Micropolar non-classical continuum theories for solids and fluids with rotational inertial physics

By

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Nomenclatures

α_1^*	Diffusion coefficient in Model RW2
α_1, α_2	Material coefficient for constitutive theory for Cauchy moment tensor
$\bar{\alpha}$	non-classicalphysics material coefficient
α_0	Dimensionless non-classicalphysics material coefficient
$\bar{\beta}$	Rotational inertial physics non-classicalmaterial coefficient
$\bar{c}_2$	Dimensionless dissipation coefficient fluids
c_d	Damping coefficient Model TW1
c_d^*	Damping coefficient Model RW1, RW2
E	Modulus of elasticity
E_0	Reference modulus of elasticity
e	Specific internal energy in Lagrangian description
$\bar{e}$	Specific internal energy in Eulerian description
ϵ, $\boldsymbol{\epsilon}$, ϵ_{ijk}	Permutation tensor
$\boldsymbol{\varepsilon}$	Strain tensor in Lagrangian description
$\bar{\boldsymbol{\varepsilon}}$	Strain tensor in Eulerian description
$\dot{\boldsymbol{\varepsilon}}$	Strain rate tensor in Lagrangian description
$\dot{\bar{\boldsymbol{\varepsilon}}}$	Strain rate tensor in Eulerian description
η	Viscosity or entropy density in Lagrangian description
$\bar{\eta}$	Viscosity or entropy density in Eulerian description
$\boldsymbol{g}$	Heat vector in Lagrangian description
$\bar{\boldsymbol{g}}$	Heat vector in Eulerian description

$\bar{h}$	Entropy flux in Eulerian description
${}^{\Theta}I$	Internal rotational inertia in Lagrangian description
${}^{\Theta}\bar{I}$	Internal rotational inertia in Eulerian description
$\boldsymbol{J}$	Deformation gradient tensor in Lagrangian description
${}_{s}\boldsymbol{J}$	Symmetric part of deformation gradient tensor in Lagrangian description
${}_{a}\boldsymbol{J}$	Skew Symmetric part of deformation gradient tensor in Lagrangian description
${}^{d}\boldsymbol{J}$	Displacement gradient tensor in Lagrangian description
${}^{d}_{s}\boldsymbol{J}$	Symmetric part of displacement gradient tensor in Lagrangian description
${}^{d}_{a}\boldsymbol{J}$	Skew symmetric part of displacement gradient tensor in Lagrangian description
L, L_0	Length, reference length
$\boldsymbol{L}$, $\dot{\boldsymbol{J}}$	Velocity gradient tensor in Lagrangian description
$\bar{\boldsymbol{L}}$	Velocity gradient tensor in Eulerian description
${}_{s}m_{11}$	Symmetric Cauchy moment tensor in Lagrangian description component 11
${}_{s}\bar{m}_{11}$	Symmetric Cauchy moment tensor in Eulerian description component 11
${}_{s}\bar{m}_{23}$	Symmetric Cauchy moment tensor in Eulerian description component 23
${}_{a}\bar{m}_{23}$	Skew symmetric Cauchy moment tensor in Eulerian description component 23
${}_{i}\boldsymbol{\omega}$, ${}_{i}\omega_k$	Internal rotational velocity or rotation rates
${}_{i}\omega_1$	Internal rotation rate in Lagrangian description about the x_1 axis
ω_0	Reference rotational velocity
${}_{i}\bar{\omega}_1$	Internal rotation rate in Eulerian description about the x_1 axis
${}_{i}\bar{\omega}_3$	Internal rotation rate in Eulerian description about the x_3 axis
$\bar{\Omega}_{xt}$	Space-time domain
$\bar{\Omega}_{xt}^{T}$	Discretization of space-time domain $\bar{\Omega}_{xt}$
$\bar{\Omega}_{xt}^{(i)}$	i^{th} space-time strip
$(\bar{\Omega}_{xt}^{(i)})^{T}$	Discretization of i^{th} space-time strip
$\bar{\Omega}_{xt}^{e}$	Space-time element e
ρ_0	Reference Density
ρ	Density in Lagrangian description

$\bar{\rho}$	Density in Eulerian description
$\bar{p}$	Pressure in Eulerian description
$\boldsymbol{\sigma}$	Cauchy stress tensor in Lagrangian description
${}_s\boldsymbol{\sigma}$	Symmetric Cauchy stress tensor in Lagrangian description
${}_a\boldsymbol{\sigma}$	Skew symmetric Cauchy stress tensor in Lagrangian description
${}^e_s\boldsymbol{\sigma}$	Symmetric equilibrium Cauchy stress tensor in Lagrangian description
${}^d_s\boldsymbol{\sigma}$	Symmetric deviatoric Cauchy stress tensor in Lagrangian description
${}^d_s\bar{\sigma}^{(0)}_{21} = {}^d_s\bar{\sigma}^{(0)}_{12}$	Symmetric deviatoric contravariant Cauchy stress in Eulerian description component 12
${}^d_a\bar{\sigma}^{(0)}_{21} = {}^d_a\bar{\sigma}^{(0)}_{12}$	Antisymmetric deviatoric contravariant Cauchy stress in Eulerian description component 12
s	Entropy source in Lagrangian description
$\bar{s}$	Entropy source in Eulerian description
t	Time
t_0	Reference time
τ_0	Reference stress
${}_i\boldsymbol{\Theta}$, ${}_i\Theta_j$, $\{{}_i\Theta\}$	Internal or classical rotations in Lagrangian description
${}_i\bar{\boldsymbol{\Theta}}$, ${}_i\bar{\Theta}_j$, $\{{}_i\bar{\Theta}\}$	Internal or classical rotations in Eulerian description
${}^r_i\boldsymbol{\Theta}$	Internal or classical rotation rates in Lagrangian description
${}^r_i\bar{\boldsymbol{\Theta}}$	Internal or classical rotation rates in Eulerian description
${}_i\Theta_1$	Internal rotation about the x_1 axis
${}_i\Theta_3$	Internal rotation about the x_3 axis
$\bar{u}$, $\bar{v}$, $\bar{w}$	Velocity components in x_1, x_2 and x_3 directions
v_0	Reference velocity
$\boldsymbol{v}$, v_i, $\{v\}$	Velocity vector in Lagrangian description
$\bar{\boldsymbol{v}}$, $\bar{v}_i$, $\{\bar{v}\}$	Velocity vector in Eulerian descriptions
$\boldsymbol{x}$, x_i, $\{x\}$	Cartesian Coordinates
$\bar{\boldsymbol{x}}$, $\bar{x}_i$, $\{\bar{x}\}$	Cartesian Coordinates

Abbreviations

BAM : Balance of angular momentum

BLM : Balance of linear momentum

BMM : Balance of moment of moments

BVP : Boundary value problem

CBL : Conservation and balance laws

CCM : Classical continuum mechanics

CCT : Classical continuum theory

CM : Conservation of mass

FLT : First law of thermodynamics

IVP : Initial value problem

NCCM : Non-classical continuum mechanics

NCCT : Non-classical continuum theory

ODE : Ordinary differential equation

PDE : Partial differential equation

SLT : Second law of thermodynamics

TV : Thermoviscous

TVES : Thermoviscoelastic solids

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Abstract

In this thesis, we derive the conservation and balance laws (CBL) and the constitutive theories for micropolar non-classical continuum theories (NCCT) for thermoviscoelastic solids (TVES) without memory in Lagrangian description and thermoviscous (TV) fluids in Eulerian description based on internal or classical rotations (${}_{i}\boldsymbol{\Theta}$) and internal or classical rotation rates (${}_{i}^{r}\bar{\boldsymbol{\Theta}}$) due to skew symmetric part of deformation gradient tensor ($\boldsymbol{J}$) and due to skew symmetric part of the velocity gradient tensor ($\bar{\boldsymbol{L}}$). Thus, these micropolar NCCT for solids and fluids incorporate $\boldsymbol{J}$ and $\bar{\boldsymbol{L}}$ in their entirety in the derivation of the conservation and balance laws. Both NCCT derived here incorporate rotational inertial physics due to the presence of microconstituents in the derivation of CBL. For micropolar TVES, we consider small deformation, small strain physics. To our knowledge, the inclusion of rotational inertial physics in micropolar NCCT presented here is the first presentation of such a micropolar NCCT.

The micropolar NCCT presented in this work for TVES considers two dissipation mechanisms. The first is due to strain rate appearing in the constitutive theory for the deviatoric Cauchy stress tensor. This mechanism is purely due to classical continuum mechanics (CCM). The second dissipation mechanisms is due to the rate of symmetric part of the internal rotation gradient tensor, appearing in the constitutive theory for the symmetric part of the Cauchy moment tensor. The first dissipation mechanisms is viscous. The second dissipation mechanisms is due to the micropolar non-classical physics and it accounts for the drag forces experienced by the microconstituents during deformation of the volume of matter. In the case of micropolar fluids, the dissipation mechanism is due to symmetric part of the velocity gradient tensor appearing in the

constitutive theory for the deviatoric Cauchy stress tensor (CCM) and also due to the symmetric part of the rotation rate gradient tensor appearing in the constitutive theory for the symmetric part of the Cauchy moment tensor (NCCM). It is shown that the mathematical models for micropolar TVES and micropolar TV fluids consisting of the conservation and balance laws and constitutive theories have closure.

It is established and demonstrated that micropolar NCCT with rotational inertial physics derived here permits coexistence of translational waves (deviatoric Cauchy stress waves) and rotational waves (Cauchy moment waves). In the case of micropolar TV fluids with rotational inertial physics, neither translational nor rotational waves can exist. This is due to the lack of elasticity in classical as well as non-classical physics, as a consequence the balance of linear momenta and balance of angular momenta are time dependent diffusion equations in translational and rotational velocities and are not wave equations.

Simple model problems describing evolutions (IVPs) in micropolar TVES and micropolar TV fluids are considered and their numerical solutions are presented (computed using space-time coupled finite element method) to specifically illustrate the influence of rotational inertial physics on the resulting evolutions.

Chapter 1

Introduction

1.1 Introduction

The displacements $\boldsymbol{u}$ and the deformation gradient tensor $\boldsymbol{J}$ are fundamental measures of deformation physics in solids. Classical continuum theories (CCT) are primarily based on $\boldsymbol{u}$ and the symmetric part of $\boldsymbol{J}$ $({}_s\boldsymbol{J})$ used to describe various strain measures. The antisymmetric part of $\boldsymbol{J}$ $({}_a\boldsymbol{J})$ containing rotations at a material point is not considered in CCT even though it exists in all deforming solids (as it is due to $\boldsymbol{J}$). Thus, in CCT the rotations in $\boldsymbol{J}$ referred to as internal rotations or classical rotations constitute a free field i.e., they exist in all deforming solids but don't influence measures of deformation. Thus, it is possible to develop a NCCT that considers rotations ${}_i\boldsymbol{\Theta}$ in the conservation and balance laws and constitutive theories. This NCCT would consider $\boldsymbol{u}$ and $\boldsymbol{J}$ in their entirety. Surana et al. [83] have shown this NCCT is ideally suited to study isotropic, homogeneous solid matter in the presence of microconstituents. When microconstituents are present, the rotations no longer constitute a free field as they are resisted by the microconstituents. Surana et al. [83] referred to this NCCT as "Micropolar NCCT based on internal or classical rotations ${}_i\boldsymbol{\Theta}$ for solid medium". Prior to publication [83] Surana et. al have presented various aspects of CBL and constitutive theories for micropolar NCCT based on rotations ${}_i\boldsymbol{\Theta}$ [84, 85, 87, 89, 93–96]. These works show necessary modification of the CBL of CCT, introducing a new balance law "balance of moment of moments" (also see Yang et. al [104]) and derivation of constitutive theories for micropolar NCCT based on rotations ${}_i\boldsymbol{\Theta}$ such that the thermodynamic and mathematical consistency [83] of the NCCT is ensured. Rotational inertial physics is not considered in these works.

In case of fluids, velocity $\bar{\boldsymbol{v}}$ and velocity gradient tensor $\bar{\boldsymbol{L}}$ are the most fundamental measures of fluid motion. CCT for fluid medium are primarily based on $\bar{\boldsymbol{v}}$ and symmetric part of $\bar{\boldsymbol{L}}$ ($\bar{\boldsymbol{D}}$), the strain rate gradient tensor. In CCT, the antisymmetric part of $\bar{\boldsymbol{L}}$ ($\bar{\boldsymbol{W}}$) is not considered in the CBL and constitutive theories, even though it exists in all deforming fluids as it is due to ($\bar{\boldsymbol{L}}$). Thus, in CCT the rotation rates ${}_i^r\bar{\boldsymbol{\Theta}}$ (in $\bar{\boldsymbol{W}}$) also called classical rotation rates constitute a free field i.e., they exist in all deforming fluid media but do not influence measures used in CCT. Thus, it is possible to develop a NCCT that considers rotation rates ${}_i^r\bar{\boldsymbol{\Theta}}$ in the CBL and constitutive theories. This NCCT would consider $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{L}}$ in their entirety. Surana et. al [79] have shown that this NCCT is ideally suited to study isotropic, homogeneous fluid medium in the presence of microconstituents. When microconstituents are present, the rotation rates ${}_i^r\bar{\boldsymbol{\Theta}}$ no longer constitute a free field as they are resisted by the microconstituents. Surana et. al [79] referred to this NCCT as "Micropolar NCCT for fluids based on internal or classical rotation rates ${}_i^r\bar{\boldsymbol{\Theta}}$". Prior to publication [79], Surana et. al have presented various aspects of CBL and constitutive theories for micropolar NCCT based on rotation rates ${}_i^r\bar{\boldsymbol{\Theta}}$ [82, 86, 88, 90, 94]. These works showed necessary modifications of the CBL of CCT, introduction of a new balance law "balance of moment of moments" and derivation of constitutive theories such that the thermodynamic and mathematical consistency [79] of the NCCT is ensured. In these works also, rotational inertial physics is not considered.

1.2 Literature review

Prior to the works in references [82, 84, 86–90, 95, 96, 104] a large number of publications have appeared (primarily related to solid continua) under couple stress theories, microtheories (micropolar, microstretch, micromorphic) and their application to beams, plates and shells. Surana et. al have clearly distinguished these published works from the works presented by them in references [80, 82, 84, 86–90, 92, 93, 95, 96, 96]. In the following, we present a brief literature review of published works on non-classical continuum theories under the titles micropolar theories, non-local theories, couple stress theories, etc. We discuss relevance and correspondence of these theories to the present work after the literature review.

A comprehensive treatment of micropolar theories can be found in references [7, 14–29, 32, 43, 44, 59]. Such theories are designed to accommodate effects at scales smaller than continuum scale. Balance laws for micromorphic materials are presented in reference [59]. The micropolar theories consider micro-deformation due to microconstituents in the continuum. In references [63, 65, 68] by Reddy et. al. and reference [48] by Zang et. al. non-local theories are presented for bending, buckling and vibration of beams, beams with nanocarbon tubes and bending of plates. The theories related to the non-local effects are believed to be originated by Eringen [17] in which a definition of non-local stress tensor is introduced through integral relationship using the product of macroscopic stress tensor and a distance kernel representing non-local effects.

The concept of couple stress was first introduced by Voigt in 1881 by assuming existence of a couple or moment per unit area on the oblique plane of the deformed tetrahedron in addition to the stress or force per unit area. The concept of couple stresses is more recently studied by Koiter [44]. Since the introduction of this concept many published works have appeared. We cite some recent works, most of which are related to micropolar couple stress theories. Authors in reference [105] report experimental study of micropolar and couple stress elasticity of compact bones in bending. Conservation integrals in couple stress elasticity are reported in reference [49]. A microstructure-dependent Timoshenko beam model based on modified couple stress theories is reported by Ma et. al. [50]. Further accounts of couple stress theories in conjunction with beams can be found in references [51, 66, 67]. Treatment of rotation gradient dependent strain energy and its specialization to Von Kármán plates and beams can be found in reference [78]. Other accounts of micropolar elasticity and Cosserat modeling of cellular solids can be found in references [57, 60, 72]. In references [49–51, 57, 60, 66, 67, 72, 78, 105] authors use Lagrangian description to derive the mathematical description of deforming solid matter purely using strain energy density functional and principle of virtual work. This approach works well for elastic solids in which mechanical deformation is reversible. Extension of these works to thermoviscoelastic solids with and without memory is not possible. In such materials, the thermal field and mechanical deformation are coupled due to the fact that the rate of work results in rate of entropy production. In reference [1]

Altenbach and Eremeyev present a linear theory for micropolar plates. Each material point is regarded as a small rigid body with six degrees of freedom. Kinematics of plates is described using the vector of translations and the vector of rotations as dependent variables. Equations of equilibrium are established in $\mathbb{R}^3$ and $\mathbb{R}^2$. Strain energy density function is used to present linear constitutive theory. The mathematical models of reference [2] are extended by the same authors to present strain rate tensors and the constitutive equations for inelastic micropolar materials. In reference [3] authors consider the conditions for the existence of the acceleration waves in thermoelastic micropolar media. The work concludes that the presence of the energy equation with Fourier heat conduction law does not influence the wave physics in thermoelastic micropolar media. Thus, from the point of view of acceleration waves in thermoelastic polar media, thermal effects i.e., temperature can be treated as a parameter. In reference [4] authors present a discussion on a collection of papers related to the mechanics of continua dealing with micro-macro aspects of deformation physics (largely related to solid matter).

The works related to micropolar theories, non-local theories, couple stress theories of fluent continua and their applications can be found in references [4, 5, 7, 10, 14–16, 18, 19, 21, 25–29, 31, 32, 43, 54, 106]. The micropolar theories consider micro-deformation of microconstituents in the continuum and associated homogenization so that the matter at macro scale is isotropic and homogeneous. The works by Eringen [21, 25–28] establish conservation and balance laws, constitutive theories, micro-mechanics considerations and their use in non-classical theories for fluent continua. Some stability and boundary considerations for non-classical theories are discussed in references [32, 43]. In reference [5] a micropolar theory is presented for binary media with applications to phase transition of fiber suspensions to show flow during the filling state of injection molding of short fiber reinforced thermoplastics. A similarity solution for boundary problem flow of a polar fluid is given in reference [10]. In specific, the paper borrows constitutive equations that are claimed to be valid for flow behavior of a suspension of very fine particles in a viscous fluid. Kinematics of micropolar continuum is presented in reference [11]. References [12, 13] consider material symmetry groups for linear Cosserat continuum and nonlinear polar elastic continuum.

Grekova et. al. [36,37,39] consider various aspects of wave processes in ferromagnetic media and elastic media with micro-rotations as well as some aspects of linear reduced Cosserat media. In references [6,9,33–35,38,40,41,46,47,52–56,58,61,104,106] various aspects of the kinematics of micropolar theories, couple stress theories, etc. are discussed and presented including some applications to plates and shells. Unfortunately, there are not many published works on micropolar NCCT that consider rotational inertial physics. References [42,45] consider flow of micropolar fluids through porous medium and dynamics of hydro-magnetic flow of micropolar fluids. There is much more similar published work regarding the polar, couple stress and non-local theories utilizing concepts similar to those already discussed here.

In summary, at present there are three micropolar NCCT being pursued in the published works. In case of solids: (1) micropolar NCCT based on internal rotations ${}_i\boldsymbol{\Theta}$, (2) micropolar NCCT based on rotations ${}_i\boldsymbol{\Theta}$ and Cosserat or external or micro-rotations ${}_e\boldsymbol{\Theta}$ with additional three unknown degrees of freedom at a material point, (3) micropolar NCCT that only considers external rotations ${}_e\boldsymbol{\Theta}$ at a material point hence does not consider ${}_i\boldsymbol{\Theta}$ even though it exists in all deforming solids. In case of fluids, similar micropolar NCCT are being pursued using rotation rates ${}_i^r\bar{\boldsymbol{\Theta}}$, ${}_e^r\bar{\boldsymbol{\Theta}}$. Surana et. al [87,88] have shown that the micropolar NCCT based on ${}_i\boldsymbol{\Theta}$ for solids and micropolar NCCT based on ${}_i^r\bar{\boldsymbol{\Theta}}$ for fluids are the only two NCCT that are thermodynamically and mathematically consistent, hence we consider only these two NCCT in extending them for rotational inertial physics.

