Finding Maximal Cohen-Macaulay and Reflexive Modules

Prashanth Sridhar

M.A. Mathematics, University of Kansas, 2018

Submitted to the graduate degree program in Department of Mathematics and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Committee members

Prof. Daniel Katz, Chairperson

Prof. Hailong Dao

Prof. Daniel Hernandez

Prof. Emily Witt

Prof. Marc Greenberg

Date defended: June 30, 2021

The Dissertation Committee for Prashanth Sridhar certifies
that this is the approved version of the following dissertation :

Finding Maximal Cohen-Macaulay and Reflexive Modules

Prof. Daniel Katz, Chairperson

Date approved: June 30, 2021

Abstract

In this work we study two classical objects in algebra - maximal Cohen-Macaulay and reflexive modules. We show the existence of a small Cohen-Macaulay module or algebra for a new class of rings in mixed characteristic. In particular, we show the existence of a birational small Cohen-Macaulay module over general biradical extensions of an unramified regular local ring of mixed characteristic and then use it to show the existence of a small Cohen-Macaulay module (algebra) under certain circumstances for general radical towers. This builds towards understanding generically Abelian extensions of an unramified regular local ring in mixed characteristic vis-à-vis Roberts (1980).

We then study the class of reflexive modules over curve singularities through the lens of I-Ulrich modules and provide applications to finite type results and strongly reflexive extensions. This is a contribution towards understanding reflexivity in the one dimensional non-Gorenstein case - the one-dimensional case is key to understanding reflexivity in higher dimensions over "nice" rings.

Acknowledgements

I extend my sincere gratitude to my PhD advisor Prof. Daniel Katz for his support, encouragement and guidance through the course of this work. I thank him for suggesting problems pertaining to maximal Cohen-Macaulay modules. I greatly appreciate the numerous discussions over the years that provided valuable insights towards this work and the talks given based on it.

Many thanks to Prof. Hailong Dao for giving me the opportunity to work with him under the Kansas extension program. The work on reflexive modules included in this thesis was an outcome of this and I thank him for introducing me to this topic, for discussions on related mathematical topics and for advice on talks. It was a good experience working with Sarasij Maitra on this project and I thank him for the collaboration.

I would like to thank Prof. Daniel Hernandez for his support during the writing of this thesis, on matters mathematical and practical. Thanks to Prof. Emily Witt and Prof. Marc Greenberg for agreeing to be on my oral comprehensive and final defense committees. Prof. Satya Mandal introduced me to DG categories and I am grateful for the opportunity to collaborate on an exposition in the area.

Much of the content in chapters 3 and 4 is borrowed from Sridhar (2021a) and Sridhar (2021b). I would like to thank Steven Dale Cutkosky, Srikanth Iyengar and the two anonymous referees for their valuable input towards improving the manuscripts and hence the thesis. Many thanks to Linquan Ma, Rebecca R.G. and V. Srinivas for helpful correspondences and suggesting future directions of research in relation to the work in this thesis.

I heartfully thank Prof. Balwant Singh and Prof. C.S. Rajan. From both I learnt a lot and my liking for mathematics is in good measure owed to them.

My PhD would not have been feasible without the opportunity given by the University of Kansas to teach. I also thank Kate Pleskac, Gloria Prothe, Lori Springs and Kerrie Brecheisen for making life easier at the mathematics department at the University of Kansas.

I am indebted to my parents, sister and wife - Malliga Sridhar, Sridhar Sundararaman, Vidhya Sridhar and Santwana Dubey for their encouragement and support at all times - in a way this would not be possible without them. Finally, I would like to dedicate this work to my four loving grandparents and uncle, all of whom passed away during the formation and writing of this thesis.

Contents

List of Figures

Chapter 1

Introduction

Small Cohen-Macaulay (CM) modules (see 2.1.1.2) and Reflexive modules (see section 2.2) are classical objects that have been studied extensively from algebraic and geometric viewpoints. However, some fundamental questions remain unanswered. Even the former's existence over a "nice" ring that is itself not Cohen-Macaulay is largely unknown! As for the latter, not much is known even in the case of one dimensional non Gorenstein rings. We are interested in themes of the following nature regarding these objects: their existence, how common they are and whether their distribution tells us something about the ring itself.

There is some interplay between these two classes of objects that we shall exploit. But in general, they are quite different and the passage from one to the other is only possible via some esoteric homological criterion that is in practice hard to compute. There are some nice implications: over Cohen-Macaulay rings of dimension at most two, reflexive modules are small CM and on the other hand, independent of dimension, over normal or Gorenstein rings, small CM modules are reflexive. However, it is seen relatively easily that these two classes or not the same. For example, over a one dimensional Cohen-Macaulay local ring, the canonical module is reflexive if and only if the ring is Gorenstein and going the other way, trivially any non-Cohen Macaulay ring is reflexive over itself.

1.1 Small Cohen-Macaulay modules

The Cohen-Macaulay property for a Noetherian ring requires the geometric notion of 'dimension' to agree with the homological notion of 'depth'. One may think of these as varieties with the property that the finite intersection of "general" hypersurfaces is unmixed or in other words, has no embedded components. These rings enjoy many nice properties stemming from the homological side. Vanishing of all intermediate local cohomology modules and a good duality theory make these rings convenient to work over. However, it is not all bad news for a ring that is not blessed with this property - it could admit a finite module with all these desirable properties, a small Cohen-Macaulay (CM) module. The existence of this object has numerous advantages - for example, the multiplicity of a small CM module is relatively easy to compute and this in turn aids in computing the multiplicity of the ring. But the most compelling consequence of its existence over a complete local domain is the fact that it implies an important statement - the positivity of Serre's intersection multiplicity conjecture. This was proved by Hochster in the 1970's. However, the problem is that this object has remained elusive - it's existence over complete local domains of dimension at least three is known only in a handful of cases and for the most part has been very difficult to construct. Hochster introduced a weaker version of a small CM module, called a big Cohen-Macaulay module, which does not imply the positivity of Serre intersection multiplicity conjecture, but does imply the other homological conjectures, see 2.1. The existence of big Cohen Macaulay modules (algebras) is now known, see Hochster & Huneke (1995) and Andre (2016).

The question of existence of small CM modules over complete local domains reduces to the integral closure of a complete regular local ring in a finite normal extension of its fraction field. So one may look at the problem in a systematic fashion indexed by Galois groups. It is then natural to first look at the case of “nice" Galois groups. Along these lines, Roberts showed in Roberts (1980) that if the Galois group is Abelian and the characteristic of the residue field of the base regular local ring does not divide the order of the group, the integral closure is Cohen-Macaulay. In particular, this applies to the equi-characteristic zero case. In the mixed characteristic p scenario, the failure of this conclusion was recorded in Koh (1986). We shall see more examples of this phenomenon in this work. In Roberts (1980), the hypothesis that the characteristic of the (algebraically closed) residue field k does not divide the order of the Galois group G critically ensures the group algebra $k[G]$ is a product of fields. There is no direct analog of the argument when $char(k)$ divides the order of G. In fact, when this hypothesis is removed, we are in an entirely new world, drawing parallels with the modular case in representation theory.

Towards understanding the obstructions in the modular case in Roberts's theorem, Katz showed in Katz (1999) that the integral closure of an unramified regular local ring of mixed characteristic $p > 0$ in an extension of its fraction field by a p-th root admits a birational small CM module. He also showed that the integral closure is not automatically Cohen-Macaulay in this case. In Katz (2021), under certain circumstances the existence of a birational small CM module in extensions obtained by adjoining the p^n-th root of a single element is shown.

Aside from these results, not much progress has been made towards the existence of small CM modules from the view point of cases indexed by the Galois group or the modular case in Roberts's theorem. We continue this endeavor by trying to understand the obstructions in the modular case of Roberts's theorem with a view towards the existence of small CM modules. As evidenced by the work in Koh (1986) and Katz (1999), radical extensions in mixed characteristic $p > 0$ obtained by adjoining n-th roots of elements of the base regular local ring with the property that p divides n are prime examples of the failure of Roberts's theorem. Moreover, the importance of radical extensions stem from Kummer theory, which says that Abelian extensions are repeated radical extensions under the presence of suitable roots of unity. We would like to know:

Question 1.1.0.1. *Does the integral closure of a regular local ring of mixed characteristic $p > 0$ in a finite Abelian extension of its fraction field admit a small CM module?*

We approach this question by studying repeated radical extensions of an unramified regular local ring, say S, of mixed characteristic $p > 0$. To make this approach one can assume that the elements whose roots we adjoin are square free, since any given multi-radical extension can be embedded in a sufficiently large square free tower, while preserving finiteness, see Huneke & Katz (2019). In the same vein, one could also impose suitable generality conditions on the elements. To this end, we can reduce to the case where the elements whose roots we adjoin lie in S^p when S is complete with perfect residue field and S^p is the subring of S obtained by lifting the Frobenius map on S/pS to S, see 3.2.1. On the other hand, the complexity of these towers increases very fast.

Towards gaining a handle, we investigate a specific situation. We consider biradical extensions obtained by adjoining p-th roots of sufficiently general square free elements say $f,g \in S$, see chapters 3 and 4. Roughly speaking, one may think of this as the case where the Galois group is $\mathbb{Z}_p \times \mathbb{Z}_p$ generically.

We now outline our principal findings in the biradical case. Fix $f,g \in S^p$ square free, non-units that form a regular sequence in S or units that are not p-th powers in S. Let $\omega^p = f$ and $\mu^p = g$. If f,g are units, assume further that $[L(\omega,\mu) : L] = p^2$. Let R be the integral closure of S in $K := L(\omega,\mu)$. Given integers $n,k \geq 1$, let $S^{p^k \wedge p^n} \subset S$ be the multiplicative subset of S consisting of elements expressible in the form $x^{p^k} + y \cdot p^n$ for some $x,y \in S$. We address the mixed characteristic two case separately for two reasons: the results are sharper in this case since such extensions are automatically Abelian and the splitting pattern of primes lying over 2 are different. When $p \geq 3$, the presence of a p-th root of unity in S necessarily ramifies p, so we do not quite have the same leg room. The first result in the biradical case is

Theorem (4.1.0.14). *Let $(S,\mathfrak{m})$ be an unramified regular local ring of mixed characteristic $p \geq 3$. Then*

1. *R is Cohen-Macaulay if*

   (a) *At least one of $S[\omega], S[\mu]$ is not integrally closed.*

   (b) *$S[\omega], S[\mu]$ are integrally closed and $fg^i \notin S^{p \wedge p^2}$ for all $1 \leq i \leq p-1$.*

2. *Let $S[\omega], S[\mu]$ be integrally closed such that $fg \in S^{p \wedge p^2}$. Then R is Cohen-Macaulay if and only if $Q := (p,f,g) \subset S$ is a two generated ideal or all of S. Moreover, $p.d._S(R) \leq 1$ and $\nu_S(R) \leq p^2 + 1$.*

3. *If $Q := (p,f,g) \subseteq S$ has grade three, R admits a birational maximal Cohen-Macaulay module.*

Surprisingly, the non Cohen-Macaulay cases occur only when the hypersurfaces $S[\omega]$ and $S[\mu]$ are both integrally closed. We will see that R is not too far from being Cohen-Macaulay, in the sense that it can be generated by p^2+1 elements over the base ring S and $p.d._S(R) \leq 1$. However, it is not as close as it appears, since if $dim(S) \geq 3$ it could be that R does not even satisfy Serre's condition (S_3). The condition on Q in item (3) can be viewed as a further generality condition on the chosen elements since S/p is a UFD.

In the mixed characteristic two case, we have a sharper result:

Theorem (4.1.1.8). *Let S be an unramified regular local ring of mixed characteristic two.*

1. *R is Cohen-Macaulay if and only if one of the following happens*
   (a) *At least one of* $S[\omega], S[\mu]$ *is not integrally closed.*
   (b) $S[\omega], S[\mu]$ *are both integrally closed and* $fg \notin S^{2\wedge 4}$.
   (c) $S[\omega], S[\mu]$ *are both integrally closed,* $fg \in S^{2\wedge 4}$ *and* $\mathscr{I} := (2, f, g) \subset S$ *is a two generated ideal or all of R.*
2. *If R is not Cohen-Macaulay, R admits a birational small CM module.*

Inspired by Katz (1999), the approach for the above theorems involves:

- Studying the conductor J of the integral closure R to a suitable complete intersection ring A.
- Since A is Gorenstein and J is unmixed, R is CM if and only if A/J is CM.
- To show that R admits a birational small CM module we choose a suitable ideal $I \subseteq A$ such that I^* is a J^*-module and $depth_S(I^*) = d$.

Along the way, we obtain examples of the existence of small CM algebras - note that if T is a non Cohen-Macaulay normal domain containing the rationals, then T cannot admit a small Cohen-Macaulay algebra B. This is because there exists a retraction from $B \to T$ using the trace map corresponding to the fraction fields which can be used to show that T is CM if B is. However, an example of the existence of a small CM algebra in mixed characteristic does not seem to be well known and we will provide definitive examples in this work (4.1.0.3). On the other hand, small Cohen-Macaulay algebras do not always exist in mixed characteristic either - any mixed characteristic local domain that is not Cohen-Macaulay after inverting p would be an example. Moreover, Bhatt's examples of non existence of small CM algebras in positive characteristic in Bhatt (2012) "deform" to mixed characteristic.

Next, in joint work with Daniel Katz, we use the intuition developed from the biradical case to general radical towers of order p and obtain patterns where the integral closure is Cohen-Macaulay. The first target here is to address the case where the p-torsion of the Abelian Galois group is annihilated by p.

Theorem (3.2.2.3). *Let S be an unramified regular local ring of mixed characteristic $p > 0$ with fraction field L. Let $f_1, \ldots, f_n \in S^{p \wedge p^2}$, square free and mutually coprime. Let $\omega_i^{n_i} = f_i$ such that $p \mid n_i$ and $p^2 \nmid n_i$ for each i. Then the integral closure of S in $L(\omega_1, \ldots, \omega_n)$ is Cohen Macaulay.*

We then use the above result to show the existence of a small CM algebra in radical towers of order p under certain circumstances. In trying to address the Abelian case via Kummer theory, one needs to adjoin a primitive p-th root of unity. We show

Theorem (3.2.3.1). *Let $T := S[\varepsilon]$ be the ramified regular local ring obtained by adjoining a primitive p-th root of unity ε to an unramified regular local subring S of mixed characteristic $p \geq 3$. Let L be the fraction field of T and ω a p-th root of a square free element of T. Then the integral closure of T in $L(\omega)$ is Cohen-Macaulay.*

We now outline the structure of chapters 2 - 4. In Chapter 2, we present the back drop of the work in the subsequent chapters. We set up some preliminary notions and look at past results in the literature that motivates us.

In Chapter 3 we begin a study of general radical towers of order p of an unramified regular local ring of mixed characteristic p with a view towards the existence of small CM modules. This is in some sense viewing the modular case of Roberts (1980) via the lens of Kummer theory. In this chapter we focus on the cases where the integral closure is automatically Cohen-Macaulay with the objective of understanding the obstructions to Roberts (1980) in the modular case and to construct small CM algebras. We first look at the biradical case in detail and then look at certain cases in general radical towers of order p.

In Chapter 4 we show that the integral closure in general square free radical towers of order p need not be Cohen-Macaulay and attempt to justify that it is quite common. We then construct (birational) small CM modules or algebras over these rings. Like chapter 3, we first study the biradical case. In trying to find small CM modules, one needs to understand the failure of Cohen-Macaulayness of the ring itself. To this end, we characterize the Cohen-Macaulayness of the integral closure in the biradical case. In the next subsection, we consider general radical towers of order p and construct small CM algebras in certain cases when the base regular local ring is complete with perfect residue field. Future directions of research and open questions are included in chapter 6.

1.2 Reflexive Modules

The study of reflexive modules presented in chapter 5 is joint work with Hailong Dao and Sarasij Maitra and is based of the paper Dao et al. (2021).

Notions of reflexivity have been studied throughout various branches of mathematics. Over a commutative ring R, a module M is called reflexive if the natural map $M \to M^{**}$ is an isomorphism, where M^* denotes $\mathrm{Hom}_R(M,R)$. If M is finitely generated, this definition is equivalent to requiring that $M \simeq M^{**}$. When R is a field, any finite dimensional vector space is reflexive, a fundamental fact in linear algebra. Over general rings, these modules were studied in the works of Dieudonné et al. (1958), Morita (1958) and Bass (1960), where the name “reflexive" first appeared, before being treated formally in Bourbaki (1965). They are now classical and ubiquitous objects in modern commutative algebra and algebraic geometry. We seek to understand the following basic question: how common are they?

Note that over an (S_2)-ring, being reflexive is equivalent to being (S_2) and reflexive in codimension one (see (Bruns & Herzog, 1998, Proposition 1.4.1)), hence understanding the one dimensional local case is key.

Assume now that $(R, \mathfrak{m})$ is a one dimensional Noetherian local Cohen-Macaulay ring. The primary examples are curve singularities or localized coordinate rings of points in projective space. It turns out that the answers to our basic question can be quite subtle. If R is Gorenstein, then any maximal Cohen-Macaulay module is reflexive (so in our dimension one situation, any torsionless module is reflexive). However, reflexive modules or even ideals are poorly understood when R is not Gorenstein. Some general facts are known, for instance if R is reduced, any second syzygy or R-dual module is reflexive. However, we found very few concrete examples in the literature: only the maximal ideal and the conductor of the integral closure of R. How many can there be? Can we classify them? When is an ideal of small colength reflexive? For instance, if R is $\mathbb{C}[[t^3, t^4, t^5]]$ then any indecomposable reflexive module is either isomorphic to R or the maximal ideal, but the reason is far from clear.

Our results will give answers to the above questions in many cases. A key point in our investigation is a systematic application of the concept of I-Ulrich modules, where I is any ideal of height one in R. A module M is called I-Ulrich if $e_I(M) = \ell(M/IM)$ where $e_I(M)$ denotes the Hilbert Samuel multiplicity of M with respect to I and $\ell(\cdot)$ denotes length. This is a straight generalization of the notion of Ulrich modules, which is just the case $I = \mathfrak{m}$ (Ulrich (1984) and Brennan et al. (1987)). Note that I is I-Ulrich simply says that I is stable, a concept heavily used in the work on Arf rings in Lipman (1971).

Of course, the study of $\mathfrak{m}$-Ulrich modules and certain variants has been an active area of research for quite some time now. The papers closest to the spirit of our work are perhaps Goto et al. (2014), Goto et al. (2016), Herzog et al. (1991), Kobayashi & Takahashi (2019b), Nakajima & Yoshida (2017) among many other sources.

We give various characterizations of I-Ulrichness (Theorem 5.1.0.6). We show the closedness of the subcategory of I-Ulrich modules under various operations, prompting the existence of a lattice like structure for I-Ulrich ideals, which can be referred to as an *Ulrich* lattice. We establish tests for I-Ulrichness using blow-up algebras and the *core* of I (recall that the core of an ideal is the intersection of all minimal reductions).

We will show that ω_R-Ulrich modules are reflexive, and they form a category critical to the abundance of reflexive modules (here ω_R is a canonical ideal of R). For instance, any maximal Cohen-Macaulay module over an ω_R-Ulrich finite extension of R is reflexive. Furthermore, a reflexive birational extension of R is Gorenstein if and only if its conductor I is I-Ulrich and ω_R-Ulrich. We make frequent use of birational extensions of R and trace ideals. This is heavily inspired by some recent interesting work in Kobayashi (2017), Goto et al. (2020), Faber (2019) and Herzog et al. (2019).

Under mild conditions, we are able to completely characterize extensions S of R that are "strongly reflexive" in the following sense: any maximal Cohen-Macaulay S-module is reflexive over R. Interestingly, in the birational case, this classification involves the core of the canonical ideal of R.

Theorem (5.2.0.2 and 5.2.0.5). *Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a module finite R-algebra such that S is a maximal Cohen-Macaulay module over R. The following are equivalent (for the last two, assume that S is a birational extension and the residue field of R is infinite):*

1. *Any maximal Cohen-Macaulay S-module is R-reflexive.*
2. *ω_S is R-reflexive.*
3. $\mathrm{Hom}_R(S,R) \cong \mathrm{Hom}_R(S,\omega_R)$.
4. *$\omega_R S \cong S$.*
5. *S is ω_R-Ulrich as an R-module.*
6. *S is R-reflexive and the conductor of S to R lies inside $(x) : \omega_R$ for some principal reduction x of ω_R.*
7. *S is R-reflexive and the conductor of S to R lies inside* $\mathrm{core}(\omega_R) :_R \omega_R$.

The theorem above extends (Kobayashi, 2017, Theorem 2.14). Also, for S satisfying one of the conditions of Section 1.2, any contracted ideal $IS \cap R$ is reflexive (Proposition 2.2.2.12). Such a statement generalizes a result by Corso-Huneke-Katz-Vasconcelos that if R is a domain and the integral closure $\overline{R}$ is finite over R, then any integrally closed ideal is reflexive (Corso et al., 2005, Proposition 2.14).

We then study when certain subcategories or subsets of $\mathrm{CM}(R)$ are finite or finite up to isomorphism. One main result roughly says that if the conductor of R has small colength, then there are only finitely many reflexive ideals that contain a regular element, up to isomorphism.

Theorem (5.3.3.4). *Let R be a one-dimensional Cohen-Macaulay local ring with infinite residue field and conductor ideal $\mathfrak{c}$. Assume that either:*

1. $\ell(R/\mathfrak{c}) \leq 3$*, or*

2. $\ell(R/\mathfrak{c}) = 4$ *and R has minimal multiplicity.*

Then the category of regular reflexive ideals of R is of finite type.

We also characterize rings with up to three trace ideals (Proposition 5.3.1.3). We observe that if $S = \mathrm{End}_R(\mathfrak{m})$ has finite representation type, then R has only finitely many indecomposable reflexive modules up to isomorphism (Proposition 5.3.4.1). In particular, seminormal singularities have "finite reflexive type" (Corollary 5.3.4.3). We give some further applications on almost Gorenstein rings. We show that in such a ring, all powers of trace ideals are reflexive (Proposition 5.4.0.3). We also characterize reflexive birational extensions of R which are Gorenstein:

Theorem (5.4.0.10). *Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a finite birational extension of R which is reflexive as an R-module. Let $I = \mathfrak{c}_R(S)$ be the conductor of S in R. The following are equivalent:*

1. *S is Gorenstein.*

2. *I is I-Ulrich and ω_R-Ulrich. That is $I \cong I^2 \cong I\omega_R$.*

We conclude chapter 5 with a number of examples. Questions and future directions of research are included in Chapter 6.

Chapter 2

Background

In this chapter we present the back drop of the work in the subsequent chapters. We look at past results in the literature that motivates us, set up some preliminary notions and prove some preparatory statements.

2.1 Small Cohen-Macaulay modules

2.1.1 The small CM module conjecture

Let $(R,\mathfrak{m})$ denote a Noetherian local ring throughout this subsection. R is said to be **Cohen-Macaulay** if some (equivalently every) system of parameters of R is a regular sequence on R. In other words, R is Cohen-Macaulay if $depth(R) = dim(R)$. In some sense, it requires the geometric notion of 'dimension' to agree with the homological notion of 'depth'. An arbitrary Noetherian ring is Cohen-Macaulay if its localization at every maximal ideal is such. One may think of these as varieties with the property that the finite intersection of "general" hypersurfaces is unmixed or in other words, has no embedded components.

Theorem 2.1.1.1 (Bruns & Herzog (1998), 2.1.6)**.** *A Noetherian ring A is Cohen-Macaulay if and only if every ideal I generated by $ht(I)$ elements is unmixed.*

These rings enjoy many nice properties stemming from the homological side. Vanishing of all intermediate local cohomology modules and a good duality theory make these rings good to work over.

The ring R could also admit a module with these nice properties:

Definition 2.1.1.2. An R-module M is a **small Cohen-Macaulay (CM) module** or a **maximal Cohen-Macaulay module** if it is finitely generated, $\mathfrak{m}M \neq M$ and every (equivalently some) system of parameters of R is a regular sequence on M. In other words, it is a finitely generated R-module such that $depth(M) = dim(M) = dim(R)$. Alternately, it is a non-zero finite module such that $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq d$, where $d = dim(R)$.

Note that a non Cohen-Macaulay local ring R can admit a maximal Cohen-Macaulay module. In this case it would be an object with "better properties" than the ring itself. A simple example of this would be if we take $R := \dfrac{k[[x,y]]}{(x^2,xy)}$ and $M := R/(x) = k[[y]]$ for some field k and indeterminates x,y over k.

Over Cohen-Macaulay rings, small CM modules are easier to find. For example, high syzygy modules are always small CM. If R were regular, small CM modules are the same as free modules. In fact, there is a rich classification theory of small CM modules over Cohen-Macaulay rings in small dimension, for references see Yoshino (1990) and Leuschke & Wiegand (2012a).

The existence of small CM modules for local rings can yield information about the structure of the ring and reveal properties of algebras over it. For example, the multiplicity of a small CM module is relatively easy to compute and this in turn aids in computing the multiplicity of the ring. But the most important consequence of its existence arises from figure 2.1 due to Hochster from the 1970's and 80's, indicating the implications amongst the homological conjectures.

The conjecture on the top right in figure 2.1 is

Conjecture 2.1.1.3 (**Small CM module** - Hochster)**.** Every complete local domain admits a small Cohen-Macaulay module.

As seen in figure 2.1, conjecture 2.1.1.3 implies nearly all of the homological conjectures. Most importantly, it implies Serre's intersection multiplicity conjecture: Assume $(S,\mathfrak{m})$ is a regular local ring of dimension n and let M,N be finitely generated non-zero modules over it such that

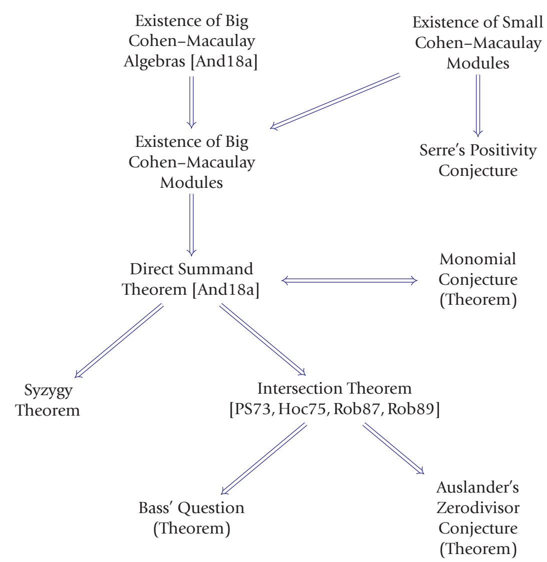


Figure 2.1: Hochster's diagram of the homological conjectures and their solutions, Ma & Schwede (2019).

$l(M \otimes_S N) < \infty$, where $l(\_)$ denotes length. Serre's intersection multiplicity is defined as

$$\chi_S(M,N) := \sum_{i=0}^{n} (-1)^i \ell(Tor_i^S(M,N)) \tag{2.1.1.3.1}$$

The primary example is when M and N are quotients of S by prime ideals P and Q such that $P+Q$ is $\mathfrak{m}$-primary. Geometrically, this is motivated by studying isolated points of intersection of two varieties. Consider the following assertions:

1. $dim(M) + dim(N) \leq n$.
2. $dim(M) + dim(N) < n \implies \chi_S(M,N) = 0$.
3. $dim(M) + dim(N) = n \implies \chi_S(M,N) > 0$.

Serre showed that all of the above are true, when S is any regular local ring containing a field or when S is a regular local ring of mixed characteristic whose completion is a power series over a DVR. He also proved assertion (1) for any regular local ring. Gillet and Soulé in Gillet (1985) and independently Roberts (1985), proved assertion (2). Gabber proved that $\chi_S(M,N) \geq 0$, see Berthelot (1997). Strict positivity of Serre's intersection multiplicity in the ramified case in mixed characteristic still remains unsolved. The problem reduces to the case where S is complete with algebraically closed residue field and M,N are quotients of S by prime ideals.

The existence of small CM modules for complete local domains would settle Serre's positivity conjecture:

Proposition 2.1.1.4 (Hochster). *Existence of small CM modules for complete local domains $\Longrightarrow$ strict positivity of Serre's intersection multiplicity.*

Proof. Let $M = S/P$ and $N = S/Q$ for prime ideals $P,Q \subseteq S$. Since $l(M \otimes_S N) < \infty$, we have that $P+Q$ is $\mathfrak{m}$-primary. Suppose that S/P and S/Q admit small CM modules U and V of ranks e and f respectively. Then U admits a finite filtration of S/P modules with e factors equal to S/P and the rest consisting of modules of strictly smaller dimension (similarly for V). Then, by the bi-additivity of Serre's intersection multiplicity (Serre (2000)) and assertion (2), $\chi_S(U,V) = ef\chi_S(S/P,S/Q)$. Since U (or V) is maximal Cohen Macaulay and V (or U) has finite S-projective dimension, $Tor_i^S(U,V) = 0$ for all $i > 0$. Thus $\chi_S(U,V) = \ell(U \otimes_S V) > 0$ and hence $\chi_S(S/P,S/Q) > 0$. ∎

Hochster introduced a weaker version of a small CM module, called a **big CM module**, which does not imply the positivity of Serre intersection multiplicity conjecture, but does imply the other homological conjectures, see 2.1. The existence of big Cohen Macaulay modules (algebras) is now known, see Hochster & Huneke (1995) and Andre (2016). On the other hand, the existence of small CM modules over non CM rings is known only in very few cases. We will now look at some of the positive results in the literature.

There are examples of non existence of small CM modules over local rings that are not catenary, but one may view this as pathological. In fact, a domain that admits a small CM module has to be universally catenary, see Hochster (1973). Hence the complete hypothesis in 2.1.1.3.

Let $(R,\mathfrak{m})$ denote a complete (Noetherian) local domain. If $dim(R) = 1$, then R is already Cohen-Macaulay, so 2.1.1.3 is true trivially. If $dim(R) = 2$, then the integral closure of R, say R', certainly satisfies Serre's criterion (S_2) (Serre's criterion for normality). Since we are in dimension two, R' is Cohen-Macaulay. Moreover, R' is a finite R-module, see Nagata (1962).

The question gets quite hard in dimension 3 and higher. In the next subsection, we provide details of a result in positive characteristic $p > 0$ due independently to Hartshorne, Hochster and Peskine-Szpiro.

Note that there are examples of non existence of small CM modules of rank one, see Dumas (1965) and Ma (2018). These may be considered as attempts towards disproving 2.1.1.3. In Hanes (1999), it is shown that the existence of a small CM module is preserved over Segre products over finitely generated positively graded algebras over a perfect field of characteristic $p > 0$. In Tavanfar (2017), it is shown that 2.1.1.3 reduces to the case of UFDs.

2.1.2 A result in char p>0

The goal of this subsection is to prove Theorem 2.1.2.7. Throughout this subsection, R will denote a ring of characteristic $p > 0$. Let $F : R \to R$ denote the Frobenius map, that is $r \to r^p$ and F^e its e-th iterate. Note that these are indeed ring homomorphisms. Although simple in definition, this map has proved to be very powerful and has been the totem of characteristic $p > 0$ methods. By restricting scalars, F_*R will denote R viewed as an R-module via the Frobenius map. F^e_*R will denote R viewed as an R-module via the e-th iterate of the Frobenius map. The ring R is said to be ***F*-finite** if F_*R is a finite R-module. This is a fairly mild condition:

Remark 2.1.2.1. If R is F-finite, so is every finitely generated algebra over R. F-finiteness is also preserved under homomorphic images and localizations. If R is F-finite, so is a power series ring over it. In particular, if K is a perfect field of characteristic $p > 0$, then a ring essentially of finite type over it is F-finite.

If M is an R-module, denote by F^e_*M the module obtained by restricting scalars via F^e.

Remark 2.1.2.2. Note that $H^i_{\mathfrak{m}}(F^e_*M) \simeq F^e_*H^i_{\mathfrak{m}}(M)$.

We recall:

Theorem 2.1.2.3 (Matlis Duality)**.** *Let $(S, \mathfrak{n}, \mathbf{k})$ be a complete local ring and E an injective hull of $\mathbf{k}$ over S. Let $\Delta_S = Hom_S(\_, E)$ be the Matlis dual functor.*

1. *Then Δ_S defines a contravariant functor from the category of modules satisfying ACC to the category of modules satisfying DCC and vice versa.*
2. *If M has ACC or DCC, then $M \simeq \Delta_S(\Delta_S(M))$.*

Theorem 2.1.2.4 (Grothendieck Vanishing Theorem)**.** *Let M be a finitely generated module of dimension d over a local ring $(S, \mathfrak{n}, \mathbf{k})$. Then $H^i_{\mathfrak{n}}(M) = 0$ for $i > d$ and $H^d_{\mathfrak{n}}(M) \neq 0$.*

We include a version of local duality that most suits our needs:

Theorem 2.1.2.5. *(Local Duality) Let $(S, \mathfrak{n}, \mathbf{k})$ be a local ring that is a homomorphic image of a Gorenstein local ring T of dimension d. Let E be an injective hull for the residue field of $\mathbf{k}$ and $\Delta_S = Hom_S(\_, E)$ be the Matlis dual functor. Then for all finitely generated modules M and all i*

$$H^i_{\mathfrak{n}}(M) \simeq \Delta_S(Ext^{d-i}_T(M, T))$$

Proposition 2.1.2.6. *Let $(S, \mathfrak{n}, \mathbf{k})$ be a local ring of dimension d that is a homomorphic image of a regular local ring T of dimension e. Let M be a finite S-module of maximal dimension that is Cohen-Macaulay on the punctured spectrum of S. Then $H^i_{\mathfrak{n}}(M)$ has finite length for all $i < d$.*

Proof. From 2.1.2.5, it suffices to show that $Ext^{e-i}_T(M, T)$ has finite length for $i < d$. Let $\mathfrak{P} \subseteq S$ be a non maximal prime ideal and $\tilde{\mathfrak{P}}$ its lift to T. $M_{\mathfrak{P}}$ is a Cohen-Macaulay $T_{\tilde{\mathfrak{P}}}$ module of dimension $dim(T_{\tilde{P}}) - e + d$. By the Auslander-Buchsbaum formula, $p.d._{T_{\tilde{\mathfrak{P}}}}(M_{\mathfrak{P}}) = e - d$. Therefore, $(Ext^{e-i}_T(M, T))_{\tilde{P}} = 0$ for all $i < d$. That is $Ext^{e-i}_T(M, T)$ has finite length as a T-module and hence as a S-module. The proof is now complete. ■

Theorem 2.1.2.7 (Hartshorne, Hochster, Peskine-Szpiro)**.** *Let K be a perfect field of char $p > 0$. Let R be a $\mathbb{N}$-graded domain that is finitely generated over $R_0 = K$. If R is Cohen-Macaulay on the punctured spectrum, then R admits a graded small CM module.*

Proof. We want to construct a finitely generated module M of maximal dimension such that $H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$, where $d = dim(R)$ and $\mathfrak{m}$ is the homogeneous maximal ideal of R. Set $z := \sum_{i<d} \ell(H^i_{\mathfrak{m}}(R))$. From 2.1.2.6, $z < \infty$. For an integer $e \geq 1$ and an integer $0 \leq s \leq p^e$ consider $M_s := \oplus_{i \equiv s \, mod \, p^e} R_i$. That is $F^e_* R = \oplus_{0 \leq s \leq p^e} M_s$. Choose e so that the number of $M_s \neq 0$ is greater than z. We have from 2.1.2.2 that

$$\oplus_{i<d} F^e_* H^i_{\mathfrak{m}}(R) = \oplus_{i<d} H^i_{\mathfrak{m}}(F^*_e(R)) = \oplus_{0 \leq s \leq p^e}(\oplus_{i<d} H^i_{\mathfrak{m}}(M_s)).$$

Since $\oplus_{i<d} F^e_* H^i_{\mathfrak{m}}(R)$ is a z-dimensional K-vector space, our choice of e implies that $H^i_{\mathfrak{m}}(M_r) = 0$ for some $0 \leq r \leq p^e$ and all $i < d$. Moreover, R is F-finite by 2.1.2.1, so that M_r is a finitely generated R-module. Thus M_r is a graded small CM module over R. ■

Corollary 2.1.2.8 (Hartshorne, Hochster, Peskine-Szpiro)**.** *Let R be a three dimensional $\mathbb{N}$-graded domain that is finitely generated over a field $R_0 = K$ with $char(K) = p > 0$. Then R admits a small CM module.*

Proof. To reduce to the case where K is perfect see Grothendieck (1971)[Chapter 0, 6.8]. The integral closure of R, say $\overline{R}$, is then $\mathbb{N}$-graded over a perfect field, see Huneke & Swanson (2006a)[Corollary 2.3.6]. Moreover, $\overline{R}$ is a finitely generated R-module, see Huneke & Swanson (2006a)[Theorem 4.6.3]. Since $\overline{R}$ satisfies Serre's criterion (S_2), it is Cohen-Macaulay on the punctured spectrum. Therefore by 2.1.2.7, $\overline{R}$ admits a small CM module and hence so does R. ■

More recently, the above result was generalized in Schoutens (2020). He first shows that the existence of a certain type of **Hasse-Schmidt derivation** implies the existence of a small CM module for complete local domains of positive characteristic with algebraically closed residue field. He then uses this to generalize 2.1.2.8 to what he calls **pseudo-graded rings** of dimension three.

Theorem 2.1.2.9 (Schoutens (2020))**.** *If R is a three dimensional pseudo-graded ring of positive characteristic, then R admits a small CM module.*

2.1.3 A viewpoint to finding small CM modules

Hochster observed that one could approach 2.1.1.3 as follows: Let R be a complete local domain and $S \subseteq R$ a regular local ring such that R is a finite S-module. Given an R-module M, it is a small CM module over R if and only if it is a small CM module over S if and only if it is a free S-module. Let $\mathbb{M}_N(S)$ denote the ring of $N \times N$ matrices with entries in S. Rephrasing 2.1.1.3, R admits a small CM module if and only if R embeds into $\mathbb{M}_N(S)$ for some $N >> 0$ such that the restriction to the subring S is the natural scalar diagonal matrix embedding.

Along these lines, note that 2.1.1.3 reduces to the integral closure of a complete regular local ring in a finite normal extension of its fraction field. So it is natural to first look at the case of extensions of the fraction field of a regular local ring with a “nice" Galois group.

Let S be an unramified regular local ring and L its quotient field. Let K be a finite extension of L and R the integral closure of S in K. It is shown in Roberts (1980), that if K/L is Abelian and $[K : L]$ is not divisible by the characteristic of the residue field of S, then R is Cohen-Macaulay. In particular, this applies to the equi-characteristic zero case. In the mixed characteristic p scenario, the conclusion fails as shown in Koh (1986) and Sridhar (2021a). In Roberts (1980), the hypothesis that the characteristic of the (algebraically closed) residue field k does not divide the order of the Galois group G critically ensures the group algebra $k[G]$ is a product of fields. There is no direct analog of the argument when $char(k)$ divides the order of G. In fact, when this hypothesis is removed, we are in an entirely new world, drawing parallels with the modular case in representation theory.

We provide a sketch of the argument in Roberts (1980) - in the sections that follow we study the failure of this phenomenon in mixed characteristic with a focus on finding small CM modules over the integral closure R. With specified notation, we need the next two lemmas to reduce to the case where S is complete with algebraically closed residue field (via standard arguments):

Lemma 2.1.3.1. *Roberts (1980) Let $S \to S'$ be a faithfully flat extension of regular local rings. Set $R' := R \otimes_S S'$ and $L' = Frac(S')$. Then:*

1. *R' is a reflexive S'-module.*

2. *If R' is finite S'-free of rank n, then R is S-free of rank n.*

Lemma 2.1.3.2. *Grothendieck (1971) Let $(A, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring. Let $K/\mathbf{k}$ be a field extension. Then there exists a flat local homomorphism of Noetherian local rings $A \to B$ such that $B/\mathfrak{m}B \simeq K$.*

We recall these basic results from group representation theory.

Theorem 2.1.3.3 (Maschke's Theorem)**.** *Let G be a finite group and K a field whose characteristic does not divide the order of G. Then the group algebra $K[G]$ is semisimple.*

Theorem 2.1.3.4 (Artin-Wedderburn Theorem)**.** *Let A be a semi-simple ring. Then $A \simeq \oplus_{i=1}^{r} \mathbb{M}_{n_i \times n_i}(D_i)$ for some division rings D_i. If A is a finite dimensional k-algebra for a field k, then each D_i is a finite dimensional division algebra over k.*

Corollary 2.1.3.5. *Let A be a semisimple ring that is a finite dimensional algebra over an algebraically closed field $\mathbf{k}$. Then $A \simeq \oplus_{i=1}^{r} \mathbb{M}_{n_i \times n_i}(\mathbf{k})$.*

Theorem 2.1.3.6 (Roberts (1980))**.** *Let S be a regular local ring, L its quotient field, and K a finite Abelian extension of L with Galois group G. Assume that the order of G is not divisible by the characteristic of the residue field of S. Then the integral closure of S in K is Cohen-Macaulay.*

Proof. (Sketch)

1. The action of the Galois group G on K gives K a $L[G]$-module structure and R a $S[G]$-module structure. The isomorphism $R \otimes_S L \simeq K$ is in fact an isomorphism of $L[G]$-modules. The normal basis theorem implies that $K \simeq L[G]$ as $L[G]$-modules, so that $R \otimes_S L \simeq L[G]$ as $L[G]$-modules.

