Topics on Coherent Sheaves for Deligne-Mumford Stacks

Promit Kundu

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Committee members

Professor Yunfeng Jiang, Chairperson

Professor Purnaprajna Bangere, Co-Chair

Professor Satya Mandal

Professor Yuanqi Wang

Professor Kyoungchul Kong

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that this is the approved version of the following dissertation :

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Professor Yunfeng Jiang, Chairperson

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Abstract

The problems studied in this thesis are problems concerned with Vafa-Witten theory in Physics which gives a relationship between Physics and mathematics through a deep relationship called S-duality and providing new relationships between various invariants. This deep study intertwines number theory, arithmetic geometry and representation theory and algebraic geometry enabling us to recover beautiful interesting formulas. Sheaves on Deligne-Mumford stacks are related to this study [16] following a large body of work of several people in the related fields [33], [32].

In this thesis we mainly study two important problems concerned with sheaves on Deligne-Mumford stacks. We mainly study sheaves on smooth toric Deligne-Mumford stacks and Bogomolov-Gieseker inequality for modified semi-stable sheaves on tame smooth Deligne-Mumford projective stacks in any dimension.

In chapter 2 we recall the definitions and preliminaries on Deligne-Mumford stacks. Then we recall the important notion of semi-stability modified to the setting of projective Deligne-Mumford stacks. We restate the notions of modified slope and define semi-stability analogously. We impose a condition §4.1.2, with which we work in the rest of chapter 4 and chapter 5.

In chapter 3 we first study toric Deligne-Mumford stacks and torsion free toric sheaves on them. We give examples of toric Deligne-Mumford stacks, torus actions on Deligne-Mumford stacks and prove a gluing formula for torsion free toric sheaves on a Toric Deligne-Mumford stack generalising [20], [9], [36] on any arbitrary toric Deligne-Mumford stack.

In chapter 4 we generalize the Bogomolov-Gieseker inequality for semistable coherent sheaves on smooth projective surfaces to smooth Deligne-Mumford surfaces. We work over positive characteristic $p > 0$ and generalize Langer's method to smooth Deligne-Mumford stacks.

In chapter 5 we generalize the Bogomolov inequality formula to higher dimensions and to Simpson Higgs sheaves on tame Deligne-Mumford stacks.

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Contents

Chapter 1

Introduction

We study two important concepts on Sheaves on Deligne-Mumford stacks in this thesis. The problems we study have deep connections to Physics and have been useful in computing Vafa-Witten invariant [33],[32]. These ideas are very useful in proving powerful relations between generating functions and modularity properties connecting number theory and algebraic geometry.

We concentrate mainly on semi-stability and recall the setting as in [28]. We fix a condition and work with it as the assumption on generating sheaves in chapter 2 handy for uses in chapter four and five.

In chapter three we discuss about Smooth Toric Deligne-Mumford stacks which are very important algebro-geometric objects as their geometric properties translate into the combinatorics of their fan structure. They provide important testing grounds for various physical theories and conjectures motivated from Physics. For introduction to Smooth Toric Deligne-Mumford stacks we refer to [4],[8]. The Torus fixed locus is important in the context of computing DT invariants and Vafa-Witten invariants [35], serving as a motivation to investigate our case. We give conditions for a *Gluing Theorem* handy for our purposes to compute *equivariant Picard Groups*.

We obtain a *characteristic function* similar to [20] by defining the dimension of the fine graded pieces of each $\{F(m), m \in X(T)\}$, which will be instrumental in the construction of the pure equivariant sheaves with a fixed characteristic function (see [9]).

Extending the work of Klyachko, Perling, Kool, we develop a combinatorial method to describe the pure equivariant sheaves of any rank and on an arbitrary smooth projective Toric Deligne-Mumford stack of any dimension. Following the methods of [20], one can construct the moduli stack of equivariant pure sheaves on $X(\Sigma)$ with fixed characteristic function χ. The torus action extends to the moduli stack of Gieseker stable sheaves on the Deligne-Mumford stack with respect to a Modified Hilbert polynomial (i.e, w.r.t a generating sheaf).

For torsion free sheaves equivariant under torus action with fixed characteristic function, we hope to match GIT semi-stability and Gieseker semi-stability . This lets us have a stratification of the stable locus of torus fixed points of Gieseker stable sheaves in terms of the stable locus of torus fixed points with fixed characteristic function which are very combinatorial in nature. Using that we hope to obtain a combinatorial description of the torus fixed μ-modified stable torsion free coherent sheaves on the Deligne-Mumford stack purely in terms of moduli stacks of pure equivariant sheaves with fixed characteristic function.

Using this decomposition as in [20], we can compute generating functions of Euler characteristics of moduli spaces of modified μ- stable sheaves with respect to a modified Hilbert polynomial on smooth proper Toric Deligne-Mumford stacks (See [36] for analogous applications on smooth projective toric $Hirzebruch\ orbifolds$.).

In the fourth chapter we discuss about the $discriminant$ of a $coherent\ sheaf$ and conditions enabling its positivity. Let X be a smooth projective surface over an algebraically closed field κ of characteristic zero. The Bogomolov-Gieseker inequality is a famous formula for slope semistable torsion free coherent sheaves on X, which says that the discriminant $\Delta(E) = 2\,\mathrm{rk}(E)c_2(E) - (\mathrm{rk}(E)-1)c_1(E)^2 \geq 0$ if E is slope semistable. This formula has many important applications such as the construction of Bridgeland stability conditions for surfaces.

If the base field κ is of positive characteristic $p > 0$, and X is a smooth projective surface over κ. Let $F : X \to X$ be the absolute Frobenius morphism of X. A torsion free coherent sheaf E is called strongly slope semistable if any Frobenius pullback F^*E is slope semistable. For a strongly slope semistable torison free coherent sheaf E on X, the Bogomolov-Gieseker inequality

$$\Delta(E) = 2\,\mathrm{rk}(E)c_2(E) - (\mathrm{rk}(E)-1)c_1(E)^2 \geq 0$$

still holds, see [22, Theorem 3.2]. In general if E is just slope semistable, the inequality has a correction term depending on the prime number p, see [22, Theorem 3.3]. The Bogomolov inequality formula in positive characteristics has applications to prove the boundedness for the moduli functor of semistable coherent sheaves on X, and the restriction theorem of slope stable torsion free coherent sheaves on X to a divisor D inside X.

In this thesis we prove the Bogomolov-Gieseker inequality for slope semistable torsion free coherent sheaves on a smooth two dimensional Deligne-Mumford stack $\mathcal{X}$ (called a surface Deligne-Mumford stack). See for instance [17]. We work on tame surface Deligne-Mumford stacks which means that the orders of the local stabilizer groups of $\mathcal{X}$ are all coprime to the character p. We use the modified slope semistability of Nironi [28] defined by generating sheaves Ξ on $\mathcal{X}$. One motivation for our study on the Bogomolov-Gieseker inequality for slope semistable torsion free coherent sheaves is the Vafa-Witten theory for projective surfaces and surface Deligne-Mumford stacks in [33], [16], where the Bogomolov-Gieseker inequality for the modified semistable sheaf E will make the moduli space of Gieseker stable sheaves on a root stack surface $\mathcal{X}$ empty for $c_2(E) < 0$. The Vafa-Witten theory for surface Deligne-Mumford stacks has applications to prove the S-duality conjecture in [35] which is a functional duality for the generating functions counting $\mathrm{SU}(r)$ and ${}^{L}\mathrm{SU}(r) = \mathrm{SU}(r)/\mathbb{Z}_r$-instantons, see [14], [15]. On the other hand, the Bogomolov-Gieseker inequality for slope semistable torsion free coherent sheaves on a surface Deligne-Mumford stack $\mathcal{X}$ is interesting in its own since it will prove some restriction theorem of slope semistable sheaves on $\mathcal{X}$ to a large degree divisor inside $\mathcal{X}$. This will have applications to the reduction of the moduli of stable Higgs bundles on surfaces to the moduli space of stable Higgs bundles on curves, which is related to the Langlands duality and mirror symmetry between the moduli spaces of SL_r and PGL_r-Higgs bundles on curves.

Let us first state our main result. We fix $\mathcal{X}$ to be a surface Deligne-Mumford stack, and a polarization $(\Xi, \mathcal{O}_X(1) = H)$ where Ξ is a generating sheaf on $\mathcal{X}$ and $\mathcal{O}_X(1)$ a polarization on the coarse moduli space X. We choose the generating sheaf Ξ to satisfy the condition that its restriction to any codimension one component in $I\mathcal{X}_1$ is a direct sum of locally free sheaves of the same rank. Here $I\mathcal{X}_1 \subset I\mathcal{X}$ is the components in the inertia stack $I\mathcal{X}$ consisting of codimension one components. Then in this case the modified slope of a torsion free coherent sheaf E is given by:

$$\mu_{\Xi}(E) = \frac{\deg(E)}{\mathrm{rk}(\Xi)\,\mathrm{rk}(E)}$$

where $\deg(E) = c_1(E)\cdot H$. The modified slope semistability of a torsion free sheaf E is equivalent to the orbifold semistability of the surface Deligne-Mumford stack $\mathcal{X}$ where the slope is given by $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$.

We can not prove a Bogomolov-Gieseker inequality formula for a modified slope semistable torsion free sheaf E for a arbitrary general generating sheaf Ξ, where the modified slope $\mu_{\Xi}(E)$ is $\frac{\mathrm{rk}(\Xi)\deg(E)}{\mathrm{rk}(E)}$ plus the contributions from codimension one components $I\mathcal{X}_1 \subset I\mathcal{X}$. We do not know if such an inequality holds for general modified slope semistable torsion free sheaves, except that we can show it holds for root stacks and a special choice of generating sheaf. This has applications to the calculation of Vafa-Witten invariants for root stack surfaces in [16]. For a root stack $\mathcal{X}$ (associated to a pair (X,D) where $D \subset X$ is a normal crossing divisor) over a smooth surface X, a choice of generating sheaf will gives the equivalence between the category of coherent sheaves on $\mathcal{X}$ and the category of rational parabolic sheaves on (X,D).

Our main result is:

Theorem 1.0.1. *(Theorem 4.1.9) Let $\mathcal{X}$ be a two dimensional smooth tame projective Deligne-Mumford stack and let Ξ be a generating sheaf such that its restriction to every component in $I\mathcal{X}_1$ is a direct sum of locally free coherent sheaves of the same rank. Then if E is a strongly modified slope semistable torsion free sheaf on $\mathcal{X}$, we have*

$$\Delta(E) \geq 0.$$

We prove Theorem 1.0.1 by a method of Moriwaki which is generalized to surface Deligne-Mumford stacks, plus the calculation using orbifold Grothendieck-Riemann-Roch formula for the surface Deligne-Mumford stack. In order to prove the theorem we review the basic knowledge of coherent sheaves over a tame Deligne-Mumford stack $\mathcal{X}$, the Frobenius morphisms and the basic properties of slope modified semistable torsion free sheaves on $\mathcal{X}$. In particular we generalize one of Langer's inequality involving maximal and minimal modified slopes in the Harder-Narasimhan filtration of a torsion free sheaf E to smooth Deligne-Mumford stacks.

As the first application of the Bogomolov-Gieseker inequality in Theorem 1.0.1, the Bogomolov-Gieseker inequality for surface Deligne-Mumford stacks in characteristic zero holds by the standard method of taking limit. The second application is to prove the Bogomolov-Gieseker inequality for rational parabolic semistable torsion free sheaves on (X,D) by relating the rational parabolic sheaves on (X,D) to torsion free sheaves on the root stack $\mathcal{X}$ associated with (X,D).

We generalize Langer's results (the four theorems in [22, §3]) for higher dimensional smooth Deligne-Mumford stacks $\mathcal{X}$ for a special choice of generating sheaf such that the modified slope is equivalent to the orbifold slope of $\mathcal{X}$. We also provide one restriction theorem for slope stable torsion free sheaves to a large degree divisor in $\mathcal{X}$. The proof is similar to [22, §3], and we put these arguments in chapter five. The restriction theorem will have applications to the reduction of the moduli space of Higgs sheaves on a surface or a surface Deligne-Mumford stack to the moduli space of Higgs bundles to a large degree curve inside the surface or the surface Deligne-Mumford stack.

We also prove the Bogomolov inequality for semistable Simpson Higgs sheaves on tame Deligne-Mumford stacks in chapter five 5.2 generalizing the method in [24]. We should point out the Higgs sheaves here are not the Higgs sheaves in [33], [16], where the Bogomolov inequality may involve correction terms.

1.1 Outline

Here is the short outline for the thesis. In chapter two we review the definitions of stacks and definitions of modified slope stability for torsion free sheaves on a tame projective smooth Deligne-Mumford stack $\mathcal{X}$.

In chapter three we define Toric Deligne-Mumford stacks and discuss toric torsion free coherent sheaves on Toric Deligne-Mumford stacks. We prove a gluing formula analogous to [20] for arbitrary smooth toric Deligne-Mumford stack

In chapter four we prove the Bogomolov-Gieseker inequality formula Theorem 1.0.1 in §4.1, and the Bogomolov inequality for rational parabolic sheaves for (X,D). Finally in chapter five we generalize the results in [22, §3, §5] to higher dimensional smooth tame Deligne-Mumford stacks. We also prove the Bogomolov inequality for Simpson Higgs sheaves on tame Deligne-Mumford stacks in chapter five, 5.2.

1.2 Convention

In the third chapter we work on $\mathbb{C}$.

While working with a torus we use T to denote orbifold torus and $\mathcal{T}$ to denote a Deligne-Mumford torus. We mention the use of T for both the cases later in chapter three explicitly stating it. G denotes an affine algebraic group and $X(G)$ denotes the character group of G.

In the second fourth and fifth chapter, we work over an algebraically closed field κ of characteristic $p > 0$ throughout of the paper unless otherwise mentioned. All Deligne-Mumford stacks are tame. We denote by $\mathbb{G}_m$ the multiplicative group over κ.

Chapter 2

Preliminaries on stacks and modified stability

2.1 Preliminaries on stacks

2.1.1 Stacks

Definition 2.1.1. *Let C be a site, ([Sta] for definitions). A category $fibered\ in\ groupoids$, $p : F \mapsto C$ is a $stack$ if for every object $Y \in C$ and covering $\{f : X \mapsto Y\}$, the functor $F(Y) \mapsto F(\{f : X \mapsto Y\})$ is an equivalence of categories [Sta]. The functor can be described as $(E) \mapsto (f^*(E), \sigma_{can})$.*

2.1.2 Algebraic Stacks

Denote a scheme by S.

Definition 2.1.2. *A stack $\mathcal{X}/S$ is an $algebraic\ stack$ if the following are satisfied.*
i) The diagonal $\Delta : \mathcal{X} \mapsto \mathcal{X} \times_S \mathcal{X}$ is representable.
ii) There exists a smooth surjective morphism $\pi : Z \mapsto \mathcal{X}$ with Z a scheme.
Often we mention $Artin\ stacks$ to relate to $Algebraic\ Stacks$.

2.1.3 Quotient Stacks

Let X be a scheme, G be a smooth group scheme over $spec(\mathbb{C})$. Define [X/G] to be the stack whose objects are given by the following.

• Let T be a scheme over $spec(\mathbb{C})$.

Define triples $(T, P, \pi : P \to X \times T)$ where π is a $G_T - equivariant$ morphism of schemes and P is a $G_T - torsor$ on the big etale site of T.

• The morphisms between two triples (T, P, π) and (T', P', π') are given by a pair of morphisms $(f : T' \to T, f^{\vee} : P' \to f^* P)$ and $f^{\vee}$ being a compatible isomorphism of pullback of $G_{T'} - torsors$.

• Examples include $[X/G]$ where X is a scheme and G acts on X.

2.1.4 Smooth Algebraic Stacks

We say $\mathcal{X}$ is $smooth$ if there exists a smooth scheme Z as in condition ii) which is $smooth$.

• Examples of main interest include:

• Smooth Artin stacks of the form $[X/G]$ where X is a smooth quasi-projective scheme and G is an affine algebraic group in characteristic 0, with a G action on X.

2.1.5 Deligne-Mumford Stacks

We say $\mathcal{X}$ is $Deligne - Mumford$ if the following is satisfied.

• In condition ii) we assume the π to be an etale covering.

• Examples of interest include the following.

• Global Quotient stacks of the form $[X/G]$.

• In particular, let X be a Noetherian scheme of finite type and let smooth G be acting on X with finite reduced stabilisers then $[X/G]$ is Deligne-Mumford.

2.1.6 Tame projective Deligne-Mumford stacks

Let $\mathcal{X}$ be a $smooth\ DM\ stack$ over an algebrically closed field k, with $finite\ diagonal$, which is equipped with a $projective\ coarse\ moduli$ which is a $scheme$ denoted by (X, π). Such stacks are described as $projective\ stacks$.

If $\pi_* : Qcoh(\mathcal{X}) \mapsto Qcoh(X)$ is exact then we say $\mathcal{X}$ is a $tame\ DM\ stack$.

• Examples of interest include *Root Stack* $\mathcal{X}$ where $gcd(r,p) = 1$ on a field of $char\ p$, obtained from $r-th$ root construction.

2.2 Preliminaries on modified stability

In this section we review the modified semistability for Deligne-Mumford surfaces in characteristic p. Using these some new results of Shapherd-Barron for Deligne-Mumford surfaces are proved in chapter 4.

2.2.1 Notations

We fix some notations for a smooth projective Deligne-Mumford stack $\mathcal{X}$ of dimension d. The Deligne-Mumford stack $\mathcal{X}$ is called *tame*, if the stabilizer groups of the Deligne-Mumford stack $\mathcal{X}$ are all linearly reductive groups. Equivalently this means the order of the stabilizer group at any geometric point of $\pi : \mathcal{X} \to X$ is relatively prime to p.

Let $\mathcal{I}$ be the index set of the components of the inertia stack $I\mathcal{X}$ such that

$$I\mathcal{X} = \bigsqcup_{g\in\mathcal{I}} \mathcal{X}_g.$$

We always use $\mathcal{X}_0 = \mathcal{X}$ to represent the trivial component. For example if $\mathcal{X} = [Z/G]$ is a global quotient stack, where Z is a quasi-projective scheme and G is a finite group acting diagonally on Z, then $I\mathcal{X} = \bigsqcup_{(g)}[Z^g/C(g)]$. Any component $\mathcal{X}_g \subset \mathcal{X}$ in the inertia stack $I\mathcal{X}$ is a closed substack of $\mathcal{X}$. We denote by $I\mathcal{X}_1 \subset I\mathcal{X}$ be the substack of $I\mathcal{X}$ consisting of components $\mathcal{X}_g$ such that their codimension in $\mathcal{X}$ is one. Let $\mathrm{pr} : I\mathcal{X} \to \mathcal{X}$ be the map from the inertia stack $I\mathcal{X}$ to $\mathcal{X}$.

For $\mathcal{X}$, we write

$$H^*_{\mathrm{CR}}(\mathcal{X}) = H^*(I\mathcal{X}) = \bigoplus_{g\in\mathcal{I}} H^*(\mathcal{X}_g)$$

to be the Chen-Ruan cohomology with $\mathbb{Q}$-coefficients. For any torsion free coherent sheaf E on $\mathcal{X}$, we use $c_i(E)$ to represent the Chern classes of E on $\mathcal{X}$, and $c_i(E) \in H^{2i}(\mathcal{X})$.

