Limit distributions for Skorohod integrals and spatial averages of the stochastic wave and heat equation

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Submitted to the graduate degree program in Department of Mathematics and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Date defended: April 26, 2021

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Abstract

In this dissertation, we study some problems related to the convergence in distribution of functionals of Gaussian processes. The approach used to address the problems presented in this thesis is based on Malliavin calculus techniques.

In Chapter 1, we prove the convergence in distribution of sequences of Itô and Skorohod integrals with integrands having singular asymptotic behavior. These sequences include stochastic convolutions and generalize the example $\sqrt{n}\int_0^1 t^n B_t dB_t$ first studied by Peccati and Yor in 2004.

In Chapter 2, we prove a functional central limit theorem for the spatial average of the mild solution to the 2D stochastic wave equation driven by a Gaussian noise, which is temporally white and spatially colored described by the Riesz kernel. We also establish a quantitative central limit theorem for the marginal and the rate of convergence is described by the total-variation distance. A fundamental ingredient in our proofs is the pointwise L^p-estimate of the Malliavin derivative, which is of independent interest.

In Chapter 3, we prove a quantitative central limit theorem for the spatial average of the mild solution to the 1D stochastic heat equation driven by space time white noise with an initial condition given by an independent white noise. As part of this chapter, we also prove the existence, uniqueness, stationarity and differentiability (in the Malliavin calculus sense) of the mild solution.

The projects in this thesis are joint work between the author and professors David Nualart, Denis Bell and Guanqu Zheng. The first chapter corresponds to the research article [3] by Denis Bell, the author and David Nualart. The second chapter consists of the manuscript [15] by the author, Guanqu Zheng and David Nualart. Lastly, chapter three represents the most recent work between the author and David Nualart.

Resumen

En esta tesis, estudiamos algunos problemas relacionados con la convergencia en distribución de funcionales de procesos Gaussianos. El enfoque utilizado para abordar los problemas presentados en esta tesis se basa en técnicas de cálculo de Malliavin.

En el Capítulo 1, probamos la convergencia en distribución de sucesiones de integrales de Itô y Skorohod donde los integrandos tienen un comportamiento asintótico singular. Estas sucesiones incluyen convoluciones estocásticas y generalizan el ejemplo $\sqrt{n}\int_0^1 t^n B_t dB_t$ primeramente estudiado por Peccati y Yor en el año 2004.

En el Capítulo 2, demostramos un teorema límite central funcional para el promedio en espacio de la solución de la ecuación de onda estocástica en dimensión dos dirigida por un ruido Gaussiano que es blanco en tiempo y espacialmente coloreado descrito por el núcleo de Riesz. En este capítulo, también establecemos un teorema límite central cuantitativo donde la convergencia es cuantificada empleando la distancia de variación total. Un ingrediente fundamental en nuestras demostraciones es la estimación puntual de la norma L^p de la derivada de Malliavin, la cual tiene un interés independiente.

En el capítulo 3, probamos un teorema límite central cuantitativo para el promedio en espacio de la solución de la ecuación de calor estocástica en dimensión dirigida por un ruido Gaussiano que es blanco en espacio y tiempo, y donde la condición inicial es dada por un ruido blanco independiente. Como parte de este capítulo, demostramos también la existencia, unicidad, estacionaridad y diferenciabilidad (en el sentido de cálculo de Malliavin) de la solución.

Los proyectos de esta tesis son un trabajo conjunto entre el autor y los profesores David Nualart, Denis Bell y Guanqu Zheng. El primer capítulo corresponde al artículo [3] de Denis Bell, el autor y David Nualart. El segundo capítulo consta del manuscrito [15] del autor, Guanqu Zheng y David Nualart. Por último, el capítulo tres contiene el trabajo más reciente entre el autor y David Nualart.

For my parents, Elena and Gregorio.

Acknowledgements

First of all, I would like to express my deep gratitude and appreciation to my advisor Professor David Nualart. I thank him for his patience, encouragement and for sharing his wisdom with me. I would not have completed this thesis without his guidance.

I also offer my thanks to Professors Zhipeng Liu, Tyrone Duncan, Xuemin Tu, and Zongwu Cai for their willingness to devote the time and effort to serve on my dissertation committee and their insightful questions.

I am most grateful to Professor Guanqu Zheng for his detailed explanations about studying central limit theorems for spatial averages of solutions to stochastic partial differential equations. Your knowledge of this area substantially improved my understanding of this topic.

I am deeply appreciative to the Faculty and Staff of the Department of Mathematics in the University of Kansas. I would like to thank Professors Jeremy Martin, Terry Soo, Atanas Stefanov, Mathew Johnson and Dionyssios Mantzavinos for their kind words of encouragement at different stages of my Ph.D. studies. I am also grateful to Kerrie Brecheisen, Gloria Prothe, Lori Springs, and Kate Pleskac for all their help during my time at KU. I would like to thank Jila Niknejad and Joseph Brennan for sharing so many teaching materials during my time as a GTA in the calculus sequence.

I would like to express my gratitude to my former professors at the Universidad de Costa Rica. The preparation I received during my time there was very important for my graduate studies. I am truly thankful to the professors Oscar Roldan, Ronald Bustamante, Santiago Cambronero, Marco Alfaro, Michael Josephy, William Alvarado, Pedro Méndez, Osvaldo Acuña, William Ugalde, Oldemar Rodríguez, Joseph Varilly, and José Alexander Ramírez for all they taught me. I am also thankful to the Universidad de Costa Rica and especially to the Escuela de Matemática for believing in me and their support during my Ph.D.

Now I want to thank my friends at KU. I am deeply thankful to Arturo Jaramillo, Hamid Mofidi, Sefika Kuzgun, Abba Ramadan, Bhargob Saikia, and Fazel Hadadifard for so many conversations, for cheering me up, and for making my time at Lawrence memorable. I would also like to thank my generation siblings Brad, Harry, Cassidy, Bibek, Jayan, Wesley, Deba, Prashanth, Yanru, and Amanda for all we shared. To all my KU friends who I did not mention here, I really appreciate our time together.

In this paragraph, I wish to show my gratitude to my friends from Costa Rica. To my oldest friends from Escuela El Carmen, to my high school friends from Liceo Jose Martí, to my dear friends from the CCC San Ramón, to my great friends from Universidad de Costa Rica, to my younger friends from the CCC San Pedro; thank you so much for supporting me and motivating me throughout my academic life. I want to especially thank my friends Alexander Peña, Michael Badilla, Osvaldo Fernández, Isaac Fernández, Gabriela Murillo, Verónica Murillo, Alberto Fonseca, Juan Gabriel Calvo, Chia-Han Chou, Adriana Sánchez, Rafael Sanabria, Iván Ramírez, and Daniel Vásquez for their spontaneous messages wishing me the best and encouraging me during my studies abroad.

Last but definitely not least, I would like to express my fondest gratitude to my family. To my parents because I owe them everything. To my wife Sandra for being by my side, having faith in my abilities, and being my comfort in times of need. To my brothers, Fabricio and Nestor, and my sister Alina, for being a fundamental part of my life and always being there when I needed them. To my nephews Heiner, Fabián, Sebastián, Luis Diego, and my nieces Yasley, Fátima, and Fabiana for all the precious memories you have given me. To my aunts Ligia and Sandra, for their help when I moved to the United States and their words of encouragement whenever I visited them. To the rest of my family, for their support and love throughout my entire life.

Contents

Introduction

The Malliavin calculus is an infinite-dimensional differential calculus whose operators act on functionals of an underlying Gaussian process. It was first introduced by Paul Malliavin in the 1970s to provide a probabilistic proof of Hörmander's hypoellipticity theorem. The theory was further developed to incorporate other significant applications including stochastic calculus for fractional Brownian motion, anticipative stochastic calculus, stochastic partial differential equations, limit theorems for functionals of Gaussian processes, and mathematical finance.

The problems studied in this dissertation correspond to particular cases of limit theorems for functionals of a fractional Brownian motion (fBm for short) and limit theorems for random fields arising from stochastic partial differential equations (SPDEs for short). The former is presented in Chapter 1, whereas the latter is developed on Chapter 2 and Chapter 3. In all three chapters, the problems are address by using an approach based on Malliavin calculus.

In this introduction, we give some motivation and a literature overview for the chapters developed in this thesis. This introduction also serves to briefly describe our projects and present the main contributions of our work.

Our discussion about Chapter 1 begins with the study of the convergence in distribution of the sequence of Skorohod integrals given by

$$F_n = \int_0^1 n^H t^n B_t^H \; \delta B_t^H,$$

where B_t^H is a fBm with Hurst parameter H in the range $(1/4, 1)$. For this sequence, it is known that F_n converges in law to a random variable of the form $\sqrt{H\Gamma(2H)}B_1^H Z$, where Z is a standard normal random variable, which is independent of the fBm B_t^H, and Γ is the Gamma function. We proceed to recall some previous work where this convergence result has been proved.

The paper [40] by Peccati and Yor was the first to prove this result in the particular case when $H = 1/2$. In this case, fBm coincides with the classical Brownian motion and the Skorohod integrals are, in fact, Itô integrals. This case was also considered by Peccati and Taqqu in [39] and by Nourdin and Nualart in [25].

The case $H \in [1/2, 1)$ was covered by Nourdin, Nualart & Peccati in [29] by means of a general theorem from Malliavin calculus. Their arguments provided quantitative bounds for the rate of convergence of the sequence F_n in terms of the Wasserstein and Kolmogorov distances. Lastly, Pratelli & Rigo studied the case $H \in (1/4, 1)$ in [42], and improved the rates of convergence previously obtained by Nourdin, Nualart & Peccati in [29].

In this context, our motivation in Chapter 1 was to develop a new approach to prove convergence in distribution of the aforementioned sequence F_n. Roughly speaking, our approach can be described in two steps. First, we introduce a new sequence of random variables, say G_n, and show that the new sequence is suitably close to F_n. Second, we deduce the convergence in distribution of F_n by using G_n instead.

As we implemented this methodology, we realized that only specific properties of the function $n^H t^n$ and the process B_t^H were necessary for our arguments to work. This motivated us to consider a more general problem and study the convergence in distribution of a sequence of Skorohod integrals given by

$$\int_0^1 \phi_n(t)\, u_t \, \delta B_t^H,$$

where the function $\phi_n(t)$ and the process u_t satisfy some suitable conditions. Essentially, this is how Chapter 1 was born.

To end our discussion about the first chapter, we proceed to record our main contributions there. These are:

(1) We provide a new approach to study the limit in distribution of the sequence of Skorohod integrals given by $F_n = \int_0^1 n^H t^n B_t^H \, \delta B_t^H$, where B_t^H is a fBm with Hurst parameter H in the range $(1/4, 1)$. More precisely, we prove Theorems 1.2.1, Theorem 1.3.1, and Theorem

1.3.3, which can be applied to the sequence F_n as a particular case.

(2) We apply our methodology to prove Theorem 1.2.5 regarding the limit in distribution of the stochastic convolution $(u *_B \psi_n)(t) := \int_0^\infty \sqrt{n}\psi(n(t-s))u_s\, dB_s$, where B is a standard Brownian motion, u_s is a suitable process and $n\psi^2(nt)$ is an approximation of the identity. We also establish Proposition 1.2.2 concerning the convergence of the finite-dimensional distribution of the stochastic convolution.

(3) We obtain Theorem 1.4.1 and Theorem 1.4.3 about the convergence in total variation of the sequences of Itô integrals studied throughout Chapter 1.

The discussion for Chapter 2 and Chapter 3 is independent of Chapter 1. Generally speaking, in these chapters, we consider specific stochastic partial differential equations and study central limit theorems for the spatial average of the solution. The methodology involved in these chapters is based on what is nowadays known as the Malliavin-Stein approach.

The Malliavin-Stein approach was introduced by Nourdin and Peccati in [26] by combining arguments from Malliavin calculus with Stein's method to, among other things, quantify Nualart and Peccati's fourth moment theorem in [33]. Roughly speaking, this approach provides bounds for the distance between the law of the standard Gaussian distribution and the law of a random variable given by a divergence also known as Skorohod integral (see e.g. Proposition 2.0.8).

Before entering into the specifics for Chapter 2 and Chapter 3, let us provide a brief overview of some previous work that motivated our results. In this sense, we must talk about the articles [12], [16], and [17].

The paper [16] by Huang, Nualart, and Viitasaari, is the first of many to study central limit theorems for spatial averages of solutions to stochastic partial differential equations. In this paper, the authors consider the stochastic heat equation with one spatial dimension driven by space-time white noise. An innovative aspect of their methodology is to take into account that the Itô–Walsh integral appearing in the solution of the SPDE corresponds to a particular case of the Skorohod integral. In this way, the authors implemented the Malliavin-Stein approach to obtain a quantitative central limit theorem, and also a functional central limit theorem for the spatial average of the solution. A fundamental ingredient in their arguments is a L^p estimate for the Malliavin derivative of the solution.

Soon after [16] was completed, the same authors and Zheng investigated the same equation in higher dimension; in their paper [17], the spatial correlation is described by the Riesz kernel. They were able to obtain similar results to those in [16] by implementing the same methodology. Finally, in the article [12], Delgado, Nualart, and Zheng considered the stochastic wave equation where spatial dimension is one and the driving Gaussian noise is white in time and fractional in space. They also were successful in obtaining similar results to those in [16] by using the same approach.

The aforementioned articles motivated many results concerning the study of limit theorems for spatial averages of solutions to SPDE's. In this context, some important results are [1], [21], [36], [38], and, of course, Chapter 2 and Chapter 3 in this dissertation.

In Chapter 2, we consider a 2D stochastic wave equation driven by a Gaussian noise, which is temporally white and spatially colored described by the Riesz kernel. Our main contributions in this chapter are:

(1) We prove Theorem 2.0.4, which gives a functional central limit theorem for the spatial average of the solution, as well as a quantitative central limit theorem for the marginal where the rate of convergence is described by the total-variation distance.

(2) We obtain Theorem 2.0.6, which provides a pointwise L^p-estimate for the Malliavin derivative of the solution.

We end our discussion about Chapter 2 by mentioning that, although, a similar problem was studied in [12], our analysis in Chapter 2 is significantly different from [12]. This happens because a few helpful properties of the fundamental wave solution in dimension one do not hold in the corresponding two-dimensional setting. For example, the fundamental wave solution in dimension one is bounded, whereas, the fundamental wave solution in dimension two has a singularity. These differences made our problem more difficult than in [12].

Finally, in Chapter 3, we consider the 1D stochastic heat equation driven by space-time white noise, with an initial condition given by an independent white noise. In this setting, our study of a central limit theorem for the solution began with proving results about the existence, uniqueness, stationarity, and differentiability (in the sense of Malliavin calculus) of the mild solution. Additionally, since there are two noises appearing in our setting, we needed to incorporate Malliavin derivatives and divergences for both noises, and we also needed to implement the Malliavin-Stein methodology for the case of the sum of two divergences. This last point is the main difference between Chapter 3 and [16]. Our contributions in this chapter are the following:

(1) We prove Theorem 3.2.1 and Theorem 3.2.2 concerning the existence, uniqueness, and stationarity of the solution to our SPDE.

(2) We establish Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.6 and Theorem 3.3.7 about the differentiability (in the sense of Malliavin calculus) of the mild solution, and estimates for the norm of the Malliavin derivative of the mild solution in terms of the fundamental heat solution.

(3) We prove Theorem 3.4.1 regarding a quantitative central limit theorem for the spatial average of the solution to our SPDE.

We conclude our introduction by bringing to the attention of the reader the website

`https://sites.google.com/site/malliavinstein/home`,

maintained by Ivan Nourdin. In this online address, the interested reader can find many research articles related to the Malliavin-Stein approach.

Chapter 1

Limit theorems for singular Skorohod integrals

In this chapter, we study the limit in distribution of sequences of random variables defined by Skorohod integrals

$$F_n = \int_0^1 \phi_n(t) u_t \delta B_t^H, \tag{1.1}$$

where B^H is fractional Brownian motion (fBm for short) with Hurst parameter H lying in the range $(1/4, 1)$, u_t is a process continuous in $L^2(\Omega)$, and ϕ_n is a sequence of deterministic kernels converging to a delta function based at 1 (hence the “singular" in the title of the chapter). We show, under suitable conditions on ϕ_n and u, that the couple (B^H, F_n) has a limit distribution of the form $(B^H, cu_1 Z)$, where Z is a $N(0,1)$ random variable, independent of B^H and c is a scaling parameter.

The study of limit problems of this type was motivated by the particular case

$$\tilde{F}_n = \int_0^1 n^H t^n B_t^H \, \delta B_t^H. \tag{1.2}$$

For this case, the limit in distribution $\tilde{F}_n$ corresponds to a random variable of the form $\sqrt{H\Gamma(2H)} B_1^H Z$, where Z is a standard normal random variable, which is independent of the fBm B_t^H, and Γ is the Gamma function. We proceed to briefly refer to some work related to the sequence $\tilde{F}_n$.

The case $H = \frac{1}{2}$, was introduced in Proposition 2.1 of [40]. In this case, B_t^H is standard Brownian motion and the integrals are of classical Itô type. The case of a fBm with $H \in [1/2, 1)$, was studied in Proposition 18 of [39], and Example 4.2 in [25]. Quantitative bounds for the rate of convergence for integrals of the form (1.2) with $H \geq \frac{1}{2}$ have been established by Nourdin, Nualart & Peccati [29] by using estimates derived from Malliavin calculus and, more recently, by Pratelli

& Rigo in [42] for $H \in (1/4, 1)$, by means of a more elementary argument.

In this chapter, we provide a new approach to study the limit distribution of (1.2). The idea is as follows. Instead of studying the convergence in distribution of F_n directly, we introduce a new sequence $G_{n,\delta}$ (or G_n), and show this new sequence is suitably close to F_n in distribution. Then, the desired convergence in law can be obtained using $G_{n,\delta}$ (or G_n) instead. Our main results to implement this methodology are Lemma 1.1.3 and Lemma 1.1.4.

This approach was based on the following observation. In the Brownian motion case $H = 1/2$, the singular asymptotic behavior of the kernels ϕ_n in (1.1) at the endpoint $t = 1$ implies that the limit distribution of the integrals F_n is determined by integration over arbitrarily small time intervals $[1-\delta, 1]$. This allows for a reduction of the problem whereby the integrals F_n are replaced by the more tractable random variables

$$G_{n,\delta} = u_{1-\delta} \int_{1-\delta}^{1} \phi_n(t)\, dB_t = u_{1-\delta} I_{n,\delta}, \tag{1.3}$$

where $\delta \in (0, 1)$, and the integral is an Itô integral with respect to the Brownian motion B. The Itô integrals $I_{n,\delta}$ in (1.3) have convergent variance and are asymptotically uncorrelated with B. Since $I_{n,\delta}$ and B are *Gaussian*, the desired result follows by first taking the limit of $G_{n,\delta}$ as $n \to \infty$ and then letting δ tend to 0.

The remaining of this chapter is organized as follows. In Section 1.1, we introduce some preliminary definitions and results which are used throughout the chapter. In particular, we recall some basic facts of fractional Brownian motion and Malliavin calculus, and we explain how the Skorohod integrals appearing in (1.1) are defined.

In Section 1.2.1, we implement the aforementioned methodology for the case $H = 1/2$ (classical Brownian motion) when the process u_t is jointly measurable and adapted. The basic result in this section is Theorem 1.2.1. As a special case of this theorem, we derive the limit law of the aforementioned sequence

$$\sqrt{n} \int_0^1 t^n B_t dB_t.$$

Theorem 1.2.1 is extended in Theorems 1.2.3 and 1.2.4 to double, and multiple, integrals respectively.

In Section 1.2.2, we apply our methodology to address a similar problem for the stochastic convolution $\int_0^\infty \sqrt{n}\psi(n(t-s))u_s\, dB_s$, where B is a standard Brownian motion, u_s is a suitable process and $n\psi^2(nt)$ is an approximation of the identity. Our main results in this section are Theorem 1.2.5 and Proposition 1.2.7.

In Section 1.3, we study the case of fractional Brownian motion ($H \neq 1/2$). Here it turns out to be more convenient to work with the approximating sequence

$$G_n = \int_0^1 \phi_n(t) u_1 \delta B_t^H.$$

As is usual in this subject, the cases $H \in (1/2, 1)$ and $H \in (1/4, 1/2)$ seem to require slightly different hypotheses and analyses, with the latter proving more involved. Analogues of Theorem 1.2.1 are presented in Theorems 1.3.1 and 1.3.3 for these two cases. The proof involves the use of the divergence operator on Wiener space and, in this sense, has a flavor of Malliavin calculus. As a special case of these theorems, we obtain a different proof for the convergence in law of the sequence $n^H \int_0^1 t^n B_t^H dB_t^H$, for H in the range $(1/4, 1)$, studied in [42].

In section 1.4, we revisit the sequence of Itô integrals introduced in Section 1.2 and study its convergence in total variation by means of Theorem 3.1 in [28] and Theorem 1 in [41].

1.1 Preliminaries

1.1.1 Fractional Brownian motion and Malliavin calculus

Fractional Brownian motion (fBm for short) with Hurst parameter $H \in (1/4, 1)$, $B^H = \{B_t^H, t \in [0,1]\}$ is a zero mean Gaussian process with a covariance function given by

$$R_H(t,s) := \mathbb{E}[B_t B_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{1.4}$$

where $s,t \in [0,1]$. In particular, when $H = 1/2$, the covariance function (1.4) reduces to $s \wedge t$ and fBm corresponds to a classical Brownian motion.

The covariance function (1.4) induces a Hilbert Space $\mathfrak{H}$ which is defined as the closure of the space of step functions $\mathscr{E}$ on $[0,1]$ endowed with the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t,s).$$

Then the mapping $B^H : \mathbf{1}_{[0,t]} \to B^H_t$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space $\mathscr{H}_1$ spanned by B^H.

The Hilbert space $\mathfrak{H}$ plays a fundamental role in this chapter. Consequently, let us record a few results regarding $\mathfrak{H}$ and the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$.

When $H = \frac{1}{2}$, B^H is just a standard Brownian motion and $\mathfrak{H} = L^2([0,1])$. In particular,

$$\langle f, g \rangle_{\mathfrak{H}} = \int_0^1 f(t)g(t)\,dt,$$

in this case.

When $H \in \left(\frac{1}{2}, 1\right)$, the inner product of two step functions $\phi, \psi \in \mathscr{E}$ can be expressed as

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = \alpha_H \int_0^1 \int_0^1 \phi(s)\psi(t)|t-s|^{2H-2} ds dt,$$

where $\alpha_H = H(2H-1)$. The space of measurable functions ϕ on $[0,1]$, such that

$$\|\phi\|^2_{|\mathfrak{H}|} := \alpha_H \int_0^1 \int_0^1 |\phi(s)||\phi(t)||t-s|^{2H-2} ds dt < \infty,$$

denoted by $|\mathfrak{H}|$, is a Banach space and we have the continuous embeddings $L^{\frac{1}{H}}([0,1]) \subset |\mathfrak{H}| \subset \mathfrak{H}$.

When $H \in (1/4, 1/2)$, the covariance of the fractional Brownian motion B^H can be expressed as

$$R_H(t,s) = \int_0^{s \wedge t} K_H(s,u) K_H(t,u) du,$$

where $K_H(t,s)$ is a square integrable kernel defined as

$$K_H(t,s) = d_H \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t v^{H-\frac{3}{2}} (v-s)^{H-\frac{1}{2}} dv \right),$$

for $0 < s < t$, with d_H being a constant depending on H. Two important properties of the kernel K_H are the following estimates

$$|K_H(t,s)| \le c_H \left((t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} \right), \tag{1.5}$$

and

$$\left| \frac{\partial K_H}{\partial t}(t,s) \right| \le c'_H (t-s)^{H-\frac{3}{2}}, \tag{1.6}$$

for all $s < t$ and for some constants c_H, c'_H.

Define a linear operator K_H^* from $\mathscr{E}$ to $L^2([0,1])$ as follows

$$(K_H^* \phi)(s) = \left(K_H(1,s)\phi(s) + \int_s^1 (\phi(t) - \phi(s)) \frac{\partial K_H}{\partial t}(t,s) dt \right). \tag{1.7}$$

The operator K_H^* can be extended to a linear isometry between the Hilbert space $\mathfrak{H}$ and $L^2([0,1])$, in other words, for any $\phi, \psi \in \mathfrak{H}$, the inner product in $\mathfrak{H}$ can be written as

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = \langle K_H^* \phi, K_H^* \psi \rangle_{L^2([0,1])}. \tag{1.8}$$

The space of Hölder continuous functions of order $\gamma > \frac{1}{2} - H$ is included in $\mathfrak{H}$.

After this brief discussion about the Hilbert space $\mathfrak{H}$, let us introduce the elements from Malliavin calculus that are used throughout the chapter. We start by introducing the derivative operator and its adjoint, the divergence.

Consider a smooth and cylindrical random variable of the form $F = f(B_{t_1}^H, \dots, B_{t_d}^H)$, where $f \in C_p^\infty(\mathbb{R}^d)$ (f and its partial derivatives have at most polynomial growth). We define its Malliavin derivative as the $\mathfrak{H}$-valued random variable DF given by

$$D_sF = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(B^H_{t_1}, \dots, B^H_{t_d})\mathbf{1}_{[0,t_i]}(s).$$

For any real number $p \geq 1$, we define the Sobolev space $\mathbb{D}^{1,p}$ as the closure of the space of smooth and cylindrical random variables with respect to the norm $\|\cdot\|_{1,p}$ given by

$$\|F\|^p_{1,p} = \mathbb{E}(|F|^p) + \mathbb{E}\left(\|DF\|^p_{\mathfrak{H}}\right).$$

Similarly, we can define the Sobolev space of $\mathfrak{H}$-valued random variables $\mathbb{D}^{1,p}(\mathfrak{H})$.

The adjoint of the Malliavin derivative operator D, denoted as δ, is called the *divergence operator*. A random element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of δ, denoted as $\mathrm{Dom}\,\delta$, if there exists a positive constant c_u depending only on u such that

$$|\mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}})| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$. If $u \in \mathrm{Dom}\,\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\left(F\delta(u)\right) = \mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}}),$$

for any $F \in \mathbb{D}^{1,2}$. We make use of the notation $\delta(u) = \int_0^1 u_t \delta B^H_t$ and call $\delta(u)$ the *Skorohod integral* of u with respect to the fractional Brownian motion B^H. The Skorohod integral satisfies the following isometry property for any element $u \in \mathbb{D}^{1,2}(\mathfrak{H}) \subset \mathrm{Dom}\delta$:

$$\mathbb{E}(\delta(u)^2) = \mathbb{E}(\|u\|^2_{\mathfrak{H}}) + \mathbb{E}(\langle Du, (Du)^*\rangle_{\mathfrak{H}\otimes\mathfrak{H}}),$$

where $(Du)^*$ is the adjoint of Du. As a consequence, we have

$$\mathbb{E}(\delta(u)^2) \leq \mathbb{E}(\|u\|^2_{\mathfrak{H}}) + \mathbb{E}(\|Du\|^2_{\mathfrak{H}\otimes\mathfrak{H}}). \tag{1.9}$$

Another important property used in this chapter is the following Lemma.

Lemma 1.1.1. *Let $F \in \mathbb{D}^{1,2}$ and let $g \in \mathfrak{H}$. Then the process Fg belongs to the domain of δ and*

$$\delta(Fg) = \int_0^1 Fg_t \delta B_t^H = F\delta(g) + \langle DF, g \rangle_{\mathfrak{H}}.$$

We refer to [30] and the references therein for a more detailed account of the properties of fractional Brownian motion and its associated Malliavin calculus (and to [2] for an introduction to the latter subject).

1.1.2 Stable convergence and technical results

Throughout the chapter we will make use of the notion of stable convergence provided in the next definition.

Definition 1.1.2. *Fix $d \geq 1$. Let F_n be a sequence of random variables with values in $\mathbb{R}^d$, all defined on the probability space $(\Omega, \mathscr{F}, P)$. Let F be a $\mathbb{R}^d$-valued random variable defined on some extended probability space $(\Omega', \mathscr{F}', P')$. That means $\Omega \in \mathscr{F}'$, and the restriction of $\mathscr{F}'$ and P' to Ω coincide with $\mathscr{F}$ and P, respectively. We say that F_n converges stably to F, if*

$$\lim_{n\to\infty} \mathbb{E}\left[Ze^{i\langle \lambda, F_n\rangle_{\mathbb{R}^d}}\right] = \mathbb{E}'\left[Ze^{i\langle \lambda, F\rangle_{\mathbb{R}^d}}\right] \tag{1.10}$$

for every $\lambda \in \mathbb{R}^d$ and every bounded $\mathscr{F}$-measurable random variable Z.

In our setting, we assume that the fractional Brownian motion B^H is defined in a probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}$ is the P-completion of the σ-field generated by B^H. Then, condition (1.10) is equivalent to saying that the couple (B^H, F_n) converges in law to (B^H, F) in the space $C([0,1]) \times \mathbb{R}^d$ (see, for instance, [18, Chapter 4]).

The following lemmas which are a consequence of Theorem 3.1 and Theorem 3.2 in [4], are the main tool for the proofs of the results in Sections 3 and 4.

Lemma 1.1.3. *Let $\{F_n\}$ be a sequence of random elements with values in a complete separable metric space (X,ρ). Assume there are X-valued random elements $F_{n,\delta}$, $\{G_{n,\delta}\}$, G_δ and G such that*

1) $\forall 0<\delta<1$*:* $\lim_{n\to\infty} \mathbb{E}(\rho(F_n, F_{n,\delta})) = 0$

2) $\lim_{\delta\to 0^+} \limsup_{n\to\infty} \mathbb{E}(\rho(F_{n,\delta}, G_{n,\delta})) = 0$

3) For any $0<\delta<1$ *we have the convergence in law $G_{n,\delta} \to G_\delta$ as $n\to\infty$.*

4) G_δ converges in law to G as $\delta \to 0^+$

Then F_n converges in law to G as $n\to\infty$

Lemma 1.1.4. *Let F_n be a sequence of random elements with values in a separable metric space (X,ρ). Assume there are X-valued random elements G_n and G such that*

1) $\lim_{n\to\infty} \mathbb{E}(\rho(F_n, G_n)) = 0.$

2) G_n converges in law to G as $n\to\infty$.

Then F_n converges in law to G as $n\to\infty$.

We conclude this subsection with the following property of the Gamma function.

Lemma 1.1.5. *For any a,b positive*

$$\lim_{n\to\infty} \frac{\Gamma(n+a)n^{b-a}}{\Gamma(n+b)} = 1.$$

Proof. This is an application of the well known limit $\lim_{z\to\infty} \frac{\Gamma(z+1)}{\sqrt{z}\left(\frac{z}{e}\right)^z} = \sqrt{2\pi}$, or in other words, an application of Stirling's formula. □

1.2 Singular limits of sequences of Itô integrals

Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion. Denote by $\mathscr{F}_t$ the natural filtration generated by B. In this section, we will study the asymptotic behavior of two types of sequences of Itô integrals. First, we discuss a class of integrals on $[0,1]$ that include a sequence of deterministic kernels ϕ_n converging to a delta function based at 1. Secondly, we apply our argument to stochastic convolutions with similar asymptotic behavior.

1.2.1 Stochastic integrals concentrating at $t = 1$

Consider a sequence of bounded, nonnegative Borel measurable functions $\phi_n(t)$ on $[0,1]$, that satisfies the following condition:

$$(\mathbf{h1}): \qquad \int_0^1 \phi_n^2(t)\,dt \to L > 0 \quad \text{and} \quad \int_0^{1-\delta} \phi_n^2(t)\,dt \to 0 \quad \text{for all } \delta \in (0,1).$$

The aim of this section is to study the asymptotic behavior of the sequence of Itô integrals

$$F_n := \int_0^1 \phi_n(t) u_t \, dB_t, \quad n \geq 1, \tag{1.11}$$

where $u = \{u_t, t \in [0,1]\}$ is an appropriate adapted and jointly measurable process.

Theorem 1.2.1. *Suppose the process u is continuous in $L^2(\Omega)$ at $t = 1$. Assume **(h1)** holds and one of the following conditions is satisfied*

(i) $\sup_{s\in[0,1]} \mathbb{E}(u_s^2) < \infty$.

(ii) $\int_0^1 \mathbb{E}(u_t^2)\,dt < \infty$ *and*

(h2): *for all* $\delta \in (0,1)$ $\sup_{t\in[0,1-\delta]} \phi_n(t) \to 0$ *as* $n \to \infty$.

Then, the sequence F_n introduced in (1.11) converges stably, as $n \to \infty$ to $\sqrt{L} u_1 Z$, where Z is a $N(0,1)$ random variable independent of the process B.

Proof. Our goal is to apply Lemma 1.1.3 with the complete, separable metric space (X,ρ) where $X := C([0,1]) \times \mathbb{R}$ and $\rho((x,y),(x',y')) = \|x-x'\|_\infty + |y-y'|$. For the sake of simplicity, we will only prove part (i), the proof of part (ii) being similar.

We divide the proof in 4 steps.

Step 1.

Take $\bar{F}_n = \left(B, \int_0^1 \phi_n(t) u_t dB_t\right)$ and for each $\delta \in (0,1)$ set $F_{n,\delta} = \left(B, \int_{1-\delta}^1 \phi_n(t) u_t dB_t\right)$. Then

$$\mathbb{E}(\rho(\bar{F}_n, F_{n,\delta})) = \mathbb{E}\left(\left|\int_0^{1-\delta} \phi_n(t) u_t dB_t\right|\right)$$

which converges to 0 as $n \to \infty$ by (i) and hypothesis **(h1)**, because

$$\mathbb{E}\left[\left(\int_0^{1-\delta} \phi_n(t) u_t dB_t\right)^2\right] \leq \sup_{s\in[0,1]} \mathbb{E}(u_s^2) \int_0^{1-\delta} \phi_n^2(t)\, dt \to 0.$$

Step 2.

Take $G_{n,\delta} = \left(B, \int_{1-\delta}^1 \phi_n(t) u_{1-\delta}\, dB_t\right)$. Note that

$$\mathbb{E}(\rho(F_{n,\delta}, G_{n,\delta})) = \mathbb{E}\left(\left|\int_{1-\delta}^1 \phi_n(t)(u_t - u_{1-\delta})\, dB_t\right|\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\left(\int_{1-\delta}^1 \phi_n(t)(u_t - u_{1-\delta})\, dB_t\right)^2\right] &= \int_{1-\delta}^1 \mathbb{E}\left[(u_t - u_{1-\delta})^2\right] \phi_n^2(t)\, dt \\ &\leq \sup_{t\in[1-\delta,1]} \mathbb{E}\left[(u_t - u_{1-\delta})^2\right] \int_0^1 \phi_n^2(t)\, dt,\end{aligned}$$

which implies $\lim_{\delta\downarrow 0^+} \limsup_{n\to\infty} \mathbb{E}(\rho(F_{n,\delta}, G_{n,\delta})) = 0$, due to **(h1)** and the $L^2(\Omega)$-continuity of u_t at 1.

Step 3.

Note that $G_{n,\delta} = \left(B, \int_{1-\delta}^1 \phi_n(t) u_{1-\delta}\, dB_t\right)$ can be written as

$$G_{n,\delta} = \left(B, u_{1-\delta} \int_{1-\delta}^1 \phi_n(t)\, dB_t\right) := (B, u_{1-\delta} Z_{n,\delta}).$$

We want to show that for each $\delta \in (0,1)$, $(B, Z_{n,\delta})$ converges in law to $(B, \sqrt{L}Z)$ as $n \to \infty$ with Z independent of B. In view of the fact that B is a Gaussian process, this will follow from the next two properties:

i) For any $\delta \in (0,1)$, $\mathbb{E}(Z_{n,\delta}^2) \to L$ as $n \to \infty$ due to **(h1)**.

ii) For any $\delta \in (0,1)$, and any $t_0 \in (0,1)$, $\lim_{n\to\infty} \mathbb{E}(Z_{n,\delta} B_{t_0}) = 0$. In fact, for $t_0 \in (0,1)$ we have, in view of **(h1)**, that

$$\mathbb{E}(Z_{n,\delta} B_{t_0}) = \int_{1-\delta}^{t_0} \phi_n(t)\, dt \leq \sqrt{t_0} \left(\int_0^{t_0} \phi_n^2(t)\, dt\right)^{1/2} \to 0, \text{ as } n \to \infty.$$

In particular Z is independent of B_{t_0} for every $0 \leq t_0 < 1$ and consequently independent also from B_1.

As a corollary, for each $\delta \in (0,1)$, $G_{n,\delta}$ converges in law to $(B, \sqrt{L} u_{1-\delta} Z)$ as $n \to \infty$ with Z independent of B.

Step 4.

Set $G_\delta = (B, \sqrt{L} u_{1-\delta} Z)$. It is clear that G_δ converges in law to $\left(B, \sqrt{L} u_1 Z\right)$ as $\delta \to 0$. It follows from Steps 1 to 3 and Lemma 1.1.3 that $(B, \int_0^1 \phi_n(t) u_t dB_t)$ converges in law in the space $C([0,1] \times \mathbb{R})$ to $(B, \sqrt{L} u_1 Z)$. This completes the proof. □

An example of a sequence of functions satisfying condition **(h1)** with $L = \frac{1}{2}$ is

$$\phi_n(t) = \sqrt{n} t^n.$$

Indeed, a direct calculation for condition **(h1)** gives

$$n\int_0^1 t^{2n}dt = \frac{n}{2n+1} \to \frac{1}{2} \quad \text{as } n\to\infty,$$

and

$$n\int_0^{1-\delta} t^{2n}dt = \frac{n(1-\delta)^{2n}}{2n+1} \to 0 \;\text{ as } n\to\infty \text{ for all } \delta\in(0,1).$$

Thus we have proved the following.

Proposition 1.2.2. *The sequence of Itô integrals*

$$\sqrt{n}\int_0^1 t^n B_t dB_t$$

converges stably, as $n\to\infty$, to $\frac{1}{\sqrt{2}}B_1 Z$, where Z is a $N(0,1)$ random variable independent of the process B.

Remark

(i) We note that Proposition 1.2.2 was obtained by Nourdin, Nualart & Peccati in [4, Proposition 3.7] as a corollary of a theorem proved by Malliavin calculus.

(ii) Theorem 1.2.1 can be extended to processes u continuous in probability at 1 that satisfy $\sup_{s\in[0,1]}|u_s|<\infty$, a.s or $\int_0^1 u_s^2 ds<\infty$ a.s and **(h2)**. Under these assumptions, the Itô integral of u is defined using the convergence in probability and the convergence in law is proved using the truncated sequence $(u_t\wedge M)\vee(-M)$, where $M>0$ is an integer.

The next result is an extension of Theorem 1.2.1 to the case of double stochastic Itô integrals, which is proved by similar arguments.

Theorem 1.2.3. *Let $u=\{u_{s,t}, 0\le s\le t\le 1\}$ be a two-parameter process continuous at $(1,1)$ in the $L^2(\Omega)$ sense. Assume that $u_{s,t}$ is $\mathscr{F}_s$-measurable for $s\le t$,* **(h1)** *holds and one of the following conditions is satisfied* *(i)* $\sup_{(s,t)\in[0,1]^2} \mathbb{E}(u_{s,t}^2) < \infty$.

(ii) $\int_0^1 \int_0^t \mathbb{E}(u_{t,s}^2)\, ds\, dt < \infty$ *and*

(h3): *for all* $\delta \in (0,1)$, $\left(\sup_{0\le t\le 1-\delta} \phi_n(t)\right)\left(\sup_{0\le t\le 1} \phi_n(t)\right) \to 0$ *as* $n \to \infty$.

