Deformations of finite morphisms and applications to moduli of surfaces of general type with $K^2 = 4p_g - 8$

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Deformations of finite morphisms and applications to moduli of surfaces of general type with $K^2 = 4p_g - 8$

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Abstract

In this thesis, we study the deformations of the canonical morphism $\varphi : X \to \mathbb{P}^N$ of irregular surfaces X of general type with at worst canonical singularities, when φ is a finite Galois morphism of degree 4 onto a smooth variety of minimal degree Y inside $\mathbb{P}^N$. These surfaces satisfy $K_X^2 = 4p_g(X) - 8$, with p_g being an even number bigger than of equal to 4. For each $p_g \geq 6$, they are classified in [GP08] into four distinct irreducible families (if $p_g = 4$, then they are classified into three distinct irreducible families). We show that, when X is general in its family, any deformation of φ has degree greater than or equal to 2 onto its image. More interestingly, we prove in addition that, with the exception of one of the families when $p_g = 4$ and of another of the families for each $p_g \geq 8$, a general deformation of φ is two–to–one onto its image, which is a surface whose normalization is a ruled surface of appropriate genus, unless it is a product of genus two curves. In the latter case, it follows that any deformation of φ is four–to–one onto its image. We also show that the deformations of a general surface X of three of the four families are unobstructed, and consequently, X belongs to a unique irreducible component of the Gieseker moduli space, which we prove is uniruled. As a consequence of our results we show the existence of the following moduli components containing irregular quadruple Galois canonical covers as proper, locally closed subloci:

(1) for any $m \geq 1$ and $p_g = 2m + 2$, a $(8m + 20)$-dimensional, uniruled, irreducible component of $\mathscr{M}_{8m,1,2m+2}$, whose general element is a canonical double cover of a non-normal surface whose normalization is an elliptic ruled surface with invariant $e = 0$; in particular a general element has a genus $m + 1$ fibration over an elliptic curve.

(2) a 28-dimensional, uniruled irreducible component of $\mathscr{M}_{16,2,6}$, whose general element is a canonical double cover of a (smooth) ruled surface over a curve of genus 2 with invariant $e = -2$.

Among other things, our results are relevant because they exhibit moduli components such that the degree of the canonical morphism jumps up at proper locally closed subloci. This is in contrast with the moduli of surfaces with $K_X^2 = 2p_g - 4$ (which are double covers of surfaces of minimal degree), studied by Horikawa (see [Hor76]) but is pleasingly similar to the moduli of smooth curves of genus $g \geq 3$.

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Chapter 1

Introduction

Canonical covers of varieties of minimal degree have a ubiquitous presence in the geometry of algebraic surfaces and higher dimensional varieties. They appear as extremal cases in a variety of geometric situations. The first, paradigmatic example of a canonical double cover is the canonical morphism of a hyperelliptic curve. In the case of surfaces, Horikawa's celebrated work (see [Hor76]) shows that minimal surfaces of general type on the Noether line $K_X^2 = 2p_g - 4$ are all canonical double covers of surfaces of minimal degree. Horikawa's results imply that the deformations of canonical double covers of surfaces of minimal degree are again canonical double covers of surfaces of minimal degree, unlike what happens for canonical double covers in the case of curves of $g \geq 3$. Therefore, a natural and intriguing question for an algebraic surface, is:

Question 1.0.1. *If $n > 2$, does the degree n of a canonical cover change when we deform the cover?*

Canonical triple covers X of surfaces of minimal degree (these covers satisfy $K_X^2 = 3p_g(X) - 6$) are very few (their geometric genus $p_g(X)$ is bounded by 5 and their images are singular surfaces; see [Hor77], [Hor82] and [Kon91]), and, when $p_g(X) \leq 4$, their deformations are again canonical triple covers of surfaces of minimal degree. In contrast, the geometry of canonical quadruple covers of minimal degree (these covers satisfy $K_X^2 = 4p_g(X) - 8$) display a wide range of behaviors. Indeed, they act like general surfaces of general type from a number of geometric perspectives (see for example [GP11]). They are the first case among low degree covers where families of irregularity $q(X) \geq 1$ appear. Moreover, as Remark 1.0.2 below indicates, quadruple canonical covers are the only ones, among covers of smooth surfaces of minimal degree, having both unbounded geometric genus and irregularity (see Theorem 1.1.1), with the possible exception of degree 6 covers. All this makes canonical quadruple covers stands out as the most interesting case among canonical covers and are natural candidates for testing Question 1.0.1.

Remark 1.0.2. It follows from a more general result (see [GP04], Theorem 3.2), that there are no odd degree canonical covers of smooth surfaces of minimal degree other than $\mathbb{P}^2$. This together with [Bea82] implies that, if $\chi(X) \geq 31$, then the degree of a canonical cover of a smooth surface of minimal degree could only be $2, 4, 6$ or 8 (if $\chi(X) \leq 30$, then $q(X)$ is bounded). Since the irregularity of degree 8 canonical covers is bounded above by 3 when $p_g \geq 115$ (see [Xiao86]), degree 4 canonical covers are the only ones, among covers of smooth surfaces of minimal degree, having unbounded irregularity (and thus unbounded geometric genus, since $p_g(X) \geq 2q(X) - 4$ by [Bea82]) except possibly the degree six canonical covers. One can show that there are no smooth regular degree 6 abelian covers of smooth surfaces of minimal degree. We strongly believe that there is no such irregular covers as well.

Thus, in this article we focus on the study of the deformations of irregular quadruple Galois finite canonical covers of smooth surfaces of minimal degree. As the table of Theorem 1.1.1 shows, Galois covers capture by themselves the complexity of behavior mentioned above. We completely figure out how the deformations are for all excepting one of the existing families for which we have some partial results. From this study it follows that the answer to Question 1.0.1 (see Table 1 and Theorem 5.1.1) is, in general, positive. Regular quadruple Galois canonical covers of surfaces of minimal degree also provide positive answers to Question 1.0.1, as the authors will show in a forthcoming article. From all this we derive interesting consequences for the moduli of surfaces of general type.

1.1 Classification of irregular quadruple Galois canonical covers of surface scrolls.

The classification of irregular quadruple Galois canonical covers of surfaces of minimal degree was done by the first two authors in [GP08]. We need the technical details of their classification results, for the purpose of this article, so we will summarize them here. The image of these covers are smooth rational normal scrolls Y. Recall that a smooth rational normal scroll is a Hirzebruch surface $\mathbb{F}_e$ $(e \geq 0)$, which is, by definition, $\mathbb{P}(\mathscr{E})$, where $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-e)$. Let $p : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ be the natural projection. The line bundles on $\mathbb{F}_e$ are of the form $\mathscr{O}_Y(aC_0 + bf)$ where $\mathscr{O}_Y(C_0) = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ and $\mathscr{O}_Y(f) = p^*\mathscr{O}_{\mathbb{P}^1}(1)$. The line bundle $\mathscr{O}_Y(aC_0 + bf)$ is very ample if $b \geq ae+1$.

Theorem 1.1.1. *([GP08], Theorem* 0.1*) Let X be an irregular canonical surface and let Y be a smooth surface of minimal degree. If the canonical bundle of X is base-point-free and $\varphi : X \to Y$ is a quadruple Galois canonical cover, then Y is the Hirzebruch surface $\mathbb{F}_0$, embedded by $|C_0 + mf|$, $(m \geq 1)$. Let G be the Galois group of φ.*

(a) If $G = \mathbb{Z}_4$, then φ is the composition of two double covers $p_1 : X_1 \to Y$ branched along a divisor D_2 and $p_2 : X \to X_1$, branched along the ramification of p_1 and $p_1^* D_1$, where D_1 is a divisor on Y and with trace zero module $p_1^*\mathscr{O}_Y(-\frac{1}{2}D_1 - \frac{1}{4}D_2)$.

(b) If $G = \mathbb{Z}^{\oplus 2}$, then X is the fiber product over Y of two double covers of Y branched along divisors D_1 and D_2, and φ is the natural morphism from the fiber product to Y.

More precisely, φ has one of the sets of invariants shown in the following table. Conversely, if $\varphi : X \to Y$ is either

(1) the composition of two double covers $p_1 : X_1 \to Y$, branched along a divisor D_2, and $p_2 : X \to X_1$, branched along the ramification of p_1 and $p_1^* D_1$, and with trace zero module $p_1^*\mathscr{O}_Y(-\frac{1}{2}D_1 - \frac{1}{4}D_2)$, with D_1 and D_2 as described in rows 1 *of the table below; or*

(2) *the fiber product over Y of two double covers $p_1 : X_1 \to Y$ and $p_2 : X_2 \to Y$, branched respectively along divisors D_2 and D_1, as described in rows* 2, 3, *and* 4 *of the table below,*

then $\varphi : X \to Y$ is a Galois canonical cover whose Galois group is $\mathbb{Z}_4$ in case 1 *and $\mathbb{Z}^{\oplus 2}$ in case* 2.

Type	$p_g(X)$	Y	G	$D_1 \sim$	$D_2 \sim$	$q(X)$
$(1)_m$	$2m+2$	$\mathbb{F}_0$	$\mathbb{Z}_4$	$(2m+4)f$	$4C_0$	1
$(1')_m$	$2m+2$	$\mathbb{F}_0$	$\mathbb{Z}_2^{\oplus 2}$	$2C_0+(2m+4)f$	$4C_0$	1
$(2)_m$ $(m \geq 2)$	$2m+2$	$\mathbb{F}_0$	$\mathbb{Z}_2^{\oplus 2}$	$(2m+2)f$	$6C_0+2f$	m
$(3)_m$	$2m+2$	$\mathbb{F}_0$	$\mathbb{Z}_2^{\oplus 2}$	$(2m+4)f$	$6C_0$	$m+3$

Table 1.1: Classification of irregular qudruple Galois canonical covers of smooth scrolls

The authors in [GP08] showed that the singularities of the general surfaces of each of these families are as follows

Type	$(1)_m$	$(1')_m$	$(2)_m$	$(3)_m$
Singularities of a general surface	A_1	smooth	smooth	smooth

Table 1.2: Singularities of a general element in the family

As previously noted, the above table shows the existence of families of quadruple Galois canonical covers with unbounded irregularity. These covers carry irrational pencils:

Remark 1.1.2. (See also [GP11], Remark 3.4) Let X be as in Theorem 1.1.1.

(1) If X is of type $(1)_m$ or $(1')_m$, then X contains an elliptic pencil of genus $m+1$ hyperelliptic curves.

(2) If X is of type $(2)_m$, then X contains a genus m pencil of genus 2 curves.

(3) If X is of type $(3)_m$, then X contains a genus $m+1$ pencil of genus 2 curves.

In addition, some of the families of Theorem 1.1.1 are extremal cases for several inequalities concerning irregular surfaces of general type, such as $K_X^2 \geq 2\chi(X)$, the slope inequality (see

[LP12]), $K_X^2 \geq 2p_g(X)$ (see [Deb82]) and $p_g(X) \geq 2q(X) - 4$ (see [Bea82]). Because of all of this, quadruple Galois covers are interesting from the perspective of the geography of irregular surfaces.

Remark 1.1.3. Although the covers of Theorem 1.1.1 are *simple iterated double covers* in the sense of [Man97], they are not *good sequences.* Moreover, the point of view of our article is to study the deformation of canonical morphisms to projective spaces, rather than the deformation of finite morphisms between two surfaces. Therefore our study distinctly differs from the study of simple iterated double covers carried out by Catanese and Manetti.

1.2 Statement of main results

Let X be a surface as in Theorem 1.1.1, and let φ be the canonical morphism of X. First we present a description of our results about the algebraic formally semiuniversal deformation space of φ (which exists by Remark 4.1.7) in the following table (see Theorem 5.1.1).

X is of type *	Degree of **any** deformation of φ	Description of φ_t for a **general** t in the deformation space of φ	Normalization of the image of φ_t for a general t in the deformation space of φ
$(1)_m$	≥ 2	Double cover onto a non-normal surface	Elliptic ruled surface with invariant 0
$(1')_m$	≥ 2	Double cover onto a non-normal surface	Elliptic ruled surface with invariant 0
$(2)_2$	≥ 2	Double cover onto a smooth surface	Ruled surface over a curve of genus 2 with invariant -2
$(2)_m$ $(m \geq 3)$	≥ 2	Unknown	Unknown
$(3)_1$	4	Quadruple cover onto a smooth surface	$\mathbb{P}^1 \times \mathbb{P}^1$
$(3)_m$ $(m \geq 2)$	≥ 2	Double cover onto a smooth surface	Product of a curve of genus two with a curve of genus $m+1$

Table 1.3: Deformations of irregular quadruple Galois canonical covers

$*$ In Table 1.3, if X is of type $(1)_m$, $(1')_m$ or $(2)_m$, then we ask X to be smooth or to have A_1 singularities (these surfaces are general in the family as in Table 1.2).

It is illustrative to compare the results summarized in the above table with the deformations of lower degree canonical covers of surfaces. As already mentioned, any deformation of a degree 2, canonical morphisms is of degree 2. More generally, deformations of double or triple canonical covers of embedded projective bundles over $\mathbb{P}^1$ of arbitrary dimension are, respectively, of degree 2 and 3 (see [GGP13b], [GGP16a] and [GGP16b]). Thus Theorem 5.1.1 is in sharp contrast with these results and, as pointed out before, quadruple covers are the lowest degree examples of canonical covers for which the degree of the canonical map of a general deformation drops down, with the possible exception of degree 3 covers with $p_g = 5$.

Remark 1.2.1. We remark that the usual obstruction spaces of $\mathbf{Def}_\varphi$ and $\mathbf{Def}_X$ are non-zero, but still we show that these functors are smooth for general surfaces of types $(1)_m$ and $(1')_m$ and $(2)_2$ (see Remark 5.2.4).

1.3 Consequences on moduli of surfaces with $K^2 = 4p_g - 8$

We outline the description of the moduli components of the surfaces in Table 1.1

X is general of type	Obstructions to deformations of X	Geometry of unique moduli component $\mathscr{M}_{[X]}$ containing X	Canonical morphism of a general surface in $\mathscr{M}_{[X]}$	Normalization of the image of the general canonical morphism
$(1)_m$ or $(1')_m$	Unobstructed	Uniruled of dimension $8m+20$	Double cover onto a non-normal surface	Elliptic ruled surface with invariant 0
$(2)_2$	Unobstructed	Uniruled of dimension 28	Double cover onto a smooth surface	Ruled surface over a curve of genus 2 with invariant -2
$(3)_1$ see also [Ops05]	Unobstructed	Uniruled of dimension 6	Quadruple cover onto $\mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{P}^1 \times \mathbb{P}^1$
$(3)_m$ see also [Ops05]	Unobstructed	Uniruled of dimension $3m+3$	Double cover onto $\mathbb{P}^1 \times C_2$ $g(C_2) = m+1$	$\mathbb{P}^1 \times C_2$

Table 1.4: Moduli of irregular quadruple Galois canonical covers

For a given m, there is a unique component of moduli of surfaces of general type that contains *all* surfaces of type $(1)_m$ and $(1')_m$ (see Theorem 5.2.1). There is also a unique moduli component that contains all surfaces of type $(2)_2$.

The fact that general surfaces X as described in the above table lie on a unique component of the moduli of surfaces of general type is a consequence of the unobstructedness of X. For general surfaces X of type $(2)_m$, we can prove the unobstructedness only when $m = 2$ as Table 2 shows (see Theorem 5.2.2).

If X is a surface of type $(3)_m$, we can say the following: the surface X is a product of smooth hyperelliptic curves of genus 2 and $m+1$. Consequently, $\mathbf{Def}_X$ is smooth, and for a fixed m, the surfaces of type $(3)_m$ lie on a unique irreducible, uniruled and connected component of the moduli space of surfaces of general type of dimension $3m+3$ (see for example [Ops05]). If $m \neq 1$, then the canonical morphism of a surface $C_1 \times C_2$, general in the moduli component, is of degree 2, and its image is isomorphic to $\mathbb{P}^1 \times C_2$. If $m = 1$, the canonical morphism is a quadruple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ that remains quadruple upon deformation.

It is indeed interesting to note that the obstruction spaces of $\mathbf{Def}_\varphi$ and $\mathbf{Def}_X$ are non-zero, but still these functors are smooth for general surfaces of types $(1)_m$ and $(1')_m$ (see Remark 5.2.4).

1.4 Comparison with known results on unobstructedness and persistence of genus two fibrations

The classification of quadruple canonical covers shows the existence of fibrations in all genus, as is illustrated in Remark 1.1.2. Thus, the results in this article apply to fibrations of all genus. Deformation and moduli of genus two fibrations have been studied in [Sei95] and [GGP13a]. Therefore, for the special case of genus two fibrations, namely, the surfaces of type $(1)_1$, $(1')_1$ and $(2)_m$, their results on unobstructedness and persistence of genus two fibrations upon deformation do apply. But in this article we show uniruledness not only of these moduli components but also of the moduli components of fibrations of all genus. We make this precise in the following remark

Remark 1.4.1. We compare our results with the existing results along this direction and highlight a few consequences.

(1) By Remark 1.1.2, our results show for any genus g, existence of moduli components whose general element has a genus g fibration over an elliptic curve.

(2) The moduli of surfaces of general type fibered by genus 2 curves have been studied by Seiler in [Sei95]. The surfaces of type $(1)_1$, $(1')_1$ and $(2)_m$ are fibred by genus 2 curves as we noted in Remark 1.1.2. In Seiler's notation ([Sei95], p. 774), surfaces of types $(1)_1$ and $(1')_1$ have invariant $(1,0,4)$ and surfaces of type $(2)_m$ have invariant $(m,0,4)$. Consequently, one can directly see the unobstructedness of X using [Sei95], Theorem 3.11, for smooth X of type $(1')_1$ or $(2)_2$.

(3) A general point of the moduli component of $\mathscr{M}_{8,1,4}$ produced in Theorem 5.2.1 is a surface with invariant $(1,0,4)$ in Seiler's notation ([Sei95], p. 774). Consequently, this is component of type I described in [Sei95], p. 809. We prove that this component is uniruled, and the image of the canonical morphism is non-normal for a general surface of this component.

(4) A general point of the moduli component of $\mathscr{M}_{16,2,5}$ produced in Theorem 5.2.2 is a surface with invariant $(2,-2,4)$ in Seiler's notation ([Sei95], p. 774). This is also a component of type I described in [Sei95], p. 809, and we prove that this component is uniruled.

(5) Notice that smooth surfaces X of type $(1')_1$ and $(2)_m$ satisfy $q(X) = q(C)$ where C is the base curve of the genus 2 fibration described in Remark 1.1.2. By [GGP13a], Theorem 2.7, the genus 2 fibration persists for any deformation of X. In this article, we reprove this result. Moreover, it is fairly straightforward to see that the normalization of the image

of the canonical morphism of a general surface of the component of $\mathscr{M}_{8,1,4}$ produced in Theorem 5.2.1 is in fact the image of its bicanonical morphism.

1.5 Techniques and brief sketch of method of proof.

In the course of our proofs, we will carry out a combination of the following seven steps (not necessarily in this order).

(1) We show that for a generic surface X of each of the four families described in Theorem 1.1.1, any deformation of X is locally trivial by Corollary 4.1.9 (this is obvious except when X is of type $(1)_m$). Consequently, we do not need to distinguish between the functors **Def** and **Def′** (see *Notations and conventions* below).

(2) To show the existence of a component in the formally semiuniversal algebraic deformation space of a generic quadruple Galois canonical cover φ of one of the four families, the degree of whose general element is ≤ 2, we deform φ to a double cover. The technique involves two main steps.

Step 1. We show that for a generic irregular quadruple Galois canonical cover $\varphi : X \to Y \hookrightarrow \mathbb{P}^N$, excepting when X is of type $(3)_1$, one can find a smooth intermediate cover $\varphi' : Y' \to Y \hookrightarrow \mathbb{P}^N$ such that φ factors as $X \to Y' \to Y \hookrightarrow \mathbb{P}^N$, and φ' can be deformed to a finite birational morphism. This is achieved by deforming the double structure on Y induced by a first order deformation of φ' to a reduced (but possibly non-normal) scheme. This is done with the aid of a fundamental theorem of the deformation theory of finite morphisms, i.e., Theorem 2.7.11, proved by the first two authors and González in [GGP10].

Step 2. We then show that each such deformation of the intermediate cover Y' is the base of a deformation of the morphism $p : X \to Y'$ with varying target.

Consequently, there is a component in the algebraic formally semiuniversal algebraic deformation space of φ the degree of whose general element is ≤ 2.

(3) Next we show with the aid of Proposition 4.1.5 (a consequence of a result of Wehler; see [Weh86]) that the degree of any deformation of φ is ≥ 2, in fact it factors through a deformation of Y'.

(4) We describe the image of a general element of this component using the results of Seiler (see [Sei92]).

(5) Let X be a generic surface of types $(1)_m, (1')_m$ or $(2)_2$. To show the smoothness of $\mathbf{Def}_\varphi$ and $\mathbf{Def}_X$, we first show that $\mathbf{Def}_{\varphi'}$ is smooth. Then we show that under some assumptionswhich are satisfied in our case, the following chain of implications holds;

$$\mathbf{Def}_{\varphi'} \text{ is smooth } \overset{\text{Thm. 4.2.1}}{\Longrightarrow} \mathbf{Def}_{p/\mathbb{P}^N} \text{ is smooth } \overset{\text{Cor. 4.2.3}}{\Longrightarrow} \mathbf{Def}_{\varphi} \text{ is smooth } \overset{\text{Cor. 4.2.4}}{\Longrightarrow} \mathbf{Def}_X \text{ is smooth}$$

(6) The uniruledness of the moduli component of X essentially follows from the uniruledness of the algebraic formally semiuniversal deformation space of $\mathbf{Def}_{p/\mathbb{P}^N}$ which we construct in Theorem 4.2.1.

1.6 Notation and conventions

- We will always work over the field of complex numbers $\mathbb{C}$ and a variety is an integral separated scheme of finite type over $\mathbb{C}$.

2. The symbols '$\sim$' denotes linear equivalence and '$\equiv$' denotes numerical equivalence.

3. We will use the multiplicative and the additive notation of line bundles interchangeably. Thus, for line bundles L_1, L_2, $L_1 \otimes L_2$ and $L_1 + L_2$ are the same. L^{-r}, $L^{\otimes -r}$ (or $-rL$) denotes $(L^\vee)^{\otimes r}$.

4. If L_i is a line bundle on the variety X_i for $i = 1,2$, $L_1 \boxtimes L_2$ is by definition, the line bundle $p_1^* L_1 \otimes p_2^* L_2$ on $X_1 \times X_2$ where $p_i : X_1 \times X_2 \to X_i$ is the i-th projection for $i = 1,2$. When $X_i = \mathbb{P}^1$ for $i = 1,2$, then $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b) := \mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(b)$.

5. For a morphism $X \to Y$ between algebraic schemes, $\Omega_{X/Y}$ (or $\Omega^1_{X/Y}$) is the sheaf of relative differentials and $T_{X/Y} = \mathscr{H}om(\Omega_{X/Y}, \mathscr{O}_X)$ is the relative tangent sheaf. By convention, Ω_X (or Ω^1_X) and T_X is obtained by taking $Y = \mathrm{Spec}(\mathbb{C})$.

6. For an alegbraic scheme X, let $T^i(X)$ and $\mathscr{T}^i(X)$ be the local and global cohomology of the cotangent complex respectively. It is well known that when X is a projective variety, the first cotangent sheaf $\mathscr{T}^1_X = \mathscr{E}xt^1(\Omega_X, \mathscr{O}_X)$ and $T^1(X) = \mathrm{Ext}^1(\Omega_X, \mathscr{O}_X)$.

7. For an algebraic scheme, $\mathbf{Def}_X$ (resp. $\mathbf{Def}'_X$) is the functor of deformations (resp. locally trivial deformations) of X.

8. For an algebraic scheme X and a line bundle L on it, $\mathbf{Def}_{(X,L)}$ (resp. $\mathbf{Def}'_{(X,L)}$) is the functor of deformations (resp. locally trivial deformations) of the scheme and the line bundle i.e. the pair (X, L).

9. For a morphism $\varphi : X \to Z$ between algebraic schemes, $\mathbf{Def}_\varphi$ (resp. $\mathbf{Def}'_\varphi$) is the functor of deformations (resp. locally trivial deformations) of φ with fixed target.

10. For morphism $X \xrightarrow{\pi} Y \to Z$ of algebraic schemes, $\mathbf{Def}_{\pi/Z}$ (resp. $\mathbf{Def}'_{\pi/Z}$) is the functor of Z-deformations (resp. locally trivial Z-deformations) of π with varying target.

For a scheme X, if $\mathbf{Def}_X = \mathbf{Def}'_X$, then $\mathbf{Def}_{(X,L)} = \mathbf{Def}'_{(X,L)}$, $\mathbf{Def}_\varphi = \mathbf{Def}'_\varphi$ and $\mathbf{Def}_{\pi/Z} = \mathbf{Def}'_{\pi/Z}$ respectively in the situation of 9, 10 and 11.

