Projective normality and normal presentation for adjoint linear series on minimal varieties

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Abstract

In this thesis, we study projective normality and normal presentation of adjoint linear series associated to an ample and globally generated line bundle on higher dimensional smooth projective varieties with nef canonical bundle. As one of the consequences of the main theorems, we give bounds on very ampleness and projective normality of pluricanonical linear systems on varieties of general type in dimensions three, four and five.

Next we concentrate on varieties with trivial canonical bundle. In the first part, we prove an effective projective normality result for an ample line bundle on regular smooth four-folds with trivial canonical bundle. In the second part, we emphasize on the projective normality of multiples of ample and globally generated line bundles on certain classes of known examples (up to deformation) of projective hyperkähler varieties. As a corollary we show that excepting two extremal cases in dimensions 4 and 6, a general curve section of any ample and globally generated linear system on the above mentioned examples is non-hyperelliptic.

This thesis is based on the articles [MR19] and [MR20] both of which are joint works with Jayan Mukherjee.

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Chapter 1

Introduction

Equations defining the embedding of a projective variety in a projective space is a topic of great interest. The study of projective normality and normal presentation dates back to the time of Italian geometers. Castelnuovo first showed that a line bundle of degree greater than $2g$ on a curve of genus g has a normal homogeneous coordinate ring and if the degree is greater than $2g+1$ then the ideal of the curve is generated by quadrics. Fujita, Saint-Donat and Mumford, among many others, rediscovered these results years later. Mumford and his school of mathematicians carried on the study of these properties on an abelian variety of arbitrary dimensions. In the early 80s, Green and Lazarsfeld showed that the results of these nature are special cases of a general N_p property (see [Gre84a], [Gre84b] and [Gre87]) for curves, projective normality being the property N_0 and normal presentation (i.e., when the ideal is generated by quadrics) being the property N_1.

It was Green who proved that a line bundle of degree $\geq 2g+p+1$ on a smooth curve of genus g satisfies the property N_p. One of the most interesting questions on surfaces concerning the N_p property that has motivated lot of work is Mukai's Conjecture that asserts that for an ample line bundle A on a smooth projective surface S, $K_S + lA$ satisfies the N_p property if $l \geq p+4$ where K_S is the canonical bundle of S. This can be thought of as an analogue of Green's result on curves for surfaces. Mukai's conjecture has not yet been proved even for $p=0$.

Another very interesting and related conjecture is the conjecture by Fujita that asserts that on a smooth projective variety of dimension n, $K_X + dA$ is globally generated for $d \geq n+1$ and $K_X + dA$ is very ample for $d \geq n+2$ where A is an arbitrary ample line bundle. Fujita's conjecture has been verified for surfaces by Reider (cf. [Rei88]) using Bogomolov's instability theorem (see

[Bog78]) on rank two vector bundles. Fujita's freeness conjecture has been proved by Ein and Lazarsfeld (see [EL93a]) for $n = 3$, by Kawamata (see [Kaw97]) for $n = 3, 4$ and by Ye and Zhu (see [YZ20]) for $n = 5$. Conjecturally, the assertion of Mukai's conjecture can be generalized in higher dimension by asserting that for a smooth projective variety of dimension n and an ample line bundle A, $K_X + lA$ satisfies the property N_p for $l \geq n + p + 2$. Progress in this direction with A just ample seems to be out of reach at this moment. A natural question to ask is what happens to the above conjecture if A is taken to be ample and base point free instead. It is a standard argument that if A is taken to be ample and base point free then Fujita's conjecture follows in its full generality by using induction and using known results for curves. Syzygies of adjunction bundles with A ample and base point free were studied in quite some detail on surfaces in a series of papers written by Gallego and Purnaprajna (see [GP01]–[GP98]).

1.1 Main result for adjoint linear series on smooth minimal varieties

In this thesis, we study the properties N_0 and N_1 of the adjoint bundle $K_X + lB$ with B ample and base point free on arbitrary dimensional smooth projective varieties X with nef canonical bundle K_X by imposing mild conditions on the line bundle B apart from the assumptions of being ample and globally generated. These are analogues for results known for surfaces. Our main result regarding projective normality on a variety X with canonical divisor K_X is the following.

Theorem 1.1.1. *(= Theorem 3.1.7) Let X be a smooth projective variety of dimension n, $n \geq 3$. Let B be an ample and base point free line bundle on X. We further assume:*

(a) K_X *is nef,* $K_X + B$ *is base point free,*

(b) $h^0(B) \geq n + 2$,

(c) $h^0(K_X + B) \geq h^0(K) + n + 1$,

(d) $B - K_X$ *is nef and effective.*

Then $K_X + lB$ is very ample and it embeds X as a projectively normal variety for all $l \geq n$.

Our result regarding normal presentation on regular varieties is the following.

Theorem 1.1.2. *(= Theorem 3.2.4) Let X be a regular smooth projective n-fold, $n \geq 3$. Let B be an ample and base point free line bundle on X. We further assume the following four conditions:*

(a) K_X *is nef,* $K_X + B$ *is base point free,*

(b) $h^0(B) \geq n+2$*,*

(c) $h^0(K_X + B) \geq h^0(K_X) + n + 1$*,*

(d) $(n-2)B - (n-1)K_X$ *is nef and non-zero effective divisor.*

Then $K_X + lB$ *will satisfy the property* N_1 *for* $l \geq n$*.*

We also prove a slightly weaker theorem regarding the N_1 property of adjoint bundles on irregular varieties (see Theorem 3.2.5). We will also briefly discuss the situation when the variety is Gorenstein and has at worst canonical singularities (see Section 3.3). Once we have these theorems, we can start looking for results using only an ample bundle if we know what multiple of that bundle is globally generated. Here solution to Fujita's freeness Conjecture comes to play an important role. Precisely by doing so, we obtain consequences on the positivity for pluricanonical series as we shall describe in the next section.

1.2 Consequences for pluricanonical linear series

The geometry of pluricanonical maps is of great importance in projective algebraic geometry. It was extensively studied by Bomberi, Catanese, Ciliberto, Kodaira (see [Bom73], [CC91], [CC93], [Kod68]) and other mathematicians. For example, Ciliberto showed that for minimal surfaces of general type nK_X is projectively normal for $n \geq 5$ (see [Cil84]), Purnaprajna produced very precise and optimal bounds for normal generation and normal presentation and higher syzygies of pluricanonical series on surfaces of general type with ample canonical bundle (see [Pur05]). In this thesis we obtain effective results on projective normality and normal presentation of pluricanonical series on smooth three-folds, four-folds and five-folds with ample canonical bundle.

Theorem 1.2.1. *(= Corollaries 4.2.3, 4.2.4, 4.2.5, 4.2.6, and 4.2.7) Let X be a smooth projective variety of dimension n with ample canonical bundle K_X.*

(1) If $n = 3$, then lK_X is very ample and embeds X as a projectively normal variety for $l \geq 12$ and normally presented for $l \geq 13$.

(2) If $n = 4$, then lK_X is very ample and embeds X as a projectively normal variety for $l \geq 24$ and normally presented for $l \geq 25$.

(3) If $n = 5$ and $p_g(X) \geq 1$, then lK_X is very ample and embeds X as a projectively normal variety for $l \geq 35$ and normally presented for $l \geq 36$.

We will also show that the above result for smooth three-folds will generalize to Gorenstein canonical three-folds (see Corollary 4.3.1).

1.3 Effective projective normality for varieties with trivial canonical bundle

Geometry of linear series on varieties X with trivial canonical bundle K_X (i.e. $K_X = 0$) is a topic that has motivated a lot of research. The question of what multiple of an ample line bundle on a K-trivial variety is very ample and projectively normal was extensively studied by many mathematicians including Gallego, Oguiso, Peternell, Purnaprajna, Saint-Donat (see [GP01], [Op95], [SD74]). Recall that a $K3$ surface is defined as a smooth projective surface S with $K_S = 0$ and $H^1(\mathscr{O}_S) = 0$. The following theorem is due to Saint-Donat.

Theorem 1.3.1. *([SD74]) Let S be a smooth projective $K3$ surface and let B be an ample line bundle on S. Then nB is very ample (in fact projectively normal) for $n \geq 3$.*

Gallego and Purnaprajna generalized Saint-Donat's result on projective normality for smooth, projective, regular ($H^1(\mathscr{O}_X) = 0$) three-fold X with trivial canonical bundle (see [GP01], Corollary 1.10). They showed that for an ample line bundle B on X, nB is projectively normal for $n = 8$ and $n \geq 10$. Moreover, if $B^3 > 1$ then nB is projectively normal if $n = 6$ and $n \geq 8$. In order to prove this theorem, Gallego and Purnaprajna studies the case when a regular, three-fold with trivial canonical bundle maps onto a variety of minimal degree by a complete linear series of an ample and globally generated line bundle. They gave a classification theorem for such situations and proved the projective normality result as a consequence by applying a theorem of Green (see Theorem 2.7.1). We remark that the varieties that appear as covers of varieties of minimal degree play an important role in the geometry of algebraic varieties. They are extremal cases in a variety of situations from algebraic curves to higher dimensional varieties (see [GP01], [Gre84a], [Gre84b], [Hor76]).

In this thesis, we prove an analogue of the classification theorem of Gallego and Purnaprajna where we study the situation when a smooth regular four-fold X with trivial canonical bundle maps to a variety of minimal degree by the complete linear system of an ample and globally generated line bundle B (see Theorem 5.1.5) and provide upper bounds to the degree of such morphisms. As a consequence of the classification theorem, Fujita's base point freeness conjecture that has been proved in dimension four by Kawamata (see [Kaw97]) and Green's theorem (c.f. Theorem 2.7.1), we prove the following effective projective normality result on regular four-folds with trivial canonical bundle.

Theorem 1.3.2. *(= Theorem 5.1.6) Let X be a smooth projective four-fold with trivial canonical bundle and let A be an ample line bundle on X.*

(1) nA is very ample and embeds X as a projectively normal variety for $n \geq 16$.

(2) If X is regular (i.e., $H^1(\mathscr{O}_X) = 0$) then nA is very ample and embeds X as a projectively normal variety for $n \geq 15$.

This theorem can be thought of as a higher dimensional analogue of Theorem 1.3.1 and Corollary 1.10, [GP01]. We remark that the standard methods of Castelnuovo-Mumford regularity (see Lemma 2.7.6) and Theorem 1.3, [GP01] yields in the situation above that nA satisfies projective normality for $n \geq 21$. One can lower this bound to 20 by using Lemma 5.1.3. On the other hand, Mukai's conjecture predicts that for a smooth projective variety X, $K_X + nA$ is projectively normal for $n \geq \dim(X) + 2$. Hence in our case we can expect nA to be projectively normal for $n \geq 6$. We also remark that Theorem 1.3.2 (2) is a consequence of Corollary 5.1.2 that says that for a smooth regular projective variety X with trivial canonical bundle and for an ample and base point free line bundle L on X, $(\dim(X)-1)L$ is projectively normal unless the morphism induced by the complete linear series $|L|$ maps X onto a variety of minimal degree. In Example 3.1.10 we provide an example of a Calabi–Yau n-fold X (in the sense of Definition 5.2.1) and an ample and globally generated line bundle B which maps X to a variety of minimal degree for which nB is not very ample. Consequently $(n-1)B$ is not projectively normal, thereby showing that the assumption on the degree of embedding of the image is sharp for Corollary 5.1.2. The above example together with [Jia16], Example 1.6 show that one cannot improve Theorem 1.3.2 beyond $n=5$ in either the regular or irregular case.

The condition that a smooth regular surface S has trivial canonical bundle is equivalent to the condition that S has a holomorphic symplectic form on it. However, in higher dimensions these two notions do not coincide which is clear from the fact that existence of a holomorphic symplectic form on a Kähler manifold demands that its dimension is even whereas there are examples of smooth projective algebraic varieties in odd dimensions with trivial canonical bundle and $H^1(\mathscr{O}_X)=0$, for example smooth hypersurfaces of degree $n+1$ in $\mathbb{P}^n$. So essentially we can have two different kinds of generalizations of $K3$ surfaces, namely Calabi–Yau and hyperkähler varieties (see Definitions 5.2.1 and 5.2.2).

The theorem of Saint-Donat that we stated (see Theorem 1.3.1) deals with ample line bundles. In the same paper, Saint-Donat proves the following theorem for ample and globally generated line bundles on $K3$ surfaces.

Theorem 1.3.3. *([SD74]) Let S be a smooth projective $K3$ surface and let B be an ample and base point free line bundle on S. Then,*

(1) $2B$ is very ample and $|2B|$ embeds S as a projectively normal variety unless the morphism given by the complete linear system $|B|$ maps S, $2:1$ onto $\mathbb{P}^2$.

(2) B is very ample and $|B|$ embeds S as a projectively normal variety unless the morphism

given by the complete linear system $|B|$ maps S, $2:1$ onto $\mathbb{P}^2$ or to a variety of minimal degree.

Gallego and Purnaprajna provide a generalization of this theorem for regular three-folds X with trivial canonical bundle in [GP01] in which they proved for an ample and globally generated line bundle B, $3B$ is projectively normal unless the morphism φ_B induced by the complete linear series of B maps X, $2:1$ onto $\mathbb{P}^3$. Moreover, they showed that $2B$ is projectively normal unless φ_B maps X, $2:1$ onto $\mathbb{P}^3$ or to a variety of minimal degree ≥ 2. They also proved that $4B$ is projectively normal on smooth, projective, regular, four-folds with trivial canonical bundle when the morphism φ_B associated to the complete linear series of an ample and globally generated line bundle B is birational onto its image and $h^0(B) \geq 7$ (see Theorem 1.11, [GP01]). More recently, Niu proved an analogue of Theorem 1.3.3 in dimension four. In fact he proved a general result for smooth, projective, regular, K_X-trivial varieties in all dimensions (see [Niu19]) with an additional assumption of $H^2(\mathcal{O}_X) = 0$ which in dimensions 2 and 3 recovers, and in dimension 4 generalizes the results of Saint-Donat and Gallego-Purnaprajna.

We see that it is a natural question to ask whether and to what extent these theorems generalize to the other class of higher dimensional analogues of $K3$ surfaces, namely hyperkähler varieties. There are many families of examples for Calabi–Yau varieties but there are only few classes of examples for hyperkähler varieties are known up to deformation. Hilbert scheme of n points on a $K3$ surface (we will denote them by $K3^{[n]}$), generalized Kummer varieties (we will denote them by $K^n(T)$) and two examples in dimension six and ten given by O'Grady (we shall denote it by $\mathcal{M}_6$ and $\mathcal{M}_{10}$ respectively) are the only known classes of examples up to deformation. Our main theorem for hyperkähler varieties is the following.

Theorem 1.3.4. *(= Theorem 5.2.17) Let X be a projective hyperkähler variety of dimension $2n \geq 4$ that is deformation equivalent to $K3^{[n]}$, $K^n(T)$ or $\mathcal{M}_6$. Let B be an ample and globally generated line bundle on X. Then the following happens;*

(1) lB is projectively normal for $l \geq 2n$.

(2) $(2n-1)B$ *is projectively normal unless:*

(a) $n=2$, $X=K3^{[2]}$ *and* φ_B *maps* X *onto a quadric (possibly singular) inside* $\mathbb{P}^5$*. In this case* $q_X(B)=2$, *deg*$(\varphi_B)=6$, *or*

(b) $n=3$, $X=K3^{[3]}$ *and* φ_B *maps* X *onto a variety of minimal degree inside* $\mathbb{P}^9$ *which is obtained by taking cones over the Veronese embedding of* $\mathbb{P}^5$ *inside* $\mathbb{P}^5$*. In this case* $q_X(B)=2$, *deg*$(\varphi_B)=30$.

Hence if X *is as above and* B *does not satisfy cases* $2(a)$ *or* $2(b)$ *then a general curve section of* $|B|$ *is non-hyperelliptic.*

As before, the study of the situation when X maps onto a variety of minimal degree by the complete linear series $|B|$ is the main ingredient of our proof. We also use two key characteristics of a hyperkähler variety which are the Riemann–Roch expression that comes from the existence of a primitive integral quadratic form on the second integral cohomology group of the variety and Matsushita's theorem on fibre space structure of a hyperkähler manifold (see Theorem 5.2.6).

1.4 Organization of this thesis

Now we provide the organization of the thesis and the description of the chapters.

In Chapter 2 we recall the basic definitions and results that we are going to use in the remaining chapters to prove our main theorems. Chapters 3, 4 and 5 contain the main results.

In Chapter 3 we study the properties N_0 and N_1 for adjoint bundles on minimal varieties. For the most part, we consider the bundles on smooth projective varieties. In the last section, we will see that the main results generalize to Gorenstein varieties with nef canonical bundle that have at worst canonical singularities.

In Chapter 4 we study those properties for pluricanonical series on three, four and five-folds using the Fujita freeness theorems. As before, we mostly deal with smooth varieties, but in the last section of this chapter, we prove a corollary for Gorenstein canonical three-folds using the results of the last section of Chapter 3.

Finally in Chapter 5, we study the effective projective normality results for certain smooth projective varieties with trivial canonical bundle. First we prove such results for smooth projective four-folds, and then we concentrate on projective hyperkähler varieties.

1.5 Notation and conventions

Throughout this thesis, we work exclusively over the field of complex numbers $\mathbb{C}$. By a variety we always mean an integral scheme of finite type over $\mathbb{C}$. For any Gorenstein variety X (see Definition 3.3.1), K_X or K will denote its canonical bundle. We will use the multiplicative and the additive notation of line bundles interchangeably. Thus, for a line bundle L, $L^{\otimes r}$ and rL are the same. We have used the notation L^{-r} for $(L^*)^{\otimes r}$. We will use L^r to denote the intersection product. The sign '$\equiv$' will be used for numerical equivalence and the sign '$\sim$' will be used for linear equivalence.

Chapter 2

Background and preliminaries

In this chapter, we recall the preliminaries that we need to state and prove our main results. Section 2.1 is devoted to discuss the basic notions of positivity of linear series on normal projective varieties. In Sections 2.2 and 2.3 we define the varieties of minimal degree and the constructions of cyclic covers respectively, and we briefly recall few related results. Section 2.4 introduces the theory of Koszul cohomology groups, the most crucial object that is central to the study that we will carry out in the later chapters. Section 2.6 introduces the projective normality, normal presentation (and N_p property), the main objects of our investigation. It will be clear from the characterization of N_p property that in order to study these properties, it is crucial to study the multiplication maps of global sections of vector bundles. In Section 2.7 we recall a few important criterions that ensures the surjectivity of such multiplication maps.

2.1 Positivity of linear series on projective varieties

We first recall the basics of linear series on normal projective varieties and we refer to [Laz04a] for more details. Let X be a normal projective variety and let L be a line bundle on X. Furthermore, let $V \subseteq H^0(L)$ be a non-zero linear subspace and we set $|V| := \mathbb{P}(V)$, the projective space of one dimensional quotients of V. We refer to $|V|$ as a *linear series* and when $V = H^0(L)$, one obtains the *complete linear series* $|L|$. Evaluations of sections gives rise to a morphism of vector bundles

$$\mathrm{ev} : V \otimes \mathscr{O}_X \to L.$$

The *base locus,* denoted as $\mathrm{Bs}(|V|)$, is by definition the set of points where all sections of $|V|$ vanish. We say that $|V|$ is base point free if the base locus $\mathrm{Bs}(|V|)$ is empty. When $V = H^0(L)$ and $|V|$ is base point free, one says that L *is base point free* or L *is globally generated.*

Example 2.1.1. ([Har77], Chapter IV, Corollary 3.2 (a)) Let C be a smooth projective curve of genus g. Then any line bundle L satisfying $\deg(L) \geq 2g$ is globally generated. To see this, recall that by [Har77], Chapter IV, Proposition 3.1, L is globally generated if and only if $h^0(L-P) = h^0(L)-1$ for all $P \in X$. The assertion follows from Riemann-Roch since $H^1(L-P) = H^1(L) = 0$ for all $P \in X$. ♠

Given a linear series $|V|$ (assume $\dim V \geq 2$), it defines a morphism

$$\varphi_{|V|} : X \backslash \mathrm{Bs}(|V|) \to \mathbb{P}(V)$$

as follows: given $x \in X\backslash \mathrm{Bs}(|V|)$, $\varphi_{|V|}(x)$ is the hyperplane in V consisting of sections vanishing at x. In general, one may view $\varphi_{|V|} : X \dashrightarrow \mathbb{P}(V)$ as a rational map. When $|V|$ is base point free, then $\varphi_{|V|}$ is a morphism.

Given a morphism $\varphi : X \to \mathbb{P}(V)$ from a normal projective variety X to the projective space of one dimensional quotients of a vector space V and assuming that the image of φ is not contained in any hyperplane (we say that $\varphi(X)$ is a *non-degenerate subvariety of* $\mathbb{P}(V)$), the pullback of sections via φ realizes V as a subspace of $H^0(\varphi^*\mathscr{O}_{\mathbb{P}(V)}(1))$ and $|V|$ is a base point free linear series on X. Further, one has $\varphi = \varphi_{|V|}$.

2.1.1 Ample and very ample line bundles

We start by defining ample and very ample line bundles on arbitrary projective varieties.

Definition 2.1.2. Let X be a projective variety and let L be a line bundle on X.

(1) L is very ample if there is an embedding $X \subseteq \mathbb{P}^N$ such that $L = \mathscr{O}_{\mathbb{P}^N}(1)|_X$.

(2) L is ample if mL is very ample for some $m > 0$.

It follows from Cartan-Serre-Grothendieck theorem (see [Laz04a], Theorem 1.2.5) that for an ample line bundle L on a projective variety X, there exists an integer $m(L)$ such that mL is very ample for all $m \geq m(L)$. It is interesting to ask for an efficient estimate for this integer $m(L)$. The situation is pretty simple for curves as the following result shows.

Example 2.1.3. ([Har77], Chapter IV, Corollary 3.2 (b), Corollary 3.3) Let C be a smooth projective curve of genus g and let L be a line bundle on C.

(1) If $\deg(L) \geq 2g+1$ the L is very ample. To see this, recall that by [Har77], Chapter IV, Proposition 3.1, L is very ample if and only if $h^0(L-P-Q) = h^0(L)-2$ for all $P, Q \in X$ (they may not be distinct). The assertion follows from Riemann-Roch since $H^1(L-P-Q) = H^1(L) = 0$ for all $P, Q \in X$.

(2) L is ample if and only if $\deg(L) > 0$. To see this, notice that the "if" part follows at once by part (1). The "only if" part holds since if L is ample then mL is very ample for some large m and consequently $\deg(mL) > 0$. ♠

To this end, we recall a proposition and one of its corollaries without the proofs. The proposition in particular implies that the restriction of an ample line bundle is ample.

Proposition 2.1.4. *([Laz04a], Proposition 1.2.9) Let $f : Y \to X$ be a finite mapping of projective varieties, and L be an ample line bundle on X. Then f^*L is an ample line bundle on Y. In particular, if $Y \subseteq X$ is a subvariety of X and L is an ample line bundle on X, then $L|_Y$ is also ample.*

Corollary 2.1.5. *([Laz04a], Corollary 1.2.11) Let L be a globally generated line bundle on X and let $\varphi = \varphi_{|L|}$ be the morphism given by the complete linear series $|L|$. Then L is ample if and only if φ is finite.*

We now recall one of the most well-known criterion of ampleness, we refer to [Laz04a] for the definition of intersection numbers.

Theorem 2.1.6. *(Nakai-Moishezon Criterion, [Laz04a], Theorem 1.2.19) Let L be a line bundle* *on a projective variety X. Then L is ample if and only if $L^{\dim V} \cdot V > 0$ for all positive dimensional irreducible subvarieties $V \subseteq X$.*

The remark after the following theorem of Miyaoka will be used in later chapters. A normal projective variety X is $\mathbb{Q}$-*Gorenstein* if the canonical divisor is $\mathbb{Q}$-Cartier.