Consider the sequence of iterated Itô integrals

$$F_n := \int_0^1 \int_0^t \phi_n(s)\phi_n(t) u_{s,t} dB_s\, dB_t, \quad n \geq 1,$$

Then F_n *converges stably, as* $n \to \infty$ *to* $\frac{1}{2} L u_{1,1} H_2(Z)$*, where* $Z \sim N(0,1)$ *is independent of the process* B*, and* $H_2(x) = x^2 - 1$ *is the second Hermite polynomial.*

Proof. We proceed as in the proof of Theorem 1.2.1. We will only prove part (ii), the proof of part (i) being similar.

Step 1:

Take $\bar{F}_n = (B, \int_0^1 \int_0^t \phi_n(s)\phi_n(t) u_{s,t} dB_s\, dB_t)$ and for $\delta \in (0,1)$ set

$$F_{n,\delta} = (B, \int_{1-\delta}^1 \int_{1-\delta}^t \phi_n(s)\phi_n(t) u_{s,t} dB_s\, dB_t).$$

Then

$$\mathbb{E}(\rho(\bar{F}_n, F_{n,\delta})) = \mathbb{E}\left(\left| \int_0^1 \int_0^{t\wedge(1-\delta)} \phi_n(t)\phi_n(s) u_{s,t} dB_s dB_t \right| \right)$$

converges to 0 as $n \to \infty$ because by condition (ii)

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1 \int_0^{t\wedge(1-\delta)} \phi_n(t)\phi_n(s) u_{s,t} dB_s dB_t \right)^2 \right] &= \int_0^1 \int_0^{t\wedge(1-\delta)} \phi_n^2(t)\phi_n^2(s) \mathbb{E}(u_{s,t}^2)\, ds\, dt \\
&\leq \left(\sup_{0\le t\le 1-\delta} \phi_n(t) \right)^2 \left(\sup_{0\le t\le 1} \phi_n(t) \right)^2 \\
&\quad \times \int_0^1 \int_0^t \mathbb{E}(u_{s,t}^2)\, ds\, dt \to 0.
\end{aligned}$$

Step 2.

Take $G_{n,\delta} = \left(B, \int_{1-\delta}^1 \int_{1-\delta}^t \phi_n(t)\phi_n(s)u_{1-\delta,1-\delta}\, dB_s dB_t\right)$. Then

$$\mathbb{E}(\rho(F_{n,\delta}, G_{n,\delta})) = \mathbb{E}\left(\left|\int_{1-\delta}^1 \int_{1-\delta}^t \phi_n(t)\phi_n(s)(u_{s,t} - u_{1-\delta.1-\delta})\, dB_s dB_t\right|\right).$$

Therefore

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{1-\delta}^1 \int_{1-\delta}^t \phi_n(t)\phi_n(s)(u_{s,t} - u_{1-\delta.1-\delta})\, dB_s dB_t\right)^2\right] \\
&= \int_{1-\delta}^1 \int_{1-\delta}^t \mathbb{E}\left[(u_{s,t} - u_{1-\delta.1-\delta})^2\right] \phi_n^2(t)\phi_n^2(s)\, ds\, dt \\
&\leq \sup_{t\in[1-\delta,1]} \mathbb{E}\left[(u_{s,t} - u_{1-\delta,1-\delta})^2\right] \left(\int_0^1 \phi_n^2(t)\, dt\right)^2,
\end{aligned}$$

which implies $\lim_{\delta\downarrow 0^+} \limsup_{n\to\infty} \mathbb{E}(\rho(F_{n,\delta}, G_{n,\delta})) = 0$, due to condition **(h1)** and the $L^2(\Omega)$- continuity of $u_{s,t}$ at $(1,1)$.

Step 3.

The term $G_{n,\delta}$ can be written as

$$G_{n,\delta} = \left(B, u_{1-\delta,1-\delta} \int_{1-\delta}^1 \int_{1-\delta}^t \phi_n(t)\phi_n(s)\, dB_s\, dB_t\right) := (B, u_{1-\delta,1-\delta} R_{n,\delta}).$$

Furthermore, in view of Proposition 4.1.2 in [31], $R_{n,\delta}$ can be expressed

$$R_{n,\delta} = \int_{1-\delta}^1 \int_{1-\delta}^t \phi_n(s)\phi_n(t)\, dB_s\, dB_t = \frac{1}{2}\|\phi_n \mathbf{1}_{[1-\delta,1]}\|_{L^2}^2\, H_2\left(\frac{\int_{1-\delta}^1 \phi_n(t) dB_t}{\|\phi_n \mathbf{1}_{[1-\delta,1]}\|_{L^2([0,1])}}\right),$$

where $H_2(x) = x^2 - 1$ is the second Hermite polynomial.

Set $S_{n,\delta} := \frac{\int_{1-\delta}^1 \phi_n(t) dB_t}{\|\phi_n \mathbf{1}_{[1-\delta,1]}\|_{L^2([0,1])}}$. A similar argument to the one used in Step 3 of Theorem 1.2.1 proves that $(B, S_{n,\delta})$ converges in law to (B, Z) with Z independent of B, and it follows that

$(B, R_{n,\delta})$ converges in law to $(B, \frac{1}{2}LH_2(Z))$. As a corollary we obtain the convergence in law of $(B, u_{1-\delta,1-\delta}R_{n,\delta})$ to $(B, \frac{1}{2}Lu_{1-\delta,1-\delta}H_2(Z))$, as n tends to ∞.

Step 4.

Set $G_\delta = (B, \frac{1}{2}Lu_{1-\delta,1-\delta}H_2(Z)$. It is clear that G_δ converges in law to $G := (B, \frac{1}{2}Lu_{1,1}H_2(Z))$ as $\delta \to 0$. Then, the conclusion follows from Steps 1 to 3 and Lemma 1.1.3. $\square$

Remarks:

(i) The previous theorem applies to the particular case $\phi_n(t) = \sqrt{n}t^n$, as before.

(ii) One can consider the more general situation of a sequence of bounded symmetric functions $\phi_n(s,t)$ on $[0,1]^2$, satisfying the following conditions:

(h12): For all $\delta \in (0,1)$

$$\lim_{n\to\infty} \int_{1-\delta}^{1} \int_{1-\delta}^{1} \phi_n^2(s,t)dsdt = \lim_{n\to\infty} \int_0^1 \int_0^1 \phi_n^2(s,t)dsdt = L > 0.$$

(h22): For any $\delta \in (0,1]$, $\sup_{0\le s\le 1-\delta, 0\le t\le 1} |\phi_n(s,t)| \to 0$ as $n \to \infty$.

In this case we need to compute the limit in law of $I_2(\phi_n \mathbf{1}_{[1-\delta,1]^2})$, which is a more complicated problem that requires additional conditions on the sequence ϕ_n. We will not treat this problem here.

Theorem 1.2.3 can be extended to higher dimensions. The proof is similar and omitted.

Theorem 1.2.4. *Let $u = \{u_{t_1,\dots,t_m}, 0 \le t_1 \le \cdots \le t_m \le 1\}$ be an m-parameter stochastic process, continuous at $(1,\dots,1) \in \mathbb{R}^m$ in the $L^2(\Omega)$ sense. Assume that $u_{t_1,\dots,t_m}$ is $\mathscr{F}_{t_1}$-measurable,* ***(h1)*** *holds and one of the following condition is satisfied:*

i) $\sup_{t_1,\dots,t_m} \mathbb{E}(u^2_{t_1,\dots,t_m}) < \infty$.

ii) $\int_0^1 \int_0^{t_m} \cdots \int_0^{t_2} \mathbb{E}\left(u^2_{t_1,\dots,t_m}\right) dt_1 dt_2 \cdots dt_m < \infty$ *and*

(h3m): *for any* $\delta \in (0,1)$, $\left(\sup_{0\le t\le 1-\delta} \phi_n(t)\right) \left(\sup_{0\le t\le 1} \phi_n(t)\right)^{m-1} \to 0$ *as* $n \to \infty$.

Consider the sequence of iterated Itô integrals

$$F_n := \int_0^1 \int_0^{t_m} \cdots \int_0^{t_2} \phi_n(t_1)\cdots\phi_n(t_m) u_{t_1,\dots,t_m} dB_{t_1} \cdots dB_{t_{m-1}} dB_{t_m}, \quad n \geq 1,$$

Then, F_n converges stably, as $n \to \infty$, to $(m!)^{-1} L^{\frac{m}{2}} u_{1,\dots,1} H_m(Z)$, where $Z \sim N(0,1)$ is independent of the process B and H_m is the mth Hermite polynomial.

1.2.2 Asymptotic behavior of stochastic convolutions

As before, let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion and set $H = L^2([0,\infty))$. Consider a nonnegative, bounded, Borel measurable function $\psi(x)$ on $\mathbb{R}$, such that $\int_{-\infty}^{\infty} \psi^2(x)dx = 1$. Let $\psi_n(x) = \sqrt{n}\psi(nx)$. Then ψ_n^2 is an approximation to the identity.

Let $u = \{u_t, t \geq 0\}$ be an adapted, jointly measurable and square integrable process. Define the stochastic convolution

$$(u *_B \psi_n)_t = \int_0^\infty u_s \psi_n(t-s) dB_s, \quad t \geq 0.$$

In this subsection, we are interested in the asymptotic behavior of $(u *_B \psi_n)_t$ as n tends to infinity. The limit in law will have the form $u_t Z_t$, where Z is a Gaussian process independent of B.

The following theorem is the main result of this subsection.

Theorem 1.2.5. *Consider a nonnegative, bounded, Borel measurable function $\psi(x)$ on $\mathbb{R}$, such that $\int_{-\infty}^{\infty} \psi^2(x)dx = 1$. Assume $u = \{u_t, t \geq 0\}$ is an adapted, jointly measurable, square integrable process and continuous at a fixed time $t > 0$ in the $L^2(\Omega)$ sense. Assume one of the following conditions is satisfied:*

i) $\lim_{|x|\to\infty} |x|\psi^2(x) = 0$

ii) $\sup_s \mathbb{E}(u_s^2) < \infty$.

Then, the stochastic convolution $(u *_B \psi_n)_t$ converges stably to $u_t Z$ as $n \to \infty$, where Z is a standard Gaussian random variable independent of B.

Proof. Our plan is to use Lemma 1.1.4. The complete, separable space (X,ρ) will be $X := C([0,\infty))\times\mathbb{R}$ with $\rho((x,y),(x',y')) = \sum_{n=0}^{\infty} 2^{-n}\sup_{t\in[n,n+1]}|x(t)-x'(t)|+|y-y'|$. We divide the proof in 2 steps.

Step 1: Fix the continuity point $t>0$. Set

$$\bar{F}_n := (B,(u*_B\psi_n)_t).$$

Let α_n be a sequence decreasing to 0 so that $n\alpha_n\to\infty$. Set

$$G_n := (B,u_{(t-\alpha_n)_+}S_n) =: (B,H_n),$$

where $S_n = \displaystyle\int_{R_n(t)}\psi_n(t-s)\,dB_s$ with $R_n(t)=\{s\geq 0 : |t-s|\leq\alpha_n\}$. Note that S_n is a centered normal random variable with variance

$$\mathbb{E}(S_n^2) = \int_{R_n(t)}\psi_n^2(t-s)\,ds = \int_{|r|\leq\alpha_n, r\leq t}\psi_n^2(r)\,dr = \int_{|z|\leq n\alpha_n, z\leq nt}\psi^2(z)\,dz \to 1 \quad \text{as } n\to\infty.$$

Our goal is to show that

$$\lim_{n\to\infty}\mathbb{E}(\rho(\bar{F}_n,G_n)) = 0 \tag{1.12}$$

for which it suffices to show $(u*_B\psi_n)_t - H_n$ converges to 0 in $L^2(\Omega)$.

Since $u_{(t-\alpha_n)_+}$ is $\mathscr{F}_{(t-\alpha_n)_+}$ measurable we can write

$$H_n = \int_{R_n(t)} u_{(t-\alpha_n)_+}\psi_n(t-s)\,dB_s$$

and therefore

$$\mathbb{E}(H_n^2) = \int_{R_n(t)}\mathbb{E}(u^2_{(t-\alpha_n)_+})\psi_n^2(t-s)\,ds = \mathbb{E}(u^2_{(t-\alpha_n)_+})\int_{|s|\leq\alpha_n, s\leq t}\psi_n^2(s)\,ds \to \mathbb{E}(u_t^2).$$

On the other hand, by Ito's isometry property we can write

$$\mathbb{E}\left((u *_B \psi_n)_t^2\right) = \int_0^\infty \mathbb{E}(u_s^2)\psi_n^2(t-s)\,ds.$$

That means, $\mathbb{E}\left((u *_B \psi_n)_t^2\right)$ is the convolution of $s \to E(u_s^2)$ with ψ_n^2 and in view of either condition i) or ii) and Theorems 9.8, 9.9 in [45], we deduce

$$\lim_{n\to\infty} \mathbb{E}\left[(u *_B \psi_n)_t^2\right] = \mathbb{E}(u_t^2).$$

Finally, by Itô's isometry and the L^2-continuity of u at t

$$\begin{aligned}\mathbb{E}\left[(u *_B \psi_n)_t H_n\right] &= \int_{R_n(t)} \mathbb{E}(u_s u_{(t-\alpha_n)_+})\psi_n^2(t-s)\,ds \\ &= \int_{R_n(t)} \mathbb{E}(u_{(t-\alpha_n)_+}(u_s - u_{(t-\alpha_n)_+}))\psi_n^2(t-s)\,ds \\ &+ \mathbb{E}(u_{(t-\alpha_n)_+}^2)\int_{R_n(t)} \psi_n^2(t-s)\,ds \to \mathbb{E}(u_t^2),\end{aligned}$$

as $n \to \infty$. Thus $(u *_B \psi_n)_t - H_n \xrightarrow{L^2(\Omega)} 0$ as $n \to \infty$, and (1.12) holds.

Step 2: For each n, $u_{(t-\alpha_n)_+}$ and S_n are independent random variables such that $u_{(t-\alpha_n)_+}$ converges to u_t and $\mathrm{Var}(S_n^2) \to 1$. This implies that G_n converges in law to $(B, u_t Z)$ with Z a standard normal random variable independent of B, and the result follows from Lemma 1.1.4. □

As in the proof of Theorem 1.2.5, if α_n is a sequence decreasing to 0 so that $n\alpha_n \to \infty$, we can consider for each $t > 0$ the sequence of random variables

$$S_n^t := \int_{|t-r|\le\alpha_n,\, r\ge 0} \psi_n(t-r)dB_r. \tag{1.13}$$

The next lemma establishes the asymptotic behavior of the sequence of processes $\{S_n^t, t > 0\}$.

Lemma 1.2.6. *The finite-dimensional distributions of the process $\{S_n^t, t > 0\}$ introduced in (1.13) converge stably to those of a centered Gaussian process $\{Z_t, t > 0\}$ independent of B and with* *covariance function given by*

$$\mathbb{E}(Z_t Z_s) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases} \tag{1.14}$$

Proof. Let $0 < t_1 < t_2 < \cdots < t_k$. We need to prove the convergence in law

$$(B, S_n^{t_1}, \ldots, S_n^{t_k}) \xrightarrow{Law} (B, Z_{t_1}, \ldots, Z_{t_k})$$

in the space $C(\mathbb{R}_+) \times \mathbb{R}^k$. We can choose N large enough so that for $n \geq N$, the Gaussian random variables $S_n^{t_i}$ become uncorrelated and hence independent. Then as in the proof of Theorem 1.2.5, it holds that

$$(S_n^{t_1}, \ldots, S_n^{t_k}) \xrightarrow{Law} (Z_{t_1}, \ldots, Z_{t_k}),$$

where the random vector $(Z_{t_1}, \ldots, Z_{t_k})$ has a standard Gaussian distribution on $\mathbb{R}^k$ and is independent of B. This completes the proof. □

Notice that we cannot expect that the convergence in Lemma 1.2.6 holds in $C([0,\infty))$. Indeed, although under some mild conditions the stochastic convolution has a continuous version, the process Z does not have a continuous version.

The following proposition establishes the convergence of the stochastic convolution as a process in the sense of the finite-dimensional distributions.

Proposition 1.2.7. *Under the assumptions of Theorem 1.2.5, suppose that the process u is continuous in $[0,\infty)$ in the $L^2(\Omega)$-sense. Then the finite-dimensional distributions of the process $\{(u *_B \psi_n)_t, t > 0\}$ converge stably to those of $\{u_t Z_t, t > 0\}$, where $\{Z_t, t > 0\}$ is a Gaussian process independent of B with covariance function given by (1.14).*

Proof. Let $0 < t_1 < t_2 < \cdots < t_k$. We want to show that

$$(B, (u *_B \psi_n)_{t_1}, (u *_B \psi_n)_{t_2}), \ldots, (u *_B \psi_n)_{t_k}) \xrightarrow{Law} (B, u_{t_1} Z_{t_1}, u_{t_2} Z_{t_2}, \ldots, u_{t_1} Z_{t_k}), \tag{1.15}$$

where the random vector $(Z_{t_1},\dots,Z_{t_k})$ has a standard Gaussian distribution on $\mathbb{R}^k$ and is independent of B. As in the proof of Theorem 1.2.5, if α_n is a sequence decreasing to 0 such that $n\alpha_n \to \infty$, we can consider for each $t>0$ the sequence of random variables S_n^t defined in (1.13). Then, we have that, by the proof of theorem 1.2.5, for each $i=1,\dots,k$,

$$(u *_B \psi_n)_{t_i} - u_{(t_i-\alpha_n)_+} S_n^{t_i} \xrightarrow{L^2} 0.$$

Also, by the L^2-continuity of u and the Cauchy-Schwartz inequality, we can write

$$u_{(t_i-\alpha_n})_+ S_n^{t_i} - u_{t_i} S_n^{t_i} \xrightarrow{L^1} 0.$$

In particular, the above convergence holds also in probability, so that

$$A_n^{t_i} := (u *_B \psi_n)_{t_i} - u_{t_i} S_n^{t_i} \xrightarrow{P} 0$$

for $i=1,\dots,k$. As a consequence,

$$(A_n^{t_1}, A_n^{t_2}, \dots, A_n^{t_k}) \xrightarrow{P} (0,0,\dots,0).$$

Then by Slutsky's theorem (1.15) follows from the convergence in law

$$(B, u_{t_1} S_n^{t_1}, \dots, u_{t_k} S_n^{t_k}) \xrightarrow{Law} (B, u_{t_1} Z_{t_1}, \dots, u_{t_k} Z_{t_k}),$$

which is a consequence of Lemma 1.2.6. This completes the proof. □

1.3 Skorohod integrals with respect to fractional Brownian Motion

Consider a fractional Brownian motion $B^H = \{B_t^H, t \in [0,1]\}$ with Hurst parameter $H \in (0,1)$. That is, B^H is a zero mean Gaussian process with covariance function (1.4). In this section, we will study the asymptotic behavior as $n \to \infty$ of a sequence of Skorohod integrals of the form

$$F_n = \int_0^1 \phi_n(t) u_t \, \delta B_t^H, \quad n \geq 1, \tag{1.16}$$

where u is a stochastic process verifying some suitable conditions. We split our study in two cases according to whether $H > 1/2$ or $H < 1/2$.

1.3.1 The case $H > 1/2$

We will assume the following conditions on the sequence ϕ_n of nonnegative and bounded functions:

(h4): $\lim_{n\to\infty} \|\phi_n\|_{\mathfrak{H}}^2 = L > 0$.

(h5): $\lim_{n\to\infty} \|\phi_n\|_r = 0$ for some $r < \frac{1}{H}$ (where here, and in the sequel, $||\cdot||_r$ denotes the L^r-norm on $[0,1]$).

We are now ready to state and prove the main results of this section.

Theorem 1.3.1. *Assume B^H is a fractional Brownian motion with Hurst parameter $H > 1/2$. Consider a sequence of nonnegative and bounded functions ϕ_n on $[0,1]$ satisfying conditions* **(h3)**, **(h4)** *and* **(h5)**. *Let u be a stochastic process satisfying the following conditions:*

(i) For any $t \in [0,1]$, $u_t \in \mathbb{D}^{1,2}$ and the mapping $t \to \|u_t\|_{1,2}$ belongs to $\mathfrak{H}$.

(ii) u_t is continuous in $\mathbb{D}^{1,2}$ at $t = 1$.

(iii) $\int_0^1 (\mathbb{E}[|D_s u_1|])^p ds < \infty$ where $\frac{1}{p} + \frac{1}{r} = 2H$, and r is the number appearing in condition **(h5)**.

Consider the sequence of Skorohod integrals introduced in (1.16). Then F_n converges stably as $n \to \infty$ to $u_1 \sqrt{L} Z$, where Z is a $N(0,1)$ random variable independent of B^H.

Proof. Notice first that conditions (i) and (ii) imply that $\phi_n(t) u_t$ belongs to $\mathbb{D}^{1,2}(\mathfrak{H}) \subset \mathrm{Dom}\delta$. In the context of Lemma 1.1.4, the complete, separable space (X, ρ) will be $X := C([0,1]) \times \mathbb{R}$ with $\rho((x,y),(x',y')) = \|x - x'\|_\infty + |y - y'|$. We divide the proof in 3 steps.

Step 1: Set $\bar{F}_n := (B^H, \int_0^1 \phi_n(t) u_t \delta B_t^H)$ and $H_n := (B^H, \int_0^1 \phi_n(t) u_1 \delta B_t^H)$. In order to show that $\lim_{n\to\infty} \mathbb{E}(\rho(\bar{F}_n, H_n)) = 0$ it suffices to prove that

$$\mathbb{E}\left[\left(\int_0^1 \phi_n(t)(u_t - u_1)\delta B_t^H\right)^2\right] \to 0.$$

Denoting $\alpha_H = H(2H-1)$, in view of (1.9) we can write

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1 \phi_n(t)(u_t - u_1)\delta B_t^H\right)^2\right] &\le \mathbb{E}(\|\phi_n(t)(u_t-u_1)\|_{\mathfrak{H}}^2) + \mathbb{E}(\|\phi_n(t) D(u_t-u_1)\|_{\mathfrak{H}\otimes\mathfrak{H}}^2) \\
&= \alpha_H \int_0^1 \int_0^1 \phi_n(t)\phi_n(s) \mathbb{E}\Big[(u_t-u_1)(u_s-u_1\Big] |t-s|^{2H-2}\, dsdt \\
&\quad + \alpha_H \int_0^1 \int_0^1 \phi_n(t)\phi_n(s) \mathbb{E}\Big[\langle D(u_t-u_1), D(u_s-u_1)\rangle_{\mathfrak{H}}\Big] |t-s|^{2H-2} dsdt \\
&= A_{1,n} + A_{2,n}. \qquad (1.17)
\end{aligned}$$

Both terms in (1.17) are handled similarly and we will show the details only for the second one. Let $0 < \delta < 1$. Then, separating the second term in two integrals, yields

$$\begin{aligned}
A_{2,n} = \alpha_H &\int_0^1 \int_0^1 \mathbf{1}_{\{s\wedge t \le 1-\delta\}} \phi_n(t)\phi_n(s) \mathbb{E}\Big[\langle D(u_t-u_1), D(u_s-u_1)\rangle_{\mathfrak{H}}\Big] |t-s|^{2H-2} dsdt \\
&+ \alpha_H \int_{1-\delta}^1 \int_{1-\delta}^1 \phi_n(t)\phi_n(s) \mathbb{E}\Big[\langle D(u_t-u_1), D(u_s-u_1)\rangle_{\mathfrak{H}}\Big] |t-s|^{2H-2} dsdt. \qquad (1.18)
\end{aligned}$$

At this step, note that by condition (i)

$$\begin{aligned}
&\int_0^1 \int_0^1 |\mathbb{E}\left(\langle D(u_t-u_1), D(u_s-u_1)\rangle_{\mathfrak{H}}\right)| |t-s|^{2H-2} dsdt \\
&\quad \le \int_0^1 \int_0^1 \|u_t - u_1\|_{1,2} \|u_s - u_1\|_{1,2} |t-s|^{2H-2} dsdt < \infty.
\end{aligned}$$

So there is a constant C such that the first term in (1.18) is bounded by

$$C \sup_{s\wedge t \le 1-\delta} \phi_n(s)\phi_n(t),$$

which converges to 0 as $n \to \infty$ by condition **(h3)**.

On the other hand, for the second term in (1.18), it follows from Cauchy-Schwartz inequality that

$$\int_{1-\delta}^{1}\int_{1-\delta}^{1} \phi_n(t)\phi_n(s)\mathbb{E}\left[\langle D(u_t-u_1), D(u_s-u_1)\rangle_{\mathfrak{H}}\right]|t-s|^{2H-2}dsdt$$
$$\leq \sup_{t\in[1-\delta,1]} \mathbb{E}\left[\|D(u_t-u_1)\|_{\mathfrak{H}}^2\right]\left(\int_0^1\int_0^1 \phi_n(s)\phi_n(t)|t-s|^{2H-2}ds\,dt\right).$$

By condition **(h4)**, the sequence $\int_0^1\int_0^1 \phi_n(s)\phi_n(t)|t-s|^{2H-2}ds\,dt$ is bounded and by condition (ii) the first factor tends to zero as $\delta \to 0$. This shows that $A_{2,n}$ tends to zero as $n \to \infty$. Repeating the same argument, we obtain that $A_{1,n}$ tends to zero as $n \to \infty$.

Step 2: Set

$$G_n := (B^H, u_1\int_0^1 \phi_n(t)\delta B_t^H) := (B^H, u_1 B_{1,n}).$$

Applying Lemma 1.1.1 we obtain

$$E(\rho(H_n, G_n)) = \alpha_H \mathbb{E}(|B_{2,n}|), \tag{1.19}$$

where $B_{2,n} = \int_0^1\int_0^1 |t-s|^{2H-2}\phi_n(t)D_s u_1\,ds\,dt$. Let p be as in the statement of the theorem and note that $p > 1$. Applying Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$, yields

$$\mathbb{E}[|B_{2,n}|] \leq \left(\int_0^1 (\mathbb{E}[|D_s u_1|])^p ds\right)^{\frac{1}{p}} \left(\int_0^1 \left(\int_0^1 |t-s|^{2H-2}\phi_n(t)dt\right)^q ds\right)^{\frac{1}{q}}.$$

The second factor is the L^q-norm of the fractional integral of order $2H-1$ of the function ϕ_n on $[0,1]$. By the Hardy-Littlewood inequality, this factor is bounded by a constant times $\|\phi_n\|_{L^r([0,1])}$, where $\frac{1}{r} = \frac{1}{q}+2H-1 = 2H-\frac{1}{p}$. Taking into account conditions (iii) and **(h5)**, we deduce that (1.19) converges to 0 as $n \to \infty$ which together with Step 1 implies $\mathbb{E}(\rho(\bar{F}_n, G_n)) \to 0$ as $n \to \infty$.

Step 3: In order to complete the proof of the theorem, we need to show G_n converges in law in the space $C([0,1]) \times \mathbb{R}$ to $G := (B^H, \sqrt{L}u_1 Z)$ where Z is a $N(0,1)$ random variable independent of B^H. To this end, it suffices to show that $(B^H, B_{1,n})$ converges in law in the space $C([0,1]) \times \mathbb{R})$ to $(B^H, \sqrt{L}Z)$. In view of the fact that B^H is a Gaussian process, this will follow from the next two properties:

(a): $\lim_{n\to\infty} \mathbb{E}[B_{1,n}^2] = L$, which follows from property **(h4)**.

(b): For any $t_0 \in [0,1]$, $\lim_{n\to\infty} \mathbb{E}[B_{1,n} B_{t_0}^H] = 0$. In fact, using property **(h5)**, we obtain for $\frac{1}{r} + \frac{1}{r'} = 1$,

$$\begin{aligned}\mathbb{E}[B_{1,n} B_{t_0}^H] &= \alpha_H \int_0^1 \int_0^{t_0} \phi_n(t) |t-s|^{2H-2} \, ds \, dt \\ &\le \alpha_n \|\phi_n\|_r \left(\int_0^1 \left(\int_0^{t_0} |t-s|^{2H-2} ds \right)^{r'} dt \right)^{1/r'} \to 0,\end{aligned}$$

as $n \to \infty$. □

Theorem 1.3.1 can be applied to the example $\phi_n(t) = n^H t^n$, and in this case, $L = H\Gamma(2H)$. Indeed, condition **(h3)** is obvious. Condition **(h4)** follows from Lemma 1.3.2 below. Condition **(h5)** holds for any $r < \frac{1}{H}$. This means that in condition (iii), it suffices to show that the integral is bounded for some $p > \frac{1}{H}$.

Lemma 1.3.2. *For any $n, m \in \mathbb{N}$ and $r > -1$*

$$\int_0^1 \int_0^1 t^n s^m |t-s|^r \, ds \, dt = \frac{\Gamma(m+1)\Gamma(r+1)}{(n+m+r+2)\Gamma(2+m+r)} + \frac{\Gamma(n+1)\Gamma(r+1)}{(n+m+r+2)\Gamma(2+n+r)}.$$

In particular for $H > 1/2$

$$\lim_{n\to\infty} n^{2H} \int_0^1 \int_0^1 x^n y^n |x-y|^{2H-2} \, dy \, dx \; = \Gamma(2H-1).$$

Proof. First of all, note that using $y = zx$ yields

$$\int_0^x y^m (x-y)^r \, dy = x^{m+1+r} \int_0^1 z^m (1-z)^r \, dz = x^{m+1+r} B(m+1, r+1),$$

where B denotes the Beta function. Then

$$\begin{aligned}
&\int_0^1 \int_0^1 t^n s^m |t-s|^r \, ds\, dt \\
&= \int_0^1 \int_0^t t^n s^m (t-s)^r \, ds\, dt + \int_0^1 \int_0^s t^n s^m (s-t)^r \, dt\, ds \\
&= \int_0^1 t^{n+m+r+1} B(m+1, r+1)\, dt + \int_0^1 s^{n+m+r+1} B(n+1, r+1) \\
&= \frac{B(m+1,r+1) + B(n+1,r+1)}{n+m+r+2},
\end{aligned}$$

The first part of the lemma now follows from the well-known relationship between the Beta and Gamma functions. The second part follows by taking $n = m$ and using Lemma 1.1.5. □

1.3.2 The case $1/4 < H < 1/2$

We assume the following conditions on the sequence ϕ_n of nonnegative and bounded functions:

(h6): $\sup_n \int_0^1 (s^{2H-1} + (1-s)^{2H-1}) \phi_n^2(s) ds < \infty$.

(h7): For any $\delta \in (0, 1]$, we have

$$\lim_{n\to\infty} \int_0^{1-\delta} \left(\int_s^{1-\delta} |\phi_n(t) - \phi_n(s)| (t-s)^{H-\frac{3}{2}} dt \right)^2 ds = 0.$$

(h8): $\lim_{n\to\infty} \int_0^1 |(K_H^* \phi_n)(s)|^p ds = 0$ for some $p > 1$.

Theorem 1.3.3. *Assume B^H is a fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$. Consider a sequence of nonnegative and bounded functions ϕ_n on $[0,1]$ satisfying conditions* **(h3)***,* **(h4)***,* **(h6)***,* **(h7)** *and* **(h8)***. Let u be a stochastic process satisfying the following conditions:*

(i) For all $t \in [0,1]$, $u_t \in \mathbb{D}^{1,2}$.

(ii) The mapping $t \to u_t$ is Hölder continuous of order $\gamma > 1/2 - H$ from $[0,1]$ into $\mathbb{D}^{1,2}$.

(iii) We have

$$\int_0^1 \mathbb{E}(|(K_H^* Du_1)(s)|^q)ds < \infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* p *is the exponent appearing in condition* **(h8)**.

Consider the sequence of Skorohod integrals introduced in (1.16). Then F_n *converges stably as* $n\to\infty$ *to* $u_1\sqrt{L}Z$, *where* Z *is a* $N(0,1)$ *random variable independent of* B^H.

Proof. As in the case $H>1/2$, we divide the proof into 3 steps using the same complete, separable space (X,ρ).

Step 1: Set $\bar{F}_n := (B^H, \int_0^1 \phi_n(t)u_t\delta B_t^H)$ and $H_n := (B^H, \int_0^1 \phi_n(t)u_1\delta B_t^H)$. In order to show that $\lim_{n\to\infty}\mathbb{E}(\rho(\bar{F}_n,H_n))=0$ it suffices to prove that

$$\mathbb{E}\left[\left(\int_0^1 \phi_n(t)(u_t-u_1)\delta B_t^H\right)^2\right]\to 0.$$

As in the proof of theorem 1.3.1, we can write

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1 \phi_n(t)(u_t-u_1)\delta B_t^H\right)^2\right] &\le \mathbb{E}(\|\phi_n(t)(u_t-u_1)\|_{\mathscr{H}}^2)+\mathbb{E}(\|\phi_n(t)D(u_t-u_1)\|_{\mathscr{H}\otimes\mathscr{H}}^2)\\ &=: C_{1,n}+C_{2,n}.\end{aligned}$$

We only work with $C_{2,n}$, the analysis of $C_{1,n}$ being similar by changing $\phi_n(t)D(u_t-u_1)$ and $\mathfrak{H}$ appropriately by $\phi_n(t)(u_t-u_1)$ and $\mathbb{R}$ in the argument below. We have, using (1.8) and (1.7),

$$\begin{aligned}C_{2,n} = \mathbb{E}\Bigg(\Bigg\| & K_H(1,s)\phi_n(s)D(u_s-u_1)\\ &+\int_s^1 (\phi_n(t)D(u_t-u_1)-\phi_n(s)D(u_s-u_1))\frac{\partial K_H}{\partial t}(t,s)dt\Bigg\|_{L^2([0,1];\mathfrak{H})}^2\Bigg).\end{aligned}$$

Since

$$(\phi_n(t)D(u_t-u_1)-\phi_n(s)D(u_s-u_1)) = \phi_n(s)D(u_t-u_s)+(\phi_n(t)-\phi_n(s))D(u_t-u_1),$$

we obtain

$$\begin{aligned}
C_{2,n} &\le 9\mathbb{E}\left(\|K_H(1,s)\phi_n(s)D(u_s-u_1)\|^2_{L^2([0,1];\mathfrak{H})}\right) \\
&+9\mathbb{E}\left(\left\|\int_s^1 \phi_n(s)D(u_t-u_s)\frac{\partial K_H}{\partial t}(t,s)dt\right\|^2_{L^2([0,1];\mathfrak{H})}\right) \\
&+9\mathbb{E}\left(\left\|\int_s^1 (\phi_n(t)-\phi_n(s))D(u_t-u_1)\frac{\partial K_H}{\partial t}(t,s)dt\right\|^2_{L^2([0,1];\mathfrak{H})}\right) \\
&=: R_{1,n}+R_{2,n}+R_{3,n}.
\end{aligned}$$

To handle the term $R_{1,n}$ we note that, by (1.5), there is a constant d_H such that

$$K(1,s)^2 \le d_H((1-s)^{2H-1}+s^{2H-1}). \tag{1.20}$$

We will denote by C a generic constant that may vary from line to line. Then by Minkowski's inequality and condition (ii) for any $\delta \in [0,1)$ we obtain

$$\begin{aligned}
R_{1,n} &\le 9\mathbb{E}\left(\int_0^1 K_H(1,s)^2\phi_n^2(s)\|D(u_s-u_1)\|^2_{\mathfrak{H}}\,ds\right) \\
&\le C\int_0^1 K_H(1,s)^2\phi_n^2(s)\|D(u_s-u_1)\|^2_{L^2[\Omega;\mathfrak{H}]}\,ds \\
&\le C\int_0^{1-\delta} K_H(1,s)^2\phi_n^2(s)(1-s)^{2\gamma}ds \\
&+C\int_{1-\delta}^1 K_H(1,s)^2\phi_n^2(s)(1-s)^{2\gamma}ds \\
&=: R_{12,n}+R_{22,n}.
\end{aligned}$$

The term $R_{12,n}$ can be estimated as follows

$$R_{12,n} \le C \sup_{0\le s\le 1-\delta} \phi_n^2(s) \int_0^1 K_H(1,s)^2(1-s)^{2\gamma} ds,$$

Taking into account that $\int_0^1 K_H(1,s)^2(1-s)^{2\gamma} ds < \infty$, we deduce from condition **(h3)** that $R_{12,n}$ converges to zero as $n \to \infty$. For $R_{22,n}$ we can write

$$R_{22,n} \le C(\delta)^{2\gamma} \int_0^1 K_H(1,s)^2 \phi_n^2(s) ds.$$

From (1.20) and condition **(h6)**, we deduce that $\sup_n R_{22,n} \to 0$ as $\delta \downarrow 0$. Therefore, we have proved that

$$\lim_{n\to\infty} R_{1,n} = 0. \tag{1.21}$$

Concerning the term $R_{2,n}$, using Minkowski's inequality, the estimate (1.6) and condition (ii), we obtain

$$\begin{aligned}
R_{2,n} &= 9\mathbb{E}\left(\left\| \int_s^1 \phi_n(s) D(u_t - u_s) \frac{\partial K_H}{\partial t}(t,s) dt \right\|^2_{L^2([0,1];\mathfrak{H})} \right) \\
&\le C \int_0^1 \left(\int_s^1 \phi_n(s) \|D(u_t - u_s)\|_{L^2(\Omega;\mathfrak{H})} \left| \frac{\partial K_H}{\partial t}(t,s) \right| dt \right)^2 ds \\
&\le C \int_0^1 \phi_n^2(s) \left(\int_s^1 (t-s)^\gamma (t-s)^{H-3/2} dt \right)^2 ds \\
&\le C \int_0^1 \phi_n^2(s)(1-s)^{2\gamma+2H-1} ds.
\end{aligned}$$

Then, for any $\delta \in (0,1]$, the integral $\displaystyle\int_0^{1-\delta} \phi_n^2(s)(1-s)^{2\gamma+2H-1} ds$ converges to zero as $n \to \infty$ due to condition **(h3)**, whereas, by condition **(h6)**,

$$\int_{1-\delta}^1 \phi_n^2(s)(1-s)^{2\gamma+2H-1} ds \le \delta^{2\gamma} \int_0^1 \phi_n^2(s)(1-s)^{2H-1} ds \le C\delta^{2\gamma} \to 0,$$

as $\delta \downarrow 0$. Therefore, we have proved that

$$\lim_{n\to\infty} R_{2,n} = 0. \tag{1.22}$$

Finally for $R_{3,n}$ taking $0<\delta<1$, it follows from Minkowski's inequality that

$$\begin{aligned}
R_{3,n} &= 9\mathbb{E}\left(\left\|\int_s^1 (\phi_n(t)-\phi_n(s))D(u_t-u_1)\frac{\partial K_H}{\partial t}(t,s)dt\right\|^2_{L^2([0,1];\mathfrak{H})}\right)\\
&\le C\int_0^1\left(\int_s^1(\phi_n(t)-\phi_n(s))\|D(u_t-u_1)\|_{L^2(\Omega;\mathfrak{H})}\left|\frac{\partial K_H}{\partial t}(t,s)\right|dt\right)^2 ds\\
&\le C\int_0^{1-\delta}\left(\int_s^{1-\delta}(\phi_n(t)-\phi_n(s))\|D(u_t-u_1)\|_{L^2(\Omega;\mathfrak{H})}\left|\frac{\partial K_H}{\partial t}(t,s)\right|dt\right)^2 ds\\
&+C\int_0^{1-\delta}\left(\int_{1-\delta}^1(\phi_n(t)-\phi_n(s))\|D(u_t-u_1)\|_{L^2(\Omega;\mathfrak{H})}\left|\frac{\partial K_H}{\partial t}(t,s)\right|dt\right)^2 ds\\
&+C\int_{1-\delta}^1\left(\int_s^1(\phi_n(t)-\phi_n(s))\|D(u_t-u_1)\|_{L^2(\Omega;\mathfrak{H})}\left|\frac{\partial K_H}{\partial t}(t,s)\right|dt\right)^2 ds\\
&=: T_{1,n}+T_{2,n}+T_{3,n}.
\end{aligned}$$

At this step, we study each term separately. For both $T_{2,n}$ and $T_{3,n}$ note that $1-\delta\le t$ so condition (ii) gives

$$\|D(u_t-u_1)\|_{L^2(\Omega;\mathfrak{H})}\le C\delta^{\gamma}.$$

Also, $\left|\dfrac{\partial K_H}{\partial t}(t,s)\right| = -\dfrac{\partial K_H}{\partial t}(t,s)$ so that

$$\begin{aligned}
T_{2,n}+T_{3,n} &\le C\delta^{2\gamma}\int_0^1\left(\int_s^1(\phi_n(t)-\phi_n(s))\left|\frac{\partial K_H}{\partial t}(t,s)\right|dt\right)^2 ds\\
&= C\delta^{2\gamma}\int_0^1\left(\int_s^1(\phi_n(t)-\phi_n(s))\frac{\partial K_H}{\partial t}(t,s)dt\right)^2 ds\\
&\le 2C\delta^{2\gamma}\int_0^1\left(K_H(1,s)\phi_n(s)+\int_s^1(\phi_n(t)-\phi_n(s))\frac{\partial K_H}{\partial t}(t,s)dt\right)^2 ds\\
&+2C\delta^{2\gamma}\int_0^1 K_H^2(1,s)\phi_n^2(s)\,ds\\
&= 2C\delta^{2\gamma}\|\phi_n\|_{\mathfrak{H}}^2+2C\delta^{2\gamma}\int_0^1 K_H^2(1,s)\phi_n^2(s)\,ds.
\end{aligned}$$

By condition **(h4)**, $\|\phi_n\|_{\mathfrak{H}}^2$ is bounded uniformly in n, and by condition **(h6)**, $\int_0^1 K_H^2(1,s)\phi_n^2(s)ds$ is bounded as well. Therefore,

$$\lim_{\delta\downarrow 0}\sup_n(T_{2,n}+T_{3,n})=0.$$

Thus, in order to show that $C_{2,n}$ converges to zero as $n\to\infty$, it suffices to show that, for a fixed $\delta\in(0,1)$,

$$\lim_{n\to\infty}T_{1,n}=0. \tag{1.23}$$

Using Minkowski's inequality, condition (ii) and the estimate (1.6), we can write

$$\begin{aligned}
T_{1,n} &= C\int_0^{1-\delta}\left(\int_s^{1-\delta}|\phi_n(t)-\phi_n(s)|\|D(u_t-u_1)\|_{L^2(\Omega;\mathfrak{H})}\left|\frac{\partial K_H}{\partial t}(t,s)\right|dt\right)^2 ds\\
&\le C\int_0^{1-\delta}\left(\int_s^{1-\delta}|\phi_n(t)-\phi_n(s)|(t-s)^{H-\frac{3}{2}}dt\right)^2 ds,
\end{aligned}$$

which converges to 0 as $n\to\infty$ by condition **(h7)**. This completes the proof of step 1.