1.3 Scope of work

The conservation and balance laws and the constitutive theories of micropolar NCCT of Surana et. al [79,82–90,93–96] derived by incorporating internal rotations or classical rotations ${}_i\boldsymbol{\Theta}$ for solids and by incorporating internal or classical rotation rates for ${}_i^r\bar{\boldsymbol{\Theta}}$ for fluids are extended to include rotational inertial physics. Derivations of micropolar NCCT for solids and fluids with rotational inertial physics are presented in chapters 2 and 3. The new physics of rotational inertia results in further modification of balance of angular momenta, balance of moment of moments and first and seconds laws of thermodynamics presented by Surana et. al [79, 82–90, 93–96] for micropolar NCCT based on internal rotations ${}_{i}\boldsymbol{\Theta}$ for solids and based on internal rotation rates ${}_{i}^{r}\bar{\boldsymbol{\Theta}}$ for fluids. The outcome is that the Cauchy stress tensor is non-symmetric (as is the case for all NCCT), but here the Cauchy moment tensor is also non-symmetric. Constitutive theories for Cauchy stress tensor, Cauchy moment tensor and heat vector are derived used conjugate pairs in entropy inequality and representation theorem. Final mathematical models consisting of CBL and the constitutive theories are shown to have closure in micropolar NCCT for solids as well as fluids incorporating rotational inertial physics. Existence of rotational waves is established for micropolar solid continua in the presence of rotational inertial physics. In case of fluids, rotational waves are shown not to exist even in the presence of rotational inertial physics due to their lack of elasticity. Influence of inertial physics is demonstrated in micropolar NCCT for solids as well as for fluids. Constitutive theories are presented for thermoviscous behavior both for solids as well as fluids. Model problem studies are presented to illustrate the influence of rotational inertial physics on their resulting solutions.

Chapter 2

Conservation and balance laws for micropolar solids

In this chapter we present the conservation and balance laws and constitutive theories for micropolar NCCT based on internal rotations or classical rotations ${}_i\boldsymbol{\Theta}$ in which rotational inertial physics is considered. These are derived and presented in Lagrangian description.

2.1 Conservation of mass

The continuity equation resulting from the principle of conservation of mass remains the same in NCCT considered here as in the case of CCT. In Lagrangian description, continuity equations [30, 64, 97] can be written in the following differential form

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x}, t) \tag{2.1}$$

For infinitesimal deformation $|J| \simeq 1$ hence $\rho_0(\boldsymbol{x}) \simeq \rho(\boldsymbol{x}, t)$, where $\rho_0(\boldsymbol{x})$ is the density of the material point at $\boldsymbol{x}$ in the reference configuration and $\rho(\boldsymbol{x}, t)$ is the Lagrangian description of the density of a material point at $\bar{\boldsymbol{x}}$ in the current configuration.

2.2 Balance of linear momenta

This balance law in micropolar NCCT remains the same as in CCT. Thus, for small deformation, small strain we can write [97] the following for balance of linear momenta

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0 \qquad \text{or} \qquad \rho_0 \frac{D\{v\}}{Dt} - \rho_0 \{F^b\} - [\sigma]^T \{\nabla\} = 0 \tag{2.2}$$

In Lagrangian description $\frac{D}{Dt} = \frac{\partial}{\partial t}$ and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ are velocities at a material point, $\boldsymbol{F}^b$ are body forces per unit mass and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. The Cauchy stress tensor is non-symmetric as its symmetry has not been established yet.

2.3 Balance of angular momenta

The principle of balance of angular momenta for a non-classical continuum incorporating internal rotations and inertial effects due to their rates can be stated as follows: The time rate of change of moment of momenta is equal to the sum of the moments of the forces and the moments. Let ${}^{\Theta}\bar{I}$ be the rotational inertia per unit mass of the deforming solid continua, then ${}^{\Theta}\bar{I}\bar{\rho}({}_i\bar{\boldsymbol{\omega}})d\bar{V}$ is the angular momenta for the elemental volume d$\bar{V}$, due to ${}^{\Theta}\bar{I}$ and angular velocity ${}_i\bar{\boldsymbol{\omega}}$ and $\bar{x} \times \bar{\rho}\bar{v}d\bar{V}$ is the moment of linear momenta for the same elemental volume d$\bar{V}$. Then according to this balance law:

$$\begin{aligned} \text{the rate of change of angular momenta} = &(\text{moments due to } \bar{P} + \bar{M}) \\ &+ \text{ moments of the body force } \bar{\rho}\,\bar{\boldsymbol{F}}^b \text{acting on } d\bar{V} \end{aligned}$$

Then, we have for the deformed volume $\bar{V}$ bounded by $\partial\bar{V}$

$$\frac{D}{Dt} \int_{\bar{V}} \left({}^{\Theta}\bar{I}\bar{\rho}\,({}_i\bar{\boldsymbol{\omega}}) + \bar{\boldsymbol{x}} \times \bar{\rho}\,\bar{\boldsymbol{v}} \right) d\bar{V} = \int_{\partial\bar{V}} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} + \bar{\boldsymbol{M}} \right) d\bar{A} + \int_{\bar{V}} \left(\bar{\boldsymbol{x}} \times \bar{\rho}\,\bar{\boldsymbol{F}}^b \right) d\bar{V} \tag{2.3}$$

We expand each term in the following

$$\begin{aligned}
\frac{D}{Dt}\int_{\bar{V}} {}^{\Theta}\bar{I}\bar{\rho}\,({}_{i}\bar{\boldsymbol{\omega}})\,d\bar{V} &= \frac{D}{Dt}\int_{\bar{V}} {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})(\bar{\rho}\,d\bar{V}) \\
&= \frac{D}{Dt}\int_{V} {}^{\Theta}I_{0}({}_{i}\boldsymbol{\omega})\rho_{0}\,dV \\
&= \int_{V} \frac{D}{Dt}({}^{\Theta}I_{0}({}_{i}\boldsymbol{\omega})\rho_{0})\,dV \\
&= \int_{V} {}^{\Theta}I_{0}\rho_{0}\frac{D({}_{i}\boldsymbol{\omega})}{Dt}\,dV \quad ; \quad \text{for constant } \rho_{0} \text{ and } {}^{\Theta}I_{0}
\end{aligned} \tag{2.4}$$

Now we consider

$$\begin{aligned}
\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}}\times\bar{\rho}\,\bar{\boldsymbol{v}}\,d\bar{V} =&\frac{D}{Dt}\int_{\bar{V}} \epsilon_{ijk}\bar{x}_{i}\bar{v}_{j}\bar{\rho}\,d\bar{V} \\
=&\frac{D}{Dt}\int_{V} \epsilon_{ijk}x_{i}v_{j}\rho_{0}\,dV \\
=&\int_{V} \rho_{0}\epsilon_{ijk}\frac{D}{Dt}(x_{i}v_{j})\,dV \\
=&\int_{V} \rho_{0}\epsilon_{ijk}(v_{i}v_{j}+x_{i}\frac{Dv_{j}}{Dt})\,dV
\end{aligned} \tag{2.5}$$

Next we consider

$$\int_{\partial\bar{V}} (\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}}+\bar{\boldsymbol{M}})\,d\bar{A} \tag{2.6}$$

Using Cauchy Principle we have

$$\boldsymbol{P}=\boldsymbol{\sigma}^{T}\cdot\boldsymbol{n}\;,\quad \boldsymbol{M}=\boldsymbol{m}^{T}\cdot\boldsymbol{n} \tag{2.7}$$

Using (2.7) in (2.6) we can write

$$\int_{\partial\bar{V}} (\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}}+\bar{\boldsymbol{M}})\,d\bar{A} = \int_{\partial\bar{V}} \bar{\boldsymbol{x}}\times\bar{\boldsymbol{\sigma}}^{T}\cdot\bar{\boldsymbol{n}}\,d\bar{A} + \int_{\partial\bar{V}} \bar{\boldsymbol{m}}^{T}\cdot\bar{\boldsymbol{n}}\,d\bar{A} \tag{2.8}$$

in which for small strain and small deformation we can write (2.8) as

$$\int_{\partial \bar{V}} (\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} + \bar{\boldsymbol{M}})\, d\bar{A} = \int_{\partial V} (\epsilon_{ijk}\, x_i\, \sigma_{mj} \cdot n_m + m_{mk} \cdot n_m)\, dA$$

Using divergence theorem we finally have

$$\begin{aligned} \int_{\partial \bar{V}} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}\right) d\bar{A} &= \int_V (\epsilon_{ijk}(x_i \sigma_{mj})_{,m} + m_{mk,m})\, dV \\ &= \int_V (\epsilon_{ijk}(\delta_{im}\sigma_{mj} + x_i \sigma_{mj,m}) + m_{mk,m})\, dV \end{aligned} \tag{2.9}$$

Consider the last term in (2.3)

$$\begin{aligned} \int_{\bar{V}} \left(\bar{\boldsymbol{x}} \times \bar{\rho}\, \bar{\boldsymbol{F}}^b\right) d\bar{V} &= \int_{\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{F}^b_j \bar{\rho}\, d\bar{V} \\ &= \int_V \epsilon_{ijk} x_i F^b_j \rho_0\, dV \end{aligned} \tag{2.10}$$

Substituting from (2.4), (2.5), (2.9) and (2.10) in (2.3) we obtain (using $\delta_{im}\sigma_{mj} = \sigma_{ij}$)

$$\begin{aligned} \int_V {}^{\Theta}I_0 \rho_0 \frac{D({}_i\omega_k)}{Dt} + \rho_0 \epsilon_{ijk} \left(v_i v_j + x_i \frac{Dv_j}{Dt}\right) dV = \int_V \Bigg(&\epsilon_{ijk}(\sigma_{ij} + x_i \sigma_{mj,m}) \\ &+ m_{mk,m}\Bigg) dV + \int_V \epsilon_{ijk} x_i F^b_j \rho_0\, dV \end{aligned} \tag{2.11}$$

Collecting terms and noting that

$$\rho_0 \epsilon_{ijk} v_i v_j = 0 \tag{2.12}$$

we can write (2.11) as

$$\begin{aligned} \int_V {}^{\Theta}I_0 \rho_0 \frac{D({}_i\omega_k)}{Dt}\, dV + \int_V \epsilon_{ijk} x_i \left(\rho_0 \frac{Dv_j}{Dt} - \sigma_{mj,m} - \rho_0 F^b_j\right) dV \\ - \int_V (\epsilon_{ijk}(\sigma_{ij}) + m_{mk,m})\, dV = 0 \end{aligned} \tag{2.13}$$

The coefficient of $\epsilon_{ijk}x_i$ in the second term of (2.13) is zero due to the balance of linear momenta, hence (2.13) reduces to

$$\int_V \left({}^{\Theta}I_0 \rho_0 \frac{D({}_i\omega_k)}{Dt} - \epsilon_{ijk}\sigma_{ij} - m_{mk,m} \right) dV = 0 \tag{2.14}$$

Since the volume V is arbitrary, we have the following from (2.14)

$${}^{\Theta}I_0 \rho_0 \frac{D({}_i\omega_k)}{Dt} - \epsilon_{ijk}\sigma_{ij} - m_{mk,m} = 0 \tag{2.15}$$

Remarks

1. If we assume the first and last term in (2.15) to be zero then we recover the balance of angular momenta for classical continuum theory.
2. If we set the first term in (2.15) to zero but retain the second and third term (due to internal rotations physics [84, 87, 89, 95] only), then we have balance of angular momenta for non-classical continuum theory that incorporates internal rotations due to ${}^{d}\boldsymbol{J}$ at the material points but ignores rotational inertial physics.
3. The appearance of the first term in (2.15) is obviously due to rotation rates and the rotational inertia ${}^{\Theta}I_0$ per unit mass (rotary inertia), the new physics considered in the current work that doesn't appear in classical continuum theories. Likewise ${}^{\Theta}\bar{I}$ is the same variable as ${}^{\Theta}I_0$ in Eulerian description.
4. Equation (2.15) is the final form of the balance of angular momenta.

2.4 Balance of moment of moments

The need for this new balance law in non-classical continuum theories was originally proposed and presented by Yang et. al [104] based on static equilibrium considerations. Later, Surana et. al advocated the need for this balance law in non-classical continuum theories and pointed out that a balance law must be derived using rate considerations. In reference [82, 96] they presented the derivation of the “balance of moment of moments" balance law for non-classical continuum theories incorporating internal rotations at material points due to displacement gradient tensor and its extension to fluent continua. In the present work, the physics under consideration is different than that in references [96], hence a rederivation of this balance law is necessary. According to this balance law the rate of change of the moment of the angular momenta due to rotation rates for a deformed volume $\bar{V}$ must be equal to the sum of the moment of moments due to antisymmetric components of the Cauchy stress tensor over the same deformed volume $\bar{V}$ and the moment of $\bar{\boldsymbol{M}}$ acting on the boundary $\partial\bar{V}$ of $\bar{V}$. Thus, we can write

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\Theta}\bar{I}\bar{\rho}({}_{i}\bar{\boldsymbol{\omega}})\, d\bar{V} = \int_{\bar{V}} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : \boldsymbol{\sigma})\, d\bar{V} + \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} \tag{2.16}$$

and in the following we simplify and/or expand each term in (2.16).

First,

$$\begin{aligned} \frac{D}{Dt}\int_{\bar{V}} (\bar{\boldsymbol{x}} \times {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}}))\bar{\rho}\; d\bar{V} &= \frac{D}{Dt}\int_{\bar{V}} {}^{\Theta}\bar{I}\bar{\rho}\epsilon_{jkl}\bar{x}_j({}_{i}\omega_k)\, d\bar{V} \\ &= \frac{D}{Dt}\int_{\bar{V}} {}^{\Theta}\bar{I}\epsilon_{jkl}\bar{x}_j({}_{i}\omega_k)\bar{\rho}\, d\bar{V} \\ &= \frac{D}{Dt}\int_{V} {}^{\Theta}I_0\epsilon_{jkl}x_j({}_{i}\omega_k)\rho_0\, dV \\ &= \int_{V} \epsilon_{jkl}\frac{D}{Dt}\left({}^{\Theta}I_0\rho_0 x_j({}_{i}\omega_k)\right) dV \end{aligned} \tag{2.17}$$

Assuming ${}^{\Theta}I_0$ and ρ_0 to be constant

$$\begin{aligned} \frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})\bar{\rho}\, d\bar{V} &= \int_{V} \epsilon_{jkl}{}^{\Theta}I_0\rho_0 \frac{D}{Dt}(x_j({}_{i}\omega_k))\, dV \\ &= \int_{V} {}^{\Theta}I_0\rho_0\epsilon_{jkl}\left(v_j({}_{i}\omega_k) + x_j\frac{D({}_{i}\omega_k)}{Dt}\right) dV \end{aligned} \tag{2.18}$$

Next, we consider the second term on the right side of (2.16)

$$\int_{\partial \bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} \, d\bar{A} = \int_{\partial \bar{V}} \epsilon_{jkl} \bar{x}_j \bar{M}_k \, d\bar{A} \tag{2.19}$$

Using Cauchy Principle for $\bar{\boldsymbol{M}}$

$$\int_{\partial \bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} \, d\bar{A} = \int_{\partial \bar{V}} \epsilon_{jkl} \bar{x}_j \bar{m}_{mk} \bar{n}_m \, d\bar{A}$$

Using Divergence Theorem

$$\begin{aligned} \int_{\partial \bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} \, d\bar{A} &= \int_{\bar{V}} \epsilon_{jkl} (\bar{x}_j \bar{m}_{mk})_{,m} \; d\bar{V} \\ &= \int_{\bar{V}} \epsilon_{jkl} (\delta_{jm} \bar{m}_{mk} + \bar{x}_j \bar{m}_{mk,m}) \, d\bar{V} \\ &= \int_{\bar{V}} \epsilon_{jkl} (\bar{m}_{jk} + \bar{x}_j \bar{m}_{mk,m}) \, d\bar{V} \end{aligned}$$

Considering small deformation and small strain ($\bar{x} \simeq x$) we can write

$$\int_{\partial \bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} \, d\bar{A} = \int_{V} \epsilon_{jkl} (m_{jk} + x_j m_{mk,m}) \, dV \tag{2.20}$$

Substituting from (2.18) and (2.20) in (2.16) $\big($and changing $\bar{V}$ to V, $\bar{\boldsymbol{x}}$ to $\boldsymbol{x}$ and $\bar{\boldsymbol{\sigma}}$ to $\boldsymbol{\sigma}$ in the first term on the right side of (2.16)$\big)$

$$\begin{aligned} \int_V {}^{\Theta}I_{0} \rho_0 \, \epsilon_{jkl} \left(v_j ({}_i\omega_k) + x_j \frac{D({}_i\omega_k)}{Dt} \right) dV = \int_V \boldsymbol{x} \times (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \, dV \\ + \int_V \epsilon_{jkl} (m_{jk} + x_j m_{mk,m}) \, dV \end{aligned} \tag{2.21}$$

We note that

$$
\begin{aligned}
m_{mk,m} &= \boldsymbol{\nabla} \cdot \boldsymbol{m} \\
\epsilon_{jkl}(x_j m_{mk,m}) &= \boldsymbol{x} \times \boldsymbol{\nabla} \cdot \boldsymbol{m} \\
{}^{\Theta}I_0 \rho_0 \epsilon_{jkl} x_j \frac{D({}_i\omega_k)}{Dt} &= {}^{\Theta}I_0 \rho_0 \, \boldsymbol{x} \times \frac{D({}_i\boldsymbol{\omega})}{Dt}
\end{aligned}
\tag{2.22}
$$

Using (2.22) in (2.21)

$$
\int_V {}^{\Theta}I_0 \rho_0 \epsilon_{jkl} v_j ({}_i\omega_k) \, dV + \int_V \boldsymbol{x} \times \left({}^{\Theta}I_0 \rho_0 \frac{D({}_i\boldsymbol{\omega})}{Dt} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} - \boldsymbol{\nabla} \cdot \boldsymbol{m} \right) dV = \int_V \epsilon_{jkl} m_{jk} \, dV \tag{2.23}
$$

Using balance of angular momenta (2.15) in (2.23), we obtain

$$
\int_V \epsilon_{jkl} \left({}^{\Theta}I_0 \rho_0 v_j ({}_i\omega_k) - m_{jk} \right) dV = 0 \tag{2.24}
$$

Since V is arbitrary, we obtain the following from (2.24)

$$
\epsilon_{jkl} \left({}^{\Theta}I_0 \rho_0 v_j ({}_i\omega_k) - m_{jk} \right) = 0 \tag{2.25}
$$

Equation (2.25) is the final form resulting from the balance of moment of moments balance law.

Remarks

1. We note that with the absence of new physics considered in this paper, i.e., when ${}^{\Theta}I_0 = 0$, (2.25) reduces to

$$
\epsilon_{jkl} m_{jk} = 0 \tag{2.26}
$$

which is the balance of moment of moments balance law in non-classical theory derived by Surana et. al in references [87, 95].

2. When ${}^{\Theta}I_0$ is not zero as the case is in the present work, (2.25) yields three equations defining antisymmetric components of the Cauchy moment tensor in terms of velocities and rotation

rates (angular velocities) and the properties of the medium.