Theorem 2.1.7. *([Miy87], Theorem 1.1) Let X a normal projective $\mathbb{Q}$-Gorenstein variety of dimension $n \geq 2$ with singular locus of codimension ≥ 3. Assume that the canonical divisor $K_X \in Pic(X) \otimes \mathbb{Q}$ is nef. Let $\rho : Y \to X$ be any resolution of the singularities. Then for arbitrary ample divisors $H_1, \ldots, H_{n-2}$, we have the following inequality:*

$$(3c_2(Y) - c_1^2(Y))\rho^*(H_1) \cdots \rho^*(H_{n-2}) \geq 0.$$

Remark 2.1.8. An obvious corollary of the theorem above is the following: let X be a smooth three (resp. four)-fold and A be an ample divisor on it. Then $A \cdot c_2(X) \geq 0$ (resp. $A^2 \cdot c_2(X) \geq 0$).

2.1.2 Nef and big line bundles

We start with the definition of nef line bundles.

Definition 2.1.9. Let X be a projective variety and let L be a line bundle on X. Then L is *numerically effective* or *nef* if $L \cdot C \geq 0$ for all irreducible curves $C \subseteq X$.

Example 2.1.10. Let X be a projective variety and let L be a globally generated line bundle on X. Then L is nef. ♠

It is easy to see that if $f : Y \to X$ is a proper morphism of projective varieties and if L is a nef line bundle on X then f^*L is a nef line bundle on Y. In particular, restrictions of nef line bundles are nef. We recall the following theorem of Kleiman.

Theorem 2.1.11. *(Kleiman's Theorem, [Laz04a], Theorem 1.4.8) Let X be a projective variety and let L be a nef line bundle on X. Then $L^{\dim V} \cdot V \geq 0$ for all irreducible subvariety $V \subseteq X$.*

Now we move on to define big line bundles. Let X be a projective variety carrying a line bundle L. The *semigroup* of L is by definition

$$\mathbf{N}(L) := \{m \geq 0 | H^0(mL) \neq 0\}.$$

Given $m \in \mathbf{N}(L)$ consider the rational map φ_m defined by the complete linear series mL.

Definition 2.1.12. Assume X is normal and $\mathbf{N}(L) \neq (0)$. Then the *Iitaka dimension of* L is by definition $\kappa(L) := \max\{\dim \varphi_m(X)\}$. If $H^0(X, mL) = 0$ for all $m > 0$, one sets $\kappa(L) = -\infty$.

Using the notion of Iitaka dimension, one defines the big line bundles on X as follows.

Definition 2.1.13. Let X be a projective variety and let L be a line bundle on X. Then L is called *big* if $\kappa(L) = \dim X$.

Example 2.1.14. An ample line bundle on X is big. Further, a nef line bundle L on X is big if and only if $L^n > 0$ where $\dim X = n$. ♠

2.1.3 Bertini's theorems

We briefly recall the following version of Bertini's theorem. We say a line bundle L on a projective variety is *fixed component free* or *movable* if the the base locus of the complete linear series $|L|$ i.e., $\mathrm{Bs}(|L|)$ is of codimension ≥ 2.

Theorem 2.1.15. *(Bertini) Let X be a projective variety and let L be a fixed component free line bundle on X. Then*

(1) if $\dim \varphi_{|L|}(X) > 1$ then the general divisors of $|L|$ are irreducible and reduced;

(2) a general divisor of $|L|$ has no singular points outside $Bs(|L|)$ and the singular points of X.

We state the classical version of the *Bertini theorem for general linear sections.*

Theorem 2.1.16. *([Laz04a], Theorem 3.3.1) Let X be an irreducible variety and let $f : X \to \mathbb{P}^r$ be a morphism. Fix an integer $d < \dim \overline{f(X)}$. If $L \subseteq \mathbb{P}^r$ is a general $(r-d)$-plane, then $f^{-1}(L)$ is irreducible.*

2.2 Varieties of minimal degree and their classification

We recall that a subvariety $X \subseteq \mathbb{P}^r$ is called *non-degenerate* if it is not contained in any hyperplane. For any non-degenerate variety $X \subseteq \mathbb{P}^r$, we have the inequality

$$\deg(X) \geq 1 + \operatorname{codim}(X).$$

Definition 2.2.1. A non-degenerate variety $X \subseteq \mathbb{P}^r$ is said to be a *variety of minimal degree* if it satisfies the equality $\deg(X) = 1 + \operatorname{codim}(X)$.

If $\operatorname{codim}(X) = 1$ then of course the variety X is a quadric hypersurface. One can completely classify the varieties of minimal degree, but before doing so, we define the rational normal curves and rational normal scrolls.

Definition 2.2.2. Consider the morphism $\mathbb{P}^1 \to \mathbb{P}^r$ defined by $(s,t) \mapsto (s^r, s^{r-1}t, \cdots, st^{r-1}, t^r)$. The image of this map is called the *standard rational normal curve* in $\mathbb{P}^r$. A *rational normal curve* in $\mathbb{P}^r$ is any curve that is obtained from the standard rational normal curve by an automorphism.

As it turns out, one may generalize this construction and define rational normal scrolls.

Definition 2.2.3. A rational normal scroll $X \subseteq \mathbb{P}^r$ of dimension n is the image of a projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ through the morphism given by the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ where the vector bundle $\mathscr{E} = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$ satisfies $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $\deg(X) = \sum a_i$.

In the situation above, if $a_1 = a_2 = \cdots = a_l = 0$ for some $0 < l < n$ then X is singular and it is a cone over a smooth rational normal scroll. The vertex or singular locus V of this cone has dimension $l - 1$ and let $X_S = X \backslash V$ be the smooth part of X. Moreover, X is normal and $\tilde{X} = \mathbb{P}(\mathscr{E}) \to X$ is a rational resolution of singularity which is called the *canonical resolution* of the rational normal scroll X.

The following theorem provides a complete classification of the varieties of minimal degree.

Theorem 2.2.4. *([EH85], Theorem 1) Let $X \subseteq \mathbb{P}^r$ is a variety of minimal degree. Then X is a cone over a smooth such variety. Moreover, if X is smooth and codim$(X) > 1$ then $X \subseteq \mathbb{P}^r$ is either a rational normal scroll or a Veronese surface $\mathbb{P}^2 \subseteq \mathbb{P}^5$.*

Figure 2.1: Real world example of scroll: Kobe Port Tower
https://en.wikipedia.org/wiki/Kobe_Port_Tower

Let X be a rational normal scroll. Let H be the class of a hyperplane section and R be the class of a general linear subspace of codimension one. We note the following fact:

Lemma 2.2.5. *([Fer01], Corollary 2.2 (2)) If codim$(V, X) = 2$ then $H \sim$ deg$(X)R$.*

We prove a lemma here that we will use in the later chapters.

Lemma 2.2.6. *Let X be a smooth projective variety and let L be an ample and globally generated line bundle on X. Let φ_L be the morphism induced by the complete linear series L. Assume φ_L maps X onto a singular rational normal scroll Y with vertex V. Then codim$(V, Y) = 2$, in other words, Y is obtained by taking cones over a rational normal curve.*

Proof. Suppose $\tilde{Y} = \mathbb{P}(\mathscr{E}) \to Y$ is the canonical resolution. If the codimension of the singular locus is > 2, then the canonical resolution is a small resolution (see [Fer01], Proposition 2.1) and hence X could be obtained by contracting subschemes of $(X \times_Y \tilde{Y})_{\text{red}}$ of codimension greater than one, which contradicts the factoriality of X. The assertion follows since $\text{codim}(V, Y) \neq 1$ as Y is normal. $\square$

2.3 Cyclic covers of projective varieties and adjunction

Quite a lot of examples in algebraic geometry comes from cyclic covers of varieties and this section is devoted to the basics of these covers, we refer to [Laz04a], 4.1. B for details. Let X be an affine bundle and $s \in \mathbb{C}[X]$ be a non-zero regular function. We take the product $X \times \mathbb{A}^1$ with the affine line and denote the coordinate of $\mathbb{A}^1$ by t. We define the subvariety $Y \subseteq X \times \mathbb{A}^1$ by $t^m - s = 0$ and denote the natural map $Y \to X$ by π. Then π can be thought of as a cyclic cover of degree m branched along the divisor $D := \text{div}(s)$. Setting $s' := t|_Y$, one obtains $s'^m = \pi^* s$. Globalizing this local description, one obtains

Proposition 2.3.1. *([Laz04a], Proposition 4.1.6) Let X be a variety and L be a line bundle on X. Given a section $s \in H^0(mL)$ defining a divisor $D \subseteq X$, there is a finite flat covering $\pi : Y \to X$ of degree m where there is an $s' \in H^0(\pi^* L)$ such that $s'^m = \pi^* s$. The divisor $D' := div(s')$ maps isomorphically to D. Moreover, if X and D are smooth, so are X' and D'.*

The divisor D in the above proposition is called the *branch divisor* of the cyclic cover. We list two properties of cyclic covers below that we will use in Examples 3.1.10 and 3.1.11.

Remark 2.3.2. (Properties of cyclic covers) Let $\pi : Y \to X$, D and D' be as above and we assume that they are smooth. Further let K_Y and K_X be the canonical bundles of Y and X respectively.

(1) The push-forward of the structure sheaf splits as follows

$$\pi_* \mathscr{O}_Y := \mathscr{O}_X \oplus (-L) \oplus \cdots \oplus (-(m-1)L).$$

(2) $K_Y \sim \pi^*(K_X + (m-1)L)$. $\diamondsuit$

We remark that any formula that connects the canonical bundles of two varieties can be thought of as an *adjunction formula* and thus the item (2) of the above remark can be regarded as an adjunction formula. We take this opportunity to state the most well-known adjunction formula below.

Remark 2.3.3. (Adjunction for divisors) Let D be a smooth divisor with canonical bundle K_D on a smooth variety X with canonical bundle K_X. Then K_D is linearly equivalent to the line bundle $(K_X + D)|_D$. $\diamondsuit$

2.4 Minimal free resolution and Koszul cohomology

In this section, we introduce the minimal free resolutions and Koszul cohomology groups. We also briefly discuss the geometric contexts in which they are used. The main references for this section are [AN10], [Eis05] and [Gre84b]. Throughout this section, we fix a complex vector space V of dimension $r+1$ and set

$$S := S(V) := \mathrm{Sym}(H^0(L)) \cong \mathbb{C}[X_0, \cdots, X_r],$$

the symmetric algebra of V. We denote by S_+ the maximal ideal $(X_0, \cdots, X_r) \subseteq S$. Let $B = \bigoplus_{q \in \mathbb{Z}} B_q$ be a finitely generated graded S module with B_q as the q-th graded piece. Further, we denote the module B shifted by a by $B(a)$, so that $B(a)_q = B_{a+q}$.

2.4.1 Minimal free resolutions of finitely generated graded modules

We aim to construct a graded free resolution of B. Given homogeneous elements $b_i \in B$ of degree a_i that generate B as an S module, we define a degree-preserving map from the graded free module $F_0 = \oplus S(-a_i) \to B$. Let $M_1 \subseteq F_0$ be the kernel of this map which is also a finitely generated S module by the Hilbert Basis Theorem. By choosing finitely many homogeneous generators of M_1, we may define a degree-preserving map from a graded free module $F_1 \to F_0$

with image M_1. Continuing in this fashion, one constructs a *graded free resolution of* B. In this situation, we recall the following theorem.

Theorem 2.4.1. *(Hilbert Syzygy Theorem, [Eis05], Theorem 1.1) B has a finite graded free resolution with $m \leq r+1$ as follows:*

$$0 \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to B \to 0. \tag{2.1}$$

Now we define what it means to say that a complex is minimal. We will see in the theorem next to the following definition that B admits a minimal graded free resolution and the minimal graded free resolutions of B are unique upto isomorphisms.

Definition 2.4.2. A complex $\mathbf{F}$ of graded S module as follows

$$\mathbf{F}: \qquad \cdots \to F_{i+1} \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \to \cdots$$

is called *minimal* if the image of δ_i is contained inside S_+F_{i-1} for all i.

Theorem 2.4.3. *([Eis05], Theorem 1.6) If $\mathbf{F}$ and $\mathbf{G}$ are minimal graded free resolutions of B, then there is a graded isomorphism $\mathbf{F} \to \mathbf{G}$ inducing identity on B. Furthermore, any graded free resolution contains the minimal graded free resolution as a direct summand.*

The following is an easy lemma that we record for future reference.

Lemma 2.4.4. *([Eis05], Proof of Proposition 1.7) Suppose (2.1) is the minimal graded free resolution of B. Then there is an isomorphism $F_p \otimes_S \mathbb{C} \cong Tor_p^S(B, \mathbb{C})$.*

Proof. Follows from the observation that the maps become zero after tensoring by $S/S_+ \cong \mathbb{C}$. □

2.4.2 Koszul cohomology and graded Betti numbers

We start by defining the Koszul complex that turns out to be one of the first examples of minimal graded free resolutions.

Definition 2.4.5. The *Koszul complex* is defined as follows

$$0 \to \wedge^{r+1} V \otimes S(-r-1) \to \cdots \to V \otimes S \to S \to \mathbb{C} \to 0 \tag{2.2}$$

where the maps $\wedge^p V \otimes S(-p) \to \wedge^{p-1} \otimes S(-p+1)$ are given by the following:

$$v_1 \wedge \cdots \wedge v_p \otimes f \mapsto \sum_{i=1}^{p} (-1)^p v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \otimes v_i f.$$

Theorem 2.4.6. *([AN10], Corollary 1.6) The Koszul complex* (2.2) *is the minimal graded free resolution of the graded S module $S/S_+ \cong \mathbb{C}$.*

Tensoring the complex (2.2) by B, we obtain the following maps.

$$\cdots \to \bigwedge^{p+1} V \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} V \otimes B_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} V \otimes B_{q+1} \to \cdots \tag{2.3}$$

Definition 2.4.7. The *Koszul cohomology group $K_{p,q}(B,V)$* is defined as follows:

$$K_{p,q}(B,V) := \operatorname{Ker}(d_{p,q})/\operatorname{Image}(d_{p+1,q-1}). \tag{2.4}$$

The elements of $K_{p,q}(B,V)$ are called the *p-th syzygies of B with respect to V of weight q*.

Lemma 2.4.8. *([AN10], Proof of Proposition 1.12) $K_{p,q}(B,V) \cong Tor_p^S(B,\mathbb{C})_{p+q}$.*

Proof. The conclusion follows from tensoring the Koszul resolution (2.2) by R and observing that the degree $(p+q)$-th part of the p-th module is $\wedge^p V \otimes B_q$. $\square$

It follows from Lemma 2.4.4 and Lemma 2.4.8 that if (2.1) is the minimal graded free resolution of B, then there are isomorphisms

$$F_p \cong K_{p,q}(B,V) \otimes S(-p-q).$$

The numbers $b_{p,q} := \dim_{\mathbb{C}} K_{p,q}(B,V)$ are called the *graded Betti numbers* of B and there is a compact way to express them using a table called the *Betti diagram* whose p-th column and q-th row contains the entry $b_{p,q}$. It is customary to put '$-$' for zero and '$*$' for potential non-zero entries in the Betti diagram.

	0	1	2	$\cdots$	p	$\cdots$
$\vdots$						
q	$b_{0,q}$	$b_{1,q}$	$b_{2,q}$	$\cdots$	$b_{p,q}$	$\cdots$
$\vdots$						

Table 2.1: Betti diagram of B with respect to V

Example 2.4.9. (Four points in $\mathbb{P}^2$) Here we encounter a geometric situation for the first time. We refer to [Eis05], 3B.2 for the details. Let $X = \{p_1, p_2, p_3, p_4\}$ be a set of four points in $\mathbb{P}^2$ and let S_X be the homogeneous coordinate ring of S_X and we set $S := \mathrm{Sym}(H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) \cong \mathbb{C}[X_0, X_1, X_2]$. There are three distinct possibilities and one needs to address them separately.

Case 1. This is the case when no three points are collinear, the picture is depicted below.

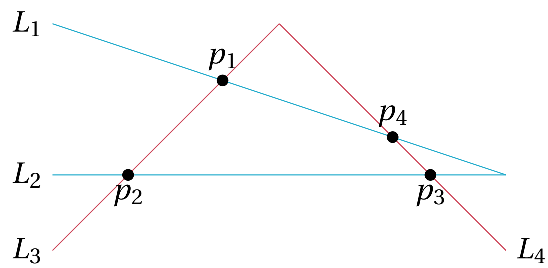

Figure 2.2: Four general points in $\mathbb{P}^2$

In this case, we set $C_1 = L_1 \cup L_2$ and $C_2 = L_3 \cup L_4$. It turns out that the homogeneous ideal I_X of $X \subseteq \mathbb{P}^2$ is the complete intersection of two conics, the defining equations of C_1 and C_2 each of which is an union of two lines. The minimal graded free resolution of S_X is as follows:

$$0 \to S(-4) \to S(-2)^{\oplus 2} \to S \to S_X \to 0.$$

The graded Betti numbers of S_X are listed in the Betti diagram below.

	0	1	2
0	1	–	–
1	–	2	–
2	–	–	1

Table 2.2: Betti diagram of four general points in $\mathbb{P}^2$

<u>Case 2.</u> When exactly three points are collinear, we have the following picture.

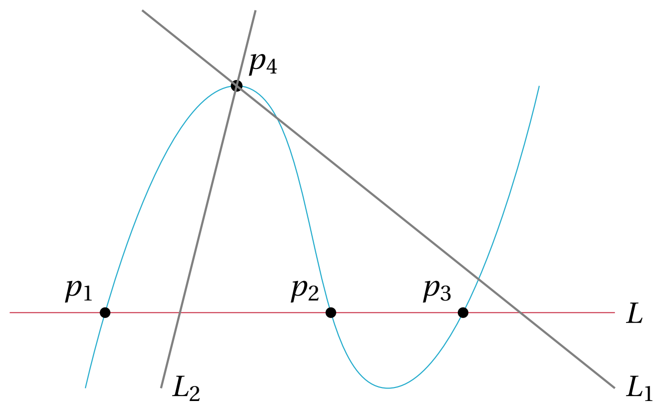

Figure 2.3: Four points in $\mathbb{P}^2$ exactly three of which are collinear

In this case, take lines L_1 and L_2 passing through p_4 that do not contain any of the three points p_1, p_2, p_3 (see picture). Set $C_1 := L \cup L_1$ and $C_2 = L \cup L_2$. It turns out that the homogeneous ideal I_X of $X \subseteq \mathbb{P}^2$ is generated by the conics defining C_1 and C_2 together with a cubic. The graded Betti numbers of S_X in this situation are listed in the Betti diagram below.

	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

Table 2.3: Betti diagram of four points in $\mathbb{P}^2$ exactly three of which are collinear

<u>Case 3.</u> The last case is when all four points lie on a line L.

Figure 2.4: Four collinear points in $\mathbb{P}^2$

It turns out that the homogeneous ideal I_X of $X \subseteq \mathbb{P}^2$ is generated by the defining equation of L and a quartic that is the product of four lines each passing through exactly one of the given points. The graded Betti numbers of S_X for this case are listed in the Betti diagram below.

	0	1	2
0	1	1	–
1	–	–	–
2	–	–	–
3	–	1	1

Table 2.4: Betti diagram of four collinear points in $\mathbb{P}^2$

♠

2.5 Koszul resolution of the ideal sheaf and vanishing theorems

We will make use of the Koszul resolution of ideal sheaves in the later chapters. We define the Koszul resolution of the ideal sheaf below in a more general set-up, namely we define the Skoda complex.

Definition 2.5.1. Let X be a smooth projective variety of dimension $n \geq 2$. Let B be a globally generated and ample line bundle on X.

(1) Take $n-1$ general sections $s_1, \ldots s_{n-1}$ of $H^0(B)$ so the intersection of the divisor of zeroes $B_i = (s_i)_0$ is a nonsingular projective curve C, that is $C = B_1 \cap \ldots \cap B_{n-1}$.

(2) Let $\mathscr{I}$ be the ideal sheaf of C and let $W = \mathrm{span}\{s_1, \ldots, s_{n-1}\} \subseteq H^0(B)$ be the subspace spanned by s_i. Note that $W \subseteq H^0(B \otimes \mathscr{I})$. For $i \geq 1$, define the *Skoda complex* $\mathbf{I}_i$ as

$$0 \to \bigwedge^{n-1} W \otimes B^{-(n-1)} \otimes \mathscr{I}^{i-(n-1)} \to \ldots \to W \otimes B^{-1} \otimes \mathscr{I}^{i-1} \to \mathscr{I}^i \to 0$$

where $\mathscr{I}^k$ stands for $\mathscr{I}^{\otimes k}$, we have used the convention that $\mathscr{I}^k = \mathscr{O}_X$ for $k \leq 0$.

In this article we will only use $\mathbf{I}_1$ which is called the *Koszul resolution* of $\mathscr{I}$ and it is exact (see [Laz04b]). We now recall two well-known vanishing theorems that are crucial for us.

Theorem 2.5.2. *(Kodaira Vanishing Theorem, [Laz04a], Theorem 4.2.1) Let X be a smooth projective variety and let L be an ample line bundle on X. Then $H^i(K_X + L) = 0$ for all $i > 0$.*

Kodaira Vanishing Theorem generalizes for nef and big line bundles as follows.

Theorem 2.5.3. *(Kawamata-Viehwag Vanishing Theorem, [Laz04a], Theorem 4.3.1) Let X be a smooth projective variety and let L be a nef and big line bundle on X. Then $H^i(K_X + L) = 0$ for all $i > 0$.*

2.6 Projective normality, normal presentation and property N_p

Throughout this section, X denotes a normal projective variety and L denotes ample and globally generated line bundle on X. The main references for this section are [Eis05], [Gre84b], [Laz89] and [Laz04a]. We set

$$R := \bigoplus_{m\geq 0} H^0(mL) \text{ and } S := \mathrm{Sym}(H^0(L) \cong \mathbb{C}[X_0, \ldots, X_s].$$

Since R is finitely generated as an S module, following the notations that we have used before, we set $K_{p,q}(X, L) := K_{p,q}(R, H^0(L))$. We have have a minimal graded free resolution of R as an S module as follows:

$$0 \longrightarrow E_{s+1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow R \longrightarrow 0, \tag{2.5}$$

Where the modules $E_i \cong \oplus S(-a_{ij})$ encodes the syzygy information as we have seen before.

Definition 2.6.1. In the situation as above,

(1) L is projectively normal (or it satisfies property N_0) if $E_0 \cong S$;

(2) L is normally presented (or it satisfies property N_1) if $E_0 \cong S$ and $E_1 \cong S(-2)$;

(3) L satisfies property N_p if $E_0 \cong S$ and $E_i \cong S(-i-1)$ for all $1 \leq i \leq p$.

We remark that L is projectively normal if and only if it is very ample and the homogeneous coordinate ring R is integrally closed. Further, L is normally presented if and only if in addition, the homogeneous ideal I_X of the embedding $X \hookrightarrow \mathbb{P}(H^0(L))$ is cut out by quadrics.

2.6.1 Characterizations of N_p property

The *syzygy bundle* M_L associated to a globally generated line bundle L is defined as follows,

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathscr{O}_X \longrightarrow L \longrightarrow 0. \tag{$*$}$$

We start by the following two lemmas that are necessary to prove the characterization i.e., Theorem 2.6.4. The proof that we include here is taken from [Laz89] and [Her06], Appendix A.

