Optimal L^2 Bounds for Certain Hamiltonian Linearizations and New Generalizations of Asymptotic Structures in Certain Banach Spaces

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Abstract

This thesis is the culmination of two distinct branches of research during my time as a graduate student. First, with my advisor Milena Stanislavova, I studied the theory of one-parameter semigroups of bounded linear operators acting on a Banach space. We used aspects of this theory to derive optimal (depending on the eigenvalue structure) L^2 bounds on the semigroup solutions to certain Hamiltonian linearized partial differential equations when it is assumed that the spectrum of the Hamiltonian linearized operator is purely imaginary. Without this assumption, our methods allowed us still to infer a priori bounds on spectrum of the linearized operator. The first chapter of this thesis is an introduction to semigroup theory and the second chapter details how we applied these methods (specifically, the Gomilko Lemma which allows us to distinguish uniform and exponential decay for the semigroup solution) to certain Hamiltonian linearized nonlinear Schródinger and Korteweg-De Vries equations.

Next, I pursued independently a research project in the geometry of Banach spaces. A Banach space X is said to have the Property of Lebesgue or to be a “PL-space" if every Riemann-integrable function $f : [0,1] \to X$ is Lebesgue almost everywhere continuous. The problem of characterizing PL-spaces in terms of their asymptotic geometry is still open. I believe that the solution to this problem will come with the advent of stronger local results which more easily allow for the inference of some amount of global asymptotic structure of X. Upgrading local asymptotic results to global ones is in general a difficult and ongoing problem in the geometry of Banach spaces.

Whether or not X is a PL-space is intimately linked to its asymptotic proximity (both global and local) to ℓ_1. In 2008, K.M. Naralenkov proved that every Banach space that is asymptotic-ℓ_1 with respect to a basis is a PL-space and he also provided details to an unpublished result of A. Pelczyński and G.C. da Rocha Filho that every spreading model of a PL-space is equivalent to ℓ_1. I generalized both of these results in my recent paper [9] so that every Banach space that

is asymptotic-ℓ_1 in a coordinate-free sense (i.e. it need not have a basis so this would include non-separable spaces) is a PL-space, and every so-called SP-asymptotic model of a PL-space is equivalent to ℓ_1. SP-asymptotic models directly generalize spreading models, and the few local-to-global results that do exist are often framed in terms of asymptotic models. The last chapter of this thesis provides some background about PL-spaces, details these new contributions to their theory, and closes with some specific ideas for future research on this topic.

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Contents

Chapter 1

Preliminary Information about Semigroups

Abstract

This chapter establishes a basic theoretical framework about one-parameter operator semigroups acting on a Banach space. These objects are important with respect to applications because they can be used to represent the solutions to various linearized PDE that are "evolution equations."

1.1 What is an Operator Semigroup?

A one-parameter family of bounded linear operators on a Banach space X is said to be an operator semigroup if, loosely speaking, it interacts well with the forward or backward evolution of the parameter. It is assumed throughout that X is a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and that $\mathscr{L}(X)$ is the set of bounded linear operators from X into itself, that is:

$$\mathscr{L}(X) = \{T : X \to X \mid T \text{ is linear and } \exists c > 0 \text{ such that } \|T(x)\| \leq c\|x\| \ \forall x \in X\}.$$

The set $\mathscr{L}(X)$ is itself a Banach space with respect to the operator norm:

$$\|T\|_{\text{op}} = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|T(x)\| = \sup_{\|x\| = 1} \|T(x)\| = \inf\{c > 0 \mid \|T(x)\| \leq c\|x\| \ \forall x \in X\}.$$

The boundedness of $T \in \mathscr{L}(X)$ is equivalent to its continuity with respect to the given norm $\|\cdot\|$ on X and the precise definition of an operator semigroup on X is now given below.

Definition 1.1.1. The one-parameter family $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ is an operator semigroup on X if the following two conditions:

1. $T(0) = I$, where I is the identity map on X
2. $T(t+s) = T(t) \circ T(s)$ for all $t, s \geq 0$

are met. If, in addition, $\limsup_{t\to 0} \|T(t)x - x\| = 0$ for all $x \in X$, then $\{T(t)\}_{t\geq 0}$ is said to be a strongly continuous, or C_0-semigroup.

Every C_0-semigroup induces another linear map, called its infinitesimal generator (or simply its generator), which is more or less its "derivative at zero" subject to initial data.

Definition 1.1.2. Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X. The generator of $\{T(t)\}_{t\geq 0}$ is the linear map $A : D(A) \subset X \to X$ defined by:

$$D(A) = \left\{ x \in X \;\middle|\; \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

where $Ax = \lim_{t\to 0} \frac{T(t)x-x}{t}$ for all $x \in D(A)$.

It is worth noting that the concept of a generator makes sense only for semigroups that are at least strongly continuous because $x \notin D(A)$ if $\limsup_{t\to 0} \|T(t)x - x\| > 0$. On the other hand, $\limsup_{t\to 0} \|T(t)x - x\| = 0$ does not guarantee that $x \in D(A)$ so $D(A)$ might well be a proper subspace of X and $A : D(A) \subset X \to X$ might well be unbounded even if $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup.

As noted in the chapter abstract, semigroups can be used to represent solutions to linearized PDE whose "Abstract Cauchy Problem" (to be defined later) has a suitable form. The classical example is the heat equation, namely,

$$\begin{cases} (\partial_t - \Delta)u(x,t) = 0 & \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R} \text{ and } t = 0 \end{cases}$$

for $u_0 \in D(A) = H^2(\mathbb{R})$ fixed. In this case, the operator $A = \partial_{xx} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ generates the

C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(\mathbb{R})$ that is defined by $T(0) = I$ and

$$u(x,t) = T(t)u_0(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} u_0(y)dy \qquad t > 0$$

where we recall that $u_0 = u(x,0)$ and that the solution to the heat equation for $x \in \mathbb{R}$ at time $t > 0$ is given by the convolution of the heat kernel $K_t(x-y) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-|x-y|^2}{4t}}$ with $u_0(y)$. It is a standard exercise to check that $\{T(t)\}_{t\geq 0}$ as above satisfies the conditions of being a C_0-semigoup on $L^2(\mathbb{R})$ that is generated by ∂_{xx}.

The next subsection collects the basic properties of C_0-semigroups that follow easily from Definitions 1.1.1 and 1.1.2. Notational conventions hereafter include that $\mathbb{N}_0$ is the set of non-negative integers and $\mathbb{N} = \mathbb{N}_0 \cap [1,\infty)$.

1.2 Basic Properties of C_0-Semigroups

The most basic property of a C_0-semigroup is that the parameter-to-orbit map is, for every fixed initial datum $x \in X$, continuous.

Theorem 1.2.1. *Let $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ be a C_0-semigroup. Then, the function $\kappa_x : [0,\infty) \to X$ defined by $\kappa_x(t) = T(t)x$ is, for all $x \in X$, continuous.*

Proof. Let $x \in X$ be arbitrary and fix $t_0 \in [0,\infty)$. If $h > 0$, then

$$\|\kappa_x(t_0+h) - \kappa_x(t_0)\| = \|T(t_0+h)x - T(t_0)x\| = \|T(t_0)\left(T(h)x - x\right)\| \leq \|T(t_0)\|_{\text{op}} \|T(h)x - x\|$$

and taking $\limsup$ as $h \to 0$ of both sides proves the right-continuity of κ. Left-continuity follows similarly if $t_0 > 0$ and hence, κ is continuous at $t_0 \in [0,\infty)$. □

If $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup, then the continuity of the parameter-to-orbit maps is more or less the assertion of pointwise boundedness for any subfamily $\{T(t)\}_{t\in I}$ where $I \subset [0,\infty)$ is compact. The Principle of Uniform Boundedness then leads to the following result.

Theorem 1.2.2. *Let $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ be a C_0-semigroup. Then, $\|T(t)\|_{op} \leq M_\omega e^{\omega t}$ for some $\omega \in \mathbb{R}$ and for some $M_\omega \geq 1$.*

Proof. Suppose, for all $n \in \mathbb{N}$, that there exists a real number $t_n \in [0, \frac{1}{n}]$ such that $\|T(t_n)\|_{\text{op}} > n$. Consider the subfamily:

$$\{T(t_n) \mid n \in \mathbb{N}\}$$

and note that, for each $x \in X$, $\|T(t_n)(x)\| \leq \sup_{t\in[0,1]} \|T(t)x\| < \infty$ by the continuity (and hence boundedness) of the parameter-to-orbit map κ_x on $[0,1]$. It follows by the Principle of Uniform Boundedness that $\sup_{n\in\mathbb{N}} \|T(t_n)\|_{\text{op}} < \infty$ and this contradicts $\|T(t_n)\|_{\text{op}} > n$ for all $n \in \mathbb{N}$. There is then a positive integer n_0 such that $\|T(t)\|_{\text{op}} \leq n_0$ for every $t \in [0, \frac{1}{n_0}]$

Let $t \geq 0$ be arbitrary and note that $t = k\frac{1}{n_0} + r$ for some $k \in \mathbb{N}_0$ and for some $r \in [0, \frac{1}{n_0})$. This obtains the estimate:

$$\begin{aligned}\|T(t)\|_{\text{op}} = \left\|T\left(k\frac{1}{n_0} + r\right)\right\|_{\text{op}} \leq \left\|T\left(\frac{1}{n_0}\right)\right\|_{\text{op}}^k \|T(r)\|_{\text{op}} \\ \leq n_0^{k+1} = n_0 e^{k\ln(n_0)} = n_0 e^{\ln(n_0)(t-r)n_0} \leq n_0 e^{n_0 \ln(n_0)t}\end{aligned}$$

so defining $\omega = n_0 \ln(n_0)$ and $M_\omega = n_0$ completes the proof. □

It is worth noting that the constants ω and M_ω in the above proof of Theorem 1.2.2 depend solely on $n_0 \in \mathbb{N}$. However, the notation M_ω is suggestive and will be explained shortly. The final theorem of this subsection provides a “toolbox" of sorts for manipulating C_0-semigroups and their generators.

Theorem 1.2.3. *Let $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ be a C_0-semigroup and let $A : D(A) \subset X \to X$ be its generator. Then,*

1. *For all $x \in X$ and for all $t \geq 0$:*
   (a) $\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x$
   (b) $\int_0^t T(s)xds \in D(A)$ *and* $A\left(\int_0^t T(s)xds\right) = T(t)x - x$

2. *For all $x \in D(A)$ and for all $t \geq 0$:*

   (a) *$T(t)x \in D(A)$ and $\frac{d}{dt}[T(t)x] = AT(t)x = T(t)Ax$*

   (b) *For all $s \in [0,t]$, $\int_s^t T(w)Axdw = \int_s^t AT(w)xdw = T(t)x - T(s)x$.*

3. *$A : D(A) \subset X \to X$ is densely-defined and closed.*

Proof. Fix $x \in X$ and let $t \geq 0$ be arbitrary. Let $\varepsilon > 0$ be given and choose $h > 0$ so small that $\|T(s)x - T(t)x\| < \varepsilon$ for $s \in [t, t+h]$ by the continuity of κ_x at t. Then,

$$\left\| \frac{1}{h} \int_t^{t+h} T(s)xds - T(t)x \right\| = \left\| \frac{1}{h} \left(\int_t^{t+h} T(s)x - T(t)x \right) ds \right\|$$
$$\leq \frac{1}{h} \int_t^{t+h} \|T(s)x - T(t)x\| ds \leq \frac{1}{h} \int_t^{t+h} \varepsilon ds = \varepsilon$$

so 1. (a) follows. The assertion 1. (b) is an application of 1. (a). Finally, the assertion 2. (a) is an application of the semigroup property (i.e. the second condition of Definition 1.1.1) and the assertion 2. (b) is nothing but the fundamental theorem of calculus.

The third assertion is perhaps the most far-reaching consequence of Theorem 1.2.3 and. For a fixed $x \in X$, define the vectors: $x_n = n \int_0^{\frac{1}{n}} T(t)xdt$ for all $n \in \mathbb{N}$. It is immediate that $x_n \in D(A)$ by 1. (b) and it is also not difficult to see that $x = \lim_{n \to \infty} x_n$ by an estimate similar the one given for the proof of 1. (a). Namely,

$$\|x - x_n\| = \left\| x - n \int_0^{\frac{1}{n}} T(t)xdt \right\| \leq n \int_0^{\frac{1}{n}} \|x - T(t)x\| dt \leq \sup_{0 \leq t \leq \frac{1}{n}} \|x - T(t)x\|$$

and the RHS goes to zero as $n \to \infty$, so that $D(A)$ is a dense subset of X. If $(x_n)_{n=1}^{\infty} \in [D(A)]^{\mathbb{N}}$, $x_n \to y \in X$, and $A(x_n) \to z$, then it lastly follows that for a fixed $h > 0$ and for all $t \in [0,h]$,

$$\|T(t)A(x_n) - T(t)z\| = \|T(t)[A(x_n) - z]\| \leq M_\omega e^{\omega h} \|A(x_n) - z\|$$

so that $T(t)A(x_n) \overset{\text{unif}}{\to} T(t)z$ on $[0,h]$ as $n \to \infty$. In particular, this implies that:

$$\frac{T(h)y-y}{h} = \lim_{n\to\infty} \frac{T(h)x_n - x_n}{h} = \lim_{n\to\infty} \frac{1}{h}\int_0^h T(t)A(x_n)dt$$
$$= \frac{1}{h}\int_0^h \lim_{n\to\infty} T(t)A(x_n)dt = \frac{1}{h}\int_0^h T(t)zdt$$

so, in turn, $\lim_{h\downarrow 0} \frac{T(h)y-y}{h} = \lim_{h\downarrow 0} \frac{1}{h}\int_0^h T(t)zdt = T(0)z = z$ because the above equality holds for all $h > 0$. It follows that $y \in D(A)$ and that $A(y) = z$, meaning that A is closed. □

One-parameter operator semigroups are purely mathematical objects that interact in a desirable manner with the forward and backward evolution of their parameter and, if they are strongly continuous, have some additional properties that are worth noting. The reason, however, for defining these objects is application-based. Namely, the “Abstract Cauchy Problem" (ACP) for the linear operator $A : D(A) \subset X \to X$ is the initial value problem:

$$\begin{cases} \frac{d}{dt}[u(t)] = A(u(t)) & \text{for all } t \geq 0 \\ u(0) = u_0 & \text{where } u_0 \in D(A) \end{cases}. \tag{1.2.1}$$

The ACP (1.2.1) is well-posed (i.e. it has a unique solution depending continuously on the initial data) if and only if A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. The solution, in this case, is given by the “semigroup representation" $u(t) = T(t)u_0$. This raises an important question: when is $A : D(A) \subset X \to X$ the generator of a C_0-semigroup?

1.3 Necessary and Sufficient Conditions for C_0-Semigroup Generation

The theory of ordinary differential equations (ODE) suggests that the semigroup $\{T(t)\}_{t\geq 0}$ whose semigroup represntation $u(t) = T(t)u_0$ solves (1.2.1) ought to behave much like the collection of linear operators $\{e^{tA}\}_{t\geq 0}$ where e^{tA} is defined by its Taylor series:

$$e^{tA} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} \tag{1.3.1}$$

which converges when $A \in \mathscr{L}(X)$ (so $\|A\|_{\text{op}} < \infty$). This leads to a preliminary question: under what condition(s) is $T(t) = e^{tA}$ for all $t \geq 0$?

Theorem 1.3.1. *Let $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ be a C_0-semigroup and let $A : D(A) \subset X \to X$ be its generator. Then, $A \in \mathscr{L}(X)$ if and only if $\limsup_{t\to 0} \|T(t) - I\|_{op} = 0$.*

Proof. Suppose first that $A \in \mathscr{L}(X)$ and define, for $t \geq 0$, the linear operators e^{tA} as in (1.3.1). It is clear that if $\|x\| = 1$, then

$$\begin{aligned}
\|e^{tA}x\| = \left\| \left(\sum_{j=0}^{\infty} \frac{(tA)^j}{j!} \right) x \right\| &= \left\| \left(\sum_{j=0}^{N} \frac{(tA)^j}{j!} + \sum_{j=N+1}^{\infty} \frac{(tA)^j}{j!} \right) x \right\| \\
&\leq \left\| \sum_{j=0}^{N} \frac{(tA)^j x}{j!} \right\| + \left\| \sum_{j=N+1}^{\infty} \frac{(tA)^j}{j!} \right\|_{\text{op}} \leq \varepsilon + \sum_{j=0}^{N} \frac{(t\|A\|_{\text{op}})^j}{j!} \leq \varepsilon + e^{t\|A\|_{\text{op}}}
\end{aligned}$$

for a given $\varepsilon > 0$ and $N = N(t, A) \in \mathbb{N}$ chosen sufficiently large. It follows that $e^{tA} \in \mathscr{L}(X)$ for each $t \geq 0$ and that $\|e^{tA}\|_{\text{op}} \leq e^{t\|A\|_{\text{op}}}$. In addition, $e^{0A} = I$ (the identity on X) and:

$$\begin{aligned}
e^{tA} \circ e^{sA} = \left(\sum_{j=0}^{\infty} \frac{(tA)^j}{j!} \right) \circ \left(\sum_{k=0}^{\infty} \frac{(sA)^k}{k!} \right) &= \sum_{i=0}^{\infty} \sum_{l=0}^{i} \frac{(tA)^l}{l!} \frac{(sA)^{i-l}}{(i-l)!} \\
&= \sum_{i=0}^{\infty} \frac{A^i}{i!} \sum_{l=0}^{i} \frac{i!}{l!(i-l)!} t^l s^{i-l} = \sum_{i=0}^{\infty} \frac{((t+s)A)^i}{i!} = e^{(t+s)A}
\end{aligned}$$

by applying the Cauchy product formula (i.e. discrete convolution) and the binomial theorem so $\{e^{tA}\}_{t\geq 0}$ is clearly a semigroup on X by Definition 1.1.1. It remains to prove that this semigroup is strongly continuous and generated by A. If $x \in X$ is fixed, then

$$\limsup_{t\to 0} \|e^{tA}x - x\| = \|e^0 x - x\| = 0 \tag{1.3.2}$$

so $\{e^{tA}\}_{t\geq 0}$ is, in fact, a C_0-semigroup. Finally,

$$\left\|\frac{e^{tA}x-x}{t}-Ax\right\| = \left\|\frac{1}{t}(e^{tA}-I-tA)x\right\| = \left\|\frac{1}{t}\left(\sum_{j=2}^{\infty}\frac{(tA)^j}{j!}\right)x\right\| = \left\|\left(\sum_{j=2}^{\infty}\frac{t^{j-1}A^j}{j!}\right)x\right\|$$
$$= \left\|\left(\sum_{j=2}^{N}\frac{t^{j-1}A^j}{j!}+\sum_{j=N+1}^{\infty}\frac{t^{j-1}A^j}{j!}\right)x\right\| \leq \varepsilon + \left\|\left(\sum_{j=2}^{N}\frac{t^{j-1}A^j}{j!}\right)x\right\| \quad (1.3.3)$$

where once again $\varepsilon > 0$ is given and $N = N(t,A) \in \mathbb{N}$ is chosen sufficiently large. It follows that $\{e^{tA}\}_{t\geq 0}$ is generated by A upon taking $\limsup_{t\to 0}$ of both sides of (1.3.3). In other words, the bounded linear operator $A \in \mathscr{L}(X)$ generates the truly exponential C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ and what is more, (1.3.2) can be improved in the following sense:

$$\|e^{tA}-I\|_{\text{op}} = \left\|\sum_{j=1}^{\infty}\frac{(tA)^j}{j!}\right\|_{\text{op}} \leq \left\|\sum_{j=1}^{N}\frac{(tA)^j}{j!}\right\|_{\text{op}} + \varepsilon \quad (1.3.4)$$

where, for a third time, $\varepsilon > 0$ is given, $N = N(t,A) \in \mathbb{N}$ is chosen sufficiently large, and applying $\limsup_{t\to 0}$ on both sides of (1.3.4) yields $\limsup_{t\to 0}\|e^{tA}-I\|_{\text{op}} = 0$.