Chapter 2

Deformation and moduli: basic results

2.1 Functor of Artin rings

Let k be an algebraically closed field. Let

(1) $\mathscr{A}$ be category of local artinian $k-$ algebras with residue field k.

(2) $\hat{\mathscr{A}}$ be the category of complete local noetherian $k-$algebras with residue field k.

(3) $\mathscr{A}^*$ be the category of local noetherian $k-$ algebras with residue field k.

Definition 2.1.1. A functor of artin rings is a covariant functor

$$F : \mathscr{A} \to \text{(Sets)}$$

For $A \in \mathscr{A}$, an element $\zeta \in F(A)$ is called an infinitesimal deformation of $\zeta_0 \in F(k)$.

Example 2.1.2. An important class of examples of functors of artin rings h_R are obtained by fixing $R \in \hat{\mathscr{A}}$ and setting

$$h_R(A) = \mathrm{Hom}_{\hat{\mathscr{A}}}(R, A)$$

Definition 2.1.3. A functor of Artin rings F is called prorepresentable if if F is isomorphic as a functor to h_R for some $R \in \hat{\mathscr{A}}$.

Remark 2.1.4. A prorepresentable functor $F = h_R$ satisifies the following properties

(H_0) $F(k)$ consists of one element (the canonical quotient $R \to R/m_R = k$ where m_R is the unique maximal ideal of R.)

Let

$$\begin{array}{ccc} A' & \xrightarrow{\Phi} & A" \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

be a diagram in $\mathscr{A}$ and consider the natural map

$$\alpha : F(A' \times_A A") \to F(A') \times_{F(A)} F(A") \tag{2.1}$$

induced by the commutative diagram

$$\begin{array}{ccc} F(A' \times_A A") & \longrightarrow & F(A") \\ \downarrow & & \downarrow \\ F(A') & \longrightarrow & F(A) \end{array}$$

then

(H_l (left exactness) α is bijective

(H_f) $F(k[\epsilon])$ has the structure of a finite dimensional $k-$ vector space where $k[\epsilon]$ is the ring of dual numbers.

A property weaker than H_l that is satisfied by a prorepresentable functor F is the following

(H_ϵ) α is surjective when $A = k$ and $A" = k[\epsilon]$.

Lemma 2.1.5. *([Ser06], Lemma* 2.2.1*) Let F be a functor of Artin rings satifying conditions H_0 and H_ϵ then the set $F(k[\epsilon])$ has the structure of a $k-$ vector space in a functorial way. This vector space is called the tangent space of the functor F and is denoted by t_F. If $F = h_R$ then let $t_R := t_F$.*

Definition 2.1.6. Every functor of Artin rings F can be extended to a functor

$$\hat{F} : \hat{\mathscr{A}} \to \text{(sets)}$$

by letting, for every $R \in \hat{\mathscr{A}}$:

$$\hat{F}(R) = \varprojlim F(R/m_R^{n+1})$$

and for every $\varphi : R \to S$:

$$\hat{F}(\varphi) : \hat{F}(R) \to \hat{F}(S)$$

induced by the maps

$$F(R/m_R^n) \to F(R/m_R^n),\ n \geq 1.$$

An element $\hat{u} \in \hat{F}(R)$ is called a **formal** element of F. By definition $\hat{u}$ can be represented by a system of elements $\{u_n \in F(R/m_R^{n+1})\}_{n\geq 0}$ such that for every $n \geq 1$, the map

$$F(R/m_R^{n+1}) \to F(R/m_R^n)$$

induced by

$$R/m_R^{n+1} \to R/m_R^n$$

sends

$$u_n \mapsto u_{n-1}$$

Lemma 2.1.7. *([Ser06], Lemma 2.2.2) Let $R \in \hat{A}$. Then there is a $1-1$ correspondence between elements of $\hat{F}(R)$ and the set of morphisms of functors*

$$h_R \to F.$$

Definition 2.1.8. Let $F \to G$ be a morhism of functors of Artin rings. f is called smooth if for every surjection $\mu : B \to A$ in $\mathscr{A}$ the natural map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

induced by the commutative diagram

$$\begin{array}{ccc} F(B) & \longrightarrow & G(B) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & G(A) \end{array}$$

is surjective. The functor F is called smooth if the morphism from F to the constant functor

$$G(A) = \{\text{one element}\} \text{ for all } A \in \mathscr{A}$$

is smooth; equivalently, if

$$F(\mu) : F(B) \to F(A)$$

is surjective for every surjection $\mu : B \to A$ in $\mathscr{A}$.

Definition 2.1.9. ([Ser06], Definition 2.2.6) Let F be a functor of Artin rings. A formal element $\hat{u} \in \hat{F}(R)$ for some $R \in \hat{\mathscr{A}}$, is called versal if the morphism defined $h_R \to F$ defined by $\hat{u}$ is smooth; $\hat{u}$ is called semiuniversal if it is versal and moreover the differential $t_R \to t_F$ is bijective.

Definition 2.1.10. ([Ser06], Section 1.1.1) Let $A \to R$ be a ring homomorphism. An $A-$ extension of R (or of R by I) is an exact sequence

$$(R', \varphi) : 0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

The set of isomorphism classes of such extensions is denoted by $\mathrm{Ex}_A(R, I)$. It has a natural $R-$ module structure. Let $A = k$. Then the isomorphism classes of extensions

$$0 \to (t) \to R' \xrightarrow{\varphi} R \to 0$$

such that $t \in m_{R'}$ is annihilated by $m_{R'}$ (so that (t) is a $k-$ vector space of dimension one) is denoted by $\mathrm{Ex}_k(R, k)$.

Definition 2.1.11. ([Ser06], Definition 2.2.9) Let F be a functor of Artin rings. Suppose that $v(F)$ is a $k-$ vector space such that for every $A \in \mathscr{A}$ and for evry object $\zeta \in F(A)$, there is a $k-$ linear map

$$\zeta_v : \text{Ex}(A,k) \to v(F)$$

with the following property: $\ker(\zeta_v)$ consists of isomorphism classes of extensions $(\widetilde{A},\varphi)$ such that

$$\zeta \in \text{Im}[F(\widetilde{A}) \to F(A)]$$

Then $v(F)$ is called an obstruction space for the functor F. If F has (0) as an obstruction space then it is called unobstructed.

2.2 Algebraic deformations and Kodaira-Spencer map

Definition 2.2.1. Let X be a projective scheme and consider a flat family of deformations η of X parametrized by an affine scheme $S = \text{Spec}(B)$, where B is a noetherian $k-$ algebra, namely a cartesian diagram as follows

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathscr{X} \\ \downarrow & & \downarrow \pi \\ \text{Spec(k)} & \xrightarrow{s} & \text{Spec(B)} \end{array}$$

where π is projective and flat and $\text{Spec}(k)$ maps to a point $s \in S$. The triple (S,s,η) is called an algebraic deformation of X.

Given such a deformation, (S,s,η), let η_n be the infinitesimal deformation induced by pulling back η under the natural closed embedding

$$\text{Spec}(\mathscr{O}_{S,s}/m^{n+1}) \to S.$$

We have $\mathscr{O}_{S,s}/m^{n+1} = \hat{\mathscr{O}}_{S,s}/\hat{m}^{n+1}$ and therefore it follows that $(\hat{\mathscr{O}}_{S,s},\eta_n)$ is a formal deformation of X defined by η.

Definition 2.2.2. ([Ser06], Definition 2.5.7) A deformation (S,s,η) is called formally universal (resp. formally semiuniversal, formally versal) if the formal deformation $(\hat{\mathscr{O}}_{S,s},\eta_n)$ is universal (resp. formally semiuniversal, formally versal)

Proposition 2.2.3. *([Ser06], Proposition* 2.5.8*) Let* (S, s, η) *be an algebraic deformation of* X*. Then*

(1) If η *is formally versal (resp. formally semiuniversal or formally universal), then the Kodaira-Spencer map*

$$k_{\pi,S} : T_s S \to \mathbf{Def}_X$$

is surjective (resp. an isomorphism)

(2) If S *is nonsingular at* s *and the Kodaira-Spencer map* $k_{\pi,S}$ *is surjective (resp. an isomorphism) then* η *is formally versal (resp. formally semiuniversal) and* X *is unobstructed, i.e, the functor* $\mathbf{Def}_X$ *is smooth.*

2.3 Algebraization and theorems of Grothendieck and Artin

Definition 2.3.1. ([Ser06], Definition 2.5.9) A formal deformation $(\bar{A}, \hat{\eta} = \{\eta_n\})$ is called algebraizable if there exists an algebraic deformation (S, s, ζ) and an isomorphism $\hat{\mathscr{O}_{S,s}} \cong \bar{A}$ sending $\eta_n \to \zeta_n$ for all n. The deformation (S, s, ζ) is called an algebraization of $(\bar{A}, \{\eta_n\})$.

Definition 2.3.2. ([Ser06], Definition 2.5.10) Let X be an algebraic scheme and $\bar{A} \in \hat{\mathscr{A}}$. A formal deformation $(\bar{A}, \{\eta_n\})$ is called effective if there exist a deformation $\bar{\eta}$:

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec(k)} & \longrightarrow & \mathrm{Spec}(\bar{A}) \end{array}$$

such that $\hat{\eta}$ is the formal deformation associated to $\bar{\eta}$.

Theorem 2.3.3. *([Ser06], Theorems* 2.5.13, 2.5.14*)*

(1) (Grothendieck) Let $\bar{\pi} : \mathscr{X} \to$ *Specf*$(\bar{A})$ *be a formal deformation of* X*. Assume that there exist a closed embedding of formal schemes* $j : \mathscr{X} \subset \mathscr{P}^r_{\bar{A}}$ *such that* $\bar{\pi} = pj$ *where* $p : \mathscr{P}^r_{\bar{A}} \to$ *Specf*$(\bar{A})$ *is the projection. Then* $\bar{\pi}$ *is effective.*

(2) *(Artin) Let X be a projective scheme and $(\bar{A}, \hat{\eta})$ be an effective formal versal deformation of X. Then $(\bar{A}, \hat{\eta})$ is algebraizable.*

2.4 Deformations of schemes

For a scheme X over an algebraically closed field k, define the functor $\mathbf{Def}_X$ and $\mathbf{Def}'_X$ of local artin $k-$ algebras as follows:

Definition 2.4.1. $\mathbf{Def}_X(A)$ consists of isomorphism classes of cartesian squares as below with $\mathscr{X} \to \operatorname{Spec}(A)$ a flat morphism.

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec(k)} & \longrightarrow & \operatorname{Spec(A)} \end{array}$$

Two deformations $\mathscr{X}$ and $\mathscr{X}'$ are isomorphic if there is an isomorphism of deformations $f : \mathscr{X} \to \mathscr{X}'$ such that we have the following cartesian square

$$\begin{array}{ccccc} X & \longrightarrow & \mathscr{X} & \xrightarrow{f} & \mathscr{X}' \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec(k)} & \longrightarrow & \operatorname{Spec(A)} & \xrightarrow{id} & \operatorname{Spec(A)} \end{array}$$

A deformation $\mathscr{X}$ is said to be trivial if $\mathscr{X} \cong X \times \operatorname{Spec}(A)$. A deformation $\mathscr{X}$ is said to be locally trivial if for every point $x \in X$ there exist an open neighbourhood $x \in U_x \subset X$ such that the induced deformation of U_x given by $\mathscr{X}$ satisfying

$$\begin{array}{ccc} U_x & \longrightarrow & \mathscr{X}|_{U_x} \\ \downarrow & & \downarrow \\ \operatorname{Spec(k)} & \longrightarrow & \operatorname{Spec(A)} \end{array}$$

is the trivial deformation of U_x. Let $\mathbf{Def}'_X$ be the subfunctor of $\mathbf{Def}_X$ consisting of locally trivial deformations of X.

An affine scheme $\operatorname{Spec}(B)$ is called rigid if any infinitesimal deformation is trivial.

Theorem 2.4.2. *([Ser06], Theorem 1.2.4) Every smooth $k-$ algebra is rigid. In particular, every* *nonsingular affine algebraic variety is rigid.*

This shows any infinitesimal deformation of a nonsingular algebraic variety is locally trivial.

Theorem 2.4.3. *([Ser06], Theorem* 2.4.1*, Proposition* 2.4.6, 2.4.8*)*

(i) *For any algebraic scheme X, the functors $\mathbf{Def}_X$ and $\mathbf{Def}'_X$ satisfy conditions H_0, $\bar{H}$ and H_ϵ of Schlessinger's theorem. Therefore if $\mathbf{Def}_X(k[\epsilon])$ (resp. $\mathbf{Def}'_X(k[\epsilon])$) is finite dimensional, then $\mathbf{Def}_X$ (resp. $\mathbf{Def}'_X$) has a semiuniversal formal element.*

(ii) *There is a canonical identification of $k-$ vector spaces*

$$\mathbf{Def}'_X(k[\epsilon]) = H^1(T_X)$$

In particular if X is nonsingular then

$$\mathbf{Def}_X(k[\epsilon]) = \mathbf{Def}'_X(k[\epsilon]) = H^1(T_X)$$

(iii) *If X is an arbitrary algebraic scheme then we have a natural identification*

$$\mathbf{Def}_X(k[\epsilon]) = Ex_k(X, \mathcal{O}_X)$$

and an exact sequence

$$0 \to H^1(X, T_X) \to \mathbf{Def}_X(k[\epsilon]) \to H^0(X, \mathcal{T}^1_X) \to H^2(X, T_X)$$

In particular, $\mathbf{Def}_{B_0}(k[\epsilon]) = T^1_{B_0}$ if $X = Spec(B_0)$ is affine.

(iv) *If X is a reduced algebraic scheme then there is an isomorphism*

$$\mathbf{Def}_X(k[\epsilon]) \cong Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

and the previous exact sequence in (iii) is isomorphic to the local to global exact sequence for $\mathscr{E}xts$:

$$0 \to H^1(X, T_X) \to Ext^1_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X) \to H^0(X, \mathscr{E}xt^1_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)) \to H^2(X, T_X)$$

(v) If X is an arbitrary algebraic scheme then then $H^2(T_X)$ is an obstruction space for the functor $\mathbf{Def}'_X$.

(vi) Let X be a reduced local complete intersection algebraic scheme and char($k = 0$), then $Ext^2_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)$ is an obstruction space for $\mathbf{Def}_X$

2.5 Deformations of line bundles

For a scheme X over an algebraically closed field k, define the functor $\mathbf{Def}_{(X,L)}$ and $\mathbf{Def}_{(X,L)}$ of local artin $k-$ algebras as follows :

Definition 2.5.1. $\mathbf{Def}_{(X,L)}(A)$ consists of isomorphism classes of the set of pairs $(\mathscr{X}, \mathscr{L})$ such that we have the following cartesian square with $\mathscr{X} \to \text{Spec}(A)$ a flat morphism.

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \text{Spec(k)} & \longrightarrow & \text{Spec(A)} \end{array}$$

and $\mathscr{L}$ is an invertible sheaf on X such that $\mathscr{L}|_X = L$. $\mathbf{Def}'_{(X,L)}(A)$ is the subfunctor of $\mathbf{Def}_{(X,L)}(A)$ consisting of isomorphism classes of pairs $(\mathscr{X}, \mathscr{L})$ such that $\mathscr{X}$ is a locally trivial deformation of X. Two deformations $(\mathscr{X}, \mathscr{L})$ and $(\mathscr{X}', \mathscr{L}')$ are isomorphic if there is an isomorphism of deformations $f : \mathscr{X} \to \mathscr{X}'$ and an isomorhism $\mathscr{L} \to f^*(\mathscr{L}')$.

For any scheme X, there is a natural map $\mathscr{O}^*_X \to \Omega_X$ defined by $u \mapsto du/u$. For a line bundle L on X, the natural map induced between the cohomology groups $H^1(\mathscr{O}^*_X) \xrightarrow{c} H^1(\Omega_X) \cong$

$\mathrm{Ext}^1(\mathscr{O}_X, \Omega_X)$ gives an extension

$$e_L: \quad 0 \to \Omega_X \to \mathscr{Q}_L \to \mathscr{O}_X \to 0.$$

We set $\mathscr{E}_L := \mathscr{H}\mathrm{om}(\mathscr{Q}_L, \mathscr{O}_X)$, and we obtain the following exact sequence which is known as the Atiyah exact sequence when X is smooth:

$$0 \to \mathscr{O}_X \to \mathscr{E}_L \to T_X \to 0. \tag{2.2}$$

Altmann and Christophersen has generalized [Ser06], Theorem 3.3.11 and showed that $H^1(\mathscr{E}_L)$ parametrizes first order locally trivial deformations of (X, L) when X is reduced.

Theorem 2.5.2. *([AC10], Theorem 3.1 (ii), [Ser06], Theorem 3.3.11) If X is a reduced projective scheme and L is an invertible sheaf on X, Then*

(i) Both $\mathbf{Def}_{X,L}$ and $\mathbf{Def}'_{X,L}$ have a semiuniversal formal element.

(ii) $\mathbf{Def}_{X,L}(k[\epsilon]) = Ext^1(\mathscr{Q}_L, \mathscr{O}_X)$ and $\mathbf{Def}'_{X,L}(k[\epsilon]) = H^1(\mathscr{E}_L)$ and there exists an exact sequence of $k-$ vector spaces

$$0 \to H^1(\mathscr{E}_L) \to Ext^1(\mathscr{Q}_L, \mathscr{O}_X) \to H^0(\mathscr{T}^1_X) \to H^2(\mathscr{E}_L)$$

(iii) The obstructions for $\mathbf{Def}_{X,L}$ lie in $H^0(\mathscr{T}^2_X)$, $H^1(\mathscr{T}^1_X)$ and $H^2(\mathscr{E}_L)$. The obstructions for $\mathbf{Def}'_{X,L}$ lie in $H^2(\mathscr{E}_L)$

(iv) Given a first order deformation of X with isomorphism class $\zeta \in Ext^1(\Omega_X, \mathscr{O}_X)$, there exists a first order deformation of L along ζ if and only if in the Yoneda product

$$Ext^1(\Omega_X, \mathscr{O}_X) \times Ext^1(\mathscr{O}_X, \Omega_X) \to Ext^2(\mathscr{O}_X, \mathscr{O}_X) = H^2(\mathscr{O}_X)$$

we have $\zeta \cdot c(L) = 0$

(v) *If L is very ample and $H^1(L) = 0$, then any formal deformation of the pair (X, L) is effective. It follows from (i) and (v) and a theorem of Artin ([Ser06], Theorem 2.5.14) that under conditions in (v), $\mathbf{Def}_{X,L}$ has an algebraic versal deformation.*

2.6 Deformations of sections

One can follow the treatment of [Ser06], Section 3.3.4 to define a map $M : \mathscr{E}_L \to H^0(L)^\vee \otimes L$ that fits into the following commutative diagram where the left vertical map is the one obtained in (2.2).

$$\begin{array}{ccc} \mathscr{O}_X & \xrightarrow{m} & H^0(L)^\vee \otimes L \\ \downarrow & & \| \\ \mathscr{E}_L & \xrightarrow{M} & H^0(L)^\vee \otimes L \end{array} \tag{2.3}$$

The proof of the following follows by repeating the argument of [Ser06], Proposition 3.3.14 word by word.

Proposition 2.6.1. *Let X be a reduced projective scheme and let L be a line bundle on X. Assume $(\mathscr{X}, \mathscr{L})$ is a first order locally trivial deformation of (X, L) defined by a cohomology class $\eta_1 \in H^1(\mathscr{E}_L)$ according to Theorem 2.5.2. Let $s \in H^0(L)$ be a section of L. Then s lifts to a section $\mathscr{L}$ if and only if $s \in ker(M_1(\eta_1))$, where $M_1 : H^1(\mathscr{E}_L) \to Hom(H^0(L), H^1(L))$ is induced by M.*

When L is base point free, in order to verify the above section-lifting-criterion, we will make use of [Ser06], (3.39) diagram with exact rows and columns, which is given for smooth case, and whose existence is a routine computation when X is reduced and projective. In particular, we have the exact sequence

$$0 \to \mathscr{E}_L \to H^0(L)^\vee \otimes L \to \mathscr{N}_{\varphi_L} \to 0, \tag{2.4}$$

where $\varphi_L : X \to \mathbb{P}(H^0(L)^\vee)$ is the morphism induced by $|L|$.

2.7 Deformations of morphisms

In this section, we provide the main technical results regarding the deformations of morphisms that are essential to carry out our study.

2.7.1 Preliminaries on deformations of morphisms

Given a morphism $\varphi : X \to Z$ of algebraic schemes, we define the functor of local artin $k-$ algebras $\mathbf{Def}_\varphi$ of deformations of φ with fixed target.

Definition 2.7.1. $\mathbf{Def}_\varphi(A)$ consists of the set of isomorphism classes of cartesian diagrams as shown below with $\mathscr{X} \to \mathrm{Spec}(A)$ a flat morphism.

$$\begin{CD} X @>>> \mathscr{X} \\ @VVV @VVV \\ Y @>>> Z \times \mathrm{Spec}(A) \\ @VVV @VVV \\ \mathrm{Spec(k)} @>>> \mathrm{Spec(A)} \end{CD}$$

Denote by $\mathbf{Def}'_\varphi$ the subfunctor consisting of isomorphism classes of cartesian diagrams as above where $\mathscr{X}$ is a locally trivial deformation of X. Two deformations $\mathscr{X} \to Z \times \mathrm{Spec}(A)$ and $\mathscr{X}' \to Z \times \mathrm{Spec}(A)$, are isomorphic if there exist an isomorphism Φ

$$\begin{array}{ccccc} & \swarrow & X & \searrow & \\ \mathscr{X} & & \xrightarrow{\quad\Phi\quad} & & \mathscr{X} \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec(k)} & & \end{array}$$

which makes the following diagram commutative

$$\begin{array}{ccccc} \mathscr{X} & & \xrightarrow{\quad\Phi\quad} & & \mathscr{X} \\ & \searrow & & \swarrow & \\ & & Z \times \mathrm{Spec}(A) & & \end{array}$$

2.7.2 Preliminaries on normal sheaves

Locally trivial deformation theory of a morphism is governed by the *normal sheaf* that we define below.

Definition 2.7.2. ([Ser06], 3.4.5) To a morphism $\varphi : X \to Z$ of algebraic schemes there exists an exact sequence of coherent sheaves which defines the sheaf $\mathcal{N}_\varphi$ called the *normal sheaf* of φ;

$$0 \to T_{X/Z} \to T_X \to \mathrm{Hom}(\varphi^*\Omega_Z, \mathcal{O}_X) \to \mathcal{N}_\varphi \to 0.$$

The morphism φ is called *non-degenerate* if $T_{X/Z} = 0$.

A morphism being non-degenerate is equivalent to being unramified in an open dense set.

Theorem 2.7.3. *([Ser06], Theorem 3.4.8) Suppose $\varphi : X \to Z$ be a morphism of algebraic schemes with X projective. If φ is non-degenerate, then $\mathbf{Def}'_\varphi$ has a formal semiuniversal deformation. Its tangent space $\mathbf{Def}'_\varphi(k[\epsilon] = H^0(N_\varphi)$ and $H^1(N_\varphi)$ is an obstruction space for $\mathbf{Def}'_\varphi$.*

Thus, a finite flat morphism between normal Cohen-Macaulay varieties is non-degenerate and so is the composition of non-degenerate morphisms between normal Cohen-Macaulay varieties. The following is the general version of [Gon06], Lemma 3.3 whose proof we omit.

Lemma 2.7.4. *Let X, Y, Z be normal Cohen-Macaulay varieties. Let $\pi : X \to Y$ be a non-degenerate morphism for which π^* is an exact functor (this happens if π is finite and flat) and let $\psi : Y \to Z$ be a non-degenerate morphism. Suppose $\varphi := \psi \circ \pi$. Then there is an exact sequence;*

$$0 \to \mathcal{N}_\pi \to \mathcal{N}_\varphi \to \pi^*\mathcal{N}_\psi \to 0.$$

2.7.3 Normal abelian covers of smooth varieties

Our objects of study are canonical morphisms that factor through abelian covers. We recall some basic facts about these covers, see [Par91] for further details.

Definition 2.7.5. Let Y be a variety and let G be a finite abelian group. A *Galois cover of Y with Galois group G* is a finite flat morphism $\pi : X \to Y$ together with a faithful action of G on X that exhibits Y as a quotient of X via G.