Lemma 2.6.2. *L satisfies N_p property if and only if $K_{p',q'}(X,L) = 0$ for all $0 \leq p' \leq p$ and $q' \geq 2$.*

Proof. We start with a minimal graded free resolution (2.5) of R as S module. Observe that L satisfies N_p property if and only if $E_{p'}$ has no generators of degree larger than $p'+1$ for all $0 \leq p' \leq p$. Since $\mathrm{Tor}^S_{p'}(R,\mathbb{C}) \cong E_{p'} \otimes \mathbb{C}$ by Lemma 2.4.4, L satisfies N_p property if and only if $\mathrm{Tor}^S_{p'}(R,\mathbb{C})_k = 0$ for all $k \geq p'+2$ and $0 \leq p' \leq p$. The conclusion follows from Lemma 2.4.8. $\square$

Lemma 2.6.3. *$K_{p,q}(X,L) \cong Ker(H^1(\wedge^{p+1} M_L \otimes (q-1)L) \longrightarrow \wedge^{p+1} H^0(L) \otimes H^1(X,(q-1)L))$.*

Proof. We start with the following commutative diagram (see [Laz89], Diagram (1.3.5))

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \wedge^{p+1} M_L \otimes (q-1)L \arrow[d] & & & \\
 & \wedge^{p+1} H^0(L) \otimes (q-1)L \arrow[d] \arrow[dr] & & 0 \arrow[d] & \\
0 \arrow[r] & \wedge^p M_L \otimes qL \arrow[r] \arrow[d] & \wedge^p H^0(L) \otimes qL \arrow[r] \arrow[dr] & \wedge^{p-1} M_L \otimes (q+1)L \arrow[r] \arrow[d] & 0 \\
 & 0 & & \wedge^{p-1} H^0(L) \otimes (q+1)L &
\end{tikzcd}$$

where the vertical and horizontal exact sequences are obtained from ($*$). Taking global sections, we obtain the following diagram.

$$
\begin{tikzcd}
& 0 \arrow[d] & & \\
& H^0(\wedge^{p+1} M_L \otimes (q-1)L) \arrow[d] & & \\
& H^0(\wedge^{p+1} H^0(L) \otimes (q-1)L) \arrow[d] \arrow[dr, "\alpha"] & & 0 \arrow[d] \\
0 \arrow[r] & H^0(\wedge^p M_L \otimes qL) \arrow[r, "\lambda"] \arrow[d, "\psi"] & \wedge^p H^0(L) \otimes H^0(qL) \arrow[r, "\mu"] \arrow[dr, "\beta"] & H^0(\wedge^{p-1} M_L \otimes (q+1)L) \arrow[d] \\
& H^1(\wedge^{p+1} M_L \otimes (q-1)L) \arrow[d, "\phi"] & & H^0(\wedge^{p-1} H^0(L) \otimes (q+1)L) \\
& \wedge^{p+1} H^0(L) \otimes H^1((q-1)L) & &
\end{tikzcd}
$$

Notice that $K_{p,q}(X,L) \cong \mathrm{Ker}(\beta)/\mathrm{Image}(\lambda)$. Also observe that λ induces an isomorphism between $H^0(\wedge^p M_L \otimes qL)$ and $\mathrm{Ker}(\beta)$. The required isomorphism between $K_{p,q}(X,L)$ and $\mathrm{Ker}(\phi)$ is given by sending x to $\psi(\lambda^{-1}x)$. $\square$

Now we are ready to state the main theorem that we will crucially use later.

Theorem 2.6.4. *Let L be an ample, globally generated line bundle on X. If the cohomology group $H^1(\bigwedge^{p'+1} M_L \otimes kL)$ vanishes for all $0 \le p' \le p$ and for all $k \ge 1$, then L satisfies the property N_p. If in addition $H^1(rL) = 0$ for all $r \ge 1$, then the above vanishing is a necessary and sufficient condition for L to satisfy N_p.*

Proof. Follows immediately from Lemma 2.6.3. $\square$

Since we are working over a field with characteristic zero, $\bigwedge^{p'+1} M_L$ is a direct summand of $M_L^{\otimes p'+1}$ (see [EL93b], Lemma 1.6). Consequently, to show that a line bundle L satisfies the property N_p, we will show that $H^1(M_L^{\otimes p'+1} \otimes kL) = 0$ for all $0 \le p' \le p$ and for all $k \ge 1$. We remark that if L is projectively normal then it is automatically very ample and we refer to [Mum70] for an overview of these circles of ideas.

Before providing some examples, we remark here that if L satisfies N_p property, then the

Betti diagram of the coordinate ring S_X looks as follows.

	0	1	2	$\cdots$	p	$p+1$	$\cdots$	$r+1$
0	1	–	–	$\cdots$	–	–	–	–
1	–	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$
2	–	–	–	$\cdots$	–	$*$	$\cdots$	$*$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Table 2.5: Betti diagram of S_X if L satisfies N_p

Example 2.6.5. (Twisted cubic curve) Let $C \subseteq \mathbb{P}^3$ be a rational normal curve of degree 3 i.e. the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ by the complete linear series $|\mathscr{O}_{\mathbb{P}^1}(3)|$. We have $S = \mathrm{Sym}(H^0(\mathscr{O}_{\mathbb{P}^1}(3)) \cong \mathbb{C}[X_0, X_1, X_2]$ where X_0, X_1, X_2 are the homogeneous coordinates of $\mathbb{P}^2$. In this case, the embedding is projectively normal, the homogeneous ideal I_C is generated by the following three quadrics

$$\Delta_{01} = X_0X_2 - X_1^2, \qquad \Delta_{02} = X_0X_3 - X_1X_2, \qquad \Delta_{12} = X_1X_3 - X_2^2$$

and the homogeneous coordinate ring $S_C = S/I_C = \oplus(H^0(\mathscr{O}_{\mathbb{P}^1}(3d))$. The relations among Δ_{ij}'s are given by the following two equations:

$$X_0\Delta_{12} - X_1\Delta_{02} + X_2\Delta_{01} = 0, \qquad X_1\Delta_{12} - X_2\Delta_{02} + X_3\Delta_{01} = 0.$$

The minimal graded free resolution of S_C takes the following shape (see [Laz89], Example 1.2.2)

$$0 \longrightarrow S(-3)^{\oplus 2} \xrightarrow{\begin{pmatrix} X_0 & X_1 \\ -X_1 & -X_2 \\ X_2 & X_3 \end{pmatrix}} S(-2)^{\oplus 3} \xrightarrow{(\Delta_{12}\ \Delta_{02}\ \Delta_{01})} S \longrightarrow S_C \longrightarrow 0.$$

Consequently, the line bundle $\mathscr{O}_{\mathbb{P}^1}(3)$ satisfies N_2. ♠

Example 2.6.6. (Rational normal curves) The above example generalizes as follows: let $C \subseteq \mathbb{P}^r$ be a rational normal curve of degree r. Using the *Eagon-Northcott complex*, one obtains that the Betti diagram of S_C is as follows (see [Eis05], Corollary 6.2).

	0	1	2	$\cdots$	$r-1$
0	1	–	–	$\cdots$	–
1	–	$\binom{r}{2}$	$2\binom{r}{3}$	$\cdots$	$(r-1)\binom{r}{r}$

Table 2.6: Betti diagram of rational normal curves

In particular, one observes from the diagram above that $\mathscr{O}_{\mathbb{P}^1}(r)$ satisfies N_r property. ♠

Example 2.6.7. (Elliptic quartic curve) Let $E \subseteq \mathbb{P}^3$ be an elliptic curve embedded by a line bundle of degree 4. Then E is a complete intersection of two quadrics Q_1 and Q_2. Further, the embedding is projectively normal and the minimal graded free resolution of the homogeneous coordinate ring S_E is as follows (see [Laz89], Example 1.2.4)

$$0 \longrightarrow S(-4) \xrightarrow{\begin{pmatrix} Q_2 \\ -Q_1 \end{pmatrix}} S(-2)^{\oplus 2} \xrightarrow{(Q_1\ Q_2)} S \longrightarrow S_E \longrightarrow 0.$$

Consequently, the line bundle satisfies N_1 but not N_2. ♠

2.6.2 Mukai's conjecture for adjoint linear series

As we discussed in the introduction, the following conjecture, known as Mukai's conjecture has motivated a lot of work regarding syzygies of projective varieties.

Conjecture 2.6.8. *Let X be a smooth projective variety of dimension n and let L be an ample line bundle on X. Then $K_X + dL$ satisfies the N_p property if $d \geq n+p+2$.*

In this section, we include a brief survey of the existing results. We first state some of the results obtained on surfaces towards this direction below.

• *Elliptic Ruled Surfaces:* Homma proved it for the case $p = 0$ for elliptic ruled surface (see [Hom80] and [Hom82]). The case $p = 1$ for elliptic ruled surfaces was proved by Gallego and Purnaprajna. In fact, they showed that the numerical classes of normally presented divisors on an elliptic ruled surface forms a convex set and as a particular case recovered Mukai's conjecture for $p = 0, 1$ and yield weaker bounds for higher syzygies (see [GP96]).

• *Surfaces with Kodaira Dimension zero:* Gallego and Purnaprajna proved Mukai's conjecture on these surfaces for $p = 0, 1$ lowering the bound by one in the latter case (see [GP98]).

• *Surfaces of General Type:* Purnaprajna proved that under mild hypothesis on an ample and globally generated line bundle A, $K_X + lA$ is projectively normal and normally presented for $l \geq 2$ where K_X is the canonical line bundle. He also obtained precise results on higher syzygies (See [Pur05]). N_p property of the adjoint bundles associated to an ample and globally generated line bundle on surfaces of general type was also studied in [BH13] where Banagere and Hanumanthu proved several interesting results in this direction.

In the following table, we list a few very important results in higher dimensional varieties.

X	L	p	$d \geq$	Reference
Arbitrary	Very ample	Arbitrary	$n+p+1$	Ein-Lazarsfeld [EL93b]
Abelian	Ample	0	3	Koizumi [Koi76]
Abelian	Ample	1	4	Kempf [Kem89]
Abelian	Ample	Arbitrary	$p+3$	Pareschi [Par00]
Hyperelliptic	Ample	Arbitrary	$p+3$	Chintapalli-Iyer [CI14]

Table 2.7: Conditions under which $K_X + dL$ satisfies N_p on X

Apart from these results, Butler proves that if E is a rank n vector bundle on a smooth projective curve C with genus $g \leq 1$ then $K_X + lA$ is projectively normal for $l \geq 2n+1$ and satisfies the property N_p for $l \geq 2n(p+1)$ where $X = \mathbb{P}(E)$ (see [But94]). Effective projective normality results and higher syzygies for ruled surfaces and higher dimensional ruled varieties were systematically studied by Park in a series of papers [Par06a]–[Par05]. Another very interesting case is the case on toric varieties; Hering, Schenck and Smith proved in [HSS06] that for an ample line bundle A on an n dimensional toric variety, lA satisfies N_p property for $l \geq n+p-1$.

2.7 Surjectivity of multiplication maps

As it will be clear to us that to study N_p property for line bundles, it is often useful to study the surjectivity for the multiplication maps of global sections of vector bundles on X. In this section, we recall some of the results that allow us to prove the surjection of such maps for curves and higher dimensional varieties. For references, see [Gre84b], [Laz04a], and [GP01]–[GP98].

2.7.1 Multiplication maps on curves

In this subsection, we recall a few important results on curves. The first result is a generalization of the base point-free pencil trick, and it is due to Green who calls it "The H^0 Lemma".

Theorem 2.7.1. *([Gre84b], Theorem (4.e.1)) Let C be a smooth, irreducible curve. Let L and M be line bundles on C. Let W be a base point free linear subsystem of $H^0(C, L)$. Then the multiplication map $W \otimes H^0(M) \to H^0(L \otimes M)$ is surjective if $h^1(M \otimes L^{-1}) \leq \dim(W) - 2$.*

Before stating the next result, we recall the notion of stability. Let C be a smooth projective curve and E be a vector bundle on C. The *slope* of E is given by $\mu(E) := \deg(E)/\operatorname{rank}(E)$. E is called *semistable* (resp. *stable*) if for every proper sub-bundle $E' \subsetneq E$, one has $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$).

Example 2.7.2. Any line bundle on C is stable. Further, when $C = \mathbb{P}^1$, by Grothendieck's Theorem, there is no stable vector bundle of rank > 1 on C and the semistable vector bundles are of the form $\bigoplus \mathcal{O}_{\mathbb{P}^1}(a)$. ♠

Proposition 2.7.3. *[But94], Proposition 2.2) Let E and F be semistable vector bundles over a curve C of genus g such that E is generated by its global sections. If*

(1) $\mu(F) > 2g$, and

(2) $\mu(F) > 2g + rank(E)(2g - \mu(E)) - 2h^1(E)$,

then the multiplication map $H^0(E) \otimes H^0(F) \to H^0(E \otimes F)$ surjects.

When we use the above proposition, one of the vector bundle is going to be the syzygy bundle M_L corresponding to a globally generated line bundle on C. Thus, we require the following theorem that gives the criterion under which the line bundle M_L is semistable on C.

Proposition 2.7.4. *([EL92], Proposition 1.5) For a smooth projective curve C of genus g, and a globally generated line bundle L on C, the syzygy bundle M_L is stable as soon as deg$(L) \geq 2g+1$.*

We remark that it is easy to see that if $\deg(L) \geq 2g+1$ then $\mu(M_L) > -2$ and we will use this fact often in the proofs of our main theorems.

2.7.2 Castelnuovo-Mumford regularity

We start by defining what it means to say that a sheaf $\mathscr{F}$ is m-regular with respect to L.

Definition 2.7.5. Let $\mathscr{F}$ be a coherent sheaf on a projective variety X and let L be an ample and globally generated line bundle on X. Then $\mathscr{F}$ is said to be *m-regular* if

$$H^i(\mathscr{F} \otimes (m-i)L) = 0 \text{ for all } i > 0.$$

It follows from the work of Mumford that if $\mathscr{F}$ is m-regular then it is $(m+k)$-regular for every $k \geq 0$. It turns out that 0-regularity provides a very effective result for surjectivity of multiplication maps as the following lemma shows.

Lemma 2.7.6. *([Mum70]) Let L be a base point free line bundle on a variety X and let $\mathscr{F}$ be a coherent sheaf on X. If $H^i(\mathscr{F} \otimes (-iL)) = 0$ for all $i \geq 1$ then the multiplication map*

$$H^0(\mathscr{F} \otimes (iL)) \otimes H^0(L) \to H^0(\mathscr{F} \otimes (i+1)L)$$

surjects for all $i \geq 0$.

2.7.3 Criterions of Gallego-Purnaprajna

The following observation of Gallego and Purnaprajna is one of the most useful ways to deal with the surjectivity of multiplication maps for global sections of tensor product of line bundles. Using the following "divide and rule" method one may prove surjectivity by proving surjectivity of simpler multiplication maps. The proof is taken from [BH13].

Observation 2.7.7. *([GP99a], Observation 1.4.1) Let E and $L_0 := \mathcal{O}_X$, L_1, $L_2, \cdots, L_r$ be coherent sheaves on a variety X. Assume the following evaluation maps*

$$\alpha_k : H^0\left(E \otimes \bigotimes_{i=0}^{k-1} L_i\right) \otimes H^0(L_k) \to H^0\left(E \otimes \bigotimes_{i=1}^{k} L_i\right)$$

surjects for all $1 \le k \le r$. Then the evaluation map $\psi : H^0(E) \otimes H^0\left(\bigotimes_{i=1}^{r} L_i\right) \to H^0\left(E \otimes \bigotimes_{i=1}^{r} L_i\right)$ also surjects.

Proof. We have the following commutative diagram.

$$\begin{tikzcd}
H^0(E) \otimes H^0(L_1) \otimes \cdots \otimes H^0(L_r) \arrow[r, "\alpha_1 \otimes \mathrm{Id}"] \arrow[d, "\phi"'] & H^0(E \otimes L_1) \otimes H^0(L_2) \otimes \cdots \otimes H^0(L_r) \arrow[d, "\alpha_2 \otimes \mathrm{Id}"] \\
H^0(E) \otimes H^0\left(\bigotimes_{i=1}^{r} L_i\right) \arrow[dd, "\psi"'] & H^0(E \otimes L_1 \otimes L_2) \otimes H^0(L_3) \otimes \cdots \otimes H^0(L_r) \arrow[d, "\alpha_3 \otimes \mathrm{Id}"] \\
 & \cdots \arrow[d, "\alpha_{r-1} \otimes \mathrm{Id}"] \\
H^0\left(E \otimes \bigotimes_{i=1}^{r} L_i\right) & H^0\left(E \otimes \bigotimes_{i=1}^{r-1} L_i\right) \otimes H^0(L_r) \arrow[l, "\alpha_r \otimes \mathrm{Id}"]
\end{tikzcd}$$

Our hypotheses implies that $\psi \circ \phi$ is surjective and thus ψ is surjective. ☐

The following two lemmas are also very important to show projective normality and higher syzygies as we will see in later chapters.

Lemma 2.7.8. *([GP99a], Observation 2.3) Let X be a regular variety (i.e. $H^1(\mathcal{O}_X) = 0$). Let E be a vector bundle and let D be a divisor such that $L = \mathcal{O}_X(D)$ is globally generated and $H^1(E \otimes L^*) = 0$. If the multiplication map $H^0(E|_D) \otimes H^0(L|_D) \to H^0((E \otimes L)|_D)$ surjects then $H^0(E) \otimes H^0(L) \to H^0(E \otimes L)$ also surjects.*

Proof. We have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(E)\otimes H^0(\mathscr{O}_X) & \longrightarrow & H^0(E)\otimes H^0(L) & \longrightarrow & H^0(E)\otimes H^0(L|_D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(E) & \longrightarrow & H^0(L\otimes E) & \longrightarrow & H^0(E\otimes L\otimes \mathscr{O}_D) & &
\end{array}$$

The top sequence is exact since X is regular. Observe that the left vertical map is surjective. We claim that the right vertical map is also surjective. Indeed, since $H^1(E\otimes L^*)=0$, the map $H^0(E)\to H^0(E\otimes \mathscr{O}_D)$ surjects. In what follows, the right vertical map is surjective by our hypothesis since it factors through $H^0(E|_D)\otimes H^0(L|_D)\to H^0(E\otimes L\otimes \mathscr{O}_D)$. The conclusion follows from a simple diagram chase. □

Lemma 2.7.9. *[GP99a], Lemma 2.9) Let X be a projective variety, let r be a non-negative integer and let F be a base-point-free line bundle on X. Let Q be an effective line bundle on X and let q be a reduced and irreducible member of $|Q|$. Let R be a line bundle and G a sheaf on X such that*

(1) $H^1(F\otimes Q^*)=0$,

(2) $H^0(M_{F\otimes\mathscr{O}_q}^{\otimes i}\otimes R\otimes \mathscr{O}_q)\otimes H^0(G)\to H^0(M_{F\otimes\mathscr{O}_q}^{\otimes i}\otimes R\otimes G\otimes \mathscr{O}_q)$ *is surjective for all* $0\le i\le r$.

Then for all $0\le i'\le r$ *and for all* $0\le k\le i'$, *the following map*

$$H^0(M_F^{\otimes k}\otimes M_{F\otimes\mathscr{O}_q}^{\otimes i'-k}\otimes R\otimes \mathscr{O}_q)\otimes H^0(G)\longrightarrow H^0(M_F^{\otimes k}\otimes M_{F\otimes\mathscr{O}_q}^{\otimes i'-k}\otimes R\otimes G\otimes \mathscr{O}_q)$$

is surjective.

Proof. We do an induction on i'. Clearly, the statement is true if $i'=0$ since it is just the condition (2) with $r=0$. Assume the statement is true for $i'-1$ and we want to prove it for i'. To this end, we use induction on k. If $k=0$, then it is again the condition (2) and we are done. Assume it is true for $k-1$ and we want to prove it for k. We have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_F \otimes \mathcal{O}_q & \longrightarrow & H^0(F) \otimes \mathcal{O}_q & \longrightarrow & F \otimes \mathcal{O}_q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{F\otimes\mathcal{O}_q} & \longrightarrow & H^0(F \otimes \mathcal{O}_q) \otimes \mathcal{O}_q & \longrightarrow & F \otimes \mathcal{O}_q & \longrightarrow & 0
\end{array}$$

Observe that the second vertical map is surjective by the condition (1) and its kernel is $H^0(F \otimes Q^*) \otimes \mathcal{O}_q$. Consequently, by Snake lemma we obtain the following exact sequence

$$0 \to H^0(F \otimes Q^*) \otimes \mathcal{O}_q \to M_F \otimes \mathcal{O}_q \to M_{F\otimes\mathcal{O}_q} \to 0. \tag{2.6}$$

Tensoring the above exact sequence by $M_F^{\otimes k-1} \otimes M_{F\otimes\mathcal{O}_q}^{\otimes i'-k} \otimes R \otimes \mathcal{O}_q$ and taking global sections, we obtain the following diagram with exact rows (the exactness of the top row can be seen by a diagram chase)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes H^0(G) & \longrightarrow & B \otimes H^0(G) & \longrightarrow & C \otimes H^0(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array} \tag{2.7}$$

where the groups A, B, C, A', B', C' are as follows:

$$\begin{aligned}
A &= H^0(F \otimes Q^*) \otimes H^0(M_F^{\otimes k-1} \otimes M_{F\otimes\mathcal{O}_q}^{\otimes i'-k} \otimes R \otimes \mathcal{O}_q), & B &= H^0(M_F^{\otimes k} \otimes M_{F\otimes\mathcal{O}_q}^{\otimes i'-k} \otimes R \otimes \mathcal{O}_q), \\
A' &= H^0(F \otimes Q^*) \otimes H^0(M_F^{\otimes k-1} \otimes M_{F\otimes\mathcal{O}_q}^{\otimes i'-k} \otimes R \otimes G \otimes \mathcal{O}_q), & C &= H^0(M_F^{\otimes k-1} \otimes M_{F\otimes\mathcal{O}_q}^{\otimes i'-k+1} \otimes R \otimes \mathcal{O}_q), \\
B' &= H^0(M_F^{\otimes k} \otimes M_{F\otimes\mathcal{O}_q}^{\otimes i'-k} \otimes R \otimes G \otimes \mathcal{O}_q), & C' &= H^0(M_F^{\otimes k-1} \otimes M_{F\otimes\mathcal{O}_q}^{\otimes i'-k+1} \otimes R \otimes G \otimes \mathcal{O}_q).
\end{aligned}$$

The left vertical map of (2.7) surjects by the induction hypothesis on i' and the right vertical map surjects by the induction hypothesis on k. Thus, the surjectivity of the middle vertical map follows from a diagram chase. $\square$

Chapter 3

Properties N_0 and N_1 for adjoint linear series on smooth minimal varieties

This chapter is devoted to the proof of Theorem 1.1.1 and Theorem 1.1.2. Throughout Section 3.1 and Section 3.2, we only work with smooth projective varieties. Finally in Section 3.2.3 we will see to what extent our results generalize for varieties with canonical singularities.

3.1 Projective normality for adjoint linear series

We first prove the projective normality results for adjoint linear series $L = K_X + lB$ for an ample and globally generated line bundle B on X, i.e., Theorem 1.1.1. The way we are going to prove this result is as follows: by Theorem 2.6.4 we need to check the surjection of $H^0(kL) \otimes H^0(L) \to H^0(k+1)L$. We first make use of the "divide and rule" method (Observation 2.7.7) to reduce the problem to checking the surjectivity of simpler multiplication maps. Then we we Koszul resolutions (see Definition 2.5.1) of the ideal sheaf of general curve sections of B and $K_X + B$ to further reduce the problem to checking such surjectivity of multiplication maps on curves and we use the results of Section 2.7.

3.1.1 Surjectivity of multiplication maps

First we prove a lemma that shows a surjection of a multiplication map of global sections as we described before. We will require this lemma crucially in the proof of our main theorem.

Lemma 3.1.1. *Let X be a smooth projective variety of dimension n, $n \geq 3$. Let B be an ample* *and base point free line bundle on X. We further assume $h^0(B) \geq n+2$. Let X_n be X, X_{n-j} be a smooth irreducible $(n-j)$-fold chosen from the complete linear system of $|B|_{X_{n-j+1}}|$ (which exists by Bertini) for all $1 \leq j \leq n-1$. Then the following will hold.*

(i) $H^1(K+lB|_{X_{n-j}}) = 0$ *for all* $0 \leq j \leq n-2$, $l \geq n-1$.

(ii) $H^0(K+nB) \otimes H^0(B) \to H^0(K+(n+1)B)$ *surjects.*

Proof. (i) By adjunction, $K_{X_{n-j}} = (K+jB)|_{X_{n-j}}$ for all $0 \leq j \leq n-1$. Thus, by Kodaira Vanishing,

$$H^1(K+lB|_{X_{n-j}}) = H^1(K_{X_{n-j}} + (l-j)B|_{X_{n-j}}) = 0$$

for all $0 \leq j \leq n-2$, $l \geq n-1$.