On the component $\mathcal{X}_g \subset I\mathcal{X}$, at a point $(x, g) \in \mathcal{X}_g$, let

$$T_x\mathcal{X} = \bigoplus_{0 \le f < 1} (T_x\mathcal{X})_{g,f}$$

be the eigenspace decomposition of $T_x\mathcal{X}$ with respect to the stabilizer action and g acts on $(T_x\mathcal{X})_{g,f}$ by $e^{2\pi i f}$.

Let $E \in \mathrm{Coh}(\mathcal{X})$ be a coherent sheaf on $\mathcal{X}$, we have an eigenbundle decomposition of $\mathrm{pr}^* E$ and on $\mathrm{pr}^* E|_{\mathcal{X}_g}$ we have

$$\mathrm{pr}^* E|_{\mathcal{X}_g} = \bigoplus_{0 \le f < 1} (\mathrm{pr}^* E)_{g,f}$$

with respect to the action of the stabilizer of $\mathcal{X}_g$, where the element g acts on $(\mathrm{pr}^* E)_{g,f}$ by $e^{2\pi i f}$. Then the orbifold Chern character is:

$$\widetilde{\mathrm{Ch}}(E) = \bigoplus_{g \in \mathcal{I}} \sum_{0 \le f < 1} e^{2\pi i f} \, \mathrm{Ch}((\mathrm{pr}^* E)_{g,f}), \tag{2.1}$$

where Ch is the general Chern character. Let $l_{g,f}$ be the rank of $(\mathrm{pr}^* E)_{g,f}$. The orbifold Todd class of $T\mathcal{X}$ is given by

$$\widetilde{\mathrm{Td}}(T\mathcal{X}) = \bigoplus_{g \in \mathcal{I}} \prod_{\substack{0 \le f < 1 \\ 1 \le i \le r_{g,f}}} \frac{1}{1 - e^{-2\pi i f} e^{-x_{g,f,i}}} \prod_{f=0} \frac{x_{g,0,i}}{1 - e^{-x_{g,0,i}}}, \tag{2.2}$$

where $(\mathrm{pr}^* T\mathcal{X})_{g,f}$ has rank $r_{g,f}$ and $x_{g,f,i}$ are Chern roots.

For any coherent sheaf E on $\mathcal{X}$, orbifold Riemann-Roch theorem [34] gives:

$$\chi(\mathcal{X}, E) = \int_{I\mathcal{X}} \widetilde{\mathrm{Ch}}(E) \cdot \widetilde{\mathrm{Td}}(T\mathcal{X}). \tag{2.3}$$

2.2.2 Modified stability

Let $\mathcal{X}$ be a smooth tame projective Deligne-Mumford stack of dimension d. We choose the polarization $\mathcal{O}_X(1)$ on its coarse moduli space $\pi : \mathcal{X} \to X$. Let $H := c_1(\mathcal{O}_X(1))$. Recall from [28],

Definition 2.2.1. *A locally free sheaf Ξ on $\mathcal{S}$ is p-very ample if for every geometric point of $\mathcal{S}$ the representation of the stabilizer group at that point contains every irreducible representation of the stabilizer group. We call Ξ a generating sheaf.*

Let Ξ be a locally free (generating) sheaf on $\mathcal{X}$. We define a functor

$$F_\Xi : D\mathrm{Coh}_{\mathcal{X}} \to D\mathrm{Coh}_X$$

by

$$F \mapsto \pi_* \mathscr{H}om_{O_{\mathcal{X}}}(\Xi, F)$$

and a functor

$$G_\Xi : D\mathrm{Coh}_X \to D\mathrm{Coh}_{\mathcal{X}}$$

by

$$F \mapsto \pi^* F \otimes \Xi.$$

From [30], the functor F_Ξ is exact since the dual $\Xi^\vee$ is locally free and the pushforward p_* is exact. The functor G_Ξ is not exact unless p is flat. For instance, if p is a flat gerbe or a root stack, it is flat.

Let us fix a generating sheaf Ξ on $\mathcal{X}$. We call the pair $(\Xi, O_S(1))$ a polarization of $\mathcal{X}$. Let E be a coherent sheaf on $\mathcal{X}$, we define the support of E to be the closed substack associated with the ideal

$$0 \to \mathcal{I} \to O_{\mathcal{X}} \to \mathscr{E}nd_{O_{\mathcal{X}}}(F).$$

So $\dim(\mathrm{Supp}\, F)$ is the dimension of the substack associated with the ideal $\mathcal{I} \subset O_{\mathcal{X}}$ since $\mathcal{X}$ is a Deligne-Mumford stack. A pure sheaf of dimension d is a coherent sheaf E such that for every non-zero subsheaf E' the support of E' is of pure dimension d. For any coherent sheaf E, we have the torsion filtration:

$$0 \subset T_0(E) \subset \cdots \subset T_l(E) = E$$

where every $T_i(E)/T_{i-1}(E)$ is pure of dimension i or zero, see [11, §1.1.4].

Definition 2.2.2. *The modified Hilbert polynomial of a coherent sheaf E on $\mathcal{X}$ is defined as:*

$$H_\Xi(E,m) = \chi(\mathcal{X}, E \otimes \Xi^\vee \otimes \pi^* O_X(m)) = H(F_\Xi(E)(m)) = \chi(X, F_\Xi(E)(m)).$$

Let E be of dimension l, then we can write:

$$H_\Xi(E,m) = \sum_{i=0}^{l} \alpha_{\Xi,i}(E)\frac{m^i}{i!}$$

which is induced by the case of schemes. The modified Hilbert polynomial is additive on short exact sequences since the functor F_Ξ is exact. If we don't choose the generating sheaf Ξ, the Hilbert polynomial H on $\mathcal{X}$ will be the same as the Hilbert polynomial on the coarse moduli space S. The *reduced modified Hilbert polynomial* for the pure sheaf E is defined as

$$h_\Xi(E) = \frac{H_\Xi(E)}{\alpha_{\Xi,d}(E)}.$$

Let E be a pure coherent sheaf. We call E Gieseker semistable if for every proper subsheaf $E' \subset E$,

$$h_\Xi(E') \leq h_\Xi(E).$$

We call E stable if $\leq$ is replaced by $<$ in the above inequality.

Definition 2.2.3. *([28, Definition 3.13]) We define the slope of E by*

$$\mu_\Xi(E) = \frac{\alpha_{\Xi,l-1}(E)}{\alpha_{\Xi,l}(E)}.$$

Then E is modified slope (semi)stable if for every proper subsheaf $F \subset E$,

$$\mu_\Xi(F)(\leq) < \mu_\Xi(E).$$

The notion of μ-stability and semistability is related to the Gieseker stability and semistability in the same way as schemes, i.e.,

$$\mu-\text{stable} \Rightarrow \text{Gieseker stable} \Rightarrow \text{Gieseker semistable} \Rightarrow \mu-\text{semistable}$$

Remark 2.2.4. *Recall that $I\mathcal{X}_1 \subset I\mathcal{X}$ is the substack of $I\mathcal{X}$ consisting of components such that the codimension of $\mathcal{X}_g \subset \mathcal{X}$ is one. If our Deligne-Mumford stack $\mathcal{X}$ is a global quotient stack, which means $\mathcal{X} = [Z/G]$ where Z is a quasi-projective scheme and G is a group scheme acting diagonally. Assume that we can choose the generating sheaf Ξ on $\mathcal{X}$ such that its restriction on any component in $I\mathcal{X}_1$ is a sum of locally free sheaves of the same rank. If the sheaf E has dimension d, then [28, Proposition 3.18] shows that*

$$\deg_{\Xi}(E) = \frac{1}{\text{rk}(\Xi)}\alpha_{\Xi,d-1}(E) - \frac{\text{rk}(E)}{\text{rk}(\Xi)}\alpha_{\Xi,d-1}(O_{\mathcal{X}}).$$

Here $\alpha_{\Xi,d-1}(E) = \int_{\mathcal{X}} c_1(E)\cdot H$.

Chapter 3

Toric Deligne-Mumford Stacks

3.1 Toric Deligne-Mumford stacks

Toric Deligne-Mumford stacks are generalisations of smooth toric varieties. We recall the construction of the Toric Deligne-Mumford stacks after reviewing basic materials on group actions [4].

3.1.1 Algebraic Group Actions on Deligne-Mumford stacks

Let $\mathcal{X}$ be a $Deligne-Mumford$ stack and let G be an algebraic group variety acting linearly on $\mathcal{X}$. We skip the definitions as they can be recalled from [9].

We mention that apart from the group acting on a stack $\phi : G \times \mathcal{X} \to \mathcal{X}$, we have two natural transformations given by:

$$\alpha : \phi \circ (id_G \times \phi) \to \phi \circ (\mu \times id_{\mathcal{X}})$$

and

$$\alpha : \phi \circ (e \times id_{\mathcal{X}}) \to id_{\mathcal{X}}$$

where μ is the group multiplication satisfying various compatibility conditions [9].

3.1.2 Definition and Construction using Gale Duality

Let $\mathcal{X}$ be a Deligne-Mumford stack.

Let $\phi : A_0 \to A_1$ be a group homomorphism of finitely generated abelian groups with free $kernel$ and finite $cokernel$.

Applying the contravariant functor $Hom(\ ,\mathbb{C}^*)$ one obtains a group homomorphism given by:

$$[\phi_G : G_{A_1} \to G_{A_0}].$$

One can associate an algebraic stack ($Picard\ Stack$) to this construction given by $[G_{A_0}/G_{A_1}]$. If ϕ has finite $cokernel$ then this is a Deligne-Mumford stack. Let us recall the notion of $Deligne-Mumford\ Tori$, which will be denoted as $Deligne-Mumford\ Tori$ [8].

Definition 3.1.1. *A Deligne-Mumford torus T is a Picard Stack over* $spec(\mathbb{C})$ *which is obtained as a quotient* $[G_{A_0}/G_{A_1}]$, *where* ϕ *is the above mentioned group homomorphism with free kernel and finite cokernel.*

Example 3.1.2. *Consider* $\phi : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}_6$ *given by* $\begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$. *One can compute the* $Deligne-Mumford$ *torus to be isomorphic to* $[(\mathbb{C}^*)^2/\mathbb{C}^* \times \mu_6] \cong \mathbb{C}^* \times B\mu_2$.

Remark 3.1.3. *Deligne Mumford tori of dimension n is always of the form* $T \times BG$ *where G is a finite abelian group and* $T = (\mathbb{C}^*)^n$ *is the coarse moduli space of the Deligne-Mumford torus. A Deligne-Mumford torus acts on itself by the multiplication map obtained from the Picard stack structure.*

.

Definition 3.1.4. *A smooth toric Deligne Mumford stack is a smooth separated Deligne Mumford stack X together with an open immersion of a Deligne-Mumford torus* $i : T \to \mathcal{X}$. *with dense image such that the action of T on itself extends to an action* $a : T \times \mathcal{X} \to \mathcal{X}$.

To a $Toric\ Deligne-Mumford\ stack$ one can associate a $stacky-fan$ (N,Σ,β) as follows [8]:

Let (N,Σ,β) be a triple consisting of.

• a finitely generated abelian group N of rank d.

• a rational simplicial fan Σ inside $\mathbb{Q}\otimes N$, consisting of n one-dimesnsional cones called rays and l top dimensional cones.

Let $\{\rho_i\}_{i=1(1)n}$ be the $rays$ given by integral lattice points v_i such that the rays span $N\otimes\mathbb{Q}$.

• $\beta:\mathbb{Z}^n\to N$ consisting of finite cokernel and free kernel.

The element $\widetilde{\beta(e_i)}$ in $N\otimes\mathbb{Q}$ is the image of $\beta(e_i)$ in $N\otimes\mathbb{Q}$ and is on the ray ρ_i where $(e_1,..,e_n)$ is the canonical basis for $\mathbb{Z}^n$, and the natural map $N\to N\otimes\mathbb{Q}$ sends $m\to m\otimes 1$.

Choose a projective resolution of N with two terms that is given below.

$$0\to\mathbb{Z}^r\to\mathbb{Z}^{d+r}\to N\to 0.$$

The first map is given by a matrix Q resolving the torsion part of N.

Let $B:\mathbb{Z}^n\to\mathbb{Z}^{d+r}$ be a lift of β.

Consider $[BQ]:\mathbb{Z}^{n+r}\to\mathbb{Z}^{d+r}$.

Consider, $DG(\beta):=coker([BQ]^*)$

Denote the composition mentioned below:

$$0\to(\mathbb{Z}^n)^*\to(\mathbb{Z}^{n+r})^*\to DG(\beta)$$

which we denote as:

$$(\beta)^\vee:(\mathbb{Z}^n)^*\to DG(\beta). \tag{3.1}$$

as the $Gale\ Dual$ of β. Now consider $Z_\Sigma := (\mathbb{C})^n - V(J_\Sigma)$, where $J_\Sigma = (\prod_{\rho_i\not\subset\sigma} z_i|\sigma\in\Sigma)$ and $\mathbb{C}[z_1,z_2,..,z_n]$ is the affine-co-ordinate ring of $\mathbb{C}^n$.

Applying $Hom(,\mathbb{C}^*)$ to the above, one obtains

$$G_\beta : G_\Sigma \to (\mathbb{C}^*)^n$$

where, $(\mathbb{C}^*)^n$ acts on Z_Σ by diagonal action.

We define the quotient stack $\mathcal{X} := [Z_\Sigma/G_\Sigma]$ the $Toric\ Deligne-Mumford\ stack$ associated to the triple (N,Σ,β).

From [4] we know that $\mathcal{X}$ is a smooth, Deligne-Mumford stack with coarse moduli space the $toric\ variety$ $X(\Sigma)$ associated to $\mathcal{X}(\Sigma) := \mathcal{X}$.

Example 3.1.5. • *$\mathcal{X}$ is a toric orbifold if and only if N is free.*

• *Consider $\beta : \mathbb{Z}^2 \to \mathbb{Z}$ given by (m,n) with $(m,n) = 1$. We obtain $[(\mathbb{C})^2 - 0/(\mathbb{C})^*]$ as the $Toric\ Deligne-Mumford\ stack$ associated to $(\mathbb{Z},\Sigma,\beta)$ where Σ is the rational simplicial fan formed by two rays given by $m,-n \in \mathbb{Z}$. This is a toric orbifold with $(\mathbb{C})^*$ acting by (m,n).*

3.1.3 GIT data

Here we mention the construction of the Toric Deligne-Mumford stack from a $GIT-data$ [7].

We associate a $GIT-data$ as follows.

• Let $K \cong (\mathbb{C})^*$ be a connected torus of rank r.

• Let $L := Hom((\mathbb{C})^*, K)$ be the $co-character\ lattice$.

• Let $D_1, D_2, .., D_m \in L^\vee \cong Hom(K,(\mathbb{C})^*)$.

The characters define a map from K to $(\mathbb{C})^*$ and hence defines a map to the torus $(\mathbb{C})^>$ and hence defines an action on $(\mathbb{C})^m$.

For a subset $I \subset \{1,2,..,m\}$ denote the set complement of I by $\bar{I}$.

Then we denote

$$L_I := \{\sum_{i\in I} a_i D_i, a_i \in R, a_i > 0\} \subset L^\vee \otimes R,$$

$$(\mathbb{C}^*)^I \times (\mathbb{C})^{\bar{I}} = \{(z_1,z_2,.,z_m), z_i \neq 0\ for\ i \in I\} \subset (\mathbb{C})^m.$$

Set, $L_\phi := \{0\}$.

We define the following objects in order to associate a toric Deligne-Mumford stack to the following data. Fix a stability condition $w \in L^\vee \otimes R$. Set,

Definition 3.1.6.

$$A_w := \{I \subset (1,2,..,m) : w \in L_I\},$$

$$U_w := \bigcup_{I \in A_w} (\mathbb{C}^*)^I \times (\mathbb{C})^{\bar{I}},$$

$$X_w := [U_w/K].$$

Note that U_w is invariant under K and hence one can form the stack quotient denoted by X_w. We denote X_w to be the associated toric stack associated to the data $(K, L, D_1, .., D_m, w)$. with particular assumptions on A_w and I [7] one can ensure that X_w is a Deligne-Mumford stack.

Let $S \subset \{(1,2,3..,m)\}$ denote the set of indices i such that $\{1,2,.,m\} - \{i\} \notin A_w$. The characters $\{D_i : i \in S\}$ are linearly independent and every element of A_w contains $S \subset A_w$.

Thus one can obtain,

$$A_w = \{I \coprod S : I \in A_{w'}\},$$

$$U_w \cong U_{w'} \times (\mathbb{C}^*)^{|S|},$$

where we have,

$$A_{w'} = 2^{\{1,2.,m\}-\{S\}}$$

and

$$U_{w'} \subset (\mathbb{C})^{m-|S|}.$$

Finally one obtains:

$$X_w = [U_{w'}/G],$$

where $U_{w'}$ is an open subset of $(\mathbb{C})^{m-|S|}$ and

$$G = ker\{K \longrightarrow (\mathbb{C}^*)^{|S|}\}.$$

Compare this with the construction using Gale duality discussed here and in [4].

3.1.4 Example

Consider, the following as in the previous discussion. Then,

$$K = (\mathbb{C})^*,\ D_1 = D_2 = 2, w = 1 \in R.$$

One obtains the weighted projective stack denoted as $P(2,2)$ as the stack quotient X_w [7].

3.1.5 Extended Stacky Fans

An $S - extended$ stacky fan consists of the following data.

Definition 3.1.7. *An $S - extended$ stacky fan consists of the following $\Sigma := (N, \Sigma, \beta, S)$.*

• *Let N be a finitely generated abelian group.*

• *Let Σ be a rational simplicial fan in $N \otimes R$.*

• *Let $\beta : (\mathbb{Z})^m \longrightarrow N$, where we denote β_i and $\widetilde{\beta_i}$ to be the images of canonical bases of $(\mathbb{Z})^m$ under β in N and $N \otimes R$ respectively.*

• *Let $S \subset \{1, 2, ., m\}$ with the following conditions.* • *Each one-dimensional cone of Σ is spanned by $\widetilde{\beta_i}$ for a unique $i \in \{1, 2., m\} - S$, each spanning a one-dimensional cone of Σ.*

• *For each $i \in S$, we have $\widetilde{b_i}$ lies in the support of support of Σ which we denote as $|\Sigma|$.*

The vectors $b_i, i \in S$ are called $extended\ vectors$. Taking $S = \phi$, one obtains the Toric Deligne-Mumford stack of [4].There is a one-one correspondence between GIT data and Extended stacky fans. For the one-one correspondence we site [7]. For extended stacky fans we refer to [13].

3.1.6 Inertia stack of a Toric Deligne-Mumford Stack

So, from now on let us fix a $stacky-fan$ (N,Σ,β). Let us denote see [4]

Definition 3.1.8. *The* $Inertia\ stack$ $(I\mathcal{X})$ *of* $\mathcal{X}$ *as:*

$$\coprod_{v\in Box(\mathcal{X})} \mathcal{X}(\Sigma/\sigma(\widetilde{v}))$$

where $\sigma(\widetilde{v})$ *is the minimal cone containing* v.

Define,

$$Box(\mathcal{X}) = \bigcup_{dim(\sigma)=d,\sigma\in\Sigma} Box(\sigma)$$

σ *being a top dimensional cone of dimension* d.

For each d dimensional cone σ *we define*

$$Box(\sigma) = \{v \in N|\widetilde{v} := v\otimes 1 = \sum_{\rho_i\subset\sigma} q_i\widetilde{b_i}, q_i \in [0,1)\}$$

where $N \to N\otimes\mathbb{Q}$ *given by* $v \to \widetilde{v} := v\otimes 1$ *is the association and* $\widetilde{b_i}$ *is the image of* $\beta(i) \in N\otimes Q$ *and lies on the ray* ρ_i.