Step 2: Set

$$G_n := (B^H,\, u_1\int_0^1\phi_n(t)\delta B_t^H) := (B^H,\, u_1 B_{1,n}).$$

Applying Lemma 1.1.1, we obtain

$$\mathbb{E}(\rho(H_n,G_n) = \mathbb{E}(|\langle \phi_n, Du_1\rangle_{\mathfrak{H}})|). \tag{1.24}$$

Our goal is to show, using condition (iii), that (1.24) converges to 0 as $n\to\infty$. Fix $\delta\in[0,1)$. We can write, by Hölder's inequality

$$\begin{aligned}\mathbb{E}(|\langle \phi_n, Du_1\rangle_{\mathfrak{H}}|) &= \mathbb{E}(|\langle (K_H^*\phi_n),(K_H^*Du_1)\rangle_{L^2([0,1])}|) \\ &\le \left(\int_0^1 |(K_H^*\phi_n)(s)|^p ds\right)^{\frac{1}{p}} \left(\int_0^1 \mathbb{E}(|(K_H^*Du_1)(s)|^q)ds\right)^{\frac{1}{q}}.\end{aligned}$$

The first factor converges to zero as $n\to\infty$ by property **(h8)** and the second one is bounded by condition (iii). Therefore, (1.24) holds, which together with Step 1 implies $E(\rho(H_n,G_n)\to 0$.

Step 3: As in the case $H>1/2$, it remains to show that $(B^H, \int_0^1 \phi_n(t)\delta B_t^H)$ converges in law in the space $C([0,1])\times\mathbb{R}$ to $(B^H,\sqrt{L}Z)$, where Z is a $N(0,1)$ random variable independent of B^H. This claim follows from the next two properties

a) $\mathrm{Var}(\int_0^1 \phi_n(t)\delta B_t^H)\to L$, which follows from condition **(h4)** because

$$\lim_{n\to\infty}\mathbb{E}\left(\left|\int_0^1 \phi_n(t)\delta B_t^H\right|^2\right) = L.$$

b) For any $t_0\in[0,1]$, $\lim_{n\to\infty}\mathbb{E}\left[B_{t_0}^H\int_0^1 \phi_n(t)\delta B_t^H\right]=0$. Indeed, we can write

$$\mathbb{E}\left[B_{t_0}^H\int_0^1 \phi_n(t)\delta B_t^H\right] = \langle \mathbf{1}_{[0,t_0]}, \phi_n\rangle_{\mathfrak{H}}$$

and

$$
\begin{aligned}
|\langle \mathbf{1}_{[0,t_0]}, \phi_n \rangle_{\mathfrak{H}}| &= \left| \int_0^1 (K_H^* \phi_n)(s)(K_H^* \mathbf{1}_{[0,t_0]})(s) ds \right| \\
&\leq \|K_H^* \phi_n\|_{L^p([0,1[)} \|K_H^* \mathbf{1}_{[0,t_0]}\|_{L^q([0,1])},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then the result follows from property **(h8)** and the fact that

$$
\|K_H^* \mathbf{1}_{[0,t_0]}\|_{L^q([0,1])} < \infty.
$$

This completes the proof. □

Theorem 1.3.3 can be applied to the example $\phi_n(t) = n^H t^n$, when $H > \frac{1}{4}$. Indeed, condition **(h4)**, again with $L = H\Gamma(2H)$, holds by Lemma 1.3.4 below. Condition **(h3)** is obvious. Property **(h6)** follows from the following computations:

$$
\begin{aligned}
&\int_0^1 ((1-s)^{2H-1} + s^{2H-1}) \phi_n^2(s)\, ds \\
&= n^{2H} \int_0^1 ((1-s)^{2H-1} s^{2n} + s^{2H-1+2n})\, ds \\
&= \frac{n^{2H} \Gamma(2H) \Gamma(2n+1)}{\Gamma(2n+2H+1)} + \frac{n^{2H} \Gamma(2n+2H)}{\Gamma(2n+2H+1)},
\end{aligned}
$$

which is uniformly bounded by Lemma 1.1.5. In order to show property **(h7)**, we write, for any $\delta \in (0,1]$,

$$
\begin{aligned}
&n^{2H}\int_0^{1-\delta}\left(\int_s^{1-\delta}(t^n-s^n)(t-s)^{H-\frac{3}{2}}dt\right)^2 ds\\
&=n^{2H}\int_0^{1-\delta}\left(\int_s^{1-\delta}\sum_{k=0}^{n-1}t^k s^{n-1-k}(t-s)^{H-\frac{1}{2}}dt\right)^2 ds\\
&\le n^{2H+2}(1-\delta)^{2n-2}\int_0^{1-\delta}\left(\int_s^{1-\delta}(t-s)^{H-\frac{1}{2}}dt\right)^2 ds\\
&=Cn^{2H+2}(1-\delta)^{2n-2}
\end{aligned}
$$

which converges to zero as $n\to\infty$.

To show property **(h8)**, we write

$$
\begin{aligned}
\int_0^1 |K_H^*\phi_n|^p(s)ds &\le Cn^{pH}\int_0^1 |K(1,s)|^p s^{np}ds + Cn^{pH}\int_0^1\left|\int_s^1 (t^n-s^n)(t-s)^{H-\frac{3}{2}}dt\right|^p ds\\
&\le Cn^{pH}\int_0^1((1-s)^{p(H-\frac{1}{2})}+s^{p(H-\frac{1}{2})})s^{np}ds\\
&+Cn^{pH+2}\int_0^1\left|\int_s^1 t^{n-1}(t-s)^{H-\frac{1}{2}}dt\right|^p ds\\
&=: B_{1,n}+B_{2,n}.
\end{aligned}
$$

For the term $B_{1,n}$, we have

$$
B_{1,n}\le C\left(\frac{n^{pH}\Gamma(p(H-\frac{1}{2})+1)\Gamma(np+1)}{\Gamma(p(H+\frac{1}{2})+2+np)}+\frac{n^{pH}}{p(H-\frac{1}{2}+n)+1}\right)
$$

By Lemma 1.1.5, this term converges to zero as $n\to\infty$, provided $p<2$. The same conclusion can be deduced for the term $B_{2,n}$ using Young's inequality.

Lemma 1.3.4. *For any $H\in(0,1/2)$, we have*

$$
\lim_{n\to\infty} n^H\|t^n\|_{\mathfrak{H}}=\sqrt{H\Gamma(2H)}.
$$

Proof. Using the operator K_H^* and integrating by parts, we can write

$$
\begin{aligned}
n^{2H}\|t^n\|_{\mathfrak{H}}^2 &= n^{2H}\|K_H^*(t^n)\|_{L^2([0,1])}^2 \\
&= n^{2H}\int_0^1 \left(K_H(1,s)s^n + \int_s^1 (t^n - s^n)\frac{\partial K_H}{\partial t}(t,s)\,dt\right)^2 ds \\
&= n^{2H}\int_0^1 \left(K_H(1,s) - n\int_s^1 t^{n-1}K_H(t,s)\,dt\right)^2 ds \\
&= n^{2H}\int_0^1 K_H(1,s)^2\,ds - 2n^{2H+1}\int_0^1\int_s^1 t^{n-1}K_Ht,s)K_H(1,s)\,dt\,ds\,dt \\
&+ n^{2H+2}\int_0^1\left(\int_s^1 t^{n-1}K_H(t,s)\,dt\right)^2 ds \\
&=: A_{1,n} + A_{2,n} + A_{3,n}.
\end{aligned}
\tag{1.25}
$$

At this step, we work each term in (1.25) separately. Since

$$
R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\,du,
$$

the first term is $A_{1,n} = n^{2H}R(1,1) = n^{2H}$. Changing the order of integration in the second term yields

$$
\begin{aligned}
A_{2,n} &= 2n^{2H+1}\int_0^1\int_0^t t^{n-1}K_H(t,s)K_H(1,s)\,ds\,dt \\
&= 2n^{2H+1}\int_0^1 t^{n-1}R(1,t)\,dt \\
&= n^{2H+1}\int_0^1 t^{n-1}(1 + t^{2H} - (1-t)^{2H})\,dt \\
&= n^{2H} + \frac{n^{2H+1}}{n+2H} - \frac{n^{2H+1}\Gamma(n)\Gamma(2H+1)}{\Gamma(n+2H+1)}.
\end{aligned}
$$

Writing the third term as a triple integral, changing the order of integration and using Lemma 1.3.2

gives

$$
\begin{aligned}
A_{3,n} &= n^{2H+2}\int_0^1\int_s^1\int_s^1 t^{n-1}K_H(t,s)u^{n-1}K_H(u,s)\,du\,dt\,ds\\
&= n^{2H+2}\int_0^1\int_0^t\int_0^u t^{n-1}K_H(t,s)u^{n-1}K_H(u,s)\,ds\,du\,dt\\
&\quad + n^{2H+2}\int_0^1\int_t^1\int_0^t t^{n-1}K_H(t,s)u^{n-1}K_H(u,s)\,ds\,du\,dt\\
&= n^{2H+2}\int_0^1\int_0^t t^{n-1}u^{n-1}R_H(t,u)\,du\,dt + n^{2H+2}\int_0^1\int_t^1 t^{n-1}u^{n-1}R_H(t,u)\,du\,dt\\
&= \frac{n^{2H+2}}{2}\int_0^1\int_0^1 t^{n-1}u^{n-1}(t^{2H}+u^{2H}-|t-u|^{2H})\,du\,dt\\
&= \frac{n^{2H+1}}{2(n+2H)}+\frac{n^{2H+1}}{2(n+2H)}-\frac{n^{2H+2}}{2}\int_0^1\int_0^1 t^{n-1}u^{n-1}|t-u|^{2H}\,du\,dt\\
&= \frac{n^{2H+1}}{(n+2H)}-\frac{n^{2H+2}\,2\,\Gamma(n)\Gamma(2H+1)}{2(2n+2H)\,\Gamma(n+1+2H)}.
\end{aligned}
$$

Thus (1.25) simplifies to

$$
\frac{n^{2H+1}\Gamma(n)\Gamma(2H+1)}{\Gamma(n+2H+1)}-\frac{n^{2H+2}\Gamma(n)\,\Gamma(2H+1)}{2(n+H)\,\Gamma(n+1+2H)},
$$

which, due to Lemma 1.1.5, converges to

$$
\Gamma(2H+1)-\frac{\Gamma(2H+1)}{2}=\frac{\Gamma(2H+1)}{2}=H\Gamma(2H).
$$

□

1.4 Convergence in total variation for the sequence of Itô integrals

In this section, we study the convergence in total variation of the sequences $\int_0^1 \phi_n(t)u_t dB_t$ and $(u *_B \psi_n)_t$ introduced in Section 3. We will do so by using Theorem 3.1 from [28], and Theorem 1 from [41], respectively.

Given two real valued random variables X, Y, we let $d_W(X,Y)$ and $d_{TV}(X,Y)$ be the Wasserstein distance and total variation distance between the laws of X and Y, respectively. More precisely,

$$d_{\mathrm{TV}}(X,Y) := \sup_A \left|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)\right|, \quad d_{\mathrm{W}}(X,Y) := \sup_h \left|\mathbb{E}[h(X) - h(Y)]\right|,$$

where the first supremum runs over all Borel subsets of $\mathbb{R}$ and the second supremum runs overs all functions $h : \mathbb{R} \to \mathbb{R}$ such that $\|h\|_{\mathrm{Lip}} \le 1$, where

$$\|h\|_{\mathrm{Lip}} = \sup_{x \neq y, x,y \in \mathbb{R}} \frac{||h(x) - h(y)|}{|x-y|}$$

We also recall the the Fortet-Mourier distance d_{FM} given by

$$d_{\mathrm{FM}}(X,Y) := \sup_h \left|\mathbb{E}[h(X) - h(Y)]\right|,$$

where the supremum runs overs all bounded functions h such that $\|h\|_\infty + \|h\|_{\mathrm{Lip}} \le 1$.

Recall that when $H = \frac{1}{2}$, the mapping $\mathbf{1}_{[0,t]} \to B_t$ can be extended to a linear isometry between $\mathfrak{H} = L^2([0,1])$ and the Gaussian space $\mathscr{H}_1$ spanned by B. Let us denote this isometry also by B. For every integer $q \ge 1$, we let $\mathscr{H}_q$ be the qth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(B(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1)\}$, where H_q is the qth Hermite polynomial defined by

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}(e^{-x^2/2}).$$

We denote by $\mathscr{H}_0$ the space of constant random variables.

In [28], the convergence in total variation of sequences living in a finite sum of Wiener chaos Theorem is studied. In particular, as a consequence of Theorem 3.1 in [28], we have the following result.

Theorem 1.4.1. *Under the assumptions of Theorem 1.2.1, assume that for each $t > 0$ $u_t \in \bigoplus_{j=0}^p \mathscr{H}_j$.*

Then, there exists a constant $c > 0$ (independent of n) such that

$$d_{TV}(F_n, \sqrt{L}u_1 Z) \leq c\, d_W(F_n, \sqrt{L}u_1 Z)^{1/(3+2p)}. \tag{1.26}$$

Furthermore, F_n converges to $\sqrt{L}u_1 Z$ in total variation as $n \to \infty$.

Proof. Notice that $F_n \in \bigoplus_{j=0}^{p+1} \mathscr{H}_j$. As a consequence of Theorem 1.2.1, the sequence F_n converges in distribution to the nonzero random variable $\sqrt{L}u_1 Z$. Then, by Theorem 3.1 in [28], there is a constant $c > 0$ such that

$$d_{TV}(F_n, \sqrt{L}u_1 Z) \leq c\, d_{FM}(F_n, \sqrt{L}u_1 Z)^{1/(1+2(p+1))},$$

Since $d_{FM} \leq d_W$ we obtain (1.26).

On the other hand, it is proved in Theorem 1.2.1 that $\sup_n \mathbb{E}(F_n^2) < \infty$, and hence F_n is uniformly integrable. Since $\mathbb{E}(|F_n|) + \mathbb{E}(|u_1 Z|) < \infty$ for all n, it follows that $d_W(F_n, \sqrt{L}u_1 Z)$ converge to 0 as $n \to \infty$ (see section 2.1 in [41]). In view of (1.26), this completes the proof. □

We remark that the rate provided by Theorem 1.4.1 is not optimal. For instance, in the special case $\phi_n(t) = \sqrt{n}t^n$ and $u = B$, the rate obtained in [42] is better.

We proceed now to study the convergence in total variation of the stochastic convolution $G_n := (u *_B \psi_n)_t$ by means of Theorem 1 from [41]. In particular, the total variation distance will be estimated in terms of the sequences

$$d_n := d_W(G_n, u_t Z) \tag{1.27}$$

and

$$I_n = 2\int_0^\infty r|\mathbb{E}\left(e^{irG_n}\right)|\, dr. \tag{1.28}$$

Theorem 1 in [41] will provide not only the convergence in total variation but also a rate of convergence.

We start by showing that under the assumptions of Theorem 1.2.5, the Wasserstein distance between G_n and $u_t Z$ converges to 0.

Lemma 1.4.2. *d_n defined in (1.27) converges to* 0 *as* $n \to \infty$.

Proof. As a result of Theorem 1.2.5, the sequence G_n converges in distribution to $u_t Z$. Furthermore, it is shown in the proof of Theorem 1.2.5, $\sup_n \mathbb{E}(G_n^2) < \infty$ and hence G_n is uniformly integrable. Then, the conclusion follows because $\mathbb{E}(|G_n|| + \mathbb{E}|u_t Z|) < \infty$ for all n (See section 2.1 in [41]). □

As a consequence of Theorem 1 in [41], we obtain our main result of this section for the stochastic convolution.

Theorem 1.4.3. *Under the conditions of Theorem 1.2.5, suppose that $C_1 = \mathbb{E}(1/|u_t|) < \infty$, ψ has compact support and u_s is measurable with respect to $\mathscr{F}_{(s-\tau)^+}$ for all $s \geq 0$ and some constant $\tau > 0$. Assume*

$$C_2 = 2 \sup_n \mathbb{E}\left(\left(\int_0^\infty \psi_n^2(t-s) u_s^2 \, ds\right)^{-1}\right) < \infty. \tag{1.29}$$

Consider the sequence defined by the stochastic convolution $G_n = \int_0^\infty \psi_n(t-s) u_s \, dB_s$. Then, there is a positive integer n_0 (depending on the support of ψ and τ) and a constant C independent of n such that

$$d_{TV}(G_n, u_t Z) \leq C d_n^{1/3} \quad \text{for all } \; n \geq n_0. \tag{1.30}$$

In particular, G_n converges in total variation to $u_t Z$.

Proof. Since ψ has compact support, there is $R > 0$ such that $\psi_n(t-s) = 0$ whenever $n|t-s| > R$. Then, we can write

$$G_n = \int_{(t-R/n)^+}^{t+R/n} \psi_n(t-s) u_s \, dB_s.$$

Choose n_0 large enough so that $R/n_0 \leq \tau$, and let $n \geq n_0$. Then, for all $s \in [t - R/n, t + R/n]$, u_s is

$\mathscr{F}_t$-measurable. Consequently,

$$\mathbb{E}(e^{irG_n}) = \mathbb{E}(E(e^{irG_n}|\mathscr{F}_t)) = \mathbb{E}\left(e^{-r^2/2\int_{(t-R/n)^+}^{t+R/n}\psi_n^2(t-s)u_s^2\,ds}\right) = \mathbb{E}\left(e^{-r^2/2\int_0^\infty \psi_n^2(t-s)u_s^2\,ds}\right).$$

Therefore

$$\begin{aligned} I_n = 2\int_0^\infty r\left|\mathbb{E}\left(e^{irG_n}\right)\right| dr = 2\int_0^\infty r\mathbb{E}\left(e^{-r^2/2\int_0^\infty \psi_n^2(t-s)u_s^2\,ds}\right) dr \\ = 2\mathbb{E}\left(\int_0^\infty re^{-r^2/2\int_0^\infty \psi_n^2(t-s)u_s^2\,ds}\,dr\right) \\ = 2\mathbb{E}\left[\left(\int_0^\infty \psi_n^2(t-s)u_s^2\,ds\right)^{-1}\right] \end{aligned}$$

In view of (1.29), this implies

$$\sup_{n\geq n_0} I_n \leq C_2. \tag{1.31}$$

On the other hand, it is proved in Theorem 1.2.5 that $\sup_n \mathbb{E}(G_n^2) < \infty$. Then, it follows from Theorem 1 in [41] that there is a constant k, independent of n, such that

$$d_{TV}(G_n, u_t Z) \leq d_n^{1/2}(1+C_1) + k\left(I_n d_n^{1/2}\right)^{2/3}. \tag{1.32}$$

Therefore (1.30) follows from (1.31) and (1.32). Finally, the convergence in total variation is a consequence of Lemma 1.4.2. □

Chapter 2

Averaging 2D stochastic wave equation

In this chapter, we consider the 2D stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \sigma(u)\dot{W}, \tag{2.1}$$

on $\mathbb{R}_+ \times \mathbb{R}^2$, where Δ is Laplacian in the space variables and $\dot{W}$ is a Gaussian centered noise with covariance given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)\|x-y\|^{-\beta} \tag{2.2}$$

for any given $\beta \in (0,2)$. In other words, the driving noise $\dot{W}$ is white in time and it has an homogeneous spatial covariance described by the Riesz kernel. Here $\dot{W}$ is a distribution-valued field and is a notation for $\frac{\partial^3 W}{\partial t \partial x_1 \partial x_2}$, where the noise W will be formally introduced later.

Throughout this chapter, we fix the boundary conditions

$$u(0,x) = 1, \quad \frac{\partial}{\partial t}u(0,x) = 0 \tag{2.3}$$

and assume $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz with Lipschitz constant $L \in (0,\infty)$ such that $\sigma(1) \neq 0$. It is well-known (see *e.g.* [11]) that equation (2.1) has a unique *mild solution*, which is adapted to the filtration generated by W, such that $\sup\left\{\mathbb{E}\left[|u(t,x)|^2\right] : (t,x) \in [0,T] \times \mathbb{R}^2\right\} < \infty$ for any finite T and

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma(u(s,y))W(ds,dy), \tag{2.4}$$

where the above stochastic integral is defined in the sense of Dalang-Walsh (see [10, 44]) and

$G_{t-s}(x-y)$ denotes the fundamental solution to the corresponding deterministic 2D wave equation, *i.e.*

$$G_t(x) = \frac{1}{2\pi\sqrt{t^2 - \|x\|^2}} \mathbf{1}_{\{\|x\|<t\}}.$$

Because of the choice of boundary conditions (2.3), $\{u(t,x) : x \in \mathbb{R}^2\}$ is strictly stationary for any fixed $t > 0$, meaning that the finite-dimensional distributions of $\{u(t,x+y) : x \in \mathbb{R}^2\}$ do not depend on y; see *e.g.* [12, Footnote 1]. Then it is natural to view the solution $u(t,x)$ as a functional over the homogeneous Gaussian random field W. Such Gaussian functional has been a recurrent topic in probability theory, for example, the celebrated Breuer-Major theorem (see *e.g.* [5, 6, 34]) provides the Gaussian fluctuation for the average of a functional subordinated to a stationary Gaussian random field. Therefore, one may wonder whether or not the spatial average of $u(t,x)$ admits Gaussian fluctuation, that is, as $R \to +\infty$

$$\textit{does} \int_{\{\|x\|\le R\}} (u(t,x)-1)\, dx \textit{ converge to } \mathscr{N}(0,1)\textit{, after proper normalization?}$$

Here $t > 0$ is fixed, $u(t,x)$ solves (2.1) and $\mathscr{N}(0,1)$ denotes the standard normal distribution.

Recently, the above question has been investigated for stochastic heat equations (see [9, 16, 17, 35]) and for the 1D stochastic wave equation (see [12]). Our work can be seen as an extension of the work [12] to the two-dimensional case. In Theorem 2.0.4 below we provide an affirmative answer to the above question.

Let us first fix some notation that will be used throughout this Chapter.

Notation. (1) The expression $a \lesssim b$ means $a \le Kb$ for some immaterial constant K that may vary from line to line.

(2) $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$ and we write $B_R = \{x : \|x\| \le R\}$. We define for each $t \in \mathbb{R}_+ := [0,\infty)$,

$$F_R(t) = \int_{B_R} (u(t,x)-1)\, dx. \tag{2.5}$$

(3) We fix $\beta \in (0,2)$ throughout this article and there are two relevant constants c_β, κ_β defined

by

$$c_\beta = \frac{\Gamma(1-\frac{\beta}{2})}{\pi 4^{\beta/2}\Gamma(\beta/2)}, \qquad \kappa_\beta = \int_{\mathbb{R}^2} d\xi \|\xi\|^{\beta-4} J_1(\|\xi\|)^2, \tag{2.6}$$

where $J_1(\cdot)$ is the Bessel function of first kind with order 1, given by (see, for instance, [22, (5.10.4)])

$$J_1(x) = \frac{x}{\pi}\int_0^\pi \sin^2\theta \cos(x\cos\theta)d\theta. \tag{2.7}$$

We point out that the quantity κ_β is finite, since $J_1(\rho)$ is uniformly bounded on $\mathbb{R}_+$ and equivalent to constant times ρ as $\rho \downarrow 0$; see *e.g.* [35, Lemma 2.1]. Furthermore, $4\pi^2 c_\beta \kappa_\beta = \int_{B_1^2} \|y-z\|^{-\beta} dydz$; see Remark 2.1.3 below.

(4) We write $\|X\|_p$ for the $L^p(\Omega)$-norm of a real random variable X.

Now we are in a position to state our main result.

Theorem 2.0.4. *Recall $F_R(t)$ defined in* (2.5). *As $R\to\infty$, the process $\{R^{\frac{\beta}{2}-2}F_R(t) : t\in\mathbb{R}_+\}$ converges in law to a centered Gaussian process $\mathscr{G}$ in the space $C(\mathbb{R}_+;\mathbb{R})$ of continuous functions, equipped with the topology of uniform convergence on compact sets, where*

$$\mathbb{E}\big[\mathscr{G}_{t_1}\mathscr{G}_{t_2}\big] = 4\pi^2 c_\beta \kappa_\beta \int_0^{t_1\wedge t_2} (t_1-s)(t_2-s)\xi^2(s)ds,$$

with $\xi(s) = \mathbb{E}[\sigma(u(s,0))]$ and c_β, κ_β being the two constants given in (2.6). *For any fixed $t>0$,*

$$d_{\mathrm{TV}}\big(F_R(t)/\sigma_R, Z\big) \lesssim R^{-\beta/2}, \tag{2.8}$$

where $Z\sim\mathscr{N}(0,1)$ and $\sigma_R := \sqrt{\mathrm{Var}(F_R(t))} > 0$ for every $R>0$.

Remark 2.0.5. *(1) The limiting process $\mathscr{G}$ has the following stochastic integral representation:*

$$\{\mathscr{G}_t : t\in\mathbb{R}_+\} \overset{(d)}{=} \left\{2\pi\sqrt{c_\beta\kappa_\beta}\int_0^t (t-s)\xi(s)dY_s : t\in\mathbb{R}_+\right\},$$

where $\{Y_t : t\in\mathbb{R}_+\}$ is a standard Brownian motion.

(2) We point out that $\sigma_R > 0$ is part of our main result. Indeed, it is a consequence of our standing assumption $\sigma(1) \neq 0$. In fact, we have the following equivalences:

$$\sigma_R = 0,\ \forall R > 0 \Leftrightarrow \exists R > 0,\ s.t.\ \sigma_R = 0 \Leftrightarrow \sigma(1) = 0 \Leftrightarrow \lim_{R\to\infty} \sigma_R^2 R^{\beta-4} = 0.$$

The proof can be done similarly as in [12, Lemma 3.4] and by using Proposition 2.32.

(3) The total-variation distance d_{TV} induces a much stronger topology than that induced by the Fortet-Mourier distance d_{FM}, where the latter is equivalent to that of convergence in law. For real random variables X,Y,

$$d_{\mathrm{TV}}(X,Y) := \sup_A \big|\mathbb{P}(X\in A) - \mathbb{P}(Y\in A)\big|, \quad d_{\mathrm{FM}}(X,Y) := \sup_h \big|\mathbb{E}[h(X) - h(Y)]\big|,$$

where the first supremum runs over all Borel subsets of $\mathbb{R}$ and the second supremum runs overs all bounded Lipschitz functions h with $\|h\|_\infty + \|h'\|_\infty \le 1$. Our quantitative CLT (3.4) *is obtained by the Malliavin-Stein approach that combines Stein's method of normal approximation with Malliavin's differential calculus on a Gaussian space; see the monograph [27] for a comprehensive treatment. One can also obtain the rate of convergence in other frequently used distances, such as the 1-Wassertein distance and Kolmogorov distance, and the corresponding bounds are of the same order as in* (3.4).

Now let us sketch a few paragraphs to briefly illustrate our methodology in proving Theorem 2.0.4. The main ingredient is the following fundamental estimate on the p-norm of the Malliavin derivative $Du(t,x)$ of the solution $u(t,x)$. It is well-known (see *e.g.* [24]) that $Du(t,x) \in L^p(\Omega;\mathfrak{H})$ for any $p \in [1,\infty)$, where $\mathfrak{H}$ is the Hilbert space associated to the noise W, defined as the completion of $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ under the inner product

$$\langle f,g\rangle_{\mathfrak{H}} := \int_{\mathbb{R}_+\times\mathbb{R}^4} f(s,y)g(s,z)\|y-z\|^{-\beta} dydzds \tag{2.9}$$

$$= c_\beta \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathscr{F}f(s,\xi)\mathscr{F}g(s,-\xi)\|\xi\|^{\beta-2} d\xi ds, \tag{2.10}$$

where c_β is given in (2.6) and $\mathscr{F}f(s,\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(s,x)dx$.

Theorem 2.0.6. *The Malliavin derivative $Du(t,x)$ is a random function denoted by $(s,y) \mapsto D_{s,y}u(t,x)$ and for any $p \in [2,\infty)$ and any $t > 0$, the following estimates hold for almost all $(s,y) \in [0,t] \times \mathbb{R}^2$:*

$$G_{t-s}(x-y)\|\sigma(u_{s,y})\|_p \le \left\|D_{s,y}u(t,x)\right\|_p \le C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y), \tag{2.11}$$

where the constants $C_{\beta,p,t,L}$ and $\kappa_{p,t}$ are given in (2.53) *and* (2.51)*, respectively.*

Remark 2.0.7. *Theorem 2.0.6 echoes the comment after [17, Lemma 2.1] and generalizes [12, Lemma 2.2] to the solution of a 2D stochastic wave equation. Although the expression in* (2.11) *looks the same as in [12, Lemma 2.2],* i.e. *L^p-norm of the Malliavin derivative is bounded by the fundamental solution to the corresponding deterministic wave equation, we would like to emphasize that the proof in the 2D setting is much more involved and requires new techniques in dealing with the singularity of $G_{t-s}(x-y)$ while in the 1D case the fundamental solution is the bounded function $\frac{1}{2}\mathbf{1}_{\{|x-y|<t-s\}}$. Modulo sophisticated integral estimates, our proof of Theorem 2.0.6 is treated through a harmonious combination of tools from Gaussian analysis (Clark-Ocone formula, Burkholder inequality) and Hardy-Littlewood-Sobolev's lemma.*

Now let us first sketch the main steps for the proof of Theorem 2.0.4 and then we will present the key steps in proving (2.11).

The typical proof of the functional CLT consists in three steps:

(**S1**) We establish the limiting covariance structure, this is the content of Section 2.2.1. In particular, the variance of the spatial average $F_R(t)$ is of order $R^{4-\beta}$, as $R \to \infty$. As one will see shortly, the important part of this step is the proof of the limit (2.34): $\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big] \to 0$ as $\|y-z\| \to \infty$. This limit is straightforward when $\sigma(u) = u$ and in the general case, we will apply the Clark-Ocone formula (see Lemma 2.1.5) to first represent $\sigma(u(s,y))$ as a stochastic integral and then apply the Itô's isometry in order to break the nonlinearity for further estimations.

(**S2**) From (**S1**), we have the covariance structure of the limiting Gaussian process $\mathscr{G}$. Then we will prove the convergence of $\{R^{\frac{\beta}{2}-2}F_R(t) : t \in \mathbb{R}_+\}$ to $\{\mathscr{G}_t : t \in \mathbb{R}_+\}$ in finite-dimensional distributions. This is made possible by the following multivariate Malliavin-Stein bound that we borrow from [16, Proposition 2.3] (see also [27, Theorem 6.1.2]). We denote by D the Malliavin derivative and by δ the adjoint operator of D that is characterized by the integration-by-parts formula (3.6). Moreover, $\mathbb{D}^{1,2}$ is the Sobolev space of Malliavin differentiable random variables $X \in L^2(\Omega)$ with $\mathbb{E}\big[\|DX\|_{\mathfrak{H}}^2\big] < \infty$ and $\mathrm{Dom}\delta$ is the domain of δ; see Section 2.1 for more details.

Proposition 2.0.8. *Let $F = (F^{(1)}, \dots, F^{(m)})$ be a random vector such that $F^{(i)} = \delta(v^{(i)})$ for $v^{(i)} \in \mathrm{Dom}\,\delta$ and $F^{(i)} \in \mathbb{D}^{1,2}$, $i = 1, \dots, m$. Let Z be an m-dimensional centered Gaussian vector with covariance matrix $(C_{i,j})_{1\le i,j\le m}$. For any C^2 function $h : \mathbb{R}^m \to \mathbb{R}$ with bounded second partial derivatives, we have*

$$\big|\mathbb{E}[h(F)] - \mathbb{E}[h(Z)]\big| \le \frac{m}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^{m} \mathbb{E}\Big[\big(C_{i,j} - \langle DF^{(i)}, v^{(j)}\rangle_{\mathfrak{H}}\big)^2\Big]}, \tag{2.12}$$

where $\|h''\|_\infty := \sup\big\{\big|\frac{\partial^2}{\partial x_i \partial x_j} h(x)\big| : x \in \mathbb{R}^m, i, j = 1, \dots, m\big\}$.

In view of (2.4), we write $u(t,x) - 1 = \delta\big(G_{t-\bullet}(x-*)\sigma(u(\bullet,*))\big)$ so that $F_R(t)$ can be represented as

$$F_R(t) = \int_{B_R} \delta\big(G_{t-\bullet}(x-*)\sigma(u(\bullet,*))\big)dx = \delta\big(\varphi_{t,R}(\bullet,*)\sigma(u(\bullet,*))\big) \tag{2.13}$$

by Fubini's theorem, with

$$\varphi_{t,R}(r,y) = \int_{B_R} G_{t-r}(x-y)dx; \tag{2.14}$$

see Section 2.1.2. Putting $V_{t,R}(s,y) = \varphi_{t,R}(s,y)\sigma(u(s,y))$, and applying the fundamental estimate (2.11), we will establish that, for any $t_1, t_2 \in (0,\infty)$,

$$R^{2\beta-8}\mathrm{Var}\big(\langle DF_R(t_1), V_{t_2,R}\rangle_{\mathfrak{H}}\big) \lesssim R^{-\beta} \text{ for } R \ge t_1 + t_2. \tag{2.15}$$

Then, we will show that Proposition 2.0.8 together with the estimate (2.15) imply the convergence in law of the finite-dimensional distributions.

The bound (2.15) for $t_1 = t_2 = t$ together with the following 1D Malliavin-Stein bound (see, *e.g.* [16, 31, 37]) will lead to the quantitative result (3.4).

Proposition 2.0.9. *Let $F = \delta(v)$ for some $\mathfrak{H}$-valued random variable $v \in \mathrm{Dom}\,\delta$. Assume $F \in \mathbb{D}^{1,2}$ and $\mathbb{E}[F^2] = 1$ and let $Z \sim \mathscr{N}(0,1)$. Then,*

$$d_{\mathrm{TV}}(F,Z) \le 2\sqrt{\mathrm{Var}\big[\langle DF, v\rangle_{\mathfrak{H}}\big]}\,. \tag{2.16}$$

(**S3**) The last step is to show tightness, which follows from the tightness of the processes restricted to $[0,T]$ for any finite T. To show the tightness of $\big\{R^{\frac{\beta}{2}-2}F_R(t) : t \in [0,T]\big\}$, in view of the well-known criterion of Kolmogorov-Chentsov (see *e.g.* [19, Corollary 16.9]), it is enough to show that for any $p \in [2,\infty)$,

$$\|F_R(t) - F_R(s)\|_p \lesssim R^{2-\frac{\beta}{2}}|t-s|^{1/2} \text{ for } s,t \in [0,T], \tag{2.17}$$

where the implicit constant does not depend on t,s or R. This will proves Theorem 2.0.4.

Finally let us pave the plan of proving the fundamental estimate (2.11). The story begins with the usual *Picard iteration*: We define $u_0(t,x) = 1$ and for $n \ge 0$,

$$u_{n+1}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y)\sigma\big(u_n(s,y)\big)W(ds,dy). \tag{2.18}$$

It is a classic result that $u_n(t,x)$ converges in $L^p(\Omega)$ to $u(t,x)$ uniformly in $x \in \mathbb{R}^2$ for any $p \ge 2$; see *e.g.* [11, Theorem 4.3]. Now it has become clear that if we assume $\sigma(1) = 0$, we will end up in the trivial case where $u(t,x) \equiv 1$, in view of the above iteration.

For each $n \ge 0$, $u_{n+1}(t,x)$ is Malliavin differentiable, as one can show by induction on n. Our strategy is to first obtain the uniform estimate of $\sup\big\{\|D_{s,y}u_n(t,x)\|_p : n \ge 0\big\}$ and then one can hope to transfer this estimate to $\|D_{s,y}u(t,x)\|_p$. As mentioned before, $Du(t,x)$ lives in the space $\mathfrak{H}$ that contains generalized functions. To overcome this, we will carefully apply the following inequality of Hardy-Littlewood-Sobolev to show $Du(t,x)$ is a random variable in $L^{\frac{4}{4-\beta}}(\mathbb{R}_+ \times \mathbb{R}^2)$, with $\beta \in (0,2)$ fixed throughout this paper.