(ii) Thanks to part (i), $H^0((K+lB)|_{X_{n-j}}) \to H^0((K+lB)|_{X_{n-j-1}})$ surjects for all $l \geq n$, $0 \leq j \leq n-2$. We have the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(L) \otimes H^0(B \otimes \mathscr{I}) & \longrightarrow & H^0(L) \otimes H^0(B) & \longrightarrow & H^0(L) \otimes V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0((L+B) \otimes \mathscr{I}) & \longrightarrow & H^0(L+B) & \longrightarrow & H^0((L+B)|_{X_1}) & \longrightarrow & 0 \end{array} \tag{3.1}$$

Here $L = K+nB$, $\mathscr{I}$ is the ideal sheaf of the curve X_1 in X and V is the cokernel of the map $H^0(B \otimes \mathscr{I}) \to H^0(B)$. The bottom row is exact by part (i) and the top row is exact by the definition of V.

Let W be the vector space corresponding to the curve X_1 on X that appears on the Koszul resolution of $\mathscr{I}$ (see Definition 2.5.1). Tensoring the following exact sequence:

$$0 \longrightarrow \bigwedge^{n-1} W \otimes B^{-(n-1)} \longrightarrow \cdots \longrightarrow \bigwedge^{2} W \otimes B^{-2} \longrightarrow W \otimes B^{-1} \longrightarrow \mathscr{I} \longrightarrow 0 \tag{3.2}$$

by $L+B$, we get the following exact sequence where $L' = L+B$,

$$0 \longrightarrow \bigwedge^{n-1} W \otimes L' \otimes B^{-(n-1)} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} W \otimes L' \otimes B^{-1} \xrightarrow{f_1} L' \otimes \mathscr{I} \longrightarrow 0.$$

To show the left most vertical map in (3.1) surjects, it is enough to prove $H^1(\mathrm{Ker}(f_1)) = 0$ as $W \subseteq H^0(B \otimes \mathscr{I})$. The following two claims prove the vanishing.

<u>*Claim 1*</u>*: $H^r(Ker(f_r)) = 0 \implies H^{r-1}(Ker(f_{r-1})) = 0$ for all $2 \le r \le n-2$.*

Proof: We have the following short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(f_r) \longrightarrow \bigwedge^r W \otimes L' \otimes B^{-r} \xrightarrow{f_r} \mathrm{Ker}(f_{r-1}) \longrightarrow 0.$$

The long exact sequence of cohomology proves the claim as $H^{r-1}(K + (n+1-r)B) = 0$ since $n+1-r > 0$ for r in the given interval. ■

<u>*Claim 2*</u>*: $H^{n-2}(L' - (n-1)B) = 0$.*

Proof: This is obvious from Kodaira Vanishing as $H^{n-2}(L' - (n-1)B) = H^{n-2}(K + 2B) = 0$. ■

Thus, in order to prove the surjectivity of the middle vertical map in (3.1), we only have to prove the surjection of the map $H^0(L|_{X_1}) \otimes V \to H^0((L+B)|_{X_1})$ as $H^0(L|_{X_{n-j}}) \to H^0(L|_{X_{n-j-1}})$ already surjects for all $0 \le j \le n-2$ by part (i). Using Lemma 2.7.1, it is enough to prove the following inequality:

$$h^1((K + (n-1)B)|_{X_1}) \le \dim(V) - 2. \tag{3.3}$$

So, first we have to find an estimate of $\dim(V)$ and we do that by the following claim.

<u>*Claim 3*</u>*: $h^0(B \otimes \mathscr{I}) = \dim(W)$.*

Proof: We tensor the exact sequence (3.2) by B and get the following exact sequence:

$$0 \longrightarrow \bigwedge^{n-1} W \otimes B^{-(n-2)} \xrightarrow{g_{n-1}} \dots \xrightarrow{g_3} \bigwedge^2 W \otimes B^{-1} \xrightarrow{g_2} W \otimes \mathcal{O}_X \xrightarrow{g_1} B \otimes \mathscr{I} \longrightarrow 0$$

So, in order to prove the claim, it is enough to show $H^0(\mathrm{Ker}(g_1)) = 0$ and $H^1(\mathrm{Ker}(g_1)) = 0$.

These two vanishing can be seen from the following four facts whose proofs we omit as they are similar to *Claim 1* and *Claim 2*.

Fact 1: $H^{r-1}(Ker(g_r)) = 0 \implies H^{r-2}(Ker(g_{r-1})) = 0$ for all $2 \le r \le n-2$,

Fact 2: $H^{n-3}(B^{-(n-2)}) = 0$,

Fact 3: $H^r(Ker(g_r)) = 0 \implies H^{r-1}(Ker(g_{r-1})) = 0$ *for all* $2 \le r \le n-2$,

Fact 4: $H^{n-2}(B^{-(n-2)}) = 0$. ■

Therefore, $\dim(V) = h^0(B) - h^0(B \otimes \mathscr{I}) \ge h^0(B) - (n-1)$ as $\dim(W) \le n-1$. Observe that the bundle $(K+(n-1)B)|_{X_1}$ is the canonical bundle of X_1 and consequently $h^1((K+(n-1)B)|_{X_1}) = 1$. Thus, the inequality (3.3) is verified thanks to $h^0(B) \ge n+2$. □

Remark 3.1.2. Since B is ample and base point free, $h^0(B) \ge n+1$. In our theorems, we are assuming that $h^0(B) \ge n+2$. Later we will give an example where $h^0(B) = 4$ and $K+3B$ does not satisfy projective normality on a regular three-fold. ♢

Now we prove another lemma that is equally crucial for the proof of Theorem 1.1.1.

Lemma 3.1.3. *Let X be a smooth projective variety of dimension n, $n \ge 3$. Let B be an ample and base point free line bundle on X. We further assume the following three conditions:*

(a) K is nef, $K+B$ is base point free,

(b) $h^0(K+B) \ge h^0(K) + n + 1$,

(c) $B-K$ is nef and effective divisor.

Let X_n be X, X_{n-j} be sufficiently general smooth irreducible $(n-j)$ fold chosen from the complete linear system of $|(K+B)|_{X_{n-j+1}}|$ for all $1 \le j \le n-1$. Then the following will hold:

(i) $H^1((2n-2)B|_{X_{n-j}}) = 0$ for all $0 \le j \le n-2$.

(ii) $H^0(K+(2n-1)B) \otimes H^0(K+B) \to H^0(2K+2nB)$ surjects.

Proof. (i) Adjunction gives us $K_{X_{n-j}} = ((j+1)K + jB)|_{X_{n-j}}$ for all $0 \le j \le n-1$. We have,

$$H^1((2n-2)B|_{X_{n-j}}) = H^1(K_{X_{n-j}} + (((2n-2j-3)B + (j+1)(B-K))|_{X_{n-j}})).$$

Note that, $2n-2j-3 \ge 1$ for all $0 \le j \le n-2$. Using Kodaira Vanishing Theorem we conclude

$$H^1((2n-2)B|_{X_{n-j}}) = 0 \text{ for all } 0 \le j \le n-2$$

as $B-K$ is nef.

Proof. (ii) Let $\mathscr{I}$ be the ideal sheaf of X_1 in X and consequently we have W as in Definition 2.5.1. We have the following diagram where $L = K+(2n-1)B$ and V is the cokernel of the map $H^0((K+B)\otimes\mathscr{I}) \to H^0(K+B)$:

$$\begin{array}{ccccccccc} 0 & \to & H^0(L)\otimes H^0((K+B)\otimes\mathscr{I}) & \to & H^0(L)\otimes H^0(K+B) & \longrightarrow & H^0(L)\otimes V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0((L+K+B)\otimes\mathscr{I}) & \longrightarrow & H^0((L+K+B)) & \longrightarrow & H^0((L+K+B)|_{X_1}) & \to & 0 \end{array} \tag{3.4}$$

The bottom row is exact by Kodaira Vanishing as $H^1((K+(2n-1)B)|_{X_{n-j}}) = 0$, the top row is exact by our construction. We have the following exact sequence:

$$0 \longrightarrow \bigwedge^{n-1} W\otimes(K+B)^{-(n-1)} \longrightarrow \ldots \longrightarrow W\otimes(K+B)^{-1} \longrightarrow \mathscr{I} \longrightarrow 0.$$

Tensoring by $L+K+B$ and taking cohomology, as in the proof of Lemma 3.1.1, we have the following two vanishings:

(V-1) $H^{r-1}(K+L+B-r(K+B)) = H^{r-1}(K+(2n-2r+1)B+(r-1)(B-K)) = 0$ for all $2\le r\le n-2$ which is obvious by Kodaira Vanishing since we have $B-K$ nef.

(V-2) $H^{n-2}(L+K+B-(n-1)(K+B)) = 0$ which comes from Kodaira Vanishing as well.

The above two vanishings show that the leftmost vertical map in (3.4) is surjective. Note that $H^0(L)\to H^0(L|_{X_1})$ is surjective by part (i). Consequently, by the application of Lemma 2.7.1, we just need the following inequality:

$$h^1((2n-2)B|_{X_1}) \le \dim(V)-2. \tag{3.5}$$

As in the proof of *Claim 3*, Lemma 3.1.1, we can see that $\dim(V)\ge h^0(K+B)-(n-1)$. Still, we have to estimate $h^1((2n-2)B|_{X_1})$. We have the short exact sequence:

$$0 \longrightarrow (-K-B)|_{X_2} \longrightarrow \mathscr{O}_{X_2} \longrightarrow \mathscr{O}_{X_1} \longrightarrow 0.$$

Tensoring this by $(2n-2)B$ and taking the long exact sequence of cohomology, we obtain

$$\dots \longrightarrow H^1((2n-2)B|_{X_2}) \longrightarrow H^1((2n-2)B|_{X_1}) \longrightarrow H^2((-K+(2n-3)B)|_{X_2}) \longrightarrow \dots$$

Since $H^1((2n-2)B|_{X_2}) = 0$ by (i), we get $h^1((2n-2)B|_{X_1}) \le h^2((-K+(2n-3)B)|_{X_2})$. Now, we have, $h^2((-K+(2n-3)B)|_{X_2}) = h^0(((n-1)K+(n-2)B+K-(2n-3)B)|_{X_2}) = h^0(K|_{X_2}-(n-1)(B-K)|_{X_2})$ where the first equality is obtained by duality.

Note that, assumption (c) gives us $h^0(K|_{X_2}-(n-1)(B-K))|_{X_2}) \le h^0(K|_{X_2})$. The long exact sequence associated to the following short exact sequence:

$$0 \longrightarrow (-B)|_{X_{n-j+1}} \longrightarrow K|_{X_{n-j+1}} \longrightarrow K_{X_{n-j}} \longrightarrow 0$$

shows us (by Kodaira Vanishing) that $h^0(K|_{X_{n-j}}) = h^0(K|_{X_{n-j+1}})$ for all $0 \le j \le n-2$. Consequently, $h^0(K|_{X_2}) = h^0(K)$. Thus, to show inequality (3.5) it is enough to show the inequality $h^0(K) \le h^0(K+B)-(n+1)$ which we have, thanks to assumption (b). □

Now we list three remarks that discuss the hypotheses of the lemma we just proved.

Remark 3.1.4. We always have $h^0(K+B) \ge h^0(K)+n$ on any smooth projective n-fold if $K+B$ and B are ample and base point free.

Proof. Note that, $h^0(K+B)-h^0(K)$ is the dimension of the cokernel V of the map $H^0(K) \to H^0(K+B)$ in $H^0((K+B)|_{X_{n-1}})$ where X_{n-1} is a smooth irreducible divisor chosen from the complete linear system of B. Notice that V is the linear subsystem of the complete linear series $|(K+B)|_{X_{n-1}}|$ obtained by pulling back the base point free complete linear series $|K+B|$ on X by the embedding $X_{n-1} \hookrightarrow X$. Consequently $|V|$ is globally generated and $\dim(V) \ge n$ as the morphism induced by $|V|$ is the composite of the embedding i and a finite morphism (given on X by $|K+B|$) and is hence finite on an $n-1$ dimensional variety X_{n-1}. ♢

Remark 3.1.5. Let X be a smooth projective variety of dimension n with nef canonical bundle K. Let B be an ample and base point free line bundle such that $B+K$ is globally generated,

$h^0(B) \geq n+2$ and $H^1(B) = 0$. Then $h^0(K+B) \geq h^0(K)+n+1$.

Proof. The assertion is trivial if $h^0(K) = 0$ or if $K = \mathscr{O}_X$. Otherwise, we have the short exact sequence:

$$0 \longrightarrow B \longrightarrow B+K \longrightarrow (B+K)|_{\mathscr{K}} \longrightarrow 0$$

where $\mathscr{K}$ is a non zero effective divisor chosen from the linear system of K. From the long exact sequence, we get that $h^0(K+B) = h^0(B)+h^0((B+K)|_{\mathscr{K}})$. But $h^0((B+K)|_{\mathscr{K}}) \geq h^0(K|_{\mathscr{K}})$. Thus, $h^0(K+B) = h^0(B)+h^0((B+K)|_{\mathscr{K}}) \geq n+2+h^0(K)-1$. $\diamondsuit$

Remark 3.1.6. Let X be a smooth projective variety of dimension n with nef canonical bundle K and $H^{n-1}(\mathscr{O}_X) = 0$. Let B be an ample and base point free line bundle on X such that $B+K$ is globally generated and $h^0(B) \geq n+2$. Then $h^0(K+B) \geq h^0(K)+n+1$.

Proof. Again, we can assume that K is a non zero effective divisor. The long exact sequence associated to the short exact sequence:

$$0 \longrightarrow K \longrightarrow B+K \longrightarrow (B+K)|_{\mathscr{B}} \longrightarrow 0$$

gives $h^0(K+B) = h^0(K)+h^0((B+K)|_{\mathscr{B}})$ (here $\mathscr{B}$ is a sufficiently general non zero effective divisor chosen from the linear system of B). Now we have $h^0((B+K)|_{\mathscr{B}}) \geq h^0(B|_{\mathscr{B}})$. Hence $h^0(K+B) \geq h^0(K)+h^0(B|_{\mathscr{B}}) \geq h^0(K)+n+1$. $\diamondsuit$

3.1.2 Proof of Theorem 1.1.1

Now we are ready to prove our main theorem and we will make use of the lemmas that we proved in the previous subsection as we promised.

Theorem 3.1.7. *Let X be a smooth projective variety of dimension n, $n \geq 3$. Let B be an ample and base point free line bundle on X. We further assume:*

(a) K is nef, $K+B$ is base point free,

(b) $h^0(B) \geq n+2$,

(c) $h^0(K+B) \geq h^0(K)+n+1$,

(d) $B-K$ *is nef and effective.*

Then $K+lB$ *is very ample and it embeds* X *as a projectively normal variety for all* $l \geq n$.

Proof. We need to prove $H^0((K+lB)^{\otimes k}) \otimes H^0(K+lB) \to H^0((K+lB)^{\otimes k+1})$ surjects $\forall k \geq 1$. We carry out the proof using several steps.

Step 1: $H^0(k(K+lB)+rB) \otimes H^0(B) \to H^0(k(K+lB)+(r+1)B)$ *surjects for* $k \geq 2$, $l \geq n$, $r \geq 0$. This comes from CM lemma (Lemma 2.7.6) once we note that,

$$H^i(k(K+lB)+(r-i)B) = H^i(K+(k-1)K+rB+(kl-i)B) = 0 \text{ for all } 1 \leq i \leq n$$

by Kodaira Vanishing.

Step 2: $H^0(k(K+lB)+(l-1)B) \otimes H^0(K+B) \to H^0((k+1)K+(kl+l)B)$ *surjects for* $k \geq 2$, $l \geq n$. This again comes from CM lemma (Lemma 2.7.6). Indeed,

$$H^i(k(K+lB)+(l-1)B-iK-iB) = H^i(K+kK+(kl+l-1-i)B-(1+i)K).$$

But $kl+l-1-i \geq 3n-1-i \geq 2n-1 > 1+n$ for all $1 \leq i \leq n$. Since assumption (d) shows us that $B-K$ is nef, Kodaira Vanishing gives

$$H^i(k(K+lB)+(l-1)B-iK-iB) = 0 \text{ for all } 1 \leq i \leq n.$$

Step 3: $H^0((K+lB)^{\otimes k}) \otimes H^0(K+lB) \to H^0((k+lB)^{\otimes k+1})$ *surjects for* $k \geq 2$, $l \geq n$, $r \geq 0$. This comes from *Steps 1*, *2* and the Observation 2.7.7.

So, we only need to prove $H^0(K+lB) \otimes H^0(K+lB) \to H^0((K+lB)^{\otimes 2})$ surjects for all $l \geq n$.

Step 4: $H^0(K+lB) \otimes H^0(B) \to H^0(K+(l+1)B)$ *surjects for* $l > n$. This comes from CM lemma (Lemma 2.7.6) once we note that $H^i(K+(l-i)B) = 0$ for all $1 \leq i \leq n$ by Kodaira Vanishing since

$l - i > 0$.

Step 5: $H^0(K+(2l-1)B) \otimes H^0(K+B) \to H^0(2K+2lB)$ surjects for $l > n$. To see this, first note that $H^i(K+(2l-1)B-iK-iB) = H^i(K+(2l-i-1)B-iK)$. Now, $2l-i-1 \geq 2n-i+1 > i$, thanks to $l \geq n+1$ and $1 \leq i \leq n$. Since $B-K$ is nef, Kodaira Vanishing implies $H^i(K+(2l-1)B-iK-iB) = 0$. Hence by CM lemma (Lemma 2.7.6), we are done.

Step 6: $H^0(K+lB) \otimes H^0(K+lB) \to H^0(2K+2lB)$ surjects for $l > n$. This comes from *Steps 4*, *5* and the Observation 2.7.7.

So, we only need to prove $H^0(K+nB) \otimes H^0(K+nB) \to H^0(2K+2nB)$ surjects which is our final step.

Step 7: $H^0(K+nB) \otimes H^0(K+nB) \to H^0(2K+2nB)$ surjects. This is because in Lemma 3.1.1, we have already proved $H^0(K+nB) \otimes H^0(B) \to H^0(K+(n+1)B)$ surjects and in *Step 4* we have showed $H^0(K+lB) \otimes H^0(B) \to H^0(K+(l+1)B)$ surjects for $l > n$. Using Observation 1.2, we only need to show the surjection of $H^0(K+(2n-1)B) \otimes H^0(K+B) \to H^0(2K+2nB)$ which we have proved in Lemma 3.1.3. □

Remark 3.1.8. Let X be a n dimensional smooth projective variety. Let B be a globally generated, ample line bundle on X. We further assume that $B-K$ is a non-zero effective divisor. If $p_g(X) \geq 2$, then $h^0(B) \geq n + p_g$.

Proof. The long exact sequence associated to the short exact sequence:

$$0 \longrightarrow K \longrightarrow B \longrightarrow B|_D \longrightarrow 0$$

shows that the cokernel of the map $H^0(K) \to H^0(B)$ is a base point free linear subsystem of the base point free complete linear system of $H^0(B|_D)$ (D is an element of the linear series of $B-K$). By the same argument used in the proof of Remark 3.4, we have $h^0(B) - p_g \geq n$. ♢

Remark 3.1.8 and Remarks 3.1.5, 3.1.6 allow us to deduce a corollary of Theorem 3.1.7 which we state below.

Corollary 3.1.9. *Let X be a smooth projective variety of dimension $n \geq 3$ with $p_g \geq 2$. Let B be an ample, globally generated line bundle on X. Assume K is nef and $B - K$ is a nef, non-zero, effective divisor. Further assume that $B + K$ is globally generated. If either $H^1(B) = 0$ or $H^{n-1}(\mathcal{O}_X) = 0$ then $K + nB$ will be very ample and it will embed X as a projectively normal variety.*

3.1.3 A discussion on optimality

In this section, we produce examples to discuss the sharpness of our conditions. Both of our examples are cyclic covers and we refer to Section 2.3 for the basics. In our first example, we construct a regular variety of general type and an ample, globally generated line bundle B on it that satisfies all the conditions of Theorem 3.1.7 except the condition (b) and show that the line bundle $K + nB$ does not satisfy the property N_0.

Example 3.1.10. Consider a double cover X of $\mathbb{P}^{n+1}$ branched along an n-fold of degree $2n+4$, $n \geq 3$. Let the natural finite morphism from X to $\mathbb{P}^{n+1}$ be f. The unique line bundle associated to this cover is $\mathcal{O}(n+2)$. We have $f_*(\mathcal{O}_X) = \mathcal{O} \oplus \mathcal{O}(-n-2)$, $H^1(\mathcal{O}_X) = 0$ and $K_X = \mathcal{O}_X$.

Consider $B = f^*\mathcal{O}(1)$. Clearly B is ample and base point free,

$$H^0(B) = H^0(f_*(B)) = H^0(\mathcal{O}(1) \oplus \mathcal{O}(-n-1)) \implies h^0(B) = n+2.$$

Kodaira Vanishing shows that $H^1(rB) = 0$ for all $r \geq 1$.

Let $Y \in |B|$ be smooth irreducible n-fold given by Bertini's Theorem. Consider the line bundle $B|_Y$ on Y. This is again ample and base point free and by adjunction $K_Y = B|_Y$. So Y is a smooth n-fold of general type. Consider the following exact sequence:

$$0 \longrightarrow B^* \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0. \tag{3.6}$$

Taking cohomology and using Kodaira Vanishing, we get $H^1(\mathcal{O}_Y) = 0$. Tensoring the sequence (2.5.1) by B and taking cohomology gives us $h^0(B|_Y) = h^0(B) - 1 = n+1$. Hence $B|_Y$ does not satisfy the condition (b) of Theorem 3.1.7.

We have that $K_Y + B|_Y = 2B|_Y$ and is hence base point free. Clearly $K_Y = B|_Y$ is nef since it is ample. Also $B|_Y - K_Y = \mathcal{O}_Y$ and is hence nef and effective. Now we show that,

$$h^0(K_Y + B|_Y) \geq h^0(K_Y) + n + 1 \text{ i.e. } h^0(2B|_Y) \geq h^0(B|_Y) + n + 1.$$

We have that $H^0(2B) = H^0(f^*\mathcal{O}(2)) = H^0(\mathcal{O}(2) \oplus \mathcal{O}(-n)) \implies h^0(2B) = h^0(\mathcal{O}(2)) = \binom{n+2}{2} + (n+2)$.

Tensoring the exact sequence (3.6) by $2B$ and taking the cohomology shows that

$$h^0(2B|_Y) = h^0(2B) - h^0(B) = \binom{n+2}{2}.$$

Now, $h^0(B|_Y) + n + 1 = 2n + 2$. Since $n \geq 3$ we have that $h^0(K_Y + B|_Y) \geq h^0(K_Y) + n + 1$. We have showed that $B|_Y$ satisfies all the conditions in Theorem 3.1.7 except (b). Now we prove that $K_Y + nB|_Y = (n+1)B|_Y$ does not satisfy property N_0.

We have that $K_Y + nB|_Y = (n+1)B|_Y$. Suppose $(n+1)B|_Y$ satisfies the property N_0. Hence for a curve section $C \in |B_Y|$ we have that $(n+1)B|_C$ is very ample. We also have that $K_C = nB|_C$ and hence $(n+1)B|_C = K_C + B|_C$.

Now $\deg(B|_C) = B^{n+1} = 2H^{n+1} = 2$ where H is a hyperplane section of $\mathbb{P}^{n+1}$ since the map f is $2:1$. But $K_C + E$ cannot be very ample if E is an effective divisor of degree 2. ♠

Now we give an example of a variety and an ample, globally generated line bundle B for which $K + nB$ does not satisfy the property N_0, where $B - K$ is neither nef nor effective although the geometric genus of the variety is large (see Corollary 3.1.9).

Example 3.1.11. Consider X a cyclic double cover of $\mathbb{P}^n$ branched along a hypersurface of degree $2r$. Denote by f the natural morphism from X to $\mathbb{P}^n$. Let $B = f^*(\mathcal{O}(1))$. We have that, $f_*(\mathcal{O}_X) = \mathcal{O} \oplus \mathcal{O}(-r)$, $K_X = f^*(\mathcal{O}(-n-1+r))$, $K_X + B = f^*(\mathcal{O}(-n+r))$, $B - K_X = f^*(\mathcal{O}(n+2-r))$.