2. Let $\widehat{S}$ denote the completion of S with respect to its maximal ideal. We have $\widehat{R} \otimes_{\widehat{S}} Frac(\widehat{S}) \simeq (R \otimes_S L) \otimes_L Frac(\widehat{S}) \simeq Frac(\widehat{S})[G]$. Combining this with 2.1.3.1, we can reduce to the case where S is complete. Further, we may assume the residue field $\mathbf{k}$ of S is algebraically closed by 2.1.3.2.

3. Since G is Abelian, the group algebra $\mathbf{k}[G]$ is commutative. Moreover, since $\mathbf{k}$ is algebraically closed, we have from 2.1.3.3 and 2.1.3.5 that $\mathbf{k}[G] \simeq \mathbf{k}^{\oplus n}$ as rings. Since S is complete, we may lift this factorization to get $S[G] \simeq S_1 \times \cdots \times S_n$ where $S_i \simeq S$. Since R is a $S[G]$-module, $R \simeq R_1 \times \cdots \times R_n$, where R_i is a S_i-module.

4. On one hand $L[G] \simeq S[G] \otimes_S L \simeq (S_1 \otimes_S L) \times \cdots \times (S_n \otimes_S L) \simeq L \times \cdots \times L$. On the other hand $L[G] \simeq K \simeq R \otimes_S L \simeq (R_1 \otimes_S L) \times \cdots \times (R_n \otimes_S L)$. Since these decompositions are canonical, $R_i \otimes_S L \simeq L$ for all i.

5. Since K/L is separable, R is a finite S-module. Since R is a (S_2) S-module, by Bruns & Herzog (1998)[Proposition 1.4.1], R is a reflexive S-module. Thus we know that the R_i are rank one reflexive S-modules. Since S is a UFD, the R_i are S-free and hence R is S-free.

■

The "modular" case, that is when the order of the Abelian Galois group is not coprime to the characteristic of the residue field is a different matter. For example, if K/L is a cyclic extension of degree p and $\text{char}(k) = p$, we have $k[G] = k[X]/(X-a)^p$. In equal characteristic $p > 0$, the problem reduces easily to the case of certain generic Artin Schreier extensions, see chapter 4. We will primarily concern ourselves with the mixed characteristic case:

Question 2.1.3.7. *What are the obstructions one faces when S is an unramified regular local ring of mixed characteristic $p > 0$ and $p \mid |G|$ in 2.1.3.6?*

Before we begin this quest, note that the proof of Roberts's theorem goes through if S is only assumed to be a UFD - in the sense that R is a free S-module. Hochster and Roberts give an example to show that the proof of the above theorem does not go through if we are in the modular case, although this does not disprove the statement in 2.1.3.6.

Example 2.1.3.8. Set $R := \widehat{\mathbb{Z}_{(2)}}[X,U,V,Y]_{(2,X,U,V,Y)}/(Y^2 - X^2V - 4U)$ where $\widehat{\mathbb{Z}_{(2)}}$ denotes the ring of 2-adic integers. R is a quadratic extension of the unramified regular local ring $S := \widehat{\mathbb{Z}_{(2)}}[X,U,V]_{(2,X,U,V)}$ obtained by adjoining the square root of $f := X^2V + 4U \in S$. Since $S/(2)$

is integrally closed and f is not a square modulo $2S$, $2 \in R$ is prime. Moreover, $R[1/2]$ is a UFD, so that by Nagata's criterion, R is a UFD.

Let $L := Frac(R)$, $K := L(\sqrt{V})$ and T the integral closure of R in K. Then

$$\omega := 2^{-1}(Y + X\sqrt{V}) \in T$$

since $\omega^2 - Y\omega + U = 0$. Suppose T admits a R-free module M. Then this defines a ring homomorphism from T to a ring of square matrices over R such that its restriction to R is the diagonal embedding. Denote by E and E' the image of $\sqrt{V}$ and ω respectively under this homomorphism. Then we get a matrix equation $Y \cdot I + X \cdot E = 2 \cdot E'$ where I is the suitable identity matrix. Looking at the diagonal entries, we get $Y \in (2, X)R$, which is a contradiction. Therefore T does not admit any module that is free over R! In particular, T is not R-free.

Remark 2.1.3.9. With notation as in 2.1.3.8, we note here that T is already Cohen-Macaulay. Indeed, T is the integral closure of $S[\sqrt{V}, \sqrt{f}]$. But $S' := S[\sqrt{V}]$ is an unramified regular local ring of mixed characteristic 2 and T is the integral closure of S' in a quadratic extension of its fraction field. In particular, it is a rank two reflexive S'-module. Since S' is a UFD, by the Direct Summand theorem Andre (2016), T is S'-free - that is T is Cohen-Macaulay. Note that one could also use 2.1.4.5 to arrive at this conclusion since $f \in S'$ is square free.

Roberts also shows that the conclusion of 2.1.3.6 is false if G is only assumed to be solvable or nilpotent. In fact, he notes that there are Galois extensions of a UFD S, such that that the minimal number of generators of the integral closure over S is arbitrarily large.

2.1.4 Generically Abelian extensions in mixed characteristic and radical extensions

The conclusion of 2.1.3.6 fails in mixed characteristic. Koh gave the only known example of this phenomenon before our work in Sridhar (2021a). We include the example in question but do not provide a proof.

Example 2.1.4.1. Koh (1986): Let S be a regular local ring of mixed characteristic 3 containing a primitive cube root of unity. Denote its fraction field by L. Assume $dim(S) \geq 3$ and that $(i\sqrt{3}, x, y)$ form part of a regular system of parameters for S. Let $a := xy^4 + 27 \in S$ and $b := x^4y + 27 \in S$. Then $a, b \in S$ are square free, relatively prime elements. Let $\omega := \sqrt[3]{ab^2}$ and R the integral closure of S in $L(\omega)$. Note that L contains a primitive third root of unity and hence $L(\omega)/L$ is Galois with Galois group $\mathbb{Z}/3\mathbb{Z}$. However R is not S-free and hence not Cohen-Macaulay.

Note that the base regular local ring is not unramified in 2.1.4.1. Katz gave an example of a p-th root extension of an unramified regular local ring such that the integral closure is not Cohen-Macaulay in Katz (1999). In this paper, he showed much more, see 2.1.4.3. In particular, in this example it is shown that the integral closure does admit a small CM module.

Example 2.1.4.2. Katz (1999)(Sketch)

1. Let $(S, \mathfrak{m})$ be an unramified regular local ring of mixed characteristic 3 with field of fractions L. Assume $dim(S) \geq 3$ and let $(3, x, y)$ be part of a regular system of parameters for S. Set $a := xy^4 + 9, b := x^4y + 9$ and $\omega := \sqrt[3]{ab^2}$. Let $K := L(\omega)$ and R the integral closure of S in K.

2. Then $P := (\omega - x^3y^2, 3) \subseteq S[\omega]$ is the unique height one prime in $S[\omega]$ containing 3. Set $Q := (\omega, b) \subseteq S[\omega]$. Then it is shown by reducing to the one dimensional case that $R = (Q \cap P)^{-1}_{S[\omega]}$.

3. R is Cohen Macaulay if and only if $J := Q \cap P$ lifts to a grade two perfect ideal to $B := S[W]_{(\mathfrak{m}, W)}$.

4. Denoting lifts by $\sim$ we have

$$0 \longrightarrow B/\tilde{J} \longrightarrow B/\tilde{Q} \oplus B/\tilde{P} \longrightarrow B/(\tilde{Q} + \tilde{P}) \longrightarrow 0 \qquad (2.1.4.2.1)$$

5. But $B/(\tilde{Q} + \tilde{P}) \simeq S/(3, x^4y, x^3y^2)$ which has depth equal to $depth(S) - 3$. Thus $depth(B/\tilde{J}) = depth(S) - 2$ and hence R is not Cohen-Macaulay.

6. However, if we set $K := b \cdot Q^{-1}$, then $M := (K \cap P)^{-1}$ is a small CM module over R !

The main result of Katz (1999) was

Theorem 2.1.4.3. *Katz (1999) Let S be an unramified regular local ring of mixed characteristic $p > 0$ with field of fractions L. Let $K := L(\omega)$ where ω is the p-th root of an arbitrary element of S. Let R be the integral closure of S in K. Then R admits a birational small CM module.*

The above result is obtained from a careful study of the conductor J of the integral closure R to a suitable complete intersection ring A (for example A could be $S[\omega]$ for $\omega \in K$ a primitive element for K/L). Much of the work in this thesis is inspired by this approach. A key technical tool in Katz (1999) and our work is the ability to describe inverses of ideals in complete intersection rings that lift to grade two perfect ideals in certain rings projecting onto it. The result below follows from work in Mond & Pellikaan (1989) or Kleiman & Ulrich (1997).

Proposition 2.1.4.4. *Katz (1999) Let A be a Noetherian domain satisfying (S_2) such that the integral closure of A is a finite A-module. Suppose that $A = B/(F)$ for $F \in B$ a prime. Let $\tilde{J} = (\Delta_1, \ldots, \Delta_n) \subseteq B$ be a grade two ideal containing F such that the Δ_i are the signed maximal minors of a $(n+1) \times n$ B-matrix ϕ. Let J denote the image of $\tilde{J}$ in A. Write $F = b_1\Delta_1 + \cdots + b_{n+1}\Delta_{n+1}$. Let ϕ' denote the $(n+1) \times (n+1)$ matrix obtained by augmenting ϕ with the column consisting of the b_i. Then J^{-1} is generated as a A-module by the set $\{\psi_{1,1}/\delta_1, \ldots, \psi_{n+1,n+1}/\delta_{n+1}\}$ where $\psi_{i,i}$ denotes the image in A of the (i,i)-th co-factor of ϕ' and δ_i denotes the image in A of Δ_i. Moreover $p.d._B(J) = p.d._B(J^{-1}) = 1$.*

In Katz (1999) it was shown that if the element whose root is adjoined in 2.1.4.3 is square free, then the integral closure is Cohen-Macaulay.

Proposition 2.1.4.5. *Assume notation as in 2.1.4.3 and let $\omega^p = f \in S$ be square free. Then R is Cohen-Macaulay. In case $S[\omega] \neq R$, then $R = P^{-1}$ where $P \subseteq S[\omega]$ is the unique height one prime in S containing p.*

Remark 2.1.4.6. The proof of 2.1.4.5 goes through if S is only assumed to be a Noetherian domain such that $p \in S$ is a prime and S/pS is integrally closed.

Remark 2.1.4.7. In Katz (2021), under certain circumstances R is shown to admit a birational small CM module in extensions obtained by adjoining the p^n-th root of a single element.

As discussed above, radical extensions in mixed characteristic $p > 0$ obtained by adjoining n-th roots of elements of the base regular local ring with the property that p divides n are prime examples of the failure of Roberts's theorem. Moreover, the importance of radical extensions stem from Kummer theory, which says that Abelian extensions are repeated radical extensions under the presence of suitable roots of unity. The cyclic case of Kummer theory has a quick elementary proof which we shall present. We first recall the classical lemma:

Lemma 2.1.4.8. *(Dedekind-Independence of Characters) Let K be a field and G a group. If $\sigma_1, \ldots, \sigma_n$ are distinct group homomorphisms from $G \to K^\times$, then they are linearly independent over K.*

Proposition 2.1.4.9. *Let L be a field containing a primitive n-th root of unity. If $L[\omega]/L$ is a field extension such that $\omega^n \in L$ and n is the smallest integer with this property, then $L[\omega]/L$ is cyclic Galois of degree n. Conversely, if K/L is a cyclic extension of degree n, then $K = L[\omega]$ for some ω such that $\omega^n \in L$.*

Proof. Consider the forward implication. ω is a root of a polynomial $X^n - f \in L[X]$. The other roots are simply $\zeta^i \omega$ for $1 \leq i \leq n-1$, where $\zeta \in L$ is a primitive n-th root of unity. Thus $L[\omega]/L$ is certainly Galois. Denote the Galois group by G and by μ_n the group of n-th roots of unity contained in $L^\times$. For all $\sigma \in G$, $\sigma(\omega) = \zeta^i \omega$ for some i, so that $\sigma(\omega)/\omega \in \mu_n$. This in fact defines a group homomorphism:

$$G \to \mu_n : \sigma \mapsto \sigma(\omega)/\omega$$

The homomorphism is clearly injective. Suppose the homomorphism is not surjective - the image is a subgroup $\mu_d \leq \mu_n$ for some $d|n$, $d < n$. For any $\sigma \in G$, we then have $(\sigma(\omega)/\omega)^d = 1$, so that $\sigma(\omega^d) = \omega^d$. This would mean $\sigma^d \in L$, which is impossible by our hypothesis. Thus the homomorphism is surjective and G is cyclic of degree n.

Now consider the backward implication. Let K/L be cyclic Galois of degree n. Let σ generate the Galois group G and let ζ be a primitive n-th root of unity in L. Then $1, \sigma, \dots, \sigma^{n-1}$ are all distinct group automorphisms of $K^\times$. By 2.1.4.8, $\sum_{i=0}^{n-1} \zeta^i \sigma^i$ is not the zero function. Choose α such that $\beta := \sum_{i=0}^{n-1} \zeta^i \sigma^i(\alpha) \neq 0$ (α can't be in $L^\times$). Then $\sigma(\beta) = \zeta^{-1}\beta$ and $\beta \notin L$. Thus $\sigma(\beta^n) = \beta^n$, so that $\beta^n \in L$ and no smaller power has this property. This completes the proof. ■

The general form of Kummer theory that is relevant to us is:

Theorem 2.1.4.10. *(Kummer Theory) Let K/L be a finite Galois extension with Galois group G. Suppose that G has exponent n and L contains a primitive n-th root of unity. Then G is Abelian if and only if there exist $a_1, \dots, a_s \in L$ such that $K = L(a_1^{1/n}, \dots, a_s^{1/n})$.*

For a proof, see Appendix A.

Remark 2.1.4.11. (Huneke & Katz (2019)) In 2.1.4.10, if L is the quotient field of a UFD S and G is Abelian, then K is contained in an extension of the form $L(p_1^{1/n}, \dots, p_t^{1/n})$ for distinct primes $p_1, \dots, p_t \in S$.

2.1.5 Motivating Questions

While chapters 3 and 4 are primarily motivated by the question of existence of small Cohen-Macaulay modules (2.1.1.3), we understand that it is an ambitious project at this time, (see section 2.1). We consider the case of generically Abelian extensions of regular local rings in mixed characteristic motivated by Roberts's theorem 2.1.3.6. We would like to understand the modular case:

Question 2.1.5.1. *What are the obstructions one faces when S is an unramified regular local ring of mixed characteristic $p > 0$ and $p \mid |G|$ in 2.1.3.6?*

Motivated by the phenomenon in Koh (1986) and Katz (1999) and the studies in Katz (1999) and Katz (2021), we consider generic extensions $K/Frac(S)$ of regular local rings S with the property that the integral closure of S in K, say R, is S-free when S contains a field but not necessarily so otherwise.

Consider the following question:

Question 2.1.5.2. *Does the integral closure of a regular local ring of mixed characteristic $p > 0$ in a finite Abelian extension of its fraction field admit a small CM module?*

Definition 2.1.5.3. A Noetherian local ring R admits a small Cohen-Macaulay (CM) algebra if there is an injective map of rings $R \hookrightarrow S$ such that every system of parameters of R becomes a regular sequence in S and S is a finite R-module.

Consider the stronger question:

Question 2.1.5.4. *Does the integral closure of a regular local ring of mixed characteristic $p > 0$ in a finite Abelian extension of its fraction field admit a small CM algebra?*

This is because there exists a retraction from $B \to T$ using the trace map corresponding to the fraction fields which can be used to show that T is CM if B is. However, an example of the existence of a small CM algebra in mixed characteristic does not seem to be well known and we will provide definitive examples in this work (4.1.0.3). On the other hand, small Cohen-Macaulay algebras do not always exist in mixed characteristic either - any mixed characteristic local domain that is not Cohen-Macaulay after inverting p would be an example. Moreover, Linquan Ma pointed out to the author that Bhatt's examples of non existence of small CM algebras in positive characteristic in Bhatt (2012) "deform" to mixed characteristic.

Katz's result 2.1.4.3 makes one wonder whether rank one MCMs are admissible:

Question 2.1.5.5. *Does the integral closure of a regular local ring of mixed characteristic $p > 0$ in a finite Abelian extension of its fraction field admit a birational small CM module?*

Kummer theory (A.0.0.2) tells us that Abelian extensions of a field of characteristic zero containing suitable roots of unity are repeated radical extensions. Thus it is natural to study repeated n-th root extensions of an unramified regular local ring of mixed characteristic $p > 0$ with the property that $p|n$. If S were to contain the rational numbers or if S were of mixed characteristic $p > 0$

and $p \nmid n$, then it follows that the integral closure of S in an arbitrary repeated radical extension is Cohen-Macaulay, see Sridhar (2021a) and Huneke & Katz (2019).

The examples 2.1.4.1 and 2.1.4.2 where obtained by adjoining a p-th root of a non-square free element of the base ring. Katz showed that in case we adjoin a p-th root of a single square free element to an unramified regular local ring of mixed characteristic $p > 0$, the integral closure is Cohen-Macaulay, see 2.1.4.5. In contrast, we will see that the integral closure need not be Cohen-Macaulay in a finite square free tower of p-th roots: in fact it could fail to be Cohen-Macaulay even if we adjoin p-th roots of two square free elements.

To take the approach through repeated radical extensions one can assume that the elements whose roots we adjoin are square free, since any given multi-radical extension can be embedded in a sufficiently large square free tower, while preserving finiteness. In the same vein, one could also impose suitable generality conditions on the elements. To this end, we can reduce to the case where the elements whose roots we adjoin lie in S^p when S is complete with perfect residue field and S^p is the subring of S obtained by lifting the Frobenius map on S/pS to S, see section 3.2.1. On the other hand, the complexity of these towers increases very fast. Towards gaining a handle, we investigate the first new case. We consider biradical extensions of an unramified regular local ring S of mixed characteristic $p > 0$ obtained by adjoining p-th roots of sufficiently general square free elements say $f, g \in S$, see chapters 3 and 4. Roughly speaking, one may think of this as the case where the Galois group is $\mathbb{Z}_p \times \mathbb{Z}_p$ generically.

2.2 Reflexive Modules

The work in this section and in chapter 5 is based of Dao et al. (2021) and is joint work with Hailong Dao and Sarasij Maitra. Throughout this section, assume that all rings are commutative with unity and are Noetherian, and that all modules are finitely generated.

2.2.1 Over general Noetherian rings

Let R denote a Noetherian ring with total ring of fractions $Q(R)$. Let $\overline{R}$ denote the integral closure of R in $Q(R)$. Let $\operatorname{Spec} R$ denote the set of prime ideals of R. For any R-module M, if the natural map $M \to M \otimes_R Q(R)$ is injective, then M is called *torsion-free*. It is called a torsion module if $M \otimes_R Q(R) = 0$. The *dual* of M, denoted M^*, is the module $\operatorname{Hom}_R(M,R)$; the *bidual* then is M^{**}. The bilinear map

$$M \times M^* \to R, \qquad (x,f) \mapsto f(x),$$

induces a natural homomorphism $h : M \to M^{**}$. We say that M is *torsionless* if h is injective, and M is *reflexive* if h is bijective.

For R-submodules M,N of $Q(R)$, we denote

$$M :_R N = \{a \in R \mid aN \subseteq M\}$$
$$M : N = \{a \in Q(R) \mid aN \subseteq M\}.$$

We will need the notion of trace ideals. We first recall the definition.

Definition 2.2.1.1. The *trace ideal* of an R-module M, denoted $\operatorname{tr}_R(M)$ or simply $\operatorname{tr}(M)$ when the underlying ring is clear, is the image of the map $\tau_M : M^* \otimes_R M \to R$ defined by $\tau_M(\phi \otimes x) = \phi(x)$ for all $\phi \in M^*$ and $x \in M$.

Say that an ideal I is a trace ideal if $I = \operatorname{tr}(M)$ for some module M. Since $\operatorname{tr}(\operatorname{tr}(M)) = \operatorname{tr}(M)$, I is a trace ideal if and only if $I = \operatorname{tr}(I)$. It is clear from the definition, that if $M \cong N$, then $\operatorname{tr}(M) = \operatorname{tr}(N)$.

There are various expositions on trace ideals scattered through the literature, see for example Herzog et al. (2019), Lindo (2017), Kobayashi & Takahashi (2019a), Goto et al. (2020), Faber (2019), etc. For our purposes, we shall mainly need the following properties of trace ideals.

Proposition 2.2.1.2. *(Kobayashi & Takahashi, 2019a, Proposition 2.4) Let M be an R-submodule of $Q(R)$ containing a nonzero divisor of R. Then the following statements hold.*

1. $\operatorname{tr}(M) = (R : M)M$.

2. *The equality $M = \mathrm{tr}(M)$ holds if and only if $M : M = R : M$ in $Q(R)$.*

Recall that a finitely generated R-submodule I of $Q(R)$ is called a *fractional ideal* and it is *regular* if it is isomorphic to an R-ideal of grade one.

Remark 2.2.1.3. Let R be any ring with total ring of fractions $Q(R)$. For any two regular fractional ideals I_1, I_2, we have $I_1 : I_2 \cong \mathrm{Hom}_R(I_2, I_1)$ where the isomorphism is as R-modules, see for example (Herzog & Kunz, 1971, Lemma 2.1).

By abuse of notation, we will identify these two R-modules and use them interchangeably.

Remark 2.2.1.4. By Remark 2.2.1.3, we can identify $I^* := \mathrm{Hom}_R(I, R)$ with $R : I$ whenever I contains a non zero divisor. This is also denoted as I^{-1}. Moreover for any non zero divisor x in I, we have $xI^* = x :_R I$. Hence $I^* \cong x :_R I$ as R-modules.

Next, we discuss some general statements about reflexive modules that will be needed. Recall that an R-module M is called *totally reflexive* if M is reflexive and $\mathrm{Ext}^i_R(M, R) = \mathrm{Ext}^i_R(M^*, R) = 0$ for all $i > 0$ (see Kustin & Vraciu (2018) for instance for more details). In the result below we need to consider modules that are locally totally reflexive on the minimal primes of R. See Remark 2.2.1.6.

Lemma 2.2.1.5. *Let R be a Noetherian ring satisfying condition (S_1). Consider modules M, N that are locally totally reflexive on the minimal primes of R.*

1. *Assume there is a short exact sequence $0 \to M \to N \to C \to 0$. If N is reflexive and C is torsionless then M is reflexive.*

2. *If $\mathrm{Hom}_R(M, N)$ is locally totally reflexive on the minimal primes of R and N is reflexive, then $\mathrm{Hom}_R(M, N)$ is reflexive.*

Proof. For (1), (Masek, 1998, Theorem 29) implies that C is locally totally reflexive on the minimal primes. Thus $\mathrm{Ext}^1_{R_P}(C_P, R_P) = 0$ for all minimal primes P and the conclusion follows from (Masek, 1998, Proposition 8).

For (2), we first prove the case $N = R$. Let $f : M^* \to M^{***}$ be the natural map. Let $g : M^{***} \to M^*$ be the dual of the natural map $M \to M^{**}$. Then $g \circ f = \mathrm{id}$, so f splits, and we get $M^{***} = M^* \oplus M_1$. But for any minimal prime P, $(M_1)_P = 0$, so M_1 has positive grade. As M^{***} embeds in a free module and R is (S_1), $M_1 = 0$.

Now, start with a short exact sequence $0 \to C \to F \to N^* \to 0$ where F is free. Dualizing, we get $0 \to N \to F^* \to D \to 0$ where D is torsionless (by (Masek, 1998, Proposition 8), a submodule of a torsionless module is torsionless). Take $\mathrm{Hom}_R(M, -)$ to get the exact sequence $0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,F^*) \to K' \to 0$ where K' is a sub-module of $\mathrm{Hom}_R(M,D)$. We can apply part (1) and the previous paragraph to get that $\mathrm{Hom}_R(M,N)$ is reflexive provided we show that K' is torsionless.

Since, D is torsionless, D embeds into D^{**} and hence into a free module say G. Thus, $\mathrm{Hom}_R(M,D)$ embeds into $\mathrm{Hom}_R(M,G)$. Finally, note that M^* is torsionless as it is a submodule of a free module and thus, by applying (Masek, 1998, Proposition 8) twice, we get that K' is torsionless. ■

Remark 2.2.1.6. We stated Lemma 2.2.1.5 in quite a general setting. One reason is in practice, as well as in this chapter, it is often applied in the following two different situations: when R is generically Gorenstein, or when M, N are locally free on the minimal primes (for instance when they are birational extensions or regular ideals of R). In such situations the totally reflexive assumptions are automatically satisfied.

The relationship between reflexive modules and birational extensions is also naturally of interest. We say that an extension $f : R \to S$ is *birational* if $S \subset Q(R)$. Equivalently $Q(R) = Q(S)$.

Proposition 2.2.1.7. *(Kobayashi & Takahashi, 2019a, Proposition 2.4) Let $M \subseteq Q(R)$ be a regular fractional ideal of R. Then M is a reflexive R-module if and only if there is an equality $M = R : (R : M)$ in $Q(R)$.*

The following lemma was stated in more generality in a recent work of S. Goto, R. Isobe, and S. Kumashiro.

Lemma 2.2.1.8. *(Goto et al., 2020, Lemma 2.6(1), Proposition 2.9). Let R satisfy* (S_1) *and* $f : R \to S$ *be a finite birational extension. Then the conductor of S to R, denoted* $\mathfrak{c}_R(S) := R : S$*, is a reflexive regular trace ideal of R. Thus, we get a bijective correspondence between reflexive regular trace ideals of R and reflexive birational extensions of R via the map* $\alpha : I \mapsto \operatorname{End}_R(I)$ *and its inverse* $\beta : S \mapsto \mathfrak{c}_R(S)$*.*

Proof. Let S be a reflexive birational extension. Then $\mathfrak{c}_R(S) \cong S^*$ is reflexive by Lemma 2.2.1.5. Next, note that $\operatorname{tr}(S) = S^*S = \mathfrak{c}_R(S)S = \mathfrak{c}_R(S)$. So $\mathfrak{c}_R(S)$ is a reflexive trace ideal. If I is a regular reflexive trace ideal, then Proposition 2.2.1.2(2) and Lemma 2.2.1.5(2) tell us that $\operatorname{End}_R(I)$ is indeed reflexive.

Finally, we have $\beta(\alpha(I)) = I$ and $\alpha(\beta(S)) = S$ by Proposition 2.2.1.2, Proposition 2.2.1.7 and the above paragraph. ■

So these birational extensions provide important sources for generating reflexive ideals. We have the following criteria for a reflexive module over R to be a module over a finite birational extension S. This was stated in (Faber, 2019, Theorem 3.5) for reduced one dimensional local rings, but the result holds for more general rings, and we restate it here with a self-contained proof:

Theorem 2.2.1.9. *Let R be a Noetherian ring and M be a finite R-module. Let S be a finite birational extension of R. Consider the following statements.*

1. *M is a module over S.*

2. $\operatorname{tr}(M) \subseteq \mathfrak{c}_R(S)$ *where* $\mathfrak{c}_R(S) = R : S$*.*

Then (1) *implies* (2)*. The converse is true if M is a reflexive R-module.*

Proof. Let M be a module over S. Then there exists a S-linear (hence R-linear) surjection $S^n \to M$, so $\operatorname{tr}(M) \subseteq \operatorname{tr}(S) = \mathfrak{c}_R(S)$.

Conversely assume $\operatorname{tr}(M) \subseteq \mathfrak{c}_R(S)$ and M reflexive. Consider $f \in M^*$ and $s \in S$, we have $s \cdot f \in M^*$ by assumption. Therefore M^* is an S-module. From the forward implication, $\operatorname{tr}(M^*) \subseteq \mathfrak{c}_R(S)$. So repeating the argument again, we get that M^{**} is an S-module. Since M is reflexive, we are done. ■

2.2.2 Dimension one

In this subsection, we prove some preliminary results that will be needed in chapter 5.

Convention 2.2.2.1. Throughout the rest of this chapter and chapter 5 (unless otherwise specified), $(R, \mathfrak{m}, k)$ will denote a Cohen-Macaulay local ring of dimension one with maximal ideal $\mathfrak{m}$ and residue field k. Denote by $Q(R)$ the total quotient ring of R. For an $\mathfrak{m}$-primary ideal I and a module M, let $e_I(M)$ denote the Hilbert-Samuel multiplicity of M with respect to I. In the case, when $I = \mathfrak{m}$, we write $e(M)$. Let $\mathfrak{c} := R : \overline{R}$ denote the conductor ideal of $\overline{R}$ to R. For an R-module M, let $\mu(M)$ and $\ell(M)$, denote the minimal number of generators of M and the length of M respectively, as an R-module.

Let $\mathrm{CM}(R)$ denote the category of maximal Cohen-Macaulay R-modules and let $\mathrm{Ref}(R)$ denote the category of reflexive R-modules. We say a category is of *finite type* if it has only finitely many indecomposable objects up to isomorphism.

Remark 2.2.2.2. Note that the following statements are true.

1. $\{\text{free } R\text{- modules}\} \subset \mathrm{Ref}(R) \subset \mathrm{CM}(R)$.
2. R is regular if and only if $\{\text{free } R\text{- modules}\} = \mathrm{CM}(R)$.
3. R is Gorenstein if and only if $\mathrm{Ref}(R) = \mathrm{CM}(R)$.

We shall be interested in the behaviour of $\mathrm{Ref}(R)$ in the case when R is “close to" being regular or Gorenstein.

The conductor and maximal ideals are natural examples of reflexive trace ideals.

Corollary 2.2.2.3. *If $\bar{R}$ is finite over R, $\mathfrak{c}$ is a regular reflexive trace ideal. If* $\dim R = 1$ *and R is Cohen Macaulay but not regular,* $\mathfrak{m}$ *is a regular reflexive trace ideal.*

Proof. The first statement follows from Lemma 2.2.1.8. For the second statement, since $\mathrm{grade}(\mathfrak{m}) = 1$, $R \subsetneq \mathfrak{m}^*$ and hence $\mathfrak{m}$ is reflexive by Proposition 2.2.1.7. Since $\mathfrak{m} \subseteq \mathrm{tr}(\mathfrak{m})$ and $\mathrm{tr}(\mathfrak{m}) = R$ if and only if $\mathfrak{m}$ is principal, $\mathfrak{m}$ is a trace ideal. ■

Support and trace

Let $\mathrm{CM}_{\mathrm{full}}(R) = \{M \in \mathrm{CM}(R) \mid \mathrm{Supp}(M) = \mathrm{Spec}\, R\}$ denote the subcategory of $\mathrm{CM}(R)$ of modules with full support.

Lemma 2.2.2.4. *Suppose that $M \in \mathrm{CM}(R)$. Then $\mathrm{tr}(M)$ is a regular ideal if and only if M_P has an R_P-free summand for each $P \in \mathrm{Min}(R)$. Thus if R is reduced then $M \in \mathrm{CM}_{\mathrm{full}}(R)$ if and only if $\mathrm{tr}(M)$ is a regular ideal.*

Proof. As we are in dimension one, clearly $\mathrm{tr}(M)$ is regular if and only if $\mathrm{tr}(M)_P = R_P$ for any $P \in \mathrm{Min}(R)$. As trace localizes, we have $\mathrm{tr}(M)_P = \mathrm{tr}_{R_P}(M_P)$, and the result follows. ■

Remark 2.2.2.5. Here we discuss why when studying $\mathrm{CM}(R)$, one can reduce to the case of $\mathrm{CM}_{\mathrm{full}}(R)$ and hence regular trace ideals thanks to Lemma 2.2.2.4. Let $\mathrm{Min}(R) = \{P_1, \dots, P_n\}$ denote the set of minimal primes of R and $(0) = \cap Q_i$ with $\sqrt{Q_i} = P_i$. For a subset $X \subset \mathrm{Min}(R)$, let $R_X = R / \cap_{P_i \in X} Q_i$. Then $\mathrm{CM}(R) = \cup_{X \subset \mathrm{Min}(R)} \mathrm{CM}_{\mathrm{full}}(R_X)$. Thus, understanding $\mathrm{CM}(R)$ amounts to understanding $\mathrm{CM}_{\mathrm{full}}(R_X)$ for all subsets X.

It is well known that $\mathfrak{c}$ and $\mathfrak{m}$ are reflexive trace ideals (Corollary 2.2.2.3). In particular, we can investigate other such ideals. We set up some further notation which we will use throughout.

$$\mathrm{T}(R) := \{I \mid I \text{ is a regular trace ideal}\}$$

$$\mathrm{RT}(R) := \{I \mid I \text{ is a regular reflexive trace ideal}\}$$

Note that if R is a complete local domain, then from (Maitra, 2020, Theorem 4.4) we get that for any ideal $I \subset R$, I^{**} is isomorphic to an ideal which contains the conductor $\mathfrak{c}$. This suggests an immediate link, relevant to our study, with the conductor ideal $\mathfrak{c}$. The following theorem gives a generalization to this fact.

Theorem 2.2.2.6. *Let R be a one-dimensional Cohen-Macaulay local ring with conductor ideal $\mathfrak{c}$. Any regular ideal I that contains a principal reduction is isomorphic to another fractional ideal J*

such that $R:J$ (which is isomorphic to I^*) contains the conductor $\mathfrak{c}$. In particular, if the residue field is infinite, any reflexive regular ideal of R is isomorphic to an ideal containing $\mathfrak{c}$.

Proof. Assume I is a regular ideal of R with a principal reduction x. Let $J = \frac{I}{x} = \{\frac{a}{x}, a \in I\}$. Clearly $I \cong J$. Since $I^{n+1} = xI^n$ for some n, we have $J \subset I^n : I^n \subset \overline{R}$. But then $R:J \supset R:\overline{R} = \mathfrak{c}$. The last statement follows by replacing I with I^* and using the fact that $I^{**} \cong I$. ■

Corollary 2.2.2.7. *Let R be as in Theorem 2.2.2.6. For any regular ideal I with a principal reduction,* $\operatorname{tr}(I) \supset \mathfrak{c}$.

Proof. Let J be the fractional ideal as in Theorem 2.2.2.6. Note that $R \subset J$. Since $R:J \supset \mathfrak{c}$, we have $\operatorname{tr}(I) = \operatorname{tr}(J) = J(R:J) \supset R:J \supset \mathfrak{c}$. ■

Lemma 2.2.2.8. *Let R be as in Theorem 2.2.2.6. Suppose that I is a regular ideal and $x \in I$ be a non zero divisor. Then* $\operatorname{tr}(I) = I((x):_R I):_R x$.

Proof. Let $J = \frac{I}{x} \cong I$. Then $\operatorname{tr}(I) = \operatorname{tr}(J) = J(R:J) = \frac{I}{x}((x):_R I)$. ■

Corollary 2.2.2.9. *Let R be as in Theorem 2.2.2.6. Suppose that I is a regular ideal and $x \in I$ be a non zero divisor. Then* $\operatorname{tr}(I) \supseteq (x):_R I$.

Proof. By Lemma 2.2.2.8, $\operatorname{tr}(I) \supset x((x):_R I):_R x = (x):_R I$. ■

Lemma 2.2.2.10. *Let R be as in Theorem 2.2.2.6. If I is a regular ideal and $I^2 = xI$ for some $x \in I$ then* $\operatorname{tr} I = (x):_R I$.

Proof. From Corollary 2.2.2.9, $(x):_R I \subseteq \operatorname{tr} I$. On the other hand, $I \operatorname{tr} I = I(II^{-1}) = I^2 I^{-1} = xII^{-1} = x \operatorname{tr} I$, so $\operatorname{tr} I \subset (x):_R I$. ■

The above allows us to classify trace ideals with reduction number one.

Corollary 2.2.2.11. *Let R be as in Theorem 2.2.2.6. Let I be a regular ideal such that $I^2 = xI$ for some $x \in I$. Then I is a trace ideal if and only if $(x):_R I = I$. In that case $I \cong I^*$ and hence I is reflexive.*

Proof. The first assertion is obvious from Lemma 2.2.2.10. For the last assertion, note that $(x) :_R I \cong I^*$. ■

It is known that under mild assumptions, all integrally closed ideals are reflexive (Corso et al., 2005, Proposition 2.14). The following proposition vastly generalizes this fact and also provides a way of generating reflexive or trace ideals by contracting ideals from certain birational extensions. For how to find such extensions see Theorem 5.2.0.5.

Proposition 2.2.2.12. *Let R be as in Theorem 2.2.2.6. Let S be a finite birational extension such that $\mathrm{CM}(S) \subset \mathrm{Ref}(R)$. Let I be a regular ideal of R. Then $IS \cap R \in \mathrm{Ref}(R)$. If I contains $\mathfrak{c}_R(S)$, then $IS \cap R \in \mathrm{RT}(R)$.*

Proof. Let $J = IS \cap R$. As $IS \in \mathrm{CM}(S)$, we have $J^{**} \subset (IS)^{**} = IS$, so $J^{**} \subset IS \cap R = J$, hence J is reflexive. If I contains $\mathfrak{c}_R(S)$ then $\mathfrak{c}_R(S) \subset J$. Now, $\mathrm{tr} J = JJ^{-1} \subset J\mathfrak{c}_R(S)^{-1} = JS$, so $\mathrm{tr} J \subset JS \cap R = J$. ■

The next two results are useful for studying colength two ideals.

Proposition 2.2.2.13. *Let $(R, \mathfrak{m})$ be as in Theorem 2.2.2.6 and further assume that R has minimal multiplicity with infinite residue field. Let I be a regular ideal of colength two. Then I is reflexive if and only if it is either integrally closed or principal.*

Proof. If I is integrally closed then it is reflexive by Proposition 2.2.2.12. Now assume I is neither integrally closed nor principal. Then necessarily $\bar{I} = \mathfrak{m}$. We can then pick a regular principal reduction x for $\mathfrak{m}$ and I. Since R has minimal multiplicity we note that $\mathfrak{m} = \mathrm{tr}(\mathfrak{m}) = (x) :_R \mathfrak{m}$ by Lemma 2.2.2.10. On the other hand $(x) :_R I \supset (x) :_R \mathfrak{m} = \mathfrak{m}$, so equality occurs. Using Remark 2.2.1.4, we get $xI^* = x\mathfrak{m}^*$ and hence $I^* = \mathfrak{m}^*$ and $I^{**} = \mathfrak{m}$. Thus I is not reflexive. ■

We classify colength two ideals that are contracted from $\mathrm{End}_R(\mathfrak{m})$.

Proposition 2.2.2.14. *Let $(R, \mathfrak{m})$ be as in Theorem 2.2.2.6. Let $S = \mathrm{End}_R(\mathfrak{m})$ and I be an ideal of colength two. Then $IS \cap R = I$ if and only if $\ell(S/IS) > \mathrm{type}(R) + 1$.*

Proof. It is clear that $IS \cap R = I$ if and only if S/IS is a faithful R/I module. As $R/I \cong k[t]/t^2$, S/IS decomposes into a direct sum of k and R/I, so it is faithful if and only if it is not a direct sum of k's, in other words the length of S/IS is strictly larger than it's number of generators. But $\mu(S) = \ell(S/\mathfrak{m}S) = \ell(S/R) + 1 = \operatorname{type} R + 1$. ■

Chapter 3

Cohen-Macaulay normalizations over radical towers in mixed characteristic

We now study general radical towers of order p of an unramified regular local ring of mixed characteristic p with a view towards 2.1.1.3. This is indeed motivated by Kummer theory (A.0.0.2) that tells us that Abelian extensions of a field of characteristic zero containing “suitable" roots of unity are repeated radical extensions. We consider the case of roots of order p inspired by the results in Katz (1999) to tackle the case where the p-torsion of the Galois group is annihilated by p.