Suppose conversely that $\limsup_{t\to 0}\|T(t)-I\|_{\text{op}} = 0$ note that for a sufficiently small $r > 0$,

$$\left\|I-\frac{1}{r}\int_0^r T(t)dt\right\|_{\text{op}} \leq \frac{1}{r}\int_0^r \|I-T(t)\|_{\text{op}}dt$$

and this implies that $I-\left(I-\frac{1}{r}\int_0^r T(t)dt\right) = \frac{1}{r}\int_0^r T(t)dt \in \mathscr{L}(X)$ is invertible. Let $x \in X$ and fix

$h \in (0,r)$. Then,

$$\begin{aligned}
\frac{T(h)x-x}{h} &= \left[\int_0^r T(t)dt\right]^{-1}\left[\int_0^r T(t)dt\right]\frac{T(h)-I}{h}x = \left[\int_0^r T(t)dt\right]^{-1}\left[\frac{1}{h}\int_0^r T(t)[T(h)-I]xdt\right] \\
&= \left[\int_0^r T(t)dt\right]^{-1}\left[\frac{1}{h}\int_0^r T(t+h)xdt - \frac{1}{h}\int_0^r T(t)xdt\right] \\
&= \left[\int_0^r T(t)dt\right]^{-1}\left[\frac{1}{h}\int_h^{r+h} T(t)xdt - \frac{1}{h}\int_0^r T(t)xdt\right] \\
&= \left[\int_0^r T(t)dt\right]^{-1}\left[\frac{1}{h}\left(\int_r^{r+h} T(t)xdt + \int_h^r T(t)xdt\right) - \frac{1}{h}\int_0^r T(t)xdt\right] \\
&= \left[\int_0^r T(t)dt\right]^{-1}\left[\frac{1}{h}\int_r^{r+h} T(t)xdt - \frac{1}{h}\int_0^h T(t)xdt\right] \qquad (1.3.5)
\end{aligned}$$

and taking $h \downarrow 0$ on both sides of (1.3.5) implies that $\lim_{h\downarrow 0}\frac{T(h)x-x}{h} = \left[\int_0^r T(t)dt\right]^{-1}(T(r)x-x)$. Then, $A : D(A) \subset X \to X$ is not only closed, but is also everywhere-defined so it follows that $A \in \mathscr{L}(X)$ as required. □

The upshot of Theorem 1.3.1 is that if $\limsup_{t\to 0}\|T(t)-I\|_{\text{op}} = 0$, then the semigroup $\{T(t)\}_{t\geq 0}$ is generated by a bounded linear operator and is therefore truly exponential. This knowledge is the first step towards a complete characterization of C_0-semigroup generators.

Definition 1.3.2. The resolvent set of a closed linear operator $A : D(A) \subset X \to X$ is the subset of the complex plane defined by $\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) : D(A) \subset X \to X \text{ is bijective}\}$.

The only assumption on $A : D(A) \subset X \to X$ beyond its linearity is that it must be closed. In particular, $D(A)$ need not be dense in X. The reason for this assumption is to ensure that, for all fixed $\lambda \in \rho(A)$, the inverse (so-called resolvent) linear operator $(\lambda I - A)^{-1} : X \to D(A)$ is closed and therefore bounded as a consequence of the closed graph theorem (because it is everywhere-defined). The resolvent operator at $\lambda \in \rho(A)$ is commonly and hereafter denoted by $R(\lambda, A)$. The cornerstone and, in some sense, first nontrivial result of basic semigroup theory is the following theorem for contraction semigroups (i.e. a C_0-semigroup satisfying $\|T(t)\|_{\text{op}} \leq 1$ for all $t \geq 0$).

Theorem 1.3.3. *The linear operator* $A : D(A) \subset X \to X$ *generates a contraction semigroup on* X *if and only if it is closed, densely-defined, and* $\|R(\lambda,A)\|_{op} \leq \frac{1}{\lambda}$ *for all* $\lambda \in (0,\infty)$.

Proof. Suppose first that $A : D(A) \subset X \to X$ generates a contraction semigroup $\{T(t)\}_{t\geq 0}$. It is immediately clear from Theorem 1.2.3 that A is closed and densely-defined. Next, define for a fixed $\lambda \in (0,\infty)$ the linear function $R_\lambda : X \to X$ by:

$$R_\lambda x = \int_0^\infty e^{-\lambda t} T(t) x dt \qquad (x \in X)$$

and notice that for every $x \in X$ with $\|x\| = 1$, $\|R_\lambda x\| \leq \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}$. It remains to show that $\lambda \in \rho(A)$ and that $R_\lambda = R(\lambda,A)$. For a fixed $x \in X$, consider

$$\begin{aligned} \frac{T(h)R_\lambda x - R_\lambda x}{h} &= \frac{1}{h} T(h) \int_0^\infty e^{-\lambda t} T(t) x dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t) x dt \\ &= \frac{1}{h} \int_h^\infty e^{-\lambda(t-h)} T(t) dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t) x dt \\ &= \frac{e^{\lambda h} - 1}{h} \int_h^\infty e^{-\lambda t} T(t) x dt - \frac{1}{h} \int_0^h e^{-\lambda t} T(t) x dt \end{aligned} \tag{1.3.6}$$

and taking $h \downarrow 0$ on both sides of (1.3.6) implies that $R_\lambda x \in D(A)$ with $A(R_\lambda x) = \lambda R_\lambda x - x$. This means in particular that for $x \in X$,

$$(\lambda I - A) R_\lambda x = \lambda R_\lambda x - A(R_\lambda x) = \lambda R_\lambda x - \lambda R_\lambda x + x = x$$

and on the other hand for $x \in D(A)$,

$$R_\lambda (\lambda I - A) x = \lambda R_\lambda x - R_\lambda A x = \lambda R_\lambda x - A R_\lambda x = \lambda R_\lambda x - \lambda R_\lambda x + x = x$$

because A is closed and commutes with $T(t)$ (and hence with R_λ) for $x \in D(A)$. It follows that $\lambda \in \rho(A)$, $R_\lambda = R(\lambda,A)$, and that $\|R(\lambda,A)\|_{\text{op}} \leq \frac{1}{\lambda}$ as required.

Suppose conversely that $A : D(A) \subset X \to X$ is closed, densely-defined, and that $\|R(\lambda,A)\|_{\text{op}} \leq$

$\frac{1}{\lambda}$ for all $\lambda \in (0,\infty)$. Define, for all $\lambda \in (0,\infty)$, the linear operator

$$A_\lambda = \lambda AR(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I$$

and observe that:

1. For $x \in D(A)$, $\|\lambda R(\lambda,A)x - x\| = \|AR(\lambda,A)x\| \leq \frac{\|Ax\|}{\lambda}$ so that $\lim_{\lambda\to\infty} \lambda R(\lambda,A)x = x$. What is more, $\lim_{\lambda\to\infty} \lambda R(\lambda,A)x = x$ for all $x \in X$ by the density of $D(A)$ in X.
2. For $\lambda \in (0,\infty)$, $A_\lambda \in \mathscr{L}(X)$ and thus generates the truly exponential semigroup $\{e^{tA_\lambda}\}_{t\geq 0}$ with $\|e^{tA_\lambda}\|_{\text{op}} = \|e^{t(\lambda^2 R(\lambda,A)-\lambda I)}\|_{\text{op}} = \|e^{t\lambda^2 R(\lambda,A)} e^{-t\lambda I}\|_{\text{op}} \leq e^{-t\lambda} e^{t\lambda^2 \|R(\lambda,A)\|_{\text{op}}} = 1$.

Let $x \in D(A)$, $\lambda, \mu > 0$, and note that by the fundamental theorem of calculus:

$$\begin{aligned}
\|e^{tA_\lambda}x - e^{tA_\mu}x\| &= \left\| \int_0^1 \frac{d}{ds}\left[e^{tsA_\lambda} e^{t(1-s)A_\mu}\right] x ds \right\| \\
&= \left\| \int_0^1 \frac{d}{ds}\left[e^{tsA_\lambda + t(1-s)A_\mu}\right] x ds \right\| = \left\| \int_0^1 e^{tsA_\lambda} e^{t(1-s)A_\mu}(tA_\lambda - tA_\mu) x ds \right\| \\
&\leq \int_0^1 \|e^{tsA_\lambda} e^{t(1-s)A_\mu}(tA_\lambda - tA_\mu)x\| ds \leq t\|A_\lambda x - A_\mu x\| \\
&= t\,\|A\,(\lambda R(\lambda,A)x - \mu R(\mu,A)x)\| \quad (1.3.7)
\end{aligned}$$

for a fixed $t \geq 0$. It follows by 1. that the RHS of (1.3.7) converges to zero as $\lambda, \mu \to \infty$ so it is allowable to define:

$$\tilde{T}(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda} x$$

for each $x \in D(A)$. This implies that the family $\{\tilde{T}(t) : D(A) \to X \mid t \geq 0\} \subset \mathscr{L}(D(A), X)$ is an operator semigroup whose strong continuity is due to the fact that $s \mapsto \tilde{T}(s)x$ is continuous for $s \in [0,t]$ as the uniform limit of continuous functions on this interval. In turn, the same is true for the continuous extensions $T(t) \in \mathscr{L}(X)$ of each $\tilde{T}(t)$ so that the family $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup on X and clearly $\|T(t)\|_{\text{op}} \leq 1$. It remains to prove that this semigroup is generated by

A. Assume to that end that $\{T(t)\}_{t\geq 0}$ is generated by $G : D(G) \subset X \to X$ and $x \in D(A)$. Then,

$$\frac{T(h)x - x}{h} = \frac{1}{h}\lim_{\lambda\to\infty}(e^{tA_\lambda}x - x) = \frac{1}{h}\lim_{\lambda\to\infty}\int_0^h e^{tA_\lambda}A_\lambda x dt = \frac{1}{h}\int_0^h T(t)Axdt \tag{1.3.8}$$

because $e^{tA_\lambda}A_\lambda x \overset{\text{unif}}{\to} T(t)Ax$ on $[0,h]$. It follows that $Gx = Ax$ by taking $h \downarrow 0$ on both sides of (1.3.8) so $A \subset G$ as operators. Finally, $1 \in \rho(A)$ by assumption so that $(I - A) : D(A) \to X$ and $(I-G)|_{D(A)} : D(A) \to X$ are both invertible (with range X by Definition 1.3.2) so that:

$$(I-G)|_{D(A)}[D(A)] = (I-A)[D(A)] = X$$

and thus $D(A) = (I-G)|_{D(A)}^{-1}[X] = (I-G)^{-1}[X] = D(G)$ so $A = G$ as required. □

Theorem 1.3.3 is the Hille-Yosida Theorem for contraction semigroups. It is the workhorse of basic semigroup theory in the sense that C_0-semigroup generators can now be fully characterized by means of rescaling and renorming arguments alone. This is the content of the full Hille-Yosida Theorem, stated below.

Theorem 1.3.4. *The linear operator $A : D(A) \subset X \to X$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\|T(t)\|_{op} \leq M_\omega e^{\omega t}$ if and only if A is closed, densely-defined, and for all $\lambda \in \mathbb{C}$ that lie in the right half-plane $\Re(\lambda) > \omega$ the estimate $\|R(\lambda,A)\|_{op} \leq \frac{M_\omega}{(\Re(\lambda)-\omega)^n}$ holds for all $n \in \mathbb{N}$.*

The full Hille-Yosida Theorem is of theoretical significance but is impractical due the need to verify $\|R(\lambda,A)\|_{\text{op}} \leq \frac{M_\omega}{(\Re(\lambda)-\omega)^n}$ for all $n \in \mathbb{N}$. Indeed, infinitely-many conditions must be checked. This difficulty is, however, largely overcome in the context of PDE because numerous linearized operators have the desirable property of dissipativity. It is a straightforward consquence of the Hahn-Banach Theorm that the set

$$\Delta(x) = \{f \in X^* \mid \|f(x)\|_{\text{op}}^2 = \|x\|^2 = f(x)\}$$

is nonempty for all $x \in X$ (where X^* denotes the continuous dual space of X). A linear operator

$A : D(A) \subset X \to X$ is then dissipative if, for all $x \in D(A)$, $\Re[f(Ax)] \leq 0$ for some $f \in \Delta(x)$. More advanced functional analytic arguments show that this dissipativity is equivalent to the condition that $\|(\lambda I - A)x\| \geq \lambda \|x\|$ for all $\lambda > 0$ and for all $x \in D(A)$. The Lumer-Philips Theorem forges a connection between dissipative operators and contraction semigroup generators.

Theorem 1.3.5. *Let $A : D(A) \subset X \to X$ be densely-defined. If A is the generator of a contraction semigroup, then A is dissipative and $(\lambda I - A)$ is a linear surjection onto X for all $\lambda > 0$. Conversely, A is the generator of a contraction semigroup if it is dissipative and if there exists $\lambda_0 > 0$ such that $(\lambda_0 I - A)$ is a linear surjection onto X.*

The proof of Theorem 1.3.5 is omitted but can be found in [25] Once again, this theorem can be adapted for general (i.e. non-contraction) C_0-semgroups by rescaling/renorming arguments.

1.4 Asymptotic Theory of Semigroup Solutions

Recall that the notation M_ω from Theorem 1.3.1 is suggestive despite the fact that the constant $M_\omega \geq 1$ produced by this theorem does not actually depend on $\omega \in \mathbb{R}$. The reason is for this notation is that it is advantageous to discuss the set of all $\omega \in \mathbb{R}$ for which there exists some constant M_ω and to then associate these M_ω with their respective real numbers ω.

Definition 1.4.1. Let $\{T(t)\}_{t \geq 0}$ be a C_0-semigroup and let $A : D(A) \subset X \to X$ be its generator. The growth bound of this semigroup is the constant $\omega_A = \inf\{\omega \in \mathbb{R} \mid \exists M_\omega \geq 1 \text{ such that } \|T(t)\|_{\text{op}} \leq M_\omega e^{\omega t}\ \forall t \geq 0\}$.

It is worth noting that $\omega_A = -\infty$ is allowable, for example if $T(t) = 0$ for all $t \geq 0$. More importantly, the hope is to prove that $\omega_A < 0$ if (1.2.1) is well-posed in which case the semigroup solution $u(t) = T(t)u_0$ decays exponentially. The growth bound of a semigroup can be represented in several different ways as a consequence of Fekete's Lemma, stated below.

Theorem 1.4.2. *Let $\eta : (0,\infty) \to (0,\infty)$ be bounded on compact subsets and subadditive. Then, $\inf_{t>0} \frac{\eta(t)}{t} = \lim_{t\to\infty} \frac{\eta(t)}{t}$ where the existence of this limit is part of the claim.*

This lemma is well-known and a proof can be found, for instance, in [6] An immediate corollary is the following theorem that yields several different characterizations of ω_A in terms of the semigroup.

Theorem 1.4.3. *Let $\{T(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $A : D(A) \subset X \to X$. Then,*

$$\omega_A = \inf_{t>0} \frac{\ln(\|T(t)\|_{op})}{t} = \lim_{t\to\infty} \frac{\ln(\|T(t))\|_{op})}{t} = \frac{\ln(rad(T(t_0)))}{t_0}$$

for all $t_0 > 0$, where $rad(\cdot)$ denotes the so-called spectral radius of a bounded linear operator.

Proof. Define $\eta : (0,\infty) \to (0,\infty)$ by $\eta(t) = \ln(\|T(t)\|_{\text{op}})$ for $t > 0$. it follows that η is bounded on compact subsets of $(0,\infty)$ and is subadditive so by Theorem 1.4.2,

$$\inf_{t>0} \frac{\eta(t)}{t} = \inf_{t>0} \frac{\ln(\|T(t)\|_{\text{op}})}{t} = \lim_{t\to\infty} \frac{\ln(\|T(t))\|_{\text{op}})}{t} = \lim_{t\to\infty} \frac{\eta(t)}{t}.$$

Write $v = \inf_{t>0} \frac{\eta(t)}{t}$ and note that for all $\omega > \omega_A$ and for all $t_0 > 0$,

$$v \leq \frac{\eta(t_0)}{t_0} \leq \frac{\ln(M_\omega e^{\omega t_0})}{t_0} = \frac{\omega t_0 + \ln(M_\omega)}{t_0} = \omega + \frac{\ln(M_\omega)}{t_0}.$$

Taking $t_0 \to \infty$ implies $v \leq \omega$ for every $\omega > \omega_A$ and then taking $\omega \downarrow \omega_A$ proves $v \leq \omega_A$. For the opposite inequality, let $d > v$ be given and note that there is a $t_0 > 0$ so that $v \leq \frac{\eta(t)}{t} < d$ for all $t \geq t_0$ because $v = \inf_{t>0} \frac{\eta(t)}{t} = \lim_{t\to\infty} \frac{\eta(t)}{t}$. This implies that $\|T(t)\|_{\text{op}} \leq e^{dt}$ for all $t \geq t_0$ and because $t \mapsto \|T(t)\|_{\text{op}}$ is bounded on $[0,t_0]$, it follows that $\|T(t)\|_{\text{op}} \leq M_0 e^{dt}$ for some $M_0 \geq 1$. In all, this yields that $v \leq \omega_A \leq d$ and because $d > v$ is arbitrary, it follows that v is not strictly less than ω_A so $v = \omega_A$ as required. Finally,

$$\frac{\ln(\text{rad}(T(t_0)))}{t_0} = \frac{\ln(\lim_{n\to\infty} \|[T(t_0)]^n\|_{\text{op}}^{\frac{1}{n}})}{t_0} = \lim_{n\to\infty} \frac{\ln(\|T(nt_0)\|_{\text{op}})}{nt_0} = \omega_A$$

where $\text{rad}(L) = \lim_{n\to\infty} \|L^n\|_{\text{op}}^{\frac{1}{n}}$ is Gelfand's formula for the spectral radius of an $L \in \mathscr{L}(X)$. It is worth noting that this formula in fact holds for the members of any Banach algebra. $\square$

Recall that the hope is to show $\omega_A < 0$ whenever (1.2.1) is well-posed. The following theorem provides several useful characterizations of this quality.

Theorem 1.4.4. *Let $\{T(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $A : D(A) \subset X \to X$. Then, the following conditions are equivalent:*

1. $\omega_A < 0$
2. $\lim_{t\to\infty} \|T(t)\|_{op} = 0$
3. *There exists $t_0 > 0$ such that $\|T(t_0)\|_{op} < 1$*
4. *There exists $t_0 > 0$ such that $rad(T(t_0)) < 1$*

Proof. The implications 1. $\implies$ 2. $\implies$ 3. $\implies$ 4. $\implies$ 1. are all either immediate in and of themselves or are trivial consequences of Theorem 1.4.3. □

Theorem 1.4.4 is a non-exhaustive list of conditions that are equivalent to $\omega_A < 0$ (for example, there is also the well-known Datko-Pazy Theorem). It is, however, more than sufficient to help prove what is essentially the second workhorse of basic semigroup theory and the main result of Section 1.4: the Gearhart-Prüss Theorem. Some basic spectral theory is also required.

1.4.1 Basic Spectral Theory of Semigroup Generators

Recall that the linear function $A : D(A) \subset X \to X$ is assumed to be closed in Definition 1.3.2. The resulting fact that $R(\lambda, A) \in \mathscr{L}(X)$ for all $\lambda \in \rho(A)$ is essential for the following properties.

Theorem 1.4.5. *Let $A : D(A) \subset X \to X$ be a closed linear operator. Then,*

1. *The resolvent set $\rho(A) \subset \mathbb{C}$ is open and, for each fixed $\lambda_0 \in \rho(A)$, the resolvent operator at $\mu \in \left\{ \lambda \in \mathbb{C} \;\middle|\; |\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0,A)\|_{op}} \right\}$ is given by $R(\mu, A) = \sum_{j=0}^{\infty} (\lambda_0 - \mu)^j R(\lambda_0, A)^{j+1}$.*
2. *The function $\delta : \rho(A) \to \mathscr{L}(X)$ defined by $\delta(\lambda) = R(\lambda, A)$ is analytic in the sense that it is locally expressable by means of a power series and $\partial_\lambda^n[\delta(\lambda)] = (-1)^n n! [\delta(\lambda)]^{n+1}$.*
3. *A fixed $\lambda_0 \in \mathbb{C}$ is a member of $\mathbb{C} \setminus \rho(A)$ if and only if $\liminf_{k\to\infty} \|R(\lambda_k, A)\|_{op} = \infty$ for all sequences $(\lambda_k)_{k=1}^{\infty} \in [\rho(A)]^{\mathbb{N}}$ convergent to λ_0.*

Proof. Let $\lambda_0 \in \rho(A)$ be given and observe that:

$$\lambda I - A = (\lambda_0 I - A) + (\lambda - \lambda_0)I = [I - (\lambda_0 - \lambda)R(\lambda_0, A)](\lambda_0 I - A). \tag{1.4.1}$$

It follows that $I - (\lambda_0 - \lambda)R(\lambda_0, A)$ is invertible (by means of Von Neumann) if the operator norm of $(\lambda_0 - \lambda)R(\lambda_0, A)$ is strictly less than 1 and because

$$\|(\lambda_0 - \lambda)R(\lambda_0, A)\|_{\text{op}} \leq |\lambda_0 - \lambda| \cdot \|R(\lambda_0, A)\|_{\text{op}}$$

this is guaranteed to occur if $|\lambda_0 - \lambda| < \frac{1}{\|R(\lambda_0, A)\|_{\text{op}}}$. In particular, this proves that the open ball centered at λ_0, $\left\{\lambda \in \mathbb{C} \;\middle|\; |\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|_{\text{op}}}\right\}$, is a subset of $\rho(A)$ and, in turn, this means that $\rho(A)$ is an open subset of $\mathbb{C}$ because λ_0 is fixed arbitrarily. Note also that for any $\lambda \in \mathbb{C}$ that satisfies $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|_{\text{op}}}$, the identity

$$\begin{aligned} R(\lambda, A) &= R(\lambda_0, A)[I - (\lambda_0 - \lambda)R(\lambda_0, A)]^{-1} \\ &= R(\lambda_0, A) \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R(\lambda_0, A)^j = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R(\lambda_0, A)^{j+1} \end{aligned} \tag{1.4.2}$$

follows by taking the inverse on both sides of (1.4.1) and expressing the more complicated inverse as a Neumann series.