3.1.7 Open substacks of a Toric Deligne-Mumford stack

Here we mention the $open\ substack$ corresponding to a $top\ dimensional\ cone\ \sigma$ is given by

$$\mathcal{U}_\sigma := [(\mathbb{C})^d/(G_\sigma)]$$

where $\beta_\sigma : (\mathbb{Z})^d \to (\mathbb{Z})^d$ given by the $d\times d$ matrix formed by the columns of β, and $G_\sigma := Hom(DG(\beta_\sigma),(\mathbb{C})^*) \cong N_\sigma$.

We define $N_\sigma := N/N\sigma$, where $N\sigma$ is the lattice generated by β_i ith column of the matrix B.

We denote the

$$\mathcal{U}_{\sigma_1,\sigma_2,..,\sigma_l} := U_{\sigma_1} \times_{\mathcal{X}} U_{\sigma_2}, .. \times_{\mathcal{X}} U_{\sigma_l}$$

is the $Deligne - Mumford\ Tori\ T$ which is the dense open substack of $\mathcal{X}$, extending the action of itself on the whole $\mathcal{X}$, l being the number of $top\ dimensional\ cones$ in Σ.

Example 3.1.9. *Let us the take the example of* $\mathbb{P}(1,2)$. *The global quotient description gives us an action of* $(\mathbb{C})^*$ *on* $(\mathbb{C})^2 - (0)$ *given by*

$$(t,(x,y)) = (tx, t^2 y).$$

On further investigation one finds the two open substacks correspond to $\{(2),(-1)\}$ *are given by* $[(\mathbb{C})/\mu_2]$ *and* $(\mathbb{C})$ *respectively.* $Deligne - Mumford\ tori$ *in this case is given by* $(\mathbb{C})^*$. *This is an example of a* $toric\ orbifold$.

3.1.8 Torus Actions

Let $T \cong (\mathbb{C}^*)^d$ be the torus of dimension d and assume it acts linearly on $(\mathbb{C})^d$. By $linear$ action we mean that there exists $co - ordinates$ given by $x_1, x_2, .., x_d$ and $characters$ given by $\chi(m_1), \chi(m_2), ..\chi(m_d)$ such that

$$\lambda.x_i = \chi(m_i)(\lambda)x_i$$

for all $i = 1, 2, .., d$ and $\lambda \in T$. We say T acts $non\ degenerately$ if $\chi(m_i)$ where $i = 1(1)d$ forms a linearly independent set in $X(T) = (\mathbb{Z})^d$. Moreover if this set does not span $X(T)$ then we have a non-$primitive$ action of T on $(\mathbb{C})^d$.

From now on, T means the $Deligne - Mumford\ tori$, we stick to the use of torus to mean $Deligne - Mumford\ Tori$.

In order to uniquely represent the elements of $X(T)$ we introduce the notion of $box\ B_T$. Each top dimensional $cone$ corresponds to a maximal open substack in $\mathcal{X}$ and let $\chi(m_i)$ be the elements in $X(T)$ that generates the non-degenerate, non-primitive action. We define

$$B_T(\sigma) := \{v \in (\mathbb{Z})^d | v = \sum_{i=1}^{d} q_i \chi(m_i) | q_i \in [0,1)\}.$$

Thus we have:

$$a = b + \sum_{i=1}^{d} l_i \chi(m_i)$$

$a, b \in X(T),\ l_i \in \mathbb{Z}$. Without loss of context we mention $\chi(m_i)$ for m_i interchangeably.

Remark 3.1.10. *We highlight that $Box(\sigma)$ and $Box_T(\sigma)$ may or may not be the same. Take*

$$\beta : (\mathbb{Z})^{n+1} \to (\mathbb{Z})^n$$

given by $\begin{pmatrix} 3 & -2 & 0 \\ 0 & 2 & -1 \end{pmatrix}$. *The $kernel$ is given by* $(2,3,6)\mathbb{Z}$.
On computing the $Gale\ Dual$ of β one obtains:

$$\beta^{\vee} : (\mathbb{Z})^{n+1} \to \mathbb{Z}$$

and hence obtains the map given by $(2,3,6)$. *We can associate the stack* $\mathbb{P}(2,3,6)$ *to this construction. $T-weight$ for each cone can be found out to be* $\{(3,0),(0,6)\}, \{(0,-6),(3,-3)\}, \{(-2,0),(-6,6)\}$. *Each of these $B_T(\sigma)$ have $cardinality$ larger than 3. But cardinality of $Box(\sigma)$ is 3 where σ is given by* $\{(3,0),(0,-1)\}$.

Let us assume that $\phi : G \times \mathcal{X} \to \mathcal{X}$ be an action on $\mathcal{X}$.
Given a sheaf F on $\mathcal{X}$ we want to understand a $G-equivariant$ structure associated to F denoted by (F, Φ)

Definition 3.1.11. *Let F be a quasi-coherent sheaf on $\mathcal{X}$. Denote by $p_2 : G \times X \to \mathcal{X}$ and $p_{23} : G \times G \times X \to G \times X$ projection on the second and last two factors. A $G-equivariant$ structure on F is an $isomorphism$*

$$\Phi : \sigma^*(F) \to p_2^*(F)$$

satisfying the following $co-cycle$ condition:

$$p_{23}^*\Phi \circ (id_G \times \sigma)^*\Phi = (\mu \times id_{\mathcal{X}})^*\Phi$$

where, μ is the multiplication in G.

Assume G and H are $diagonalisable\ algebraic\ groups$. with $character$ groups denoted by $X(G)$ and $X(H)$ respectively. Let $\mathcal{X}$ be a given global quotient Deligne-Mumford stack i.e given by $[X/H]$ where H acts on X with reduced and finite stabilisers and X is a $scheme$. Then $[X/H]$ is a Deligne-Mumford-stack.

Any G action on X commuting with the action of H induces a canonical action on Deligne-Mumford stack $\mathcal{X}$.

3.2 Coherent sheaves on Toric Deligne-Mumford Stacks

Let F be a coherent sheaf on $[X/H]$. See [9] for the next proposition.

Proposition 3.2.1. *The category of quasi-coherent sheaves on $[X/H]$ is equivalent to the category of $H-equivariant$ coherent sheaves on X.*

We come to the problem of describing $G-equivariant$ sheaves on $[X/H]$ where G action commutes with H. Then,

Proposition 3.2.2. *The category of $G-equivariant$ sheaves on $[X/H]$ correspond to the category of coherent sheaves on X with commuting $G,H-equivariant$ structures.*

Now, we look at the previous setting with $X = spec(R)$ where R is a $commutative\ Noetherian\ ring$, equipped with a $G,H-equivariant$ action. Then the action can be extended to a linear G,H commuting action on $H^0(X,F)$. Then

$$H^0(X,F) = \bigoplus_{\chi \in X(G)} H^0(X,F)_\chi$$

yielding a $X(G)$ graded decomposition into $G-eigenspaces$. Each $H^0(X,F)_\chi$ has an induced H action and can be further written down into $H-eigenspaces$:

$$H^0(X,F)_\chi = \bigoplus_{\psi\in X(H)} H^0(X,F)_{\chi,\psi}$$

The $functor\ H^0(X,.)$ gives a decomposition of a $G,H-equivariant$ coherent sheaf on $spec(R)$ into:

$$H^0(X,F) = \bigoplus_{\chi\in X(G)} \bigoplus_{\psi\in X(H)} H^0(X,F)_{\chi,\psi}.$$

Proposition 3.2.3. *Let $\mathcal{X} = [spec(R)/H]$, then the category of quasi-coherent sheaves on $\mathcal{X}$ is equivalent to $X(G),X(H)-$ $graded\ R\ modules$.*

3.2.1 S-family of Toric Sheaves on $[(\mathbb{C})^d/H]$

We fix a toric Deligne-Mumford stack $\mathcal{X}$ and T acts on it $regularly$. If F is a $T-equivariant$ sheaf on $\mathcal{X}$ then we denote it as a $toric\ sheaf$. From the previous discussion as in [9], [20], we obtain a description of a coherent $toric\ sheaf$ in terms of vector spaces and morphisms between them. Let T act on $(\mathbb{C})^d$ non-degenerately, then for $H^0((\mathbb{C})^d,O_{\mathbb{C}^d})$ we obtain its decomposition into $eigenspaces$ with $dimension$ 1 and 0 otherwise.

Let $S_T = \{m \in X(T)|H^0((\mathbb{C})^d,O_{\mathbb{C}^d})_m \neq 0\}$ be the set of $weights$ for which the $weight\ spaces$ are non-zero. S_T forms a semi-group in $X(T)$.

If T acts with the abovesaid characters m_i then S_T is generated by $\{m_i\}_{i=1(1)d}$. Given a $T-equivariant$ quasi-coherent sheaf F on $(\mathbb{C})^d$ we have,

$$H^0((\mathbb{C})^d,F) = \bigoplus_{m\in X(T)} F(m)$$

be the graded-module decomposition. This lets us associate *vector spaces* given by $F(m)$ and x_i being the co-ordinates we get the *multiplication by* x_i gives by denoting $\chi_i(m) := x_i$:

$$\chi_i(m) : F(m) \to F(m+m_i).$$

These maps satisfy:

$$\chi_j(m+m_i) \circ \chi_i(m) = \chi_i(m+m_j) \circ \chi_j(m).$$

We define an $S-family$ associated to F.

Definition 3.2.4. *An S-family of F is given by.*

• $F(m)_{m\in X(T)}$ *a collection of vector spaces.*

• *Morphisms given by* $\chi_i(m)$ *satisfying the above co-cycle condition.*

• *Morphisms between S-families are given by: Let* $\hat{F}$ *and* $\hat{G}$ *be two S-family then, it is a family of morphisms:*

$$\{\phi(m) : F(m) \to G(m)\}_{m\in X(T)}$$

commuting with χ_i, $i, j = 1(1)d$

3.2.2 Stacky S-family on $[(\mathbb{C})^d/H]$

In the same way as above we obtain a *stacky S-family* associated to a *T-equivariant* coherent sheaf F. In this section we take care of the *fine-grading* involved due to the action of H through its *characters* on each *T-weight space*.

Let H act through $\{n_i\}_{i=1(1)d} \in X(H)$.

We associate a decomposition on each

$$F(m) = \bigoplus_{\Psi\in X(H)} F(m)_\psi.$$

and *multiplication* as above gives us:

$$x_i =: \chi_m : F(m)_n \to F(m+m_i)_{n+n_i}.$$

We associate a collection $\hat{F}$ to denote the Stacky-s family associated to F containing the following information.

Definition 3.2.5. *Associate to F the following data.*

• *A collection of vector spaces $(F(m)_n)_{m\in X(T), n\in X(H)}$.*

• *A collection of linear morphisms between vector spaces given by*

$$\chi_i(m) : F(m) \to F(m+m_i)$$

satisfying

$$\chi_i(m) : F(m)_n \to F(m+m_i)_{n+n_i}$$

and satisfying the $co-cycle$ condition:

$$\chi_j(m+m_i) \circ \chi_i(m) = \chi_i(m+m_j) \circ \chi_j(m)$$

for all $i, j = 1(1)d, m \in X(T), n \in X(H)$.

• *Morphisms are analogously defined as in the case of $S-family$.*

3.2.3 Gluing Formula for torsion free sheaves on $\mathcal{X}(\Sigma)$

In this section we precisely prove the conditions for which a $torsion\ free$ coherent sheaf F admits the gluing conditions on each open $substack$ given by the cones of $maximal\ dimesnsion\ d$. So we fix σ_1 and σ_2. Let the cones match on $d-p$ rays denoted by $\lambda_1, \lambda_2, .., \lambda_{d-p}$ and the other p rays are different.

Let us first state the $Gluing\ Formula$ given by:

Proposition 3.2.6. *Let $\mathcal{X}$ be a Deligne-Mumford stack with $G-action$ and let U_i be a $G-$ invariant cover by $open\ substacks$. Let (F_i,ϕ_i) be a collection of $G-equivariant$ quasi-coherent sheaves on the U_i and assume there are $G-equivariant\ isomorphisms$ given by*

$$\phi_{ij}: F_i|_{ij} \to F_j|_{ij}$$

satisfying $\phi_{ii} = id$ and the $cocycle\ identity$ given by:

$$\phi_{ik}|_{ijk} = \phi_{jk}|_{ijk} \circ \phi_{ij}|_{ijk}.$$

Then there exists a unique $G-equivariant$ quasi-coherent sheaf (F,Φ) on $\mathcal{X}$ together with $G-$ $equivariant\ isomorphisms$ given by:

$$\phi_i : F|_i \to F_i$$

satisfying

$$\phi_j|_{ij} = \phi_{ij} \circ \phi_i|_{ij}.$$

Denote the $open\ substacks$ correspondingly by $\mathcal{U}_{\sigma_1}$ and $\mathcal{U}_{\sigma_2}$. Denote the fiber product

$$\mathcal{U}_{\sigma_1 \times_{\mathcal{X}(\Sigma)}} \mathcal{U}_{\sigma_2} := [Z_{12}/(G_{\sigma_1} \times G_{\sigma_2})],$$

where from [4] one can show that

$$\mathcal{U}_{\sigma_i} = [Z_1/G_{\sigma_1})] := [(\mathbb{C})^d/G_{\sigma_1})],\ Z_{12} = (\mathbb{C}^*)^p \times (\mathbb{C})^{d-p}.$$

where $Z_1 := (\mathbb{C}^d)$ The open immersion $\psi_{12}: \mathcal{U}_{\sigma_{12}} \to \mathcal{U}_{\sigma_1}$ is given by $T-equivariant$ map from

$$\phi_1 : Z_{12} \to Z_1$$

which is also *equivariant* under the projection given by:

$$p_i : G_{\sigma_1} \times G_{\sigma_2} \to G_{\sigma_i},$$

for $i = 1, 2$.

The $T-weights$ can be computed on each Z_i from the $T-action$ of it onto itself. $g \in G_\Sigma$ belongs to G_{σ_1}, iff $g.Z_1 \cap Z_1$ is non-empty. This gives the G_{σ_1} action on Z_1.

In order to glue sheaf F we try to produce a gluing formula in terms of the $stacky\ S-families\ \hat{F}$ purely in terms of $DG(\beta_{\sigma_1}) \times DG(\beta_{\sigma_2})$ representations.

Let us denote for $i = 1(1)2$

$$Box_T(\sigma_i) = [0, c_{1,i}] \times [0, c_{1,i}] \times \ldots [0, c_{1,i}].$$

Let T act on Z_i by $(m_{1,i}, ..m_{d,i})$.

We denote F_1 and F_2 to denote the restrictions of F on each open toric substack. Next we denote by $F_{1,12}$ the pullback of F_1 to Z_{12}. We want to compute the associated $S-family\ \hat{F}(m)$ where we give

$$m = (\frac{i_1}{c_1} + l_1)m_{1,i} + (\frac{i_2}{c_2} + l_2)m_{2,i} + ..(\frac{i_d}{c_d} + l_d)m_{d,i}$$

where,

$$0 \le i_j < c_j,\ j = 1(1)d,\ i = 1,2$$

We want to compute

$${}_b\hat{F}_{1,12}(l_1, l_2, .., l_d) := F_{1,12}(b + l_1 m_{1,1} + .. + l_d m_{d,1})$$

In this notation we describe

$$F(m) = F((\frac{i_1}{c_1} + l_1)m_{1,i} + (\frac{i_2}{c_2} + l_2)m_{2,i} + ..(\frac{i_d}{c_d} + l_d)m_{d,i})$$

We define the restriction of

$$F_1|_{\phi_1(Z_{12})} := G_1.$$

Let us fix a $box-element$ $b := (\frac{i_1}{c_1}, \frac{i_2}{c_2} .., \frac{i_d}{c_d}) \in X(T) \cong (\mathbb{Z})^d$.

We first restrict F_1 to $(\mathbb{C}^*)^p \times (\mathbb{C})^{d-p} \subset (\mathbb{C})^d$.

For a fixed $(l_{p+1}, .., l_d)$ we have ${}_bF_1(l_1, , l_d)$ stabilise for $l_1, l_2, , , l_p >> 0$.

Thus $multiplication$ by x_i (being co-ordinates of Z_1) is $isomorphism$ for $i = 1(1)p$, altering the $fine-grading$. We denote the limit by identifying the $isomorphism$ as equality. Thus, identifying $F_1|_{\phi(Z_{12})} := G_1$ we obtain using graded tensor products a $S-family$ given by $\hat{G}_1$, where:

$${}_bG_1(l_1, .., l_d) =_b F_1(\infty, \infty, .., l_{p+1}, , l_d).$$

For all $l_1, l_2, .., l_p >> 0$ such that

$${}_bF_1(l_1, l_2, ..l_d) =_b F_1(\infty, \infty, .., l_{p+1}, ..l_d),$$

we have

$${}_bF_1(\infty, \infty, l_{p+1}, l_d)_l =_b G_1(l_1, l_2, ..l_d)_l,$$

where we have $l \in DG(\beta_{\sigma_1})$.

We observe that the $fine-grading$ of ${}_bG_1(l_1, ..l_d)_l$ is determined by that of ${}_bG_1(0, 0, .., l_{p+1}, , l_d)$ given by:

$${}_bG_1(l_1, ..l_d)_l =_b G_1(0, ..l_{p+1}, ., l_d)_{l-\sum_{i=1}^p l_i(n_i)} \bigotimes \prod_{i=1}^p (\chi)^{l_i n_i}$$

where $(\chi)^{l_i n_i} \in DG(\beta_{\sigma_1})$, for $i = 1(1)p$.

The action of $DG(\beta_{\sigma_1})$ on Z_1 is given by $(n_{1,i}, n_{2,i}, , n_{d,i}), i = 1(1)2$. We shall suppress the i in this notation in the calculation of the first part.

We define the $fine-grading$ of the limit vector space by ${}_bF_1(\infty, ., l_{p+1}, .l_d)_l :=_b G_1(0, .., l_{p+1}, ., l_d)_l$.