Lemma 2.0.10 (Hardy-Littlewood-Sobolev). *If $1 < p < p_0 < \infty$ with $p_0^{-1} = p^{-1} - \alpha n^{-1}$, then there is some constant C that only depends on p, α and n, such that*

$$\|I^{\alpha} g\|_{L^{p_0}(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)},$$

for any locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$, where with $\alpha \in (0,n)$,

$$\left(I^{\alpha} g\right)(x) := \int_{\mathbb{R}^n} \|x-y\|^{\alpha-n} g(y)dy.$$

For our purpose, with $n = 2$, $\alpha = 2-\beta$, $p = 2q = 4/(4-\beta)$ and $p_0 = 4/\beta$, we deduce from Hölder's inequality that

$$\begin{aligned} \langle f, g\rangle_{\mathfrak{H}_0} &:= \int_{\mathbb{R}^2} f(x)g(y)\|x-y\|^{-\beta} dxdy \qquad (2.19)\\ &\leq \|f\|_{L^{2q}(\mathbb{R}^2)} \|I^{2-\beta} g\|_{L^{4/\beta}(\mathbb{R}^2)} \\ &\leq C_{\beta} \|f\|_{L^{2q}(\mathbb{R}^2)} \|g\|_{L^{2q}(\mathbb{R}^2)}, \qquad (2.20) \end{aligned}$$

for any $f, g \in L^{2q}(\mathbb{R}^2)$; see *e.g.* [43, pages 119-120].

Once we obtain the uniform estimate of $\sup\left\{\|D_{s,y} u_n(t,x)\|_p : n \geq 0\right\}$ and prove $Du(t,x) \in L^{\frac{4}{4-\beta}}(\mathbb{R}_+ \times \mathbb{R}^2)$, that is, $(s,y) \longmapsto D_{s,y}u(t,x)$ is indeed a random function, we proceed to the proof of (2.11). In view of the Clark-Ocone formula (see Lemma 2.1.5), we have $\mathbb{E}[D_{s,y}u_{t,x}|\mathscr{F}_s] = G_{t-s}(x-y)\sigma(u(s,y))$ almost surely, where $\{\mathscr{F}_s : s \in \mathbb{R}_+\}$ is the filtration generated by the noise; see Section 2.1.2. Then, the lower bound in (2.11) follows immediately from the conditional Jensen inequality. The upper bound follows from the uniform estimates of $\|D_{s,y}u_n(t,x)\|_p$ by a standard argument.

Before we end this introduction, let us point out another technical difficulty in this paper. After the application of Lemma 2.0.10 during the process of estimating $\|D_{s,y}u_n(t,x)\|_p$, we will encounter integrals of the form

$$\int_s^t \left(\int_{\mathbb{R}^2} G_{t-r}^{2q}(x-z) G_{r-s}^{2q}(z) dz \right)^{\delta} dr \tag{2.21}$$

where $q \in (1/2, 1)$ and $\delta \in [1, 1/q]$. In the case of stochastic heat equation, the estimation of the above integrals is straightforward due to the semi-group property. However, for the wave equation the kernel G_t *does not* satisfy the semi-group property and the estimation of the above integrals is quite involved. For the case of the 1D stochastic wave equation, as one can see from the paper [12], the computations take advantage of the simple form of the fundamental solution (*i.e.* $\frac{1}{2}\mathbf{1}_{\{|x-y|<t-s\}}$). For our 2D case, the singularity within the fundamental solution $G_{t-s}(x-y)$ puts the technicality to another level and we have to estimate the convolution $G_{t-r}^{2q} * G_{r-s}^{2q}$ by exact computations. A basic technical tool used in this problem is the following lemma.

Lemma 2.0.11. *For $0 \le s < t < \infty$, with $\|z\| = \mathbf{w} > 0$ and $q \in (1/2, 1)$, we have*

$$\begin{aligned} G_t^{2q} * G_s^{2q}(z) \lesssim \mathbf{1}_{\{\mathbf{w}<s\}} \left[t^2 - (s-\mathbf{w})^2\right]^{1-2q} + \left[t^2 - (s+\mathbf{w})^2\right]^{1-2q} \mathbf{1}_{\{t>s+\mathbf{w}\}} \\ + \mathbf{1}_{\{|s-\mathbf{w}|<t<s+\mathbf{w}\}} \left[(\mathbf{w}+s)^2 - t^2\right]^{-q+\frac{1}{2}} \left[t^2 - (s-\mathbf{w})^2\right]^{-q+\frac{1}{2}}, \end{aligned} \tag{2.22}$$

where the implicit constant only depends on q.

The rest of this article is organized as follows: Section 2 collects some preliminary facts for our proofs, Section 3 contains the proof of Theorem 2.0.4 and Section 4 is devoted to proving the fundamental estimate (2.11).

2.1 Preliminaries

This section provides some preliminary results that are required for further sections. It consists of two subsections: Section 2.1.1 contains several important facts on the function $G_{t-s}(x-y)$ and Section 2.1.2 is devoted to a minimal set of results from stochastic analysis, notably the tools from Malliavin calculus.

2.1.1 Basic facts on the fundamental solution

Let us fix some more notation here.

Notation. For $p \in \mathbb{R}$, we write $(v)_+^p = v^p$ if $v > 0$ and $(v)_+^p = 0$ if $v \le 0$. Then, we can write

$$G_t(x) = \frac{1}{2\pi}(t^2 - \|x\|^2)_+^{-1/2}.$$

Recall the function $\varphi_{t,R}(r,y)$ introduced in (3.31):

$$\varphi_{t,R}(s,y) = \int_{B_R} G_{t-r}(x-y)dx.$$

In what follows, we put together several useful facts on the function $G_t(z)$.

Lemma 2.1.1. (1) *For any $p \in (0,1)$ and $t > 0$.*

$$\int_{\mathbb{R}^2} G_t^{2p}(z)dz = \frac{(2\pi)^{1-2p}}{2-2p} t^{2-2p}. \tag{2.23}$$

(2) *For $t > s$, we have $\varphi_{t,R}(s,y) \le (t-s)\mathbf{1}_{\{\|y\| \le R+t\}}$ and $\int_{\mathbb{R}^2} \varphi_{t,R}(s,y)dy = (t-s)\pi R^2$.*

The proof of Lemma 2.1.1 is omitted, as it follows from simple and exact computations. As a consequence of Lemma 2.1.1-(2), we have

$$\int_{\mathbb{R}^2} \varphi_{t,R}(s,z+\xi)\varphi_{t,R}(s,z)dz \le \pi(t-s)^2R^2. \tag{2.24}$$

The following lemma is also a consequence of Lemma 2.1.1.

Lemma 2.1.2. *For $t_1, t_2 \in (0,\infty)$, we put*

$$\Psi_R(t_1,t_2;s) := R^{\beta-4}\int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\|y-z\|^{-\beta}dydz.$$

Then

(i) $\Psi_R(t_1,t_2;s)$ *is uniformly bounded over* $s \in [0, t_2 \wedge t_1]$ *and* $R > 0$*;*

(ii) For any $s \in [0, t_2 \wedge t_1]$*,* $\Psi_R(t_1,t_2;s)$ *converges to* $4\pi^2 c_\beta \kappa_\beta (t_1 - s)(t_2 - s)$*, as* $R \to \infty$*.*

Here the quantities c_β *and* κ_β *are given in* (2.6)*.*

Proof. By using Fourier transform as in (2.10), we can write

$$\begin{aligned}\Psi_R(t_1,t_2;s) &= R^{\beta-4}\int_{B_R^2} dxdx' \int_{\mathbb{R}^4} G_{t_1-s}(x-y)G_{t_2-s}(x'-z)\|y-z\|^{-\beta}dydz\\
&= c_\beta R^{\beta-4}\int_{B_R^2} dxdx' \int_{\mathbb{R}^2} d\xi e^{-i(x-x')\cdot\xi}\left(\frac{\sin((t_1-s)\|\xi\|)}{\|\xi\|}\frac{\sin((t_2-s)\|\xi\|)}{\|\xi\|}\right)\|\xi\|^{\beta-2}\\
&= c_\beta \int_{B_1^2} dxdx' \int_{\mathbb{R}^2} d\xi e^{-i(x-x')\cdot\xi}\ \frac{\sin((t_1-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\frac{\sin((t_2-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\|\xi\|^{\beta-2},\end{aligned}$$

where in the last equality we made the change of variables $\xi \to \xi R^{-1}$.

The Fourier transform of $x \in \mathbb{R}^2 \longmapsto \mathbf{1}_{\{\|x\|\le 1\}}$ is $\xi \in \mathbb{R}^2 \longmapsto 2\pi\|\xi\|^{-1}J_1(\|\xi\|)$ (see, for instance, Lemma 2.1 in [35]), where J_1 is the Bessel function of first kind with order 1 introduced in (2.7). Then, we can rewrite $\Psi_R(t_1,t_2;s)$ as

$$c_\beta \int_{\mathbb{R}^2}\left[2\pi\|\xi\|^{-1}J_1(\|\xi\|)\right]^2\left(\frac{\sin((t_1-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\frac{\sin((t_2-s)\|\xi\|R^{-1})}{\|\xi\|R^{-1}}\right)\|\xi\|^{\beta-2}d\xi.$$

Since $\sin((t-s)\|\xi\|R^{-1})/(\|\xi\|R^{-1})$ is uniformly bounded over $s \in (0,t]$ and converges to $t-s$ as $R \to \infty$, then the statement (i) holds true and

$$\Psi_R(t_1,t_2;s) \xrightarrow{R\to\infty} 4\pi^2 c_\beta \kappa_\beta (t_1-s)(t_2-s).$$

by the dominated convergence theorem with the dominance condition $\kappa_\beta < \infty$. □

Remark 2.1.3. *By inverting the Fourier transform, we have*

$$(2\pi)^2 c_\beta \kappa_\beta = c_\beta \int_{\mathbb{R}^2}(2\pi)^2 J_1(\|\xi\|)^2\|\xi\|^{-2}\|\xi\|^{\beta-2}d\xi = \int_{B_1^2}\|y-z\|^{-\beta}dydz.$$

2.1.2 Basic stochastic analysis

Let $\mathfrak{H}$ be defined (see (2.9) and (2.10)) as the completion of $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ under the inner product

$$\langle f, g\rangle_{\mathfrak{H}} = \int_{\mathbb{R}_+\times\mathbb{R}^4} f(s,y)g(s,z)\|y-z\|^{-\beta} dydzds \text{ for } f,g \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2).$$

Consider an isonormal Gaussian process associated to the Hilbert space $\mathfrak{H}$, denoted by $W = \{W(\phi) : \phi \in \mathfrak{H}\}$. That is, W is a centered Gaussian family of random variables such that $\mathbb{E}\big[W(\phi)W(\psi)\big] = \langle \phi, \psi\rangle_{\mathfrak{H}}$ for any $\phi, \psi \in \mathfrak{H}$. As the noise is white in time, a martingale structure naturally appears. First we define $\mathscr{F}_t$ to be the σ-algebra generated by $\mathbb{P}$-null sets and $\big\{W(\phi) : \phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ has compact support contained in $[0,t] \times \mathbb{R}^2\big\}$, so we have a filtration $\mathbb{F} = \{\mathscr{F}_t : t \in \mathbb{R}_+\}$. If $\big\{\Phi(s,y) : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\big\}$ is an $\mathbb{F}$-adapted random field such that $\mathbb{E}\big[\|\Phi\|_{\mathfrak{H}}^2\big] < +\infty$, then

$$M_t = \int_{[0,t]\times\mathbb{R}^2} \Phi(s,y)W(ds,dy),$$

interpreted as the Dalang-Walsh integral ([10, 44]), is a square-integrable $\mathbb{F}$-martingale with quadratic variation given by

$$\langle M\rangle_t = \int_{[0,t]\times\mathbb{R}^4} \Phi(s,y)\Phi(s,z)\|y-z\|^{-\beta} dydzds = \big\|\Phi(\bullet,*)\mathbf{1}_{\{\bullet\le t\}}\big\|_{\mathfrak{H}}^2.$$

Let us record a suitable version of Burkholder-Davis-Gundy inequality (BDG for short); see *e.g.* [20, Theorem B.1].

Lemma 2.1.4 (BDG). *If* $\big\{\Phi(s,y) : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\big\}$ *is an adapted random field with respect to* $\mathbb{F}$ *such that* $\|\Phi\|_{\mathfrak{H}} \in L^p(\Omega)$ *for some* $p \ge 2$*, then*

$$\left\| \int_{[0,t]\times\mathbb{R}^2} \Phi(s,y)W(ds,dy) \right\|_p^2 \le 4p \left\| \int_{[0,t]\times\mathbb{R}^4} \Phi(s,z)\Phi(s,y)\|y-z\|^{-\beta} dydzds \right\|_{p/2}. \tag{2.25}$$

We refer interested readers to the book [20] for a nice introduction to Dalang-Walsh's theory. For our purpose, we will often apply BDG as follows. If Φ is $\mathbb{F}$-adapted and $\|G_{t-\bullet}(x-$

$*)\Phi(\bullet,*)\|_{\mathfrak{H}} \in L^p(\Omega)$ for some $p \geq 2$, then BDG implies

$$\left\| \int_{[0,t]\times\mathbb{R}^2} G_{t-s}(x-y)\Phi(s,y)W(ds,dy) \right\|_p^2$$
$$\leq 4p \left\| \int_{[0,t]\times\mathbb{R}^4} G_{t-s}(x-z)G_{t-s}(x-y)\Phi(s,y)\Phi(s,z)\|y-z\|^{-\beta} dsdzdy \right\|_{p/2}, \qquad (2.26)$$

by viewing $\int_{[0,t]\times\mathbb{R}^2} G_{t-s}(x-y)\Phi(s,y)W(ds,dy)$ as the martingale

$$\left\{ \int_{[0,r]\times\mathbb{R}^2} G_{t-s}(x-y)\Phi(s,y)W(ds,dy) : r \in [0,t] \right\} \text{ evaluated at at time } t.$$

Now let us recall some basic facts on the Malliavin calculus associated with W. For any unexplained notation and result, we refer to the book [30]. We denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let $\mathscr{S}$ be the space of simple functionals of the form $F = f(W(h_1),\dots,W(h_n))$ for $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$, $1 \leq i \leq n$. Then, the Malliavin derivative DF is the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),\dots,W(h_n))h_i\,.$$

The derivative operator D is closable from $L^p(\Omega)$ into $L^p(\Omega;\mathfrak{H})$ for any $p \geq 1$ and we define $\mathbb{D}^{1,p}$ to be the completion of $\mathscr{S}$ under the norm $\|F\|_{1,p} = \left(\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\|DF\|_{\mathfrak{H}}^p\right]\right)^{1/p}$.

The *chain rule* for D asserts that if $F_1, F_2 \in \mathbb{D}^{1,2}$ and $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are Lipschitz, then $h_1(F_1)h_2(F_2) \in \mathbb{D}^{1,1}$ and $h_i(F_i) \in \mathbb{D}^{1,2}$ with

$$D\big(h_1(F_1)h_2(F_2)\big) = h_2(F_2)Y_1DF_1 + h_1(F_1)Y_2DF_2, \qquad (2.27)$$

where Y_i is some $\sigma\{F_i\}$-measurable random variable bounded by the Lipschitz constant of h_i for $i = 1,2$; ; when the h_i are differentiable, we have $Y_i = h_i'(F_i)$, $i = 1,2$ (see, for instance, [30, Proposition 1.2.4]).

We denote by δ the adjoint of D given by the duality formula

$$\mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF\rangle_{\mathfrak{H}}] \tag{2.28}$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom}\delta \subset L^2(\Omega;\mathfrak{H})$, the domain of δ. The operator δ is also called the Skorohod integral and in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod (see *e.g.* [14, 32]). In our context, the Dalang-Walsh integral coincides with the Skorohod integral: Any adapted random field Φ that satisfies $\mathbb{E}\big[\|\Phi\|_{\mathfrak{H}}^2\big] < \infty$ belongs to the domain of δ and

$$\delta(\Phi) = \int_0^\infty \int_{\mathbb{R}^2} \Phi(s,y) W(ds,dy).$$

The proof of this result is analogous to the case of integrals with respect to the Brownian motion (see [30, Proposition 1.3.11]), by just replacing real processes by $\mathfrak{H}_0$-valued processes, where $\mathfrak{H}_0$ is defined in (2.19). As a consequence, the equation (2.4) can be written as

$$u(t,x) = 1 + \delta\big(G_{t-\bullet}(x-*)\sigma(u(\bullet,*))\big).$$

The operators D and δ satisfy the commutation relation

$$[D,\delta]V := (D\delta - \delta D)(V) = V. \tag{2.29}$$

By Fubini's theorem and the duality formula (3.6), we can interchange the Skorohod integral and Lebesgue integral: Suppose $f_x \in \operatorname{Dom}\delta$ is adapted for each x in some finite measure space (E,μ) such that $\int_E f_x \mu(dx)$ also belongs to $\operatorname{Dom}\delta$ and $\mathbb{E}\int_E \|f_x\|_{\mathfrak{H}}^2 \mu(dx) < \infty$, then

$$\delta\left(\int_E f_x \mu(dx)\right) = \int_E \delta(f_x)\mu(dx) \text{ almost surely.} \tag{2.30}$$

Indeed, for any $F \in \mathscr{S}$,

$$\mathbb{E}\left[F\delta\left(\int_E f_x\mu(dx)\right)\right] = \mathbb{E}\big\langle DF, \int_E f_x\mu(dx)\big\rangle_{\mathfrak{H}} = \int_E \mathbb{E}\big\langle DF, f_x\big\rangle_{\mathfrak{H}}\mu(dx)$$
$$= \int_E \mathbb{E}\big[F\delta(f_x)\big]\mu(dx) = \mathbb{E}\left[F\int_E \delta(f_x)\mu(dx)\right],$$

which gives us (2.30). In particular, the equalities in (2.13) are valid.

With the help of the derivative operator, we can represent $F \in \mathbb{D}^{1,2}$ as a stochastic integral. This is the content of the following two-parameter Clark-Ocone formula, see *e.g.* [8, Proposition 6.3] for a proof.

Lemma 2.1.5 (Clark-Ocone formula)**.** *Given $F \in \mathbb{D}^{1,2}$, we have almost surely*

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathbb{E}\big[D_{s,y}F|\mathscr{F}_s\big]W(ds,dy).$$

We end this section with the following useful fact: If $\{\Phi_s : s \in \mathbb{R}_+\}$ is a jointly measurable and integrable process satisfying $\int_{\mathbb{R}_+}\big(\mathrm{Var}(\Phi_s)\big)^{1/2}ds < \infty$, then

$$\sqrt{\mathrm{Var}\left(\int_{\mathbb{R}_+}\Phi_s ds\right)} \le \int_{\mathbb{R}_+}\sqrt{\mathrm{Var}(\Phi_s)}ds. \tag{2.31}$$

2.2 Gaussian fluctuation of the spatial averages

We follow the three steps described in our introduction.

2.2.1 Limiting covariance structure

Proposition 2.2.1. *Suppose $t_1, t_2 \in (0,\infty)$. We have, with $\xi(s) = \mathbb{E}\big[\sigma(u(s,0))\big]$,*

$$\frac{\mathbb{E}\big[F_R(t_1)F_R(t_2)\big]}{R^{4-\beta}} \xrightarrow{R\to\infty} 4\pi^2 c_\beta \kappa_\beta \int_0^{t_1\wedge t_2}(t_1-s)(t_2-s)\xi^2(s)ds \tag{2.32}$$

with $\kappa_\beta = \int_{\mathbb{R}^2} d\xi \|\xi\|^{\beta-4} J_1(\|\xi\|)^2 \in (0,\infty)$. *In particular, for any* $t > 0$,

$$\mathrm{Var}\big(F_R(t)\big) R^{\beta-4} \xrightarrow{R\to\infty} 4\pi^2 c_\beta \kappa_\beta \int_0^t (t-s)^2 \xi^2(s) ds.$$

Proof. Recall that $F_R(t) = \int_0^t \int_{\mathbb{R}^2} \varphi_{t,R}(s,y)\sigma(u(s,y))W(ds,dy)$. Then, by Itô's isometry,

$$\mathbb{E}\big[F_R(t_1)F_R(t_2)\big] = \int_0^{t_1\wedge t_2} \int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\|y-z\|^{-\beta}\mathbb{E}\big[\sigma(u(s,y))\sigma(u(s,z))\big] dydzds.$$

We claim that, as $R \to \infty$,

$$R^{\beta-4} \int_0^{t_1\wedge t_2} \int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big] dydzds \to 0. \tag{2.33}$$

Assuming (2.33), we can deduce from Lemma 2.1.2, the stationarity of the process $\{u(t,x) : x \in \mathbb{R}^2\}$ and dominated convergence that

$$\lim_{R\to\infty} \frac{\mathbb{E}\big[F_R(t_1)F_R(t_2)\big]}{R^{4-\beta}} = \lim_{R\to\infty} \int_0^{t_1\wedge t_2} \xi^2(s)\Psi_R(t_1,t_2;s)ds = \text{RHS of (2.32)},$$

where $\xi(s) = \mathbb{E}[\sigma(u(s,0))]$ is uniformly bounded over $s \in [0, t_1 \wedge t_2]$.

We need to prove (2.33) now and it is enough to show for any $s \in (0, t_1 \wedge t_2]$

$$\lim_{\|y-z\|\to\infty} \mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big] = 0. \tag{2.34}$$

Indeed, if (2.34) holds for any given $s \in (0, t_1 \wedge t_2]$, then for arbitrarily small $\varepsilon > 0$, there is some $K = K(\varepsilon, s)$ such that $\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big] < \varepsilon$, for $\|y-z\| \geq K$. By Lemma 2.1.2, we deduce

$$\begin{aligned} &R^{\beta-4} \int_{\|y-z\|\geq K} \varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,,z))\big] dydz \\ &\leq \varepsilon \Psi_R(t_1,t_2;s) \lesssim \varepsilon, \end{aligned}$$

while using the uniform L^2-boundedness of $u(t,x)$, we get

$$
\begin{aligned}
&R^{\beta-4}\int_{\|y-z\|<K}\varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big]dydz\\
&\lesssim R^{\beta-4}\int_{\|y-z\|<K}\varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}dydz\\
&=R^{\beta-4}\int_{\|\xi\|<K}d\xi\|\xi\|^{-\beta}\left(\int_{\mathbb{R}^2}\varphi_{t,R}(s,z+\xi)\varphi_{t,R}(s,z)dz\right)\lesssim R^{\beta-2}\int_{\|\xi\|<K}d\xi\|\xi\|^{-\beta}\text{ by (2.24)}\\
&\lesssim R^{\beta-2}\xrightarrow{R\to\infty}0.
\end{aligned}
$$

That is, we just proved for any $s\in(0,t_1\wedge t_2]$,

$$
R^{\beta-4}\int_{\mathbb{R}^4}\varphi_{t,R}(s,y)\varphi_{t,R}(s,z)\|y-z\|^{-\beta}\mathrm{Cov}\big[\sigma(u(s,y)),\sigma(u(s,z))\big]dydz\xrightarrow{R\to\infty}0,
$$

where the LHS is uniformly bounded in $R>0$ and $s\in(0,t_1\wedge t_2]$ in view of Lemma 2.1.2. Then the claim (2.33) follows from the dominated convergence.

It remains to verify (2.34). By Theorem 2.0.6, for any $0<s<t$,

$$
\|D_{s,y}u(t,x)\|_p\lesssim G_{t-s}(x-y).
$$

By Lemma 2.1.5,

$$
\sigma(u(s,y))=\mathbb{E}\big[\sigma(u(s,y))\big]+\int_0^s\int_{\mathbb{R}^2}\mathbb{E}\Big[D_{r,\gamma}\big(\sigma(u(s,y))\big)|\mathscr{F}_r\Big]W(dr,d\gamma).
$$

As a consequence,

$$
\mathbb{E}\big[\sigma(u(s,y))\sigma(u(s,z))\big]=\xi^2(s)+T(s,y,z),
$$

where

$$
T(s,y,z)=\int_0^s\int_{\mathbb{R}^4}\mathbb{E}\Big(\mathbb{E}\big[D_{r,\gamma}(\sigma(u(s,y)))|\mathscr{F}_r\big]\mathbb{E}\big[D_{r,\gamma'}\big(\sigma(u(s,z))\big)|\mathscr{F}_r\big]\Big)\|\gamma-\gamma'\|^{-\beta}d\gamma d\gamma' dr.
$$

By the chain-rule (3.4) for the derivative operator,

$$D_{r,\gamma}\big(\sigma(u(s,y))\big) = \Sigma_{s,y} D_{r,\gamma} u(s,y)$$

with $\Sigma_{s,y}$ an adapted random field uniformly bounded by L, where we recall that L is the Lipschitz constant of σ. This implies,

$$\begin{aligned}\Big|\mathbb{E}\Big(\mathbb{E}\big[D_{r,\gamma}(\sigma(u(s,y)))|\mathscr{F}_r\big]\mathbb{E}\big[D_{r,\gamma'}\big(\sigma(u(s,z))\big)|\mathscr{F}_r\big]\Big)\Big| &\lesssim \big\|D_{r,\gamma}u(s,y)\big\|_2\big\|D_{r,\gamma'}u(s,z)\big\|_2 \\ &\lesssim G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z).\end{aligned}$$

Thus,

$$|T(s,y,z)| \lesssim \int_0^s \int_{\mathbb{R}^4} G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z)\|\gamma-\gamma'\|^{-\beta} d\gamma d\gamma' dr.$$

Suppose $\|y-z\| > 2s$, then

$$G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z)\|\gamma-\gamma'\|^{-\beta} \leq G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z)\big(\|y-z\|-2s\big)^{-\beta}$$

from which we get

$$|T(s,y,z)| \lesssim \big(\|y-z\|-2s\big)^{-\beta} \int_0^s \int_{\mathbb{R}^4} G_{s-r}(\gamma-y)G_{s-r}(\gamma'-z) d\gamma d\gamma' dr \xrightarrow{\|y-z\|\to\infty} 0.$$

This implies (2.34) and hence concludes our proof. □

2.2.2 Convergence of finite-dimensional distributions

As it was explained in the introduction, a basic ingredient for the convergence of finite-dimensional distributions is the following estimate

$$R^{2\beta-8}\mathrm{Var}\big(\langle DF_R(t_1),V_{t_2,R}\rangle_{\mathfrak{H}}\big)\lesssim R^{-\beta}\text{ for }R\geq t_1+t_2, \tag{2.35}$$

where we recall that $V_{t,R}(s,y)=\varphi_{t,R}(s,y)\sigma(u(s,y))$ and $\varphi_{t,R}$ is defined in (3.31).

Note that the Malliavin-Stein bound (2.16) and the above bound (2.35) with $t_1=t_2=t$ lead to the quantitative CLT in (3.4). In fact, from (2.35) and (2.16), we have for any fixed $t>0$ and $Z\sim\mathscr{N}(0,1)$,

$$d_{\mathrm{TV}}\big(F_R(t)/\sigma_R,Z\big)\leq\frac{2}{\sigma_R^2}\sqrt{\mathrm{Var}\big(\langle DF_R(t),V_{t,R}\rangle_{\mathfrak{H}}\big)}\lesssim\frac{1}{\sigma_R^2}R^{4-\frac{3\beta}{2}},\ R\geq 2t;$$

by Proposition 2.2.1, $\sigma_R^2R^{\beta-4}$ converges to some explicit positive constant, see (2.32). So we can write, for all $R\geq R_t$

$$d_{\mathrm{TV}}\big(F_R(t)/\sigma_R,Z\big)\leq CR^{-\beta/2},$$

where R_t is some constant that does not depend on R. As the total variation distance is aways bounded by 1, we can write for $R\leq R_t$,

$$d_{\mathrm{TV}}\big(F_R(t)/\sigma_R,Z\big)\leq 1\leq (R_t)^{\beta/2}R^{-\beta/2},\forall R\leq R_t.$$

Therefore, the bound (3.4) follows.

Note that (2.35), together with Proposition 2.0.8, implies the convergence in law of the finite dimensional distributions. In fact, fix any integer $m\geq 1$ and choose m points $t_1,\dots,t_m\in(0,\infty)$, then consider the random vector $\Phi_R=\big(F_R(t_1),\dots,F_R(t_m)\big)$ and let $\mathbf{G}=(\mathscr{G}_1,\dots,\mathscr{G}_m)$ denote a centered Gaussian random vector with covariance matrix $(C_{i,j})_{1\le i,j\le m}$ given by

$$C_{i,j} := 4\pi^2 c_\beta \kappa_\beta \int_0^{t_i\wedge t_j} (t_i - s)(t_j - s)\xi^2(s)ds.$$

Recall from (2.13) that $F_R(t_i) = \delta(V_{t_i,R})$ for all $i = 1,\dots,m$. Then, by (2.12) we can write

$$\left|\mathbb{E}(h(R^{\frac{\beta}{2}-2}\Phi_R)) - \mathbb{E}(h(\mathbf{G}))\right| \le \frac{m}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^m \mathbb{E}\left(\left|C_{i,j} - R^{\beta-4}\langle DF_R(t_i), V_{t_j,R}\rangle_{\mathfrak{H}}\right|^2\right)} \tag{2.36}$$

for every $h \in C^2(\mathbb{R}^m)$ with bounded second partial derivatives. Thus, in view of (2.36), in order to show the convergence in law of $R^{\frac{\beta}{2}-2}\Phi_R$ to $\mathbf{G}$, it suffices to show that for any $i, j = 1,\dots,m$,

$$\lim_{R\to\infty} \mathbb{E}\left(\left|C_{i,j} - R^{\beta-4}\langle DF_R(t_i), V_{t_j,R}\rangle_{\mathfrak{H}}\right|^2\right) = 0. \tag{2.37}$$

Notice that, by the duality relation (3.6) and the convergence (2.32), we have

$$\begin{aligned} R^{\beta-4}\mathbb{E}\Big(\big\langle DF_R(t_i), V_{t_j,R}\big\rangle_{\mathfrak{H}}\Big) &= R^{\beta-4}\mathbb{E}\big[F_R(t_i)\delta(V_{t_j,R})\big] \\ &= R^{\beta-4}\mathbb{E}\big[F_R(t_i)F_R(t_j)\big] \xrightarrow{R\to\infty} C_{i,j}. \end{aligned} \tag{2.38}$$

Therefore, the convergence (2.37) follows immediately from (2.38) and (2.35). Hence the finite-dimensional distributions of $\{R^{\frac{\beta}{2}-2}F_R(t) : t \in \mathbb{R}_+\}$ converge to those of $\mathscr{G}$ as $R \to \infty$.

The rest of this subsection is then devoted to the proof of (2.35).

Proof of (2.35). Recall from (2.13) that

$$F_R(t) = \int_{B_R} (u(t,x) - 1)dx = \delta(V_{t,R}) \quad \text{with} \quad V_{t,R}(s,y) = \varphi_{t,R}(s,y)\sigma(u(s,y)).$$

The commutation relation (3.7) implies for $s \le t$,

$$D_{s,y}F_R(t) = D_{s,y}\delta(V_{t,R}) = V_{t,R}(s,y) + \delta(D_{s,y}V_{t,R}). \tag{2.39}$$

By the chain rule for the derivative operator (see (3.4))

$$D_{s,y}[V_{t,R}(r,z)] = \varphi_{t,R}(r,z)D[\sigma(u(r,z))] = \varphi_{t,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z), \tag{2.40}$$

where $\Sigma_{r,z}$ is an adapted random field bounded by the Lipschitz constant of σ. Substituting (2.56) into (2.39), yields, for $s \le t$,

$$D_{s,y}F_R(t) = \varphi_{t,R}(s,y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}^2} \varphi_{t,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z)W(dr,dz).$$

Then, for $t_1, t_2 \in (0,\infty)$, we can write $\langle DF_R(t_1), V_{t_2,R}\rangle_{\mathfrak{H}} = A_1 + A_2$, with

$$A_1 = \langle V_{t_1,R}, V_{t_2,R}\rangle_{\mathfrak{H}} = \int_0^{t_1\wedge t_2}\int_{\mathbb{R}^4} \varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\sigma(u(s,y))\sigma(u(s,z))\|y-z\|^{-\beta}dydzds$$

and

$$A_2 = \int_0^{t_1\wedge t_2}\int_{\mathbb{R}^4}\left(\int_s^{t_1}\int_{\mathbb{R}^2}\varphi_{t_1,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z)W(dr,dz)\right) \times \|y-y'\|^{-\beta}V_{t_2,R}(s,y')dsdydy'.$$

(i) *Estimation of* $\mathrm{Var}(A_1)$. From (2.31), we deduce that $\mathrm{Var}(A_1)$ is bounded by

$$\left(\int_0^{t_2\wedge t_1}\left(\mathrm{Var}\int_{\mathbb{R}^4}\varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\sigma(u(s,y))\sigma(u(s,z))\|y-z\|^{-\beta}dydz\right)^{1/2}ds\right)^2. \tag{2.41}$$

Note that the variance term in (2.41) is equal to

$$\int_{\mathbb{R}^8}\varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\varphi_{t_1,R}(s,y')\varphi_{t_2,R}(s,z')\|y-z\|^{-\beta}\|y'-z'\|^{-\beta} \times \mathrm{Cov}\Big[\sigma(u(s,y))\sigma(u(s,z)), \sigma(u(s,y'))\sigma(u(s,z'))\Big]dydzdy'dz'. \tag{2.42}$$

To estimate the covariance term, we apply the Clark-Ocone formula (see Lemma 2.1.5) to write

$$
\begin{aligned}
&\sigma(u(s,y))\sigma(u(s,z)) - \mathbb{E}[\sigma(u(s,y))\sigma(u(s,z))] \\
&\quad = \int_0^s \int_{\mathbb{R}^2} \mathbb{E}\Big\{ D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big) | \mathscr{F}_r \Big\} W(dr, d\gamma).
\end{aligned}
$$

Then we apply Itô's isometry to obtain

$$
\begin{aligned}
&\mathrm{Cov}\Big[\sigma(u(s,y))\sigma(u(s,z)), \sigma(u(s,y'))\sigma(u(s,z'))\Big] \qquad (2.43)\\
&= \int_0^s \int_{\mathbb{R}^4} \mathbb{E}\Big[\mathbb{E}\big\{ D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big) | \mathscr{F}_r \big\} \mathbb{E}\big\{ D_{r,\gamma'}\big(\sigma(u(s,y'))\sigma(u(s,z'))\big) | \mathscr{F}_r \big\} \Big] \\
&\quad \times \|\gamma - \gamma'\|^{-\beta} d\gamma d\gamma' dr,
\end{aligned}
$$

where, by the chain rule (3.4),

$$
D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big) = \sigma(u(s,y))\Sigma_{s,z} D_{r,\gamma} u(s,z) + \sigma(u(s,z))\Sigma_{s,y} D_{r,\gamma} u(s,y).
$$

Then by Cauchy-Schwarz inequality and Theorem 2.0.6, we can see that the above covariance term (2.43) is bounded by

$$
\begin{aligned}
&\int_0^s \int_{\mathbb{R}^4} \Big\| D_{r,\gamma}\big(\sigma(u(s,y))\sigma(u(s,z))\big) \Big\|_2 \Big\| D_{r,\gamma'}\big(\sigma(u(s,y'))\sigma(u(s,z'))\big) \Big\|_2 \|\gamma - \gamma'\|^{-\beta} d\gamma d\gamma' dr \\
&\lesssim \int_0^s dr \int_{\mathbb{R}^4} d\gamma d\gamma' \|\gamma - \gamma'\|^{-\beta} \Big(\|D_{r,\gamma} u(s,z)\|_4 + \|D_{r,\gamma} u(s,y)\|_4 \Big) \\
&\qquad \times \Big(\|D_{r,\gamma'} u(s,z')\|_4 + \|D_{r,\gamma'} u(s,y')\|_4 \Big) \\
&\lesssim \int_0^s dr \int_{\mathbb{R}^4} d\gamma d\gamma' \|\gamma - \gamma'\|^{-\beta} \big(G_{s-r}(z-\gamma) + G_{s-r}(y-\gamma) \big) \big(G_{s-r}(z'-\gamma') + G_{s-r}(y'-\gamma') \big).
\end{aligned}
$$

Now we can plug the last estimate into (2.42) for further computations:

$$
\begin{aligned}
&\operatorname{Var}\left(\int_{\mathbb{R}^4}\varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\sigma(u(s,y))\sigma(u(s,z))\|y-z\|^{-\beta}dydz\right)\\
&\lesssim\int_0^s dr\int_{\mathbb{R}^{12}}\varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\varphi_{t_1,R}(s,y')\varphi_{t_2,R}(s,z')\|y-z\|^{-\beta}\|y'-z'\|^{-\beta}\|\gamma-\gamma'\|^{-\beta}\\
&\times\big(G_{s-r}(z-\gamma)+G_{s-r}(y-\gamma)\big)\big(G_{s-r}(z'-\gamma')+G_{s-r}(y'-\gamma')\big)d\gamma d\gamma' dydzdy'dz'. \qquad (2.44)
\end{aligned}
$$

In order to obtain $\operatorname{Var}(A_1)\lesssim R^{8-3\beta}$, it is enough to show $\sup_{s\le t_1\wedge t_2}\mathscr{T}_s\lesssim R^{8-3\beta}$ with

$$
\begin{aligned}
\mathscr{T}_s:=&\int_0^s dr\int_{\mathbb{R}^{12}}\varphi_{t_1,R}(s,y)\varphi_{t_2,R}(s,z)\varphi_{t_1,R}(s,y')\varphi_{t_2,R}(s,z')\|y-z\|^{-\beta}\|y'-z'\|^{-\beta}\\
&\times\|\gamma-\gamma'\|^{-\beta}G_{s-r}(z-\gamma)G_{s-r}(z'-\gamma')d\gamma d\gamma' dydzdy'dz'
\end{aligned}
$$

as other terms from (2.44) can be estimated *in the same way* with *the same bound*.