The second assertion follows immediately from (1.4.2) and differentiating n times with respect to λ. Now, fix $\lambda_0 \in \mathbb{C}$ and suppose that $\liminf_{k\to\infty} \|R(\lambda_k, A)\|_{\text{op}} = \infty$ for all sequences $(\lambda_k)_{k=1}^{\infty} \in [\rho(A)]^{\mathbb{N}}$ convergent to λ_0. If $\lambda_0 \in \rho(A)$, then the set $\{\lambda_k \mid k \in \mathbb{N}\}$ is a compact subset of $\rho(A)$ for all sequences $(\lambda_k)_{k=1}^{\infty} \in [\rho(A)]^{\mathbb{N}}$ convergent to λ_0. The analytic function $\delta : \rho(A) \to \mathscr{L}(X)$ defined by $\delta(\lambda) = R(\lambda, A)$ is therefore bounded on this set meaning $\liminf_{k\to\infty} \|R(\lambda_k, A)\|_{\text{op}} = \infty$ cannot be true. Conversely,

$$|\lambda_k - \lambda_0| \geq \frac{1}{\|R(\lambda_k, A)\|_{\text{op}}}$$

for all sequences $(\lambda_k)_{k=1}^{\infty} \in [\rho(A)]^{\mathbb{N}}$ convergent to λ_0 if $\lambda_0 \in \mathbb{C} \setminus \rho(A)$ (else λ_0 is a member of

$\rho(A)$). It follows that $\|R(\lambda_k,A)\|_{\text{op}} \geq \frac{1}{|\lambda_k-\lambda_0|}$ and taking the $\liminf_{k\to\infty}$ of both sides proves that $\liminf_{k\to\infty}\|R(\lambda_k,A)\|_{\text{op}} = \infty$ since $\lambda_k \to \lambda_0$. □

The set $\mathbb{C}\setminus\rho(A)$ is typically denoted by $\sigma(A)$ and is called spectrum of the closed operator $A : D(A) \subset X \to X$. The set $\sigma(A) \subset \mathbb{C}$ is always closed (because $\rho(A)$ is open) and, if $A \in \mathscr{L}(X)$, bounded (a less trivial consequence of Liouville's Theorem) and, as the set of $\lambda \in \mathbb{C}$ where $\lambda I - A$ fails to be invertible, it loosely represents the eigenvalues of A. This set can be decomposed by looking at exactly how $\lambda I - A$ fails to be bijective.

Definition 1.4.6. Let $A : D(A) \subset X \to X$ be a closed linear operator. Then, the point spectrum of A is the set

$$P\sigma(A) = \{\lambda \in \sigma(A) \mid \lambda I - A \text{ is not } 1-1\}$$

consisting of true eigenvalues in the sense that $\ker(\lambda I - A) \neq \emptyset$. The approximate point spectrum of A is the set:

$$AP\sigma(A) = \{\lambda \in \sigma(A) \mid \lambda I - A \text{ is not } 1-1 \text{ or the range of } \lambda I - A \text{ is not closed in } X\}.$$

It is clear that $P\sigma(A) \subset AP\sigma(A)$ and, in addition, the members of $AP\sigma(A)$ are approximate in the sense that for each fixed $\lambda \in AP\sigma(A)$, there exists a sequence of vectors $(x_n)_{n=1}^{\infty} \in [D(A)]^{\mathbb{N}}$ such that $0 = \limsup_{n\to\infty}\|(\lambda I - A)x_n\|$. The residual spectrum of A is the set:

$$R\sigma(A) = \{\lambda \in \sigma(A) \mid \text{the range of } \lambda I - A \text{ is not dense in } X\}$$

and it is evident that $\sigma(A) = AP\sigma(A) \cup R\sigma(A)$, though this union need not be disjoint.

Returning to the asymptotic theory of C_0-semigroup generators, it is clear from Theorem 1.3.4 that $s_A = \sup\{\Re(\lambda) \mid \lambda \in \sigma(A)\} \leq \omega_A$. There is then hope to ascertain the decay of the semigroup solution by investigating the spectrum of the generator, particularly if the opposite inequality $s(A) \geq \omega_A$ holds. Whether or not $s_A = \omega_A$ holds can be investigated through so-called spectral mapping theorems. For instance, it can be shown that if $\{T(t)\}_{t\geq 0}$ is the C_0-semigroup generated by $A : D(A) \subset X \to X$, then

$$e^{t\sigma(A)} = \{e^{t\lambda} \in \mathbb{C} \mid \lambda \in \sigma(A)\} \subset \sigma(T(t))$$

and moreover, that $e^{tP\sigma(A)} = P\sigma(T(t)) \setminus \{0\}$ and $e^{tR\sigma(A)} = R\sigma(T(t)) \setminus \{0\}$ for all $t \geq 0$.

1.4.2 The Gearhart-Prüss Theorem

It turns out that there exist necessary and sufficient conditions to guarantee $\omega(A) < 0$ based on the decay of $\|R(\lambda,A)\|_{\text{op}}$ for $\lambda \in \mathbb{C}_+ = \{\lambda \mid \Re(\lambda) > 0\}$ if X is a Hilbert space. This is what might be considered the cornerstone of the asymptotic theory of strongly continuous semigroups. Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ for this and the next subsection.

Theorem 1.4.7. *Let $\{T(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $A : D(A) \subset X \to X$. It follows that $\omega_A < 0$ if and only if $\mathbb{C}_+ \subset \rho(A)$ and $M = \sup_{\lambda \in \mathbb{C}_+} \|R(\lambda,A)\|_{op} < \infty$.*

Proof. Suppose first that $\omega_A < 0$ and fix $\omega \in (\omega_A, 0)$. It follows by the definition of ω_A that there exists a constant $M_\omega \geq 1$ such that $\|T(t)\|_{\text{op}} \leq M_\omega e^{\omega t}$ and applying Theorem 1.3.4, this implies that $\mathbb{C}_+ \subset \{\lambda \mid \Re(\lambda) > \omega\} \subset \rho(A)$ and that

$$\|R(\lambda,A)\|_{\text{op}} \leq \frac{M_\omega}{(\Re(\lambda) - \omega)^n}$$

for all $n \in \mathbb{N}$ and for all λ in the half-plane $\Re(\lambda) > \omega$. Taking $n = 1$ yields that

$$\|R(\lambda,A)\|_{\text{op}} \leq \frac{M_\omega}{(\Re(\lambda) - \omega)} \leq \frac{M_\omega}{-\omega}$$

for all $\lambda \in \mathbb{C}_+$ and therefore that $\sup_{\lambda \in \mathbb{C}_+} \|R(\lambda,A)\|_{\text{op}} \leq \frac{M_\omega}{-\omega} < \infty$.

Suppose conversely that $\mathbb{C}_+ \subset \rho(A)$ and $M = \sup_{\lambda \in \mathbb{C}_+} \|R(\lambda,A)\|_{\text{op}} < \infty$. These hypotheses in concert with Theorem 1.4.5 *3* imply that $i\mathbb{R} = \{\lambda \mid \Re(\lambda) = 0\} \subset \rho(A)$ so it follows that, in fact,

$i\mathbb{R} \cup \mathbb{C}_+ \subset \rho(A)$ and the uniform bound on the operator norm of the resolvent at $\lambda \in \mathbb{C}_+$ extends to $\lambda \in i\mathbb{R}$ by continuity. Now, define $\omega = |\omega_A| + 1$ and note that the shifted semigroup:

$$T_{-\omega}(t) = e^{-\omega t} T(t) \qquad t \geq 0$$

is strongly continuous, is generated by $A - \omega I$, and has a negative growth bound. The key to this proof is then to represent the resolvent operator of A along a vertical line in $\mathbb{C}_+$ in terms of the resolvent operator of $A - \omega I$. Namely,

$$R(\omega + is, A)x = \int_0^\infty e^{-(\omega+is)t} T(t)x dt = \int_0^\infty e^{-ist} T_{-\omega}(t)x dt = R(is, A - \omega I)x$$

for all $s \in \mathbb{R}$ and for all $x \in X$. This is to say (setting $T_{-\omega}(t) = 0$ for all $t < 0$) that $R(\omega + is, A)x$ is the Fourier Transform of $T_{-\omega}(t)x \in L^2(\mathbb{R}, X)$. Plancharel's Theorem then obtains:

$$\int_{-\infty}^\infty \|R(\omega + is, A)x\|^2 ds = 2\pi \int_{-\infty}^\infty \|T_{-\omega}(t)x\|^2 dt \leq L\|x\|^2$$

where the constant $L \geq 0$ is derived by means of the negative growth bound on the shifted semigroup. The resolvent identity shows that $R(is, A) = [I - \omega R(is, A)]R(\omega + is, A)$ so that

$$\int_{-\infty}^\infty \|R(is, A)x\|^2 ds \leq (1 + \omega M)^2 \int_{-\infty}^\infty \|R(\omega + is, A)x\|^2 ds \leq (1 + \omega M)^2 L\|x\|^2$$

and an identical estimate follows for the adjoint semigroup $\{[T(t)]^*\}_{t\geq 0}$ on X that is generated by A^*. Applying the semigroup inversion formula established by [6] in the case $j = 2$ now obtains:

$$\begin{aligned} |\langle tT(t)x, y\rangle| &= \left| \left\langle \frac{1}{2\pi i} \lim_{N\to\infty} \int_{-N}^N e^{(\omega+is)t} R(\omega + is, A)^2 x ds, y \right\rangle \right| \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \left| \int_N^N e^{(\omega+is)t} \langle R(\omega + is, A)^2 x, y\rangle ds \right| \end{aligned} \tag{1.4.3}$$

for all $x \in D(A^2)$, for all $t > 0$, and for all $y \in X$, where viewing the inner product against a fixed

$y \in X$ as a bounded linear functional on X allows for the limit to be pulled out of the inner product and for the inner product to be pushed under the integral. Cauchy's Formula now asserts that

$$(1.4.3) \leq \frac{1}{2\pi} \limsup_{N\to\infty} \left| \int_{-N}^{N} e^{ist} \langle R(is,A)^2 x, y \rangle ds \right| + \frac{1}{2\pi} \limsup_{N\to\infty} \left| \int_{0}^{\omega} e^{r+iN} \langle R(r+in,A)^2 x, y \rangle dr \right| + \frac{1}{2\pi} \limsup_{N\to\infty} \left| \int_{0}^{\omega} e^{r-iN} \langle R(r-in,A)^2 x, y \rangle dr \right| \quad (1.4.4)$$

and noting that $\|R(\lambda,A)x\| = \frac{1}{|\lambda|}\|\lambda R(\lambda,A)x\| = \frac{1}{|\lambda|}\|R(\lambda,A)Ax + x\| \leq \frac{M\|Ax\|+\|x\|}{|\lambda|}$ for all $x \in D(A^2)$ and for all nonzero $\lambda \in i\mathbb{R} \cup \mathbb{C}_+$,

$$(1.4.4) \leq \frac{1}{2\pi} \limsup_{N\to\infty} \left| \int_{-N}^{N} e^{ist} \langle R(is,A)^2 x, y \rangle ds \right| + \limsup_{N\to\infty} \frac{M\omega \|y\| e^{t\omega}(M\|Ax\| + \|x\|)}{N\pi}$$
$$= \frac{1}{2\pi} \limsup_{N\to\infty} \int_{-N}^{N} |\langle R(is,A)x, R(-is,A^*)y \rangle| ds \leq \| \|R(is,A)x\| \cdot \|R(-is,A^*)y\| \|_{L^1_s(\mathbb{R})} \quad (1.4.5)$$

because $]R(is,A)]^* = R(-is,A^*)$. Finally, an application of Hölder's Inequality shows that

$$(1.4.5) \leq \| \|R(is,A)x\| \|_{L^2_s(\mathbb{R})} \| \|R(-is,A^*)y\| \|_{L^2_s(\mathbb{R})} \leq \frac{1}{2\pi} \sqrt{(1+\omega M)L\|x\|^2} \sqrt{(1+\omega M)L\|y\|^2} \quad (1.4.6)$$

and because $D(A^2)$ is dense in X, it follows from (1.4.6) that

$$\|tT(t)\|_{\text{op}} = \sup_{\substack{x,y \in D(A^2) \\ \|x\|=\|y\|=1}} |\langle tT(t)x, y \rangle| \leq \frac{(1+\omega M)L}{2\pi}$$

which implies that $\|T(t)\|_{\text{op}} \leq \frac{(1+\omega M)L}{2\pi t} \to 0$ as $t \to \infty$ so $\omega_A < 0$ by Theorem 1.4.4. □

The difficulty with using Theorem 1.4.7 directly in applications is that $i\mathbb{R} \subset \rho(A)$ as a byproduct of the hypotheses. Many interesting PDEs are not "spectrally stable" in this sense so there is the need for a finer tool that distinguishes between semigroups that decay exponentially and those that merely satisfy a uniform bound of the form $\sup_{t>0} \|T(t)\|_{\text{op}} < \infty$ and may have some purely imaginary spectrum. The result that accomplishes this task is the Gomilko Lemma.

1.4.3 The Gomilko Lemma

The proof of the Gomilko Lemma follows closely that of Theorem 1.4.7.

Theorem 1.4.8. *Let $\{T(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $A : D(A) \subset X \to X$. It follows that $\|T(t)\|_{op} \leq M < \infty$ for all $t \geq 0$ if and only if $\mathbb{C}_+ \subset \rho(A)$ and the estimate*

$$\sup_{\delta>0} \delta \int_{-\infty}^{\infty} \left(\|R(\delta+is,A)x\|^2 + \|R(\delta+is,A^*)x\|^2\right) ds < C\|x\|^2$$

is valid for all $x \in X$.

Proof. Suppose first that $\|T(t)\|_{\text{op}} \leq M < \infty$ for all $t \geq 0$. Notice that $M = M_0 e^{0t}$ so $\omega_A \leq 0$. This implies as in the proof of Theorem 1.4.7 that $\mathbb{C}_+ \subset \rho(A)$ and as before, write

$$R(\omega+is)x = \int_0^{\infty} e^{-(\omega+is)t} T(t)x dt = \int_0^{\infty} e^{-ist} T_{-\omega}(t)x dt = R(is, A-\omega I)x$$

where $T_{-\omega}(t) = e^{-\omega t} T(t)$ for each $t \geq 0$, and where $\omega > 0$ and $s \in \mathbb{R}$ are arbitrary. Plancharel's Theorem then obtains

$$\int_{-\infty}^{\infty} \|R(\omega+is,A)x\|^2 ds = 2\pi \int_{-\infty}^{\infty} \|T_{-\omega}(t)x\|^2 dt \leq 2\pi \int_0^{\infty} \frac{M^2\|x\|^2}{e^{\omega t}} dt = \frac{2\pi M^2\|x\|^2}{\omega}$$

for all $x \in X$ where $T_{-\omega}(t) = 0$ for all $t < 0$ as in Theorem 1.4.7. An identical estimate holds for the 2-norm of the resolvent of the adjoint A^* so that

$$\omega \int_{-\infty}^{\infty} \left(\|R(\omega+is,A)x\|^2 + \|R(\omega+is,A^*)x\|^2\right) ds \leq \omega \left(\frac{4\pi M^2\|x\|^2}{\omega}\right) = C\|x\|^2 \qquad (1.4.7)$$

where $C = 4\pi M^2 \geq 0$. Taking the supremum over all $\omega > 0$ of (1.4.7) then produces the desired estimate.

Suppose conversely that $\mathbb{C}_+ \subset \rho(A)$ and that the estimate

$$\sup_{\delta>0} \delta \int_{-\infty}^{\infty} \left(\|R(\delta+is,A)x\|^2 + \|R(\delta+is,A^*)x\|^2\right) ds < C\|x\|^2$$

is valid for all $x \in X$. Consider, for a fixed $x \in X$, the function $f_x : \mathbb{C}_+ \to X$ defined by $f_x(\lambda) = R(\lambda,A)x$. Complex differentiating yields

$$\frac{d}{d\lambda}[f_x(\lambda)] = -R(\lambda,A)^2x = \frac{d}{d\lambda}\int_0^{\infty} e^{-t\lambda}T(t)xdt = -\int_0^{\infty} e^{-t\lambda}tT(t)xdt = -\mathbf{L}(tT(t)x)$$

where $\mathbf{L}$ denotes the Laplace Transform, by applying Theorem 1.4.5 *2* and moving the derivative under the integral. In particular, $\mathbf{L}$ can be inverted at the function $tT(t)x$ by the formula:

$$tT(t)x = \mathbf{L}^{-1}[R(\lambda,A)^2x] = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\alpha+is)t}R(\alpha+is,A)^2xds$$

where $\alpha = \Re(\lambda)$. It follows that for all $t > 0$ and for all $y \in X$,

$$\begin{aligned} |\langle T(t)x,y\rangle| &= \left|\left\langle \frac{1}{2\pi it}\int_{-\infty}^{\infty} e^{(\alpha+is)t}R(\alpha+is,A)^2xds, y\right\rangle\right| \\ &= \frac{1}{2\pi t}\lim_{N\to\infty}\left|\int_{-N}^{N} e^{(\alpha+is)t}\langle R(\alpha+is,A)^2x,y\rangle ds\right| \quad (1.4.8) \end{aligned}$$

by the same logic as in the proof of Theorem 1.4.7. Then,

$$\begin{aligned} (1.4.8) &\leq \frac{e^{t\alpha}}{2\pi t}\limsup_{N\to\infty}\int_{-N}^{N} |\langle R(\alpha+is,A)^2x,y\rangle| ds \\ &= \frac{e^{t\alpha}}{2\pi t}\limsup_{N\to\infty}\int_{-N}^{N} |\langle R(\alpha+is,A)x, R(\alpha-is,A^*)y\rangle| ds \\ &\leq \frac{e^{t\alpha}}{4\pi t}\limsup_{N\to\infty}\int_{-N}^{N} \left(\|R(\alpha+is,A)x\|^2 + \|R(\alpha-is,A^*)y\|^2\right) ds \quad (1.4.9) \end{aligned}$$

where the Cauchy-Schwarz inequality is used, followed by Young's inequality ($2ab \leq a^2+b^2$ for $a,b \geq 0$). Now, because $\|R(\alpha-is,A^*)y\| = \|R(\alpha+is,A)y\|$ and because (1.4.9) is valid for all

$\alpha = \Re(\lambda)$ with $\lambda \in \mathbb{C}_+$, take $y = x$ and $\alpha = \frac{1}{t}$ to obtain

$$(1.4.9) \leq \frac{\alpha}{4\pi} \limsup_{N\to\infty} \int_{-N}^{N} \left(\|R(\alpha+is,A)x\|^2 + \|R(\alpha+is,A)x\|^2\right) ds$$
$$\leq \frac{1}{4\pi} \sup_{\alpha>0} \int_{-\infty}^{\infty} \left(\|R(\alpha+is,A)x\|^2 + \|R(\alpha+is,A)x\|^2\right) ds \quad (1.4.10)$$

which finally implies that

$$|\langle T(t)x,x\rangle| \leq \frac{C\|x\|^2}{4\pi} = \tilde{C}\|x\|^2$$

for all $x \in X$ so that $\|T(t)\|_{\text{op}} \leq \tilde{C}$. This estimate is valid for all fixed $t > 0$ so the semigroup is uniformly bounded as required. □

1.5 Closing Remarks

The projects outlined in the next two chapters detail attempts to derive uniform bounds on the L^2 norms of the semigroup solutions to certain linearized PDEs by means of Theorem 1.4.8. Recall that Theorem 1.3.5 is the tool most frequently used in practice to ascertain whether or not a closed linear operator generates a C_0-semigroup. In particular, if X is a Hilbert space and if $x \in X$, then $f_x = \langle \cdot, x\rangle \in \Delta(x) \subset X^*$ and the densely-defined linear operator $A : D(A) \subset X \to X$ is dissipative provided that

$$0 \geq \Re[f_x(Ax)] = \langle Ax, x\rangle \quad (1.5.1)$$

for all $x \in D(A)$. It follows from (1.5.1) and density arguments that $0 \geq \langle A^*x, x\rangle$ for all $x \in D(A^*)$ so that $A^* : D(A^*) \subset X \to X$ is dissipative as well and this is sufficient by [25, Corollary 4.4, Ch. 2] to conclude that A generates a contraction semigroup by Theorem 1.3.5. An immediate corollary is that if $B : D(B) \subset X \to X$ satisfies the estimate

$$\omega\|x\|^2 \geq \Re[f_x(Bx)] = \langle Bx, x\rangle$$

for all $x \in D(B)$ (dense in X), then $0 \geq \langle Bx, x\rangle - \omega\|x\|^2 = \langle Bx, x\rangle - \omega\langle x, x\rangle = \langle (B-\omega I)x, x\rangle$ so it follows by the above logic that $(B-\omega I) : D(B) \subset X \to X$ generates a contraction semigroup. Then, [25, Theorem 1.1, Ch. 3] (the bounded perturbation theorem) implies that $B : D(B) \subset X \to X$ generates a C_0-semigroup with growth bound $\omega_B \leq \omega$.