We can see that for $r \geq n+3$, $B - K_X$ is not nef. However by making r large enough we can make p_g as large as we wish to and in particular make $p_g \geq 2$. We also have $H^1(B) = 0$. We now show that for $r \geq n+3$, $K_X + nB$ is not projectively normal. Indeed,

$$K_X + nB = f^*(\mathcal{O}(r-1)) \implies H^0(K+nB) = H^0(\mathcal{O}(r-1) \oplus \mathcal{O}(-1)) = H^0(\mathcal{O}(r-1)).$$

Now $H^0(2K_X + 2nB) = H^0(f^*(\mathcal{O}(2r-2))) = H^0(\mathcal{O}(2r-2)) \oplus H^0(\mathcal{O}(r-2))$. If $r \geq 2$ we can clearly see that $K + nB$ is not projectively normal. Hence we can see that the condition $B - K_X$ nef and effective is essential in Corollary 3.1.9. ♠

3.2 Normal presentation for adjoint linear series

In this section, we will prove the normal presentation result for adjoint linear series. We will follow the same procedure described at the beginning of Section 3.1.

3.2.1 Surjectivity of multiplication maps

We first verify the surjectivity of the multiplication maps of global sections in the following three lemmas that will arise after we perform the "divide and rule".

Lemma 3.2.1. *Let X be a regular smooth projective variety of dimension n, $n \geq 3$. Let B be an ample and base point free line bundle on X. We further assume $h^0(B) \geq n+2$. Let X_n be X and let X_{n-j} be a smooth irreducible $(n-j)$-fold chosen from the complete linear system $|B|_{X_{n-j+1}}|$ (which exists by Bertini) for all $1 \leq j \leq n-1$. Then the map*

$$H^0((K+nB)|_{X_{n-j}}) \otimes H^0(B|_{X_{n-j}}) \to H^0((K+(n+1)B)|_{X_{n-j}})$$

surjects for $0 \leq j \leq n-1$.

Proof. To start with, notice that X_{n-j} is regular for all j. Indeed, it can easily be seen by taking cohomology of the exact sequence

$$0 \longrightarrow -B|_{X_{n-j+1}} \longrightarrow \mathcal{O}_{X_{n-j+1}} \longrightarrow \mathcal{O}_{X_{n-j}} \longrightarrow 0.$$

Because of the vanishing Lemma 3.1.1 (i), by the repeated application of Lemma 2.7.8, it is enough to prove $H^0((K+nB)|_{X_1}) \otimes H^0(B|_{X_1}) \to H^0((K+(n+1)B)|_{X_1})$ surjects. To show this using Lemma 2.7.1, we have to prove the inequality $h^1((K+(n-1)B)|_{X_1}) \leq h^0(B|_{X_1}) - 2$ which follows directly from our assumption that $h^0(B) \geq n+2$. □

Lemma 3.2.2. *Let X be a regular smooth projective n-fold, $n \geq 3$. Let B be an ample and base point free line bundle on X. We further assume:*

(a) K is nef, $K+B$ is base point free,

(b) $h^0(K+B) \geq h^0(K) + n + 1$,

(c) $(n-2)B - (n-1)K$ is nef and effective.

Let X_n be X and let X_{n-j} be a sufficiently general smooth irreducible $(n-j)$-fold chosen from the complete linear system $|(K+B)|_{X_{n-j+1}}|$ for all $1 \leq j \leq n-1$. Then the following will hold:

(i) $H^1((2n-3)B|_{X_{n-j}}) = 0$ for all $0 \leq j \leq n-2$,

(ii) $H^0((K+(2n-2)B)|_{X_{n-j}}) \otimes H^0((K+B)|_{X_{n-j}}) \to H^0((2K+(2n-1)B)|_{X_{n-j}})$ surjects for all $0 \leq j \leq n-1$.

Proof. (i) Adjunction gives us $K_{X_{n-j}} = ((j+1)K + jB)|_{X_{n-j}}$ for all $0 \leq j \leq n-1$. We have

$$H^1((2n-3)B|_{X_{n-j}}) = H^1(K_{X_{n-j}} + (B + (2n-4-j)B - (j+1)K)|_{X_{n-j}}).$$

Note that,

$$B - \frac{j+1}{2n-4-j}K = B - \frac{n-1}{n-2}K + \frac{n-1}{n-2}K - \frac{j+1}{2n-4-j}K.$$

We have $n-1 \geq j+1$ and $n-2 \leq 2n-4-j$ for all $0 \leq j \leq n-2$. Consequently, $(2n-4-j)B-(j+1)K$ is nef as K and $B - \dfrac{n-1}{n-2}K$ are nef. Using Kodaira Vanishing we conclude $H^1((2n-3)B|_{X_{n-j}}) = 0$ for all $0 \leq j \leq n-2$.

(ii) As in the proof of Lemma 3.2.1, X_{n-j} is regular forr all j. Repeated application of Lemma 2.7.8 shows that it is enough to prove the lemma for $j = n-1$. Hence, we have to prove the surjection of

$$H^0((K+(2n-2)B)|_{X_1}) \otimes H^0((K+B)|_{X_1}) \to H^0((2K+(2n-1)B)|_{X_1}).$$

Application of Lemma 2.7.1 shows us it is enough to check the following inequality:

$$h^1((2n-3)B|_{X_1}) \leq h^0((K+B)|_{X_1}) - 2. \tag{3.7}$$

We have the short exact sequence:

$$0 \longrightarrow (-K-B)|_{X_2} \longrightarrow \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_{X_1} \longrightarrow 0.$$

Tensoring this by $(2n-3)B$ and taking the long exact sequence, we obtain the following.

$$\ldots \longrightarrow H^1((2n-3)B|_{X_2}) \longrightarrow H^1((2n-3)B|_{X_1}) \longrightarrow H^2((-K+(2n-4)B)|_{X_2}) \longrightarrow \ldots$$

Since $H^1((2n-3)B|_{X_2}) = 0$ by (i), we get $h^1((2n-3)B|_{X_1}) \leq h^2((-K+(2n-4)B)|_{X_2})$. We have,

$$\begin{aligned} h^2((-K+(2n-4)B)|_{X_2}) &= h^0(((n-1)K+(n-2)B+K-(2n-4)B)|_{X_2}) \\ &= h^0((K+(n-1)K-(n-2)B)|_{X_2}). \end{aligned}$$

Note that, assumption (c) gives us $h^0((K+(n-1)K-(n-2)B)|_{X_2}) \leq h^0(K|_{X_2})$. The long exact sequence associated to the following short exact sequence

$$0 \longrightarrow (-B)|_{X_{n-j+1}} \longrightarrow K|_{X_{n-j+1}} \longrightarrow K|_{X_{n-j}} \longrightarrow 0$$

shows us (by Kodaira Vanishing) that $h^0(K|_{X_{n-j}}) = h^0(K|_{X_{n-j+1}})$ for all $0 \leq j \leq n-2$. Consequently we get that $h^0(K|_{X_2}) = h^0(K)$. Thus, in order to show (3.7) it is enough to show the inequality $h^0(K) \le h^0((K+B)|_{X_1})-2$ which comes from assumption (b). Indeed, tensoring the above exact sequence by $B|_{X_{n-j+1}}$ and taking cohomology (recall that X_{n-j+1} is regular), one sees easily that $h^0((K+B)|_{X_{n-j}}) = h^0((K+B)|_{X_{n-j+1}})-1$. □

Lemma 3.2.3. *Let X be a regular smooth projective n-fold, $n \ge 3$. Let B be an ample and base point free line bundle on X. We further assume:*

(a) K is nef, $K+B$ is base point free,

(b) $h^0(B) \ge n+2$,

(c) $h^0(K+B) \ge h^0(K)+n+1$,

(d) $(n-2)B-(n-1)K$ is nef and non-zero effective divisor.

Let X_n be X and let X_{n-j} be a sufficiently general smooth irreducible $(n-j)$ fold chosen from the complete linear system of $|B|_{X_{n-j+1}}|$ for all $1 \le j \le n-1$. Let L be $K+lB$ where $l \ge n$. Then

$$H^0(M_{L|_{X_{n-j}}} \otimes L|_{X_{n-j}}) \otimes H^0(B|_{X_{n-j}}) \longrightarrow H^0(M_{L|_{X_{n-j}}} \otimes L|_{X_{n-j}} \otimes B|_{X_{n-j}})$$

surjects for all $0 \le j \le n-1$.

Proof. For simplicity, we only prove the assertion for $l = n$ that is for $L = K+nB$. The proof for $l > n$ is similar. We prove the case for $L = K+nB$ by induction on j. Before starting the induction, we first prove the following claim.

Claim: In the context of our theorem we have $B^n \ge 4$.

Proof of the Claim: Let $h^0(B) = r+1$. Let f be the finite morphism induced by the ample and base point free line bundle B. We have that $B^n = \deg(f)\cdot\deg(Y)$ where Y is the scheme theoretic image. Now, the codimension of Y in $\mathbb{P}^r \ge 1$ and hence $\deg(Y) \ge 2$. So the only way $B^n < 4$ is when the following happens.

• *Case 1:* $\deg(f) = 1$ and $\deg(Y) = 2$ and hence $\operatorname{codim}(Y) = 1$. In this case we have that Y is a variety of minimal degree and it is either a smooth quadric hypersurface or a cone over a smooth rational normal scroll or a cone over the Veronese embedding of $\mathbb{P}^2$ (see [EH85]). In all three cases Y is normal. Indeed, the first case is trivial. The second and third case follows from the fact that a cone over a projectively normal embedding is normal. Now f is a finite birational map between normal varieties and is hence an isomorphism. Consequently, the image is a smooth rational normal scroll whose canonical divisor is negative ample. This contradicts the hypothesis (a).

• *Case 2:* $\deg(f) = 1$ and $\deg(Y) = 3$ and $\operatorname{codim}(Y) = 2$. In this case again Y is a variety of minimal degree and hence a normal variety and we have that f is an isomorphism which leads to a contradiction as before.

• *Case 3:* $\deg(f) = 1$ and $\deg(Y) = 3$ and $\operatorname{codim}(Y) = 1$. In this case consider a general curve section C of $|B|$ in X. It is the pullback of a general curve section D of $\mathscr{O}(1)$ in Y. By Bertini we have that C can be taken to be smooth and irreducible and since f is surjective we have that D is reduced and irreducible. Notice that D is a plane curve since the codimension of Y was 1. Moreover, D is a plane curve of degree 3 and hence we have that $p_a(D) = 1$. C is the normalization of D and hence $g(C) \le 1$. But we have that $2g(C) - 2 = (n-1)B^n + B^{n-1}K_C$ and hence $g(C) \ge 4$ since $B^n = 3$, $n \ge 3$ and K_C is nef. So we have a contradiction. ■

Now we start our induction on j. We aim to show the following.

<u>Induction Step</u>: *Suppose* $H^0(M_{L|_{X_{n-j}}} \otimes L|_{X_{n-j}}) \otimes H^0(B|_{X_{n-j}}) \to H^0(M_{L|_{X_{n-j}}} \otimes L|_{X_{n-j}} \otimes B|_{X_{n-j}})$ *surjects for some* $1 \le j \le n-1$. *Then*

$$H^0(M_{L|_{X_{n-j+1}}} \otimes L|_{X_{n-j+1}}) \otimes H^0(B|_{X_{n-j+1}}) \to H^0(M_{L|_{X_{n-j+1}}} \otimes L|_{X_{n-j+1}} \otimes B|_{X_{n-j+1}})$$

surjects.

Proof of Induction Step: First we prove

$$H^1(M_{L|_{X_{n-j+1}}} \otimes L|_{X_{n-j+1}} \otimes (B|_{X_{n-j+1}})^*) = 0 \text{ for } 1 \le j \le n-1. \tag{3.8}$$

We have the short exact sequence:

$$0 \longrightarrow M_{L|_{X_{n-j+1}}} \otimes L'|_{X_{n-j+1}} \longrightarrow H^0(L|_{X_{n-j+1}}) \otimes L'|_{X_{n-j+1}} \longrightarrow (L+L')|_{X_{n-j+1}} \longrightarrow 0$$

where $L' = K + (n-1)B$. In order to prove (3.8), it is enough to prove

$$H^0(L|_{X_{n-j+1}}) \otimes H^0(L'|_{X_{n-j+1}}) \to H^0((L+L')|_{X_{n-j+1}})$$

surjects since according to Lemma 3.1.1 (i), $H^1(L'|_{X_{n-j+1}}) = 0$.

To prove this surjection with the help of Observation 2.7.7, we need to prove the following:

$$H^0((K+lB)|_{X_{n-j+1}}) \otimes H^0(B|_{X_{n-j+1}}) \to H^0((L+(l+1)B)|_{X_{n-j+1}}) \text{ surjects for all } l > n. \tag{3.9}$$

$$H^0((K+nB)|_{X_{n-j+1}}) \otimes H^0(B|_{X_{n-j+1}}) \to H^0((L+(n+1)B)|_{X_{n-j+1}}) \text{ surjects.} \tag{3.10}$$

$$H^0((K+(2n-2)B)|_{X_{n-j+1}}) \otimes H^0((K+B)|_{X_{n-j+1}}) \to H^0((2K+(2n-1)B)|_{X_{n-j+1}}) \text{ surjects.} \tag{3.11}$$

We use CM Lemma (Lemma 2.7.6) to prove (3.9). Recall that $K_{X_{n-j+1}} = (K+(j-1)B)|_{X_{n-j+1}}$. Hence by Kodaira Vanishing,

$$H^i((K+(l-i)B)|_{X_{n-j+1}}) = H^i((K+(j-1)B)|_{X_{n-j+1}} + ((l-i-j+1)B)|_{X_{n-j+1}}) = 0$$

for all $1 \le i \le n-j+1$.

We have already proved (3.10) in Lemma 3.2.1.

For simplicity we do some re-indexing to prove (3.11) only. We will show that

$$H^0((K+(2n-2)B)|_{X_{n-j}}) \otimes H^0((K+B)|_{X_{n-j}}) \to H^0((2K+(2n-1)B)|_{X_{n-j}})$$

surjects for all $0 \le j \le n-2$. We have already proved the surjection when $j = 0$ in Lemma 3.2.2. So, we assume $1 \le j \le n-2$. Our obvious choice is to use the CM Lemma (Lemma 2.7.6).

For all $1 \le i \le n-j-1$, the following holds:

$$H^i(((1-i)K+(2n-2-i)B)|_{X_{n-j}}) = H^i((K+jB+(2n-2-2i-j)B+i(B-K))|_{X_{n-j}}) = 0$$

as $B-K$ is nef and $2n-2-2i-j>0$ for i in the given range.

Notice that, $H^{n-j}(((n+j-2)B-(n-j-1)K)|_{X_{n-j}}) = H^0(((n-1)K-(n-2)B-(j-1)K)|_{X_{n-j}})$. Now, $((n-1)K-(n-2)B-(j-1)K)|_{X_{n-j}}$ is negative nef and $(n-1)K-(n-2)B$ is negative of a non-zero effective divisor and consequently $H^{n-j}(((n+j-2)B-(n-j-1)K)|_{X_{n-j}}) = 0$.

Since we have proved (3.8), to finish the proof of *Induction Step* using Lemma 1.3, it is enough to prove the following map

$$H^0(M_{L|_{X_{n-j+1}}} \otimes L|_{X_{n-j+1}} \otimes \mathcal{O}_{X_{n-j}}) \otimes H^0(B|_{X_{n-j+1}} \otimes \mathcal{O}_{X_{n-j}}) \to H^0(M_{L|_{X_{n-j+1}}} \otimes L|_{X_{n-j+1}} \otimes B|_{X_{n-j+1}} \otimes \mathcal{O}_{X_{n-j}})$$

surjects for all $1 \le j \le n-1$. Now we use the vector bundle technique (Lemma 2.7.9) by taking $F = L|_{X_{n-j+1}}$, $R = L|_{X_{n-j+1}}$, $Q = \mathcal{O}_{X_{n-j+1}}(B|_{X_{n-j+1}})$, $r=1$, $G = B|_{X_{n-j}}$. We need to show the following:

$$H^1(F \otimes Q^*) = 0; \text{ this comes from Lemma 3.1.1 (i).} \tag{3.12}$$

$$H^0(M_{L|_{X_{n-j}}} \otimes L|_{X_{n-j}}) \otimes H^0(B|_{X_{n-j}}) \to H^0(M_{L|_{X_{n-j}}} \otimes L|_{X_{n-j}} \otimes B|_{X_{n-j}}) \text{ surjects; it is our hypothesis.} \tag{3.13}$$

$$H^0(L|_{X_{n-j}}) \otimes H^0(B|_{X_{n-j}}) \to H^0((L+B)|_{X_{n-j}}) \text{ surjects; this comes from Lemma 3.1.} \tag{3.14}$$

That concludes the proof of the *Induction Step*. Now we have to prove the base case.

<u>Base Case</u>: We have to prove $H^0(M_{L|_{X_1}} \otimes L|_{X_1}) \otimes H^0(B|_{X_1}) \to H^0(M_{L|_{X_1}} \otimes L|_{X_1} \otimes B|_{X_1})$ *surjects.*

Proof of Base Case: Notice, $\deg(L|_{X_1}) = (K+nB)\cdot B^{n-1}$, We have,

$$2g-2 = (B|_{X_2})^2 + (B|_{X_2})\cdot K_{X_2} = B^n + (K+(n-2)B)\cdot B^{n-1} \text{ where } g = p_g(X_1).$$

$$\implies \deg(L|_{X_1}) > 2g, \text{ thanks to } B^n > 2 \implies M_{L|_{X_1}} \text{ is semistable and } \mu(M_{L|_{X_1}}) > -2 \text{ (see [But94]).}$$

We will use Proposition 1.4 to prove the required surjection of *Base Case*. We need to check:

$$\mu(M_{L|_{X_1}} \otimes L|_{X_1}) > 2g. \tag{3.15}$$

$$\mu(M_{L|_{X_1}} \otimes L|_{X_1}) > 4g - \deg(B|_{X_1}) - 2h^1(B|_{X_1}). \tag{3.16}$$

To prove (3.15), we have to show $(K+nB)\cdot B^{n-1} - 2 \geq B^n + B^{n-1}\cdot(K+(n-2)B)+2$ which follows since $B^n \geq 4$.

Showing (3.16) is equivalent to proving $2h^1(B|_{X_1}) \geq (n-3)B^n + B^{n-1}\cdot K + 6$. Riemann-Roch gives the following,

$$2h^1(B|_{X_1}) = 2h^0(B|_{X_1}) + (n-3)B^n + B^{n-1}\cdot K.$$

That finishes the proof since $h^0(B|_{X_1}) \geq 3$, thanks to $h^0(B) \geq n+2$. □

3.2.2 Proof of Theorem 1.1.2

Now we are ready to prove our main theorem regarding normal presentation of adjoint bundles on regular minimal varieties.

Theorem 3.2.4. *Let X be a regular smooth projective n-fold, $n \geq 3$. Let B be an ample and base point free line bundle on X. We further assume the following four conditions:*

(a) *K is nef, $K+B$ is base point free,*

(b) *$h^0(B) \geq n+2$,*

(c) *$h^0(K+B) \geq h^0(K)+n+1$,*

(d) *$(n-2)B-(n-1)K$ is nef and non-zero effective divisor.*

Then $K+lB$ will satisfy the property N_1 for $l \geq n$.

Proof. We prove the assertion only for $l=n$, the case $l>n$ is similar. Let $L = K+nB$. Since we already know that $H^1(M_L \otimes L) = 0$ which comes from the projective normality of L, we only have to prove for all $k \geq 1$,

$$H^1(M_L^{\otimes 2} \otimes L^{\otimes k}) = 0. \tag{3.17}$$

We omit the proof when $k \geq 2$ which follows easily from CM Lemma (Lemma 2.7.6). Here we only prove the key case $k = 1$ that is $H^1(M_L^{\otimes 2} \otimes L) = 0$. We have the short exact sequence:

$$0 \longrightarrow M_L^{\otimes 2} \otimes L \longrightarrow H^0(L) \otimes M_L \otimes L \longrightarrow M_L \otimes L^{\otimes 2} \longrightarrow 0.$$

It is enough to prove that $H^0(L) \otimes H^0(M_L \otimes L) \to H^0(M_L \otimes L^{\otimes 2})$ surjects as $H^1(M_L \otimes L) = 0$. We use Observation 2.7.7; it is enough to prove the following:

$$H^0(M_L \otimes L) \otimes H^0(B) \to H^0(M_L \otimes L \otimes B) \text{ surjects.} \tag{3.18}$$

$$H^0(M_L \otimes L) \otimes H^0(lB) \to H^0(M_L \otimes L \otimes lB) \text{ surjects for all } l \geq 2. \tag{3.19}$$

$$H^0(M_L \otimes (K + (2n-1)B) \otimes H^0(K + B) \to H^0(M_L \otimes (2K + 2nB)) \text{ surjects.} \tag{3.20}$$

We have proved (3.18) in Lemma 3.2.3.

In order to prove (3.19), we again use Observation 2.7.7. Therefore it is enough to prove that $H^0(M_L \otimes (K + lB)) \otimes H^0(B) \to H^0(M_L \otimes (K + (l+1)B))$ surjects for $l > n$. To prove this our obvious choice is to use CM lemma (Lemma 2.7.6). First, we want to show that $H^1(M_L \otimes (K + (l-1)B)) = 0$ which is equivalent to showing the surjection of the following map:

$$H^0(L) \otimes H^0(K + (l-1)B) \to H^0(L + K + (l-1)B). \tag{3.21}$$

If $l = n + 1$ then this has already been proved in Theorem 3.1.7, Step 7. If $l > n + 1$ then in order to show the surjection of (3.21), it is enough to prove

$$H^0(2K + 2nB + rB) \otimes H^0(B) \to H^0(2K + 2nB + (r+1)B)$$

surjects for all $r \geq 0$. This is *Step* 1 in Theorem 3.1.7 with $k = 2$. Now we will show that, for all $2 \leq i \leq n$, $H^i(M_L \otimes (K + (l-i)B)) = 0$. We have the short exact sequence:

$$0 \longrightarrow M_L \otimes (K + (l-i)B) \longrightarrow H^0(L) \otimes (K + (l-i)B) \longrightarrow 2K + (l+n-i)B \longrightarrow 0.$$

It gives us the long exact sequence:

$$\ldots \to H^{i-1}(2K + (l+n-i)B) \to H^i(M_L \otimes (K + (l-i)B)) \to H^0(L) \otimes H^i(K + (l-i)B) \to \ldots$$

Since the first and the last terms are zero by Kodaira Vanishing, hence $H^i(M_L \otimes (K + (l-i)B)) = 0$ for all $2 \leq i \leq n$.

We are left to prove (3.20). Again we are going to use CM Lemma (Lemma 2.7.6). We have to prove the following three things:

$$H^1(M_L \otimes (2n-2)B) = 0. \tag{3.22}$$

$$H^j(M_L \otimes ((2n-1-j)B - (j-1)K)) = 0 \text{ for all } 2 \leq j \leq n-1. \tag{3.23}$$

$$H^n(M_L \otimes ((n-1)B - (n-1)K)) = 0. \tag{3.24}$$

We observe that (3.22) is equivalent to showing $H^0(L) \otimes H^0((2n-2)B) \to H^0(L + (2n-2)B)$ surjects. Using Observation 2.7.7, this is equivalent to showing $H^0(K + lB) \otimes H^0(B) \to H^0(K + (l+1)B)$ surjects for all $l \geq n$. This follows from Lemma 3.1.1 and Theorem 3.1.7, *Step* 4.

To prove (3.23), we write down the short exact sequence:

$$0 \to M_L \otimes (a_jB - b_jK) \to H^0(L) \otimes (a_jB - b_jK) \to L \otimes (a_jB - b_jK) \to 0 \tag{3.25}$$

where $a_j = 2n-1-j$, $b_j = j-1$. The long exact sequence corresponding to it is:

$$\ldots \to H^{j-1}(L \otimes (a_jB - b_jK)) \to H^j(M_L \otimes (a_jB - b_jK)) \to H^0(L) \otimes H^j(a_jB - b_jK) \to \ldots$$

Now $H^{j-1}(L \otimes (a_j B - b_j K)) = H^{j-1}(K + (3n-2j)B + (j-1)(B-K)) = 0$ as $3n-2j > 0$ for all $j < n$ and $B - K$ is nef. Also, $H^j(a_j B - b_j K) = H^j(K + (2n-2j-1)B + j(B-K)) = 0$ as $2n-2j-1 > 0$ for all $j < n$ and $B - K$ is nef.

