_{k=1}^{n} \beta(f_k g_k) 1_{\underline{\gamma_k}^{-1}} = \sum_{k=1}^{n} \beta(f_k g_k) \star 1_{\underline{\gamma_k}^{-1}} = \tau^{-1}\left(\sum_{k=1}^{n} [T_{f_k g_k}][V_{\gamma_k}]\right) = 0.$$

It follows that each $f_k g_k = 0$ due to $\{\underline{\gamma_k}^{-1}\}_k$ being distinct. So, $f_k = 0$ because $g_k \in C^{-1}$. $\square$

4.5 Fredholm operators generated by quasi-continuous Toeplitz operators and a rational rotation

Throughout this section, fix a rational number p/q in lowest terms with $p \neq 0$, $q \geq 2$, and define the rotation $\gamma \in \mathrm{Aut}(\mathbb{D})$, $\gamma(z) = \lambda z$ where $\lambda = e^{i2\pi p/q}$. Relative to the orthonormal basis $\{1, z, z^2, ...\}$ of H^2, the unitary diagonal composition operator

$$C_\gamma = V_\gamma = \mathrm{diag}(1, \lambda, ..., \lambda^{q-1}, 1, \lambda, ..., \lambda^{q-1}, 1, ...),$$

from which it is clear the spectra $\sigma(C_\gamma) = \sigma_e(C_\gamma) = \{1, \lambda, ..., \lambda^{q-1}\} = \{1, e^{i2\pi/q}, ..., e^{i2\pi(q-1)/q}\}$.

This section characterizes the Fredholm operators in $\mathcal{TC}(QC, \gamma) = \mathcal{TC}(QC, \Gamma_\gamma)$ where $\Gamma_\gamma = \{\gamma^k : k \in \mathbb{Z}\} \cong \mathbb{Z}/q\mathbb{Z} =: \{0, 1, ..., q-1\}$ is the cyclic group of order q generated by γ. Note that γ^k is the k-fold operation on $\mathbb{D}$ or $\partial\mathbb{D}$ depending on the context, and that addition in $\{0, 1, ..., q-1\}$ is modulo q. For the C*-dynamical system (QC, Γ_γ) with the action $k \in \{0, 1, ..., q-1\} \mapsto \Phi_{\gamma^{-k}} \in \mathrm{Aut}(QC)$ by composition, consider the faithful representation $f \in QC \mapsto M_f \in \mathcal{L}(L^2)$ of QC as multiplication operators on L^2 over $\partial\mathbb{D}$. This system has its *regular* covariant representation on $\bigoplus_{j=0}^{q-1} L^2$ as

$$\begin{cases} f \in QC \mapsto \bigoplus_{j=0}^{q-1} M_{f\circ\gamma^j} \\ k \in \{0, 1, ..., q-1\} \mapsto S_k \end{cases}$$

where S_k is the k-step shift operator, $S_k(\bigoplus_{j=0}^{q-1} g_j) = \bigoplus_{j=0}^{q-1} g_{j-k}$, $g_j \in L^2$. The corresponding

*-representation of the algebraic crossed product takes a generic element $(f_k : 0 \le k \le q-1)$ to

$$\sum_{k=0}^{q-1}\left(\bigoplus_{j=0}^{q-1} M_{f_k\circ\gamma^j}\right) S_k = \begin{bmatrix} M_{f_0} & M_{f_{q-1}} & \dots & M_{f_1} \\ M_{f_1\circ\gamma} & M_{f_0\circ\gamma} & \dots & M_{f_2\circ\gamma} \\ \dots & \dots & \dots & \dots \\ M_{f_{q-1}\circ\gamma^{q-1}} & M_{f_{q-2}\circ\gamma^{q-1}} & \dots & M_{f_0\circ\gamma^{q-1}} \end{bmatrix}. \tag{4.5.1}$$

Also, the faithful representation $f \in QC \mapsto M_f \in \mathcal{L}(L^2)$ induces a *-isomorphism of the $q \times q$ matrix C*-algebra $M_q \bigotimes QC$ over QC into $\mathcal{L}(\bigoplus_{j=0}^{q-1} L^2)$ as

$$[f_{j,k}]_{0\le j,k\le q-1} \mapsto [M_{f_{j,k}}]_{0\le j,k\le q-1}, \qquad f_{j,k} \in QC. \tag{4.5.2}$$

Now, in view of (4.5.1) and (4.5.2), the results surrounding Theorem 4.4.4 specialize to Γ_γ and assert that an isometric *-isomorphism on $\mathcal{TC}(QC,\gamma)/\mathcal{K}$ is the norm-closure of the map

$$\begin{aligned}
\sum_{k=0}^{q-1}[T_{f_k}][C_{\gamma^k}] = \sum_{k=0}^{q-1}[T_{f_{-k}}][V_{\gamma^{-k}}] \in \mathcal{TC}(QC,\gamma)/\mathcal{K} &\mapsto (f_{-k} : 0 \le k \le q-1) \\
&\mapsto \begin{bmatrix} M_{f_0} & M_{f_1} & \dots & M_{f_{q-1}} \\ M_{f_{q-1}\circ\gamma} & M_{f_0\circ\gamma} & \dots & M_{f_{q-2}\circ\gamma} \\ \dots & \dots & \dots & \dots \\ M_{f_1\circ\gamma^{q-1}} & M_{f_2\circ\gamma^{q-1}} & \dots & M_{f_0\circ\gamma^{q-1}} \end{bmatrix} \in \mathcal{L}\left(\bigoplus_{j=0}^{q-1} L^2\right) \\
&\mapsto \begin{bmatrix} f_0 & f_1 & \dots & f_{q-1} \\ f_{q-1}\circ\gamma & f_0\circ\gamma & \dots & f_{q-2}\circ\gamma \\ \dots & \dots & \dots & \dots \\ f_1\circ\gamma^{q-1} & f_2\circ\gamma^{q-1} & \dots & f_0\circ\gamma^{q-1} \end{bmatrix} \\
&=: [f_0, f_1, \dots, f_{q-1}] \in M_q \bigotimes QC.
\end{aligned} \tag{4.5.3}$$

Write $A_\gamma := \{[f_0, f_1, ..., f_{q-1}] : f_0, f_1, ..., f_{q-1} \in QC\}$. It follows from (4.5.3) that A_γ is a *-subalgebra of $M_q \bigotimes QC$, and it is closed by a per-entry consideration. Therefore, due to density (Corollary 4.4.5), $\mathcal{TC}(QC,\gamma)/\mathcal{K}$ consists exclusively of these sums in (4.5.3) and the map actually defines a *-isomorphism from $\mathcal{TC}(QC,\gamma)/\mathcal{K}$ onto A_γ. In particular, one can identify

$$\mathcal{TC}(QC,\gamma) = \left\{ \sum_{k=0}^{q-1} T_{f_k} C_\gamma^k + K : f_k \in QC, K \in \mathcal{K} \right\}. \tag{4.5.4}$$

Let $\det : M_q \bigotimes QC \to QC$ be the surjective *-homomorphic determinant map. In the following theorem we use the notation $[f_0, f_1, ..., f_{q-1}]$ at the end of (4.5.3) and similarly use

$$[\hat{f}_0, \hat{f}_1, ..., \hat{f}_{q-1}] := \begin{bmatrix} \hat{f}_0 & \hat{f}_1 & ... & \hat{f}_{q-1} \\ \hat{f}_{q-1} \circ \gamma & \hat{f}_0 \circ \gamma & ... & \hat{f}_{q-2} \circ \gamma \\ ... & ... & ... & ... \\ \hat{f}_1 \circ \gamma^{q-1} & \hat{f}_2 \circ \gamma^{q-1} & ... & \hat{f}_0 \circ \gamma^{q-1} \end{bmatrix}$$

for the harmonic extensions in $\mathbb{D}$. We arrive at the main result of this section.

Theorem 4.5.1. *Let $T = \sum_{k=0}^{q-1} T_{f_k} C_\gamma^k + K$, where $f_k \in QC$ and $K \in \mathcal{K}$, be an arbitrary element of the operator algebra $\mathcal{TC}(QC, \gamma)$. Then T is Fredholm if and only if $g := \det[f_0, f_1, ..., f_{q-1}] \in QC^{-1}$, or equivalently, $\psi := \det[\hat{f}_0, \hat{f}_1, ..., \hat{f}_{q-1}]$ is bounded away from zero over the annulus $\{1 - \delta < |z| < 1\}$ for some $0 < \delta < 1$.*

Proof. One has $(M_q \bigotimes QC)^{-1} = \det^{-1}(QC^{-1})$ by matrix multiplication rules, and that $A_\gamma^{-1} = A_\gamma \bigcap (M_q \bigotimes QC)^{-1}$ by spectral permanence for the C*-subalgebra A_γ. These observations and the *-isomorphism from $\mathcal{TC}(QC, \gamma)/\mathcal{K}$ onto A_γ give the first equivalence.

For the second equivalence, let $\hat{g}$ be the continuous extension (Section 1.9) of g on $M(H^\infty)$. Then $g \in QC^{-1}$ if and only if $\hat{g}$ is nonvanishing on $M(H^\infty + C)$, while the latter is equivalent to

$$\inf\{|\hat{g}(z)| : 1 - \epsilon < |z| < 1\} > 0, \quad \exists \epsilon \in (0, 1) \tag{4.5.5}$$

by the dense embedding $\mathbb{D} \hookrightarrow M(H^\infty)$. Since the harmonic extensions in $\mathbb{D}$ of QC functions are asymptotically multiplicative [49] and the harmonic extension of $f_k \circ \gamma^j$ in $\mathbb{D}$ equals $\hat{f}_k \circ \gamma^j$, $\hat{g}$ and ψ are asymptotically equal in $\mathbb{D}$. That is,

$$\lim_{|z| \to 1} |\hat{g}(z) - \psi(z)| = 0$$

from which the conclusion follows in view of (4.5.5). □

The Fredholm index for operators in $\mathcal{TC}(C, \gamma)$ is computed by Jury [86, Theorem 3.1] via K-theory and homotopy. Unlike C, the C*-algebra QC is not singly generated, not even separable, and its K-theory seems intractable. The rest of this section gives a certain indication that Jury's index formula should extend from C to QC symbols. That is, we tend to think the following is true although we have been unable to prove it. *Continuing from Theorem 4.5.1, is it true that*

$$\mathrm{ind}T = q^{-1}(\mathrm{ind}T_g) = -q^{-1} \cdot \text{winding number of } \psi|(r\partial\mathbb{D}) \tag{4.5.6}$$

for $r \in (0,1)$ *large enough?*

We are led to define $QC_\gamma := \{f \in QC : f \circ \gamma = f\}$, the C*-subalgebra of *periodic* QC functions on q cycles forming a partition of $\partial\mathbb{D}$. Recall that T_f for $f \in QC$ is Fredholm if and only if $f \in QC^{-1}$.

Lemma 4.5.2. *There exists a *-isomorphism* Λ *from* QC_γ *onto* QC *such that for any* $f \in QC_\gamma^{-1}$

$$\mathrm{ind}T_f = q(\mathrm{ind}T_{\Lambda f}). \tag{4.5.7}$$

Proof. Set $\lambda_1 = e^{i2\pi/q}$ and define the rotation $\gamma_1 \in \mathrm{Aut}(\mathbb{D})$, $\gamma_1(z) = \lambda_1 z$. For every $f \in QC_\gamma$, due to $QC = VMO \bigcap L^\infty$ and periodicity $f \circ \gamma_1 = f \circ \gamma^l = f$ where $pl = 1 \mod q$, the restriction $f|(1, \lambda_1)$ is of VMO on the subarc $(1, \lambda_1)$ of $\partial\mathbb{D}$ and

$$\left|\frac{1}{\delta}\int_1^{1+\delta} f d\theta - \frac{1}{\delta}\int_{\lambda_1-\delta}^{\lambda_1} f d\theta\right| = \left|\frac{1}{\delta}\int_{\lambda_1}^{\lambda_1+\delta} f d\theta - \frac{1}{\delta}\int_{\lambda_1-\delta}^{\lambda_1} f d\theta\right| \to 0, \quad \delta \downarrow 0. \tag{4.5.8}$$

Define $g(e^{it}) = f(e^{it/q})$, $0 < t < 2\pi$, to be the dilation of $f|(1, \lambda_1)$ to $\partial\mathbb{D} \setminus \{1\}$. Then g is of VMO on $\partial\mathbb{D} \setminus \{1\}$ while

$$\left|\frac{1}{\delta}\int_1^{1+\delta} g d\theta - \frac{1}{\delta}\int_{1-\delta}^{1} g d\theta\right| = \left|\frac{q}{\delta}\int_1^{1+\delta/q} f d\theta - \frac{q}{\delta}\int_{\lambda_1-\delta/q}^{\lambda_1} f d\theta\right| \to 0, \quad \delta \downarrow 0$$

by (4.5.8). So, g is of VMO on $\partial\mathbb{D}$ and $g \in QC$. Therefore, we have constructed a map

$\Lambda : f \in QC_\gamma \mapsto g \in QC$ which is clearly a *-isomorphism into QC. Conversely, take any $g \in QC$, compress it to the subarc $(1, \lambda_1)$, and then replicate the compression periodically onto a perforated circle. By a similar argument, the resulting periodic function f is of VMO on each of the q subarcs $(\lambda_1^k, \lambda_1^{k+1})$ with zero integral gaps at the end points λ_1^k. So, $f \in VMO$ and $f \in QC_\gamma$. Reversing this procedure, one sees $\Lambda f = g$. Thus Λ is also surjective.