Next we compute the $S-family$ for the $toric$ sheaf

$$ {}_bF_{1,12}(l_1,l_2,.l_d). $$

This can be computed by using the graded tensor product on ϕ_1 and the $\mathbb{C}[x_1^{\pm1},x_2^{\pm1},x_p^{\pm1},x_{p+1}..,x_d]$ module G_1 one obtains an unique expression in the module $G_1\otimes\mathbb{C}[\gamma_1^{\pm1},.,\gamma_p^{\pm1},\gamma_{p+1}.,\gamma_d]$, given by:

$$ \bigoplus_R v_{i'}\otimes\alpha_{i'}, $$

where,

$$ i' := (\frac{i'_1}{c_1},\frac{i'_2}{c_2},..,\frac{i'_p}{c_p},\frac{i_{p+1}}{c_{p+1}},.\frac{i_d}{c_d})\in R, $$

$$ R = [0,c_1]\times[0,c_2]\times..[0,c_p], $$

$$ v_{i'}\in_{(\frac{i'_1}{c_1},\frac{i'_2}{c_2},..,\frac{i'_p}{c_p},\frac{i_{p+1}}{c_{p+1}},.\frac{i_d}{c_d})} G_1(l_1,..l_d)_l, $$

$$ \alpha_{i'} = (\frac{i'_1-i_1}{c_1},.\frac{i'_p-i_p}{c_p},.0,0), $$

the weight of the element

$$ \alpha_{i'}\in\mathbb{C}[\gamma_1^{\pm1},.,\gamma_p^{\pm1},,\gamma_{p+1}^1,\gamma_d^1], $$

satisfies such that

$$ wt(v_{i'})+wt(\alpha_{i'}) = b+l_1m_1,..l_dm_d\in X(T). $$

Next we determine the $fine-grading$. Denoting the $G_{\sigma_1}\times G_{\sigma_2}$ action on $\alpha_{i'}$ by $(t_{i'},s_{i'})\in X(DG(\beta_{\sigma_1}))\times X(DG(\beta_{\sigma_2}))$, and assuming that the $weight$ of $v_{i'}$ given by

$$ l+t_{i'}\in DG(\beta_{\sigma_1}) $$

we obtain that the $fine-grading$ of

$$ \bigoplus_R v_{i'}\otimes\alpha_{i'} $$

is given by:

$$(l, s_{i'}) \in DG(\beta_{\sigma_1}) \times DG(\beta_{\sigma_2}).$$

Thus the $fine-grading$ of $\hat{F_{1,12}}$ at the point $(b+l_1m_1+..+l_dm_d) \in X(T)$ is equal to:

$$\bigoplus_{\substack{i'\in R \\ l\in X(DG(\beta_{\sigma_1}))}} {}_{(\frac{i'_1}{c_1},\frac{i'_2}{c_2},..,\frac{i'_p}{c_p},\frac{i_{p+1}}{c_{p+1}},.,\frac{i_d}{c_d})} G_1(l_1,l_2,,l_d)_{l-t_{i'}} \otimes t_{i'} \otimes s_{i'}$$

and then using the earlier definition of fine grading one obtains,

$$\bigoplus_{\substack{i'\in R \\ l\in X(DG(\beta_{\sigma_1}))}} {}_{(\frac{i'_1}{c_1},\frac{i'_2}{c_2},..,\frac{i'_p}{c_p},\frac{i_{p+1}}{c_{p+1}},.,\frac{i_d}{c_d})} F_1(\infty,.\infty,,l_{p+1},l_d)_{l-t_{i'}} \otimes t_{i'} \otimes s_{i'} \bigotimes \prod_{i=1}^{p} (\chi)^{(l_i n_i)}.$$

We have suppressed the notation so far but now to find the $Gluing\ formula$ we have to make them resurface. So to denote the final formula where we use index 1 and 2 in the subscript to denote the cones,$(\lambda_1,..\lambda_{d-p})$ the rays on which they match, we have to observe that $F_{1,12}$ lives on ${}_{b_1}(l_1,l_2,..l_d)$ and $F_{2,21}$ lives on ${}_{b_2}(l_1,l_2,.l_d)$. So multiplying by the element in $\mathbb{C}[\gamma_1^{\pm 1},\gamma_2^{\pm 1},\gamma_{p+1},\gamma_d]$ of $weight$

$${}_{b_1-b_2}(\sum_{i=1}^{d} l_i(m_{1,i}-m_{2,i})) \in X(T))$$

let us denote its fine grading by

$$(r_1,r_2) \in DG(\beta_{\sigma_1}) \times DG(\beta_{\sigma_2})$$

one finally obtains the desired $Gluing-Formula$. We finally obtain,

Theorem 3.2.7. *Let F be a torsion free toric sheaf on $\mathcal{X}$. Then the category of torsion free toric sheaves is equivalent to category of l-tuples of $Stacky\ S-family$ given by $\{\hat{F}_i\}$, with $i,j = 1(1)l$*

satisfying the following equality of $DG(\beta_{\sigma_i}) \times DG(\beta_{\sigma_j})$ representations:

$$\bigoplus_{\substack{i' \in R_1 \\ l \in X(DG(\beta_{\sigma_1}))}} {}_{i'}F_1(\infty,.\infty,,l_{p+1},l_d)_{l-t_{i'}} \otimes s_{i'} \otimes A_{i',1} \cong \bigoplus_{\substack{i' \in R_2 \\ l \in X(DG(\beta_{\sigma_2}))}} {}_{i'}F_2(\infty,.\infty,,l_{p+1},l_d)_{l-t_{i'}} \otimes (t_{i'}+r_1) \otimes A_{i',2}$$

where

$$A_{i',1} := \prod_{i=1}^{p} (\chi)^{(l_i n_{i,1}+t_{i'})} \in X(DG(\beta_{\sigma_1}))$$

,

$$A_{i',2} := \prod_{i=1}^{p} (\chi)^{(l_i n_{i,2}+s_{i'}+r_2)} \in X(DG(\beta_{\sigma_1}))$$

and i' defined as before.

As a final application in our upcoming work we compute the $T-equivariant\ Picard\ Groups$ of $toric\ Deligne-Mumford$ stacks. Note that the $T-equivariant$ group is different from $Picard\ Group$. See for instance [36].

Chapter 4

Bogomolov Gieseker Inequality

4.1 Bogomolov-Gieseker inequality

We generalise Bogomolov's Inequality to the case of higher dimensional $Deligne-Mumford$ stacks. We start from a few results in characteristic 0 concerning $modified\ strongly\ semi-stable$ sheaves. Let W denote a modified semi-stable sheaf and semi-stable here refers to $modified$ under condition *, §4.1.2 [28],[21] and $semi-stable$ for projective varieties is defined as in [11].

Theorem 4.1.1. *Let $\mathcal{X}$ be a smooth projective generically tame Deligne-Mumford stack with finite diagonal. Let us fix a polarisation satisfying the above condition *, §4.1.2, and let W be a μ semi-stable torsion free coherent sheaf on $\mathcal{X}$. There exists a smooth projective scheme Y, and a finite flat cover of the Deligne-Mumford stack $\mathcal{X}$ which we denote by $p : Y \mapsto \mathcal{X}$. Then p^*W is μ semi-stable on Y if and only if W is μ semi-stable.*

Remark 4.1.2. *We observe that the proof runs for $\mathcal{X}$ normal, projective.*

On observation one finds that in characteristic 0, μ semi-stable sheaves are closed under tensor products. Hence from the above theorem we have, semi-stable sheaves on $\mathcal{X}$ are also closed under tensor product.

In characteristic p, strongly $\mu-$semi-stable coherent sheaves are defined to be those whose $Frobenius\ twists$ denoted as $F^m(W)$ for a coherent sheaf W, are $\mu-$semi-stable. From the proof we also find that if W is strongly $\mu-$semi-stable sheaf then $p^*(W)$ is again strongly $\mu-$semi-stable where we use Frobenius commutes with pullback. Thus one can establish the semi-stability of strongly semi-stable tensor products of coherent torsion free sheaves to be so on smooth Deligne-Mumford *stacks of any rank. Neitherless we generalise [27] method to our case in this chapter where we prove Bogomolov-Gieseker inequality for surface Deligne-Mumford stacks.*

We first fix some notations in characteristic $p > 0$. We assume $\mathcal{X}$ a smooth tame projective Deligne-Mumford stack of dimension d. Let

$$\mathcal{X}^{(i)} = \mathcal{X} \times_{\operatorname{Spec}\kappa} \operatorname{Spec}(\kappa)$$

be the Deligne-Mumford stack obtained from $\mathcal{X}$ by applying the i-th power of absolute Frobenius morphism on $\operatorname{Spec}\kappa$. The geometric Frobenius morphism $F_g : \mathcal{X} \to \mathcal{X}^{(1)}$ is defined by the fiber product $\mathcal{X}^{(1)} = \mathcal{X} \times_{\operatorname{Spec}\kappa} \operatorname{Spec}(\kappa)$ and the absolute Frobenius morphism $F : \mathcal{X} \to \mathcal{X}$.

We review a bit on the coherent sheaves with connections in [18], where the theory is for schemes, but in étale topology it works for Deligne-Mumford stacks. Following N.Katz [18], a connection on a quasi-coherent sheaf E is a homomorphism

$$\nabla : E \to E \otimes \Omega_{\mathcal{X}}$$

such that $\nabla(ge) = g\nabla(e) + e \otimes dg$ where g and e are sections of $O_{\mathcal{X}}$ and E respectively over an open subset of $\mathcal{X}$, and dg denotes the image of g under the canonical exterior differentiation $d : O_{\mathcal{X}} \to \Omega_{\mathcal{X}}$. The kernel $K := K(E, \nabla)$ of ∇ is the $O_{\mathcal{X}}$-linear map $K = \nabla_1 \circ \nabla : E \to E \otimes \Omega^2_{\mathcal{X}}$ where $\nabla_1 : E \otimes \Omega_{\mathcal{X}} \to E \otimes \Omega^2_{\mathcal{X}}$ is the map defined by $\nabla_1(e \otimes \omega) = e \otimes d\omega - \nabla(e) \wedge \omega$. A connection $\nabla : E \to E \otimes \Omega_{\mathcal{X}}$ is *integrable* if its kernel $K = 0$. Then let $\mathbf{MIC}(\mathcal{X})$ be the category of pairs (E, ∇) where E is a quasi-coherent $O_{\mathcal{X}}$-module and ∇ is an integrable connection.

The p-curvature of an integrable connection ∇ is given by a morphism

$$\psi : \operatorname{Der}_{\kappa}(\mathcal{X}) \to \operatorname{End}(E),$$

by

$$\psi(D) = (\nabla(D))^p - \nabla(D^p),$$

where D is a section of $\mathrm{Der}_\kappa(\mathcal{X})$.

For the geometric Frobenius morphism $F_g : \mathcal{X} \to \mathcal{X}^{(1)}$, in [18, Theorem 5.1], N. Katz constructed a canonical connection ∇_{can} on F_g^*E which is determined in the following Cartier theorem:

There is an equivalence of categories between the category of quasi-coherent sheaves on $\mathcal{X}^{(1)}$ and the full subcategory of $\mathbf{MIC}(\mathcal{X})$ consisting of objects (E,∇) whose p-curvature is zero. The equivalence is given by: $E \mapsto (F_g^*E, \nabla_{\mathrm{can}})$ and $(E,\nabla) \mapsto E^\nabla$. Here the unique ∇_{can} on F_g^*E makes $E \cong (F_g^*E)^{\nabla_{\mathrm{can}}}$, and for a sheaf E, E^∇ is the kernel $\ker(\nabla)$ of ∇.

We first have a result of generating sheaves under Frobenius pullbacks:

Lemma 4.1.3. *Let $F_g : \mathcal{X} \to \mathcal{X}^{(1)}$ be the geometric Frobenius morphism. Then a locally free sheaf Ξ is a generating sheaf on $\mathcal{X}^{(1)}$ if and only if $F_g^*\Xi$ is a generating sheaf on $\mathcal{X}$.*

Proof. That Ξ is a generating sheaf means it contains all the irreducible representations of the stabilizer group of $\mathcal{X}^{(1)}$. Then the pullback $F_g^*\Xi$ contains all the irreducible representations of the stabilizer group of $\mathcal{X}$ since p is prime to all the orders of the stabilizer groups of $\mathcal{X}$. □

Thus from the Cartier's theorem above, the following is a generalization of [22, Proposition 2.2]:

Proposition 4.1.4. *A coherent sheaf E on $\mathcal{X}^{(1)}$ is slope semistable with respect to (H,Ξ) if and only of F_g^*E is slope semistable with respect to $(F_g^*H, F_g^*\Xi)$.*

Proof. This is from the modified slope of E is the product of some p-th power with the modified slope of F_g^*E. □

The following lemma is the generalization of [22, Lemma 2.3]. We recall that a sheaf E is ∇-semistable if the inequality of modified slopes is satisfied for all nonzero ∇-preserved subsheaves of E.

Lemma 4.1.5. *Consider a torsion free sheaf E on $\mathcal{X}$ with a κ-connection ∇, and assume that E is ∇-semistable. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the usual Harder-Narasimhan filtration, then the induced morphisms $E_i \to (E/E_i) \otimes \Omega_{\mathcal{X}}$ are nonzero $\mathcal{O}_{\mathcal{X}}$-morphisms.*

Proof. If the induced morphisms $E_i \to (E/E_i)\otimes\Omega_{\mathcal{X}}$ are zero, then it induces a Harder-Narasimhan filtration for ∇-connections which contradicts the ∇-semistability. □

Now we fix some notations for the slopes of a coherent sheaf E on $\mathcal{X}$ with respect to the polarization (Ξ, H). Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m := E$$

be the Harder-Narasimhan filtration of E with respect to modified slope stability. We denote by $\mu_{\Xi,\max}(E), \mu_{\Xi,\min}(E)$ the maximal and minimal modified slope of E respectively. We let

$$L_{\Xi,\max}(E) = \lim_{k\to\infty} \frac{\mu_{\Xi,\max}((F^k)^*E)}{p^k}; \quad L_{\Xi,\min}(E) = \lim_{k\to\infty} \frac{\mu_{\Xi,\min}((F^k)^*E)}{p^k}.$$

The sequence $\frac{\mu_{\Xi,\max}((F^k)^*E)}{p^k}$ is weakly increasing and $\frac{\mu_{\Xi,\min}((F^k)^*E)}{p^k}$ is weakly decreasing. Also $L_{\Xi,\max}(E) \geq \mu_{\max}(E)$ and $L_{\Xi,\min}(E) \leq \mu_{\min}(E)$. Suppose that E is slope semistable, then $L_{\Xi,\max}(E) = \mu_{\Xi}(E)$ (or $L_{\Xi,\min}(E) = \mu_{\Xi}(E)$) if and only if E is strongly slope semistable.

First we have:

Lemma 4.1.6. *Let E be a slope semistable torsion-free sheaf on $\mathcal{X}$ such that F^*E is unstable. Then in the Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_m := F^*E$, the $\mathcal{O}_{\mathcal{X}}$-morphisms $E_i \to (F^*E/E_i)\otimes\Omega_{\mathcal{X}}$ induced by ∇_{can} are nonzero.*

Proof. Note that this is Proposition 1 in 2.2.1 where Shepherd-Barron dealt with general slope stability for schemes. The result for modified slope stability is from Proposition 4.1.4 and Lemma 4.1.5. □

For any divisors H, A, $H^i \cdot A^{d-i}$ denotes the integration $\int_{\mathcal{X}} H^i \cdot A^{d-i}$. Here We still denote H and A for π^*H, π^*A where $\pi : \mathcal{X} \to X$ is the map to its coarse moduli space.

Corollary 4.1.7. *Let us define*

$$\alpha_{\Xi}(E) := \max(L_{\Xi,\max}(E) - \mu_{\Xi,\max}(E), \mu_{\Xi,\min}(E) - L_{\Xi,\min}(E)).$$

Then let A be a nef divisor for X such that $\pi_*(T_X \otimes \sum_{k=1}^{m}(L^k))(A)$ is globally generated, L is π ample line bundle on $\mathcal{X}$, with index $m \in Pic(\mathcal{X})$ and we have

$$\alpha_{\Xi}(E) \leq \frac{\mathrm{rk}(E)-1}{p-1}(max_{1\leq k\leq m-1}(\mu_{\Xi}(A\otimes(L^k)))).$$

Proof. This is the proof we modify [22, Cor 2.5]. We modify our case by observing that $\sum_{k=1}^{m-1} L^k$ is a generating sheaf on $\mathcal{X}$. So we have

$$\pi^*(\pi_*(T\mathcal{X}\otimes\sum_{k=1}^{m-1}L^k))\otimes\sum_{k=1}^{m-1}L^{-k}\mapsto T\mathcal{X}$$

which is a surjection. From our assumption $\pi_*(T\mathcal{X}\otimes\sum_{k=1}^{m-1}L^k)(A)$ is globally generated. Using the right exactness of pullback we obtain,

$$((O_{\mathcal{X}}(-A))^r\otimes\sum_{k=1}^{m-1}L^{-k})\mapsto\pi^*\pi_*(T\mathcal{X}\otimes\sum_{k=1}^{m-1}L^k)\otimes\sum_{k=1}^{m-1}L^{-k}\mapsto T\mathcal{X}$$

which is again a surjection.

Dualising we obtain,

$$0\mapsto\Omega_{\mathcal{X}}\mapsto((O_{\mathcal{X}}(A))^r\otimes\sum_{k=1}^{m-1}L^k)$$

Tensoring by $F^*(E)/E_i$ and taking $\mu_{\Xi,max}$ we obtain,

$$\mu_{\Xi,max}((F^*(E)/E_i)\otimes\Omega_{\mathcal{X}})\leq\mu_{\Xi,max}(((F^*(E)/E_i)\otimes(O_{\mathcal{X}}(A))^r\otimes\sum_{k=1}^{m-1}L^k))$$

We now use

$$\mu_{\Xi,max}(E\oplus F)=max(\mu_{\Xi,max}(E),\mu_{\Xi,max}(F))$$

where E, F are torsion free coherent sheaves.

Using the above the right hand side reduces to,

$$max_{1\leq k\leq m-1}(\mu_{\Xi,max}(F^*(E)/E_i\otimes A\otimes L^k)).$$

Now we estimate,

$$\mu_{\Xi,max}(F^*E/E_i\otimes A\otimes L^k)=\mu_{\Xi}(E_{i+1}/E_i)+\mu_{\Xi}(A\otimes L^k).$$

Taking max we obtain,

$$max_{1\leq k\leq m-1}(\mu_{\Xi,max}(F^*(E)/E_i\otimes A\otimes L^k))=max_{1\leq k\leq m-1}(\mu_{\Xi}(A\otimes L^k))+\mu_{\Xi}(E_{i+1}/E_i).$$

Rest of the proof follows [22, Cor 2.5]. □

4.1.1 fdHN property

As in [22, §2.6], a torsion free coherent sheaf E on $\mathcal{X}$ has an fdHN property (finite determinacy of the Harder-Narasimhan filtration) if there exists a k_0 such that all quotients in the Harder-Narasimhan filtration of $(F^{k_0})^*E$ are strongly modified slope semistable with respect to Ξ. E is fdHN of it has an fdHN property. Similar to the proof in [22, Theorem 2.7], we have

Proposition 4.1.8. *Let E be a torsion free sheaf on $\mathcal{X}$, the E is fdHN.*

Proof. Let

$$0=E_0\subset E_1\subset\cdots\subset E_m=E$$

be the Harder-Narasimhan filtration of E. We have the HNP (Harder-Narasimhan polygon) associated with E defined by connecting the points $p(E_0),p(E_1),\cdots,p(E_m)$ by successively line segments and connecting the last one with the first one. Here $p(G)=(\alpha_{\Xi,d-1}(G),\alpha_{\Xi,d}(G))$. Since the base field κ has characteristic $p>0$, there is a sequence of polygons $HNP_k(E)$, where $HNP_k(E)$ is defined by contracting $HPN((F^k)^*E)$ along the $\alpha_{\Xi,d-1}$ axis by the factor $1/p^k$. All of the polygons are bounded and $HNP_k(E)$ is contained in $HNP_{k+1}(E)$. Then the proof of [22, Theorem 2.7] goes through without major changes. We omit the details. □

In this section we always fix $\mathcal{X}$ as a smooth tame projective Deligne-Mumford stack of dimension 2. Interesting two dimensional Deligne-Mumford stacks are reviewed in [16], where the name surface Deligne-Mumford stack is used. We always fix a generating sheaf Ξ on $\mathcal{X}$ and talk about modified semi-stable sheaves in characteristic p and assume the existence of a π ample line bundle on $\mathcal{X}$.

4.1.2 Condition ⋆

We say a generating sheaf Ξ on $\mathcal{X}$ satisfying Condition ⋆ if either $\Xi = \mathcal{O}_{\mathcal{X}}$ or its restriction on any component in $I\mathcal{X}_1$ is a sum of locally free sheaves of the same rank as in [28, Proposition 3.18] and Remark 2.2.4.