We are left to prove (3.24) only. The long exact sequence associated to (3.25) corresponding to $j = n$ gives the required vanishing for the following reasons:

$$H^{n-1}((2n-1)B - (n-2)K) = H^{n-1}(K + nB + (n-1)(B-K)) = 0.$$

$$H^n((n-1)(B-K)) = H^0(nK - (n-1)B) = H^0((n-1)K - (n-2)B + K - B) = 0.$$

The last equality comes from the fact that $(n-1)K - (n-2)B$ is negative effective and $K - B$ is negative nef. That concludes the proof. □

Now we prove a weaker result for the normal presentation of the adjunction bundle associated to an ample, globally generated line bundle on an irregular variety of dimension n. Here we have to use Koszul resolution to restrict ourselves to the multiplication map on the curve section as the variety is not regular. We include only a sketch of the proof as it is very similar to what we have done thus far.

Theorem 3.2.5. *Let X be an irregular smooth projective variety of dimension n, $n \geq 3$. Let B be an ample and base point free line bundle on X. We further assume:*

(a) K is nef, B' and $K + B'$ is base point free whenever $B \equiv B'$,

(b) $h^0(B) \geq n+2$,

(c) $h^0(K+B) \geq h^0(K') + n + 1$ whenever $K \equiv K'$,

(d) $(n-2)B - (n-1)K$ is nef and non-zero effective divisor.

Then $K + lB$ will satisfy the property N_1 for $l \geq n$.

Proof. As before, we just give the sketch for $L = K + nB$. We have to prove that $H^1(M_L^{\otimes 2} \otimes L^{\otimes k}) = 0$. Again, we just discuss the case when $k = 1$. It is enough to prove the surjection of the following multiplication map of global sections,

$$H^0(M_L \otimes L) \otimes H^0(L) \to H^0(M_L \otimes L^{\otimes 2}).$$

Let E be a torsion line bundle in $\mathrm{Pic}^0(X)$ which is not n torsion. Such an E exists as $\mathrm{Pic}^0(X)$ is an abelian variety when X is irregular. Note that $B+E$ is globally generated by assumption (a). Observation 2.7.7 tells us it is enough to check the following three maps surject:

$$H^0(M_L \otimes (K+nB)) \otimes H^0(B+E) \to H^0(M_L \otimes ((n+1)B+E)). \tag{3.26}$$

$$H^0(M_L \otimes (K+rB+E)) \otimes H^0(B) \to H^0(M_L \otimes ((r+1)B+E)) \text{ for } n+1 \le r \le 2n-2. \tag{3.27}$$

$$H^0(M_L \otimes (K+(2n-1)B+E)) \otimes H^0(K+B-E) \to H^0(M_L \otimes (2K+2nB)). \tag{3.28}$$

To show (3.26) surjects, we use CM Lemma (Lemma 2.7.6). We have to prove the following,

$$H^i(M_L \otimes (K+nB-iB-iE) = 0 \text{ for all } 1 \le i \le n.$$

When $2 \le i \le n-1$ this follows easily by multiplying the exact sequence $(*)$ by suitable line bundle and then taking the cohomology.

When $i = n$, same computation shows the vanishing once we see that $H^0(nE) = 0$.

To prove the vanishing for $i = 1$, we need to show the surjection of the following map:

$$H^0(L) \otimes H^0(K+(n-1)B-E) \to H^0(2K+(2n-1)B-E).$$

By Observation 2.7.7, Lemma 3.1.1 and Theorem 3.1.7, *Step* 4, it is enough to prove the surjection of $H^0(K+(2n-2)B) \otimes H^0(K+B-E) \to H^0(2K+(2n-1)B-E)$. Now, $K+B-E$ is base point free by our assumption. Let C be a curve section of $K+B-E$. Using Koszul resolution

(Definition 2.5.1) and Lemma 2.7.1, it is enough to check,

$$h^1(((2n-3)B+E)|_C) \leq h^0(K+B)-(n+1).$$

Now, $h^1(((2n-3)B+E)|_C) = h^0((nK-(n-2)B-nE)|_C) = h^0(nK-(n-2)B-nE) \leq h^0(K-nE)$ thanks to assumption (d). So, the inequality follows thanks to assumption (c).

(3.27) and (3.28) follows from CM Lemma (Lemma 2.7.6) as well. □

Remark 3.2.6. We always have $h^0(K+B) \geq h^0(K')+n$ on any smooth projective n-fold if $K+B$, $K'+B$, B are ample, base point free and $K \equiv K'$.

Proof. We have $K' = K+\delta$ where δ is a numerically trivial line bundle. By Riemann-Roch, we obtain that $h^0(K+B) = h^0(K+B+\delta)$. The assertion follows from an argument similar to the proof of Remark 3.1.4. ♢

3.3 Properties N_0 and N_1 on varieties with canonical singularities

In this section we recall some relevant results for canonical varieties, we refer to [Rei79] and [Rei87] for details. For a normal projective variety X, the canonical divisor K_X is a Weil divisor and not necessarily Cartier (it is so if X is smooth for example). We set

$$\omega_X^{[r]} := \mathcal{O}_X(rK_X)$$

and this is a divisorial sheaf. First we define Gorenstein varieties.

Definition 3.3.1. Let X be a normal projective variety. If X is Cohen-Macaulay and K_X is Cartier, then X is called *Gorenstein*.

Now we define what it means to say that X has canonical singularities.

Definition 3.3.2. Let X be a normal projective variety as above. Then X is said to have *canonical singularities* if the following two conditions hold:

(1) $\omega_X^{[r]}$ is locally free for some $r \geq 1$,

(2) for some resolution $f : Y \to X$ and r as in (1), $f_*\omega_Y^{\otimes r} = \omega_X^{[r]}$.

The smallest integer r for which (1) holds is called the *canonical index* of X. If moreover $\omega_X^{[r]}$ is ample then X is called *canonical variety*.

There is a more general class of singular varieties and they are called varieties with Du-Bois singularity. It follows from the works of Kollár and Kovács, it follows that (log-) canonical singularities are Du-Bois (see [KK10]). Now we are ready to make the following

Remark 3.3.3. Theorems 3.1.7, 3.2.4, 3.2.5 and 4.2.1 go through for X Gorenstein with Du-Bois singularities. Indeed, we required smooth hyperplane sections of the ample and base point free line bundle B. The smoothness is used to justify Kodaira Vanishing (both on the general member of $|B|$ and on X) and to apply Green's result (Lemma 2.7.1) on the smooth curve section.

We observe that if X is Gorenstein with Du-Bois singularities and B is Cartier, the general member of $|B|$ is Cohen-Macaulay as well. Also since X is nonsingular in codimension 1 and $|B|$ is base point free, a general member of $|B|$ is smooth outside the singular locus of X (by Bertini's theorem) and is hence nonsingular in codimension 1. The above two observations show that the general member is normal. Now the general hyperplane section of $|B|$ also has Du-Bois singularities (see [Kol13], Proposition 6.20). We also have (see [Kol13], Theorem 10.42) that for a projective, Cohen-Macaulay variety with Du-Bois singularities, Kodaira Vanishing theorem holds for an ample line bundle. Now the complete intersection surface that we get is a normal surface and hence singularities are isolated. So Bertini's Theorem gives us a smooth curve section and we can apply Lemma 2.7.1. $\diamondsuit$

Chapter 4

Properties N_0 and N_1 for pluricanonical linear series on canonical varieties

The goal of this section is to prove effective projective normality and normal presentation results for pluricanonical series on smooth canonical varieties in dimension ≤ 5. Since we are interested in working with only ample canonical bundles instead of ample and globally generated canonical bundles, it is important for us to know effective global generation results for adjoint bundles. Precisely for this reason, the posidive evidences of a conjecture of Fujita that we will describe below are of great interest for us.

4.1 Fujita's conjecture for adjoint bundles

We state the conjecture of Fujita regarding the positivity of adjoint bundles formulated in 1985.

Conjecture 4.1.1. *Let X be a smooth projective variety of dimension n with canonical bundle K_X and let L be an ample line bundle on X.*

(1) (Fujita Freeness Conjecture) $K_X + mL$ is globally generated for $m \geq n+1$.

(2) (Fujita Very Ampleness Conjecture) $K_X + mL$ is very ample if $m \geq n+2$.

Observe that for smooth projective curves both Fujita freeness and Fujita very ampleness follow thanks to Example 2.1.1 and Example 2.1.3. Our main interest lies in the Fujita freeness and we review a few results in that direction. After the case for curves, the next situation to consider is that for surfaces. In this case both parts of the conjecture follows from the work of Reider that we are going to describe next.

4.1.1 Reider's theorems for surfaces

We state the theorem of Reider for surfaces.

Theorem 4.1.2. *([Rei88], Theorem 1.1) Let X be a smooth projective surface with canonical bundle K_X and let L be a nef line bundle on X.*

(a) If $L^2 \geq 5$ and $p \in X$ is a base point of $|K_X + L|$ then there is an effective divisor E passing through p for which one of the following happens:

(1) $L \cdot E = 0$ and $E^2 = -1$;

(2) $L \cdot E = 1$ and $E^2 = 0$.

(b) If $L^2 \geq 10$ and $p, q \in X$ that are not separated by $|K_X + L|$ then there is an effective divisor E passing through p and q for which one of the following happens:

(1) $L \cdot E = 0$ and $E^2 = -1$ or -2;

(2) $L \cdot E = 1$ and $E^2 = -1$ or 0;

(3) $L \cdot E = 2$ and $E^2 = 0$.

4.1.2 Fujita freeness for higher dimensional varieties

In this section we provide the results in the direction of Fujita freeness in higher dimensions. The conjecture is proven for three-fold by Ein and Lazarsfeld ([EL93a], in dimension three and four by Kawamata ([Kaw97]), and in dimension five by Ye and Zhu ([YZ20]). The precise statements are provided below.

Theorem 4.1.3. *(See [Kaw97], Theorem 3.1) Let X be a normal projective variety of dimension 3, L an ample Cartier divisor, and $x_0 \in X$ a smooth point. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions:*

(1) $\sqrt[p]{L^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0,

(2) $\sigma_1 \geq 3$, $\sigma_2 \geq 3$ *and* $\sigma_3 > 3$.

Then $|K_X + L|$ *is free at* x_0.

Corollary 4.1.4. *(See [Kaw97], Corollary 3.2) Let X be a smooth projective variety of dimension 3, and H an ample divisor. Then* $|K_X + mH|$ *is base point free if* $m \geq 4$. *Moreover, if* $H^3 \geq 2$, *then* $|K_X + 3H|$ *is also base point free.*

Theorem 4.1.5. *(See [Kaw97], Corollary 4.2 and [YZ20], Main Theorem) Let X be a smooth projective variety of dimension 4 (resp. 5), and H an ample divisor. Then* $|K_X + mH|$ *is base point free if* $m \geq 5$ *(resp.* $m \geq 6$*).*

We remark here that an effective freeness for adjoint bundles with quadratic bounds has been proven by Anghern and Siu ([AS95]).

4.2 Proof of Theorem 1.2.1

In this section, we will concentrate on the behavior of pluricanonical series. First, we will prove a theorem whose corollaries will give us effective results on three and four-folds. We follow all the conventions and notation described in the first paragraph of Chapter 3. In particular, by a variety we mean a smooth projective variety.

Theorem 4.2.1. *Let X be an n dimensional smooth projective variety and let B be an ample, globally generated line bundle on X. Let L be a nef line bundle on X. Moreover, assume:*

(a) $(n-1)(B-L) - K$ *is ample,*

(b) $B - K$ *is ample,*

(c) $B + L$ *is globally generated,*

(d) $h^0(K - L) \leq h^0(B) - (n+1)$.

Then $nB + L$ *will be very ample and it will embed X as a projectively normal variety.*

Proof. Let X_n be X and let X_{n-j} be a smooth irreducible $(n-j)$ fold chosen from the complete linear system of $|B|_{X_{n-j+1}}|$ by Bertini, for all $1 \le j \le n-1$. By adjunction, $K_{X_{n-j}} = (K + jB)|_{X_{n-j}}$ for all $0 \le j \le n-1$. We have to prove $H^0(k(nB+L)) \otimes H^0(nB+L) \to H^0((k+1)(nB+L))$ surjects. Here we show the key case that is the case when $k=1$. The proof for $k \ge 2$ is similar. We break the proof into a few steps.

Step 1: $H^0(nB+L)\otimes H^0(B) \to H^0((n+1)B+L)$ surjects. We have the following diagram where $\mathscr{I}$ is the ideal sheaf of X_1 in X, V is the cokernel of $H^0(B\otimes\mathscr{I}) \to H^0(B)$:

$$\begin{tikzcd} 0 \arrow[r] & H^0(nB+L)\otimes H^0(B\otimes\mathscr{I}) \arrow[r] \arrow[d] & H^0(nB+L)\otimes H^0(B) \arrow[r] \arrow[d] & H^0(nB+L)\otimes V \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(((n+1)B+L)\otimes\mathscr{I}) \arrow[r] & H^0((n+1)B+L) \arrow[r] & H^0(((n+1)B+L)|_{X_1}) \arrow[r] & 0 \end{tikzcd} \tag{4.1}$$

Note that $H^0((rB+L)|_{X_{n-j}}) \to H^0((rB+L)|_{X_{n-j-1}})$ surjects for all $0 \le j \le n-2$, $r \ge n$ because of the vanishing $H^1(((r-1)B+L)|_{X_{n-j}}) = 0$. Therefore the bottom horizontal sequence is exact. Note that the top row is exact as well. The leftmost vertical map is surjective. Indeed, tensoring the following exact sequence (recall that W is the span of $n-1$ general sections of B)

$$0 \to \bigwedge^{n-1} W \otimes B^{-(n-1)} \to \cdots \to \bigwedge^{2} W\otimes B^{-2} \to W\otimes B^{-1} \to \mathscr{I} \to 0 \tag{4.2}$$

by $(n+1)B+L$, we get the following exact sequence where $L' = (n+1)B+L$

$$0 \xrightarrow{f_{n-2}} \bigwedge^{n-1} W\otimes L'\otimes B^{-(n-1)} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} W\otimes L'\otimes B^{-1} \xrightarrow{f_1} L'\otimes\mathscr{I} \longrightarrow 0.$$

Therefore, to see that the leftmost vertical map surjects, we just need $H^1(\mathrm{Ker}(f_1)) = 0$. The proof of this vanishing is similar to that of Lemma 3.1.1. It comes from the following two facts:

Fact 1: $H^j(Ker(f_j)) = 0 \implies H^{j-1}(Ker(f_{j-1})) = 0$ for all $2 \le j \le n-2$ (proof similar to Claim 1, Lemma 3.1.1).

Fact 2: $H^{n-2}(L'-(n-1)B) = H^{n-2}(2B+L) = 0$ (using $B-K$ is ample and Kodaira Vanishing).

Therefore, to prove the assertion of this step, it is enough to show the surjection of the rightmost vertical map. We use Lemma 1.6 to prove that, we need the following inequality:

$$h^1((n-1)B+L)|_{X_1}) \leq h^0(B)-(n+1).$$

But $h^1((n-1)B+L)|_{X_1}) = h^0((K-L)|_{X_1}) = h^0(K-L)$ (the last equality is due to the ampleness of $B+L-K$) which proves the assertion of this step because of our assumption (d).

Step 2: $H^0(rB+L)\otimes H^0(B) \to H^0((r+1)B+L)$ *surjects for all* $r \geq n+1$. This comes from CM Lemma (Lemma 2.7.6).

Step 3: $H^0((2n-1)B+L)\otimes H^0(B+L) \to H^0(2nB+2L)$ *surjects*. This comes from CM Lemma (Lemma 2.7.6) as well thanks to assumption (a). □

In the following corollary, we assume Fujita freeness, at least for pluricanonical series.

Corollary 4.2.2. *Let X be an n dimensional smooth projective variety, $n \geq 3$, with ample canonical bundle K. We further assume that lK is globally generated for all $l \geq n+2$. Then the following statements will hold.*

(i) *If* $h^0(K) \leq h^0((n+2)K)-(n+1)$ *then* $n(n+2)K$ *is very ample and it embeds* X *as a projectively normal variety.*

(ii) *If* $h^0((n+2)K) > n+1$ *then* $(n(n+2)+1)K$ *is very ample and it embeds* X *as a projectively normal variety.*

(iii) $(n(n+2)+m)K$ *is very ample and it embeds* X *as a projectively normal variety for all* $m \geq 2$.

Proof of (i), (ii). Follows directly from Theorem 4.2.1 with $B = (n+2)K$ and $L = 0, K$ respectively.

Proof of (iii). Let $s = n+2$. The proof is entirely based on CM Lemma (Lemma 2.7.6). We give an outline here. We divide the proof into a few steps.

Step 1: $H^0((ns+m)K)\otimes H^0(sK) \to H^0(((n+1)s+m)K)$ *surjects for all* $m \geq 2$. This comes from CM Lemma (Lemma 2.7.6).

Step 2: $H^0(((2n-1)s+m)K)\otimes H^0((s+m)K) \to H^0((2ns+2m)K)$ *surjects for all* $m \geq 2$. To prove this, first notice that if $m \geq s$, then $m = as+b$ where $a \geq 1$ and $b < s$. In that case, by Observation

2.7.7, it is enough to show $H^0(((2ns+m+(a-1)s)K)\otimes H^0((s+b)K)\to H^0((2ns+2m)K)$ surjects for all $m\geq 2$ which comes from CM Lemma (Lemma 1.7). If $m<s$, we can directly use CM Lemma (Lemma 2.7.6).

The above two steps show that $H^0((ns+m)K)\otimes H^0((ns+m)K)\to H^0((2ns+2m)K)$ surjects. Similar calculation shows $H^0(k(ns+m)K)\otimes H^0((ns+m)K)\to H^0((k+1)(ns+m)K)$ surjects for all $k\geq 2$. □

Now we combine our results with the base point freeness theorems on three and four-folds (see [EL93a] and [Kaw97]). In particular, we will use Theorems 4.1.3, 4.1.5 and Corollary 4.1.4. For the statement of the Riemann-Roch formula, we refer to [Har77], Appendix A.

Corollary 4.2.3. *Let X be a smooth projective three-fold with ample canonical bundle K. Then nK is very ample and embeds X as a projectively normal variety for all $n\geq 12$.*

Proof. We have by Riemann-Roch that $\chi(D)+\chi(-D)=\frac{-K\cdot D^2}{2}+2\chi(\mathcal{O}_X)$. Hence, $K\cdot D^2$ is even for any divisor D. In particular K^3 is even. By Corollary 4.1.4 we have that $4K$ is base point free.

By CM Lemma (Lemma 2.7.6), the corollary is obvious for $n\geq 14$ since K is ample and we have Kodaira Vanishing. For $n=13$ we use Theorem 4.2.1 with $L=K$. Conditions (a), (b), (c) are easily satisfied. We need to check that $h^0(4K)\geq 5$. We note that by Riemann-Roch (see the formula given in [Har77], Appendix A, Exercise 6.7) and Remark 2.1.8 we have $h^0(4K)\geq 6+h^0(2K)$ and hence we are done.

For the case $n=12$ we again apply Theorem 4.2.1 but now with $L=0$. Here we need to check the fact that $h^0(4K)\geq h^0(K)+4$. If $h^0(K)=0$ then we are done trivially since $4K$ is ample and base point free. If not then we know that K is effective and hence $h^0(K)\leq h^0(2K)$. The required inequality comes from the inequality $h^0(4K)\geq 6+h^0(2K)$. □

Corollary 4.2.4. *Let X be a smooth projective three-fold with ample canonical bundle K. Then we have that the embedding by nK for $n\geq 13$ is normally presented.*

Proof. Suppose that $L=nK$. We note that the cases $n=3l+1$ with $l\geq 4$ normal presentation of nK directly follows from Riemann-Roch and Theorem 3.2.4 for regular threefolds and 3.2.5

for irregular threefolds using $B = lK$ respectively. While using Theorem 3.2.5 we need to check the conditions. We only check the conditions (a) and (c) below. The other conditions follow directly from the Riemann-Roch formula ([Har77], Appendix A, Exercise 6.7) once we note that $K \cdot c_2(X) \geq 0$.

First we show that condition (a) holds. We have $B = lK$, $l \geq 4$. Suppose $B' \equiv B$, then we have that $B' - K$ is ample and $(B' - K)^3 > 27$ (Using $K^3 \geq 2$) and $(B' - K) \cdot C \geq 3$ and $(B' - K)^2 \cdot S \geq 9$ for any curve C and surface S respectively. Hence B' is base point free by Theorem 4.1.3. Similar reasoning will show that $K + B'$ is base point free as well.

Now show that condition (c) holds. Since $K + B$ is ample and base point free, if $h^0(K') = 0$ we are done. Otherwise K' is effective and $h^0(K') \leq h^0(2K')$. Note that $h^0(4K) \leq h^0((l+1)K)$. But since all higher cohomology of $2K'$ vanishes by Kodaira Vanishing we have by Riemann-Roch that $h^0(2K')$ depends only on the numerical class of K' and hence $h^0(2K) = h^0(2K')$. So it is enough to show that $h^0(4K) - h^0(2K) \geq 4$ which we have shown in the proof of Corollary 4.2.3 (in fact ≥ 6).

For other cases, it is enough to show that $H^1(M_L^{\otimes 2} \otimes L^{\otimes k}) = 0$ since we have already shown projective normality for nK for $n \geq 13$. We only show the case $k = 1$ since for $k \geq 2$ the proof follows from CM Lemma (Lemma 2.7.6). We have the following exact sequence

$$0 \longrightarrow M_L^{\otimes 2} \otimes L \longrightarrow H^0(L) \otimes M_L \otimes L \longrightarrow M_L \otimes L^{\otimes 2} \longrightarrow 0.$$

Taking cohomology we have the following

$$\ldots \longrightarrow H^0(L) \otimes H^0(M_L \otimes L) \longrightarrow H^0(M_L \otimes L^{\otimes 2}) \longrightarrow H^1(M_L^{\otimes 2} \otimes L) \longrightarrow \ldots$$

It is enough to show that $H^0(L) \otimes H^0(M_L \otimes L) \to H^0(M_L \otimes L^{\otimes 2})$ is surjective. Now $4K$ is base point free. We first show that $H^0(M_L \otimes L) \otimes H^0(4K) \to H^0(M_L \otimes L + 4K)$ is surjective. To do this it is enough to show (by Lemma 2.7.6) the following three vanishings:

(i) $H^1(M_L \otimes L - 4K) = 0$,

(ii) $H^2(M_L \otimes L - 8K) = 0$,

(iii) $H^3(M_L \otimes L - 12K) = 0$.

Now $L = nK$ with $n \geq 14$ (the case when $L = 13K$ has already been taken care of). By tensoring the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

by $L-8K$ and $L-12K$ respectively and using Kodaira Vanishing theorem we can see that (ii) and (iii) follow immediately. Now we note that to show (i) we need to to show that the following map

$$H^0(L) \otimes H^0(L-4K) \to H^0(2L-4K)$$

is surjective. We now note that $L-4K = mK$ where $m \geq 10$. Using Observation 2.7.7 we can keep on showing surjectivity of multiplication maps by $H^0(4K)$ until we are left with lK where $0 \leq l \leq 7$. Hence $H^0(L) \otimes H^0(L-4K) \to H^0(2L-4K)$ is surjective for $L = nK$ and $n \geq 19$. We need to check separately from $14 \leq n \leq 18$.

Case n=14. We need to show the surjectivity of $H^0(14K) \otimes H^0(10K) \to H^0(24K)$. We have by Lemma 2.7.6 the surjectivity of $H^0(14K) \otimes H^0(4K) \to H^0(18K)$. We have the surjectivity of $H^0(18K) \otimes H^0(6K) \to H^0(24K)$ using *Step 1*, Theorem 4.2.1 with $B = 6K$ and $L = 0$.