2.5 First law of thermodynamics

The sum of work and heat added to a deforming volume of matter must result in increase of the energy of the system. This is expressed as a rate equation in Eulerian description as

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{2.27}$$

where $\bar{E}_t$, $\bar{Q}$, and $\bar{W}$ are total energy, heat added and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} + \frac{1}{2}{}^{\Theta}\bar{I}\,({}_i\bar{\boldsymbol{\omega}}\cdot{}_i\bar{\boldsymbol{\omega}}) - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right) d\bar{V} \tag{2.28}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\, d\bar{A} \tag{2.29}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}} (\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}_i\dot{\bar{\boldsymbol{\Theta}}})\, d\bar{A} \tag{2.30}$$

Here $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ is body force vector per unit mass and $\bar{\boldsymbol{q}}$ is rate of heat. The third term in the integrand is due to additional rate of work due to rotation rates. Note that the additional term $\bar{\boldsymbol{M}}\cdot{}_i\dot{\bar{\boldsymbol{\Theta}}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of internal rotations. Expanding integrals and following reference [97] one can show that

$$\begin{aligned}\frac{D\bar{E}}{Dt} =& \frac{D}{Dt}\int_{\bar{V}} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} + \frac{1}{2}{}^{\Theta}\bar{I}\,({}_i\bar{\boldsymbol{\omega}}\cdot{}_i\bar{\boldsymbol{\omega}}) - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right) d\bar{V} \\ =& \int_V \left(\rho_0\frac{De}{Dt} + \rho_0\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt} + \rho_0{}^{\Theta}I_0({}_i\boldsymbol{\omega})\cdot\frac{D({}_i\boldsymbol{\omega})}{Dt} - \rho_0\boldsymbol{F}^b\cdot\boldsymbol{v}\right) dV\end{aligned} \tag{2.31}$$

Using

$$\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} = \boldsymbol{q} \cdot \boldsymbol{n}\, dA$$
$$\bar{\rho}\, d\bar{V} = \rho_0\, dV \tag{2.32}$$
$$d\bar{V} = |J|\, dV$$

and applying the divergence theorem, we obtain

$$-\int_{\partial\bar{V}} \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} = -\int_{\partial V} \boldsymbol{q} \cdot \boldsymbol{n}\, dA = -\int_{V} \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV \tag{2.33}$$

Using Cauchy stress tensor $\boldsymbol{\sigma}$ and Cauchy moment tensor $\boldsymbol{m}$, following reference [97], one can show that

$$\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}}\, d\bar{A} = \boldsymbol{v} \cdot (\boldsymbol{\sigma})^T \cdot \boldsymbol{n}\, dA = \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} \tag{2.34}$$

$$\bar{\boldsymbol{M}} \cdot {}_i\dot{\bar{\boldsymbol{\Theta}}}\, d\bar{A} = \left({}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot \boldsymbol{n}\, dA = \left({}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A} \tag{2.35}$$

Thus, we can write the following for (2.27)

$$\begin{aligned}
&\int_V \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} + \rho_0^{\Theta} I_0({}_i\boldsymbol{\omega}) \cdot \frac{D({}_i\boldsymbol{\omega})}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV \\
&\qquad = -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV + \int_{\partial V} \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} + \int_{\partial V} \left({}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A} \\
&\qquad = -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV + \int_V \boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) dV + \int_V \boldsymbol{\nabla} \cdot \left({}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) dV
\end{aligned} \tag{2.36}$$

Following reference [97], we can also show that

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) + \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \tag{2.37}$$

$$\boldsymbol{\nabla} \cdot \left({}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) = {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) + \boldsymbol{m} \cdot (\boldsymbol{\nabla} \cdot {}_i\dot{\boldsymbol{\Theta}}) \tag{2.38}$$

Substituting (2.37) and (2.38) in (2.36)

$$\int_V \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} + \rho_0^{\Theta} I_0 ({}_i\boldsymbol{\omega}) \cdot \frac{D({}_i\boldsymbol{\omega})}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v} \right) dV$$
$$= -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{q} \, dV + \int_V \left(\boldsymbol{v} \cdot \left(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \right) + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + {}_i\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\nabla} \cdot \boldsymbol{m} \right) + m_{kj} \frac{\partial ({}_i\dot{\Theta}_j)}{\partial x_k} \right) dV \quad (2.39)$$

or

$$\int_V \boldsymbol{v} \cdot (\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \, dV + \int_V \left(\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} \right.$$
$$\left. + \rho_0^{\Theta} I_0 ({}_i\boldsymbol{\omega}) \cdot \frac{D({}_i\boldsymbol{\omega})}{Dt} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{kj} \frac{\partial ({}_i\dot{\Theta}_j)}{\partial x_k} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \right) dV = 0 \quad (2.40)$$

Using balance of linear momenta, (2.40) reduces to

$$\int_V \left(\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} \right.$$
$$\left. + \rho_0^{\Theta} I_0 ({}_i\boldsymbol{\omega}) \cdot \frac{D({}_i\boldsymbol{\omega})}{Dt} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{kj} \frac{\partial ({}_i\dot{\Theta}_j)}{\partial x_k} - {}_i\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\nabla} \cdot \boldsymbol{m} \right) \right) dV = 0 \quad (2.41)$$

Since volume V is arbitrary, the following holds:

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho_0^{\Theta} I_0 ({}_i\boldsymbol{\omega}) \cdot \frac{D({}_i\boldsymbol{\omega})}{Dt} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{kj} \frac{\partial ({}_i\dot{\Theta}_j)}{\partial x_k} - {}_i\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\nabla} \cdot \boldsymbol{m} \right) = 0 \quad (2.42)$$

The first law of thermodynamics given by (2.42) can be further simplified as shown below.

First we note that

$$\begin{aligned} \sigma_{ji} \frac{\partial v_i}{\partial x_j} &= \mathrm{tr}([\sigma][L]) = \boldsymbol{\sigma} : \boldsymbol{L} \\ m_{jk} \frac{\partial ({}_i\dot{\Theta}_j)}{\partial x_k} &= \mathrm{tr}([m][{}^{i\Theta}\dot{J}]) = \boldsymbol{m} : {}^{i\Theta}\dot{\boldsymbol{J}} \end{aligned} \quad (2.43)$$

Furthermore

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{D} + \boldsymbol{W} = \dot{\boldsymbol{\varepsilon}} + {}_a\dot{\boldsymbol{J}} \\ {}^{i\Theta}\dot{\boldsymbol{J}} &= {}^{i\Theta}_{s}\dot{\boldsymbol{J}} + {}^{i\Theta}_{a}\dot{\boldsymbol{J}} \end{aligned} \tag{2.44}$$

Consider decomposition of Cauchy stress and Cauchy moment tensor into symmetric and antisymmetric components

$$\begin{aligned} \boldsymbol{\sigma} &= {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} \\ \boldsymbol{m} &= {}_s\boldsymbol{m} + {}_a\boldsymbol{m} \end{aligned} \tag{2.45}$$

Using (2.44) and (2.45) in (2.43)

$$\begin{aligned} \boldsymbol{\sigma} : \boldsymbol{L} &= {}_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + {}_a\boldsymbol{\sigma} : {}_a\dot{\boldsymbol{J}} \\ \boldsymbol{m} : {}^{i\Theta}\dot{\boldsymbol{J}} &= {}_s\boldsymbol{m} : {}^{i\Theta}_{s}\dot{\boldsymbol{J}} + {}_a\boldsymbol{m} : {}^{i\Theta}_{a}\dot{\boldsymbol{J}} \end{aligned} \tag{2.46}$$

Also from balance of angular momenta (2.15)

$${}^{\Theta}I_0 \rho_0 \frac{D({}_i\boldsymbol{\omega})}{Dt} - \epsilon_{ijk}\sigma_{ij} = m_{mk,m} = \boldsymbol{\nabla} \cdot \boldsymbol{m} \tag{2.47}$$

hence

$${}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) = {}^{\Theta}I_0 \rho_0 \left({}_i\dot{\boldsymbol{\Theta}} \cdot \frac{D({}_i\boldsymbol{\omega})}{Dt} \right) - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \tag{2.48}$$

and since

$$\begin{aligned} {}_i\dot{\boldsymbol{\Theta}} &= {}_i\boldsymbol{\omega} \\ \epsilon_{ijk} &= \boldsymbol{\epsilon} \\ {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) &= {}_a\boldsymbol{\sigma} : {}_a\dot{\boldsymbol{J}} \end{aligned} \tag{2.49}$$

Using (2.46), (2.48) and (2.49) we can write (2.42) as

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - {}_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_s\boldsymbol{m} : {}^{i\Theta}_{s}\dot{\boldsymbol{J}} - {}_a\boldsymbol{m} : {}^{i\Theta}_{a}\dot{\boldsymbol{J}} = 0 \tag{2.50}$$

Remarks

1. Equations (2.50) is the final form of the first law of thermodynamics (energy equation) for the physics considered in this work.

2. For the non-classical continuum theory in references [87, 95] that only considers internal rotations and ${}^{\Theta}I_0 = 0$: (i) the last term in (2.50) obviously becomes zero as ${}^{\Theta}I_0 = 0$ (ii) the fifth term in (2.50) also becomes zero due to the fact that balance of moment of moments in this case yields $\epsilon_{ijk} m_{ij} = 0$ implying that $\boldsymbol{m}^T = \boldsymbol{m}$, hence ${}_a\boldsymbol{m} = 0$.

2.6 Second law of thermodynamics

If $\bar{\eta}$ is the entropy density in volume $\bar{V}$, $\bar{h}$ is the entropy flux between $\bar{V}$ and the volume of matter surrounding it, and $\bar{s}$ is the source of entropy in $\bar{V}$ due to non-contacting bodies, then the rate of increase of entropy in volume $\bar{V}$ is at least equal to that supplied to $\bar{V}$ from all contacting and non-contacting sources [97]. Thus

$$\frac{D}{Dt} \int_{\bar{V}} \bar{\eta} \bar{\rho} \, d\bar{V} \geq \int_{\partial \bar{V}} \bar{h} \, d\bar{A} + \int_{\bar{V}} \bar{s} \bar{\rho} \, d\bar{V} \tag{2.51}$$

Using Cauchy's postulate for $\bar{h}$

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \tag{2.52}$$

Using (2.52) in (2.51)

$$\frac{D}{Dt} \int_{\bar{V}} \bar{\eta} \bar{\rho} \, d\bar{V} \geq - \int_{\partial \bar{V}} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \, d\bar{A} + \int_{\bar{V}} \bar{s} \bar{\rho} \, d\bar{V} \tag{2.53}$$

Inequality (2.53) needs to be transformed to Lagrangian description. This can be done using

$$
\begin{aligned}
d\bar{V} &= |J| dV \\
\rho_0 &= |J| \rho \\
\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \, d\bar{A} &= \boldsymbol{\psi} \cdot \boldsymbol{n} \, dA
\end{aligned}
\tag{2.54}
$$

Using (2.54) in (2.53)

$$
\frac{D}{Dt} \int_V \eta \rho_0 \, dV \geq - \int_{\partial V} \boldsymbol{\psi} \cdot \boldsymbol{n} \, dA + \int_V s \rho_0 \, dV \tag{2.55}
$$

Using Gauss's divergence theorem for the terms over ∂V gives (noting that $\boldsymbol{\psi}$ is a tensor of rank one)

$$
\frac{D}{Dt} \int_V \eta \rho_0 \, dV \geq - \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\psi} \, dV + \int_V s \rho_0 \, dV \tag{2.56}
$$

or

$$
\int_V \left(\rho_0 \frac{D\eta}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\psi} - \rho_0 s \right) dV \geq 0 \tag{2.57}
$$

and since volume V is arbitrary

$$
\rho_0 \frac{D\eta}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\psi} - \rho_0 s \geq 0 \tag{2.58}
$$

Inequality (2.58) is the entropy inequality and is the most fundamental form resulting from the second law of thermodynamics (Clausius Duhem inequality). Inequality (2.58) is strictly a statement that contains entropy terms and hence contains no information regarding reversible deformation processes such as in case of elastic solids. Thus, it provides no information or mechanisms for deriving the constitutive theories for such solids. Only when the mechanical rate of work results in rate of entropy production will inequality (2.58) have some information regarding the associated conjugate pairs that result in rate of entropy production. One can also note that (2.58) in its present form does not provide any information regarding constitutive theory for heat vector $\boldsymbol{q}$.

Another form of the entropy inequality is possible using a relationship between $\boldsymbol{\psi}$ and $\boldsymbol{q}$ and the energy equation. Since the energy equation has all possible mechanisms that result in energy storage and dissipation, the form of entropy inequality derived using the energy equation is expected to be helpful in the derivations of constitutive theories. Using

$$\boldsymbol{\psi} = \frac{\boldsymbol{q}}{\theta}\,, \qquad s = \frac{r}{\theta} \tag{2.59}$$

where θ is absolute temperature, $\boldsymbol{q}$ is heat vector and r is a suitable potential, then

$$\boldsymbol{\nabla}\cdot\boldsymbol{\psi} = \psi_{i,i} = \frac{q_{i,i}}{\theta} - \frac{q_i\theta_{,i}}{\theta^2} = \frac{q_{i,i}}{\theta} - \frac{q_i g_i}{\theta^2} = \frac{\boldsymbol{\nabla}\cdot\boldsymbol{q}}{\theta} - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta^2} \tag{2.60}$$

Substituting from (2.60) into (2.58) and multiplying throughout by θ yields

$$\rho_0\theta\frac{D\eta}{Dt} + \big(\boldsymbol{\nabla}\cdot\boldsymbol{q} - \rho_0 r\big) - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \geq 0 \tag{2.61}$$

From the energy equation (2.50) (after inserting $\rho_0 r$ term)

$$\boldsymbol{\nabla}\cdot\boldsymbol{q} - \rho_0 r = -\rho_0\frac{De}{Dt} + {}_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + {}_s\boldsymbol{m} : {}^{i\Theta}_{s}\dot{\boldsymbol{J}} + {}_a\boldsymbol{m} : {}^{i\Theta}_{a}\dot{\boldsymbol{J}} \tag{2.62}$$

Substituting from (2.62) into (2.61), using $\Phi = e - \eta\theta$ and regrouping terms

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_s\boldsymbol{m} : {}^{i\Theta}_{s}\dot{\boldsymbol{J}} - {}_a\boldsymbol{m} : {}^{i\Theta}_{a}\dot{\boldsymbol{J}} \leq 0 \tag{2.63}$$

Equation (2.63) is the final form of the second law of thermodynamics.

2.7 Mathematical model consisting of CBL of NCCM for solids

The equations resulting from the conservation of mass (CM), balance of linear momenta (BLM), balance of angular momenta (BAM), balance of moment of moments (BMM), first law of thermodynamics (FLT) and the second law of thermodynamics (SLT) are summarized in the following.

Using $v_i = \frac{Du_i}{Dt}$ and ${}_i\omega_k = \frac{D({}_i\Theta_k)}{Dt}$ have

$$\frac{Dv_i}{Dt} = \frac{D^2 u_i}{Dt^2} \quad ; \quad \frac{D({}_i\omega_k)}{Dt} = \frac{D^2({}_i\Theta_k)}{Dt^2} \tag{2.64}$$

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x}, t) \qquad \text{(CM)} \quad (2.65)$$

$$\rho_0 \frac{D^2 u_k}{Dt^2} - \rho_0 F_k{}^b - \frac{\partial \sigma_{jk}}{\partial x_j} = 0 \qquad \text{(BLM)} \quad (2.66)$$

$${}^{\Theta}I_0 \rho_0 \frac{D^2({}_i\Theta_k)}{Dt^2} - \epsilon_{ijk}\sigma_{ij} - \frac{\partial m_{jk}}{\partial x_j} = 0 \qquad \text{(BAM)} \quad (2.67)$$

$$\epsilon_{jkl}\left({}^{\Theta}I_0 \rho_0 v_j({}_i\omega_k) - m_{jk}\right) = 0 \qquad \text{(BMM)} \quad (2.68)$$

$$\rho_0 \frac{De}{Dt} + \frac{\partial q_i}{\partial x_i} - \text{tr}([{}_s\sigma][\dot{\varepsilon}]) - \text{tr}([{}_sm][{}^{i\Theta}_{s}\dot{J}]) - \text{tr}([{}_am][{}^{i\Theta}_{a}\dot{J}]) = 0 \qquad \text{(FLT)} \quad (2.69)$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt}\right) + \frac{q_i \cdot g_i}{\theta} - \text{tr}([{}_s\sigma][\dot{\varepsilon}]) - \text{tr}([{}_sm][{}^{i\Theta}_{s}\dot{J}]) - \text{tr}([{}_am][{}^{i\Theta}_{a}\dot{J}]) \leq 0 \qquad \text{(SLT)} \quad (2.70)$$

Remarks

1. This mathematical model consists of ten equations: BLM(3), BAM(3), BMM(3), FLT(1) in 25 dependent variables $\boldsymbol{u}(3)$, $\boldsymbol{\sigma}(9)$, $\boldsymbol{m}(9)$, $\boldsymbol{q}(3)$, $\theta(1)$. Thus we need additional fifteen equations for this mathematical model to have closure. These additional equations are obtained from the derivations of the constitutive theories.

2. We shall see that Φ, η and e are not dependent variables in the mathematical model as they can be expressed as functions of the other dependent variables in item (1).

3. From the entropy inequality we can conclude the following

   (a) From the term $\frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta}$, $q_i g_i$ is a conjugate pair.

   (b) The term $\text{tr}([{}_s\sigma][\dot{\varepsilon}])$ suggests that $[{}_s\sigma]$ and $[\dot{\varepsilon}]$ are rate of work (mechanical) conjugate pair. This is obviously due to classical mechanics.

   (c) The term $\text{tr}([{}_sm][{}^{i\Theta}_{s}\dot{J}])$ suggests $[{}_sm]$ and $[{}^{i\Theta}_{s}\dot{J}]$ are also rate of work conjugate pair. This is due to the contribution of non-classical mechanics based on internal rotations.

(d) From the term tr$([{}_a m][{}^{i\Theta}_{\;a}\dot{J}])$ it can also be concluded that ${}_a\boldsymbol{m}$, ${}^{i\Theta}_{\;a}\dot{\boldsymbol{J}}$ are a rate of work (mechanical) conjugate. However, based on Surana et. al [96], in the non-classical mechanics theories the constitutive theory for ${}_a\boldsymbol{m}$ (when ${}_a\boldsymbol{m}$ is a possible choice of constitutive variable) leads to deformation physics that is non-physical. In [96] authors present derivation of constitutive theory for ${}_a\boldsymbol{m}$ (in the absence of BMM balance law) as well as model problem studies to substantiate this issue. Hence, $[{}_a m]$, $[{}^{i\Theta}_{\;a}\dot{J}]$ is not conjugate pair, therefore ${}_a\boldsymbol{m}$ is not a constitutive variable, thus we do not have constitutive theory for ${}_a\boldsymbol{m}$. Therefore tr$([{}_a m][{}^{i\Theta}_{\;a}\dot{J}]) = 0$ must be used as a constraint equation.

4. From remark 3 we can conclude that it is possible to obtain the following additional equations through constitutive theories

   (i) constitutive theories for ${}_s\boldsymbol{\sigma}$ (6)

   (ii) constitutive theories for ${}_s\boldsymbol{m}$ (6)

   (iii) constitutive theories for $\boldsymbol{q}$ (3)

   This provides us with additional fifteen equations needed to provide closure to the mathematical model consisting of (2.65) - (2.70)

5. In the present work, we only consider small strain, small deformation physics. Thus, density is constant and the conservation of mass is not part of the final mathematical model.

6. We remark that even in finite deformation case when using Lagrangian description, density can always be determined using ρ_0 and $|J|$ once the deformation is known. Hence, this is a post processing operation. For this reason, even in this case, ρ is not a dependent variable in the mathematical model in Lagrangian description. This eliminates (2.65) as part of the final mathematical model.

2.8 Constitutive theories

In this section, we present constitutive theories for thermoelastic solids without memory. We consider small deformation, small strain physics as considered in the conservation and balance laws. In the derivation of constitutive theories, we begin with entropy inequality. Conjugate pairs in the entropy inequality (2.70) facilitate the initial choice of constitutive variables. First, we must perform additive decomposition of ${}_s\boldsymbol{\sigma}$ into ${}^e_s\boldsymbol{\sigma}$, equilibrium Cauchy stress tensor and ${}^d_s\boldsymbol{\sigma}$, deviatoric Cauchy stress tensor. Mutually exclusive volumetric and distortional deformation physics are addressed by the constitutive theories for ${}^e_s\boldsymbol{\sigma}$ and ${}^d_s\boldsymbol{\sigma}$.

$$
{}_s\boldsymbol{\sigma} = {}^e_s\boldsymbol{\sigma} + {}^d_s\boldsymbol{\sigma} \tag{2.71}
$$

Substituting (2.71) into (2.70), we obtain

$$
\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}^e_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}^d_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_s\boldsymbol{m} : {}^{i\Theta}_{s}\dot{\boldsymbol{J}} - {}_a\boldsymbol{m} : {}^{i\Theta}_{a}\dot{\boldsymbol{J}} \leq 0 \tag{2.72}
$$

It is well known [97] that constitutive theory for ${}^e_s\boldsymbol{\sigma}$ can not be derived using entropy inequality in Lagrangian description. Thus, we must consider (2.72) in Eulerian description. Using contravariant Cauchy stress measures $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\bar{\boldsymbol{m}}^{(0)}$ we can write

$$
\begin{aligned}
\bar{\rho} \left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}^e_s\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}^d_s\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} \\
- {}_s\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}} - {}_a\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\ a}\bar{\boldsymbol{J}} \leq 0
\end{aligned} \tag{2.73}
$$

From (2.73) we can write

$$
{}^e_s\bar{\boldsymbol{\sigma}}^{(0)} = {}^e_s\bar{\boldsymbol{\sigma}}^{(0)} \left(\bar{\rho}, \bar{\theta} \right) \tag{2.74}
$$

$$
{}^d_s\bar{\boldsymbol{\sigma}}^{(0)} = {}^d_s\bar{\boldsymbol{\sigma}}^{(0)} \left(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\theta} \right) \tag{2.75}
$$

$$
{}_s\bar{\boldsymbol{m}}^{(0)} = {}_s\bar{\boldsymbol{m}}^{(0)} \left(\bar{\rho}, {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}, \bar{\theta} \right) \tag{2.76}
$$