We have

$$\deg_{\Xi}(E) = \frac{1}{\mathrm{rk}(\Xi)}\alpha_{\Xi,d-1}(E) - \frac{\mathrm{rk}(E)}{\mathrm{rk}(\Xi)}\alpha_{\Xi,d-1}(\mathcal{O}_{\mathcal{X}}).$$

Here $\deg(E) = \int_{\mathcal{X}} c_1(E)\cdot H^{d-1} = \alpha_{\Xi,d-1}(E)$. Also $\alpha_{\Xi,d}(E) = \mathrm{rk}(E)\,\mathrm{rk}(\Xi)H^d$. Since $\frac{\mathrm{rk}(E)}{\mathrm{rk}(\Xi)}\alpha_{\Xi,d-1}(\mathcal{O}_{\mathcal{X}})$ is constant for a fixed polarization $\mathcal{O}_X(1) = H, \Xi$, it is reasonable to use $\mu_{\Xi}(E) = \frac{\mathrm{rk}(\Xi)\deg_{\Xi}(E)}{\mathrm{rk}(E)}$ as the definition of modified slope.

For any torsion free coherent sheaf E on $\mathcal{X}$ of rank rk, recall the Chern character morphism in (2.1), we rewrite it here

$$\widetilde{\mathrm{Ch}}(E) = \bigoplus_{g\in\mathcal{I}} \sum_{0\le f<1} e^{2\pi i f}\,\mathrm{Ch}((\mathrm{pr}^* E)_{g,f}).$$

We set:

$$\Delta(E) := \mathrm{Ch}_1(E)^2 - 2\,\mathrm{Ch}_0(E)\cdot\mathrm{Ch}_2(E) = 2\,\mathrm{rk}(E)c_2(E) - (\mathrm{rk}(E)-1)c_1^2(E).$$

Let $F : \mathcal{X} \to \mathcal{X}$ be the absolute Frobenius map and it is identity in characteristic zero. Note that a coherent sheaf E on $\mathcal{X}$ is strongly modified slope semistable if and only if it is semistable under Frobenius pullbacks. From [31] we know $L_{\Xi,\max}(E)$ and $L_{\Xi,\min}(E)$ are well defined rational numbers. Also we choose a nef divisor A on the scheme X such that $\pi_*(T_{\mathcal{X}} \otimes \Xi^{\vee})(A)$ is globally generated, and let:

$$\beta_{\mathrm{rk}(E)}(A,H) = (\frac{rk(E).(rk(E)-1)(max_{1\leq k\leq m-1}(\mu_{\Xi}((A)\otimes L^k))}{p-1})^2$$

- L being the $\pi-$ ample line bundle on $\mathcal{X}$.
- m denotes the index of the torsion element $L \in \ Pic(\mathcal{X})$.
- A is a nef divisor in X such that $\pi_*(T(\mathcal{X}) \otimes \sum_{i=1}^{m-1} L^{-i})(A)$ is globally generated on X.

4.1.3 Two dimensional case

In this section we prove the following result:

Theorem 4.1.9. *Let $\mathcal{X}$ be a two dimensional smooth projective Deligne-Mumford stack. Then if E be a strongly modified slope semistable torsion free sheaf on $\mathcal{X}$, we have*

$$\Delta(E) \geq 0.$$

We use the method of [27]. Let us first prove a lemma.

Lemma 4.1.10. *Suppose that E is strongly modified slope semistable torsion free coherent sheaf of rank $\mathrm{rk}(E)$ on $\mathcal{X}$ and $c_1(E)\cdot H = 0$. Let L be a line bundle on $\mathcal{X}$, then there exists a positive number M such that*

$$h^0(\mathcal{X},(F^n)^*(E)\otimes L) \leq M\cdot p^n$$

for sufficiently large integers n.

Proof. First we take a positive integer m such that $L\cdot H - mH^2 < 0$. Therefore we have $c_1((F^n)^*(E)\otimes L\otimes H^{-m}) < 0$. Since $(F^n)^*(E)\otimes L\otimes H^{-m}$ is modified slope semistable, we have $H^0(\mathcal{X},(F^n)^*(E)\otimes L\otimes H^{-m}) = 0$ (otherwise it will contradicts with the modified semistability). Then we choose a general element $C \in |mH|$ and consider $\mathcal{C} = \pi^{-1}(C)$, and the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-\mathcal{C}) \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{C}} \to 0.$$

Tensor with $(F^n)^*(E) \otimes L$ we get an inequality

$$h^0(\mathcal{X}, (F^n)^*(E) \otimes L) \leq h^0(\mathcal{C}, (F^n_{\mathcal{C}})^*(E|_{\mathcal{C}}) \otimes L|_{\mathcal{C}})$$

where $F_{\mathcal{C}}$ is the absolute Frobenius morphism of $\mathcal{C}$, and $E|_{\mathcal{C}}, L|_{\mathcal{C}}$ are the restrictions to $\mathcal{C}$.

We show that there exists a positive number M such that

$$h^0(\mathcal{C}, (F^n_{\mathcal{C}})^*(E|_{\mathcal{C}}) \otimes L|_{\mathcal{C}}) \leq M \cdot p^n.$$

We prove it for any vector bundle E on $\mathcal{C}$ of rank $\mathrm{rk}(E)$ and a line bundle L on $\mathcal{C}$. The rank one case of E is obvious since $(F^n)^*E = E^{\otimes p^n}$ and by orbifold Riemann-Roch theorem [34]. General case is proved by induction on the rank $\mathrm{rk}(E)$. Consider an exact sequence

$$0 \to G \to E \to Q \to 0$$

where G is a line bundle and Q is a rank $\mathrm{rk}-1$ vector bundle, and the exact sequence:

$$0 \to (F^n)^*G \otimes L \longrightarrow (F^n)^*E \otimes L \longrightarrow (F^n)^*Q \otimes L \to 0.$$

Thus we have

$$h^0(\mathcal{C}, (F^n_{\mathcal{C}})^*(E) \otimes L) \leq h^0(\mathcal{C}, (F^n_{\mathcal{C}})^*(G) \otimes L) + h^0(\mathcal{C}, (F^n_{\mathcal{C}})^*(Q) \otimes L).$$

We are done. □

Proof of Theorem 4.1.9

We claim that we can assume $c_1(E) = 0$. Let $\mathrm{rk} := l \cdot p^j$ for some integers l and j, then l is prime to p. Then from [3, Lemma 2.1], since X is a scheme, there is a separable finite morphism $\phi : Y \to X$ such that $c_1(\phi^* E)$ is divisible by l. Note that [3, Lemma 2.1] proved the statement for smooth schemes, but since we are in two dimensional, the argument works for surfaces with at most of quotient singularities. The coarse moduli space X always has quotient singularities. Let $f := \phi \circ F^j$, then $c_1(f^* E)$ is divisible by rk. We form the cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\widetilde{f}} & \mathcal{X} \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array} \tag{4.1}$$

where $\mathcal{Y}$ is the fibre product. Then $\mathcal{Y}$ is a Deligne-Mumford stack with the stacky locus all pullback from the stacky locus of $\mathcal{X}$. We set:

$$\widetilde{E} := \widetilde{f}^*(E) \otimes \mathcal{O}_{\mathcal{Y}}\left(-\frac{c_1(\widetilde{f}^*(E))}{\mathrm{rk}}\right).$$

Then we calculate

$$c_1(\widetilde{E}) = 0; \quad c_2(\widetilde{E}) = \widetilde{f}^*\left(c_2(E) - \frac{\mathrm{rk}-1}{2\,\mathrm{rk}} c_1(E)^2\right).$$

Since E is strongly modified slope semistable with the slope given by $\mu_\Xi(E) = \frac{\deg(E)}{\mathrm{rk}(E)\cdot \mathrm{rk}(\Xi)}$, from [10, Lemma 1.1], $\widetilde{f}^* E$ is modified slope semistable. Therefore $c_2(\widetilde{E}) \geq 0$ implies that $\Delta(E) \geq 0$.

Then we assume $c_1(E) = 0$. By Lemma 4.1.10, there exists positive numbers M_1, M_2 such that

$$h^0(\mathcal{X}, (F^n)^*(E)) \leq M_1 \cdot p^n; \quad h^0(\mathcal{X}, (F^n)^*(E^\vee) \otimes \omega_{\mathcal{X}}) \leq M_2 \cdot p^n$$

for sufficiently large integers n. In the inertia stack $I\mathcal{X}$, $\mathcal{I}_1$ denotes the index set such that $\mathcal{X}_g \subset \mathcal{X}$ has codimension one for $g \in \mathcal{I}_1$. We let $\mathcal{I}_2$ denotes the index set such that $\mathcal{X}_g \subset \mathcal{X}$ has codimension two, i.e., the stacky locus consisting of points in $\mathcal{X}$. Then by orbifold Riemann-Roch theorem (2.3) from [34],

$$\begin{aligned}
\chi(\mathcal{X},(F^n)^*(E)) &= \int_{I\mathcal{X}} \widetilde{\mathrm{Ch}}((F^n)^*(E))\cdot\widetilde{\mathrm{Td}}(T_{\mathcal{X}})\\
&= \int_{\mathcal{X}} \mathrm{Ch}((F^n)^*(E))\cdot\mathrm{Td}(T_{\mathcal{X}}) +\\
&+ \sum_{g\in\mathcal{I}_1}\int_{\mathcal{X}_g}\sum_{0\le f<1} e^{2\pi i f}\,\mathrm{Ch}(((F^n)^*(E)_{g,f})\cdot\mathrm{Td}(T_{\mathcal{X}})_{g,f}\\
&+ \sum_{g\in\mathcal{I}_2}\int_{\mathcal{X}_g}\sum_{0\le f<1} e^{2\pi i f}\,\mathrm{Ch}(((F^n)^*(E)_{g,f})\cdot\mathrm{Td}(T_{\mathcal{X}})_{g,f}\\
&= -c_2((F^n)^*(E)) + \int_{\mathcal{X}} \mathrm{rk}\cdot\mathrm{Td}(T_{\mathcal{X}})\\
&+ \sum_{g\in\mathcal{I}_1}\left(\int_{\mathcal{X}_g}\sum_{0\le f<1} e^{2\pi i f} c_1(((F^n)^*(E)_{g,f}) + \int_{\mathcal{X}_g}\sum_{0\le f<1} e^{2\pi i f}\cdot\mathrm{Td}(T_{\mathcal{X}})_{g,f}\right)\\
&+ \sum_{g\in\mathcal{I}_2}\int_{\mathcal{X}_g}\sum_{0\le f<1} e^{2\pi i f}\cdot\mathrm{Td}(T_{\mathcal{X}})^0_{g,f}
\end{aligned}$$

where $\mathrm{Td}(T_{\mathcal{X}})^0_{g,f}$ is the constant term of $\mathrm{Td}(T_{\mathcal{X}})_{g,f}$. By the Frobenius pullback property of Chern classes we have

$$\begin{aligned}
\chi(\mathcal{X},(F^n)^*(E)) &= -p^{2n}c_2(E) + \int_{\mathcal{X}} \mathrm{rk}\cdot\mathrm{Td}(T_{\mathcal{X}})\\
&+ \sum_{g\in\mathcal{I}_1}\left(p^n\int_{\mathcal{X}_g}\sum_{0\le f<1} e^{2\pi i f} c_1(((E)_{g,f}) + \int_{\mathcal{X}_g}\left(\sum_{0\le f<1} e^{2\pi i f}\right)\cdot\mathrm{Td}(T_{\mathcal{X}})_{g,f}\right)\\
&+ \sum_{g\in\mathcal{I}_2}\int_{\mathcal{X}_g}\left(\sum_{0\le f<1} e^{2\pi i f}\right)\cdot\mathrm{Td}(T_{\mathcal{X}})^0_{g,f}\\
&\le (M_1+M_2)p^n
\end{aligned}$$

Hence for large n, to ensure the above inequality we must have $c_2(E)\ge 0$. We are done. □.

Remark 4.1.11. *If the base field κ has character zero, and E is a modified semistable torsion free sheaf on $\mathcal{X}$, then the Bogomolov inequality $\Delta(E)\ge 0$ also holds by the standard method of taking limit, see [22]. We omit the details here, and note that in the character zero case Bogomolov* *inequality for orbifold semistable torsion free sheaves is proved in [19, Lemma 2.5] for surface Deligne-Mumford stacks with only isolated quotient singularities.*

4.2 Bogomolov inequality for parabolic sheaves

In this section we give an application of the Bogomolov inequality in Theorem 4.1.9 to rational parabolic sheaves on a surface X.

4.2.1 Root surfaces

Let X be a smooth projective surface and $D \subset X$ is a simple normal crossing Cartier divisor. Let $r \in \mathbb{Z}_{>0}$ be a positive integer. The line bundle with the section $(\mathcal{O}_X(D), s_D)$ defines a morphism

$$S \to [\mathbb{A}^1/\mathbb{G}_m].$$

Let $\Theta_r : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism of stacks given by the morphism

$$x \in \mathbb{A}^1 \mapsto x^r \in \mathbb{A}^1; \quad t \in \mathbb{G}_m \mapsto t^r \in \mathbb{G}_m,$$

which sends $(\mathcal{O}_X(D), s_D)$ to $(\mathcal{O}_X(D)^{\otimes r}, s_D^r)$.

Definition 4.2.1. *Let $\mathcal{X} := \sqrt[r]{(X,D)}$ be the stack obtained by the fibre product*

$$\begin{array}{ccc} \sqrt[r]{(X,D)} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow \pi & & \downarrow \Theta_r \\ X & \xrightarrow{(\mathcal{O}_X(D), s_D)} & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

We call $\mathcal{X} = \sqrt[r]{(X,D)}$ the root stack obtained from X by the r-th root construction.

The stack $\mathcal{X} = \sqrt[r]{(X,D)}$ is a smooth Deligne-Mumford stack with stacky locus $\mathcal{D} := \pi^{-1}(D)$, and $\mathcal{D} \to D$ is a μ_r-gerbe over X coming from the line bundle $\mathcal{O}_S(D)|_D$.

Remark 4.2.2. *The general root stacks over a logarithmic scheme X is constructed in [5], and the pair (X, D) defines a canonical log structure on X. Since we don't need the abstract language of log schemes we refer the reader to [5] for details.*

Theorem 4.1.9 implies the following result:

Proposition 4.2.3. *Let $\mathcal{X} = \sqrt[r]{(X, D)}$ be the r-th root stack corresponding to the pair (X, D), and let E be a strongly slope semistable torsion free coherent sheaf on $\mathcal{X}$ with respect to the polarization $(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_X(1))$. Then we have*

$$\Delta(E) = 2\,\mathrm{rk}(E)c_2(E) - (\mathrm{rk}(E) - 1)c_1(E)^2 \geq 0.$$

4.2.2 Parabolic sheaves

Definition 4.2.4. *([25]) Let E be a torsion-free coherent sheaf on X. A* parabolic structure *on E is given by a length d-filtration*

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_r(E) \supset F_{r+1}(E) = E(-D),$$

together with a system of weights

$$0 \leq \alpha_1, \alpha_1, \cdots, \alpha_r < 1.$$

We call $E_\bullet = (E, F_i(E))$ a rational parabolic sheaf associated with the divisor D if all the weights $\alpha_1, \alpha_1, \cdots, \alpha_r$ are all rational. Let $G_i(E) = F_i(E)/F_{i+1}(E)$. The Hilbert polynomial $\chi(G_i(E)(m))$ is called the i-th multiplicity polynomial of the weight α_i.

As in [25, Definition 1.8], the parabolic Euler characteristic pa$-\chi(E_\bullet)$ of $E_\bullet$ is defined as:

$$\chi(E(-D)) + \sum_{i=0}^{d-1} \alpha_i \chi(G_i). \tag{4.2}$$

The polynomial $\text{pa}-\chi(E_\bullet(m))$ is called the parabolic Hilbert polynomial of $E_\bullet$ and the polynomial $\text{pa}-\chi(E_\bullet(m))/\text{rk}(E)$ is denoted by $\text{pa}-p_{E_\bullet}(m)$.

Definition 4.2.5. *The parabolic sheaf of $E_\bullet$ is said to be parabolic Gieseker stable (resp. parabolic semistable) if for every parabolic subsheaf $F_\bullet$ of $E_\bullet$ with*

$$0 < \text{rk}(F) < \text{rk}(E)$$

we have

$$\text{pa}-p_{F_\bullet}(m) < \text{pa}-p_{E_\bullet}(m), \quad (\textit{resp.}\, \text{pa}-p_{F_\bullet}(m) \leq \text{pa}-p_{E_\bullet}(m)).$$

The parabolic degree pa-deg$(E_\bullet)$ *is defined by*

$$\text{pa-deg}(E_\bullet) = \int_0^1 \deg(E_\alpha)d\alpha + \text{rk}(E)\cdot\deg(D).$$

We set the parabolic slope as $\text{pa}-\mu(E_\bullet) = \frac{\text{pa-deg}(E_\bullet)}{\text{rk}(E)}$. *Then $E_\bullet$ is parabolic slope stable (resp. parabolic semistable) if for every parabolic subsheaf $F_\bullet \subset E_\bullet$ with*

$$0 < \text{rk}(F) < \text{rk}(E)$$

we have

$$\text{pa}-\mu(F_\bullet) < \text{pa}-\mu(E_\bullet), \quad (\textit{resp.}\, \text{pa}-\mu(F_\bullet) \leq \text{pa}-\mu(E_\bullet)).$$

4.2.3 Equivalence of categories

For the root stack $\mathcal{X} = \sqrt[r]{(X,D)}$, we choose the generating sheaf $\Xi = \oplus_{i=0}^r \mathcal{O}_\mathcal{X}(\mathcal{D}^{\frac{i}{r}})$. Let Coh$(\mathcal{X})$ be the abelian category of coherent sheaves on $\mathcal{X}$, and $\text{Par}_{\frac{1}{r}}(X,D)$ the abelian category of rational parabolic sheaves on (X,D) with length r. There exist two functors:

$$\mathcal{F}_\mathcal{X} : \text{Coh}(\mathcal{X}) \to \text{Par}_{\frac{1}{r}}(X,D); \quad E \mapsto \mathcal{F}_\mathcal{X}(E)$$

where $\mathcal{F}_{\mathcal{X}}(E)_l = \pi_*(\mathsf{E}\otimes\mathcal{O}_{\mathcal{X}}(-l\mathcal{D}))$; and

$$G_{\mathcal{X}} : \mathrm{Par}_{\frac{1}{r}}(X,D) \to \mathrm{Coh}(\mathcal{X}); \quad E_\bullet \mapsto \int^{\mathbb{Z}} g_{\mathcal{X}}(E_\bullet)(l,l)$$

where $\int^{\mathbb{Z}} g_{\mathcal{X}}(E_\bullet)(l,l)$ is the colimit of wedges:

$$\begin{array}{ccc} g_{\mathcal{X}}(E_\bullet)(l,m) & \xrightarrow{f_{d,m}} & g_{\mathcal{X}}(E_\bullet)(l,l) \\ {\scriptstyle h_{l,m}}\downarrow & & \downarrow{\scriptstyle w(l)} \\ g_{\mathcal{X}}(E_\bullet)(m,m) & \xrightarrow{w(m)} & \mathcal{G} \end{array}$$

where

1. $g_{\mathcal{X}}(E_\bullet) : \mathbb{Z}^0 \times \mathbb{Z} \to \mathrm{Coh}(\mathcal{X})$ is a map given by:

$$(l,m) \mapsto \mathcal{O}_{\mathcal{X}}(l\mathcal{D}) \otimes p^* E_m;$$

2. $m \geq l$ is an arrow in $\mathbb{Z}$, and the arrow $h_{l,m}$ is induced by the canonical section of the divisor, the arrow $f_{l,m}$ is induced by the filtration $p^*E_\bullet$, the arrow $w(l)$ is a dinatural transformation and $\mathcal{G}$ is a sheaf in $\mathrm{Coh}(\mathcal{X})$.