In this chapter we look at cases where the integral closure is Cohen-Macaulay and hope that with "appropriate generality" these constructions will prove to be a small Cohen-Macaulay algebra for an arbitrary radical tower of order p. Towards gaining a handle, we look at the first case in section 3.1. We consider biradical extensions of an unramified regular local ring S of mixed characteristic $p > 0$ obtained by adjoining p-th roots of sufficiently general square free elements say $f, g \in S$. Roughly speaking, one may think of this as the case where the Galois group is $\mathbb{Z}_p \times \mathbb{Z}_p$ generically. We can reduce to the case where the elements whose roots we adjoin lie in S^p when S is complete with perfect residue field and S^p is the subring of S obtained by lifting the Frobenius map on S/pS to S, see 3.2.1. In section 3.2, in joint work with Daniel Katz, we use this developed intuition and apply it to more general settings.

3.1 The Biradical case and preliminary observations

In this section we fix notation and record some observations that will be used subsequently. Throughout this chapter all rings considered are commutative and Noetherian.

Convention 3.1.0.1.

- Let $(R, \mathfrak{m})$ be a local ring of dimension d. A nonzero R-module M is a **small CM module** if it is finitely generated and every system of parameters of R (equivalently some system of parameters) is a regular sequence on M. If R is an arbitrary Noetherian ring, then an R-module M is a **small CM module** if for all maximal ideals $\mathfrak{m} \subseteq R$, $M_{\mathfrak{m}}$ is a small CM module over $R_{\mathfrak{m}}$.

- A Noetherian ring R admits a **small CM algebra** S if there is an injective, module finite map of rings $R \to S$ such that S is Cohen-Macaulay.

- For R any commutative ring and M an R-module, we will denote by M_R^*, the dual module $Hom_R(M,R)$. If R is clear from the context, we will simply denote it by M^*. In particular, if R is a domain with field of fractions K and $M \subseteq K$, we use M^* to also denote $(R :_K M)$ (see Huneke & Swanson (2006a)[2.4.2] for example).

- Let R be a Noetherian ring and M an R-module. For $G \subseteq M$ a subset, the notation $M = \langle G \rangle_R$ means that M is generated as a R-module by G.

- For a Noetherian ring R of dimension at least one, we use the notation

$$NNL_1(R) := \{P \in Spec(R) \,|\, height(P) = 1, R_P \text{ is not a DVR}\}$$

- Suppose S is a ring and let $p \in \mathbb{Z}$ be a prime such that $p \in S$ is a non-unit. Let $F : S/pS \to S/pS$ be the Frobenius map. Let S^p denote the subring of S obtained by lifting the image of F to S. Define $S^{p^k \wedge p^n}$ for $k,n \geq 1$ to be the multiplicative subset of S of elements expressible in the form $x^{p^k} + y \cdot p^n$ for some $x,y \in S$. In particular, $S^{p \wedge p} = S^p$.

- For a local ring R and an R-module M, denote $\nu_R(M)$ for the minimal number of generators of M over R.

- For a Noetherian local ring R and M a finite R-module, denote by $Syz^i_R(M)$ the i-th syzygy of M in a minimal free resolution of M over R.

Remark 3.1.0.2. Suppose that $R \to T$ is a module finite extension of domains with R integrally closed. Set $d := [Frac(T) : Frac(R)]$ and suppose $d \in R$ is a unit. Then if T is Cohen-Macaulay, so is R. To see this, note that the trace map of their corresponding fraction fields gives an R-linear retraction $d^{-1}Tr : T \to R$. This ensures that for every ideal $I \subseteq R$, $IT \cap R = I$. Hence if R is not Cohen-Macaulay, T is not Cohen-Macaulay.

Remark 3.1.0.3. Recall the general fact: Let R be a domain with field of fractions L and let K be a finite field extension of L. Then if the monic minimal polynomial $f(X)$ of $\gamma \in K$ over L is such that $f(X) \in R[X]$, then $R[\gamma] \simeq R[X]/(f(X))$.

Remark 3.1.0.4. We make use of the following observation later (Vasconcelos (1991)[Theorem 2.4]): Let $S \subseteq C \subseteq D$ be an extension of Noetherian domains such that S is integrally closed, D is module finite over S and D is birational to C. Then if C satisfies Serre's condition R_1, so does D. To see this, assume C satisfies R_1. Let $P \subseteq D$ be any height one prime. Since going down holds for the extension $S \subseteq D$, $Q := P \cap C$ is a height one prime in C. Since $C_Q \subseteq D_P$ is a birational extension and C_Q is a DVR, we have $C_Q = D_P$. Thus D satisfies R_1.

Remark 3.1.0.5. Let $S \subseteq D$ be an extension of Noetherian domains such that going down holds. Let $\overline{D}$ denote the integral closure of D in its field of fractions K and assume $\overline{D}$ is finite over D. If $c \cdot u, c \cdot v \in D :_K \overline{D}$ with $c \in D$ and $u.v \in S$ such that there exists no height one prime of S containing both of them, then $NNL_1(D) \subseteq V(c)$.

Remark 3.1.0.6. Let $S \subseteq R$ be a finite extension of Noetherian local rings such that S is Gorenstein and R is Cohen-Macaulay. Then for any finite R-module M, $Hom_R(M, \omega_R) \simeq Hom_S(M,S)$ as R-modules (and S-modules), where ω_R is the canonical module of R. Indeed, we have $\omega_R \simeq Hom_S(R,S)$, so that by Hom-tensor adjointness, we have what we want.

Proposition 3.1.0.7. *Let S be a regular local ring and L its fraction field. Let $K := L(\sqrt[n_1]{a_1}, \dots, \sqrt[n_k]{a_k})$ with $a_i \in S$ and the n_i positive integers that are units in S for $1 \le i \le k$. Then the integral closure of S in K is Cohen-Macaulay. In particular*

1. *If S contains the rational numbers, the above conclusion holds for arbitrary integers n_i.*

2. *If S has mixed characteristic $p > 0$, the above conclusion holds for integers n_i with the property that $p \not| n_i$ for all $1 \le i \le k$.*

Proof. First assume that the $a_i \in S$ are square free, mutually coprime and $n = n_i$ for all $1 \le i \le k$. Then, from Huneke & Katz (2019)[Prop 5.3], $R = S[\sqrt[n]{a_1}, \dots, \sqrt[n]{a_k}]$ is integrally closed. Moreover, R is Cohen-Macaulay from 3.1.0.3. So the conclusion holds in this case.

Suppose the a_i and n_i are arbitrary. Set $n := \prod_{i=1}^{k} n_i$. Since S is a UFD, by 2.1.4.11, there exist $b_1, \dots, b_m \in S$ square free and mutually coprime such that $K \hookrightarrow \mathscr{K} := L(\sqrt[n]{b_1}, \dots, \sqrt[n]{b_m})$. But the integral closure of S in $\mathscr{K}$ is Cohen-Macaulay as observed above. By 3.1.0.2, the integral closure of S in K is Cohen-Macaulay.

(1) and (2) now follow immediately. ■

Convention 3.1.0.8. We will use the following notation for the remainder of this section. Let S denote a Noetherian integrally closed domain of dimension d and L its field of fractions. Assume $Char(L) = 0$. Let $p \in \mathbb{Z}$ be a prime such that $p \in S$ is a principal prime and S/pS is integrally closed. An unramified regular local ring of mixed characteristic p satisfies the above hypothesis, though not all results in this paper require this specific setting. The assumptions stated above will stand throughout the paper unless otherwise specified.

An element $x \in S$ is said to be square free if for all height one primes $Q \subset S$ containing x, $QS_Q = (x)S_Q$. Say that a subset $W \subset S$ satisfies $\mathscr{A}_1$ if for all distinct $x, y \in W$, there exists no height one prime $Q \subset S$ such that $x, y \in Q$.

Fix $f, g \in S$ such that they are not p-th powers in S, are square free and satisfy $\mathscr{A}_1$. Let W, U be indeterminates over S. We have the monic irreducible polynomials $F(W) := W^p - f \in S[W]$ and $G(U) := U^p - g \in S[U]$. Let $K := L(\omega, \mu)$ where ω and μ are roots of $F(W)$ and $G(U)$

respectively and assume that $G(U)$ is irreducible over $L(\omega)$, so that $[K:L]=p^2$. Denote by R the integral closure of S in K. That is, R is the integral closure of $A:=S[\omega,\mu]$.

Remark 3.1.0.9. It follows from 3.1.0.3 that $A\simeq S[W,U]/(F(W),G(U))$, $S[\omega]\simeq S[W]/(F(W))$ and $S[\mu]\simeq S[U]/(G(U))$.

We need a couple of technical lemmas:

Lemma 3.1.0.10 (Katz (1999))**.** *Let $p=2k+1$ and $h\in S\setminus pS$. Let W be an indeterminate over S. If*

$$C:=(W^p-h^p)-(W-h)^p=\sum_{j=1}^{k}(-1)^{j+1}\binom{p}{j}(W\cdot h)^j[W^{p-2j}-h^{p-2j}] \qquad (3.1.0.10.1)$$

$C':=(p(W-h))^{-1}\cdot C$ and $\tilde{P}:=(p,W-h)S[W]$, then $C'\notin\tilde{P}$.

Lemma 3.1.0.11. *Let $p=2k+1$ and $h\in S\setminus pS$. Let W be an indeterminate over S. Suppose C' is as defined in 3.1.0.10. Then $C'\equiv h^{p-1}$ mod $(p,W-h)S[W]$.*

Proof. We have in $S[W]$

$$\begin{aligned}C' &= \sum_{j=1}^{k}(-1)^{j+1}j^{-1}\binom{p-1}{j-1}(W\cdot h)^j[W^{p-2j-1}+\cdots+h^{p-2j-1}]\\ &\equiv \sum_{j=1}^{k}(-1)^{j+1}j^{-1}\binom{p-1}{j-1}h^{2j}\cdot(p-2j)\cdot h^{p-2j-1} \bmod (p,W-h)\\ &\equiv -2h^{p-1}\sum_{j=1}^{k}(-1)^{j+1}\binom{p-1}{j-1} \bmod (p,W-h)\\ &\equiv -h^{p-1}(-1)^{k+1}\binom{2k}{k} \bmod (p,W-h)\\ &\equiv h^{p-1} \bmod (p,W-h)\end{aligned}$$

■

Convention 3.1.0.12. Suppose $h_1,h_2\in S\setminus pS$. In this case, let C'_1 and C'_2 denote respectively the elements in the rings $S[W]$ and $S[U]$ obtained by setting $h=h_1$ and $h=h_2$ in 3.1.0.10. Denote by c'_1 and c'_2 their respective images in the rings $S[\omega]$ and $S[\mu]$ respectively. If $h_1=0$ ($h_2=0$), simply set $c'_1=0$ ($c'_2=0$). Denote by d_i the corresponding element in $S[\omega\mu^i]$ for $1\le i\le p-1$.

We extract what we need from Katz (1999):

Proposition 3.1.0.13. *With notation as specified above, $S[\omega]$ is integrally closed if and only if $f \notin S^{p \wedge p^2}$. Further, if $S[\omega]$ is not integrally closed, write $f = h^p + a \cdot p^2$ for some $a, h \in S$, $h \neq 0$. Then*

(a) $\overline{S[\omega]} = P^*_{S[\omega]} = S[\omega, \tau]$ *where* $\tau = p^{-1} \cdot (\omega^{p-1} + h\omega^{p-2} + \cdots + h^{p-1})$ *and* $P := (p, \omega - h)$ *is the unique height one prime in* $S[\omega]$ *containing* p.

(b) *If* $p \geq 3$, *there are exactly two height one primes in* $\overline{S[\omega]}$ *containing* p, Q_1 *and* Q_2 *satisfying* ${Q_1}_{Q_1} = (\omega - h)_{Q_1}$ *and* ${Q_2}_{Q_2} = (p)_Q$. *If* $p = 2$, $2 \in S[\omega, \tau] = S[\tau]$ *is square free.*

(c) *If* $p \geq 3$, τ *satisfies* $l(T) := T^2 - c_1' T - a \cdot (\omega - h)^{p-2}$ *over* $S[\omega]$ *where* $c_1' \in S[\omega]$ *is as in 3.1.0.10. If* $p = 2$, τ *satisfies* $l(T) := T^2 - hT - a$ *over* S.

(d) $\overline{S[\omega]}$ *is S-free with a basis given by the set* $\{1, \omega, \ldots, \omega^{p-2}, \tau\}$.

The next proposition characterizes when A is integrally closed.

Proposition 3.1.0.14. *With established notation, the following hold:*

1. *There exists a unique height one prime $P \subseteq A$ containing p.*

2. *The ring A is integrally closed if and only if A_P is a DVR.*

3. *The ring A is integrally closed if and only if $f \notin S^p$, $g \notin S[\omega]^{p \wedge p^2}_{(p)}$ (or vice versa).*

Proof. For (1), let $\phi : B := S[W, U] \to A$ be the natural projection map. Height one primes in A pull back to height three primes in B containing $Ker(\phi) = (F(W), G(U))B$. First assume that $f, g \in S^p$. Write $f = h_1^p + a \cdot p$ and $g = h_2^p + b \cdot p$ for some $h_1, h_2, a, b \in S$. It is then clear that the only height three prime in B containing $Ker(\phi)$ and p is $\tilde{P} \subset B$, given by $\tilde{P} := (p, W - h_1, U - h_2)B$. Therefore $P := (p, \omega - h_1, \mu - h_2)A$ is the unique height one prime in A containing p in this case.

Now let $f = h_1^p + a \cdot p$ and $g \notin S^p$. From 3.1.0.13, $S[\mu]$ is integrally closed. Since S/pS is integrally closed and $g \notin S^p$, $p \in S[\mu]$ is a principal prime. Since $A \simeq S[\mu][W]/(F(W))$, it is now clear that $P := (p, \omega - h_1)A$ is the unique height one prime in A containing p.

Now assume that $f, g \notin S^p$. As noted above, p is a principal prime in the integrally closed rings $S[\omega]$ and $S[\mu]$. We need to show that there exists a unique height three prime ideal of B minimal over $(p, F(W), G(U))B$ or equivalently a unique height one prime minimal in $C[U]$ over $(G(U))C$ where $C := S[\omega]/(pS[\omega])$ is a domain. Let Q be the fraction field of C. If $G(U)$ is irreducible over Q, then from 3.1.0.3 we get $C[\mu] \simeq C[U]/(G(U))$ so that p is a principal prime in A. If $G(U)$ is reducible over Q, then $G(U) = (U-r)^p$ in $Q[U]$ for some $r \in Q$, so that $(U-r)Q[U]$ is the unique minimal prime over $G(U)Q[U]$. Since every prime $T \subseteq C[U]$ minimal over $G(U)C[U]$ intersects trivially with C, there is a unique height one prime $P \subseteq A$ containing p.

For (2), observe that A satisfies S_2 since it is S-free. To show the reverse implication, we see from part (1) that it suffices to show that $A[1/p]$ is integrally closed. Applying Huneke & Katz (2019)[Proposition 5.3] to the ring $S[1/p]$, we see that $A[1/p]$ is indeed integrally closed. This completes the proof of the backward direction of (2). The forward direction is obvious.

For the backward direction of (3), assume that $f \notin S^p$ and $g \notin S[\omega]_{(p)}^{p \wedge p^2}$. As noted in the proof of part (1), $p \in S[\omega]$ is a prime. Suppose $g \notin S[\omega]_{(p)}^p$. Then $PA_P = pA_P$ and (2) implies that A is integrally closed. Next, suppose that $g - h^p \in pS[\omega]_{(p)}$ for some $h \in S[\omega]_{(p)}$. Since $A_P \simeq S[\omega][U]_{(p,U-h)}/(G(U))$, A_P is a DVR if and only if $G(U) \notin (p, U-h)^2 S[\omega][U]_{(p,U-h)}$. Since

$$U^{p-1} + \cdots + h^{p-1} \in (p, U-h)S[\omega][U] \tag{3.1.0.14.1}$$

we see from our hypothesis that $G(U) \notin (p, U-h)^2 S[\omega][U]_{(p,U-h)}$. Thus A_P is a DVR and the conclusion follows from part (2).

For the forward direction of (3), we prove the contrapositive. As a first case, suppose $f, g \in S^p$ and $f = h_1^p + ap$, $g = h_2^p + bp$ with $h_1, h_2, a, b \in S$. For any $1 \le i \le p-1$, notice that the element $\eta_i := p^{-1}(\omega - h_1)^i(\mu - h_2)^{p-i} \in K$ satisfies $\eta_i^p \in A$ since $(\omega - h_1)^p, (\mu - h_2)^p \in pA$. But $\eta_i \notin A$ since A is S-free with basis $\{\omega^i \mu^j \mid 0 \le i, j \le p-1\}$. Thus A is not integrally closed.

Now suppose that $f \notin S^p$ and $g \in S[\omega]_{(p)}^{p \wedge p^2}$. Let $g - h^p \in pS[\omega]_{(p)}$ for some $h \in S[\omega]_{(p)}$. We have $A_P \simeq S[\omega][U]_{(p,U-h)}/(G(U))$ since $p \in S[\omega]$ is prime. From our assumption and (3.1.0.14.1), it now follows that A_P is not a DVR and hence A is not integrally closed. This finishes the proof of the forward implication of (3). ■

We note down a natural extension of Katz (1999), 3.2:

Proposition 3.1.0.15. *With established notation, R is S-free if $f \notin S^p$ and $g \in S[\omega]^p$. In particular, if S is Cohen-Macaulay, then R is Cohen-Macaulay.*

Proof. Since $f \notin S^p$, $S[\omega]$ is integrally closed by 3.1.0.13 and $p \in S[\omega]$ is a principal prime. Moreover, 3.1.0.14(3) allows us to assume that $g \in S[\omega]_{(p)}^{p \wedge p^2}$. Write $g = h^p + pb$, with $h, b \in S[\omega]$. Note that $g \in S[\omega]_{(p)}^{p \wedge p^2}$ implies that $b \in pS[\omega]_{(p)} \cap S = pS[\omega]$. That is $g \in S[\omega]^{p \wedge p^2}$. In this case, the proof of 2.1.4.5 goes through, so that R is $S[\omega]$-free and hence S-free. Thus the proof is complete. ■

Remark 3.1.0.16. We will see in Chapter 4 (Sridhar (2021a)[Example 2.12]) that R need not be S-free when $f, g \notin S^p$. However, to construct a small CM module over R it suffices to consider the case $f, g \in S^p$ when S is a complete unramified regular local ring with perfect residue field, see 3.2.1. This motivates us to understand the case $f, g \in S^p$.

Convention 3.1.0.17. We maintain notation established in 3.1.0.8 and make the additional assumptions that $f, g \in S^p$ for the rest of this section.

Write $f = h_1^p + p \cdot a$ and $g = h_2^p + p \cdot b$ with $h_1, h_2, a, b \in S$. Note that under these assumptions, if $f \in pS$, $f = p \cdot a$ for some $a \notin pS$. This is because $f \in S$ is square free. So in this case $S[\omega]$ is necessarily integrally closed by 3.1.0.13.

We now identify scenarios where R is S-free - we handle the case $p \geq 3$ first. The splitting patterns of the primes lying over p are different in the case $p = 2$. Moreover, the extensions we consider are automatically Abelian in the mixed characteristic 2 case.

Proposition 3.1.0.18. *Assume $p \geq 3$. R is S-free if at least one of the rings $S[\omega], S[\mu]$ is not integrally closed.*

Proof. We organize the proof as follows:

1. Assume $S[\omega]$ and $S[\mu]$ are both not integrally closed. We then

(a) Identify a finite birational overring $A \hookrightarrow \mathscr{R}A$ such that $\mathscr{R}A$ satisfies R_1.

(b) Identify a “natural" finite birational overring $\mathscr{R}A \hookrightarrow Z$ such that Z is S-free, so that $R = Z$ is S-free.

2. Assume exactly one of the rings $S[\omega]$, $S[\mu]$ is integrally closed. We then take an identical path as indicated in (1) above.

1. (a) From 3.1.0.13, $f,g \in S^{p \wedge p^2}$. Write $f = h_1^p + a' \cdot p^2$ and $g = h_2^p + b' \cdot p^2$ for some $a', b' \in S$. Note that $h_1, h_2 \neq 0$ since f,g are square free. We have from 3.1.0.13 that $S[\omega, \tau_1], S[\mu, \tau_2]$ are the respective normalizations of $S[\omega]$ and $S[\mu]$ where

$$\tau_1 = p^{-1} \cdot (\omega^{p-1} + h_1 \omega^{p-2} + \cdots + h_1^{p-1})$$

$$\tau_2 = p^{-1} \cdot (\mu^{p-1} + h_2 \mu^{p-2} + \cdots + h_2^{p-1})$$

Set $E := A[\tau_1, \tau_2]$. Let X,Y be indeterminates over A and let $\phi : A[X,Y] \to E$ be the projection map sending $X \tau_1$ and $Y \tau_2$. From 3.1.0.13:

$$X^2 - c_1'X - a'(\omega - h_1)^{p-2}, Y^2 - c_2'Y - b'(\mu - h_2)^{p-2} \in Ker(\phi)$$

Height one primes in E containing p correspond to height three primes in $A[X,Y]$ containing $Ker(\phi)$ and p. Since $P := (p, \omega - h_1, \mu - h_2)$ is the unique height one prime in A containing p, any such height three prime in $A[X,Y]$ has to contain either X or $X - c_1'$. Likewise it contains either Y or $Y - c_2'$. Therefore if $Q \subseteq A[X,Y]$ is a height three prime containing p and $Ker(\phi)$, it must be that $(p, \omega - h_1, \mu - h_2, X - m, Y - n) \subseteq Q$ for some $m,n \in A$ and hence the containment must be an equality. Moreover, there is at least one height one prime in E containing p, since p is not a unit in S. Therefore, the only possibilities for height one primes in E containing p are

$$P_1 := (p, \omega - h_1, \mu - h_2, \tau_1, \tau_2)$$

$$P_2 := (p, \omega - h_1, \mu - h_2, \tau_1, \tau_2 - c_2')$$

$$P_3 := (p, \omega - h_1, \mu - h_2, \tau_1 - c_1', \tau_2)$$

$$P_4 := (p, \omega - h_1, \mu - h_2, \tau_1 - c_1', \tau_2 - c_2')$$

We have $\omega \cdot F'(\omega) = p \cdot f \in (S[\mu, \tau_2, \omega] :_K R)$ and identically $p \cdot g \in (S[\omega, \tau_1, \mu] :_K R)$ (see for example Huneke & Swanson (2006a)[Theorem 12.1.1]) and hence $p \cdot f, p \cdot g \in (E :_K R)$. From 3.1.0.5, $NNL_1(E) \subseteq V(p)$. But the localizations of E at P_2, P_3 and P_4 are regular with uniformizing parameters being the images of $\omega - h_1, \mu - h_2$ and p respectively. For example, consider P_{2P_2}. Let $Q_1 := (p, \omega - h_1, \tau_1)S[\omega, \tau_1]$ and $Q_2 := (p, \mu - h_2, \tau_2 - c_2')S[\mu, \tau_2]$. From 3.1.0.13, $Q_{1Q_1} = (\omega - h_1)_{Q_1}$ and $Q_{2Q_2} = (p)_{Q_2}$. Thus $P_{2P_2} = (\omega - h_1)_{P_2}$. The P_3 and P_4 cases are similar. Note that however $P_{1P_1} = (\omega - h_1, \mu - h_2)_{P_1}$.

Set $\eta_1 = p^{-1}(\omega - h_1)(\mu - h_2)^{p-2} \in K$. Let X be an indeterminate over E. Then η_1 satisfies $l(X) \in E[X]$ where $l(X) := X^{p-1} - (\tau_1 - c_1')(\tau_2 - c_2')^{p-2}$ since $p \cdot \tau_1 = (\omega - h_1)^{p-1} + p \cdot c_1'$ (similarly for $p \cdot \tau_2$). We claim that $\mathscr{R}A := E[\eta_1]$ is regular in codimension one. From 3.1.0.4, $NNL_1(\mathscr{R}A) \subseteq V(P_1 \mathscr{R}A)$. Denote by $\overline{l(X)}$ the image of $l(X)$ in $(E/P_1E)[X]$. From 3.1.0.11 we have $c_1' \equiv h_1^{p-1}$ and $c_2' \equiv h_2^{p-1}$ in the ring E/P_1E, and thus [1]

$$\overline{l(X)} = X^{p-1} - (h_1 h_2^{p-2})^{p-1} = \prod_{k=1}^{p-1} (X + k h_1 h_2^{p-2}) \in (E/P_1E)[X]$$

Thus, the only possibilities for height one primes in $\mathscr{R}A$ lying over P_1 are

$$Q_k = (p, \omega - h_1, \mu - h_2, \tau_1, \tau_2, \eta_1 + k h_1 h_2^{p-2})\mathscr{R}A$$

[1] If R is a ring of characteristic p and X, Y indeterminates over R, then for $X^{p-1} - Y^{p-1} \in R[X,Y]$, $X^{p-1} - Y^{p-1} = \prod_{i=1}^{p-1}(X + iY)$.

for $1 \leq k \leq p-1$ and we have

$$Q_{kQ_k} = (\omega - h_1, \mu - h_2, \eta_1 + kh_1 h_2^{p-2})_{Q_k}$$

Since $(\mu - h_2)\eta_1 = (\tau_2 - c_2')(\omega - h_1)$ and

$$\eta_1, \tau_2 - c_2', \prod_{j=1, j\neq k}^{p-1} (\eta_1 + jh_1 h_2^{p-2}) \notin Q_k$$

we have $Q_{kQ_k} = (\mu - h_2)_{Q_k} = (\omega - h_1)_{Q_k}$. Therefore $\mathscr{R}A$ is regular in codimension one.

(b) Set $Z := \langle T \rangle_S$ where $T := T_1 \cup T_2 \cup T_3$ and

$$T_1 := \{(\omega - h_1)^i (\mu - h_2)^j \,|\, 0 \leq i, j \leq p-1, i + j < p-1\}$$

$$T_2 := \{p^{-1}(\omega - h_1)^i (\mu - h_2)^j \,|\, 0 \leq i, j \leq p-1, i + j \geq p-1, i + j \neq 2p-2\}$$

$$T_3 := \{p^{-2}(\omega - h_1)^{p-1} (\mu - h_2)^{p-1}\}$$

For every choice of $0 \leq i, j \leq p-1$, there is a unique element in T with "leading coefficient" $\omega^i \mu^j$. Therefore the order of T is p^2. Moreover, since A is S-free with a basis given by $D := \{(\omega - h_1)^i (\mu - h_2)^j \,|\, 0 \leq i, j \leq p-1\}$, the elements of T are linearly independent over S. From the relations

$$(\omega - h_1)^p = a' p^2 - pc_1'(\omega - h_1)$$

$$(\mu - h_2)^p = b' p^2 - pc_2'(\mu - h_2)$$

we get that Z is a ring. Since Z satisfies S_2, if we show $\mathscr{R}A \subseteq Z$, then from 3.1.0.4 $Z = R$. Since $D \subseteq T$, we have $A \subseteq Z$. Since $\eta_1 = p^{-1}(\omega - h_1)(\mu - h_2)^{p-2} \in T$, it only remains to be seen that $\tau_1, \tau_2 \in Z$. But this is clear from the relation $\tau_1 =$

$p^{-1}(\omega - h_1)^{p-1} + c_1'$ (analogously for τ_2). Thus $Z = R$ and R is S-free.

2. (a) Assume without loss of generality $\overline{S[\omega]} = S[\omega, \tau]$ where $\tau = p^{-1}(\omega^{p-1} + \cdots + h_1^{p-1})$ and that $S[\mu]$ is integrally closed. Notice that if $P := (p, \mu - h_2) \subseteq S[\mu]$ is the unique height one prime in $S[\mu]$ containing p, then $P_P = (\mu - h_2)_P$ since $(\mu - h_2)(\mu^{p-1} + \cdots + h_2^{p-1}) = bp$ and $b \notin pS$. From 3.1.0.13, $f \in S^{p \wedge p^2}$, so write $f = h_1^p + a'p^2$.

   Set $E := S[\omega, \mu, \tau]$. From 3.1.0.13, it follows that there are precisely two height one primes in E containing p, namely $P_1 := (p, \omega - h_1, \tau, \mu - h_2)$ and $P_2 := (p, \omega - h_1, \tau - c_1', \mu - h_2)$. From 3.1.0.5, E is regular in codimension one outside of P_1 and P_2. It follows from 3.1.0.13(b) that $P_{1P_1} = (\omega - h_1, \mu - h_2)_{P_1}$ and $P_{2P_2} = (p, \mu - h_2)_{P_2} = (\mu - h_2)_{P_2}$.

   Set $\eta_1 := p^{-1}(\omega - h_1)(\mu - h_2)^{p-1} \in K$. $\eta_1 \in R$ since it satisfies

$$l(X) := X^{p-1} - (\tau - c_1')k_2^{p-2}(\mu - h_2) \in E[X] \qquad (3.1.0.18.1)$$

   where $k_2 = p^{-1}(\mu - h_2)^p \in A \setminus P$. Set $\mathscr{R}A := E[\eta_1]$. From 3.1.0.4, $\mathscr{R}A$ is regular in codimension one if height one primes in $\mathscr{R}A$ lying over P_1E are regular. From the above integral equation for η_1 over E, it is clear that the only such height one prime in $\mathscr{R}A$ is $Q_1 := (p, \omega - h_1, \tau, \mu - h_2, \eta_1)$. Now $Q_{1Q_1} = (\omega - h_1, \mu - h_2, \eta_1)_{Q_1}$. But $\eta_1(\mu - h_2) = k_2(\omega - h_1)$ and $k_2 \notin Q_1$. Further, since $\tau - c_1' \notin Q_1$, $Q_{1Q_1} = (\eta_1)_{Q_1}$. Therefore $\mathscr{R}A$ is regular in codimension one.

   (b) Set $Z := < T >_S$ where $T = T_1 \cup T_2 \cup T_3$ and

$$T_1 := \{(\omega - h_1)^i (\mu - h_2)^j \mid i + j < p, 0 \le i \le p-2, 0 \le j \le p-1\}$$

$$T_2 := \{p^{-1}(\omega - h_1)^i (\mu - h_2)^j \mid i + j \ge p, 1 \le i \le p-1, 1 \le j \le p-1\}$$

$$T_3 := \{\tau - c_1' = p^{-1}(\omega - h_1)^{p-1}\}$$

For every $0 \leq i, j \leq p-1$, there is a unique element in T with "leading coefficient" $\omega^i \mu^j$. Therefore the order of T is p^2. Moreover since A is S-free with a basis given by $D := \{(\omega - h_1)^i (\mu - h_2)^j \,|\, 0 \leq i, j \leq p-1\}$, the elements of T are linearly independent over S. From the relations

$$(\omega - h_1)^p = a' p^2 - p c'_1 (\omega - h_1)$$

$$(\mu - h_2)^p = bp - p c'_2 (\mu - h_2)$$

we see that Z is a ring. Since Z satisfies S_2, if we show that $\mathscr{R}A \subseteq Z$, then from 3.1.0.4 $Z = R$. It now suffices to note that $D \subseteq Z$, $\eta_1 \in T$ and $\tau := p^{-1}(\omega - h_1)^{p-1} + c'_1 \in Z$, so that $\mathscr{R}A \subseteq Z$. Thus $R = Z$ is S-free.

■

Lemma 3.1.0.19. *With established notation, assume that $S[\omega]$ and $S[\mu]$ are integrally closed. The following hold*

1. $(P^{(p-1)})^*_A = \langle T \rangle_S$, *where $P^{(p-1)}$ denotes the $(p-1)$-th symbolic power of the unique height one prime $P \subseteq A$ containing p and $T := T_1 \cup T_2$, with*

$$T_1 := \{(\omega - h_1)^i (\mu - h_2)^j |\, 0 \leq i, j \leq p-1 \,,\, i + j < p\}$$

$$T_2 := \{p^{-1}(\omega - h_1)^i (\mu - h_2)^j |\, 0 \leq i, j \leq p-1 \,,\, i + j \geq p\}$$

2. *The ring $A/P^{(p-1)}$ is Cohen-Macaulay.*

Proof. For every $0 \leq i, j \leq p-1$, there is a unique element in T with "leading coefficient" $\omega^i \mu^j$. Therefore the order of T is p^2. Moreover since A is S-free with a basis given by $D := \{(\omega - h_1)^i (\mu - h_2)^j \,|\, 0 \leq i, j \leq p-1\}$, the elements of T are linearly independent over S. From the relations

$$(\omega - h_1)^p = ap - p c'_1 (\omega - h_1)$$

$$(\mu - h_2)^p = bp - pc_2'(\mu - h_2)$$

we see that $\langle T \rangle_S$ is a ring (in particular it is an A-module). Moreover, since it is Cohen-Macaulay, (2) immediately follows from (1) by using **??**. Therefore, only (1) remains to be shown. Since $(P^{(p-1)})^*$ and $\langle T \rangle_S$ are birational, S_2 A-modules, it suffices to show their equality in codimension one. If $Q \subseteq A$, $Q \neq P$ is a height one prime, the equality is clear. So localize A at P and assume (A,P) local for the rest of the proof. Then $\langle T \rangle_S = A[\eta]$ where $\eta := (\omega - h_1)^{-1}(\mu - h_2)$. Note that $A = B'/(G(U))$, where $B' := S[\omega][U]$. Set $\tilde{I} := (\omega - h_1, U - h_2)^{p-1} \subseteq B'$. We have $G(U) \in \tilde{I}$:

$$\begin{aligned} U^p - g &= (U - h_2)^p + p(C_2'(U - h_2) - b) \\ &= -k_1^{-1}(\omega - h_1)(C_2'(U - h_2) - b) \cdot \Delta_1 + 0 \cdot \Delta_2 + \cdots + 0 \cdot \Delta_{p-1} - (U - h_2) \cdot \Delta_p \end{aligned}$$

where $\Delta_i = (-1)^i e_i$ with $e_i = (\omega - h_1)^{p-i}(U - h_2)^{i-1}$ and $k_1 = p^{-1}(\omega - h_1)^p$. That is $\tilde{I}$ is the lift to B' of P^{p-1}. Further, $\tilde{I}$ is grade two perfect since it arises as the ideal of maximal minors of the $p \times (p-1)$ matrix M :

$$M = \begin{bmatrix} U - h_2 & 0 & \dots & 0 & 0 \\ \omega - h_1 & U - h_2 & 0 & \dots & 0 \\ 0 & \omega - h_1 & U - h_2 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \omega - h_1 & U - h_2 \\ 0 & \dots & 0 & 0 & \omega - h_1 \end{bmatrix}$$

Let M' be the $p \times p$ matrix obtained by adjoining M with the column of coefficients of $G(U)$:

$$M' = \begin{bmatrix} U-h_2 & 0 & \dots & 0 & 0 & -k_1^{-1}(\omega-h_1)(C_2'(U-h_2)-b) \\ \omega-h_1 & U-h_2 & 0 & \dots & 0 & 0 \\ 0 & \omega-h_1 & U-h_2 & \dots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \omega-h_1 & U-h_2 & 0 \\ 0 & \dots & 0 & 0 & \omega-h_1 & -(U-h_2) \end{bmatrix}$$

By Kleiman & Ulrich (1997)[Lemma 2.5] (or Katz (1999)[Prop 2.1]), $(P^{p-1})^*$ is generated as an A-module by the set $\{\delta_i^{-1}M'_{i,i} \mid 1 \le i \le p\}$, where δ_i denotes the image of Δ_i in A and $M'_{i,i}$ the image in A of the (i,i)-th cofactor of M'. This is exactly the set $\{\eta^{p-1}, \eta^{p-2}, \dots, \eta, 1\}$. Since η satisfies the integral equation $X^p - k_1^{-1}k_2 \in A[X]$ (with $k_2 = p^{-1}(\mu-h_2)^p$), this implies $(P^{p-1})^* = A[\eta] = \langle T \rangle_S$. Thus the proof is complete. ∎

Proposition 3.1.0.20. *With established notation, R is S-free if $S[\omega]$ and $S[\mu]$ are integrally closed and $fg^i \notin S^{p\wedge p^2}$ for $1 \le i \le p-1$. Further in this case, $P^{(p-1)}$ is the conductor of R to A where P is the unique height one prime in A containing p and $P^{(p-1)}$ denotes the $(p-1)$-th symbolic power of P.*

Proof. Since $S[\omega]$ and $S[\mu]$ are integrally closed, we have from 3.1.0.13 that $f, g \notin S^{p\wedge p^2}$. Write $f = h_1^p + ap$ and $g = h_2^p + bp$ with $a, b \notin pS$. We first note the following: The condition $fg^i \notin S^{p\wedge p^2}$ for all $1 \le i \le p-1$ is equivalent to the condition $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$. This follows since for $1 \le i \le p-1$

$$\begin{aligned} fg^i &= (h_1^p + ap)(h_2^p + bp)^i \\ &= (h_1h_2^i)^p + h_2^{p(i-1)}(ah_2^p + ibh_1^p) \cdot p + q \cdot p^2 \end{aligned}$$

for some $q \in S$. We organize the proof as follows:

- We first construct the normalization of A locally at $NNL_1(A)$. Since $NNL_1(A) = \{P\}$, we only need to construct R_P.

- Using (1), identify a finite birational overring $A \hookrightarrow \mathscr{R}A$ such that $\mathscr{R}A$ satisfies R_1. Then choose a suitable finite birational overring $\mathscr{R}A \hookrightarrow Z$ such that Z is S-free. This would show $R = Z$ is S-free.

1. From 3.1.0.5, A is regular in codimension one outside of $V(p)$. Moreover $P := (p, \omega - h_1, \mu - h_2)$ is the only height one prime in A containing p. Localize at P and assume (A, P) local for part (1). Now $(\omega - h_1)^p = pk_1$ and $(\mu - h_2)^p = pk_2$ where $k_1 = a - c_1'(\omega - h_1)$ and $k_2 = b - c_2'(\mu - h_2)$. Since $k_1, k_2 \notin P$, we have $P = (\omega - h_1, \mu - h_2)$. The element $\eta = (\omega - h_1)^{-1}(\mu - h_2) \in K$ satisfies the integral equation $l(X) := X^p - k_1^{-1}k_2 \in A[X]$. If $\eta \in A$, then $P_P = (\omega - h_1)_P$ so A is integrally closed. But from **?**[Proposition 2.7] this is impossible. Therefore $A[\eta]$ is a proper birational extension of A. We claim that $E := A[\eta]$ is regular when $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$. We will observe that E is local with maximal ideal $Q \subseteq E$ either of the form $Q = PE$ or $Q = (P, \eta - r)E$ for some suitable $r \in S \setminus pS$. To see this, let $\phi : \tilde{E} := S[W,U]_{\tilde{P}}[X] \longrightarrow E$ be the natural projection map sending $W \mapsto \omega, U \mapsto \mu, X \mapsto \eta$, where W, U, X are indeterminates over S and $\tilde{P} := (p, W - h_1, U - h_2)$. Let $Q \subseteq E$ be any maximal ideal and let $\tilde{Q}$ be the preimage of Q under ϕ. Now

$$l(X) \equiv X^p - ba^{-1} \in (A/P)[X] \simeq (S/pS)[X] \tag{3.1.0.20.1}$$

Since S/pS is a field, if $l(X)$ is an irreducible polynomial over $(S/pS)[X]$ then

$$(p, W - h_1, U - h_2, l(X)) \subseteq \tilde{Q}$$

is a height four prime containing p. Therefore the above inclusion must be an equality. If on the other hand $l(X)$ is reducible over S/pS then $l(X) \equiv (X - r)^p \in (S/pS)[X]$ for some $r \in S \setminus pS$. So in this case

$$(p, W - h_1, U - h_2, X - r) \subset \tilde{Q}$$

is a height four prime. Again, the above inclusion must then be an equality. Therefore in either case E is local and the maximal ideal is either of the form $PE = (\omega - h_1, \mu - h_2)E$ or $Q := (\omega - h_1, \mu - h_2, \eta - r)E$. In the first case, since $\eta \cdot (\omega - h_1) = \mu - h_2$, E is a DVR. In the second case, we have $Q_Q = (\omega - h_1, \eta - r)_Q$. We now show that $Q_Q = (\eta - r)_Q$. We have for some $m \in E$:

$$\begin{aligned}(\eta - r)^p &= \eta^p - ba^{-1} + pm \\ &= k_1^{-1} k_2 - ba^{-1} + pm \\ &= k_1^{-1}[-c_2'(\mu - h_2) + ba^{-1}c_1'(\omega - h_1) + pmk_1] \\ &= k_1^{-1}(\omega - h_1)[-c_2'\eta + ba^{-1}c_1' + (\omega - h_1)^{p-1}m]\end{aligned}$$

So Q is principal if $\alpha := c_2'\eta - ba^{-1}c_1'$ is invertible in E. To show $\alpha \in E$ is a unit, it suffices to show $(ac_2'\eta - bc_1')^p \in E$ is invertible. From 3.1.0.11, $c_1' \equiv h_1^{p-1} \bmod Q$ and $c_2' \equiv h_2^{p-1} \bmod Q$. We then have

$$\begin{aligned}(ac_2'\eta - bc_1')^p &\equiv a^p(c_2')^p\eta^p - b^p(c_1')^p \bmod Q \\ &\equiv b(a^{p-1}(c_2')^p - b^{p-1}(c_1')^p) \bmod Q \\ &\equiv b[a^{p-1}(h_2^{p-1})^p - b^{p-1}(h_1^{p-1})^p] \bmod Q \\ &\equiv b\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \bmod Q\end{aligned}$$

Thus $\alpha \in E$ is a unit. Hence $E = A[\eta]$ is regular, that is $E = R$.