$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\left(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{2.77}$$

$$\bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{D}}, {}^{r\Theta}_{i\;\,s}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{2.78}$$

$$\bar{\eta} = \bar{\eta}\left(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{D}}, {}^{r\Theta}_{i\;\,s}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{2.79}$$

using (2.78) we can obtain $\frac{D\bar{\Phi}}{Dt}$

$$\frac{D\bar{\Phi}}{Dt} = \frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\dot{\bar{\rho}} + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{D}}} : \dot{\bar{\boldsymbol{D}}} + \frac{\partial\bar{\Phi}}{\partial({}^{r\Theta}_{i\;\,s}\bar{\boldsymbol{J}})} : ({}^{r\Theta}_{i\;\,s}\dot{\bar{\boldsymbol{J}}}) + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}} \tag{2.80}$$

using (2.80) in (2.73) and using

$$\dot{\bar{\Phi}} = \bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\left(-\boldsymbol{\delta} : \bar{\boldsymbol{D}}\right)$$

from continuity equation, we have by regrouping terms and setting coefficients of $\dot{\bar{\boldsymbol{D}}}$, ${}^{r\Theta}_{i\;\,s}\dot{\bar{\boldsymbol{J}}}$, $\dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$ to zero (implying that (2.80) is satisfied for arbitrary but admissible choices of $\dot{\bar{\boldsymbol{D}}}$, ${}^{r\Theta}_{i\;\,s}\dot{\bar{\boldsymbol{J}}}$, $\dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$)

$$\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} - {}^{e}_{s}\bar{\boldsymbol{\sigma}}^{(0)}\right) : \bar{\boldsymbol{D}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}^{d}_{s}\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}_{s}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;\,s}\bar{\boldsymbol{J}} - {}_{a}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;\,a}\bar{\boldsymbol{J}} \leq 0 \tag{2.81}$$

setting coefficient of $\bar{\boldsymbol{D}}$ in the first term to zero gives

$$ {}^{e}_{s}\bar{\boldsymbol{\sigma}}^{(0)} = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} = \bar{p}\left(\bar{\rho}, \bar{\theta}\right)\boldsymbol{\delta}; \quad \bar{p}(\bar{\rho}, \bar{\theta}) = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \tag{2.82}$$

Equation (2.82) is the constitutive theory for ${}^{e}_{s}\bar{\boldsymbol{\sigma}}^{(0)}$ for compressible matter. Using (2.82), (2.81) reduces to

$$\frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}^{d}_{s}\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}_{s}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;\,s}\bar{\boldsymbol{J}} - {}_{a}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;\,a}\bar{\boldsymbol{J}} \leq 0 \tag{2.83}$$

The constitutive theory for incompressible case is derived by introducing incompressibility condition

$$\bar{p}(\bar{\theta})\boldsymbol{\delta} : \bar{\boldsymbol{D}} = 0 \tag{2.84}$$

in (2.84) we obtain

$$ {}^{e}_{s}\bar{\boldsymbol{\sigma}}^{(0)} = \bar{p}(\bar{\theta})\boldsymbol{\delta} \tag{2.85}$$

and (2.86) reduces to

$$ \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}^{d}_{s}\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}_{s}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;s}\bar{\boldsymbol{J}} - {}_{a}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;a}\bar{\boldsymbol{J}} \leq 0 \tag{2.86}$$

constitutive theories (2.82) and (2.85) in Lagrangian description can be written as

$$ {}^{e}_{s}\boldsymbol{\sigma} = p(\rho,\theta)\boldsymbol{\delta} \quad ; \quad p(\rho,\theta) \text{ is thermodynamic pressure} \tag{2.87}$$

for the compressible case and for the incompressible case we have

$$ {}^{e}_{s}\boldsymbol{\sigma} = p(\theta)\boldsymbol{\delta} \quad ; \quad p(\theta) \text{ is thermodynamic pressure} \tag{2.88}$$

The Lagrangian form of (2.86) can be written as

$$ \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{s}\boldsymbol{m} : {}^{i\Theta}_{\;s}\dot{\boldsymbol{J}} - {}_{a}\boldsymbol{m} : {}^{i\Theta}_{\;a}\dot{\boldsymbol{J}} \leq 0 \tag{2.89}$$

Conjugate pairs in the entropy inequality suggest (for thermoviscous elastic physics without memory)

$$ {}^{d}_{s}\boldsymbol{\sigma} = {}^{d}_{s}\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon},\theta\right) \quad ; \quad \text{in the absence of dissipation} \tag{2.90}$$

$$ {}^{d}_{s}\boldsymbol{\sigma} = {}^{d}_{s}\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon},\dot{\boldsymbol{\varepsilon}},\theta\right) \quad ; \quad \text{in the presence of dissipation} \tag{2.91}$$

$$ {}_{s}\boldsymbol{m} = {}_{s}\boldsymbol{m}\left({}^{i\Theta}_{\;s}\boldsymbol{J},\theta\right) \quad ; \quad \text{in the absence of dissipation} \tag{2.92}$$

$$ {}_{s}\boldsymbol{m} = {}_{s}\boldsymbol{m}\left({}^{i\Theta}_{\;s}\boldsymbol{J},{}^{i\Theta}_{\;s}\dot{\boldsymbol{J}},\theta\right) \quad ; \quad \text{in the presence of dissipation} \tag{2.93}$$

$$ \boldsymbol{q} = \boldsymbol{q}\left(\boldsymbol{g},\theta\right) \tag{2.94}$$

2.8.1 Constitutive theory for ${}_s^d\boldsymbol{\sigma}$ using representation theorem

In this section, we derive constitutive theories for ${}_s^d\boldsymbol{\sigma}$ based on (2.90) using representation theorem [8,62,69–71,73–77,99–103,107,108] in the absence of dissipation. Since ${}_s^d\boldsymbol{\sigma}$ is a symmetric tensor of rank two, integrity based on the combined generators (symmetric tensors of rank two) of $[\varepsilon]$ and θ (symmetric tensor of rank two and tensor of rank zero) consists of tensors $[I]$, $[\varepsilon]$ and $[\varepsilon]^2$. Thus, we can represent $[{}_s^d\sigma]$ by a linear combination of the combined generators.

$$[{}_s^d\sigma] = {}^{\sigma}\underset{\sim}{\alpha}^0[I] + {}^{\sigma}\underset{\sim}{\alpha}^1[\varepsilon] + {}^{\sigma}\underset{\sim}{\alpha}^2[\varepsilon]^2 \tag{2.95}$$

in which

$${}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta); \quad i = 0,1,2 \tag{2.96}$$

We introduce new notations in (2.95) and (2.96) to facilitate the subsequent details of the derivation. We define $[{}^{\sigma}\underset{\sim}{G}^1] = [\varepsilon]$, $[{}^{\sigma}\underset{\sim}{G}^2] = [\varepsilon]^2$, i.e., $[{}^{\sigma}\underset{\sim}{G}^i]$; $i = 1, 2, \cdots, N (N = 2)$ as the combined generators due to argument tensors $[\varepsilon]$ and θ and ${}^{\sigma}\underset{\sim}{I}^1 = I_\varepsilon$, ${}^{\sigma}\underset{\sim}{I}^2 = II_\varepsilon$, ${}^{\sigma}\underset{\sim}{I}^3 = III_\varepsilon$ i.e., ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \cdots, M (M = 3)$ as the combined invariants of the same argument tensors.

Then, (2.95) and (2.96) can be written as

$$[{}_s^d\sigma] = {}^{\sigma}\underset{\sim}{\alpha}^0[I] + \sum_{i=1}^{N} {}^{\sigma}\underset{\sim}{\alpha}^i[{}^{\sigma}\underset{\sim}{G}^i] \tag{2.97}$$

$${}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i\left({}^{\sigma}\underset{\sim}{I}^j, \theta\right); \quad i = 1, 2, \cdots, N; \quad j = 1, 2, \cdots, M \tag{2.98}$$

The material coefficients in the constitutive theory for $[{}_s^d\sigma]$ are determined by considering Taylor series expansion of ${}^{\sigma}\underset{\sim}{\alpha}^i$; $i = 0, 1, \cdots, N$ in ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \cdots, M$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \cdots, M$ and θ.

$$
{}^{\sigma}\underset{\sim}{\alpha}^{i} = {}^{\sigma}\underset{\sim}{\alpha}^{i}\Big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial^{\sigma}\underset{\sim}{I}^{j}}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underline{\Omega}}\right) + \frac{\partial^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}}\right)[I]; \quad i = 0, 1, \cdots, N \tag{2.99}
$$

Substituting (2.99) in (2.97), collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients, we can obtain

$$
\begin{aligned}
[{}^{d}_{s}\sigma] = \underset{\sim}{\sigma}^{0}\Big|_{\underline{\Omega}}[I] + \sum_{j=1}^{M} {}^{\sigma}\underset{\sim}{a}_{j}\, {}^{\sigma}\underset{\sim}{I}^{j}[I] + \sum_{i=1}^{N} {}^{\sigma}\underset{\sim}{b}_{i}[{}^{\sigma}\underset{\sim}{G}^{i}] + \sum_{i=1}^{N}\sum_{j=1}^{M} {}^{\sigma}\underset{\sim}{c}_{ij}\, {}^{\sigma}\underset{\sim}{I}^{j}[{}^{\sigma}\underset{\sim}{G}^{i}] \\
- \sum_{i=1}^{N} {}^{\sigma}\underset{\sim}{d}_{i}\left(\theta - \theta|_{\underline{\Omega}}\right)[{}^{\sigma}\underset{\sim}{G}^{i}] - \alpha_{tm}\left(\theta - \theta|_{\underline{\Omega}}\right)[I]
\end{aligned} \tag{2.100}
$$

Coefficients ${}^{\sigma}\underset{\sim}{a}_{j}$, ${}^{\sigma}\underset{\sim}{b}_{j}$, ${}^{\sigma}\underset{\sim}{c}_{ij}$, ${}^{\sigma}\underset{\sim}{d}_{i}$, α_{tm}; $i = 1, 2, \cdots, N$; $j = 1, 2, \cdots, M$ are functions of ${}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$ $j = 1, 2, \cdots, M$. These are the material coefficients.

Remarks

1. This constitutive theory contains $(N = 2,\ M = 3)$ fourteen material coefficients and contains up to fifth degree terms in the components of $[\varepsilon]$, but is linear in temperature θ.
2. A linear constitutive theory in which the products of ${}^{\sigma}\underset{\sim}{I}^{j}$, $[{}^{\sigma}\underset{\sim}{G}^{i}]$ and $\left(\theta - \theta|_{\underline{\Omega}}\right)$ are neglected and only up to linear terms in $[\varepsilon]$ are retained is given by

$$
[{}^{d}_{s}\sigma] = \underset{\sim}{\sigma}^{0}\Big|_{\underline{\Omega}}[I] + {}^{\sigma}\underset{\sim}{a}_{1}\, {}^{\sigma}\underset{\sim}{I}^{1}[I] + {}^{\sigma}\underset{\sim}{b}_{1}[{}^{\sigma}\underset{\sim}{G}^{1}] - \alpha_{tm}|_{\underline{\Omega}}\left(\theta - \theta|_{\underline{\Omega}}\right)[I] \tag{2.101}
$$

Using the notation ${}^{\sigma}\underset{\sim}{b}_{1} = 2\,\mu|_{\underline{\Omega}}$, ${}^{\sigma}\underset{\sim}{a}_{1} = \lambda|_{\underline{\Omega}}$ and using ${}^{\sigma}\underset{\sim}{I}^{1} = \text{tr}[\varepsilon]$, $[{}^{\sigma}\underset{\sim}{G}^{1}] = [\varepsilon]$. We can write (2.101) as

$$
[{}^{d}_{s}\sigma] = \underset{\sim}{\sigma}^{0}\Big|_{\underline{\Omega}}[I] + 2\,\mu|_{\underline{\Omega}}[\varepsilon] + \lambda|_{\underline{\Omega}}(\text{tr}[\varepsilon])[I] - \alpha_{tm}|_{\underline{\Omega}}\left(\theta - \theta|_{\underline{\Omega}}\right)[I] \tag{2.102}
$$

This is Hooke's Law in which μ and λ are Lame's constants defined in a known configuration $\underline{\Omega}$.

3. In the presence of dissipation dependent on strain rate tensor ($\dot{\boldsymbol{\varepsilon}}$), the constitutive theory for ${}_s^d\boldsymbol{\sigma}$ changes to

$$[{}_s^d\sigma] = \underset{\sim}{\sigma}^0\big|_{\underline{\Omega}}[I] + 2\,\mu|_{\underline{\Omega}}[\varepsilon] + \lambda|_{\underline{\Omega}}(\mathrm{tr}[\varepsilon])[I] + 2\,\eta|_{\underline{\Omega}}[\dot{\varepsilon}] \\ + \kappa|_{\underline{\Omega}}(\mathrm{tr}[\dot{\varepsilon}])[I] - \alpha_{tm}|_{\underline{\Omega}}\left(\theta - \theta|_{\underline{\Omega}}\right)[I] \tag{2.103}$$

2.8.2 Constitutive theory for ${}_s\boldsymbol{m}$ using representation theorem

We consider constitutive theory for ${}_s\boldsymbol{m}$ using (2.92) in the absence of dissipation. This derivation is exactly the same as for ${}_s^d\boldsymbol{\sigma}$. Based on representation theorem [8, 62, 69–71, 73–77, 99–103, 107, 108], we begin with

$$[{}_sm] = {}^m\underset{\sim}{\alpha}^0[I] + \sum_{i=1}^{N} {}^m\underset{\sim}{\alpha}^i[{}^m\underset{\sim}{G}^i] \tag{2.104}$$

$${}^m\underset{\sim}{\alpha}^i = {}^m\underset{\sim}{\alpha}^i\left({}^m\underset{\sim}{I}^j, \theta\right); \quad i = 1, 2, \cdots, N; \quad j = 1, 2, \cdots, M \tag{2.105}$$

in which

$$[{}^m\underset{\sim}{G}^1] = [{}_s^{i\Theta}J]\,,\ [{}^m\underset{\sim}{G}^2] = [{}_s^{i\Theta}J]^2\,; \quad [{}^m\underset{\sim}{G}^i]\,;\ i = 1, 2, \cdots, N; \quad N = 2 \\ {}^m\underset{\sim}{I}^1 = I_{({}_s^{i\Theta}J)}\,,\ {}^m\underset{\sim}{I}^2 = II_{({}_s^{i\Theta}J)}\,,\ {}^m\underset{\sim}{I}^3 = III_{({}_s^{i\Theta}J)}\,; \quad [{}^m\underset{\sim}{I}^j]\,;\ j = 1, 2, \cdots, M; \quad M = 3 \tag{2.106}$$

Material coefficients in (2.104) are determined using Taylor series expansion of ${}^m\underset{\sim}{\alpha}^i; i = 0, 1, 2$ in ${}^m\underset{\sim}{I}^j; j = 1, 2, \cdots, M$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^m\underset{\sim}{I}^j; j = 1, 2, \cdots, M$ and θ (for simplicity).

$$ {}^m\underset{\sim}{\alpha}^i = {}^m\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial {}^m\underset{\sim}{\alpha}^i}{\partial {}^m\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j - {}^m\underset{\sim}{I}^j\big|_{\underline{\Omega}}\right) + \frac{\partial {}^m\underset{\sim}{\alpha}^i}{\partial \theta}\bigg|_{\underline{\Omega}} \left(\theta - \theta\big|_{\underline{\Omega}}\right)[I]; \quad i = 0, 1, 2 \tag{2.107} $$

Substituting (2.107) in (2.104), collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients, we can write

$$ \begin{aligned} [{}_s m] = {} & \underset{\sim}{m}^0[I] + \sum_{j=1}^{M} {}^m\underset{\sim}{a}_j \, {}^m\underset{\sim}{I}^j[I] + \sum_{i=1}^{N} {}^m\underset{\sim}{b}_i[{}^m\underset{\sim}{G}^i] + \sum_{i=1}^{N}\sum_{j=1}^{M} {}^m\underset{\sim}{c}_{ij} \, {}^m\underset{\sim}{I}^j[{}^m\underset{\sim}{G}^i] \\ & - \sum_{i=1}^{N} {}^m\underset{\sim}{d}_i \left(\theta - \theta\big|_{\underline{\Omega}}\right)[{}^m\underset{\sim}{G}^i] - {}^m\alpha_{tm}\left(\theta - \theta\big|_{\underline{\Omega}}\right)[I] \end{aligned} \tag{2.108} $$

Coefficients ${}^m\underset{\sim}{a}_j$, ${}^m\underset{\sim}{b}_i$, ${}^m\underset{\sim}{c}_{ij}$, ${}^m\underset{\sim}{d}_i$, ${}^m\alpha_{tm}$ are material coefficients that can be functions of ${}^m\underset{\sim}{I}^j\big|_{\underline{\Omega}}$ and $\theta\big|_{\underline{\Omega}}$.

Remarks

1. This constitutive theory also contains $(N = 2,\ M = 3)$ fourteen material coefficients and contains up to fifth degree terms in the components of $[{}^{i\Theta}_{\ s}J]$, but is linear in θ.

2. A linear constitutive theory in which the products of ${}^m\underset{\sim}{I}^j$, $[{}^m\underset{\sim}{G}^i]$ and $\left(\theta - \theta\big|_{\underline{\Omega}}\right)$ are neglected and only up to linear terms in $[{}^{i\Theta}_{\ s}J]$ are retained is given by

$$ [{}_s m] = \underset{\sim}{m}^0\big|_{\underline{\Omega}}[I] + {}^m\underset{\sim}{a}_1 \, {}^m\underset{\sim}{I}^1[I] + {}^m\underset{\sim}{b}_1[{}^{i\Theta}_{\ s}J] - {}^m\alpha_{tm}\big|_{\underline{\Omega}}\left(\theta - \theta\big|_{\underline{\Omega}}\right)[I] \tag{2.109} $$

Using the notation ${}^m\underset{\sim}{b}_1 = 2\mu_m$ and noting that ${}^m\underset{\sim}{I}^1 = \text{tr}[{}^{i\Theta}_{\ s}J] = 0$, we can write the following from (2.109)

$$ [{}_s m] = \underset{\sim}{m}^0\big|_{\underline{\Omega}}[I] + 2\mu_m[{}^{i\Theta}_{\ s}J] - {}^m\alpha_{tm}\big|_{\underline{\Omega}}\left(\theta - \theta\big|_{\underline{\Omega}}\right)[I] \tag{2.110} $$

A further simplified theory in which the first and the last term in (2.110) are neglected is given by

$$[{}_s m] = 2\mu_m [{}_s^{i\Theta} J] \tag{2.111}$$

in which the material coefficient $2\mu_m$ can be dependent on the invariant of $[{}_s^{i\Theta} J]$ and θ in a known configuration $\underline{\Omega}$.

$$\mu_m = \mu_m \left(I_{({}_s^{i\Theta} J)}\Big|_{\underline{\Omega}} , II_{({}_s^{i\Theta} J)}\Big|_{\underline{\Omega}} , III_{({}_s^{i\Theta} J)}\Big|_{\underline{\Omega}} , \theta|_{\underline{\Omega}} \right) \tag{2.112}$$

3. In the presence of dissipation dependent on the rate of symmetric part of the internal rotation gradient tensor (${}_s^{i\Theta} \dot{\boldsymbol{J}}$), the constitutive theory for ${}_s\boldsymbol{m}$ changes to

$$[{}_s m] = 2\mu_m [{}_s^{i\Theta} J] + 2\eta_m [{}_s^{i\Theta} \dot{J}] \tag{2.113}$$

2.8.3 Constitutive theory for $\boldsymbol{q}$

We consider $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta)$ and use representation theorem [8,62,69–71,73–77,99–103,107,108]. The only tensor of rank one from the combined generators of the argument tensors $\boldsymbol{g}$ and θ is just $\boldsymbol{g}$ and the combined invariant is $\boldsymbol{g} \cdot \boldsymbol{g}$ (or ${}^q\!I$). Thus, we can write based on [97]

$$\{q\} = -{}^q\alpha\{q\} \tag{2.114}$$

in which ${}^q\!\underset{\sim}{\alpha} = {}^q\!\underset{\sim}{\alpha}({}^q\!I, \theta)$.

Material coefficients in the constitutive theory for $\boldsymbol{q}$ given by (2.114) are obtained by considering Taylor series expansion of ${}^q\!\underset{\sim}{\alpha}$ in ${}^q\!I$ and θ about a known configuration $\underline{\Omega}$ and retaining up to linear terms in ${}^q\!I$ and θ (for simplicity).

$${}^q\!\underset{\sim}{\alpha} = {}^q\!\underset{\sim}{\alpha}\Big|_{\underline{\Omega}} + \frac{\partial({}^q\alpha)}{\partial({}^q\!I)}\bigg|_{\underline{\Omega}} \left({}^q\!I - {}^q\!I|_{\underline{\Omega}}\right) + \frac{\partial({}^q\alpha)}{\partial\theta}\bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}}\right) \tag{2.115}$$

Substituting (2.115) in (2.114) and collecting coefficients gives

$$\{q\} = -k_1\big|_{\underline{\Omega}}\{g\} - k_2\big|_{\underline{\Omega}}(\{g\}^T\{g\})\{g\} - k_3\big|_{\underline{\Omega}}\left(\theta - \theta\big|_{\underline{\Omega}}\right)\{g\} \tag{2.116}$$

This constitutive theory is based on integrity, hence uses complete basis. From (2.116) we can derive a linear constitutive theory for $\{q\}$.

$$\{q\} = -k_1\big|_{\underline{\Omega}}\{g\} \tag{2.117}$$

The material coefficients k_1, k_2 and k_3 are in a known configuration $\underline{\Omega}$ and can be functions of ${}^q\!I$, i.e., $\{g\}^T\{g\}$ and temperature θ in $\underline{\Omega}$.