Let $\pi : X \to Y$ be a Galois G cover of a smooth variety Y with X normal. Then $\pi_* \mathcal{O}_X$ splits as a direct sum indexed by the characters. More precisely,

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} L_\chi^{-1}.$$

Let D be the branch divisor of π. Let $\mathscr{C}$ be the set of cyclic subgroups of G and for all $H \in \mathscr{C}$, denote by S_H the set of generators of the group of characters H^*. Then, we may write

$$D = \sum_{H \in \mathscr{C}} \sum_{\psi \in S_H} D_{H,\psi}$$

where $D_{H,\psi}$ is the sum of all the components of D that have inertia group H and character ψ. The sheaves L_χ and the divisors $D_{H,\psi}$ is called the *building data* of the cover. For every pair $\chi, \chi' \in G^*$, for every $H \in \mathscr{C}$ and for every $\psi \in S_H$, one may write $\chi|_H = \psi^{i_\chi}$ and $\chi'|_H = \psi^{i_{\chi'}}$, $i_\chi, i_{\chi'} \in \{0, \ldots, m_H - 1\}$ where m_H is the order of H. Let $S_\chi = \{(H, \psi) : \chi|_H \neq \psi^{m_H - 1}\}$.

Recall that for a variety Z, T_Z denotes its tangent sheaf. Furthermore, if Z is smooth and D is a divisor on Z, $\Omega_Z^p(\log D)$ denotes the sheaf of logarithmic differential p–forms. The following is a generalization of [Par91], Corollary 4.1 to the case X normal.

Proposition 2.7.6. *Let $\pi : X \to Y$ be an abelian cover with Galois group G, X normal and Y smooth. Assume the branch divisor D of π is normal crossing. Then,*

$$(\pi_* \mathcal{N}_\pi)^\chi \cong \bigoplus_{(H,\psi) \in S_\chi} \mathcal{O}_{D_{H,\psi}}(D_{H,\psi}) \otimes L_\chi^{-1}.$$

Proof. We have an exact sequence

$$0 \to T_X \to \pi^* T_Y \to \mathcal{N}_\pi \to 0.$$

Now since π is finite and all sheaves in the exact sequence are quasi-coherent we have that the pushforward is an exact functor and consequently we get the following exact sequence.

$$0 \to \pi_* T_X \to \pi_* \pi^* T_Y \to \pi_* \mathcal{N}_\pi \to 0$$

Now observe that $(\pi_* \pi^* T_Y)^\chi = T_Y \otimes L_\chi^{-1}$. Also note that $\pi_* T_X$ is a reflexive sheaf and hence it is enough to determine $\pi_* T_X$ in an open subset of codimension ≥ 2. Consider the set S which is the image of the singular locus of X under π. Since X is normal we have that S is of codimension at least two in Y. Removing S and $\pi^{-1}(S)$ (which is also of codimension 2 in X since π is finite) from X and Y respectively, we can assume that X and Y are smooth hence by [Par91], Proposition 4.1(b) we have that $(\pi_* T_X)^\chi = T_Y(-\log D_{H,\psi}; (H,\psi) \in S_\chi) \otimes L_\chi^{-1}$. Since Y is smooth and $D_{H,\psi}$ is a normal crossing divisor (since D is normal crossing) we have that $T_Y(-\log D_{H,\psi}; (H,\psi) \in S_\chi)$ is locally free and the conclusion follows. □

2.7.4 Multiple structrues on reduced connected schemes

One of the central technique for deforming a finite morphism to a morphism of smaller degree is to construct a suitable multiple structure on the image of the morphism which are called ropes.

Definition 2.7.7. Let Y be a reduced connected scheme and let $\mathcal{E}$ be a vector bundle of rank $m-1$ on Y. A *rope of multiplicity m on Y with conormal bundle $\mathcal{E}$* is a scheme $\widetilde{Y}$ with $\widetilde{Y}_{\text{red}} = Y$ such that $\mathcal{I}^2_{Y/\widetilde{Y}} = 0$, and $\mathcal{I}_{Y/\widetilde{Y}} = \mathcal{E}$ as $\mathcal{O}_Y$ modules. If $\mathcal{E}$ is a line bundle then $\widetilde{Y}$ is called a *ribbon* on Y.

A rope $\widetilde{Y}$ on Y with conormal bundle $\mathcal{E}$ is parametrized by its extension class $[e_{\widetilde{Y}}] \in \mathrm{Ext}^1(\Omega_Y, \mathcal{E})$

of its restricted cotangent sequence, the lower exact sequence in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{O}_{\widetilde{Y}} & \longrightarrow & \mathscr{O}_Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{E} & \longrightarrow & \Omega_{\widetilde{Y}|_Y} & \longrightarrow & \Omega_Y & \longrightarrow & 0 \end{array}$$

Let φ be a morphism from an integral cohen macaulay variety X to a smooth irreducible variety Z. Let Y be the scheme theoretic image of φ. Let $Y \overset{i}{\hookrightarrow} Z$ denote the closed embedding. Assume that Y is smooth and φ induces a finite morphism from $X \xrightarrow{\pi} Y$. Under these conditions π is surjective and flat. $\pi_*(\mathscr{O}_X)$ is a locally free $\mathscr{O}_Y$ module of some rank n. The trace map gives a splitting of the injective map $\mathscr{O}_Y \to \pi_*(\mathscr{O}_X)$. Hence $\pi_*(\mathscr{O}_X)$ is a direct sum of $\mathscr{O}_Y$ and a locally free $\mathscr{O}_Y$ module $\mathscr{E}$ of rank $n-1$.

Proposition 2.7.8. *([Gon06], Proposition 3.7) Let X be an integral Cohen-Macaulay projective variety and let Z be a smooth irreducible variety. Let $\varphi : X \to Z$ be a morphism that factors as $\varphi = i \circ \pi$, where π is a finite cover of a smooth variety Y and $i : Y \hookrightarrow Z$ is an embedding. Let $\mathscr{E}$ be the trace zero module of π and let $\mathscr{I}$ be the ideal sheaf of $i(Y)$. There exists a homomorphism*

$$H^0(\mathscr{N}_\varphi) \xrightarrow{\psi} Hom(\pi^*(\mathscr{I}/\mathscr{I}^2), \mathscr{O}_X)$$

that appears when taking cohomology on the commutative diagram [Gon06] (3.3.2). *Since*

$$Hom(\pi^*(\mathscr{I}/\mathscr{I}^2), \mathscr{O}_X) = H^0(\mathscr{N}_{Y/Z}) \oplus H^0(\mathscr{N}_{Y/Z} \otimes \mathscr{E}),$$

the homomorphism ψ has two components;

$$H^0(\mathscr{N}_\varphi) \xrightarrow{\psi_1} H^0(\mathscr{N}_{Y/Z}) \text{ and } H^0(\mathscr{N}_\varphi) \xrightarrow{\psi_2} H^0(\mathscr{N}_{Y/Z} \otimes \mathscr{E}).$$

Proposition 2.7.9. *([Gon06], Proposition 2.1) Let Y be a smooth irreducible closed subvariety of* *a smooth irreducible variety Z. Let $Y \overset{i}{\hookrightarrow} Z$ be the closed immersion. Let $\mathscr{E}$ be a locally free sheaf of rank $n-1$ on Y. Then*

(i) There is a $1-1$ correspondence between pairs $(\widetilde{Y},\widetilde{i})$ where $\widetilde{Y}$ is a rope on Y with conormal bundle $\mathscr{E}$ and $\widetilde{i}:\widetilde{Y}\to Z$ is a morphism that extends i and elements $\tau \in H^0(\mathscr{N}_{Y/Z}\otimes\mathscr{E})$

(ii) $\widetilde{i}$ is a closed immersion if and only if τ is surjective.

2.7.5 Two fundamental theorems of Gallego-González-Purnaprajna

Theorem 2.7.10. *([Gon06], Theorem 3.8) Let φ be a morphism from an integral cohen macaulay variety X to a smooth irreducible variety Z. Let Y be the scheme theoretic image of φ. Assume that Y is smooth and φ induces a finite morphism from $X \xrightarrow{\pi} Y$ with trace zero module $\mathscr{E}$. Let $(\widetilde{X},\widetilde{\varphi})$ be a first-order locally trivial infinitesimal deformation of $X \xrightarrow{\varphi} Z$ defined by a global section ν of $\mathscr{N}_{\varphi}$.*

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{X} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\widetilde{\varphi}} \\ Z & \longrightarrow & Z \times Spec(k[\epsilon]) \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & Spec(k[\epsilon]) \end{array}$$

Taking image of $\widetilde{\varphi}$ inside $Z \times Spec(k[\epsilon])$ we have a cartesian diagram

$$\begin{array}{ccc} Z & \hookrightarrow & Z \times Spec(k[\epsilon]) \\ \uparrow & & \uparrow \\ (Im(\Phi))_0 & \longrightarrow & Im(\Phi) \\ \downarrow & & \downarrow \\ Spec(k) & \hookrightarrow & T \end{array}$$

Then

(i) The central fibre $(Im(\Phi))_0$ contains Y and is contained in the first order infinitesimal neighbourhood of Y and is equal to the image of the morphism $\widetilde{Y}\to Z$ obtained from $\Psi_2(\nu)$. More precisely the ideal of both the central fibre of the image of $\widetilde{\varphi}$ and the image of $\widetilde{Y}\to Z$

inside Z is the kernel of the composite homomorphism

$$\mathscr{I}_{Y,Z} \to \mathscr{I}_{Y,Z}/\mathscr{I}^2_{Y,Z} \xrightarrow{\Psi_2(v)} \mathscr{E}$$

(ii) *The image of $\widetilde{\varphi}$ is the scheme theoretic union of its central fibre and the flat deformation of Y defined by $\Psi_1(v)$.*

We will make use of the the following theorem of the deformation theory of finite morphisms to reduce the degree of a general deformation of the canonical morphism.

Theorem 2.7.11. *([GGP10], Theorem 1.4) Let X be a smooth irreducible projective variety and let $\varphi : X \to \mathbb{P}^N$ be a morphism that factors through an embedding $Y \hookrightarrow \mathbb{P}^N$ with Y smooth and let $\pi : X \to Y$ be the induced morphism which we assume to be finite of degree $n \geq 2$. Let $\widetilde{\varphi} : \widetilde{X} \to \mathbb{P}^N_\Delta$ ($\Delta = Spec\left(\frac{\mathbb{C}[\epsilon]}{\epsilon^2}\right)$) be a first order infinitesimal deformation of φ and let $v \in H^0(\mathscr{N}_\varphi)$ be the class of $\widetilde{\varphi}$. If*

(a) *the homomorphism $\psi_2(v)$ has rank $k > \frac{n}{2} - 1$ (resp. $\psi_2(v)$ is a surjective homomorphism), and*

(b) *there exists an algebraic formally semiuniversal deformation of $\mathbf{Def}_\varphi$ and $\mathbf{Def}_\varphi$ is smooth,*

then there exists a flat family of morphisms, $\Phi : \mathscr{X} \to \mathbb{P}^N_T$ over T, where T is a smooth irreducible algebraic curve with a distinguished point 0, such that

(1) *$\mathscr{X}_t$ is a smooth irreducible projective variety,*

(2) *the restriction of Φ to the first order infinitesimal neighbourhood of 0 is $\widetilde{\varphi}$, and*

(3) *for $t \neq 0$, Φ_t is finite and one-to-one onto its image in $\mathbb{P}^N$ (resp. for $t \neq 0$, Φ_t is an embedding inside $\mathbb{P}^N$) .*

(4) *If* $\psi_2(v)$ *is a surjective homomorphism, then the central fibre* $Im(\Phi)_0$ *of* $Im(\Phi)$ *given by the following cartesian diagram*

$$\begin{array}{ccc} (Im(\Phi))_0 & \longrightarrow & Im(\Phi) \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & T \end{array}$$

is a rope $\widetilde{Y}$ *with* $\widetilde{Y}_{red} = Y$ *with conormal bundle* $\mathscr{E}$.

We shall use the theorem above to study the deformations of the canonical morphisms of the varieties we are interested in. However, we are also interested in the degree of the canonical morphisms of the moduli components of these varieties. We remark that a general deformation of the canonical morphism of a regular variety remains canonical by [GGP10], Lemma 2.4 (the statement requires smoothness, but it holds for varieties with canonical singularities as well, see [BCG21], proof of Proposition 2.9). Since the varieties we are interested in are not regular, we need the following proposition.

Chapter 3

Deformations and moduli of covers of rational normal scrolls

3.1 Varieties of minimal degree and rational normal scrolls

We recall that a subvariety $X \subseteq \mathbb{P}^r$ is called *non-degenerate* if it is not contained in any hyperplane. For any non-degenerate variety $X \subseteq \mathbb{P}^r$, we have the inequality

$$\deg(X) \geq 1 + \operatorname{codim}(X).$$

Definition 3.1.1. A non-degenerate variety $X \subseteq \mathbb{P}^r$ is said to be a *variety of minimal degree* if it satisfies the equality $\deg(X) = 1 + \operatorname{codim}(X)$.

If $\operatorname{codim}(X) = 1$ then of course the variety X is a quadric hypersurface. One can completely classify the varieties of minimal degree, but before doing so, we define the rational normal curves and rational normal scrolls.

Definition 3.1.2. Consider the morphism $\mathbb{P}^1 \to \mathbb{P}^r$ defined by $(s,t) \mapsto (s^r, s^{r-1}t, \cdots, st^{r-1}, t^r)$. The image of this map is called the *standard rational normal curve* in $\mathbb{P}^r$. A *rational normal curve* in $\mathbb{P}^r$ is any curve that is obtained from the standard rational normal curve by an automorphism.

As it turns out that one may generalize this construction and define rational normal scrolls.

Definition 3.1.3. A rational normal scroll $X \subseteq \mathbb{P}^r$ of dimension n is the image of a projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ through the morphism given by the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ where the vector bundle $\mathscr{E} = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$ satisfies $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $\deg(X) = \sum a_i$.

In the situation above, if $a_1 = a_2 = \cdots = a_l = 0$ for some $0 < l < n$ then X is singular and it is a cone over a smooth rational normal scroll. The vertex or singular locus V of this cone has dimension $l-1$ and let $X_S = X \backslash V$ be the smooth part of X. Moreover, X is normal and $\tilde{X} = \mathbb{P}(\mathscr{E}) \to X$ is a rational resolution of singularity which is called the *canonical resolution* of the rational normal scroll X.

The following theorem provides a complete classification of the varieties of minimal degree.

Theorem 3.1.4. *([EH85], Theorem 1) Let $X \subseteq \mathbb{P}^r$ is a variety of minimal degree. Then X is a cone over a smooth such variety. Moreover, if X is smooth and codim$(X) > 1$ then $X \subseteq \mathbb{P}^r$ is either a rational normal scroll or a Veronese surface $\mathbb{P}^2 \subseteq \mathbb{P}^5$.*

3.2 Canonical morphism of smooth projective curves of genus $g \geq 2$ and their deformations

We recall a classical situation.

Theorem 3.2.1. *(known classically) For a smooth projective curve C of genus $g \geq 2$, the canonical bundle K_C is ample and base point free. Let $\varphi : C \to \mathbb{P}^N$ be the morphism induced by the complete linear series $|K_C|$ (hence $N + 1 = h^0(K_C) = g$). Then one of the following happens :*

(1) K_C is very ample and hence φ is an embedding. Then C is called non-hyperelliptic.

(2) φ is a double cover of its image Y which is a rational normal curve, i.e, $Y \cong \mathbb{P}^1$, φ factors as

$$\begin{array}{ccc} C & & \\ \downarrow \pi & \overset{\varphi}{\searrow} & \\ Y & \xrightarrow{i} & \mathbb{P}^{g-1} \end{array}$$

where $deg(\pi) = 2$. Then C is called hyperelliptic.

Theorem 3.2.2. *(known classically)*

(1) Since very ampleness is an open condition, for a non-hyperelliptic curve, a general deformation of φ is an embedding.

(2) Any curve of genus $g = 2$ *is hyperelliptic, which means for any deformation*

$$\begin{array}{ccc} C & \longrightarrow & \mathscr{C}_T \\ \downarrow \varphi & & \downarrow \Phi_T \\ \mathbb{P}^N & \longrightarrow & \mathbb{P}^{g-1}_T \end{array}$$

$\Phi_t : \mathscr{C}_t \to \mathbb{P}^{g-1}$ *is a two-to-one morphism onto its image for* $t \neq 0$

(3) For a hyperelliptic curve of genus $g > 2$*, there exist a deformation*

$$\begin{array}{ccc} C & \longrightarrow & \mathscr{C}_T \\ \downarrow \varphi & & \downarrow \Phi_T \\ \mathbb{P}^N & \longrightarrow & \mathbb{P}^{g-1}_T \end{array}$$

$\Phi_t : \mathscr{C}_t \to \mathbb{P}^{g-1}$ *is an embedding* $t \neq 0$.

(4) (known classically and can be seen more systematically using Theorem 2.7.11) that one can choose Φ_T *such that* $(Im(\Phi_T))_t$ *satisfying*

$$\begin{array}{ccc} (Im(\Phi_T))_0 & \longrightarrow & Im(\Phi_T) \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & T \end{array}$$

satisfies,

(i) $(Im(\Phi_T))_t$ *is canonically embedded non-hyperelliptic curve*

(ii) $(Im(\Phi_T))_0$ *is an embedded rope* $\widetilde{Y}$ *of multiplicity two on* $Y \cong \mathbb{P}^1$ *where the latter is embedded as a rational normal curve. Such a double structure is also known as a rational ribbon. These are proper and local complete intersection schemes over k and hence by [Kle80], (*7*,* $pg\,46$*,), [Har66], (*V*,* $9.3, 9.7$*,* VII*,* 3.4*) and [Con00], (*$pg\,157$*) has a dualizing sheaf which is very ample which in fact gives the embedding in this case.*

Remark 3.2.3. Hence a general deformation of a hyperelliptic curve of genus $g \geq 3$ is non-hyperelliptic and it hence it forms a proper locally closed loci in the moduli M_g of smooth curves of genus g.

3.3 Canonical morphisms of surfaces of general type and their deformations

We look at the analogous situation for surfaces of general type.

3.3.1 Noether's inequality and Beauville's result

A minimal surface of general type X satisfies an important inequality

$$K_X^2 \geq 2p_g - 4$$

known as Noether's inequality. Together with Bogomolov-Miyaoka-Yau inequality the geography looks like the following

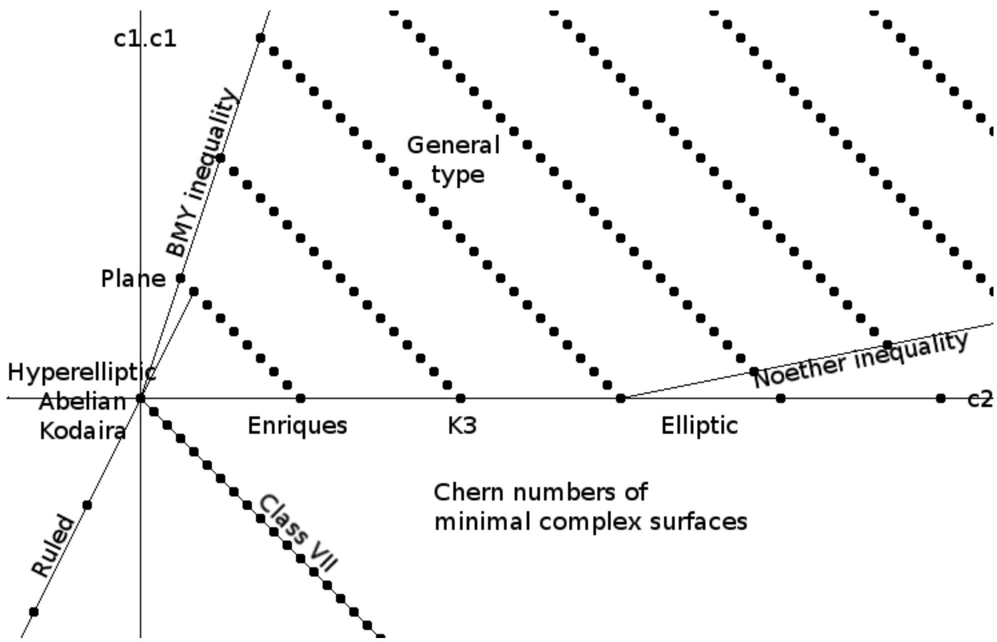

Figure 3.1: Geography of surfaces of general type

Theorem 3.3.1. *Let X be a smooth minimal complex surface of general type and let $\varphi_{|K_X|} : X \dashrightarrow \mathbb{P}^{p_g(X)-1}$ be its canonical map*

(i) ([Bea82]) If the image of $\varphi_{|K_X|}$ is a surface, then the degree d of the canonical map is less than or equal to 36 *and for $\chi(X) \geq 31$ geometric genus it is bounded by* 9.

(ii) ([Xiao86]) Xiao showed that if $p_g(X) \geq 132$, the degree is bounded by 8.

3.3.2 Deformations and moduli of surfaces of general type

In [Gie77], David Gieseker showed the existence of a quasiprojective coarse moduli of canonical models of surfaces of general type.

Theorem 3.3.2. *([Gie77])*

(i) For any pair of positive integers (x, y) there exist a (possibly empty) quasiprojective variety $M = M_{x,y}$ which is a coarse moduli space of canonical models X of surfaces of general type with $\chi(\mathcal{O}_X) = x$ and $K_X^2 = y$.

(ii) Two minimal surfaces of general type are deformation equivalent if and only if the isomorphism classes of their canonical models belongs to the same connected componenets of the moduli space.

Let X the canonical model of a surface of general type. Then X has a formally universal algebraic deformation with base $\mathrm{Def}(X)$ and finite automorphism group $\mathrm{Aut}(X)$

Theorem 3.3.3. *([Man97], Corollary* 2.4*) Let X be the canonical model of a surface of general type. Then the germ of M at $[X]$ is analytically isomorphic to the quotient $Def(X)/Aut(X)$*

3.3.3 Canonical double covers, moduli of surfaces with $K_X^2 = 2p_g - 4$ and Horikawa's results

Horikawa in his well-known work [Hor76] (Annals of Mathematics, 1976) showed

Theorem 3.3.4. *([Hor76]) Let X be the canonical model of a smooth minimal algebraic surface of general type with $K_X^2 = 2p_g - 4$, then K_X is base point free and the morphism induced by $|K_X|$ is a ramified double cover over a surface of minimal degree inside $\mathbb{P}^{p_g - 1}$.*

Surfaces on the Noether line are also called Horikawa surfaces. In the same paper Horikawa classified the deformation types and studied the moduli of surfaces on the Noether line.

Theorem 3.3.5. *(Deformations of Horikawa surfaces, [Hor76]) A general deformation of a Horikawa surface (resp. its canonical model) is again a Horikawa surface (resp. the canonical model of a Horikawa surface).*

Remark 3.3.6. The result in fact holds for more general "Horikawa varieties" or "hyperelliptic varieties" in higher dimensions (by the work of Fujita, Kobayashi, Bangere, Chen, Gallego , see [Fu83], [Ko92], [BCG21]).

Remark 3.3.7. One can see that deformations of canonical morphisms of Horikawa surfaces or for more general Horikawa varieties are similar to those of smooth curves of genus $g = 2$.

3.3.4 Canonical covers of rational normal scrolls and non-existence of triple covers

Set-up 3.3.8. *Suppose X is a projective surface with worst canonical singularities and ample and base point free canonical bundle K_X. Let $\varphi : X \to \mathbb{P}^{h^0(K_X)-1}$, be the morphism induced by the complete linear series of K_X. Let $Im(\varphi) = Y$ be smooth. Assume that φ induces a finite morphism $\pi : X \to Y$ and that $i : Y \hookrightarrow \mathbb{P}^{h^0(K_X)-1}$ be an embedding of minimal degree.*

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\varphi} & \\ Y & \longrightarrow & \mathbb{P}^N \end{array}$$

We want to study deformations of φ.

Theorem 3.3.9. *([Kon91]) There exist no canonical triple cover over a smooth surface of minimal degree.*

Theorem 3.3.10. *([GP04], Theorem 3.2) There are no odd degree canonical covers over a smooth surface of minimal degree other than $\mathbb{P}^2$*

3.3.5 Quadruple Galois canonical covers over smooth scrolls and moduli of surfaces with $K_X^2 = 4p_g - 8$

Our aim is to study deformations of φ when φ is as in Set-up 3.3.8 and

(i) X is irregular and

(ii) φ is a degree four Galois cover onto its image

Remark 3.3.11. As the first two authors show in [GP08], there exist both irregular and regular surfaces satisfying the above conditions unlike Horikawa surfaces. We refer to section 1.1 for a detailed discussion on irregular quadruple Galois canonical covers of smooth rational normal scrolls.

Chapter 4

Deformations of iterated double covers of embedded varieties

4.1 Deformations of iterated double covers of embedded varieties

Throughout this subsection, we will work with the following diagram where X, Y and Z are normal local complete intersection (abbreviated as lci) projective varieties, $i : Z \hookrightarrow \mathbb{P}^N$ is an embedding, and $g : X \to Z$ is a twice iterated double cover:

$$X \xrightarrow{\pi} Y \xrightarrow{p} Z \hookrightarrow \mathbb{P}^N, \qquad g = p \circ \pi : X \to Z$$

We set $\psi := i \circ p$, and $\varphi := \psi \circ \pi$. Notice that $\mathbf{Def}'_\varphi$ has a formal semiuniversal deformation space by [Ser06], Theorem 3.4.8, and our objective is to determine the degree of a general locally trivial deformation of φ. We will show that under suitable hypothesis, φ can be locally trivially deformed to a two-to-one morphism onto its image. We first need the following technical fact that we will put as a remark for future reference.