2. Set $\mathscr{R}A := A[k_1\eta]$ for $\eta = (\omega - h_1)^{-1}(\mu - h_2)$ and $k_1 = p^{-1}(\omega - h_1)^p$. Note that $k_1\eta = p^{-1}(\omega - h_1)^{p-1}(\mu - h_2)$ and that it satisfies the integral equation $X^p - k_1^{p-1}k_2 \in A[X]$ for $k_2 := p^{-1}(\mu - h_2)^p$. Since $k_1 \notin P$, by 3.1.0.4 and part (1) of the proof, $\mathscr{R}A$ is regular in codimension one.

   Set $Z := \langle T \rangle_S$ where T is as in the statement of 3.1.0.19. We see that Z is a ring from the relations

$$(\omega-h_1)^p = ap - pc_1'(\omega-h_1)$$

$$(\mu-h_2)^p = bp - pc_2'(\mu-h_2)$$

Moreover it is a free S-module of rank p^2. Clearly $\mathscr{R}A \subseteq Z$, so Z inherits R_1 from $\mathscr{R}A$. Thus $Z = R$ and R is S-free.

Finally, from 3.1.0.19(1) $P^{(p-1)}$ is contained in the conductor J of R to A. Since A_P is a one dimensional Gorenstein local ring, $J_P = (P^{p-1})_P$ and thus $J \subseteq P^{(p-1)}$. Thus $P^{(p-1)}$ is the conductor of R to A.

■

Remark 3.1.0.21. The condition $fg^i \notin S^{p\wedge p^2}$ for $1 \leq i \leq p-1$ in 3.1.0.20 is saying that some suitable subrings of A are integrally closed. As noted in the proof of 3.1.0.20, the condition is equivalent to $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$. Let $1 \leq k, i \leq p-1$ and $1 \leq i(k) \leq p-1$ be such that $i(k) - ik \in p\mathbb{Z}$. The condition $\prod_{i=1}^{p-1}(ah_2^p + ibh_1^p) \notin pS$ is saying that for all $1 \leq i \leq p-1$, $A_i := S[\omega\mu^{i(1)}, \ldots, \omega^j\mu^{i(j)}, \ldots, \omega^{p-1}\mu^{i(p-1)}]$ is integrally closed. Indeed

$$(\omega\mu^i)^p = fg^i = (h_1h_2^i)^p + (ah_2^{ip} + ibh_1^p h_2^{ip-p})p + p^2q$$

for some $q \in S$. If $i = 1$, we have that fg is squarefree in S and by 3.1.0.13 $S[\omega\mu]$ is integrally closed. If $i \neq 1$, the given condition is equivalent to saying that

$$NNL_1(S[\omega\mu^i]) \cap V(p) = \emptyset$$

Moreover $NNL_1(S[\omega\mu^i]) \subseteq V(g)$. Choose k such that $i(k) = 1$, so that $S[\omega\mu^i, \omega^k\mu]$ is a finite birational extension of both $S[\omega\mu^i]$ and $S[\omega^k\mu]$. Now $NNL_1(S[\omega^k\mu]) \subseteq V(f)$. So by 3.1.0.4, $S[\omega\mu^i, \omega^k\mu]$ is regular in codimension one since $f, g \in S$ satisfy $\mathscr{A}_1$. Since A_i is a finite birational extension of $S[\omega\mu^i, \omega^k\mu]$, it is regular in codimension one for the same reason. The remark follows since it is easily checked that A_i is S-free.

Remark 3.1.0.22. The powers of the prime $P^{p-1} \subseteq A$ in 3.1.0.20 are not P-primary in general. For example if $p = 3$, observe that $3a, 3b \in P^2$. However, it holds that $P^{(p-1)} = (p) + P^{p-1}$.

3.1.1 Mixed characteristic two case

As noted earlier, the extensions we consider are automatically Abelian in mixed characteristic two and hence the results we obtain here are sharper. However, the reason to treat it independently in this chapter is that the splitting pattern of primes lying over the principal ideal generated by p are different if $p = 2$. For this subsection, we maintain notation established in 3.1.0.17 and additionally assume $p = 2$.

Proposition 3.1.1.1. *R is S-free if at least one of the rings $S[\omega], S[\mu]$ is not integrally closed.*

Proof. First assume that both $S[\omega]$ and $S[\mu]$ are not integrally closed. We have that $S[\tau_1]$ is integrally closed from 3.1.0.13, for $\tau_1 := 2^{-1}(\omega + h_1)$. Further τ_1 satisfies $l_1(T) := T^2 - h_1T - a' \in S[T]$ where $a = 2a'$ for some $a' \in S$ and T is an indeterminate over S. Further, $2 \in S[\tau_1]$ is square free since $l_1(T)$ and $l_1'(T)$ are relatively prime over the quotient field of $S/2S$. Writing $b = 2b'$ for some $b' \in S$, we also have that $l_2(T) := T^2 - h_2T - b'$ and $l_2'(T)$ are relatively prime over the quotient field of $S[\tau_1]/Q$ for all height one primes $Q \subseteq S[\tau_1]$ containing 2. Therefore $2 \in E = S[\tau_1, \tau_2]$ is square free as well. Applying Huneke & Katz (2019)[Proposition 5.3] to the ring $S[1/2]$, we see that $R[1/2] = A[1/2] \subseteq E[1/2] \subseteq R[1/2]$. Therefore $NNL_1(E) \subseteq V(2)$. Since $2 \in E$ is square free, E is regular in codimension one. Clearly E is generated over S by $\{1, \tau_1, \tau_2, \tau_1\tau_2\}$ and hence E is S-free of rank four. Thus E satisfies Serre's criterion S_2 and is integrally closed, that is $E = R$.

Next, without loss of generality assume $S[\mu]$ is integrally closed and $S[\omega]$ is not. From 3.1.0.13, we have $S[\tau_1]$ is integrally closed for $\tau_1 := 2^{-1}(\omega + h_1)$ and that $2 \in S[\tau_1]$ is square free. Since $E := S[\tau_1, \mu] \simeq S[\tau_1][U]/(G(U))$, height one primes in E containing 2 are of the form $(Q, \mu - h_2)E$ where $Q \subseteq S[\tau_1]$ is a height one prime containing 2. By 3.1.0.14(2) and 3.1.0.4, $NNL_1(E) \subseteq V(2)$. But for any height one prime $\mathscr{P} := (Q, \mu - h_2) \subseteq E$ containing 2, $\mathscr{P}_{\mathscr{P}} = (\mu - h_2)_{\mathscr{P}}$. This is because going down holds for the extension $S \subseteq S[\mu] \subseteq E$, so $\mathscr{P}$ contracts back to the height one prime $P := (2, \mu - h_2) \subseteq S[\mu]$. Since $(\mu - h_2)(\mu + h_2) = 2b$ and $b \notin P$ by 3.1.0.13, we have $PS[\mu]_P = (\mu - h_2)_P$. Thus, $\mathscr{P}_{\mathscr{P}} = (\mu - h_2)_{\mathscr{P}}$ and E is regular in codimension one. Clearly E is generated over S by $\{1, \mu, \tau_1, \mu\tau_1\}$. Thus E is S-free of rank four and hence satisfies Serre's criterion S_2. So $E = R$ and this completes the proof. ■

Proposition 3.1.1.2. *With established notation, R is S-free if $S[\omega]$ and $S[\mu]$ are integrally closed and $fg \notin S^{2\wedge 4}$. Further, in this case $P^*_A = R$ so that P is the conductor of R to A, where P is the unique height one prime in A containing* 2.

Proof. Since $S[\omega]$ and $S[\mu]$ are integrally closed, we have from 3.1.0.13 that $f, g \notin S^{2\wedge 4}$. Write $f = h_1^2 + 2 \cdot a$ and $g = h_2^2 + 2 \cdot b$ with $a, b \notin 2S$. The condition $fg \notin S^{2\wedge 4}$ is equivalent to the condition $(ah_2^2 + bh_1^2) \notin 2S$. This follows since

$$\begin{aligned} fg &= (h_1^2 + 2a)(h_2^2 + 2b) \\ &= (h_1 h_2)^2 + (ah_2^2 + bh_1^2) \cdot 2 + 4ab \end{aligned} \tag{3.1.1.2.1}$$

Note that the above is equivalent to requiring that $S[\omega\mu]$ be integrally closed. This is because, since $f, g \in S$ satisfy $\mathscr{A}_1$, $fg \in S$ is square free. Following this, $S[\omega\mu]$ is integrally closed if and only if $fg \notin S^{2\wedge 4}$ by 3.1.0.13.

Let $\tau = 2^{-1}(\mu - h_2)(\omega - h_1) \in K$. We see that τ satisfies

$$l(T) := T^2 - k_1 k_2 \in A[T]$$

where $k_1 := 2^{-1}(\omega - h_1)^2 = h_1^2 + a - \omega h_1$, $k_2 := 2^{-1}(\mu - h_2)^2 = h_2^2 + b - \mu h_2$ and T is an indeterminate over A. Note that

$$2 = k_1^{-1}(\omega - h_1)^2 = k_2^{-1}(\mu - h_2)^2 \tag{3.1.1.2.2}$$

and $k_1, k_2 \notin P$. We claim that $C := S[\omega, \mu, \tau]$ is integrally closed under the given hypothesis. The unique height one prime $P \subseteq A$ containing 2 is $P := (2, \omega - h_1, \mu - h_2)$. Now

$$l(T) \equiv T^2 - ab \in (A/P)[T] \simeq (S/2S)[T]$$

There exists a unique height one prime containing 2 in C and since $S/2S$ is integrally closed, the only possible forms for this unique height one prime are $Q_1 := PC$ or $Q_2 := (2, \omega - h_1, \mu - h_2, \tau - m)C$ for some $m \in S$ satisfying $m^2 - ab \in 2S$. In the first case, Q_{1Q_1} is principal due to (3.1.1.2.2) and

$$(\mu - h_2)\tau = k_2(\omega - h_1) \tag{3.1.1.2.3}$$

Now let $Q_2 \subseteq C$ be the unique height one prime in question. We will show that $Q_{2Q_2} = (\tau - m)_{Q_2}$. First from (3.1.1.2.2) and (3.1.1.2.3), we have $Q_{2Q_2} = (\tau - m, \mu - h_2)_{Q_2}$. By definition, we have in C_{Q_2}

$$\begin{aligned} (\tau - m)^2 &\equiv (k_1k_2 + ab) \bmod (2) \\ &\equiv (ah_2(\mu - h_2) + bh_1(\omega - h_1) + h_1h_2(\omega - h_1)(\mu - h_2)) \bmod (2) \\ &\equiv (\mu - h_2)(ah_2 + bh_1k_2^{-1}\tau + h_1h_2(\omega - h_1)) \bmod (2) \end{aligned} \tag{3.1.1.2.4}$$

where the last equivalence follows from (3.1.1.2.3). We claim that $ah_2 + bh_1k_2^{-1}\tau$ is a unit in C_{Q_2}. If the claim holds, it follows from (3.1.1.2.2) that $Q_{2Q_2} = (\tau - m)_{Q_2}$. To show the claim, assume on the contrary that $ah_2 + bh_1k_2^{-1}\tau \in Q_{2Q_2}$. Then

$$k_2(a^2h_2^2 + b^2h_1^2k_2^{-2}\tau^2) = k_2a^2h_2^2 + b^2h_1^2k_1 \in Q_{2Q_2}$$

By definition of k_1, k_2 we get $(ah_2^2 + bh_1^2)ab \in Q_{2Q_2}$ and hence $(ah_2^2 + bh_1^2) \in Q_{2Q_2} \cap S = 2S$. This contradicts our hypothesis. Thus the claim is true and Q_{2Q_2} is principal. From 3.1.0.14(2) and 3.1.0.4, C is regular in codimension one. Let D denote the S-module generated by $G := \{1, \omega, \mu, \tau\}$. Note that D is in fact a ring and is S-free of rank four. Then $D \subseteq C \subseteq D$. Thus $C = D$ satisfies S_2 and hence $C = R$ is S-free.

We now show that P is the conductor of R to A. Since A is not integrally closed, A_P is not a DVR by 3.1.0.14(2). Therefore the conductor is contained in P. On the other hand, since $P \cdot \tau \subseteq A$ and $R = A + S \cdot \tau$, P conducts R into A. Thus P is the conductor of R to A and the proof is complete. ■

3.2 General Radical towers

The work in this section is based of joint work with Daniel Katz in Katz & Sridhar (2021). Here we search for patterns that ensure the integral closure of an unramified regular local ring of mixed characteristic $p > 0$ in a finite general radical tower of order $p > 0$ of its quotient field is a free module.

The first good scenario occurs when the elements whose roots we adjoin all are chosen generally from the multiplicative subset $S^{p \wedge p^2}$ (see 3.1.0.1), where S is the regular local ring. This is a generalization of 3.1.0.18 and 3.1.1.1.

Convention 3.2.0.1.

- For a prime integer p, $\zeta_p(X) \in \mathbb{Z}[X]$ will denote the p-th cyclotomic polynomial.
- For a Noetherian ring R and a prime ideal $Q \subseteq R$, the notation $(Q \mid q_1, \ldots, q_r)$ for $q_i \in R$ means $Q_Q = (q_1, \ldots, q_r)_Q$.

We include the following two results from Huneke & Katz (2019) for convenience:

Proposition 3.2.0.2 (Huneke & Katz (2019))**.** *Let S be an integrally closed Noetherian domain and $n \in S$ a unit for some positive integer n. Let $a_1, \ldots, a_r \in S$ be square free elements satisfying $\mathscr{A}_1$. Then $a_2, \ldots, a_r$ are square free in $S[\sqrt[n]{a_1}]$.*

Proposition 3.2.0.3 (Huneke & Katz (2019))**.** *Let S be an integrally closed Noetherian domain and $n \in S$ a unit for some positive integer n. Let $a_1, \ldots, a_r \in S$ be square free elements satisfying $\mathscr{A}_1$. Then $R = S[\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_r}]$ is integrally closed.*

We set up notation for this section now. We work over a regular local ring for simplicity, although results may apply in more generality.

Convention 3.2.0.4. Throughout this section $(S,\mathfrak{m},\mathbf{k})$ will denote an unramified regular local ring of mixed characteristic $p>0$ and L its field of fractions. Let $\mathscr{E}:=\{r_1,\ldots,r_n\}\subseteq S$ consist of square free elements that are not p-th powers in S. Assume that $\mathscr{E}$ satisfies $\mathscr{A}_1$. Let $X_1,\ldots,X_n$ be indeterminates over S. We have for each $1\le i\le n$, the monic irreducible polynomial $E_i:=X_i^p-r_i\in S[X_i]$. For each i, let κ_i be a root of E_i. Let $K:=L(\kappa_1,\ldots,\kappa_n)$ and let R denote the integral closure of S in K. That is R is the integral closure of

$$A:=S[\kappa_1,\ldots,\kappa_n]\simeq\frac{S[X_1,\ldots,X_n]}{(X^p-r_1,\ldots,X^p-r_n)}$$

.

Our conditions ensure that $[K:L]$ has the correct degree:

Remark 3.2.0.5. The conditions in 3.2.0.4 ensure that $[K:L]=p^n$. To see this, first note that $Tr_{K/L}(y)=0$ for $y:=\prod_{i=1}^n\kappa_i^{t_i}\in K$ and integers $0\le t_i\le p-1$ not all zero. Indeed, since $y^p\in L$, we only need to show that $y\notin L$. But if $y\in L$, then $y\in S$ since S is integrally closed. Therefore $\prod_{i=1}^n r_i^{t_i}\in S$ is a p-th power. This is impossible since the r_i are relatively prime and not p-th powers themselves. Thus $Tr_{K/L}(y)=0$.

Suppose that E_n is reducible over $L(\kappa_1,\ldots,\kappa_{n-1})$. Then necessarily $\kappa_n\in L(\kappa_1,\ldots,\kappa_{n-1})$. Write $\kappa_n=\sum a_{(i_1,\ldots,i_{n-1})}\kappa_1^{i_1}\ldots\kappa_{n-1}^{i_{n-1}}$ for $0\le i_j\le p-1$. Multiplying by $\kappa_1^{p-i_1}\ldots\kappa_{n-1}^{p-i_{n-1}}$ and taking trace, we see that $a_{(i_1,\ldots,i_{n-1})}=0$. This is impossible, so that E_n is irreducible over $L(\kappa_1,\ldots,\kappa_{n-1})$. Thus $[K:L]=p^n$.

Remark 3.2.0.6. It follows from Proposition 3.2.0.3 that $A[1/p]$ is integrally closed if $p\nmid r_i$ for all i. In particular $NNL_1(A)\subseteq V(p)$.

We motivate why it suffices to consider the case $\mathscr{E}\subseteq S^p$ in our search for small CM modules (algebras) over R, at least when S is complete with perfect residue field. In other words, we may assume the elements whose roots we adjoin have p-th roots mod p in S.

3.2.1 Reducing to S^p

Throughout this subsection, assume notation as in 3.2.0.4 and additionally assume that S is complete and $\mathbf{k}$ is perfect.

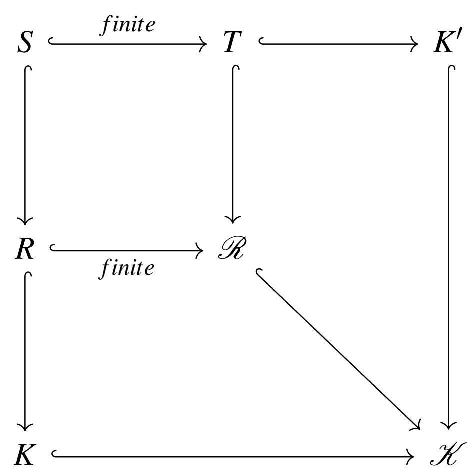
Remark 3.2.1.1. Let $(T, \mathfrak{n})$ be a complete regular local ring of positive characteristic $p > 0$. Then the image of the Frobenius map on T is a closed subspace of T in the $\mathfrak{n}$-adic topology.

Suppose $\mathfrak{m} = (p, \tau_2, \ldots, \tau_d)$ and set $T := S[\chi_2, \ldots, \chi_d]$, where $\chi_i^p = \tau_i$. We claim:

1. T is an unramified regular local ring of mixed characteristic $p > 0$.

2. $S \subseteq T^p$.

Assume both the claims hold. Let K' be the fraction field of T and set $\mathscr{K} := K[K']$. Let $\mathscr{R}$ be the integral closure of S in $\mathscr{K}$. We have



If $\mathscr{R}$ admits a small CM module, then R admits a small CM module. Replacing S with T, we may then assume $\mathscr{E} \subseteq S^p$ as long as the elements of $\mathscr{E}$ remain square free in T. Note that we may assume that p or none of the τ_i divide any of the elements of $\mathscr{E}$. In that case, from 3.2.0.2 and 3.2.0.3 it follows that the elements of $\mathscr{E}$ are square free in T. Therefore only claims (1) and (2) remain to be shown.

For (1), we have

$$S_2 := S[\chi_2] \simeq S[W_2]_{(\mathbf{m}, W_2)}/(W_2^p - \tau_2)$$

is unramified regular local with regular system of parameters $(p,\chi_2,\tau_3,\dots,\tau_d)\subseteq S_2$. Now let $2\le i\le d$, and assume that $S_i:=S[\chi_2,\dots,\chi_i]$ is unramified regular local with a regular system of parameters given by $(p,\chi_2,\dots,\chi_i,\tau_{i+1},\dots,\tau_d)$. The polynomial $W_{i+1}^p-\tau_{i+1}\in S_i[W_{i+1}]$ is irreducible since τ_{i+1} is part of a regular system of parameters for S_i. Therefore, $S_{i+1}:=S[\chi_2,\dots,\chi_{i+1},\tau_{i+2},\dots,\tau_d]$ is an unramified regular local ring with maximal ideal given by $(p,\chi_2,\dots,\chi_{i+1},\tau_{i+2},\dots,\tau_d)$. Therefore inductively $T=S[\chi_2,\dots,\chi_d]$ is unramified regular local with maximal ideal $\mathfrak{n}:=(p,\chi_2,\dots,\chi_d)$.

For (2), set $E:=T/(p)$, so that $C:=S/pS\hookrightarrow E$. Let $f\in S$ be arbitrary and $g\in E$ be its natural image. Let x_i denote the image in E of τ_i. Let $g=\sum_{i=0}^{\infty}g_i$ where g_i is a homogeneous polynomial in $x_2,\dots,x_d$ with coefficients in $\mathbf{k}$. Let G_i denote the set of monomials in $x_2,\dots,x_d$ of degree i. Define the set:

$$\left(\frac{E}{\langle G_i\rangle_C}\right)^p:=\{x\in E\,|\,x=h^p+m,m\in\langle G_i\rangle_C\}$$

Now, $g\in\left(\frac{E}{\langle G_1\rangle_C}\right)^p$ since $\mathbf{k}$ is perfect. Assume $g\in\left(\frac{E}{\langle G_n\rangle_C}\right)^p$ for $n\ge1$. Write $g=h^p+\sum_i v_i\alpha_i+m$ where $v_i\in\mathbf{k}$, α_i monomials in $x_2,\dots,x_d$ of degree n and $m\in\langle G_{n+1}\rangle_C$. Since $\mathbf{k}$ is perfect and $\alpha_i\in E^p$, $g\in\left(\frac{E}{\langle G_{n+1}\rangle_C}\right)^p$. Thus g lies in the closure of the subspace E^p in E. By 3.2.1.1, $g\in E^p$ and hence $f\in T^p$. The claim has been proved.

For some properties of the subring S^p, see Katz & Sridhar (2021).

3.2.2 Class 1 towers

Motivated by subsection 3.2.1, we assume the following notation:

Convention 3.2.2.1. Assume notation as in 3.2.0.4. Additionally assume that $\mathscr{E}\subseteq S^p$. Let h_i denote a p-th root modulo p of r_i, that is $r_i-h_i^p\in pS$. If $h_i\in S\setminus pS$, let L_i denote the element in the ring $S[X_i]$ obtained by setting $h=h_i$ in 3.1.0.10. Denote by l_i the image of L_i in the ring $S[\kappa_i]$. If $h_i=0$, simply set $l_i=0$.

Say that $\mathscr{E}$ defines a **class 1 tower** if $\mathscr{E} \subseteq S^{p \wedge p^2}$. We first show that if $\mathscr{E}$ defines a class one tower, then R is Cohen Macaulay.

Theorem 3.2.2.2. *With established notation, if $\mathscr{E} \subseteq S^{p \wedge p^2}$ and $p \geq 3$, then R is Cohen Macaulay.*

Proof. Let R_i be the integral closure of $S[\kappa_i]$. Since $r_i \in S^{p \wedge p^2}$, we have from 2.1.4.5 that $R_i = S[\kappa_i, \psi_i]$, where $\psi_i = p^{-1} h_i^{p-1} \zeta_p(\kappa_i / h_i)$. Further ψ_i satisfies the integral equation

$$u_i(X) := X^2 - l_i X - p^{-2}(r_i - h_i^p)(\kappa_i - h_i)^{p-2} \in S[\kappa_i][X] \tag{3.2.2.2.1}$$

There are precisely two height one primes lying over pS in R_i:

$$p \nearrow Q_{u_i} := (p, \kappa_i - h_i, \psi_i - l_i)$$
$$p \searrow Q_{r_i} := (p, \kappa_i - h_i, \psi_i)$$

From 2.1.4.5, we have $(Q_{u_i} \mid p)$ and $(Q_{r_i} \mid \kappa_i - h_i)$. Hence the choice of "$u_i$" for "unramified" and "$r_i$" for "ramified". We also know that the R_i are all Cohen-Macaulay. Let $V \subseteq R$ denote the join of the R_i. Then V is Cohen-Macaulay since $[K : L] = p^n$, see 3.2.0.5. That is V is S-free of rank p^n. There are 2^n height one primes in V containing p. If $Q \subseteq V$ is a height one prime containing p, then the non singular primes are either of the form $(Q \mid p)$ or $(Q \mid \kappa_i - h_i)$. The (possibly) singular ones are of the form $Q_{(i_1,\ldots,i_l)} := (Q \mid \kappa_{i_1} - h_{i_1}, \ldots, \kappa_{i_l} - h_{i_l})$ for some $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\}$, $i_1 < \cdots < i_l$ and $l \geq 2$.

Note that from Remark 3.2.0.6 and Remark 3.1.0.4, V is regular in codimension outside of the $Q_{(i_1,\ldots,i_l)}$. There are utmost $2^n - n - 1$ singularities in codimension one in V. We now identify an $\mathbf{R}_1$-ification of V. For $i, j \in \{1, \ldots, n\}$, $i < j$, define

$$\eta_{ij} := p^{-1}(\kappa_i - h_i)^{p-2}(\kappa_j - h_j) \in K$$

Then η_{ij} satisfies the integral equation

$$v_{ij}(X) := X^{p-1} - (\psi_i - l_i)^{p-2}(\psi_j - l_j) \in V[X]$$

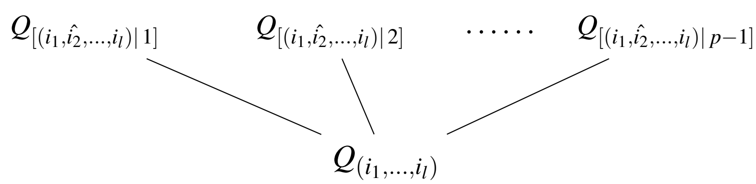
To de-singularize $Q_{(i_1,\ldots,i_l)}$ consider the finite birational extension $V \hookrightarrow V_{(i_1,\ldots,i_l)} := V[\eta_{i_1 i_2},\ldots,\eta_{i_1 i_l}]$. From 3.1.0.11 we have [2]

$$\overline{v_{i_1 i_2}(X)} \equiv X^{p-1} - (h_{i_1}^{p-2} h_{i_2})^{p-1} = \prod_{k=1}^{p-1} (X + k h_{i_1}^{p-2} h_{i_2}) \in \frac{V_{(i_1,\ldots,i_l)}}{Q_{(i_1,\ldots,i_l)} V_{(i_1,\ldots,i_l)}}[X]$$

Therefore height one primes in $V[\eta_{i_1 i_2}]$ lying over $Q_{(i_1,\ldots,i_l)}$ are of the form

$$Q_{[(i_1,\hat{i_2},\ldots,i_l)\,|\,k]} := (Q_{(i_1,\ldots,i_l)}, \eta_{i_1 i_2} + k h_{i_1}^{p-2} h_{i_2})$$

for $1 \le k \le p-1$.



The point is that $Q_{[(i_1,\hat{i_2},\ldots,i_l)\,|\,k]}$ locally has one less generator: it is of the form

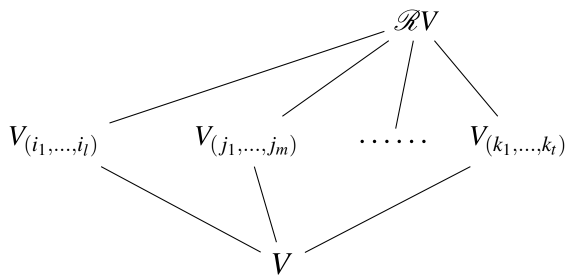
$$(Q_{[(i_1,\tilde{i_2},\ldots,i_l)\,|\,k]} \,|\, (\kappa_{i_1} - h_{i_1}, \widehat{\kappa_{i_2} - h_{i_2}}, \ldots, \kappa_{i_l} - h_{i_l}))$$

To see this note that $\prod_{i=1, i\neq k}^{p-1} (\eta_{i_1 i_2} + i h_{i_1}^{p-2} h_{i_2}) \notin Q_{[(i_1,\hat{i_2},\ldots,i_l)\,|\,k]}$ so that $\eta_{i_1 i_2} + k h_{i_1}^{p-2} h_{i_2}$ is locally a redundant generator. Also, since $(\kappa_{i_1} - h_{i_1})\dot{\eta}_{i_1 i_2} = (\kappa_{i_2} - h_{i_2})(\psi_1 - l_1)$ and $(\psi_1 - l_1) \notin Q_{[(i_1,\hat{i_2},\ldots,i_l)\,|\,k]}$, $\kappa_{i_2} - h_{i_2}$ is a redundant generator locally.

Proceeding inductively, it is clear that in $V_{(i_1,\ldots,i_l)}$ all height one primes lying over $Q_{i_1,\ldots,i_l)}$ (utmost $(p-1)^{l-1}$) are non-singular. Now set

$$\mathscr{R}V := V[\{p^{-1}(\kappa_{i_1} - h_{i_1})^{p-2}(\kappa_{i_2} - h_{i_2})\}_{\{i_1,i_2 \in \{1,\ldots,n\}\,|\,i_1 < i_2\}}]$$

[2] If R is a ring of characteristic p and X,Y indeterminates over R, then for $X^{p-1} - Y^{p-1} \in R[X,Y]$, $X^{p-1} - Y^{p-1} = \prod_{i=1}^{p-1}(X + iY)$.



For all possible $\{i_1,\dots,i_l\} \subseteq \{1,\dots,n\}$, $i_1 < \cdots < i_l$ and $l \geq 2$, $V \hookrightarrow V_{(i_1,\dots,i_l)} \hookrightarrow \mathscr{R}V$ are finite birational extensions. From 3.1.0.4, $\mathscr{R}V$ is an R_1-ification for V.

We now identify a finite birational overring of $\mathscr{R}V$ that is S-free. This ring would then inherit R_1 from $\mathscr{R}V$ by 3.1.0.4 and the proof would be complete. The rest of the proof concerns identifying this overring.

Note that $A = S[\kappa_1,\dots,\kappa_n]$ is S-free of rank p^n with a basis given by

$$F := \{\prod_{i=1}^{n}(\kappa_i - h_i)^{j_i} \,|\, 0 \leq j_i \leq p-1\}$$

For each $1 \leq i \leq n$, define $\Gamma_i : F \to \mathbb{N} \cup \{0\}$ by $\Gamma((\kappa_1 - h_1)^{j_1} \dots (\kappa_i - h_i)^{j_i} \dots (\kappa_n - h_n)^{j_n}) = j_i$ and

$$\Gamma : F \to (\mathbb{N} \cup \{0\})^n,\ f \mapsto (\Gamma_1(f),\dots,\Gamma_n(f))$$

Let $\gamma : (\mathbb{N} \cup \{0\})^n \to \mathbb{N} \cup \{0\}$ be the map sending $(x_1,\dots,x_n) \mapsto x_1 + \cdots + x_n$. For every $0 \leq k \leq n$, set

$$\mathscr{S}_k := \{p^{-k} \cdot m \,|\, m \in (\gamma\Gamma)^{-1}([(p-1)k,(p-1)(k+1)))\}$$

and $\mathscr{S} := \cup_{0 \leq k \leq n} \mathscr{S}_k$. By definition, the sets $\mathscr{S}$ and F are in bijection and it follows that $E := \langle \mathscr{S} \rangle_S$ is S-free of rank p^n. We are through if we show the following

1. E is a S-algebra.

2. $\mathscr{R}V \subseteq E$.

3. $\mathscr{R}V \hookrightarrow E$ is a finite map.

Suppose (1) holds. Then (2) follows immediately. To show (3), it suffices to show for each $1 \leq k \leq n$, $\mathscr{S}'_k \subseteq R$ for

$$\mathscr{S}'_k = \{p^{-k} \cdot m \,|\, m \in (\gamma\Gamma)^{-1}((p-1)\cdot k)\}$$

Consider $\alpha \in \mathscr{S}'_k$. Let $\alpha = p^{-k}(\kappa_1 - h_1)^{i_1} \ldots (\kappa_n - h_n)^{i_n}$ such that $\sum_{j=1}^{n} i_j = k\cdot(p-1)$. Then

$$\begin{aligned}
\alpha^{p-1} &= p^{-k(p-1)} \cdot \prod_{j=1}^{n} [(\kappa_j - h_j)^{p-1}]^{i_j} \\
&= \prod_{j=1}^{n} p^{-i_j}[(\kappa_j - h_j)^{p-1}]^{i_j} \\
&= \prod_{j=1}^{n} (\tau_j - l_j)^{i_j} \in V
\end{aligned}$$

Therefore $\mathscr{S}'_k \subset R$ and hence $E \hookrightarrow R$. Thus (3) holds.

Only (1) remains to be shown. We first show that E is an A-module. For $0 \leq k \leq n$, let

$$\mathscr{S}^p_k := \cup_{k \leq j \leq n}\{p^{j-k} \cdot m \,|\, m \in \mathscr{S}_j\}$$

For example, $\mathscr{S}^p_0 = F$ and hence $\langle \mathscr{S}^p_0 \rangle_A = A \subseteq E$. Now assume for some $0 \leq k \leq n-1$ that $\langle \cup_{0 \leq j \leq k} \mathscr{S}^p_j \rangle_A \subseteq E$. We will show that $\langle \cup_{0 \leq j \leq k+1} \mathscr{S}^p_j \rangle_A \subseteq E$. Consider $\beta := p^{-(k+1)} \cdot y \in \mathscr{S}^p_{k+1}$ with $y \in F$. If $x \in F$ is such that for all $1 \leq i \leq n$, $\Gamma_i(x) + \Gamma_i(y) \leq p-1$, then $xy \in F$ and $\gamma\Gamma(xy) \geq k+1$. So $x \cdot \beta \in E$.

Now suppose that for some $1 \leq i \leq n$, $\Gamma_i(x) + \Gamma_i(y) \geq p$. We have

$$\begin{aligned}
(\kappa_i - h_i)^p &= \kappa_i^p - h_i^p + l_i p(\kappa_i - h_i) \\
&\equiv l_i p(\kappa_i - h_i) \, mod\, (p^2 A) \qquad (3.2.2.2.2)
\end{aligned}$$

Thus in this case, we have $x\beta \in \langle \cup_{0 \leq j \leq k} \mathscr{S}^p_j \rangle_A \subseteq E$. Therefore, $\langle \cup_{0 \leq j \leq k+1} \mathscr{S}^p_j \rangle_A \subseteq E$. Induction on k gives $\langle \cup_{0 \leq j \leq n} \mathscr{S}^p_j \rangle_A \subseteq E$. For all $0 \leq j \leq n$, $\mathscr{S}_j \subseteq \mathscr{S}^p_j$, so that

$$\langle \mathscr{S} \rangle_A = \langle \cup_{0 \leq j \leq n} \mathscr{S}_j \rangle_A \subseteq \langle \cup_{0 \leq j \leq n} \mathscr{S}^p_j \rangle_A \subseteq E$$

Thus E is an A-module. Further, $E = \langle \cup_{0\le j\le n} \mathscr{S}'_j \rangle_A$. If we show for arbitrary $x, y \in \cup_{0\le j\le n}\mathscr{S}'_j$, $x \cdot y \in E$, then E is a birational A- algebra. Pick $x \in \mathscr{S}'_l$ and $y \in \mathscr{S}'_m$ and set $x' = p^l x$ and $y' = p^m y$.

Case 1: Suppose $l+m \le n$. If for all $1 \le i \le n$, $\Gamma_i(x') + \Gamma_i(y') \le p-1$, then $xy \in \mathscr{S}'_{l+m}$ and we are done. Suppose there exists $1 \le k \le n$ such that for all $1 \le i \le k$, $\Gamma(x') + \Gamma(y') \ge p$ and for $k < i \le n$, $\Gamma(x') + \Gamma(y') \le p-1$. If $k \ge l+m$, then by 3.2.2.2.8, $xy \in A \subseteq E$. Therefore assume $k < l+m$. We have

$$xy = p^{-(l+m-k)} (\prod_{i=1}^{k} \alpha_i) \prod_{j=1}^{n} (\kappa_i - h_i)^{c_i} \tag{3.2.2.2.3}$$

where $\alpha_i \equiv l_i(\kappa_i - h_i) \, mod \, pA$, $c_i = \Gamma_i(x') + \Gamma_i(y') - p$ for $1 \le i \le k$ and $c_i = \Gamma_i(x') + \Gamma_i(y')$ for $k < i \le n$. Note here that

$$\sum_{i=1}^{n} c_i = (m+l)(p-1) - kp = (m+l-k)(p-1) - k \tag{3.2.2.2.4}$$

Set $\varepsilon := (\prod_{i=1}^{k} \alpha_i)^{-1} xy$. Let δ be an arbitrary monomial of $(\prod_{i=1}^{k} \alpha_i)$. We will show $\varepsilon \cdot \delta \in E$ so that $xy \in E$. For some $0 \le j \le k$, $p^j (\kappa_{i_i} - h_{i_1}) \dots (\kappa_{i_{k-j}} - h_{i_{k-j}})$ divides δ in A for some $\{i_1, \dots, i_{k-j}\} \subseteq \{1, \dots, k\}$. Then $\delta \cdot \varepsilon \in \langle p^{-(m+l-k-j)} z \rangle_A$ for some $z \in F$. Here $z \in F$ since $c_i \le p-2$ if $1 \le i \le k$. If $j \ge m+l-k$, then $\delta\varepsilon \in A$ and we are done. Assume $j < m+l-k$, we have

$$\begin{aligned} \gamma\Gamma(z) &= (m+l-k)(p-1) - k + (k-j) \\ &\ge (m+l-k-j)(p-1) \end{aligned} \tag{3.2.2.2.5}$$

This shows $\delta\varepsilon \in E$ and the proof of case 1 is complete.

Case 2: Suppose that $l+m > n$. So, $0 < l+m-n \le n$. We claim that there exists $\{i_1, \dots, i_{l+m-n}\} \subseteq \{1, \dots, n\}$ distinct such that for $1 \le s \le l+m-n$, $\Gamma_{i_s}(x') + \Gamma_{i_s}(y') \ge p$. Suppose the claim does not hold. We have

$$\begin{aligned}\gamma\Gamma(x') + \gamma\Gamma(y') &\leq [n-(l+m-n)+1](p-1)+(2p-2)(l+m-n-1)\\ \Longrightarrow (m+l)(p-1) &\leq [2n-(l+m)+1](p-1)+(2p-2)(l+m-n-1)\\ \Longrightarrow 0 &\leq (p-1)(2n+1)-(p-1)(2n+2) \end{aligned} \quad (3.2.2.2.6)$$

which is absurd. Thus the claim holds. Without loss of generality, assume that $q \geq l+m-n$ is such that for $1 \leq i \leq q$, $\Gamma_i(x') + \Gamma_i(y') \geq p$ and for $q < i \leq n$ (if $q \neq n$), $\Gamma_i(x') + \Gamma_i(y') \leq p-1$. Then

$$xy = p^{-(l+m-q)}(\prod_{i=1}^{q} \alpha_i)(\prod_{j=1}^{n}(\kappa_j - h_j)^{c_j}) \quad (3.2.2.2.7)$$

where $\alpha_i \equiv -l_i(\kappa_i - h_i) \, mod \, (pA)$, $c_j = \Gamma_j(x') + \Gamma_j(y') - p$ for $1 \leq j \leq q$ and $c_j = \Gamma_j(x') + \Gamma_j(y')$ for $q < s \leq n$. We have

$$\sum_{i=1}^{n} c_i = (m+l)(p-1) - qp = (m+l-q)(p-1) - q$$

If $q \geq l+m$, $xy \in A \subseteq E$, so we are done. Assume $q < l+m$. Set $\varepsilon := xy(\prod_{i=1}^{q} \alpha_i)^{-1}$ and let δ be an arbitrary monomial of $\prod_{i=1}^{q} \alpha_i$. We will show $\varepsilon \cdot \delta \in E$ so that $xy \in E$. Certainly δ is a multiple in A of an element of the form $p^e(\kappa_{i_1} - h_{i_1}) \ldots (\kappa_{i_{q-e}} - h_{i_{q-e}})$ for some $0 \leq e \leq q$ and $\{i_1, \ldots, i_{q-e}\} \subseteq \{1, \ldots, q\}$. Therefore $\delta \cdot \varepsilon \in \langle p^{-(l+m-q-e)} \cdot z \rangle_A$ for some $z \in F$. Here $z \in F$ since $c_i \leq p-2$ for $1 \leq i \leq q$. But

$$\begin{aligned}\gamma\Gamma(z) &= (m+l-q)(p-1) - q + (q-e)\\ &\geq (m+l-q-e)(p-1)\end{aligned} \quad (3.2.2.2.8)$$

therefore $p^{-(l+m-q-e)} \cdot z \in E$ and since E is an A-module $\delta \cdot \varepsilon \in E$. This completes the proof of case 2 and hence we have shown that E is a S-algebra. This shows claim (1). Since E satisfies Serre's criterion (S_2), $E = R$. In particular, R is S-free, or in other words R is Cohen-Macaulay. ■

Corollary 3.2.2.3. *Let S be an unramified regular local ring of mixed characteristic $p > 0$ with fraction field L. Let $f_1, \ldots, f_n \in S^{p \wedge p^2}$, square free and mutually coprime. Let $\omega_i^{d_i} = f_i$ such that $p \mid d_i$ and $p^2 \nmid d_i$ for each i. Then the integral closure of S in $L(\omega_1, \ldots, \omega_n)$ is Cohen Macaulay.*

Proof. Let $\kappa_i := \sqrt[p]{f_i} \in L(\omega_1, \ldots, \omega_n)$ be p-th roots. Let $\mathscr{R}$ denote the integral closure of S in the subfield $L(\kappa_1, \ldots, \kappa_n)$. By 3.2.2.2, $\mathscr{R}$ is a free S-module. Since the f_i are squarefree and lie in $S^{p \wedge p^2}$, $p \nmid f_i$. Therefore, applying 3.2.0.2 to $S[1/p]$, we see that the $f_i \in \mathscr{R}$ are square free. They satisfy $\mathscr{A}_1$ in $\mathscr{R}$ since they do so in S. If R is the integral closure of S in $L(\omega_1, \ldots, \omega_n)$, we then see by 3.2.0.3 that $R = \mathscr{R}[\sqrt[p^{-1}d_1]{\kappa_1}, \ldots, \sqrt[p^{-1}d_n]{\kappa_n}]$. In particular, R is $\mathscr{R}$-free and hence is S-free. Thus R is Cohen-Macaulay. ■

We will use the above result to show the existence of small CM algebras in radical towers of order p under certain circumstances in chapter 4.