2.9 Complete mathematical model resulting from NCCM for solids

In the following, we present the complete mathematical model consisting of CBL (2.66)-(2.69) and the constitutive theories (2.103), (2.113) and (2.117). For small strain, small deformation physics the CBL and constitutive theories with dissipation due to rate of strain (CCM) and rate of symmetric part of rotation gradient tensor (micropolar NCCT) are given by

$$\rho_0 \frac{D^2 u_k}{Dt^2} - \rho_0 F_k{}^b - \frac{\partial \sigma_{jk}}{\partial x_j} = 0 \qquad \text{(BLM)} \tag{2.118}$$

$${}^\Theta I_0 \rho_0 \frac{D^2({}_i\Theta_k)}{Dt^2} - \epsilon_{ijk}\sigma_{ij} - \frac{\partial m_{jk}}{\partial x_j} = 0 \qquad \text{(BAM)} \tag{2.119}$$

$$\epsilon_{jkl}\left({}^\Theta I_0 \rho_0 v_j ({}_i\omega_k) - m_{jk}\right) = 0 \qquad \text{(BMM)} \tag{2.120}$$

$$\rho_0 \frac{De}{Dt} + \frac{\partial q_i}{\partial x_i} - \text{tr}([{}_s\sigma][\dot{\varepsilon}]) - \text{tr}([{}_s m][{}^{i\Theta}_{\ s}\dot{J}]) - \text{tr}([{}_a m][{}^{i\Theta}_{\ a}\dot{J}]) = 0 \qquad \text{(FLT)} \tag{2.121}$$

$$\text{Additive decomposition of } \boldsymbol{\sigma} \quad : \quad \boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} \quad ; \quad {}_s\boldsymbol{\sigma} = {}^d_s\boldsymbol{\sigma} + {}^e_s\boldsymbol{\sigma} \tag{2.122}$$

$$ {}_s^d\boldsymbol{\sigma} = \sigma_0|_{\underline{\Omega}}\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} + \lambda\left(\mathrm{tr}(\boldsymbol{\varepsilon})\right)\boldsymbol{I} + 2\eta\dot{\boldsymbol{\varepsilon}} + \kappa\mathrm{tr}(\dot{\boldsymbol{\varepsilon}})\boldsymbol{I} \tag{2.123} $$

$$ {}_s\boldsymbol{m} = 2\mu_m\left({}_s^{i\Theta}\boldsymbol{J}\right) + 2\eta_m\left({}_s^{i\Theta}\dot{\boldsymbol{J}}\right) \tag{2.124} $$

$$ \boldsymbol{q} = -k\boldsymbol{g} \tag{2.125} $$

In the mathematical model we have a total of 25 equations BLM(3), BAM(3), BMM(3), FLT(1), constitutive theories for ${}_s^d\boldsymbol{\sigma}(6)$, ${}_s\boldsymbol{m}(6)$, $\boldsymbol{q}(3)$ in a total of 25 dependent variables $\boldsymbol{u}(3)$, ${}_s^d\boldsymbol{\sigma}(6)$, ${}_a\boldsymbol{\sigma}(3)$, ${}_s\boldsymbol{m}(6)$, ${}_a\boldsymbol{m}(3)$, $\boldsymbol{q}(3)$, $\theta(1)$ hence the model has closure. The last term in the entropy inequality

$$ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s^d\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_s\boldsymbol{m} : {}_s^{i\Theta}\dot{\boldsymbol{J}} - {}_a\boldsymbol{m} : {}_a^{i\Theta}\dot{\boldsymbol{J}} \leq 0 \tag{2.126} $$

must be set to zero to ensure that (2.126) is always satisfied. Then, in addition to (2.118)-(2.125) we must also satisfy

$$ {}_a\boldsymbol{m} : {}_a^{i\Theta}\dot{\boldsymbol{J}} = 0 \tag{2.127} $$

Hence equation (2.127) serves as a constraint on the mathematical model that must be satisfied to guarantee that the entropy inequality is not violated. The final mathematical model consists of equations (2.118)-(2.125), (2.127).

Chapter 3

Conservation and balance laws for micropolar fluids

In this chapter, we present the CBL and constitutive theories for micropolar NCCT based on internal or classical rotation rates ${}_{i}^{r}\bar{\Theta}$ in Eulerian description in which rotational inertial physics is considered.

3.1 Conservation of mass

The continuity equation resulting from the principle of conservation of mass remains the same in micropolar NCCT considered here as in case of CCT. The differential form of the continuity equation in Eulerian description for a compressible fluid is given by [97, 98]

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\boldsymbol{\nabla}} \cdot (\bar{\rho}\bar{\boldsymbol{v}}) = 0 \quad \text{or} \quad \frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\text{div}(\bar{\boldsymbol{v}}) = 0 \tag{3.1}$$

For incompressible matter $\rho_0 = \bar{\rho}$; hence (3.1) reduces to

$$\text{div}(\bar{\boldsymbol{v}}) = 0 \tag{3.2}$$

3.2 Balance of linear momenta

Balance of linear momenta in micropolar NCCT also remains the same as in CCT. We have the following equations for BLM in Eulerian description where ${}^{(0)}\bar{\boldsymbol{\sigma}}$ is basis independent Cauchy stress tensor [97, 98].

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}}\cdot\left({}^{(0)}\bar{\boldsymbol{\sigma}}\right) = 0 \quad \text{or} \quad \bar{\rho}\frac{\partial \bar{v}_i}{\partial t} + \bar{\rho}\bar{v}_j\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho}\bar{F}_i^b - \frac{\partial\left({}^{(0)}\bar{\sigma}_{ji}\right)}{\partial \bar{x}_j} = 0 \tag{3.3}$$

in which $\bar{\boldsymbol{F}}^b$ is body force per unit mass.

3.3 Balance of angular momenta

The principle of balance of angular momenta for non-classical continuum mechanics (NCCM) incorporating internal rotation rates, their spatial and temporal derivatives and inertial effects can be stated as: The time rate of change of moment of moments is equal to the sum of moments of the forces and the moments in the current configuration at any time t. Let ${}^{\Theta}\bar{I}$ be the rotational inertia per unit mass of the deforming fluent continua then ${}^{\Theta}\bar{I}\bar{\rho}({}_i\bar{\boldsymbol{\omega}})d\bar{V}$ is the angular momenta per unit mass of the fluent continua for the elemental volume $d\bar{V}$ due to ${}^{\Theta}\bar{I}$ and angular velocity ${}_i\bar{\boldsymbol{\omega}}$. The moment of linear momenta for the same volume $d\bar{V}$ is $\bar{\boldsymbol{x}} \times \bar{\rho}\bar{v}d\bar{V}$. Then, according to this balance law:

$$\begin{aligned} &\text{rate of change of angular momenta} \;=\; (\text{moments due to } \bar{P} + \bar{M}) \\ &+ \text{moments of the body forces } \bar{\rho}\bar{\boldsymbol{F}}^b \text{acting on } d\bar{V} \end{aligned}$$

Thus, for the deformed volume $\bar{V}$ bounded by $\partial\bar{V}$ we can write:

$$\begin{aligned} \frac{D}{Dt}\int_{\bar{V}} \left({}^{\Theta}\bar{I}({}_i\bar{\boldsymbol{\omega}})\bar{\rho} + \bar{\boldsymbol{x}} \times \rho\bar{\boldsymbol{v}}\right) d\bar{V} = &\int_{\partial\bar{V}} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}\right) d\bar{A} \\ &+ \int_{\bar{V}} \left(\bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b\right) d\bar{V} \end{aligned} \tag{3.4}$$

We consider each term of (3.4)

$$\begin{aligned}\frac{D}{Dt}\int_{\bar{V}} {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})\bar{\rho}\, d\bar{V} &= \frac{D}{Dt}\int_{V} {}^{\Theta}I({}_{i}\boldsymbol{\omega})\rho_{0}\, dV \\ &= \int_{V} \frac{D}{Dt}\left({}^{\Theta}I({}_{i}\boldsymbol{\omega})\right)\rho_{0}\, dV \\ &= \int_{\bar{V}} \frac{D}{Dt}\left({}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})\right)\bar{\rho}\, d\bar{V}\end{aligned} \tag{3.5}$$

if ${}^{\Theta}\bar{I}$ is constant, then (3.5) reduces to

$$\frac{D}{Dt}\int_{\bar{V}} {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})\bar{\rho}\, d\bar{V} = \int_{\bar{V}} {}^{\Theta}\bar{I}\bar{\rho}\frac{D}{Dt}({}_{i}\bar{\boldsymbol{\omega}})d\bar{V} \tag{3.6}$$

and

$$\begin{aligned}\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times \rho\bar{\boldsymbol{v}} d\bar{V} &= \frac{D}{Dt}\int_{\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{v}_j\bar{\rho} d\bar{V} \\ &= \frac{D}{Dt}\int_{V} \epsilon_{ijk}x_i v_j \rho_{0}\, dV \\ &= \int_{\bar{V}} \frac{D}{Dt}\left(\epsilon_{ijk}\bar{x}_i\bar{v}_j\right)\bar{\rho} d\bar{V} \\ &= \int_{\bar{V}} \epsilon_{ijk}\left(\bar{v}_i\bar{v}_j + \bar{x}_i\frac{D\bar{v}_j}{Dt}\right)\bar{\rho} d\bar{V}\end{aligned} \tag{3.7}$$

since $\epsilon_{ijk}\bar{v}_i\bar{v}_j = 0$, (3.7) reduces to

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times \rho\bar{\boldsymbol{v}} d\bar{V} = \int_{\bar{V}} \epsilon_{ijk}\bar{x}_i\frac{D\bar{v}_j}{Dt} d\bar{V} \tag{3.8}$$

and

$$\begin{aligned}\int_{\partial\bar{V}} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}\right) d\bar{A} &= \int_{\partial\bar{V}} \left(\bar{\boldsymbol{x}} \times {}^{(0)}\, \bar{\boldsymbol{\sigma}}^{T} \cdot \bar{\boldsymbol{n}} + {}^{(0)}\bar{\boldsymbol{m}}^{T} \cdot \bar{\boldsymbol{n}}\right) d\bar{A} \\ &= \int_{\partial\bar{V}} \left(\epsilon_{ijk}\bar{x}_i({}^{(0)}\bar{\sigma}_{mj})\bar{n}_m + {}^{(0)}\bar{m}_{mj}\bar{n}_m\right) d\bar{A}\end{aligned} \tag{3.9}$$

using Divergence Theorem

$$\int_{\partial\bar{V}} (\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}) d\bar{A} = \int_{\bar{V}} \left(\epsilon_{ijk}(\bar{x}_i({}^{(0)}\bar{\sigma}_{mj}))_{,m} + ({}^{(0)}\bar{m}_{mj})_{,m}\right) d\bar{V} \tag{3.10}$$

we note the following

$$\epsilon_{ijk}\left(\bar{x}_i({}^{(0)}\bar{\sigma}_{mj})\right)_{,m} = \epsilon_{ijk}\left(\delta_{im}{}^{(0)}\bar{\sigma}_{mj} + \bar{x}_i{}^{(0)}\bar{\sigma}_{mj,m}\right) \tag{3.11}$$

using (3.11) in (3.10) we can write

$$\int_{\partial\bar{V}} (\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} + \bar{\boldsymbol{M}}) d\bar{A} = \int_{\bar{V}} \left(\epsilon_{ijk}(\bar{x}_i({}^{(0)}\bar{\sigma}_{mj}))_{,m} + ({}^{(0)}\bar{m}_{mj})_{,m}\right) d\bar{V} \tag{3.12}$$

and

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{F}}^b d\bar{V} = \int_{\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{F}_j^b\bar{\rho}d\bar{V} \tag{3.13}$$

substituting from (3.6), (3.8), (3.12) and (3.13) in (3.4) and rearranging terms, we obtain

$$\begin{aligned}\int_{\bar{V}} {}^{\Theta}\bar{I}\bar{\rho}\frac{D}{Dt}({}_i\bar{\omega}_k)d\bar{V} + \int_{\bar{V}} \epsilon_{ijk}\bar{x}_i\left(\bar{\rho}\frac{D\bar{v}_j}{Dt} - {}^{(0)}\bar{\sigma}_{mj,m} - \bar{\rho}\bar{F}_j^b\right) d\bar{V} \\ - \int_{\bar{V}} \left(\epsilon_{ijk}{}^{(0)}\bar{\sigma}_{ij} + {}^{(0)}\bar{m}_{mk,m}\right) d\bar{V} = 0\end{aligned} \tag{3.14}$$

The coefficient of $\epsilon_{ijk}\bar{x}_i$ in the second term in (3.14) is zero due to balance of linear momenta, hence (3.14) reduces to

$$\int_{\bar{V}} \left({}^{\Theta}\bar{I}\bar{\rho}\frac{D}{Dt}({}_i\bar{\omega}_k) - \epsilon_{ijk}{}^{(0)}\bar{\sigma}_{ij} - {}^{(0)}\bar{m}_{mk,m}\right) d\bar{V} = 0 \tag{3.15}$$

For isotropic homogeneous matter $\bar{V}$ is arbitrary hence we can obtain differential form of (3.15)

$${}^{\Theta}\bar{I}\bar{\rho}\frac{D}{Dt}({}_i\bar{\omega}_k) - \epsilon_{ijk}{}^{(0)}\bar{\sigma}_{ij} - {}^{(0)}\bar{m}_{mk,m} = 0 \tag{3.16}$$

Remarks

1. If we set the first and the last term in (3.16) to zero, then we recover balance of angular momenta for classical continuum mechanics in Eulerian description.

2. If we set the first term in (3.16) to zero but retain second and third order terms, then we

have balance of angular momenta for NCCM incorporating internal rotation rates without the rotational inertial physics.

3. Appearance of the first term in (3.16) is due to consideration of time varying rotation rates and rotational inertia ${}^{\Theta}\bar{I}$. This is new physics considered in the present work that neither appears in CCM nor NCCM published works.

4. Equation (3.16) is the final form of balance of angular momenta.

3.4 Balance of moment of moments

This is a new balance law originally proposed by Yang et. al [104] for NCCM which was derived based on static considerations (hence cannot be referred to as a balance law). Later, Surana et. al explained the rationale for this balance law and pointed out that a balance law must be derived using rate considerations. In references [82, 93, 96] they presented derivation of the "balance of moment of moments" balance law for NCCM for fluent and solid continua in the presence of internal rotation rates and internal rotations. In the work presented in this dissertation, the physics considered is different than in reference [82], hence a rederivation of this balance law is necessary. According to this balance law the rate of change of moment of angular momenta due to rotation rates in a deformed volume $\bar{V}$ must be equal to the sum of the moment of moments due to the antisymmetric components of the Cauchy stress tensor over the same deformed volume $\bar{V}$ and the moment of $\bar{\boldsymbol{M}}$ acting on boundary $\partial\bar{V}$ of $\bar{V}$.

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})\bar{\rho}\, d\bar{V} = \int_{\bar{V}} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}})\, d\bar{V} + \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} \tag{3.17}$$

we expand each term of (3.17) in the following

$$\begin{aligned}\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})\bar{\rho}\, d\bar{V} &= \frac{D}{Dt}\int_{\bar{V}} {}^{\Theta}\bar{I}\, \epsilon_{jkl}\bar{x}_j({}_{i}\bar{\omega}_k)\bar{\rho}\, d\bar{V} \\ &= \frac{D}{Dt}\int_{V} {}^{\Theta}I\, \epsilon_{jkl}x_j({}_{i}\omega_k)\, \rho_0\, dV \\ &= \int_{\bar{V}} \frac{D}{Dt}\left({}^{\Theta}\bar{I}\, \epsilon_{jkl}\bar{x}_j({}_{i}\bar{\omega}_k)\bar{\rho}\right) d\bar{V} \end{aligned} \tag{3.18}$$

Assuming ${}^{\Theta}\bar{I}$ to be constant

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\boldsymbol{x}} \times {}^{\Theta}\bar{I}({}_{i}\bar{\boldsymbol{\omega}})\bar{\rho}\, d\bar{V} = \int_{\bar{V}} {}^{\Theta}\bar{I}\epsilon_{jkl}\left(\bar{v}_j({}_{i}\bar{\omega}_k) + \bar{x}_j\frac{D}{Dt}({}_{i}\bar{\omega}_k)\right)\bar{\rho} d\bar{V} \tag{3.19}$$

Expanding the last term

$$\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} = \int_{\partial\bar{V}} \epsilon_{jkl}\bar{x}_j\bar{M}_k d\bar{A} \tag{3.20}$$

using Cauchy principle for $\bar{\boldsymbol{M}}$

$$\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} = \int_{\partial\bar{V}} \epsilon_{jkl}\bar{x}_j({}^{(0)}\bar{m}_{mk})\bar{n}_m d\bar{A} \tag{3.21}$$

using Divergence Theorem

$$\begin{aligned}\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int_{\bar{V}} \epsilon_{jkl}\left(\bar{x}_j\left({}^{(0)}\bar{m}_{mk}\right)\right)_{,m}\, d\bar{V} \\ &= \int_{\bar{V}} \epsilon_{jkl}\left(\delta_{jm}\, {}^{(0)}\bar{m}_{mk} + \bar{x}_j\left({}^{(0)}\bar{m}_{mk,m}\right)\right)\, d\bar{V} \\ &= \int_{\bar{V}} \epsilon_{jkl}\left({}^{(0)}\bar{m}_{jk} + \bar{x}_j\, {}^{(0)}\bar{m}_{mk,m}\right)\, d\bar{V} \end{aligned} \tag{3.22}$$

substituting from (3.19) and (3.22) in (3.17)

$$\begin{aligned}\int_{\bar{V}} {}^{\Theta}\bar{I}\,\bar{\rho}\,\epsilon_{jkl}\left(\bar{v}_j({}_{i}\bar{\omega}_k) + \bar{x}_j\,\frac{D({}_{i}\bar{\omega}_k)}{Dt}\right) d\bar{V} &= \int_{\bar{V}} \bar{\boldsymbol{x}} \times \boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} d\bar{V} \\ &+ \int_{\bar{V}} \epsilon_{jkl}\left({}^{(0)}\bar{m}_{jk} + \bar{x}_j\left({}^{(0)}\bar{m}_{mk,m}\right)\right)\, d\bar{V} \end{aligned} \tag{3.23}$$

we note that

$$
\begin{aligned}
{}^{(0)}\bar{m}_{mk,m} &= \bar{\boldsymbol{\nabla}} \boldsymbol{\cdot} {}^{(0)}\bar{\boldsymbol{m}} \\
\epsilon_{jkl}\left(\bar{x}_j({}^{(0)}\bar{m}_{mk,m})\right) &= \bar{\boldsymbol{x}} \times \bar{\boldsymbol{\nabla}} \boldsymbol{\cdot} {}^{(0)}\bar{\boldsymbol{m}} \\
{}^{\Theta}\bar{I}\bar{\rho}\epsilon_{jkl}\frac{D({}_i\bar{\omega}_k)}{Dt} &= {}^{\Theta}\bar{I}\bar{\rho}\,\bar{\boldsymbol{x}} \times \frac{D({}_i\bar{\boldsymbol{\omega}})}{Dt}
\end{aligned}
\tag{3.24}
$$

using (3.24) in (3.23) and regrouping terms

$$
\begin{aligned}
\int_{\bar{V}} {}^{\Theta}\bar{I}\bar{\rho}\epsilon_{jkl}\bar{v}_j\left({}_i\omega_k\right)d\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}}\times\left({}^{\Theta}\bar{I}\bar{\rho}\frac{D\left({}_i\bar{\boldsymbol{\omega}}\right)}{Dt} - \boldsymbol{\epsilon} : {}^{(0)}\boldsymbol{\sigma} - \bar{\boldsymbol{\nabla}} \boldsymbol{\cdot} {}^{(0)}\bar{\boldsymbol{m}}\right)d\bar{V} \\
= \int_{\bar{V}} \epsilon_{jkl}{}^{(0)}\bar{m}_{jk}d\bar{V}
\end{aligned}
\tag{3.25}
$$

using balance of angular momenta (3.15) in (3.25), we obtain

$$
\int_{\bar{V}} \epsilon_{jkl}\left({}^{\Theta}\bar{I}\bar{\rho}\bar{v}_j\left({}_i\bar{\omega}_k\right) - {}^{(0)}\bar{m}_{jk}\right)d\bar{V} = 0 \tag{3.26}
$$

For homogeneous, isotropic continua, $\bar{V}$ is arbitrary, hence we obtain the following from (3.26)

$$
\epsilon_{jkl}\left({}^{\Theta}\bar{I}\bar{\rho}\bar{v}_j\left({}_i\bar{\omega}_k\right) - {}^{(0)}\bar{m}_{jk}\right) = 0 \tag{3.27}
$$

Equation (3.27) is the final form resulting from the balance of moment of moments balance law.