Chapter 1 of this thesis offers only the briefest of introductions to the basic theory of operator semigroups. There exist a number of important classes of semigroups (e.g. analytic, differentiable, compact, etc..) that are not discussed here because they are not necessarily relevant to the material of Chapters 2 and 3. Additional information regarding these semigroups and semigroup/operator theory in general can be found in [6] and additional information regarding underlying functional analytic concepts can be found in [4].

Chapter 2

L^2 Operator Norm Estimates for Semigroup Solutions to Certain Linearized NLS and KdV Equations

Abstract

This chapter shows that the spectrum away from $i\mathbb{R}$ for certain nonlinear Schrödinger (NLS) and Korteweg-de Vries (KdV) linearizations is necessarily contained in a strip. If, moreover, the spectrum is assumed to be purely imaginary, then it is possible to derive optimal L^2 operator norm bounds on the semigroups generated by these linearizations. The latter result is achieved by using the Gomilko Lemma and splitting the integral into “high" and “low" energy pieces, each of which admits (by different methods) a suitable upper bound.

2.1 Introduction

There is some hope to use Theorem 1.4.8 if the generator $A : D(A) \subset X \to X$ of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ can be decomposed as the composition of linear operators whose adjoints have desirable properties. In particular, the methods of this (and the next) chapter are meant to address PDEs whose linearizations about known standing wave solutions (i.e. solutions of the form $\Phi(x,t) = e^{i\omega t}\phi(x)$ where ϕ is real-valued) or travelling wave solutions can be expressed as

$$\partial_t[v(x,t)] = \mathscr{J}\mathscr{L}v(x,t)$$

where $\mathscr{J}$ is anti-self-adjoint (i.e. $\mathscr{J}^* = -\mathscr{J}$) and $\mathscr{L}$ is self-adjoint (i.e. $\mathscr{L}^* = \mathscr{L}$), and where (in the standing wave case) $u(x,t) = e^{i\omega t}(\phi(x) + v(x,t))$ is a perturbation of the ansatz wave. Problems of this sort (particularly with Hamiltonian generators) have been well-studied, for example, by Grillakis, Shatah, and Strauss in [15]. Of course, GSS methods cover far more than the $\mathscr{J}\mathscr{L}$ operators covered here and in our paper [27], though the spectral symmetry in both cases is important. Later, in [18] and its addendum, Kapitula, Kevrekidis, and Sandstede introduced eigenvalue counting methods to study the generalized eigenvalue problem and therefore spectral stability for these sorts of nonlinear waves. There were also concurrent studies of the orbital stability of nonlinear waves for Hamiltonian systems, such as [11, 12] for NLS. However, standard energy methods fail in this case for subharmonic perturbations, so additional techniques including inverse scattering methods and making use of other conserved quantities is needed. The goal of [27] was not to repeat the techniques of these various studies but, under the assumption of spectral stability, determine optimal bounds for the semigroups generated by certain $\mathscr{J}\mathscr{L}$ operators. The most straightforward situation in which this is possible is the NLS equation in the form

$$iu_t(x,t) + \Delta u(x,t) + f(|u(x,t)|^2)u(x,t) = 0 \tag{2.1.1}$$

where $t > 0$, $x \in [-\pi, \pi]$, and $u \in H^2_{\text{per}}[-\pi, \pi] \subset L_2[-\pi, \pi]$. A brief integration-by-parts shows that Δ is in this situation self-adjoint (with respect to the L^2 inner product), and linearization of (2.1.1) about the standing wave $e^{i\omega t}\phi(x)$ obtains a system of the form

$$\partial_t \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathscr{L}_1 & 0 \\ 0 & \mathscr{L}_2 \end{pmatrix} \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} := \mathscr{J}\mathscr{L} \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} \tag{2.1.2}$$

where $v = v_1 + iv_2 \in \mathbb{C}$ and where $\mathscr{L}_{1,2} = -\Delta - V_{1,2}$ for some potentials $V_{1,2}$. The next subsection is devoted to providing an a priori bound on the location of $\sigma(\mathscr{J}\mathscr{L})$ for $\mathscr{J}\mathscr{L}$ as in (2.1.2) and, if $\sigma(\mathscr{J}\mathscr{L}) \subset i\mathbb{R}$ is assumed, an optimal upper bound for the L^2 operator norm of the semigroup $e^{t(\mathscr{J}\mathscr{L})}$ as well. It should be noted that these methods still work if the semigroup acts on the more general domain $H^2_{\text{per}}[-M,M]^d$ for any $M>0$ or on the unbounded domain $H^2(\mathbb{R}^d)$. Finally, linearization of the KdV equation

$$u_t(x,t)+u_{xxx}(x,t)+\partial_x[f(u^2)u]=0 \tag{2.1.3}$$

where $t>0$, $x\in[-\pi,\pi]$, and $u\in H^3_{\text{per}}[-\pi,\pi]\subset L^2[-\pi,\pi]$ about a known travelling wave solution $\phi(x-\omega t)$ produces a system of the form $v_t=\mathscr{J}\mathscr{L}v$ where $\mathscr{J}=\partial_x$ and where $\mathscr{L}=-\Delta-V$ for some potential V. After dealing with NLS, we will show that there are analogous results for this linearized PDE and, as in the NLS setup, these results can be extended to slightly more general domains, but we consider the basic cases only in this work.

Finally, we hope to be able to apply similar methods to the Dirac equation in the form:

$$i\partial_t u(x,t)=\begin{pmatrix}1 & \partial_x\\ -\partial_x & -1\end{pmatrix}u(x,t)-f\left(\bar{u}(x,t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}u(x,t)\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}u(x,t) \tag{2.1.4}$$

where $x\in[-\pi,\pi]$, $t\geq 0$, $u(x,t)=(u_1,u_2)\in\mathbb{C}^2$, and $f\in C^\infty(\mathbb{R})$. This is a simple (low-dimensional) version of the Dirac $\mathscr{J}\mathscr{L}$ system found in [5]. We proceeded in a manner similar to (2.1.1) where we re-wrote (2.1.4) in an equivalent and more manageable form, at which point we were able to derive suitable Gomilko bounds for the constant coefficient case. We believe that it is possible to determine similar bounds for not only small, but actually bounded potentials, as is the situation for NLS and KdV in our first paper [27]. This has proved more difficult than anticipated and is an ongoing project. In particular, the behavior at the Fourier mode level of the “bad" part of the linearized operator is not as straightforward as in the previous two situations.

2.2 NLS

Consider the closed linear operator $\mathscr{J}\mathscr{L} : H^2_{\text{per}}[-\pi,\pi]^2 \to L^2[-\pi,\pi]^2$ defined by:

$$\mathscr{J}\mathscr{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathscr{L}_1 & 0 \\ 0 & \mathscr{L}_2 \end{pmatrix}$$

where $\mathscr{L}_{1,2} = -\Delta - V_{1,2}$ for some bounded potentials $V_{1.2}$. First, it is not difficult to show that the operator $\mathscr{J}\begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}$ generates a C_0-semigroup on $L^2[-\pi,\pi]^2$ with dense domain $H^2_{\text{per}}[-\pi,\pi]^2$ and consequently, $\mathscr{J}\mathscr{L}$ itself (as a bounded perturbation) generates a C_0-semigroup on the same ambient space with the same dense domain. Now, the application of Gomilko's Lemma requires suitable bounds on the resolvents of $\mathscr{J}\mathscr{L}$ and $(\mathscr{J}\mathscr{L})^*$ and, because

$$\begin{aligned} ((\delta+i\mu)-\mathscr{J}\mathscr{L})^{-1} &= (\mathscr{J}(-(\delta+i\mu)\mathscr{J}-\mathscr{L}))^{-1} = (\mathscr{L}+\mathscr{J}(\delta+i\mu))^{-1}\mathscr{J} \\ ((\delta+i\mu)-(\mathscr{J}\mathscr{L})^*)^{-1} &= ((\delta+i\mu)+\mathscr{L}\mathscr{J})^{-1} \\ &= ((-(\delta+i\mu)\mathscr{J}+\mathscr{L})\mathscr{J})^{-1} = -\mathscr{J}(\mathscr{L}-\mathscr{J}(\delta+i\mu))^{-1} \end{aligned}$$

where $\mathscr{J}^* = \mathscr{J}^{-1} = -\mathscr{J}$ and $\mathscr{L}^* = \mathscr{L}$, it suffices to ascertain the invertibility of $\mathscr{L} \pm \mathscr{J}(\delta + i\mu)$, along with a suitable bound.

2.2.1 Construction of NLS Resolvent

The goal of this subsection is, for $\delta > 0$ and $\mu \in \mathbb{R}$, to invert the operator $\mathscr{L} \pm \mathscr{J}(\delta+i\mu)$. Write formally that $z = (\mathscr{L} \pm \mathscr{J}(\delta+i\mu))^{-1} f$ where $f \in L^2[-\pi,\pi]^2$ and note then that

$$\begin{aligned} f &= (\mathscr{L} \pm \mathscr{J}(\delta+i\mu))z \\ &= \left[\begin{pmatrix} -\Delta - V_1 & 0 \\ 0 & -\Delta - V_2 \end{pmatrix} \pm S \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} S^{-1}(\delta+i\mu) \right] z \end{aligned}$$

where $S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ diagonalizes $\mathscr{J}$. This is equivalent to the system

$$f = \begin{pmatrix} -\Delta - V \pm (\mu - \delta i) & -V \\ V & -\Delta - V \mp (\mu - \delta i) \end{pmatrix} z$$

after taking $V = \frac{1}{2}(V_1 + V_2)$. It is enough (by symmetry) to consider now only the case:

$$f = \begin{pmatrix} -\Delta - V - \mu + i\delta & -V \\ V & -\Delta - V + \mu - i\delta \end{pmatrix} z \tag{2.2.1}$$

where $\mu > 0$. This is equivalent to the system

$$\begin{cases} f_1 = (-\Delta - V - \mu + i\delta) z_1 - V z_2 \\ f_2 = (-\Delta - V + \mu - i\delta) z_2 + V z_1 \end{cases} \tag{2.2.2}$$

and let us now write $H_0 = -\Delta - V$ where, being self-adjoint, $\sigma(H_0) \subset \mathbb{R}$. It follows that

$$z_2 = (H_0 + \mu - i\delta)^{-1}(f_2 - V z_1)$$

and plugging this into the first equation, there is

$$f_1 = (H_0 - \mu + i\delta) z_1 - V(H_0 + \mu - i\delta)^{-1}(f_2 - V z_1)$$

or, equivalently,

$$[(H_0 - \mu + i\delta) + V(H_0 + \mu - i\delta)^{-1} V] z_1 = f_1 + V(H_0 + \mu - i\delta)^{-1} f_2 \tag{2.2.3}$$

Now, $(H_0+\mu)^{-1} = ((-\Delta - V) - (-\mu))^{-1}$ exists for $\mu > \|V\|_\infty$ because $\sigma(H_0)$ is not only real but is in fact a subset of $[-\|V\|_\infty, \infty)$, so from the resolvent identity, there is

$$R(H_0, i\delta - \mu) - R(H_0, -\mu) = (i\delta - \mu - (-\mu))R(H_0, i\delta - \mu)R(H_0, -\mu)$$

or, equivalently,

$$(H_0 + \mu - i\delta)^{-1} = (H_0 + \mu)^{-1} + i\delta(H_0 + \mu - i\delta)^{-1}(H_0 + \mu)^{-1}$$

so that (2.2.3) reads

$$(H_0 + V((H_0 + \mu)^{-1} + i\delta(H_0 + \mu - i\delta)^{-1}(H_0 + \mu)^{-1})V - \mu + i\delta)z_1 = f_1 + V(H_0 + \mu - i\delta)^{-1} f_2$$

or, equivalently,

$$(H_0 + V(H_0 + \mu)^{-1}V + \mu - i\delta)z_1 = f_1 - i\delta V(H_0 + \mu - i\delta)^{-1}(H_0 + \mu)^{-1}V z_1 + V(H_0 + \mu - i\delta)^{-1} f_2.$$

Finally, taking $H = H_0 + V(H_0 + \mu)^{-1}V$ (so that H is also self-adjoint), there is the following system of equations for z_1 and z_2:

$$\begin{cases} (H + \mu - i\delta)z_1 = f_1 - i\delta V(H_0 + \mu - i\delta)^{-1}(H_0 + \mu)^{-1}V z_1 + V(H_0 + \mu - i\delta)^{-1} f_2 \\ z_2 = (H_0 + \mu - i\delta)^{-1}(f_2 - V z_1) \end{cases} \tag{2.2.4}$$

where H exists for $\mu > \|V\|_\infty$ and also $(H + \mu - i\delta)^{-1}$ exists because $\sigma(H) \subset \mathbb{R}$. Then, (2.2.4) is equivalent to the system

$$\begin{cases} (I + i\delta(H + \mu - i\delta)^{-1}V(H_0 + \mu - i\delta)^{-1}(H_0 + \mu)^{-1}V)z_1 = (H + \mu - i\delta)^{-1}(f_1 + V(H_0 + \mu - i\delta)^{-1} f_2) \\ z_2 = (H_0 + \mu - i\delta)^{-1}(f_2 - V z_1) \end{cases}$$

and, in particular, note that because $\|R(\lambda,A)\|_{\text{op}} \leq \frac{1}{\text{dist}(\lambda,\sigma(A))}$ if A is a bounded and self-adjoint operator on a Hilbert space, it follows that

$$\|(H+\mu-i\delta)^{-1}\|_{\text{op}} \leq \frac{1}{\text{dist}(i\delta-\mu,\sigma(H))} \leq \frac{1}{\delta}$$
$$\|(H_0+\mu-i\delta)^{-1}\|_{\text{op}} \leq \frac{1}{\text{dist}(i\delta-\mu,\sigma(H_0))} < \frac{1}{\mu-\|V\|_\infty} \leq \frac{1}{\|V\|_\infty}$$
$$\|(H_0+\mu)^{-1}\|_{\text{op}} \leq \frac{1}{\text{dist}(-\mu,\sigma(H_0))} < \frac{1}{\mu-\|V\|_\infty} \leq \frac{1}{\|V\|_\infty}$$

for $\mu > 2\|V\|_\infty$. Then, the operator applied to z_1 in the above system is invertible (by Von Neumann) and it follows that

$$\|z_1\|_2 \leq \frac{1}{1-r}\|(H+\mu-i\delta)^{-1}(f_1+V(H_0+\mu-i\delta)^{-1}f_2)\|_2 \leq \frac{C_1(\|f_1\|_2+\|f_2\|_2)}{\delta}$$

where $0 \leq \|i\delta(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}(H_0+\mu)^{-1}V\|_2 < r < 1$ for $\mu > 2\|V\|_\infty$. In particular, it is possible to solve for z_1 and for z_2 (with $\|z_2\|_2 \leq \frac{C_2(\|f_1\|_2+\|f_2\|_2)}{\delta}$) if $\mu > 2\|V\|_\infty$ and this leads to the following observation.

Theorem 2.2.1. *Let $\mathscr{J}\mathscr{L}$ be as in Section 2.2. Then, either $\Re(\lambda)=0$ or $\Re(\lambda)\neq 0$ and $|\Im(\lambda)| \leq 2\|V\|_\infty$ if $\lambda \in \sigma(\mathscr{J}\mathscr{L})$.*

Proof. The operator $\mathscr{J}\mathscr{L}-(\delta+i\mu)$ is invertible if $\mathscr{L}\pm\mathscr{J}(\delta+i\mu)$ is invertible and this is the case (by the above argument) if $\mu > 2\|V\|_\infty$ and $\delta > 0$. □

It remains to control the Gomilko quantities (under the assumption that $\sigma(\mathscr{J}\mathscr{L}) \subset i\mathbb{R}$) in order to ascertain an optimal L^2 bound on the semigroup generated by $\mathscr{J}\mathscr{L}$. The next section begins by showing that it is enough to consider these bounds for small δ.