3.2.3 On p-th roots over certain ramified regular local rings of mixed characteristic $p > 0$

In trying to address the case where the p-torsion of the Abelian Galois group is annihilated by p in 2.1.5.2 via Kummer theory, one needs to adjoin a primitive p-th root of unity. We show:

Theorem 3.2.3.1. *Let $T := S[\varepsilon]$ be the ramified regular local ring obtained by adjoining a primitive p-th root of unity ε to an unramified regular local subring S of mixed characteristic $p \geq 3$. Let L be the fraction field of T and ω a p-th root of a square free element of T. Then the integral closure of T in $L(\omega)$ is Cohen-Macaulay.*

Proof. See Katz & Sridhar (2021). ■

Chapter 4

On constructing small Cohen-Macaulay modules and algebras over radical towers in mixed characteristic

In this chapter we continue the study of general radical towers of order p of an unramified regular local ring of mixed characteristic p initiated in Chapter 2. In the previous chapter, we looked at the cases where the integral closure is already Cohen-Macaulay. In this chapter, we show the integral closure need not be Cohen-Macaulay and attempt to justify that it is quite common. We then construct (birational) small CM modules or algebras over these rings. Like in chapter 2 we first study the biradical case in section 4.1. In trying to find small CM modules one needs to understand the failure of Cohen-Macaulayness of the ring itself. To this end, we characterize the Cohen-Macaulayness of the integral closure in the biradical case. In section 4.2, in joint work with Daniel Katz, we consider the general case and in future work hope to generalize these results towards answering 2.1.5.2 when the base regular local ring is complete with perfect residue field.

4.1 The Biradical case

For this section we maintain notation set up in 3.1.0.8, 3.1.0.1 and 3.1.0.12. From 3.1.0.15 we know that if exactly one of $f,g \in S^p$, then R is Cohen-Macaulay. We will see that that R is not automatically Cohen-Macaulay in case $f,g \in S^2$ or if $f,g \notin S^2$. However, we can reduce to the case $f,g \in S^p$ when S is complete with perfect residue field, where S^p is the subring of S obtained by lifting the Frobenius map on S/pS to S, see 3.2.1.

We need the following proposition for 4.1.0.2. The form given here is a bit more general than we actually need.

Proposition 4.1.0.1. *Let T be any Gorenstein local domain such that its integral closure T' is a finite T-module. Let J denote the conductor ideal of T. Then T' is Cohen-Macaulay if and only if T/J is Cohen-Macaulay.*

Proof. Let E denote the field of fractions of T. Since $End(J) := (J :_E J)$ is a ring, we have $T' \subseteq J^* = End(J) \subseteq T'$, so that $J^* = T'$. For any $0 \neq x \in J$, set $J' := (x :_T J)$. Note that J is height one unmixed (see for example Katz (1999)[Proposition 2.1(2)]). Also, J is principal if and only if T is integrally closed, so we may assume J is not principal. Since T is Gorenstein, T/J is Cohen-Macaulay if and only if T/J' is Cohen-Macaulay (see Huneke & Ulrich (1987)[Proposition 2.5] for example). From the depth lemma, T/J' is a Cohen-Macaulay ring if and only if $J' \simeq T'$ is a Cohen-Macaulay T-module. This completes the proof since T' is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay T-module. ■

When $f, g \notin S^p$, R is not necessarily Cohen-Macaulay, as shown in the following example.

Example 4.1.0.2. Set $S := \mathbb{Z}[X,Y,V]_{(2,X,Y,V)}$. Let $f = XV^2 + 4$ and $g = XY^2 + 4$. Then f, g are square free, form a regular sequence in S and do not lie in S^2. Note that $2 \in S[\omega]$ is a prime. Set $C := S[\omega]/(2) \simeq (S/2S)[\gamma]$ where $\gamma = \sqrt{xv^2}$ and x, y, v denote the respective images in $S/2S$. Since $g = XY^2 + 4 = (V^{-1}Y\omega)^2 + 4(1 - V^{-2}Y^2)$, from 3.1.0.14(3), A is not integrally closed. Moreover from Katz (1999)[Lemma 3.2], $(P^*)_P = R_P$. From 3.1.0.14(2), for all height one primes $Q \subseteq A$, $Q \neq P$, $(P^*)_Q = R_Q = A_Q$. Since P^* and R are birational S_2 A-modules, $P^* = R$. From 3.1.0.14(2), the conductor of A is contained in P and hence is equal to P. We now show that A/P is not Cohen-Macaulay, so that by 4.1.0.1, R is not Cohen-Macaulay.

Let $Q \subseteq C[U]$ denote the unique height one prime minimal over the image of $G(U)$. Set $\varepsilon := v^{-1}y\gamma$. Then $Q = (U - \varepsilon)Frac(C)[U] \cap C[U]$ is the kernel of the natural surjection $C[U] \to C[\varepsilon]$. Thus $A/P \simeq D := C[\varepsilon]$.

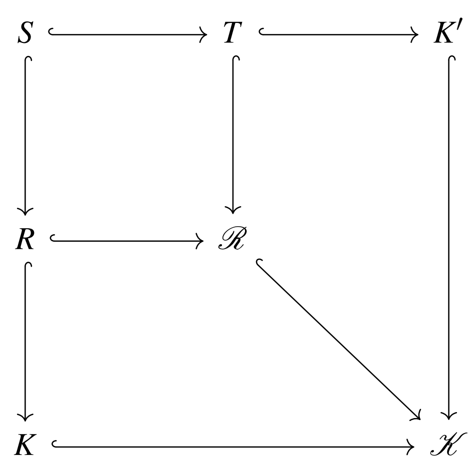
Since D is module finite over the regular local ring $S/2S$, it is Cohen-Macaulay if and only if it is $S/2S$-free. Certainly D is generated over $S/2S$ by the set $\{1, \gamma, \varepsilon, \gamma\varepsilon\}$. Since $\varepsilon \cdot \gamma = xyv \in S/2S$, we can trim this set to $G := \{1, \gamma, \varepsilon\}$. But $p.d._{S/2S}(D) = 1$, since D admits the minimal free resolution

$$0 \longrightarrow S/2S \xrightarrow{\psi^T} (S/2S)^3 \xrightarrow{\phi} D \longrightarrow 0$$

where ϕ is the natural projection corresponding to the ordered set G and $\psi = [0\ y\ -v]$. Thus A/P is not Cohen-Macaulay and hence R is not Cohen-Macaulay.

However the example in 4.1.0.3 admits a small CM algebra:

Example 4.1.0.3. We assume notation as in 4.1.0.2, so that R is a non Cohen-Macaulay normal domain of mixed characteristic 2. Set $K' := L(\sqrt{X})$ and $T := S[\sqrt{X}]$. Note that T is an unramified regular local ring of mixed characteristic 2 and $f, g \in T^{2\wedge 4}$. We claim that $f, g \in T$ are square free. To show this, we can assume that $2 \in S$ is a unit since $f, g \notin 2S$. Then, by Huneke & Katz (2019)[Proposition 5.2] $f, g \in T$ are square free. Clearly, $f, g \in T$ satisfy $\mathscr{A}_1$. Therefore by 3.1.1.1, the integral closure of T in $\mathscr{K} := K'(\omega, \mu)$, say $\mathscr{R}$, is Cohen-Macaulay. Therefore, $\mathscr{R}$ is a small CM algebra over R.



Example 4.1.0.4. We note here that the existence of a birational small CM module does not obstruct the existence of a small CM algebra. Assume notation as in Example 2.1.4.2. Then R is known to admit a birational small CM module. We will now show that it admits a small CM algebra.

Set $T := S[\sqrt[3]{x}, \sqrt[3]{y}]$. Note that T is an unramified regular local ring with regular system of parameters $(3, \sqrt[3]{x}, \sqrt[3]{y})$. Let E be the integral closure of $T[\sqrt[3]{a}, \sqrt[3]{b}]$. Then certainly $R \hookrightarrow E$ and E is a finite R-module. Note that $a, b \in T$ satisfy $\mathscr{A}_1$ since they do so in S. Since $a, b \in S$ are square free, they are square free in T as well by Huneke & Katz (2019)[Proposition 5.2]. Finally, $a, b \in T^{3\wedge 9}$, so that by 3.1.0.18, E is Cohen-Macaulay. Thus E is a small CM algebra for R.

In general if $f, g \notin S^p$, we have not been able to construct a birational small CM module. However when S is complete with perfect residue field, we can reduce to the case $f, g \in S^p$ if we relax the birationality constraint, see section 1.2.1. Motivated by this, we focus on the case $f, g \in S^p$ for the remainder of this section and hence assume notation as set up in 3.1.0.17.

We now look in detail at cases where R is not S-free, when $f, g \in S^p$. That is in the primary case of interest, when S is an unramified regular local ring of mixed characteristic p, we look at non-Cohen-Macaulay integral closures R. More specifically, we will show that $p.d._S(R) = 1$ under some natural conditions and show that in this case R admits a birational small CM module.

From 3.1.0.18 and 3.1.0.20, if we are looking for a non S-free R, we must have that $S[\omega]$ and $S[\mu]$ are integrally closed such that there exists an $1 \leq i \leq p-1$ satisfying $fg^i \in S^{p\wedge p^2}$. The reader can easily see that if it exists, such an "i" is unique. We start by identifying an ideal $I \subseteq A$, such that $I^* = R$ under this circumstance.

Convention 4.1.0.5. For the remainder of this section make the additional assumption that $f, g \notin S^{p\wedge p^2}$. Write $f = h_1^p + ap$, $g = h_2^p + bp$ with $a, b \notin pS$. Assume further that $h_1, h_2 \neq 0$. Note here that if $h_1 = 0$ (or $h_2 = 0$), we have by 3.1.0.20 that R is S-free. Let $P := (p, \omega - h_1, \mu - h_2) \subseteq A$ denote the unique height one prime in A containing p.

Lemma 4.1.0.6. *For $H := (p, \omega\mu^i - h_1 h_2^i) \subseteq A$, $H_P^* = \langle 1, \tau_i \rangle_{A_P}$ where*

$$\tau_i = p^{-1}[(\omega\mu^i)^{p-1} + h_1 h_2^i (\omega\mu^i)^{p-2} + \cdots + (h_1 h_2^i)^{p-1}] \in K$$

Proof. Localize A at P and assume (A,P) local. Consider the ideal

$$\tilde{H} := (p, W\mu^i - h_1 h_2^i) \subseteq S[\mu][W]$$

We have $F(W) \in \tilde{H}$:

$$F(W) - h_2^{-ip}[(W\mu^i)^{p-1} + h_1 h_2^i (W\mu^i)^{p-2} + \cdots + (h_1 h_2^i)^{p-1}] \cdot (W\mu^i - h_1 h_2^i) \in pS[\mu][W] \tag{4.1.0.6.1}$$

Clearly $\tilde{H}$ is a grade two perfect ideal in $S[\mu][W]$ and is the ideal of maximal minors of the matrix E:

$$E = \begin{bmatrix} W\mu^i - h_1 h_2^i \\ p \end{bmatrix}$$

Adjoining the column of coefficients from (4.1.0.6.1) appropriately, we have for some $\alpha \in S[\mu[[W]$ the matrix E':

$$E' = \begin{bmatrix} W\mu^i - h_1 h_2^i & \alpha \\ p & h_2^{-ip}[(W\mu^i)^{p-1} + h_1 h_2^i (W\mu^i)^{p-2} + \cdots + (h_1 h_2^i)^{p-1}] \end{bmatrix}$$

From Katz (1999)[Proposition 2.1], $H^* = \langle E'_{11}/\delta_1, E'_{22}/\delta_2 \rangle_A$ where E'_{ii} and δ_i denote the image in A of the (i,i)-th cofactor of E' and the i-th (signed) minor of E respectively. Thus $H^* = \langle 1, \tau_i \rangle_A$ ∎

Lemma 4.1.0.7. *With established notation, let $fg^i \in S^{p \wedge p^2}$. Then for*

$$I := pA + P^{p-2} \cdot (\omega\mu^i - h_1 h_2^i)A$$

we have $I_A^* = R$.

Proof. Since I_A^* and R are birational S_2 A-modules, it suffices to show the desired equality in codimension one. If $Q \neq P$ is a height one prime in A, $I_Q^* = R_Q = A_Q$. Therefore localize A at P and assume (A,P) and (S,pS) are one dimensional local rings for the remainder of the proof.

We have $A = S[\mu, \omega\mu^i]$. Note that $g, fg^i \in S$ are units and therefore trivially are square free and satisfy $\mathscr{A}_1$. Moreover, $S[\mu]$ is integrally closed and $S[\omega\mu^i]$ is not. Thus we are in the setting of 3.1.0.18(2). From the proof of 3.1.0.18(2)(a), we get that $R = A[\tau, \eta]$ where $\tau = p^{-1}[(\omega\mu^i)^{p-1} + h_1h_2^i(\omega\mu^i)^{p-2} + \cdots + (h_1h_2^i)^{p-1}]$ and $\eta = p^{-1}(\omega\mu^i - h_1h_2^i)(\mu - h_2)^{p-1}$. Since $\tau\eta \in A$, we see from 3.1.0.13(c) and equation (3.1.0.18.1) that $R = \langle 1, \eta, \ldots, \eta^{p-2}, \tau\rangle_A$.

Since $p \in P^{p-1}$, a straightforward calculation gives

$$P^{p-1} \cap (p, \omega\mu^i - h_1h_2^i) = (p) + P^{p-1} \cap (\omega\mu^i - h_1h_2^i) = (p) + P^{p-2} \cdot (\omega\mu^i - h_1h_2^i) = I$$

From 3.1.0.19, $(P^{p-1})^* = A[\eta] \subseteq R$. Since A is Gorenstein, $A :_K R \subseteq P^{p-1}$. Let H be as in 4.2.0.2. Combining 4.2.0.2 and 3.1.0.13(c), we get that $H^* = A[\tau] \subseteq R$. Again since H is reflexive, $A :_K R \subseteq H$. Therefore, $A :_K R \subseteq P^{p-1} \cap H = I$.

To show $IR \subseteq A$, note that $I\eta^i \subseteq A$ for $1 \leq i \leq p-2$, since $I \subseteq P^{p-1} = A[\eta]^*$. Similarly, $I\tau \subseteq A$ since $I \subseteq H = A[\tau]^*$. Thus we have shown $I = A :_K R$ and the proof is complete. ∎

We now set out to show R need not be Cohen-Macaulay - again we handle the mixed characteristic two case later.

Lemma 4.1.0.8. *Assume $p \geq 3$ and let $S[\omega], S[\mu]$ be integrally closed such that $fg \in S^{p \wedge p^2}$. Then*

1. *$R \subseteq \langle \{1\} \cup p^{-1} \cdot (\omega - h_1, \mu - h_2)^{p-1} \rangle_A$.*
2. *Consider $y = p^{-1}(\sum_{i=1}^{p} a_i(\mu - h_2)^{p-i}(\omega - h_1)^{i-1}) \in K$ with the $a_i \in A$. Then $y \in R$ if and only if for all $2 \leq i \leq p$, $a_{i-1}h_2 + a_ih_1 \in P$.*

Proof. From 4.1.1.2, $p \cdot R \subseteq A$, so consider an arbitrary element $y := p^{-1} \cdot x \in R$ with $x \in A$. From 4.1.1.2, $x \cdot (\omega - h_1)^{p-2}(\omega\mu - h_1h_2) \in pA$. Lifting to $B := S[W, U]$ and denoting lifts by $\sim$

$$\tilde{x}(W - h_1)^{p-2}(WU - h_1h_2) \in (p, F(W), G(U)) \tag{4.1.0.8.1}$$

Write

$$\omega\mu - h_1h_2 = (\omega - h_1)(\mu - h_2) + h_2(\omega - h_1) + h_1(\mu - h_2) \tag{4.1.0.8.2}$$

Lifting the identity in (4.1.0.8.2) to B we see that $\tilde{x} \in (p, W-h_1, (U-h_2)^{p-1})$. By symmetry $\tilde{x} \in (p, (W-h_1)^{p-1}, U-h_2)$ and hence

$$\tilde{x} \in (p, (U-h_2)^{p-1}, (W-h_1)^{p-1}, (W-h_1)(U-h_2))$$

This is because for a regular sequence $(q, y, z) \subseteq B$

$$(q, y, z^n) \cap (q, y^n, z) = (q, y^n, z^n, yz) \tag{4.1.0.8.3}$$

Since $1 \in R$, towards describing A-module generators for R we may assume that $y = p^{-1}x$ with

$$x = a_1 \cdot (\mu - h_2)^{p-1} + a_2 \cdot (\omega - h_1)^{p-1} + a_3 \cdot (\omega - h_1)(\mu - h_2) \tag{4.1.0.8.4}$$

for some $a_1, a_2, a_3 \in A$. Suppose we can write

$$y = p^{-1}[a_1(\mu - h_2)^{p-1} + a_2(\omega - h_1)(\mu - h_2)^{p-2} + \cdots + a_p(\omega - h_1)^{p-1} + b \cdot (\omega - h_1)^i(\mu - h_2)^i] \tag{4.1.0.8.5}$$

with $1 \leq i < (p-1)/2$ and $a_i, b \in A$. By (4.1.0.8.4), we can do this for $i = 1$, where $a_j = 0$ for $2 \leq j \leq p-1$. Now using (4.1.0.8.2) we get that $y \cdot (\omega - h_1)^{p-1-i}(\mu - h_2)^{i-1}(\omega\mu - h_1 h_2) \in pA$ if and only if

$$a_{i+1}h_1(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} + a_i h_2(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} + bh_1(\omega - h_1)^{p-1}(\mu - h_2)^{2i} \in pA$$

Pulling back to B

$$\tilde{b}h_1(U-h_2)^{2i} + h_2\tilde{a}_i(U-h_2)^{p-1} + h_1\tilde{a}_{i+1}(U-h_2)^{p-1} \in (p, (U-h_2)^p, W-h_1) \tag{4.1.0.8.6}$$

and thus $\tilde{b} \in (p, W-h_1, (U-h_2)^{p-2i-1})$. By symmetry, $\tilde{b} \in (p, U-h_2, (W-h_1)^{p-2i-1})$ and hence by (4.1.0.8.3)

$$\tilde{b} \in (p, (U-h_2)^{p-2i-1}, (W-h_1)^{p-2i-1}, (W-h_1)(U-h_2))$$

Therefore

$$py \in (p, (\omega-h_1)^{i+1}(\mu-h_2)^{i+1}) + (\omega-h_1, \mu-h_2)^{p-1}$$

Starting from (4.1.0.8.4) and iterating the argument from (4.1.0.8.5) to this point sufficiently many times, we see that

$$R \subseteq \langle \{1\} \cup p^{-1} \cdot (\omega-h_1, \mu-h_2)^{p-1} \rangle_A$$

Consider

$$y = p^{-1}(\sum_{i=1}^{p} a_i(\mu-h_2)^{p-i}(\omega-h_1)^{i-1}) \in K \tag{4.1.0.8.7}$$

with the $a_i \in A$. From 4.1.1.2, $y \in R$ if and only if for all $2 \le i \le p$

$$y \cdot (\omega-h_1)^{p-i}(\mu-h_2)^{i-2}(\omega\mu - h_1h_2) \in A \tag{4.1.0.8.8}$$

From (4.1.0.8.2) the above statements are equivalent to

$$(a_{i-1}h_2 + a_ih_1)(\mu-h_2)^{p-1}(\omega-h_1)^{p-1} \in pA \tag{4.1.0.8.9}$$

for each $2 \le i \le p$. Lifting to B, we see that (4.1.0.8.9) is equivalent to

$$a_{i-1}h_2 + a_ih_1 \in P \tag{4.1.0.8.10}$$

Thus the proof is complete. ■

Lemma 4.1.0.9. *Assume S is an unramified regular local ring of mixed characteristic $p \ge 3$. Let $S[\omega], S[\mu]$ be integrally closed such that $fg \in S^{p\wedge p^2}$. Then $\nu_S(R) \le p^2+1$. More explicitly, set*

$\eta_i := p^{-1}(\omega - h_1)^i(\mu - h_2)^{p-i}$ for $1 \leq i \leq p-1$. We have $R = \langle A[\eta_1, \ldots, \eta_{p-1}] \cup \{\varepsilon\}\rangle_S$ for:

$$\varepsilon := p^{-1} \sum_{i=1}^{p} (-1)^i c^{p-i} e^{i-1} (\mu - h_2)^{p-i} (\omega - h_1)^{i-1}$$

where $h_1 \equiv zc \bmod pS$, $h_2 \equiv ze \bmod pS$ for some $z \in S \setminus pS$ and $c, e \in S$ relatively prime.

Proof. Suppose $\langle T\rangle_S$ is as in 3.1.0.19. Note that it is just the ring $A[\eta_1, \ldots, \eta_{p-1}]$. Since $\langle T\rangle_S$ is S-free of rank p^2, the assertion $\nu_S(R) \leq p^2 + 1$ follows from the second assertion.

Since S/pS is regular local (a UFD), $h_1 \equiv (zc) \bmod pS$, $h_2 \equiv (ze) \bmod pS$ for some $z \in S \setminus pS$ and $c, e \in S$ relatively prime. First suppose c or e is a unit in S. Then it follows from 4.1.0.8(1) and (2) that $R = \langle A[\eta_1, \ldots, \eta_{p-1}] \cup \varepsilon\rangle_A$. Notice that if $i + j > 0$, then $(\omega - h_1)^i(\mu - h_2)^j \in (A[\eta_1, \ldots, \eta_{p-1}] :_K \varepsilon)$. Thus $R = \langle A[\eta_1, \ldots, \eta_{p-1}] \cup \{\varepsilon\}\rangle_S$ in this case.

Next assume that neither c or e is a unit, so that $(p, c, e) \subseteq S$ forms a regular sequence. Now $\varepsilon \in R$ from 4.1.0.8(2). From 4.1.0.8(1), it suffices to look at elements of the form

$$y = p^{-1}(\sum_{i=1}^{p} a_i(\mu - h_2)^{p-i}(\omega - h_1)^{i-1}) \in R$$

In view of 4.1.0.8(2), the condition $a_1h_2 + a_2h_1 \in P$ upon lifting to $B := S[W,U]$ (denoting lifts by $\sim$) tells us that $\tilde{a_1}, \tilde{a_2}$ arise from the first syzygy of the grade five complete intersection B-ideal, $\tilde{Q} := (p, c, e, W - h_1, U - h_2)$. In particular

$$\tilde{a_2} \in (p, c, W - h_1, U - h_2) \cap (p, e, W - h_1, U - h_2) = (p, ce, W - h_1, U - h_2)$$

since $a_2h_2 + a_3h_1 \in P$ as well. Since $A[\eta_1, \ldots, \eta_{p-1}] \subseteq R$, towards describing A-module generators for R we may assume $\tilde{a_2} = \alpha ce$ for some $\alpha \in B$ and consequently that $\tilde{a_1} = -\alpha c^2$ and $\tilde{a_3} = -\alpha e^2$.

Now let $3 \leq i < p$ be such that for all $1 \leq k \leq i$, $a_k = (-1)^k \alpha c^{i-k} e^{k-1}$ for some $\alpha \in A$. Lifting $a_ih_2 + a_{i+1}h_1 \in P$ to B, we have $(-1)^i \alpha e^i + a_{i+1}c \in \tilde{P}$. Since $A[\eta_1, \ldots, \eta_{p-1}] \subseteq R$, we may assume that $\alpha = \alpha' c$ for some $\alpha' \in B$ and hence that $a_{i+1} = (-1)^{i+1}\alpha' \cdot e^i$. Iterating this argument, we get that $R = \langle A[\eta_1, \ldots, \eta_{p-1}] \cup \{\varepsilon\}\rangle_A$. Finally, if $i + j > 0$ then $(\omega - h_1)^i(\mu - h_2)^j \in (A[\eta_1, \ldots, \eta_{p-1}] :_K$

ε) and the conclusion follows. ∎

Proposition 4.1.0.10. *Let $(S,\mathfrak{m})$ be an unramified regular local ring of mixed characteristic $p \geq 3$ such that $S[\omega]$ and $S[\mu]$ are integrally closed and $fg \in S^{p \wedge p^2}$. Then R is Cohen-Macaulay if and only if $Q := (p,h_1,h_2) \subset S$ is a two generated ideal or all of S. Moreover, $p.d_S(R) \leq 1$.*

Proof. Since S/pS is a UFD, $h_1 \equiv zc \bmod pS$, $h_2 \equiv ze \bmod pS$ for some $z \in S \setminus pS$ and $c,e \in S$ relatively prime. Then Q is a two generated ideal or all of R if and only if c or e is a unit. From 4.1.0.9, $R = \langle A[\eta_1,\ldots,\eta_{p-1}] \cup \{\varepsilon\}\rangle_S$. Suppose that c is a unit. Then

$$(\mu - h_2)^{p-1} \in \langle \varepsilon, (\omega - h_1)(\mu - h_2)^{p-2}, \ldots, (\omega - h_1)^{p-1}\rangle_S.$$

Thus R is S-free of rank p^2 and hence is Cohen-Macaulay.

Now assume neither c nor e is a unit, that is Q is either grade three perfect or grade two and not perfect. We know from 4.1.0.9 that $R = \langle A[\eta_1,\ldots,\eta_{p-1}] \cup \{\varepsilon\}\rangle_S$. With T as in 3.1.0.19, define $\Gamma : T \to \mathbb{Z}$, $\Gamma' : T \to \mathbb{Z}$ by

$$\Gamma(p^{-k}(\omega - h_1)^i(\mu - h_2)^j) = i + j$$

$$\Gamma'(p^{-k}(\omega - h_1)^i(\mu - h_2)^j) = i$$

Define a total ordering on T as follows: for $x,y \in T$, if $\Gamma(x) \geq \Gamma(y)$ then $x \geq y$ and if $\Gamma(x) = \Gamma(y)$, then $x \geq y$ if $\Gamma'(x) \geq \Gamma'(y)$. Let $\alpha : S^{p^2+1} \to R$ be the S-projection map defined by the generating set $T \cup \{\varepsilon\}$ such that the basis element e_{p^2+1} maps to ε and the image of the basis elements e_i, $i \neq p^2+1$ is defined by the ordered set T. Consider $U = [u_i] \in Ker(\alpha)$. Since A is S-free with a basis given by $\{(\omega - h_1)^i(\mu - h_2)^j \,|\, 0 \leq i,j \leq p-1\}$, we get that that $u_i = 0$ for $m \leq i \leq p^2$, where $m = p^2 - 2^{-1}(p-1)p + 1$. Let $p\varepsilon = \sum_{i=1}^{m-1} v_i x_i$ with the $v_i \in S$ and x_i from the ordered set T. Since $p \in S$ is prime we get the following free resolution of R over S:

$$0 \longrightarrow S \xrightarrow{\psi^T} S^{p^2+1} \xrightarrow{\alpha} R \longrightarrow 0 \qquad (4.1.0.10.1)$$

where $\psi = [v_1 \ldots v_{m-1}\ 0 \ldots 0\ -p]$. The above resolution is minimal since $\psi^T(S) \subseteq \mathfrak{m}S^{p^2+1}$, so that $p.d_S(R) = 1$. The proof is now complete. ■

Remark 4.1.0.11. Let S be an unramified regular local ring of mixed characteristic p. Note that the ideal $(p, h_1, h_2) \subseteq S$ is a two generated ideal or all of R if and only if the same property holds for the ideal $(p, f, g) \subseteq S$. Similarly, the ideal (p, h_1, h_2) has grade three if and only if the ideal (p, f, g) has the same property.

Example 4.1.0.12. The conditions in 4.1.0.10 give a non-empty class of non Cohen-Macaulay integral closures R. For an example where $Q = (p, h_1, h_2)$ has grade three, consider $S = \mathbb{Z}[X, Y]_{(3,X,Y)}$ where X, Y are indeterminates over $\mathbb{Z}_{(3)}$. Let

$$f = -5X^3 + 9 = X^3 + 3(3 - 2X^3)$$

$$g = -2Y^3 + 9 = Y^3 + 3(3 - Y^3)$$

and $\omega^3 = f, \mu^3 = g$. Then f, g are square free elements that form a regular sequence in S. It is easily checked that $[K : L] = 9$ and that this choice satisfies the hypothesis of 4.1.0.10, so that $p.d._S(R) = 1$.

For an example where Q has grade two but $p.d_S(S/Q) = 3$, let $S = \mathbb{Z}[X, Y]_{(p,X,Y)}$ for some prime number $p \geq 3$. Set

$$f = (1 - p)X^{2p} + p^2 = (X^2)^p + p(p - X^{2p})$$

$$g = (1 + p)(XY)^p + p^2 = (XY)^p + p(p + (XY)^p)$$

Then $f, g \in S$ are square free and form a regular sequence in S. It is easily verified that $[K : L] = p^2$ and that the choice satisfies the hypothesis of 4.1.0.10, so that $p.d._S(R) = 1$.

Lemma 4.1.0.13. *With established notation, the following holds*

$$P_A^* = \langle 1, p^{-1}(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} \rangle_A$$

Proof. Set $P_1 := (p, \omega - h_1)$ and $P_2 := (p, \mu - h_2)$, so that $P^* = P_1^* \cap P_2^*$. Let $\tilde{P}_1 := (p, W - h_1) \subseteq S[\mu][W]$. It is the maximal minors of

$$E = \begin{bmatrix} W - h_1 \\ p \end{bmatrix}$$

We have $F(W) \in \tilde{P}_1$, $F(W) = a \cdot (-p) + (W^{p-1} + h_1 W^{p-2} + \cdots + h_1^{p-1})(W - h_1)$. Adjoining the appropriate column of coefficients

$$E' = \begin{bmatrix} W - h_1 & a \\ p & W^{p-1} + \cdots + h_1^{p-1} \end{bmatrix}$$

From Kleiman & Ulrich (1997)[Lemma 2.5] $P_1^* = \langle E'_{11}/\delta_1, E'_{22}/\delta_2 \rangle_A$ where E'_{ii} and δ_i denote the image in A of the (i,i)-th cofactor of E' and the i-th (signed) minor of E. Therefore

$$P_1^* = \langle 1, p^{-1}(\omega^{p-1} + \cdots + h_1^{p-1}) \rangle_A$$

Identically

$$P_2^* = \langle 1, p^{-1}(\mu^{p-1} + \cdots + h_2^{p-1}) \rangle_A$$

Now consider $y \in P^* = P_1^* \cap P_2^*$. Write for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in A$

$$py = p\alpha_1 + \beta_1(\omega^{p-1} + \cdots + h_1^{p-1}) = p\alpha_2 + \beta_2(\mu^{p-1} + \cdots + h_2^{p-1})$$

Lifting to $B := S[W, U]$ and denoting lifts by $\sim$

$$p(\tilde{\alpha}_1 - \tilde{\alpha}_2) + \tilde{\beta}_1(W^{p-1} + \cdots + h_1^{p-1}) - \tilde{\beta}_2(U^{p-1} + \cdots + h_2^{p-1}) \in (F(W), G(U))$$

Writing $W^{p-1}+\cdots+h_1^{p-1}=(W-h_1)^{p-1}+p\cdot C_1'$ (respectively for $U^{p-1}+\cdots+h_2^{p-1}$),

$$\tilde{\beta}_1(W-h_1)^{p-1}-\tilde{\beta}_2(U-h_2)^{p-1}\in(p,F(W),G(U))$$

This gives $\tilde{\beta}_1\in(p,W-h_1,(U-h_2)^{p-1})$. Since $1\in P^*$ and $(\omega-h_1)(\omega^{p-1}+\cdots+h_1^{p-1})\in pA$, we get $P^*\subseteq\langle 1,p^{-1}(\omega-h_1)^{p-1}(\mu-h_2)^{p-1}\rangle_A$. Since the reverse inclusion is obvious, the proof is complete. ■

Theorem 4.1.0.14. *Let $(S,\mathfrak{m})$ be an unramified regular local ring of mixed characteristic $p\geq 3$. Then*

1. *R is Cohen-Macaulay if*

   (a) *At least one of $S[\omega],S[\mu]$ is not integrally closed.*

   (b) *$S[\omega],S[\mu]$ are integrally closed and $fg^i\notin S^{p\wedge p^2}$ for all $1\leq i\leq p-1$.*

2. *Let $S[\omega],S[\mu]$ be integrally closed and $fg\in S^{p\wedge p^2}$. Then R is Cohen-Macaulay if and only if $Q:=(p,f,g)\subseteq S$ is a two generated ideal or all of S. Moreover, $p.d_S(R)\leq 1$ and $\nu_S(R)\leq p^2+1$.*

3. *If $Q:=(p,f,g)\subseteq S$ has grade three, R admits a birational maximal Cohen-Macaulay module.*

Proof. We have shown 1(a) in 3.1.0.18 and 1(b) in 3.1.0.20. The proof of (2) follows from 4.1.0.10, 4.1.0.9 and 4.1.0.11.

Now assume Q has grade three. From part (1), we may assume that $S[\omega],S[\mu]$ are integrally closed and $fg^i\in S^{p\wedge p^2}$ for some (unique) $1\leq i\leq p-1$. From 4.1.1.2, $I^*=R$ for $I:=pA+(\omega\mu^i-h_1h_2^i)\cdot P^{p-2}$. Set $M:=(IP)^*$. Then M is an R-module since $(A:_K IP)=((A:_K I):_K P)=(R:_K P)$. We will show $depth_S(M)=d$, so that M is a small CM module over R. By definition

$$M=(IP)^*=(p\cdot P+(\omega\mu^i-h_1h_2^i)\cdot P^{p-1})^*=F_1\cap F_2.$$

where $F_1 = p^{-1}P^*$ and $F_2 = (\omega\mu^i - h_1h_2^i)^{-1}(P^{p-1})^*$. This is because for ideals $H,N \subseteq A$, $(A :_K H+N) = (A :_K H) \cap (A :_K N)$ as A-modules. Now $A/P \simeq S/pS$ as S-modules, therefore by the depth lemma P is S-free. By 3.1.0.6, $Hom_A(P,A) \simeq Hom_S(P,S)$ as S-modules and hence P^* is Cohen-Macaulay. On the other hand, since $(P^{p-1})^*$ and $(P^{(p-1)})^*$ are birational S_2 modules that agree in codimension one, we have $(P^{p-1})^* = (P^{(p-1)})^*$. From 3.1.0.19(2) and **??** we then have that $(P^{p-1})^*$ is Cohen-Macaulay. Therefore F_1 and F_2 are Cohen-Macaulay since $F_1 \simeq P^*$ and $F_2 \simeq (P^{p-1})^*$ as A-modules and S-modules. We have the natural short exact sequence of S-modules

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0 \tag{4.1.0.14.1}$$

To complete the proof it suffices to show that $depth_S(F_1+F_2) \geq d-1$. Set

$$\mathscr{F} := p(\omega\mu^i - h_1h_2^i)\cdot(F_1+F_2) = \mathscr{F}_1 + \mathscr{F}_2.$$

where $\mathscr{F}_1 := (\omega\mu^i - h_1h_2^i)P^*$ and $\mathscr{F}_2 := p(P^{p-1})^*$. Clearly $F_1+F_2 \simeq \mathscr{F}$ as A-modules and hence as S-modules. From 3.1.0.19(1), $\mathscr{F}_2 = (p)+P^p$ and from 4.1.0.13

$$\mathscr{F}_1 = (\omega\mu^i - h_1h_2^i, p^{-1}(\omega\mu^i - h_1h_2^i)(\omega-h_1)^{p-1}(\mu-h_2)^{p-1})A. \tag{4.1.0.14.2}$$

Set $m := \omega\mu^i - h_1h_2^i$. We make the following two claims:

1. $\mathscr{F} = \mathscr{F}_2 + (m)$.

2. $(\mathscr{F}_2 :_A m) = (p)+P^{p-1}$.

Assume both claims hold. Since $(\mathscr{F}_2 :_A m) \simeq \mathscr{F}_2 \cap (m)$ as A-modules and hence S-modules, we have a natural short exact sequence of S-modules

$$0 \longrightarrow (p)+P^{p-1} \longrightarrow \mathscr{F}_2 \oplus (m) \longrightarrow \mathscr{F} \longrightarrow 0 \tag{4.1.0.14.3}$$

If $depth_S((p)+P^{p-1}) = d$, then $depth_S(\mathscr{F}) \geq d-1$ and we are done. But $depth_S((p)+P^{p-1}) = d$

if and only if $A/((p)+P^{p-1})$ is Cohen-Macaulay. For $B := S[W,U]_{(\mathfrak{m},W-h_1,U-h_2)}$ we have as B-modules

$$A/((p)+P^{p-1}) \simeq B/((p)+(W-h_1,U-h_2)^{p-1}).$$

Since B/pB is regular local and any power of a complete intersection B-ideal is perfect, we are through. Therefore only the claims remain to be proved.

Set $\mathscr{Q} := \mathscr{F}_2 + (m)$. For claim (1), from (4.1.0.14.2) we only need to show

$$s := p^{-1} m(\omega - h_1)^{p-1}(\mu - h_2)^{p-1} \in \mathscr{Q}.$$

Since $fg^i \in S^{p \wedge p^2}$, we get $ah_2^p + ibh_1^p \in pS$. Moreover, $grade(Q) = 3$ implies $a - qh_1^p \in pS$ and $b + i^{-1}qh_2^p \in pS$ for some $q \in S$. Write

$$m = (\omega - h_1)(\mu^i - h_2^i) + h_2^i(\omega - h_1) + h_1(\mu^i - h_2^i). \tag{4.1.0.14.4}$$

and recall that $(\omega - h_1)^p = p(a - c_1'(\omega - h_1))$ and $(\mu - h_2)^p = p(b - c_2'(\mu - h_2))$. Then

$$\begin{aligned} s &\equiv p^{-1}[h_2^i(\omega - h_1) + h_1(\mu^i - h_2^i)](\omega - h_1)^{p-1}(\mu - h_2)^{p-1} \bmod \mathscr{Q} \\ &\equiv ah_2^i(\mu - h_2)^{p-1} + bh_1(\mu^{i-1} + \cdots + h_2^{i-1})(\omega - h_1)^{p-1} \bmod \mathscr{Q} \\ &\equiv ah_2^i(\mu - h_2)^{p-1} + ibh_1h_2^{i-1}(\omega - h_1)^{p-1} \bmod \mathscr{Q} \\ &\equiv qh_1h_2^i[h_1^{p-1}(\mu - h_2)^{p-1} - h_2^{p-1}(\omega - h_1)^{p-1}] \bmod \mathscr{Q}. \end{aligned} \tag{4.1.0.14.5}$$

Now $(\omega\mu^i - h_1h_2^i) \cdot P^{p-2} \subseteq \mathscr{Q}$, (4.1.0.14.4) and $P^p \subseteq \mathscr{Q}$ imply

$$h_2^i(\omega - h_1) + h_1(\mu^i - h_2^i) \in (\mathscr{Q} :_A P^{p-2}).$$

Therefore for all $0 \leq j \leq p-2$

$$\begin{aligned}
&(h_2^i(\omega-h_1)+h_1(\mu^i-h_2^i))\cdot h_2^{p-2-j}h_1^j(\omega-h_1)^{p-2-j}(\mu-h_2)^j \\
=&h_2^{i-1}[(h_2(\omega-h_1))^{p-j-1}(h_1(\mu-h_2))^j]+ \\
&(h_1(\mu-h_2))^{j+1}(\mu^{i-1}+\cdots+h_2^{i-1})(h_2(\omega-h_1))^{p-2-j} \in \mathscr{Q}.
\end{aligned}$$

Thus

$$h_2^{i-1}[(h_2(\omega-h_1))^{p-j-1}(h_1(\mu-h_2))^j] \equiv -ih_2^{i-1}(h_1(\mu-h_2))^{j+1}(h_2(\omega-h_1))^{p-2-j} \bmod \mathscr{Q}.$$

It then follows that for any $1 \leq k \leq p-j-1$

$$h_2^{i-1}[(h_2(\omega-h_1))^{p-j-1}(h_1(\mu-h_2))^j] \equiv (-i)^k h_2^{i-1}(h_1(\mu-h_2))^{j+k}(h_2(\omega-h_1))^{p-j-1-k} \bmod \mathscr{Q}.$$

In particular for $j=0$ and $k=p-1$ we get

$$h_2^{i-1}(h_2(\omega-h_1))^{p-1} \equiv h_2^{i-1}(h_1(\mu-h_2))^{p-1} \bmod \mathscr{Q}. \tag{4.1.0.14.6}$$

Combining (4.1.0.14.6) and (4.1.0.14.5), we see that $s \in \mathscr{Q}$ and thus claim (1) holds.