To prove (4.5.7), $f \in QC_\gamma^{-1}$ and $g := \Lambda f \in QC^{-1}$ imply as before

$$\epsilon := \inf\{|\hat{f}(z)|, |\hat{g}(z)| : 1-\delta < |z| < 1\} > 0 \tag{4.5.9}$$

for some $0 < \delta < 1$. By the index formula in [49, Theorem 7.36], $\mathrm{ind}T_f$ equals minus the winding number $w(f, r)$ (about the origin) of the curve $t \in [0,1] \mapsto \hat{f}(re^{i2\pi t})$ for any $1-\delta < r < 1$, and the same applies to $\mathrm{ind}T_g$. For $k = 0, ..., q-1$ and $1-\delta < r < 1$, let $w(f, r, k)$ be the winding number of $t \in [k/q, (k+1)/q] \mapsto \hat{f}(re^{i2\pi t})$. Since

$$w(f, r) = \sum_{k=0}^{q-1} w(f, r, k),$$

it suffices to show that for some $1-\delta < r', r'' < 1$ and every $k = 0, ..., q-1$

$$w(f, r', k) = w(g, r''). \tag{4.5.10}$$

To this end, we apply [120, Lemma 5] on the QC functions f, g to assert for some $1-\delta < r'' < r' < 1$ satisfying $1-r'' = q(1-r')$, $\forall k = 0, ..., q-1$ and $\forall t \in [k/q, (k+1)/q]$, that

$$\left| \hat{f}(r'e^{i2\pi t}) - \frac{1}{1-r'} \int_{e^{i2\pi t}-(1-r')/2}^{e^{i2\pi t}+(1-r')/2} f d\theta \right| < \epsilon/4, \tag{4.5.11}$$

$$\left| \hat{g}(r''e^{i2\pi(qt-k)}) - \frac{1}{1-r''} \int_{e^{i2\pi(qt-k)}-(1-r'')/2}^{e^{i2\pi(qt-k)}+(1-r'')/2} g d\theta \right| < \epsilon/4. \tag{4.5.12}$$

Note that the first integral for instance is taken over the subarc of $\partial\mathbb{D}$ centered at $e^{i2\pi t}$ with θ-measure $1-r'$ while [120, Lemma 5] uses the arc-length measure (not normalized). By the construction of $g = \Lambda f$, $1 - r'' = q(1 - r')$, and periodicity of f,

$$\frac{1}{1-r''} \int_{e^{i2\pi(qt-k)}-(1-r'')/2}^{e^{i2\pi(qt-k)}+(1-r'')/2} g d\theta = \frac{q}{1-r''} \int_{e^{i2\pi(t-k/q)}-(1-r'')/(2q)}^{e^{i2\pi(t-k/q)}+(1-r'')/(2q)} f d\theta$$
$$= \frac{1}{1-r'} \int_{e^{i2\pi t}-(1-r')/2}^{e^{i2\pi t}+(1-r')/2} f d\theta \tag{4.5.13}$$

if $(1-r'')/2 < qt - k < 1 - (1-r'')/2$. If not, then either $0 \le qt - k \le (1-r'')/2$ or $1 - (1-r'')/2 \le qt - k \le 1$. In the former case, write

$$\delta' := (1 - r'')/2 - (qt - k)$$

and (4.5.13) still goes through after splitting the first integral in two:

$$\frac{1}{1-r''} \int_{e^{i2\pi(qt-k)}-(1-r'')/2}^{e^{i2\pi(qt-k)}+(1-r'')/2} g d\theta = \frac{q}{1-r''} \left(\int_{1}^{e^{i2\pi(t-k/q)}+(1-r'')/(2q)} f d\theta + \int_{\lambda_1-\delta'/q}^{\lambda_1} f d\theta \right)$$
$$= \frac{1}{1-r'} \left(\int_{e^{i2\pi k/q}}^{e^{i2\pi t}+(1-r')/2} f d\theta + \int_{e^{i2\pi k/q}-\delta'/q}^{e^{i2\pi k/q}} f d\theta \right)$$
$$= \frac{1}{1-r'} \int_{e^{i2\pi t}-(1-r')/2}^{e^{i2\pi t}+(1-r')/2} f d\theta$$

where the last equality is due to $t - k/q + \delta'/q = (1-r'')/(2q) = (1-r')/2$. The latter case is similar. So it follows from (4.5.11), (4.5.12), (4.5.13), and then (4.5.9) that

$$|\hat{f}(r' e^{i2\pi t}) - \hat{g}(r'' e^{i2\pi(qt-k)})| < \epsilon/2,$$
$$|(1-s)\hat{f}(r' e^{i2\pi t}) + s\hat{g}(r'' e^{i2\pi(qt-k)})| \ge |\hat{f}(r' e^{i2\pi t})| - s|\hat{f}(r' e^{i2\pi t}) - \hat{g}(r'' e^{i2\pi(qt-k)})| > \epsilon/2$$

for every $(s,t) \in [0,1] \times [k/q, (k+1)/q]$. Therefore, $t \in [k/q, (k+1)/q] \mapsto \hat{f}(r' e^{i2\pi t})$ is homotopic in $\mathbb{C} \setminus 0$ to $t \in [k/q, (k+1)/q] \mapsto \hat{g}(r'' e^{i2\pi(qt-k)})$, and the winding number $w(f, r', k)$ of the former curve equals that of the latter which in turn equals $w(g, r'')$. That is, we have shown (4.5.10) for every $k = 0, ..., q-1$, as required to conclude the proof of (4.5.7). $\square$

A key observation is that $\det(A_\gamma) \subset QC_\gamma$.

Lemma 4.5.3. $(\det[f_0, f_1, ..., f_{q-1}]) \circ \gamma = \det[f_0, f_1, ..., f_{q-1}]$.

Proof. Let $\mathcal{S}_q$ be the permutation group on $\{0, 1, ..., q-1\}$ and define $s \in \mathcal{S}_q$ by $s_0 = 1, ..., s_{q-2} = q-1, s_{q-1} = 0$. Write

$$[f_0, f_1, ..., f_{q-1}] = [a(i,j)]_{0 \le i,j \le q-1}, \quad [f_0 \circ \gamma, f_1 \circ \gamma, ..., f_{q-1} \circ \gamma] = [b(i,j)]_{0 \le i,j \le q-1}$$

and use the pattern of these matrices to verify that

$$b(i,j) = a(i,j) \circ \gamma = a(s_i, s_j). \tag{4.5.14}$$

Since the sign group homomorphism $\mathrm{sgn} : \mathcal{S}_q \to \{\pm 1\}$ is invariant under conjugation by s, one deduces from (4.5.14) that

$$\begin{aligned}
\det[b(i,j)]_{0 \le i,j \le q-1} &= \sum_{\sigma \in \mathcal{S}_q} \mathrm{sgn}(\sigma) \prod_{i=0}^{q-1} b(i, \sigma_i) = \sum_{\sigma \in \mathcal{S}_q} \mathrm{sgn}(\sigma) \prod_{i=0}^{q-1} a(s_i, s_{\sigma_i}) \\
&= \sum_{\sigma \in \mathcal{S}_q} \mathrm{sgn}(\sigma) \prod_{k=0}^{q-1} a(k, s\sigma s^{-1}(k)), \quad (k = s_i) \\
&= \sum_{\omega \in \mathcal{S}_q} \mathrm{sgn}(\omega) \prod_{k=0}^{q-1} a(k, \omega_k), \quad (\omega = s\sigma s^{-1}) \\
&= \det[a(i,j)]_{0 \le i,j \le q-1}
\end{aligned}$$

as required, since $(\det[f_0, f_1, ..., f_{q-1}]) \circ \gamma = \det[f_0 \circ \gamma, f_1 \circ \gamma, ..., f_{q-1} \circ \gamma]$. $\square$

Therefore, if $g := \det[f_0, f_1, ..., f_{q-1}] \in QC^{-1}$, then $g \in QC_\gamma^{-1}$ with $q^{-1}(\mathrm{ind} T_g) = \mathrm{ind} T_{\Lambda g}$, so that (4.5.6) can be reduced to the question $\mathrm{ind} T = \mathrm{ind} T_{\Lambda g}$? Equivalently, can one find a continuous path of Fredholm operators joining T and $T_{\Lambda g}$?

4.6 Toeplitz-composition algebras generated by PQC symbols and a linear fractional non-automorphism fixing a boundary point

Fix a linear fractional non-automorphism $\phi : \mathbb{D} \to \mathbb{D}$ satisfying $\phi(\zeta) = \zeta$ for a unique $\zeta \in \partial\mathbb{D}$. This section uses notations and basic facts from Section 1.12 to derive some preliminary results common to the two subsections.

First we need to obtain an explicit description of the maximal ideal space $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ of the commutative Toeplitz C*-subalgebra $\mathcal{T}(PQC(\zeta))/\mathcal{K}$ of the Calkin algebra. This is done by describing its fiber structure over the circle, on the basis of Sarason's work in [120] on that of $M(\mathcal{T}(PQC)/\mathcal{K})$. The key to the leverage on Sarason's result, Theorem 4.3.1, lies in a lemma under the following setup: Let

$$r : M(\mathcal{T}(PQC)/\mathcal{K}) \to M(\mathcal{T}(PQC(\zeta))/\mathcal{K}), \quad r_1 : M(\mathcal{T}(PQC(\zeta))/\mathcal{K}) \to M(\mathcal{T}(C)/\mathcal{K})$$

be the surjective restriction maps. Then, $r_1 \circ r$ is the restriction map from $M(\mathcal{T}(PQC)/\mathcal{K})$ onto $M(\mathcal{T}(C)/\mathcal{K})$. Here $M(\mathcal{T}(C)/\mathcal{K}) \cong M(C)$ under the isomorphism $f \in C \mapsto [T_f] \in \mathcal{T}(C)/\mathcal{K}$, and

$$r(M_\lambda(\mathcal{T}(PQC)/\mathcal{K})) = r_1^{-1}(\lambda) = M_\lambda(\mathcal{T}(PQC(\zeta))/\mathcal{K}), \quad \forall \lambda \in \partial\mathbb{D} \cong M(C). \tag{4.6.1}$$

Lemma 4.6.1. *The map r on the fiber $M_\zeta(\mathcal{T}(PQC)/\mathcal{K})$ is a constant, denoted by $\langle\zeta\rangle$ and determined by*

$$\langle\zeta\rangle([T_f]) = f(\zeta), \qquad f \in PQC(\zeta). \tag{4.6.2}$$

For every $\lambda \neq \zeta \in \partial\mathbb{D}$, the map r on the fiber $M_\lambda(\mathcal{T}(PQC)/\mathcal{K})$ is a homeomorphism onto $M_\lambda(\mathcal{T}(PQC(\zeta))/\mathcal{K})$.

Proof. Let $\xi \in M_\zeta(\mathcal{T}(PQC)/\mathcal{K})$, so that $\xi([T_z]) = \zeta$. It suffices to prove the first part by showing $\xi([T_f]) = f(\zeta)$ for every $f \in PQC(\zeta)$. Consider the Gelfand transform of $\phi(z) := z - \zeta$ on $M(PQC)$ with zero set precisely $M_\zeta(PQC)$. Then $f \in PQC(\zeta)$ implies that the zero set of $f - f(\zeta)$ on $M(PQC)$ contains $M_\zeta(PQC)$ which is also the zero set of the closed principal ideal

$\overline{\phi PQC}$ in PQC. Thus $f - f(\zeta) \in \overline{\phi PQC}$ by radicality of the ideals in PQC. Hence, for every $\epsilon > 0$, there exists $g \in PQC$ with $\|f - f(\zeta) - \phi g\|_\infty \le \epsilon$. It follows that

$$\begin{aligned}|\xi([T_f]) - f(\zeta)| = |\xi([T_{f-f(\zeta)}])| &\le |\xi([T_\phi])\xi([T_g])| + \epsilon \\ &= |(\xi([T_z]) - \zeta)\xi([T_g])| + \epsilon = \epsilon.\end{aligned}$$

So, $\xi([T_f]) = f(\zeta)$ as required.