2.2.2 Optimal L^2 Bounds for the NLS Semigroup

Assume that $\mathscr{J}\mathscr{L}$ is spectrally stable in the sense that $\sigma(\mathscr{J}\mathscr{L}) \subset i\mathbb{R}$ and consider the shifted generator $\mathscr{J}\mathscr{L} - \delta_0$ for some $\delta_0 > 0$. Clearly, it follows that $\mathbb{C}_+ \subset \rho(\mathscr{J}\mathscr{L} - \delta_0)$ and, moreover,

$$\begin{aligned}\sup_{\mu\in\mathbb{R}} \|R(i\mu, \mathscr{J}\mathscr{L} - \delta_0)\| &\leq \sup_{|\mu|\leq 4\|V\|_\infty} \|R(i\mu, \mathscr{J}\mathscr{L} - \delta_0)\| + \sup_{|\mu|\geq 4\|V\|_\infty} \|R(i\mu, \mathscr{J}\mathscr{L} - \delta_0)\| \\ &\leq C_* + \sup_{|\mu|\geq 4\|V\|_\infty} \|R(i\mu - \delta_0, \mathscr{J}\mathscr{L})\| \\ &\leq C_* + \frac{C_{\mu,V}}{\delta_0}(\|f_1\|_2 + \|f_2\|_2)\end{aligned}$$

where $C_* \geq 0$ is the supremum of the (continuous) resolvent of the operator $\mathscr{J}\mathscr{L} - \delta_0$ on the compact set $\{i\mu \in \mathbb{C} \mid |\mu| \leq 4\|V\|_\infty\}$ and where $\limsup_{\mu\to\infty} C_{\mu,V} < \infty$. It follows from Theorem 1.4.2 (Gearhart-Prüss, where we note that establishing the uniform resolvent bound on $i\mathbb{R}$ is equivalent to the given statement in this work) that the semigroup generatd by $\mathscr{J}\mathscr{L} - \delta_0$ has a negative growth bound and is therefore also bounded uniformly in time. Applying the sufficient condition from Gomilko's Lemma then obtains the bound:

$$\sup_{\delta\geq\delta_0} \delta \int_{-\infty}^{\infty} \left(\|R(\delta + i\mu, \mathscr{J}\mathscr{L})f\|_2^2 + \|R(\delta + i\mu, (\mathscr{J}\mathscr{L})^*)f\|_2^2\right) d\mu \leq C_{\delta_0}\|f\|_2^2$$

so it remains, as promised, to control this integral for small δ, say $0 < \delta < 1$. To do this, it is enough to establish control in the form:

$$\int_{-\infty}^{\infty} (\|z_1(\mu)\|_2^2 + \|z_2(\mu)\|_2^2) d\mu \leq \frac{C}{\delta}(\|f_1\|_2^2 + \|f_2\|_2^2) \tag{2.2.5}$$

and, of course, this reduces to control for $\mu \geq 0$ only. Splitting the resulting integral into "low" and "high" energies, it follows that (2.2.5) is bounded above by:

$$\int_{\mu\leq\mu_*} (\|z_1(\mu)\|_2^2 + \|z_2(\mu)\|_2^2) d\mu + \int_{\mu>\mu_*} (\|z_1(\mu)\|_2^2 + \|z_2(\mu)\|_2^2) d\mu := I_{low} + I_{high}$$

where $\mu_* := \max\{2+\|V\|_\infty, 2\|V\|_\infty\}$. Note that in the high energies case, there is:

$$\begin{aligned}
I_{high} &= \int_{\mu^*}^{\infty} \|z_1(\mu)\|_2^2 d\mu + \int_{\mu_*}^{\infty} \|z_2(\mu)\|_2^2 d\mu \\
&= \int_{\mu_*}^{\infty} \|z_1(\mu)\|_2^2 d\mu + \int_{\mu_*}^{\infty} \|(H_0+\mu-\delta)^{-1}(f_2 - Vz_1)\|_2^2 d\mu \\
&\lesssim \int_{\mu_*}^{\infty} \|z_1(\mu)\|_2^2 d\mu + \int_{\mu_*}^{\infty} \|(H_0+\mu-i\delta)^{-1} f_2\|_2^2 d\mu + \int_{\mu_*}^{\infty} \frac{\|V\|_\infty^2}{(\mu-\|V\|_\infty)^2} \|z_1(\mu)\|_2^2 d\mu
\end{aligned}$$

and control of the middle integral in the form $\frac{C}{\delta}$ follows because the self-adjoint operator iH_0 generates (by Stone's Theorem) the group of isometries $\|e^{itH_0}\|_{\text{op}} = 1$, to which the backwards implication from Gomilko's Lemma can then be applied. The derivation of a suitable bound for I_{high} therefore reduces to control in the form

$$\int_{\mu_*}^{\infty} \|z_1(\mu)\|_2^2 d\mu \leq \frac{C}{\delta}(\|f_1\|_2^2 + \|f_2\|_2^2)$$

because $\frac{\|V\|_\infty^2}{(\mu-\|V\|_\infty)^2} < 1$ for $\mu > \mu_*$. Note that:

$$\begin{aligned}
&\int_{\mu_*}^{\infty} \|z_1(\mu)\|_2^2 d\mu \\
&= \int_{\mu_*}^{\infty} \|(I+i\delta(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}(H_0+\mu)^{-1}V)^{-1}(H+\mu-i\delta)^{-1}(f_1+V(H_0+\mu-i\delta)^{-1}f_2)\|_2^2 d\mu
\end{aligned}$$

and, since $\mu > \mu_*$, this integral is bounded above by

$$\begin{aligned}
2\int_{\mu_*}^{\infty} &\|(H+\mu-i\delta)^{-1}(f_1+V(H_0+\mu-i\delta)^{-1}f_2)\|_2^d d\mu \\
&\lesssim \int_{\mu_*}^{\infty} \|(H+\mu-i\delta)^{-1} f_1\|_2^2 d\mu + \int_{\mu_*}^{\infty} \|(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2\|_2^2 d\mu
\end{aligned}$$

where control in the form $\frac{C}{\delta}\|f_1\|_2^2$ is established for the first of these integrals by means of the Stone's Theorem$\to$Gomilko argument used above, and where similar control of the second of these integrals requires somewhat more care to be taken. Note that if $n_* = \lfloor \mu_* \rfloor$, then

$$\int_{\mu_*}^{\infty} \|(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2\|_2^2 d\mu \leq \sum_{n=n_*}^{\infty} \int_n^{n+1} \|(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2\|_2^2 d\mu$$

and, in particular, consider the orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of eigenvectors of H with eigenvalues $\mu_j \in \mathbb{C}$. If $P_{[n,n+1]}$ is the projection onto the closed subspace spanned by the eigenvectors corresponding to the (finitely-many) eigenvalues $\mu_j \in [n,n+1]$, then the above integral admits (modulo a fixed constant) the upper bound:

$$\sum_{n=n_*}^{\infty} \Big(\int_n^{n+1} \|P_{[n-1,n+2]}(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2\|_2^2 d\mu + \int_n^{n+1} \|(P_{<n-1}+P_{>n+2})(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2\|_2^2 d\mu \Big).$$

Looking at the second integral first, note that:

$$\|(P_{<n-1}+P_{>n+2})(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2\|_2^2 = \sum_{j:\mu_j\notin[n-1,n+2]} |\langle (H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2, e_j\rangle|^2$$

and, because $(H+\mu-i\delta)^{-1}$ is also self-adjoint (with eigenvalues $\frac{1}{\mu_j-\mu+i\delta}$), this is equal to

$$\sum_{j:\mu_j\notin[n-1,n+2]} |\langle V(H_0+\mu-i\delta)^{-1}f_2, (H+\mu-i\delta)^{-1}e_j\rangle|^2 \leq \sum_{\mu\notin[n-1,n+2]} \frac{1}{|\mu_j-\mu+i\delta|^2} |\langle V(H_0+\mu-i\delta)^{-1}f_2, e_j\rangle|^2$$

meaning that the second integral boils down to estimating

$$\|V\|_\infty^2 \int_n^{n+1} \sum_{j:\mu_j\notin[n-1,n+2]} |\langle (H_0+\mu-i\delta)^{-1}f_2, e_j\rangle|^2 \lesssim \int_n^{n+1} \|(H_0+\mu-i\delta)^{-1}f_2\|_2^2$$

and it follows from the Stone's Theorem→Gomilko argument that

$$\sum_{n=n_*}^{\infty}\int_{n}^{n+1}\|(H_0+\mu-i\delta)^{-1}f_2\|_2^2d\mu \leq \int_{-\infty}^{\infty}\|(H_0+\mu-i\delta)^{-1}f_2\|_2^2d\mu \leq \frac{C}{\delta}\|f_2\|_2^2$$

so it remains to control the quantity

$$\sum_{n=n_*}^{\infty}\int_{n}^{n+1}\|P_{[n-1,n+2]}(H+\mu-i\delta)^{-1}V(H_0+\mu-i\delta)^{-1}f_2\|_2^2d\mu$$

by $\frac{C}{\delta}\|f_2\|_2^2$.As in the previous case, this integral is equal to

$$\int_{n}^{n+1}\sum_{j:\mu_j\in[n-1,n+2]}|\langle V(H_0+\mu-i\delta)^{-1}f_2,e_j\rangle|^2\frac{1}{|\mu_j-\mu+i\delta|^2}d\mu$$

due to the fact that $(H+\mu-i\delta)^{-1}$ is self-adjoint with eigenvalues $\frac{1}{\mu_j-\mu+i\delta}$. Now, Theorem 1.4.5 (1) and the fact that $\frac{1}{\|R(i\delta-n,H_0)\|_{\text{op}}}>n-\|V\|_\infty\geq n_*-\|V\|_\infty\geq 1>|\mu-n|$ give that:

$$(H_0+\mu-i\delta)^{-1}=\sum_{l=0}^{\infty}(n-\mu)^l[(H_0+n-i\delta)^{-1}]^{l+1}$$

which, in turn, implies that

$$\begin{aligned}
|\langle V(H_0+\mu-i\delta)^{-1}f_2,e_j\rangle|^2 &= \left|\left\langle V\left(\sum_{l=0}^{\infty}(n-\mu)^l[(H_0+n-i\delta)^{-1}]^{l+1}\right)f_2,e_j\right\rangle\right|^2\\
&= \left|\sum_{l=0}^{\infty}\left\langle\left(V(n-\mu)^l[(H_0+n-i\delta)^{-1}]^{l+1}\right)f_2,e_j\right\rangle\right|^2\\
&\leq \left(\sum_{l=0}^{\infty}\left|\left\langle V[(H_0+n-i\delta)^{-1}]^{l+1}f_2,e_j\right\rangle\right|\right)^2\\
&= \left(\sum_{l=0}^{\infty}(1+l^2)^{\frac{1}{2}}(1+l^2)^{-\frac{1}{2}}\left|\left\langle V[(H_0+n-i\delta)^{-1}]^{l+1}f_2,e_j\right\rangle\right|\right)^2
\end{aligned}$$

and this last quantity is, from Cauchy-Schwarz, bounded above by the product:

$$\left(\sum_{l=0}^{\infty}(1+l^2)\left|\left\langle V[(H_0+n-i\delta)^{-1}]^{l+1}f_2,e_j\right\rangle\right|^2\right)\left(\sum_{l=0}^{\infty}(1+l^2)^{-1}\right)$$

so, defining $F_{n,l}=V[(H_0+n-i\delta)^{-1}]^{l+1}f_2$, it follows that $\|F_{n,l}\|_2\leq\frac{\|V\|_\infty}{(n-\|V\|_\infty)^{l+1}}\|f_2\|_2$ and that there is control of the desired integral in the form

$$\sum_{n=n_*}^{\infty}\int_n^{n+1}\sum_{j:\mu_j\in[n-1,n+2]}\sum_{l=0}^{\infty}(1+l^2)|\langle F_{n,l},e_j\rangle|^2\sum_{l=0}^{\infty}(1+l^2)^{-1}\frac{1}{|\mu_j-\mu+i\delta|^2}d\mu$$
$$\lesssim\sum_{l=0}^{\infty}(1+l^2)\sum_{n=n_*}^{\infty}\sum_{j:\mu_j\in[n-1,n+2]}|\langle F_{n,l},e_j\rangle|^2\int_n^{n+1}\frac{1}{|\mu_j-\mu+i\delta|^2}d\mu \quad (2.2.6)$$

where we have used the fact that $\sum_{l=0}^{\infty}(1+l^2)^{-1}<\infty$, the independence from μ of $\sum_{l=0}^{\infty}(1+l^2)|\langle F_{n,l},e_j\rangle|^2$, and also Fubini's Theorem to move the integral inside the sum in j. Now, for each of the (finitely-many) j such that $\mu_j\in[n-1,n+2]$, there is

$$\int_n^{n+1}\frac{1}{|\mu_j-\mu+i\delta|^2}d\mu\leq\int_0^{\infty}\frac{1}{(\mu_j-\mu)^2+\delta^2}d\mu\leq\pi\leq\frac{\pi}{\delta}$$

and consequently, we have

$$\begin{aligned}2.2.6&\leq\frac{C}{\delta}\sum_{l=0}^{\infty}(1+l^2)\sum_{n=n_*}^{\infty}\sum_{j:\mu_j\in[n-1,n+2]}|\langle F_{n,l},e_j\rangle|^2\\&\leq\frac{C}{\delta}\sum_{l=0}^{\infty}(1+l^2)\sum_{n=n_*}^{\infty}\|F_{n,l}\|_2^2\\&\leq\frac{C}{\delta}\sum_{l=0}^{\infty}(1+l^2)\sum_{n=n_*}^{\infty}\frac{\|V\|_\infty^2\|f_2\|_2^2}{(n-\|V\|_\infty)^{2l+2}}\leq\frac{C'\|f_2\|_2^2}{\delta}\sum_{l=0}^{\infty}(1+l^2)\sum_{n=n_*}^{\infty}\frac{1}{(n-\|V\|_\infty)^{2l}(n-\|V\|_\infty)^2}\\&\leq\frac{C'\|f_2\|_2^2}{\delta}\sum_{l=0}^{\infty}\frac{(1+l^2)}{2^{2l}}\sum_{n=n_*}^{\infty}\frac{1}{(n-\|V\|_\infty)^2}=\frac{C''}{\delta}\|f_2\|_2^2\end{aligned}$$

so it follows finally that we have control in the form $I_{high}\leq\frac{C}{\delta}(\|f_1\|_2^2+\|f_2\|_2^2)$ where $C\geq0$ is a fixed constant independend of δ. It remains to derive similar control of $I_{low}=\int_0^{\mu_*}(\|z_1(\mu)\|_2^2+\|z_2(\mu)\|_2^2)d\mu$ where we recall that that the resolvent $z=(z_1,z_2)$ exists for such μ by means of the spectral stability assumption $\sigma(\mathscr{J}\mathscr{L}) \subset i\mathbb{R}$. Now, fix $N > \mu_*$ and such that $\pm iN \notin \sigma(JL)$ (this is possible because all of the eigenvalues of $\mathscr{J}\mathscr{L}$ are isolated). In particular, there are finitely-many paired eigenvalues $\pm i\mu_j$ of the Hamiltonian $\mathscr{J}\mathscr{L}$ operator that are contained by the interval $[-iN, iN]$, say, $\{\pm i\mu_j\}_{j=1}^{J_N}$. Define for each pair $\pm i\mu_j$ the Riesz projection

$$P_j = \frac{1}{2\pi i}\int_{\gamma_j} R(z, \mathscr{J}\mathscr{L})dz$$

where γ_j is a closed curve of index one the encircles both eigenvalues $\pm i\mu_j$ but no other spectrum of $\mathscr{J}\mathscr{L}$. The operator P_j can be represented as a matrix of dimension n_j whose Jordon canonical form has l_j distinct blocks, each of dimension n_j^l for $l = 1, \ldots, l_j$. It is a well-known fact that P_j commutes with $\mathscr{J}\mathscr{L}$ and also that $\|P_j e^{t(\mathscr{J}\mathscr{L})}\|_{\text{op}} \leq Ct^{n_j^{l_j}-1}$. Define then the operators $P_N = \sum_{j=1}^{J_N} P_j$ and $Q_N = I - P_N$, and note that

$$\|e^{t\mathscr{J}\mathscr{L}P_N}\|_{\text{op}} = \|e^{tP_N\mathscr{J}\mathscr{L}}\|_{\text{op}} = \|P_N e^{t\mathscr{J}\mathscr{L}}\|_{\text{op}} \leq Ct^{\max_{1\leq j\leq J_N}\{n_j^{l_j}-1\}}.$$

In addition, $Q_N\mathscr{J}\mathscr{L}$ has no spectrum in a neighborhood of $[-iN, iN]$ so that $\limsup_{\delta\to 0}\|R(\delta + i\mu, Q_N\mathscr{J}\mathscr{L}) - R(i\mu, Q_N\mathscr{J}\mathscr{L})\|_{\text{op}} = 0$ by the continuity of the resolvent of $Q_N\mathscr{J}\mathscr{L} = \mathscr{J}\mathscr{L}Q_N$ and then

$$\limsup_{\delta\to 0}\int_{-N}^{N}\|R(\delta + i\mu, \mathscr{J}\mathscr{L})Q_N f\|_2^2 d\mu = \int_{-N}^{N}\|R(i\mu, \mathscr{J}\mathscr{L})Q_N f\|_2^2 d\mu \leq C_N\|f\|_2^2$$

by dominated convergence. Note that $C_N \geq 0$ depends on the choice of N only, and thus, ultimately only on $\|V\|_\infty$. Taken together with the high energies estimate which still holds for $Q_N f$, there is

$$\limsup_{\delta\to 0}\int_{-\infty}^{\infty}\|R(\delta + i\mu, \mathscr{J}\mathscr{L})Q_N f\|_2^2 d\mu \leq \frac{C}{\delta}\|Q_N f\|_2^2$$

and applying the Gomilko Lemma to the semigroup $e^{t\mathscr{J}\mathscr{L}}$ restricted to its invariant subspace $Q_N(L^2[-\pi,\pi]^2)$, it follows that $\sup_{t>0}\|e^{t\mathscr{J}\mathscr{L}}Q_nf\|_2 \le C\|f\|_2^2$, and consequently,

$$\|e^{t\mathscr{J}\mathscr{L}}f\|_2 = \|e^{t\mathscr{J}\mathscr{L}}(P_N+Q_N)f\|_2 \le \|e^{t\mathscr{J}\mathscr{L}}P_Nf\|_2 + \|e^{t\mathscr{J}\mathscr{L}}Q_Nf\|_2$$
$$\le Ct^{\max_{1\le j\le J_N}\{n_j^{l_j}-1\}}\|f\|_2^2 + C\|f\|_2^2 \le Ct^{\max_{1\le j\le J_N}\{n_j^{l_j}-1\}}\|f\|_2^2$$

because the semigroup clearly admits a uniform bound for $t\in[0,1]$. We have therefore shown the main result for the NLS semigroup generated by $\mathscr{J}\mathscr{L}$ under the assumption $\sigma(\mathscr{J}\mathscr{L})\subset i\mathbb{R}$, stated formally below.

Theorem 2.2.2. *Let $\mathscr{J}\mathscr{L}$ be as in Section 2.2 and suppose that V_1, V_2 are bounded and real-valued potential functions with $V := \frac{1}{2}(V_1+V_2)$. The spectrum of $\mathscr{J}\mathscr{L}$ consists only of eigenvalues with finite multiplicity, accumulating only at infinity, and assume in addition that $\sigma(\mathscr{J}\mathscr{L})\subset i\mathbb{R}$. It follows that*

$$\|e^{t\mathscr{J}\mathscr{L}}f\|_2 \le Ct^{\max_{1\le j\le J_N}\{n_j^{l_j}-1\}}\|f\|_2^2 \tag{2.2.7}$$

where there are J_N possibly non-simple eigenvalue pairs $\pm i\mu_j$ in $[-iN, iN]$ for N sufficiently large (depending on $\|V\|_\infty$. Note that if all the eigenvalues of $\mathscr{J}\mathscr{L}$ are simple, then the bound in (2.2.7) *is uniform.*

Let us now in the next section demonstrate analogous results for the KdV equation (2.1.3).

2.3 KdV

Recall that we are interested in the KdV equation (2.1.3), which reads:

$$u_t(x,t) + u_{xxx}(x,t) + \partial_x[f(u^2)u] = 0$$

where $t>0$, $x\in[-\pi,\pi]$, and $u\in H^3_{\text{per}}[-\pi,\pi]\subset L^2[-\pi,\pi]$. Linearizing about a known travelling wave solution $\phi(x-\omega t)$ produces a system of the form $v_t = \mathscr{J}\mathscr{L}v$ where $\mathscr{J} = \partial_x$ and where

$\mathscr{L} = -\Delta - V$ for some potential V. It is convenient at this point to remind the reader of the following standard facts from Fourier analysis:

- $f \in L^2[-\pi,\pi] \implies f(x) = \sum_{k=-\infty}^{\infty} e^{ikx}\hat{f}(k)$ where $\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)dx$
- $\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$. This result is called Parseval's Theorem and it was actually used implicitly in the previous NLS section.
- The projection operator $P_k(f) = e^{ikx}\hat{f}(k)$ is well-defined and, as a result, so are the operators $P_{\neq k} = I - P_k$ and $P_{|k|} = P_k + P_{-k}$.

and now, we consider the closed operator $\mathscr{J}\mathscr{L} = \partial_x(-\Delta - V) : H^3_{\text{per}}[-\pi,\pi] \to L^2_{\text{per}}[-\pi,\pi]$. This operator is densely-defined and generates C_0-semigroup due to the bounded perturbation theorem (see Section 1.5). As before, let $f = [\partial_x(-\Delta - V) - (\delta + i\mu)]^{-1} g$ and note that an application of the Gomilko Lemma will require an estimate of the form

$$\sup_{0<\delta<1} \delta \int_{-\infty}^{\infty} \|f(\mu)\|_2^2 d\mu \leq C\|g\|_2^2$$

for some fixed $C \geq 0$, and where the estimate boils down (again, as before) to small δ. Let us now construct the resolvent $f(\mu)$ starting from the fact that

$$g = [\partial_x(-\Delta - V) - (\delta + i\mu)]f \tag{2.3.1}$$

and looking at the Fourier coefficients on both sides. Note first that

$$\hat{g}(k) = [-(ik)^3 - ik\hat{V}(k) - (\delta + i\mu)]\hat{f}(k) \implies \hat{g}(0) = -(\delta + i\mu)\hat{f}(0)$$

or, in other words, it suffices to assume that g and f are mean-free (i.e. $\hat{g}(0) = \hat{f}(0) = 0$). Next, take ∂_x^{-1} (which exists on $L^2_{\text{per}}[-\pi,\pi]$) on both sides of (2.3.1) to obtain that

$$\partial_x^{-1} g = (-\Delta - V - \delta\partial_x^{-1} - i\mu\partial_x^{-1})f \tag{2.3.2}$$

and it is worth noting that the operators $-\Delta - V - i\mu\partial_x^{-1}$ and $\delta\partial_x^{-1}$ are self-adjoint and skew-symmetric, respectively, on $H^3_{\text{per}}[-\pi,\pi]$. Let $\nu^3 = \mu \in \mathbb{R}$ for convenience and note that the dispersion of the first operator $-\Delta - V - i\mu\partial_x^{-1}$ is of the form

$$k^2 - \frac{\mu}{k} = \frac{1}{k}(k^3 - \nu^3) = \frac{1}{k}(k-\nu)(k^2 + k\nu + \nu^2)$$

where we are ignoring the potential V for the moment. This shows that the modulus of k^{th} dispersion mode is small only if k is close to ν. We will exploit this fact in our calculations. Let $k_0 = k_0(\nu)$ be the integer closest to $\nu \in \mathbb{R}$ and note that for $k \neq k_0$, there is $|k-\nu| \geq \frac{1}{2}$ so that

$$\left|k^2 - \frac{\mu}{k}\right| \geq \frac{1}{2k}(k^2 + k\nu + \nu^2) \geq \frac{1}{4k}\max\{k^2, \nu^2\} \tag{2.3.3}$$

where the $k = 0$ case is irrelevant since f, g are mean-free. It follows that $-\Delta - V - i\mu\partial_x^{-1}$ should be invertible mode-by-mode for all $k \neq k_0$. The aim, then, is to deal with both the critical ($k = k_0$) and non-critical modes of (2.3.2) to invert the operator on the RHS and obtain suitable Gomilko bounds for f.