To show one containment in claim (2), note that $(p)+P^{p-1} \subseteq (\mathscr{F}_2 :_A m)$ since $p+P^p = \mathscr{F}_2$. For the reverse inclusion, consider $y \in (\mathscr{F}_2 :_A m)$. Lifting to B and denoting lifts by $\sim$

$$\tilde{y}(WU^i-h_1h_2^i) \in (p,F(W),G(U))+(W-h_1,U-h_2)^p = (p)+(W-h_1,U-h_2)^p. \tag{4.1.0.14.7}$$

Using (4.1.0.14.4) we have $\tilde{y} \in (p,(W-h_1)^{p-1},U-h_2)$. Similarly $\tilde{y} \in (p,W-h_1,(U-h_2)^{p-1})$ and from (4.1.0.8.3)

$$\tilde{y} \in (p,(W-h_1)^{p-1},(U-h_2)^{p-1},(W-h_1)(U-h_2)). \tag{4.1.0.14.8}$$

Now assume for some $1 \leq i \leq 2^{-1}(p-1)-1$,

$$\tilde{y} \in (p,(W-h_1)^i(U-h_2)^i)+(W-h_1,U-h_2)^{p-1}. \tag{4.1.0.14.9}$$

Write $\tilde{y} = \alpha \cdot (W-h_1)^i(U-h_2)^i + \beta$ for some $\alpha \in B$ and $\beta \in (p)+(W-h_1,U-h_2)^{p-1}$. From (4.1.0.14.7):

$$\alpha \cdot (W-h_1)^i(U-h_2)^i(WU^i - h_1h_2^i) \in (p)+(W-h_1,U-h_2)^p. \tag{4.1.0.14.10}$$

Using the regular sequence $(p,(U-h_2)^{i+1},(W-h_1)^{i+1},h_2^i) \subseteq B$, we get

$$\alpha \in (p,(W-h_1)^{p-2i-1},U-h_2).$$

Similarly, using the regular sequence $(p,(W-h_1)^{i+1},(U-h_2)^{i+1},h_1(U^{i-1}+\cdots+h_2^{i-1})) \subseteq B$ we get $\alpha \in (p,W-h_1,(U-h_2)^{p-2i-1})$. Thus by (4.1.0.8.3):

$$\alpha \in (p,(W-h_1)^{p-2i-1},(U-h_2)^{p-2i-1},(W-h_1)(U-h_2))$$

and hence

$$\tilde{y} \in (p,(W-h_1)^{i+1}(U-h_2)^{i+1})+(W-h_1,U-h_2)^{p-1}.$$

Thus starting from (4.1.0.14.8) we may induct on i to get

$$\tilde{y} \in (p,(W-h_1)^{2^{-1}(p-1)}(U-h_2)^{2^{-1}(p-1)})+(W-h_1,U-h_2)^{p-1} = (p)+(W-h_1,U-h_2)^{p-1}.$$

This shows $(\mathscr{F}_2 :_A m) = (p)+P^{p-1}$ and all claims have been proved. Thus R admits a birational small CM module.

■

Remark 4.1.0.15. If $grade(Q)=2$ and $p.d_S(S/Q)=3$ in the context of 4.1.0.14(3), we are not able to construct a birational small CM module over R at present. However, if we allow an extension of the quotient field, then constructing a small CM module over R may be possible in this case, see section 3.2.

Remark 4.1.0.16. By a vector bundle on the punctured spectrum of a regular local ring $(S,\mathfrak{m})$ or simply a bundle on S we mean a finitely generated reflexive S-module M such that M_P is S_P-free for all non maximal ideals $P \subseteq S$. One could use 4.1.0.14(2) to generate examples of non-trivial bundles M on localizations of polynomial rings or power series rings over $\mathbb{Z}_{(p)}$ of dimension d at least three such that $rank_S(M) = p^2 + d - 3$. Moreover, these bundles would satisfy $p.d._S(M) = 1$.

Let $d \geq 3$ and $(T,\mathfrak{n})$ be a d-dimensional unramified regular local ring of mixed characteristic p. Choose $(S,\mathfrak{m}) \subseteq T$ a three dimensional subring of T that is an unramified regular local ring of mixed characteristic p and a quotient of T by a regular sequence (such a choice is possible for example when T is a localization of a polynomial ring over $\mathbb{Z}_{(p)}$). Let $S \subseteq (E,\mathfrak{n}') \subseteq T$ be such that E is regular local and $S = E/(t)$ for some $0 \neq t \in E$. Using 4.1.0.14(2), with the base ring as S, construct R such that it is not S-free. Choose a minimal S-free resolution

$$0 \longrightarrow S \xrightarrow{\psi^T} S^{p^2+1} \longrightarrow R \longrightarrow 0 \qquad (4.1.0.16.1)$$

Let M' be the cokernel of the E-matrix $\phi := \left[\begin{array}{c|c} \psi^T & t \end{array}\right]$

$$0 \longrightarrow E \xrightarrow{\phi} E^{p^2+2} \longrightarrow M' \longrightarrow 0 \qquad (4.1.0.16.2)$$

so that $p.d._E(M) = 1$. The ideal of maximal minors of ψ^T is $\mathfrak{m}$-primary since R is a bundle over S. Therefore the ideal of maximal minors of ϕ is $\mathfrak{n}'$-primary and hence it is free on the punctured spectrum of E. Proceeding this way, we can construct a finite module M over T that is free on the punctured spectrum of T and $p.d._T(M) = 1$. Moreover, since M is an S_2 T-module, it is T-reflexive (see Bruns & Herzog (1998)[Proposition 1.4.1] for example).

4.1.1 Mixed Characteristic two case

In this section we identify what it means for R to be Cohen-Macaulay when S is an unramified regular local ring of mixed characteristic two and $f,g \in S^2$. When R is not Cohen-Macaulay, we show the existence of a birational small CM module.

Towards this, from 3.1.1.2 and 3.1.1.1, if we seek a non S-free R, we must have that $S[\omega]$ and $S[\mu]$ are integrally closed such that $S[\omega\mu] \cong S[X]/(X^2 - fg)$ is not integrally closed. This scenario is very much possible, see 4.1.0.12. In this situation, we start by identifying an ideal $J \subseteq A$ such that $J^* = R$.

Convention 4.1.1.1. For this section, we maintain notation as set up in 3.1.0.17 and additionally assume $p = 2$. In case we are in the situation $f,g \notin S^{2\wedge 4}$, assume that $f,g \notin 2S$. This is justified, since if exactly one of $f,g \in 2S$, then by 3.1.1.2, R is S-free. The case $f,g \in 2S$ is not possible since f,g satisfy $\mathscr{A}_1$.

Proposition 4.1.1.2. *With established notation, let $S[\omega], S[\mu]$ be integrally closed and $fg \in S^{2\wedge 4}$, so that $S[\omega\mu]$ is not integrally closed. Then for $J := (2, \omega\mu - h_1h_2)A$, we have $J_A^* = R$.*

Proof. Since J^* and R are birational S_2 A-modules, it suffices to show $J_Q^* = R_Q$ for all height one primes $Q \subseteq A$. From 3.1.0.14(2), $J_Q^* = R_Q = A_Q$ for all height one primes $Q \neq P$. So we may assume $(S, 2S)$ and (A, P) are one dimensional local rings.

Note that $A = S[\omega, \omega\mu]$ and that $\{f, fg\}$ satisfy $\mathscr{A}_1$ since they are both units. Since $S[\omega]$ is integrally closed and $S[\omega\mu]$ is not, the description of R from the proof of 3.1.1.1 applies. We have that R is generated over S by the set $\{1, \mu, \tau, \mu\tau\}$ where $\tau = 2^{-1}(\omega\mu + h_1h_2)$. This immediately implies J conducts R into A.

Let $\phi : B := S[W,T] \to A$ be the projection map defined by $W \mapsto \omega$ and $T \mapsto \omega\mu$, where W,T are indeterminates over S. Note that $Ker(\phi) := (W^2 - f, T^2 - fg)$. Suppose $l \in A$ conducts R to A. Since A_P is not regular, $l \in P = (2, \omega - h_1, \omega\mu - h_1h_2)$. Write $l = x \cdot 2 + y \cdot (\omega - h_1) + z \cdot (\omega\mu - h_1h_2)$

for some $x,y,z \in A$. Viewing $l \cdot \tau \in A$ in B and denoting lifts by $\sim$, we get

$$\tilde{y} \cdot (W-h_1)(T-h_1h_2) \in (2,(W-h_1)^2,(T-h_1h_2)^2) \qquad (4.1.1.2.1)$$

By a standard regular sequence argument, $\tilde{y} \in (2, W-h_1, T-h_1h_2)$ and so $y \in P$. Since $(\omega-h_1)^2 \in 2A$, we have $l \in J$. Thus J is the conductor of R to A and the proof is complete. ∎

Lemma 4.1.1.3. *With established notation, set*

$$I := (2, \omega\mu - h_1h_2, h_2\omega - h_1\mu)A = (2, \omega\mu - h_1h_2, (\omega+h_1)(\mu+h_2))A$$

Then $p.d._S(I) \leq 1$. More precisely, $I \simeq S^2 \oplus_S C$ for some S-module C that admits the free resolution

$$0 \longrightarrow S \xrightarrow{\psi^T} S^3 \xrightarrow{\phi} C \longrightarrow 0$$

where ϕ is given by $\phi(e_1) = 2\omega$, $\phi(e_2) = 2\mu$ and $\phi(e_3) = h_2\omega - h_1\mu$ and $\psi = [-h_2\ h_1\ 2]$.

Proof. We claim that I is generated over S by the set $G := \{2, 2\omega, 2\mu, \omega\mu - h_1h_2, h_2\omega - h_1\mu\}$. To see this, note that $2\omega\mu \in (\omega\mu - h_1h_2) \cdot S + 2 \cdot S$. Next $\omega \cdot (\omega\mu - h_1h_2) = a \cdot 2\mu - h_1(h_2\omega - h_1\mu)$. A symmetric argument takes care of $\mu \cdot (\omega\mu - h_1h_2)$. We also have $\omega\mu \cdot (\omega\mu - h_1h_2) = -h_1h_2(\omega\mu - h_1h_2) + 4 \cdot e$ for some $e \in S$. Finally, since $(\omega, \mu) \subseteq ((2, \omega\mu - h_1h_2) :_A h_2\omega - h_1\mu)$, the claim holds.

Now, let $\phi' : S^5 \to I$ be the projection map defined by the ordered generating set G. If $[x_1 x_2 x_3 x_4 x_5]^T \in Ker(\phi)$, then since A is S-free with a basis given by $\{1, \omega, \mu, \omega\mu\}$, we have that $x_1 = x_4 = 0$. Therefore $I \simeq S^2 \bigoplus_S C$, where C is the S-module generated by $\{2\omega, 2\mu, h_2\omega - h_1\mu\}$. Now C admits the above resolution for if $E := [s_1\ s_2\ s_3]^T \in S^3$, then $E \in Ker(\phi)$ if and only if $2s_1 + h_2s_3 = 0$ and $2s_2 - h_1s_3 = 0$. Thus, if $E \in Ker(\phi)$, then there exists $k \in S$ such that $s_1 = -h_2k$, $s_2 = h_1k$ and $s_3 = 2k$, so that $E \in Im(\psi^T)$.

∎

Proposition 4.1.1.4. *Let $(S, \mathfrak{m})$ be an unramified regular local ring of mixed characteristic two. Let $S[\omega], S[\mu]$ be integrally closed rings and $fg \in S^{2\wedge 4}$. Then*

1. *If $f,g \in \mathfrak{m}$, $R \simeq S^2 \bigoplus_S Syz_S^2(S/Q)$ where $Q := (2,h_1,h_2) \subset S$.*

2. *$p.d._S(R) \leq 1$.*

3. *R is Cohen-Macaulay if and only if Q is a two generated ideal or all of R.*

Proof. We have from 4.1.1.2 that $J_A^* = R$, for $J := (2, \omega\mu - h_1h_2)A$. Let $I \subseteq A$ be as in 4.1.1.3 and $P := (2, \omega - h_1, \mu - h_2)$ be the unique height one prime containing 2 in A. Now $IA_P = JA_P$ since $\omega \notin P$ and $\omega \cdot (h_2\omega - h_1\mu) \in J$. Clearly $rad(I) = P$. Therefore, by Katz (1999)[Prop 2.1], $I^* = J^* = R$. From 3.1.0.6, $R \simeq Hom_S(I,S)$ as S-modules. Now, if $f,g \in \mathfrak{m}$, then it is clear from the free resolution for I over S in 4.1.1.3, that (1) holds.

If f were a unit say, then so is h_1, so the resolution in 4.1.1.3 is not minimal. In this case, R is Cohen-Macaulay. Therefore, for the proof of (2) and (3) we may assume that $f,g \in \mathfrak{m}$. Since $S/2S$ is regular local (a UFD), we have $Q = (2, zc, ze)$ for some $z \notin 2S$. Then S/Q admits the following free S-resolution:

$$0 \longrightarrow S \xrightarrow{[-e,c,-2]^T} S^3 \xrightarrow{\Phi} S^3 \xrightarrow{\psi} S \longrightarrow S/Q \longrightarrow 0$$

where $\psi(e_1) = 2$, $\psi(e_2) = zc$, $\psi(e_3) = ze$ and

$$\Phi = \begin{bmatrix} zc & ze & 0 \\ -2 & 0 & e \\ 0 & -2 & -c \end{bmatrix}$$

Note that this is indeed a resolution by the Buchsbaum-Eisenbud criterion (Buchsbaum & Eisenbud (1973)[Cor 1]). Thus $p.d_S(R) = p.d._S(Syz_S^2(S/Q)) \leq 1$ and (2) holds. Finally, R is Cohen-Macaulay if and only if c or e is a unit. The latter is clearly equivalent to Q being a two generated ideal and thus (3) holds. ■

Remark 4.1.1.5. In the context of 4.1.1.4, if $Q := (2,h_1,h_2)S$ is a complete intersection ideal of grade three, then the conductor of R to A is the ideal I in 4.1.1.3.

To show this, since the only element of $NNL_1(A)$ is $P = (2, \omega - h_1, \mu - h_2)$ and since the conductor of a ring that satisfies S_2 is unmixed, it suffices to show that I is P-primary. Let $x \cdot y \in I$

such that $y \in A$ and $x \in A \setminus P$. Certainly $y \in P$, so write $y = 2 \cdot a_1 + (\omega - h_1) \cdot a_2 + (\mu - h_2) \cdot a_3$ for some $a_i \in A$. Since $x \cdot y \in I$ if and only if $x \cdot (a_2(\omega - h_1) + a_3(\mu - h_2)) \in I$, it suffices to show $a_2(\omega - h_1) + a_3(\mu - h_2) \in I$. Lifting to $B := S[W,U]_{(\mathfrak{m}, W-h_1, U-h_2)}$ and denoting lifts by $\sim$, we have for some $\tilde{b_i} \in B$,

$$\tilde{a_2} \cdot \tilde{x}(W - h_1) + \tilde{a_3} \cdot \tilde{x}(U - h_2) + 2 \cdot \tilde{b_1} + (WU - h_1 h_2) \cdot \tilde{b_2} + (W - h_1)(U - h_2) \cdot \tilde{b_3} \in (F(W), G(U)) \tag{4.1.1.5.1}$$

Writing $WU - h_1 h_2 = (W - h_1)(U - h_2) + h_2(W - h_1) + h_1(U - h_2)$, we have that $\tilde{a_2} \cdot \tilde{x} + \tilde{b_2} \cdot h_2 \in \tilde{P}$ where $\tilde{P} := (2, W - h_1, U - h_2)$. Similarly $\tilde{a_3} \cdot \tilde{x} + \tilde{b_2} \cdot h_1 \in \tilde{P}$. Hence $h_1 \tilde{a_2} \tilde{x} - h_2 \tilde{a_3} \tilde{x} \in \tilde{P}$ and therefore $h_1 \tilde{a_2} - h_2 \tilde{a_3} \in \tilde{P}$. Since $\tilde{P} + (h_1, h_2)B$ is a grade five complete intersection ideal, $\tilde{a_2} \equiv h_2 \cdot z \bmod P$ and $\tilde{a_3} \equiv h_1 \cdot z \bmod P$ for some $z \in A$. We have $P \subseteq (I :_A (\omega - h_1, \mu - h_2))$, so

$$a_2(\omega - h_1) + a_3(\mu - h_2) \equiv [h_2(\omega - h_1) + h_1(\mu - h_2)] \cdot z \bmod I$$

Since $h_2(\omega - h_1) + h_1(\mu - h_2) \in I$, we are done. Thus, I is P-primary and hence is the conductor of R to A.

Example 4.1.1.6. The conditions in 4.1.1.4 produce a non-empty class of non Cohen-Macaulay integral closures R. In fact they are quite abundant. From 4.1.1.4, there are two classes of examples, the first one being the case where $Q := (2, h_1, h_2)$ is grade two with $p.d_S(S/Q) = 3$ and the other when Q is grade three perfect. For an example of the first kind, set $S := \mathbb{Z}[X,Y,V]_{(2,X,Y,V)}$ where X, Y, V are indeterminates over $\mathbb{Z}_{(2)}$ and let

$$f = V^2X^2 - 2X^2 + 4 = (VX)^2 + 2(2 - X^2)$$

$$g = V^2Y^2 - 2Y^2 + 4 = (VY)^2 + 2(2 - Y^2)$$

and $\omega^2 = f, \mu^2 = g$. Then f, g are square free elements that form a regular sequence in S. It is straightforward to check that $[L(\omega, \mu) : L] = 4$. The hypersurface rings $S[\omega]$ and $S[\mu]$ are integrally closed, but the hypersurface ring $S[\omega\mu]$ is not. Since $(2,VX,VY) \subseteq S$ is a grade two ideal such that $p.d_S(S/(2,VX,VY)) = 3$, by 4.1.1.4, $p.d._S(R) = 1$.

For an example of the second kind, let $S = \mathbb{Z}[X,Y]_{(2,X,Y)}$, where X,Y are indeterminates over $\mathbb{Z}_{(2)}$ and take

$$f = -X^2+4 = X^2+2(2-X^2)$$

$$g = -Y^2+4 = Y^2+2(2-Y^2)$$

We now get to our main theorem showing that R always admits a birational small CM module when S is an unramified regular local ring of mixed characteristic two and $f,g \in S^2$. By the reduction in section 1.2.1, if S is complete with perfect residue field, then this would show that R always admits a small CM module, when $f,g \in S$ are square free and form a regular sequence.

Lemma 4.1.1.7. *With established notation, P_A^* is generated as an A-module by $\{1,\eta\}$, where $\eta := 2^{-1}(\omega+h_1)(\mu+h_2) \in K$.*

Proof. Set $P_1 := (2,\omega-h_1)$ and $P_2 := (2,\mu-h_2)$, so that $P^* = P_1^* \cap P_2^*$. Let $\tilde{P}_1 := (2,W-h_1) \subseteq S[\mu][W]$. It is the maximal minors of

$$M = \begin{bmatrix} W-h_1 \\ 2 \end{bmatrix}$$

We have $F(W) \in \tilde{P}_1$, $F(W) = a\cdot(-2)+(W+h_1)(W-h_1)$. Adjoining the appropriate column of coefficients we get

$$M' = \begin{bmatrix} W-h_1 & a \\ 2 & W+h_1 \end{bmatrix}$$

From Kleiman & Ulrich (1997)[Lemma 2.5], P_1^* is generated as A-module by $\{M'_{11}/\delta_1, M'_{22}/\delta_2\}$ where M'_{ii} and δ_i denote the image in A of the (i,i)-th cofactor of M' and the i-th (signed) minor of M respectively. Therefore, P_1^* is generated as a A-module by $\{1,2^{-1}(\omega+h_1)\}$. Identically, P_2^* is generated over A by $\{1,2^{-1}(\mu+h_2)\}$. Now consider $y \in P^* = P_1^* \cap P_2^*$. Lifting to $B := S[W,U]$

and denoting lifts by $\sim$

$$2\tilde{y} \in (2, W+h_1) \cap (2, U+h_2) + (F(W), G(U)) = (2, (W+h_1)(U+h_2), F(W), G(U))$$

Thus $2y \in (2, (\omega+h_1)(\mu+h_2))A$ and hence this shows $P^* \subseteq A + A \cdot \eta$. The reverse inclusion is clear since $\eta \cdot P \subseteq A$. Thus the proof is complete. ■

Theorem 4.1.1.8. *Let S be an unramified regular local ring of mixed characteristic two and $f, g \in S^2$.*

1. *R is Cohen-Macaulay if and only if one of the following happens*
   (a) *At least one of $S[\omega], S[\mu]$ is not integrally closed.*
   (b) *$S[\omega], S[\mu]$ are both integrally closed and $fg \notin S^{2\wedge 4}$.*
   (c) *$S[\omega], S[\mu]$ are both integrally closed, $fg \in S^{2\wedge 4}$ and $\mathscr{I} := (2, f, g) \subset S$ is a two generated ideal or all of R.*
2. *If R is not Cohen-Macaulay, R admits a birational small CM module.*

Proof. For (1), we have already shown that the conditions in (a) and (b) imply R is Cohen-Macaulay in 3.1.1.1 and 3.1.1.2 respectively. From 4.1.1.4 and 4.1.0.11, we see that the condition in 1(c) implies that R is Cohen-Macaulay. For the forward implication of (1), the contrapositive follows from 4.1.0.11 and 4.1.1.4.

For (2), by (1) it only remains to be shown that R admits a birational small CM module when $S[\omega], S[\mu]$ are both integrally closed, $fg \in S^{2\wedge 4}$ and $p.d._S(S/Q) > 2$. Therefore, assume all of these conditions for the remainder of the proof.

We have from 4.1.1.2 that for $I := (2, \omega\mu - h_1h_2)A$, $I^* = R$. Set $M = (IP)^*$, where $P = (2, \omega - h_1, \mu - h_2)$ is the unique height one prime containing 2 in A. Then $(IP)^*$ is an R-module since

$$(IP)^* = A :_K IP = ((A :_K I) : P) = (R :_K P)$$

We now show that $depth_S(M) = d$. By definition,

$$(IP)_A^* = (2 \cdot P + (\omega\mu - h_1h_2) \cdot P)_A^* = F_1 \cap F_2$$

where $F_1 = 2^{-1}P^*$ and $F_2 = (\omega\mu - h_1h_2)^{-1}P^*$. This is because for ideals $J, J' \subseteq A$, $(A :_K J + J') = (A :_K J) \cap (A :_K J')$ as A-modules.

Now P is S-free since $A/P \simeq S/2S$ as S-modules and by the depth lemma, $depth_S(P) = d$. By 3.1.0.6, $P_A^* \simeq P_S^*$ as S-modules, so P^* is Cohen-Macaulay as well and hence F_1 and F_2 are Cohen-Macaulay. We have the natural short exact sequence of S-modules

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0$$

By the depth lemma, it suffices to show $depth_S(F_1 + F_2) \geq d - 1$. Clearly $F_1 + F_2 \simeq F_1' + F_2'$ as A-modules and hence S-modules where $F_1' = (\omega\mu - h_1h_2)P^* \subseteq A$ and $F_2' = 2P^* \subseteq A$. We claim that $F_1' + F_2' = F_2' + (\omega\mu - h_1h_2)$ as ideals of A. By 4.1.1.7, we only need to show that

$$v := 2^{-1}(\omega + h_1)(\mu + h_2)(\omega\mu - h_1h_2) \in H := F_2' + (\omega\mu - h_1h_2)$$

Writing

$$(\omega\mu - h_1h_2) = (\omega - h_1)(\mu - h_2) + h_2(\omega - h_1) + h_1(\mu - h_2),$$

we have

$$\begin{aligned} v &\equiv 2^{-1}(\omega + h_1)(\mu + h_2)(h_2(\omega - h_1) + h_1(\mu - h_2)) \bmod(H) \\ &\equiv ah_2(\mu + h_2) + bh_1(\omega + h_1) \bmod(H) \end{aligned} \tag{4.1.1.8.1}$$

Since $S/2S$ is regular local, $h_1 \equiv (zc) \bmod (2)$, $h_2 \equiv (ze) \bmod (2)$ for some $z \notin 2S$ and c, e such that $(2, c, e) \subseteq S$ form a regular sequence. From (1) in 3.1.1.2, $fg \in S^{2\wedge 4}$ implies $ah_2^2 + bh_1^2 \in 2S$

and hence $a-qc^2 \in 2S$ and $b+qe^2 \in 2S$ for some $q \in S$. Therefore, (4.1.1.8.1) implies

$$\begin{aligned} v &\equiv qc^2h_2(\mu+h_2) - qe^2h_1(\omega+h_1) \bmod(H) \\ &\equiv qce(h_1(\mu+h_2) - h_2(\omega+h_1)) \bmod(H) \end{aligned} \tag{4.1.1.8.2}$$

But $\omega\mu - h_1h_2 - (\omega-h_1)(\mu-h_2) = h_2(\omega-h_1)+h_1(\mu-h_2) \in H$. Since $2 \in H$, $h_1(\mu+h_2) - h_2(\omega+h_1) \in H$ and from (4.1.1.8.2), $v \in H$. Therefore $F_1'+F_2' = H = (2,(\omega+h_1)(\mu+h_2),\omega\mu - h_1h_2)$ by 4.1.1.7. From 4.1.1.3, $p.d_S(H) \leq 1$ so that $depth_S(H) \geq d-1$. This completes the proof, and hence $M = (IP)^*$ is a small CM module over R. ■

4.2 On small CM modules over general radical towers

The work in this section is joint work with Prof. Daniel Katz and is based of Katz & Sridhar (2021). We begin with the following observation:

Lemma 4.2.0.1. *Let $\psi : A \to B$ be a finite homomorphism of unital commutative rings. Suppose B admits a finite module M such that M is A-free of rank n. Let N be any A-module. Then B admits a module C such that $C \simeq N^{\oplus n}$ as A-modules.*

Proof. Note that M defines a ring homomorphism $\phi : B \to \mathbb{M}_{n\times n}(A)$ such that $\phi(\psi(A))$ consists of scalar matrices. The map is injective if and only if M is faithful over B. Set $C := \mathbb{M}_{n\times 1}(N)$. Then C clearly admits a B-module structure via ϕ and the claim holds. ■

Recall for a Noetherian domain A, an element $x \in A$ is said to be square free if for all height one primes $Q \subset A$ containing x, $QA_Q = (x)A_Q$. Say that a subset $W \subset A$ satisfies $\mathscr{A}_1$ if for all distinct $x, y \in W$, there exists no height one prime $Q \subset A$ such that $x, y \in Q$. Consider the following:

Lemma 4.2.0.2. *Let A be a normal local domain of mixed characteristic $p > 0$ with field of fractions F. Let $r_1, \ldots, r_n \in A$ denote square free elements satisfying $\mathscr{A}_1$ such that $r_i \notin pA$ for all i. Let $q_i = p^{e_i}d_i$ be arbitrary integers for $1 \leq i \leq n$ and $p \nmid d_i$. Then the integral closure of A*

in $E := F(\sqrt[q_1]{r_1}, \ldots, \sqrt[q_n]{r_n})$ *admits a small CM module if and only if the integral closure of* A *in* $F(\sqrt[p^{e_1}]{r_1}, \ldots, \sqrt[p^{e_n}]{r_n})$ *admits one.*

Proof. Since A is a normal domain, the integral closure of A in a finite field extension of its fraction field is a finite A-module. The forward implication is then clear. Let $\mathscr{R}$ denote the integral closure of A in $F(\sqrt[p^{e_1}]{r_1}, \ldots, \sqrt[p^{e_n}]{r_n})$. Since $r_i \notin pA$ and they satisfy $\mathscr{A}_1$, we get from 3.2.0.2 and 3.2.0.3 that $w_i := \sqrt[p^{e_i}]{r_i} \in \mathscr{R}$ is square free. Since the $w_i \in \mathscr{R}$ satisfy $\mathscr{A}_1$, another application of 3.2.0.3 gives that $\mathscr{R}[\sqrt[d_1]{w_1}, \ldots, \sqrt[d_n]{w_n}]$ is integrally closed. In particular, the integral closure of A in E, say B, is $\mathscr{R}$-free. By 4.2.0.1 if $\mathscr{R}$ admits a small CM module, so does B. ■

Proposition 4.2.0.3. *Let* A *be an integrally closed domain of characteristic zero such that* $p \in A$ *is an odd prime element. Write* ζ_p *for the pth cyclotomic polynomial, so that* $\zeta_p \in \mathbb{Z}[x] \subseteq A[x]$. *Then:*

(i) ζ_p *is irreducible over* A.

(ii) If A *is an unramified local ring of mixed characteristic* p, *and* ε *is a primitive pth root of unity, then* $A[\varepsilon]$ *is a ramified regular local ring.*

Proof. For (i) taking $h = 1$, let $C' \in \mathbb{Z}[x]$ denote the polynomial in 3.1.0.10. Then $C' \notin (x-1, p)\mathbb{Z}[x]$, so $C' \notin (x-1,p)A[x]$. Suppose $\zeta_p = f(x)g(x)$, with $f(x), g(x) \in A[x]$. Then $f(x)g(x) \equiv \zeta_p \equiv (x-1)^{p-1}$, modulo pA, so in $A[x]$, we can write $f(x) = (x-1)^r + pa(x)$ and $g(x) = (x-1)^s + pb(x)$, where $r+s = p-1$ and $a(x), b(x) \in A[x]$.

On the one hand, $\zeta_p = (x-1)^{p-1} + pC'$, while on the other hand

$$f(x)g(x) = (x-1)^{p-1} + pa(x)(x-1)^s + pb(x)(x-1)^r + p^2a(x)b(x)).$$

It follows that $C' = a(x)(x-1)^s + b(x)(x-1)^r + pa(x)b(x) \in (x-1,p)A[x]$, a contradiction. Thus, ζ_p is irreducible over A (and its quotient field). In particular, $A[\varepsilon]$ is a free A-module of rank $p-1$.

Finally, suppose A is an unramified regular local ring of mixed characteristic p and $\mathfrak{n}$ is its maximal ideal. By what we have just shown, $A[\varepsilon] = A[x]/(\zeta_p)$. In $A[x]$, $M = (\mathfrak{n}, x-1)A[x]$ is the unique maximal ideal containing $\mathfrak{n}$ and ζ_p. Since $C' \notin M$, in $A[x]_M$, $p = (C')^{-1}(\zeta_p - (x-$

$1)^{p-1})$. Thus, in $A[\varepsilon] = A[\varepsilon]_{MA[\varepsilon]}$, p is a redundant generator. It follows that $A[\varepsilon]$ is a regular local ring. More over, $p \in (\varepsilon - 1)^{p-1}A[\varepsilon]$, so that $A[\varepsilon]$ is a ramified regular local ring of mixed characteristic. ■

See Katz & Sridhar (2021) for more results.

Chapter 5

On Reflexive and I-Ulrich Modules over Curve Singularities

The content of this chapter is based of joint work with Hailong Dao and Sarasij Maitra in Dao et al. (2021). In this chapter we study reflexive modules over one dimensional Cohen-Macaulay rings. Our key technique exploits the concept of I-Ulrich modules.

We describe the structure of this chapter. For background results on reflexive modules, trace ideals and birational extensions see 2.2. Section 5.1 develops the concept of I-Ulrich modules for any ideal I of height one in R, see Definition 5.1.0.1. We give various characterizations of I-Ulrichness (Theorem 5.1.0.6). We show the closedness of the subcategory of I-Ulrich modules under various operations, prompting the existence of a lattice like structure for I-Ulrich ideals, which can be referred to as an *Ulrich* lattice. We establish tests for I-Ulrichness using blow-up algebras and the *core* of I. Finally, we show that an ω_R-Ulrich M satisfies $\mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_R(M,\omega_R)$, and that such a module is reflexive.

The later sections deal with applications. In Section 5.2, under mild conditions, we are able to completely characterize extensions S of R that are “strongly reflexive" in the following sense: any maximal Cohen-Macaulay S-module is reflexive over R. Interestingly, in the birational case, this classification involves the core of the canonical ideal of R. Theorem 5.2.0.5 extends (Kobayashi, 2017, Theorem 2.14). Also, for S satisfying one of the conditions of 5.2.0.5, any contracted ideal $IS \cap R$ is reflexive (Proposition 2.2.2.12). Such a statement generalizes a result by Corso-Huneke-Katz-Vasconcelos that if R is a domain and the integral closure $\overline{R}$ is finite over R, then any integrally closed ideal is reflexive (Corso et al., 2005, Proposition 2.14).

Section 5.3 deals with various “finiteness results", where we study when certain subcategories or subsets of $\mathrm{CM}(R)$ are finite or finite up to isomorphism. One main result roughly says that if the conductor of R has small colength, then there are only finitely many reflexive ideals that contain a regular element, up to isomorphism. We also characterize rings with up to three trace ideals in Proposition 5.3.1.3. We observe that if $S = \mathrm{End}_R(\mathfrak{m})$ has finite representation type, then R has only finitely many indecomposable reflexive modules up to isomorphism (Proposition 5.3.4.1). In particular, seminormal singularities have "finite reflexive type" (Corollary 5.3.4.3).

In Section 5.4 we give some further applications on almost Gorenstein rings. We show that in such a ring, all powers of trace ideals are reflexive (Proposition 5.4.0.3). We also characterize reflexive birational extensions of R which are Gorenstein and conclude with a number of examples.

5.1 *I*-Ulrich modules

Throughout this section, we maintain notation as set up in 2.2.2.1. Let $I \subseteq R$ be an ideal of finite colength and x a principal reduction. This section grew out of the realization that the equality $xM = IM$ for certain modules M appears in many situations related to our investigation. For instance, it will turn out that when I is a canonical ideal, such a module is reflexive and the finite extensions satisfying such conditions are "strongly reflexive", see Definition 5.2.0.3.

We shall call these modules I-Ulrich, and define them slightly more generally without using principal reductions. Obviously the name and definition are inspired by the very well-studied notion of Ulrich modules, which are $\mathfrak{m}$-Ulrich in our sense. Note that our definition is very much a straight generalization of an Ulrich module, and not as restrictive as those studied in Goto et al. (2014) and Goto et al. (2016).

Definition 5.1.0.1. We say that $M \in \mathrm{CM}(R)$ is I-Ulrich if $e_I(M) = \ell(M/IM)$. Let $\mathrm{Ul}_I(R)$ denote the category of I-Ulrich modules.

Note that if $M \cong N$ in $\mathrm{CM}(R)$, then the same isomorphism takes IM to IN, so $\ell(M/IM) = \ell(N/IN)$ for any ideal I and so Ulrich condition is preserved under isomorphism.

Example 5.1.0.2. Let $M \in \mathrm{CM}(R)$. As $\ell(I^nM/I^{n+1}M) = e_I(M)$ for $n \gg 0$, it follows that I^nM is I-Ulrich for $n \gg 0$.

Definition 5.1.0.3. Let $B(I)$ denote the blow-up of I, namely the ring $\bigcup_{n\geq 0}(I^n : I^n)$. Let $b(I) = \mathfrak{c}_R(B(I))$, the conductor of $B(I)$ to R.

Remark 5.1.0.4. If x is a principal reduction of I, then it is well-known that $B(I) = R[\frac{I}{x}]$, (Barucci & Pettersson, 1995, Theorem 1).

We shall use some standard properties of Hilbert Samuel multiplicity in the proof of the next proposition. The reader can refer to various resources like, Serre (1965), Serre (1997), (Bruns & Herzog, 1998, mainly Corollary 4.7.11), (Huneke & Swanson, 2006b, mainly Proposition 11.1.0, 11.2.1) for further details on multiplicity.

Proposition 5.1.0.5. *Let R be a one-dimensional Cohen-Macaulay local ring. Suppose that $x \in I$ is a (principal) reduction and $M \in \mathrm{CM}(R)$. The following are equivalent:*

1. *M is I-Ulrich.*
2. *$IM = xM$.*
3. *$IM \subseteq xM$.*
4. *$IM \cong M$.*
5. *$M \in \mathrm{CM}(B(I))$ (see Remark 5.1.0.8).*
6. *M is I^n-Ulrich for all $n \geq 1$.*
7. *M is I^n-Ulrich for infinitely many n.*
8. *M is I^n-Ulrich for some $n \geq 1$.*

Proof. As x is a reduction of I, $\ell(M/xM) = e_I(M)$. So (1) is equivalent to $\ell(M/xM) = \ell(M/IM)$, or $IM = xM$. The equivalence of (2) and (3) is obvious. Clearly (2) implies (4). Assuming (4), then $M \cong I^n M$ for $n \gg 0$, so M is I-Ulrich by Example 5.1.0.2. We have established the equivalence of (1) through (4).

Next, (3) is equivalent to $\frac{I}{x}M \subseteq M$. In other words (3) implies that $M \in \mathrm{CM}(B(I))$, since $R[\frac{I}{x}] = B(I)$. Since $B(I) = B(I^n)$ for any $n \geq 1$, we have $(5) \Rightarrow (6)$. Clearly $(6) \Rightarrow (7) \Rightarrow (8)$. Finally, assume (8). We have $\ell(M/I^n M) = e_{I^n}(M) = ne_I(M)$. Note that for each i, $I^i M$ is in $\mathrm{CM}(R)$ and hence, using properties of multiplicities, we get $\ell(I^i M/I^{i+1}M) \leq \ell(I^i M/xI^i M) = e_I(I^i M) = e_I(M)$ for each i. Thus, equality must occur for each i; in particular, it occurs for $i = 0$, which shows that M is I-Ulrich. ■

Without any assumption on the existence of a principal reduction, the following still holds:

Theorem 5.1.0.6. *Let R be a one-dimensional Cohen-Macaulay local ring. Let I be a regular ideal and $M \in \mathrm{CM}(R)$. The following are equivalent:*

1. *$IM \cong M$.*
2. *M is I-Ulrich.*
3. *M is I^n-Ulrich for all $n \geq 1$.*
4. *M is I^n-Ulrich for infinitely many n.*
5. *M is I^n-Ulrich for some $n \geq 1$.*
6. *$M \in \mathrm{CM}(B(I))$ (see Remark 5.1.0.8).*

Proof. Assume (1), then $M \cong I^n M$ for $n \gg 0$, so M is I-Ulrich by Example 5.1.0.2. If (2) holds, then we may pass to a local faithfully flat extension of R possessing an infinite residue field and apply Proposition 5.1.0.5, followed by (Grothendieck, 1967, Proposition 2.5.8) to see that (1) holds. The statements (2) to (5) are unaffected by local faithfully flat extensions, so we can enlarge the residue field and apply Proposition 5.1.0.5. As I^n contains a principal reduction for $n \gg 0$ and $B(I) = B(I^n)$ for all $n \geq 1$, Proposition 5.1.0.5 implies that (4) and (6) are equivalent. ■

Corollary 5.1.0.7. *Let R be as in Theorem 5.1.0.6. Let I be a regular ideal. Then R is I-Ulrich if and only if I is principal.*

Proof. Theorem 5.1.0.6 implies that $IR \cong R$, so I is principal. ■

Remark 5.1.0.8. Note that if $M \in \mathrm{CM}(R)$ is I-Ulrich, the proofs of Proposition 5.1.0.5 and Theorem 5.1.0.6 show that the action of $B(I)$ on M extends the action of R on M. In other words, there is an action of $B(I)$ on M which when restricted to R yields the original action of R on M. In particular, if $M \subseteq Q(R)$, multiplication in $Q(R)$ gives an action of $B(I)$ on M.