Case n=15. We need to show the surjectivity of $H^0(15K) \otimes H^0(11K) \to H^0(26K)$. We have that $H^0(15K) \otimes H^0(5K) \to H^0(20K)$ surjects by Theorem 4.2.1 with $B = 5K$ and $L = 0$. We also have the surjectivity of $H^0(20K) \otimes H^0(6K) \to H^0(26K)$ by Lemma 2.7.6.

Case n=16. Obvious.

Case n=17. We need to show the surjectivity of $H^0(17K) \otimes H^0(13K) \to H^0(30K)$. This case is easy and follows from Lemma 2.7.6.

Case n=18. We need to show the surjectivity of $H^0(18K) \otimes H^0(14K) \to H^0(32K)$. This case follows directly from Lemma 2.7.6.

The algorithmic nature of the proof shows that we have actually proved the surjectivity of the map $H^0(M_L \otimes (L+4lK)) \otimes H^0(4K) \to H^0(M_L \otimes (L+4(l+1)K))$. Since $L = nK$ where $n \geq 14$, to complete the proof we just need to prove the surjection of the multiplication map

$$H^0(M_L \otimes (L+4lK)) \otimes H^0(pK) \to H^0(M_L \otimes (L+(4l+p)K))$$

where $l \geq 2$ and $p \leq 7$. Moreover if $n \geq 16$ we have that $l \geq 3$. So for $n \geq 16$, using Lemma 2.7.6 we see that it is enough to prove the surjection of $H^0(L+mK) \otimes H^0(L) \to H^0(2L+mK)$ where $m \geq 5$. But we have the surjection of $H^0(L) \otimes H^0(L) \to H^0(2L)$. Thus, using Observation 2.7.7, we only need to prove the surjection of $H^0(lK) \otimes H^0(mK) \to H^0((m+l)K)$ where $l \geq 32$. Since $4K$ is base point free, we have the above surjection by Lemma 2.7.6 and Observation 2.7.7.

To finish the proof we need to handle the two following cases separately.

L=14K: We need to show the surjection of $H^0(M_L \otimes (L+8K)) \otimes H^0(6K) \to H^0(M_L \otimes (L+14K))$. By lemma 2.7.6 we notice that it is enough to show the surjection of $H^0(16K) \otimes H^0(14K) \to H^0(30K)$ which is clear by the same lemma.

L=15K: We need to show the surjection of $H^0(M_L \otimes (L+8K)) \otimes H^0(7K) \to H^0(M_L \otimes (L+15K))$. By Lemma 2.7.6, we notice that it is enough to show that $H^0(16K) \otimes H^0(15K) \to H^0(31K)$ surjects which is again clear by the same lemma. □

Corollary 4.2.5. *Let X be a smooth projective four dimensional variety with ample canonical bundle K. Then nK is very ample and it will embed X as a projectively normal variety for all $n \geq 24$.*

Proof. It comes from Corollary 4.2.2. The following is the Riemann-Roch formula a line bundle B,

$$\chi(B) = -\frac{1}{720}(K^4 - 4K^2 \cdot c_2 - 3c_2^2 + K \cdot c_3 + c_4) - \frac{1}{24}B \cdot K \cdot c_2 + \frac{1}{24}B^2 \cdot (K^2 + c_2) - \frac{1}{12}B^3 \cdot K + \frac{1}{24}B^4.$$

It is enough to show that $h^0(2K) \leq h^0(6K) - 5$ which can be seen easily, thanks to the fact $K^2 \cdot c_2 \geq 0$ (see Remark 2.1.8). In fact, $h^0(2K) \leq h^0(6K) - 6$ which verifies condition (ii). □

Corollary 4.2.6. *Let X be a smooth projective 4-fold with ample canonical bundle K we have that the embedding by nK for $n \geq 25$ is normally presented.*

Proof. We use the same argument as in Corollary 4.2.5, but now using the fact that $6K$ is globally generated (see Theorem 4.1.5). □

Corollary 4.2.7. *Let X be a smooth projective 5-fold with ample canonical bundle K with an additional property that $p_g(X) \geq 1$. Then the embedding by nK for $n \geq 35$ is projectively normal and the embedding by nK for $n \geq 36$ is normally presented.*

Proof. We know that nK is globally generated for $n \geq 7$ (see Theorem 4.1.5). Let $\mathscr{K}$ be a smooth divisor chosen from the linear system of $|K|$. The corollary follows if we notice the fact that $h^0(7K) - h^0(6K) = h^0((7K)|_{\mathscr{K}})$ and apply Riemann-Roch formula on $\mathscr{K}$ to verify conditions (i) and (ii) of Corollary 4.2.2. Normal Presentation follows from the similar arguments used before.

□

4.3 Properties N_0 and N_1 for pluicanonical series on canonical threefolds

In this section, we prove a corollary for pluricanonical series on canonical threefolds (in the sense of Definition 3.3.2)

Corollary 4.3.1. *If X is a projective three-fold with canonical Gorenstein singularities and ample canonical bundle K then we have that nK is projectively normal for $n \geq 12$ and satisfies N_1 if $n \geq 13$.*

Proof. Follows from Remark 3.3.3 since $4K$ is base point free (see [Lee00], Theorem 3.2). □

Chapter 5

Effective projective normality for certain varieties with $K_X = 0$

The main aim of this chapter is to prove results on effective very ampleness and projective normality on a four dimensional variety with trivial canonical bundle.

5.1 Projective normality for regular four-folds

One can obtain effective projective normality results in these cases just using Theorem 3.1.7 and Theorem 3.2.4 (see Lemma 5.1.3). In order to improve the bounds, one can use a result of Green as we are going to describe next.

5.1.1 A result of Green and its consequence

Theorem 5.1.1. *([Gre84a], p.1089, (3)) Let X be a regular projective variety of dimension n for which the canonical bundle K is ample and base point free. Moreover, assume $h^0(K) \geq n+2$. Let φ_K be the morphism induced by the complete linear series $|K|$. If $\varphi_K(X)$ is not a variety of minimal degree then the multiplication map $H^0(K) \otimes H^0(K^{\otimes l}) \to H^0(K^{\otimes l+1})$ surjects for $l \geq n$.*

The following corollary is the precise version of what we will use in the subsequent sections.

Corollary 5.1.2. *Let X be a regular variety of dimension $n \geq 3$ with trivial canonical bundle. Let L be an ample and globally generated line bundle on X and φ_L be the morphism induced by the complete linear series $|L|$. Assume φ_L does not map X onto a variety of minimal degree. Then $L^{\otimes n-1}$ is projectively normal.*

Proof. We use Theorem 2.6.4 and Observation 2.7.7 to notice that to prove the projective normality of $L^{\otimes n-1}$, it is enough to show the surjectivity of the map $H^0(lL)\otimes H^0(L)\longrightarrow H^0((l+1)L)$ for $l \geq n-1$. We just prove the case for $l = n-1$ here, the rest follow similarly.

Choose a smooth section T of the ample and base point free line bundle L. We have the following commutative diagram,

$$\begin{array}{ccccccccc}
0 & \to & H^0((n-1)L)\otimes H^0(\mathscr{O}_X) & \to & H^0((n-1)L)\otimes H^0(L) & \to & H^0((n-1)L)\otimes H^0(K_T) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0((n-1)L) & \longrightarrow & H^0(nL) & \longrightarrow & H^0(nK_T) & \longrightarrow & 0
\end{array}$$

where $K_T = L|_T$ (by adjunction) denotes the canonical bundle of T. Since the leftmost map surjects, the middle vertical map surjects if and only if rightmost vertical map surjects. By Kodaira Vanishing we have that $H^1((n-2)L) = 0$ and hence we have a surjection,

$$H^0((n-1)L) \to H^0((n-1)L|_T) = H^0((n-1)K_T).$$

Hence the rightmost vertical map surjects if and only if we have the surjection of

$$H^0((n-1)K_T)\otimes H^0(K_T) \longrightarrow H^0(nK_T).$$

We notice that T is a smooth irreducible regular variety of general type and hence by Theorem 2.7.1, the above map surjects unless T is mapped to a variety of minimal degree by the complete linear series of $L|_T$. But the latter is equivalent to saying that X is mapped by the complete linear series of $|L|$ to a variety of minimal degree. $\square$

5.1.2 Proof of Theorem 1.3.2

We start with a general statement on projective normality and normal presentation.

Lemma 5.1.3. *Let X be a smooth, projective n-fold with trivial canonical bundle. Let B be an* *ample and base point free line bundle on X with $h^0(B) \geq n+2$. Then lB satisfies the property N_0 for all $l \geq n$. Moreover, if X is Calabi–Yau, then lB satisfies the property N_1 for all $l \geq n$.*

Proof. Follows immediately from Theorem 3.1.7 and Theorem 3.2.4. ☐

Now we want to find out what multiple of an ample line bundle is very ample on a four dimensional variety with trivial canonical bundle. We will use the Fujita freeness on four-folds that has been proved by Kawamata in [Kaw97]. We begin with a lemma.

Lemma 5.1.4. *Let X be a smooth projective four-fold with trivial canonical bundle. Let A be an ample line bundle and let $B = nA$ for $n \geq 5$. Then the multiplication map $H^0(3B+kA) \otimes H^0(B) \to H^0(4B+kA)$ is surjective for $k \geq 1$.*

Proof. Note that B is base point free by Kawamata's proof of Fujita's base point freeness theorem on four-folds (see Theorem 4.1.5). We prove the statement for $k = 1$. For $k > 1$ the proof is exactly the same.

Let C be a smooth and irreducible curve section of the linear system $|B|$ and let $\mathscr{I}$ be the ideal sheaf of C in X. We have the following commutative diagram with the two horizontal rows exact. Here V is the cokernel of the map $H^0(B \otimes \mathscr{I}) \to H^0(B)$.

$$\begin{tikzcd}
0 \arrow[r] & H^0(B \otimes \mathscr{I}) \otimes H^0(3B+A) \arrow[r] \arrow[d] & H^0(B) \otimes H^0(3B+A) \arrow[r] \arrow[d] & V \otimes H^0(3B+A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0((4B+A) \otimes \mathscr{I}) \arrow[r] & H^0((4B+A)) \arrow[r] & H^0(4B+A|_C) \arrow[r] & 0
\end{tikzcd}$$

Now we claim that the leftmost vertical map is surjective. Consider the Koszul resolution,

$$0 \to \bigwedge^3 W \otimes B^{-3} \to \bigwedge^2 W \otimes B^{-2} \to W \otimes B^* \to \mathscr{I} \to 0. \tag{5.1}$$

Tensor it with $4B + A$ to get the following,

$$0 \to \bigwedge^3 W \otimes (B+A) \xrightarrow{f_3} \bigwedge^2 W \otimes (2B+A) \xrightarrow{f_2} W \otimes (3B+A) \xrightarrow{f_1} (4B+A) \otimes \mathscr{I} \to 0 \tag{5.2}$$

That gives us two short exact sequences.

$$0 \to \operatorname{Ker}(f_1) \to W \otimes (3B + A) \xrightarrow{f_1} (4B + A) \otimes \mathscr{I} \to 0, \tag{5.3}$$

$$0 \to \bigwedge^3 W \otimes (B + A) \xrightarrow{f_3} \bigwedge^2 W \otimes (2B + A) \xrightarrow{f_2} \operatorname{Ker}(f_1) \to 0. \tag{5.4}$$

Taking long exact sequence of cohomology in the second sequence we get the following,

$$\bigwedge^2 W \otimes H^1(2B + A) \to H^1(\operatorname{Ker}(f_1)) \to \bigwedge^3 W \otimes H^2(B + A). \tag{5.5}$$

Hence $H^1(\operatorname{Ker}(f_1)) = 0$ since the other terms of the exact sequence vanish by Kodaira Vanishing. The long exact sequence of cohomology associated to the first sequence is the following,

$$W \otimes H^0(3B + A) \to H^0((4B + A) \otimes \mathscr{I})) \to H^1(\operatorname{Ker}(f_1)). \tag{5.6}$$

We showed that the last term is zero, thus $W \otimes H^0(3B + A) \to H^0((4B + A) \otimes \mathscr{I}))$ surjects. Consequently, $H^0(B \otimes \mathscr{I}) \otimes H^0(3B + A) \to H^0((4B + A) \otimes \mathscr{I})$ surjects since $W \subseteq H^0(B \otimes \mathscr{I})$.

In order to prove the lemma we are left to show that $V \otimes H^0(3B + A) \to H^0((4B + A)|_C)$ surjects. Since we have the surjection of $H^0(3B + A) \to H^0((3B + A)|_C)$, it is enough to show the surjection of $V \otimes H^0(3B + A|_C) \to H^0((4B + A)|_C)$. Thus, using Lemma 2.7.1 we need to prove the following inequality

$$h^1(2B + A|_C) \leq \dim V - 2.$$

To prove this inequality, first we tensor the Koszul resolution by B to obtain,

$$0 \to \bigwedge^3 W \otimes B^{-2} \xrightarrow{f_3} \bigwedge^2 W \otimes B^* \xrightarrow{f_2} W \otimes \mathscr{O}_X \xrightarrow{f_1} B \otimes \mathscr{I} \to 0. \tag{5.7}$$

As before, we end up getting two short exact sequences,

$$0 \to \mathrm{Ker}(f_1) \to W \otimes \mathscr{O}_X \xrightarrow{f_1} B \otimes \mathscr{I} \to 0 \tag{5.8}$$

$$0 \to \bigwedge^3 W \otimes B^{-2} \xrightarrow{f_3} \bigwedge^2 W \otimes B^* \xrightarrow{f_2} \mathrm{Ker}(f_1) \to 0 \tag{5.9}$$

The long exact sequence of cohomology associated to the second sequence gives,

$$\bigwedge^2 W \otimes H^0(B^*) \to H^0(\mathrm{Ker}(f_1)) \to \bigwedge^3 W \otimes H^1(B^{-2}). \tag{5.10}$$

Consequently, $H^0(\mathrm{Ker}(f_1)) = 0$ since $H^1(B^{-2}) = 0$ by Kodaira Vanishing and $H^0(B^*) = 0$.

Taking cohomology once more we have the following exact sequence,

$$\bigwedge^2 W \otimes H^1(B^*) \to H^1(\mathrm{Ker}(f_1)) \to \bigwedge^3 W \otimes H^2(B^{-2}). \tag{5.11}$$

Hence $H^1(\mathrm{Ker}(f_1)) = 0$ since the other terms of the exact sequence vanish by Kodaira Vanishing. The long exact sequence of cohomology associated to the first sequence is the following.

$$H^0(\mathrm{Ker}(f_1)) \to W \otimes H^0(\mathscr{O}_X) \to H^0(B \otimes \mathscr{I})) \to H^1(\mathrm{Ker}(f_1)) \tag{5.12}$$

But the first and last terms are zero by Kodaira Vanishing and hence $h^0(B \otimes \mathscr{I}) = \dim W \leq 3$. Thus we obtain the inequality $\dim V - 2 \geq h^0(B) - 5$.

On the other hand the canonical bundle of C is given by $3B|_C$. Applying Serre Duality it is enough to prove that $h^0(B - A) \leq h^0(B) - 5$ i.e. $h^0((n-1)A) \leq h^0(nA) - 5$.

Applying Riemann–Roch for nA and $(n-1)A$ and subtracting the equations we obtain,

$$h^0(nA) - h^0((n-1)A) = \frac{n^4 - (n-1)^4}{24} A^4 + \frac{n^2 - (n-1)^2}{24} A^2 \cdot c_2. \tag{5.13}$$

The result of Miyaoka (see Remark 2.1.8) shows that $A^2 \cdot c_2 \geq 0$ which gives,

$$h^0(nA) - h^0((n-1)A) \geq \frac{n^4 - (n-1)^4}{24} A^4 \geq 5 \text{ if } n \geq 5 \tag{5.14}$$

and that concludes the proof. $\square$

Now we give a classification theorem in which we classify the varieties which come as an image of a regular four-fold with trivial canonical bundle by an ample, globally generated line bundle with an additional property of being a variety of minimal degree.

Theorem 5.1.5. *Let X be a regular smooth projective four-fold with trivial canonical bundle. Let φ be the morphism induced by the complete linear series of an ample and base point free line bundle B on X with $h^0(B) = r+1$ and let d be the degree of φ. If φ maps X to a variety of minimal degree Y then,*

$$d \leq \frac{24(r-1)}{r-3}.$$

(a) Assume Y is smooth. Then one of the following happens:

(1) $Y = \mathbb{P}^4$.

(2) Y is a smooth quadric hypersurface in $\mathbb{P}^5$.

(3) Y is a smooth rational normal scroll of dimension 4 in $\mathbb{P}^6$ or $\mathbb{P}^7$ and X is fibered over $\mathbb{P}^1$. Moreover, the general fibre is a smooth threefold G with $K_G = 0$ and the degree d satisfies the following bounds;

$$2 \leq d \leq min\left\{6h^0(B|_G), \frac{24(r-1)}{r-3}\right\}.$$

If in addition G is regular we have the following;

$$2h^0(B|_G) - 6 \leq d \leq min\left\{6(h^0(B|_G)-1), \frac{24(r-1)}{r-3}\right\}, \text{ if } d \text{ is even and}$$
$$2h^0(B|_G) - 5 \leq d \leq min\left\{6(h^0(B|_G)-1), \frac{24(r-1)}{r-3}\right\}, \text{ if } d \text{ is odd.}$$

(4) Y is a smooth rational normal scroll in $\mathbb{P}^r$ for $r \geq 8$ and X is fibered over $\mathbb{P}^1$ and the general fibre is a three-fold G with $K_G = 0$ and the degree d of φ_B satisfies $2 \leq d \leq 18$.

(b) Assume Y is singular. Then one of the following happens:

(1) Y is a singular quadric hypersurface.

(2) Y is a singular four-fold which is either a triple cone over a rational normal curve in $\mathbb{P}^r$ where $6 \leq r \leq 8$ or a double cone over the Veronese surface in $\mathbb{P}^5$.

Proof. We first prove the inequality. Using Riemann–Roch we can see that,

$$h^0(B) = \frac{1}{24}B^4 + \frac{1}{24}B^2 \cdot c_2 + 2. \tag{5.15}$$

and we also have that $B^4 = d(r-3)$ since Y is a variety of minimal degree. By Miyaoka's result (see Remark 2.1.8) we have that $B^2 \cdot c_2 \geq 0$ and hence we have the inequality $d \leq \frac{24(r-1)}{r-3}$.

(a) We now describe the cases when Y is a smooth variety of minimal degree. We have that $r \geq 4$.

Case 1. If $r = 4$, we have that $Y = \mathbb{P}^4$.

Case 2. If $r = 5$, we have that codimension of Y is one and degree is 2 which implies that Y is a smooth quadric hypersurface.

Suppose $r \geq 6$, we have that Y is a smooth rational normal scroll (which is abstractly a projective bundle over $\mathbb{P}^1$) and is hence fibered over $\mathbb{P}^1$. Let this map from Y to $\mathbb{P}^1$ be ϕ. Composing this with φ we get a map $\phi \circ \varphi : X \to \mathbb{P}^1$. Therefore, X is fibered over $\mathbb{P}^1$. The general fibre is the inverse image of the general linear fiber of the smooth scroll and is hence irreducible by Bertini's theorem. This along with generic smoothness implies that the general fibre of ϕ is a smooth threefold G with $K_G = 0$ by adjunction. Let the general fibre of Y be denoted by R and that of X is denoted by G. We have the following exact sequence of cohomology of line bundles on X.

$$0 \to H^0(B(-G)) \to H^0(B) \to H^0(B \otimes \mathscr{O}_G) \to H^1(B(-G)). \tag{5.16}$$

Notice that $H - R$ is a base point free divisor in Y where H is a hyperplane section in Y. We have that Y is $S(a_1, a_2, a_3, a_4)$ i.e, Y is the image of $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is the following vector bundle,

$$\mathbb{P}(\mathscr{E}) = \mathscr{O}(a_1) \oplus \mathscr{O}(a_2) \oplus \mathscr{O}(a_3) \oplus \mathscr{O}(a_4)$$

mapped to the projective space by $|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)|$.

<u>*Case 3.*</u> For the cases $r = 6$ or $r = 7$ we use the fact that degree of φ is equal to the degree of $\varphi|_G$ and then use Riemann–Roch theorem on the threefold G (see [Har77], Appendix A, Exercise 6.7) noticing the fact that $K_G = 0$ and that $B|_G$ is ample and base point free. This gives the upper bound $6h^0(B|_G)$ since we have that $B|_G \cdot c_2 \geq 0$ (see Remark 2.1.8). The lower bound 2 is due to the fact that G cannot be birational to $\mathbb{P}^3$.

Assuming G is regular and hence Calabi–Yau we have that $h^0(B|_G) \geq \frac{1}{6}(B|_G)^3 + 1$ and hence we have $d \leq 6(h^0(B|_G) - 1)$. The lower bound is obtained by Proposition 2.2, part (1) of [KW14].

<u>*Case 4.*</u> Suppose $r \geq 8$. Recall that $H - R$ is base point free. We compute $(H-R)^4 = H^4 - 4H^3R$;

$$H^4 = \sum_{i=0}^{3} a_i H^3 R \text{ and } r = \sum_{i=0}^{3} a_i + 3. \tag{5.17}$$

So, $r \geq 8$ gives $\sum_{i=0}^{3} a_i \geq 5$ which gives $(H-R)^4 > 0$ as H is ample. Hence, $H - R$ is nef and big and consequently $B - G$ is nef and big as well. Thus by Kawamata-Viehwag Vanishing, we have that $H^1(B - G) = 0$. Hence $\varphi|_G$ is given by the complete linear system $|B|_G|$.

Since G maps to $F = \mathbb{P}^3$ we have that $h^0(B|_G) = 4$. Now, the degree of φ is also the degree of $\varphi|_G$ for a general fibre G. Hence by a result of Gallego and Purnaprajna (see [GP01], Theorem 1.6) we have that $2 \leq d \leq 18$.

(b) Now we assume that Y is singular.

Case 1. As before, if $r = 5$ then Y is a singular quadric.

Case 2. Suppose the image Y of X under the morphism defined by $|B|$ is a singular variety. Then by Theorem 2.2.4, Y is a cone over a smooth variety of minimal degree with vertex V. Moreover, $\operatorname{codim}(V, Y) = 2$ by Lemma 2.2.6. Hence Y can be either a triple cone over a rational normal curve or a double cone over the Veronese surface in $\mathbb{P}^5$.

Suppose Y is a cone over a rational normal curve. Let R be a general linear subspace of Y and G its inverse image. By Bertini, G is irreducible and $d = B^3 \cdot G$. Moreover, using the fact that the codimension of the singular locus of Y is exactly 2, we have $B = \deg(Y) \cdot G = (r-3)G$ by Lemma 2.2.5. Hence $d \geq (r-3)^3$. Using the previously proved upper bound we have,

$$(r-3)^3 \leq 24(r-1)$$

and hence $r \leq 8$. □

Now we prove our main result using the previous theorem. We notice that part (2) of the next theorem holds for both hyperkähler and Calabi–Yau four-folds (see Definitions 5.2.1 and 5.2.2) since it only requires a regular four-fold with trivial canonical bundle.

Theorem 5.1.6. *Let X be a smooth projective four-fold with trivial canonical bundle and let A be an ample line bundle on X. then,*

(1) nA is very ample and embeds X as a projectively normal variety for $n \geq 16$.

(2) If $H^1(\mathscr{O}_X) = 0$ then nA is very ample and embeds X as a projectively normal variety for $n \geq 15$.

Proof. (1) By the result of Kawamata (see Theorem 4.1.5), we have that on a four-fold with trivial canonical bundle if A is ample then nA is base point free for $n \geq 5$. Now using CM lemma (see Lemma 2.7.6) we can easily prove that nA satisfies the property N_0 for $n \geq 21$.

If we set $B = 5A$ then $20A = 4B$ and it satisfies the property N_0 by Lemma 5.1.3.

Using Lemma 5.1.4, Lemma 2.7.6 and Observation 2.7.7, we can see that the following map

$$H^0(nkA) \otimes H^0(nA) \longrightarrow H^0((nk+n)A)$$

is surjective for $k \geq 2$ and $16 \leq n \leq 19$. So we are left to check the surjectivity of the multiplication map $H^0(nA) \otimes H^0(nA) \longrightarrow H^0(2nA)$ for $16 \leq n \leq 19$. We just prove it for $n = 16$. The proof is similar for the other three cases.