Remark 4.1.1. Let Y be a normal lci projective variety. Let L be a line bundle on Y and B be a divisor in $H^0(L)$. Assume $H^1(L) = 0$. Let $f : \mathscr{Y} \to T$ be a deformation of Y over a smooth projective pointed variety $(T, 0)$ (f is assumed to be proper and flat). Assume that L lifts to a line bundle $\mathscr{L}$ on $\mathscr{Y}$. Then (possibly after shrinking T) $f_*(\mathscr{L})$ is locally free of rank $h^0(L)$ on T. We have a Cartesian diagram as shown below.

$$\begin{array}{ccc} \mathscr{Y} \times_T \mathbb{P}(f_*(\mathscr{L})) & \xrightarrow{p} & \mathbb{P}(f_*(\mathscr{L})) \\ \downarrow q & & \downarrow g \\ \mathscr{Y} & \xrightarrow{f} & T \end{array}$$

Consequently, $p : \mathscr{Y} \times_T \mathbb{P}(f_*(\mathscr{L})) \to \mathbb{P}(f_*(\mathscr{L}))$ is a deformation of Y with $q^*(\mathscr{L})$ and the incidence divisor $\mathscr{B} \in H^0(q^*(\mathscr{L}))$ giving natural lifts of L and B respectively on $\mathscr{Y} \times_T \mathbb{P}(f_*(\mathscr{L}))$. Now since $f_*(\mathscr{L})$ is locally free, by shrinking T, we can always construct a section $s : T \to \mathbb{P}(f_*(\mathscr{L}))$ and $\mathscr{B} \times_{\mathbb{P}(f_*(\mathscr{L}))} T$ is a lift of the divisor B to $\mathscr{Y}$. Conversely any lift $\mathscr{B}$ of B on $\mathscr{Y}$ is obtained by a pullback induced by a section $s : T \to \mathbb{P}(f_*(\mathscr{L}))$. ■

Remark 4.1.2. Let X and Y be normal lci projective varieties with Y smooth and consider morphisms $X \xrightarrow{\pi} Y \xrightarrow{\psi} \mathbb{P}^N$. Let π be a finite $2:1$ morphism with trace zero module L^* on Y. Let $\varphi = \psi \circ \pi$. Let $L^* = \omega_Y \otimes \psi^*(\mathscr{O}_{\mathbb{P}^N}(-1))$. Then $\varphi^*(\mathscr{O}_{\mathbb{P}^N}(1)) = \omega_X$. ■

Theorem 4.1.3. *Let X be a normal lci projective variety and let Y and Z be smooth projective varieties. Let $\pi : X \to Y$ be a finite, flat morphism of degree two onto Y with trace zero module $\mathscr{E}_\pi = L^*$ and branched along a divisor $B \in H^0(L^{\otimes 2})$, and let $p : Y \to Z$ be a finite (hence flat) morphism of degree two onto Z with trace zero module $\mathscr{E}_p$. Let $i : Z \hookrightarrow \mathbb{P}^N$ be an embedding, $\varphi = i \circ p \circ \pi$ and $\psi = i \circ p$. Suppose*

(a) $H^2(\mathscr{O}_Y) = 0$,

(b) $\mathbf{Def}_\psi$ *is smooth,*

(c) $\psi_2 : H^0(\mathscr{N}_\psi) \to H^0(\mathscr{N}_{Y/\mathbb{P}^N} \otimes \mathscr{E}_p)$ *is non-zero.*

(d) $H^1(L^{\otimes 2}) = 0$

Then there exist a flat family $\mathscr{X} \to T$ of deformations of X over a smooth pointed affine algebraic curve $(T, 0)$ and a T-morphism $\Phi : \mathscr{X} \to \mathbb{P}^N_T$ satisfying:

(1) $\Phi = \Psi \circ \Pi$, *where* $\mathscr{Y} \to T$ *is a flat family,* $\Psi : \mathscr{Y} \to \mathbb{P}^N_T$, *and* $\Pi : \mathscr{X} \to \mathscr{Y}$ *are* T*-morphisms with* $\mathscr{Y}_0 = Y$, $\Pi_0 = \pi$, *and* $\Psi_0 = \psi$,

(2) Π_t *is a finite morphism of degree 2 for all* t, *and* Ψ_t *is birational onto its image for all* $t \in T - \{0\}$,

(3) Suppose that φ *is the canonical morphism of* X *and that* $\mathscr{E}_\pi = L^* = \omega_Y \otimes \psi^*(\mathscr{O}_{\mathbb{P}^N}(-1))$, *then* Φ_t *can be taken to be the canonical morphism of* $\mathscr{X}_t$

Proof. We will prove the assertions (1) and (2) in two steps.

Step 1. In this step, we deform ψ into a birational morphism. Notice that $\mathbf{Def}_\psi$ is unobstructed, and has an algebraic formally semiuniversal deformation by [BGG20], Proposition 1.5. Moreover, ψ_2 is non-zero, hence we apply Theorem 2.7.11 and one gets that there exists a family $\mathscr{Y}$ of smooth projective varieties, proper and flat over a smooth pointed affine algebraic curve $(T, 0)$ and a T-morphism $\Psi : \mathscr{Y} \to \mathbb{P}^N_T$ with:

(1) $\Psi_0 = \psi$,

(2) Ψ_t is birational onto its image for all $t \in T - \{0\}$,

Step 2. We construct a deformation $\mathscr{X} \to \mathscr{Y} \to \mathbb{P}^N_T \to T$ of φ. For this we need to construct a deformation $\Pi : \mathscr{X} \to \mathscr{Y} \to T$ of the finite morphism $\pi : X \to Y$. Let $q : \mathscr{Y} \to T$ be the deformation obtained by applying the forgetful map to $\mathscr{Y} \to \mathbb{P}^N_T \to T$. We need to construct lifts $\mathscr{L}^{\otimes 2}$ and $\mathscr{B}$ of the line bundle $L^{\otimes 2}$ and the divisor B respectively on $\mathscr{Y}$. Note that since $H^2(\mathscr{O}_Y) = 0$, we have that the map $\mathbf{Def}_{(Y,L)} \to \mathbf{Def}_Y$ is smooth by ([Ser06], Proposition 2.3.6). Hence by [Ser06], Proposition 2.2.5, (iv), we have a lift $\mathscr{L}$ of L on $\mathscr{Y}$. The conclusion follows from Remark 4.1.1. This proves statements (1) and (2).

For part (3) we note that $\omega_{\mathscr{Y}/T} \otimes \Psi^*(\mathscr{O}_{\mathbb{P}^N_T}(-1))$ is a deformation of L^*, since $L^* = \omega_Y \otimes \psi^*(\mathscr{O}_{\mathbb{P}^N}(-1))$. Hence $-2(\omega_{\mathscr{Y}/T} \otimes \Psi^*(\mathscr{O}_{\mathbb{P}^N_T}(-1))$ is a lift of $L^{\otimes 2}$. Now we apply Remark 4.1.1 to construct a relative cover using a lift $\mathscr{B} \in H^0(-2(\omega_{\mathscr{Y}/T} \otimes \Psi^*(\mathscr{O}_{\mathbb{P}^N_T}(-1))))$ of the divisor $B \in H^0(L^{\otimes 2})$. Since for each

t, the trace zero module $L_t^* = \omega_{Y_t} \otimes \Psi_t^*(\mathscr{O}_{\mathbb{P}^N_t}(-1))$, we have that for each t, Φ_t is the canonical morphism of X_t by Remark 4.1.2. □

Remark 4.1.4. Let X be a normal lci projective variety with ample and base point free canonical bundle K. Let T be a smooth affine curve and $\mathscr{Y} \xrightarrow{\Phi} \mathbb{P}^N_T \to T$ be a deformation of the canonical morphism of X, i.e, for all $t \in T$, Φ_t is given by the complete linear series $\omega_{\mathscr{Y}_t}$. Suppose that the degree of the finite morphism Φ_t is d for a general $t \in T$. Then there exist an irreducible component U_X of the universal deformation space of X such that for a general closed point $u \in U_X$, the canonical morphism of the fibre $\mathscr{X}_u$ of the universal family over U_X has degree less than or equal to d.

Proof. Choose an irreducible component U_X containing T. Since T is smooth this embedding factors through the reduced induced structure U_X^0 of U_X. Consider the pullback $\mathscr{X} \xrightarrow{p} U_X^0$ of the universal family to U_X^0. Since $p_g(\mathscr{X}_s)$ is constant for $s \in U_X^0$, we have that $p_*(\omega_{\mathscr{X}/U_X^0})$ is locally free of rank $h^0(\omega_X)$. Then $\mathscr{X} \xrightarrow{\Psi} \mathbb{P}(p_*(\omega_{\mathscr{X}/U_X^0})) \to U_X^0$ is a deformation of the canonical morphism of X such that for each $s \in U_X^0$, Ψ_s is the canonical morphism of $\mathscr{X}_s$. Since U_X^0 is integral, we have that degree of Ψ_s is upper semicontinuous. Now $\mathscr{Y} \xrightarrow{\Phi} \mathbb{P}^N_T \to T$ is obtained by pulling back $\mathscr{X} \xrightarrow{\Psi} \mathbb{P}(p_*(\omega_{\mathscr{X}/U_X^0})) \to U_X^0$ to T by the embedding $T \hookrightarrow U_X^0$. This shows that degree of Ψ_s is less than or equal to d for a general $s \in U_X^0$. Since closed points of U_X are the same as closed points of U_X^0, we are done. ■

Now we will find the the condition following the proof of [Weh86], Proposition 1.10, under which any deformation of φ factors through a deformation of π (with varying target).

Proposition 4.1.5. *Let $\pi : X \to Y$ be a finite, flat morphism with trace zero module $\mathscr{E}$ between projective varieties with X normal lci, and Y smooth. Let $\psi : Y \to Z$ be a non-degenerate morphism to a smooth projective variety Z. Let $\varphi = \psi \circ \pi$ be the composed morphism. Assume that any deformation of X is locally trivial (e.g. when X is smooth) and $H^0(\mathscr{N}_\psi \otimes \mathscr{E}) = 0$. Then the natural map between the functors $\mathbf{Def}_{\pi/Z} \to \mathbf{Def}_\varphi$ is smooth.*

Proof. Note that $\mathbf{Def}_{\pi/Z}$ and $\mathbf{Def}_{\varphi}$ has formal semiuniversal deformations. Because of the following diagram

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & H^0(\mathcal{N}_\pi) & \longrightarrow & T^1(\pi/Z) & \longrightarrow & H^0(\mathcal{N}_\psi) & \longrightarrow & H^1(\mathcal{N}_\pi) & \longrightarrow & T^2(\pi/Z) & \longrightarrow & H^1(\mathcal{N}_\psi) & \longrightarrow & \dots \\
 & & \| & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \| & & \downarrow \alpha_2 & & \downarrow \beta_2 & & \\
\dots & \longrightarrow & H^0(\mathcal{N}_\pi) & \longrightarrow & H^0(\mathcal{N}_\varphi) & \longrightarrow & H^0(\pi^*\mathcal{N}_\psi) & \longrightarrow & H^1(\mathcal{N}_\pi) & \longrightarrow & H^1(\mathcal{N}_\varphi) & \longrightarrow & H^1(\pi^*\mathcal{N}_\psi) & \longrightarrow & \dots
\end{array}$$

the map between the functors $\mathbf{Def}_{\pi/Z} \to \mathbf{Def}_{\varphi}$ is smooth if α_1 is surjective and α_2 is injective (see [Ser06], Proposition 2.3.6). But that happens if β_1 is surjective and β_2 is injective. The assertion follows since β_2 is always injective and β_1 is surjective if $H^0(\mathcal{N}_\psi \otimes \mathcal{E}) = 0$. □

The following is the main result that we will use to prove Theorem 5.1.1. The prove of this result is an immediate consequence of Theorem 4.1.3 and Corollary 4.1.5.

Corollary 4.1.6. *Assume the hypotheses (a), (b), (c) and (d) of Theorem 4.1.3. Furthermore, assume* $\mathbf{Def}_{\varphi}$ *has an algebraic formally semiuniversal deformation space, any deformation of X is locally trivial, and* $H^0(\mathcal{N}_\psi \otimes \mathcal{E}) = 0$. *Then a general deformation of* φ *is a composition of a double cover over a deformation* Y' *of* Y *followed by a morphism of* $Y' \to \mathbb{P}^N$ *that is birational onto its image, consequently, it is a two-to-one morphism onto its image.*

Remark 4.1.7. Let X be a surface with ample and globally generated canonical bundle ω_X with at worst canonical singularities. Let φ be the canonical morphism of X. Then $H^0(T_X) = 0$. Furthermore, $\mathbf{Def}_X$, $\mathbf{Def}_{\varphi}$, and $\mathbf{Def}_{(X,\omega_X)}$ have algebraic formally universal deformation spaces. ■

We have assumed in Proposition 4.1.5 and Corollary 4.1.6 that any deformation of X is locally trivial. This hypothesis is satisfied if X is smooth. We will show the hypothesis to be true in more generality, namely, if we allow X to have A_1 singularities. We shall establish this fact using a corollary of the following proposition.

Proposition 4.1.8. *Let X be a normal lci projective variety. Assume* $\pi : X \to Y$ *is double cover of a smooth projective variety Y with ramification divisor R and branch divisor* $B \in H^0(L^{\otimes 2})$. *Then* *we have the following four-term exact sequence*

$$0 \to H^0(\mathcal{N}_\pi) \to H^0(\pi^*\mathcal{O}_Y(B)|_R) \to H^0(\mathcal{T}^1_X) \to H^1(\mathcal{N}_\pi) \tag{4.1}$$

Proof. Let $Y' := \mathbb{V}(L^{-1}) := \operatorname{Spec}(\operatorname{Sym}(L^{-1}))$ denote the total space of the line bundle L. Let $p' : Y' \to Y$ be the projection. We have an embedding of $i : X \hookrightarrow Y'$ as a divisor in Y' such that $\pi = p' \circ i$. The conormal sheaf $\mathcal{N}^*_{X/Y'} := \mathcal{I}/\mathcal{I}^2$ of X in Y' is given by $\pi^*(-\mathcal{O}_Y(B))$ (since X is defined as the scheme of zeroes of $t^n - p^*B$ where $t \in p^*(L)$). Since X is a local complete intersection, we have an exact sequence

$$0 \to \pi^*(-\mathcal{O}_Y(B)) \to \Omega_{Y'} \otimes \mathcal{O}_X \to \Omega_X \to 0. \tag{4.2}$$

Now since $p' : Y' \to Y$ is a smooth morphism, we have that $\Omega_{Y'/Y}$ is an invertible sheaf isomorphic to $p'^*(-L)$ and we have an exact sequence

$$0 \to \pi^*(\Omega_Y) \to \Omega_{Y'} \otimes \mathcal{O}_X \to \pi^*(-L) \to 0. \tag{4.3}$$

We also have another exact sequence as follows

$$0 \to \pi^*(-\mathcal{O}_Y(B)) \to \pi^*(-L) \to \pi^*(-L) \otimes \mathcal{O}_R \to 0. \tag{4.4}$$

Now apply snake lemma to the following diagram where the first row is (4.2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*(-B) & \longrightarrow & \Omega_{Y'} \otimes \mathcal{O}_X & \longrightarrow & \Omega_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^*(-L) & \longrightarrow & \pi^*(-L) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

and use the previous two short exact sequences (4.3) and (4.4) to get the following exact sequence.

$$0 \to \pi^*(\Omega_Y) \to \Omega_X \to \pi^*(-L) \otimes \mathcal{O}_R \to 0$$

Since π is non-degenerate, we get that $T_{X/Y} = 0$ and hence by dualizing the above sequence we have the following exact sequence

$$0 \to T_X \to \pi^* T_Y \to \mathscr{E}xt^1(\pi^*(-L) \otimes \mathscr{O}_R, \mathscr{O}_X) \to \mathscr{T}_X^1 \to 0. \tag{4.5}$$

Notice that $\mathscr{E}xt^1(\pi^*(-L) \otimes \mathscr{O}_R, \mathscr{O}_X) = \pi^*(L \otimes \mathscr{O}_R(R)) = \pi^*(\mathscr{O}_Y(B)) \otimes \pi^*\mathscr{O}_R = \pi^*(\mathscr{O}_Y(B)) \otimes \mathscr{O}_R$, and consequently (4.5) becomes

$$0 \to T_X \to \pi^* T_Y \to \pi^*(\mathscr{O}_Y(B)) \otimes \mathscr{O}_R \to \mathscr{T}_X^1 \to 0. \tag{4.6}$$

The exact sequence of the proposition follows from the fact that $\mathscr{N}_\pi = \ker(\pi^*(\mathscr{O}_Y(B)) \otimes \mathscr{O}_R \to \mathscr{T}_X)$. □

Corollary 4.1.9. *In the set-up of Proposition 4.1.8, assume that B has normal crossing and $H^1(\mathscr{N}_\pi) = 0$. Then $\mathbf{Def}'_X = \mathbf{Def}_X$.*

Proof. Since B has normal crossing, it follows from Proposition 2.7.6 that $H^0(\mathscr{N}_\pi) \cong H^0(\pi^*\mathscr{O}_Y(B)|_R)$. Consequently, $H^0(\mathscr{T}_X^1) = 0$ by the exact sequence of Proposition 4.1.8. The assertion then follows from the five-term exact sequence

$$0 \to H^1(T_X) \to \mathrm{Ext}^1(\Omega_X, \mathscr{O}_X) \to H^0(\mathscr{T}_X^1) \to H^2(T_X) \to T^2(X)$$

associated to the spectral sequence $E_2^{p,q} = H^p(\mathscr{T}_X^q) \implies T^{p+q}(X)$ and general deformation theory. □

Remark 4.1.10. Assume all the hypotheses of Corollary 4.1.9. Further assume X is a surface with at worst canonical singularities and $\mathbf{Def}_X$ is smooth. Then it follows from the work of Burns and Wahl (see [BW74]) that if X' is the minimal resolution of X, then $\mathbf{Def}_{X'}$ is smooth.

4.2 Geometry of deformation spaces of iterated double covers

One of our objective is to describe the moduli components of surfaces of type $(1)_m$ and $(1')_m$. We will see that for a fixed m, there is a unique component of the moduli space that contains all surfaces of both types, and that this component is uniruled. The proof of this fact is based on the following result.

Theorem 4.2.1. *Assume the hypothesis (a), (b) and (d) of Theorem 4.1.3. Further assume B is normal crossing (in particular any deformation of X is locally trivial by Proposition 2.7.6 and Corollary 4.1.9). Then $\mathbf{Def}_{\pi/\mathbb{P}^N}$ has a smooth uniruled algebraic formally semiuniversal deformation space $V_{\pi/\mathbb{P}^N}$.*

Proof. We construct an algebraic formally semiuniversal family of deformations of the functor $\mathbf{Def}_{\pi/\mathbb{P}^N}$

$$\mathscr{X}_{\pi/\mathbb{P}^N} \to \mathbb{P}^N_{V_{\pi/\mathbb{P}^N}} \to V_{\pi/\mathbb{P}^N}$$

over a smooth pointed irreducible base $(V_{\pi/\mathbb{P}^N}, 0)$.

Let $\mathscr{Y}_\psi \to \mathbb{P}^N_{U_\psi} \to U_\psi$ be the algebraic formally semiuniversal family of deformations of the functor $\mathbf{Def}_\psi$ (this space exists, see for example the proof of Theorem 4.1.3).

Let $(\mathscr{Y}_L, \mathscr{L}) \to U_L$ be the algebraic formally semiuniversal deformation space of the functor $\mathbf{Def}_{(Y,L)}$. Let $\mathscr{Y} \to U$ be the algebraic formally semiuniversal deformation space of the functor $\mathbf{Def}_Y$. Forgetful maps between functors induce a cartesian diagram, which in turn induces a cartesian diagram of algebraic formally semiuniversal deformation spaces as shown below.

$$\begin{array}{ccc} \mathbf{Def}_\psi \times_{\mathbf{Def}_Y} \mathbf{Def}_{(Y,L)} & \longrightarrow & \mathbf{Def}_{(Y,L)} \\ \downarrow & & \downarrow \\ \mathbf{Def}_\psi & \longrightarrow & \mathbf{Def}_Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} U_\psi \times_U U_L & \longrightarrow & U_L \\ \downarrow & & \downarrow \\ U_\psi & \longrightarrow & U \end{array}$$

Since $H^2(\mathscr{O}_Y) = 0$, we have that the forgetful map $\mathbf{Def}_{(Y,L)} \to \mathbf{Def}_Y$ is smooth (see [Ser06], Proposition 2.3.6) and hence the map $\mathbf{Def}_\psi \times_{\mathbf{Def}_Y} \mathbf{Def}_{(Y,L)} \to \mathbf{Def}_\psi$ is smooth. Now using the fact that $\mathbf{Def}_\psi$ is smooth, we have that $\mathbf{Def}_\psi \times_{\mathbf{Def}_Y} \mathbf{Def}_{(Y,L)}$ is smooth and hence $U_\psi \times_U U_L$ is smooth. We set $U_{(\psi,L)} := U_\psi \times_U U_L$. The semiuniversal families form the follow form the following cartesian diagram

$$\begin{array}{ccc}(\mathscr{Y}_\psi \times_{\mathscr{Y}} \mathscr{Y}_L \to \mathbb{P}^N_{U_{(\psi,L)}}, \mathscr{L}_\psi) & \longrightarrow & (\mathscr{Y}_L, \mathscr{L}) \\ \downarrow & & \downarrow \\ (\mathscr{Y}_\psi \to \mathbb{P}^N_{U_\psi}) & \longrightarrow & \mathscr{Y}\end{array}$$

Hence $(\mathscr{Y}_\psi \times_{\mathscr{Y}} \mathscr{Y}_L \to \mathbb{P}^N_{U_{(\psi,L)}} \to U_{(\psi,L)}, \mathscr{L}_\psi)$ is a smooth algebraic formally semiuniversal deformation of $\mathbf{Def}_\psi \times_{\mathbf{Def}_Y} \mathbf{Def}_{(Y,L)}$ where $\mathscr{L}_\psi$ is the pullback of $\mathscr{L}$ under the morphism $\mathscr{Y}_\psi \times_{\mathscr{Y}} \mathscr{Y}_L \to \mathscr{Y}_L$. Let the map $\mathscr{Y}_{\psi,L} := \mathscr{Y}_\psi \times_{\mathscr{Y}} \mathscr{Y}_L \to U_{(\psi,L)}$ be denoted by $p_{(\psi,L)}$. Since $H^1(L^{\otimes 2}) = 0$, we have that $p_{(\psi,L)*}(\mathscr{L}_\psi{}^{\otimes 2})$ is free after possibly shrinking $U_{(\psi,L)}$. Let $V_{\pi/\mathbb{P}^N} := \mathbb{P}(p_{(\psi,L)*}(\mathscr{L}_\psi{}^{\otimes 2}))$ and consider the Cartesian diagram

$$\begin{array}{ccc}\mathscr{Y}_{\psi,L} \times_{U_{\psi,L}} V_{\pi/\mathbb{P}^N} & \longrightarrow & V_{\pi/\mathbb{P}^N} \\ \downarrow & & \downarrow \\ \mathscr{Y}_{\psi,L} & \longrightarrow & U_{\psi,L}\end{array}$$

Choose a basis $\bigoplus_{i=0}^{M} \mathscr{O}_{U_{(\psi,L)}} s_i$ of $p_{(\psi,L)*}(\mathscr{L}_\psi{}^{\otimes 2})$. Let $X_i \in H^0(\mathscr{O}_{V_{\pi/\mathbb{P}^N}}(1)) = H^0(p_{(\psi,L)*}(\mathscr{L}_\psi{}^{\otimes 2})^*)$, with $0 \le i \le M$ be the dual basis. Now on $\mathscr{Y}_{\psi,L} \times_{U_{(\psi,L)}} V_{\pi/\mathbb{P}^N}$, consider the divisor $\mathscr{B} = \sum_{i=0}^{M} X_i s_i$. One can construct a relative Galois double cover $\mathscr{X}_{\pi/\mathbb{P}^N} \to \mathscr{Y}_{\psi,L} \times_{U_{(\psi,L)}} V_{\pi/\mathbb{P}^N}$ given by the equation $t^2 - \mathscr{B}$ in the total space of $q^*(\mathscr{L}_\psi)$ where t is the tautological section of $q^*(\mathscr{L}_\psi)$ and $q : \mathscr{Y}_{\psi,L} \times_{U_{(\psi,L)}} V_{\pi/\mathbb{P}^N} \to \mathscr{Y}_{\psi,L}$. The fibre of this relative double cover over a point $(u,[r]) \in V_{\pi/\mathbb{P}^N}$ with $u \in U_{(\psi,L)}$ and $r \in H^0(\mathscr{L}^{\otimes 2}_{\psi,u})$ is the double cover $\mathscr{X}_{\pi/\mathbb{P}^N,u} \to \mathscr{Y}_{\psi,L,u}$ given by the line bundle $L_{\psi,u}$ and the divisor $r \in H^0(\mathscr{L}^{\otimes 2}_{\psi,u})$. This is therefore a smooth algebraic deformation of the functor $\mathbf{Def}_{\pi/\mathbb{P}^N}$.