We say that an extension $f : R \to S$ is *birational* if $S \subset Q(R)$. Equivalently $Q(R) = Q(S)$. Also such an f induces a bijection on the sets of minimal primes of S and R and f_P is an isomorphism at all minimal primes P of R. Let $\mathrm{Bir}(R)$ denote the set of finite birational extensions of R.

Corollary 5.1.0.9. *Let R be as in Theorem 5.1.0.6. Let $R \subseteq S$ be a finite birational extension of rings. Then S is I-Ulrich if and only if $B(I) \subseteq S$.*

Proof. Follows immediately from Theorem 5.1.0.6 and Remark 5.1.0.8. ■

Corollary 5.1.0.10. *Let R be as in Theorem 5.1.0.6. Let I be a regular ideal. If $\bar{R}$ is a finitely generated R-module then $\bar{R}$ and the conductor $\mathfrak{c}$ are I-Ulrich.*

Proof. As $B(I) \subseteq \bar{R}$, $\bar{R} \in \mathrm{CM}(B(I))$ and so by Theorem 5.1.0.6, $\bar{R}$ is I-Ulrich. Since $\mathfrak{c} \in \mathrm{CM}(\bar{R}) \subseteq \mathrm{CM}(B(I))$, the conclusion follows. ■

Corollary 5.1.0.11. *Let R be as in Theorem 5.1.0.6. If $M \in \mathrm{CM}(R)$ is I-Ulrich, then $\mathrm{tr}\, M \subseteq b(I)$. If $M \in \mathrm{Ref}(R)$, then the converse holds.*

Proof. This follows from Theorem 5.1.0.6 and Theorem 2.2.1.9. ■

Lemma 5.1.0.12. *Let R be as in Theorem 5.1.0.6. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{CM}(R)$. If B is I-Ulrich then so are A, C.*

Proof. We may enlarge the residue field if necessary and assume that I has a principal reduction x. Then x is a regular element and hence induces an exact sequence

$$0 \to A/xA \to B/xB \to C/xC \to 0.$$

B is I-Ulrich if and only if I kills the middle module, but if that's the case then I kills the other two as well. ■

Corollary 5.1.0.13. *Let R be as in Theorem 5.1.0.6. Let $M \in \mathrm{Ul}_I(R)$. For any $f \in M^*, \mathrm{Im}(f) \in \mathrm{Ul}_I(R)$.*

Corollary 5.1.0.14. *Let R be as in Theorem 5.1.0.6. If ideals J, L are in $\mathrm{Ul}_I(R)$, then $J+L, J\cap L \in \mathrm{Ul}_I(R)$.*

Proof. The assertion follows from the short exact sequence $0 \to J\cap L \to J \oplus L \to J+L \to 0$. ■

Lemma 5.1.0.15. *Let R be as in Theorem 5.1.0.6. If $M \in \mathrm{Ul}_I(R)$, then $\mathrm{Hom}_R(M,N) \in \mathrm{Ul}_I(R)$ for any module $N \in \mathrm{CM}(R)$.*

Proof. As above, we can assume there is a principal reduction x of I. Note that there is an embedding

$$\mathrm{Hom}_R(M,N) \otimes_R R/xR \to \mathrm{Hom}_R(M/xM, N/xN)$$

and the latter is killed by I since $M \in \mathrm{Ul}_I(R)$. This shows that $\mathrm{Hom}_R(M,N) \otimes_R R/xR$ is killed by I and this finishes the proof. ■

Proposition 5.1.0.16. *Let R be as in Theorem 5.1.0.6. If $M \in \mathrm{Ul}_I R$, then $\mathrm{tr}(M) \in \mathrm{Ul}_I(R)$.*

Proof. Since, $\mathrm{tr}(M)$ is the sum of all images of elements in M^*, the proof follows immediately from Corollary 5.1.0.13 and Corollary 5.1.0.14. ■

Corollary 5.1.0.17. *Let R be as in Theorem 5.1.0.6. The set of I-Ulrich ideals is a lattice under addition and intersection. The largest element is $b(I)$.*

Proof. That this set forms a lattice follows from Corollary 5.1.0.14. For the last assertion, first note that $b(I)$ is a module over $B(I)$, and then apply Theorem 5.1.0.6, Proposition 5.1.0.16 and Corollary 5.1.0.11. ■

Remark 5.1.0.18. Aberbach and Huneke Aberbach & Huneke (1996) defined the *coefficient ideal* of I relative to a principal reduction x as the largest ideal J such that $xJ = IJ$. It follows that the coefficient ideal is just $b(I)$.

From now on we assume that I contains a principal reduction. Recall that the core of I, denoted $\operatorname{core}(I)$, is defined as the intersection of all (minimal) reductions of I.

Proposition 5.1.0.19. *Let R be as in Theorem 5.1.0.6. Assume that I has a principal reduction x. Consider $M \in \mathrm{Ul}_I(R)$. We have,*

$$\operatorname{tr}(M) \subseteq (x) :_R I \subseteq \operatorname{tr}(I).$$

If the residue field of R is infinite, then:

$$\operatorname{tr}(M) \subseteq \operatorname{core}(I) :_R I \subseteq (x) :_R I \subseteq \operatorname{tr}(I).$$

Proof. Let $J = \operatorname{tr}(M)$. Note that by Proposition 5.1.0.16, $J \in \mathrm{Ul}_I(R)$ and so $IJ = xJ \subset (x)$ for any principal reduction x. So $J \subset \cap((x) :_R I) = \operatorname{core}(I) :_R I$. The last inclusion comes from Corollary 2.2.2.9. If the residue field of R is infinite, then the core is the intersection of all principal reductions, and the second part follows. ■

Corollary 5.1.0.20. *Let R be as in Theorem 5.1.0.6. Suppose that I is a regular ideal with a principal reduction x. Then*

$$b(I) = \operatorname{tr}(b(I)) \subseteq (x) :_R I \subseteq \operatorname{tr}(I).$$

If the residue field of R is infinite, then:

$$b(I) = \operatorname{tr}(b(I)) \subseteq \operatorname{core}(I) :_R I \subseteq (x) :_R I \subseteq \operatorname{tr}(I).$$

Proof. Since $b(I) = \operatorname{tr}(B(I)) \in \operatorname{Ul}_I(R)$, the conclusion follows from Proposition 5.1.0.19. ■

Corollary 5.1.0.21. *Let R be as in Theorem 5.1.0.6. Assume that the residue field of R is infinite. Let $M \in \operatorname{Ref}(R)$. The following are equivalent.*

1. *M is I-Ulrich.*
2. $\operatorname{tr}(M) \subseteq b(I)$.
3. $\operatorname{tr}(M) \subseteq (x) :_R I$ *for some principal reduction x of I.*
4. $\operatorname{tr}(M) \subseteq (x) :_R I$ *for any principal reduction x of I.*
5. $\operatorname{tr}(M) \subseteq \operatorname{core}(I) :_R I$.

Proof. Combining Corollary 5.1.0.11 and Proposition 5.1.0.19, we see that the proof would be complete if we show (3) implies (1). If (3) holds, then for any $f \in M^*$, we have that $I \cdot f \subseteq \operatorname{Hom}_R(M, xR) = xM^*$. Therefore by Proposition 5.1.0.5, M^* is I-Ulrich. Since $M \in \operatorname{Ref} R$, by Lemma 5.1.0.15 M is I-Ulrich. ■

In light of the above results, it is natural to ask when $b(I) = \operatorname{tr}(I)$. Note that if this is the case, then $(x) :_R I$ is independent of the principal reduction.

Proposition 5.1.0.22. *Let R be as in Theorem 5.1.0.6. If I is a regular reflexive trace ideal such that $xI = I^2$ for some $x \in I$, then $b(I) = \operatorname{tr}(I)$.*

Proof. As $I^n \cong I$ for all $n > 0$. we get that $B(I) = I : I$. By Lemma 2.2.1.8, $b(I) = I = \operatorname{tr}(I)$. ■

We can moreover relate $b(I)$ with $\operatorname{core} I :_R I$ for a large number of cases.

Theorem 5.1.0.23. *Let R be a reduced one dimensional ring with infinite residue field k. Let I be a regular ideal with reduction number r. Assume that* $\operatorname{char}(k) = 0$ *or* $\operatorname{char}(k) > r$. *Then*

$$b(I) = \operatorname{core}(I) :_R I.$$

Proof. By (Polini & Ulrich, 2005, Theorem 3.4 b) we have that,

$$\mathrm{core}(I) = x^{n+1} :_R I^n$$

for suitably large n, where x is a principal reduction of I. Thus,

$$\mathrm{core}(I) :_R I = x^{n+1} :_R I^{n+1} \cong (I^{n+1})^*$$

Now for large n, $(I^{n+1})^*$ is I-Ulrich by Lemma 5.1.0.15 and hence $b(I) = \mathrm{core}(I) :_R I$ by Corollary 5.1.0.17. ■

The next proposition will help in establishing some finiteness results.

Proposition 5.1.0.24. *Let R be as in Theorem 5.1.0.6. Let I, J be regular ideals. Consider the following statements.*

1. $\mathrm{Ul}_I(R) = \mathrm{Ul}_J(R)$.

2. $B(I) = B(J)$.

3. $b(I) = b(J)$.

Then $(1) \iff (2) \implies (3)$. *If R is Gorenstein, then all three are equivalent.*

Proof. Recall that $\mathrm{Ul}_I(R) = \mathrm{CM}(B(I))$ by Theorem 5.1.0.6. So if $B(I) = B(J)$ then $\mathrm{Ul}_I(R) = \mathrm{Ul}_J(R)$. Now assume (1). Let $S = B(I)$ and $T = B(J)$. Then $S \in \mathrm{Ul}_J(R) = \mathrm{CM}(T)$. Thus $T \subset TS \subset S$. By symmetry $S \subset T$, so $S = T$. If $S = T$, then $\mathfrak{c}_R(S) = \mathfrak{c}_R(T)$, so (3) follows from (2). If R is Gorenstein, then $B(I), B(J)$ are reflexive, so $b(I) = b(J)$ implies $B(I) = B(J)$ by Lemma 2.2.1.8. ■

Theorem 5.1.0.25. *Let R be as in Theorem 5.1.0.6. Let $\mathfrak{c}$ be the conductor and I be a regular ideal. If $\mathfrak{c} \cong I^s$ for some s then* $\mathrm{Ul}_I(R) = \mathrm{CM}(\overline{R})$. *If furthermore R is complete and reduced, then* $\mathrm{Ul}_I(R)$ *has finite type.*

Proof. As $\mathfrak{c}$ is a regular ideal, $\overline{R}$ is R-finite. As $\mathfrak{c} \cong \mathfrak{c}^n$ for all n, $B(\mathfrak{c}) = \overline{R}$. On the other hand $B(I) = B(I^s) = B(\mathfrak{c})$, proving the first claim. If R is complete and reduced, then $\overline{R}$ is a product of DVRs, so $\mathrm{Ul}_I(R) = \mathrm{CM}(\overline{R})$ has finite type. ■

Proposition 5.1.0.26. *Let R be as in Theorem 5.1.0.6. Assume that I is a regular ideal. Let $S =$ $\mathrm{End}_R(I)$ (which is a birational extension of R). If M is I-Ulrich, then* $\mathrm{Hom}_R(M,I) \cong \mathrm{Hom}_R(M,S)$.

Proof. We have an exact sequence $0 \to L \to I \otimes M \to IM \to 0$ where L has finite length. Take $\mathrm{Hom}_R(-,I)$ we get an isomorphism $\mathrm{Hom}_R(IM,I) \cong \mathrm{Hom}_R(I \otimes M,I)$. The first is isomorphic to $\mathrm{Hom}_R(M,I)$ as $IM \cong M$, and the second is isomorphic to $\mathrm{Hom}_R(M,\mathrm{Hom}_R(I,I)) = \mathrm{Hom}_R(M,S)$ by Hom-tensor adjointness. ■

Corollary 5.1.0.27. *Let R be as in Theorem 5.1.0.6 and further assume that R has a canonical ideal ω_R. The following are equivalent:*

1. $M \in \mathrm{Ul}_{\omega_R}(R)$
2. $\mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_R(M,\omega_R)$.

Proof. (1) implies (2) by Proposition 5.1.0.26. Conversely, note that $R \cong \mathrm{End}_R(\omega_R)$. Hence, using Hom-Tensor adjointness, statement (2) is the same as $\mathrm{Hom}_R(\omega_R M,\omega_R) \cong \mathrm{Hom}_R(M,\omega_R)$. Hence dualizing with respect to ω_R and using Theorem 5.1.0.6 finishes the proof. ■

Corollary 5.1.0.28. *Let R be as in Theorem 5.1.0.6. Assume that R has a canonical ideal ω_R and $M \in \mathrm{Ul}_{\omega_R}(R)$. Then M is reflexive.*

Proof. Corollary 5.1.0.27 implies that $M^* \cong M^\vee$, where $M^* = \mathrm{Hom}_R(M,R)$ and $M^\vee = \mathrm{Hom}_R(M,\omega_R)$. By Lemma 5.1.0.15, M^* is still in $\mathrm{Ul}_{\omega_R}(R)$, so we have $M^{**} \cong M^{*\vee} \cong M^{\vee\vee} \cong M$, as desired. ■

Corollary 5.1.0.29. *Let R be as in Theorem 5.1.0.6. Suppose that R has a canonical ideal ω_R. Then for large enough n, the ideal $I = \omega_R^n$ is reflexive and satisfies $I^* \cong I^\vee$.*

We end this section by looking into the question when the maximal ideal $\mathfrak{m}$ is I-Ulrich. This will be applied when we discuss almost Gorenstein rings in the last section.

Proposition 5.1.0.30. *Let $(R,\mathfrak{m},k)$ be as in Theorem 5.1.0.6. Suppose there is an exact sequence $0 \to R \to I \to k^{\oplus s} \to 0$. Then $\mathfrak{m}$ is I-Ulrich.*

Proof. We can assume that R is not regular, for if R is regular the conclusion follows easily. The assumption is equivalent to $I\mathfrak{m} \subset (a)$ for some $a \in I$. We need to show that $I\mathfrak{m} = a\mathfrak{m}$. As $a\mathfrak{m} \subset I\mathfrak{m} \subset (a)$ and $\ell((a)/a\mathfrak{m}) = 1$, it is enough to show that $I\mathfrak{m}$ is not equal to (a). Suppose that $I\mathfrak{m} = (a)$. Then $E = I\mathfrak{m} : I\mathfrak{m} = (a) : (a) = R$. As $\mathfrak{m} : \mathfrak{m} \subset E$, it follows that $\mathfrak{m} : \mathfrak{m} = R$. But $\mathfrak{m} : \mathfrak{m} \cong \mathfrak{m}^*$, and as $\mathfrak{m}$ is reflexive, it follows that $\mathfrak{m} \cong R$, which is impossible if R is not regular. ■

5.2 Strongly Reflexive Extensions

Throughout this section, we maintain notation established in 2.2.2.1. Suppose that R has a canonical ideal ω_R. In this section we are interested in the following question. Let S be a finite extension of R. When is any CM S-module R-reflexive? Of course if $S = R$, this is equivalent to R being Gorenstein, or $\omega_R \cong R$. It turns out that there is a pleasant generalization to any finite extension S that is in $\mathrm{CM}(R)$: $\mathrm{CM}(S) \subset \mathrm{Ref}(R)$ if and only if $\omega_R S \cong S$, in other words S is ω_R-Ulrich.

We start with a useful lemma that will be used repeatedly in the proof of our main theorem.

Lemma 5.2.0.1. *Let $(R,\mathfrak{m},k)$ be a one-dimensional Cohen-Macaulay local ring. Let S be a module finite R-algebra such that $S \in \mathrm{CM}(R)$. Let $M \in \mathrm{CM}(S)$.*

1. *The map*

$$F : M \mapsto \mathrm{Hom}_R(M,R)$$

is an S-linear, contravariant functor from $\mathrm{CM}(S)$ to $\mathrm{CM}(S)$.

2. *If $M \in \mathrm{Ref}(R)$, then $M \cong F(F(M))$ in $\mathrm{CM}(S)$.*

Proof. Note that $\mathrm{Hom}_R(M,R)$ is naturally an S-module via the action

$$(s \cdot f)(m) := f(sm), \qquad s \in S, m \in M, f \in \mathrm{Hom}_R(M,R)$$

and that this extends the action of R. Thus, the conclusion of part (1) follows. For part (2), notice that the canonical R-linear map $M \to M^{**}$ is S-linear with respect to the action above. ■

Theorem 5.2.0.2. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring. Assume that R has a canonical ideal ω_R. Let S be a module finite R-algebra such that $S \in \mathrm{CM}(R)$. The following are equivalent:*

1. $\mathrm{CM}(S) \subset \mathrm{Ref}(R)$.
2. $\omega_S \in \mathrm{Ref}(R)$.
3. $\omega_S \cong S^*$.
4. $\omega_R S \cong S$.
5. *S is ω_R-Ulrich.*
6. *S is reflexive and* $\mathrm{tr}(S) \subset b(\omega_R)$.

Proof. Clearly (1) implies (2). For the converse, assume that $\omega_S \in \mathrm{Ref}(R)$. Take $M \in \mathrm{CM}(S)$. Then $M^\vee = \mathrm{Hom}_R(M, \omega_R) \in \mathrm{CM}(S)$. Take a free S-cover of $M^\vee$ and apply $^\vee$, we obtain an exact sequence $0 \to M \to (S^n)^\vee \to N \to 0$ in $\mathrm{CM}(R)$. Thus $M \in \mathrm{Ref}(R)$ by Lemma 2.2.1.5.

That (3) implies (2) follows by Lemma 2.2.1.5. For the converse, assume that $\omega_S \in \mathrm{Ref}(R)$. Take a free S-cover of ω_S^* and apply *. From Lemma 5.2.0.1 we get an exact sequence in $\mathrm{CM}(S)$: $0 \to \omega_S \to (S^n)^* \to N \to 0$. (Note that N is in $\mathrm{CM}(S)$ as it is a submodule of a torsionfree R-module and also has an S-module structure.) This has to split in $\mathrm{CM}(S)$ (since $\mathrm{Ext}^1_S(N, \omega_S) = 0$), so that ω_S is a direct summand of $(S^n)^*$ in $\mathrm{CM}(S)$. Since $S \in \mathrm{Ref}(R)$ by (2) implies (1), ω_S^* is a direct summand of S^n in $\mathrm{CM}(S)$ using Lemma 5.2.0.1(2). Thus ω_S^* is S-projective. Since S is semi-local and ω_S^* is locally free of constant rank, it is S-free of rank one (Stacks project authors, 2021, Tag 02M9). Now applying Lemma 5.2.0.1(1) one gets that ω_S^{**} is isomorphic to S^* as S-modules, so as R-modules as well and hence (3) follows.

The equivalence of (3), (4) and (5) follows from Theorem 5.1.0.6 and Corollary 5.1.0.27. The equivalence of (5) and (6) follows from Corollary 5.1.0.21. ■

Definition 5.2.0.3. We shall call an extension of R satisfying the equivalent conditions of Theorem 5.2.0.2 a *strongly reflexive* extension.

Remark 5.2.0.4. The notions of reflexive extensions and totally reflexive extensions, over not necessarily commutative rings, have been defined and studied by X. Chen in (Chen, 2013, Definition 2.3, 3.3). They are related to but not the same as ours. For instance, a reflexive extension in Chen's notion would require $S \in \operatorname{Ref}(R)$ and $\operatorname{Hom}_R(S,R) \cong S$, and would imply $\operatorname{Ref}(S) \subset \operatorname{Ref}(R)$.

Strongly reflexive birational extensions satisfy even more interesting characterizations.

Theorem 5.2.0.5. *Let $(R,\mathfrak{m},k)$ be a one-dimensional Cohen-Macaulay local ring. Let $S \in \operatorname{Bir}(R)$. Assume R has a canonical ideal ω_R admitting a principal reduction. Let a be an arbitrary principal reduction of ω_R and set $K := a^{-1}\omega_R$ (see Remark 5.2.0.6). The following are equivalent:*

1. $\operatorname{CM}(S) \subset \operatorname{Ref}(R)$.
2. $\omega_S \in \operatorname{Ref}(R)$.
3. $\omega_S \cong S^*$.
4. *S is ω_R-Ulrich.*
5. $S = KS$.
6. $K \subseteq S$.
7. $S \in \operatorname{Ref}(R)$ *and* $\mathfrak{c}_R(S) = \operatorname{tr}(S) \subseteq R : K = (a) : \omega_R$.
8. $S \in \operatorname{Ref}(R)$ *and* $\mathfrak{c}_R(S) = \operatorname{tr}(S) \subseteq \operatorname{core}(\omega_R) :_R \omega_R$ *(assuming the residue field is infinite).*

Proof. We already know that (1) through (4) are equivalent from Theorem 5.2.0.2. The equivalence of (4) and (5) follows from Proposition 5.1.0.5 and that of (4) and (6) from Remark 5.1.0.4 and Corollary 5.1.0.9. Since a is an arbitrary principal reduction of ω_R we see that (7) holds if and only if (8) holds, as the core is the intersection of all principal reductions if the residue field is infinite.

The equivalence of (4), (7), (8) follows from Corollary 5.1.0.21. ■

Remark 5.2.0.6. The condition that R has a canonical ideal with principal reduction is satisfied for instance when $\hat{R}$ is generically Gorenstein with infinite residue field, see Goto et al. (2013).

Corollary 5.2.0.7. *Let R be as in Theorem 5.2.0.2. Assume that R has a canonical ideal ω_R. Let $Q(R) \hookrightarrow A$ be an extension of the total quotient ring of R. Assume that the integral closure of R in A, say $\bar{R}^A$, is a finite R-module. Then $\bar{R}^A \in \mathrm{Ref}(R)$.*

Proof. From Corollary 5.1.0.10, $\bar{R} \in \mathrm{Ul}_{\omega_R}(R)$. Since $\bar{R}^A \in \mathrm{CM}(\bar{R})$, by Theorem 5.2.0.2 $\bar{R}^A \in \mathrm{Ref}(R)$. ■

Corollary 5.2.0.8. *Let $R \to S$ be a finite extension of rings such that R is a generically Gorenstein (S_2) ring of arbitrary dimension and S is (S_1). If the extension $R \to S$ is strongly reflexive in codimension one, then any finite (S_2) S-module M is R-reflexive.*

Proof. Since R satisfies (S_2) and M is a (S_2) R-module, M is R-reflexive if and only if this is true in codimension one. Since R is generically Gorenstein and S is (S_1), we may apply Theorem 5.2.0.2 to see M is R-reflexive in codimension one. ■

Corollary 5.2.0.9. *Let R be a generically Gorenstein (S_2) ring of arbitrary dimension. Let $Q(R) \hookrightarrow A$ be an extension of the total quotient ring of R. Assume that the integral closure of R in A, say $\bar{R}^A$, is a finite R-module. Then $\bar{R}^A \in \mathrm{Ref}(R)$.*

Proof. Since $R \to \bar{R}$ is strongly reflexive in codimension one, by Corollary 5.2.0.8 $\bar{R}^A \in \mathrm{Ref}(R)$. ■

5.3 Some finite type results

We maintain notation established in 2.2.2.1 for this section. Here we study when certain subsets of interesting ideals and modules are "finite". We say that a subset $\mathscr{S}$ of $\mathrm{mod}(R)$ is *of finite type* if any element of $\mathscr{S}$ is isomorphic to a direct sum of modules from a finite set in $\mathrm{mod}(R)$. Note that since we sometimes consider sets that are not subcategories which are closed under isomorphism, this notion is a bit broader than the usual notion of "finite representation type". Representation finiteness of subcategories of $\mathrm{CM}(R)$ have been studied heavily, and many beautiful connections to the singularities of R have been discovered over the years. Our study suggests that the same promise could hold for reflexive modules.

Consider the following classes of ideals of R:

$$\begin{aligned}
\mathscr{I}(R) &:= \{I \mid I \text{ is an integrally closed regular ideal}\} \\
\mathscr{I}_{\mathfrak{c}}(R) &:= \{I \mid I \text{ is an integrally closed regular ideal and } \mathfrak{c} \subseteq I\} \\
\mathrm{Ref}_1(R) &:= \{I \mid I \text{ is a reflexive regular ideal}\}. \\
\mathrm{T}(R) &:= \{I \mid I \text{ is a regular trace ideal}\}
\end{aligned}$$

We shall look at the finiteness of these classes of ideals and the interaction between them. Note that from Proposition 2.2.2.12, we have that $\mathscr{I} \subseteq \mathrm{Ref}_1(R)$ and that $\mathscr{I}_{\mathfrak{c}} \subseteq \mathrm{RT}(R) := \mathrm{Ref}_1(R) \cap \mathrm{T}(R)$.

5.3.1 Finiteness of $\mathrm{T}(R)$

We begin by answering the following question raised by E. Faber in (Faber, 2019, Question 3.7).

Question 5.3.1.1. *Let R be a one-dimensional complete local or graded ring. Are the following equivalent?*

1. $\mathrm{CM}(R)$ *is of finite type.*
2. *There are only finitely many possibilities for* $\mathrm{tr}(M)$*, where* $M \in \mathrm{CM}(R)$.

The answer to this question is negative. Consider the following example.

Example 5.3.1.2. Let $R = k[[t^e, \ldots, t^{2e-1}]]$ where $\overline{R} = k[[t]]$, k infinite and $e \geq 4$. Then the set of trace ideals is finite but $\mathrm{CM}(R)$ is infinite.

Proof. Here $\mathfrak{c} = \mathfrak{m}$. By Corollary 2.2.2.7, there are exactly two trace ideals, R and $\mathfrak{m}$. Since $\overline{R}$ is an $\mathfrak{m}$-Ulrich R-module, $\mu_R(\overline{R}) = e(R)$. However, since $e(R) = e \geq 4$, $\mathrm{CM}(R)$ is infinite by (Leuschke & Wiegand, 2012b, Theorem 4.2). ■

Note here that finitely many trace ideals in a ring can raise some natural classification questions. Of course, a single trace ideal characterizes a DVR. The following proposition provides a strong motivation to classify such rings.

Proposition 5.3.1.3. *Let $(R,\mathfrak{m},k)$ be a complete local one-dimensional domain containing an infinite field k, so that $\overline{R}=k[[t]]$. Let $e(R)=e$ and let v be the valuation defining $\overline{R}$.*

1. *$\#\mathrm{T}(R)=2$ if and only if $R=k[[t^e,\ldots,t^{2e-1}]]$.*
2. *The following are equivalent*
   (a) *$\#\mathrm{T}(R)=3$*
   (b) *$R=k[[\alpha t^e,t^c,t^{c+1},\ldots,\widehat{t^{2e}},\ldots,t^{c+e-1}]]$ where α is a unit of $k[[t]]$ and $e+2\le c\le 2e$.*
   (c) *$\ell(R/\mathfrak{c})=2$*

Proof. Note that every integrally closed ideal in R is of the form $I_f:=\{r\in R\mid v(r)\ge f\}$ where $f\in\mathbb{N}$. Then $\mathfrak{m}=I_e$ and let $\mathfrak{c}=I_c$ where c is chosen maximally, that is $\mathfrak{c}\neq I_{c+1}$.

For (1), first assume $\#\mathrm{T}(R)=2$. Here $\mathfrak{m}$ and R are the only trace ideals and so $\mathfrak{c}=\mathfrak{m}$. Hence, $e=c$ and choose $t^e+\sum_i\beta_i t^i\in\mathfrak{m}$, $\beta_i\in k$, so that it is part of a minimal generating set for $\mathfrak{m}$. Since $\mathfrak{c}=\mathfrak{m}$, we have that $t^{e+j}+t^j\sum_i\beta_i t^i\in\mathfrak{m}$ for all $j\ge 1$. Since R is complete, we have that $t^{e+j}\in\mathfrak{m}$ for all $j\ge 0$ and thus $R=k[[t^e,\ldots,t^{2e-1}]]$. The other direction is clear using Corollary 2.2.2.7.

For (2), first assume $\#\mathrm{T}(R)=3$. By Proposition 2.2.2.12, we get that there are no integrally closed ideals strictly between $\mathfrak{c}$ and $\mathfrak{m}$. In other words there does not exist $r\in R$ such that $v(r)=f$ for all $e<f<c$. Since R is complete, $\mathfrak{c}=(t^c,t^{c+1},\ldots,t^{c+e-1})$. Thus we can choose a principal reduction $x=t^e+\sum_{i=1}^{c-e-1}k_i t^{e+i}\in R$ of $\mathfrak{m}$ where $k_i\in k$. Consider the ideal $I:=(x)+\mathfrak{c}$. We claim that $I=\mathfrak{m}$. Take any element $r\in\mathfrak{m}$. If $v(r)>e$, then $r\in\mathfrak{c}\subseteq I$. If $v(r)=e$, to show $r\in I$, after multiplication by a suitable element of k we may assume that $r=t^e+\sum_{i=1}^{c-e-1}b_i t^{e+i}$ where $b_i\in k$. If $v(r-x)\neq 0$, then necessarily $e<v(r-x)<c$, which is impossible. Therefore $r=x$ and $I=\mathfrak{m}$. Finally we have $2e\ge c$ since there does not exist any element in R with valuation strictly between e and c. Since $\mathfrak{c}\neq\mathfrak{m}$, we have that $c\ge e+2$.

To show (b) implies (c), assume R has the specified form. Then since $\mathfrak{c} = (t^c, t^{c+1}, \ldots, t^{c+e-1})$, we have that $\mathfrak{m}/\mathfrak{c}$ is a cyclic R-module. Moreover since $c \leq 2e$, $\mathfrak{m}^2 \subseteq \mathfrak{c}$ and $\mathfrak{m}/\mathfrak{c}$ is a k-vector space. Thus $\ell(\mathfrak{m}/\mathfrak{c}) = 1$, that is $\ell(R/\mathfrak{c}) = 2$.

(c) implies (a) is clear from Corollary 2.2.2.7. ■

5.3.2 Finiteness of $\mathscr{I}$ and $\mathscr{I}_{\mathfrak{c}}$

Proposition 5.3.2.1. *Let R be a one-dimensional Cohen-Macaulay local ring. Suppose $\overline{R}$ is a finite R-module. Then $\mathscr{I}_{\mathfrak{c}}$ is a finite set. Moreover if R is a complete local domain, $\mathscr{I}$ is of finite type.*

Proof. Let $\operatorname{MaxSpec}(\overline{R}) = \{\mathfrak{n}_1, ..., \mathfrak{n}_s\}$. Since $\mathfrak{c}$ is a regular ideal of $\overline{R}$, choose an irredundant primary decomposition in $\overline{R}$, $\mathfrak{c} = \cap_{i=1}^{s} \mathfrak{n}_i^{(r_i)}$ where $I^{(n)}$ denotes the n^{th} symbolic power of an ideal I. Since any $J \in \mathscr{I}_{\mathfrak{c}}$ is the contraction to R of an ideal of $\overline{R}$ containing $\mathfrak{c}$, $J = \cap_{i=1}^{s} (\mathfrak{n}_i^{(s_i)} \cap R)$ where $1 \leq s_i \leq r_i$ for each i. Thus $\mathscr{I}_{\mathfrak{c}}$ is a finite set.

Now assume R is a complete local domain, so that $\overline{R}$ is a DVR; thus the elements of $\mathscr{I}$ are totally ordered by inclusion. From the first part of this proposition, it suffices to consider $I \in \mathscr{I}$, $I \subseteq \mathfrak{c}$. Now $I = I\overline{R} \cap R$, but $I\overline{R} \subseteq \mathfrak{c}\overline{R} = \mathfrak{c}$. Therefore I is also an ideal of $\overline{R}$ and there exists $0 \neq a \in R$ such that $a\overline{R} = I$. Therefore $I \cong \overline{R}$ as R-modules and $\mathscr{I}$ has finite type. ■

5.3.3 Finiteness of $\operatorname{Ref}_1(R)$

We first note that $\operatorname{Ref}(R)$ (in fact $\operatorname{Ref}_1(R)$) is of infinite type if $\overline{R}$ is not finitely generated over R.

Lemma 5.3.3.1. *Let R be a one-dimensional Cohen-Macaulay local ring. Let I be a regular ideal. Then $B(I)$ is a finite R-module.*

Proof. From Example 5.1.0.2, I^n is I-Ulrich for sufficiently large n. Thus $I^{n+1} \simeq I^n$ for such n by Theorem 5.1.0.6. Therefore $\operatorname{End}_R(I^n)$ stabilizes and $B(I)$ is a finite R-module. ■

Lemma 5.3.3.2. *Let R be a one-dimensional Cohen-Macaulay local ring. Assume* $\operatorname{Ref}_1(R)$ *has finite type and that R admits a canonical ideal ω_R. Then $\overline{R}$ is a finite R-module. In particular, R is reduced.*

Proof. It suffices to show that $\overline{R}$ is a finite R-module. Suppose on the contrary that it is not. By Lemma 5.3.3.1, $\overline{R}$ is not a finite $B(\omega_R)$-module. Thus, we can find an infinite chain of rings S_i inside $\overline{R}$, $B(\omega_R) \subsetneq S_1 \subsetneq \cdots \subsetneq S_i \subsetneq \dots$ such that each S_i is a finite R-module. From Corollary 5.1.0.9, the S_i are ω_R-Ulrich and hence by Theorem 5.2.0.2, they are R-reflexive. Consider $S_i \subsetneq S_j$, and let if possible $S_i \simeq S_j$ as R-modules. Then they are isomorphic as S_i-modules as well. By Theorem 2.2.1.9, $S_i = \operatorname{tr}_{S_i}(S_j) \subseteq \mathfrak{c}_{S_i}(S_j)$. So $S_i = S_j$, a contradiction. Therefore the S_i's are indecomposable and mutually non-isomorphic and hence, $\operatorname{Ref}_1(R)$ is not of finite type. ■

We prove next that $\operatorname{Ref}_1(R)$ is of finite type when the conductor has small colength. Before stating Theorem 5.3.3.4, we summarize the cases that we will always reduce to in the proof.

Lemma 5.3.3.3 (Reduction Lemma)**.** *Let $(R,\mathfrak{m},k)$ be a one-dimensional Cohen-Macaulay local ring. Assume $\overline{R}$ is finite over R and let $\mathfrak{c}$ be the conductor ideal. Further assume that k is infinite. For any ideal I, consider the following conditions:*

1. $\mathfrak{c} \subset I$,

2. $\mathfrak{c} \subsetneq x :_R \overline{I} \subsetneq x :_R I \subsetneq \mathfrak{m}$ *where x is a principal reduction of I.*

Let $\operatorname{Ref}_1'(R) := \{I \in \operatorname{Ref}_1(R) \mid I \text{ satisfies (1) and (2) above}\}$. *Then* $\operatorname{Ref}_1(R)$ *is of finite type if and only if* $\operatorname{Ref}_1'(R)$ *is of finite type.*

Proof. If $\operatorname{Ref}_1(R)$ is of finite type then certainly $\operatorname{Ref}_1'(R)$ is of finite type. Conversely assume that $\operatorname{Ref}_1'(R)$ is of finite type. Let $I \in \operatorname{Ref}_1(R)$ and I not principal. From Theorem 2.2.2.6 we may assume that $\mathfrak{c} \subseteq I$. By Corollary 5.1.0.10, we have

$$\mathfrak{c} \subseteq x :_R \overline{I} \subseteq x :_R I \subseteq \mathfrak{m}$$

If $\mathfrak{c} = x :_R \overline{I}$, then by Remark 2.2.1.4, $(\overline{I})^* \cong \mathfrak{c}$ and hence $\mathfrak{c} \cong \overline{I}$. But both are trace ideals, and hence $\mathfrak{c} = I = \overline{I}$. Similarly if $x :_R I = \mathfrak{m}$, by Remark 2.2.1.4 we have $I \cong \mathfrak{m}^*$. Finally, if $x :_R \overline{I} = x :_R I$, we get $I^* = (\overline{I})^*$ by Remark 2.2.1.4. Thus $I = \overline{I}$ and hence $I \in \mathscr{I}_{\mathfrak{c}}$. Combining the above observations and finally using Proposition 5.3.2.1, we have that $\operatorname{Ref}_1(R)$ is of finite type. ■

Theorem 5.3.3.4. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring. Let $\overline{R}$ be finite over R and let $\mathfrak{c}$ be the conductor ideal. Further assume that k is infinite. Consider the following.*

1. $\ell(R/\mathfrak{c}) \leq 3$

2. $\ell(R/\mathfrak{c}) = 4$ *and R has minimal multiplicity.*

Then in all the above cases, $\mathrm{Ref}_1(R)$ *is of finite type.*

Proof. (1) follows immediately from Lemma 5.3.3.3.

Suppose (2) holds. Note that by Lemma 5.3.3.3, we can assume that $I \in \mathrm{Ref}_1'(R)$ and $\ell(R/x :_R I) = 2$ where x is a principal reduction of I.

Since $J = x :_R I$ is reflexive and R has minimal multiplicity, by Proposition 2.2.2.13 we get that J is integrally closed. Thus, $I \cong J^*$ where $J \in \mathscr{I}_{\mathfrak{c}}$ and the proof is now complete by Proposition 5.3.2.1. ■

Corollary 5.3.3.5. *Let R be a complete one-dimensional local domain containing an infinite field such that* $\#\mathrm{T}(R) = 3$. *Then* $\mathrm{Ref}_1(R)$ *is of finite type.*

Proof. This follows from Proposition 5.3.1.3 and Theorem 5.3.3.4. ■

Remark 5.3.3.6. Theorem 5.3.3.4 is true if we only assume that $|\mathrm{Min}(\hat{R})| \leq |k|$. To see this, first note that since R is one dimensional and CM, $\bar{R}$ is a finite R-module if and only if R is analytically unramified (see for example (Leuschke & Wiegand, 2012b, Theorem 4.6)). In this case, $\bar{\hat{R}} = \bar{R} \otimes_R \hat{R}$, so $\mathfrak{c}\hat{R} \subseteq \mathfrak{c}_{\hat{R}}$. Therefore $l(\hat{R}/\mathfrak{c}_{\hat{R}}) \leq l(\hat{R}/\mathfrak{c}\hat{R}) = l(R/\mathfrak{c})$. Since $\hat{R}$ is reduced, the number of maximal ideals in its integral closure is equal to its number of minimal primes. By (Fouli & Olberding, 2018, Corollary 3.3), every ideal of $\hat{R}$ admits a principal reduction. Then from the argument in Theorem 5.3.3.4, $\mathrm{Ref}_1(\hat{R})$ has finite type. By (Grothendieck, 1967, Proposition 2.5.8), $\mathrm{Ref}_1(R)$ has finite type.

Remark 5.3.3.7. Since $\mathfrak{c} \in \mathscr{I}_{\mathfrak{c}}$, by Theorem 2.2.2.6 and (Goto et al., 2003, Proposition 2.9) if $R/\mathfrak{c}$ is Gorenstein, then $\mathrm{Ref}_1(R)$ is of finite type. Moreover in this case, by (Corso et al., 1998, Theorem 3.7), we have that $\mu(\mathfrak{c}) = \mu(\mathfrak{m})$, so that R necessarily has minimal multiplicity here. In particular finiteness of $\mathrm{Ref}_1(R)$ in the cases $\ell(R/\mathfrak{c}) \leq 2$ follows from this as well and in these cases, R necessarily has minimal multiplicity.

5.3.4 Finiteness of $\mathrm{Ref}(R)$

In this subsection, we give a criterion for finiteness of $\mathrm{Ref}(R)$ and derive that over seminormal singularities, the category of reflexive modules is of finite type.

Proposition 5.3.4.1. *Let $(R,\mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Let $S = \mathrm{End}_R(\mathfrak{m})$. If $\mathrm{CM}(S)$ is of finite type, then $\mathrm{Ref}(R)$ is of finite type.*

Proof. Let M be an indecomposable, non-free reflexive module over R. Then $\mathrm{tr}(M) \subset \mathfrak{m} = \mathfrak{c}_R(S)$, so $M \in \mathrm{CM}(S)$ by Theorem 2.2.1.9. Finally, note that if two S-modules are isomorphic, then they are also isomorphic as R-modules. ■

Corollary 5.3.4.2. *Assume that $(R,\mathfrak{m})$ is complete, reduced, one-dimensional and the conductor $\mathfrak{c}$ of R is equal to $\mathfrak{m}$. Then $\mathrm{Ref}(R)$ is of finite type.*

Proof. By assumption $S = \mathrm{End}_R(\mathfrak{m}) = \bar{R}$, which is a product of DVRs and therefore $\mathrm{CM}(S)$ is of finite type. Thus, Proposition 5.3.4.1 applies. ■

As a consequence of the above, we can study finiteness of $\mathrm{Ref}(R)$ for 'seminormal' rings. R. Swan Swan (1980) defined a seminormal ring as a reduced ring R such that whenever $b, c \in R$ satisfy $b^3 = c^2$, there is an $a \in R$ with $a^2 = b, a^3 = c$. For a detailed exposition on various results related to seminormality (including generalizations to the above definition), we refer the reader to Vitulli (2011). Seminormality has also been studied in the context of studying F-singularities in characteristic $p > 0$. For instance, F-injective rings constitute a class of examples for seminormal rings (Datta & Murayama, 2019, Corollary 3.6).