2.3.1 Construction of KdV Resolvent

Note that ∂_x and ∂_x^{-1} both commute with the projection P_k, and let us first deal with the non-critical Fourier modes of (2.3.2). Evidently, for a fixed $\delta > 0$ and $\nu \in \mathbb{R}$, there is

$$P_{\neq k_0}[\partial_x^{-1} g] = P_{\neq k_0}[(-\Delta - V - i\mu\partial_x^{-1} - \delta\partial_x^{-1})f]$$

and this is equivalent, by commutativity and the fact that $P_k[Vf] = P_k[V(P_k + P_{\neq k})f] = P_k[VP_k(f)] + P_k[VP_{\neq k}(f)]$ for any integer k, to the equation

$$\begin{aligned}\partial_x^{-1} P_{\neq k_0}[g] &= (-\Delta - i\mu\partial_x^{-1} - \delta\partial_x^{-1})P_{\neq k_0}[f] - P_{\neq k_0}[Vf] \\ &= (-\Delta - i\mu\partial_x^{-1} - \delta\partial_x^{-1})P_{\neq k_0}[f] - P_{\neq k_0}[VP_{k_0}(f)] - P_{\neq k_0}[VP_{\neq k_0}(f)]\end{aligned}$$

so consequently,

$$\partial_x^{-1} P_{\neq k_0}[g] + P_{\neq k_0}[VP_{k_0}(f)] = (-\Delta - i\mu\partial_x^{-1} - \delta\partial_x^{-1})P_{\neq k_0}[f] - P_{\neq k_0}[VP_{\neq k_0}(f)] \tag{2.3.4}$$

which is expressable in the form

$$\partial_x^{-1} P_{\neq k_0}[g] + P_{\neq k_0}[VP_{\neq k_0}(f)] + \delta\partial_x^{-1} P_{\neq k_0}[f] = (-\Delta - i\mu\partial_x^{-1} - P_{\neq k_0}[VP_{\neq k_0}(\cdot)])P_{\neq k_0}[f].$$

Now, the operator $\mathscr{M} = \mathscr{M}_{V,\nu} = -\Delta - i\mu\partial_x^{-1} - P_{\neq k_0}[VP_{\neq k_0}(\cdot)]$ is self-adjoint and acts invariantly on the subspace $P_{\neq k_0}\left(H^3_{\text{per}}[-\pi,\pi]\right)$. Moreover, one has

$$\mathscr{M} = (-\Delta - i\mu\partial_x^{-1})(I - (-\Delta - i\mu\partial_x^{-1})^{-1} P_{\neq k_0}[VP_{\neq k_0}(\cdot)])$$

implying that it is invertible (as the product of invertible operators) if there is the upper bound $\|(-\Delta - i\mu\partial_x^{-1})^{-1} P_{\neq k_0}[VP_{\neq k_0}(\cdot)]\| < 1$. Note that from (2.3.3), there is

$$\|(-\Delta - i\mu\partial_x^{-1})^{-1} P_{\neq k_0}[VP_{\neq k_0}(\cdot)]\| \leq \frac{Ck\|V\|_\infty}{\max\{k^2,\nu^2\}} \leq \frac{C\|V\|_\infty}{\nu} < \frac{1}{2}$$

for $\nu > C\|V\|_\infty$ where $C \geq 0$ is a fixed constant, and consequently, $\mathscr{M}^{-1}$ has the Von Neumann expansion

$$\mathscr{M}^{-1} = \sum_{l=0}^{\infty} [(-\Delta - i\mu\partial_x^{-1})^{-1} P_{\neq k_0}[VP_{\neq k_0}(\cdot)]]^l (-\Delta - i\mu\partial_x^{-1})^{-1} \tag{2.3.5}$$

where $\|\mathscr{M}^{-1}\|_{\text{op}} \leq \frac{Ck}{\max\{k^2,\nu^2\}}$ as a result. Analogous to the non-critical mode equation (2.3.4) there is the projection onto the critical mode k_0 expressable as:

$$(-\Delta - i\mu\partial_x^{-1} - \delta\partial_x^{-1})P_{k_0}[f] - P_{k_0}[VP_{k_0}(f)] = \partial_x^{-1} P_{k_0}[g] + P_{k_0}[VP_{\neq k_0}(f)]. \tag{2.3.6}$$

Now under the “high energies" assumption $\nu \geq C\|V\|_\infty$ from before, $\mathscr{M}^{-1}$ exists and we may apply it in (2.3.4) to obtain that

$$P_{\neq k_0}[f] = \delta \mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0}[f] + \mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0}[g] + \mathscr{M}^{-1}P_{\neq k_0}[VP_{\neq k_0}(f)] \tag{2.3.7}$$

which we may then plug back into (2.3.6) to find:

$$\begin{aligned}(-\Delta - i\mu\partial_x^{-1} - P_{k_0}[VP_{k_0}(\cdot)] - P_{k_0}[VP_{\neq k_0}\mathscr{M}^{-1}P_{\neq k_0}(VP_{k_0}(\cdot))]P_{k_0}[f] \\ = \delta\partial_x^{-1}P_{k_0}[f] + \partial_x^{-1}P_{k_0}[g] + P_{k_0}V(\delta\mathscr{M}^{-1}(\partial_x^{-1}P_{\neq k_0}[f])) + P_{k_0}V\mathscr{M}^{-1}(\partial_x^{-1}P_{\neq k_0}[g])\end{aligned}$$

after re-arranging. Let $\mathscr{Q} = (-\Delta - i\mu\partial_x^{-1} - P_{k_0}[VP_{k_0}(\cdot)] - P_{k_0}[VP_{\neq k_0}\mathscr{M}^{-1}P_{\neq k_0}(VP_{k_0}(\cdot))]$ which is self-adjoint and acts invariantly on $P_{k_0}\left(H^3_{\text{per}}[-\pi,\pi]\right) \subset P_{k_0}\left(L^2[-\pi,\pi]\right)$. The above equation then has the form:

$$(\mathscr{Q} + i\frac{\delta}{k_0})P_{k_0}[f] = \delta\partial_x^{-1}P_{k_0}[f] + \partial_x^{-1}P_{k_0}[g] + P_{k_0}V(\delta\mathscr{M}^{-1}(\partial_x^{-1}P_{\neq k_0}[f])) + P_{k_0}V\mathscr{M}^{-1}(\partial_x^{-1}P_{\neq k_0}[g])$$

and, because $\mathscr{Q}$ is self-adjoint, the LHS resolvent operator is invertible with $\|R(-i\frac{\delta}{k_0},\mathscr{Q})\|_{\text{op}} \leq \frac{k_0}{\delta}$ so putting everything from this section together there is the following result.

Theorem 2.3.1. *The resolvent $f = [\partial_x(-\Delta - V) - (\delta + i\mu)]^{-1}g$ can now be constructed mode-by-mode. In particular, we have that*

- $P_{\neq k_0}[f] = \delta\mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0}[f] + \mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0}[g] + \mathscr{M}^{-1}P_{\neq k_0}[VP_{\neq k_0}(f)]$
- $P_{k_0}[f] = \left(\mathscr{Q} + i\frac{\delta}{k_0}\right)^{-1}\left(\delta\partial_x^{-1}P_{k_0}[f] + \partial_x^{-1}P_{k_0}[g] + P_{k_0}V(\delta\mathscr{M}^{-1}(\partial_x^{-1}P_{\neq k_0}[f])) + P_{k_0}V\mathscr{M}^{-1}(\partial_x^{-1}P_{\neq k_0}[g])\right)$

or, equivalently, $f = \mathscr{R}f + \mathscr{T}g = P_{k_0}[\mathscr{R}f + \mathscr{T}g] + P_{\neq k_0}[\mathscr{R}f + \mathscr{T}g]$ where

- $P_{k_0}[\mathscr{R}f] = \left[\delta\left(\mathscr{Q} + i\frac{\delta}{k_0}\right)^{-1}P_{k_0}VP_{\neq k_0}\mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0}\right]f$
- $P_{k_0}[\mathscr{T}g] = \left[\left(\mathscr{Q} + i\frac{\delta}{k_0}\right)^{-1}\partial_x^{-1}P_{k_0} + \left(\mathscr{Q} + i\frac{\delta}{k_0}\right)^{-1}P_{k_0}VP_{\neq k_0}\mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0}\right]g$
- $P_{\neq k_0}[\mathscr{R}f] = (I - \delta\mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0})^{-1}\mathscr{M}^{-1}P_{\neq k_0}[VP_{k_0}(\cdot)]f$
- $P_{\neq k_0}[\mathscr{T}g] = [(I - \delta\mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0})^{-1}\mathscr{M}^{-1}\partial_x^{-1}P_{\neq k_0}(\cdot)]g$

where the final two bullet points folow from (2.3.7). *Moreover, we have from the above formulae that*

- $\|P_{\neq k_0}[\mathscr{R}f]\|_2 \leq \frac{C\|V\|_\infty\|f\|_2}{\nu}$
- $\|P_{\neq k_0}[\mathscr{T}g]\|_2 \leq \frac{C\|g\|_2}{\nu^2}$
- $\|P_{k_0}[\mathscr{R}f]\|_2 \leq \frac{C\|V\|_\infty\|f\|_2}{\nu}$

and finally, that $\|P_{k_0}[\mathscr{T}g]\|_2 \leq \frac{C\|g\|_2}{\delta}$ *from the bound on* $\mathscr{M}^{-1}$ *from* (2.3.5).

lIt follows that $\|\mathscr{R}\|_2 < 1$ for $\mu \gg \mu_* = \max\{\|V\|_\infty^3, 1\}$ so we may write in this case $f = (I-\mathscr{R})^{-1}\mathscr{T}g$ by Von Neumann. In particular, there is $\|f\|_2 \leq \frac{C\|g\|_2}{\delta}$ by looking at the k_0 and $\neq k_0$ cases for $\mathscr{T}$ so that the resolvent f not only exists for $\mu \gg \mu_*$ but it also satisfies the estimate $\sup_{\mu\in\mathbb{R}} \|R(\delta + i\mu, \partial_x(-\Delta - V))\|_{\text{op}} \leq \frac{C}{\delta}$. This of course shows an absence of spectrum for this KdV generator if μ is large enough, but the bound blows up as $\delta \to 0$. Let us now assume spectral stability $\sigma(\mathscr{J}\mathscr{L}) \subset i\mathbb{R}$ and derive proper Gomilko bounds to obtain an optimal L^2 estimate on the KdV semigroup.

2.3.2 Optimal L^2 Bounds for the KdV Semigroup

Note that the assumption $\partial_x(-\Delta - V) := \mathscr{J}\mathscr{L} : H^3_{\text{per}}[-\pi,\pi] \to L^2_{\text{per}}[-\pi,\pi]$ has purely imaginary spectrum taken together with the (uniform in μ) estimate $\sup_{\mu\in\mathbb{R}} \|R(\delta + i\mu, \partial_x(-\Delta - V))\|_2 \leq \frac{C}{\delta}$ implies that

$$\sup_{\mu\in\mathbb{R}} \|R(\delta + i\mu, \partial_x(-\Delta - V)\|_2 \leq C_\delta$$

for all $\delta > 0$. It follows by continuity of $\mu \mapsto R(\delta + i\mu, \partial_x(-\Delta - V))$ on compact intervals that there is a uniform bound along the vertical line $\{\Re(z) = \delta\}$ for each $\delta > 0$, and consequently, there are sub-exponential bounds of the form $\|e^{t\mathscr{J}\mathscr{L}}\|_2 \leq C_\delta e^{\delta t}$ for each $\delta > 0$. We will now proceed as in the NLS case, where we show uniform Gomilko bounds for the “high energies" and project away from isolated imaginary eigenvalues for the “low energies." Looking at the high energies first, this amounts to bounds of the form:

$$\delta \int_{|\mu|>N^3} \|\partial_x(-\Delta - V) - (\delta + i\mu))^{-1} g\|_2^2 d\mu \leq C\|g\|_2^2 \tag{2.3.8}$$

$$\delta \int_{|\mu|>N^3} \|(-\Delta - V)\partial_x - (\delta + i\mu)^{-1} g\|_2^2 d\mu \leq C\|g\|_2^2 \tag{2.3.9}$$

by considering both $\mathscr{J}\mathscr{L}$ and $(\mathscr{J}\mathscr{L})^*$ for the Gomilko estimates with $N > \max\{\|V\|_\infty^3, 1\}$. An analogous argument to the NLS case shows that we need only establish the above bounds for $0 < \delta < 1$. The cases $\pm\mu$ are symmetric so we need only consider $\mu > 0$, and first:

$$\int_1^\infty |\hat{f}(0)|^2 d\mu \leq \int_1^\infty \frac{|\hat{g}(0)|^2}{|\delta + i\mu|^2} d\mu \leq \int_1^\infty \frac{|\hat{g}(0)|^2}{\delta^2 + \mu^2} \leq C\|g\|_2^2 \leq \frac{C}{\delta}\|g\|_2^2$$

because $0 < \delta < 1$. It then suffices as previously to assume that f, g are mean-free. Letting $\nu^3 = \mu$ as before, the integral to control (2.3.8) becomes (for mean-free f, g)

$$\delta \int_{\nu > N} \|(\partial_x(-\Delta - V) - (\delta + i\nu^3))^{-1} g\|_2^2 \nu^2 d\nu$$

and because ν is so large that $f = \mathscr{R} f + \mathscr{T} g = (I - \mathscr{R})^{-1}\mathscr{T} g$, this integral is equal to

$$\int_{\nu > N} \|(I - \mathscr{R})^{-1} P_{\neq k_0}[\mathscr{T} g]\|_2^2 \nu^2 d\nu + \int_{\nu > N} \|(I - \mathscr{R})^{-1} P_{k_0}[\mathscr{T} g]\|_2^2 \nu^2 d\nu.$$

These two integrals may be controlled as follows: first,

$$\int_{\nu > N} \|(I - \mathscr{R})^{-1} P_{\neq k_0}[\mathscr{T} g]\|_2^2 \nu^2 d\nu \leq C \int_{\nu > N} \nu^{-2} \|g\|_2^2 d\nu \leq C\|g\|_2^2 \leq \frac{C}{\delta}\|g\|_2^2$$

and for the critical mode, we note that the operator $P_{k_0}[\mathscr{T} g]$ has the form (mode-by-mode):

$$\left(\mathscr{Q} + i\frac{\delta}{k_0}\right)^{-1} P_{k_0}[\mathscr{T} g] = \frac{1}{k_0^2 - \frac{\nu^3}{k_0} - c_\nu + i\frac{\delta}{k_0}} \widehat{\mathscr{T} g}(k_0) e^{ik_0 x}.$$

Looking at the real part of the dispersion with $\tilde{k} = k_0^3 - c_\nu k_0$, it follows that

$$k_0^2 - \frac{\nu^3}{k_0} - c_\nu = \frac{\tilde{k} - \nu^3}{k_0} = (\tilde{k} - \nu)\frac{\tilde{k}^2 + \tilde{k}\nu + \nu^2}{k_0} \sim (\tilde{k} - \nu)\nu$$

as $c_\nu = O(1)$ and $\tilde{k} = k_0 + O(\nu^{-1})$. Now, it suffices for $P_{k_0}[\mathscr{T} g]$ to bound:

$$\int_{\nu > N} \left\| \left(\mathscr{Q} + i\frac{\delta}{k_0}\right)^{-1} \partial_x^{-1} P_{k_0}[g] \right\|_2^2 \nu^2 d\mu \leq C \sum_{l=N}^{\infty} \int_{l-\frac{1}{2}}^{l+\frac{1}{2}} \left\| \left(\mathscr{Q} + i\frac{\delta}{l}\right)^{-1} \partial_x^{-1} P_{k_0}[g] \right\|_2^2 \nu^2 d\nu$$

because $k_0 = k_0(\nu) = l$ for $\nu \in (l - \frac{1}{2}, l + \frac{1}{2})$. Let $g_l = \partial_x^{-1} P_{k_0}[g]$ and partition the interval $(l - \frac{1}{2}, l + \frac{1}{2})$ as

$$\left(l - \frac{1}{2}, l + \frac{1}{2}\right) \subset \left\{\nu \,\middle|\, |\nu - \tilde{k}| < \frac{\delta}{l^2}\right\} \cup \left(\bigcup_{m=1}^{\infty} \left\{\nu \,\middle|\, |\nu - \tilde{k}| \geq \frac{\delta}{2^m l^2}\right\}\right) := \mathscr{A}_0 \cup \bigcup_{m=1}^{\infty} \mathscr{A}_m$$

Then, $\nu \in \mathscr{A}_0$ implies $\frac{1}{|k_0^2 - \frac{\nu^3}{k_0} + c_\nu + i\frac{\delta}{k_0}|} \sim \frac{l}{\delta}$ and $\nu \in \mathscr{A}_m$ implies $\frac{1}{|k_0^2 - \frac{\nu^3}{k_0} + c_\nu + i\frac{\delta}{k_0}|} \sim \frac{l}{2^m \delta}$ so that

$$\begin{aligned} \int_{l-\frac{1}{2}}^{l+\frac{1}{2}} \left\| \left(\mathscr{Q} + i\frac{\delta}{l}\right)^{-1} \partial_x^{-1} P_{k_0}[g] \right\|_2^2 \nu^2 d\nu &\leq C \sum_{m=0}^{\infty} 2^{-2m} l^2 \delta^{-2} \|g_l\|_2^2 \int_{\mathscr{A}_m} d\nu \\ &\leq \frac{C}{\delta} \|g_l\|_2^2 \sum_{m=0}^{\infty} 2^{-m} \leq \frac{C}{\delta} \|g\|_2^2 \end{aligned}$$

because $\sum_l \|g_l\|_2^2 \leq \|g\|_2^2$. The adjoint Gomilko integral (2.3.9) is handled in a completely analogous manner (with some addition steps to favorably manipulate the resolvent of $(\mathscr{J}\mathscr{L})^*$), so the high energy bounds $\mu > \max\{\|V\|_\infty^3, 1\}$ are completed. The “low energy" Gomilko bounds work in the same way (projecting away from small and purely imaginary eigenvalues) as the NLS case and, in all, we have the following result.

Theorem 2.3.2. *Let $\mathscr{J}\mathscr{L} = \partial_x(-\Delta - V)$ bet he linearized KdV operator associated to* (2.1.3) *and assume that $\sigma(\mathscr{J}\mathscr{L}) \subset i\mathbb{R}$. Then, there is the bound*

$$\|e^{t\mathscr{J}\mathscr{L}} f\|_2 \leq C t^{N-1} \|f\|_2$$

where N is the size of the largest Jordan block associated to any purely imaginary eigenvalue of $\mathscr{J}\mathscr{L}$. As before, this bound is uniform in time if all the eigenvalues are simple (i.e. $N = 1$).

Chapter 3

Towards a Characterization of the Property of Lebesgue in Banach Spaces

Abstract

This chapter provides basic background information about the so-called Property of Lebesgue in Banach spaces. Moreover, it details two new generalizations of exist results concerning the asymptotic structure of these spaces. These results bring us closer to solving the problem of characterizing Banach space that have the Property of Lebesgue in terms of their asymptotic geometry.