To prove the second part, let $g \in C$ with $g(\lambda) = 1$, $g(\zeta) = 0$. For $\xi_1 \neq \xi_2 \in M_\lambda(\mathcal{T}(PQC)/\mathcal{K})$, there must exist $f \in PQC$ such that $\xi_1([T_f]) \neq \xi_2([T_f])$. Consider the product function gf. Evidently, $gf \in PQC(\zeta)$ with $(gf)(\zeta) = 0$. Since for $k = 1, 2$

$$r(\xi_k)([T_{gf}]) = \xi_k([T_{gf}]) = \xi_k([T_g])\xi_k([T_f]) = g(\lambda)\xi_k([T_f]) = \xi_k([T_f]),$$

one has $r(\xi_1) \neq r(\xi_2)$. That is, the continuous map r on $M_\lambda(\mathcal{T}(PQC)/\mathcal{K})$ is injective and hence a homeomorphism onto $M_\lambda(\mathcal{T}(PQC(\zeta))/\mathcal{K})$. □

Note in passing that the Toeplitz symbol map ([49]) $\tau : \mathcal{T}(L^\infty) \to L^\infty$ takes $\mathcal{T}(PQC(\zeta))$ to $PQC(\zeta)$ and satisfies $\tau(T)(\zeta) = \langle\zeta\rangle([T])$, $\forall T \in \mathcal{T}(PQC(\zeta))$. The fibers of $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ over $\partial\mathbb{D}$ are described by the following theorem, whose proof is immediate from Theorem 4.3.1, (4.6.1) and the preceding lemma. See the beginning of Section 4.2 for notations in Sarason's description of $M(QC)$ and $M(PQC)$.

Theorem 4.6.2. *The fiber $M_\zeta(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ consists only of the functional $\langle\zeta\rangle$. For every $\lambda \neq \zeta \in \partial\mathbb{D}$ the fiber $M_\lambda(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ is the disjoint union of a family $\{F_y : y \in M_\lambda(QC)\}$ of closed subsets, where F_y for $y \in M_\lambda^\pm(QC) \setminus M_\lambda^0(QC)$ consists of a single functional assuming the value $f(y\pm)$ at $[T_f]$ for every $f \in PQC(\zeta)$, and where F_y for $y \in M_\lambda^0(QC)$ is homeomorphic to $[0,1]$ via the *-isomorphism between $C(F_y)$ and $C[0,1]$ determined by*

$$[T_f]|F_y \mapsto (t \in [0,1] \mapsto tf(y+) + (1-t)f(y-)), \ \ f \in PQC(\zeta).$$

Next recall the key relation between Toeplitz operators and the composition operator C_ϕ

$$[T_f][C_\phi] = [C_\phi][T_f] = f(\zeta)[C_\phi], \qquad f \in PQC(\zeta). \tag{4.6.3}$$

Lemma 4.6.3. *For every $T \in \mathcal{T}(PQC(\zeta))$,*

$$[T][C_\phi] = [C_\phi][T] = \langle\zeta\rangle([T])[C_\phi].$$

Proof. If T is a finite product of Toeplitz operators with $PQC(\zeta)$ symbols, then the equalities follow from (4.6.3) and (4.6.2). One completes the proof by passing to sums of products and then to the closure. $\square$

4.6.1 The parabolic case

Assume in this subsection ϕ is parabolic, that is, $\phi'(\zeta) = 1$ at the fixed boundary point. Then one has a commutative C*-subalgebra $\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K}$ of the Calkin algebra. The goal is to obtain an explicit description of the maximal ideal space $M(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$, so that the essential spectrum and norm of certain operators of interest can be computed, and that the Fredholm index can be determined. These results extend Corollary 6.5 in [109] which addresses the smaller and simpler subalgebra $\mathcal{TC}(C, \phi)/\mathcal{K}$.

First note the essential spectrum $\sigma_e(C_\phi)$ has an explicit form. For, conjugating with a rotation, we may take

$$\phi(z) = \frac{(2-\alpha)z + \alpha}{-\alpha z + (2+\alpha)}, \qquad \phi(1) = 1$$

where α, $\Re\alpha > 0$, is the translation number of ϕ. By the half-plane version ($\theta = \pi/2$) of Corollary 7.42 in [42], $\sigma(C_\phi)$ is a logarithmic spiral in the disc from 1 to 0

$$\sigma(C_\phi) = \{e^{-\alpha t} : t \geq 0\} \bigcup 0 =: e^{-\alpha[0,\infty]}.$$

Since such a spiral has empty interior in $\mathbb{C}$ and no isolated points, Theorem 37.8 in [35] gives

$$\sigma_e(C_\phi) = \sigma(C_\phi) = e^{-\alpha[0,\infty]}. \tag{4.6.4}$$

Note that $\sigma_e(C_\phi)$ is homeomorphic to $[0, 1]$ via the modulus map $z \mapsto |z|$.

In view of Theorem 4.6.2, it remains to identify the fibers of $M(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$ over $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$. The fibers over $\xi \neq \langle\zeta\rangle \in M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ are all singletons.

Theorem 4.6.4. *For $\xi \neq \langle\zeta\rangle \in M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$, the fiber $M_\xi(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$ consists of a single functional vanishing at $[C_\phi]$.*

Proof. Since $\xi \neq \langle\zeta\rangle$, $\lambda := \xi|(\mathcal{T}(C)/\mathcal{K}) \neq \zeta \in \partial\mathbb{D}$ and $\xi([T_g]) = g(\lambda)$ for every $g \in C$. Choose $g \in C$ with $g(\zeta) = 1$ while $g(\lambda) = 0$. Applying any $x \in M_\xi(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$ on the equality (4.6.3) with $f = g$ gives $x([C_\phi]) = 0$. Since the functionals in the nonempty fiber are determined by their value at the generator $[C_\phi]$, the statement is proved. □

The fiber $M_{\langle\zeta\rangle}(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$ will be identified after several lemmas. Consider the isomorphism $f \in C(e^{-\alpha[0,\infty]}) \mapsto f([C_\phi]) \in \mathcal{C}^*([C_\phi])$ via the continuous functional calculus for the essentially normal operator C_ϕ. Write

$$C_0(e^{-\alpha[0,\infty]}) := \{f \in C(e^{-\alpha[0,\infty]}) : f(0) = 0\}, \qquad C_0([C_\phi]) := \{f([C_\phi]) : f \in C_0(e^{-\alpha[0,\infty]})\}.$$

Lemma 4.6.5. *For every $T \in \mathcal{T}(PQC(\zeta))$ and $a \in C_0([C_\phi])$,*

$$[T]a = a[T] = \langle\zeta\rangle([T])a.$$

Proof. Since the maximal ideal $C_0(e^{-\alpha[0,\infty]})$ is singly generated by the identity function, $f \in C_0(e^{-\alpha[0,\infty]})$ can be uniformly approximated by zg, $g \in C(e^{-\alpha[0,\infty]})$. That is, $a = f([C_\phi])$ can be approximated in the Calkin algebra by $[C_\phi]g([C_\phi]) = g([C_\phi])[C_\phi]$. By Lemma 4.6.3,

$$[T][C_\phi]g([C_\phi]) = \langle\zeta\rangle([T])[C_\phi]g([C_\phi]), \qquad g([C_\phi])[C_\phi][T] = \langle\zeta\rangle([T])g([C_\phi])[C_\phi].$$

The proof is complete after passing to the limit. $\square$

Lemma 4.6.6. *There exists $x \in M_{\langle\zeta\rangle}(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$ such that $x([C_\phi]) = 0$.*

Proof. Choose a sequence $\lambda_n \to \zeta$ on the circle, $\lambda_n \neq \zeta$. Next choose $\xi_n \in M_{\lambda_n}(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ for every n. According to the previous theorem, let x_n be the functional in $M_{\xi_n}(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$ with $x_n([C_\phi]) = 0$. The sequence x_n in $M(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$ has a cluster point x, and

$$x_{n_\omega} \to x$$

for a subnet indexed by ω. Evidently, $x([C_\phi]) = 0$. Applying the convergence on $[T_f]$, $f \in C$, yields $f(\lambda_{n_\omega}) \to x([T_f])$ while $f(\lambda_{n_\omega}) \to f(\zeta)$. Therefore, $x([T_f]) = f(\zeta)$ for every $f \in C$, so

$$x|(\mathcal{T}(PQC(\zeta))/\mathcal{K}) \in M_\zeta(\mathcal{T}(PQC(\zeta))/\mathcal{K}) = \langle\zeta\rangle$$

by Lemma 4.6.1. Such x fulfills the requirement. $\square$

The next lemma gives an essential norm relation and expresses $\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K}$ as the direct sum of its Toeplitz C*-subalgebra and a closed ideal of its composition C*-subalgebra.

Lemma 4.6.7. *For every $T \in \mathcal{T}(PQC(\zeta))$ and $a \in C_0([C_\phi])$,*

$$\|[T] + a\| \geq \|[T]\|. \tag{4.6.5}$$

Consequently, one has the Banach space direct-sum decomposition

$$\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K} = \mathcal{T}(PQC(\zeta))/\mathcal{K} \bigoplus C_0([C_\phi]).$$

Proof. Since the Gelfand transform of the commutative C*-algebra $\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K}$ is isometric, one has the representations

$$\|[T]+a\| = \max\{|x([T])+x(a)| : x \in M(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})\}, \tag{4.6.6}$$

$$\|[T]\| = \max\{|x([T])| : x \in M(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})\}. \tag{4.6.7}$$

We proceed by partitioning $M(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$ in fibers over $M(\mathcal{T}(PQC(\zeta)/\mathcal{K})$.

As before, the fiber $M_\xi(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$ for any $\xi \neq \langle\zeta\rangle \in M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ consists of a single functional x vanishing at $[C_\phi]$. Thus, $[C_\phi]$ lies in the maximal ideal $\ker(x)\bigcap \mathcal{C}^*([C_\phi])$ of $\mathcal{C}^*([C_\phi])$. Under the continuous functional calculus for $[C_\phi]$, $C_0([C_\phi])$ is the closed principal (and maximal) ideal of $\mathcal{C}^*([C_\phi])$ generated by $[C_\phi]$ because the function ideal is so. Thus, $a \in C_0([C_\phi]) \subset \ker(x)$ and $|x([T])+x(a)| = |x([T])|$ for such x. For every $x \in M_{\langle\zeta\rangle}(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$, $x([T]) \equiv \langle\zeta\rangle([T])$, while $x([C_\phi]) = 0$ hence $x(a) = 0$ for at least one x in this fiber by the previous lemma. Combining these two cases that exhaust $M(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$, the norm inequality follows from (4.6.6) and (4.6.7).

It immediately follows from (4.6.5) that $\mathcal{T}(PQC(\zeta))/\mathcal{K}\bigcap C_0([C_\phi]) = \{0\}$ and that the direct-sum is a norm-closed, self-adjoint linear subspace of $\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K}$ containing all of its generators. The sum is also closed under multiplication, by Lemma 4.6.5, and is therefore a C*-subalgebra of the Calkin algebra containing $\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K}$. The proof is complete. $\square$

Remark 4.6.8. Incidentally, the two maximal ideals $\ker(x)\bigcap \mathcal{C}^*([C_\phi]) = C_0([C_\phi])$ for every $x \in M(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$ vanishing at $[C_\phi]$.

Theorem 4.6.9. *The fiber $M_{\langle\zeta\rangle}(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K})$ is homeomorphic to $e^{-\alpha[0,\infty]}$ via the map*

$$x \in M_{\langle\zeta\rangle}(\mathcal{TC}(PQC(\zeta),\phi)/\mathcal{K}) \mapsto x([C_\phi]) \in e^{-\alpha[0,\infty]}.$$

Proof. The continuous map is certainly injective on the generator $[C_\phi]$ with range contained in $\sigma_e(C_\phi) = e^{-\alpha[0,\infty]}$, (4.6.4), by spectral permanence in the C*-subalgebra of the Calkin algebra. It remains only to show surjectivity. To this end, fix an arbitrary $\lambda \in e^{-\alpha[0,\infty]}$ and consider the multiplicative linear functional

$$v_\lambda : a = f([C_\phi]) \in C_0([C_\phi]) \mapsto f(\lambda).$$

We claim that the direct-sum linear functional $\langle\zeta\rangle \bigoplus v_\lambda$ defined on

$$\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K} = \mathcal{T}(PQC(\zeta))/\mathcal{K} \bigoplus C_0([C_\phi])$$

is also multiplicative. For, given $T, S \in \mathcal{T}(PQC(\zeta))$ and $a = f([C_\phi]), b = g([C_\phi]) \in C_0([C_\phi])$,

$$([T]+a)([S]+b) = [T][S] + [T]b + a[S] + ab = [T][S] + \langle\zeta\rangle([T])b + \langle\zeta\rangle([S])a + ab$$

by Lemma 4.6.5, so that

$$\begin{aligned}\langle\zeta\rangle \bigoplus v_\lambda(([T]+a)([S]+b)) &= \langle\zeta\rangle([T])\langle\zeta\rangle([S]) + \langle\zeta\rangle([T])g(\lambda) + \langle\zeta\rangle([S])f(\lambda) + f(\lambda)g(\lambda) \\ &= (\langle\zeta\rangle([T]) + f(\lambda))(\langle\zeta\rangle([S]) + g(\lambda)) \\ &= (\langle\zeta\rangle \bigoplus v_\lambda([T]+a))(\langle\zeta\rangle \bigoplus v_\lambda([S]+b))\end{aligned}$$

as desired. Therefore, $x := \langle\zeta\rangle \bigoplus v_\lambda \in M_{\langle\zeta\rangle}(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$ satisfies $x([C_\phi]) = \lambda$, and the proof is complete. $\square$