We define that a parabolic sheaf $E_\bullet \in \mathrm{Par}_{\frac{1}{r}}(X,D)$ to be torsion free if E_0 is torsion free. Then we have:

Theorem 4.2.6. *([5, Theorem 6.1]) The functor $G_{\mathcal{X}}$ maps torsion free sheaves on X to torsion free sheaves on $\mathcal{X}$. Moreover, $\mathcal{F}_{\mathcal{X}}$ and $G_{\mathcal{X}}$ are inverse to each other when applied to torsion free sheaves.*

4.2.4 Parabolic Bogomolov inequality

In this section we apply Proposition 4.2.3 and Kawamata cover of (X,D) constructed in [2] to deduce the parabolic Bogomolov inequality for parabolic semistable sheaves.

Let $Y \to X$ be the Galois cover with Galois group $G = \mathrm{Gal}(\mathrm{Rat}(Y)/\mathrm{Rat}(X))$ as in [2]. Let $r||G|$, then by base change of the root stack in the fibre product diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\widetilde{f}} & \mathcal{X} \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

in Diagram 4.1, the Deligne-Mumford stack $\mathcal{Y}$ is the root stack of the pullback line bundle $(f^*\mathcal{O}_X(D), f^*s_D)$. Since the degree of $f^*\mathcal{O}_X(D)$ is divided by r, the root stack associated this line bundle is trivial. Therefore we have $\mathcal{Y} \cong Y$ is a scheme and we have the following diagram:

$$\begin{array}{ccc} & & \mathcal{X} \\ & {\scriptstyle \widetilde{f}}\nearrow & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

and the quotient stack $[Y/G] \cong \mathcal{X}$.

We are ready to state the parabolic Bogomolov inequality. We set up some notations. First let W be a coherent sheaf on $\mathcal{X}$. Then from Theorem 4.2.6, there exists a rational parabolic sheaf (E, F_*, α_*) such that $F_i = \pi_*(W \otimes \mathcal{O}_{\mathcal{X}}(-i\mathcal{D}))$ and $\alpha_i = \frac{i}{r}$.

Let $D = \sum_{\lambda=1}^{h} D_\lambda$ be the decomposition of D into smooth irreducible components. Let $\widetilde{D} = (f^*D)_{\mathrm{red}}$ which is normal crossing. Then $f^*D_\lambda = k_\lambda r (f^*D_\lambda)_{\mathrm{red}}$, where $1 \le \lambda \le h$, and $k_\lambda \ge 1$ are integers. The torsion free coherent sheaf W on $\mathcal{X} = [Y/G]$ gives a G-equivariant torsion free coherent sheaf on Y which we still denote by W. Let

$$\iota : \widetilde{D} \to Y$$

be the inclusion and $\iota^\lambda : \widetilde{D}_\lambda \to Y$ be the inclusion of the component $\widetilde{D}_\lambda$, where $\widetilde{D}_\lambda = (f^*D_\lambda)_{\mathrm{red}}$. Let $H_i = E|_D / F_i(E|_D)$ be the sheaf on D. Then

$$H_j = \bigoplus_{\lambda=1}^{h} \bar{\iota}_*^\lambda H_j^\lambda,$$

where $\bar{\iota}^{\lambda} : D_{\lambda} \hookrightarrow D$ denotes the inclusion. Define $G_{i,\lambda} = H_i^{\lambda}/H_{i+1}^{\lambda}$ and

$$\widetilde{G}_{i,\lambda} := (f\iota^{\lambda})^{*} G_{i,\lambda}.$$

Then from [2, Formula (3.15)] we have:

$$W = f^{*}E + \sum_{i=1}^{r}\sum_{\lambda=1}^{h}\sum_{j=1}^{k_{\lambda}m_i} \iota_{*}^{\lambda}\left(\widetilde{G}_{i,\lambda} \otimes N_{\widetilde{D}_{\lambda}}^{j}\right) \tag{4.3}$$

in the K-theory $K_0(Y)$, where $N_{\widetilde{D}_{\lambda}} = \mathcal{O}_Y(\widetilde{D}_{\lambda})|_{\widetilde{D}_{\lambda}}$ is the normal bundle to the divisor $\widetilde{D}_{\lambda}$.

From equivalence between the categories $\mathrm{Coh}(\mathcal{X})$ and $\mathrm{Par}_{\frac{1}{r}}(X, D)$ in Theorem 4.2.6, the the main result in [5], by choosing the generating sheaf $\Xi = \bigoplus_{i=0}^{r} \mathcal{O}_{\mathcal{X}}(i\mathcal{D})$, W is modified semistable if and only if the corresponding rational parabolic sheaf (E, F_{*}, α_{*}) is parabolic semistable. But from [2, Lemma 3.13], the rational parabolic sheaf (E, F_{*}, α_{*}) is parabolic semistable with respect to $\mathcal{O}_X(1)$ if and only if the corresponding sheaf W is orbifold semistable with respect to $f^{*}\mathcal{O}_X(1)$. From Proposition 4.2.3, if W is strongly semistable, then

$$\Delta(W) \geq 0.$$

Thus we have

Theorem 4.2.7. *([2]) Let W be an orbifold strongly semistable torsion free coherent sheaf on the root stack $\mathcal{X}$ such that its corresponding rational parabolic sheaf (E, F_{*}, α_{*}) is parabolic strongly semistable, then we have*

$$\begin{aligned}
&c_2(E) + c_1(E) \cup \left(\sum_{i=1}^{r}\sum_{\lambda=1}^{h} \alpha_i \cdot r_{i\lambda}[D_{\lambda}]\right) + \frac{1}{2}\left(\sum_{i=1}^{r}\sum_{\lambda=1}^{h} \alpha_i \cdot r_{i\lambda}[D_{\lambda}]\right)^2 \\
&- \sum_{i=1}^{r}\sum_{\lambda=1}^{h}\left(\frac{\alpha_i^2 \cdot r_{i\lambda} \cdot D_{\lambda} \cdot D_{\lambda}}{2} + \alpha_i \cdot d_{i\lambda}\right) \\
&\geq \frac{\mathrm{rk}(E) - 1}{2\,\mathrm{rk}(E)}\left(c_1(E) + \sum_{i=1}^{r}\sum_{\lambda=1}^{h} \alpha_i \cdot r_{i\lambda} \cdot [D_{\lambda}]\right)^2
\end{aligned}$$

where $r_{i\lambda} = \mathrm{rk}(G_{i,\lambda})$ *and* $d_{i\lambda} = \deg(G_{i,\lambda})$.

Proof. The Chern class formula $c_2(W), c_1(W)$ are calculated in [2, §4]. Plug these into the inequality $\Delta(W) \geq 0$ we get the formula in the theorem. □

Remark 4.2.8. *[2] proves that the orbifold semistability of W is equivalent to the parabolic semistability of the corresponding parabolic sheaf* (E, F_*, α_*)*, and the parabolic semistability is equivalent to the modified semistability of W again for the generating sheaf* $\Xi = \oplus_{i=0}^{r} \mathcal{O}_{\mathcal{X}}(\mathcal{D}^{\frac{i}{r}})$.

4.3 A K theoretic decomposition of W on root stack $\mathcal{X}$

We end this topic by trying to derive a $K-theoretic$ decomposition of a $locally-free$ coherent sheaf W on the root stack $\mathcal{X}$. We denote the $canonical\ line\ bundle$ on $\mathcal{X}$ by N.

Denote by R, the $\mu_r - gerbe$ over a simple divisor D in X, the coarse moduli which is a smooth surface in our case. Denote by π and π_R the corresponding coarse moduli maps to X and D respectively. Denote the $parabolic\ bundle$ underlying by E_* [12]. This construction does not depend on the dimension and can be extended to SNC case. As in [12] we introduce an elementary transformation on the bundle W on the root stack $\mathcal{X}$. We note that R is a μ_r gerbe on D and hence $W|_R$ decomposes into the disjoint sum of weight spaces $W_{i,R}$ that is:

$$W|_R = \bigoplus_{i=0}^{r-1} W_{i,R} = \bigoplus_{i=0}^{r-1} Hom_{O_R}((N_R)^i, \pi_R^* \pi_{R,*}(W|_R \otimes (N_R^i)^\vee)) \tag{4.4}$$

where i varies from 1 to r and $N_R = O(R)|_R$. The twisted sectors corresponding to 1 corresponds to the fixed part as in [12] and the other summands are the twisted sectors of this bundle $W|_R$ over R. We write this as

$$W|_R = W_{R,fix} \bigoplus W_{R,var}$$

where,

$$W_{R,fix} = \pi_R^* \pi_{R,*}(W|_R)$$

We introduce the exact sequences as:

$$0 \mapsto N^{-(l+1)} \mapsto N^{-l} \mapsto N_R^{-l} \mapsto 0 \tag{4.5}$$

where N is the line bundle on the root stack serving as to the r-th root to $\pi^* D$ where,

$$N_R = O(R)|_R$$

Tensoring with W and taking push forward is exact (π is a finite map in the etale topology) we obtain

$$0 \mapsto E_{(l+1)} \mapsto E_{(l)} \mapsto gr_l(E) \mapsto 0 \tag{4.6}$$

where $gr_l(E)$ is the l th grade of the parabolic bundle with weight l.
Observing and denoting π_R as the coarse moduli map for R we have ,

$$\bigoplus_{l=0}^{r-1} \pi_R^* gr_{[l]}(E) \bigotimes N_R^l \simeq W_R$$

which agrees with the decomposition of the vector bundle W restricted to the gerbe in terms of the primitive characters of $\mathbb{Z}/r\mathbb{Z}$ obtaining a decomposition of W in $K_{0,et}(R)$ in terms of the parabolic components.
In order to obtain a $K-theoretic$ decomposition of W in $K_{0,et}(\mathcal{X})$ in terms of the parabolic bundle $\{E_{*,1/r}\}$, we follow [12] in order to do the necessary computations. In order to do so we use the following exact sequence:

$$0 \mapsto (N^{-(1+i)} \otimes W) \mapsto (N^{-i} \otimes W) \mapsto (N_R^{i+1} \otimes W|_R) \mapsto 0 \tag{4.7}$$

4.3.1 Elementary Transformation of Vector Bundles

We follow [12] to construct a K theoretic decomposition of the bundle W. and mention the necessary exact sequences:

$$0 \mapsto e(W) \mapsto W \mapsto W|_{R,var} \mapsto 0 \tag{4.8}$$

$$0 \mapsto W|_{R,var} \otimes N_R^\vee \mapsto e(W)|_R \mapsto W|_{R,fix} \mapsto 0 \tag{4.9}$$

where $W|_{R,fix}$ and $W|_{R,var}$ are the components of $W|_R$ on the gerbe over D.

Define $\rho(W)$ to be the largest non-zero integer k such that

$$((N_R)^k \otimes \pi_R^* \pi_{R,*}(W|_R \otimes (N_R^k)^\vee))$$

is non-zero.

We also we observe from [12] that if,

$$\rho(W) = 0,$$

then

$$W \simeq \pi^*(\pi_*(W))$$

on $\mathcal{X}$.

We begin with a lemma, If $\rho(W) > 0$, then $\rho(e(W)) < \rho(W)$.

We further observe applying π_*, and using the fact that

$$\pi_* W_{R,var} = 0,$$

for any coherent sheaf W on $\mathcal{X}$ and hence obtain:

$$[e^r(W)] = [e^{r-1}(W)] = [e^{r-2}(W)] \ldots = [(W)]$$

where $[V] := \pi_*(V)$ is the corresponding push-forward on X.

For a torsion free coherent sheaf V on $\mathcal{X}$ if $\rho(V) = 0$, then V descends i.e

$$V \simeq \pi^* \pi_* V.$$

and the fact that $\rho(W)$ decreases on application of it at most $r-$ times one obtains:

$$e^r(W) = \pi^* \pi_* W = \pi^*(E)$$

where $E = E_0$ is the torsion free coherent sheaf underlying $\{E_{*,1/r}\}$. One obtains the K group decomposition of W as:

Denoting $[V]$ for the isomorphism class of V in $K_{0,et}(\mathcal{X})$, we denote $e^i(W)$ by the $1-st$ elementary transformation applied to $e^{i-1}(W)$ which is $i-th$ iteration on W:

$$[e^i(W)] = [e^{i-1}(W)] + j_*([\sum_{i=0}^{r-1} e^i(W)|_{R,var}]),$$

where $[V] \in K_{0,et}(\mathcal{X})$ corresponding to a coherent sheaf V on $\mathcal{X}$ ($\mathcal{X}$ *is smooth*) and $j : R \mapsto \mathcal{X}$ being the inclusion of the reduced divisor and $j_* : K_{0,et}(R) \mapsto K_{0,et}(\mathcal{X})$.

Summing over i, one obtains the required K group decomposition of W:

$$[W] = [e^r(W)] + \sum_{i=0}^{r-1} j_*([e^i(W)|_{R,var}]).$$

Hence,

$$[W] = [\pi^*(E)] + \sum_{i=0}^{r-1} j_*([e^i(W)|_{R,var}]).$$

4.3.2 Example

Define $\rho(W)$ to be the largest non-zero integer k such that

$$((N_R)^k \otimes \pi_R^* \pi_{R,*}(W|_R \otimes (N_R^k)^\vee))$$

is non-zero.

We also we observe from [12] that if,

$$\rho(W) = 0,$$

then

$$W \simeq \pi^*(\pi_*(W))$$

on $\mathcal{X}$.

Let us compute $\rho(W \otimes N^{-r})$ using the decomposition abovesaid.

We compute,

$$(W \otimes N^{-r})_{R,var}$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}(W|_R \otimes N_R^{-r} \otimes N_R^{-i}) \otimes N_R^i$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}((W|_{R,fix} \oplus W|_{R,var}) \otimes N_R^{-r} \otimes N_R^{-i}) \otimes N_R^i.$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}((W|_{R,fix}) \otimes N_R^{-r} \otimes N_R^{-i}) \oplus (W|_{R,var} \otimes N_R^{-r} \otimes N_R^{-i})) \otimes N_R^i.$$

Using, the decompositions of $W_{R,fix}$ and observing $\pi_R^*(O(-D)|_D) = N_R^{-r}$ on R, one obtains for the first summand:

$$\bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}((W|_{R,fix}) \otimes N_R^{-r} \otimes N_R^{-i}) \otimes N_R^i.$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}(\pi_R^* \pi_{R,*}(W|_R) \otimes N_R^{-r} \otimes N_R^{-i}) \otimes N_R^i.$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}(\pi_R^* \pi_{R,*}(W|_R) \otimes \pi_R^*(O(-D)|_D) \otimes N_R^{-i}) \otimes N_R^i.$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}(\pi_R^*(\pi_{R,*}(W|_R) \otimes (O(-D)|_D)) \otimes N_R^{-i}) \otimes N_R^i.$$

Observing R is a gerbe over $spec\ \mathbb{C}$ with coarse moduli scheme D, we have:

$$\pi_{R,*}(N_R^{-i}) = 0 \tag{4.10}$$

for $i > 0 \in \mathbb{N}$ and using the projection formula we observe that the first summand is zero.

Let us treat the second summand. We observe:

$$\bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}(W|_{R,var} \otimes N_R^{-r} \otimes N_R^{-i}) \otimes N_R^i.$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}(\sum_{j=1}^{r-1} (\pi_R^* \pi_{R,*}(W|_R \otimes N_R^{-j}) \otimes N_R^j) \otimes N_R^{-r} \otimes N_R^{-i})) \otimes N_R^i.$$

$$= \bigoplus_{i=1}^{r-1} \pi_R^* \pi_{R,*}(\sum_{j=1}^{r-1} (\pi_R^* \pi_{R,*}(W|_R \otimes N_R^{-j}) \otimes N_R^{-r} \otimes N_R^{j-i})) \otimes N_R^i.$$

1) If $0 \le j - i \le r$ then $j - i - r \le 0$, and hence using projection formula and the above discussion we see that the summand is zero for the case mentioned.

2) In the other case when $j - i \le 0$ we have $\pi_R^*(O(-D)|_D) = N_R^{-r}$ and using projection formula we have the other term is also 0.

Hence we obtain, $\rho(W \otimes N^{-r}) = 0$ from which we conclude,

$$W \otimes N^{-r} = \pi^*(E(-D)).$$

4.3.3 K-theoretic decomposition in terms of Parabolic components

We compute

$$j_*(\sum_{i=0}^{r-1} e^i(W)|_{R,var})$$

in terms of the $gr_i(E) := \frac{E_i}{E_{i+1}}$ on X, where $j : R \to \mathcal{X}$ is the inclusion.

Following the argument of the above computation we establish a recurrence relation between a vector bundle V on the gerbe R and $(V_{R,var} \otimes N_R^{\vee})_{R,var}$.

$$(V_{R,var} \otimes N_R^{\vee})_{R,var}$$

$$= \sum_{k=1}^{r-1} \pi_R^* \pi_{R,*}(V_{R,var} \otimes N_R^{\vee} \otimes N_R^{-k}) \otimes N_R^k$$

$$= \sum_{k=1}^{r-1} \pi_R^* \pi_{R,*}(\sum_{j=1}^{r-1} \pi_R^* \pi_{R,*}(V_R \otimes N_R^{-j}) \otimes N_R^j \otimes N_R^{\vee} \otimes N_R^{-k}) \otimes N_R^k$$

$$= \sum_{k=1}^{r-1} \pi_R^* \pi_{R,*}(\sum_{j=1}^{r-1} \pi_R^* \pi_{R,*}(V_R \otimes N_R^{-j}) \otimes N_R^{j-1-k}) \otimes N_R^k.$$

$$= \sum_{k=1}^{r-2} \pi_R^* \pi_{R,*}(V_R \otimes N_R^{-(k+1)}) \otimes N_R^k.$$

Thus, one obtains the recurrence relation:

$$(e^i(V)|_{R,var})_j = (e^{i-1}(V)|_{R,var})_{j+1}, \tag{4.11}$$

$$1 \leq i+j \leq \rho(V), 1 \leq j \leq (r-1), 0 \leq i \leq (r-1).$$

We observe from the example that if $j-1-k=0$ then the summand survives and contributes to the bundle $(V_{R,var} \otimes N_R^{-1})_{R,var}$ serving as an explanation for the recurrence relation. Replacing W as the vector bundle on $\mathcal{X}$ and W_R being the restricted bundle on the gerbe R, $\rho(W) = r-1$ we obtain:

$$\sum_{i=0}^{r-1} e^i(W)|_{R,var} = \sum_{j=1}^{r-1} (\sum_{i=0}^{r-j-1} \pi_R^*(gr_{i+j}(E)|_D)) \otimes N_R^j. \tag{4.12}$$

Thus we have:

$$[W] = [\pi^*(E)] + \sum_{j=1}^{r-1} j_*[\sum_{i=0}^{r-j-1} (\pi_R^*(gr_{i+j}(E)|_D)) \otimes N_R^j] \tag{4.13}$$

Chapter 5

Higher dimensional case of Bogomolov-Gieseker Inequality

5.1 Higher dimension case

In this last chapter, we generalize Langer's argument to higher dimension case. For simplicity of the calculation of modified slopes, we restrict to a special case of smooth Deligne-Mumford stacks $\mathcal{X} = [Z/G]$ which is a quotient stack such that the action of G is diagonalizable. We still let the generating sheaf Ξ on $\mathcal{X}$ satisfying Condition 4.1.2. Still let $d = \dim(\mathcal{X})$ be the dimension of $\mathcal{X}$.