For $s\in(0,t_1\wedge t_2]$, we write, using (3.31),

$$
\begin{aligned}
\mathscr{T}_s=&\int_0^s dr\int_{B_R^4}\int_{\mathbb{R}^{12}}G_{t_1-s}(x_1-y)G_{t_1-s}(x_1'-y')G_{t_2-s}(x_2-z)G_{t_2-s}(x_2'-z')G_{s-r}(z-\gamma)\\
&\times G_{s-r}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta}\|y-z\|^{-\beta}\|y'-z'\|^{-\beta}d\gamma d\gamma' dydzdy'dz'dx_1dx_1'dx_2dx_2'.
\end{aligned}
$$

Making the change of variables

$$
(\gamma,\gamma',y,z,y',z',x_1,x_1',x_2,x_2')\to R(\gamma,\gamma',y,z,y',z',x_1,x_1',x_2,x_2')
$$

and using $G_t(Rz)=R^{-1}G_{tR^{-1}}(z)$ for every $t,R>0$ yields

$$
\begin{aligned}
R^{-14+3\beta}\mathscr{T}_s=&\int_0^s dr\int_{B_1^4}\int_{\mathbb{R}^{12}}G_{\frac{t_1-s}{R}}(x_1-y)G_{\frac{t_1-s}{R}}(x_1'-y')G_{\frac{t_2-s}{R}}(x_2-z)G_{\frac{t_2-s}{R}}(x_2'-z')\\
&\times G_{\frac{s-r}{R}}(z-\gamma)G_{\frac{s-r}{R}}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta}\|y-z\|^{-\beta}\|y'-z'\|^{-\beta}d\gamma d\gamma' dydzdy'dz'dx_1dx_1'dx_2dx_2'.
\end{aligned}
$$

Using the fact (2.23), we can integrate out x_1, x_1', x_2, x_2' to bound $R^{-14+3\beta}\mathscr{T}_s$ by

$$R^{-10+3\beta}(t_1-s)^2(t_2-s)^2\int_0^s dr\int_{\mathbb{R}^{12}}\mathbf{1}_{\{\|y\|\vee\|y'\|\vee\|z\|\vee\|z'\|\vee\|\gamma\|\vee\|\gamma'\|\le 1+(t_1+t_2)R^{-1}\}}$$
$$\times G_{\frac{s-r}{R}}(z-\gamma)G_{\frac{s-r}{R}}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta}\|y-z\|^{-\beta}\|y'-z'\|^{-\beta}d\gamma d\gamma' dy dz dy' dz'. \qquad (2.45)$$

Suppose $R\ge t_1+t_2$ and notice that

$$\sup_{z\in B_2}\int_{B_2}\|y-z\|^{-\beta}dy\le\int_{B_4}\|y\|^{-\beta}dy=\frac{2\pi}{2-\beta}4^{2-\beta}<\infty.$$

Therefore, integrating out y, y' in (2.45), we obtain

$$\mathscr{T}_s\lesssim R^{10-3\beta}\int_0^s dr\int_{\mathbb{R}^8}\mathbf{1}_{\{\|z\|\vee\|z'\|\vee\|\gamma\|\vee\|\gamma'\|\le 2\}}G_{\frac{s-r}{R}}(z-\gamma)G_{\frac{s-r}{R}}(z'-\gamma')\|\gamma-\gamma'\|^{-\beta}d\gamma d\gamma' dz dz'.$$

We further integrate out z, z' and use (2.23) again to write

$$\sup_{s\le t_1\wedge t_2}\mathscr{T}_s\lesssim R^{8-3\beta}\int_{\mathbb{R}^8}\mathbf{1}_{\{\|\gamma\|\vee\|\gamma'\|\le 2\}}\|\gamma-\gamma'\|^{-\beta}d\gamma d\gamma'\lesssim R^{8-3\beta}.$$

So we obtain $\mathrm{Var}(A_1)\lesssim R^{8-3\beta}$ for $R\ge t_1+t_2$, where the implicit constant does not depend on R.

Next we estimate the variance of A_2.

(ii) *Estimate of* $\mathrm{Var}(A_2)$. Using again (2.31), we write

$$\mathrm{Var}(A_2)\le\left(\int_0^{t_1\wedge t_2}\left\{\mathrm{Var}\int_{\mathbb{R}^4}\left(\int_s^{t_1}\int_{\mathbb{R}^2}\varphi_{t_1,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z)W(dr,dz)\right)\|y-y'\|^{-\beta}\right.\right.$$
$$\left.\left.\times\varphi_{t_2,R}(s,y')\sigma(u(s,y'))dydy'\right\}^{1/2}ds\right)^2=:\left(\int_0^{t_1\wedge t_2}\sqrt{\mathscr{U}_s}ds\right)^2.$$

As before, we will show $\sup_{s\le t_2\wedge t_1}\mathscr{U}_s\lesssim R^{8-3\beta}$.

First note that

$$\int_s^{t_1}\int_{\mathbb{R}^2}\varphi_{t_1,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z)W(dr,dz)=\mathfrak{M}_{s,y}(t_1),$$

where $\{\mathfrak{M}_{s,y}(\tau) : \tau \in [s,t_1]\}$ is the square-integrable martingale given by

$$\mathfrak{M}_{s,y}(\tau) := \int_s^\tau \int_{\mathbb{R}^2} \varphi_{t_1,R}(r,z)\Sigma_{r,z}D_{s,y}u(r,z)W(dr,dz).$$

Then we deduce from the martingale property that

$$\mathbb{E}\big[\sigma(u(s,y'))\mathfrak{M}_{s,y}(t_1)\big] = \mathbb{E}\big[\sigma(u(s,y'))\mathbb{E}(\mathfrak{M}_{s,y}(t_1)|\mathscr{F}_s)\big] = 0,$$

that is, $\mathfrak{M}(t_1)$ and $\sigma\big(u(s,y')\big)$ are uncorrelated. Moreover, by Itô's formula,

$$\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,\widetilde{y}}(t_1) = \underbrace{\int_s^{t_1} \mathfrak{M}_{s,y}(\tau)d\mathfrak{M}_{s,\widetilde{y}}(\tau) + \int_s^{t_1} \mathfrak{M}_{s,\widetilde{y}}(\tau)d\mathfrak{M}_{s,y}(\tau)}_{\text{martingale}-\text{part}} + \langle \mathfrak{M}_{s,y}, \mathfrak{M}_{s,\widetilde{y}}\rangle_{t_1},$$

where the bracket $\langle \mathfrak{M}_{s,y}, \mathfrak{M}_{s,\widetilde{y}}\rangle_{t_1}$ between both martingales is equal to

$$\int_s^{t_1} \int_{\mathbb{R}^4} \varphi_{t_1,R}(r,z)\Sigma_{r,z}\big(D_{s,y}u(r,z)\big)\varphi_{t_1,R}(r,\widetilde{z})\Sigma_{r,\widetilde{z}}\big(D_{s,\widetilde{y}}u(r,\widetilde{z})\big)\|z-\widetilde{z}\|^{-\beta}dzd\widetilde{z}dr.$$

So, using the estimate (2.11), we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,\widetilde{y}}(t_1)\sigma(u(s,y'))\sigma(u(s,\widetilde{y'}))\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\big(\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,\widetilde{y}}(t_1)|\mathscr{F}_s\big)\sigma(u(s,y'))\sigma(u(s,\widetilde{y'}))\Big] \lesssim \big\|\langle \mathfrak{M}_{s,y}, \mathfrak{M}_{s,\widetilde{y}}\rangle_{t_1}\big\|_2 \\
&\lesssim \int_s^{t_1} \int_{\mathbb{R}^4} \varphi_{t_1,R}(r,z)\|D_{s,y}u(r,z)\|_4\varphi_{t_1,R}(r,\widetilde{z})\|D_{s,\widetilde{y}}u(r,\widetilde{z})\|_4\|z-\widetilde{z}\|^{-\beta}dzd\widetilde{z}dr \\
&\lesssim \int_s^{t_1} \int_{\mathbb{R}^4} \varphi_{t_1,R}(r,z)G_{r-s}(y-z)\varphi_{t_1,R}(r,\widetilde{z})G_{r-s}(\widetilde{y}-\widetilde{z})\|z-\widetilde{z}\|^{-\beta}dzd\widetilde{z}dr.
\end{aligned}$$

As a consequence, the variance-term $\mathscr{U}_s$ is indeed a second moment and

$$\begin{aligned}\mathscr{U}_s &= \int_{\mathbb{R}^8} dydy'd\widetilde{y}d\widetilde{y'}\|y-y'\|^{-\beta}\|\widetilde{y}-\widetilde{y'}\|^{-\beta}\varphi_{t_2,R}(s,y')\varphi_{t_2,R}(s,\widetilde{y'}) \\ &\quad \times \mathbb{E}\Big[\mathfrak{M}_{s,y}(t_1)\mathfrak{M}_{s,y'}(t_1)\sigma(u(s,y'))\sigma(u(s,\widetilde{y'}))\Big] \\ &\lesssim \int_s^{t_1} dr \int_{\mathbb{R}^{12}} dzd\widetilde{z}dydy'd\widetilde{y}d\widetilde{y'}\|y-y'\|^{-\beta}\|\widetilde{y}-\widetilde{y'}\|^{-\beta}\|z-\widetilde{z}\|^{-\beta} \\ &\quad \times \varphi_{t_2,R}(s,y')\varphi_{t_2,R}(s,\widetilde{y'})\varphi_{t_1,R}(r,z)\varphi_{t_1,R}(r,\widetilde{z})G_{r-s}(y-z)G_{r-s}(\widetilde{y}-\widetilde{z}),\end{aligned}$$

which has the same kind of expression as $\mathscr{T}_s$. The same arguments that led to the uniform estimate of $\mathscr{T}_s$ yields

$$\sup_{s\le t_1\wedge t_2} \mathscr{U}_s \lesssim R^{8-3\beta},$$

for $R \ge t_1+t_2$, thus we obtain $\mathrm{Var}(A_2) \lesssim R^{8-3\beta}$ for $R \ge t_1+t_2$. Hence, for $R \ge t_1+t_2$,

$$R^{2\beta-8}\mathrm{Var}\big(\langle DF_R(t_1), V_{t_2,R}\rangle_{\mathfrak{H}}\big) \lesssim R^{2\beta-8}\big[\mathrm{Var}(A_2)+\mathrm{Var}(A_1)\big] \lesssim R^{-\beta}.$$

This completes the proof of (2.35). □

2.2.3 Tightness

Set $q = \frac{2}{4-\beta} \in (1/2, 1)$. As explained in the introduction, by the Kolmogorov-Chentsov criterion for tightness, it is enough to prove the inequality (2.17): For any $T > 0$, $p \ge 2$ and for any $0 \le s < t \le T \le R$,

$$\big\|F_R(t)-F_R(s)\big\|_p \lesssim R^{1/q}\sqrt{t-s}, \tag{2.46}$$

where the implicit constant does not depend on t, s or R.

Proof of (2.46). Recall that $F_R(t) = \int_0^t \int_{\mathbb{R}^2} \varphi_{t,R}(s,y)\sigma(u(s,y))W(ds,dy)$. Then by BDG inequality

(2.25) and (2.20) we have, with the convention that $\varphi_{s,R}(r,y) = 0$ if $r > s$,

$$\begin{aligned}\big\|F_R(t) - F_R(s)\big\|_p^2 &\lesssim \bigg\| \int_{[0,t]\times\mathbb{R}^4} \big(\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\big)\sigma(u(r,y))\big(\varphi_{t,R}(r,z) - \varphi_{s,R}(r,z)\big) \\ &\qquad \times \sigma(u(r,z))\|y-z\|^{-\beta}dydzdr\bigg\|_{p/2} \\ &\lesssim \bigg\| \int_0^t dr \left(\int_{\mathbb{R}^2} \Big| \big(\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\big)\sigma(u(r,y))\Big|^{2q} dy\right)^{1/q} \bigg\|_{p/2}.\end{aligned}$$

Applying Minkowski's inequality yields

$$\begin{aligned}\big\|F_R(t) - F_R(s)\big\|_p^2 &\lesssim \int_0^t dr \left(\int_{\mathbb{R}^2} \big|\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\big|^{2q} \|\sigma(u(r,y))\|_p^{2q} dy\right)^{1/q} \\ &\lesssim \int_0^t dr \left(\int_{\mathbb{R}^2} \big|\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\big|^{2q} dy\right)^{1/q}. \qquad (2.47)\end{aligned}$$

Note that

$$\begin{aligned}\big|\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\big| &= \mathbf{1}_{\{r\geq s\}} \int_{B_R} G_{t-r}(x-y)dx \\ &\quad + \mathbf{1}_{\{r<s\}} \int_{B_R} \mathbf{1}_{\{\|x-y\|<s-r\}} \big[G_{s-r}(x-y) - G_{t-r}(x-y)\big]dx \\ &\quad + \mathbf{1}_{\{r<s\}} \int_{B_R} \mathbf{1}_{\{\|x-y\|\geq s-r\}} G_{t-r}(x-y)dx \\ &=: S_1 + S_2 + S_3.\end{aligned}$$

The first summand S_1 is bounded by $\mathbf{1}_{\{r\geq s\}}(t-r)\mathbf{1}_{\{\|y\|\leq R+t\}} \leq (t-s)\mathbf{1}_{\{\|y\|\leq R+t\}}$, in view of Lemma

2.1.1-(2). For the second summand, we can write

$$
\begin{aligned}
S_2 &\leq \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+s\}}\int_{B_R}\mathbf{1}_{\{\|x\|<s-r\}}\big[G_{s-r}(x)-G_{t-r}(x)\big]dx \\
&\leq \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+s\}}\int_{\{\|x\|<s-r\}}\left(\frac{1}{2\pi\sqrt{(s-r)^2-\|x\|^2}}-\frac{1}{2\pi\sqrt{(t-r)^2-\|x\|^2}}\right)dx \\
&= \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+s\}}\sqrt{t-s}\Big(\sqrt{t+s-2r}-\sqrt{t-s}\Big) \quad \text{by explicit computation} \\
&\lesssim \sqrt{t-s}\mathbf{1}_{\{\|y\|\leq R+s\}};
\end{aligned}
$$

In the same way, the third summand can be bounded as follows

$$
S_3 \leq \mathbf{1}_{\{r<s\}}\mathbf{1}_{\{\|y\|\leq R+t\}}\int_{\mathbb{R}^2}\mathbf{1}_{\{s-r\leq\|x\|<t-r\}}G_{t-r}(x)dx \lesssim \mathbf{1}_{\{\|y\|\leq R+t\}}\sqrt{t-s}.
$$

Therefore, we can continue with (2.47) to write

$$
\big\|F_R(t)-F_R(s)\big\|_p^2 \lesssim \int_0^t dr\left(\int_{\mathbb{R}^2}(t-s)^q\mathbf{1}_{\{\|y\|\leq R+t\}}dy\right)^{1/q} \lesssim (t-s)(R+t)^{2/q}.
$$

This implies (2.46). □

2.3 Fundamental estimate on the Malliavin derivative

This section is devoted to the proof of Theorem 2.0.6. After a useful lemma, we study the convergence and moment estimates for the Picard approximation in Section 3.15. The main body of the proof of Theorem 2.0.6 is given in Section 2.3.2 and we leave proofs of two technical lemmas to Section 2.3.3. Recall that $\beta \in (0,2)$ is fixed throughout this paper.

Lemma 2.3.1. *Given any random field* $\{\Phi(r,z) : (r,z) \in \mathbb{R}_+ \times \mathbb{R}^2\}$*, we have for any* $x \in \mathbb{R}^2$, $0 \leq$

$s < t < \infty$ and $p \geq 2$,

$$\left\| \int_s^t dr \int_{\mathbb{R}^4} dydz\, G_{t-r}(x-y) G_{t-r}(x-z) \Phi(r,z) \Phi(r,y) \|y-z\|^{-\beta} \right\|_{p/2}$$
$$\leq K_\beta t^{\frac{(2-2q)^2}{2q}} \int_s^t dr \int_{\mathbb{R}^2} dz\, G_{t-r}^{2q}(x-z) \big\| \Phi(r,z) \big\|_p^2, \tag{2.48}$$

where $q = \frac{2}{4-\beta} \in (1/2, 1)$ and the constant K_β only depends on β.

Proof. By (2.20), there exists some constant C_β that only depends on β such that

$$\begin{aligned}
&\int_{\mathbb{R}^4} dydz\, G_{t-r}(x-y) G_{t-r}(x-z) \Phi(r,z) \Phi(r,y) \|y-z\|^{-\beta} \\
&\leq C_\beta \left(\int_{\mathbb{R}^2} dy\, G_{t-r}^{2q}(x-y) |\Phi(r,y)|^{2q} \right)^{1/q} \\
&\leq C_\beta \left(\frac{(2\pi)^{1-2q}}{2-2q} (t-r)^{2-2q} \right)^{\frac{1}{q}-1} \int_{\mathbb{R}^2} dy\, G_{t-r}^{2q}(x-y) |\Phi(r,y)|^2 \\
&\leq K_\beta t^{\frac{(2-2q)^2}{2q}} \int_{\mathbb{R}^2} dy\, G_{t-r}^{2q}(x-y) |\Phi(r,y)|^2,
\end{aligned}$$

where we have used the fact that $G_{t-r}^{2q}(y)dy$, with $2q < 2$, is a finite measure on $\mathbb{R}^2$ with total mass $\frac{(2\pi)^{1-2q}}{2-2q}(t-r)^{2-2q}$ in view of (2.23) and we have put $K_\beta = C_\beta \big(\frac{(2\pi)^{1-2q}}{2-2q} \big)^{\frac{1}{q}-1}$. Therefore, a further application of Minkowski's inequality yields the bound in (2.48). □

2.3.1 Moment estimates for the Picard approximation

Recall the Picard iteration introduced in (2.18): $u_0(t,x) = 1$ and

$$u_{n+1}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y) \sigma\big(u_n(s,y)\big) W(ds,dy) \text{ for } n \geq 0. \tag{2.49}$$

Using the estimates (2.26) and (2.48), we can write with $2q = \frac{4}{4-\beta} \in (1,2)$, $p \geq 2$ and $n \geq 1$,

$$
\begin{aligned}
&\|u_n(t,x)\|_p^2 \leq 2 + 8p \\
&\quad \times \left\| \int_{[0,t]\times\mathbb{R}^4} G_{t-s}(x-z)G_{t-s}(x-y)\sigma(u_n(s,y))\sigma(u_n(s,z))\|y-z\|^{-\beta} dsdzdy \right\|_{p/2} \\
&\qquad \leq 2 + 8pK_\beta t^{\frac{(2-2q)^2}{2q}} \int_0^t ds \int_{\mathbb{R}^2} G_{t-s}^{2q}(x-y)\|\sigma(u_{n-1}(s,y))\|_p^2 dy.
\end{aligned}
$$

Then, using (2.23), we can write

$$
\begin{aligned}
\|u_n(t,x)\|_p^2 &\leq 2 + 8pK_\beta t^{\frac{(2-2q)^2}{2q}} \int_0^t ds \int_{\mathbb{R}^2} G_{t-s}^{2q}(x-y)\Big(2\sigma(0)^2 + 2L^2\|u_{n-1}(s,y)\|_p^2\Big) dy \\
&\leq 2 + \frac{16pK_\beta(2\pi)^{1-2q}}{(2-2q)(3-2q)} t^{\frac{(2-2q)^2}{2q}+3-2q}\sigma(0)^2 \\
&\quad + 16pK_\beta t^{\frac{(2-2q)^2}{2q}} L^2 \int_0^t ds \int_{\mathbb{R}^2} G_{t-s}^{2q}(x-y)\|u_{n-1}(s,y)\|_p^2 dy,
\end{aligned}
$$

where L is the Lipschitz constant of σ. This leads to

$$
H_n(t) \leq c_1 + c_2 \int_0^t ds H_{n-1}(s), \tag{2.50}
$$

where $H_n(t) = \sup_{x\in\mathbb{R}^2} \|u_n(t,x)\|_p^2$,

$$
c_1 := 2 + \frac{pK_\beta^*\sigma(0)^2}{3-2q} t^{\frac{(2-2q)^2}{2q}+3-2q} \quad \text{and} \quad c_2 := pK_\beta^* L^2 t^{\frac{(2-2q)^2}{2q}+2-2q},
$$

where $K_\beta^* = \frac{16K_\beta(2\pi)^{1-2q}}{2-2q} = 16C_\beta\big(\frac{(2\pi)^{1-2q}}{2-2q}\big)^{1/q}$ is a constant depending only on β. Therefore, by iterating the inequality (2.50) and taking into account that $H_0(t) = 1$, yields

$$
H_n(t) \leq c_1 \exp(c_2 t).
$$

In what follows, we will denote by C_β^* a generic constant that only depends on β and may be different from line to line. In this way, we obtain

$$\|u_n(t,x)\|_p \le \big(\sqrt{2}+\sqrt{p}C_\beta^* t^{\frac{3-\beta}{2}}|\sigma(0)|\big)\exp\big(pC_\beta^* t^{2-\beta}L^2\big).$$

As a consequence,

$$\|\sigma(u_n(t,x))\|_p \le |\sigma(0)| + L\big(\sqrt{2}+\sqrt{p}C_\beta^* t^{\frac{3-\beta}{2}}|\sigma(0)|\big)\exp\big(pC_\beta^* t^{2-\beta}L^2\big) =: \kappa_{p,t}. \tag{2.51}$$

2.3.2 Proof of Theorem 2.0.6

The proof will be done in several steps.

Step 1. In this step, we will establish the following estimate (2.52) for the p-norm of the Malliavin derivative of the Picard iteration.

Proposition 2.3.2. *For any $n \ge 3$ and any $p \ge 2$*

$$\big\|D_{s,y}u_{n+1}(t,x)\big\|_p \le C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y), \tag{2.52}$$

for almost all $(s,y) \in [0,t]\times\mathbb{R}^2$, where $\kappa_{p,t}$ is defined in (2.51) *and the constant $C_{\beta,p,t,L}$ is given by*

$$C_{\beta,p,t,L} := 1+\sqrt{p}LC_\beta^* t^{\frac{1}{q}-\frac{1}{2}} + pC_\beta^* L^2 t^{\frac{2}{q}-1} + \sum_{k=3}^{\infty}\frac{(pC_\beta^* L^2)^{k/2}}{\sqrt{(k-2)!}}t^{k(\frac{1}{q}-\frac{1}{2})}, \tag{2.53}$$

with C_β^* a constant only depending on β.

One key ingredient for proving Proposition 2.3.2 is the following Lemma 2.3.3, which is a consequence of the technical Lemma 2.0.11. Both Lemma 2.0.11 and Lemma 2.3.3 will be proved in Section 2.3.3.

Lemma 2.3.3. *For $q \in (1/2,1)$, $\delta \in [1,1/q]$ and $s<t$, we have*

$$K_{s,t}(z) := \int_s^t dr \big[G_{t-r}^{2q} * G_{r-s}^{2q}(z)\big]^{\delta} \lesssim (t-s)^{1-\delta(2q-1)} G_{t-s}^{\delta(2q-1)}(z). \tag{2.54}$$

where the implicit constant only depends on q.

Proof of Proposition 2.3.2. Fix $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$ and $p \geq 2$. Let us first establish the following weaker estimate:

$$u_n(t,x) \in \mathbb{D}^{1,p} \text{ and } \|D_{s,y}u_n(t,x)\|_p \leq C G_{t-s}(x-y), \tag{2.55}$$

for almost all $(s,y) \in [0,t] \times \mathbb{R}^2$, where the constant C may depend on n. It follows from (2.49) that the claim (2.55) holds true for $n = 0,1$, because $D_{s,y}u_0(t,x) = 0$ and $D_{s,y}u_1(t,x) = \sigma(1)G_{t-s}(x-y)$. Now suppose the claim (2.55) holds true for $n \geq 1$, then taking the Malliavin derivative in both sides of equality (2.49) and using the commutation relationship (3.7) and the chain rule (3.4), we obtain

$$D_{s,y}u_{n+1}(t,x) = G_{t-s}(x-y)\sigma\big(u_n(s,y)\big) + \int_s^t \int_{\mathbb{R}^2} G_{t-r}(x-z)\Sigma_{r,z}^{(n)} D_{s,y}u_n(r,z) W(dr,dz),$$

where $\big\{\Sigma_{s,y}^{(n)} : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\big\}$ is an adapted random field that is uniformly bounded by L, for each n. We recall that the constant L is the Lipschitz constant of the function σ appearing in (2.1). It follows that

$$\begin{aligned}
\big\|D_{s,y}u_{n+1}(t,x)\big\|_p^2 &\leq 2\kappa_{p,t}^2 G_{t-s}^2(x-y) + 8p\bigg\| \int_s^t \int_{\mathbb{R}^4} G_{t-r}(x-z)G_{t-r}(x-z')\Sigma_{r,z}^{(n)} \\
&\qquad \times D_{s,y}u_n(r,z)\Sigma_{r,z'}^{(n)} D_{s,y}u_n(r,z')\|z-z'\|^{-\beta} dz dz' dr \bigg\|_{p/2} \quad \text{by BDG (2.26)} \\
&\leq 2\kappa_{p,t}^2 G_{t-s}^2(x-y) + 8pL^2C_n^2 \int_s^t \big\|G_{t-r}(x-\bullet)G_{r-s}(y-\bullet)\big\|_{\mathfrak{H}_0}^2 dr,
\end{aligned}$$

by applying Minkowski's inequality and using the induction hypothesis, where $\kappa_{p,t}$ is defined in

(2.51) and $\mathfrak{H}_0$ has been introduced in (2.19). Note that Lemma 2.0.10 (see (2.20)) implies

$$\int_s^t \left\|G_{t-r}(x-\bullet)G_{r-s}(y-\bullet)\right\|_{\mathfrak{H}_0}^2 \le C_\beta \int_s^t dr \big(G_{t-r}^{2q} * G_{r-s}^{2q}\big)^{1/q}(x-y) \le C_\beta^* t^{\frac{1}{q}-1} G_{t-s}^{2-\frac{1}{q}}(x-y),$$

where the last inequality follows from Lemma 2.3.3 with $\delta = 1/q$ and C_β^* is a constant that only depends on β. Finally, using

$$G_{t-s}^{2-\frac{1}{q}}(x-y) \le \big[2\pi(t-s)\big]^{\frac{1}{q}} G_{t-s}^2(x-y), \tag{2.56}$$

we get $\|D_{s,y}u_{n+1}(t,x)\|_p \le C_{n+1}G_{t-s}(x-y)$ with $C_{n+1} = \sqrt{2\kappa_{p,t}^2 + pL^2C_n^2C_\beta^* t^{\frac{2}{q}-1}}$ and thus by routine computations, we can show $u_{n+1}(t,x) \in \mathbb{D}^{1,p}$; see also **Step 2**. This shows (2.55) for each n. Moreover, we point out that $D_{s,y}u_{n+1}(t,x) = 0$ if $s \ge t$.

To obtain the uniform estimate in (2.52), we proceed with the finite iterations

$$\begin{aligned} D_{s,y}u_{n+1}(t,x) &= G_{t-s}(x-y)\sigma\big(u_n(s,y)\big) \\ &+ \int_s^t \int_{\mathbb{R}^2} G_{t-r_1}(x-z_1)\Sigma_{r_1,z_1}^{(n)} G_{r_1-s}(z_1-y)\sigma(u_{n-1}(s,y))W(dr_1,dz_1) \\ &+ \sum_{k=2}^n \int_s^t \cdots \int_s^{r_{k-1}} \int_{\mathbb{R}^{2k}} G_{r_k-s}(z_k-y)\sigma\big(u_{n-k}(s,y)\big) \\ &\quad\times \prod_{j=1}^k G_{r_{j-1}-r_j}(z_{j-1}-z_j)\Sigma_{r_j,z_j}^{(n+1-j)} W(dr_j,dz_j) =: \sum_{k=0}^n T_k^{(n)}, \end{aligned} \tag{2.57}$$

where $T_k^{(n)}$ denotes the kth item in the sum and $r_0 = t, z_0 = x$. For example, $T_0^{(n)} = G_{t-s}(x-y)\sigma\big(u_n(s,y)\big)$ and

$$T_1^{(n)} = \int_s^t \int_{\mathbb{R}^2} G_{t-r_1}(x-z_1)\Sigma_{r_1,z_1}^{(n)} G_{r_1-s}(z_1-y)\sigma(u_{n-1}(s,y))W(dr_1,dz_1).$$

We are going to estimate the p-norm of each of term $T_k^{(n)}$ for $k = 0,\dots,n$.

Case $k = 0$: It is clear that

$$\|T_0^{(n)}\|_p \le \kappa_{p,t}G_{t-s}(x-y), \tag{2.58}$$

where $\kappa_{p,t}$ is the constant defined in (2.51).

Case $k=1$: Applying (2.26), Minkowski's inequality and (2.20), we can write

$$\begin{aligned}
\|T_1^{(n)}\|_p^2 &\le 4p\Bigg\| \int_s^t \int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)G_{t-r_1}(x-z_1')G_{r_1-s}(z_1-y)G_{r_1-s}(z_1'-y) \\
&\qquad \times \|z_1-z_1'\|^{-\beta}\Sigma_{r_1,z_1}^{(n)}\Sigma_{r_1',z_1'}^{(n)}\sigma^2(u_{n-1}(s,y))dz_1dz_1'dr_1\Bigg\|_{p/2} \\
&\le 4pL^2\kappa_{p,t}^2\int_s^t \big\|G_{t-r_1}(x-\bullet)G_{r_1-s}(y-\bullet)\big\|_{\mathfrak{H}_0}^2 dr_1 \quad \text{with } \mathfrak{H}_0 \text{ introduced in (2.19)} \\
&\le 4pL^2\kappa_{p,t}^2C_\beta\int_s^t\left(\int_{\mathbb{R}^2}G_{t-r_1}^{2q}(x-z_1)G_{r_1-s}^{2q}(z_1-y)dz_1\right)^{1/q}dr_1,
\end{aligned}$$

with $q=2/(4-\beta)$. Then, we can deduce immediately from Lemma 2.3.3 (with $\delta=1/q$) that

$$\|T_1^{(n)}\|_p^2 \le pL^2\kappa_{p,t}^2C_\beta^* t^{\frac{1}{q}-1}G_{t-s}^{2-\frac{1}{q}}(x-y), \tag{2.59}$$

for some generic constant C_β^*, which only depends on β. Taking (2.56) into account, we obtain

$$\|T_1^{(n)}\|_p \le \sqrt{p}L\kappa_{p,t}C_\beta^* t^{\frac{1}{q}-\frac{1}{2}}G_{t-s}(x-y). \tag{2.60}$$

Case $k=2$: We can write

$$T_2^{(n)} = \int_s^t\int_{\mathbb{R}^2}W(dr_1,dz_1)G_{t-r_1}(x-z_1)\Sigma_{r_1,z_1}^{(n)}N_{r_1,z_1}$$

with N_{r_1,z_1} defined to be

$$N_{r_1,z_1} = \int_s^{r_1}\int_{\mathbb{R}^2}G_{r_2-s}(z_2-y)\sigma\big(u_{n-2}(r_2,z_2)\big)G_{r_1-r_2}(z_1-z_2)\Sigma_{r_2,z_2}^{(n-1)}W(dr_2,dz_2),$$

which is clearly $\mathscr{F}_{r_1}$-measurable. Applying again (2.26), Minkowski's inequality and (2.20), we can bound $\|T_2^{(n)}\|_p^2$ by

$$
\begin{aligned}
&4p\left\| \int_s^t \int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)G_{t-r_1}(x-z_1')\|z_1-z_1'\|^{-\beta}\Sigma^{(n)}_{r_1,z_1}\Sigma^{(n)}_{r_1',z_1'}N_{r_1,z_1}N_{r_1,z_1'}dz_1dz_1'dr_1\right\|_{p/2} \\
&\leq 4pL^2\int_s^t\int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)G_{t-r_1}(x-z_1')G_{r_1-s}(z_1-y)G_{r_1-s}(z_1'-y) \\
&\qquad \times \|N_{r_1,z_1}\|_p\|N_{r_1,z_1'}\|_p\|z_1-z_1'\|^{-\beta}dz_1dz_1'dr_1 \\
&\leq 4pL^2C_\beta\int_s^t\left(\int_{\mathbb{R}^2} G^{2q}_{t-r_1}(x-z_1)\|N_{r_1,z_1}\|_p^{2q}dz_1\right)^{1/q}dr_1. \qquad (2.61)
\end{aligned}
$$

The same arguments used to obtain the bound (2.60) for $\|T_1^{(n)}\|_p$ yield

$$
\|N_{r_1,z_1}\|_p \leq \sqrt{p}L\kappa_{p,t}C^*_\beta t^{\frac{1}{q}-\frac{1}{2}}G_{r_1-s}(z_1-y). \qquad (2.62)
$$

Substituting (2.62) into (2.61) and applying Lemma 2.3.3 with $\delta = 1/q$, we obtain

$$
\begin{aligned}
\|T_2^{(n)}\|_p^2 &\leq 4pL^2C_\beta(\sqrt{p}L\kappa_{p,t}C^*_\beta t^{\frac{1}{q}-\frac{1}{2}})^2\int_s^t\left(\int_{\mathbb{R}^2} G^{2q}_{t-r_1}(x-z_1)G^{2q}_{r_1-s}(z_1-y)dz_1\right)^{1/q}dr_1 \\
&\leq p^2L^4\kappa^2_{p,t}C^*_\beta t^{\frac{3}{q}-2}G^{2-\frac{1}{q}}_{t-s}(x-y),
\end{aligned}
$$

which implies

$$
\|T_2^{(n)}\|_p \leq pL^2\kappa_{p,t}C^*_\beta t^{\frac{3}{2q}-1}G^{1-\frac{1}{2q}}_{t-s}(x-y). \qquad (2.63)
$$

In view of (2.56), we obtain

$$
\|T_2^{(n)}\|_p \leq pL^2\kappa_{p,t}C^*_\beta t^{\frac{2}{q}-1}G_{t-s}(x-y). \qquad (2.64)
$$

Case $3 \leq k \leq n$: The strategy to handle these cases will be slightly different. We need to get rid of the power $\frac{1}{q}$ in order to iterate the integrals in the time variables and obtain a summable series. We can write

$$
T_k^{(n)} = \int_s^t\int_{\mathbb{R}^2} W(dr_1,dz_1)G_{t-r_1}(x-z_1)\Sigma^{(n)}_{r_1,z_1}\widehat{N}_{r_1,z_1}
$$

with $\widehat{N}_{r_1,z_1}$ defined to be

$$\widehat{N}_{r_1,z_1} = \int_{s<r_k<\cdots<r_2<r_1} \int_{\mathbb{R}^{2k-2}} G_{r_k-s}(z_k-y)\sigma\big(u_{n-k}(s,y)\big) \times \prod_{j=2}^{k} G_{r_{j-1}-r_j}(z_{j-1}-z_j)\Sigma_{r_j,z_j}^{(n+1-j)} W(dr_j,dz_j),$$

which is $\mathscr{F}_{r_1}$-measurable. Then, we deduce from (2.26) and (2.48) that

$$\begin{aligned}\|T_k^{(n)}\|_p^2 &\leq 4p \bigg\| \int_s^t dr_1 \int_{\mathbb{R}^4} G_{t-r_1}(x-z_1)\Sigma_{r_1,z_1}^{(n)} \widehat{N}_{r_1,z_1} G_{t-r_1}(x-z_1')\Sigma_{r_1,z_1'}^{(n)} \widehat{N}_{r_1,z_1'} \\ &\qquad \times \|z_1'-z_1\|^{-\beta} dz_1 dz_1' \bigg\|_{p/2} \\ &\leq 4pK_\beta L^2 t^{\frac{(2-2q)^2}{2q}} \int_s^t dr_1 \int_{\mathbb{R}^2} dz_1 G_{t-r_1}^{2q}(x-z_1)\|\widehat{N}_{r_1,z_1}\|_p^2.\end{aligned}$$

Now we can iterate the above process to obtain

$$\begin{aligned}\|T_k^{(n)}\|_p^2 \leq \Big(4pL^2K_\beta t^{\frac{(2-2q)^2}{2q}}\Big)^{k-1} \int_s^t dr_1 \int_s^{r_1} dr_2 \cdots \int_s^{r_{k-2}} dr_{k-1} \int_{\mathbb{R}^{2k-2}} dz_1 \cdots dz_{k-1} \\ \times G_{t-r_1}^{2q}(x-z_1) G_{r_1-r_2}^{2q}(z_1-z_2)\cdots G_{r_{k-2}-r_{k-1}}^{2q}(z_{k-2}-z_{k-1}) \big\|\widetilde{N}_{r_{k-1},z_{k-1}}\big\|_p^2, \end{aligned} \tag{2.65}$$

where $\widetilde{N}_{r_{k-1},z_{k-1}}$ is defined to be

$$\int_s^{r_{k-1}} \int_{\mathbb{R}^2} W(dr_k,dz_k)\sigma\big(u_{n-k}(s,y)\big) G_{r_{k-1}-r_k}(z_{k-1}-z_k)\Sigma_{r_k,z_k}^{(n+1-k)} G_{r_k-s}(z_k-y).$$

Therefore, the same arguments for estimating $\|T_1^{(n)}\|_p^2$ (see (2.59)), lead to

$$\big\|\widetilde{N}_{r_{k-1},z_{k-1}}\big\|_p^2 \leq p\kappa_{p,t}^2 L^2 C_\beta^* t^{\frac{1}{q}-1} G_{r_{k-1}-s}^{2-\frac{1}{q}}\big(z_{k-1}-y\big), \tag{2.66}$$

with C_β^* being a generic constant that only depends on β. On the other hand, applying Lemma

2.3.3 with $\delta = 1$, we can write

$$\int_{r_{k-1}}^{r_{k-3}} dr_{k-2} \int_{\mathbb{R}^2} dz_{k-2} G^{2q}_{r_{k-3}-r_{k-2}}(z_{k-3}-z_{k-2}) G^{2q}_{r_{k-2}-r_{k-1}}(z_{k-2}-z_{k-1})$$
$$\lesssim t^{2-2q} G^{2q-1}_{r_{k-3}-r_{k-1}}(z_{k-3}-z_{k-1}), \tag{2.67}$$

with the convention $z_0 = x$ and $r_0 = t$. Plugging the estimates (2.66) and (2.67) into (2.65), yields

$$\begin{aligned}\|T_k^{(n)}\|_p^2 \leq \kappa_{p,t}^2 &\left(pL^2 C_\beta^* t^{\frac{2(1-q)^2}{q}}\right)^k t^{5-4q-\frac{1}{q}} \\ &\times \int_s^t dr_1 \int_s^{r_1} dr_2 \cdots \int_s^{r_{k-3}} dr_{k-1} \int_{\mathbb{R}^{2k-4}} dz_1 \cdots dz_{k-3} dz_{k-1} \\ &\times G^{2q}_{t-r_1}(x-z_1) \cdots G^{2q}_{r_{k-4}-r_{k-3}}(z_{k-4}-z_{k-3}) \\ &\times G^{2q-1}_{r_{k-3}-r_{k-1}}(z_{k-3}-z_{k-1}) G^{2-\frac{1}{q}}_{r_{k-1}-s}(z_{k-1}-y)\end{aligned}$$

By Cauchy-Schwartz inequality and (2.23),

$$\begin{aligned}&\int_{\mathbb{R}^2} G^{2q-1}_{r_{k-3}-r_{k-1}}(z_{k-3}-z_{k-1}) G^{2-\frac{1}{q}}_{r_{k-1}-s}(z_{k-1}-y) dz_{k-1} \\ &\quad \leq \left[\int_{\mathbb{R}^2} G^{4q-2}_{r_{k-3}-r_{k-1}}(z) dz \int_{\mathbb{R}^2} G^{4-\frac{2}{q}}_{r_{k-1}-s}(z) dz\right]^{1/2} \leq C_\beta^* t^{1-2q+\frac{1}{q}}.\end{aligned}$$

In this way, we obtain

$$\begin{aligned}\|T_k^{(n)}\|_p^2 \leq \kappa_{p,t}^2 &(pL^2 C_\beta^* t^{\frac{2(1-q)^2}{q}})^k t^{6(1-q)} \mathbf{1}_{\{\|x-y\|<t-s\}} \\ &\times \int_s^t dr_1 \int_s^{r_1} dr_2 \cdots \int_s^{r_{k-3}} dr_{k-1} \int_{\mathbb{R}^{2k-6}} dz_1 \cdots dz_{k-3} \\ &\times G^{2q}_{t-r_1}(x-z_1) \cdots G^{2q}_{r_{k-4}-r_{k-3}}(z_{k-4}-z_{k-3})\end{aligned} \tag{2.68}$$

The indicator function $\mathbf{1}_{\{\|x-y\|<t-s\}}$ appears in (2.68), because

$$\mathbf{1}_{\{\|z_{k-1}-y\|<r_{k-1}-s, \|z_{k-3}-z_{k-1}\|<r_{k-3}-r_{k-1}, \ldots, \|x-z_1\|<t-r_1\}} \leq \mathbf{1}_{\{\|x-y\|<t-s\}}.$$

Now, we can perform the integration with respect to $dz_{k-3},\dots,dz_1$ one by one to get

$$\int_{\mathbb{R}^{2k-6}} dz_1\cdots dz_{k-3} G^{2q}_{t-r_1}(x-z_1)G^{2q}_{r_1-r_2}(z_1-z_2)\cdots G^{2q}_{r_{k-4}-r_{k-3}}(z_{k-4}-z_{k-3}) \\ = \left(\frac{(2\pi)^{1-2q}}{2-2q}\right)^{k-3} \times \prod_{j=1}^{k-3}\left(r_{j-1}-r_j\right)^{2-2q} \le \left(\frac{(2\pi)^{1-2q}}{2-2q}t^{2-2q}\right)^{k-3},$$

in view of the equality (2.23). Together with the integration on the simplex $\{t>r_1>\cdots>r_{k-3}>r_{k-1}>s\}$, we get

$$\left\|T_k^{(n)}\right\|_p^2 \le \frac{(pC^*_\beta L^2)^k}{(k-2)!}\kappa^2_{p,t}t^{k(\frac{2}{q}-1)-2}\mathbf{1}_{\{\|x-y\|<t-s\}}.$$

Thus, taking into account that

$$\mathbf{1}_{\{\|x-y\|<t-s\}} \le \left[2\pi(t-s)\right]^2 G^2_{t-s}(x-y),$$

we obtain for $k\in\{3,\dots,n\}$,

$$\left\|T_k^{(n)}\right\|_p \le \kappa_{p,t}\frac{(pC^*_\beta L^2)^{k/2}}{\sqrt{(k-2)!}}t^{\,k(\frac{1}{q}-\frac{1}{2})}G_{t-s}(x-y). \tag{2.69}$$

Hence, we deduce from (2.58), (2.60) and (2.69) that for any $n\ge 3$,

$$\left\|D_{s,y}u_{n+1}(t,x)\right\|_p \le \sum_{k=0}^{n}\left\|T_k^{(n)}\right\|_p \le C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y),$$

where the constant $C_{\beta,p,t,L}$ is defined in (2.53). This proves Proposition 2.3.2. □

Step 2. We are going to show that $D_{s,y}u(t,x)$ is a real-valued random variable. As a consequence of (2.20), (2.52) and (2.23), we have for any $p \geq 2$ and with $q = 2/(4-\beta)$

$$\begin{aligned}
&\mathbb{E}\Big[\|Du_{n+1}(t,x)\|_{\mathfrak{H}}^p\Big]^{2/p} = \Big\|\int_{\mathbb{R}_+} ds \big\|D_{s,\bullet}u_{n+1}(t,x)\big\|_{\mathfrak{H}_0}^2\Big\|_{p/2} \\
&\lesssim \Big\|\int_{\mathbb{R}_+} ds\left(\int_{\mathbb{R}^2}|D_{s,y}u_{n+1}(t,x)|^{2q}dy\right)^{1/q}\Big\|_{p/2} \\
&\lesssim \int_{\mathbb{R}_+} ds\left(\int_{\mathbb{R}^2}\big\|D_{s,y}u_{n+1}(t,x)\big\|_p^{2q}dy\right)^{1/q} \text{by applying Minkowski twice} \\
&\lesssim \int_{\mathbb{R}_+} ds\left(\int_{\mathbb{R}^2} G_{t-s}^{2q}(x-y)dy\right)^{1/q} \lesssim \int_0^t (t-s)^{\frac{2-2q}{q}}ds \lesssim 1.
\end{aligned}$$

One can first read from the above estimates that $\{Du_{n+1}(t,x), n \geq 1\}$ is uniformly bounded in $L^p\big(\Omega;\mathfrak{H}\big)$, which together with the L^p-convergence of $u_n(t,x)$ to $u(t,x)$ implies the convergence of $Du_{n+1}(t,x)$ to $Du(t,x)$ in the weak topology on $L^p\big(\Omega;\mathfrak{H}\big)$ up to a subsequence; this fact is well-known in the literature, see for instance [24]. One can deduce from the same arguments that $\{Du_{n+1}(t,x), n\geq 1\}$ is uniformly bounded in $L^p\big(\Omega; L^{2q}(\mathbb{R}_+\times\mathbb{R}^2)\big)$:

$$\begin{aligned}
&\big\|Du_{n+1}(t,x)\big\|_{L^p(\Omega;L^{2q}(\mathbb{R}_+\times\mathbb{R}^2))}^p = \Big\|\int_{\mathbb{R}_+\times\mathbb{R}^2}|D_{s,y}u_{n+1}(t,x)|^{2q}dyds\Big\|_{\frac{p}{2q}}^{\frac{p}{2q}} \\
&\quad\leq \left(\int_{\mathbb{R}_+\times\mathbb{R}^2}\big\|D_{s,y}u_{n+1}(t,x)\big\|_p^{2q}dyds\right)^{\frac{p}{2q}} \lesssim \left(\int_{\mathbb{R}_+\times\mathbb{R}^2} G_{t-s}^{2q}(x-y)dyds\right)^{\frac{p}{2q}} \lesssim 1.
\end{aligned}$$

So up to a subsequence, $Du_n(t,x)$ also converges to $Du(t,x)$ in the weak topology on $L^p\big(\Omega;L^{2q}(\mathbb{R}_+\times\mathbb{R}^2)\big)$. In particular, we have $(2q < 2 \leq p < \infty)$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^2}\Big\|\int_{\mathbb{R}_+\times\mathbb{R}^2}|D_{s,y}u(t,x)|^{2q}dyds\Big\|_{\frac{p}{2q}} < +\infty \tag{2.70}$$

and $D_{s,y}u(t,x)$ is a real function in (s,y).