Theorem 4.6.2, 4.6.4, and 4.6.9 together determine the behavior of cosets in $\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K}$ on its maximal ideal space, from which essential spectrum and norm formulas are derived in the following theorem for certain combinations of Toeplitz and composition operators. For a bivariate polynomial $p(z, w) = \sum_{0 \le j+k \le n} \beta_{j,k} z^j w^k$, $\beta_{j,k} \in \mathbb{C}$, denote the operator

$$p(C_\phi, C_\phi^*) := \beta_{0,0} I + \sum_{1 \le j+k \le n} \beta_{j,k} \prod_{l=1}^{j+k} S_{j,k,l}$$

where each $(j+k)$-tuple $\{S_{j,k,l} : 1 \le l \le j+k\}$ is a permutation of j occurences of C_ϕ and k of

C_ϕ^*. Note that for any $x \in M(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$

$$x([p(C_\phi, C_\phi^*)]) = p(x([C_\phi]), \overline{x([C_\phi])}). \tag{4.6.8}$$

Let $(\langle\zeta\rangle, \lambda)$ be the functional $x \in M_{\langle\zeta\rangle}(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})$ with $x([C_\phi]) = \lambda$. By Theorem 4.6.4

$$x([C_\phi]) = 0, \quad \forall x \in M_0 := \bigsqcup_{\lambda \neq \zeta \in \partial\mathbb{D}} M_\lambda(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K}) \bigsqcup (\langle\zeta\rangle, 0). \tag{4.6.9}$$

Theorem 4.6.10. *For $f \in PQC(\zeta)$ and a bivariate polynomial p with $p(0,0) = 0$,*

$$\sigma_e(T_f + p(C_\phi, C_\phi^*)) = \sigma_e(T_f) \bigcup \{f(\zeta) + p(\lambda, \bar{\lambda}) : \lambda \in e^{-\alpha[0,\infty]}\},$$
$$\|T_f + p(C_\phi, C_\phi^*)\|_e = \|f\|_\infty \vee \max\{|f(\zeta) + p(\lambda, \bar{\lambda})| : \lambda \in e^{-\alpha[0,\infty]}\},$$

$$\sigma_e(T_f) = \{f(\zeta)\} \bigcup_{\lambda \neq \zeta \in \partial\mathbb{D}} \{f(y\pm) : y \in M_\lambda^\pm(QC) \setminus M_\lambda^0(QC)\}$$
$$\bigcup \{tf(y+) + (1-t)f(y-) : y \in M_\lambda^0(QC), t \in [0,1]\}.$$

Proof. By spectral permanence, (4.6.8), (4.6.9), $p(0,0) = 0$, Theorem 4.6.9 and (4.6.2),

$$\begin{aligned} \sigma_e(T_f + p(C_\phi, C_\phi^*)) &= \{x([T_f]) + p(x([C_\phi]), \overline{x([C_\phi])}) : x \in M(\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K})\} \\ &= \{x([T_f]) : x \in M_0\} \bigcup \left\{f(\zeta) + p(\lambda, \bar{\lambda}) : \lambda \in e^{-\alpha[0,\infty]}\right\} \\ &= \sigma_e(T_f) \bigcup (f(\zeta) + \sigma_e(p(C_\phi, C_\phi^*))). \end{aligned} \tag{4.6.10}$$

The essential norms equal the essential spectral radii in this case, while $\|T_f\|_e = \|f\|_\infty$ for any L^∞ symbol ([49], Chapter 7). So the essential norm formula follows from (4.6.10).

Considering the fibers of $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ over $\partial\mathbb{D}$, one finds $\sigma_e(T_f)$ by Theorem 4.6.2. $\square$

Corollary 4.6.11. *For $f \in PQC(\zeta)$ and a polynomial p, $\sigma_e(T_f + p(C_\phi, C_\phi^*))$ is connected.*

Proof. Subtracting a constant from p translates the essential spectrum. So we assume $p(0,0) = 0$.

By a theorem of R. G. Douglas ([49], Theorem 7.45), $\sigma_e(T_f)$ is connected for any L^∞ symbol. In view of the homeomorphism $z \mapsto |z|$ between $e^{-\alpha[0,\infty]}$ and $[0,1]$, the second set in the union in (4.6.10) is also connected. Since the two connected sets intersect (both containing $f(\zeta)$), the union $\sigma_e(T_f + p(C_\phi, C_\phi^*))$ is connected. $\square$

We note in passing that there are harmonic Toeplitz operators on the Bergman space with disconnected essential spectra [133].

The next result states that the index of a Fredholm operator in $\mathcal{TC}(PQC(\zeta), \phi)$ is the same as that of its Toeplitz component. The interest consists in the fact that the latter index can be expressed in terms of winding numbers [120, Theorem 4].

Theorem 4.6.12. *If $T \in \mathcal{T}(PQC(\zeta))$ and a bivariate polynomial p with $p(0,0) = 0$ are such that $T + p(C_\phi, C_\phi^*)$ is Fredholm, then* $\operatorname{ind}(T + p(C_\phi, C_\phi^*)) = \operatorname{ind}(T)$.

Proof. Consider the homeomorphism $\tau : z \in e^{-\alpha[0,\infty]} \mapsto |z| \in [0,1]$. To each $t \in [0,1]$, define $k_t : r \in [0,1] \mapsto (1-t)r \in [0,1]$ and put

$$h_t = \tau^{-1} \circ k_t \circ \tau \in C_0(e^{-\alpha[0,\infty]}).$$

Then $h_0(z) = z$, $h_1(z) \equiv 0$, and h_t depends continuously on t for k_t does so. Put

$$a_t = [T] + p(h_t([C_\phi]), \overline{h_t}([C_\phi])),$$

and one has a continuous path $t \mapsto a_t$ in $[\mathcal{TC}(PQC(\zeta), \phi)]$ joining $a_0 = [T] + p([C_\phi], [C_\phi^*])$ to $a_1 = [T]$. Since $x(h_t([C_\phi])) = h_t(x([C_\phi]))$ while $x([C_\phi]) \in e^{-\alpha[0,\infty]}$ for each $x \in M([\mathcal{TC}(PQC(\zeta), \phi)])$, and since h_t for each $t \in [0,1]$ maps $e^{-\alpha[0,\infty]}$ into itself while fixing 0, a deduction similar to the proof of Theorem 4.6.10 reveals that the range of the Gelfand transform $\hat{a}_t$ is contained in that of $\hat{a}_0$, the latter being disjoint from the origin by hypothesis. Thus $[T + p(C_\phi, C_\phi^*)]$ and $[T]$ belong to the same component of $[\mathcal{TC}(PQC(\zeta), \phi)]^{-1}$, *a fortiori* to the same component of the group of invertible elements in the Calkin algebra. The index equality, as determined by the abstract index in the Calkin algebra (cf. [49, Theorem 5.35]), then follows. $\square$

Recall the closed subset $M_0 \subset M([\mathcal{TC}(PQC(\zeta), \phi)])$ defined in (4.6.9). The first Čech cohomology group of a compact Hausdorff space Ω is written $H^1(\Omega)$.

Proposition 4.6.13. $H^1(M([\mathcal{TC}(PQC(\zeta), \phi)])) \cong H^1(M_0)$.

Proof. Consider the homomorphism from the abstract index group of $[\mathcal{T}(PQC(\zeta))]$ into that of $[\mathcal{TC}(PQC(\zeta), \phi)]$, mapping every component of $[\mathcal{T}(PQC(\zeta))]^{-1}$ to that of $[\mathcal{TC}(PQC(\zeta), \phi)]^{-1}$ which contains the former. The map is surjective. For, by Lemma 4.6.7 let $[T] + f([C_\phi]) \in [\mathcal{TC}(PQC(\zeta), \phi)]^{-1}$ where $T \in \mathcal{T}(PQC(\zeta))$ and

$$f \in C_0(e^{-\alpha[0,\infty]}) = \overline{zC(e^{-\alpha[0,\infty]})} = \overline{\{zp(z, \bar{z}) : p \text{ a bivariate polynomial}\}}$$

in uniform closures over $e^{-\alpha[0,\infty]}$. Then the component of $[\mathcal{TC}(PQC(\zeta), \phi)]^{-1}$ containing $[T] + f([C_\phi])$ contains $[T] + p([C_\phi], [C_\phi^*])$ for some p with $p(0,0) = 0$, and in turn contains $[T]$ together with its component of $[\mathcal{T}(PQC(\zeta))]^{-1}$, by the proof of Theorem 4.6.12. This verifies surjectivity.

To see that the map is also injective, suppose $[T_1], [T_2] \in [\mathcal{T}(PQC(\zeta))]^{-1}$ share a common component of $[\mathcal{TC}(PQC(\zeta), \phi)]^{-1}$. That is

$$[T_1] = [T_2]e^a$$

for some $a \in [\mathcal{TC}(PQC(\zeta), \phi)]$. It follows that the Gelfand transform $\hat{a}$ assumes discrete values on the fiber over $\langle\zeta\rangle$ because $e^{\hat{a}}$ is constant there. Since this fiber is homeomorphic to $e^{-\alpha[0,\infty]}$ (Theorem 4.6.9) and to $[0,1]$, the range of the continuous function $\hat{a}$ on the fiber must be connected hence constant. Since all other fibers of $M([\mathcal{TC}(PQC(\zeta), \phi)])$ over $M([\mathcal{T}(PQC(\zeta))])$ are singletons, we assert $a \in [\mathcal{T}(PQC(\zeta))]$. So, $[T_1], [T_2]$ share a common component of $[\mathcal{T}(PQC(\zeta))]^{-1}$, and the map is injective.

Now the two abstract index groups are isomorphic, so are the groups $H^1(M([\mathcal{TC}(PQC(\zeta), \phi)]))$ and $H^1(M([\mathcal{T}(PQC(\zeta))]))$. But the latter group is isomorphic to $H^1(M_0)$ in view of the natural homeomorphism from M_0 onto $M([\mathcal{T}(PQC(\zeta))])$ given by restriction. This completes the proof. $\square$

4.6.2 The non-parabolic case

Assume in this subsection ϕ is non-parabolic, that is, $\phi'(\zeta) \neq 1$ at the fixed boundary point. The C*-algebra $\mathcal{TC}(PQC(\zeta), \phi)/\mathcal{K}$ is not commutative because C_ϕ is not essentially normal. Let $\mathcal{A}$ be the (non-self-adjoint) norm-closed algebra generated by $\{C_\phi, T_f : f \in PQC(\zeta)\}$. Since $\mathcal{A} \supset \mathcal{T}(C) \supset \mathcal{K}$, the quotient algebra $\mathcal{A}/\mathcal{K}$ equals the closed subalgebra of the Calkin algebra generated by $\{[C_\phi], [T_f] : f \in PQC(\zeta)\}$, a commutative Banach algebra containing the C*-subalgebra $\mathcal{T}(PQC(\zeta))/\mathcal{K}$.

Lemma 4.6.14. $\sigma_e(C_\phi) = \{|z| \leq \sqrt{s}\}$.

Proof. It is known that $\phi'(\zeta) > 0$ (cf. [109], p. 744). If $\phi'(\zeta) < 1$, then ζ is the Denjoy-Wolff point [42, p. 59] of ϕ, and [42, Theorem 7.26] ([40, Corollary 4.8]) asserts that $\sigma(C_\phi) = \{|z| \leq \sqrt{s}\}$. And, since ζ is the only point on $\partial\mathbb{D}$ with $|\phi(\zeta)| = 1$, [42, Lemma 7.25] (also [40, Theorem 4.6 and p. 97]) asserts that every z, $0 < |z| < \sqrt{s}$, is an eigenvalue of C_ϕ of infinite multiplicity. Therefore,

$$\{0 < |z| < \sqrt{s}\} \subset \sigma_e(C_\phi) \subset \{|z| \leq \sqrt{s}\}$$

yields $\sigma_e(C_\phi) = \{|z| \leq \sqrt{s}\}$.

If otherwise $\phi'(\zeta) > 1$, then $\psi'(\zeta) = 1/\phi'(\zeta) < 1$ for the Krein adjoint ψ of ϕ [88, Prop. 3.4], and the derivation above applies to ψ at its Denjoy-Wolff point ζ to give

$$\sigma_e(C_\psi) = \{|z| \leq 1/\sqrt{s}\}.$$

Since $\sigma_e(C_\phi^*) = s\sigma_e(C_\psi)$ due to $[C_\phi^*] = s[C_\psi]$, multiplying the equality above by s and taking complex conjugates yield $\sigma_e(C_\phi) = \{|z| \leq \sqrt{s}\}$ again. □

Remark 4.6.15. It follows from the lemma and the essential norm formula [29] in terms of Aleksandrov-Clark measures that

$$\|[C_\phi]\| = \sqrt{s} = \rho([C_\phi])$$

for the non-normal $[C_\phi]$. It would be interesting to know if this equality, or equivalence, between essential norm and essential spectral radius extends to polynomials of C_ϕ.