Remarks

1. We note that in the absence of rotational inertia ${}^{\Theta}\bar{I}$ (new physics considered in this work), i.e., when ${}^{\Theta}\bar{I} = 0$, (3.27) reduces to

$$
\epsilon_{jkl}\,{}^{(0)}\bar{m}_{jk} = 0 \tag{3.28}
$$

   This is same as the BMM balance law introduced in references [82, 96, 104].

2. When ${}^{\Theta}\bar{I}$ is not zero, (3.27) yields three equations defining the antisymmetric parts of the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ in terms of velocities, rotation rates (angular velocities) and the

properties $\bar{\rho}$ and ${}^{\Theta}\bar{I}$ of the continua.

3.5 First law of thermodynamics

The sum of work and heat added to a volume of matter must result in an increase of the energy of the volume. This can be expressed as a rate equation in Eulerian description.

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{dt} + \frac{D\bar{W}}{Dt} \tag{3.29}$$

where $\bar{E}_t$, $\bar{Q}$ and $\bar{W}$ are total energy, heat added and work done. Their rates can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} + \frac{1}{2}{}^{\Theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\cdot{}_i\bar{\boldsymbol{\omega}}\right) - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} \tag{3.30}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\; d\bar{A} \tag{3.31}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}} \left(\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}_i^r\bar{\boldsymbol{\Theta}}\right)d\bar{A} \tag{3.32}$$

where $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ are body forces per unit mass and $\bar{\boldsymbol{g}}$ is heat vector. The third term in the integrand is due to additional rate of work due to rotation rates. We expand integrals in (3.30)-(3.32). Following reference [97] we can show

$$\begin{aligned}\frac{D\bar{E}_t}{Dt} &= \frac{D}{Dt}\int_{\bar{V}} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} + \frac{1}{2}{}^{\Theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\cdot{}_i\bar{\boldsymbol{\omega}}\right) - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} \\ &= \int_{V}\left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\rho}\bar{\boldsymbol{v}}\frac{D\bar{\boldsymbol{v}}}{Dt} + \bar{\rho}{}^{\Theta}\bar{I}\left({}_i\bar{\boldsymbol{\omega}}\right)\cdot\frac{D({}_i\bar{\boldsymbol{\omega}})}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{v}}\right)d\bar{V}\end{aligned} \tag{3.33}$$

using Divergence Theorem (3.31) can be written as

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}d\bar{A} = \int_{\bar{V}} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}\; d\bar{V} \tag{3.34}$$

using Cauchy principle for $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{M}}$ we can show that

$$
\begin{aligned}
\frac{D\bar{W}}{Dt} &= \int_{\partial\bar{V}} \left(\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} + \bar{\boldsymbol{M}} \cdot {}^{r}_{i}\bar{\boldsymbol{\Theta}}\right) d\bar{A} \\
&= \int_{\partial\bar{V}} \left(\bar{\boldsymbol{v}} \cdot ({}^{(0)}\bar{\boldsymbol{\sigma}})^{T} \cdot \bar{\boldsymbol{n}} + {}^{r}_{i}\bar{\boldsymbol{\Theta}} \cdot ({}^{(0)}\bar{\boldsymbol{m}}) \cdot \bar{\boldsymbol{n}}\right) d\bar{A} \\
&= \int_{\partial\bar{V}} \left(\bar{\boldsymbol{v}} \cdot ({}^{(0)}\bar{\boldsymbol{\sigma}})^{T} + {}^{r}_{i}\bar{\boldsymbol{\Theta}} \cdot ({}^{(0)}\bar{\boldsymbol{m}})\right) d\bar{\boldsymbol{A}}
\end{aligned} \tag{3.35}
$$

using Divergence Theorem

$$
\frac{D\bar{W}}{Dt} = \int_{\bar{V}} \left(\bar{\boldsymbol{\nabla}} \cdot \left(\bar{\boldsymbol{v}} \cdot ({}^{(0)}\bar{\boldsymbol{\sigma}})^{T}\right) + \bar{\boldsymbol{\nabla}} \cdot \left({}^{r}_{i}\bar{\boldsymbol{\Theta}} \cdot ({}^{(0)}\bar{\boldsymbol{m}})^{T}\right)\right) d\bar{V} \tag{3.36}
$$

following reference [97], we can show

$$
\bar{\boldsymbol{\nabla}} \cdot \left(\bar{\boldsymbol{v}} \cdot ({}^{(0)}\bar{\boldsymbol{\sigma}})^{T}\right) = \bar{\boldsymbol{v}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right) + {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} \tag{3.37}
$$

$$
\bar{\boldsymbol{\nabla}} \cdot \left({}^{r}_{i}\bar{\boldsymbol{\Theta}} \cdot ({}^{(0)}\bar{\boldsymbol{m}})^{T}\right) = {}^{r}_{i}\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) + {}^{(0)}\bar{\boldsymbol{m}} : {}^{r\Theta}_{i}\bar{\boldsymbol{J}} \tag{3.38}
$$

substituting (3.37) and (3.38) in (3.36)

$$
\begin{aligned}
\frac{D\bar{W}}{Dt} = \int_{\bar{V}} (\bar{\boldsymbol{v}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right) + {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} \\
+ {}^{r}_{i}\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) + {}^{(0)}\bar{\boldsymbol{m}} : {}^{r\Theta}_{i}\bar{\boldsymbol{J}}) d\bar{V}
\end{aligned} \tag{3.39}
$$

Substituting (3.30), (3.34) and (3.39) in (3.29)

$$
\begin{aligned}
&\int_{\bar{V}} \bar{\boldsymbol{v}} \cdot \left(\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^{b} - \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right) d\bar{V} + \int_{\bar{V}} \left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}}\right. \\
&\left. - {}^{(0)}\bar{\boldsymbol{m}} : {}^{r\Theta}_{i}\bar{\boldsymbol{J}} - {}^{r}_{i}\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) + \bar{\rho}\left({}^{\Theta}\bar{I}\right)\left({}_{i}\bar{\boldsymbol{\omega}} \cdot \frac{D({}_{i}\bar{\boldsymbol{\omega}})}{Dt}\right)\right) d\bar{V} = 0
\end{aligned} \tag{3.40}
$$

Using balance of linear momenta (3.3) in (3.40) and grouping last two terms in the integrand we obtain (noting that ${}^{r}_{i}\bar{\boldsymbol{\Theta}} = {}_{i}\bar{\boldsymbol{\omega}}$)

$$\int_{\bar{V}} (\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}^{r}_{i}\Theta\bar{\boldsymbol{J}} + {}_{i}\bar{\boldsymbol{\omega}} \cdot \left(\bar{\rho}({}^{\Theta}\bar{I}) \frac{D({}_{i}\bar{\boldsymbol{\omega}})}{Dt} - \bar{\boldsymbol{\nabla}} \cdot \left({}^{(0)}\bar{\boldsymbol{m}}\right) \right)) d\bar{V} = 0 \tag{3.41}$$

For isotropic, homogeneous continua, $\bar{V}$ is arbitrary, hence we can set the integrand in (3.41) to zero.

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}^{r}_{i}\Theta\bar{\boldsymbol{J}} + {}_{i}\bar{\boldsymbol{\omega}} \cdot \left(\bar{\rho}({}^{\Theta}\bar{I}) \frac{D({}_{i}\bar{\boldsymbol{\omega}})}{Dt} - \bar{\boldsymbol{\nabla}} \cdot \left({}^{(0)}\bar{\boldsymbol{m}}\right) \right) = 0 \tag{3.42}$$

From balance of angular momenta

$$\bar{\rho}({}^{\Theta}\bar{I}) \frac{D({}_{i}\bar{\boldsymbol{\omega}})}{Dt} - \bar{\boldsymbol{\nabla}} \cdot \left({}^{(0)}\bar{\boldsymbol{m}}\right) = \boldsymbol{\epsilon} : \left({}^{(0)}\bar{\boldsymbol{m}}\right) \tag{3.43}$$

Substituting from (3.43) into (3.42)

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}^{r}_{i}\Theta\bar{\boldsymbol{J}} + {}_{i}\bar{\boldsymbol{\omega}} \cdot (\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}) = 0 \tag{3.44}$$

Let

$$\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} = {}^{(0)}\bar{\boldsymbol{\tau}} \tag{3.45}$$

in which ${}^{(0)}\bar{\boldsymbol{\tau}}$ is a vector, containing three components, and noting that

$${}_{i}\bar{\boldsymbol{\omega}} \cdot {}^{(0)}\bar{\boldsymbol{\tau}} = {}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_{i}\bar{\boldsymbol{\omega}} \tag{3.46}$$

using (3.45) and (3.46) in (3.44) we obtain

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} - {}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r\Theta}\bar{\boldsymbol{J}} + {}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_{i}\bar{\boldsymbol{\omega}} = 0 \tag{3.47}$$

the energy equation (3.47) resulting from the first law of thermodynamics can be further simplified (shown below). We note the following,

$$\bar{\boldsymbol{L}} = \bar{\boldsymbol{D}} + \bar{\boldsymbol{W}} \tag{3.48}$$

$${}_{i}^{r\Theta}\bar{\boldsymbol{J}} = {}_{i\ s}^{r\Theta}\bar{\boldsymbol{J}} + {}_{i\ a}^{r\Theta}\bar{\boldsymbol{J}} \tag{3.49}$$

We consider decomposition of ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ into symmetric and antisymmetric parts

$$\begin{aligned} {}^{(0)}\bar{\boldsymbol{\sigma}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} + {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} \\ {}^{(0)}\bar{\boldsymbol{m}} &= {}_{s}^{(0)}\bar{\boldsymbol{m}} + {}_{a}^{(0)}\bar{\boldsymbol{m}} \end{aligned} \tag{3.50}$$

for which

$$\begin{aligned} {}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{L}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} + {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{W}} \\ {}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r\Theta}\bar{\boldsymbol{J}} &= {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i\ s}^{r\Theta}\bar{\boldsymbol{J}} + {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i\ a}^{r\Theta}\bar{\boldsymbol{J}} \end{aligned} \tag{3.51}$$

Substituting (3.51) in (3.47) we can obtain

$$\begin{aligned} \bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{W}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i\ s}^{r\Theta}\bar{\boldsymbol{J}} \\ - {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i\ a}^{r\Theta}\bar{\boldsymbol{J}} + {}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_{i}\bar{\boldsymbol{\omega}} = 0 \end{aligned} \tag{3.52}$$

We can show that

$${}^{(0)}\bar{\boldsymbol{\tau}} \cdot {}_{i}\bar{\boldsymbol{\omega}} = {}_{a}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{W}} \tag{3.53}$$

using (3.53) in (3.52)

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i\ s}^{r\Theta}\bar{\boldsymbol{J}} - {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i\ a}^{r\Theta}\bar{\boldsymbol{J}} = 0 \tag{3.54}$$

This is the final form of the energy equation resulting from the first law of thermodynamics.

3.6 Second law of thermodynamics

If $\bar{\eta}$ is the entropy density in the volume $\bar{V}$, $\bar{h}$ is the entropy flux between $\bar{V}$ and the volume of matter surrounding it and $\bar{s}$ is the source of entropy in $\bar{V}$ due to non contacting sources (bodies), then the rate of increase of entropy in volume $\bar{V}$ is at least equal to that applied to $\bar{V}$ from all contacting and non-contacting sources [97]. Thus

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\eta}\bar{\rho}d\bar{V} \geq \int_{\partial\bar{V}} \bar{h}d\bar{A} + \int_{\bar{V}} \bar{s}\bar{\rho}d\bar{V} \tag{3.55}$$

using Cauchy's postulate for $\bar{h}$

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \tag{3.56}$$

using (3.56) in (3.55)

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\eta}\bar{\rho}d\bar{V} \geq -\int_{\partial\bar{V}} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}}d\bar{A} + \int_{\bar{V}} \bar{s}\bar{\rho}d\bar{V} \tag{3.57}$$

using Gauss' Divergence Theorem for the terms over $\partial\bar{V}$ gives (noting that $\bar{\boldsymbol{\psi}}$ is a tensor of rank one)

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\eta}\bar{\rho}d\bar{V} \geq -\int_{\bar{V}} \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\psi}}d\bar{V} + \int_{\bar{V}} \bar{s}\bar{\rho}d\bar{V} \tag{3.58}$$

we note that

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\eta}(\bar{\rho}d\bar{V}) = \frac{D}{Dt}\int_{V} \eta\rho_0 dV = \int_{V} \frac{D\eta}{Dt}\rho_0 dV = \int_{\bar{V}} \frac{D\bar{\eta}}{Dt}\bar{\rho}d\bar{V} \tag{3.59}$$

using (3.59) in (3.58) we obtain

$$\int_{\bar{V}} \left(\bar{\rho}\frac{D\bar{\eta}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\psi}} - \bar{s}\bar{\rho}\right) d\bar{V} \geq 0 \tag{3.60}$$

For homogeneous. isotropic matter volume $\bar{V}$ is arbitrary hence we can write the following from (3.60)

$$\bar{\rho}\frac{D\bar{\eta}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\psi}} - \bar{s}\bar{\rho} \geq 0 \tag{3.61}$$

Equation (3.61) is the most fundamental form of the SLT or entropy inequality (Clausius Duhem inequality). We note that entropy inequality is strictly a statement that contains entropy terms, hence contains no information regarding reversible deformation physics. In this form (3.61) the entropy inequality provides no mechanism(s) for deriving constitutive theories. Only when the mechanical rate of work that results in rate of entropy production is introduced in the entropy inequality, will the entropy inequality contain information regarding conjugate pairs resulting in rate of entropy production. We also note entropy inequality (3.61) does not provide any information regarding constitutive theory for heat vector $\bar{\boldsymbol{q}}$. In the following we derive another form of the entropy inequality using a relationship between $\bar{\boldsymbol{\psi}}$ and $\bar{\boldsymbol{q}}$ and relationship between $\bar{\Phi}$, $\bar{e}$ and $\bar{\eta}$. Since the energy equation has all possible mechanisms that result in energy storage and dissipation, the form of entropy inequality derived using energy equation is expected to be helpful in the derivation of the constitutive theories. Using

$$\bar{\boldsymbol{\psi}} = \frac{\bar{\boldsymbol{q}}}{\bar{\theta}} \quad , \quad \bar{s} = \frac{\bar{r}}{\bar{\theta}} \tag{3.62}$$

where $\bar{\theta}$ is absolute temperature and $\bar{r}$ is a suitable potential

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\psi}} = \bar{\psi}_{i,i} = \frac{\bar{q}_{i,i}}{\bar{\theta}} - \frac{\bar{q}_i \bar{\theta}_{,i}}{\bar{\theta}^2} = \frac{\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}}}{\bar{\theta}} - \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}^2} \tag{3.63}$$

substituting from (3.63) into (3.61) and multiplying through by $\bar{\theta}$

$$\bar{\rho}\bar{\theta}\frac{D\bar{\eta}}{Dt} + \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \bar{\rho}\bar{r}\right) - \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \geq 0 \tag{3.64}$$

From energy equation (3.54) (after including $\bar{s}\bar{\rho}$) term)

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \bar{s}\bar{\rho} = -\bar{\rho}\frac{D\bar{e}}{Dt} + {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} + {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}} + {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}} \tag{3.65}$$

substituting (3.65) into (3.64), using $\bar{\Phi} = \bar{e} - \bar{\eta}\bar{\theta}$ and regrouping terms

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} \\ - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}} - {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}} \leq 0 \tag{3.66}$$

Equation (3.66) is the final form of the entropy inequality resulting from the second law of thermodynamics.

3.7 Mathematical model consisting of CBL of NCCM for fluids

The system of partial differential equations and algebraic equations resulting from the conservation and balance laws of NCCM incorporating internal rotation rates, their material derivatives and rotational inertial effects are given by: conservation of mass (CM), balance of linear momenta (BLM), balance of angular momenta (BAM), balance of moment of moments (BMM), first law of thermodynamics (FLT) and the second law of thermodynamics (SLT). These are listed in the following using:

$$ {}_{i}\bar{\boldsymbol{\omega}} = {}_{i}^{r}\bar{\boldsymbol{\Theta}} \quad , \quad \frac{D\left({}_{i}\bar{\boldsymbol{\omega}}\right)}{Dt} = {}_{i}^{r}\dot{\bar{\boldsymbol{\Theta}}} \tag{3.67}$$

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\mathrm{div}(\bar{\boldsymbol{v}}) = 0 \qquad \text{(CM)} \tag{3.68}$$

$$\bar{\rho}\frac{\partial\bar{\boldsymbol{v}}}{\partial t} + \bar{\rho}\left(\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{\nabla}}\right)\bar{\boldsymbol{v}} - \bar{\rho}\bar{\boldsymbol{F}}^{b} - \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \qquad \text{(BLM)} \tag{3.69}$$

$${}^{\Theta}\bar{I}\bar{\rho}\frac{D}{Dt}({}_{i}\bar{\omega}_{k}) - \epsilon_{ijk}{}^{(0)}\bar{\sigma}_{ij} - {}^{(0)}\bar{m}_{mk,m} = 0 \qquad \text{(BAM)} \tag{3.70}$$

$$\epsilon_{jkl}\left({}^{\Theta}\bar{I}\bar{\rho}\bar{v}_{j}\left({}_{i}\bar{\omega}_{k}\right) - {}^{(0)}\bar{m}_{jk}\right) = 0 \qquad \text{(BMM)} \tag{3.71}$$

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}} - {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}} = 0 \qquad \text{(FLT)} \tag{3.72}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}} - {}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}} \leq 0 \qquad \text{(SLT)} \tag{3.73}$$

Remarks

1. The mathematical model consists of eleven equations: CM (1), BLM (3), BAM (3), BMM(3), FLT (1) in twenty six dependent variables: $\bar{\rho}$ (1), $\bar{\boldsymbol{v}}$ (3), ${}^{(0)}\bar{\boldsymbol{\sigma}}$ (9), ${}^{(0)}\bar{\boldsymbol{m}}$ (9), $\bar{\boldsymbol{q}}$ (3), $\bar{\theta}$ (1), thus we need an additional fifteen equations for the mathematical model to have closure. These additional equations are obtained from the constitutive theories.

2. We shall see that $\bar{\Phi}$, $\bar{\eta}$ and $\bar{e}$ are not dependent variables in the mathematical model as they can be expressed in terms of other dependent variables in remark (1).

3. From entropy inequality we can conclude the following.

   (a) From the term $\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}$, we conclude that $\bar{\boldsymbol{q}}, \bar{\boldsymbol{g}}$ is a conjugate pair.

   (b) The term ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}}$, suggests that ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{D}}$ are rate of work (mechanical) conjugate pair. This is obviously due to classical continuum mechanics.

   (c) The term ${}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}}$ suggests that ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ and ${}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}}$ are also rate of work (mechanical) conjugate pair. This is the contribution of non-classical continuum mechanics incorporating internal rotation rates.

   (d) From the term ${}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}}$ it can be concluded that ${}_{a}^{(0)}\bar{\boldsymbol{m}}$, ${}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}}$ are a rate of work (mechanical) conjugate pair. However, based on Surana et. al [82] in non-classical continuum mechanics the constitutive theory for ${}_{a}^{(0)}\bar{\boldsymbol{m}}$ (when ${}_{a}^{(0)}\bar{\boldsymbol{m}}$ is a possible choice of constitutive variable) leads to deformation physics that is non-physical. In reference [82] authors present constitutive theory for ${}_{a}^{(0)}\bar{\boldsymbol{m}}$ (in the absence of BMM balance law) as well as for ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ and model problem studies to substantiate this issue. Based on reference [82], ${}_{a}^{(0)}\bar{\boldsymbol{m}}$ and ${}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}}$ are not a conjugate pair, therefore ${}_{a}^{(0)}\bar{\boldsymbol{m}}$ is not a constitutive tensor. Thus, ${}_{a}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{a}^{\Theta}\bar{\boldsymbol{J}} = 0$ must be used as a constraint equation in the mathematical model.

4. From remark (3) we can conclude that it is possible to obtain the following additional equations through constitutive theories

   (a) Constitutive theory for ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$ (6)

(b) Constitutive theory for ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ (6)

(c) Constitutive theory for $\bar{\boldsymbol{q}}$ (3)

This provides us with additional fifteen equations needed to provide closure to the mathematical model consisting of equations (3.68)-(3.73).