3.1 Introduction to PL-Spaces

Let μ be the usual Lebesgue measure. A time-honored analysis exercise is to prove, for real-valued functions on $[0,1]$, that boundedness and μ-almost everywhere (μ-a.e.) continuity is equivalent to both Riemann and Darboux integrability. It is interesting to note, as L.M. Graves did in his 1927 paper [14], that this is not in general true for functions of the form $f:[0,1]\to X$ where X is an infinite-dimensional Banach space. Indeed, if $r_1,r_2,\ldots$ is a listing of $\mathbb{Q}\cap[0,1]$ and if $p>1$, then $f:[0,1]\to\ell_p$ defined by $f(s)=0$ for s irrational and $f(r_n)=e_n$ where $e_1,e_2,\ldots$ is the canonical unit vector basis has Riemann sums of the form

$$\begin{aligned}\left\|\sum_{i=1}^{d}(p_i-p_{i-1})f(t_i)\right\|_p^p &= \sum_{i=1}^{d}|(p_i-p_{i-1})f(t_i)|^p \\ &\le \sum_{i=1}^{d}(p_i-p_{i-1})^p = \sum_{i=1}^{d}(p_i-p_{i-1})(p_i-p_{i-1})^{p-1} \\ &\le \max_{1\le i\le d}(p_i-p_{i-1})^{p-1}\sum_{i=1}^{d}(p_i-p_{i-1}) = \max_{1\le i\le d}(p_i-p_{i-1})^{p-1}\end{aligned}$$

and this quantity goes to zero as we take finer and finer partitions of $[0,1]$. In other words, this above $f:[0,1]\to\ell_p$ is Riemann-integrable with integral zero whenever $p>1$, yet it is clearly everywhere discontinuous on $[0,1]$ at the same time. A similar argument applies for $X=c_0$. However, it is crucial to see that this argument *will not* work for $X=\ell_1$. This is because there is no "extra" of the quantity (p_i-p_{i-1}) to take outside the sum if $p=1$ and consequently, this discontinuous function is *not* Riemann-integrable if $X=\ell_1$. In fact, ℓ_1 is the prototypical Banach space for which boundedness and μ-a.e. continuity is equivalent to Riemann integrability. This was first proved in the1970s by Nemirovskiĭ, Očan, and Redžuani in [22], and independently also by G.C. da Rocha Filho in [7]. The existence of such a Banach space made it necessary to define the class of Banach spaces with this property.

Definition 3.1.1. The Banach space X is said to have the Property of Lebesgue (i.e. to be a "PL-space") if every Riemann-integrable function $f:[0,1]\to X$ is μ-a.e. continuous.

The problem of characterizing PL-spaces in terms of their intrinsic geometry is still open, but the field of Banach space geometry began in the 1970s to exhibit objects and techniques that have helped to make considerable progress in this area. At the time, Banach space geometers were interested in whether or not there existed a Banach space containing no isomorphic copy of either ℓ_p $(1\le p<\infty)$ or c_0. This problem was solved in 1974 when B.S. Tsirelson constructed such a space. Soon after, Figiel and Johnson published a useful construction of the continuous dual to Tsirelson's original space and today, one normally refers to this dual space itself as Tsirelson's space. Both Tsirelson's original space and its Figiel-Johnson dual $\mathscr{T}$ contain no isomorphic copy of either ℓ_p $(1 \leq p < \infty)$ or c_0.

Another key open problem in the geometry of Banach spaces between roughly 1980 and the mid-1990s was to determine the distortability of the ℓ_p spaces for $1 < p < \infty$. This was solved by E. Odell and Th. Schlumprecht in their famous paper [24], but they left unanswered the question of whether or not there is a distortable Banach space that is not arbitrarily distortable. The Banach space $\mathscr{T}$ is still the prime candidate for such a space, and one of the early papers in trying to address this problem was the seminal work [20] by N. Tomczak-Jaegermann and V. Milman. In this work, they introduced the concept of a Banach space that behaves in a limiting sense like ℓ_p.

Definition 3.1.2. Let X be a Banach space with the basis $(e_i)_{i=1}^{\infty}$. If there exist $\alpha_1, \alpha_2 > 0$ such that for every $N \in \mathbb{N}$, there is an $M = M_N \in \mathbb{N}$ so that

$$\alpha_1 \left(\sum_{i=1}^{N} \|x_i\|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{i=1}^{N} x_i \right\| \leq \alpha_2 \left(\sum_{i=1}^{N} \|x_i\|^p \right)^{\frac{1}{p}} \tag{3.1.1}$$

for all block sequences $(x_i)_{i=1}^{N}$ of the basis with $\text{supp}(x_1) \geq M$, then X is said to be asymptotic-ℓ_p with respect to $(e_i)_{i=1}^{\infty}$.

It is well-known that $\mathscr{T}$ and a number of its arbitrarily distortable variants are asymptotic-ℓ_1 with respect to their canonical bases. What is more, R. Gordon proved in his survey paper [13] that $\mathscr{T}$ is a PL-space and his argument was precisely the same the one used in the ℓ_1 case, but it also made it clear that a global estimate of the form (3.1.1) was likely required for a Banach space to be a PL-space. The nature of the proofs for ℓ_1 and $\mathscr{T}$ led to the 2008 proof by K.M. Naralenkov that every asymptotic-ℓ_1 Banach space is a PL-space, again by essentially the same argument.

Concurrently with the distortability and embedding problems of the 1970s, A. Brunel and L. Sucheston in [3] initiated the study of local asymptotic structure in Banach spaces with their development of spreading models. Spreading models are local in the sense that they correspond to specific normalized basic sequences in a Banach space X in the following asymptotic manner.

Definition 3.1.3. Let $(x_i)_{i=1}^{\infty}$ be a normalized basic sequence in X and let $(v_i)_{i=1}^{\infty}$ be a normalized

1-spreading basis for a Banach space $(V, \|\cdot\|_V)$. If there exist $\varepsilon_N \downarrow 0$ such that

$$\left| \left\| \sum_{k=1}^{N} \lambda_k x_{i_k} \right\| - \left\| \sum_{i=1}^{N} \lambda_i v_i \right\|_V \right| < \varepsilon_N$$

for all $N \leq i_1 < i_2 < \ldots < i_N$ and for all scalars $|\lambda_i| \leq 1$, then $(v_i)_{i=1}^{\infty}$ is said to be a spreading model of X that is generated by $(x_i)_{i=1}^{\infty}$.

The result of A. Brunel and L. Sucheston is that every normalized basic sequence in X has a subsequence that generates a spreading model of X. This was the first of many deep applications of infinitary combinatorics (Ramsey theory) to the geometry of Banach spaces. In the same paper [21] as his asymptotic-ℓ_1 theorem, K.M. Naralenkov gave a proof that every spreading model of a PL-space is equivalent to the canonical ℓ_1 unit vector basis (though he credited the proof to A. Pelczyński and G.D. da Rocha Filho as an unpublished result). PL-spaces therefore necessitate at least some form of local ℓ_1 asymptotic structure. Unfortunately, Schur spaces also have this spreading model property and even they need not be PL-spaces (see [21]) despite their structural proximity to ℓ_1.

Until recently, the asymptotic results of K.M. Naralenkov represented essentially what was known about the asymptotic geometry of PL-spaces. Where these local and global asymptotic results “meet in the middle" to fully characterize PL-spaces is still not clear, however, I was able to generalize both them in my paper [9] that is scheduled to appear later this year in *Real Analysis Exchange*. In what follows, I will introduce some preliminary information about PL-spaces, prove in detail my generalizations of the K.M. Naralenkov results, and then discuss what realistically can be done on this problem in the near future. The language and proofs (where included) of the following sections are taken essentially verbatim from [9].

3.2 Preliminary Information

Let us begin with a familiar definition. All of the proofs of this section are more or less analysis exercises and can be found in [9, 13], for example.

Definition 3.2.1. A finite and strictly increasing sequence of real numbers $P = (p_i)_{i=0}^d$ is said to be a partition of $[0,1]$ if $p_0 = 0$ and if $p_d = 1$.

A partition $P = (p_i)_{i=0}^d$ of $[0,1]$ specifies, for every $i \in \{1,\ldots,d\}$, the non-negative real number $\Delta_P(i) = p_i - p_{i-1} = \mu([p_{i-1}, p_i]) = \mu((p_{i-1}, p_i))$ and the maximum $\pi(P)$ of these numbers is said to be its mesh size. If $\Delta_P(i) = \Delta_P(j) = \Delta_P$ for all $i \neq j$, then P is said to be regular. Any partition of $[0,1]$ whose range contains $\operatorname{ran}(P) = \{p_0,\ldots,p_d\}$ as a subset is said to refine P and to every finite collection of partitions of $[0,1]$, there corresponds a unique coarsest partition of $[0,1]$ called the common refinement that refines all of them simultaneously.

The Darboux integrability of a real-valued function on $[0,1]$ is characterized by the convergence of its upper and lower Darboux sums to the same value (e.g. [26, Theorem 6.6]). These sums can be defined only if it makes sense to discuss infima and suprema, and this is not necessarily the case for subsets of X. If, however, $f : [0,1] \to \mathbb{R}$ is bounded, then

$$\sup_{s\in I} f(s) - \inf_{s\in I} f(s) = \sup_{s,s'\in I} |f(s) - f(s')|$$

for all non-empty and compact sub-intervals $I \subset [0,1]$. This motivates a more nuanced approach to Darboux integrability. For convenience, let $\mathscr{B}([0,1],X) = \left\{ f : [0,1] \to X \;\middle|\; \sup_{s\in[0,1]} \|f(s)\| < \infty \right\}$ be the collection of bounded X-valued functions on $[0,1]$.

Definition 3.2.2. Let $f \in \mathscr{B}([0,1],X)$, $s_0 \in [0,1]$, and $\delta > 0$. The non-negative real number defined by

$$\omega_f[\mathscr{N}_\delta(s_0)] = \sup\left\{ \|f(s) - f(s')\| \;\middle|\; s,s' \in \mathscr{N}_\delta(s_0)] \right\}$$

is said to be the oscillation of f with respect to the sub-interval $\mathscr{N}_\delta(s_0) = [s_0 - \delta, s_0 + \delta] \cap [0,1]$.

This definition now offers a natural generalization of [26, Theorem 6.6].

Definition 3.2.3. Let $f \in \mathscr{B}([0,1],X)$. If, for all $\varepsilon > 0$, there exists a partition $P_\varepsilon = P = (p_i)_{i=0}^d$ of $[0,1]$ such that

$$\sum_{i=1}^d \Delta_P(i)\,\omega_f[\mathscr{N}_{\delta_i}(s_i)] \leq \varepsilon \tag{3.2.1}$$

where $\delta_i = \frac{\Delta_P(i)}{2}$ and $s_i = p_{i-1} + \delta_i$ for each $i \in \{1, \ldots, d\}$, then f is said to be Darboux-integrable. The above definition reduces to [26, Theorem 6.6] if $X = \mathbb{R}$ and is in this case equivalent to boundedness in μ-a.e. continuity. If $f \in \mathscr{B}([0,1], X)$, then:

- $\inf_{\delta>0} \omega_f[\mathscr{N}_\delta(s)] = 0$ if and only if f is continuous at s
- $\Omega_f(\lambda) = \left\{ s \in [0,1] \;\middle|\; \inf_{\delta>0} \omega_f[\mathscr{N}_\delta(s)] < \frac{1}{\lambda} \right\}$ is a relatively open (and therefore μ-measurable) subset of $[0,1]$ for all $\lambda > 0$
- If $H \subset \mathbb{R}$, then $\mu(H) = 0$ if and only if, for all $\varepsilon > 0$, there exist open intervals $U_1, U_2, \ldots$ that both cover H and satisfy $\sum_{j=1}^{\infty} \mu(U_j) < \varepsilon$

permit now a characterization of the Darboux-integrable X-valued functions on $[0,1]$, $\mathscr{D}([0,1], X)$, that mirrors the real-valued situation.

Theorem 3.2.4. *Let $f \in \mathscr{B}([0,1], X)$. Then, $f \in \mathscr{D}([0,1], X)$ if and only if it is μ-a.e. continuous.*

The proof of this theorem is the same as in the usual real-valued situation and it does not require the completeness of X. On the other hand, the assumption that X is a Banach space is essential for defining the actual Darboux integral of a given $f \in \mathscr{D}([0,1], X)$. This can be done by means of the Riemann integral, and note lastly that $\mathscr{D}([0,1], X) \subset \mathscr{B}([0,1], X)$ as a subspace.

The Riemann integrability of a real-valued function on $[0,1]$ is characterized by the convergence of its Riemann sums to the same value (e.g. [17, Definition 11.56]). Recall that the ordered pair (P, T) is said to be a tagged partition (resp. interior tagged partition) of $[0,1]$ if $P = (p_i)_{i=0}^{d}$ is a partition of $[0,1]$ and if $T = (t_i)_{i=1}^{d}$ is such that $t_i \in [p_{i-1}, p_i]$ (resp. $t_i \in (p_{i-1}, p_i)$) for all $i \in \{1, \ldots, d\}$. In the case that $f : [0,1] \to X$, the vector $\sum_{i=1}^{d} \Delta_P(i) f(t_i) = S_f(P, T)$ is said to be the Riemann sum of f with respect to (P, T) and the Riemann integrability of f may be assessed in the following familiar manner.

Definition 3.2.5. A function $f : [0,1] \to X$ is Riemann-integrable if there exists a vector $x_f \in X$ such that for all $\varepsilon > 0$, there is a $\delta = \delta_\varepsilon > 0$ so that

$$\|x_f - S_f(P, T)\| \leq \varepsilon$$

for all tagged partitions (P,T) of $[0,1]$ that satisfy $\pi(P) < \delta$.

The set $\mathscr{R}([0,1],X)$ of Riemann-integrable X-valued functions on $[0,1]$ is quite clearly a subspace of $\mathscr{B}([0,1],X)$ and the vector $x_f \in X$ is also uniquely determined. It follows that $f \mapsto x_f$ is a well-defined linear function and this function is called the Riemann integral of the X-valued functions on $[0,1]$. What is more, Definition 3.2.5 is independent of the distinction between tagged and interior tagged partitions by means of a straightforward argument. The following straightforward lemma proved in [9, 21] asserts this fact.

Lemma 3.2.6. *A function $f : [0,1] \to X$ is Riemann-integrable if and only if there exists an $x_f \in X$ such that for all $\varepsilon > 0$, there is a $\delta = \delta_\varepsilon > 0$ so that*

$$\|x_f - S_f(P,T)\| \leq \varepsilon$$

for all interior tagged partitions (P,T) of $[0,1]$ that satisfy $\pi(P) < \delta$.

Definition 3.2.5 now admits several more equivalent reformulations, where the only non-triviality is the notion of the convex hull of a general subset of X.

Theorem 3.2.7. *Let $f : [0,1] \to X$. Then, the following claims are equivalent.*

(i) The function f is Riemann-integrable.

(ii) There exists a vector $x_f \in X$ such that for all $\varepsilon > 0$, there is a partition P_ε of $[0,1]$ so that $\|x_f - S_f(P,T)\| \leq \varepsilon$ for all tagged partitions (P,T) of $[0,1]$ where P refines P_ε.

(iii) For all $\varepsilon > 0$, there exists a partition P_ε of $[0,1]$ such that $\|S_f(P_1,T_1) - S_f(P_2,T_2)\| \leq \varepsilon$ for all tagged partitions (P_1,T_1) and (P_2,T_2) where P_1 and P_2 refine P_ε.

(iv) For all $\varepsilon > 0$, there exists a partition P_ε of $[0,1]$ such that $\|S_f(P_1,T_1) - S_f(P_2,T_2)\| \leq \varepsilon$ for all tagged partitions (P_1,T_1) and (P_2,T_2) where $P_1 = P_2 = P_\varepsilon$.

The implications $1 \implies 2 \implies 3 \implies 4$ are obvious, and the convex hull argument is only required for $4 \implies 3$. Then, $3 \implies 2$ is simple but requires the completeness of X, and $2 \implies 1$

is straightforward as well. Finally, it can be seen in the proof of Theorem 3.2.7 and Lemma 3.2.6 that the same statements where tagged partitions are replaced by interior tagged partitions are also equivalent to Riemann integrability. The inclusion $\mathscr{D}([0,1],X) \subset \mathscr{R}([0,1],X)$ is now an obvious corollary of Theorem 3.2.7 (iv) and Definition 3.2.3. This begs the question: when is this inclusion strict? In other words, when does X have the property that every $f \in \mathscr{R}([0,1],X)$ is μ-a.e. continuous?

3.3 Asymptotic Geometry of PL-Spaces

To say that $\mathscr{D}([0,1],X) \subset \mathscr{R}([0,1],X)$ holds with equality is to say that X has the Property of Lebesgue. This condition can be equivalently formulated in terms of Theorem 3.2.4 as is noted in the introduction to this chapter and restated below.

Definition 3.3.1. The Banach space X is said to be a PL-space if every $f \in \mathscr{R}([0,1],X)$ is μ-a.e. continuous.

All finite-dimensional normed vector spaces are PL-spaces while infinite-dimensional Banach spaces may or may not have the Property of Lebesgue. The direct proof that a Banach space X is a PL-space ordinarily consists of showing that $f \in \mathscr{B}([0,1],X)$ is not Riemann-integrable if it is discontinuous on a set of positive Lebesgue measure, and this is the strategy that is used by [13, Theorems 26 and 27], [21, Theorem 6], and by [9, Theorem 1.0.2]. It is used once more here by the generalization of these asymptotic results that is proved in the following subsection.

3.3.1 A Global Result

The concept of X being asymptotic-ℓ_p with respect to a basis (Definition 3.1.2) is a global one. However, it is restricted to those Banach spaces that have bases and therefore cannot apply to say, non-separable Banach spaces. This was remedied shortly after the original paper by N. Tomczak-Jaegermann and V. Milman by means of a coordinate-free generalization that can be defined in terms of an asymptotic game of two players. Let S denote the "subspace" player, V denote the “vector" player, and let $\mathrm{cof}(X) = \{Y \subset X \mid Y \text{ a closed subspace and } \dim(X/Y) < \infty\}$. For every $N \in \mathbb{N}$, we may define the game $G(S,V,N)$ as follows:

Turn 1: S chooses $Y_1 \in \mathrm{cof}(X)$ and V chooses any $y_1 \in Y_1$

Turn 2: S chooses $Y_2 \in \mathrm{cof}(X)$ and V chooses any $y_2 \in Y_2$

$\vdots$

Turn N: S chooses $Y_N \in \mathrm{cof}(X)$ and V chooses any $y_N \in Y_N$

and S wins $G(S,V,N)$ if it is possible to force the inequality

$$\alpha_1 \left(\sum_{i=1}^{N} \|y_1\|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{i=1}^{N} y_i \right\| \leq \alpha_2 \left(\sum_{i=1}^{N} \|y_i\|^p \right)^{\frac{1}{p}}$$

for some $\alpha_1, \alpha_2 > 0$ (i.e. $(y_i)_{i=1}^N \sim \ell_P^N$). Now, if the constants $\alpha_1, \alpha_2 > 0$ are uniform for all $N \in \mathbb{N}$, then we can make a definition.

Definition 3.3.2. The Banach space X is said to be coordinate-free (CF) asymptotic-ℓ_p if there exist $\alpha_1, \alpha_2 > 0$ (uniform in N) such that S wins $G(S,V,N)$ for every $N \in \mathbb{N}$.

The definition in terms of an asymptotic game of X being CF-asymptotic-ℓ_p is indispensible because it allows for the direct generalization of Definition 3.1.2 where the chosen subspaces of finite codimension are of the form $\overline{\mathrm{span}\{e_i \mid i \geq M\}}$. Choosing subspaces and vectors alternately allows one to recover (3.1.1) as the winning outcome of $G(S,V,N)$ in an asymptotic-ℓ_p space. It is also important to note that every $Y \in \mathrm{cof}(X)$ is complemented in X, that is,

$$Y \in \mathrm{cof}(X) \implies X = Y \oplus Z \text{ where } Z \cong X/Y \text{ and } \dim(Z) < \infty$$

and there are consequently (bounded linear) projections $\mathscr{P}_Y$ and $\mathscr{P}_Z$ from X onto Y and Z respectively, such that every $x \in X$ can be uniquely represented by $x = \mathscr{P}_Y(x) + \mathscr{P}_Z(x)$. This brings us to the generalization of Naralenkov’s Theorem that I recently proved, and I will include the detailed proof below.