The fibers of $M(\mathcal{A}/\mathcal{K})$ over $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ can be similarly described as in Theorem 4.6.4 and 4.6.9 for the C*-algebra case. The counterpart to Theorem 4.6.4 is true for $\mathcal{A}/\mathcal{K}$ with an identical proof.

Theorem 4.6.16. *For $\xi \neq \langle\zeta\rangle \in M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$, the fiber $M_\xi(\mathcal{A}/\mathcal{K})$ consists of a single functional vanishing at $[C_\phi]$.*

To prove the counterpart to Theorem 4.6.9, we need some variants of the three lemmas. We shall only outline their proof, if not omitting it. Let $P_0([C_\phi])$ be the norm-closure of

$$\{[C_\phi]p([C_\phi]) : p \text{ is a polynomial}\}.$$

Lemma 4.6.17. *For every $T \in \mathcal{T}(PQC(\zeta))$ and $b \in P_0([C_\phi])$,*

$$[T]b = b[T] = \langle\zeta\rangle([T])b.$$

Lemma 4.6.18. *There exists $y \in M_{\langle\zeta\rangle}(\mathcal{A}/\mathcal{K})$ such that $y([C_\phi]) = 0$.*

Proof. Such y arises as a cluster point as before, using Theorem 4.6.16 instead. □

Lemma 4.6.19. *For every $T \in \mathcal{T}(PQC(\zeta))$ and $b \in P_0([C_\phi])$,*

$$\|[T] + b\| \geq \|[T]\|. \tag{4.6.11}$$

Consequently, one has the Banach space direct-sum decomposition

$$\mathcal{A}/\mathcal{K} = \mathcal{T}(PQC(\zeta))/\mathcal{K} \bigoplus P_0([C_\phi]).$$

Proof. Relative to the commutative C*-algebra $\mathcal{T}(PQC(\zeta))/\mathcal{K}$ one has

$$\|[T]\| = \max\{|\xi([T])| : \xi \in M(\mathcal{T}(PQC(\zeta))/\mathcal{K})\},$$

while relative to the commutative Banach algebra $\mathcal{A}/\mathcal{K}$ one has

$$\|[T]+b\| \geq \max\{|y([T])+y(b)| : y \in M(\mathcal{A}/\mathcal{K})\}.$$

Partitioning $M(\mathcal{A}/\mathcal{K})$ into fibers over $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$ and using Theorem 4.6.16 and Lemma 4.6.18, one verifies that the second maximum is no less than the first. Note that $y([C_\phi]) = 0$ implies $y(b) = 0$ by multiplicativity and continuity. The direct-sum decomposition follows from (4.6.11) as usual, noting Lemma 4.6.17 and the fact that $P_0([C_\phi])$ is norm-closed. □

Theorem 4.6.20. *The fiber $M_{\langle\zeta\rangle}(\mathcal{A}/\mathcal{K})$ is homeomorphic to $\{|z| \leq \sqrt{s}\}$ via the map*

$$y \in M_{\langle\zeta\rangle}(\mathcal{A}/\mathcal{K}) \mapsto y([C_\phi]) \in \{|z| \leq \sqrt{s}\}.$$

Proof. Let $A([C_\phi])$ be the non-self-adjoint closed Calkin subalgebra singly generated by $[C_\phi]$. The injective continuous map has its range in

$$\sigma([C_\phi], \mathcal{A}/\mathcal{K}) \subset \sigma([C_\phi], A([C_\phi])) = \operatorname{hull}(\sigma_e(C_\phi)) = \{|z| \leq \sqrt{s}\}$$

by Lemma 4.6.14. Conversely, every $\lambda \in \sigma([C_\phi], A([C_\phi])) = \{|z| \leq \sqrt{s}\}$ equals $m([C_\phi])$ for some multiplicative linear functional m on the commutative Banach algebra $A([C_\phi])$. Letting $m' := m|P_0([C_\phi])$, one directly verifies that the direct-sum linear functional $\langle\zeta\rangle \bigoplus m'$ on $\mathcal{A}/\mathcal{K} = \mathcal{T}(PQC(\zeta))/\mathcal{K} \bigoplus P_0([C_\phi])$ is multiplicative using Lemma 4.6.17. Thus, $y := \langle\zeta\rangle \bigoplus m' \in M_{\langle\zeta\rangle}(\mathcal{A}/\mathcal{K})$ with $y([C_\phi]) = m([C_\phi]) = \lambda$, establishing surjectivity and completing the proof. □

The Shilov boundary $\partial(\mathcal{A}/\mathcal{K})$ can be explicitly identified. Denote by $(\langle\zeta\rangle, \lambda)$ the functional $y \in M_{\langle\zeta\rangle}(\mathcal{A}/\mathcal{K})$ with $y([C_\phi]) = \lambda$.

Theorem 4.6.21. *With ξ ranging over $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$,*

$$\partial(\mathcal{A}/\mathcal{K}) = \bigsqcup_{\xi \neq \langle\zeta\rangle} M_\xi(\mathcal{A}/\mathcal{K}) \bigsqcup \left\{(\langle\zeta\rangle, \lambda) : |\lambda| = 0, \sqrt{s}\right\}.$$

Proof. Throughout the proof denote by F the union on the right side. F is a closed subset of $M(\mathcal{A}/\mathcal{K})$ because it is the pre-image of a closed set under a continuous function:

$$F = \{y \in M(\mathcal{A}/\mathcal{K}) : |y([C_\phi])| = 0, \sqrt{s}\},$$

noting Theorem 4.6.16.

We shall first show F is a boundary for $\mathcal{A}/\mathcal{K}$. Let $T \in \mathcal{T}(PQC(\zeta))$ and let p be a polynomial. Write $a = [T] + [C_\phi]p([C_\phi])$ with Gelfand transform $\hat{a}$ on $M(\mathcal{A}/\mathcal{K})$. By the maximum modulus principle for analytic functions,

$$\max_{|\lambda|\le\sqrt{s}} |\langle\zeta\rangle([T]) + \lambda p(\lambda)| = \max_{|\lambda|=0,\sqrt{s}} |\langle\zeta\rangle([T]) + \lambda p(\lambda)|. \tag{4.6.12}$$

Partitioning $M(\mathcal{A}/\mathcal{K})$ into fibers over $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$, we have by their descriptions

$$\|\hat{a}\|_\infty = \sup_{\xi\neq\langle\zeta\rangle} |\xi([T])| \vee \max_{|\lambda|\le\sqrt{s}} |\langle\zeta\rangle([T]) + \lambda p(\lambda)|$$
$$\|\hat{a}|F\|_\infty = \sup_{\xi\neq\langle\zeta\rangle} |\xi([T])| \vee \max_{|\lambda|=0,\sqrt{s}} |\langle\zeta\rangle([T]) + \lambda p(\lambda)|$$

and obtain by (4.6.12) $\|\hat{a}\|_\infty = \|\hat{a}|F\|_\infty$. This norm equality on $M(\mathcal{A}/\mathcal{K})$ extends to every element of $\mathcal{A}/\mathcal{K}$ due to the decomposition in Lemma 4.6.19 and density of $[C_\phi]p([C_\phi])$ in $P_0([C_\phi])$. That is, the closed subset F is a boundary, hence

$$F \supset \partial(\mathcal{A}/\mathcal{K}).$$

To show the reverse, consider the restriction map $r : M(\mathcal{A}/\mathcal{K}) \to M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$. Since

$$\partial(\mathcal{T}(PQC(\zeta))/\mathcal{K}) = M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$$

for the C*-algebra, Theorem 1.3.7 implies that the singleton fiber (Theorem 4.6.16) $M_\xi(\mathcal{A}/\mathcal{K})$ lies in $\partial(\mathcal{A}/\mathcal{K})$ for every $\xi \neq \langle\zeta\rangle$. Next, using a cluster-point argument as before, $(\langle\zeta\rangle, 0)$ lies in the closure of the union of these fibers, and hence also in the closed set $\partial(\mathcal{A}/\mathcal{K})$. It remains only to show for an arbitrary λ, $|\lambda| = \sqrt{s}$, that $(\langle\zeta\rangle, \lambda) \in \partial(\mathcal{A}/\mathcal{K})$. To this end let

$$b := \lambda[I] + [C_\phi] \in \mathcal{A}/\mathcal{K}.$$

Since $y(b) \equiv \lambda$ on $\bigsqcup_{\xi \neq \langle\zeta\rangle} M_\xi(\mathcal{A}/\mathcal{K})$ and $(\langle\zeta\rangle, \mu)(b) = \lambda + \mu$, $|\mu| \leq \sqrt{s}$, $|\hat{b}|$ peaks at $(\langle\zeta\rangle, \lambda)$ with $\|\hat{b}\|_\infty = 2\sqrt{s}$. This is because for $|\lambda| = \sqrt{s}$ and $|\mu| \leq \sqrt{s}$,

$$|\lambda + \mu| = 2\sqrt{s} \Longleftrightarrow \mu = \lambda.$$

The peak point $(\langle\zeta\rangle, \lambda)$ for $\mathcal{A}/\mathcal{K}$ must lie on $\partial(\mathcal{A}/\mathcal{K})$, as required. □

One should note that Corollary 6.6 in [109] shows the C*-algebra $\mathcal{TC}(C, \phi)/\mathcal{K}$ isomorphic to a direct sum involving a crossed product of continuous functions by $\mathbb{Z}$, under which essential spectra of operators in $\mathcal{TC}(C, \phi)$ unfortunately remain elusive. Letting U be the Calkin algebra and $B = \mathcal{A}/\mathcal{K}$ in (1.3.1), one has for every $A \in \mathcal{A}$ the essential spectral inclusions

$$\{y([A]) : y \in \partial(\mathcal{A}/\mathcal{K})\} \subset \sigma_e(A) \subset \{y([A]) : y \in M(\mathcal{A}/\mathcal{K})\}. \tag{4.6.13}$$

However, a more careful argument captures the entire essential spectrum as follows.

Theorem 4.6.22. *For $T \in \mathcal{T}(PQC(\zeta))$ and $b \in P_0([C_\phi])$, one has in the Calkin algebra that*

$$\sigma([T] + b) = \sigma([T]) \bigcup (\langle\zeta\rangle([T]) + \sigma(b)).$$

Proof. Putting $t = [T] - \langle\zeta\rangle([T])$, we shall prove the equivalent statement that

$$\sigma(t + b) = \sigma(t) \bigcup \sigma(b). \tag{4.6.14}$$

Let $\mathscr{U}$ be a maximal commutative subalgebra of the Calkin algebra containing $\mathcal{A}/\mathcal{K}$. Since $tb = 0$ by Lemma 4.6.17, $m(t) = 0$ or $m(b) = 0$ for every $m \in M(\mathscr{U})$. Then every $\lambda \in \sigma(t + b) =$

$\sigma(t+b,\mathscr{U})$ equals $m(t+b)$ for some $m \in M(\mathscr{U})$, which is either $m(t) \in \sigma(t)$ if $m(b) = 0$, or $m(b) \in \sigma(b)$ if $m(t) = 0$. That is, $\sigma(t+b) \subset \sigma(t) \bigcup \sigma(b)$.

Conversely, let $0 \neq \lambda \in \sigma(t) \bigcup \sigma(b)$. Then $\lambda = m_1(t)$ or $\lambda = m_2(b)$ for some $m_1, m_2 \in M(\mathscr{U})$. Since $\lambda \neq 0$, either $m_1(b) = 0$ giving $\lambda = m_1(t+b) \in \sigma(t+b)$, or $m_2(t) = 0$ giving $\lambda = m_2(t+b) \in \sigma(t+b)$. To see also $0 \in \sigma(t+b)$, note that for all polynomials p

$$(\langle\zeta\rangle, 0)([C_\phi]p([C_\phi])) = 0,$$

so that $(\langle\zeta\rangle, 0)(b) = 0$ by continuity. But $(\langle\zeta\rangle, 0) \in \partial(\mathcal{A}/\mathcal{K})$ (Theorem 4.6.21) implies that it extends to some $m_0 \in M(\mathscr{U})$ with

$$m_0(b) = (\langle\zeta\rangle, 0)(b) = 0, \qquad m_0(t) = \langle\zeta\rangle(t) = 0,$$

hence $\sigma(t+b) \ni m_0(t+b) = 0$ as required. So, $\sigma(t) \bigcup \sigma(b) \subset \sigma(t+b)$ and (4.6.14) is proved. $\square$

The essential spectrum formula for a typical operator in $\mathcal{A}$ follows immediately.