Step 3. Let us prove the lower bound. By Lemma 2.1.5, we can write

$$u(t,x) - 1 = \int_0^t\int_{\mathbb{R}^2}\mathbb{E}\big[D_{s,y}u(t,x)|\mathscr{F}_s\big]W(ds,dy),$$

so that a comparison with (2.4) yields $\mathbb{E}\big[D_{s,y}u(t,x)|\mathscr{F}_s\big] = G_{t-s}(x-y)\sigma(u(s,y))$ almost everywhere in $\Omega \times \mathbb{R}_+ \times \mathbb{R}^2$. It follows that

$$\big\|\mathbb{E}[D_{s,y}u(t,x)|\mathscr{F}_s]\big\|_p = G_{t-s}(x-y)\big\|\sigma(u_{s,y})\big\|_p,$$

thus by conditional Jensen, we have

$$\big\|D_{s,y}u(t,x)\big\|_p \geq G_{t-s}(x-y)\big\|\sigma(u_{s,y})\big\|_p,$$

which is exactly the lower bound in (2.11).

Step 4. We are finally in a position to prove the upper bound in (2.11). Put $p^\star = p/(p-1)$, which is the conjugate exponent for p. Let us pick a nonnegative function $M \in C_c(\mathbb{R}_+ \times \mathbb{R}^2)$ and random variable $\mathscr{Z} \in L^{p^\star}(\Omega)$ with $\|\mathscr{Z}\|_{p^\star} \leq 1$. Since $Du_n(t,x)$ converges to $Du(t,x)$ in the weak topology on $L^p\big(\Omega; L^{2q}(\mathbb{R}_+ \times \mathbb{R}^2)\big)$ along some subsequence (say $Du_{n_k}(t,x)$), we have, in view of (2.52)

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}^2} M(s,y)\mathbb{E}\big[ZD_{s,y}u(t,x)\big]dsdy &= \lim_{k\to\infty}\int_{\mathbb{R}_+\times\mathbb{R}^2} M(s,y)\mathbb{E}\big[ZD_{s,y}u_{n_k}(t,x)\big]dsdy \\ &\leq C_{\beta,p,t,L}\kappa_{p,t}\int_{\mathbb{R}_+\times\mathbb{R}^2} M(s,y)G_{t-s}(x-y)dsdy.\end{aligned}$$

This implies that for almost all $(s,y) \in [0,t \times \mathbb{R}^2$,

$$\mathbb{E}\big[ZD_{s,y}u(t,x)\big] \leq C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y)$$

Taking the supremum over $\{\mathscr{Z} : \|\mathscr{Z}\|_{p^\star} \leq 1\}$ yields

$$\|D_{s,y}u(t,x)\|_p \leq C_{\beta,p,t,L}\kappa_{p,t}G_{t-s}(x-y),$$

which finishes the proof.

2.3.3 Proof of technical lemmas

For convenience, let us recall Lemma 2.0.11 below.

Lemma 2.0.11. For $t > s$, with $\|z\| = \mathbf{w} > 0$ and $q \in (1/2, 1)$

$$G_t^{2q} * G_s^{2q}(z) \lesssim \mathbf{1}_{\{\mathbf{w}<s\}} \left[t^2 - (s-\mathbf{w})^2\right]^{1-2q} + \left[t^2 - (s+\mathbf{w})^2\right]^{1-2q} \mathbf{1}_{\{t>s+\mathbf{w}\}} \\ + \mathbf{1}_{\{|s-\mathbf{w}|<t<s+\mathbf{w}\}} \left[(\mathbf{w}+s)^2 - t^2\right]^{-q+\frac{1}{2}} \left[t^2 - (s-\mathbf{w})^2\right]^{-q+\frac{1}{2}}, \tag{2.22}$$

where the implicit constant depends only on q.

Proof of Lemma 2.0.11. We are interested in estimating

$$\mathbf{I} = \int_{\mathbb{R}^2} \left(t^2 - \|x\|^2\right)_+^{-q} \left(s^2 - \|x-z\|^2\right)_+^{-q} dx,$$

where $(v)_+^{-q} = v^{-q}$ for $v > 0$ and $(v)_+^{-q} = 0$ for $v \le 0$. Because the convolution of two radial functions is radial, the quantity $\mathbf{I}$ depends only on s, t and $\|z\|$. Hence, we can assume additionally that $z = (\mathbf{w}, 0)$, where $\mathbf{w} > 0$. Note that the integral $\mathbf{I}$ vanishes if $t + s < \mathbf{w}$ and we can write, putting $x = (\xi, \eta)$,

$$\mathbf{I} = \int_{\mathbb{R}^2} \left(t^2 - \xi^2 - \eta^2\right)_+^{-q} \left(s^2 - (\xi - \mathbf{w})^2 - \eta^2\right)_+^{-q} d\xi d\eta.$$

Making the change of variables $(x, y) = \left(\xi^2 + \eta^2, (\mathbf{w} - \xi)^2 + \eta^2\right)$ yields

$$\mathbf{I} = \frac{1}{2} \int_D (t^2 - x)^{-q} (s^2 - y)^{-q} \left[(\sqrt{x} + \mathbf{w})^2 - y\right]^{-1/2} \left[y - (\sqrt{x} - \mathbf{w})^2\right]^{-1/2} dx dy, \tag{2.71}$$

where

$$D = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < t^2, 0 < y < s^2, \left(\sqrt{x} - \mathbf{w}\right)^2 < y < \left(\sqrt{x} + \mathbf{w}\right)^2\right\}.$$

To derive the expression (2.71) for $\mathbf{I}$, we have used the fact that the Jacobian of the change of variables is

$$\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = 4\mathbf{w}|\eta| = 2\big[(\sqrt{x}+\mathbf{w})^2 - y\big]^{1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{1/2}.$$

Then, integrating first in the variable y yields

$$\begin{aligned}\mathbf{I} &= \frac{1}{2}\int_0^{t^2} dx(t^2-x)^{-q}\int_{D(x)} dy\ (s^2-y)^{-q}\big[(\sqrt{x}+\mathbf{w})^2 - y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2} \\ &=: \frac{1}{2}\int_0^{t^2}(t^2-x)^{-q}\mathscr{S}_q(x)dx,\end{aligned}$$

where

$$D(x) = \big\{y\in\mathbb{R} : (x,y)\in D\big\} = \Big\{y\in\mathbb{R} : y<s^2, \big(\sqrt{x}-\mathbf{w}\big)^2 < y < \big(\sqrt{x}+\mathbf{w}\big)^2\Big\}$$

and

$$\mathscr{S}_q(x) = \int_{D(x)} dy\ (s^2-y)^{-q}\big[(\sqrt{x}+\mathbf{w})^2 - y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2}. \tag{2.72}$$

Let us first deal with $\mathscr{S}_q(x)$ for every $x\in(0,t^2)$. There are two possible cases, depending on the value of x:

(A) When $(\sqrt{x}-\mathbf{w})^2 < s^2 < (\sqrt{x}+\mathbf{w})^2$,

$$\begin{aligned}\mathscr{S}_q(x) &= \int_{(\sqrt{x}-\mathbf{w})^2}^{s^2}(s^2-y)^{-q}\big[(\sqrt{x}+\mathbf{w})^2-y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2}dy \\ &\le \mathrm{Beta}(1/2,1-q)\big[(\sqrt{x}+\mathbf{w})^2-s^2)\big]^{-1/2}\big[s^2-(\sqrt{x}-\mathbf{w})^2\big]^{-q+\frac{1}{2}} \\ &\lesssim \big[(\sqrt{x}+\mathbf{w})^2-s^2)\big]^{-1/2}\big[s^2-(\sqrt{x}-\mathbf{w})^2\big]^{-q+\frac{1}{2}}.\end{aligned} \tag{2.73}$$

Throughout this section, $\mathrm{Beta}(a,b)$ denotes the usual beta function:

$$\mathrm{Beta}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,\ a,b\in(0,\infty).$$

(B) When $(\sqrt{x}-\mathbf{w})^2 < (\sqrt{x}+\mathbf{w})^2 < s^2$,

$$\begin{aligned}\mathscr{S}_q(x) &= \int_{(\sqrt{x}-\mathbf{w})^2}^{(\sqrt{x}+\mathbf{w})^2} (s^2-y)^{-q}\big[(\sqrt{x}+\mathbf{w})^2-y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2}dy \\ &\le (s^2-(\sqrt{x}+\mathbf{w})^2)^{-q}\int_{(\sqrt{x}-\mathbf{w})^2}^{(\sqrt{x}+\mathbf{w})^2} \big[(\sqrt{x}+\mathbf{w})^2-y\big]^{-1/2}\big[y-(\sqrt{x}-\mathbf{w})^2\big]^{-1/2}dy \\ &= \mathrm{Beta}(1/2,1/2)\big[s^2-(\sqrt{x}+\mathbf{w})^2\big]^{-q} \lesssim \big[s^2-(\sqrt{x}+\mathbf{w})^2\big]^{-q}.\end{aligned}$$

Note that three positive numbers a,b,c can form sides of a triangle if and only if *the sum of any two of them is strictly bigger than the third one*, which is equivalent to saying that $|a-b|<c<a+b$. It follows that

$$\begin{aligned}(\sqrt{x}-\mathbf{w})^2 < s^2 < (\sqrt{x}+\mathbf{w})^2 &\Leftrightarrow \sqrt{x},\mathbf{w},s \text{ can be the sides of a triangle} \\ &\Leftrightarrow (s-\mathbf{w})^2 < x < (s+\mathbf{w})^2.\end{aligned}$$

Furthermore, it is trivial that $(\sqrt{x}-\mathbf{w})^2 < (\sqrt{x}+\mathbf{w})^2 < s^2 \Leftrightarrow x < (s-\mathbf{w})^2$ and $s > \mathbf{w}$.

Now we decompose the integral $2\mathbf{I} = \int_0^{t^2}(t^2-x)^{-q}\mathscr{S}_q(x)dx$ into two parts corresponding to the cases **(A)** and **(B)**:

$$2\mathbf{I} = \mathbf{I_A} + \mathbf{I_B},$$

where

$$\mathbf{I_A} = \int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\mathscr{S}_q(x)dx \quad \text{and} \quad \mathbf{I_B} = \int_0^{(s-\mathbf{w})^2\wedge t^2}(t^2-x)^{-q}\mathscr{S}_q(x)dx.$$

Estimation of $\mathbf{I_A}$. We first write, using (2.73),

$$\begin{aligned}\mathbf{I_A} &\lesssim \int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\big[(\sqrt{x}+\mathbf{w})^2-s^2)\big]^{-1/2}\big[s^2-(\sqrt{x}-\mathbf{w})^2\big]^{-q+\frac{1}{2}}dx \\ &= \int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\big[(\mathbf{w}+s)^2-x\big]^{-q+\frac{1}{2}}\big[x-(\mathbf{w}-s)^2\big]^{-q+\frac{1}{2}}\big[(\sqrt{x}+\mathbf{w})^2-s^2\big]^{q-1}dx.\end{aligned}$$

Recall in this case $\sqrt{x}+\mathbf{w}>s$, which implies $(\sqrt{x}+\mathbf{w})^2-s^2>x-(s-\mathbf{w})^2>0$. Therefore,

$$\mathbf{I_A}\lesssim\int_{(s-\mathbf{w})^2}^{t^2\wedge(s+\mathbf{w})^2}(t^2-x)^{-q}\big[(\mathbf{w}+s)^2-x\big]^{-q+\frac{1}{2}}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx.$$

Now we consider the following two sub-cases:

(A1) If $s+\mathbf{w}<t$, then for $(s-\mathbf{w})^2<x<(s+\mathbf{w})^2<t$, we have, with $\gamma=2-q^{-1}$,

$$\begin{aligned}(t^2-x)^{-q}&\le\big[t^2-(s+\mathbf{w})^2\big]^{-q\gamma}\big[(s+\mathbf{w})^2-x\big]^{-q+q\gamma}\\&=\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}\big[(s+\mathbf{w})^2-x\big]^{q-1}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{I_A}&\lesssim\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}\int_{(s-\mathbf{w})^2}^{(s+\mathbf{w})^2}\big[(\mathbf{w}+s)^2-x\big]^{-1/2}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx\\&=\mathrm{Beta}(1/2,1/2)\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}.\end{aligned}$$

(A2) If $(s-\mathbf{w})^2<t^2<(s+\mathbf{w})^2$ (*i.e.* $s,\mathbf{w},t$ form triangle sides), then

$$\begin{aligned}\mathbf{I_A}&\lesssim\int_{(s-\mathbf{w})^2}^{t^2}(t^2-x)^{-q}\big[(\mathbf{w}+s)^2-x\big]^{-q+\frac{1}{2}}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx\\&\le\big[(\mathbf{w}+s)^2-t^2\big]^{-q+\frac{1}{2}}\int_{(s-\mathbf{w})^2}^{t^2}\big(t^2-x\big)^{-q}\big[x-(\mathbf{w}-s)^2\big]^{-1/2}dx\\&\lesssim\big[(\mathbf{w}+s)^2-t^2\big]^{-q+\frac{1}{2}}\big[t^2-(s-\mathbf{w})^2\big]^{-q+\frac{1}{2}}\end{aligned}$$

because $\int_a^b(b-x)^{-q}(x-a)^{-1/2}dx=\mathrm{Beta}(1/2,1-q)(b-a)^{-q+\frac{1}{2}}$ for any $0\le a<b<\infty$ and for any $q<1$.

Combining **(A1)** and **(A2)**, we have obtained

$$\mathbf{I_A}\lesssim\big[t^2-(s+\mathbf{w})^2\big]^{1-2q}\mathbf{1}_{\{t>s+\mathbf{w}\}}+\mathbf{1}_{\{|s-\mathbf{w}|<t<s+\mathbf{w}\}}\big[(\mathbf{w}+s)^2-t^2\big]^{\frac{1-2q}{2}}\big[t^2-(s-\mathbf{w})^2\big]^{\frac{1-2q}{2}}.\quad(2.74)$$

Estimation of $\mathbf{I_B}$. In this case, $\sqrt{x} < s - \mathbf{w}$ and $\mathbf{w} < s$, then

$$s^2 - (\sqrt{x} + \mathbf{w})^2 > (s - \mathbf{w})^2 - x > 0.$$

Therefore, $\mathscr{S}_q(x) \lesssim \left[(s-\mathbf{w})^2 - x\right]^{-q}$ and the quantity $\mathbf{I_B}$ can be bounded as follows

$$\begin{aligned}\mathbf{I_B} &= \int_0^{(s-\mathbf{w})^2} (t^2 - x)^{-q} \mathscr{S}_q(x) dx \lesssim \int_0^{(s-\mathbf{w})^2} (t^2 - x)^{-q} \left[(s-\mathbf{w})^2 - x\right]^{-q} dx \\ &\lesssim \left[t^2 - (s-\mathbf{w})^2\right]^{1-2q},\end{aligned} \tag{2.75}$$

because for any $0 < a < b < \infty$ and any $p, q \in (1/2, 1)$

$$\begin{aligned}\int_0^a (b-x)^{-p}(a-x)^{-q} dx &= \int_0^a (b-a+y)^{-p} y^{-q} dy = (b-a)^{1-p-q} \int_0^{\frac{a}{b-a}} y^{-q}(1+y)^{-p} dy \\ &\le (b-a)^{1-p-q} \int_0^{\infty} y^{-q}(1+y)^{-p} dy \lesssim (b-a)^{1-p-q}.\end{aligned}$$

Our proof is done by combining the estimates (2.74) and (2.75) to get (2.22). □

Now let us apply Lemma 2.0.11 to prove Lemma 2.3.3.

Proof of Lemma 2.3.3. Put $\mu = (t-r) \wedge (r-s)$ and $\nu = (t-r) \vee (r-s)$ and assume $\mu \neq \nu$. We apply Lemma 2.0.11 to write

$$\begin{aligned}\left(G_{t-r}^{2q} * G_{r-s}^{2q}(z)\right)^{\delta} &\lesssim \Big(\mathbf{1}_{\{\mathbf{w}<\mu\}} \left[\nu^2 - (\mu - \mathbf{w})^2\right]^{1-2q} + \left[\nu^2 - (\mu + \mathbf{w})^2\right]^{1-2q} \mathbf{1}_{\{\nu > \mu + \mathbf{w}\}} \\ &\quad + \mathbf{1}_{\{|\mu - \mathbf{w}| < \nu < \mu + \mathbf{w}\}} \left[(\mathbf{w} + \mu)^2 - \nu^2\right]^{-q+\frac{1}{2}} \left[\nu^2 - (\mu - \mathbf{w})^2\right]^{-q+\frac{1}{2}} \Big)^{\delta} \\ &\lesssim \mathbf{1}_{\{\mathbf{w}<\mu\}} \left[\nu^2 - (\mu - \mathbf{w})^2\right]^{\delta(1-2q)} + \left[\nu^2 - (\mu + \mathbf{w})^2\right]^{\delta(1-2q)} \mathbf{1}_{\{\nu > \mu + \mathbf{w}\}} \\ &\quad + \mathbf{1}_{\{|\mu - \mathbf{w}| < \nu < \mu + \mathbf{w}\}} \left[(\mathbf{w} + \mu)^2 - \nu^2\right]^{\delta(\frac{1}{2}-q)} \left[\nu^2 - (\mu - \mathbf{w})^2\right]^{\delta(\frac{1}{2}-q)},\end{aligned}$$

where $\mathbf{w} = \|z\| > 0$ and $0 > \delta(1-2q) \geq \frac{1}{q} - 2 > -1$. Define

$$\begin{aligned} K_{s,t}^{(1)}(z) :&= \int_s^t dr \mathbf{1}_{\{\mathbf{w}<\mu\}} \left[\nu^2 - (\mu - \mathbf{w})^2\right]^{\delta(1-2q)} \\ &= \int_s^t dr \mathbf{1}_{\{\mathbf{w}<\mu\}} \left[(\nu + \mu - \mathbf{w})(\nu - \mu + \mathbf{w})\right]^{\delta(1-2q)} \end{aligned}$$

and note that $t - r > r - s$ if and only if $r < \frac{t+s}{2}$. Then, by exact computations and decomposing the integral in the intervals $[s, (t+s)/2]$ and $[(t+s)/2, t]$, yields

$$\begin{aligned} K_{s,t}^{(1)}(z) &= \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}} \int_{s+\mathbf{w}}^{(t+s)/2} (t-s-\mathbf{w})^{\delta(1-2q)} (t+s+\mathbf{w}-2r)^{\delta(1-2q)} dr \\ &\quad + \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}} \int_{(t+s)/2}^{t-\mathbf{w}} (t-s-\mathbf{w})^{\delta(1-2q)} (2r+\mathbf{w}-t-s)^{\delta(1-2q)} dr \\ &= 2 \times \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}} (t-s-\mathbf{w})^{\delta(1-2q)} \frac{1}{2(\delta(1-2q)+1)} \\ &\quad \times \left[(t-s-\mathbf{w})^{\delta(1-2q)+1} - \mathbf{w}^{\delta(1-2q)+1}\right] \\ &\leq \frac{(t-s)^{\delta(1-2q)+1}}{\delta(1-2q)+1} \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}} (t-s-\mathbf{w})^{\delta(1-2q)} \\ &\lesssim (t-s)^{\delta(1-2q)+1} (t-s)^{\delta(1-2q)} \mathbf{1}_{\{\mathbf{w}<\frac{t-s}{2}\}} \\ &\lesssim (t-s)^{\delta(1-2q)+1} \left[(t-s)^2 - \|z\|^2\right]^{\delta(\frac{1}{2}-q)} \mathbf{1}_{\{\|z\|<t-s\}}. \end{aligned} \tag{2.76}$$

By the same arguments, we can get

$$\begin{aligned} K_{s,t}^{(2)}(z) :&= \int_s^t dr \left[\nu^2 - (\mu + \mathbf{w})^2\right]^{\delta(1-2q)} \mathbf{1}_{\{\nu>\mu+\mathbf{w}\}} \\ &= \int_s^t dr \left[(\nu + \mu + \mathbf{w})(\nu - \mu - \mathbf{w})\right]^{\delta(1-2q)} \mathbf{1}_{\{\nu>\mu+\mathbf{w}\}} \\ &= \mathbf{1}_{\{t-s>\mathbf{w}\}} (t-s+\mathbf{w})^{\delta(1-2q)} \int_s^{(t+s-\mathbf{w})/2} \left(t+s-2r-\mathbf{w}\right)^{\delta(1-2q)} dr \\ &\quad + \mathbf{1}_{\{t-s>\mathbf{w}\}} (t-s+\mathbf{w})^{\delta(1-2q)} \int_{(t+s+\mathbf{w})/2}^t \left(2r-s-t-\mathbf{w}\right)^{\delta(1-2q)} dr \\ &= \mathbf{1}_{\{t-s>\mathbf{w}\}} (t-s+\mathbf{w})^{\delta(1-2q)} \frac{1}{2(\delta(1-2q)+1)} (t-s-\mathbf{w})^{\delta(1-2q)+1} \times 2 \\ &\lesssim (t-s)^{\delta(1-2q)+1} \left[(t-s)^2 - \|z\|^2\right]^{\delta(\frac{1}{2}-q)} \mathbf{1}_{\{\|z\|<t-s\}}. \end{aligned} \tag{2.77}$$

Similarly, we first write

$$\begin{aligned}K_{s,t}^{(3)}(z) :&= \int_s^t dr \mathbf{1}_{\{|\mu-\mathbf{w}|<\nu<\mu+\mathbf{w}\}} \left[(\mathbf{w}+\mu)^2-\nu^2\right]^{\delta(\frac{1}{2}-q)} \left[\nu^2-(\mu-\mathbf{w})^2\right]^{\delta(\frac{1}{2}-q)} \\ &= \int_s^t dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}} \left[(\mu+\nu)^2-\mathbf{w}^2\right]^{\delta(\frac{1}{2}-q)} (\mathbf{w}+\mu-\nu)^{\delta(\frac{1}{2}-q)} (\mathbf{w}+\nu-\mu)^{\delta(\frac{1}{2}-q)} \\ &= \left[(t-s)^2-\mathbf{w}^2\right]^{\delta(\frac{1}{2}-q)} \int_s^t dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}} (\mathbf{w}+\mu-\nu)^{\delta(\frac{1}{2}-q)} (\mathbf{w}+\nu-\mu)^{\delta(\frac{1}{2}-q)}.\end{aligned}$$

Recall $t-r > r-s$ if and only if $r < \frac{t+s}{2}$. Then

$$\begin{aligned}&\int_s^{(t+s)/2} dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}} (\mathbf{w}+\mu-\nu)^{\delta(\frac{1}{2}-q)} (\mathbf{w}+\nu-\mu)^{\delta(\frac{1}{2}-q)} \\ &= \mathbf{1}_{\{\mathbf{w}<t-s\}} \int_{\frac{t+s-\mathbf{w}}{2}}^{\frac{t+s}{2}} dr\, (\mathbf{w}-t-s+2r)^{\delta(\frac{1}{2}-q)} (\mathbf{w}+t+s-2r)^{\delta(\frac{1}{2}-q)} \\ &= \mathbf{1}_{\{\mathbf{w}<t-s\}} 2^{\delta(1-2q)} \int_a^b (r-a)^{-\delta(\frac{1}{2}-q)} (c-r)^{\delta(\frac{1}{2}-q)} dr,\end{aligned}$$

where $a = \dfrac{t+s-\mathbf{w}}{2} < b = \dfrac{t+s}{2} < c = \dfrac{t+s+\mathbf{w}}{2}$. It is easy to show that

$$\begin{aligned}\int_a^b (r-a)^{\delta(\frac{1}{2}-q)} (c-r)^{\delta(\frac{1}{2}-q)} dr &= (c-a)^{\delta(1-2q)+1} \int_0^{\frac{b-a}{c-a}} t^{\delta(\frac{1}{2}-q)} (1-t)^{\delta(\frac{1}{2}-q)} dt \\ &\leq (c-a)^{\delta(1-2q)+1} \int_0^1 t^{\delta(\frac{1}{2}-q)} (1-t)^{\delta(\frac{1}{2}-q)} dt \\ &= \mathrm{Beta}(\delta(\frac{1}{2}-q)+1, \delta(\frac{1}{2}-q)+1)) (c-a)^{\delta(1-2q)+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_s^{(t+s)/2} dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}} (\mathbf{w}+\mu-\nu)^{\delta(\frac{1}{2}-q)} (\mathbf{w}+\nu-\mu)^{\delta(\frac{1}{2}-q)} \\ &\quad \lesssim \mathbf{1}_{\{\mathbf{w}<t-s\}} \mathbf{w}^{\delta(1-2q)+1} \leq (t-s)^{\delta(1-2q)+1} \mathbf{1}_{\{\|z\|<t-s\}}.\end{aligned}$$

In the same manner, we can get

$$
\begin{aligned}
&\int_{(t+s)/2}^{t} dr \mathbf{1}_{\{\nu-\mu<\mathbf{w}<\mu+\nu\}} (\mathbf{w}+\mu-\nu)^{\delta(\frac{1}{2}-q)} (\mathbf{w}+\nu-\mu)^{\delta(\frac{1}{2}-q)} \\
&\quad = \mathbf{1}_{\{\mathbf{w}<t-s\}} \int_{\frac{t+s}{2}}^{\frac{t+s+\mathbf{w}}{2}} dr\, (\mathbf{w}-t-s+2r)^{\delta(\frac{1}{2}-q)} (\mathbf{w}+t+s-2r)^{\delta(\frac{1}{2}-q)} \\
&\quad = \mathbf{1}_{\{\mathbf{w}<t-s\}} 2^{\delta(1-2q)} \int_{b}^{c} (c-r)^{\delta(\frac{1}{2}-q)} (r-a)^{\delta(\frac{1}{2}-q)} dr \qquad (2.78) \\
&\quad \le \mathbf{1}_{\{\mathbf{w}<t-s\}} 2^{\delta(1-2q)} (c-a)^{\delta(1-2q)+1} \mathrm{Beta}(\delta(\frac{1}{2}-q)+1, \delta(\frac{1}{2}-q)+1) \\
&\quad \lesssim \mathbf{1}_{\{\mathbf{w}<t-s\}} \mathbf{w}^{\delta(1-2q)+1} \le (t-s)^{\delta(1-2q)+1} \mathbf{1}_{\{\|z\|<t-s\}},
\end{aligned}
$$

where $a = \dfrac{t+s-\mathbf{w}}{2} < b = \dfrac{t+s}{2} < c = \dfrac{t+s+\mathbf{w}}{2}$. Thus, we obtain

$$
K_{s,t}^{(3)}(z) \lesssim (t-s)^{\delta(1-2q)+1} \left[(t-s)^2 - \|z\|^2\right]_+^{\delta(\frac{1}{2}-q)} \mathbf{1}_{\{\|z\|<t-s\}}, \qquad (2.79)
$$

with $\delta(q-\frac{1}{2}) \le 1-\frac{1}{2q} \in (0,\frac{1}{2})$. Combining the estimates (2.76), (2.77) and (2.79) allows us to finish the proof. □

Chapter 3

A central limit theorem for the stochastic heat equation with random initial condition

In this chapter, we consider the following nonlinear stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \sigma(u)\frac{\partial^2}{\partial t \partial x}\eta \quad \text{for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = \xi(x), \end{cases} \tag{3.1}$$

where σ is a Lipschitz function with constant $L \in (0,\infty)$, η is a space-time Gaussian white noise and ξ is a Gaussian white noise. We consider that both ξ and η are defined on the same probability space $(\Omega, \mathscr{F}, P)$ and are independent of each other.

As in [44], by a mild solution to (3.1) we mean a random field $u = \{u(t,x),\ t > 0,\ x \in \mathbb{R}\}$ satisfying some measurability conditions which will be specified later, and the following stochastic integral equation,

$$u(t,x) = \int_{\mathbb{R}} p_t(x-y)\xi(dy) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))\eta(ds,dy), \ \text{ a.s. for } t > 0, x \in \mathbb{R}, \tag{3.2}$$

where $p_t(x) = (2\pi t)^{-1/2} e^{-x^2/(2t)}$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. As usual in this framework, we set $p_t(\cdot) := 0$ for $t \leq 0$.

Our main purpose in this chapter is to establish a quantitative central limit theorem for the spatial average of the mild solution to (3.1). Our main result, Theorem 3.4.1, basically says that if $u(t,x)$ is the mild solution to 3.1, then, after proper normalization, $\int_{-R}^{R} u(t,x)\,dx$ converges in total variation to a standard normal random variable when $R \to \infty$.

Although the main result in this chapter is Theorem 3.4.1, there are other important contributions throughout chapter. We proceed now to briefly refer to those. First of all, the existence and uniqueness of the mild solution in our setting is not directly covered by the work of Le Chen in [7] or by the work of Dalang in [10]. For this reason, our study of the quantitative central limit theorem, starts by proving existence and uniqueness of the mild solution. Additionally, we prove that the mild solution to (3.1) is stationary. Furthermore, since our problem involves two independent noises, some modifications to the usual Malliavin-Stein approach are necessary. To starters, we need to introduce Malliavin derivatives with respect to both noises, and prove the differentiability (in the sense of Malliavin calculus) of the mild solution. Similarly, we need to introduce two divergences, and include both of them in the implementation of the Malliavin-Stein methodology.

The rest of the chapter is organized as follows. In Section 3.1, we define define the noise η and ξ, and collect the elements of stochastic analysis, Malliavin calculus and Stein's method that are necessary for the rest of the chapter.

In Section 3.2, we state and prove Theorem 3.2.1 and Theorem 3.2.2 concerning the existence, uniqueness and stationarity of the solution to (3.1).

In Section 3.3, we establish Theorem 3.3.1 and Theorem 3.3.2 about the differentiability (in the sense of Malliavin calculus) of the solution to (3.1). In the second part of this section, we obtain estimates for norm of the Malliavin derivative of the mild solution in terms of the fundamental heat solution. These estimates correspond to Theorem 3.3.6, and Theorem 3.3.7, and represent a key ingredient in the proof of Theorem 3.4.1.

Finally, in Section 3.4, we present our main result which is the quantitative central limit theorem for the spatial average of the solution to (3.1).

Throughout the Chapter we denote by C a generic constant which can vary from line to line. However, we will specify dependence where we feel it may be relevant. We also use the following notation introduced in Chapter 2.

(1) The expression $a \lesssim b$ means $a \leq Kb$ for some immaterial constant K that may vary from line to line.

(2) We write $\|X\|_p$ for the $L^p(\Omega)$-norm of a real random variable X.

3.1 Preliminaries

This section provides some preliminary results that are required for further sections. It consists of three subsections: Subsection 3.1.1 contains several important facts from stochastic analysis, Subsection 3.1.2 is devoted to introduce concepts and results from Malliavin calculus and Subsection 3.1.3 details the so called, Malliavin-Stein methodology. It is worth mentioning that the application of Malliavin calculus and the Malliavin-Stein approach in this chapter is slightly different from Chapter 2 because there are two independent isonormal Gaussian processes involved.

3.1.1 Stochastic Analysis

We start by introducing the white noise in $\mathbb{R}_+ \times \mathbb{R}$. Denote by $\mathscr{B}_f^2(\mathbb{R}_+ \times \mathbb{R})$ the collection of Borel sets $A \subset \mathbb{R}_+ \times \mathbb{R}$ with finite Lebesgue measure, denoted by $|A|$. Consider a centered Gaussian family of random variables $\eta = \{\eta(A),\, A \in \mathscr{B}_f^2\}$, defined on a complete probability space $(\Omega, \mathscr{F}, P)$, with covariance given by

$$\mathbb{E}\left[\eta(A)\eta(B)\right] = |A \cap B|.$$

Note that the mapping $1_A \to \eta(A)$ for $A \in \mathscr{B}_f^2$ can be extended to a linear isometry between $L^2(\mathbb{R}_+ \times \mathbb{R})$ and the Gaussian space spanned by η. In this way, $\eta = \{\eta(h),\ h \in L^2(\mathbb{R}_+ \times \mathbb{R})\}$ is a centered Gaussian family of random variables satisfying

$$\mathbb{E}[\langle \eta(h)\eta(g)\rangle] = \langle h, g\rangle_{L^2(\mathbb{R}_+ \times \mathbb{R})},$$

for all $h, g \in L^2(\mathbb{R}_+ \times \mathbb{R})$. This makes η an isonormal Gaussian process.

Let us now define the white noise on $\mathbb{R}$. Denote by $\mathscr{B}_f^1(\mathbb{R})$ the collection of Borel sets $A \subset \mathbb{R}$ with finite Lebesgue measure. Let $\xi = \{\xi(A),\, A \in \mathscr{B}_f^1\}$ be a centered Gaussian family of random variables defined on the same probability space $(\Omega, \mathscr{F}, P)$ with covariance

$$\mathbb{E}\left[\xi(A)\xi(B)\right] = |A \cap B|,$$

and independent of η. By proceeding similarly to what we did for η, we obtain a centered Gaussian family of random variables $\xi = \{\xi(g),\ g \in L^2(\mathbb{R})\}$ such that $\mathbb{E}[\langle \xi(h)\xi(g)\rangle] = \langle h, g\rangle_{L^2(\mathbb{R})}$, for all $g, h \in L^2(\mathbb{R})$. In this way, ξ is also an isonormal Gaussian process.

At this point, we proceed to set up the filtered probability space. In other words, we introduce the filtration $\mathscr{F}_t$.

For all $t > 0$, we take $\mathscr{F}_t^{\eta}$ to be the σ-algebra generated by the P-null sets and $\{\eta(1_{[0,s]\times A}), 0 \leq s \leq t, A \in \mathscr{B}_f^2\}$. Although $\{\mathscr{F}_t^{\eta}\}_{t>0}$ defines a filtration, this is not the appropriate filtration for our purpose because we need to include ξ as well. To fix this, we denote by $\mathscr{F}^{\xi}$ the σ-algebra generated by ξ and we set $\mathscr{F}_t := \mathscr{F}_t^{\eta} \vee \mathscr{F}^{\xi}$. Then, we have constructed a filtration $\mathbb{F} = \{\mathscr{F}_t : t \in \mathbb{R}_+\}$ suitable for our purposes. Equipped with this filtration, we proceed now to recall some facts about stochastic integration.