5. In this work, we consider compressible as well as incompressible thermoviscous fluids.

3.8 Constitutive theories

In this section, we present constitutive theories for thermoviscous compressible and incompressible fluids. As in the case of solids, here also we begin with entropy inequality. The conjugate pairs in the entropy inequality (3.73) expressed in terms of Helmholtz free energy density are instrumental in determining the constitutive variables, their argument tensors as well as derivation of some constitutive theories. Choice of $\bar{\Phi}$, $\bar{\eta}$, ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$ as constitutive variables based on axioms of constitutive theories [30, 97], entropy inequality as well as the other balance laws is straightforward. The choice of some argument tensors of ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}_{s}^{(0)}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$ can be made based on conjugate pairs in the SLT. Additionally, temperature $\bar{\theta}$ is also required to be an argument tensor of all constitutive variables due to non-isothermal physics.
For compressible continua density varies during evolution. Based on conservation of mass in Lagrangian description, changing density is defined by changing $[J]$, deformation gradient tensor.

$$|\bar{J}| = \frac{\rho_0}{\rho(\boldsymbol{x}, t)}$$

Thus, $|J|$ or $\rho_0/\rho(\boldsymbol{x}, t)$ or $1/\rho(\boldsymbol{x}, t)$ must be argument tensor of the constitutive variables in Lagrangian description. In Eulerian description, choice of $1/\rho(\boldsymbol{x}, t)$ is replaced by $1/\bar{\rho}(\bar{\boldsymbol{x}}, t)$ hence at the onset we begin with

$${}_{s}^{(0)}\bar{\boldsymbol{\sigma}} = {}_{s}^{(0)}\bar{\boldsymbol{\sigma}}(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{D}}, \bar{\theta}) \tag{3.74}$$

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = {}^{(0)}_{s}\bar{\boldsymbol{m}}(\frac{1}{\bar{\rho}}, {}^{r\Theta}_{i}{}_{s}\bar{\boldsymbol{J}}, \bar{\theta}) \tag{3.75} $$

$$ \bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{g}}, \bar{\theta}) \tag{3.76} $$

The argument tensors of $\bar{\Phi}$ and $\bar{\eta}$ at this stage can be chosen using principle of equipresence [30, 97]. We remark that principle of equipresence is not used in (3.74)-(3.76) as the conjugate pairs in entropy inequality specifically dictate the choice of argument tensors used along with the additionally required $1/\bar{\rho}$ and $\bar{\theta}$.

$$ \bar{\Phi} = \bar{\Phi}(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{D}}, {}^{r\Theta}_{i}{}_{s}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}) \tag{3.77} $$

$$ \bar{\eta} = \bar{\eta}(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{D}}, {}^{r\Theta}_{i}{}_{s}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}) \tag{3.78} $$

The argument tensors of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ can be enhanced to permit more comprehensive physics. Let $\boldsymbol{\gamma}_{(i)}$; $i = 1, 2, \dots, n$ be the convected time derivatives of the Green's strain tensor $\bar{\boldsymbol{\varepsilon}}_{(0)}$ (covariant basis) up to order n and let $\boldsymbol{\gamma}^{(i)}$; $i = 1, 2, \dots, n$ be the convected time derivatives of the Almansi strain tensor $\bar{\boldsymbol{\varepsilon}}^{[0]}$ (contravariant basis) up to order n (see reference [97] for details). Then, we find that

$$ \boldsymbol{\gamma}_{(1)} = \boldsymbol{\gamma}^{(1)} = \bar{\boldsymbol{D}} \tag{3.79} $$

i.e., $\bar{\boldsymbol{D}}$ is basis independent, however $\boldsymbol{\gamma}_{(i)}$; $i = 2, 3, \dots, n$ and $\boldsymbol{\gamma}^{(i)}$; $i = 2, 3, \dots, n$ are in covariant and in contravariant basis. Thus, we note that the first convected time derivative of $\bar{\boldsymbol{\varepsilon}}_{[0]}$, i.e., $\bar{\boldsymbol{D}}$ or $\boldsymbol{\gamma}_{(1)}$, is an argument tensor of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$. The first convected time derivative of $\bar{\boldsymbol{\varepsilon}}^{[0]}$ i.e., $\boldsymbol{\gamma}_{(1)}$ is also equal to $\bar{\boldsymbol{D}}$. This suggests that perhaps a constitutive theory that considers convected time derivatives of $\bar{\boldsymbol{\varepsilon}}_{[0]}$ or $\bar{\boldsymbol{\varepsilon}}^{[0]}$ up to order n is worthy of consideration. Thus $\bar{\boldsymbol{D}}$ can also be replaced by $\boldsymbol{\gamma}^{(i)}$; $i = 1, 2, \dots, n$ or $\boldsymbol{\gamma}_{(i)}$; $i = 1, 2, \dots, n$. The choice of $\boldsymbol{\gamma}_{(i)}$; $i = 1, 2, \dots, n$ (covariant basis) or $\boldsymbol{\gamma}^{(i)}$; $i = 1, 2, \dots, n$ (contravariant basis) depends upon whether ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is chosen to be ${}_{s}\bar{\boldsymbol{\sigma}}^{(0)}$

(contravariant measure) or ${}_s\bar{\boldsymbol{\sigma}}_{(0)}$ (covariant measure). To make the derivation basis independent we replace $\bar{\boldsymbol{D}}$ by ${}^{(i)}\boldsymbol{\gamma}$; $i = 1, 2, \dots, n$ convected time derivative of the desired strain tensor. More specifically when

$$ {}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}} = {}_s\bar{\boldsymbol{\sigma}}^{(0)} \quad ; \quad {}^{(i)}\boldsymbol{\gamma} = \boldsymbol{\gamma}_{(i)}\, ; \; i = 1, 2, \dots, n \tag{3.80} $$

and when

$$ {}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}} = {}_s\bar{\boldsymbol{\sigma}}_{(0)} \quad ; \quad {}^{(i)}\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(i)}\, ; \; i = 1, 2, \dots, n \tag{3.81} $$

When we replace $\bar{\boldsymbol{D}}$ in (3.74), (3.77) and (3.78) by ${}^{(i)}\boldsymbol{\gamma}$; $i = 1, 2, \dots, n$ the resulting constitutive theory for ${}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}$ is referred to as ordered rate constitutive theory of order n. Thus now we have

$$ {}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}(\frac{1}{\bar{\rho}}, {}^{(i)}\boldsymbol{\gamma}, \bar{\theta})\; ; \; i = 1, 2, \dots, n \tag{3.82} $$

$$ {}^{(0)}_{\ s}\bar{\boldsymbol{m}} = {}^{(0)}_{\ s}\bar{\boldsymbol{m}}(\frac{1}{\bar{\rho}}, {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}, \bar{\theta}) \tag{3.83} $$

$$ \bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\frac{1}{\bar{\rho}}, \bar{\boldsymbol{g}}, \bar{\theta}) \tag{3.84} $$

$$ \bar{\Phi} = \bar{\Phi}(\frac{1}{\bar{\rho}}, {}^{(j)}\boldsymbol{\gamma}, {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta})\; ; \; j = 1, 2, \dots, n \tag{3.85} $$

$$ \bar{\eta} = \bar{\eta}(\frac{1}{\bar{\rho}}, {}^{(j)}\boldsymbol{\gamma}, {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta})\; ; \; j = 1, 2, \dots, n \tag{3.86} $$

In (3.82)-(3.86) we have the final choice of argument tensors of the constitutive variables. We can now obtain the material derivative of $\bar{\Phi}$ using (3.85)

$$\frac{D\bar{\Phi}}{Dt} = \dot{\bar{\Phi}} = \frac{\partial\bar{\Phi}}{\partial(1/\bar{\rho})}\frac{-1}{\bar{\rho}^2}\dot{\bar{\rho}} + \sum_{j=1}^{n}\frac{\partial\bar{\Phi}}{\partial\left({}^{(j)}\boldsymbol{\gamma}\right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}}\right) + \frac{\partial\bar{\Phi}}{\partial\left({}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}}\right)} : \left({}_{i}^{r}{}_{s}^{\Theta}\dot{\bar{\boldsymbol{J}}}\right) + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}}\cdot\dot{\bar{\boldsymbol{g}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}} \tag{3.87}$$

From continuity equation (3.68)

$$\dot{\bar{\rho}} = -\bar{\rho}\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}} = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}\bar{D}_{kl}\delta_{lk} = -\bar{\rho}\boldsymbol{\delta} : \bar{\boldsymbol{D}} \tag{3.88}$$

Substituting from (3.88) in (3.87)

$$\frac{D\bar{\Phi}}{Dt} = \frac{\partial\bar{\Phi}}{\partial(1/\bar{\rho})}\frac{1}{\bar{\rho}}\boldsymbol{\delta} : \bar{\boldsymbol{D}} + \sum_{j=1}^{n}\frac{\partial\bar{\Phi}}{\partial\left({}^{(j)}\boldsymbol{\gamma}\right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}}\right) + \frac{\partial\bar{\Phi}}{\partial\left({}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}}\right)} : \left({}_{i}^{r}{}_{s}^{\Theta}\dot{\bar{\boldsymbol{J}}}\right) + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}}\cdot\dot{\bar{\boldsymbol{g}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}} \tag{3.89}$$

we note that

$$\frac{\partial\bar{\Phi}\left(\frac{1}{\bar{\rho}}\right)}{\partial\left(\frac{1}{\bar{\rho}}\right)} = \frac{\partial\bar{\Phi}(\bar{\rho})}{(\partial\bar{\rho})}(-\bar{\rho}^2) \tag{3.90}$$

we can also make substitution from (3.90) in (3.89). After this substitution, $\bar{\Phi} = \bar{\Phi}(\bar{\rho}, {}^{(j)}\boldsymbol{\gamma}, {}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta})$,

$$\frac{D\bar{\Phi}}{Dt} = -\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} : \bar{\boldsymbol{D}} + \sum_{j=1}^{n}\frac{\partial\bar{\Phi}}{\partial\left({}^{(j)}\boldsymbol{\gamma}\right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}}\right) + \frac{\partial\bar{\Phi}}{\partial\left({}_{i}^{r}{}_{s}^{\Theta}\bar{\boldsymbol{J}}\right)} : \left({}_{i}^{r}{}_{s}^{\Theta}\dot{\bar{\boldsymbol{J}}}\right) + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}}\cdot\dot{\bar{\boldsymbol{g}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}} \tag{3.91}$$

Substituting (3.91) in the entropy inequality (3.73) and regrouping terms

$$
\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}}\right) : \bar{\boldsymbol{D}} + \bar{\rho} \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\boldsymbol{\gamma}\right)} : \left({}^{(j)}\dot{\boldsymbol{\gamma}}\right)
+ \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}_{i}^{r}{}_{s}^{\Theta} \bar{\boldsymbol{J}}\right)} : \left({}_{i}^{r}{}_{s}^{\Theta} \dot{\bar{\boldsymbol{J}}}\right) + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \frac{\partial \bar{\Phi}}{\partial \dot{\bar{\theta}}} \dot{\bar{\theta}}
+ \bar{\rho} \left(\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta}\right) \dot{\bar{\theta}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{s}^{\Theta} \bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{3.92}
$$

For arbitrary but admissible ${}^{(j)}\dot{\boldsymbol{\gamma}}$; $j = 1, 2, \ldots, n$, ${}_{i}^{r}{}_{s}^{\Theta} \dot{\bar{\boldsymbol{J}}}$, $\dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$ the entropy inequality (3.93) is satisfied if the following hold (i.e., their coefficients are set to zero).

$$
\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\boldsymbol{\gamma}\right)} = 0 \implies \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\boldsymbol{\gamma}\right)} = 0
\implies \bar{\Phi} \neq \bar{\Phi}\left({}^{(j)}\boldsymbol{\gamma}\right) \; ; \; j = 1, 2, \ldots, n \tag{3.93}
$$

$$
\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}_{i}^{r}{}_{s}^{\Theta} \bar{\boldsymbol{J}}\right)} = 0 \implies \frac{\partial \bar{\Phi}}{\partial \left({}_{i}^{r}{}_{s}^{\Theta} \bar{\boldsymbol{J}}\right)} = 0 \implies \bar{\Phi} \neq \bar{\Phi}\left({}_{i}^{r}{}_{s}^{\Theta} \bar{\boldsymbol{J}}\right) \tag{3.94}
$$

$$
\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} = 0 \implies \frac{\partial \bar{\Phi}}{\partial \bar{\boldsymbol{g}}} = 0 \implies \bar{\Phi} \neq \bar{\Phi}(\bar{\boldsymbol{g}}) \tag{3.95}
$$

$$
\bar{\rho} \left(\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta}\right) = 0 \implies \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta} = 0 \implies \bar{\eta} = -\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \tag{3.96}
$$

From (3.93)-(3.95) we can conclude that $\bar{\rho}$ and $\bar{\theta}$ are the only argument tensors of $\bar{\Phi}$. From (3.96) we conclude that $\bar{\eta}$ is not a constitutive variable as it is deterministic using $\partial \bar{\Phi}/\partial \bar{\theta}$. Using (3.93)-(3.96) the entropy inequality (3.92) reduces to

$$
\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_{s}^{(0)}\bar{\boldsymbol{\sigma}}\right) : \bar{\boldsymbol{D}} - {}_{s}^{(0)}\bar{\boldsymbol{m}} : {}_{i}^{r}{}_{s}^{\Theta} \bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{3.97}
$$

We remark that by setting the coefficient of $\bar{\boldsymbol{D}}$ in (3.97) to zero (for arbitrary but admissible $\bar{\boldsymbol{D}}$)

we obtain

$$-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = 0 \tag{3.98}$$

$$-{}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{3.99}$$

which are inappropriate due to the fact that (3.98) implies that ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is not a function of ${}^{(j)}\boldsymbol{\gamma}$; $j = 1, 2, \ldots, n$, as $\bar{\Phi}$ is only a function of $\bar{\rho}$ and $\bar{\theta}$, which is invalid based on (3.82). Thus, at this stage we must maintain entropy inequality in the form stated in (3.97). In order to proceed further, we consider decomposition of Cauchy stress tensor ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ into equilibrium ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ and deviatoric tensor ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ where ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ causes change of volume without distortion and ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ causes distortion of volume without change of volume.

$${}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{es}\bar{\boldsymbol{\sigma}} + {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} \tag{3.100}$$

Thus, we consider

$$\begin{aligned} {}^{(0)}_{es}\bar{\boldsymbol{\sigma}} &= {}^{(0)}_{es}\bar{\boldsymbol{\sigma}}(\bar{\rho}, 0, \bar{\theta}) \\ {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} &= {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}(\bar{\rho}, {}^{(j)}\boldsymbol{\gamma}, \bar{\theta}) \ ; \ j = 1, 2, \ldots, n \\ \text{and} \ \ {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} &= {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}(\bar{\rho}, 0, \bar{\theta}) = 0 \end{aligned} \tag{3.101}$$

The remaining constitutive variables and their argument tensors remain the same

$${}^{(0)}_{s}\bar{\boldsymbol{m}} = {}^{(0)}_{s}\bar{\boldsymbol{m}}(\bar{\rho}, {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}, \bar{\theta}) \tag{3.102}$$

$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta}) \tag{3.103}$$

$$\bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\theta}\right) \tag{3.104}$$

3.8.1 Constitutive theory for ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$: Compressible continua

Substituting (3.101) in entropy inequality (3.97) and regrouping terms

$$\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} - {}^{(0)}_{es}\bar{\boldsymbol{\sigma}}\right) : \bar{\boldsymbol{D}} - {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{r}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{3.105}$$

Since $\bar{\Phi}$ is a function of $\bar{\rho}$ and $\bar{\theta}$ so is ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$. Thus ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ can be determined by setting the coefficient of $\bar{D}_{kl}$ in the first term of (3.105) to zero

$${}^{(0)}_{es}\bar{\boldsymbol{\sigma}} = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} \qquad \text{or} \qquad [{}^{(0)}_{es}\bar{\sigma}] = \bar{p}(\bar{\rho},\bar{\theta})[I] \tag{3.106}$$

in which

$$\bar{p}\left(\bar{\rho},\bar{\theta}\right) = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \tag{3.107}$$

where $\bar{p}\left(\bar{\rho},\bar{\theta}\right)$ is thermodynamic pressure for compressible fluent continua and is defined by equation of state. The entropy inequality (3.105) reduces to

$$-{}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{r}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{3.108}$$

Entropy inequality (3.108) is satisfied if

$${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} \geq 0 \tag{3.109}$$

$${}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{r}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}} \geq 0 \tag{3.110}$$

and

$$\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{3.111}$$

Inequalities (3.109) and (3.110) require that rate of work due to ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ be positive and (3.111) serves as restriction on the constitutive theory for $\bar{\boldsymbol{q}}$.

3.8.2 Constitutive theory for ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$: Incompressible continua

For incompressible matter $\bar{\rho} = \rho_0$, constant, hence $\frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0$. Thus the Constitutive theory for ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ cannot be derived using (3.105). The incompressibility condition must be enforced in the derivation of the constitutive theory for ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ by incorporating it in the entropy inequality. The incompressibility condition is given by continuity equation.

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = \text{tr}(\bar{\boldsymbol{D}}) = \bar{D}_{kk} = \bar{D}_{kl}\delta_{lk} = \boldsymbol{\delta} : \bar{\boldsymbol{D}} = 0 \tag{3.112}$$

Thus, we can add the following to the entropy inequality (3.105)

$$\bar{p}(\bar{\theta})\boldsymbol{\delta} : \bar{\boldsymbol{D}} = 0 \tag{3.113}$$

$$\left(\bar{p}(\bar{\theta})\boldsymbol{\delta} - {}^{(0)}_{es}\bar{\boldsymbol{\sigma}}\right) : \bar{\boldsymbol{D}} - {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}^{(0)}_{s}\bar{\boldsymbol{m}} : {}^{r}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \tag{3.114}$$

Setting the coefficient of $\bar{\boldsymbol{D}}$ to zero in the first term of (3.114)

$$ {}^{(0)}_{es}\bar{\boldsymbol{\sigma}} = \bar{p}(\bar{\theta})\boldsymbol{\delta} \qquad \text{or} \qquad [{}^{(0)}_{es}\bar{\sigma}] = \bar{p}(\bar{\theta})[I] \tag{3.115}$$

where $\bar{p}(\bar{\theta})$ is mechanical pressure. Since $\bar{p}(\bar{\theta})$ is an arbitrary Lagrange multiplier, it is independent of the deformation field. The entropy inequality (3.114) reduces to (3.108) with conditions (3.109)-(3.111) that must be satisfied by the constitutive theories for ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$.

3.8.3 Constitutive theory for ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$

We consider (3.101)

$$ {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}(\bar{\rho}, {}^{(j)}\boldsymbol{\gamma}, \bar{\theta}) \ ; \ j = 1, 2, \dots, n \tag{3.116}$$

Pairs in (3.109) from entropy inequality confirm that ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ and ${}^{(j)}\boldsymbol{\gamma}$; $j = 1, 2, \dots, n$ are rate of work conjugate. We derive constitutive theory for ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ using representation theorem [8, 62, 69–71, 73–77, 99–103, 107, 108]. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}$; $i = 1, 2, \dots, N_\sigma$ be the combined generators of the argument tensors of ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ that are symmetric tensors of rank two, then ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ can be expressed using linear combination of $\boldsymbol{I}$ and ${}^{(j)}\boldsymbol{\gamma}$; $i = 1, 2, \dots, N_\sigma$ in the current configuration.

$$
{}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} = {}^{\sigma}\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N_\sigma} {}^{\sigma}\alpha^i \left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \right) \tag{3.117}
$$

In the linear combination (3.117), coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, N_\sigma$ are functions of the combined invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_\sigma$ of the same argument tensors of ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ in (3.116), $\bar{\rho}$ and $\bar{\theta}$.

$$
{}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i \left(\bar{\rho}, {}^{\sigma}\underset{\sim}{I}^j, \bar{\theta} \right) \ ; \ j = 1, 2, \dots, M_\sigma \ ; \ i = 0, 1, \dots, N_\sigma \tag{3.118}
$$

The material coefficients in the constitutive theory for ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ given by (3.117) are determined by considering Taylor series expansion of ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, N_\sigma$ in ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_\sigma$ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_\sigma$ (for simplicity). Taylor series expansion in $\bar{\theta}$ is not considered as the influence of thermal field on stress tensor has already been considered in the constitutive theory for ${}^{(0)}_{es}\bar{\boldsymbol{\sigma}}$ stress tensor.

$$
{}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M_\sigma} \frac{\partial\, {}^{\sigma}\alpha^i}{\partial \left({}^{\sigma}\underset{\sim}{I}^j \right)}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right) \ ; \ i = 0, 1, \dots, N_\sigma \tag{3.119}
$$

Substituting ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, N_\sigma$ into (3.117), collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients.

$$
\begin{aligned}
{}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} = {} & \underset{\sim}{\sigma}^{(0)}\Big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{j=1}^{M_\sigma} {}^{\sigma}\underset{\sim}{a}_j ({}^{\sigma}\underset{\sim}{I}^j) \boldsymbol{I} + \sum_{i=1}^{N_\sigma} {}^{\sigma}\underset{\sim}{b}_i ({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i) \\
& + \sum_{i=1}^{N_\sigma} \sum_{j=1}^{M_\sigma} {}^{\sigma}\underset{\sim}{c}_{ij} ({}^{\sigma}\underset{\sim}{I}^j) {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i
\end{aligned} \tag{3.120}
$$

Coefficients ${}^{\sigma}\underset{\sim}{a}_j$, ${}^{\sigma}b_i$ and ${}^{\sigma}\underset{\sim}{c}_{ij}$; $i = 1, 2, \dots, N_\sigma$, $j = 1, 2, \dots, M_\sigma$ and functions of $\bar{\rho}|_{\underline{\Omega}}$, ${}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}$ and $\bar{\theta}|_{\underline{\Omega}}$; $j = 1, 2, \dots, M_\sigma$. These are material coefficients.