Theorem 3.3.3. *Every CF-asymptotic-ℓ_1 Banach space is a PL-space.*

Proof. Let $f \in \mathscr{B}([0,1],X)$ and suppose that it is discontinuous on a set $H \subset [0,1]$ that has positive Lebesgue measure. It suffices as noted to prove that $f \notin \mathscr{R}([0,1],X)$ and this can be done by showing that there exists a constant $c_f > 0$ so that for every partition P of $[0,1]$,

$$\|S_f(P,T_1) - S_f(P,T_2)\| \geq c_f$$

for some sequences T_1 and T_2. Define, for all $N \in \mathbb{N}$, the subspaces $\{Y_i^N\}_{i=1}^N \subset \operatorname{cof}(X)$ as in Definition 3.3.2 and consider the subsets

$$G_i^N = \{s \in [0,1] \mid (\mathscr{P}_{Z_i^N} \circ f) : [0,1] \to Z_i^N \text{ is discontinuous at } s\}$$

where $X = Y_i^N \oplus Z_i^N$ for all $i \in \{1,\ldots,N\}$. Note that $(\mathscr{P}_{Z_i^N} \circ f) \notin \mathscr{R}([0,1],Z_i^N)$ if $\mu(G_i^N) > 0$ because

$$\sup_{s\in[0,1]} \|(\mathscr{P}_{Z_i^N} \circ f)(s)\| \leq \|\mathscr{P}_{Z_i^N}\|_{X\to X} \sup_{s\in[0,1]} \|f(s)\| < \infty$$

since f is bounded and $\mathscr{P}_{Z_i^N}$ is a bounded linear projection, and because Z_i^N is a PL-space (being finite-dimensional). It follows that $f \notin \mathscr{R}([0,1],X)$ in this situation either, otherwise its composition with $\mathscr{P}_{Z_i^N}$ is Riemann-integrable. Assume then that $\mu(G_i^N) = 0$ for all $N \in \mathbb{N}$ and for all $i \in \{1,\ldots,N\}$. This implies that $\mu(G) = 0$ where $G = \bigcup_{N=1}^\infty \bigcup_{i=1}^N G_i^N$ as it is the countable union of μ-null sets. Next, observe that there exists $n_0 \in \mathbb{N}$ so that

$$H_{n_0} = \left\{ s \in [0,1] \;\middle|\; \inf_{\delta>0} \omega_f[\mathscr{N}_\delta(s)] \geq \frac{1}{n_0} \right\}$$

has positive Lebesgue measure, or else $\mu(H) = 0$ as in Theorem 3.2.4 also as the countable union of μ-null sets. Let $P = (p_k)_{k=0}^d$ be a partition of $[0,1]$ and consider the non-empty set

$$A_{n_0} = \{k \mid \mu((p_{k-1}, p_k) \cap (H_{n_0} \setminus G)) > 0\}$$

whose members are $k_1, \ldots, k_r$ $(r \leq d)$. Note that $(\mathscr{P}_{Z_i^r} \circ f)$ is continuous on $H_{n_0} \setminus G$ by construction and let $\varepsilon > 0$ be given.

Let $s_1 \in (p_{k_1-1}, p_{k_1}) \cap (H_{n_0} \setminus G)$ and, as $\inf_{\delta>0} \omega_f[\mathscr{N}_\delta(s_1)] \geq \frac{1}{n_0}$, it follows that there exist $u_1, v_1 \in \mathscr{N}_{\delta_1}(s_1)$ so that

$$\|z_1\| \geq \frac{1}{2n_0} \text{ and } \|\mathscr{P}_{Z_1^r}(z_1)\| < \varepsilon$$

where $z_1 = f(u_1) - f(v_1)$ and $\delta_1 > 0$ is sufficiently small. Next, let $s_2 \in (p_{k_2-1}, p_{k_2}) \cap (H_{n_0} \setminus G)$ and note that there exist $u_2, v_2 \in \mathscr{N}_{\delta_2}(s_2)$ so that

$$\|z_2\| \geq \frac{1}{2n_0} \text{ and } \|\mathscr{P}_{Z_2^r}(z_2)\| < \frac{\varepsilon}{2}$$

where $z_2 = f(u_2) - f(v_2)$ and $\delta_2 > 0$ is sufficiently small. It follows in particular that for all $i \in \{1, \ldots, r\}$, there exist $u_i, v_i \in \mathscr{N}_{\delta_i}(s_i)$ so that

$$\|z_i\| \geq \frac{1}{2n_0} \text{ and } \|\mathscr{P}_{Z_i^r}(z_i)\| < \frac{\varepsilon}{2^{i-1}}$$

where $z_i = f(u_i) - f(v_i)$ and $\delta_i > 0$ is sufficiently small. Define the sequences $T_1 = (t_{1,k})_{k=1}^d$ and $T_2 = (t_{2,k})_{k=1}^d$ such that $t_{1,k_i} = u_i$ and $t_{2,k_i} = v_i$ for all $i \in \{1, \ldots, r\}$ and such that $t_{1,k} = t_{2,k} \in [p_{k-1}, p_k]$ for all $k \notin A_{n_0}$. It now follows that (P, T_1) and (P, T_2) are tagged partitions of $[0,1]$ such that

$$\begin{aligned}
\|S_f(P,T_1) - S_f(P,T_2)\| &= \left\| \sum_{i=1}^r \Delta_P(k_i) z_i \right\| \\
&= \left\| \sum_{i=1}^r \Delta_P(k_i) [\mathscr{P}_{Y_i^r}(z_i) + \mathscr{P}_{Z_i^r}(z_i)] \right\| \qquad (3.3.1) \\
&\geq \left\| \sum_{i=1}^r \Delta_P(k_i) \mathscr{P}_{Y_i^r}(z_i) \right\| - \left\| \sum_{i=1}^r \Delta_P(k_i) \mathscr{P}_{Z_i^r}(z_i) \right\|
\end{aligned}$$

and (3.3.1) is then bounded below by

$$\left\|\sum_{i=1}^{r}\Delta_P(k_i)\mathscr{P}_{Y_i^r}(z_i)\right\| - \sum_{i=1}^{r}\Delta_P(k_i)\frac{\varepsilon}{2^{i-1}} \geq \left\|\sum_{i=1}^{r}\Delta_P(k_i)\mathscr{P}_{Y_i^r}(z_i)\right\| - 2\varepsilon$$

and this quantity is, in turn, bounded below by

$$\zeta_1\sum_{i=1}^{r}\Delta_P(k_i)\|\mathscr{P}_{Y_i^r}(z_i)\| - 2\varepsilon = \zeta_1\sum_{i=1}^{r}\Delta_P(k_i)\|z_i - \mathscr{P}_{Z_i^r}(z_i)\| - 2\varepsilon \tag{3.3.2}$$

with the CF-asymptotic-ℓ_1 condition. Finally, (3.3.2) is bounded below by

$$\begin{aligned}\zeta_1\sum_{i=1}^{r}\Delta_P(k_i)(\|z_i\| - \|\mathscr{P}_{Z_i^r}(z_i)\|) - 2\varepsilon &\geq \frac{\zeta_1\mu(H_{n_0}\setminus G)}{2n_0} - 2\varepsilon(\zeta_1+1)\\ &= \frac{\zeta_1\mu(H_{n_0}\setminus G)}{4n_0} = c_f > 0\end{aligned}$$

for $\varepsilon = \frac{\zeta_1\mu(H_{n_0}\setminus G)}{8n_0(\zeta_1+1)} > 0$, and this completes the proof of Theorem 3.3.3 □

This generalization of [21, Theorem 6] is noteworthy because there are CF-asymptotic-ℓ_1 Banach spaces that do not have a basis (e.g. $\ell_1(\Gamma)$, Γ uncountable). Note, however, that the inescapability of the asymptotic-ℓ_1 condition persists in the sense that it is necessary to bound from below those arbitrary sequences of the form $y_i \in Y_i$ where $\{Y_i\}_{i=1}^N \subset \mathrm{cof}(X)$ is as in Definition 3.3.2. It is reasonable to think that perhaps this global asymptotic requirement cannot be relaxed as a sufficient condition for the Property of Lebesgue.

3.3.2 A Local Result

The Property of Lebesgue conversely influences the local asymptotic structure of X. An easy result is [10, Theorem 4.0.2] which asserts that certain unconditional spreading models of a PL-space are necessarily equivalent to the canonical ℓ_1 unit vector basis. This result depends fundamentally upon the well-known dichotomy theorem that an unconditional (and non-trivial) spreading model is either equivalent to the canonical ℓ_1 unit vector basis or it is norm-Cesáro summable to zero. The use of this dichotomy theorem can with some effort seen to be superfluous, as is described in [21]. Similar techniques are now used below to replicate the results of [21] for 1-spreading asymptotic models. If, for all $k \in \mathbb{N}$, the sequence $(x_i^k)_{i=1}^\infty \in X^{\mathbb{N}}$ is normalized and L-basic and if, in addition, the diagonal sequence $(x_{i_k}^k)_{k=1}^\infty \in X^{\mathbb{N}}$ is L-basic for all $k \le i_1 < i_2 < \ldots$, then $(x_i^k)_{i=1,k\in\mathbb{N}}^\infty$ is said to be an L-basic array and there is the following analogue of Definition 3.1.3.

Definition 3.3.4. Let $(x_i^k)_{i=1,k\in\mathbb{N}}^\infty$ be an L-basic array in X and let $(v_i)_{i=1}^\infty$ be a normalized basis for a Banach space $(V, \|\cdot\|_V)$. If there exist positive real numbers $\varepsilon_N \downarrow 0$ such that for all

$$\left| \left\| \sum_{k=1}^N \lambda_k x_{i_k}^k \right\| - \left\| \sum_{i=1}^N \lambda_i v_i \right\|_V \right| < \varepsilon_N$$

for all $N \le i_1 < \ldots < i_N$ and for all scalars $(\lambda_i)_{i=1}^N \in [-1,1]^N$, then $(v_i)_{i=1}^\infty$ is said to be an asymptotic model of X generated by $(x_i^k)_{i=1,k\in\mathbb{N}}^\infty$.

The L-basic array $(x_i^k)_{i=1,k\in\mathbb{N}}^\infty$ is said to be good if it generates an asymptotic model $(v_i)_{i=1}^\infty$ and, as in the spreading model case, there is $\lim_{i_1\to\infty} \left\| \sum_{k=1}^N \lambda_k x_{i_k}^k \right\| = \left\| \sum_{i=1}^N \lambda_i v_i \right\|_V$ for all scalar sequences $(\lambda_i)_{i=1}^N$. However, the asymptotic model $(v_i)_{i=1}^\infty$ need not be 1-spreading and, while there is $\lim_{i_1\to\infty} \left\| \sum_{j=1}^N \lambda_{k_j} x_{i_{k_j}}^{k_j} \right\| = \left\| \sum_{i=1}^N \lambda_{k_i} v_{k_i} \right\|_V$ for the column sub-array $(x_i^{k_j})_{i=1,j\in\mathbb{N}}^\infty$, this sub-array need not be good because the integers $N \le i_1 < \ldots < i_N$ can be selected "above the diagonal" so that a bound of the form given in Definition 3.3.4 need not exist. These diagonal sequences also need not even be L-basic anymore. If in contrast $(x_i^{k_j})_{i=1,j\in\mathbb{N}}^\infty$ is good, then it automaticallly generates $(v_{k_i})_{i=1}^\infty$ as an asymptotic model but with respect to a sequence $\varepsilon_N \downarrow 0$ of positive real numbers that is perhaps different from the analogous sequence for the original good array $(x_i^k)_{i=1,k\in\mathbb{N}}^\infty$. These observations lead directly to the stipulation that the good array $(x_i^k)_{i=1,k\in\mathbb{N}}^\infty$ generates an SP-asymptotic model if:

- The asymptotic model $(v_i)_{i=1}^\infty$ is 1-spreading
- Every sub-array of the form $(x_i^{k_j})_{i=1,j\in\mathbb{N}}^\infty$ is good (and consequently generates $(v_{k_i})_{i=1}^\infty$ as an asymptotic model) and is also L-basic

- $\sup\left\{\varepsilon_1 = \varepsilon_1\left((x_i^{k_j})_{i=1,j\in\mathbb{N}}^\infty\right) \,\middle|\, k_1 < k_2 < \ldots\right\} < \infty$

which is to say that finite diagonal sequences of sufficiently high index are close to the corresponding finite linear combinations of a fixed spreading sequence. As a concrete example, if $(x_i^k)_{i=1}^\infty = (x_i^j)_{i=1}^\infty$ for all $k \neq j$, then $(v_i)_{i=1}^\infty$ is an SP-asymptotic model and is also the spreading model generated by this single column sequence. Now, let $\mathrm{dy}(\mathbb{Q})$ be as in [21] the set of dyadic rational numbers, that is, real numbers of the form $\frac{a}{2^b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, and note that the set

$$\Lambda = \mathrm{dy}(\mathbb{Q}) \cap (0,1) = \left\{ r_{kj} = \frac{2j-1}{2^k} \,\middle|\, k \in \mathbb{N} \text{ and } j \in \{1,\ldots,2^{k-1}\} \right\}$$

is enumerable by $r_{kj} = r_{2^{k-1}+j-1} = r_N$ specifically. Consider now a technical lemma that relates Riemann integrability to SP-asymptotic models.

Lemma 3.3.5. *Let $(v_i)_{i=1}^\infty$ be an SP-asymptotic model of X generated by the good L-basic array $(x_i^k)_{i=1,k\in\mathbb{N}}^\infty$. Then,*

$$\limsup_{n\to\infty} \frac{\|v_1 - v_2 + \ldots - v_{2^n}\|_V}{2^n} = \delta_0 > 0 \tag{3.3.3}$$

if $f : [0,1] \to X$ defined by $f(r_n) = x_{4n}^{n^2} - x_{4n+1}^{n^2+1}$ and $f(s) = 0$ otherwise is not Riemann-integrable.

Proof. Let $\varepsilon > 0$ be given and choose $N = N_\varepsilon \in \mathbb{N}$ such that $\frac{CL}{2^N} < \varepsilon$ and such that, for a contradiction, $\frac{\|v_1 - v_2 + \ldots - v_{2^N}\|_V}{2^N} < \varepsilon$. Next, let $(p_k)_{i=0}^{2^N}$ be a regular partition of $[0,1]$ (that is, $\Delta_P(k) = \Delta_P = \frac{1}{2^N}$) and let $(t_k)_{k=1}^{2^N}$ be such that (P,T) is an *interior* tagged partition of $[0,1]$. Then,

$$\|S_f(P,T)\| = \frac{1}{2^N}\left\|\sum_{k\in A} f(t_k)\right\| = \frac{1}{2^N}\left\|\sum_{k\in A} x_{4n_k}^{n_k^2} - x_{4n_k+1}^{n_k^2+1}\right\| \leq \frac{L}{2^N}\left\|\sum_{k=1}^{2^N} x_{4n_k}^{n_k^2} - x_{4n_k+1}^{n_k^2+1}\right\|$$

where $A = \{k \mid t_k \in \Lambda \cap (p_{k-1}, p_k)\}$ and where $r_{n_1} = r_{lj} \in \left(\frac{k-1}{2^N}, \frac{k}{2^N}\right)$ is such that $n_1 = 2^{l-1} + j - 1 \geq 2^{N-1}$ because $\mathrm{ran}(P) \cap (0,1) = \{r_{lj} \in \Lambda \mid l \leq N\}$ so in particular, $4n_1 \geq 2^{N+1}$. That is to say, the first 2^N enumerants of Λ are precisely the partition points of P so, by restricting our attention to an interior tagged partition, the tags are guaranteed to be "farther than 2^{N+1}" down their columns. What is more, $n_k^2 + 1 < n_{k+1}^2$ necessarily so that the members of the above sum form an alternating signs diagonal sequence of 2^{N+1} terms in some sub-array of the original good array. The hypothesis that $(v_i)_{i=1}^{\infty}$ is an SP-asymptotic model then implies that this norm quantity admits the upper bound:

$$\frac{L}{2^N}\left(\varepsilon_{2^{N+1}} + \left\|\sum_{i=1}^{2^N} v_{n_i^2} - v_{n_i^2+1}\right\|_V\right) \leq \frac{CL}{2^N} + \frac{2\|v_1 - v_2 + \ldots - v_{2^N}\|_V}{2^N} < \varepsilon + 2\varepsilon = 3\varepsilon$$

where we have used the fact that $(v_i)_{i=1}^{\infty}$ is 1-spreading as well. It follows that f is Riemann-integrable with integral zero so the proof is complete. □

The above proof follows closely the analogous proof in [21] for spreading models, and specifically we needed to impose the SP requirement on $(v_i)_{i=1}^{\infty}$ so that it would not only be 1-spreading, but also so that an estimate of the form found in Definition 3.3.4 can be used - interior tags ensure that the diagonal sequence in the Riemann sum starts far enough down the array, but there is a priori no guarantee that this sequence is close to the corresponding linear combination of v_i's or is even L-basic. Finally, it is well-known (see [2, 9, 21]) that $\limsup_{n\to\infty} \frac{\|v_1 - v_2 + \ldots - v_{2^n}\|_V}{2^n} = \delta_0 > 0$ is a sufficient condition for a 1-spreading sequence $(v_i)_{i=1}^{\infty}$ to be equivalent to the canonical ℓ_1 unit vector basis, so the argument that every SP-asymptotic model of a PL-space is equivalent to the canonical ℓ_1 unit vector basis is complete, omitting further details.

3.4 Future Research

The problem of deriving a full characterization of the Property of Lebesgue in Banach spaces is still open, and future research on it will likely center around upgrading local results to the global condition of being CF-asymptotic-ℓ_1. A recent result due to C. Krause in [19] already makes this possible in Banach spaces that have well-behaved bases with respect to their blocking structure. Namely, it is a corollary of his results that: if X is a Banach space with a basis $(e_i)_{i=1}^{\infty}$ such that every normalized block basis of $(e_i)_{i=1}^{\infty}$ generates a spreading model of X, then X is a PL-space if and only if it is asymptotic-ℓ_1 with respect to $(e_i)_{i=1}^{\infty}$. A proof of this result can be found in [9], along with examples showing that this result applies not only to ℓ_p $(1 \leq p < \infty)$ and c_0, but also to the sequence Lorentz space $d(w,1)$.

An alternative approach was suggested by B. Sari and Th. Schlumprecht at the end of my recent Banach space webinar talk, where they noted that methods similar to those used in Lemma 3.3.5 can likely be used to prove that reflexive and separable PL-spaces satisfy so-called asymptotic-(ℓ_1,ℓ_1) estimates on normalized weakly null trees (see [23]) and consequently embed as a subspace into a Banach space that is asymptotic-ℓ_1 with respect to a basis. This is probably the most immediate and feasible direction for future research, but unfortunately, the most it can tell us is whether or not a PL-space embed into an asymptotic-ℓ_1 space and it also only applies to PL-spaces such as $\mathscr{T}$ that are known to be both reflexive and (because it has a basis) separable. Of course, it is well-known due to P. Enflo that not every separable Banach space has a basis, and it is not clear how the Property of Lebesgue interacts with such spaces.

I believe that a full asymptotic characterization of PL-spaces will require the development sufficient local conditions (e.g. spreading models, asymptotic models, normalized weakly null trees, etc...) for X itself to be CF-asymptotic-ℓ_1. This is an extremely difficult problem and the subject of recent research due to Argyros et. al. in [1] where it was shown that even if every normalized weakly null good array in X generates an asymptotic model $(v_i)_{i=1}^{\infty} \sim \ell_1$, it is not guaranteed that X itself will be CF-asymptotic-ℓ_1. In fact, they constructed a reflexive space X with a basis such that the closed linear span of *every* infinite subsequence of the basis is not CF-asymptotic-ℓ_1, while X itself has the property that every normalized weakly null good array generates an ℓ_1 asymptotic model. This is not to say that all hope for upgrading local asymptotic results to global ones is lost. In particular, D. Freeman et. al. proved in [8] that if X is separable, contains no isomorphic copy of ℓ_1, and has the property that every normalized weakly null good array generates an asymptotic model $(v_i)_{i=1}^{\infty} \sim c_0$, then X itself must be CF-asymptotic-ℓ_∞ (that is, the estimate of Definition 3.3.4 holds with the usual c_0 norm rather than $\|\cdot\|_p$). The solution (if it exists) to the problem of finding an asymptotic characterization of the Property of Lebesgue will no doubt be uncovered by means of future work done on the subtle and varied local asymptotic structure of Banach spaces.

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