As proved in [44], for any jointly measurable, $\mathbb{F}$-adapted random field $\{X(s,t),\ (s,t) \in \mathbb{R}_+ \times \mathbb{R}\}$ such that,

$$\int_0^{\infty} \int_{\mathbb{R}} \mathbb{E}\left[X(s,y)^2\right] dy\, ds < \infty,$$

the following stochastic integral

$$\int_0^{\infty} \int_{\mathbb{R}} X(s,y)\, \eta(ds, dy)$$

interpreted as the Dalang-Walsh integral ([10, 44]), is well-defined.

We have now all the necessary concepts to give a precise definition to what is means to be a mild solution to (3.1).

Definition 3.1.1. *A random field $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is a mild solution to (3.1) if*

(1) u is adapted, i.e. for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $u(t,x)$ is $\mathscr{F}_t$-measurable;

(2) u is jointly measurable with respect to $\mathscr{B}(\mathbb{R}_+ \times \mathbb{R}) \times \mathscr{F}$*;*

(3) u satisfies (3.2) a.s., for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$*.*

Let us now recall a fundamental inequality for estimating the L^p-norm of a Dalang-Walsh integral. More precisely, let us record a suitable version of Burkholder-Davis-Gundy inequality (BDG for short); see *e.g.* [20, Theorem B.1].

Lemma 3.1.2 (BDG)**.** *If* $\{\Phi(s,y) : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\}$ *is an adapted random field with respect to* $\mathbb{F}$ *such that* $\|\Phi\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \in L^p(\Omega)$ *for some* $p \geq 2$*, then*

$$\left\| \int_{[0,t]\times\mathbb{R}} \Phi(s,y)\eta(ds,dy) \right\|_p^2 \leq 4p \left\| \int_{[0,t]\times\mathbb{R}} \Phi^2(s,z)dzds \right\|_{p/2}.$$

As a manner of fact, we will often apply BDG inequality together with Minkowski integral inequality. Informally, Minkowski integral inequality says that the norm of an integral is less than the integral of the norm (see *e.g.* [43, A.1]. The precise statement from combining BDG with Minkowski's inequality is the content of the following Lemma.

Lemma 3.1.3. *If* $\{\Phi(s,y) : (s,y) \in \mathbb{R}_+ \times \mathbb{R}^2\}$ *is an adapted random field with respect to* $\mathbb{F}$ *such that* $\|\Phi\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \in L^p(\Omega)$ *for some* $p \geq 2$*, then*

$$\left\| \int_{[0,t]\times\mathbb{R}} \Phi(s,y)\eta(ds,dy) \right\|_p^2 \leq 4p \int_{[0,t]\times\mathbb{R}} \|\Phi^2(s,z)\|_{p/2} dzds = 4p \int_{[0,t]\times\mathbb{R}} \|\Phi(s,z)\|_p^2 dzds. \quad (3.3)$$

Most of the discussion in Section 3.3 and Section 3.4 relies on Malliavin calculus. For this reason, we will introduce some basic elements of this topic in the next subsection. We point out that the content of Subsection 3.1.2 is slightly different from what is discussed in Chapter 2 because now we need to incorporate the two independent isonormal Gaussian processes involved. For any unexplained notation and result, we refer to the book [30] (see also [27, Chapter 2]).

3.1.2 Malliavin Calculus

Set $\mathfrak{H}_1 := L^2(\mathbb{R})$ and $\mathfrak{H}_2 := L^2(\mathbb{R}_+ \times \mathbb{R})$. As seen in Subsection 3.1.1 $\xi = \{\xi(h), h \in \mathfrak{H}_1\}$ and $\eta = \{\eta(g), g \in \mathfrak{H}_2\}$ are independent isonormal Gaussian processes on the same probability space $(\Omega, \mathscr{F}, P)$. As it happens, the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ can be combined into another separable Hilbert space, denoted by $\mathfrak{H} := \mathfrak{H}_1 \oplus \mathfrak{H}_2$ consisting of the set of all pairs (h_1, h_2) where $h_i \in \mathfrak{H}_i$, $i = 1, 2$, and inner product given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{\mathfrak{H}} = \langle x_1, y_1 \rangle_{\mathfrak{H}_1} + \langle x_2, y_2 \rangle_{\mathfrak{H}_2} .$$

Then, the process $X = \{X(h,g) = \xi(h) + \eta(g), (h,g) \in \mathfrak{H}\}$ turns out to be an isonormal Gaussian process.

Denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let $\mathscr{S}$ be the space of simple functionals of the form $F = f(X(h_1), \ldots, X(h_n))$ for $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}$, $1 \leq i \leq n$. Then, the Malliavin derivative DF is the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), \ldots, X(h_n)) h_i .$$

Note that since $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, we can project DF onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$. In this way, we define $D^\xi F$ and $D^\eta F$ as the projections of DF onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively; see *e.g.* section 5 in [13] for a similar discussion.

The derivative operators D^η is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathfrak{H}_2)$ for any $p \geq 1$ and we define $\mathbb{D}_\eta^{1,p}$ to be the completion of $\mathscr{S}$ under the norm $\|F\|_{1,p} = \left(\mathbb{E}\big[|F|^p\big] + \mathbb{E}\big[\|D^\eta F\|_{\mathfrak{H}_2}^p\big] \right)^{1/p}$.

The *chain rule* for D^η asserts that if $F \in \mathbb{D}_\eta^{1,2}$ and $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $h(F) \in \mathbb{D}_\eta^{1,2}$ with

$$D^\eta \big(h(F) \big) = Y \, D^\eta F, \tag{3.4}$$

where Y is some $\sigma\{F\}$-measurable random variable bounded by the Lipschitz constant of h. In fact, when h is differentiable, we have $Y = h'(F)$ (see, for instance, [30, Proposition 1.2.4]).

The following Lemma (see, for instance, [30, Proposition 1.5.3]), provides a useful criteria to show a random variable F belongs to $\mathbb{D}_\eta^{1,p}$.

Lemma 3.1.4. *Let $\{F_n, n \geq 1\}$ be a sequence of random variables converging to F in $L^p(\Omega)$ for some $p > 1$. Suppose that the sequence $\{a_n, n \geq 1\}$ given by*

$$a_n = \left(\mathbb{E}[\|D^\eta F_n\|_{\mathfrak{H}_2}^p]\right)^{1/p} = \|D^\eta F_n\|_{L^p(\Omega;\mathfrak{H}_1)},$$

is bounded. Then $F \in \mathbb{D}_\eta^{1,p}$.

The divergence operator δ^η is introduced as the adjoint of the derivative operator D^η. More precisely, an element u in $L^2(\Omega, \mathfrak{H}_2)$, belongs to the domain of δ^η, if there is a constant $c_u > 0$ satisfying

$$\left|\mathbb{E}\left[\langle D^\eta F, u\rangle_{\mathfrak{H}_2}\right]\right| \leq c\sqrt{\mathbb{E}|F^2|} \text{ for all } F \in \mathbb{D}_\eta^{1,2}. \tag{3.5}$$

In particular, for any $u \in \text{Dom}\,\delta^\eta$, condition (3.5) means that the linear operator $F \to \mathbb{E}\left[\langle D^\eta F, u\rangle_{\mathfrak{H}_2}\right]$ is continuous from $\mathbb{D}_\eta^{1,2}$, equipped with the $L^2(\Omega)$ norm, into $\mathbb{R}$. Thus, we can extend this linear operator to a linear operator from $L^2(\Omega)$ to $\mathbb{R}$. Consequently, Riesz representation theorem gives the existence of a unique element in $L^2(\Omega)$, denoted by $\delta^\eta(u)$, such that $\mathbb{E}[\delta^\eta(u)F] = \mathbb{E}[\langle u, D^\eta F\rangle_{HH_2}]$ for all $F \in \mathbb{D}_\eta^{1,2}$. In other words, for $u \in \text{Dom}\,\delta^\eta$, $\delta^\eta(u)$ is the unique element of $L^2(\Omega)$ characterized by the duality formula

$$\mathbb{E}[\delta^\eta(u)F] = \mathbb{E}[\langle u, D^\eta F\rangle_{\mathfrak{H}_2}] \tag{3.6}$$

for any $F \in \mathbb{D}_\eta^{1,2}$.

The aforementioned definitions for $\mathbb{D}_\eta^{1,p}$, δ^η are minimally modified to define $\mathbb{D}_\xi^{1,p}$ and δ^ξ. Similarly, the chain rule and Lemma 3.1.4 also hold when η is replaced by ξ.

The operators D^η and δ^η satisfy the commutation relation

$$(D^\eta \delta^\eta - \delta^\eta D^\eta)(V) = V, \tag{3.7}$$

which is useful when one needs to take the derivative of an element given by a divergence. The same relation holds for ξ as well.

In our context, the Dalang-Walsh integral coincides with δ^η. More precisely, we have the following Lemma

Lemma 3.1.5. *Any adapted random field Φ that satisfies $\mathbb{E}\left[\|\Phi\|_{\mathfrak{H}_2}^2\right] < \infty$ belongs to the domain of δ^η and*

$$\delta^\eta(\Phi) = \int_0^\infty \int_{\mathbb{R}} \Phi(s,y)\eta(ds,dy).$$

The proof of this result is analogous to the case of integrals with respect to the Brownian motion (see [30, Proposition 1.3.11]), by just replacing real processes by $\mathfrak{H}_2$-valued processes.

With the help of the derivative operator D^η, we can represent $F \in \mathbb{D}_\eta^{1,2}$ as a stochastic integral. To be precise, we have the the following two-parameter Clark-Ocone formula, see *e.g.* [8, Proposition 6.3] for a proof.

Lemma 3.1.6 (Clark-Okone formula)**.** *Given $F \in \mathbb{D}_\eta^{1,2}$, we have almost surely*

$$F = \mathbb{E}(F) + \int_0^\infty \int_{\mathbb{R}} \mathbb{E}\left[D_{s,y}^\eta F | \mathscr{F}_s\right] \eta(ds,dy).$$

Using Jensen's inequality for conditional expectation, Clark-Okone formula leads to the following following Poincaré type inequality.

Corollary 3.1.7 (Poincaré type formula)**.** *If $F,G \in \mathbb{D}_\eta^{1,2}$, then*

$$|Cov[F,G]| \le \int_0^\infty \int_{\mathbb{R}} \|D_{s,y}^\eta F\|_2 \|D_{s,y}^\eta G\|_2 \, dy\, ds.$$

We finish this subsection with a simple, but important Remark (see Section 3.4).

Remark 3.1.8.

(a) If $G(x) \in \mathfrak{H}_1$, then $\delta^\xi(G) = \int_{\mathbb{R}} G(x)\xi(dx)$.

(b) For a process $\Phi = \{\Phi(s),\ s \in [0,t]\}$ such that $\sqrt{\mathrm{Var}(\Phi_s)}$ is integrable on $[0,t]$, we have

$$\sqrt{\mathrm{Var}\left(\int_0^t \Phi(s)\,ds\right)} \leq \int_0^t \sqrt{\mathrm{Var}(\Phi_s)}\,ds.$$

3.1.3 Malliavin-Stein methodology

Theorem 3.4.1 in Section 5 relies on a combination of Malliavin calculus and Stein's method. Hence, in this subsection we will introduce basic elements of this methodology. We refer the interested reader to the monograph [27] for a comprehensive treatment.

Stein's method is a probabilistic technique which allows one to measure the distance between a probability distribution and a target distribution, which for our purpose will be the normal distribution. Recall that the total variation distance between two random variables F and G is defined by

$$d_{TV}(F,G) := \sup_{B \in \mathscr{B}(\mathbb{R})} |P(F \in B) - P(G \in B)|, \tag{3.8}$$

where $\mathscr{B}(\mathbb{R})$ is the collection of all Borel sets in $\mathbb{R}$. We point out that $d_{TV}(F,G)$ only depends on the laws of F and G and defines a metric on the set of probability measures on $\mathbb{R}$. Furthermore, the topology induced by d_{TV} is strictly stronger than the topology of convergence in distribution, see *e.g.* [27, Proposition C.3.1].

The following theorem provides an upper bound for the total variation distance between any random variable and a random variable with standard normal distribution. We refer the reader to [27, Theorem 3.3.1] for a proof.

Theorem 3.1.9. *For $Z \sim \mathscr{N}(0,1)$ and for any random variable F,*

$$d_{TV}(F,Z) \leq \sup_{f \in \mathfrak{F}_{TV}} \left|\mathbb{E}[f'(F)] - \mathbb{E}[Ff(F)]\right|, \tag{3.9}$$

where $\mathfrak{F}_{TV}$ is the class of continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_\infty \leq \sqrt{\pi/2}$ and $\|f'\|_\infty \leq 2$.

By borrowing ideas from the Malliavin-Stein approach introduced by Nourdin and Pecatti in [26], Theorem 3.1.9 can be combined with Malliavin calculus to get the following proposition.

Proposition 3.1.10. *Let $F = \delta^{\xi}(v) + \delta^{\eta}(u)$ for some $\mathfrak{H}_1$-valued random variable $v \in Dom\delta^{\xi}$ and some $\mathfrak{H}_2$-valued random variable $u \in Dom\delta^{\eta}$. Assume $F \in \mathbb{D}^{1,2}_{\xi}$, $F \in \mathbb{D}^{1,2}_{\eta}$ and $E[F^2] = 1$ and let $Z \sim \mathcal{N}(0,1)$. Then we have*

$$d_{TV}(F,Z) \leq 2\sqrt{2}\sqrt{Var\left[\langle D^{\xi}F, v\rangle_{\mathfrak{H}_1}\right] + Var\left[\langle D^{\eta}F, u\rangle_{\mathfrak{H}_2}\right]} \tag{3.10}$$

Proof. Let $f \in \mathfrak{F}_{TV}$. By our assumption on F and (3.6) applied to δ^{ξ} and δ^{η}, we have

$$\begin{aligned}\mathbb{E}(Ff(F)) &= \mathbb{E}\left[\left(\delta^{\xi}(v) + \delta^{\eta}(u)\right) f(F)\right] \\ &= \mathbb{E}\left[f'(F)\left\langle v, D^{\xi}F\right\rangle_{\mathfrak{H}_1}\right] + \mathbb{E}\left[f'(F)\langle u, D^{\eta}F\rangle_{\mathfrak{H}_2}\right].\end{aligned}$$

Consequently, by Theorem 3.1.9

$$\begin{aligned}d_{TV}(F,Z) &\leq \sup_{f\in\mathfrak{F}_{TV}} \left|\mathbb{E}\left(f'(F) - Ff(F)\right)\right| \\ &= \sup_{f\in\mathfrak{F}_{TV}} \left|\mathbb{E}\left[f'(F)\left(1 - \left\langle v, D^{\xi}F\right\rangle_{\mathfrak{H}_1} - \langle u, D^{\eta}F\rangle_{\mathfrak{H}_2}\right)\right]\right| \\ &\leq 2\mathbb{E}\left[\left|1 - \left\langle v, D^{\xi}F\right\rangle_{\mathfrak{H}_1} - \langle u, D^{\eta}F\rangle_{\mathfrak{H}_2}\right|\right].\end{aligned} \tag{3.11}$$

By using the duality relation (3.6), we have

$$\mathbb{E}\left[\left\langle v, D^{\xi}F\right\rangle_{\mathfrak{H}_1} + \langle u, D^{\eta}F\rangle_{\mathfrak{H}_2}\right] = \mathbb{E}\left[\delta^{\xi}(v)F + \delta^{\eta}(u)F\right] = \mathbb{E}(F^2) = 1.$$

Thus, (3.11) implies

$$d_{TV}(F,Z) \leq 2\sqrt{\mathrm{Var}\left(\langle v, D^{\xi}F\rangle_{\mathfrak{H}_1} + \langle u, D^{\eta}F\rangle_{\mathfrak{H}_2}\right)}.$$

The desired result follows from the well known inequality

$$\mathrm{Var}\left(\left\langle v, D^{\xi} F\right\rangle_{\mathfrak{H}_1} + \langle u, D^{\eta} F\rangle_{\mathfrak{H}_2}\right) \leq 2\,\mathrm{Var}\left(\left\langle v, D^{\xi} F\right\rangle_{\mathfrak{H}_1}\right) + 2\,\mathrm{Var}\left(\langle u, D^{\eta} F\rangle_{\mathfrak{H}_2}\right).$$

□

3.2 Existence, uniqueness and stationarity.

In this section, we state and prove the existence, uniqueness and strict stationarity of the mild solution to (3.1). The precise statements of our main results are the following.

Theorem 3.2.1 (Existence and uniqueness)**.** *There is a unique mild random-field solution $u(t,x)$ to (3.1) such that for all $p \geq 2$, $T > 0$ and $t \in (0,T]$*

$$\sup_{x\in\mathbb{R}} \|u(t,x)\|_p^2 \leq Cb(t), \tag{3.12}$$

where $b(t) = \frac{1}{\sqrt{t}}$, and the constant C depends on T, p and the function $\sigma(x)$.

Theorem 3.2.2. *Let $u(t,x)$ be the random field given by Theorem 3.2.1. Then $\{u(t,x) : x \in \mathbb{R}\}$ is stationary for any fixed $t > 0$.*

The proof of existence and uniqueness follows the standard Picard's iteration scheme, while the proof of stationarity uses the same ideas of the proof of Lemma 7.1 in [8]. We postpone the proof of these Theorems until Subsection 3.2.2. We do so, to present some results, which are heavily used in the proof of Theorem 3.2.1 and in Section 3.3. This is the content of the next subsection.

3.2.1 Some basic results

We start this subsection with three simple, but important, observations. We omit their proofs as they follow explicit calculations and straightforward arguments.

Observation 3.2.3.

(a) The Gaussian kernel $p_t(x)$ satisfies that for any $t > 0$

$$\int_{\mathbb{R}} p_t^2(z)\,dz = \frac{1}{\sqrt{4\pi t}}.$$

(b) Since the function $\sigma(x)$ is assumed to be Lipschitz with Lipschitz constant L, then for any $p \geq 1$ there exist a constant $C > 0$ (depending on p, L and $\sigma(0)$) such that if $z(t,y)$ is a random field in $L^p(\Omega)$, then

$$\|\sigma(z(t,y))\|_p^2 \leq C(1 + \|z(t,y)\|_p^2). \tag{3.13}$$

(c) Let $T > 0$. The function $b(t) = \frac{1}{\sqrt{t}}$ satisifes $1 \leq \sqrt{T}b(t)$ for any $t \in (0,T]$.

A key ingredient when working with Picard iterations is a Gronwall type inequality. This is the content of the following results. We point out that Proposition 3.2.4 and Lemma 3.2.5 are a consequence of Theorem 1 in [46]. Hence, we refer the reader to [46] for the corresponding proofs.

Proposition 3.2.4. *Let $T > 0$. Suppose $a(t)$ is a nonnegative function locally integrable on $0 \leq t \leq T$ and $g(t)$ is a nonnegative, bounded, nondecreasing continuous function defined on $0 \leq t \leq T$,and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t \leq T$ satisfying*

$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{-1/2}u(s)\,ds$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(1/2))^n}{\Gamma(n/2)}(t-s)^{n/2-1}a(s)\right] ds, \text{ for all } 0 \leq t \leq T.$$

Lemma 3.2.5. *Let g and u be as in Theorem 3.2.4. Define for $0 \leq t \leq T$*

$$[Au](t) = g(t)\int_0^t (t-r)^{-1/2}u(r)\,dr.$$

Then for all $n \in \mathbb{N}$

$$[A^n u](t) \leq \frac{(g(t)\Gamma(1/2))^n}{\Gamma(n/2)} \int_0^t (t-r)^{n/2-1} u(r)\,dr \tag{3.14}$$

One immediate consequence of Lemma 3.2.5, is the following summability condition satisfied by $[A^n u](t)$.

Corollary 3.2.6. *Let $r \geq 0$. Then,*

$$\sum_{n=1}^{\infty} \left([A^n u](t)\right)^{1/r}$$

converges uniformly on $[0,T]$.

Proof. For n sufficiently large $n/2-1>0$. Then (3.14) implies

$$(A^n u(t))^{1/r} \leq \left(\frac{(g(t)\Gamma(1/2))^n}{\Gamma(n/2)} \int_0^t (t-s)^{n/2-1} u(s)\,ds\right)^{1/r} \leq \left(\frac{(g(T)\Gamma(1/2))^n}{\Gamma(n/2)} T^{n/2-1} \int_0^T u(r)\,dr,\right)^{1/r}.$$

The desired conclusion follows because

$$\sum_{n=1}^{\infty} \frac{(g(T)\Gamma(1/2))^{n/r}}{(\Gamma(n/2))^{1/r}} T^{(n/2-1)/r} < \infty,$$

by, say, Stirling's formula and the root test for series. □

3.2.2 Proof of main results

Proof of Theorem 3.2.1

The proof consist of three steps.

Step 1:

Consider the Picard iteration scheme

$$
\begin{aligned}
u_0(t,x) &= \int_{\mathbb{R}} p_t(x-y)\xi(dy) \\
u_{n+1}(t,x) &= u_0(t,x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))\,\eta(ds,dy) \quad \text{for } n \geq 0.
\end{aligned} \tag{3.15}
$$

We prove that the following statements hold for all $n \in \mathbb{N}$

(i) $u_n(t,x)$ is well defined and adapted to $\mathscr{F}_t$.

(ii) For any $p \geq 2$ and $T > 0$, the following estimate holds for all $0 < t \leq T$

$$
\sup_{x\in\mathbb{R}} \|u_n(t,x)\|_p^2 \leq C\,b(t),
$$

where the constant C depends on T, p and the function $\sigma(x)$.

(iii) $I_n(t,x) := \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))\,\eta(ds,dy)$ is well defined. Further, there is a version of $I_n(t,x)$ which is jointly measurable.

Naturally, the proof will be done by induction.

For the case $n = 0$, statement (i) is clear from the definition of $u_0(t,x)$. Let $p \geq 2$. Note that Itô isommetry and Observation 3.2.3(a) imply

$$
\mathbb{E}[u_0^2(t,x)] = \int_{\mathbb{R}} p_t^2(x-y)\,dy = (4\pi)^{-1/2} b(t).
$$

Actually, since $u_0(t,x)$ is Gaussian, there exists $C > 0$ (depending only p) so that $\mathbb{E}[u_0^p(t,x)] \leq C\mathbb{E}[u_0^2(t,x)]^{p/2}$. We conclude

$$
\|u_0(t,x)\|_p^2 \leq Cb(t) \quad \text{for all } x \in \mathbb{R}.
$$

In particular, for all $t > 0$

$$\sup_{x\in\mathbb{R}} \|u_0(t,x)\|_p^2 \leq C\, b(t),$$

which proves statement (ii) holds for $n = 0$. For statement (iii), we note that Observation 3.2.3(a),(b), and the previous inequality imply

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)\|\sigma(u_0(s,y))\|_2^2\, ds\, dy &\leq C\int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)(1+\|u_0(s,y)\|_2^2)\, ds\, dy \\ &\leq C\int_0^t \frac{1}{\sqrt{t-s}}(1+b(s))\, ds.\end{aligned}$$

However, explicit calculations show that $\int_0^t \frac{1}{\sqrt{t-s}}(1+b(s))\, ds < \infty$. Then

$$\int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)\|\sigma(u_0(s,y))\|_2^2\, ds\, dy < \infty. \tag{3.16}$$

In particular, the stochastic integral

$$\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_0(s,y))\,\eta(dy, ds),$$

is well defined. In fact, condition (3.16) implies also the existence of a version of $I_n(t,x)$ which is jointly measurable. This verifies statement (iii) holds true for the case $n = 0$.

Assume there is $n \in \mathbb{N}$ so that statements (i), (ii) and (iii) hold true for any $k \in \mathbb{N}$ with $0 \leq k \leq n$. Since $u_{n+1}(t,x) = u_0(t,x) + I_n(t,x)$, it follows that $u_{n+1}(t,x)$ is well defined. Further, $u_{n+1}(t,x)$ is adapted to $\mathscr{F}_t$ because both $u_0(t,x)$ and $I_n(t,x)$ are adapted. This verifies statement (i) for $n+1$.

Let $k \in \mathbb{N}$ with $0 \le k \le n$. Let $p \ge 2$ and $T > 0$. An application of (3.3), implies that for some constants C_p

$$\|u_{k+1}(t,x)\|_p^2 \le C_p\|u_0(t,x)\|_p^2 + C_p\left\|\int_0^t\int_{\mathbb{R}} p_{t-r}^2(x-z)\sigma^2(u_k(r,z))\,dr\,dz\right\|_{p/2}$$
$$\le C_p\|u_0(t,x)\|_p^2 + C_p\int_0^t\int_{\mathbb{R}} p_{t-r}^2(x-z)\left\|\sigma^2(u_k(r,z))\right\|_{p/2}\,dr\,dz$$
$$\le C_p\|u_0(t,x)\|_p^2 + C_p\int_0^t\int_{\mathbb{R}} p_{t-r}^2(x-z)\left\|\sigma(u_k(r,z))\right\|_p^2\,dr\,dz.$$

At this step, we will introduce some notation. Set

$$\tilde{u}_k(t) := \sup_{x\in\mathbb{R}}\|u_k(t,x)\|_p^2.$$

Then, Observation 3.2.3(a),(b),(c) together with the inequality $\|u_0(t,x)\|_p^2 \le Cb(t)$ imply

$$\tilde{u}_{k+1}(t) \le Cb(t) + C\int_0^t \frac{1}{\sqrt{t-r}}\tilde{u}_k(r)\,dr$$

for some constant C depending on p, T, and the function $\sigma(x)$. In fact, we can rewrite the previous inequality in the form

$$\tilde{u}_{k+1}(t) \le Cb(t) + [A\tilde{u}_k](t), \tag{3.17}$$

for the operator A introduced in Lemma 3.2.5 with $g(t) = C$. By iterating (3.17), we obtain

$$\tilde{u}_{n+1}(t) \le Cb(t) + C\sum_{k=1}^{n} A^k b(t) + A^{n+1}\tilde{u}_0(t).$$

However, since $\tilde{u}_0(t) \lesssim b(t)$, we conclude

$$\tilde{u}_{n+1}(t) \le Cb(t) + C\sum_{k=1}^{n+1}[A^k b](t), \tag{3.18}$$

for some constant $C > 0$ depending only on p, T and the function σ.

At this point, we note $b(t)$ is integrable on $(0,T]$. Then Corollary 3.2.6 (taking $r=1$), implies $\sum_{k=1}^{\infty}[A^k b](t)$ converges uniformly on $(0,T]$. In particular, there is a constant C_T depending only on T so that $\sum_{k=1}^{\infty} A^k b(t) < C_T$ for all $t \in (0,T]$. However, Observation 3.2.3(c) implies $C_T \lesssim b(t)$. Therefore, we conclude from (3.18) that for any $0 < t \leq T$

$$\tilde{u}_{n+1}(t) \leq Cb(t),$$

where the constant C depends on T, p and the function $\sigma(x)$. This proves statement (ii) holds for $n+1$. Finally, statement (iii) holds for $n+1$ because

$$\int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)\|\sigma(u_{n+1}(s,y))\|_2^2\, ds\, dy \leq C\int_0^t \frac{1}{\sqrt{t-s}}(1+b(s))\, ds < \infty.$$

This finishes step 1.

Step 2:

We prove the existence of the random field $u(t,x)$ which is jointly measurable, $\mathscr{F}_t$-adapted, and satisfies (3.1) and (3.12). This is done by proving the $L^2(\Omega)$-convergence of the Picard iteration scheme.

Let $n \in \mathbb{N}$. For each $k \in \mathbb{N}$ with $0 \leq k \leq n$, set

$$H_k(t) = \sup_{x \in \mathbb{R}} \|u_{k+1}(t,x) - u_k(t,x)\|_p^2.$$

In view of statement (ii) in Step 1, $H_k(t)$ is well defined. Moreover, $H_0(t) \lesssim b(t)$ and similar arguments to the ones used to obtain (3.17) yield the existence of a constant $C > 0$ so that

$$H_k(t) \leq C\int_0^t \frac{1}{\sqrt{t-r}} H_{k-1}(r) = [AH_{k-1}](t).$$

We conclude

$$H_n(t) \le A^n H_0(t) \lesssim [A^n b](t).$$

In view of Corollary 3.2.6), we have $\sum_{n=1}^{\infty} H_n(t)^{1/2}$ converges uniformly on $(0,T]$ for any $T>0$. Consequently, $u_n(t,x)$ converges in $L^p(\Omega)$ uniformly on $(0,T]\times\mathbb{R}$ for any $T>0$. Let us denote by $u(t,x)$ the L^p limit of $u_n(t,x)$. Then, we note the following

(1) Each $u_n(t,x)$ is jointly measurable and adapted to $\mathscr{F}_t$, so the same holds for $u(t,x)$.

(2) $u(t,x)$ satisfies (3.12). Indeed, this is a consequence of statement (ii) in Step 1 and Fatou's lemma.

(3) $u(t,x)$ satisfies (3.2). Indeed, this follows from the definition

$$u_{n+1}(t,x) = \int_{\mathbb{R}} p_t(x-y)\xi(dy) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x-y)\sigma(u_n(r,y))\,\eta(dr,dy),$$

and the fact that $u_n(t,x)$ converges to $u(t,x)$ in $L^p(\Omega)$.

This finishes Step 2.

Step 3:

We prove the uniqueness (up to a modification) of the solution by using Proposition 3.2.4. More precisely, assume $u(t,x)$ and $v(t,x)$ are two jointly measurable, $\mathscr{F}_t$-adapted random fields satisfying (3.2) and (3.12). Let $T>0$ and $p\ge 2$. Set $D(t)=\sup_{x\in\mathbb{R}}\|u(t,x)-v(t,x)\|_p^2$. This is well define because both u,v satisfy (3.12). Similar arguments to the ones used to obtain (3.17) imply the existence of a constant $C>0$ so that

$$D(t) \le C\int_0^t \frac{1}{\sqrt{t-r}} D(r)\,dr \quad \text{for all } 0\le t<T.$$

It follows from Theorem 3.2.4 that $R(t)=0$ for all $0\le t<T$. This proves the uniqueness of the solution and finishes the proof.

Proof of Theorem 3.2.2

Fix $t > 0$. We start by noticing the following facts

(i) The shift noise ξ_y defined by

$$\xi_y(\phi) = \int_{\mathbb{R}} \phi(x-y)\,\xi(dx), \quad \phi \in \mathfrak{H}_1$$

has the same distribution as the noise ξ.

(ii) The shift noise η_y defined by

$$\eta_y(\phi) = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi(s,x-y)\,\eta(ds,dx) \quad \phi \in \mathfrak{H}_2$$

has the same distribution as the noise η.

Furthermore, for each $y \in \mathbb{R}$, uniqueness of the mild solution imply that $\{u(t,x+y),\ x \in \mathbb{R}\}$ coincides almost surely with the random field u driven by ξ_y and η_y. We conclude that the finite dimensional distributions of $\{u(t,x+y),\ x \in \mathbb{R}\}$ do not depend on y, in other words, we have shown the strict stationarity of $u(t,x)$ for $t > 0$ fixed (see *e.g.* Lemma 7.1 in [8] for similar arguments).

Before moving to the next section, where we concern ourselves with the differentiability in the sense of Malliavin calculus of the random field $u(t,x)$, let us record a Remark which will be important in Section 3.4. We point out that the first part of the Remark is a consequence of Theorem 3.2.1 and Observation 3.2.3(b), while the second is a consequence of Theorem 3.2.2.

Remark 3.2.7.

(i) The following estimate holds for all $p \geq 2$, $T > 0$ and $t \in (0,T]$

$$\sup_{x\in\mathbb{R}} \|\sigma(u(t,x))\|_p^2 \leq C b(t),$$

where $b(t) = \frac{1}{\sqrt{t}}$, and the constant C depends on T, p and the function $\sigma(x)$.

(ii) For any $r > 0$, $\mathbb{E}(\sigma^2(u(r,y)))$ does not depend on y.

3.3 Malliavin derivatives of $u(t,x)$.

In this section, we prove results regarding the Malliavin derivatives of the mild solution to (3.1. We point out that the estimates given by Theorem 3.3.6 and Theorem 3.3.7 are key ingredients in the proof of Theorem 3.4.1 in Section 3.4.

Theorem 3.3.1. *Let $u(t,x)$ be the random field given by Theorem 3.2.1. Then $u(t,x) \in \mathbb{D}^{1,2}_{\xi}$, and the derivative $D^{\xi}u(t,x)$ satisfies*

$$D^{\xi}_y u(t,x) = p_t(x-y) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_{\xi}(r,z)D^{\xi}_y u(r,z)\,\eta(dr\,dz),$$

where $\Sigma_{\xi}(r,z)$ is an adapted random field bounded by the Lipschtiz constant of $\sigma(x)$.

Proof. Consider the Picard iteration scheme (3.15). Since $u_0(t,x) = \int_{\mathbb{R}} p_t(x-z)\xi(dz)$, it is clear that $u_0(t,x) \in \mathbb{D}^{1,2}_{\xi}$ and $D^{\xi}_y u_0(t,x) = p_t(x-y)$. Similarly, if we assume $u_n(t,x) \in \mathbb{D}^{1,2}_{\xi}$ for some $n \in \mathbb{N}$, then we can apply D^{ξ} to

$$u_{n+1}(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))\,\eta(ds,dy),$$

to obtain that $u_{n+1}(t,x) \in \mathbb{D}^{1,2}_{\xi}$ and

$$D^{\xi}_y u_{n+1}(t,x) = p_t(x-y) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_{\xi,n}(r,z)D^{\xi}_y u_n(r,z)\eta(dr,dz), \tag{3.19}$$

where $\Sigma_{\xi,n}(r,z)$ is a random field uniformly bounded by L. In this way, we conclude that $u_n(t,x) \in \mathbb{D}^{1,2}_{\xi}$ for any $t > 0$, $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Now, we will prove that for any $t > 0$ and $x \in \mathbb{R}$

$$\sup_{n\in\mathbb{N}} \|D^{\xi}u_n(t,x)\|^2_{L^2(\Omega;\mathfrak{H}_1)} < \infty.$$

As a manner of fact, we will prove the stronger result

$$\sup_{n\in\mathbb{N}} \Psi_n(t) < \infty, \tag{3.20}$$

where $\Psi_n(t) := \sup_{x\in\mathbb{R}} \|D^\xi u_n(t,x)\|^2_{L^2(\Omega;\mathfrak{H}_1)}$.

Let us start by computing $\Psi_0(t)$ explicitly. In this case, $D^\xi_y u_0(t,x) = p_t(x-y)$ and by Observation 3.2.3(a), we have

$$\begin{aligned}
\|D^\xi u_0(t,x)\|^2_{L^p(\Omega;\mathfrak{H}_1)} &= \mathbb{E}\left[\int_{\mathbb{R}} \left(D^\xi_y u_0(t,x)\right)^2 dy\right] \\
&= \int_{\mathbb{R}} p_t^2(x-y)\, dy \\
&= \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{t}}.
\end{aligned}$$

Hence, $\Psi_0(t) = \dfrac{1}{\sqrt{4\pi}} \dfrac{1}{\sqrt{t}}$.

Let $n \in \mathbb{N}$. It follows from (3.19)

$$\begin{aligned}
\|D^\xi u_{n+1}(t,x)\|^2_{L^2(\Omega;\mathfrak{H}_1)} &\leq 2\, \|p_t(x-\bullet)\|^2_{L^p(\Omega;\mathfrak{H}_1)} \\
&\quad + 2 \left\| \int_0^t \int_{\mathbb{R}} p_{t-r}(x-z) \Sigma_{\xi,(n)}(r,z) D^\xi_\bullet u_n(r,z) \eta(dr,dz) \right\|^2_{L^2(\Omega;\mathfrak{H}_1)}.
\end{aligned}$$

For the moment, let us focus on the second expression. Combining the isometry with the uniform boundedness of $\Sigma_{\xi,n}(r,z)$, we have

$$\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_{\xi,(n)}(r,z) D^{\xi}_{\bullet} u_n(r,z)\eta(dr,dz) \right\|^2_{L^2(\Omega;\mathfrak{H}_1)} \\
&= \mathbb{E}\left[\int_{\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_{\xi,(n)}(r,z) D^{\xi}_{y} u_n(r,z)\eta(dr,dz) \right)^2 dy \right] \\
&\leq L^2 \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x-z)\mathbb{E}\left[|D^{\xi}_y u_n(r,z)|^2 \right] dz\,dr\,dy \\
&= L^2 \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x-z) \left(\mathbb{E}\left[\int_{\mathbb{R}} |D^{\xi}_y u_n(r,z)|^2 dy \right] \right) dz\,dr \\
&= L^2 \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x-z) \|D^{\xi}_{\bullet} u_n(r,z)\|^2_{L^2(\Omega;\mathfrak{H}_1} dz\,dr.
\end{aligned}$$

Then

$$\|D^{\xi} u_{n+1}(t,x)\|^2_{L^2(\Omega;\mathfrak{H}_1)} \leq 2\Psi_0(t) + 2L^2 \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x-z)\|D^{\xi}_{\bullet} u_n(r,z)\|^2_{L^2(\Omega;\mathfrak{H}_1} dz\,dr.$$

In view of Observation 3.2.3(a), we conclude

$$\Psi_{n+1}(t) \leq 2\Psi_0(t) + A\psi_n(t), \tag{3.21}$$

for the operator A introduced in Lemma 3.2.5 with $g(t) = L^2/\sqrt{\pi}$. We iterate (3.21) to obtain

$$\Psi_{n+1}(t) \leq 2\Psi_0(t) + 2\sum_{k=1}^{n} [A^k\Psi_0](t) + [A^{n+1}\Psi_0](t) \leq 2\Psi_0(t) + 2\sum_{k=1}^{n+1} [A^k\Psi_0](t).$$

In view of Corollary 3.2.6, we have $\sum_{k=1}^{\infty} [A^k\Psi_0](t) < \infty$. Finally, since $n \in \mathbb{N}$ was arbitrary, we conclude

$$\sup_{n\in\mathbb{N}} \Psi_{n+1}(t) \leq 2\Psi_0(t) + 2\sum_{k=1}^{\infty} [A^k\Psi_0](t),$$

which proves (3.20).

Taking into account that $u_n(t,x)$ converges to $u(t,x)$ in $L^2(\Omega)$ for all $p \geq 1$, we deduce from Lemma 3.1.4 $u(t,x) \in \mathbb{D}^{1,2}_{\xi}$. In fact, $D^{\xi} u_n(t,x)$ converges to $D^{\xi} u(t,x)$ in the weak topology of

$L^2(\Omega;\mathfrak{H}_1)$ see *e.g.* [30, Proposition 1.2.3]. Finally, applying the operator D^ξ to both members of (3.2), we deduce the desired formula for $D^\xi u(t,x)$. □

Theorem 3.3.2. *Let $u(t,x)$ be the random field given by Theorem 3.2.1. Then $u(t,x) \in \mathbb{D}_\eta^{1,p}$ for all $p \geq 2$, and the derivative $D^\eta u(t,x)$ satisfies*

$$D^\eta_{s,y}u(t,x) = p_{t-s}(x-y) + \int_s^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_\eta(r,z)D^\eta_{s,y}u(r,z)\,\eta(dr\,dz).$$

if $s < t$, and $D^\eta_{s,y}u(t,x) = 0$ for $s > t$, where $\Sigma_\eta(r,z)$ is an adapted random field bounded by L.