Now note that given a flat family of polarized schemes $f : (\mathscr{C}, \mathscr{M}) \to S$ over an affine scheme S with $f_*(\mathscr{M})$ free, giving a divisor $\mathscr{D} \in H^0(\mathscr{M})$ flat over S is equivalent to giving a unique S - valued point in $\mathbb{P}(f_*(\mathscr{M}))$ and hence a section $S \to \mathbb{P}(f_*(\mathscr{M}))$. This along with the fact that $U_{(\psi,L)}$ is formally semiuniversal implies that $V_{\pi/\mathbb{P}^N}$ is a smooth algebraic formally semiuniversal deformation of the functor $\mathbf{Def}_{\pi/\mathbb{P}^N}$. Also since it is a projective bundle over a smooth affine scheme, it is uniruled. □

Remark 4.2.2. Under the assumptions of Theorem 4.2.1, it is easy to prove the smoothness of $\mathbf{Def}_{\pi/\mathbb{P}^N}$ only using the existence of a formally semiuniversal deformation, without the explicit construction. Indeed, since $H^1(\mathcal{N}_\pi) = 0$ by Proposition 2.7.6, and the assumptions $H^1(L^{\otimes 2}) = H^2(\mathcal{O}_Y) = 0$, it follows from the following exact sequence

$$\cdots \to H^0(\mathcal{N}_\pi) \to T^1(\pi/\mathbb{P}^N) \to H^0(\mathcal{N}_\psi) \to H^1(\mathcal{N}_\pi) \to T^2(\pi/\mathbb{P}^N) \to H^1(\mathcal{N}_\psi) \to \cdots$$

Since $H^1(\mathcal{N}_\pi) = 0$, and [Ser06], Proposition 2.3.6 that the forgetful map $\mathbf{Def}_{\pi/\mathbb{P}^N} \to \mathbf{Def}_\psi$ is smooth. Consequently $\mathbf{Def}_{\pi/\mathbb{P}^N}$ is smooth as $\mathbf{Def}_\psi$ is smooth by hypothesis. ■

The following corollary shows that if $\mathbf{Def}_\varphi$ has an algebraic formally universal deformation space, then that space is also smooth and uniruled under suitable assumptions. In fact, one can also expect to determine the degree of a general deformation of φ.

Corollary 4.2.3. *Assume the hypotheses (a), (b), (c), and (d) of Theorem 4.1.3. Further assume B is normal crossing, and*

(a) $\mathbf{Def}_\varphi$ *has an algebraic formally universal deformation space,*

(b) $H^0(N_\psi \otimes \mathcal{E}_\pi) = 0$.

Then the the following happens:

(1) the natural forgetful map $\mathbf{Def}_{\pi/\mathbb{P}^N} \to \mathbf{Def}_\varphi$ *is smooth,*

(2) the algebraic formally universal deformation space of $\mathbf{Def}_\varphi$ *is smooth and uniruled, and*

(3) a general deformation of φ is a composition of a double cover over a deformation Y' of Y followed by a morphism of $Y' \to \mathbb{P}^N$ that is birational onto its image.

Proof. The smoothness of $\mathbf{Def}_{\pi/\mathbb{P}^N} \to \mathbf{Def}_\varphi$ is a consequence of Proposition 4.1.5, thanks to assumption (b). Moreover, $\mathbf{Def}_\varphi$ is smooth since $\mathbf{Def}_{\pi/\mathbb{P}^N}$ is smooth (see [Ser06], Proposition 2.2.5 (iii)), thanks to Theorem 4.2.1.

Now we show that the algebraic formally semiuniversl deformations space of $\mathbf{Def}_\varphi$, which we denote by U_φ, is uniruled. In the notation of Theorem 4.2.1, after possibly shrinking $V_{\pi/\mathbb{P}^N}$

we can assume that $V_{\pi/\mathbb{P}^N} = U_{(\psi,L)} \times \mathbb{P}^m$ where over a point $u \in U_{(\psi,L)}$, the fibre which is a projective space that parametrizes the divisors in the linear system of $\mathscr{L}_{\psi,u}$ which are branch divisors of the finite morphism $X_u \to Y_u$. The conclusion follows since a branch divisor is uniquely determined by the finite morphism.

Finally, part (3) follows from Corollary 4.1.6. □

Now we provide the consequences of the above results on the deformations of X.

Corollary 4.2.4. *Assume all the hypotheses of Corollary 4.2.3. Furthermore assume* $\mathbf{Def}_X$ *has an algebraic formally semiuniversal deformation space. If the natural forgetful map* $\mathbf{Def}_\varphi \to \mathbf{Def}_X$ *has surjective differential map then* $\mathbf{Def}_X$ *is smooth and the algebraic formally semiuniversal deformation space of* X *is uniruled.*

Proof. Since $\mathbf{Def}_\varphi$ is smooth by Corollary 4.2.3, the smoothness of $\mathbf{Def}_X$ follows from [Ser06], Proposition 2.3.7. Composing by the smooth surjection $V_{\pi/\mathbb{P}^N} \to U_\varphi$, we have a smooth surjection $V_{\pi/\mathbb{P}^N} \to U$ where U is the algebraic formally universal deformation space of $\mathbf{Def}_X$. Lastly, U is uniruled since X is normal and for a normal abelian cover, the branch divisors are uniquely determined by X. □

Chapter 5

Deformations and moduli of irregular quadruple Galois canonical covers

5.1 Deformations of irregular quadruple Galois canonical covers of surface scrolls

The objective of this section is to study the deformations of the canonical morphisms of surfaces of each of the four types $(1)_m$, $(1')_m$, $(2)_m$ and $(3)_m$ (general in the first three cases and arbitrary in the fourth) described in Theorem 1.1.1. In particular, we aim to prove the following

Theorem 5.1.1. *Let X be an irregular surface with at worst canonical singularities. Assume the canonical bundle ω_X is ample and globally generated, and the canonical morphism φ is a quadruple Galois cover onto a smooth surface of minimal degree, i.e, X belongs to one of the four families described in Theorem 1.1.1. If X is of type $(1)_m$, $(1')_m$ or $(2)_m$, assume furthermore that X is smooth or has A_1 singularities. Then we have the following description of the algebraic formally semiuniversal deformation space of φ (that exists by Remark 4.1.7).*

(1) If X belongs to the family of type $(1)_m$ $(m \geq 1)$, $(1')_m$ $(m \geq 1)$, $(2)_2$ or $(3)_m$ $(m \geq 2)$, then there exist an irreducible component in the algebraic formally semiuniversal deformation space of φ, such that its general element is a two-to-one morphism onto its image

(I) which is a non-normal variety whose normalization is an elliptic ruled surface with invariant $e = 0$, if X is a surface of type $(1)_m$ or $(1')_m$ (we will show in Theorem 5.2.1 that in this case φ is unobstructed and hence this is the only component);

(II) *which is a smooth surface ruled over a smooth curve of genus* 2 *with invariant* $e = -2$, *if* X *is a surface of type* $(2)_2$ *(we will show in Theorem 5.2.1 that* φ *is unobstructed and hence this is the only component);*

(III) *which is a product of a smooth curve of genus* 2 *with a smooth non-hyperelliptic curve of genus* $m+1$ *if* X *is a surface of type* $(3)_m$ $(m \geq 2)$*;*

and is induced by the complete linear series of a line bundle numerically equivalent to ω_X.

(2) *If* X *is of type* $(3)_1$, *any deformation of* φ *is a morphism of degree four onto its image which is* $\mathbb{F}_0$ *and is induced by the complete linear series of a line bundle numerically equivalent to* ω_X.

(3) *Any deformation of* φ *is of degree* ≥ 2. *In particular, there do not exist any irreducible component in the algebraic formally semiuniversal deformation space of* φ, *such that its general element is birational onto its image.*

First we fix the notations that we are going to use throughout this section. It follows from Theorem 1.1.1 that if $\pi : X \to Y$ is an irregular quadruple Galois canonical cover of a smooth variety of minimal degree Y with trace zero module $\mathscr{E}$, then $i : Y = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ and the embedding is given by the complete linear series $|\mathscr{O}_Y(C_0 + mf)|$. We have $N = 2m+1$ and we identify $\mathscr{O}_Y(C_0)$ with $\mathscr{O}_Y(1,0)$ and $\mathscr{O}_Y(f)$ with $\mathscr{O}_Y(0,1)$. Notice that $T_Y = \mathscr{O}_Y(2,0) \oplus \mathscr{O}_Y(0,2)$, whence $h^0(T_Y) = 6$. One has the following two exact sequences;

$$0 \to \mathscr{O}_Y(2,0) \oplus \mathscr{O}_Y(0,2) \to T_{\mathbb{P}^N}|_Y \to \mathscr{N}_{Y/\mathbb{P}^N} \to 0, \tag{5.1}$$

$$0 \to \mathscr{O}_Y \to \mathscr{O}_Y(1,m)^{\oplus N+1} \to T_{\mathbb{P}^N}|_Y \to 0. \tag{5.2}$$

Lemma 5.1.2. *Let* $Y = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ *be the embedding is given by the complete linear series* $|\mathscr{O}_Y(C_0 + mf)|$.

(1) $H^0(T_{\mathbb{P}^N}|_Y) = (N+1)^2 - 1$, $H^0(\mathcal{N}_{Y/\mathbb{P}^N}) = (N+1)^2 - 7$.

(2) $H^1(T_{\mathbb{P}^N}|_Y) = 0$, $H^1(\mathcal{N}_{Y/\mathbb{P}^N}) = 0$.

Proof. Since Y is regular with $H^2(\mathcal{O}_Y) = 0$, the assertions about $H^j(T_{\mathbb{P}^N}|_Y)$ for $j = 0,1$ follows from (5.2). Consequently, it is easy to compute $H^j(\mathcal{N}_{Y/\mathbb{P}^N})$ for $j = 0,1$ using (5.1). □

Now we fix our notations for nonrational ruled surfaces. A nonrational ruled surface over a nonrational smooth curve C of genus $g \neq 0$ is by definition a projective bundle $W = \mathbb{P}(\mathcal{E}')$ where $\mathcal{E}'$ is a rank 2 vector bundle on C. We will always assume that $\mathcal{E}'$ is normalized, i.e., $\mathcal{E}'$ has sections, but any twist of $\mathcal{E}'$ by any line bundle of negative degree has no section. By definition $e := -\deg(\det(\mathcal{E}'))$ is the invariant of W. A section of $p' : W \to C$ determines a sectional curve C_0' with self intersection $-e$, and let f' be the numerical class of a fiber of p'. It is known that $\mathrm{Pic}(W) = \mathbb{Z}C_0' \oplus p'^*\mathrm{Pic}(C)$. In particular, the Néron-Severi group $\mathrm{NS}(W) = \mathbb{Z}C_0' \oplus \mathbb{Z}f'$ satisfying

$$C_0'^2 = -e, \quad C_0'f' = 1, \quad f'^2 = 0.$$

If $\mathbf{a}$ is a divisor on E', $\mathbf{a}f'$ denotes the pull-back $p'^*\mathbf{a}$. The canonical bundle $\omega_W = -2C_0' + (\mathbf{e} + \omega_C)f'$, where $\mathbf{e} := \det(\mathcal{E}')$, consequently

$$\omega_W \equiv -2C_0' - (e + 2 - 2g)f'.$$

5.1.1 Deformations of canonical morphisms for types $(1)_m$ and $(1')_m$

For these surfaces, we have the following diagram.

$$X \xrightarrow{\pi_1} X_1 \xrightarrow{p_1} Y \overset{i}{\hookrightarrow} \mathbb{P}^N, \qquad \pi = p_1 \circ \pi_1 : X \to Y$$

We also know that $p_{1*}\mathcal{O}_{X_1} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-2C_0)$. Since we have identified $\mathcal{O}_Y(C_0)$ with $\mathcal{O}_Y(1,0)$, we can write $p_{1*}\mathcal{O}_{X_1} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-2,0)$. It is easy to see that $X_1 = E \times \mathbb{P}^1$ where $\psi : E \to \mathbb{P}^1$ is a smooth double cover, with $\psi_*\mathcal{O}_E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, i.e. E is a smooth elliptic curve. We set $\varphi_1 := i \circ p_1$ and call B the branch divisor of π_1.

Proposition 5.1.3. *Let X be a surface of type $(1)_m$ or $(1')_m$. Then the following happens:*

(1) $h^1(\mathcal{O}_{X_1}) = 1$ *and* $h^2(\mathcal{O}_{X_1}) = 0$*,*

(2) $h^0(\mathcal{N}_{p_1}) = 4$ *and* $h^1(\mathcal{N}_{p_1}) = 0$*,*

(3) $h^0(T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-2,0)) = 1$ *and* $h^1(T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-2,0)) = 0$*,*

(4) $h^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(-2C_0)) = 3$ *and* $h^1(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(-2C_0)) = 0$*,*

(5) $h^0(\mathcal{N}_{\varphi_1}) = (N+1)^2$ *and* $h^1(\mathcal{N}_{\varphi_1}) = 0$*; consequently* φ_1 *is unobstructed.*

Proof. (1) Follows from $h^j(\mathcal{O}_{X_1}) = h^j(\mathcal{O}_Y) \oplus h^j(\mathcal{O}_Y(-2,0)$ and Künneth formula.

(2) We apply Proposition 2.7.6. Since Y is regular, it follows that $h^0(\mathcal{N}_{p_1}) = h^0(\mathcal{O}_Y(4,0)) - 1 = 4$. Furthermore, since $H^2(\mathcal{O}_Y) = 0$, we obtain $h^1(\mathcal{N}_{p_1}) = h^1(\mathcal{O}_Y(4,0)) = 0$.

(3) Tensor (5.2) by $\mathcal{O}_Y(-2,0)$ to obtain the following exact sequence

$$0 \to \mathcal{O}_Y(-2,0) \to \mathcal{O}_Y(-1,m)^{\oplus N+1} \to T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-2,0) \to 0.$$

It follows that $h^0(T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-2,0)) = 1$, $h^1(T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-2,0)) = 0$.

(4) This is a consequence of the long exact sequence associated to the exact sequence (5.1) tensored by $\mathcal{O}_Y(-2,0)$ and part (3).

(5) We have the following short exact sequence by lemma 2.7.4

$$0 \to \mathcal{N}_{p_1} \to \mathcal{N}_{\varphi_1} \to p_1^*\mathcal{N}_{Y/\mathbb{P}^N} \to 0. \tag{5.3}$$

It follows that $h^0(\mathcal{N}_{\varphi_1}) = h^0(\mathcal{N}_{p_1}) + h^0(p_1^*\mathcal{N}_{Y/\mathbb{P}^N}) = h^0(\mathcal{N}_{p_1}) + h^0(\mathcal{N}_{Y/\mathbb{P}^N}) + h^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(-2C_0))$, since $h^1(\mathcal{N}_{p_1}) = 0$ by part (2). We obtain $h^0(\mathcal{N}_{\varphi_1}) = (N+1)^2$ thanks to part (2) and Lemma 5.1.2.

The fact $h^1(\mathcal{N}_{\varphi_1}) = 0$ follows from the vanishings of $h^1(\mathcal{N}_{p_1})$ (proven in part (2)), $h^1(\mathcal{N}_{Y/\mathbb{P}^N})$ (proven in Lemma 5.1.2), and $h^1(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(-2C_0))$ (proven in part (4)). ☐

Corollary 5.1.4. *Let X be a surface of type $(1)_m$ or $(1')_m$. If X is smooth or has A_1 singularities, then there exists a smooth, affine irreducible algebraic curve T for which the following happens;*

(a) $\Phi_t : \mathscr{X}_t \to \mathbb{P}^{2m+1}$ *is a morphism of degree two from a normal projective surface with at worst canonical singularities for all $t \in T - \{0\}$. Further for any $t \in T - \{0\}$, the normalization of Im(φ_t) is an elliptic ruled surface which is the projectivization of a rank two split vector bundle on the elliptic curve and has invariant $e = 0$. Further one can take Φ_t to be the canonical morphism of $\mathscr{X}_t$.*

(b) $\Phi_0 : \mathscr{X}_0 \to \mathbb{P}^{2m+1}$ *is the canonical morphism $\varphi : X \to \mathbb{P}^N$.*

Moreover the forgetful map from $\mathbf{Def}_{\pi_1/\mathbb{P}^N} \to \mathbf{Def}_{\varphi}$ is smooth and hence any deformation of φ is a morphism of degree ≥ 2, and a general deformation of φ is a morphism of degree 2 onto its image. Hence in particular φ cannot be deformed to a birational morphism.

Proof. We check the hypotheses of Theorem 4.1.3. Hypothesis (a) has been checked in Proposition 5.1.3 (1). To check hypothesis (b), we need to prove that $\mathbf{Def}_{\varphi_1}$ is smooth, which we have showed in Proposition 5.1.3 (5). To check hypothesis (c), we need to check $H^0(\mathscr{N}_{\varphi_1}) \to H^0(\mathscr{N}_{Y/\mathbb{P}^N} \otimes \mathscr{O}_Y(-2,0))$ is non-zero. This is a consequence of the long exact sequence associated to (5.3), and the facts that $h^1(\mathscr{N}_{p_1}) = 0$ (proven in Proposition 5.1.3 (2)), and $h^0(\mathscr{N}_{Y/\mathbb{P}^N} \otimes \mathscr{O}_Y(-2,0)) \neq 0$ (proven in Proposition 5.1.3 (4)). The fact that $\mathscr{X}_t$ is a normal projective surface with at worst canonical singularities follow thanks to [Kaw99]. Now note that since $h^1(T_{\mathbb{P}^N}|_Y \otimes \mathscr{O}_Y(-2,0)) = h^1(T_{\mathbb{P}^N}|_Y) = 0$ (by Proposition 5.1.3 (3) and Lemma 5.1.2 (2)), we have that $h^1(\varphi_1^*(T_{\mathbb{P}^N})) = 0$ and $\mathbf{Def}_{\varphi_1} \to \mathbf{Def}_{X_1}$ is smooth. Hence the map $H^0(\mathscr{N}_{\varphi_1}) \to H^1(T_{X_1})$ is surjective. By [Sei92], Lemma 12, there exist an open set in $H^1(T_{X_1})$ such that for a smooth curve along a first order deformation belonging to the open set a general deformation of X_1 along the curve is an elliptic ruled surface which is the projectivization of a split rank two vector bundle with invariant $e = 0$. Also $H^0(\mathscr{N}_{\varphi_1}) \to H^0(\mathscr{N}_{Y/\mathbb{P}^N} \otimes \mathscr{O}_Y(-2,0))$ is surjective and there exist an open set of non-zero elements in $H^0(\mathscr{N}_{Y/\mathbb{P}^N} \otimes \mathscr{O}_Y(-2,0))$. Hence one can choose an element (in fact an open set of elements) from $H^0(\mathscr{N}_{\varphi_1})$ such that it maps to a non-zero element in $H^0(\mathscr{N}_{Y/\mathbb{P}^N} \otimes \mathscr{O}_Y(-2,0))$ and the general induced deformation $\mathscr{X}_{1t}$ of X_1 is an elliptic ruled surface which is the projectivization of a split rank two vector bundle with invariant $e = 0$. Finally to check hypothesis (d), note that $H^1(L^{\otimes 2}) = H^1(B) = 0$ by Proposition 5.2.3, (1).

Note that $\mathscr{E}_{\pi_1} = p_1^*(\mathscr{O}_Y(-C_0-(m+2)f))$. Let $\mathscr{E}_{p_1} = \mathscr{O}_Y(-2C_0)$. Then

$$\omega_{X_1} = p_1^*(\omega_Y \otimes \mathscr{O}_Y(2C_0)) = p_1^*(\mathscr{O}_Y(-2C_0-2f)\otimes\mathscr{O}_Y(2C_0)) = p_1^*(-2f).$$

Thus, we obtain $\omega_{X_1}\otimes\varphi_1^*(\mathscr{O}_{\mathbb{P}^N}(-1)) = p_1^*(-2f)\otimes p_1^*(-C_0-mf) = p_1^*(\mathscr{O}_Y(-C_0-(m+2)f)) = \mathscr{E}_{\pi_1}$. Hence by Theorem 4.1.3, we can take Φ_t to be the canonical morphism of X_t.

The second assertion follows from Corollary 4.1.6. The existence of an algebraic formally semiuniversal deformation space of φ follows from Remark 4.1.7. Since X has at worst A_1 singularities, B has normal crossings, so it is easy to check that $H^1(\mathscr{N}_{\pi_1}) = 0$ (which we have verified Proposition 5.2.3 (1)). Consequently, Corollary 4.1.9 shows that any deformation of X is locally trivial. To finish the proof, we need to show that $H^0(\mathscr{N}_{\varphi_1}\otimes\mathscr{E}_{\pi_1}) = 0$ where $\mathscr{E}_{\pi_1}$ is the trace zero module of π_1. We make use of the fact that $X_1 = E\times\mathbb{P}^1$ for an elliptic curve E. Recall that $\psi : E\to\mathbb{P}^1$ is the morphism induced by the restriction of p_1, satisfies $\psi_*\mathscr{O}_E = \mathscr{O}_{\mathbb{P}^1}\oplus\mathscr{O}_{\mathbb{P}^1}(-2)$. We have

$$\pi_{1*}\mathscr{O}_X = \mathscr{O}_{X_1}\oplus(\psi^*\mathscr{O}_{\mathbb{P}^1}(-1)\boxtimes\mathscr{O}_{\mathbb{P}^1}(-m-2)).$$

Also recall that $T_{X_1} = (\mathscr{O}_E\boxtimes\mathscr{O}_{\mathbb{P}^1}(2))\oplus\mathscr{O}_{X_1}$ and $\mathscr{E}_{\pi_1} = \psi^*\mathscr{O}_{\mathbb{P}^1}(-1)\boxtimes\mathscr{O}_{\mathbb{P}^1}(-m-2)$. It is easy to check that $H^1(T_{X_1}\otimes\mathscr{E}_{\pi_1}) = 0$. One has the following pullback of the Euler sequence;

$$0\to\mathscr{O}_{X_1}\to p_1^*\mathscr{O}_Y(1,m)^{\oplus N+1}\to\varphi_1^*T_{\mathbb{P}^N}\to 0. \tag{5.4}$$

We tensor (5.4) by $\mathscr{E}_{\pi_1}$ and take the long exact sequence of cohomology. Notice that $H^1(\mathscr{E}_{\pi_1}) = 0$, and $H^0(p_1^*\mathscr{O}_Y(1,m)\otimes\mathscr{E}_{\pi_1}) = 0$, consequently $H^0(\varphi_1^*T_{\mathbb{P}^N}\otimes\mathscr{E}_{\pi_1}) = 0$. Now consider the exact sequence;

$$0\to T_{X_1}\to\varphi_1^*T_{\mathbb{P}^N}\to\mathscr{N}_{\varphi_1}\to 0. \tag{5.5}$$

It follows from the long exact sequence of cohomology that $H^0(\mathcal{N}_{\varphi_1} \otimes \mathcal{E}_{\pi_1}) = 0$. □

Before moving on to the next case, we make a remark, that will help us to see that for these surfaces $H^1(\mathcal{N}_\varphi) \neq 0$.

Remark 5.1.5. It follows from the vanishing of $H^1(T_{X_1} \otimes \mathcal{E}_{\pi_1})$ and the long exact sequence associated to (5.5) that $h^1(\mathcal{N}_{\varphi_1} \otimes \mathcal{E}_{\pi_1}) \geq h^1(\varphi_1^* T_{\mathbb{P}^N} \otimes \mathcal{E}_{\pi_1}) \geq h^1(T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-1,-m-2)) = N+1$, where the last equality follows from (5.2). ■

Corollary 5.1.6. *Let X be a surface of type $(1)_m$ or $(1')_m$. If X is smooth or has A_1 singularities, then there exists an irreducible component U_φ of the of the algebraic formally semiuniversal deformation space of φ (that exists by Remark 4.1.7) whose general elements are two-to-one morphisms onto their image whose normalization is an elliptic ruled surface with invariant $e = 0$. Further, there does not exist any component of the algebraic formally semiuniversal deformation space of φ whose general elements are morphisms that are birational onto their image.*

Proof. Since the curve constructed in Corollary 5.1.4 is irreducible, it is contained in an irreducible component. Now the assertion follows by applying semicontinuity to the reduced induced structure of the irreducible component (note that a general closed point of an irreducible scheme is the same as a general closed point of its reduced induced structure). □

The corollary of the following proposition show that the image of a general morphism in the irreducible component U_φ constructed above is necessarily non-normal. We remark that what we prove in the following proposition is a slightly stronger statement than what we need in order to prove Corollary 5.1.8; to prove Corollary 5.1.8, we only need the conclusion of the following proposition for $e = 0$.