For $n = 16$, the following maps are surjective by Lemma 5.1.4 and Lemma 2.7.6,

$$H^0(16A) \otimes H^0(5A) \to H^0(21A) \text{ and } H^0(21A) \otimes H^0(5A) \to H^0(26A). \tag{5.18}$$

Therefore, by Observation 2.7.7 we need to show that $H^0(26A) \otimes H^0(6A) \longrightarrow H^0(32A)$ is surjective which follows from Lemma 2.7.6 as well.

(2) Suppose $H^1(\mathscr{O}_X) = 0$. We just need to show that $15A$ satisfies the property N_0. Let $B = 5A$ which is ample and base point free (see Theorem 4.1.5). By Corollary 5.1.2, we know that $3B$ is projectively normal unless the image of the morphism induced by the complete linear series $|B|$ is a variety of minimal degree. Thus, we aim to show that the image of the morphism induced by the complete linear series $|5A|$ is not a variety of minimal degree. Applying Riemann–Roch we get,

$$h^0(5A) = \frac{625}{24}A^4 + \frac{25}{24}A^2 \cdot c_2 + 2 \geq 28. \tag{5.19}$$

Now suppose that the image is a variety of minimal degree. However, since the codimension of the image is ≥ 24, by Theorem 5.1.5, we have that the image cannot be a quadric hypersurface or a cone over the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ or a cone over a rational normal curve. Hence the image is a smooth rational normal scroll.

Let $h^0(B) = r + 1$. Hence the degree of the image is $r - 3$. Also, let the degree of the finite morphism given by the complete linear series of B be d. We know by Theorem 5.1.5 that

$$d \leq \frac{24(r-1)}{r-3}.$$

Using $h^0(B) \geq 28$ we have that $r \geq 27$ and hence $d \leq 26$.

Since the image of the morphism is a smooth rational normal scroll of dimension 4, we can choose a general $\mathbb{P}^3 = R$ and take the pullback of the divisor R under the morphism induced by the complete linear series $|B|$ and call it G. The degree of the morphism restricted to G is again d. Since the degree of R in the image is 1 we have that $d = B^3 \cdot G = 125 A^2 \cdot G \geq 125$ (since A is ample and G is effective) contradicting $d \leq 26$. Hence the image cannot be a variety of minimal degree. $\square$

5.2 Projective normality for certain hypekähler varieties

As we discussed before, K3 surfaces can be generalized in higher dimensions in two different ways and they give rise to Calabi–Yau and hyperkähler manifolds. We first define Calabi–Yau manifolds.

Definition 5.2.1. A compact Kähler manifold M of dimension $n \geq 3$ is called *Calabi–Yau* if it has trivial canonical bundle and the hodge numbers $h^{p,0}(M)$ vanish for all $0 < p < n$.

With this definition, Calabi–Yau manifolds are necessarily projective. We define the hyperkähler manifolds in the next section and provide the four known classes of examples. We remark that the definition of hyperkähler manifolds (Definition 5.2.2) does not imply projectivity in general, and if it is then we call it a *hyperkähler variety*.

The decomposition theorem of Bogomolov (see [Bog78]) says, any complex manifold with trivial first Chern class admits a finite étale cover isomorphic to a product of complex tori, Calabi–Yau manifolds and hyperkähler manifolds. Thus, these spaces can be thought of as the "building blocks" for manifolds with trivial first Chern class.

5.2.1 Definition and examples of hyperkähler varieties

We start with the definition.

Definition 5.2.2. A compact Kähler manifold M is called *hyperkähler* if it is simply connected and its space of global holomorphic two forms is spanned by a symplectic form.

Recall that a hyperkähler variety X is a projective hyperkähler manifold. The symplectic form ensures that K_X is trivial and $\dim(X)$ is even. It is also known that $\chi(\mathscr{O}_X) = n+1$ and the following are the values of $h^p(X, \mathscr{O}_X)$,

$$h^p(X, \mathscr{O}_X) = \begin{cases} 1 & p \text{ is even,} \\ 0 & p \text{ is odd.} \end{cases}$$

Only a few classes of examples of hyperkähler varieties are known. Beauville first gave examples of two distinct deformation classes of compact hyperkähler manifolds in all even dimensions greater than or equal to 2 (see [Bea84]). The first example is the Hilbert scheme $S^{[n]}$ of length n subschemes on a $K3$ surface S. The second one is the generalized Kummer variety $K^n(T)$ which is the fibre over the 0 of an Abelian surface T under the morphism $\phi \circ \psi$ (see the diagram below)

$$T^{[n+1]} \xrightarrow{\psi} T^{(n+1)} \xrightarrow{\phi} T$$

where $T^{[n+1]}$ Hilbert scheme of length $n+1$ subschemes on the Abelian variety T, $T^{(n+1)}$ is the symmetric product, ψ is the Hilbert chow morphism and ϕ is the addition on T. Two other distinct deformation classes of hyperkahler manifolds $\mathscr{M}_6$ and $\mathscr{M}_{10}$ are given by O'Grady in dimensions 6 and 10 respectively which appear as desingularizations of certain moduli spaces of sheaves over symplectic surfaces (see [O'Gr99], [O'Gr03]). All other known examples are deformation equivalent to one of these.

5.2.2 Properties of hyperkähler varieties

We start by the following theorem of Beauville and Fujiki (see [Bea84] and [Fuj85]).

Theorem 5.2.3. *Let X be a hyperkähler variety of dimension $2n$. There exists a quadratic form $q_X : H^2(X,\mathbb{C}) \to \mathbb{C}$ and a positive constant $c_X \in \mathbb{Q}_+$ such that for all α in $H^2(X,\mathbb{C})$, $\int_X \alpha^{2n} = c_X \cdot q_X(\alpha)^n$. The above equation determines c_X and q_X uniquely if one assumes the following two conditions.*

(I) q_X is a primitive integral quadratic form on $H^2(X,\mathbb{Z})$;

(II) $q_X(\sigma,\bar{\sigma}) > 0$ for all $0 \neq \sigma \in H^{2,0}(X)$.

Here q_X and c_X are called the Beauville form and Fujiki constant respectively.

The Beauville form and Fujiki constants are fundamental invariants of a hyperkähler variety. They play an important role in determining the intersections on X as the following theorem shows (See [Fuj85], [GH03]).

Theorem 5.2.4. *Let X be a hyperkähler variety of dimension $2n$. Assume that $\alpha \in H^{4j}(X,\mathbb{C})$ of type $(2j,2j)$ on all small deformations of X. Then there exists a constant $C(\alpha) \in \mathbb{C}$ depending on α such that $\int_X \alpha \cdot \beta^{2n-2j} = C(\alpha) \cdot q_X(\beta)^{n-j}$ for all $\beta \in H^2(X,\mathbb{C})$.*

As a consequence of the theorem above, we get the following form of the Riemann–Roch formula for a line bundle L on a hyperkähler variety of dimension $2n$ (see [Huy99]),

$$\chi(X,L) = \sum_{i=0}^{n} \frac{a_i}{(2i)!} q_X(c_1(L))^i \tag{5.20}$$

where $a_i = C(td_{2n-2i}(X))$. Here a_i's are constants depending only on the topology of X.

Elingsrad-Gottsche-Lehn computes the rational constants of the Riemann–Roch expression for hyperkähler manifolds of deformation type $K3^{[n]}$ (See [EGL01]) and Nieper computes the same for generalized Kummer varieties $K^n(T)$ of dimension $2n$ (see [Nie03]). If X is of $K3^{[n]}$

type we have,

$$\chi(L) = \binom{\frac{1}{2}q(L)+n+1}{n} \quad and \quad c_X = \frac{(2n)!}{n!2^n}. \tag{5.21}$$

For a generalized Kummer variety of dimension $2n$ we have the following Riemann–Roch formula,

$$\chi(L) = (n+1)\binom{\frac{1}{2}q(L)+n}{n} \quad and \quad c_X = (n+1)\frac{(2n)!}{n!2^n} \tag{5.22}$$

The Riemann–Roch formula and Fujiki constant for $\mathscr{M}_6$ are the same as that of $K^3(T)$.

We will use Matsushita's theorem on fibre space structure of hyperkähler varieties. We recall the definition and the main theorem.

Definition 5.2.5. Let X be an algebraic variety. A fibre space structure of X is a proper surjective morphism $f : X \to S$ that satisfies the following two conditions:

(1) X and S are normal varieties with $0 < \dim(S) < \dim(X)$.

(2) A general fibre of f is connected.

Theorem 5.2.6. *([Mat99], Theorem 2, (3)) Let $f : X \to B$ be a fibre space structure on a projective hyperkähler variety X of dimension 2n with projective base B. Then* $\dim(B) = n$.

5.2.3 Proof of Theorem 1.3.4

Let X be a hyperkähler variety of dimension $2n$ and let B be an ample and globally generated line bundle on X. In this section, φ_B will always denote the morphism induced by the complete linear series $|B|$. The aim is to study the projective normality of $B^{\otimes 2n-1}$. We do this by Corollary 5.1.2 that requires us to analyze the case when φ_B maps X onto a variety of minimal degree. Recall that a variety of minimal degree is either (1) a quadric hypersurface, or (2) a smooth rational normal scroll, or (3) cone over a smooth rational normal scroll, or (4) cone over the Veronese embedding of $\mathbb{P}^2$ inside $\mathbb{P}^5$ ([EH85]). The following lemma eliminates a few cases.

Lemma 5.2.7. *Let B be an ample and globally generated line bundle on a hyperkähler variety X of dimension $2n$. Suppose the morphism φ_B induced by the complete linear series $|B|$ maps X onto a variety of minimal degree Y.*

(1) If Y is a quadric hypersurface then $h^0(B) = 2n+2$.

(2) Y can never be a smooth rational normal scroll.

(3) If Y is a cone over a smooth rational normal scroll then the codimension of its singular locus is two i.e. Y is obtained by taking cones over a smooth rational normal curve.

(4) If Y is a cone over the Veronese embedding of $\mathbb{P}^2$ inside $\mathbb{P}^5$ then $h^0(B) = 2n+4$.

Proof. (1) and (4) are obvious. (3) comes from Lemma 2.2.6. We give the proof of (2) below.

To prove (2) we argue by contradiction. Suppose the image is a smooth rational normal scroll. Since a smooth scroll admits a morphism to $\mathbb{P}^1$ we have a composed morphism from X to $\mathbb{P}^1$. Take the Stein factorization of this morphism which has connected fibres and notice that since X is smooth this further factors through a normalization. So we get a morphism from X to a normal base of dimension 1 (hence smooth in this case) with connected fibres which contradicts Matsushita's result on the fibre space structure of a holomorphic symplectic manifold (see Theorem 5.2.6). □

We give two definitions below that we will use later in this note.

Definition 5.2.8. For a given hyperkähler variety X, we define the following two polynomials,

$$RR_X(x) = \sum_{i=0}^{n} \frac{a_i}{(2i)!}(x)^i \quad \text{and} \quad R_X(x) = RR_X(x) - \frac{a_n}{(2n)!}x^n$$

where $a_i = C(td_{2n-2i})$. (Note: these polynomials depend only on the deformation type of X.)

Definition 5.2.9. For a given hyperkähler variety X with Beauville form q_X, we define the constant α_X as below,

$$\alpha_X = \min\left\{ q_X(A) \,\middle|\, \text{A is an ample line bundle on } X \right\}.$$

Our next task is to find an upper bound for $\deg(\varphi_B)$ for an ample and base point free line bundle B. The next result in fact just uses the fact that B is base point free and the induced morphism φ_B is generically finite.

Lemma 5.2.10. *Let X be a hyperkähler manifold of dimension $2n$. Assume $R_X|_{\mathbb{Z}\geq 0}$ is increasing and $R_X(\alpha_X) > 2n$. Then, for any globally generated line bundle B on X, such that φ_B is generically finite, deg$(\varphi_B) < (2n)!$.*

Proof. Let $Y = \mathrm{Im}(\varphi_B)$. We have the equation $B^{2n} = \deg(\varphi_B)\cdot\deg(Y)$ to begin with. Note that,

$$\deg(Y) \geq 1 + \mathrm{codim}(Y) \implies \deg(Y) \geq h^0(B) - 2n \tag{5.23}$$

since $\mathrm{codim}(Y) = h^0(B) - 2n$. Thus, we get,

$$B^{2n} \geq \deg(\varphi_B)(h^0(B) - 2n). \tag{5.24}$$

Recall that $B^{2n} = c_X(q_X(B))^n$ and notice that $h^0(B) = RR_X(q_X(B)) = R_X(q_X(B)) + \frac{a_n}{(2n)!}(q_X(B))^n$. Since, $a_n = c_X$, using 5.24 we get,

$$c_X(q_X(B))^n \geq \deg(\varphi_B)\left(R_X(q_X(B)) + \frac{a_n}{(2n)!}(q_X(B))^n - 2n\right)$$

$$\implies c_X(q_X(B))^n\left(1 - \frac{\deg(\varphi_B)}{(2n)!}\right) \geq \deg(\varphi_B)(R_X(q_X(B)) - 2n) \geq \deg(\varphi_B)(R_X(\alpha_X) - 2n).$$

That concludes the proof since the last term is strictly greater than zero by hypothesis and $q_X(B) > 0$ since B is nef and big. □

Remark 5.2.11. If all Todd classes of the hyperkähler variety X is fakely effective then $R_X|_{\mathbb{Z}\geq 0}$ is increasing. In particular, it is satisfied for all known examples of hyperkähler varieties, except O'Grady's 10 dimensional example $\mathscr{M}_{10}$ (see [CJ20], Theorem 1.8) remaining unknown, which is also clear from their explicit Riemann–Roch expressions. ◇

The remark above leads to the following consequence.

Remark 5.2.12. The hypothesis of Lemma 5.2.10 are satisfied for all known examples of hyper-kähler varieties X of dimension $2n \geq 4$ (see [CJ20], Theorem 1.8), except O'Grady's 10 dimensional example $\mathscr{M}_{10}$.

Proof. Thanks to the previous remark, it is enough to show that $R_X(\alpha_X) > 2n$.

If X is either type $K3^{[n]}$ or $K^n(T)$ then $\alpha_X \geq 2$. Since $R_X|_{\mathbb{Z}\geq 0}$ is increasing, we have,

$$R_X(\alpha_X) \geq RR_X(2) - \frac{c_X 2^n}{(2n)!}. \tag{5.25}$$

If X is of type $K3^{[n]}$ (resp. $K^n(T)$), using the Riemann–Roch expression 5.21 (resp. 5.22) and $n \geq 2$ we have the following,

$$RR_X(2) - \frac{c_X 2^n}{(2n)!} = \binom{n+2}{n} - \frac{1}{n!} > 2n \quad \left(\text{resp. } RR_X(2) - \frac{c_X 2^n}{(2n)!} = (n+1)^2 - \frac{1}{n!} > 2n\right). \tag{5.26}$$

Same argument works for $\mathscr{M}_6$ as well since its Riemann–Roch expression is the same as that of $K^3(T)$. That concludes the proof of the assertion. ♢

The upper bound for the degree has the following consequence on the secant lines of an embedding of a $K3$ surface. Even though it has nothing to do with our present purpose, it still might be of independent interest.

Corollary 5.2.13. *Let S be a $K3$ surface and B be a very ample line bundle. Consider the projective embedding of S in $\mathbb{P}^{h^0(B)-1}$ and the closed subvariety of $Gr(2, h^0(B))$ consisting of lines that intersect the $K3$ surface at a subscheme of length at least 2. Then a general such line intersects S at a subscheme of length ≤ 7.*

Proof. Given the above conditions we construct a generically finite morphism f from $X = S^{[2]}$ to $Gr(2, h^0(B))$. Given a point on X we take the length 2 subscheme it defines on S and send it to the linear span of the length two subscheme inside $\mathbb{P}^{h^0(B)-1}$. Since a general such line does not lie on S, it intersects S at finitely many points. So a general point in the image of the morphism f has got finite fibers. Hence f is a generically finite morphism.

Observe that this morphism is given by the complete linear series $B-\delta$ where 2δ is the class of the divisor in $S^{[2]}$ that parametrizes non-reduced subschemes of length 2 on the $K3$ surface S. By Lemma 5.2.10 and Remark 5.2.12, we have $\deg(f) \leq 23$. If a line intersects S at k points then the line has $\binom{k}{2}$ preimages under the morphism f. Thus for a general line $\binom{k}{2} \leq 23$ and hence $k \leq 7$. □

In the next Lemma, we will use the upper bound for $\deg(\varphi_B)$ to put more restrictions on the image of φ_L when it maps onto a variety of minimal degree.

Lemma 5.2.14. *Let X be a hyperkähler manifold of dimension $2n \geq 4$ for which $R_X|_{\mathbb{Z}\geq 0}$ is increasing, $R_X(\alpha_X) > 2n$ and $RR_X(\alpha_X) \geq 4n$. Then for any ample and globally generated line bundle B on X, φ_B can never map X onto a variety that is obtained by taking cones over a rational normal curve.*

Proof. Suppose $Y = \mathrm{Im}(\varphi_B)$ is a variety of minimal degree that is obtained by taking cones over a rational normal curve and $d = \deg(\varphi_B)$. Therefore Y is singular in codimension two. Let G be the inverse image of a general linear subspace of R of codimension 1 in Y. Notice that G is irreducible by Bertini's theorem.

We have $B^{2n-1} \cdot G = d$. Using Lemma 2.2.5, we deduce that B can be written as the pullback of $\deg(Y) \cdot R$. Thus, G is ample and $d = \deg(Y)^{2n-1} \cdot G^{2n} \geq \deg(Y)^{2n-1}$.

Now we use the fact that $\deg(Y) = \big(RR_X(q_X(B)) - 2n\big)$ and $d < (2n)!$ (see Lemma 5.2.10 and Remark 5.2.12) that leads us to the following inequality,

$$\big(RR_X(q_X(B)) - 2n\big)^{2n-1} < (2n)! \tag{5.27}$$

which is absurd. Indeed, by our assumption, $RR_X(q_X(B)) - 2n \geq RR_X(\alpha_X) - 2n \geq 2n$. □

Notice that the proof above also shows the following.

Remark 5.2.15. Let X be a hyperkähler manifold of dimension $2n \geq 4$ for which $R_X|_{\mathbb{Z}\geq 0}$ is increasing and $R_X(\alpha_X) > 2n$. Let B be an ample and globally generated line bundle for which either $h^0(B) \geq 4n$, or $B^{2n} < \alpha_X^n (h^0(B) - 2n)^{2n}$. Then φ_B can never map X onto a variety that is obtained by taking cones over a smooth rational normal curve. $\diamondsuit$

Combining Lemmas 5.2.7, 5.2.14 and Remarks 5.2.12 and 5.2.15 we get the following.

Proposition 5.2.16. *Let X be a hyperkähler manifold of dimension $2n \geq 4$ and let B be an ample and globally generated line bundle on X. If X is deformation equivalent to either $K^n(T)$ or $\mathscr{M}_6$ then the morphism φ_B given by the complete linear series $|B|$ will never map X onto a variety of minimal degree. If X is of type $K3^{[n]}$ and φ_B maps X onto a variety Y of minimal degree then either,*

(1) X is of type $K3^{[2]}$, $q_X(B) = 2$, deg$(\varphi_B) = 6$ and Y is a quadric hypersurface (possibly singular) inside $\mathbb{P}^5$ which can not be obtained by taking cones over any rational normal scroll, or

(2) X is of type $K3^{[3]}$, $q_X(B) = 2$, deg$(\varphi_B) = 30$ and Y is a variety embedded inside $\mathbb{P}^9$ that is obtained by taking cones over the Veronese embedding of $\mathbb{P}^2$ inside $\mathbb{P}^5$.

Proof. To start with, note that $\alpha_X \geq 2$. Suppose X is of type $K^n(T)$ or $\mathscr{M}_6$. By Riemann–Roch, we get $h^0(B) = RR_X(q_X(B)) \geq RR_X(2) > 2n+4$. Consequently, by Lemma 5.2.7, Im(φ_B) can only be a variety that is obtained by taking cones over a rational normal curve. But that is impossible by Lemma 5.2.14 since $RR_X(2) > 4n$.

Now, assume X is of type $K3^{[n]}$. We can argue exactly like the previous paragraph to conclude that φ_B will never map X onto a variety of minimal degree if $n \geq 5$.

We deal with the case $n = 2, 3$ and 4 separately and we will use Lemma 5.2.7. Note that $q_X(B)$ is even, say $q_X(B) = 2k$ for some positive integer k.

Suppose $n = 2$. Note that $h^0(B) > 8$ if $q(B) \geq 4$ and $B^{2n} < 2^2(h^0(B) - 4)^4$ if $q(B) = 2$. Consequently by Remark 5.2.15, Y can not be obtained by taking cones over rational normal curve. The equation $RR_X(2k) = 2n+2$ has only one positive even integer solution $k = 1$ in which case $q_X(B) = 2$, deg$(\varphi_B) = 6$ and Y is a quadric hypersurface in $\mathbb{P}^5$. $RR_X(2k) = 2n+4$ has no integer solution.

Suppose $n = 3$. Argument similar to that of the case $n = 2$ yields that Y can not be obtained by taking cones over rational normal curve. Moreover, $RR_X(2k) = 2n + 2$ has no solution and $RR_X(2k) = 2n + 4$ has only one positive integer solution $k = 1$ in which case $q_X(B) = 2$, $\deg(\varphi_B) = 30$ and Y is a variety of minimal degree in $\mathbb{P}^9$ obtained by taking Veronese embedding of $\mathbb{P}^2$ inside $\mathbb{P}^5$.

Similar argument shows that φ_B can not map $K3^{[4]}$ onto a variety of minimal degree. □

Now we are ready to give the proof of our main theorem.

Theorem 5.2.17. *Let X be a projective hyperkähler variety of dimension $2n \geq 4$ that is deformation equivalent to $K3^{[n]}$, $K^n(T)$ or $\mathscr{M}_6$. Let B be an ample and globally generated line bundle on X. Then the following happens;*

(1) $B^{\otimes l}$ is projectively normal for $l \geq 2n$.

(2) $B^{\otimes 2n-1}$ is projectively normal unless:

(a) $n = 2$, $X = K3^{[2]}$ and φ_B maps X onto a quadric (possibly singular) inside $\mathbb{P}^5$. In this case $q_X(B) = 2$, $\deg(\varphi_B) = 6$, or

(b) $n = 3$, $X = K3^{[3]}$ and φ_B maps X onto a variety of minimal degree inside $\mathbb{P}^9$ which is obtained by taking cones over the Veronese embedding of $\mathbb{P}^5$ inside $\mathbb{P}^5$. In this case $q_X(B) = 2$, $\deg(\varphi_B) = 30$.

Hence if X is as above and B does not satisfy cases $2(a)$ or $2(b)$ then a general curve section of $|B|$ is non-hyperelliptic.

Proof. To prove (1) we simply notice that $h^0(B) \geq 2n + 2$ by the Riemann–Roch formula on X. The assertion follows by Lemma 5.1.3. (2) follows directly by Corollary 5.1.2, and Proposition 5.2.16. The statement on non-hyperellipticity of a general curve section C follows from the fact that $B^{d-1}|_C = K_C$ by adjunction and that a very ample line bundle restricts to a very ample line bundle on a closed immersion. □

We finish by the following example of Debarre. It shows the existence of an ample and globally generated line bundle on a hyperkähler variety of $K3^{[2]}$ type that induces a 6-1 map onto a variety of minimal degree by its complete linear series.

Example 5.2.18. ([Deb18]) Let (S, L) be a polarized $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z}L$ and $L^2 = 4$. Then L is very ample and consequently we get a morphism $\phi : S^{[2]} \to Gr(2,4)$ to the Grassmannian.

Now, L induces a line bundle L_2 on $S^{[2]}$ and it is known that $\mathrm{Pic}(S^{[2]}) = \mathbb{Z}L_2 \oplus \mathbb{Z}\delta$. Moreover, the pullback of the Plücker line bundle on the Grassmannian has class $L_2 - \delta$ on $S^{[2]}$. Therefore, if (S, L) is general then it contains no line and consequently ϕ will be finite of degree $\binom{4}{2} = 6$. ♠

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