Proof. Let $p \geq 2$. As in the proof the proof of Theorem 3.3.1, we will use the Picard's iteration scheme (3.15). By using similar arguments, we conclude $u_n(t,x) \in \mathbb{D}_\eta^{1,p}$ for any $t > 0$, $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Since $u_n(t,x)$ converges to $u(t,x)$ in $L^p(\Omega)$ and given Lemma 3.1.4, it suffices to prove that for all $t > 0$, $x \in \mathbb{R}$

$$\sup_{n\in\mathbb{N}} \|D^\eta_{\bullet,*}u_n(t,x)\|_{L^p(\Omega,\mathfrak{H}_2)} < \infty.$$

In fact, we will prove that for any $T > 0$

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \|D^\eta_{\bullet,*}u_n(t,x)\|^2_{L^p(\Omega,\mathfrak{H}_2)} < \infty. \tag{3.22}$$

Before entering in the details, we make the following important observations.

(i) For all n, $u_n(t,x)$ is adapted and satisfies $\mathbb{E}(\|u_n(t,x)\|^2_{\mathfrak{H}_2}) < \infty$. Consequently, $D^\eta_{s,y}u_n(t,x) = 0$ if $s > t$. The proof of this result is analogous to the case of integrals with respect to the Brownian motion (see *e.g.* [31, Lemma 3.4.1]).

(ii) If $s < t$ and $n \geq 0$, then

$$\begin{aligned} D^\eta_{s,y}u_{n+1}(t,x) &= p_{t-s}(x-y)\sigma(u_n(s,y)) \\ &\quad + \int_s^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_\eta(r,z)D^\eta_{s,y}u_n(r,z)\,\eta(dr,dz). \end{aligned}$$

Indeed, this is a consequence of (3.7) combined with Observation (i) and the fact that the Dalang-Walsh integral appearing in $u_{n+1}(t,x)$ can be written as a divergence. Nevertheless, it will be more convenient to write

$$D^{\eta}_{s,y}u_{n+1}(t,x) = p_{t-s}(x-y)\sigma(u_n(s,y)) \\ + \int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_\eta(r,z)D^{\eta}_{s,y}u_n(r,z)\,\eta(dr,dz).$$

As the expression for $D^{\eta}_{s,y}u_{n+1}(t,x)$ contains two terms, we will study each of them separately. Let $T>0$. In order to control the term $p_{t-s}(x-y)\sigma(u_n(s,y))$, we will prove

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\|p_{t-\bullet}(x-*)\sigma(u_n(\bullet,*))\|^2_{L^p(\Omega;\mathfrak{H}_2)} < \infty. \tag{3.23}$$

Indeed, by Minkowski's integral inequality

$$\|p_{t-\bullet}(x-*)\sigma(u_n(\bullet,*))\|^2_{L^p(\Omega;\mathfrak{H}_2)} = \left\|\int_0^t \int_{\mathbb{R}} p^2_{t-s}(x-y)\sigma^2(u_0(s,y))\,dy\,ds\right\|_{p/2} \\ \leq \int_0^t \int_{\mathbb{R}} p^2_{t-s}(x-y)\|\sigma(u_0(s,y))\|^2_p\,dy\,ds,$$

As shown in the proof of Theorem 3.2.1, there is a constant C depending on T, p and the function $\sigma(x)$, so that $\|u_0(s,y)\|^2_p \leq Cb(s)$, for all $y\in\mathbb{R}$, where $b(s)=\frac{1}{\sqrt{s}}$. Then, Observation 3.2.3(b) implies the same inequality holds for $\|\sigma(u_0(s,y))\|^2_p$. In this way, Observation 3.2.3(a) implies

$$\|p_{t-\cdot}(x-*)\sigma(u_n(\cdot,*))\|^p_{L^p(\Omega;\mathfrak{H}_2)} \leq C\int_0^t \frac{b(s)}{\sqrt{4\pi}\sqrt{t-s}}\,ds.$$

Since explicit calculation show $\int_0^t \frac{b(s)}{\sqrt{4\pi}\sqrt{t-s}}\,ds$ is bounded for $t\in[0,T]$, we deduce (3.23).

The term $\int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_\eta(r,z)D^{\eta}_{s,y}u_n(r,z)\,\eta(dr,dz)$ is a little bit different to control. Burkholder's inequality for Hilbert-valued process (see *e.g* [23, page 212]) and Minkowski's integral inequality, yield

$$\left\| \int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_\eta(r,z) D^\eta_{\bullet,*} u_n(r,z)\, \eta(dr,dz) \right\|^2_{L^p(\Omega;\mathfrak{H}_2)} \leq C_p L^2 \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x-z) \times \left\| D^\eta_{s,y} u_n(r,z) \right\|^2_{L^p(\Omega;\mathfrak{H}_2)} dz\, dr. \tag{3.24}$$

Then

$$\left\| D^\eta_{\bullet,*} u_{n+1}(t,x) \right\|^2_{L^p(\Omega;\mathfrak{H}_2)} \leq 2 \| p_{t-\bullet}(x-*)\sigma(u_n(\bullet,*)) \|^2_{L^p(\Omega;\mathfrak{H}_2)} + \left\| \int_0^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma_\eta(r,z) D^\eta_{s,y} u_n(r,z)\, \eta(dr,dz) \right\|^2_{L^p(\Omega;\mathfrak{H}_2)}.$$

In particular, (3.23) and (3.24) imply

$$\left\| D^\eta_{\bullet,*} u_{n+1}(t,x) \right\|^2_{L^p(\Omega;\mathfrak{H}_2)} \leq C + C \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x-z) \left\| D^\eta_{s,y} u_n(r,z) \right\|^2_{L^p(\Omega;\mathfrak{H}_2)} dz\, dr \tag{3.25}$$

for all $t \in [0,T]$, where the constant C depends on T, p and the function $\sigma(x)$. Set

$$V_n(t) := \sup_{x \in \mathbb{R}} \left\| D^\eta_{\bullet,*} u_n(t,x) \right\|^2_{L^p(\Omega;\mathfrak{H}_2)}.$$

Note that (3.23) implies $V_1(t)$ is uniformly bounded. Furthermore, (3.25) and Observation 3.2.3(a) imply

$$V_{n+1}(t) \leq C + C \int_0^t \frac{1}{\sqrt{t-r}} V_n(t)\, dr.$$

Iterating the above inequality, and using Corollary 3.2.6 shows that $\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} V_n(t) < \infty$. This proves (3.22). Therefore, $u(t,x) \in \mathbb{D}^{1,p}_\eta$ for all $p \geq 2$. The desired formula for $D^\eta u_(t,x)$ follows from applying D^η to both sides of (3.2) together with similar arguments to those used in Observation (i) and (ii) at the beginning of this proof. □

The remaining of this section is dedicated to establishing fundamental estimates for the Malliavin derivatives of the mild solution to (3.1). However, let us make one important comment about these upcoming results. Until now, we have relied on Lemma 3.2.5 and Corollary 3.2.6 to iterate recursive inequalities and obtain the summability of the expressions appearing there. Well, obtaining the L^p estimates for the Malliavin derivative will continue with this sort of argument, but we will need other results to replace Lemma 3.2.5 and Corollary 3.2.6. These results correspond to Proposition 3.3.4 and Corollary 3.3.5.

Let us start by recalling a simple Lemma involving the Gamma function.

Lemma 3.3.3. *Let $\alpha,\beta > 0$. Then,*

$$\int_r^t (t-s)^{\alpha-1}(s-r)^{\beta-1}\,ds = (t-r)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Proof. The proof follows from using the change of variables $s = r+z(t-r)$, the definition of the beta function

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \;\text{ for } a,b>0,$$

and the well known relation $\beta(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. □

By combining mathematical induction, the previous Lemma and the following property of the Gaussian kernel

$$\int_{\mathbb{R}} p_{t-r}^2(x-z)p_{r-s}^2(z-y)\,dz = \frac{1}{\sqrt{4\pi}\sqrt{(t-r)}\sqrt{r-s}}\sqrt{t-s}\,p_{t-s}^2(x-y),$$

for any $0 \le s < r < t$ and $x,y \in \mathbb{R}$, we can obtain the following Proposition. We leave the proof as an exercise to the reader.

Proposition 3.3.4. *Set $h(s,y,t,x) = p_{t-s}^2(x-y)$ for $0 \le s < t$ and $x,y \in \mathbb{R}$. Define*

$$[Lh](s,y,t,x) = \int_s^t \int_{\mathbb{R}} p_{t-r}^2(x-z)h(s,y,r,z)\,dz\,dr,$$

and $[L^{n+1}h](s,y,t,x) = \int_s^t \int_{\mathbb{R}} p_{t-r}^2(x-z)[L^n h](s,y,r,z)\,dz\,dr,$ *for* $n \in \mathbb{N}$, $n \geq 1$. *Then, for all* $n \in \mathbb{N}$

$$[L^n h](s,y,t,x) = (\sqrt{4\pi})^{-n/2}(t-s)^{n/2}\frac{\Gamma(1/2)^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}h(s,y,t,x).$$

An important consequence of Proposition 3.3.4, is the following Corollary.

Corollary 3.3.5. *Fix* $T > 0$. *Let* $C \in \mathbb{R}$. *Then for any* $0 \leq s < t \leq T$ *and* $x, y \in \mathbb{R}$

$$\sum_{n=1}^{\infty} C^n [L^n h](s,y,t,x) \leq C_T h(s,y,t,x),$$

where $C_T = \sum_{n=1}^{\infty} (\sqrt{4\pi})^{-n/2} T^{n/2} C^n \frac{\Gamma(1/2)^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}$.

Proof. Note that for $0 \leq s < t < T$, we have $t - s \leq T$. Then, Proposition 3.3.4 implies

$$[L^n h](s,y,t,x) \leq (\sqrt{4\pi})^{-n/2} T^{n/2} \frac{\Gamma(1/2)^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)} h(s,y,t,x).$$

Thus we only need to show that

$$C_T = \sum_{n=1}^{\infty} (\sqrt{4\pi})^{-n/2} T^{n/2} C^n \frac{\Gamma(1/2)^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)},$$

is finite. However, this is an immediate consequence of say, the root test and Stirling's formula. □

We are finally ready to prove the estimates for the Malliavin derivative of $u(t,x)$.

Theorem 3.3.6. *For any* $p \in [2,\infty)$, $0 \leq t \leq T$ *and* $x \in \mathbb{R}$, *we have that for almost every* $(s,y) \in [0,T] \times \mathbb{R}$,

$$\left\| D_{s,y}^{\eta} u(t,x) \right\|_p \leq C p_{t-s}(x-y)\sqrt{b(s)}, \tag{3.26}$$

for some constant C *which depends on* T, p *and the function* σ.

Proof. The proof will be done in two steps.

Step 1:

Let $p \in [2,\infty)$. For $n \in \mathbb{N}$, set $g_n(s,y,t,x) := \|D^{\eta}_{s,y}u_n(t,x)\|_p^2$ where u_n is defined by the Picard iteration scheme introduced in (3.15). As seen in the proof of Theorem 3.3.1, we have $D^{\eta}_{s,y}u_0(t,x) = 0$, $D^{\eta}_{s,y}u_1(t,x) = p_{t-s}(x-y)\sigma(u_0(s,y))$. Therefore $g_0(s,y,t,x) = 0$ and $g_1(s,y,t,x) = p^2_{t-s}(x-y)\|\sigma(u_0(s,y))\|_p^2$. We deduce from Observation 3.2.3(b) and Step 1 in the proof of Theorem 3.2.1, the existence of a constant $C > 0$ depending on p and the function $\sigma(x)$ so that

$$g_1(s,y,t,x) \leq Cp^2_{t-s}(x-y)b(s).$$

Let $n \in \mathbb{N}$. We know

$$D^{\eta}_{s,y}u_{n+1}(t,x) = p_{t-s}(x-y)\sigma(u_n(s,y)) + \int_s^t \int_{\mathbb{R}} p_{t-r}(x-z)\Sigma^{(n)}_{\eta}(r,z)D^{\eta}_{s,y}u_n(r,z)\,\eta(dr\,dz),$$

where $\Sigma^{(n)}_{\eta}(r,z)$ is an adapted random field bounded by L. This, together with (3.3) imply the existence of a constant $C > 0$ (depending on p and the function $\sigma(x)$) such that

$$\|D^{\eta}_{s,y}u_{n+1}(t,x)\|_2^2 \leq Cp^2_{t-s}(x-y)\|\sigma(u_n(s,y))\|_p^2 + C\int_s^t \int_{\mathbb{R}} p^2_{t-r}(x-z)\|D^{\eta}_{s,y}u_n(r,z)\|_p^2\,dz\,dr.$$

Consequently, we deduce

$$\|D^{\eta}_{s,y}u_{n+1}(t,x)\|_2^2 \leq Cp^2_{t-s}(x-y)b(s) + C\int_s^t \int_{\mathbb{R}} p^2_{t-r}(x-z)\|D^{\eta}_{s,y}u_n(r,z)\|_p^2\,dz\,dr.$$

At this point, we rewrite the above inequality as

$$g_{n+1}(s,z,t,x) \leq Ch(s,y,t,x)b(s) + C[Lg_n](s,z,t,x), \tag{3.27}$$

where h and L are defined in Lemma 3.3.4 and $C > 0$ is a constant depending on T, p and the function $\sigma(x)$. Iterating (3.27) and using Observation 3.2.3(c) together with the fact that

$g_1(s,y,t,x) \lesssim h(s,y,t,x)$, we conclude

$$g_{n+1}(s,y,t,x) \leq Ch(s,y,t,x)b(s) + Cb(s)\sum_{j=1}^{n+1} C^j [L^j h](s,y,t,x),$$

where C depends only on T, p and the function $\sigma(x)$. Consequently, Corollary 3.3.5 implies

$$g_n(s,y,t,x) \leq C\, h(s,y,t,x)b(s) \text{ for all } n \in \mathbb{N}.$$

Substituting $g_n(s,y,t,x)$ and $h(s,y,t,x)$ for their equivalent expressions, yield

$$\|D^{\eta}_{s,y}u_n(t,x)\|_p^2 \leq Cp^2_{t-s}(x-y)b(s). \tag{3.28}$$

for a constant C depending on T,p and the function $\sigma(x)$. In this way, we have shown the respective version to Theorem 3.3.6 for the Picard iterations. This finishes Step 1.

Step 2:

Put $q = p/(p-1)$ which is the conjugate exponent for p. Let us pick a nonnegative function $M \in C_c(\mathbb{R}_+ \times \mathbb{R})$ and random variable $\mathscr{Z} \in L^q(\Omega)$ with $\|\mathscr{Z}\|_q \leq 1$. Since $D^{\eta}u_n(t,x)$ converges to $D^{\eta}u(t,x)$ in the weak topology on $L^p\big(\Omega;\mathfrak{H}_2\big)$, we have, in view of Step 1

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} M(s,y)\mathbb{E}\big[ZD^{\eta}_{s,y}u(t,x)\big]\,dy\,ds &= \lim_{n\to\infty}\int_{\mathbb{R}_+\times\mathbb{R}} M(s,y)\mathbb{E}\big[ZD^{\eta}_{s,y}u_n(t,x)\big]\,dy\,ds \\ &\leq C\int_{\mathbb{R}_+\times\mathbb{R}} M(s,y)p_{t-s}(x-y)\sqrt{b(s)}\,dy\,ds.\end{aligned}$$

This implies that for almost all $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\mathbb{E}\big[ZD^{\eta}_{s,y}u(t,x)\big] \leq Cp_{t-s}(x-y)\sqrt{b(s)}$$

Taking the supremum over $\{\mathscr{Z} : \|\mathscr{Z}\|_q \leq 1\}$ yields

$$\|D^{\eta}_{s,y}u(t,x)\|_p \leq Cp_{t-s}(x-y)\sqrt{b(s)},$$

which finishes the proof. □

We end this section with the corresponding estimate for the L^2-norm of D^{ξ}.

Theorem 3.3.7. *For any* $0 \leq t \leq T$ *and* $x \in \mathbb{R}$*, we have for almost every* $z \in \mathbb{R}$*,*

$$\left\|D^{\xi}_{z}u(t,x)\right\|_2 \leq Cp_t(x-z), \tag{3.29}$$

for some constant C *which depends on* T *and* L*.*

Proof. The proof is analogous to the proof of Theorem 3.3.6, with only minor modifications. For example, the Malliavin derivative D^{ξ} does not have s, but one can follow the same proof as in Theorem 3.3.6 by taking $s = 0$. More precisely, set $g_n(0,z,t,x) := \|D^{\xi}_{z}u_n(r,y)\|_2^2$. Then, $g_0(0,z,t,x) = p_t^2(x-z) = h(0,z,t,x)$. Furthermore, the same arguments leading to (3.27), now imply

$$g_{n+1}(0,z,t,x) \leq Ch(0,z,t,x) + C[Lg_n](0,z,t,x)$$

This leads to the desired inequality for the Picard iterations. From there, similar arguments to those in Step 2 from the proof of Theorem 3.3.6, yield the desired result. We leave the remaining details to the reader. □

3.4 Quantitative Central Limit Theorem

This section is dedicated to proving a quantitative central limit theorem for the process $F_R(t)$, where

$$F_R(t) := \int_{-R}^{R} u(t,x)\,dx, \tag{3.30}$$

and $u(t,x)$ is the random field from Theorem 3.2.1. Our precise statement is the following.

Theorem 3.4.1. *For every $t>0$, there exist a constant $c(t)$, such that*

$$d_{TV}\left(\frac{F_R(t)}{\sigma_R}, Z\right) \leq c(t) R^{-1/2},$$

where $\sigma_R = \sqrt{\text{Var}(F_R(t))}$ and Z has law $N(0,1)$.

The proof of this Theorem is postponed to Subsection 3.4.2. We do so, to introduce some technical results, which are used when proving Theorem 3.4.1. These results are presented in the next subsection.

3.4.1 Some Prerequisites

First, we need to establish the behavior of σ_R as $R\to\infty$. To this end, let us introduce some notation. For $R>0$ and $y\in\mathbb{R}$, we define

$$\varphi_{t,R}(s,y) := \int_{-R}^{R} p_{t-s}(x-y)\,dx. \tag{3.31}$$

As it happens, studying σ_R as $R\to\infty$ can be achieved by understanding $\langle \varphi_{t,R}(s,\bullet), \varphi_{t,R}(s,\bullet)\rangle_{L^2(\mathbb{R})}$. This is the content of Lemma 3.4.2.

Lemma 3.4.2. *Let $t_1,t_2>0$ and let $0\leq s<t_1\wedge t_2$. We have*

$$\int_{\mathbb{R}} \varphi_{t_1,R}(s,y)\varphi_{t_2}(s,y)\,dy = 2\int_0^{2R} p_{t_1+t_2-2s}(z)(2R-z)\,dz.$$

Proof. We start by noticing that $p_{t_2-s}(x'-y) = p_{t_2-s}(y-x')$. Then, Tonelli's theorem and the semigroup property of the Gaussian kernel, imply

$$\begin{aligned}\int_{\mathbb{R}} \varphi_{t_1,R}(s,y)\varphi_{t_2}(s,y)\,dy &= \int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t_1-s}(x-y)p_{t_2-s}(x'-y)dx\,dx'dy \\ &= \int_{-R}^{R}\int_{-R}^{R}\int_{\mathbb{R}} p_{t_1-s}(x-y)p_{t_2-s}(y-x)dy\,dx\,dx' \\ &= \int_{-R}^{R}\int_{-R}^{R} p_{t_1+t_2-2s}(x-x')dx\,dx'.\end{aligned}$$

By using the change of variables $u = x - x'$ and $v = x + x'$, we obtain

$$\int_{\mathbb{R}} \varphi_{t_1,R}(s,y)\varphi_{t_2}(s,y)\,dy = \frac{1}{2}\int\int_{R_{uv}} p_{t_1+t_2-2s}(u)\,dvdu,$$

where R_{uv} is the square with vertices $(2R,0)$, $(0,2R)$, $(-2R,0)$ and $(0,-2R)$. At this point, we note that both the region R_{uv} and the function $p_{t_1+t_2-2s}(u)$ are symmetric. Hence, we can integrate only in the part where $u,v \geq 0$ and multiply by 4. Thus,

$$\begin{aligned}\int_{\mathbb{R}} \varphi_{t_1,R}(s,y)\varphi_{t_2}(s,y)\,dy &= 4\cdot\frac{1}{2}\int_0^{2R}\int_0^{2R-u} p_{t_1+t_2-2s}(u)\,dv\,du \\ &= 2\int_0^{2R} p_{t_1+t_2-2s}(u)(2R-u)\,du.\end{aligned}$$

□

A simple application of Lemma 3.4.2, leads to the following Proposition.

Proposition 3.4.3. *For any* $t_1, t_2 > 0$

$$\lim_{R\to\infty}\frac{1}{R}\mathbb{E}\left[F_R(t_1)F_R(t_2)\right] = 2 + 2\int_0^{t_1\wedge t_2}\rho(r)\,dr,$$

where $\rho(r) = \mathbb{E}(\sigma^2(u(r,y)))$*. In particular, for all* $t > 0$ *we have that* $\sigma_R^2 \sim R$ *as* $R \to \infty$*.*

Proof. Recall that

$$F_R(t_1) = \int_{\mathbb{R}} \varphi_{t_1,R}(0,y)W(dy) + \int_0^{t_1}\int_{\mathbb{R}} \varphi_{t_1,R}(r,z)\sigma(u(r,z))\eta(dr\,dz),$$

and similarly $F_R(t_2)$. Further, each stochastic integral satisfies an isometry and they are uncorrelated because η and W were independent. It follows

$$\frac{1}{R}\mathbb{E}[F_R(t_1)F_R(t_2)] = \frac{1}{R}\int_{\mathbb{R}} \varphi_{t_1,R}(0,y)\varphi_{t_2,R}(0,y)\,dy + \int_0^{t_1\wedge t_2}\left(\frac{1}{R}\int_{\mathbb{R}} \varphi_{t_1,R}(0,y)\varphi_{t_2,R}(0,y)\,dy\right)\mathbb{E}[\sigma^2(u(s,y))]\,ds,$$

where we are using Remark 3.2.7(ii) to move $\mathbb{E}[\sigma^2(u(s,y))]$ outside the integral in y. Moreover, by our previous Lemma

$$\lim_{R\to\infty}\frac{1}{R}\int_{\mathbb{R}} \varphi_{t_1,R}(0,y)\varphi_{t_2,R}(0,y)\,dy = 2\int_0^\infty p_{t_1+t_2-2s}(u)\cdot 2\,du = 2.$$

This together with the dominated convergence theorem gives the first assertion of the result. The second assertion follows directly from the first. □

We end this Subsection with the following Lemma, which is a technical result whose purpose is to simplify some calculations in the proof of Theorem 3.4.1.

Lemma 3.4.4. *Let* $0 \le s < r < t$. *Set*

$$S_{s,r,t}(R) := \int_{\mathbb{R}^2}\int_{\mathbb{R}} \varphi_{t,R}(s,y)\varphi_{t,R}(s,y')\varphi_{t,R}^2(r,z)p_{r-s}(z-y)p_{r-s}(z-y')dzdydy'.$$

Then, $S_{s,r,t}(R) \le 2R$ *for all* $R > 0$. *In fact, the same inequality holds for*

$$T_{r,s,t}(R) := \int_{\mathbb{R}^2}\int_{\mathbb{R}} \varphi_{t,R}^2(s,y)\varphi_{t,R}^2(s,y')p_{s-r}(y-z)p_{s-r}(y'-z)dzdydy',$$

when $0 \le r < s < t$.

Proof. For the sake of simplicity, we will only prove the inequality for $S_{s,r,t}(R)$, the proof of

$T_{r,s,t}(R)$ being similar. In view of (3.31), we can write

$$S_{s,r,t}(R) = \int_{\mathbb{R}^2}\int_{[-R,R]^4}\int_{\mathbb{R}} p_{t-s}(x-y)p_{t-s}(x'-y')p_{t-r}(\tilde{x}-z)$$
$$\times\, p_{t-r}(\tilde{x}'-z)p_{r-s}(z-y)p_{r-s}(z-y')dzdxdx'd\tilde{x}d\tilde{x}'dydy'.$$

Now, we interchange the order of integration to $d\tilde{x}d\tilde{x}'dy'dydzdxdx'$. Then, we make the following observations

(i) The integrals of $p_{t-r}(\tilde{x}-z)$ and $p_{t-r}(\tilde{x}'-z)$ with respect to $\tilde{x}, \tilde{x}'$ are bounded by 1 because we can replace $[-R,R]$ by $\mathbb{R}$.

(ii) When we integrate y, y', we can use the semigroup property. In fact, we can use this property again when we integrate z.

In this way,

$$S_{s,r,t}(R) \leq \int_{[-R,R]^2} p_{2t+2r-4s}(x-x')\,dx\,dx'.$$

Integrating x over $\mathbb{R}$ and then x' over $[-R,R]$ yields the desired result. □

We are finally ready to prove our main result from this Section.

3.4.2 Proof of Theorem 3.4.1.

Let $t>0$.Thanks to Fubini's theorem for stochastic integration, and in view of (3.31), we can write

$$F_R(t) = \int_{\mathbb{R}} \varphi_{t,R}(0,y)\xi(dy) + \int_0^t \int_{\mathbb{R}} \varphi_{t,R}(r,z)\sigma(u(r,z))\eta(dr\,dz).$$

In fact, by Lemma 3.1.5 and Remark 3.1.8(a), we can express

$$F_R(t) = \delta^{\xi}(v_R(t)) + \delta^{\eta}(u_R(t)),$$

where $v_R(t) = \varphi_{t,R}(0,y)$ and $u_R(t) = 1_{[0,t]}\varphi_{t,R}(r,z)\sigma(u(r,z))$. Proposition 3.1.10, implies

$$d_{TV}\left(\frac{F_R(t)}{\sigma_R}, Z\right) \leq \frac{2\sqrt{2}}{\sigma_R^2}\sqrt{\mathrm{Var}\left[\left\langle D^\xi F_R, v_R\right\rangle_{\mathfrak{H}_1}\right] + \mathrm{Var}\left[\langle D^\eta F_R, u_R\rangle_{\mathfrak{H}_2}\right]}. \tag{3.32}$$

In view of the basic inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, we deduce from (3.32) that

$$d_{TV}\left(\frac{F_R(t)}{\sigma_R}, Z\right) \leq \frac{2\sqrt{2}}{\sigma_R^2}\left(\sqrt{\mathrm{Var}\left[\left\langle D^\xi F, v\right\rangle_{\mathfrak{H}_1}\right]} + \sqrt{\mathrm{Var}\left[\langle D^\eta F, u\rangle_{\mathfrak{H}_2}\right]}\right).$$

On the other hand, Proposition 3.4.3 implies $\sigma_R^2 \sim R$ as $R \to \infty$. Therefore, it suffices to show the existence of two constants $c_1(t)$ and $c_2(t)$ so that

$$\sqrt{\mathrm{Var}\left[\left\langle D^\xi F_R(t), v_R(t)\right\rangle_{\mathfrak{H}_1}\right]} \leq c_1(t)\sqrt{R}, \tag{3.33}$$

$$\sqrt{\mathrm{Var}\left[\langle D^\eta F_R(t), u_R(t)\rangle_{\mathfrak{H}_2}\right]} \leq c_2(t)\sqrt{R}, \tag{3.34}$$

Let us start with the part corresponding to D^ξ. For this case, $v_R(t) = \varphi_{t,R}(0,y)$ and

$$D_z^\xi F_R(t) = \varphi_{t,R}(0,z) + \int_0^t \int_{\mathbb{R}} \varphi_{t,R}(s,y)\Sigma^\xi(s,y) D_z^\xi u(s,y)\eta(ds,dy),$$

Consequently,

$$\begin{aligned}\left\langle D^\xi F_R(t), v_R(t)\right\rangle_{\mathfrak{H}_1} &= \int_{\mathbb{R}} \varphi_{t,R}^2(0,z)\,dz \\ &\quad + \int_{\mathbb{R}} \varphi_{t,R}(0,z)\left(\int_0^t \int_{\mathbb{R}} \varphi_{t,R}(s,y)\Sigma^\xi(s,y) D_z^\xi u(s,y)\eta(ds,dy)\right) dz.\end{aligned}$$

Note that since the first term is deterministic, it has variance equal to 0. Thus, we only need to consider

$$\mathrm{Var}\left(\int_{\mathbb{R}} \varphi_{t,R}(0,z)\left(\int_0^t \int_{\mathbb{R}} \varphi_{t,R}(r,y)\Sigma_\xi(r,y) D_z^\xi u(r,y)\eta(dr,dy)\right) dz\right).$$

Then, the isometry of the stochastic integral together with Cauchy-Schwartz and the estimate

$\|D_z^\xi u(r,y)\|_2 \lesssim p_r(y-z)$ (Theorem 3.3.7), imply

$$
\begin{aligned}
&\mathrm{Var}\left(\int_{\mathbb{R}} \varphi_{t,R}(0,z)\left(\int_0^t \int_{\mathbb{R}} \varphi_{t,R}(r,y)\Sigma_\xi(r,y)D_z^\xi u(r,y)\eta(dr,dy)\right)dz\right)\\
&= \int_{\mathbb{R}^2} \varphi_{t,R}(0,z)\varphi_{t,R}(0,z')\int_0^t\int_{\mathbb{R}} \varphi_{t,R}^2(r,y)\mathbb{E}\left[\left|\Sigma_\xi^2(r,y)D_z^\xi(r,y)D_{z'}^\xi(r,y)\right|\right] dy\,dr\,dz\,dz'\\
&\le L^2\int_{\mathbb{R}^2} \varphi_{t,R}(0,z)\varphi_{t,R}(0,z')\int_0^t\int_{\mathbb{R}} \varphi_{t,R}^2(r,y)\|D_z^\xi(r,y)\|_2\|D_{z'}^\xi(r,y)\|_2 dy\,dr\,dz\,dz'\\
&\le C\int_0^t\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}} \varphi_{t,R}(0,z)\varphi_{t,R}(0,z')\varphi_{t,R}^2(r,y)p_r(y-z)p_r(y-z')dy\,dz\,dz'\right)ds
\end{aligned}
$$

for some constant C depending on t and the function $\sigma(x)$. Since the expression inside the parenthesis corresponds to the function $S_{0,r,t}(R)$ introduced in Lemma 3.4.4, this same Lemma implies

$$
\mathrm{Var}\left(\int_{\mathbb{R}} \varphi_{t,R}(0,z)\left(\int_0^t \int_{\mathbb{R}} \varphi_{t,R}(s,y)\Sigma_\xi(s,y)D_z^\xi u(s,y)\eta(ds,dy)\right)dz\right) \le 2CRt,
$$

which leads to (3.33).

We now study the term corresponding to D^η. The proof is similar to the proof of Theorem 1.1 in [16], although some adjustments are necessary. First, unlike [16], we do not have the condition

$$
\sup_{0\le s\le t}\sup_{y\in\mathbb{R}}\|\sigma(u(s,y))\|_p < \infty,
$$

in our setting. Additionally, the function $b(s) = \frac{1}{\sqrt{s}}$ appears in our estimates for the L^p-norm of $D_{s,y}u(t,x)$, whereas the estimate corresponding estimate in [16] (Theorem A.1) involves only the fundamental heat solution.

Recall that for this case, $v_R(s,y) = 1_{[0,t]}(s)\varphi_{t,R}(s,y)\sigma(u(s,y))$ and

$$
D_{s,y}^\eta F_R = \varphi_{t,R}(s,y) + \int_s^t \int_{\mathbb{R}} \varphi_{t,R}(s,z)\Sigma^\eta(s,z)D_{s,y}^\eta u(s,z)\eta(ds,dz),
$$

where $\Sigma^\eta(r,z)$ is a bounded random field. It follows,

$$\langle D^\eta F_R, v_R\rangle_{\mathfrak{H}_2} = \int_0^t \int_{\mathbb{R}} \varphi_{t,R}(s,z)^2 \sigma^2(u(s,z))\, dz ds$$
$$+ \int_0^t \int_{\mathbb{R}} \varphi_{t,R}(s,y)\sigma(u(s,y)) \left(\int_s^t \int_{\mathbb{R}} \varphi_{t,R}(r,z)\Sigma^\eta(r,z) D^\eta_{s,y}u(r,z)\eta(dr,dz)\right) dyds.$$

Then Remark 3.1.8(b), implies

$$\sqrt{\mathrm{Var}\left[\langle D^\eta F_R, v_R\rangle_{\mathfrak{H}_2}\right]} \le A_1 + A_2, \tag{3.35}$$

where

$$A_1 = \int_0^t \left(\int_{\mathbb{R}^2} \varphi_{t,R}(s,y)^2 \varphi_{t,R}(s,y') \mathrm{Cov}[\sigma^2(u(s,y))\sigma^2(u(s,y'))]\, dy dy'\right)^{1/2} ds$$

and

$$A_2 = \int_0^t \left(\int_{\mathbb{R}^2} \varphi_{t,R}(s,y)\varphi_{t,R}(s,y') \right.$$
$$\left. \times \int_s^t \int_{\mathbb{R}} \varphi^2_{t,R}(r,z)\mathbb{E}[\sigma(u(s,y))\sigma(u(s,y'))\Sigma^2_\eta(r,z) D^\eta_{s,y}u(r,z) D^\eta_{s,y'}u(r,z)] dzdrdydy'\right)^{1/2} ds.$$

At this point, we divide the proof in two steps.

Step 1:

Let us estimate the term A_1. In fact, we start by estimating $\mathrm{Cov}[\sigma^2(u(s,y))\sigma^2(u(s,y'))$. It is for this term that we need the Poincaré type inequality and the chain rule. More precisely, Corollary 3.1.7, implies

$$\mathrm{Cov}[\sigma^2(u(s,y))\sigma^2(u(s,y')) \le \int_0^\infty \int_{\mathbb{R}} \|D^\eta_{r,z}\sigma^2(u(s,y)\|_2 \|D^\eta_{r,z}\sigma^2(u(s,y')\|_2\, dzdr.$$

However, by the chain rule

$$D^{\eta}_{r,z}\sigma^2(u(s,y) = 2\sigma(u(s,y))\Sigma^{\eta}(s,y)D^{\eta}_{r,z}u(s,y),$$

where the random field is uniformly bounded. Additionally, we have a similar expression for $D^{\eta}_{r,z}\sigma^2(u(s,y'))$. An application of Cauchy-Schwartz leads to

$$\begin{aligned}
&\mathrm{Cov}[\sigma^2(u(s,y))\sigma^2(u(s,y'))\\
&\leq L^2\int_0^{\infty}\int_{\mathbb{R}}\|\sigma(u(s,y))\|_4\|\sigma(u(s,y'))\|_4\|D^{\eta}_{r,z}u(s,y)\|_4\|D^{\eta}_{r,z}u(s,y)\|_4\,dz\,dr\\
&\leq L^2\int_0^{s}\int_{\mathbb{R}}\|\sigma(u(s,y))\|_4\|\sigma(u(s,y'))\|_4\|D^{\eta}_{r,z}u(s,y)\|_4\|D^{\eta}_{r,z}u(s,y)\|_4\,dz\,dr,
\end{aligned}$$

where for the last step, we used $D_{r,z}u(s,y)=0$ if $r>s$. Note that in the context of the term A_1, we have $s<t$. Therefore, for the term A_1, we are considering $0\leq r<s<t$. Then, in view of Remark 3.2.7 and Theorem 3.3.6, we conclude

$$\mathrm{Cov}[\sigma^2(u(s,y))\sigma^2(u(s,y'))\leq Cb(s)\int_0^s\int_{\mathbb{R}}p_{s-r}(y-z)p_{s-r}(y'-z)b(r)\,dz\,dr. \tag{3.36}$$

where the constant C depends on t and the function $\sigma(x)$. We are finally ready to estimate A_1.

In view of (3.36) and Lemma 3.4.4, we have

$$\begin{aligned}
A_1 &= \int_0^t\left(\int_{\mathbb{R}^2}\varphi^2_{t,R}(s,y)\varphi^2_{t,R}(s,y')\mathrm{Cov}[\sigma^2(u(s,y))\sigma^2(u(s,y'))]\,dy\,dy'\right)^{1/2}ds\\
&\leq C\int_0^t\left(\int_{\mathbb{R}^2}\varphi_{t,R}(s,y)^2\varphi_{t,R}(s,y')b(s)\int_0^s\int_{\mathbb{R}}p_{s-r}(y-z)p_{s-r}(y'-z)b(r)\,dz\,dr\,dy\,dy'\right)^{1/2}ds\\
&= C\int_0^t\sqrt{b(s)}\left(\int_0^s b(r)T_{r,s,t}\,dr\right)^{1/2}ds\leq C\sqrt{2R}\int_0^t\sqrt{b(s)}\left(\int_0^s b(r)\,dr\right)^{1/2}ds.
\end{aligned}$$

As a manner of fact, explicit calculations show that

$$\int_0^t\sqrt{b(s)}\left(\int_0^s b(r)\,dr\right)^{1/2}ds<\infty.$$

This leads to $A_1 \lesssim \sqrt{R}$ as desired.

Step 2:

Now we estimate the term A_2. We start by applying Hölder's inequality to obtain

$$\begin{aligned}&\mathbb{E}[\sigma(u(s,y))\sigma(u(s,y'))\Sigma_\eta^2(r,z)D^\eta_{s,y}u(r,z)D^\eta_{s,y'}u(r,z)]\\&\leq L^2\|\sigma(u(s,y))\|_4\|\sigma(u(s,y'))\|_4\|D^\eta_{s,y}u(r,z)\|_4\|D^\eta_{s,y'}u(r,z)\|_4.\end{aligned}$$

Then,

$$\begin{aligned}A_2 \leq &L\int_0^t \left(\int_s^t \int_{\mathbb{R}^2}\int_{\mathbb{R}} \varphi_{t,R}(s,y)\varphi_{t,R}(s,y')\varphi^2_{t,R}(r,z)\right.\\&\left.\times\|\sigma(u(s,y))\|_4\|\sigma(u(s,y'))\|_4\|D^\eta_{s,y}u(r,z)\|_4\|D^\eta_{s,y'}u(r,z)\|_4 dzdydy'\,dr\right)^{1/2} ds.\end{aligned}$$

We note that since $0 \leq s < r < t$, Remark 3.2.7 and Theorem 3.3.6, imply

$$\mathbb{E}[\sigma(u(s,y))\sigma(u(s,y'))\Sigma_\eta^2(r,z)D^\eta_{s,y}u(r,z)D^\eta_{s,y'}u(r,z)] \leq Cb^2(s)p_{r-s}(y-z)p_{r-s}(z-y').$$

for a constant C depending on t and the function $\sigma(x)$. It follows,

$$A_2 \lesssim \int_0^t b(s)\left(\int_s^t \left[\int_{\mathbb{R}^2}\int_{\mathbb{R}} \varphi_{t,R}(s,y)\varphi_{t,R}(s,y')\varphi^2_{t,R}(r,z)p_{r-s}(y-z)p_{r-s}(z-y')dzdydy'\right] dr\right)^{1/2} ds.$$

Again, we use Lemma 3.4.4 to obtain

$$A_2 \lesssim \sqrt{2R}\int_0^t b(s)\sqrt{t-s}\,ds.$$

Explicit calculations show $\int_0^t b(s)\sqrt{t-s}\,ds < \infty$. Consequently, $A_2 \lesssim \sqrt{R}$. This, together with Step 1 and (3.35), prove (3.34). The proof is now complete.

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