Corollary 4.6.23. *For $T \in \mathcal{T}(PQC(\zeta))$ and a polynomial p with $p(0) = 0$,*

$$\sigma_e(T + p(C_\phi)) = \sigma_e(T) \bigcup \{\langle\zeta\rangle([T]) + p(\lambda) : |\lambda| \leq \sqrt{s}\}.$$

Proof. Simply take $b = p([C_\phi])$ in the preceding theorem, and notice in the Calkin algebra

$$\sigma(b) = p(\sigma_e(C_\phi)) = \{p(\lambda) : |\lambda| \leq \sqrt{s}\} \tag{4.6.15}$$

by spectral mapping and Lemma 4.6.14. $\square$

Corollary 4.6.24. *For $f \in PQC(\zeta)$ and a polynomial p, $\sigma_e(T_f + p(C_\phi))$ is connected.*

Proof. Take $p(0) = 0$ and note $\langle\zeta\rangle([T_f]) \in \sigma_e(T_f)$. The rest follows from the formula above and the fact that $\sigma_e(T_f)$ is connected [49]. $\square$

It is interesting to observe that the subalgebra $\mathcal{A}/\mathcal{K}$ of the Calkin algebra preserves spectra, thus is a *full* subalgebra. In other words, the second inclusion in (4.6.13) is indeed an equality.

Proposition 4.6.25. *The algebra $\mathcal{A}/\mathcal{K}$ is a full subalgebra of the Calkin algebra.*

Proof. Elements of $\mathcal{A}/\mathcal{K}$ have the form $[T]+b$, $T \in \mathcal{T}(PQC(\zeta))$, $b \in P_0([C_\phi])$, by Lemma 4.6.19.

By Theorem 4.6.16 and continuity, with ξ ranging over $M(\mathcal{T}(PQC(\zeta))/\mathcal{K})$,

$$y(b) = 0, \quad \forall y \in N_0 := \bigsqcup_{\xi \neq \langle\zeta\rangle} M_\xi(\mathcal{A}/\mathcal{K}) \bigsqcup (\langle\zeta\rangle, 0). \tag{4.6.16}$$

Let p be a polynomial. Then as in (4.6.15) in the Calkin algebra,

$$\sigma(p([C_\phi])) = \{p(\lambda) : |\lambda| \leq \sqrt{s}\} = \{(\langle\zeta\rangle, \lambda)(p([C_\phi]) : |\lambda| \leq \sqrt{s}\}.$$

Passing from $p([C_\phi])$ to b in the norm limit while noting commutativity, the sets on the left and right converge in the Hausdorff metric in the space of compact planar sets to $\sigma(b)$ and $\{(\langle\zeta\rangle, \lambda)(b) : |\lambda| \leq \sqrt{s}\}$, respectively, making

$$\sigma(b) = \{(\langle\zeta\rangle, \lambda)(b) : |\lambda| \leq \sqrt{s}\}. \tag{4.6.17}$$

Now it follows from the structure of $M(\mathcal{A}/\mathcal{K})$ and (4.6.16), (4.6.17), that

$$\begin{aligned}\sigma([T]+b, \mathcal{A}/\mathcal{K}) &= \{y([T]) : y \in N_0\} \bigcup \{\langle\zeta\rangle([T]) + (\langle\zeta\rangle, \lambda)(b) : |\lambda| \leq \sqrt{s}\} \\ &= \sigma([T]) \bigcup (\langle\zeta\rangle([T]) + \sigma(b)).\end{aligned}$$

Comparing with Theorem 4.6.22, one has $\sigma([T]+b, \mathcal{A}/\mathcal{K}) = \sigma([T]+b)$ proving spectral permanence for $\mathcal{A}/\mathcal{K}$ in the Calkin algebra. □

While the closed algebra $\{[T_f] : f \in H^\infty + C\}$ is maximal commutative in the Calkin algebra [45, Corollary 2], the C*-algebra $\mathcal{T}(PQC)/\mathcal{K}$ is not [120]. Using Sarason's construction [118, 120], we shall show the algebra $\mathcal{A}/\mathcal{K}$ generated by $\mathcal{T}(PQC(\zeta))/\mathcal{K}$ and $[C_\phi]$ is not maximal commutative either. That is, its commutant in the Calkin algebra satisfies $(\mathcal{A}/\mathcal{K})' \supsetneq \mathcal{A}/\mathcal{K}$. We are however unable to answer the much harder question about its double commutant: $(\mathcal{A}/\mathcal{K})'' \supsetneq \mathcal{A}/\mathcal{K}$?

Proposition 4.6.26. *The algebra $\mathcal{A}/\mathcal{K}$ is not maximal commutative in the Calkin algebra.*

Proof. Choose $\lambda \in \partial\mathbb{D}$ with $\lambda \neq \zeta$. By the argument in the last paragraph of [120, p. 837], and with λ in place of $1 \in \partial\mathbb{D}$, there exists a real function $v \in L^\infty$ continuous at $\zeta \in \partial\mathbb{D}$ such that $[T_v]$ commutes with $\mathcal{T}(PQC)/\mathcal{K}$ and yet

$$[T_v] \notin \mathcal{T}(PQC)/\mathcal{K}. \tag{4.6.18}$$

It follows from continuity at ζ that $[T_v][C_\phi] = v(\zeta)[C_\phi] = [C_\phi][T_v]$, so that the closed algebra generated by $\mathcal{A}/\mathcal{K}$ and $[T_v]$ is commutative. It remains to show $[T_v] \notin \mathcal{A}/\mathcal{K}$.

Assume on the contrary $[T_v] \in \mathcal{A}/\mathcal{K}$ in the sequel. Then by Lemma 4.6.19,

$$[T_v] = [T] + b$$

for some $T \in \mathcal{T}(PQC(\zeta))$ and $b \in P_0([C_\phi])$. Let $\mathcal{U}$ be the closed subalgebra of $\mathcal{A}/\mathcal{K}$ generated by $\mathcal{T}(PQC(\zeta))/\mathcal{K}$ and $[T_v]$. Then $\mathcal{U}$ is self-adjoint since v is real. Putting $u = [T] - [T_v] \in \mathcal{U}$, one has $u + b = 0$. Considering the fiber structure of $M(\mathcal{A}/\mathcal{K})$ over $M(\mathcal{U})$ of the C*-subalgebra $\mathcal{U}$, one can apply techniques used earlier to deduce $u = 0$.

To that end, we first show the fiber $M_\zeta(\mathcal{U})$ over $\zeta \in M(\mathcal{T}(C)/\mathcal{K})$ is a singleton, say $\langle\langle\zeta\rangle\rangle$. For, the generic generator for $\mathcal{U}$ has the form $[T_f]$, either $f \in PQC(\zeta)$ or $f = v$. Let $\kappa \in M_\zeta(\mathcal{U})$ and $\epsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $\|f|(\zeta - \delta, \zeta + \delta) - f(\zeta)\|_\infty < \epsilon$, due to continuity of f at ζ, and let $g \in C$ be $[0,1]$-valued with $g(\zeta) = 1$ and $g \equiv 0$ on $\partial\mathbb{D} \setminus (\zeta - \delta, \zeta + \delta)$. Then

$$\begin{aligned}|\kappa([T_f]) - f(\zeta)| &= |\kappa(([T_f] - f(\zeta)[I])[T_g])| \leq \|([T_f] - f(\zeta)[I])[T_g]\| \\ &= \|[T_{(f-f(\zeta))g}]\| \leq \|(f - f(\zeta))g\|_\infty < \epsilon.\end{aligned}$$

That is, $\kappa([T_f]) = f(\zeta)$, which proves the uniqueness of κ. Consequently, if $\kappa \neq \langle\langle\zeta\rangle\rangle$ in $M(\mathcal{U})$, then $\kappa|(\mathcal{T}(C)/\mathcal{K}) \neq \zeta$ on $\partial\mathbb{D}$, and an argument similar to that used to prove Theorem 4.6.4 shows that $M_\kappa(\mathcal{A}/\mathcal{K})$ consists of a single functional vanishing at $[C_\phi]$. Next, a cluster-point argument shows the existence of a functional in $M_{\langle\langle\zeta\rangle\rangle}(\mathcal{A}/\mathcal{K})$ vanishing at $[C_\phi]$. Such functionals must vanish at $b \in P_0([C_\phi])$ as well. Now one has the following implications, the last of which due to $\mathcal{U} \cong C(M(\mathcal{U}))$ for the C*-subalgebra $\mathcal{U}$,

$$u + b = 0 \implies (u + b)(M(\mathcal{A}/\mathcal{K})) = \{0\} \implies u(M(\mathcal{U})) = \{0\} \implies u = 0.$$

Therefore,

$$[T_v] = [T] \in \mathcal{T}(PQC(\zeta))/\mathcal{K} \subset \mathcal{T}(PQC)/\mathcal{K}.$$

This contradiction against (4.6.18) completes the proof. □

The following result is the counterpart to Theorem 4.6.12. The proof is similar and hence omitted, using instead the continuous path of Fredholm operators

$$t \in [0,1] \mapsto T + p((1-t)C_\phi)$$

in view of Corollary 4.6.23 applied to the polynomials $z \mapsto p((1-t)z)$ fixing 0.

Theorem 4.6.27. *If $T \in \mathcal{T}(PQC(\zeta))$ and a polynomial p with $p(0) = 0$ are such that $T + p(C_\phi)$ is Fredholm, then* $\mathrm{ind}(T + p(C_\phi)) = \mathrm{ind}(T)$.

We mention that in view of the Arens-Royden theorem the counterpart to Proposition 4.6.13, with the compact subset $N_0 \subset M([\mathcal{A}])$ in (4.6.16) replacing M_0, would hold if only the algebra singly generated by $[C_\phi]$ were semi-simple (see Remark 4.6.15).

Chapter 5

Further Problems

5.1 Lifting subalgebras of commutants of analytic Toeplitz operators

Commutant lifting/extension problems constitute a common theme in Hilbert space operator theory. Recall that $S \in \mathcal{L}(H)$ on a Hilbert space H is *subnormal* if it admits a normal extension $N \in \mathcal{L}(K)$ on a Hilbert space K of which H is a subspace. Such a normal extension N is *minimal* if K is the smallest reducing subspace for N containing H. Minimal normal extensions exist uniquely. It is known that not all operators in the commutant $\{S\}'$ of a general subnormal S *lift* to an operator (i.e. admit an extension) in $\{N\}'$ of its minimal normal extension N, and that the ones which do lift do so uniquely due to a classical result of J. Bram. See [33] for basic facts about subnormal operators. Write $\mathcal{A} \subset \mathcal{L}(K)$ for the weakly closed unital subalgebra of operators on K for which H is invariant. Then the restriction map

$$\alpha : T \in \{N\}' \bigcap \mathcal{A} \mapsto T|H \in \{S\}'$$

is an injective and contractive Banach algebra homomorphism whose range, called the *lifting subalgebra* of $\{S\}'$, consists precisely of those operators in $\{S\}'$ which lift. By the open mapping theorem, the lifting subalgebra is norm closed if and only if α is bounded below.

Analytic Toeplitz operators T_f on $H^2(\partial\mathbb{D})$ defined by nonconstant symbols $f \in H^\infty \subset L^\infty$ are subnormal with minimal normal extensions the multiplication operators M_f on $L^2(\partial\mathbb{D})$. It is of interest to identify the lifting subalgebra of the commutant $\{T_f\}'$, that is, the range of

$$\alpha = \alpha_f : T \in \{M_f\}' \bigcap \mathcal{A} \mapsto T|H^2 \in \{T_f\}'.$$

Note that every $T_h \in \{T_f\}'$, $h \in H^\infty$, lifts to $M_h \in \{M_f\}'$. The special case of when the lifting subalgebra actually equals $\{T_f\}'$ is characterized in [38, Theorem 2], from which it is further shown [38, Theorem 3] that all covering maps have this remarkable property besides inner functions. Parallel to [38, Cor., p. 2] on lifting commutants, the compatibility result [38, Prop., p. 2], which amounts to

$$\{T_f\}' \bigcap \operatorname{ran}(\alpha_g) = \{T_g\}' \bigcap \operatorname{ran}(\alpha_f), \qquad f, g \in H^\infty,$$

has another immediate corollary on lifting subalgebras.

Proposition 5.1.1. *If $f, g \in H^\infty$ are such that $\{T_f\}' \subset \{T_g\}'$, then the lifting subalgebra of $\{T_f\}'$ is contained in that of $\{T_g\}'$, and the lifting map α_f^{-1} equals the restriction of α_g^{-1}.*

Consider the inner-outer factorization $f = ug$ with a nonconstant inner factor u (so that T_u is a pure isometry), and let $\{h_k\}_{k\in\mathbb{N}}$ be a sequence of H^∞ functions satisfying $h_k g \to 1$ pointwise in $\mathbb{D}$ and $\sup_k \|h_k g\|_\infty \le 1$ (see [38, pp. 4-5]). For $A \in \{T_f\}'$, put

$$C_A := \sup_{n\ge 0, k\ge 1} \|T_{h_k}^n A T_g^n\| \in [\|A\|, \infty].$$

An inspection of the proof of [38, Theorem 2] yields a more general result characterizing the lifting subalgebra itself.