Proposition 5.1.7. *There does not exist a surface of general type X' with at worst canonical singularities and $K_{X'}^2 = 4p_g(X') - 8$ that satisfies both of the following properties.*

(1) There exist an ample and base point free line bundle $K \equiv \omega_{X'}$ with $h^0(K) = p_g(X')$.

(2) *The morphism φ' induced by the complete linear series $|K|$ is two-to-one onto its image which is a smooth elliptic ruled surface with invariant $e \geq 0$.*

Proof. Suppose there exist such a surface X' with a numerically canonical bundle K satisfying the properties in the proposition. Let the image of the morphism φ given by $|K|$ be W so that the morphism φ factors as

$$X' \xrightarrow{\pi'} W \hookrightarrow \mathbb{P}^{N'}$$

where $N'+1 = p_g(X')$. Let the very ample line bundle on W be denoted by $aC_0' + bf'$. Note that we have $\varphi'^*(\mathcal{O}_{\mathbb{P}^{N'}}(1)) = K$. The morphism is induced by the complete linear series and hence

$$h^0(K) = h^0(aC_0' + bf') + h^0((aC_0' + bf') \otimes \mathcal{E}_{\pi'}) = N' + 1,$$

where $\mathcal{E}_{\pi'}$ is the trace zero module of π'. But now $\mathcal{E}_{\pi'} \equiv K_W \otimes (-aC_0' - bf')$. Hence $h^0((aC_0' + bf') \otimes \mathcal{E}_{\pi'}) = 0$ which gives $h^0(aC_0' + bf') = p_g(X')$. Since $h^i(aC_0' + bf') = 0$ for $i = 1, 2$ (see for example [GP96], Proposition 3.1), we obtain by Riemann-Roch

$$\frac{1}{2}(-a^2e - ae + 2ab + 2b) = p_g(X').$$

Now note that $K = \varphi^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \pi^*(aC_0' + bf')$. Hence $K^2 = 2(aC_0' + bf')^2 = 2(-a^2e + 2ab)$. Using the relation $K^2 = 4p_g(X') - 8$, we obtain

$$2(-a^2e + 2ab) = 2(-a^2e - ae + 2ab + 2b) - 8.$$

This gives $2b - 4 = ae$. But very ampleness of $aC_0' + bf'$ implies $a \geq 1$, $b \geq ae + 3$ which implies $-ae \geq 2$ which is a contradiction if $e \geq 0$. □

Corollary 5.1.8. *Consider the irreducible component U_φ obtained in Corollary 5.1.6. There exist an open set $U_\varphi^0 \subseteq U_\varphi$ such that for a closed point $t \in U_\varphi^0$, $Im(\varphi_t)$ is non-normal, whose normalization is an elliptic ruled surface with $e = 0$ which is the projectivization of a rank two split vector* *bundle on the elliptic curve.*

Proof. Since we are concerned with closed points $t \in U_\varphi$, we can take the reduced induced structure of U_φ and consider the pullback of the formally semiuniversal family. Thus, without loss of generality, one can assume that U_φ is integral. Let $\mathscr{X} \xrightarrow{\Phi} \mathbb{P}^N_{U_\varphi} \to U_\varphi$ be the algebraic formally semiuniversal family of φ. Since the forgetful map $\mathbf{Def}_{\pi_1/\mathbb{P}^N} \to \mathbf{Def}_\varphi$ is smooth we have that the above deformation factors as $\mathscr{X} \xrightarrow{\Pi_1} \mathscr{X}_1 \xrightarrow{\Phi_1} \mathbb{P}^N_{U_\varphi} \to U_\varphi$ $(\Phi = \Pi_1 \circ \Phi_1)$. Let $\mathscr{Y} = \mathrm{Im}(\Phi) = \mathrm{Im}(\Phi_1)$. Since $\mathscr{X}$ is integral, $\mathscr{Y}$ is integral. Since U_φ is integral, we have by generic flatness that $\mathscr{Y} \to U_\varphi$ is flat (after possibly shrinking U_φ). Consider the induced deformation $\mathscr{X}_1 \to U_\varphi$. By our choice of U_φ, we have that there exist $t \in U_\varphi$ such that $\mathscr{X}_{1t}$ is an elliptic ruled surface which is the projectivization of a rank two split vector bundle on the elliptic curve with invariant $e = 0$. Then by [Sei92], Lemma 12, we have that for a general $t \in U_\varphi$, $\mathscr{X}_{1t}$ has the same property. Now for a general $t \in U_\varphi$, $\mathrm{Im}(\Phi_t) = \mathrm{Im}(\Phi_{1t}) = \mathscr{Y}_t$. Also for a general $t \in U_\varphi$, we have that $\mathscr{X}_{1t} \to \mathscr{Y}_t$ is the normalization map. Assume for a general t, that $\mathscr{Y}_t$ is smooth. Then $\mathscr{Y}_t \cong \mathscr{X}_{1t} = \mathrm{Im}(\varphi_t)$. But this is contradiction to Proposition 5.1.7. ☐

5.1.2 Deformation of canonical morphism for type $(2)_m$ $(m \geq 2)$

Now we do analogous calculations for surfaces type $(2)_m$. In this case, We have the following diagram where $p_{1*}\mathscr{O}_{X_1} = \mathscr{O}_Y \oplus \mathscr{O}_Y(-3,-1)$ and $p_{2*}\mathscr{O}_{X_2} = \mathscr{O}_Y \oplus \mathscr{O}_Y(0,-m-1)$.

$$\begin{array}{ccccc} X & \xrightarrow{\pi_1} & X_1 & & \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle p_1} & & \\ X_2 & \xrightarrow[p_2]{} & Y & \underset{i}{\hookrightarrow} & \mathbb{P}^N \end{array} \tag{5.6}$$

Notice $X_2 = \mathbb{P}^1 \times C$ for a smooth curve C that is a double cover $\psi : C \to \mathbb{P}^1$ with $\psi_*\mathscr{O}_C = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-m-1)$. Set $\varphi_j = p_j \circ i$ for $j = 1, 2$ and call B the branch divisor of π_2.

Proposition 5.1.9. *Let X be a surface of type $(2)_m$ $(m \geq 2)$. Then the following happens;*

(1) $h^1(\mathscr{O}_{X_2}) = m$ *and* $h^2(\mathscr{O}_{X_2}) = 0$,

(2) $h^0(\mathcal{N}_{p_2}) = 2m+2$ *and* $h^1(\mathcal{N}_{p_2}) = 0$,

(3) $h^0(T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(0,-m-1)) = h^1(\mathcal{O}_Y(0,-m-1)) = m$ *and* $h^1(T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(0,-m-1)) = 0$,

(4) $h^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(0,-m-1)) = 5m-2$ *and* $h^1(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(0,-m-1)) = 0$,

(5) $h^0(\mathcal{N}_{\varphi_2}) = (N+1)^2 + 7m - 7$ *and* $h^1(\mathcal{N}_{\varphi_2}) = 0$ *where* $\varphi_2 = p_2 \circ i$*; consequently,* φ_2 *is unobstructed.*

Proof. (1) Follows from $h^j(\mathcal{O}_{X_2}) = h^j(\mathcal{O}_Y) \oplus h^j(\mathcal{O}_Y(0,-m-1)$ and Künneth formula.

(2) We apply Proposition 2.7.6. Since Y is regular, it follows that $h^0(\mathcal{N}_{p_0}) = h^0(\mathcal{O}_Y(0,2m+2)) - 1 = 2m+2$. Furthermore, since $H^2(\mathcal{O}_Y) = 0$, we obtain $h^1(\mathcal{N}_{p_2}) = h^1(\mathcal{O}_Y(0,2m+2)) = 0$.

(3) The assertion follows by tensoring the exact sequence (5.2) by $\mathcal{O}_Y(0,-m-1)$ and taking the long exact sequence of cohomology.

(4) This is obtained by tensoring (5.1) by $\mathcal{O}_Y(0,-m-1)$ and taking cohomology.

(5) As before, the assertion follows from the following exact sequence (see Lemma 2.7.4)

$$0 \to \mathcal{N}_{p_2} \to \mathcal{N}_{\varphi_2} \to p_2^*\mathcal{N}_{Y/\mathbb{P}^N} \to 0.$$

Since $h^1(\mathcal{N}_{p_2}) = 0$, we obtain $h^0(\mathcal{N}_{\varphi_2}) = h^0(\mathcal{N}_{p_2}) + h^0(\mathcal{N}_{Y/\mathbb{P}^N}) + h^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(0,-m-1))$. We get the value of $h^0(\mathcal{N}_{\varphi_2})$ from part (2), (4) and Lemma 5.1.2. Finally, $h^1(\mathcal{N}_{\varphi_2}) = 0$ by part (2), (4) and Lemma 5.1.2. □

Corollary 5.1.10. *Let X be a surface of type* $(2)_m$ *($m \geq 2$), which is smooth or has A_1 singularities.*

(1) *Suppose $m = 2$, then there exists a smooth, affine algebraic curve T for which the following happens;*

   (a) $\Phi_t : \mathcal{X}_t \to \mathbb{P}^5$ *is a morphism of degree two from a normal projective surface with at worst canonical singularities for all $t \in T - \{0\}$. Further for any $t \in T - \{0\}$, the normalization of Im(φ_t) is a ruled surface over a smooth curve of genus 2 and has invariant $e = -2$. Further one can take Φ_t to be the canonical morphism of $\mathcal{X}_t$.*

(b) $\Phi_0 : \mathscr{X}_0 \to \mathbb{P}^5$ *is the canonical morphism* $\varphi : X \to \mathbb{P}^N$.

(2) *The forgetful map from* $\mathbf{Def}_{\pi_2/\mathbb{P}^N} \to \mathbf{Def}_\varphi$ *is smooth and hence any deformation of* φ *will be a morphism of degree* ≥ 2 *onto its image. Hence in particular*

(a) φ *cannot be deformed to a birational morphism;*

(b) *a general deformation of* φ *is a morphism of degree* 2 *onto its image if* $m = 2$.

Proof. (1) The existence of the curve T so that for a general $t \in T$, Φ_t has degree two follows from Theorem 4.1.3, Proposition 5.1.9 and [Kaw99]. Note that since $h^1(T_{\mathbb{P}^5}|_Y \otimes \mathscr{O}_Y(0,-3)) = h^1(T_{\mathbb{P}^5}|_Y) = 0$ (by Proposition 5.1.9 (3), Lemma 5.1.2 (2)), we have that $h^1(\varphi_2^*(T_{\mathbb{P}^5})) = 0$ and $\mathbf{Def}_{\varphi_2} \to \mathbf{Def}_{X_2}$ is smooth. Hence the map $H^0(\mathscr{N}_{\varphi_2}) \to H^1(T_{X_2})$ is surjective. By [Sei92], Lemma 12, there exist an open set in $H^1(T_{X_2})$ such that for a smooth curve along a first order deformation belonging to the open set a general deformation of X_2 along the curve is a ruled surface over a smooth curve of genus 2 with invariant $e = -2$. Also $H^0(\mathscr{N}_{\varphi_2}) \to H^0(\mathscr{N}_{Y/\mathbb{P}^5} \otimes \mathscr{O}_Y(0,-3))$ is surjective and there exist an open set of non-zero elements in $H^0(\mathscr{N}_{Y/\mathbb{P}^5} \otimes \mathscr{O}_Y(0,-3))$. Hence one can choose an element (in fact an open set of elements) from $H^0(\mathscr{N}_{\varphi_2})$ such that it maps to a non-zero element in $H^0(\mathscr{N}_{Y/\mathbb{P}^5} \otimes \mathscr{O}_Y(0,-3))$ and the general induced deformation $\mathscr{X}_{2t}$ of X_2 is a ruled surface over a smooth curve of genus 2 with invariant $e = -2$. The trace zero module of the finite map π_2 is $\mathscr{E}_{\pi_2} = \omega_{X_2} \otimes \varphi_2^*(\mathscr{O}_{\mathbb{P}^N}(-1))$; the proof of this statement follows exactly as in Proposition 5.1.3, (6). The fact that Φ_t can be taken to be the canonical morphism of X_t follows from Theorem 4.1.3.

(2) Recall that $X_2 = \mathbb{P}^1 \times C$ for a smooth curve C that is a double cover $\psi : C \to \mathbb{P}^1$ with $\psi_*\mathscr{O}_C = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-m-1)$. We have the following splitting of $\pi_{2*}\mathscr{O}_X$:

$$\pi_{2*}\mathscr{O}_X = \mathscr{O}_{X_2} \oplus (\mathscr{O}_{\mathbb{P}^1}(-3) \boxtimes \psi^*\mathscr{O}_{\mathbb{P}^1}(-1)).$$

Also, $T_{X_2} = (\mathscr{O}_{\mathbb{P}^1} \boxtimes \psi^*\mathscr{O}_{\mathbb{P}^1}(1-m)) \oplus \mathscr{O}_{\mathbb{P}^1}(2) \boxtimes \mathscr{O}_C$, and $\mathscr{E}_{\pi_2} = \mathscr{O}_{\mathbb{P}^1}(-3) \boxtimes \psi^*\mathscr{O}_{\mathbb{P}^1}(-1)$ is the trace zero module of π_2. It is easy to check that $H^1(T_{X_2} \otimes \mathscr{E}_{\pi_2}) = 0$. One has the following pullback of the

Euler sequence;

$$0 \to \mathscr{O}_{X_2} \to p_2^*\mathscr{O}_Y(1,m)^{\oplus N+1} \to \varphi_2^* T_{\mathbb{P}^N} \to 0.$$

By tensoring the above exact sequence by $\mathscr{E}_{\pi_1}$ and taking cohomology, we obtain $H^0(\varphi_2^* T_{\mathbb{P}^N} \otimes \mathscr{E}_{\pi_2}) = 0$. Consequently, the exact sequence;

$$0 \to T_{X_2} \to \varphi_2^* T_{\mathbb{P}^N} \to \mathscr{N}_{\varphi_2} \to 0.$$

shows that $H^0(\mathscr{N}_{\varphi_2} \otimes \mathscr{E}_{\pi_2}) = 0$, and the assertion follows from Proposition 4.1.5 (and Corollary 4.1.6). □

The following corollary is analogous to Corollary 5.1.6 and follows immediately from Corollary 5.1.10.

Corollary 5.1.11. *Let X be a surface of type $(2)_m$ $(m \geq 2)$ which is smooth or has A_1 singularities.*

(1) If $m = 2$, there exists an irreducible component U_φ of the algebraic formally semiuniversal deformation space of φ (that exists by Remark 4.1.7) whose general elements are a two-to-one morphisms onto their image, whose normalization is a ruled surface over a smooth curve of genus 2 and has invariant $e = -2$.

(2) There does not exist any component of the algebraic formally semiuniversal deformation space of φ whose general elements ara morphisms that are birational onto their image.

The following propositions and corollary show that the image of a general morphism in the irreducible component constructed above is smooth. It also shows that this open set intersects the locally closed subloci where the deformed morphism is again the canonical morphism.

Recall that for a ruled surface $X \to C$ of invariant e over a smooth curve of genus $g \geq 1$, we denote C_0 and f denote the numerical classes of a section and a fibre respectively satisfying $C_0'^2 = -e, f'^2 = 0$ and $C_0' \cdot f' = 1$.

Proposition 5.1.12. *Suppose that there exist a flat family $(\mathscr{X}_2 \to T, \mathscr{L})$ of polarized surfaces ruled over a curve C of genus m over a smooth one dimensional base T with*

(1) $\mathscr{X}_{20}$ *has invariant* 0 *and* $\mathscr{L}_0 \equiv C_0' + 2mf'$

(2) $\mathscr{X}_{2t}$ *has invariant* $-m$ *or* $-(m-1)$ *accordingly as* m *is even or odd.*

Then after possibly shrinking T, $\mathscr{L}_t \equiv C_0' + \frac{3m}{2}f'$ *if* m *is even and* $\mathscr{L}_t \equiv C_0' + \frac{3(m-1)}{2}f'$ *if* m *is odd.*

Proof. We prove the statement for m even. The proof is identical for m odd. Let $\mathscr{X}_{20} = X_2$. Let $X_2 \xrightarrow{q} C$ be the smooth morphism. Consider the following commutative diagram.

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & T^1(X_2/C) = H^1(T_{X_2/C}) & \longrightarrow & T^1(q) & \longrightarrow & H^1(T_C) & \longrightarrow & T^2(X_2/C) = H^2(T_{X_2/C}) & \longrightarrow & T^2(q) & \longrightarrow & H^2(T_C) & \longrightarrow \cdots \\
& \| & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \| & & \downarrow \alpha_2 & & \downarrow \beta_2 & \\
\cdots \longrightarrow & T^1(X_2/C) = H^1(T_{X_2/C}) & \longrightarrow & H^1(T_{X_2}) & \longrightarrow & H^1(q^*(T_C)) & \longrightarrow & T^2(X_2/C) = H^2(T_{X_2/C}) & \longrightarrow & H^2(T_{X_2}) & \longrightarrow & H^2(q^*(T_C)) & \longrightarrow \cdots
\end{array}$$

Note that since $q_*(\mathscr{O}_{X_2}) = \mathscr{O}_C$, $H^i(q^*(T_C)) = H^i(T_C)$ for $i = 1,2$. Hence the maps β_1 is surjective and the map β_2 is injective and therefore α_1 is surjective and α_2 is injective. This implies that the forgetful map $\mathbf{Def}_q \to \mathbf{Def}_{X_2}$ is smooth. Hence there exist a deformation of $\mathscr{C} \to T$ of C so that $\mathscr{X}_2 \to T$ factors as $\mathscr{X}_2 \xrightarrow{Q} \mathscr{C} \to T$. Fix a line bundle $\mathscr{O}_C(1)$ of degree one on C. Since $H^2(\mathscr{O}_C) = 0$, we have that the line bundle $\mathscr{O}_C(1)$ lifts to a line bundle $\mathscr{O}_{\mathscr{C}}(1)$. Now the numerical class of $Q^*(\mathscr{O}_{\mathscr{C}}(1))$ restricts to the numerical class of f' on the central fibre and it is the pullback of a degree one line bundle on $\mathscr{C}_t$ on a general fibre. Hence the numerical class of f' on X_2 deforms to the numerical class of f' on $\mathscr{X}_{2t}$. Now suppose that a line bundle of numerical class C_0' in X_2 deforms to a line bundle of numerical class $aC_0' + bf'$ on $\mathscr{X}_{2t}$ for $t \neq 0$. Using the fact that their self intersections are the same and noting that $C_0'^2 = 0$ on X_2 while $C_0'^2 = m$ on $\mathscr{X}_{2t}$, we have that $a^2m + 2ab = 0$. Suppose that $a = 0$. Then for sufficiently large k, $C_0' + kf'$ which is very ample on X_{20} deforms to $(k+b)f'$ which is not ample. Hence $a \neq 0$ and $b = \frac{-am}{2}$. Then C_0' on X_{20} deforms to $a(C_0' - \frac{m}{2}f')$ on $\mathscr{X}_{2t}$. Considering that on $\mathscr{X}_{20}$, $C_0' \cdot f' = 1$, we have that $a = 1$. Hence our statement is proven. □

We prove a slightly stronger version of a result we need to prove Corollary 5.1.14. More precisely we will use the result proven below for $m = 2$.

Proposition 5.1.13. *Suppose* X *is a ruled surface over a curve* C *of genus* m *with invariant* $-m$

or $-(m-1)$ *accordingly as* m *is even or odd. Then a line bundle* $L \equiv C_0 + \frac{3m}{2} f$ *is very ample if* m *is even and a line bundle* $L \equiv C_0 + \frac{3(m-1)}{2} f$ *is very ample if* m *is odd.*

Proof. We use the following criterion for very ampleness (see [LM05] Corollary 2.13): let $|H|$ be the complete linear series of a line bundle $H \equiv C_0 + bf$. Then $|H|$ is very ample if and only for any two points P and Q on C, $h^0(H - (P+Q)f) = h^0(H) - 4$.

Let m be even and consider $L \equiv C_0 + \frac{3m}{2} f$ on a ruled surface with invariant $-m$ over a curve of genus m with $m \geq 2$. We need to show, for any two points P and Q on C,

$$h^0(L - (P+Q)f) = h^0(L) - 4.$$

Let $X = \mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is normalized and since $e < 0$, we have that $\mathscr{E}$ is stable. Since higher pushforward of any bundle of numerical equivalence class $C_0 + bf$ is zero we can compute the above cohomology by pushing forward to the base curve of genus m. Notice that $L - (P+Q)f \equiv C_0 + (\frac{3m}{2} - 2)f$. Hence it is enough to show that for any two line bundles L_1 and L_2 on C with degree $3m/2 - 2$ and $3m/2$ respectively, we have

$$h^0(L_1 \otimes \mathscr{E}) = h^0(L_2 \otimes \mathscr{E}) - 4.$$

For simplicity, let us denote L_1 by $\mathscr{O}_C(\frac{3m}{2} - 2)$, and L_2 by $\mathscr{O}_C(\frac{3m}{2})$. Applying Riemann-Roch to the vector bundles $\mathscr{O}_C(\frac{3m}{2} - 2) \otimes \mathscr{E}$ and $\mathscr{O}_C(\frac{3m}{2}) \otimes \mathscr{E}$ and taking subtracting we get

$$h^0\left(\mathscr{E}\left(\frac{3m}{2}\right)\right) - h^0\left(\mathscr{E}\left(\frac{3m}{2} - 2\right)\right) = h^1\left(\mathscr{E}\left(\frac{3m}{2}\right)\right) - h^1\left(\mathscr{E}\left(\frac{3m}{2} - 2\right)\right) + c_1\left(\mathscr{E}\left(\frac{3m}{2}\right)\right) - c_1\left(\mathscr{E}\left(\frac{3m}{2} - 2\right)\right).$$

Since $\mathscr{E}$ is of rank two we have that $c_1(\mathscr{E}(\frac{3m}{2}) - c_1(\mathscr{E}(\frac{3m}{2} - 2)) = 4$. Hence we are done if we show

$$h^1\left(\mathscr{E}\left(\frac{3m}{2}\right)\right) = h^1\left(\mathscr{E}\left(\frac{3m}{2} - 2\right)\right) = 0.$$

Note that $h^1(\mathscr{E}(\frac{3m}{2})) = h^0(\mathscr{E}^*(\frac{m}{2} - 2)$. Now the slope of the vector bundle $\mu(\mathscr{E}^*(\frac{m}{2} - 2)) = -2$.

Also since $\mathscr{E}$ is stable we have that $\mathscr{E}^*(\frac{m}{2}-2)$ is stable. Since its slope is negative we have that $h^0(\mathscr{E}^*(\frac{m}{2}-2))=0$.

Now note that $h^1(\mathscr{E}(\frac{3m}{2}-2)) = h^0(\mathscr{E}^*(\frac{m}{2}))$. Note that $\deg(\mathscr{E}^*(\frac{m}{2}))=0$ and $\mathscr{E}^*(\frac{m}{2})$ is stable since $\mathscr{E}$ is stable. Then $h^0(\mathscr{E}^*(\frac{m}{2}))=0$ since for degree 0 vector bundles the existence of a section contradicts stability. The proof for the case m odd follows exactly along the same lines. □

Corollary 5.1.14. *In Corollary 5.1.10 (1), we can choose the curve T so that after possibly shrinking T, for $t \in T$, $t \neq 0$, $Im(\Phi_t)$ is smooth and Φ_t can be taken to be the canonical morphism of $\mathscr{X}_t$.*

Proof. We resume notations of Corollary 5.1.10. Consider the factorization $X \xrightarrow{\pi_2} X_2 \xrightarrow{p_2} Y \xrightarrow{i} \mathbb{P}^N$. Let $\varphi_2 = i \circ p_2$. Note that since $X_2 = C \times \mathbb{P}^1$, where C is a smooth curve of genus 2, we have that it is a ruled surface over C and has invariant $e=0$. Consider $L = \varphi_2^*(\mathscr{O}_{\mathbb{P}^N}(1)) = p_2^*(\mathscr{O}_Y(C_0+2f))$ (recall C_0 and f are the classes of a section and fibre of Y). Since X_2 is a double cover branched along $8f$, we have that $p_2^*(\mathscr{O}_Y(C_0)) \equiv aC_0'$ and $p_2^*(\mathscr{O}_Y(f)) \equiv bf'$. We have $ab=2$. Then setting $h^0(p_2^*\mathscr{O}_Y(C_0))=2$ we have that $a=1$ and hence $b=2$. Hence $p_2^*(\mathscr{O}_Y(C_0+2f)) \equiv C_0'+4f'$.