We state several theorems generalizing Langer [22, §3].

Theorem 5.1.1. *Let D_1 be a very ample divisor on X and $\mathcal{D}_1 := \pi^{-1}(D_1)$ and $\mathcal{D} = \pi^{-1}(D)$ for D a general element $D \in |D_1|$. If the restriction of a coherent sheaf E on $\mathcal{X}$ to $\mathcal{D}$ is not modified slope semistable with respect to $H|_D$ and $\Xi|_D$, then let $\mu_{\Xi,i}, r_i$ denote the modified slopes and ranks respectively in the Harder-Narasimhan filtration of $E|_D$, we have*

$$\sum_{i<j} r_i r_j (\mu_{\Xi,i} - \mu_{\Xi,j})^2 \leq H^d \Delta(E) + 2\,\mathrm{rk}(E)^2 (L_{\Xi,\max}(E) - \mu_\Xi(E), \mu_\Xi(E) - L_{\Xi,\min}(E)). \tag{5.1}$$

Theorem 5.1.2. *If a torsion free sheaf E on $\mathcal{X}$ is strongly modified slope semistable, we have*

$$\Delta(E) \cdot H^{d-2} \geq 0.$$

Theorem 5.1.3. *If a torsion free sheaf E on $\mathcal{X}$ is just modified slope semistable, then we have*

$$H^d \cdot \Delta(E) \cdot H^{d-2} + \mathrm{rk}(\Xi)^2 \beta_{\mathrm{rk}(E)} \geq 0.$$

Before stating the last theorem, we introduce some notations. First for torsion free sheaves G', G on $\mathcal{X}$, we set

$$\xi^{\Xi}_{G',G} = \frac{c_1(G')}{\mathrm{rk}(\Xi)\,\mathrm{rk}(G')} - \frac{c_1(G)}{\mathrm{rk}(\Xi)\,\mathrm{rk}(G)}.$$

We also set

$$K^+ := \{D \in \mathrm{Num}(X) | D^2 H^{d-2} > 0, DH^{d-1} \geq 0 \text{ for all nef } H\}.$$

where $\mathrm{Num}(X) = \mathrm{Pic}(X) \otimes \mathbb{R}/\sim$ and $\sim$ is an equivalence relation meaning $L_1 \sim L_2$ if and only if $L_1 A H^{d-2} = L_2 A H^{d-2}$ for all divisors A on X.

Theorem 5.1.4. *If we have*

$$H^d \cdot \Delta(E) \cdot H^{d-2} + \mathrm{rk}(\Xi)^2 \beta_{\mathrm{rk}(E)} < 0,$$

then there exists a saturated subsheaf $E' \subset E$ such that $\xi^{\Xi}_{E',E} \in K^+$.

We prove these theorems by induction on the rank $\mathrm{rk}(E)$, and following Langer's method. We only state the parts of the proof which are different to Langer's method in smooth case and refer to [22, §3] for detailed arguments in the proof which is the same as Langer. For the induction process, let $\mathrm{Thm}^i(\mathrm{rk})$ represent the statement that Theorem 4.i holds for ranks $\leq$ rk for $i = 1, 2, 3, 4$ and $\mathrm{Thm}^5(\mathrm{rk})$ represents that Theorem 5.1.2 holds for $\mathrm{rk}(E) \leq$ rk.

5.1.1 $\mathrm{Thm}^1(\mathrm{rk})$ implies $\mathrm{Thm}^5(\mathrm{rk})$

Suppose that the torsion free sheaf E is strongly modified slope semistable with respect to (H, Ξ), and $\Delta(E) \cdot H^{d-2} < 0$. We have $L_{\Xi,\max}(E) = L_{\Xi,\min} = \mu_{\Xi}(E)$. Theorem $\mathrm{Thm}^1(\mathrm{rk})$ implies that the restriction of E to H is still modified slope semistable. Since E is strongly modified slope semistable, $(F^k)^*E$ is also strongly modified slope semistable, and its restriction to a very general element in $|H|$ is strongly modified slope semistable. Therefore by induction the restriction of $(F^k)^*E$ to a very general element in $H_1 \cap \cdots \cap H_{d-1}$ for $H_1, \cdots, H_{d-1} \in |H|$ is strongly modified slope semistable. Therefore we are reduced to the two dimensional Deligne-Mumford stack case.

Then this is Theorem 4.1.9.

5.1.2 $\mathrm{Thm}^5(\mathrm{rk})$ implies $\mathrm{Thm}^3(\mathrm{rk})$

First note that in this case,

$$\beta_{\mathrm{rk}} = \left(\frac{\mathrm{rk}(\mathrm{rk}(E)-1)}{p-1}\left(\frac{H^{d-1}\cdot A}{H^d}\right)\right)^2$$

since $M=0$. Our polarization is (H,Ξ), we first have the following inequality:

$$H^d\cdot\Delta(E)H^{d-2}+\mathrm{rk}(E)^2\,\mathrm{rk}(\Xi)^2(L_{\Xi,\max}(E)-\mu_\Xi(E))(\mu_\Xi(E)-L_{\Xi,\min}(E))\geq 0 \tag{5.2}$$

To prove this inequality, first from the finite property fdHPN in §4.1.8 there exists a positive integer k such that all the quotients in the Harder-Narasimhan filtration of $(F^k)^*E$ are strongly modified slope semistable. Consider the Harder-Narasimhan filtration

$$0=E_0\subset E_1\subset\cdots\subset E_m=(F^k)^*E$$

and let $F_i=E_i/E_{i-1}$, $r_i=\mathrm{rk}(F_i)$, $\mu_i=\mu_\Xi(F_i)$. The Hodge index theorem (holds for smooth Deligne-Mumford stacks) implies that

$$\begin{aligned}\frac{\Delta((F^k)^*E)H^{d-2}}{\mathrm{rk}(E)} &= \sum_i\frac{\Delta(F_i)H^{d-2}}{r_i}-\frac{1}{\mathrm{rk}(E)}\sum_{i<j}r_ir_j\left(\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j}\right)^2H^{d-2}\\ &\geq\sum_i\frac{\Delta(F_i)H^{d-2}}{r_i}-\frac{\mathrm{rk}(\Xi)^2}{H^d\,\mathrm{rk}(E)}\sum_{i<j}r_ir_j(\mu_i-\mu_j)^2\end{aligned}$$

$\mathrm{Thm}^5(\mathrm{rk})$ implies that $\Delta(F_i)H^{d-2}\geq 0$. Therefore by [22, Lemma 1.4], we have

$$\begin{aligned}&\frac{H^d\cdot\Delta(E)H^{d-2}}{\mathrm{rk}(E)}\geq\\ &\qquad-\mathrm{rk}(E)\,\mathrm{rk}(\Xi)^2\left(\mu_{\Xi,\max}((F^k)^*E)-\mu_\Xi((F^k)^*E)\right)\left(\mu_\Xi((F^k)^*E)-\mu_{\Xi,\min}((F^k)^*E)\right)\end{aligned}$$

Both sides are divided by p^{2k}, we get:

$$H^d \cdot \Delta(E)H^{d-2} + \mathrm{rk}(E)^2\,\mathrm{rk}(\Xi)^2(L_{\Xi,\max}(E) - \mu_\Xi(E))(\mu_\Xi(E) - L_{\Xi,\min}(E)) \geq 0.$$

It is ready to prove $\mathrm{Thm}^3(\mathrm{rk})$. Suppose that E is just modified slope semistable. We aim to use (5.2) and Corollary 4.1.7. The method is the same as in [22, §3.6], and we take an ample divisor D on X and set $H(t) = H + tD$. Similar method shows that the Harder-Narasimhan filtration of E with respect to $(H(t), \Xi)$ is independent of t when t is positively small. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the Harder-Narasimhan filtration with respect to $(H(t), \Xi)$. We have (since E is modified slope semistable)

$$\mu_{\Xi,H}(E) \geq \mu_{\Xi,H}(E_1) = \lim_{t\to 0} \mu_{\Xi,H(t)}(E_1) \geq \lim_{t\to 0} \mu_{\Xi,H(t)}(E) = \mu_{\Xi,H}(E).$$

Hence

$$\lim_{t\to 0} \mu_{\Xi,\max,H(t)}(E) = \lim_{t\to 0} \mu_{\Xi,\max,H(t)}(E_1) = \mu_{\Xi,H}(E),$$

and similarly,

$$\lim_{t\to 0} \mu_{\Xi,\min,H(t)}(E) = \mu_{\Xi,H}(E).$$

Thus we can apply (5.2) and Corollary 4.1.7, and note $M = 0$, we get the result

$$H^d \cdot \Delta(E)H^{d-2} + \mathrm{rk}(\Xi)^2 \beta_{\mathrm{rk}(E)} \geq 0.$$

5.1.3 $\mathrm{Thm}^3(\mathrm{rk})$ implies $\mathrm{Thm}^4(\mathrm{rk})$

If we have the condition in $\mathrm{Thm}^4(\mathrm{rk})$, i.e., $H^d \cdot \Delta(E)H^{d-2} + \mathrm{rk}(\Xi)^2\beta_{\mathrm{rk}(E)} < 0$. Then from $\mathrm{Thm}^3(\mathrm{rk})$, E is not modified slope semistable. Let $E' \subset E$ be the maximal destabilizing subsheaf of E and set $E'' = E/E'$, $r' = \mathrm{rk}(E')$, $r'' = \mathrm{rk}(E'')$. First we calculate:

$$\frac{\Delta(E)H^{d-2}}{\mathrm{rk}(E)} + \frac{rr'\xi^2_{E',E}H^{d-2}}{r''} = \frac{\Delta(E')H^{d-2}}{r'} + \frac{\Delta(E'')H^{d-2}}{r''}.$$

We also have $\mathrm{rk}(\Xi)^2\frac{\beta_{\mathrm{rk}(E)}}{\mathrm{rk}(E)} \geq \mathrm{rk}(\Xi)^2(\frac{\beta_{r'}}{r'}+\frac{\beta_{r''}}{r''})$. Since $H^d \cdot \Delta(E)H^{d-2}+\mathrm{rk}(\Xi)^2\beta_{\mathrm{rk}(E)} < 0$, and we require $H^d > 0$, either $\xi^2_{E',E} > 0$ or at least one of the $H^d \cdot \Delta(E')H^{d-2}+\mathrm{rk}(\Xi)^2\beta_{r'}$ and $H^d \cdot \Delta(E'')H^{d-2}+\mathrm{rk}(\Xi)^2\beta_{r''}$ is negative. Therefore the same argument as in [11, Theorem 7.3.3] gives the result.

5.1.4 $\mathrm{Thm}^4(\mathrm{rk})$ implies $\mathrm{Thm}^2(\mathrm{rk})$

Suppose that $\Delta(E)H^{d-2} < 0$. The condition in $\mathrm{Thm}^4(\mathrm{rk})$, applying to $(F^l)^*E$ (since E is strongly modified slope semistable by the condition in $\mathrm{Thm}^2(\mathrm{rk})$), is:

$$H^d \cdot \Delta((F^l)^*E)H^{d-2}+\mathrm{rk}(\Xi)^2\beta_{\mathrm{rk}(E)} < 0$$

which is equivalent to

$$l > \frac{1}{2}\log_p\left(-\frac{\mathrm{rk}(\Xi)^2\beta_{\mathrm{rk}(E)}}{H^d \cdot \Delta(E)H^{d-2}}\right).$$

Then for large l, there exists a saturated torsion free subsheaf $E' \subset (F^l)^*E$ such that $\xi_{E',(F^l)^*E} \in K^+$. By “self-duality" property of K^+ we have $\xi_{E',(F^l)^*E}H^{d-1} > 0$, which means that the sheaf E is not strongly modified semistable, a contradiction.

5.1.5 $\mathrm{Thm}^2(\mathrm{rk}-1)$ implies $\mathrm{Thm}^1(\mathrm{rk})$

We use $\Pi = |H|$ to denote the complete linear system, and let $Z := \{(D,x) \in \Pi \times X : x \in D\}$ be the incidence variety. Let

$$p : Z \to \Pi; \quad q : Z \to X$$

be the corresponding projections. For each $s \in \Pi$, let Z_s be the scheme theoretic fiber of p over the point s. Consider the following cartisian diagram:

$$\begin{array}{ccccc} & & \mathcal{Z} & \xrightarrow{q} & \mathcal{X} \\ & \swarrow^{p\circ\pi} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Pi & \xleftarrow{p} & Z & \xrightarrow{q} & X \end{array}$$

Then $\mathcal{Z}$ is a Deligne-Mumford stack which is given by $\{(\pi^{-1}(D),x):(D,x)\in\Pi\times X, x\in D\}$. The generating sheaf Ξ on $\mathcal{X}$ is pullback under q and gives a generating sheaf $q^*\Xi$ on $\mathcal{Z}$ which is relative to Π.

We work on the sheaf q^*E for a torsion free sheaf E on $\mathcal{X}$, and let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = q^*E$$

be the relative Harder-Narasimhan filtration with respect to $p\circ\pi$. This means that there exists an open subset $U\subset\Pi$ such that all $F_i = E_i/E_{i-1}$ are flat over U and for each $s\in U$ the fibers $(E_\bullet)_s$ is the Harder-Narasimhan filtration of $E_s = q^*E|_{\mathcal{Z}_s}$ for $\mathcal{Z}_s = \pi^{-1}(Z_s)$. From the proof of in [22, §3.9], the relative Harder-Narasimhan filtration is actually the Harder-Narasimhan filtration of q^*E with respect to

$$(p^*\mathcal{O}_\Pi(1)^{\dim(\Pi)}q^*H, q^*\Xi).$$

By the finite property in §4.1.1, for the sheaf q^*E, there exists a positive integer k such that all the quotients in the Harder-Narasimhan filtration of $(F^k)^*(q^*E) = q^*((F^k)^*E)$ are strongly modified semistable. We will prove the inequality (5.1), and from [22, Lemma 1.5], when applying to the polygons of the Harder-Narasimhan filtration for modified slopes, we just prove the case that all the graded pieces F_i's are strongly modified slope semistable with respect to $(p^*\mathcal{O}_\Pi(1)^{\dim(\Pi)}q^*H, q^*\Xi)$.

We perform the same argument as in [22, §3.9], and let $\Lambda\subset\Pi$ be a pencil. Set $Y = p^{-1}(\Lambda)$, and $\mathcal{Y} = (p\circ\pi)^{-1}(\Lambda)\subset\mathcal{Z}$. Since $q|_Y : Y\to X$ is the blow up of X along the base locus B of Λ, we can view $q|_{\mathcal{Y}} : \mathcal{Y}\to\mathcal{X}$ to be the stacky blow up of $\mathcal{X}$ along the locus $\mathcal{B} = \pi^{-1}(B)$. If $d\geq 3$, then B is a smooth connected variety. So $\mathcal{B}$ is a smooth connected substack, and there is only one exceptional divisor $\mathcal{N}$ for $q|_{\mathcal{Y}}$. We write down

$$c_1(F_i|_Y) = q|_{\mathcal{Y}}^*\mathcal{M}_i + b_i\mathcal{N}$$

where $\mathcal{M}_i$ are divisors on $\mathcal{X}$ which are pullbacks of divisors M_i on X and b_i are rational numbers. If the dimension $d = 2$, then B consists of $N = H^d$ distinct points and $\mathcal{B}$ consists of N distinct stacky points. Let $\mathcal{N}_1, \cdots, \mathcal{N}_N$ be the exceptional divisors of $q|_{\mathcal{Y}}$. There exist rational numbers b_{ij} and divisors $\mathcal{M}_i$ such that

$$c_1(F_i|_Y) = q|_{\mathcal{Y}}^*\mathcal{M}_i + \sum_j b_{ij}\mathcal{N}_j.$$

Let $b_i = (\sum_j b_{ij})/N$. We have

$$\mu_{\Xi,i} = \frac{c_1(F_i|_Y)p^*\mathcal{O}_\Pi(1)q^*H^{d-2}}{r_i \operatorname{rk}(\Xi)} = \frac{\mathcal{M}_i \cdot H^{d-1} + b_i N}{r_i \operatorname{rk}(\Xi)}.$$

Thm^2(rk-1) implies that $\Delta(F_j|_Y)p^*\mathcal{O}_\Pi(1)q^*H^{d-2} \geq 0$ for every j. We calculate

$$\begin{aligned}
\frac{N\Delta(E)q|_{\mathcal{Y}}^*H^{d-2}}{\operatorname{rk}(E)} &= \sum_i \frac{N\Delta(F_i|_{\mathcal{Y}})q|_{\mathcal{Y}}^*H^{d-2}}{r_i} - \frac{N}{\operatorname{rk}(E)}\sum_{i<j} r_i r_j \left(\frac{c_1(F_i|_{\mathcal{Y}})}{r_i} - \frac{c_1(F_j|_{\mathcal{Y}})}{r_j}\right)^2 \cdot q|_{\mathcal{Y}}^*H^{d-2} \\
&\geq \frac{N}{\operatorname{rk}(E)}\sum_{i<j} r_i r_j \left(N\left(\frac{b_i}{r_i} - \frac{b_j}{r_j}\right)^2 - \left(\frac{\mathcal{M}_i}{r_i} - \frac{\mathcal{M}_j}{r_j}\right)^2 H^{d-2}\right) \\
&\geq \frac{1}{\operatorname{rk}(E)}\sum_{i<j} r_i r_j \left((N)^2\left(\frac{b_i}{r_i} - \frac{b_j}{r_j}\right)^2 - \left(\frac{\mathcal{M}_i H^{d-1}}{r_i} - \frac{\mathcal{M}_j H^{d-1}}{r_j}\right)^2\right).
\end{aligned}$$

The last inequality is from Hodge index theorem for smooth Deligne-Mumford stacks, and from the slope $\mu_{\Xi,i}$, the last expression above gives

$$2\sum_i N b_i \mu_{\Xi,i} - \frac{1}{\operatorname{rk}(E)}\sum_{i<j} r_i r_j (\mu_{\Xi,i} - \mu_{\Xi,j})^2.$$

To prove the claim, first $(q|_{\mathcal{Y}})_*(E_i|_{\mathcal{Y}}) \subset E$ implies that

$$\frac{\sum_{j\leq i}\mathcal{M}_j H^{d-1}}{\operatorname{rk}(\Xi)\sum_{j\leq i} r_j} \leq \mu_{\Xi,\max}(E)$$

which gives the inequality:

$$\sum_{j\le i} b_j N \ge \sum_{j\le i} \mathrm{rk}(\Xi) r_j (\mu_{\Xi,j} - \mu_{\Xi,\max}(E)) \tag{5.3}$$

Therefore

$$\begin{aligned}\sum_i N b_i \mu_{\Xi,i} &= \sum_i \left(\sum_{i<j} N b_j\right)(\mu_{\Xi,i} - \mu_{\Xi,i+1}) \\ &\ge \sum_i \left(\sum_{j\le i} \mathrm{rk}(\Xi) r_j (\mu_{\Xi,j} - \mu_{\Xi,\max}(E))\right)(\mu_{\Xi,i} - \mu_{\Xi,i+1}) \\ &= \mathrm{rk}(\Xi)\cdot \sum_{i<j} \frac{r_i r_j}{\mathrm{rk}(E)}(\mu_{\Xi,i} - \mu_{\Xi,j})^2 + \mathrm{rk}(E)(\mu_\Xi(E) - \mu_{\Xi,\max}(E))(\mu_\Xi(E) - \mu_{\Xi,\min}(E)).\end{aligned}$$