Corollary 5.3.4.3. *Suppose that $(R,\mathfrak{m})$ is a seminormal complete reduced local ring of dimension one. Then $\mathrm{Ref}(R)$ is of finite type.*

Proof. By (Vitulli, 2011, Proposition 2.10(1)) (with $A = R, B = \overline{R}$), we get that $\mathfrak{c} = \mathfrak{m}$, so Corollary 5.3.4.2 applies. ■

5.4 Further applications and examples

Maintaining notation established in 2.2.2.1, we discuss notions of being 'close to Gorenstein' as promised in Remark 2.2.2.2.

Definition 5.4.0.1. R is called **almost Gorenstein** if $a :_R \omega_R \supseteq \mathfrak{m}$ for some principal reduction a of ω_R. R is called **nearly Gorenstein** if $\operatorname{tr}(\omega_R) \supseteq \mathfrak{m}$.

These classes of rings have attracted a lot of attention lately, the reader can refer to Herzog et al. (2019), Barucci & Fröberg (1997), Goto et al. (2013), Goto et al. (2015), Dao et al. (2020) amongst other sources.

In our language:

Proposition 5.4.0.2. *Assume that $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring which has a canonical ideal ω_R with some principal reduction a. R is almost Gorenstein if and only if $\mathfrak{m}$ is ω_R-Ulrich.*

Proof. This follows from Proposition 5.1.0.30. ■

It is clear from Corollary 2.2.2.9 that in this situation we get $\operatorname{tr}(\omega_R) \supseteq \mathfrak{m}$. This provides a proof for a well-known fact that almost Gorenstein rings are nearly Gorenstein.

One would expect that for rings close to Gorenstein, it would be easier to find reflexive modules. We now give supporting evidence for that statement.

Proposition 5.4.0.3. *Let $(R, \mathfrak{m})$ be almost Gorenstein and let I be a regular ideal with $S = \operatorname{End}_R(I)$.*

(1) If S is reflexive and strictly larger than R, then $\operatorname{CM}(S) \subseteq \operatorname{Ref}(R)$. Thus $IM \in \operatorname{Ref}(R)$ for any $M \in \operatorname{CM}(R)$. In particular all powers of I are reflexive.

(2) If I is a proper trace ideal, then IM is reflexive for any $M \in \operatorname{CM}(R)$. In particular I and all of its powers are reflexive.

Proof. As S is reflexive and $\mathfrak{c}_R(S) \subset \mathfrak{m} \subseteq R : \frac{\omega_R}{a}$ by hypothesis, we are done by Theorem 5.2.0.5. Since $IM \in \mathrm{CM}(S)$, the proof of (1) is complete. For part (2), just note that $S = I^*$ is reflexive and not equal to R. ■

In particular, if R is almost Gorenstein, $\mathfrak{m}^n$ is reflexive for all n.

Remark 5.4.0.4. Suppose R is almost Gorenstein but not Gorenstein, and take $I = \omega_R$. I is not reflexive. Note that here $\mathrm{End}_R(I)$ is reflexive but does not contain R properly. Thus the conditions on I in Proposition 5.4.0.3 are needed.

The following example shows that in general, $\mathfrak{m}^2$ can fail to be reflexive.

Example 5.4.0.5. Let $R = k[[t^5,t^6,t^{14}]]$. Then $\mathfrak{m}^2 = (t^{10},t^{11},t^{12},t^{15},t^{16},\dots)$. Thus $(\mathfrak{m}^2)^* = (1,t^4,t^5,t^6,\dots)$ and $t^{14} \in (\mathfrak{m}^2)^{**}$. But $t^{14} \notin \mathfrak{m}^2$.

With a bit more work one can even find an example where none of $\mathfrak{m}^n$, $n \geq 2$ is reflexive.

Example 5.4.0.6. Let $R = k[[t^6,t^8,t^{11},t^{13},t^{15}]]$. Then $\mathfrak{m}^2 = (t^{12},t^{14},t^{16},\dots)$. Thus, $(\mathfrak{m}^2)^* = t^{-12}(t^{11},t^{12},t^{13},t^{14},t^{15},t^{16}) = t^{-12}\mathfrak{c}$. Thus, $t^{13} \in (\mathfrak{m}^2)^{**}$ but $t^{13} \notin \mathfrak{m}^2$, so $\mathfrak{m}^2$ is not reflexive. However, note that t^6 is a minimal reduction of $\mathfrak{m}$ and $t^6\mathfrak{m}^2 = \mathfrak{m}^3$. Thus, none of the higher powers of $\mathfrak{m}$ can be reflexive.

Next, we classify when $\mathrm{Ref}(R)$ is of finite type for almost Gorenstein rings.

Proposition 5.4.0.7. *Suppose that $(R,\mathfrak{m})$ is almost Gorenstein. Let $S = \mathrm{End}_R(\mathfrak{m})$. Then $\mathrm{Ref}(R)$ is of finite type if and only if $\mathrm{CM}(S)$ is of finite type.*

Proof. The 'if' direction is Proposition 5.3.4.1. The other direction follows from Proposition 5.4.0.3(1). ■

Remark 5.4.0.8. Let $S = \mathrm{End}_R(\mathfrak{m})$. Using the notations in Kobayashi (2017), we thus get $\mathrm{CM}(S) \subset \mathrm{Ref}(R) = \Omega\mathrm{CM}(R)$, so $\mathrm{CM}(S) = \Omega\mathrm{CM}'(R)$ by Theorem 2.2.1.9. It follows that $\Omega\mathrm{CM}'(R)$ has finite type if and only if $\mathrm{CM}(S)$ has finite type. This recovers results by T. Kobayashi (Kobayashi, 2017, Corollary 1.3).

Proposition 5.4.0.9. *Let (R, m, k) be a one-dimensional, reduced, complete local ring containing $\mathbb{Q}$ and further assume that k is algebraically closed. Consider the following statements.*

(a) $\mathrm{CM}(R)$ *is of finite type.*

(b) $\mathrm{Ref}(R)$ *is of finite type.*

(c) $\mathrm{Ref}_1(R)$ *is of finite type.*

(d) $\mathrm{RT}(R)$ *is finite.*

If R is Gorenstein, then all the four statements are equivalent. If R is an almost Gorenstein domain, then $(b), (c)$ and (d) are equivalent.

Proof. Assume first that R is Gorenstein. Clearly, $(a) \implies (b) \implies (c) \implies (d)$. Now suppose $\mathrm{CM}(R)$ is not of finite type. Then by (Leuschke & Wiegand, 2012b, Theorem 4.13(ii)(a)) and Lemma 2.2.1.8, we get $\mathrm{RT}(R)$ is not of finite type. This completes the first part of the proof.

Next assume that R is an almost Gorenstein domain. We only need to show $(d) \implies (b)$. Assume that $\mathrm{Ref}(R)$ is of infinite type, and hence $\mathrm{CM}(S)$ is of infinite type by Proposition 5.4.0.7, where $S = \mathrm{End}_R(\mathfrak{m})$. Thus, there are infinitely many non-isomorphic finite reflexive birational extensions of R by (Leuschke & Wiegand, 2012b, Theorem 4.13(ii)(a)) and by Proposition 5.4.0.3(1). The proof is now complete using Lemma 2.2.1.8. ■

Next, we classify birational reflexive extensions of R that are Gorenstein. Our result was inspired by and extends (Goto et al., 2013, Theorem 5.1).

Theorem 5.4.0.10. *Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let $S \in \mathrm{Ref}(R)$ be a birational extension of R. Let $I = \mathfrak{c}_R(S)$. The following are equivalent:*

1. *S is Gorenstein.*

2. *I is I-Ulrich and ω_R-Ulrich. That is $I \cong I^2 \cong I\omega_R$.*

Proof. Note that I is a trace ideal by Lemma 2.2.1.8. Suppose (1) holds. Then $S = \mathrm{End}_R(I) = \mathrm{End}_S(I)$ so $I \cong S$ by (Kobayashi, 2017, Lemma 2.9). Thus $I = aS$ for some $a \in I$, necessarily a regular element, and since $S = S^2$, we have $aI = I^2$. By Corollary 2.2.2.9, we have $I = \mathrm{tr}(I) \supseteq (a) :_R I \supseteq I$, so $I = (a) :_R I$. Thus $I \cong I^*$. However, since S is Gorenstein, we have $S \cong S^\vee$, so $S^* \cong S^\vee$. By Corollary 5.1.0.27, S, and hence I is ω_R-Ulrich.

Assume (2). We can assume that R has infinite residue field and thus I has a reduction a. Thus $I^2 = aI$, and the same argument in the preceding paragraph shows that $I \cong I^* \cong S$. So S is ω_R-Ulrich, which (by Corollary 5.1.0.27) implies $S^\vee \cong S^* \cong S$, thus S is Gorenstein. ■

Corollary 5.4.0.11. *(Goto et al., 2013, Theorem 5.1) Suppose that $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let $S = \mathrm{End}_R(\mathfrak{m})$. The following are equivalent:*

1. *S is Gorenstein.*

2. *R has minimal multiplicity and is almost Gorenstein.*

Proof. Note that minimal multiplicity and almost Gorenstein are just $\mathfrak{m}$ being $\mathfrak{m}$-Ulrich and ω_R-Ulrich, respectively. ■

Example 5.4.0.12. (trace ideals are not always reflexive) Let $R = k[[t^5, t^6, t^7]]$. Here $\mathfrak{c} = \mathfrak{m}^2$. Let $I = (t^5, t^7)$. Then $I \in \mathrm{T}(R)$ but $I^{**} \cong (t^5, t^6, t^7)$ and hence $I \notin \mathrm{RT}(R)$.

Proof. $\mathfrak{c} = \mathfrak{m}^2$ is clear. A straight forward computation gives $J := (t^5 : (t^5, t^7)) = (t^5, t^{13}, t^{14})$, so that $\mathrm{tr}(I) = II^* = I(t^5)^{-1}J = (t^5, t^7)$. Therefore $I \in \mathrm{T}(R)$.

However, $((t^5) :_R J) = \mathfrak{m}$. So I is not reflexive. ■

Example 5.4.0.13. Let $R = k[[t^4, t^5, t^6]]$, which is a complete intersection domain of multiplicity 4. The conductor $\mathfrak{c}$ is $\mathfrak{m}^2$, with colength 4. The set $\mathrm{RT}(R)$ is infinite and classified by (Goto et al., 2020, Example 3.4(i)). We note a few features that illustrate our results:

1. It shows that the category of reflexive (regular) ideals is not of finite type, so our Theorem 5.3.3.4 is sharp, as the conductor has colength 4 but R does not have minimal multiplicity.

2. The finite set of integrally closed ideals are $\{(t^n)_{n\geq 4} = \mathfrak{m}, (t^n)_{n\geq 5}, (t^n)_{n\geq 6}, (t^n)_{n\geq 8} = \mathfrak{c}\}$.

3. The rest is the infinite family $\{I_a = (t^4 - at^5, t^6), a \in k\}$. We have $t^{10} = (t^4 - at^5)t^6 + at^5t^6$ which shows that $t^5 \in \overline{I_a}$ and so $\overline{I_a} = \mathfrak{m}$. So none of the I_a are integrally closed. However let $S = \mathrm{End}_R(\mathfrak{m}) = k[[t^4, t^5, t^6, t^7]]$. Then $\ell(S/I_a) = 3$ and $\ell(R/I_a) = 2$, so $I_aS \cap R = I_a$ by Proposition 2.2.2.14. In other words all trace ideals in R are contracted from S.

Example 5.4.0.14. Let $R = k[[t^4, t^6, t^7, t^9]]$. Then clearly $\mathfrak{c} = (t^6, t^7, t^8, t^9)$ and therefore $\mathrm{T}(R) = \{\mathfrak{c}, \mathfrak{m}, R\}$. Thus we are in the situation of Corollary 5.3.3.5.

Next note that $\mathrm{End}_R(\mathfrak{m}) = k[[t^2, t^3]]$. By (Leuschke & Wiegand, 2012b, Theorem 4.18), $\mathrm{CM}(\mathrm{End}(\mathfrak{m}))$ is of finite type. Hence $\mathrm{Ref}(R)$ is of finite type by Proposition 5.3.4.1.

Example 5.4.0.15 (Reflexivity is not preserved under going modulo a non-zero divisor in general). Let $M \in \mathrm{Ul}(R) \cap \mathrm{Ref}(R)$. Let l be a principal reduction of $\mathfrak{m}$. Then M/lM is a finite dimensional k-vector space. Since, R/l is Artinian, k is reflexive if and only if R is Gorenstein (recall that $\mathrm{Hom}_R(k, R)$ is the non-zero socle if R is Artinian). So, if R is not Gorenstein, then $M/lM \notin \mathrm{Ref}(R/l)$.

Chapter 6

Future Outlook

Abstract

In this chapter, we look at some future directions of research motivated by this work.

Broadly speaking, the themes explored in this work are:

1. Existence of maximal (small) Cohen Macaulay modules (in mixed characteristic).
2. Reflexivity in codimension one.

The future directions of research may be classified as:

1. Maximal Cohen-Macaulay modules
   - Existence of MCMs over generically Abelian extensions of regular local rings in mixed characteristic.
   - Existence/ nonexistence of small Cohen Macaulay algebras in mixed characteristic.
   - Existence of MCMs over Artin-Schreier extensions of regular local rings in characteristic $p > 0$.
   - Existence/ Non-existence of rank one MCMs over "geometric local rings".
2. Reflexive Modules
   - Finite representation type of the class of reflexive modules in codimension one.
   - The interplay between reflexive modules, I-Ulrich modules, conductors, trace and core of ideals over curve singularities.
3. Mixed Characteristic Commutative Algebra

- Exploring the situation in mixed characteristic of the result in Huneke (1982), which gives a tight connection between the (S_n)-property, multiplicity and the Cohen-Macaulay property for a complete local ring containing a field.
- Understanding the failure of the Cohen-Macaulay property in generically Abelian extensions of regular local rings in mixed characteristic.

4. Exploring the connection between representations of the Galois group of a generic extension of a regular local ring and the singularities of the integral closure.

We now look at the above items in a little more detail.

6.1 Small Cohen-Macaulay modules and mixed characteristic commutative algebra

Of course, the fundamental conjecture of Hochster remains:

Conjecture 6.1.0.1 (**Small CM module** - Hochster)**.** Every complete local domain admits a small Cohen Macaulay module.

Although there has been much progress in constructing a maximal Cohen-Macaulay module if the finite generation hypothesis is dropped, the conjecture as stated remains a mystery. Not much is known even in the case of finitely generated algebras over an algebraically closed field. It is now generally believed that the conjecture may be false and a good place to start would be to look at examples of non-existence of rank one MCMs over finitely generated $\mathbb{C}$-algebras, for example non Cohen-Macaulay UFDs, see Dumas (1965), Kiehl (1974), Mori (1977) and Marcelo & Schenzel (2011).

"Classifying" the approach to the existence of small CM modules by Galois groups seems like a worthwhile approach. Along these lines, we would like to understand better the underlying reasons for the failure of Roberts's theorem in mixed characteristic:

Question 6.1.0.2. *What are the obstructions one faces when S is an unramified regular local ring of mixed characteristic $p > 0$ and $p \mid |G|$ in 2.1.3.6?*

More generally we would like to understand:

Question 6.1.0.3. *Let S be a regular local ring and L its fraction field. Let K/L be a Galois extension and G be the Galois Group. Let R be the integral closure of S in K. Is there a relation between the representations G admits over the (algebraically closed) residue field of S and the singularities of R ? If so, is this quantifiable?*

The motivating question for our work in Chapters 2 and 3 is still unanswered:

Question 6.1.0.4. *Does the integral closure of a regular local ring of mixed characteristic $p > 0$ in a finite Abelian extension of its fraction field admit a small CM module?*

The indication is that this may be true and we intend to approach it by considering the following two questions:

Question 6.1.0.5. *Let S be an unramified regular local ring of mixed characteristic $p > 0$ and L its fraction field. Does the integral closure of S in $L(\sqrt[n]{f_1}, \dots, \sqrt[n]{f_m})$ admit a small CM module for an arbitrary integer $n \geq 2$ and elements $a_1, \dots, a_m \in S$?*

Question 6.1.0.6. *Let S be a ramified regular local ring of mixed characteristic p obtained by adjoining a primitive n-th root of unity to an unramified regular local subring for some integer n that is a multiple of p. Let $L = Frac(S)$ be its fraction field. Does the integral closure of S in $L(\sqrt[n]{f_1}, \dots, \sqrt[n]{f_m})$ admit a small CM module for an arbitrary integer $n \geq 2$ and elements $a_1, \dots, a_m \in S$?*

The phenomenon in 4.1.0.3 raises many interesting questions. Firstly one would want to understand when small CM algebras exist in mixed characteristic. Note that if R is a non CM normal domain containing the rationals, then R cannot admit a small CM algebra S. This is because there exists a retraction from $S \hookrightarrow R$ using the trace map corresponding to the fraction fields.
It certainly does not happen always - any mixed characteristic domain that is not Cohen-Macaulay after inverting p would be an example. Moreover, Bhatt's examples of non existence of small CM algebras in positive characteristic in Bhatt (2012) deform to mixed characteristic.

Secondly, one may ask if in 6.1.0.5 and 6.1.0.6 we can even a construct a small CM algebra. This holds some promise as seen in section 4.2.

As a first step towards answering 6.1.0.4, we consider the following in Katz & Sridhar (2021):

Question 6.1.0.7. *Can we generalize 4.1.0.14 and 4.1.1.8 to address the case when the p-torsion of the Galois group in 6.1.0.4 is annihilated by p ?*

See sections 3.2 and 4.2 for some reductions and partial results. We surmise that answers to the following questions will take us closer to answering 6.1.0.4. We maintain the following notation: Let S denote an unramified regular local ring of mixed characteristic $p > 0$ and dimension $d \geq 3$. Let L be its quotient field and K/L a finite field extension. Let R be the integral closure of S in K.

Question 6.1.0.8. *Let K/L be Abelian. For any integer $0 \leq r \leq d$, does there exist R such that $p.d_S(R) = r$? In particular, is $depth(R) \geq d-1$?*

So far, as evidenced by 4.1.0.14 and 4.1.1.8, $depth(R) \geq d-1$. Moreover all examples in multi-radical extensions constructed thus far seem to enjoy this property.

Question 6.1.0.9. *Let K/L be Abelian. For arbitrary d, can we construct a R such that $p.d_S(R) = 1$ and R is free on the punctured spectrum of S?*

If one were able to do this, we would have a counterexample to the analog of Huneke (1982) in mixed characteristic.

Question 6.1.0.10. *Let K' be the Galois closure of K and R' the integral closure of S in K'. If R admits a small CM module, does R' admit a small CM module?*

Question 6.1.0.11. *Let B be a normal domain and p a prime integer such that $p \in B$ is a non-unit. Let $C := A[\varepsilon]$ where epsilon is a primitive p^n-th root of unity for $n \geq 1$. Is the integral closure of C a free B-module?*

Let $A \hookrightarrow B$ be such that A is a regular local ring and B a normal domain that is module finite over A. The following question is motivated by phenomena observed in 3.1.0.18 and 3.1.1.1.

Question 6.1.0.12. *Let $e \in A$ be a principal prime such that $B[1/e]$ is Cohen Macaulay. If there exists a height one prime $P \subseteq B$ containing e, such that $A_{(e)} \hookrightarrow B_P$ is unramified, is B Cohen Macaulay?*

The notion of a Lim-Cohen Macaulay sequence was introduced by Hochster, see Hochster (2017). The existence of such a sequence for a complete local domain implies positivity of Serre's intersection multiplicity conjecture. Define:

Definition 6.1.0.13. Let R be a complete local domain of mixed characteristic p. Let $S \subseteq R$ be an unramified regular local subring of mixed characteristic p with a system of parameters given by $(p, \underline{x})$. Set $\mathscr{B}^n_S(R) := \overline{R[\sqrt[p^n]{\underline{x}}]}$, where $\overline{}$ denotes normalization.

Question 6.1.0.14. *Let R be a complete local domain of mixed characteristic $p > 0$ and $S \subseteq R$ an unramified regular local subring. Is $\{\mathscr{B}^n_S(R)\}_{n \geq 0}$ a Lim Cohen-Macaulay Sequence for R?*

The study of Ulrich modules has generated a lot of interest in the last three decades. More recently the notion of a Lim Ulrich sequence was introduced in Ma (2020). We would like to determine the following:

Question 6.1.0.15. *Are the small CM modules constructed in 4.1.0.14 and 4.1.1.8 Ulrich?*

Another avenue of study would be to explore the interaction between the trace of maximal Cohen Macaulay modules and the singularities of the ring inspired by Pérez & G. (2021) and Benali et al. (2021). This was pointed out to the author by Rebecca R.G.

Finally, inspired by 2.1.3.6, one could study the situation in characteristic $p > 0$. Namely, the question of existence of small CM modules over Abelian extensions of regular local rings of characteristic $p > 0$. This easily reduces to the case of certain generically Artin-Schreier extensions. As a first step:

Question 6.1.0.16. *Is the integral closure of a regular local ring of positive char $p > 0$ in an Artin-Schreier extension of its fraction field Cohen-Macaulay ? If not, does it admit a small CM module?*

6.2 Reflexive and I-Ulrich modules

As evidenced by the study in Chapter 5, there is a lot to be understood about the class of reflexive modules even in dimension one. Recently, there has been quite a few works along these lines, see Dao et al. (2021), Dao (2021), Dao & Lindo (2021) and Herzog et al. (2021). The representation theory of Cohen-Macaulay modules over Cohen-Macaulay rings is well developed, see Yoshino (1990) and Leuschke & Wiegand (2012b) for example. Our study in Chapter 5 encourages us to ask:

Question 6.2.0.1. *Let R be a complete local ring of dimension one.*

1. *Can we classify when* $\mathrm{Ref}(R)$ *has finite type ?*
2. *Can we classify when* $\mathrm{Ref}_1(R)$ *is of finite type?*
3. *Can we classify when R has finite trace type?*

A related question in positive characteristic is:

Question 6.2.0.2. *Suppose R is a one dimensional Cohen-Macaulay ring of dimension one and characteristic $p > 0$. When does $R^{1/q}$ belong to* $\mathrm{Ref}(R)$ *for $q = p^i$ large enough?*

Some of the first obstructions in the dimension one study of reflexive modules would be cleared up if we have answers to

Question 6.2.0.3. *Let R be a complete local ring of dimension one. If an ideal $I \subseteq R$ is reflexive, is* $\mathrm{tr}(I)$ *reflexive?*

Question 6.2.0.4. *Let R be a complete local ring of dimension one. When is an ideal of colength* 2 *a trace ideal? When is it reflexive?*

Since one of our main tools in section 4 was the concept of I-Ulrich modules we are led to ask:

Question 6.2.0.5. *What is the correct generalization of the definition of I-Ulrich modules to higher dimensions ?*

References

Aberbach, I. & Huneke, C. (1996). A theorem of Briançon-Skoda type for regular local rings containing a field. *Proceedings of the American Mathematical Society*, 124(3), 707–713.

Andre, Y. (2016). La conjecture du facteur direct.

Barucci, V. & Fröberg, R. (1997). One-dimensional almost gorenstein rings. *Journal of Algebra*, 188(2), 418–442.

Barucci, V. & Pettersson, K. (1995). On the biggest maximally generated ideal as the conductor in the blowing up ring. *manuscripta mathematica*, 88(1), 457–466.

Bass, H. (1960). Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.*, 95, 466–488.

Benali, J., Pothagoni, S., & G., R. R. (2021). Cohen-macaulay test ideals over rings of finite and countable cohen-macaulay type.

Berthelot, P. (1997). Altérations de variétés algébriques. In *Séminaire Bourbaki : volume 1995/96, exposés 805-819*, number 241 in Astérisque. Société mathématique de France. talk:815.

Bhatt, B. (2012). On the non-existence of small cohen-macaulay algebras. *Journal of Algebra*, 411.

Bourbaki, N. (1965). Diviseurs, in "Algèbre Commutative". *Chap. VII. Hermann, Paris*.

Brennan, J. P., Herzog, J., & Ulrich, B. (1987). Maximally generated cohen-macaulay modules. *Mathematica Scandinavica*, 61, 181–203.

Bruns, W. & Herzog, H. J. (1998). *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition.

Buchsbaum, D. A. & Eisenbud, D. (1973). What makes a complex exact? *Journal of Algebra*, 25(2), 259–268.

Chen, X.-W. (2013). Totally reflexive extensions and modules. *J. Algebra*, 379, 322–332.

Corso, A., Huneke, C., Katz, D., & Vasconcelos, W. V. (2005). Integral closure of ideals and annihilators of homology. *Commutative algebra*, 244, 33–48.

Corso, A., Huneke, C., & Vasconcelos, W. V. (1998). On the integral closure of ideals. *manuscripta mathematica*, 95(1), 331–347.

Dao, H. (2021). Reflexive modules, self-dual modules and arf rings.

Dao, H., Kobayashi, T., & Takahashi, R. (2020). Trace of canonical modules, annihilator of ext, and classes of rings close to being gorenstein. *arXiv preprint arXiv:2005.02263*.

Dao, H. & Lindo, H. (2021). Stable trace ideals and applications.

Dao, H., Maitra, S., & Sridhar, P. (2021). On reflexive and i-ulrich modules over curve singularities.

Datta, R. & Murayama, T. (2019). Permanence properties of f-injectivity. *arXiv preprint arXiv:1906.11399*.

Dieudonné, J. et al. (1958). Remarks on quasi-frobenius rings. *Illinois Journal of Mathematics*, 2(3), 346–354.

Dumas, M. (1965). Sous-anneaux dinvariants danneaux de polynomes. *COMPTES RENDUS HEBDOMADAIRES DES SEANCES DE L ACADEMIE DES SCIENCES*, 260(22), 5655.

Faber, E. (2019). Trace ideals, Normalization Chains, and Endomorphism Rings. *arXiv preprint arXiv:1901.04766*.

Fouli, L. & Olberding, B. (2018). Generators of reductions of ideals in a local noetherian ring with finite residue field. *Proceedings of the American Mathematical Society*, 146(12), 5051–5063.

Gillet, H. (1985). K-th 'e orie and nullit é of intersection multiplicities. *CR Acad. Sci. Paris S 'e r. I Math.*, 300, 71–74.

Goto, S., Hayasaka, F., & Kasuga, S. (2003). Towards a theory of Gorenstein $\mathfrak{m}$-primary integrally closed ideals. In *Commutative algebra, singularities and computer algebra (Sinaia, 2002)*, volume 115 of *NATO Sci. Ser. II Math. Phys. Chem.* (pp. 159–177). Kluwer Acad. Publ., Dordrecht.

Goto, S., Isobe, R., & Kumashiro, S. (2020). Correspondence between Trace Ideals and birational extensions with application to the analysis of the Gorenstein property of rings. *J. Pure Appl. Algebra*, 224(2), 747–767.

Goto, S., Matsuoka, N., & Phuong, T. T. (2013). Almost Gorenstein rings. *Journal of Algebra*, 379, 355–381.

Goto, S., Ozeki, K., Takahashi, R., Watanabe, K.-I., & Yoshida, K.-I. (2014). Ulrich ideals and modules. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 156 (pp. 137–166).: Cambridge University Press.

Goto, S., Ozeki, K., Takahashi, R., Watanabe, K.-i., & Yoshida, K.-i. (2016). Ulrich ideals and modules over two-dimensional rational singularities. *Nagoya Mathematical Journal*, 221(1), 69–110.

Goto, S., Takahashi, R., & Taniguchi, N. (2015). Almost gorenstein rings–towards a theory of higher dimension. *Journal of Pure and Applied Algebra*, 219(7), 2666–2712.

Grothendieck, A. (1967). Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32), 361.

Grothendieck, A. A. (1971). *Éléments de géométrie algébrique [par] A. Grothendieck [et] J.A.*

Dieudonné. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 166. Berlin: Springer-Verlag.

Hanes, D. (1999). *Special conditions on maximal Cohen-Macaulay modules, and applications to the theory of multiplicities*. PhD thesis, University of Michigan.

Herzog, J., Hibi, T., & Stamate, D. I. (2019). The trace of the canonical module. *Israel Journal of Mathematics*, 233(1), 133–165.

Herzog, J., Kumashiro, S., & Stamate, D. I. (2021). The tiny trace ideals of the canonical modules in cohen-macaulay rings of dimension one.

Herzog, J. & Kunz, E. (1971). Der kanonische modul eines Cohen-Macaulay-Rings (sem., regensburg, 1970/71). *Lecture Notes in Math*, 238.

Herzog, J., Ulrich, B., & Backelin, J. (1991). Linear maximal cohen-macaulay modules over strict complete intersections. *Journal of Pure and Applied Algebra*, 71(2-3), 187–202.

Hochster, M. (1973). Cohen-macaulay modules. In *Lecture Notes in Mathematics* (pp. 120–152). Springer Berlin Heidelberg.

Hochster, M. (2017). Homological conjectures and lim cohen-macaulay sequences. In *Homological and Computational Methods in Commutative Algebra* (pp. 173–197). Springer International Publishing.

Hochster, M. & Huneke, C. (1995). Applications of the existence of big cohen-macaulay algebras. *Advances in Mathematics*, 113, 45–117.

Huneke, C. (1982). A remark concerning multiplicities. *Proceedings of the American Mathematical Society*, 85(3), 331–331.

Huneke, C. & Katz, D. (2019). Uniform symbolic topologies in abelian extensions. *Transactions of the American Mathematical Society*, 372(3), 1735–1750.

Huneke, C. & Swanson, I. (2006a). *Integral closure of ideals, rings, and modules*, volume 13. Cambridge University Press.

Huneke, C. & Swanson, I. (2006b). *Integral Closure of Ideals, Rings, and Modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.

Huneke, C. & Ulrich, B. (1987). The structure of linkage. *Annals of Mathematics*, 126(2), 277–334.

Katz, D. (1999). On the existence of maximal cohen-macaulay modules over p-th root extensions. *Proceedings of the American Mathematical Society*, 127(9), 2601–2609.

Katz, D. (2021). Conductors in mixed characteristic. *Journal of Algebra*, 571, 350–375. Commutative Algebra and its Interactions with Algebraic Geometry: A volume in honor of Craig Huneke on the occasion of his 65th birthday.

Katz, D. & Sridhar, P. (2021). Small CM modules over repeated radical extensions in mixed characteristic. *in preparation*.

Kiehl, Reinhardt, F. E. (1974). Algebraische eigenschaften der lokalen ringe in den spitzen der hilbertschen modulgruppen. *Inventiones mathematicae*, 24, 121–148.

Kleiman, S. & Ulrich, B. (1997). Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers. *Transactions of the American Mathematical Society*, 349(12), 4973–5000.

Kobayashi, T. (2017). Syzygies of Cohen-Macaulay modules over one dimensional Cohen-Macaulay local rings. *arXiv preprint arXiv:1710.02673*.

Kobayashi, T. & Takahashi, R. (2019a). Rings whose ideals are isomorphic to trace ideals. *Mathematische Nachrichten*, 292(10), 2252–2261.

Kobayashi, T. & Takahashi, R. (2019b). Ulrich modules over cohen–macaulay local rings with minimal multiplicity. *Quarterly Journal of Mathematics*, 70(2), 487–507.

Koh, J. (1986). Degree p extensions of an unramified regular local ring of mixed characteristicp. *Journal of Algebra*, 99(2), 310–323.

Kustin, A. R. & Vraciu, A. (2018). Totally reflexive modules over rings that are close to gorenstein. *Journal of Algebra*.

Leuschke, G. J. & Wiegand, R. (2012a). *Cohen-Macaulay representations*, volume 181 of *Mathematical Surveys and Monographs*. Providence, RI: American Mathematical Society.

Leuschke, G. J. & Wiegand, R. (2012b). *Cohen-Macaulay representations*. Number 181. American Mathematical Soc.

Lindo, H. (2017). Trace ideals and centers of endomorphism rings of modules over commutative rings. *J. Algebra*, 482, 102–130.

Lipman, J. (1971). Stable ideals and arf rings. *American Journal of Mathematics*, 93(3), 649–685.

Ma, L. (2018). Maximal cohen-macaulay modules over certain segre products.

Ma, L. (2020). Lim ulrich sequence: proof of lech's conjecture for graded base rings.

Ma, L. & Schwede, K. (2019). Recent applications of \$p\$-adic methods to commutative algebra. *Notices of the American Mathematical Society*, 66, 820–831.

Maitra, S. (2020). Partial trace ideals and Berger's Conjecture. *arXiv preprint arXiv:2003.11648*.

Marcelo, A. & Schenzel, P. (2011). Non-cohen–macaulay unique factorization domains in small dimensions. *Journal of Symbolic Computation*, 46(5), 609–621. Groebner Bases and Applications.

Masek, V. (1998). Gorenstein dimension of modules. *arXiv preprint math.AC/9809121*.

Mond, D. & Pellikaan, R. (1989). Fitting ideals and multiple points of analytic mappings. In *Lecture Notes in Mathematics* (pp. 107–161). Springer Berlin Heidelberg.

Mori, S. (1977). Graded factorial domains. *Japanese journal of mathematics. New series*, 3(2), 223–238.

Morita, K. (1958). Duality for modules and its applications to the theory of rings with minimum condition. *Science Reports of the Tokyo Kyoiku Daigaku, Section A*, 6(150), 83–142.

Nagata, M. (1962). *Local rings*. Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley
Sons New York-London.

Nakajima, Y. & Yoshida, K.-i. (2017). Ulrich modules over cyclic quotient surface singularities. *Journal of Algebra*, 482, 224–247.

Polini, C. & Ulrich, B. (2005). A formula for the core of an ideal. *Mathematische Annalen*, 331(3), 487–503.

Pérez, F. & G., R. R. (2021). Characteristic-free test ideals.

Roberts, P. (1980). Abelian extensions of regular local rings. *Proceedings of the American Mathematical Society*, 78(3), 307–310.

Roberts, P. (1985). The vanishing of intersection multiplicities of perfect complexes. *Bulletin (New Series) of the American Mathematical Society*, 13(2), 127 – 130.

Schoutens, H. (2020). A differential-algebraic criterion for obtaining a small cohen-macaulay module. *Proceedings of the American Mathematical Society*, 148(10), 4165–4177.

Serre, J.-P. (1965). Algèbre locale. multiplicités, volume 11 of cours au collège de france, 1957–1958, rédigé par pierre gabriel. seconde édition, 1965. *Lecture Notes in Mathematics. Springer-Verlag, Berlin*, 6.

Serre, J.-P. (1997). *Algèbre locale, multiplicités: cours au Collège de France, 1957-1958*, volume 11. Springer Science & Business Media.

Serre, J.-P. (2000). *Local Algebra*. Springer Berlin Heidelberg.

Sridhar, P. (2021a). Existence of birational small cohen-macaulay modules over biquadratic extensions in mixed characteristic. *Journal of Algebra*, 582, 100–116.

Sridhar, P. (2021b). On the existence of birational maximal cohen-macaulay modules over biradical extensions in mixed characteristic. *Journal of Pure and Applied Algebra*, 225(12), 106790.

Stacks project authors, T. (2021). The stacks project. `https://stacks.math.columbia.edu`.

Swan, R. G. (1980). On seminormality. *Journal of Algebra*, 67(1), 210–229.

Tavanfar, E. (2017). Reduction of the small cohen–macaulay conjecture to excellent unique factorization domains. *Archiv der Mathematik*, 109, 429–439.

Ulrich, B. (1984). Gorenstein rings and modules with high numbers of generators. *Mathematische Zeitschrift*, 188(1), 23–32.

Vasconcelos, W. V. (1991). Computing the integral closure of an affine domain. *Proceedings of the American Mathematical Society*, 113(3), 633–638.

Vitulli, M. A. (2011). Weak normality and seminormality. In *Commutative Algebra* (pp. 441–480). Springer.

Yoshino, Y. (1990). *Maximal Cohen-Macaulay Modules over Cohen-Macaulay Rings*. London Mathematical Society Lecture Note Series. Cambridge University Press.

Appendix A

Kummer Theory

We look at a proof of the general case of Kummer Theory. The version 2.1.4.10 presented in chapter 2 follows from this. Before we begin, we recall Hilbert's famous theorem:

Theorem A.0.0.1. *(Hilbert's Theorem 90) Let K/L be a Galois extension of fields with Galois group G. Then $H^1(G, K^\times) = 0$.*

Theorem A.0.0.2. *(Kummer Theory) Let L be a field containing a primitive n-th root of unity and Ω an algebraic closure of L. Let $A^n_\Omega(L)$ denote the set of finite Abelian extensions of L of exponent n contained in Ω and $S_n(L)$ the subgroups of $L^\times$ containing $L^{\times n}$ as a subgroup of finite index. Then the map*

$$A^n_\Omega(L) \to S_n(L) : K \to L^\times \cap K^n$$

defines a bijective correspondence. The inverse of the above map is given by

$$S_n(L) \to A^n_\Omega(L) : B \to L[B^{1/n}] \subseteq \Omega$$

where $L[B^{1/n}] \subseteq \Omega$ denotes the smallest subfield of Ω containing L and a n-th root of every element of B. If $K \leftrightarrow B$, then $[K : L] = (B : L^{\times n})$.

Proof. Let K/L be a finite Galois extension and set $\alpha(K) := L^\times \cap K^n$. We have

$$[K : L] \geq [L[\alpha(K)^{1/n}] : L] = (\alpha(L[\alpha(K)^{1/n}]) : L^{\times n}) \geq (\alpha(K) : L^{\times n})$$

Consider the short exact sequence of groups

$$1 \longrightarrow \mu_n \longrightarrow K^\times \xrightarrow{x \mapsto x^n} K^{\times n} \longrightarrow 1 \tag{A.0.0.2.1}$$

Considering the long exact sequence in cohomology with respect to G and applying A.0.0.1 we get

$$1 \longrightarrow \mu_n \longrightarrow L^\times \xrightarrow{x \mapsto x^n} L^\times \cap K^{\times n} \longrightarrow H^1(G, \mu_n) \longrightarrow 1 \tag{A.0.0.2.2}$$

Since G acts trivially on μ_n, $H^1(G,\mu_n) = Hom(G,\mu_n)$. Thus, $(L^\times \cap K^{\times n})/L^{\times n} \simeq Hom(G,\mu_n)$. Now assume G is Abelian of exponent n. It is then isomorphic to some subgroup of a finite direct sum of copies of $\mathbb{Z}/n\mathbb{Z}$. It is then clear that $|Hom(G,\mu_n)| = |G| = [K : L]$. Thus equalities hold in the above inequalities when G is Abelian of exponent n. In particular $K = L[\alpha(K)^{1/n}]$.

Conversely consider $B \in S_n(L)$ and set $\beta(B) := L[B^{1/n}]$. The extension $\beta(B)/L$ is Galois since L contains a primitive n-th root of unity. Suppose $b_1, \ldots, b_m$ are a set of generators for $B/L^\times$, then $G := Gal(\beta(B)/L)$ embeds in $\prod_{i=1}^m Gal(L[a_i^{1/n}]/L)$ since $\beta(B)$ is a composite of the $L[a_i^{1/n}]$. Therefore $\beta(B)/L$ is Abelian of exponent n. Therefore as observed in the forward direction of the proof, we have $\alpha(\beta(B)) \simeq Hom(G,\mu_n)$. Explicitly, the map sends

$$a \mapsto (\sigma \mapsto \sigma(a^{1/n})/a^{1/n})$$

This maps the subgroup $B/F^{\times n}$ isomorphically onto $Hom(G/H,\mu_n)$ where H consists of all $\sigma \in G$ such that $\sigma(a^{1/n})/a^{1/n} = 1$ for all $a \in B$. Such a σ necessarily is the identity map on $\beta(B)$. Thus $B/F^{\times n} \simeq Hom(G,\mu_n)$ and $\alpha(\beta(B)) = B$. Thus the sets $A^n_\Omega(L)$ and $S_n(L)$ are in bijection via the maps α and β. The proof already shows that if $K \leftrightarrow B$, then $[K : L] = (B : L^{\times n})$. ■