Note that the pair (X_2, L) is unobstructed since $h^2(\mathscr{E}_L)=0$ (since $h^2(\mathscr{O}_{X_2}) = h^2(T_{X_2})=0$). Also since $h^2(\mathscr{O}_{X_2})=0$, $\mathbf{Def}_{(X_2,L)} \to \mathbf{Def}_{X_2}$ is smooth. Choose a smooth curve T from the smooth versal deformation space of (X_2, L). Let $(\mathscr{X}_2 \xrightarrow{\sigma} T, \mathscr{L})$ be the family obtained. Then for a general such curve, for $t \neq 0$, $\mathscr{X}_{2t}$ has invariant -2. By Proposition 5.1.12, $\mathscr{L}_t \equiv C_0'+3f'$ which is very ample by Proposition 5.1.13. Since $H^1(L)=0$, (easy to check by projection formula) we have that (after shrinking T), $\sigma_*(\mathscr{L})$ is locally free of rank $h^0(L)$ and we get a morphism $\mathscr{X}_2 \xrightarrow{\Phi_2} \mathbb{P}(\sigma_*(\mathscr{L})) \to T$ which is an embedding for $t \neq 0$ since it is given by the complete linear series of a line bundle numerically equivalent to $C_0'+3f'$, which is very ample. Note that $\mathscr{E}_{\pi_2} = \omega_{X_2} \otimes \varphi_2^*(L^*)$ and let $B \in H^0(-2(\omega_{X_2} \otimes \varphi_2^*(L^*)))$ be the divisor giving π_2. Note that $-2(\omega_{\mathscr{X}_2/T} \otimes \Phi_2^*(\mathscr{O}_{\mathbb{P}^N_T}(-1)))$ is a lift of $-2(\omega_{X_2} \otimes \varphi_2^*(L^*))$. Then by Remark 4.1.1 one can construct a lift $\mathscr{B}$ of B and hence a relative double cover Π_2 since $H^1(\mathscr{O}_{X_2}(B))=0$ (easy to check, see the proof of Proposition 5.2.5 (1)). Consider $\Phi = \Pi_2 \circ \Phi_2$. For $t \neq 0$, Φ_t is the composition of a double cover Π_{2t} followed by an embedding Φ_{2t} of a smooth surface given by the complete linear series of a very ample line bundle $\mathscr{L}_t$. Moreover Π_{2t} is branched along $-2(\omega_{\mathscr{X}_{2t}} \otimes \mathscr{L}_t^*)$. Hence Φ_t is the canonical morphism of $\mathscr{X}_t$ (by Remark 4.1.2) and its image is smooth. $\square$

5.1.3 Deformation of canonical morphism for type $(3)_m$

Note that the surfaces of type $(3)_m$ are of the form $C_1 \times C_2$ where C_1 is a smooth hyperelliptic curve of genus 2 and C_2 is a smooth hyperelliptic curve of genus $m+1$. Let ψ_1 and ψ_2 be the canonical morphisms of C_1 and C_2 respectively. Notice that the morphism φ is the morphism $\psi : C_1 \times C_2 \to \mathbb{P}^1 \times \mathbb{P}^m$, where $\psi = \psi_1 \times \psi_2$, composed by the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{2m+1}$ given by the complete linear series $|\mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^m}(1)|$. Thus, we have the following diagram.

$$\begin{array}{ccccc} X = C_1 \times C_2 & & & & \\ \downarrow \pi & \searrow^{\psi} & & & \\ Y = \mathbb{P}^1 \times \mathbb{P}^1 & \overset{}{\underset{i_1}{\hookrightarrow}} & Z = \mathbb{P}^1 \times \mathbb{P}^m & \underset{i_2}{\hookrightarrow} & \mathbb{P}^N \end{array} \tag{5.7}$$

We have $\psi_{1*}\mathscr{O}_{C_1} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-3)$. Further, $\psi_2 = i' \circ \psi'_2$ where $\psi'_2 : C_2 \to \mathbb{P}^1$ is a double cover satisfying $\psi'_{2*}\mathscr{O}_{C_2} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-m-2)$, and $i' : \mathbb{P}^1 \hookrightarrow \mathbb{P}^m$ is embedding. We also have the following diagram

$$X \xrightarrow{\pi_2} X_2 \xrightarrow{p_2} Y \hookrightarrow \mathbb{P}^N. \tag{5.8}$$

Notice that $X_2 = \mathbb{P}^1 \times C_2$, and set $\varphi_2 := i \circ p_2$.

Corollary 5.1.15. *Let X be a (smooth) surface of type $(3)_m$ and let U_φ is the algebraic formally semiuniversal deformation space of* $\mathbf{Def}_\varphi$.

(1) Assume $m = 1$. Then any deformation of φ is morphism of degree 4 onto its image which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

(2) If $m \geq 2$ then there exists a smooth, affine algebraic curve T for which;

(a) $\Phi_t : \mathscr{X}_t \to \mathbb{P}^{2m+1}$ is a morphism of degree 2 onto its image which is isomorphic to $\mathbb{P}^1 \times C$ for a smooth non-hyperelliptic curve C of genus $m+1$ for all $t \in T - \{0\}$,

(b) $\Phi_0 : \mathscr{X}_0 \to \mathbb{P}^{2m+1}$ *is the canonical morphism* $\varphi : X \to \mathbb{P}^N$.

Moreover, the forgetful map from $\mathbf{Def}_{\pi_2/\mathbb{P}^N} \to \mathbf{Def}_\varphi$ *is smooth and hence any deformation of* φ *will be a morphism of degree* ≥ 2 *onto its image.*

Proof. (1) Notice that when $m = 1$, Y is a quadric hypersurface in $\mathbb{P}^3$ (i.e., $N = 3$), and $\mathscr{N}_{Y/\mathbb{P}^3} = \mathscr{O}_Y(2,2)$. Notice that $\mathbf{Def}_\varphi$ has an algebraic formally semiuniversal deformation space. It is enough to prove that $\mathbf{Def}_{\pi/\mathbb{P}^3} \to \mathbf{Def}_\varphi$ is smooth. By Proposition 4.1.5, we just need the vanishing of $H^0(\mathscr{N}_{Y/\mathbb{P}^3} \otimes \mathscr{E})$. It is easy to see this vaninishing since $\mathscr{E} = \mathscr{O}_Y(-3,0) \oplus \mathscr{O}_Y(0,-3) \oplus \mathscr{O}_Y(-3,-3)$.

(2) Choose a smooth algebraic curve T for which there is a deformation $\Psi : \mathscr{C} \to \mathbb{P}^m_T$ satisfying the following two conditions (it exists because $m \geq 2$);

(a) $\Psi_t : \mathscr{C}_t \to \mathbb{P}^m$ is the canonical embedding of a smooth non-hyperelliptic curve of genus $m+1$ for $t \neq 0$.

(b) $\Psi_0 : \mathscr{C}_0 \to \mathbb{P}^m$ is the canonical morphism $\psi : C_2 \to \mathbb{P}^m$.

We define Φ to be the composition $C_1 \times \mathscr{C} \xrightarrow{\psi_1 \times \Psi} \mathbb{P}^1 \times \mathbb{P}^m_T \to \mathbb{P}^{2m+1}_T$ where the last morphism is given by the relatively very ample line bundle $\mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^m_T}(1)$, and the first assertion follows.

To see the second assertion, we need to show that the map $\mathbf{Def}_{\pi_2/\mathbb{P}^N} \to \mathbf{Def}_\varphi$ is smooth, which follows (thanks to Proposition 4.1.5) if $H^0(\mathscr{N}_{\varphi_2} \otimes \mathscr{E}_{\pi_2}) = 0$, where $\mathscr{E}_{\pi_2} = \mathscr{O}_{X_2}(-3,0)$ is the trace zero module of π_2. Notice that we have the following exact sequence

$$0 \to T_{X_2} \to \varphi_2^* T_{\mathbb{P}^N} \to \mathscr{N}_{\varphi_2} \to 0.$$

Identify T_{X_2} with $(\mathscr{O}_{\mathbb{P}^1}(2) \boxtimes \mathscr{O}_{C_2}) \oplus (\mathscr{O}_{\mathbb{P}^1} \boxtimes (\psi_2'^* \mathscr{O}_{\mathbb{P}^1}(-m)))$. One checks that $H^1(T_{X_2} \otimes \mathscr{E}_{\pi_2}) = 0$ using Künneth formula.

Now we aim to show that $H^0(\varphi_2^* T_{\mathbb{P}^N} \otimes \mathscr{E}_{\pi_2}) = 0$. By projection formula, we need to check the vanishings of $H^0(T_{\mathbb{P}^N}|_Y \otimes \mathscr{O}_Y(-3,0))$ and $H^0(T_{\mathbb{P}^N}|_Y \otimes \mathscr{O}_Y(-3,-m-2))$. Let C be a general hyperplane section of Y; it is enough to show that $H^0(T_{\mathbb{P}^N}|_Y \otimes \mathscr{O}_Y(-3,0)|_C)$ and $H^0(T_{\mathbb{P}^N}|_Y \otimes$

$\mathcal{O}_Y(-3,-m-2)|_C)$. We have the following diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^{N-1}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^{N-1}}(1)^{\oplus N} & \longrightarrow & T_{\mathbb{P}^{N-1}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^{N-1}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^{N-1}}(1)^{\oplus N+1} & \longrightarrow & T_{\mathbb{P}^N}|_{\mathbb{P}^{N-1}} & \longrightarrow & 0
\end{array}$$

By snake lemma and the splitting of the middle vertical map, we obtain $T_{\mathbb{P}^N}|_{\mathbb{P}^{N-1}} = T_{\mathbb{P}^{N-1}} \oplus \mathcal{O}_{\mathbb{P}^{N-1}}(1)$. Since hyperplane section of Y is a rational normal curve of degree $N-1$, we get

$$T_{\mathbb{P}^N}|_C = \mathcal{O}_{\mathbb{P}^1}(N)^{\oplus N-1} \oplus \mathcal{O}_{\mathbb{P}^1}(N-1).$$

Thus, we have the following to equalities:

(I) $T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-3,0)|_C = \mathcal{O}_{\mathbb{P}^1}(N-3m)^{\oplus N-1} \oplus \mathcal{O}_{\mathbb{P}^1}(N-3m-1)$, and

(II) $T_{\mathbb{P}^N}|_Y \otimes \mathcal{O}_Y(-3,-m-2)|_C = \mathcal{O}_{\mathbb{P}^1}(N-4m-2)^{\oplus N-1} \oplus \mathcal{O}_{\mathbb{P}^1}(N-4m-3)$.

The conclusion follows since $N = 2m+1$. The proof is now complete. □

Corollary 5.1.16. *Let X be a (smooth) surface of type $(3)_m$.*

(1) If $m=1$, a general element of any irreducible component of U_φ is a four to one morphism onto its image which is $\mathbb{P}^1 \times \mathbb{P}^1$.

(2) If $m \geq 2$, there exist a component in the algebraic formally semiuniversal deformation space of φ whose general element is a two-to-one morphism onto its image which is a product of a smooth curve of genus 2 with a smooth non-hyperelliptic curve of genus $m+1$. Further, there does not exist any component in the algebraic formally semiuniversal deformation space of φ whose general element is a morphism birational onto its image.

Proof. Since the curve constructed in Corollary 5.1.15 is irreducible, it is contained in an irreducible component. Now the assertion follows by applying semicontinuity to the reduced induced structure of the irreducible component. □

Proof of Theorem 5.1.1. It follows immediately from Corollary 5.1.6, Corollary 5.1.8, Corollary 5.1.11, Corollary 5.1.14, and Corollary 5.1.16. □

5.2 Moduli of quadruple Galois canonical covers

In this section we will study the moduli components of irregular quadruple covers of minimal degree. Furthermore, if the cover is unobstructed, we know that there is a unique component of the moduli of surfaces of general type; in that case we would like to understand the geometry of this moduli component. Regarding surfaces X of type $(1)_m$ and $(1')_m$, our result is as follows.

Theorem 5.2.1. *Let X be a surface of type $(1)_m$ or $(1')_m$. If X is smooth or has A_1 singularities then* $\mathbf{Def}_\varphi$ *and* $\mathbf{Def}_X$ *are smooth, where φ is the canonical morphism of X, in particular X is contained in a unique irreducible component of the moduli of surfaces of general type. Furthermore, for a given m,*

(1) There exists a unique irreducible component of the moduli of surfaces of general type $\mathscr{M}_{8m,1,2m+2}$ containing all surfaces of both types. This component is uniruled of dimension $8m+20$, and

(2) the canonical morphism of a general element of this component is a two-to-one morphism onto its image which is a non-normal variety whose normalization is an elliptic ruled surface which is the projectivization of a rank two split vector bundle over an elliptic curve and has invariant $e = 0$.

The situation is not as clean as the previous theorem for general surfaces of type $(2)_m$ for $m \geq 3$, even though the result is neat for $m = 2$. In particular, we show the following.

Theorem 5.2.2. *Let X be a surface of type $(2)_m$. If X is smooth or has A_1 singularities, then the following happens.*

(1) If $m = 2$, then $\mathbf{Def}_\varphi$ *and* $\mathbf{Def}_X$ *are smooth, in particular X is contained in a unique irreducible component of the moduli of surfaces of general type. Furthermore,*

(a) there exists a unique irreducible component of the moduli space of surfaces of general type $\mathcal{M}_{16,2,6}$ containing X (and all other surfaces of type $(2)_2$). This component is uniruled of dimension 28, and

(b) the canonical morphism of a general element in that component is double cover onto its image whose normalization is a ruled surface over a smooth curve of genus 2 and has invariant $e = -2$.

(2) If $m \geq 3$, there do not exist an irreducible component of the moduli of surfaces of general type $\mathcal{M}_{8m,m,2m+2}$ containing X, such that the canonical morphism of a general element in that component is birational onto its image.

This section is devoted to the proofs of Theorem 5.2.1 and Theorem 5.2.2.

5.2.1 Description of moduli components of surfaces of types (1) and (1′)

First we aim to prove Theorem 5.2.1. Throughout this subsection, we work with the notations of § 5.1.1. Let $B \in |\psi^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(2m+4)|$ be the branch divisor of π_1. In order to do that, we need the following cohomology computations.

Proposition 5.2.3. *Let X be a surface of type $(1)_m$ or $(1')_m$. If X is smooth or has A_1 singularities, then the following happens:*

(1) $h^0(\mathcal{O}_{X_1}(B)) = h^0(\mathcal{N}_{\pi_1}) = 8m+20$, $h^1(\mathcal{O}_{X_1}(B)) = h^1(\mathcal{N}_{\pi_1}) = 0$ and $h^2(\mathcal{O}_{X_1}(B)) = h^2(\mathcal{N}_{\pi_1}) = 0$,

(2) $h^0(\pi_1^* T_{X_1}) = 4$, $h^1(\pi_1^* T_{X_1}) = 4$ and $h^2(\pi_1^* T_{X_1}) = 4m$,

(3) $h^1(T_X) = 8m+20$, $h^2(T_X) = 4m$,

(4) $h^0(\mathcal{N}_\varphi) = 4m^2+16m+24$, $h^1(\mathcal{N}_\varphi) \geq 2m+2$.

Proof. (1) It is easy to see from Künneth formula, and projection formula that

$$h^0(\mathcal{O}_{X_1}(B)) = 4m+20,\ h^1(\mathcal{O}_{X_1}(B)) = 0,\ \text{and}\ h^2(\mathcal{O}_{X_1}(B)) = 0.$$

The remaining assertions follow from the following exact sequence

$$0 \to \mathscr{O}_{X_1} \to \mathscr{O}_{X_1}(B) \to \mathscr{O}_B(B) \to 0$$

Proposition 2.7.6 and Proposition 5.1.3 (1).

(2) One checks this readily by Proposition 5.1.3 (1), Künneth formula, and projection formula since $h^j(\pi_1^* T_{X_1})$ is nothing but the following sum

$$h^j(\mathscr{O}_{X_1}) + h^j(\mathscr{O}_E \boxtimes \mathscr{O}_{\mathbb{P}^1}(2)) + h^j(\psi^* \mathscr{O}_{\mathbb{P}^1}(-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-m-2)) + h^j(\psi^* \mathscr{O}_{\mathbb{P}^1}(-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-m)).$$

(3) We use the following two exact sequence

$$0 \to T_X \to \pi_1^* T_{X_1} \to \mathscr{N}_{\pi_1} \to 0. \tag{5.9}$$

Since $h^0(T_X) = 0$, and $h^1(\mathscr{N}_{\pi_1}) = h^2(\mathscr{N}_{\pi_1}) = 0$ by part (1), we get the following two exact sequences:

$$0 \to H^0(\pi_1^* T_{X_1}) \to H^0(\mathscr{N}_{\pi_1}) \to H^1(T_X) \to H^1(\pi_1^* T_{X_1}) \to 0,$$

$$0 \to H^2(T_X) \to H^2(\pi_1^* T_{X_1}) \to 0.$$

The conclusion now follows from part (2).

(4) We get the following exact sequence from Lemma 2.7.4:

$$0 \to \mathscr{N}_{\pi_1} \to \mathscr{N}_{\varphi} \to \pi_1^* \mathscr{N}_{\varphi_1} \to 0. \tag{5.10}$$

We first compute $h^0(\mathscr{N}_{\varphi}$. From part (1), we get $h^1(\mathscr{N}_{\pi_1}) = 0$. It follows from projection formula that

$$h^0(\mathscr{N}_{\varphi}) = h^0(\mathscr{N}_{\pi_1}) + h^0(\mathscr{N}_{\varphi_1}) + h^0(\mathscr{N}_{\varphi_1} \otimes \mathscr{E}_{\pi_1}).$$

Recall that we have checked the vanishing of $h^0(\mathcal{N}_{\varphi_1}\otimes\mathcal{E}_{\pi_1})$ in the proof of Corollary 5.1.4. Thus,

$$h^0(\mathcal{N}_\varphi) = 8m+20+(N+1)^2$$

by part (1) and Proposition 5.1.3 (5). The conclusion follows since $N+1=2m+2$.

Notice that $h^2(\mathcal{N}_{\pi_1})=0$ by part (1). From (5.10), we obtain that $h^1(\mathcal{N}_\varphi)=h^1(\mathcal{N}_{\varphi_1})+h^1(\mathcal{N}_{\varphi_1}\otimes\mathcal{E}_{\pi_1})$. The conclusion follows from Remark 5.1.5. ☐

Proof of Theorem 5.2.1. We resume the notations of § 5.1.1. Fix a surface X of type $(1)_m$ or $(1')_m$ smooth or with A_1 singularities; then B has normal crossing. First note that $\mathbf{Def}_\varphi$ and $\mathbf{Def}_X$ has algebraic formally semiuniversal deformation space by Remark 4.1.7.

We apply Corollary 4.2.3. All the hypotheses have been verified in the proofs of Proposition 5.2.3 and Corollary 5.1.4. It follows that $\mathbf{Def}_\varphi$ is unobstructed.

To show $\mathbf{Def}_X$ is unobstructed, we use Corollary 4.2.4. It remains to verify that the differential of the map $\mathbf{Def}_\varphi\to\mathbf{Def}_X$ is surjective. Since the canonical bundle ω_X lifts to any first order deformation of X, it is enough to show that any section of ω_X lifts to any first order deformation of (X,ω_X). We aim to use the section lifting criterion (Proposition 2.6.1). Consider the generalized Atiyah extension (2.2)

$$0\to\mathcal{O}_X\to\mathcal{E}_{\omega_X}\to T_X\to 0. \tag{5.11}$$

Since the canonical bundle ω_X lifts to any first order deformation of X, $H^1(\mathcal{E}_{\omega_X})\to H^1(T_X)$ is surjective. This, together with $H^0(T_X)=0$ implies that

$$h^0(\mathcal{E}_{\omega_X})=1 \text{ and } h^1(\mathcal{E}_{\omega_X})=h^1(\mathcal{O}_X)+h^1(T_X)=8m+21$$

where the last equality follows from Proposition 5.2.3 (3). Now consider the following exact sequence (2.4)

$$0\to\mathcal{E}_{\omega_X}\to H^0(\omega_X)^\vee\otimes\omega_X\to\mathcal{N}_\varphi\to 0. \tag{5.12}$$

Recall that a section in $H^0(\omega_X)$ lifts to a first order deformation $\eta \in H^1(\mathscr{E}_{\omega_X})$ of the pair (X, ω_X) if and only if its image under the map $H^1(\mathscr{E}_{\omega_X}) \to \mathrm{Hom}(H^0(\omega_X), H^1(\omega_X))$ induced from (5.12) is zero (see Proposition 2.6.1). Thus, it is enough to show that the map $H^1(\mathscr{E}_{\omega_X}) \to H^0(\omega_X)^\vee \otimes H^1(\omega_X))$ induced from (5.12) is zero. Now, (5.12) gives rise to the following exact sequence

$$0 \to H^0(\mathscr{E}_{\omega_X}) \to H^0(\omega_X)^\vee \otimes H^0(\omega_X) \to H^0(\mathscr{N}_\varphi) \to H^1(\mathscr{E}_{\omega_X}). \tag{5.13}$$

Thus, the dimension of the image of $H^0(\mathscr{N}_\varphi) \to H^1(\mathscr{E}_{\omega_X})$ is

$$h^0(\mathscr{E}_{\omega_X}) - (h^0(\omega_X))^2 + h^0(\mathscr{N}_\varphi) = 1 - (2m+2)^2 + 4m^2 + 16m + 24 = 8m + 21$$

where the last equality follows from Proposition 5.2.3. But this dimension is same as $h^1(\mathscr{E}_{\omega_X})$. This shows that (5.13) is surjective on the right, and consequently any section of ω_X lifts to any first order deformation of (X, ω_X). Thus the differential of $\mathbf{Def}_\varphi \to \mathbf{Def}_X$ is surjective. Thus $\mathbf{Def}_X$ is unobstructed, and the algebraic formally semiuniversal (in fact universal) deformation space U_X of this functor is smooth, irreducible and uniruled.

(1) We show that there exist a unique component of moduli of surfaces of general type containing *all* surfaces of both types $(1)_m$ and $(1')_m$. We do this by the following few steps.

Step 1. We claim that *all* bidouble covers i.e, the family $(1')_m$ is contained in an irreducible component of the moduli. We will show this by showing that the bidouble covers in $(1')_m$ are parametrized by an open set of $\mathbb{P}(H^0(2C_0 + (2m+4)f)) \times \mathbb{P}(H^0(4C_0))$ which is irreducible.

Let $L_1 = -C_0 - (m+2)f$ and $L_2 = -2C_0$ on $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Note that Y is rigid and consider the following Cartesian square.

$$\begin{array}{ccc} Y \times \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2})) & \longrightarrow & \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2})) \\ \downarrow q & & \downarrow \\ Y & \xrightarrow{\quad p \quad} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Furthermore, consider the divisor $\mathscr{B}_1 = \{(x, r, s) \in Y \times \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2})) | r(x) = 0\} =$

$q^*(L_1^{\otimes -2})$ and $\mathscr{B}_2 = \{(x,r,s) \in Y \times \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2})) | s(y) = 0\} = q^*(L_2^{\otimes -2})$. Let

$$T = Y \times \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2})).$$

Let $\mathscr{L}_1^{\otimes -1} \xrightarrow{f_1} T$ and $\mathscr{L}_2^{\otimes -1} \xrightarrow{f_2} T$ denote the total space of the line bundles $q^*(L_1^{\otimes -1})$ and $q^*(L_2^{\otimes -1})$ on T. Moreover, let $t_1 \in H^0(f_1^* q^*(L_1^{\otimes -1}))$ and $t_2 \in H^0(f_2^* q^*(L_2^{\otimes -1}))$ be the corresponding tautological sections. Then one can consider relative double covers on T given as the zero locus of $t_i^2 - f_i^* \mathscr{B}_i$ inside $\mathscr{L}_i^{\otimes -1}$.

$$\begin{array}{ccc} T_1 = (t_i^2 - f_i^* \mathscr{B}_i)_0 & \hookrightarrow & \mathscr{L}_i^{\otimes -1} \\ & \searrow & \downarrow \\ & & T \end{array}$$

Now consider the fibre product of the relative double covers T_1 and T_2 over T. Consider the flat family $T_1 \times_T T_2 \to T \to \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2}))$. Pulling back the composed morphism at a point $(r,s) \in \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2}))$ we have the following Cartesian square

$$\begin{array}{ccc} Y_r \times_Y Y_s & \hookrightarrow & T_1 \times_T T_2 \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & T \\ \downarrow & & \downarrow \\ (r,s) & \hookrightarrow & \mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2})) \end{array}$$

where Y_r and Y_s denote the double covers constructed by $r \in H^0(L_1^{\otimes -2})$ and $s \in H^0(L_2^{\otimes -2})$. We now know by classification of bidouble canonical covers that smooth bidouble covers are parametrized an open set of $\mathbb{P}(p_*(L_1^{\otimes -2})) \times_k \mathbb{P}(p_*(L_2^{\otimes -2}))$.