Theorem 5.1.2. *For $A \in \{T_f\}'$, A lifts if and only if $C_A < \infty$, in which case $\|\alpha^{-1}(A)\| = C_A$ and $\alpha^{-1}(A)(\bar{u}^n H^2) \subset \bar{u}^n H^2$, $n = 0, 1, \ldots$.*

Proof. First suppose A lifts. Since $\alpha^{-1}(T_{h_k}^n A T_g^n) = M_{h_k}^n \alpha^{-1}(A) M_g^n$, $\|h_k g\|_\infty \le 1$, and since M_u^n

is unitary on $L^2(\partial\mathbb{D})$,

$$\|T_{h_k}^n A T_g^n\| \leq \|M_{h_k}^n \alpha^{-1}(A) M_g^n\| = \|M_{h_k}^n \alpha^{-1}(A) M_g^n M_u^n\|$$
$$= \|M_{h_k}^n \alpha^{-1}(A) M_f^n\| = \|M_{(h_k g u)^n} \alpha^{-1}(A)\| \leq \|\alpha^{-1}(A)\|.$$

That is, $C_A \leq \|\alpha^{-1}(A)\| < \infty$.

Next suppose $C_A < \infty$. Following exactly Cowen's proof [38, p. 5], one has for any $n \geq 0$ and $\phi \in H^2$ that $g^{-n}A(g^n\phi) \in H^2$ with $\|g^{-n}A(g^n\phi)\|_2 \leq C_A\|\phi\|_2$, from which an extension $\tilde{A} \in \{M_f\}'$ of A can be constructed satisfying $\|\tilde{A}\| \leq C_A$ and $\tilde{A}(\bar{u}^n H^2) \subset \bar{u}^n H^2$ [38, p. 4]. That is, A lifts with $\|\alpha^{-1}(A)\| = \|\tilde{A}\| = C_A$ and the said invariant subspaces. □

The first part of the following corollary is due to the preceding theorem and an earlier remark, while the second part can be proved in the same manner as [38, Theorem 1].

Corollary 5.1.3. *The lifting subalgebra of $\{T_f\}'$ is norm closed if and only if*

$$C := \sup\{C_A/\|A\| : 0 \neq A \in \{T_f\}', C_A < \infty\} < \infty,$$

in which case the lifting subalgebra does not contain nonzero compact operators.

Remark 5.1.4. For any nonconstant $f \in H^\infty$ and $\zeta \in \mathbb{D}$, one has $\{T_f\}' = \{T_{f-f(\zeta)}\}'$ where $f - f(\zeta)$ has a nonconstant inner factor hence satisfying the hypothesis for Theorem 5.1.2, while Proposition 5.1.1 asserts that the two lifting subalgebras are identical as well as the lifting maps.

Motivated by Cowen's result on covering maps (ie. automorphic symbols), one may attempt to apply Theorem 5.1.2 to determine the lifting subalgebra for a larger class of symbols. Recall that $f \in H^\infty$ is called *semi-automorphic* [39] if $f = \pi \circ \xi$ for a covering map π on $\mathbb{D}$ and a univalent self map ξ of $\mathbb{D}$. For such f, Cowen [39] obtains a closed-form description of $\{T_f\}'$ and asks what is the lifting subalgebra of $\{T_f\}'$ under the description.

5.2 Multiplication operators defined by analytic covering maps

Let $\phi \in H^\infty(\mathbb{D})$ be a covering map onto a planar region Ω with nontrivial subgroup $G \subset \mathrm{Aut}(\mathbb{D})$ of deck transformations, and consider the multiplication operator T_ϕ on the Hardy space $H^2(\mathbb{D})$ and Bergman space $A^2(\mathbb{D})$, respectively. Since ϕ is $H^2(\mathbb{D})$-ancestral [36, p. 22], T_ϕ on $H^2(\mathbb{D})$ commutes with no nonzero compact operators [36, p. 27]. Do these properties also hold for $A^2(\mathbb{D})$? In particular, since the inner factor of $\phi - \phi(\alpha)$, $\forall \alpha \in \mathbb{D}$, is an interpolating Blaschke product, ϕ would be $A^2(\mathbb{D})$-ancestral should the following question have a positive answer:

If a function $f \in A^2(\mathbb{D})$ vanishes at the zeros $\{z_n\}_n$ in $\mathbb{D}$ of an interpolating Blaschke product b, is it true that $f \in \overline{bA^2(\mathbb{D})}$?

Consider the zero-based invariant subspace $M_b := \{f \in A^2(\mathbb{D}) : f(z_n) \equiv 0\}$. By the deep result in [4], the wandering subspace $M_b \ominus zM_b$ generates M_b as invariant space. Hence the original question of $M_b \subset \overline{bA^2(\mathbb{D})}$ reduces to if $M_b \ominus zM_b \subset \overline{bA^2(\mathbb{D})}$? Since $M_b \ominus zM_b$ is well known to be one-dimensional, it further suffices to construct a nonzero element in the former and prove it contained in the latter.

It is well known that each $L \in G$ induces a unitary operator $U_L := T_{L'^{1/2}} C_L$ on $H^2(\mathbb{D})$ commuting with T_ϕ, where the square root $L'^{1/2} \in H^\infty(\mathbb{D})$ of the non-vanishing derivative $L' \in H^\infty(\mathbb{D})$ is so chosen that

$$(L \circ J)'^{1/2} = (L'^{1/2} \circ J) J'^{1/2}, \qquad \forall L, J \in G.$$

This is possible because G is isomorphic to the fundamental group of Ω, thus a free group, so that one just needs to randomly pick the square roots for the generators and then extend the selection to G accordingly. The map $L \in G \mapsto U_{L^{-1}} \in \mathcal{L}(H^2(\mathbb{D}))$ is a unitary representation of G. Likewise, $L \in G \mapsto V_{L^{-1}} \in \mathcal{L}(A^2(\mathbb{D}))$ is a unitary representation of G commuting with T_ϕ on $A^2(\mathbb{D})$, where $V_L := T_{L'} C_L$. In the latter case, every unitary operator commuting with T_ϕ is "almost" given by such V_L [72, Th. 2.2], and the von Neumann algebra $\{T_\phi, T_\phi^*\}'$ equals the weak-operator closure of the linear span $\mathrm{sp}\{V_L : L \in G\}$ [72, Cor. 2.5]. Is this also true on $H^2(\mathbb{D})$ with $\{U_L : L \in G\}$?

5.3 Spectral theory of multiplication operators

For a non-constant function $f \in H^\infty(\Omega)$ and $X = H^p(\Omega, \omega)$ or $A^p(\Omega, wda)$, the point spectrum $\sigma_p(T_f)$ of $T_f \in \mathcal{L}(X)$ is empty, so that right away T_f has the SVEP (single-valued extension property). Also, using the facts $\sigma(T_f) = \sigma(f) = \overline{f(\Omega)}$ and $\|T_f\| = \|f\|_\infty$ obtained in the process of extending the analytic functional calculus for T_f in Section 2.5, one explicitly computes the norm of the resolvent of T_f as follows

$$\|(\lambda I - T_f)^{-1}\| = \|T_{1/(\lambda - f)}\| = \|1/(\lambda - f)\|_\infty = 1/d(\lambda, \overline{f(\Omega)}), \quad \forall \lambda \notin \sigma(T_f) = \overline{f(\Omega)}.$$

Therefore, growth rate conditions on $\|(\lambda I - T_f)^{-1}\|$ are easy to check. These considerations appear to suggest an interesting local spectral theory for T_f. For instance, one may ask whether T_f, f a univalent function, satisfies Dunford's property (C), Bishop's properties (α), (β), the property (β^*), or the Albrecht-Eschmeier property (δ), all key ingredients in the theory of spectral operators and more generally decomposable operators on Banach spaces (cf. [91]).

Recall that the *approximate point spectrum* of $T \in \mathcal{L}(X)$ is defined as

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

Proposition 5.3.1. *For $f \in H^\infty(\Omega)$ non-constant, $\partial \overline{f(\Omega)} \subset \sigma_{ap}(T_f) \subset \sigma_e(T_f) \subset f(\partial\Omega)$.*

Proof. Since $\sigma(T_f) = \overline{f(\Omega)}$, $\partial \overline{f(\Omega)} = \partial\sigma(T_f) \subset \sigma_{ap}(T_f)$ by standard facts about Banach space operators. Next, if $\lambda I - T_f$ has closed range, then it is bounded below by the open mapping theorem since $\lambda I - T_f = T_{\lambda - f}$ is injective. This implies $\sigma_{ap}(T_f) \subset \sigma_e(T_f)$. Lastly, $\sigma_e(T_f) \subset f(\partial\Omega)$ due to Lemma 2.2.5 (see Remark 2.2.6). □

The following corollary seems to supplement certain results of [9, 34] on the essential spectrum $\sigma_e(T_f)$. Note that the property $\overline{\Omega}^o = \Omega$ is not a conformal property, that is, $\overline{f(\Omega)}^o = f(\Omega)$ does not necessarily follow from $\overline{\Omega}^o = \Omega$ and f being univalent on Ω (counterexamples abound, e.g. by the Riemann mapping theorem).

Corollary 5.3.2. *If $f \in H^\infty(\Omega)$ is univalent and $\overline{f(\Omega)}^o = f(\Omega)$, then $\sigma_{ap}(T_f) = \sigma_e(T_f) = f(\partial\Omega)$.*

Proof. If $f \in H^\infty(\Omega)$ is univalent, then $f(\partial\Omega) \bigcap f(\Omega) = \emptyset$, see the proof of Corollary 2.3.10. Thus $f(\partial\Omega) \subset \overline{f(\Omega)} \setminus f(\Omega)$, which together with (2.2.5) gives $f(\partial\Omega) = \overline{f(\Omega)} \setminus f(\Omega)$, while $\partial\overline{f(\Omega)} = \overline{f(\Omega)} \setminus \overline{f(\Omega)}^o = \overline{f(\Omega)} \setminus f(\Omega)$ by hypothesis. Now $\partial\overline{f(\Omega)} = f(\partial\Omega)$ forces equalities in the preceding proposition. □

5.4 Compact perturbations of Toeplitz and Hankel operators

Let $A \in \mathcal{L}(H^2)$. A consideration of the matrix forms yields the well-known fact that A is a Toeplitz operator if and only if $T_{\bar{z}}AT_z = A$, a Hankel operator if and only if $T_{\bar{z}}A = AT_z$. When these characterizing equalities are relaxed modulo $\mathcal{K}$, one has the *essentially* Toeplitz and Hankel operators, respectively.

The questions of characterizing *compact perturbations* of Toeplitz and respectively Hankel operators in $\mathcal{L}(H^2)$ are more involved and can not be based on matrix considerations. The essentially Toeplitz operators coincide with the essential commutant of $\mathcal{T}(C)$ and constitute a much larger class than the compact perturbations of Toeplitz operators. In the first edition of his book [49], which appeared in 1972, R. G. Douglas asked whether

$$T_{\bar{u}}AT_u - A \in \mathcal{K}, \ \forall \text{ inner function } u$$

implies $A = T_f + K$ for some $f \in L^\infty$, $K \in \mathcal{K}$? This long-standing open problem was solved in the affirmative by J. Xia [143] in 2009 (although a weaker version was solved by Davidson [45] in 1977). Since inner functions can be uniformly approximated by Blaschke products (Frostman's theorem), the condition is the same as if u exhausts all Blaschke products. Does the result remain true if u is further restricted to certain subsets of inner functions which are not uniformly dense?

Analogous to the Toeplitz case, the class of essentially Hankel operators is strictly larger [97] than that of the compact perturbations of Hankel operators. In view of the results above (especially Davidson's result), it seems reasonable to ask: If

$$T_{\tilde{b}}A - AT_b \in \mathcal{K},\ \forall \text{ interpolating Blaschke product } b, \tag{5.4.1}$$

or if

$$T_{\tilde{f}}A - AT_f \in \mathcal{K},\ \forall f \in QC, \tag{5.4.2}$$

does it follow that $A = H_g + K$ for some $g \in L^\infty$, $K \in \mathcal{K}$? Notice that (5.4.1) is the same as if $T_{\tilde{f}}A - AT_f \in \mathcal{K}, \forall f \in H^\infty$ by the Garnett-Nicolau theorem. Moreover, $T_{\bar{z}}A - AT_z \in \mathcal{K}$ implies $T_zA - AT_{\bar{z}} \in \mathcal{K}$, and these two conditions together give $T_{\tilde{f}}A - AT_f \in \mathcal{K}, \forall f \in C$. That is, (5.4.1) is indeed the same as if $T_{\tilde{f}}A - AT_f \in \mathcal{K}, \forall f \in H^\infty + C$ which subsumes (5.4.2). Conversely, Hartman's theorem ensures

$$T_{\tilde{f}}(H_g + K) - (H_g + K)T_f \in \mathcal{K},\ \forall f \in H^\infty + C.$$

Thus either implication, if true, would characterize compact perturbations of Hankel operators.