So we get:

$$\frac{N\Delta(E) q|_{\mathcal{Y}}^* H^{d-2}}{\mathrm{rk}(E)} \ge \sum_{i<j} \frac{2\,\mathrm{rk}(\Xi) - 1}{\mathrm{rk}(E)} r_i r_j (\mu_{\Xi,i} - \mu_{\Xi,j})^2 + 2\,\mathrm{rk}(E)(\mu_\Xi(E) - \mu_{\Xi,\max}(E))(\mu_\Xi(E) - \mu_{\Xi,\min}(E)).$$

We generalize the restriction theorem of Langer to smooth Deligne-Mumford stacks in higher dimensions. We give a general statement for the (5.2). Recall from Corollary 4.1.7,

$$\alpha_\Xi(E) := \max(L_{\Xi,\max}(E) - \mu_{\Xi,\max}(E), \mu_{\Xi,\min}(E) - L_{\Xi,\min}(E)).$$

Let A be a nef divisor for X such that $\pi_*(T_{\mathcal{X}} \otimes \Xi)(A)$ is globally generated, then there exists a large number $M > 0$ (depending on the data (H, Ξ)) such that

$$\alpha_\Xi(E) \le \frac{\mathrm{rk}(E) - 1}{p-1}\left(\frac{H^{d-1}\cdot A}{H^d} + M\right).$$

Theorem 5.1.5. *Let E be a torsion free sheaf on a smooth Deligne-Mumford stack $\mathcal{X}$. Then we have*

$$H^d \cdot \Delta(E) H^{d-2} + \mathrm{rk}(E)^2 \,\mathrm{rk}(\Xi)^2 (L_{\Xi,\max}(E) - \mu_\Xi(E))(\mu_\Xi(E) - L_{\Xi,\min}(E)) \ge 0 \tag{5.4}$$

and

$$H^d \cdot \Delta(E)H^{d-2} + \mathrm{rk}(E)^2\,\mathrm{rk}(\Xi)^2(\mu_{\Xi,\max}(E) - \mu_\Xi(E))(\mu_\Xi(E) - \mu_{\Xi,\min}(E)) \geq 0 \qquad (5.5)$$

Proof. The proof of (5.4) is the same as in Claim (5.2). The proof of formula (5.5) is the same as [22, Theorem 5.1]. □

Theorem 5.1.6. *Let E be a torsion free sheaf of rank $\mathrm{rk}(E) \geq 2$ on a smooth Deligne-Mumford stack $\mathcal{X}$. Suppose that E is slope modified stable with respect to $(\Xi, \mathcal{O}_\mathcal{X}(1) = H)$. Let $D \subset |mH|$ be a normal divisor such that $E|_\mathcal{D}$ has no torsion where $\mathcal{D} = \pi^{-1}(D))$. If*

$$m > \lfloor \frac{\mathrm{rk}(E)-1}{\mathrm{rk}(E)}\Delta(E)H^{d-2} + \frac{1}{H^d\,\mathrm{rk}(E)(\mathrm{rk}(E)-1)} + \frac{(\mathrm{rk}(E)-1)\beta_{\mathrm{rk}}}{H^d\,\mathrm{rk}(E)} \rfloor$$

then $E|_\mathcal{D}$ is slope modified stable with respect to $(\Xi|_\mathcal{D}, H|_\mathcal{D})$.

Proof. The proof is the same as [22, Theorem 5.2]. □

5.2 Bogomolov's inequality for Higgs sheaves

Let k be an algebraically closed field of characteristic $p \geq 0$ and $\mathcal{X}$ be a smooth tame projective Deligne-Mumford stack of dimension d over k with coarse moduli space $\pi : \mathcal{X} \to X$. Let Ξ be a generating sheaf on $\mathcal{X}$ satisfying Condition $\star$ in 4.1.2, and let H an ample divisor on X.

Definition 5.2.1. *A Higgs sheaf (E,θ) is a pair consisting of a coherent sheaf $E \in Coh(\mathcal{X})$ and an $\mathcal{O}_\mathcal{X}$-homomorphism $\theta : E \to E \otimes \Omega_\mathcal{X}$ satisfying the integrability condition $\theta \wedge \theta = 0$. We say a Higgs sheaf (E,θ) a system of Hodge sheaves if there is a decomposition $E = \oplus E^i$ such that $\theta : E^i \to E^{i-1} \otimes_{\mathcal{O}_\mathcal{X}} \Omega_\mathcal{X}$*

1. *We say that (E,θ) is slope semistable if $\mu_\Xi(E') \leq \mu_\Xi(E)$ for every Higgs subsheaf (E',θ') of (E,θ).*

2. *A system of Hodge sheaves (E,θ) is slope semistable if the inequality $\mu_\Xi(E') \leq \mu_\Xi(E)$ is satisfied for every subsystem of Hodge sheaves (E',θ') of (E,θ).*

We recall the main results of Ogus and Vologodsky [29], where the theory is for schemes, but in étale topology it works for Deligne-Mumford stacks.

Assume that $p > 0$. Let S be a scheme over k and $f : \mathcal{X} \to S$ be a morphism of stacks over k. A lifting of $\mathcal{X}/S$ modulo p^2 is a morphism $\widetilde{f} : \widetilde{\mathcal{X}} \to \widetilde{S}$ of flat $\mathbb{Z}/p^2\mathbb{Z}$-stacks such that f is the base change of $\widetilde{f}$ by the closed embedding $S \to \widetilde{S}$ defined by p. Let $\mathrm{MIC}_{p-1}(\mathcal{X}/S)$ be the category of $O_{\mathcal{X}}$-modules with an integrable connection whose p-curvature is nilpotent of level $\leq p-1$. Let $\mathrm{HIG}_{p-1}(\mathcal{X}^{(1)}/S)$ denote the category of Higgs $O_{\mathcal{X}^{(1)}}$-modules with a nilpotent Higgs field of level $\leq p-1$. We have the following theorem of Ogus and Vologodsky ([29, Theorem 2.8])

Theorem 5.2.2. *If $f : \mathcal{X} \to S$ is a smooth morphism with a lifting $\widetilde{\mathcal{X}}^{(1)} \to \widetilde{S}$ of $\mathcal{X}^{(1)} \to S$ modulo p^2, then the Cartier transform*

$$C_{\mathcal{X}/S} : MIC_{p-1}(\mathcal{X}/S) \to HIG_{p-1}(\mathcal{X}^{(1)}/S)$$

defines an equivalence of categories with quasi-inverse

$$C^{-1}_{\mathcal{X}/S} : HIG_{p-1}(\mathcal{X}^{(1)}/S) \to MIC_{p-1}(\mathcal{X}/S).$$

Lemma 5.2.3. *Let $(E,\theta) \in HIG_{p-1}(\mathcal{X}^{(1)}/S)$. Then we have $[C^{-1}_{\mathcal{X}/S}(E)] = F_g^*[E]$, where $[\cdot]$ denotes the class of a coherent sheaf in the Grothendieck group $K_0(\mathcal{X})$.*

Proof. See [24, Lemma 2]. □

Corollary 5.2.4. *Assume $S = \operatorname{Spec} k$, and let $(E,\theta) \in HIG_{p-1}(\mathcal{X}^{(1)}/S)$. Then (E,θ) is slope semistable with respect to (H,Ξ) iff $C^{-1}_{\mathcal{X}/S}(E)$ is slope ∇-semistable with respect to $(F_g^*H, F_g^*\Xi)$.*

Proof. The proof is the same as that of [24, Corollary 1]. □

Lemma 5.2.5. *Let (E,θ) be a torsion free slope semistable Higgs sheaf on $\mathcal{X}$. Then there exists an $\mathbb{A}^1$-flat family of Higgs sheaves $(\widetilde{E},\widetilde{\theta})$ on $\mathcal{X}\times\mathbb{A}^1$ such that the restriction $(\widetilde{E}_t,\widetilde{\theta}_t)$ to the fiber over any closed point $t \in \mathbb{A}^1$ is isomorphic to (E,θ) and (E_0,θ_0) is a slope semistable system of Hodge sheaves.*

Proof. See [23, Corollary 5.7]. □

Proposition 5.2.6. *Let V be a torsion free sheaf on $\mathcal{X}$, then*

$$H^{d-2}\Delta(V) \geq -\frac{(\operatorname{rank} V \operatorname{rank} \Xi)^2}{H^d}(L_{\Xi,\max}(V) - \mu_\Xi(V))(\mu_\Xi(V) - L_{\Xi,\min}(V))$$

Proof. The proposition follows from Theorem 5.1.2 by the same arguments as in the proof of Theorem 5.1.3. □

Theorem 5.2.7. *Assume $p = 0$. Let (E,θ) be a slope semistable Higgs sheaf with respect to (H,Ξ). Then we have $H^{d-2}\Delta(E) \geq 0$.*

Proof. Deforming (E,θ) to a system of Hodge sheaves (see Lemma 5.2.5) we can assume that (E,θ) is nilpotent. Now we use the standard reduction to positive characteristic technique, which we recall for the convenience of the reader (see [26, section 2]). There exists a finitely generated $\mathbb{Z}$-algebra $R \subset k$ and a tame smooth Deligne-Mumford stack $\widetilde{\mathcal{X}} \to S = \operatorname{Spec} R$ such that $\mathcal{X} = \widetilde{\mathcal{X}} \times_S \operatorname{Spec} k$. Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \widetilde{X}$ be its coarse moduli space. We can assume that $X = \widetilde{X} \times_S \operatorname{Spec} k$, π is induced by $\widetilde{\pi}$ after base change, and there exists an ample divisor $\widetilde{H}$ on $\widetilde{X}$ extending H and a generating sheaf $\widetilde{\Xi}$ on $\widetilde{\mathcal{X}}$ extending Ξ such that its restriction to every component in $I\widetilde{\mathcal{X}}_1$ is a direct sum of locally free coherent sheaves of the same rank. We can also assume that there exists an S-flat family of Higgs sheaves $(\widetilde{E},\widetilde{\theta})$ on $\widetilde{\mathcal{X}}$ extending (E,θ).

Shrinking S, by openness of semistability we can assume that $(\widetilde{E}_s,\widetilde{\theta}_s)$ is slope semistable with respect to $(\widetilde{\Xi}_s,\widetilde{H}_s)$ for any $s \in S$. Choose a closed point $s \in S$ such that the characteristic q of the residue field $k(s)$ is $\geq \operatorname{rank} E$. Then the stack $\widetilde{\mathcal{X}} \times_S \operatorname{Spec}(R/m_s^2)$ is a lifting of $\widetilde{\mathcal{X}}_s$ modulo q^2. By Corollary 5.2.4, one can associate to $(\widetilde{E}_s,\widetilde{\theta}_s)$ a slope ∇-semistable sheaf with integrable connection (V_s,∇_s) with respect to $(F_g^*\widetilde{H}_s, F_g^*\widetilde{\Xi}_s)$. From Lemma 5.2.3, it follows that

$$\widetilde{H}_s^{d-2}\Delta(V_s) = q^2\widetilde{H}_s^{d-2}\Delta(\widetilde{E}_s). \tag{5.6}$$

Let $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V_s$ be the usual Harder-Narasimhan filtration of V_s, then by [16, Lemma 2.7], the induced morphisms $V_i \to (V_s/V_i) \otimes \Omega_{\widetilde{X}_s}$ are nonzero $O_{\widetilde{X}_s}$-morphisms. Take a nef divisor A on $\widetilde{X}_s$ such that $\pi_*(T_{\widetilde{X}_s} \otimes \widetilde{\Xi}_s)(A)$ is globally generated. From [16, Proposition 2.10 and Corollary 2.11], it follows that

$$\mu_{\widetilde{\Xi}_s,\max}(V_s) - \mu_{\widetilde{\Xi}_s,\min}(V_s) \leq (\operatorname{rank} V_s - 1)\left(\frac{\widetilde{H}_s^{d-1}A}{\widetilde{H}_s^d} + M\right)$$

and

$$\max(L_{\widetilde{\Xi}_s,\max}(V_s) - \mu_{\widetilde{\Xi}_s,\max}(V_s), \mu_{\widetilde{\Xi}_s,\min}(V_s) - L_{\widetilde{\Xi}_s,\min}(V_s)) \leq \frac{\operatorname{rank} V_s - 1}{q-1}\left(\frac{\widetilde{H}_s^{d-1}A}{\widetilde{H}_s^d} + M\right),$$

for some positive constant M. They imply that

$$(L_{\widetilde{\Xi}_s,\max}(V_s) - L_{\widetilde{\Xi}_s,\min}(V_s)) \leq \frac{\operatorname{rank} V_s - 1}{1 - \frac{1}{q}}\left(\frac{\widetilde{H}_s^{d-1}A}{\widetilde{H}_s^d} + M\right).$$

Hence Proposition 5.2.6 gives

$$q^2 \widetilde{H}_s^{d-2}\Delta(\widetilde{E}_s) = \widetilde{H}_s^{d-2}\Delta(V_s) \geq -\frac{(\operatorname{rank} V_s \operatorname{rank} \Xi)^2}{\widetilde{H}_s^d}(\frac{\operatorname{rank} V_s - 1}{1 - \frac{1}{q}})^2\left(\frac{\widetilde{H}_s^{d-1}A}{\widetilde{H}_s^d} + M\right)^2.$$

Taking sufficiently large q, one obtains

$$H^{d-2}\Delta(E) = \widetilde{H}_s^{d-2}\Delta(\widetilde{E}_s) \geq 0.$$

□

Using the same argument in [24, section 6], one can recover [6, Theorem 1.1]:

Theorem 5.2.8. *Assume that $p = 0$, $d = 2$, and the canonical line bundle K_X is nef, then we have*

$$c_1^2(T_X) \leq 3c_2(T_X).$$

References

[Sta] *Stack Project: https://stacks.math.columbia.edu/download/stacks-morphisms.pdf.*

[2] Biswas, I. (1997). Parabolic Bundles as Orbifold Bundles. *Duke Math.J.*, 88(2), 305–325.

[3] Bloch, S. & Gieseker, D. (1971). The positivity of the Chern classes of an ample vector bundle. *Invent. Math.*, (12), 112–117.

[4] Borisov, L. A., Chen, L., & Smith, G. G. (2005). The Orbifold Chow Ring of Toric Deligne-Mumford stacks. *Journal of the American Mathematical Society*, (8), 193–215.

[5] Borne, N. & Vistoli, A. (2012). Parabolic sheaves on logarithmic schemes. *Advances in Mathematics*, (231), 1327–1363.

[6] Chen, J. & Tseng, H. (2020). On the Bogomolov-Miyaoka-Yau Inequality for Stacky Surfaces. *Taiwan. J. Math.*, 24(4), 841–853.

[7] Coates, T., Iritani, H., & Jiang, Y. (2018). The Crepant Transformation Conjecture for Toric Complete Intersections. *Advances in Mathematics*, 329, 1002–1087.

[8] Fantechi, B., Mann, E., & Nironi, F. (2010). Smooth Toric Deligne-Mumford Stacks. *J.reine angew.Math.*, 648, 201–244.

[9] Gholampour, A., Jiang, Y., & Kool, M. (2017). Sheaves on weighted projective spaces and modular forms. *Adv. Theor. Math. Phys. 21*, 21, 1455–1524.

[10] Gieseker, D. (1979). On a theorem of Bogomolov on Chern classes of stable bundles. *American J. Math.*, 101(1), 77–85.

[11] Huybrechts, D. & Lehn, M. (1997). *The geometry of moduli spaces of sheaves*. Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR MR1450870 (98g:14012).

[12] Iyer, J. N. & Simpson, C. T. (2006). The Chern character of a parabolic bundle, and a parabolic Reznikov theorem in the case of finite order at infinity. *preprint, arXiv:math/0612144*.

[13] Jiang, Y. (2008). The orbifold cohomology ring of simplicial toric stack bundles. *Illinois J.Math.*, 52(2), 493–514.

[14] Jiang, Y. (2019a). *Counting twisted sheaves and S-duality*. preprint, arXiv:1909.04241.

[15] Jiang, Y. (2019b). *The Vafa-Witten invariants for surface Deligne-Mumford stacks and S-duality*. Survey for ICCM-.

[16] Jiang, Y. & Kundu, P. (2021). The Tanaka-Thomas's Vafa-Witten invariants for surface Deligne-Mumford stacks. *Pure and Appl. Math. Quaterly*, 17(1), 503–573.

[17] Jiang, Y., Kundu, P., & Sun, H. M. (2021). On the Bogomolov-Gieseker inequality for tame Deligne-Mumford surfaces. *preprint,arXiv:2104.10614*.

[18] Katz, N. (1970). Nilpotent connections and the monodromy theorem: applications of a result of Turrittin. *Publ. Math. IHES*, 39, 175–232.

[19] Kawamata, Y. (1992). Abundance theorem for minimal threefolds. *Invent. Math*, 2, 229–246.

[20] Kool, M. (2011). Fixed point loci of moduli spaces of sheaves on toric varieties. *Advances in Mathematics*, (227), 1700–1755.

[21] Kresch, A. & Vistoli, A. (2003). On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map. *arXiv:math/0301249*.

[22] Langer, A. (2004). Semistable sheaves in positive characteristic. *Ann. Math.*, 159, 251–276.

[23] Langer, A. (2014). Semistable modules over Lie algebroids in positive characteristic. *Doc. Math.*, 19(88), 509–540.

[24] Langer, A. (2015). Bogomolov's inequality for Higgs sheaves in positive characteristic. *Invent. Math.*, 199(88), 889–920.

[25] Maruyama, M. & Yokogawa, K. (1992). Moduli of parabolic stable sheaves. *Math. Ann*, 293(1), 77–99.

[26] Matsuki, K. & Olsson, M. (2002). Kawamata-Viehweg vanishing as Kodaira vanishing for stacks. *pre-print, arXiv:math/0212259*.

[27] Moriwaki, A. (1991). A note on Bogomolov-Gieseker's inequality in positive characteristic. *Duke Math. J.*, 64, 361–375.

[28] Nironi, F. (2008). Moduli Spaces of Semistable Sheaves on Projective Deligne-Mumford Stacks, arXiv:0811.1949.

[29] Ogus, O. & Vologodsky, V. (2007). Nonabelian Hodge theory in characteristic p. *Publ. Math. Inst. Hautes etudes Sci.*, 106, 1–138.

[30] Olsson, M. & Starr, J. (2003). Quot functors for Deligne-Mumford stacks. *Comm. Algebra*, 31(8), 4069–4096.

[31] Shepherd-Barron, N. I. (1991). Unstable vector bundles and linear systems on surfaces in characteristic p. *Invent. Math.*, 106, 243–262.

[32] Tanaka, Y. & Thomas, R. P. (2017). Vafa-Witten invariants for projective surfaces II: semistable case. *Pure Appl. Math. Q.*, 13, 517–562.

[33] Tanaka, Y. & Thomas, R. P. (2020). Vafa-Witten invariants for projective surfaces I: stable case. *J. Alg. Geom. to appear*.

[34] Toen, B. (1999). Théorémes de Riemann-Roch pour les champs de Deligne-Mumford. *K-Theory*, 18(1), 33–76.

[35] Vafa, C. & Witten, E. (1994). A strong coupling test of S-duality. *Nucl. Phys.B.431, hep-th/9408074*, (pp. 3–77).

[36] Wang, W. (2020). Toric Sheaves on Hirzebruch Orbifolds. *Doc. Math.*, 25, 655–699.