Step 2. In this step we note that there is an *unique* irreducible component, say D, of the moduli containtaing *all* surfaces of type $(1')_m$. This comes from Step 1 and the unobstructedness of the smooth surfaces of type $(1')_m$.

Step 3 We claim that *any* cyclic cover in $(1)_m$ can be can be deformed to a smooth bidouble cover along an irreducible curve whose general fibre parametrizes smooth surfaces of general type. In particular given any cyclic cover, there exist an irreducible component D' containing the cyclic cover and a smooth bidouble cover. Indeed, take the cyclic cover X. Its intermediate cover X_1 is smooth. The cyclic cover is obtained by a (special) choice of branch divisor from $p_1^*(2C_0+2(m+2)f)$. Since X_1 is smooth, by a different (special) choice of branch divisor, one can construct a smooth bidouble cover over X_1. Since $\mathbb{P}(H^0(p_1^*(2C_0+2(m+2)f)))$ is irreducible and a general member is smooth by Bertini, we have that one can deform the cyclic cover to a smooth bidouble cover along a curve whose general member is a smooth surface of general type (in this case a smooth double cover over an elliptic ruled surface).

Step 4. We claim that $D' = D$. Indeed, if $D' \neq D$, the smooth bidouble cover lies in both D and D' contradicting its unobstructedness.

The dimension of the moduli component containing surfaces of type $(1)_m$ and $(1')_m$ follows from Proposition 5.2.3 (3) and the unobstructedness of $\mathbf{Def}_X$. That completes the proof of part (1).

(2) Since there is a unique component of the moduli space containing all surfaces of type $(1)_m$ and $(1')_m$, therefore to describe the canonical morphism of a general surface in this component, it is enough to start with a general surface X of either types. It follows from Corollary 5.1.4, Remark 4.1.4, and the fact that X is unobstructed, that a for a general surface of the algebraic formally universal deformation space of X, the canonical morphism is of degree two onto its image which is non-normal and whose normalization is an elliptic ruled surface which is the projectivization of a split vector bundle over an elliptic curve with invariant $e = 0$. Since X is a smooth surface with ample canonical bundle we have that the same holds for its unique irreducible moduli component. That completes the proof. □

Remark 5.2.4. It is interesting to note that $H^1(\mathcal{N}_\varphi) \neq 0$ by Proposition 5.2.3, but $\mathbf{Def}_\varphi$ is still unobstructed by the above proof. Another example of such an instance is when $\phi : H \to \mathbb{P}^L$ is a morphism that is finite onto its image where H is a hyperkähler variety. It has been proven in [MR20], Lemma 3.1, that in this case $\mathbf{Def}_\phi$ is unobstructed but $H^1(\mathcal{N}_\phi) \cong H^2(T_H)$ which is non-zero in general. ■

5.2.2 Description of moduli components of surfaces of type $(2)_m$

Now we aim to prove Theorem 5.2.2. Throughout this subsection, we work with the notations of § 5.1.2. Recall that $B \in |\mathcal{O}_{\mathbb{P}^1}(6) \otimes \psi^*\mathcal{O}_{\mathbb{P}^1}(2)|$ is the branch divisor of π_2. In order to do that, we need the following cohomology computations.

Proposition 5.2.5. *Let X be a surface of type $(2)_2$. If X is smooth or has A_1 singularities, then the following happens:*

(1) $h^0(\mathcal{N}_{\pi_2}) = 22$ *and* $h^1(\mathcal{N}_{\pi_2}) = 0$,

(2) $h^0(\pi_2^* T_{X_2}) = 3$ *and* $h^1(\pi_2^* T_{X_2}) = 9$,

(3) $h^1(T_X) = 28$,

(4) $h^0(\mathcal{N}_{\varphi}) = 65$.

Proof. (1) It is easy to see from Künneth formula, and projection formula that

$$h^0(\mathcal{O}_{X_2}(B)) = 21, \text{ and } h^1(\mathcal{O}_{X_2}(B)) = 0.$$

The remaining assertions follow from the following exact sequence

$$0 \to \mathcal{O}_{X_2} \to \mathcal{O}_{X_2}(B) \to \mathcal{O}_B(B) \to 0$$

Proposition 2.7.6 and the fact that $h^1(\mathcal{O}_{X_2}) = 2$ (see Proposition 5.1.9 (1)).

(2) This one follows from Proposition 5.1.9 (1), Künneth formula, and projection formula since $h^j(\pi_2^* T_{X_2})$ is is the following sum

$$h^j(\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_C) + h^j(\mathcal{O}_{\mathbb{P}^1} \boxtimes \psi^*\mathcal{O}_{\mathbb{P}^1}(-1)) + h^j(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \psi^*\mathcal{O}_{\mathbb{P}^1}(-1)) + h^j(\mathcal{O}_{\mathbb{P}^1}(-3) \boxtimes \psi^*\mathcal{O}_{\mathbb{P}^1}(-2)).$$

(3) We use the following two exact sequence

$$0 \to T_X \to \pi_2^* T_{X_2} \to \mathscr{N}_{\pi_2} \to 0. \tag{5.14}$$

Since $h^0(T_X) = 0$, and $h^1(\mathscr{N}_{\pi_2}) = 0$ by part (1), we get the following exact sequence:

$$0 \to H^0(\pi_2^* T_{X_2}) \to H^0(\mathscr{N}_{\pi_2}) \to H^1(T_X) \to H^1(\pi_2^* T_{X_2}) \to 0.$$

The conclusion now follows from part (2).

(4) We get the following exact sequence from Lemma 2.7.4:

$$0 \to \mathscr{N}_{\pi_2} \to \mathscr{N}_{\varphi} \to \pi_2^* \mathscr{N}_{\varphi_2} \to 0. \tag{5.15}$$

We first compute $h^0(\mathscr{N}_{\varphi})$. From part (1), we get $h^1(\mathscr{N}_{\pi_2}) = 0$. It follows from projection formula that

$$h^0(\mathscr{N}_{\varphi}) = h^0(\mathscr{N}_{\pi_2}) + h^0(\mathscr{N}_{\varphi_2}) + h^0(\mathscr{N}_{\varphi_2} \otimes \mathscr{E}_{\pi_2}).$$

The vanishing of $h^0(\mathscr{N}_{\varphi_2} \otimes \mathscr{E}_{\pi_2})$ has been shown in the proof of Corollary 5.1.10. Thus,

$$h^0(\mathscr{N}_{\varphi}) = 22 + 36 + 7 = 65$$

by part (1) and Proposition 5.1.9 (5). □

Proof of Theorem 5.2.2. It is enough to show that show that, for a surface X of type $(2)_2$ such that B has normal crossings, any section of $H^0(\omega_X)$ lifts to any first order deformation of X. The remaining argument is identical to the Proof of Theorem 5.2.1; all the cohomological criteria have been verified in Proposition 5.1.9, Corollary 5.1.10 and in Proposition 5.2.5. The proof of assertion 1 (b) and (2) are consequences of Corollary 5.1.10, Remark 4.1.4 and the unobstructedness of X.

Using the generalized Atiyah sequence (5.11), and arguing as in the proof of Theorem 5.2.1, we obtain

$$h^0(\mathscr{E}_{\omega_X}) = 1 \text{ and } h^1(\mathscr{E}_{\omega_X}) = h^1(\mathscr{O}_X) + h^1(T_X) = 30$$

thanks to Proposition 5.2.5 (3). Now, arguing as in the proof of Theorem 5.2.1, we get that the dimension of the image of $H^0(\mathscr{N}_\varphi) \to H^1(\mathscr{E}_{\omega_X})$ is

$$h^0(\mathscr{E}_{\omega_X}) - (h^0(\omega_X))^2 + h^0(\mathscr{N}_\varphi) = 1 - 36 + 65 = 30$$

thanks to Proposition 5.2.5 (4). Thus, the map $H^1(\mathscr{E}_{\omega_X}) \to H^0(\omega_X)^\vee \otimes H^1(\omega_X)$ is zero. □

We end this section by asking the following natural questions, concerning the deformations of surfaces of type $(2)_m$, as it is evident that our technique of showing unobstructedness does not work for them.

Question 5.2.6. *Let X be a smooth surface of type $(2)_m$.*

(1) Are $\mathbf{Def}_\varphi$ and $\mathbf{Def}_X$ unobstructed if $m \geq 3$? The problem that we face for these surfaces is that $H^1(\mathscr{N}_{\pi_2}) \neq 0$. Thus, for these surfaces, we know that the forgetful map $\mathbf{Def}_{\pi_2/\mathbb{P}^N} \to \mathbf{Def}_\varphi$ is smooth; however the smoothness of $\mathbf{Def}_{\pi_2/\mathbb{P}^N}$ is unknown despite knowing the smoothness of $\mathbf{Def}_{\varphi_2}$, since the forgetful map $\mathbf{Def}_{\pi_2/\mathbb{P}^N} \to \mathbf{Def}_{\varphi_2}$ is not smooth.

(2) Can surfaces of type $(2)_m$, for $m \geq 3$ be deformed to canonical double covers over surfaces ruled over smooth curves of genus m ?

5.3 Description of non-general locus

Remark 5.3.1. In order to compare with the hyperelliptic loci in the moduli of smooth curves of genus $g \geq 3$, it is interesting to know the dimension of the locally closed sublocus in the moduli of X, where the canonical morphism is once again a quadruple cover onto its image. Note that they are indeed subloci of $\mathscr{M}_{[X]}$ because of the unobstructedness of X. The following table gives the dimension of $\mathscr{M}^G_{[X]}$:= the locally closed sublocus in the moduli of X, where the canonical morphism is once again a quadruple Galois cover onto its image with Galois group G and also that of $\mathscr{M}^{\mathrm{Nat}}_{[X]}(G)$, a subspace of $\mathscr{M}_{[X]}$ containing $\mathscr{M}^{G}_{[X]}$ called natural deformations where the canonical morphism is again four–to–one but not Galois in general (see [Par91], Definition 5.1). Both of these loci are uniruled and we present their dimensions below. We omit the proof as it follows using methods same as Theorem 4.2.1 from under some vanishing conditions .

Proposition 5.3.2. *In the notation of Theorem 1.1.1, let*

(1) For family $(1)_m$*, let* $D_1 \in |L_1^{\otimes 2} \otimes L_2^{\otimes -1}|$*,* $D_2 \in |L_2^{\otimes 2}|$*. Then*

(i) $\mathscr{M}^{G}_{[X]}$ *is a uniruled subvariety of dimension* $h^0(\mathscr{N}_{Y/\mathbb{P}^N}) + \sum_{i=1}^{2} h^0(\mathscr{O}_{D_i}(D_i)) - (N+1)^2$

(ii) $\mathscr{M}^{Nat}_{[X]}$ *is an uniruled subvariety of dimension* $h^0(\mathscr{N}_{Y/\mathbb{P}^N}) + (h^0(\mathscr{O}_{D_1}(D_1)) + h^0(\mathscr{O}_Y(D_1 - L_2))) + (h^0(\mathscr{O}_{D_2}(D_2)) + h^0(\mathscr{O}_Y(D_2 - L_2)) + h^0(\mathscr{O}_Y(D_2 - L_1))) - (N+1)^2$

(2) For family $(1')_m$ *and* $(2)_2$*, let* $D_2 \in |L_1^{\otimes 2}|$*,* $D_1 \in |L_2^{\otimes 2}|$ *Then*

(i) $\mathscr{M}^{G}_{[X]}$ *is a uniruled subvariety of dimension* $h^0(\mathscr{N}_{Y/\mathbb{P}^N}) + \sum_{i=1}^{2} h^0(\mathscr{O}_{D_i}(D_i)) - (N+1)^2$

(2) $\mathscr{M}^{Nat}_{[X]}$ *is an uniruled subvariety of dimension* $h^0(\mathscr{N}_{Y/\mathbb{P}^N}) + \sum_{i=1}^{2} (h^0(\mathscr{O}_{D_i}(D_i)) + h^0(\mathscr{O}_Y(D_i - L_i))) - (N+1)^2$

TABLE 3. Dimension estimates

X is general of type	Dimension of $\mathscr{M}^{G}_{[X]}$	Dimension of $\mathscr{M}^{\mathrm{Nat}}_{[X]}(G)$
$(1)_m$	$2m+1$	$2m+4$
$(1')_m$	$6m+13$	$6m+18$
$(2)_2$	25	25

Table 5.1: Non-general locus in the moduli

Chapter 6

Infinitesimal Torelli Theorem

The goal of this section is to prove the infinitesimal Torelli theorem for some smooth families of quadruple covers. Let X be a smooth algebraic variety of dimension n with ample canonical bundle $\omega_X = \Omega^n_X$. Let $p : \mathscr{X} \to U_X$ be a semiuniversal deformation of X, and we assume that S is smooth. The infinitesimal Torelli problem for weight n Hodge structure asks how far the complex structure of X is determined by the decreasing Hodge filtration

$$\left(F^p = \bigoplus_{i \geq p} H^{n-i}(X, \Omega^i_X)\right)_{p \in \mathbb{N}}.$$

The Hodge filtrations on the fibres glue together to give a subbundle $\mathscr{F}^p$ of $\mathscr{O}_{U_X} \otimes R^n p_* \mathbb{Z}_{\mathscr{X}}$ and define the period map $\Phi_n : U_X \to D_n$ to the space D_n parametrizing Hodge filtrations of weight n. The tangent map $T\Phi_n$ at the special point is the composition of an injective map and the sum of the linear maps

$$\lambda_i : H^1(X, T_X) \to \mathrm{Hom}(\mathbb{C}; H^{n-i}(X, \Omega^i_X), H^{k-i+1}(X, \Omega^{i-1}_X)), \quad n \geq i \geq 1$$

induced by from the contraction maps. Hence $T\Phi_n$ is injective if λ_i is injective for some i, in which case Φ_n is an immersion. We say that

(1) the infinitesimal Torelli theorem holds (for weight n Hodge structures) for X if Φ_n is an immersion;

(2) the infinitesimal Torelli theorem for periods of n forms holds for X if λ_n is an injection.

The infinitesimal Torelli theorem for periods of n forms holds for X if and only if the following map is a surjection

$$H^{n-1}(X,\Omega_X)\otimes H^0(X,\Omega_X^n)\to H^{n-1}(\Omega_X\otimes\Omega_X^n).$$

Classically, a curve of genus $g\geq 2$ satisfies the infinitesimal Torelli theorem if and only if $g=2$ or it is non-hyperelliptic. When $n=2$ i.e., when X is a surface, the infinitesimal Torelli theorem for periods of 2 forms holds for X if and only if the infinitesimal Torelli theorem for weight 2 Hodge structures holds for X. The infinitesimal Torelli problem for abelian covers were studied by Pardini in [Par98] in a very general setting. We refer to the article of Catanese (see [Cat84]) for counterexamples of Torelli problems, the article of Bauer and Catanese (see [BC04]) for counterexamples of the infinitesimal Torelli theorems with ω_X quasi-very ample.

The following criterion under which the infinitesimal Torelli theorem for weight n Hodge structures holds for X was developed by Flenner.

Theorem 6.0.1. *([Fle86], Theorem 1.1) Let X be a compact n-dimensional Kähler manifold and assume the existence of a resolution of Ω_X by vector bundles*

$$0\to\mathscr{G}\to\mathscr{F}\to\Omega_X\to 0.$$

If both conditions are satisfied:

(a) $H^{j+1}(S^j\mathscr{G}\otimes\bigwedge^{n-j-1}\mathscr{F}\otimes\omega_X^{-1})=0$ *for all* $0\leq j\leq n-2$;

(b) The pairing $H^0(S^{n-p}\mathscr{G}^{-1}\otimes\omega_X)\otimes H^0(S^{p-1}\mathscr{G}^{-1}\otimes\omega_X)\to H^0(S^{n-1}\mathscr{G}^{-1}\otimes\omega_X^{\otimes 2})$ *is surjective for a suitable* $p\in\{1,2,\dots,n\}$

then the canonical map $\lambda_p: H^1(X,T_X)\to Hom(\mathbb{C};H^{n-p}(X,\Omega_X^p),H^{n+1-p}(X,\Omega_X^{p-1}))$ *is injective.*

Notice that if X is a smooth quadruple Galois canonical cover of a smooth surface of minimal degree with irregularity one, then X is necessarily of type $(1')_m$. In order to prove the theorem, we are going to invoke the theorem of Flenner i.e., Theorem 6.0.1.

Theorem 6.0.2. *Let X be a smooth surface with irregularity one. Assume the canonical bundle ω_X is ample and globally generated, and the canonical morphism φ is a quadruple Galois canonical cover onto a smooth surface of minimal degree. Then the infinitesimal Torelli theorem holds for X.*

Proof. We work with the notations of § 5.1.1. We have the following commutative diagram

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & Z \\ & \searrow^{\pi_1} & \downarrow p \\ & & X_1 = E \times \mathbb{P}^1 \end{array}$$

where $Z = \mathbb{V}(\mathscr{E}_{\pi_1})$ is the affine bundle over X_1 with the natural projection p. We have the following short exact sequence (4.2)

$$0 \to \pi_1^* \mathscr{E}_{\pi_1}^{\otimes 2} \to \Omega_Z|_X \to \Omega_X \to 0.$$

We will use the criterion of Flenner (Theorem 6.0.1).

(a) Since $\dim(X) = 2$, we need to check $H^1(\Omega_Z|_X \otimes \omega_X^{-1}) = 0$. We also have the following exact sequence (4.3)

$$0 \to \pi_1^* \Omega_{X_1} \to \Omega_Z|_X \to \pi_1^* \mathscr{E}_{\pi_1} \to 0. \tag{6.1}$$

Now, $\Omega_{X_1} = (\mathscr{O}_E \boxtimes \mathscr{O}_{\mathbb{P}^1}(-2)) \oplus (\mathscr{O}_E \boxtimes \mathscr{O}_{\mathbb{P}^1})$. Consequently, we obtain:

$$H^1(\pi_1^* \Omega_{X_1} \otimes \omega_X^{-1}) = H^1(\pi_1^*(\Omega_{X_1} \otimes (\psi^* \mathscr{O}_{\mathbb{P}^1}(-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-m))))$$

Now, the last term by projection formula is just

$$H^1(\psi^* \mathscr{O}_{\mathbb{P}^1}(-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-m-2)) \oplus H^1(\psi^* \mathscr{O}_{\mathbb{P}^1}(-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-m)) \oplus$$
$$H^1(\psi^* \mathscr{O}_{\mathbb{P}^1}(-2) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-2m-4)) \oplus H^1(\psi^* \mathscr{O}_{\mathbb{P}^1}(-2) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-2m-2)).$$

This term is zero by Künneth formula and the projection formula. Notice that we also have

$$H^1(\pi_1^*\mathscr{E}_{\pi_1}\otimes\omega_X^{-1}) = H^1(\pi_1^*(\psi^*\mathscr{O}_{\mathbb{P}^1}(-2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(-2m-2)))$$

This term is just $H^1(\psi^*\mathscr{O}_{\mathbb{P}^1}(-2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(-2m-2))\oplus H^1(\psi^*\mathscr{O}_{\mathbb{P}^1}(-3)\boxtimes\mathscr{O}_{\mathbb{P}^1}(-3m-4))$ which is zero by Künneth formula and the projection formula. Thus, it follows from (6.1) that $H^1(\Omega^1_Z|_X\otimes\omega_X^{-1}) = 0$.

(b) Now we check the surjection corresponding to $p=1$, i.e.,

$$H^0(\pi_1^*(\mathscr{E}_{\pi_1}^*)^{\otimes 2}\otimes\omega_X)\otimes H^0(\omega_X)\to H^0(\pi_1^*(\mathscr{E}_{\pi_1}^*)^{\otimes 2}\otimes\omega_X^{\otimes 2}).$$

Thus we need to check the surjection of

$$H^0(\pi_1^*(\psi^*\mathscr{O}_{\mathbb{P}^1}(2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(2m+4))\otimes\omega_X)\otimes H^0(\omega_X)\to H^0(\pi_1^*(\psi^*\mathscr{O}_{\mathbb{P}^1}(2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(2m+4))\otimes\omega_X^{\otimes 2}).$$

To check this surjection, we use Castelnuovo-Mumford regularity (see [Mum70]). By projection formula,

$$H^1(\pi_1^*(\psi^*\mathscr{O}_{\mathbb{P}^1}(2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(2m+4))) = H^1(\psi^*\mathscr{O}_{\mathbb{P}^1}(2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(2m+4))\oplus H^1(\psi^*\mathscr{O}_{\mathbb{P}^1}(1)\boxtimes\mathscr{O}_{\mathbb{P}^1}(m+2))$$

and it is easy to check that both terms are zero by Künneth formula and the projection formula. Now we compute the following cohomology group

$$H^2(\pi_1^*(\psi^*\mathscr{O}_{\mathbb{P}^1}(2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(2m+4))\otimes\omega_X^{-1}) = H^2(\psi^*\mathscr{O}_{\mathbb{P}^1}(1)\boxtimes\mathscr{O}_{\mathbb{P}^1}(m+4))\oplus H^2(\psi^*\mathscr{O}_{\mathbb{P}^1}\boxtimes\mathscr{O}_{\mathbb{P}^1}(2))$$

and one checks that both terms are zero. That concludes the proof. □

Remark 6.0.3. Let X be a surface of type $(3)_m$. It is easy to see in these cases that infinitesimal Torelli theorem holds only if X is a surface of type $(3)_1$. We give a brief explanation following [BC04] for the sake of completeness. We resume the notations of § 5.1.3. Since $\dim(X) = 2$, infinitesimal Torelli theorem holds $\iff$ infinitesimal Torelli theorem for periods of 2 forms holds $\iff$ $H^1(\Omega^1_X)\otimes H^0(\Omega^2_X)\to H^1(\Omega^1_X\otimes\Omega^2_X)$ is surjective.

Using $\Omega^1_X=(\Omega^1_{C_1}\boxtimes\mathscr{O}_{C_2})\oplus(\mathscr{O}^1_{C_1}\boxtimes\Omega_{C_2})$, $\Omega^2_X=\Omega^1_{C_1}\boxtimes\Omega^1_{C_2}$, and Künneth formula, we have

$$H^1(\Omega^1_X)=\Big(H^0(\Omega^1_{C_1})\otimes H^1(\mathscr{O}_{C_2})\Big)\oplus\Big(H^1(\mathscr{O}_{C_1})\otimes H^0(\Omega^1_{C_2})\Big),$$

$$H^0(\Omega^2_X)=H^0(\Omega^1_{C_1})\otimes H^0(\Omega^1_{C_2}),$$

$$H^1(\Omega^1_X\otimes\Omega^2_X)=H^1((\Omega^1_{C_1})^{\otimes 2}\boxtimes\Omega^1_{C_2})\oplus H^1(\Omega^1_{C_1}\boxtimes(\Omega^1_{C_2})^{\otimes 2})=\Big(H^0((\Omega^1_{C_1})^{\otimes 2})\otimes H^1(\Omega^1_{C_2})\Big)\oplus\Big(H^1(\Omega^1_{C_1})\times H^0((\Omega^1_{C_2})^{\otimes 2})\Big),$$

where the last equality is obtained by using the fact that $H^1((\Omega^1_{C_j})^{\otimes 2})=0$ for $j=1,2$. Notice also that $H^1(\mathscr{O}^1_{C_j})\otimes H^0(\Omega^1_{C_j})\to H^1(\Omega^1_{C_j})$ is surjective for $j=1,2$ by the non-degeneracy of Serre duality. Thus, $H^1(\Omega^1_X)\otimes H^0(\Omega^2_X)\to H^1(\Omega^1_X\otimes\Omega^2_X)$ is surjective if and only if

$$H^0(\Omega^1_{C_j})\times H^0(\Omega^1_{C_j})\to H^0(\Omega^{\otimes 2}_{C_j}) \tag{6.2}$$

is surjective for $j=1,2$. Notice that (6.2) is surjective for $j=1$ since C_1 is a hyperelliptic curve of genus 2. Since C_2 is also hyperelliptic, (6.2) is surjective only when $m=1$. ■

We end this article by asking the natural question regarding the infinitesimal Torelli theorem for smooth surfaces of type $(2)_m$.

Question 6.0.4. *Let X be a smooth surface of type $(2)_m$. Does the infinitesimal Torelli theorem hold for X? Let us resume the notations of § 5.1.2. Let $Z:=\mathbb{V}(\mathscr{E}_{\pi_2})$ be the affine bundle over X_2. It is easy to verify that $H^1(\Omega^1_Z|_X\otimes\omega_X^{-1})=0$. However, to apply the criterion of Flenner (i.e., Theorem 6.0.1), we need the surjectivity of the following multiplication map:*

$$H^0(\pi_2^*(\mathscr{E}^*_{\pi_2})^{\otimes 2}\otimes\omega_X)\otimes H^0(\omega_X)\to H^0(\pi_2^*(\mathscr{E}^*_{\pi_2})^{\otimes 2}\otimes\omega_X^{\otimes 2}).$$

It follows from [GP11], Lemma 2.1 that this map is not surjective.

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