These questions may be investigated from the perspective of operator equations. For $f \in L^\infty$, consider the elementary operators $V_f, W_f \in \mathcal{L}(\mathcal{L}(H^2))$ acting on the operator Banach space $\mathcal{L}(H^2)$ as follows:

$$V_f(A) = T_{\bar{f}}AT_f - A$$
$$W_f(A) = T_{\tilde{f}}A - AT_f.$$

Then, the kernels of V_z, W_z are precisely the Toeplitz and Hankel operators, the pre-images $V_z^{-1}(\mathcal{K}), W_z^{-1}(\mathcal{K})$ the essentially Toeplitz and Hankel operators, and $V_z^{-1}(V_z(\mathcal{K})), W_z^{-1}(W_z(\mathcal{K}))$ the compact perturbations of Toeplitz and Hankel operators, respectively. Although $W_z(\mathcal{K})$ is uniformly dense in $\mathcal{K}$ [97], neither $V_z(\mathcal{K})$ nor $W_z(\mathcal{K})$ exhausts $\mathcal{K}$. Notice that the compact perturbation problems can be rephrased as: If $V_f(A) \in \mathcal{K}$ (resp. $W_f(A) \in \mathcal{K}$) for a natural class of functions $f \in L^\infty$ including $f(z) = z$, is it true that $V_z(A) \in V_z(\mathcal{K})$ (resp. $W_z(A) \in W_z(\mathcal{K})$)?

This formulation calls for an understanding of the compact operator subspaces $V_z(\mathcal{K}), W_z(\mathcal{K})$ relative to these conditions.

Another type of characterizations is in terms of convergence of certain operator sequences. [58, Theorem 4.1] states that A is a compact perturbation of a Toeplitz operator if and only if $T_{\bar{z}}^n A T_z^n$ converges in norm. This result is a special case of a general convergence theorem [100, Theorem 2.1]. Also, a mixed-type result [100, Corollary 4.5] asserts that A is a compact perturbation of a Toeplitz operator if and only if A is essentially Toeplitz *and* $n^{-1}\sum_{k=1}^n T_{\bar{z}}^k A T_z^k$ converges in norm, which is again a special case of [100, Theorem 4.3]. The question arises if similar results hold for compact perturbations of Hankel operators?

5.5 Hankel operators in certain C*-algebras and a problem of Barría and Halmos

Barría and Halmos [16] asked, on the Hardy space H^2, whether Hankel operators in the essential commutant $\mathcal{T}(C)'_e = \{T_z\}'_e$ must be in the Toeplitz algebra $\mathcal{T}(L^\infty)$? The answer turned out to be negative [26, Theorem 4.6], and several concrete constructions of $f \in QC$, $g \in L^\infty$, are known such that the commutators $[T_z, H_g] \in \mathcal{K}$, $[T_f, H_g] \notin \mathcal{K}$, so that $H_g \in \mathcal{T}(C)'_e$ but $H_g \notin \mathcal{T}(QC)'_e \supset \mathcal{T}(L^\infty)$. A naturally revised question would then ask if Hankel operators in $\mathcal{T}(QC)'_e$ must be in the Toeplitz algebra? The question is of interest for $\mathcal{T}(QC)'_e \supsetneq \mathcal{T}(L^\infty)$ [142]. Also notice that the characterization of $\mathcal{T}(QC)'_e$ in [142, Theorem 1.4] does not readily characterize the symbol functions of Hankel operators in $\mathcal{T}(QC)'_e$.

There is an equivalent formulation in terms of Douglas algebras. Define

$$B_T = \{f \in L^\infty : H_f \in \mathcal{T}(L^\infty)\},$$
$$B_{QC} = \{f \in L^\infty : H_f \in \mathcal{T}(QC)'_e\},$$
$$B_C = \{f \in L^\infty : H_f \in \mathcal{T}(C)'_e\},$$

and notice that inclusions between these Douglas algebras correspond to implications of membership of Hankel operators in the respective C*-subalgebras of $\mathcal{L}(H^2)$.

Proposition 5.5.1. $H^\infty + C \subsetneq B_T \subset B_{QC} \subsetneq B_C \subsetneq L^\infty$.

Proof. $\mathcal{T}(L^\infty) \subset \mathcal{T}(QC)'_e \subset \mathcal{T}(C)'_e$ gives $B_T \subset B_{QC} \subset B_C$. $H_f \in \mathcal{K} \subset \mathcal{T}(L^\infty)$ for all $f \in H^\infty + C$ gives $H^\infty + C \subset B_T$, and the inclusion is proper due to the construction in [27, Section 3] of a noncompact Hankel operator in $\mathcal{T}(L^\infty)$. B_{QC} is proper in B_C as in [26]. To show B_C is proper in L^∞, there exists $g \in L^\infty$ with $(\bar{z} - z)g \notin H^\infty + C$ because $H^\infty + C$ does not contain non-trivial ideals of L^∞ (Prop. 1.8.8), and $[T_z, H_g] \notin \mathcal{K}$ implies $g \notin B_C$. □

So, the revised Barría-Halmos question asks *if* $B_T = B_{QC}$? It follows from the Chang-Marshall theorem that B_T, B_{QC}, B_C are completely determined by the inner functions invertible in these algebras. In this regard, one has the following characterizations [147] for B_{QC} and B_C, while that for B_T seems beyond reach at this point.

Proposition 5.5.2. *Let $u \in H^\infty$ be an inner function. Then*

(i) $\bar{u} \in B_C$ *if and only if* $u|M_\lambda(L^\infty)$ *is constant for every* $\lambda \in \partial\mathbb{D} \setminus \pm 1$.

(ii) $\bar{u} \in B_{QC}$ *if and only if* $u|M_y(L^\infty)$ *is constant for every* $y \in M(QC) \setminus M^0_{\pm 1}(QC)$.

Proof. Since $\pi[T_f, H_{\bar{u}}] = \pi H_{(\tilde{f}-f)\bar{u}}$ for every $f \in QC$, $\bar{u} \in B_C$ (resp. B_{QC}) if and only if

$$(\tilde{f} - f)\bar{u} \in H^\infty + C, \ \forall f \in C \text{ (resp. } QC). \tag{5.5.1}$$

By the Bishop-Glicksberg theorem, the first case of (5.5.1) is equivalent to $(\tilde{f}-f)\bar{u}|M_\lambda(L^\infty) \in H^\infty|M_\lambda(L^\infty)$ for every $f \in C$ and $\lambda \in \partial\mathbb{D} \setminus \pm 1$, due to $\tilde{f}(\pm 1) = f(\pm 1)$ for $f \in C$. The latter is in turn equivalent to $\bar{u}|M_\lambda(L^\infty) \in H^\infty|M_\lambda(L^\infty)$ for every $\lambda \in \partial\mathbb{D} \setminus \pm 1$, that is, $u|M_\lambda(L^\infty)$ is constant by the Clancey-Gosselin property of such fibers $M_\lambda(L^\infty)$. This proves (i).

Again by the Bishop-Glicksberg theorem, the second case of (5.5.1) is equivalent to $(\tilde{f} - f)\bar{u}|M_y(L^\infty) \in H^\infty|M_y(L^\infty)$ for every $f \in QC$ and $y \in M(QC)\setminus M^0_{\pm 1}(QC)$, due to $\tilde{f}(y) = f(\bar{y}) = f(y)$ for $f \in QC$ and $y \in M^0_{\pm 1}(QC)$. The latter is equivalent to $\bar{u}|M_y(L^\infty) \in H^\infty|M_y(L^\infty)$ for every $y \in M(QC) \setminus M^0_{\pm 1}(QC)$, that is, $u|M_y(L^\infty)$ is constant by the Clancey-Gosselin property of such fibers $M_y(L^\infty)$. This proves (ii). □

Remark 5.5.3. Recall that for $f \in L^\infty$ and $\lambda \in \partial\mathbb{D}$, $f|M_\lambda(L^\infty)$ is constant if and only if f is continuous at λ. Thus for $u = bs$ where b is a Blaschke product with zero sequence $\{z_n\}_n$ in $\mathbb{D}$, and where s is a singular inner function with support set $S \subset \partial\mathbb{D}$ of its singular measure, the condition of Proposition 5.5.2(i) amounts to both b and s being continuous at every point of the circle other than ± 1. That is, $(\overline{\{z_n\}_n} \bigcap \partial\mathbb{D}) \bigcup S \subset \{\pm 1\}$ [147], which, in case $u = b$, reduces to [26, Prop. 3.5] (see also [15, Prop., p. 1508]), and in case $u = s$ amounts to

$$s(z) = \exp\left(r_1 \frac{z+1}{z-1}\right) \exp\left(r_2 \frac{z-1}{z+1}\right), \quad z \in \mathbb{D},\ r_1, r_2 \geq 0.$$

Also, it follows from Lemma 3.5.1 and the Clancey-Gosselin property that $u|M_\lambda(L^\infty)$ is constant if and only if $u|M_y(L^\infty)$ is so for every $y \in M_\lambda(QC)$, so that the added requirement in (ii) is precisely that $u|M_y(L^\infty)$ be constant for every $y \in M_{\pm 1}(QC) \setminus M^0_{\pm 1}(QC)$.

Specializing Proposition 5.5.2(ii) to interpolating Blasche products and singular inner functions, respectively, we have obtained the following results [147].

Theorem 5.5.4. *If b is an interpolating Blaschke product with zero sequence $\{z_n\}_n$ in $\mathbb{D}$, then $\bar{b} \in B_{QC}$ if and only if $\{z_n\}_n$ lies in the union of a Stolz angle at 1 and another at -1.*

Theorem 5.5.5. *The only singular inner functions invertible in B_{QC} are the constants.*

Note that Theorem 5.5.4 and the full strength of the Chang-Marshall theorem characterize B_{QC}, and that Theorem 5.5.5 equivalently states that the inner functions invertible in B_{QC} are all Blaschke products.

Interpolating sequences for certain subalgebras of H^∞ are characterized by Sundberg and Wolff [132]. For $QA := QC \bigcap H^\infty$ and a sequence $\{z_n\}_n$ in $\mathbb{D}$, their key result is that the map

$$f \in QA \mapsto (f(z_n))_n \in l^\infty$$

is surjective if and only if the sequence $\{z_n\}_n$ is distinct and sparse. Using this and other results, [27, Theorem 3] shows that for a class $\mathscr{S}$ of sparse Blaschke product b, $H_{\bar{b}}$ is a compact perturbation of a product of two Hankel operators and thus lies in the Toeplitz algebra $\mathcal{T}(L^\infty)$.

In view of distance localization in the Calkin C*-subalgebra $[\mathcal{TH}(L^\infty)]$ with respect to the center $[\mathcal{T}(QC_s)] \cong QC_s$, one would have an affirmative answer to the revised Barría-Halmos question if only one could approximate interpolating Blaschke products of the type as in Theorem 5.5.4 by sparse products in $\mathscr{S}$ *uniformly on each fiber* $M_y(L^\infty)$, $\forall y \in M^0_{\pm 1}(QC)$. Uniform approximation results on fibers of $M(L^\infty)$ over $M(QC)$ are seldom found in the literature.

5.6 Toeplitz operators on the Hardy space versus the Bergman space

The differences between Toeplitz operator theory on the Hardy space H^2 versus the Bergman space $A^2(\mathbb{D})$ are reflected in many well-known results. For instance, Beurling's classical theorem states that the wandering subspace $M \ominus T_z M$ of any nonzero invariant subspace M for the shift operator T_z on H^2 is spanned by an inner function, and that $M \ominus T_z M$ generates M as invariant subspace. While the latter remains true [4] for all invariant subspaces of the Bergman shift, their wandering subspaces can have arbitrary dimension [77]. The full Toeplitz algebra on H^2 properly contains its commutator ideal [49], but on $A^2(\mathbb{D})$ the two coincide [131]. Yet another example is that although the essential spectrum of every Toeplitz operator on the Hardy space is connected [49], there are ones on the Bergman space with disconnected essential spectra [133].

One the other hand, for $f \in L^\infty$ with harmonic extension $\hat{f} \in h^\infty(\mathbb{D})$, write T_f for the Toeplitz operator on H^2 and $T_{\hat{f}}$ the Toeplitz operator on $A^2(\mathbb{D})$. Then it is a natural question to ask about the relation between T_f and $T_{\hat{f}}$, although very few results of this kind, if any, are found in the literature. Let $f, g \in L^\infty$. Compact semicommutators and commutators of T_f, T_g are characterized respectively in [11, 139, 154] and [67], and those of $T_{\hat{f}}, T_{\hat{g}}$ are characterized respectively in [153] and [127]. Also see [152]. These characterizations appear in various equivalent forms, some of which are in terms of analyticity of L^∞ functions on support sets S_m for the Hardy space case, or analyticity of $h^\infty(\mathbb{D})$ functions on (Gleason) parts P_m for the Bergman space case. While the former for $f \in L^\infty$ always implies the latter for $\hat{f} \in h^\infty(\mathbb{D})$, the converse is false in general. Thus it is of interest to consider certain special classes of symbols for which the converse is in fact true.

Motivated by characteristic symbols and PQC symbols, let Λ be a Borel subset of $\partial\mathbb{D}$. The question is if $\hat{1}_\Lambda \equiv c \in [0,1]$ on a nontrivial part P_m in $M(H^\infty) \setminus \mathbb{D}$, must the constant $c \in \{0,1\}$? Note that since $\overline{P_m} \bigcap M(L^\infty) = \emptyset$, there does not appear to be an immediate affirmative answer.

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