Remarks

1. This constitutive theory for ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ given by (3.120) contains $(N + M + NM)$ material coefficients. This is a non-linear ordered rate constitutive theory of order n for ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ and is based on integrity.

2. A simple linear constitutive theory in which products of ${}^{\sigma}\underset{\sim}{I}^{j}$, ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}$ and $(\bar{\theta} - \bar{\theta}\big|_{\underline{\Omega}})$ are neglected is given by

$$ {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} = \underset{\sim}{\sigma}^{0}\Big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{i=1}^{n} 2\mu_i {}^{(i)}\boldsymbol{\gamma} + \sum_{i=1}^{n} \lambda_i {}^{(i)}\boldsymbol{\gamma} : \boldsymbol{I} \tag{3.121} $$

   μ_i and λ_i are material coefficients for convected time derivative ${}^{(i)}\boldsymbol{\gamma}$ of the corresponding strain tensor. The constitutive theory (3.121) is also ordered rate constitutive theory of order n, but is linear in the components of $[{}^{i}\gamma]$; $i = 1, 2, \dots, n$.

3. From (3.122) we can obtain the most simplified constitutive theory for ${}^{(0)}_{ds}\bar{\boldsymbol{\sigma}}$ if we choose $n = 1$ (rate constitutive theory of order one)

$$ {}^{(0)}_{ds}\bar{\boldsymbol{\sigma}} = \underset{\sim}{\sigma}^{0}\Big|_{\underline{\Omega}} \boldsymbol{I} + 2\mu_1 {}^{(1)}\boldsymbol{\gamma} + \lambda_1 {}^{(1)}\boldsymbol{\gamma} : \boldsymbol{I} \tag{3.122} $$

   The constitutive theory (3.122) is Newton's law of viscosity for thermoviscous compressible fluids. We note that ${}^{(1)}\boldsymbol{\gamma} = \bar{\boldsymbol{D}}$, symmetric part of the velocity gradient tensor. Initial stress field, $\underset{\sim}{\sigma}^{(0)}[I]$, and the last term is due to thermal expansion or contraction. The first and second viscosities are μ_1 and λ_1. For incompressible fluent continua, ${}^{(1)}\boldsymbol{\gamma} = 0$ due to continuity, hence the third term in (3.122) becomes zero.

3.8.4 Constitutive theory for ${}^{(0)}_{s}\bar{\boldsymbol{m}}$

We consider (3.102)

$$ {}^{(0)}_{s}\bar{\boldsymbol{m}} = {}^{(0)}_{s}\bar{\boldsymbol{m}} \left(\bar{\rho}, {}^{r}_{i}{}^{\Theta}_{s} \bar{\boldsymbol{J}}, \bar{\theta} \right) \tag{3.123} $$

Let ${}^m\underset{\sim}{\boldsymbol{G}}^i$; $i = 1, 2, \dots, N_m$ be the combined generators of the argument tensors of ${}^{(0)}\bar{\boldsymbol{m}}$ in (3.123) that are symmetric tensors of rank two. Then, based on representation theorem [8, 62, 69–71, 73–77, 99–103, 107, 108] we can express ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ as a linear combination of $\boldsymbol{I}$ and ${}^m\underset{\sim}{\boldsymbol{G}}^i$; $i = 1, 2, \dots, N_m$ in the current configuration.

$$
{}^{(0)}_{s}\bar{\boldsymbol{m}} = ({}^m\alpha^0)\boldsymbol{I} + \sum_{i=1}^{N_m} ({}^m\alpha^i){}^m\underset{\sim}{\boldsymbol{G}}^i \tag{3.124}
$$

The coefficients in the linear combination (3.124) are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_m$, the combined invariants of the same argument tensors of ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ in (3.124).

In this particular case $N_m = 2$ and $M_m = 3$

$$
{}^m\underset{\sim}{\boldsymbol{G}}^1 = {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}} \quad ; \quad {}^m\underset{\sim}{\boldsymbol{G}}^2 = {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}^2 \tag{3.125}
$$

and

$$
{}^m\underset{\sim}{I}^1 = I_{\left({}^{r\Theta}_{i\ s}\bar{J}\right)} \quad {}^m\underset{\sim}{I}^2 = II_{\left({}^{r\Theta}_{i\ s}\bar{J}\right)} \quad {}^m\underset{\sim}{I}^3 = III_{\left({}^{r\Theta}_{i\ s}\bar{J}\right)} \tag{3.126}
$$

The material coefficients in the constitutive theory (3.124) for ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ are determined by considering Taylor series expansion of ${}^m\alpha^i$; $i = 0, 1, \dots, N_m$ in ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_m$ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_m$.

$$
{}^m\alpha^i = {}^m\alpha^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M_m} \left.\frac{\partial\left({}^m\alpha^i\right)}{\partial\left({}^{r\Theta}_{i\ s}\bar{J}\right)}\right|_{\underline{\Omega}} \left({}^{r\Theta}_{i\ s}\bar{J} - {}^{r\Theta}_{i\ s}\bar{J}\Big|_{\underline{\Omega}}\right) \; ; \; i = 0, 1, \dots, N_m \tag{3.127}
$$

Substituting (3.127) in (3.124) and collecting coefficients of the terms defined in the current configuration and introducing new notation for the coefficients, we can write

$$
\begin{aligned}
{}^{(0)}_{\ s}\bar{\boldsymbol{m}} = \underset{\sim}{m}^0\big|_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M_m} {}^m\underset{\sim}{\alpha}_j({}^m\underset{\sim}{I}^j)\boldsymbol{I} + \sum_{i=1}^{N_m} {}^m\underset{\sim}{b}_i({}^m\underset{\sim}{\boldsymbol{G}}^i) \\
+ \sum_{i=1}^{N_m}\sum_{j=1}^{M_m} {}^m\underset{\sim}{c}_{ij}({}^m I^j){}^m\underset{\sim}{\boldsymbol{G}}^i
\end{aligned}
\tag{3.128}
$$

Coefficients ${}^m\underset{\sim}{\alpha}_j$, ${}^m\underset{\sim}{b}_i$ and ${}^m\underset{\sim}{c}_{ij}$ are $(N + M + NM)$ material coefficients. These can be function of $\bar{\rho}|_{\underline{\Omega}}$, ${}^m\underset{\sim}{I}^j$ and $\bar{\theta}|_{\underline{\Omega}}$; $j = 1, 2, \ldots, M_m$.

Remarks

1. This constitutive theory (3.128) is obviously a non-linear constitutive theory based on integrity.
2. Since $N_m = 2$ and $M_m = 3$, this constitutive theory requires eleven material coefficients.
3. This constitutive theory contains up to fifth degree terms of the components of ${}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}$.
4. A linear constitutive theory in the components of ${}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}$ in which products of ${}^m\underset{\sim}{I}^j$ and ${}^m\underset{\sim}{\boldsymbol{G}}^i$ are neglected is given by

$$
{}^{(0)}_{\ s}\bar{\boldsymbol{m}} = \underset{\sim}{m}^0\big|_{\underline{\Omega}}\boldsymbol{I} + {}^m\underset{\sim}{\alpha}^1\left({}^m\underset{\sim}{I}^1\right)\boldsymbol{I} + {}^m\underset{\sim}{b}^1\left({}^m\underset{\sim}{G}^1\right)
\tag{3.129}
$$

Since

$$
{}^m\underset{\sim}{I}^1 = \operatorname{tr}\left({}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}\right) = 0
\tag{3.130}
$$

the constitutive theory (3.128) reduces to

$$
{}^{(0)}_{\ s}\bar{\boldsymbol{m}} = \underset{\sim}{m}^0\big|_{\underline{\Omega}}\boldsymbol{I} + {}^m\underset{\sim}{b}^1\left({}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}}\right)
\tag{3.131}
$$

A further simplified theory in which first term in (3.131) is neglected is given by (defining

$\mu_m = {}^m\underset{\sim}{b}_i)$

$$ {}^{(0)}_{\ s}\bar{\boldsymbol{m}} = \mu_m \, {}^{r\Theta}_{i\ s}\bar{\boldsymbol{J}} \tag{3.132}$$

in which

$$\mu_m = \mu_m\left(\bar{\rho}\big|_{\underline{\Omega}}, I_{\left({}^{r\Theta}_{i\ s}\bar{J}\right)}\big|_{\underline{\Omega}}, II_{\left({}^{r\Theta}_{i\ s}\bar{J}\right)}\big|_{\underline{\Omega}}, III_{\left({}^{r\Theta}_{i\ s}\bar{J}\right)})\big|_{\underline{\Omega}}, \bar{\theta}\big|_{\underline{\Omega}}\right) \tag{3.133}$$

3.8.5 Constitutive theory for $\bar{\boldsymbol{q}}$

We consider $\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\bar{\boldsymbol{g}}, \bar{\theta})$ and use representation theorem [8,62,69–71,73–77,99–103,107,108]. The combined generators of the argument tensors $\bar{\boldsymbol{g}}$ and $\bar{\theta}$ that are tensors of rank one is just $\bar{\boldsymbol{g}}$ and the combined invariant is $\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$ (or ${}^qI$). Thus, the constitutive theory for $\bar{\boldsymbol{q}}$ in the current configuration can be written as

$$\bar{\boldsymbol{q}} = -{}^q\alpha\bar{\boldsymbol{g}} \tag{3.134}$$

in which

$${}^q\alpha = {}^q\alpha\left(\bar{\rho}, {}^qI, \theta\right) \tag{3.135}$$

The material coefficients in the constitutive theory for $\bar{\boldsymbol{q}}$ given by (3.134) are obtained by considering Taylor series expansion ${}^q\alpha$ in ${}^qI$ and $\bar{\theta}$ in a known configuration $\underline{\Omega}$ and retaining up to linear terms in ${}^qI$ and $\bar{\theta}$

$${}^q\alpha = {}^q\alpha\big|_{\underline{\Omega}} + \left.\frac{\partial^q\alpha}{\partial\left({}^qI\right)}\right|_{\underline{\Omega}} \left({}^qI - {}^qI\big|_{\underline{\Omega}}\right) + \left.\frac{\partial\left({}^q\alpha\right)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}\big|_{\underline{\Omega}}\right) \tag{3.136}$$

Substituting (3.136) in (3.134) and collecting coefficients of the terms defined in current configuration gives the following (after introducing new coefficients)

$$\bar{\boldsymbol{q}} = -k\big|_{\underline{\Omega}}\bar{\boldsymbol{g}} - k_1\big|_{\underline{\Omega}}\left(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}\right)\bar{\boldsymbol{g}} - k_2\big|_{\underline{\Omega}}\left(\bar{\theta} - \bar{\theta}\big|_{\underline{\Omega}}\right)\bar{\boldsymbol{g}} \tag{3.137}$$

the materials coefficients k, k_1 and k_2 can be functions of $\bar{\rho}\big|_{\underline{\Omega}}$, ${}^qI\big|_{\underline{\Omega}}$ and $\bar{\theta}\big|_{\underline{\Omega}}$ This constitutive theory (3.137) based on integrity is a non-linear constitutive theory in temperature gradient (contains up to cubic terms of temperature gradients). A linear constitutive theory for $\bar{\boldsymbol{q}}$ is given by

$$\bar{\boldsymbol{q}} = -\, k\big|_{\underline{\Omega}}\, \bar{\boldsymbol{g}} \tag{3.138}$$

This is Fourier heat conduction law in which $k = k(\,\bar{\rho}\big|_{\underline{\Omega}}\,,\, {}^{q}I\big|_{\underline{\Omega}}\,,\, \bar{\theta}\big|_{\underline{\Omega}})$ still holds.

3.9 Complete mathematical model resulting from NCCM for fluids

In the following we present complete mathematical model consisting of CBL (3.68)-(3.72) and constitutive theories (3.106), (3.122), (3.132) and (3.138). If we consider contravariant Cauchy stress and Cauchy moment tensors then we can use ${}_{s}\boldsymbol{\sigma}^{(0)}$ and ${}_{s}\boldsymbol{m}^{(0)}$ instead of ${}^{(0)}_{s}\boldsymbol{\sigma}$ and ${}^{(0)}_{s}\boldsymbol{m}$. The complete mathematical model consisting of the CBL and the constitutive theories is listed below where ${}_{i}\bar{\boldsymbol{\omega}} = {}^{r}_{i}\bar{\boldsymbol{\Theta}}$, $\frac{D({}_{i}\bar{\boldsymbol{\omega}})}{Dt} = {}^{r}_{i}\dot{\bar{\boldsymbol{\Theta}}}$. We consider incompressible micropolar thermoviscous fluid medium.

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\mathrm{div}(\bar{\boldsymbol{v}}) = 0 \qquad \text{(CM)} \tag{3.139}$$

$$\bar{\rho}\frac{\partial\bar{\boldsymbol{v}}}{\partial t} + \bar{\rho}\left(\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{\nabla}}\right)\bar{\boldsymbol{v}} - \bar{\rho}\bar{\boldsymbol{F}}^{b} - \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\sigma}}^{(0)} = 0 \qquad \text{(BLM)} \tag{3.140}$$

$${}^{\Theta}\bar{I}\bar{\rho}\frac{D}{Dt}({}_{i}\bar{\omega}_{k}) - \epsilon_{ijk}\bar{\sigma}^{(0)}_{ij} - \bar{m}^{(0)}_{mk,m} = 0 \qquad \text{(BAM)} \tag{3.141}$$

$$\epsilon_{jkl}\left({}^{\Theta}\bar{I}\bar{\rho}\bar{v}_{j}\left({}_{i}\bar{\omega}_{k}\right) - \bar{m}^{(0)}_{jk}\right) = 0 \qquad \text{(BMM)} \tag{3.142}$$

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}_{s}\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}_{s}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;\,s}\bar{\boldsymbol{J}} - {}_{a}\bar{\boldsymbol{m}}^{(0)} : {}^{r\Theta}_{i\;\,a}\bar{\boldsymbol{J}} = 0 \qquad \text{(FLT)} \tag{3.143}$$

$$\text{Additive decomposition of } \bar{\boldsymbol{\sigma}}^{(0)}: \quad \bar{\boldsymbol{\sigma}}^{(0)} = {}_{s}\bar{\boldsymbol{\sigma}}^{(0)} + {}_{a}\bar{\boldsymbol{\sigma}}^{(0)} \quad ; \quad {}_{s}\bar{\boldsymbol{\sigma}}^{(0)} = {}^{d}_{s}\bar{\boldsymbol{\sigma}}^{(0)} + {}^{e}_{s}\bar{\boldsymbol{\sigma}}^{(0)} \tag{3.144}$$

$${}^{d}_{s}\bar{\boldsymbol{\sigma}}^{(0)} = \bar{\boldsymbol{\sigma}}\big|_{\underline{\Omega}} + 2\eta\bar{\boldsymbol{D}} + \kappa(\mathrm{tr}(\bar{\boldsymbol{D}}))\boldsymbol{I} \tag{3.145}$$

$${}^{e}_{s}\bar{\boldsymbol{\sigma}} = \bar{p}\left(\bar{\rho}, \bar{\theta}\right) \tag{3.146}$$

$${}_{s}\bar{\boldsymbol{m}}^{(0)} = 2\eta_{m}\left({}^{r\Theta}_{i\;\,s}\bar{\boldsymbol{J}}\right) \tag{3.147}$$

$$\bar{\boldsymbol{q}} = -k\bar{\boldsymbol{g}} \tag{3.148}$$

In this mathematical model we have CM(1), BLM(3), BAM(3), BMM(3), FLT(1) and the constitutive theories for ${}_s^d\bar{\boldsymbol{\sigma}}^{(0)}(6)$, ${}_s\bar{\boldsymbol{m}}^{(0)}(6)$, $\bar{\boldsymbol{q}}(3)$, a total of 26 equations in 26 dependent variables $\bar{\rho}(1)$, $\bar{\boldsymbol{v}}(3)$, ${}_s^d\bar{\boldsymbol{\sigma}}^{(0)}(6)$, ${}_a\bar{\boldsymbol{\sigma}}^{(0)}(3)$, ${}_s\bar{\boldsymbol{m}}^{(0)}(6)$, ${}_a\bar{\boldsymbol{m}}^{(0)}(3)$, $\bar{\boldsymbol{q}}(3)$, $\bar{\theta}(1)$. The last term in the entropy inequality

$$\frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}_s^d\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}_s\bar{\boldsymbol{m}}^{(0)} : {}_i^r{}_s^{\Theta}\bar{\boldsymbol{J}} - {}_a\bar{\boldsymbol{m}}^{(0)} : {}_i^r{}_a^{\Theta}\bar{\boldsymbol{J}} \leq 0 \tag{3.149}$$

must be set to zero to ensure that (3.149) is not violated. Then, in addition to (3.139)-(3.148) we must also satisfy

$${}_a\bar{\boldsymbol{m}}^{(0)} : {}_i^r{}_a^{\Theta}\bar{\boldsymbol{J}} = 0 \tag{3.150}$$

Hence equation (3.150) serves as a constraint on the mathematical model that must be satisfied to guarantee that the entropy inequality is not violated. The final mathematical model consists of equations (3.139)-(3.148), (3.150).

Chapter 4

Finite element formulations for obtaining solutions of IVPs in Lagrangian and Eulerian description

4.1 Introduction

The CBL and constitutive theories for solid and fluent continua have been presented in Lagrangian and Eulerian description. The finite element method is ideally suited for obtaining numerical solutions of the IVPs described by these mathematical models. We could consider two finite element methodologies: (i) space-time coupled methods (ii) space-time decoupled methods. Both methods have their strengths and short comings. Space-time coupled methods maintain simultaneous dependence of all dependent variables on space and time. This is obviously needed to support true physical evolution in IVPs. When minimally conforming spaces are used in space and time, this approach permits calculation of solution error in terms of L_2-norm of the residual functional(s). In space-time decoupled methods, simultaneous dependence of dependent variables on space and time is not maintained. One considers a discretization in space followed by integral form in space using methods like GM/WF. In the local approximations, the approximation functions are functions of space and the degrees of freedom are functions of time. Substitution of the local approximation in the integral form for space decouples space and time and results in a system of linear or nonlinear ODEs in time which are then integrated using direct, explicit or implicit time integration methods. Decoupling of space and time is obviously contrary to true physics described by the IVPs, hence results in approximation. For IVPs in $\mathbb{R}^2$ and $\mathbb{R}^3$, the space-time decoupled finite element method with time integration is meritorious in avoiding complexities of space-time coupled methods. In the following, we consider both space-time coupled and space-time decoupled finite element methods for the mathematical models in Lagrangian as well as Eulerian description.

4.2 Space-time coupled finite element method

In case of IVPs, the space-time differential operators are either linear or nonlinear but not symmetric. In Lagrangian description, the CBL and constitutive theories considered yield linear space-time differential operator. In case of Eulerian description the space-time differential operator corresponding to the mathematical model is obviously nonlinear (primarily due to mathematical description in deformed or current material coordinates). Since the space-time operator is not self adjoint only space-time integral form based on space-time residual function (STRF) is space-time variationally consistent and would yield unconditionally stable computational processes [91]. We consider the mathematical models in Lagrangian and Eulerian description in the following, i.e., IVPs in Lagrangian and Eulerian description in the following.

4.2.1 IVPs in Lagrangian description

We consider CBL and constitutive theories in Lagrangian description. There are many choices in casting the CBL and constitutive theories in various different differential forms. The basic methodology of STRF based on integral forms is not effected by this choice. We consider the following PDEs describing IVPs in Lagrangian description.

BLM:

$$\left.\begin{aligned}
\rho_0 \frac{\partial^2(u_1)}{\partial t^2} - \rho_0 F_1^b - \left(\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3}\right) = 0\\
\rho_0 \frac{\partial^2(u_2)}{\partial t^2} - \rho_0 F_2^b - \left(\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3}\right) = 0\\
\rho_0 \frac{\partial^2(u_3)}{\partial t^2} - \rho_0 F_3^b - \left(\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3}\right) = 0
\end{aligned}\right\} \tag{4.1}$$

BAM:

$$\left.\begin{aligned}
{}^{\Theta}I_0 \rho_0 \frac{\partial^2({}_i\Theta_1)}{\partial t^2} - 2({}_a\sigma_{23}) - \left(\frac{\partial m_{11}}{\partial x_1} + \frac{\partial m_{21}}{\partial x_2} + \frac{\partial m31}{\partial x_3}\right) = 0\\
{}^{\Theta}I_0 \rho_0 \frac{\partial^2({}_i\Theta_2)}{\partial t^2} - 2({}_a\sigma_{31}) - \left(\frac{\partial m_{12}}{\partial x_1} + \frac{\partial m_{22}}{\partial x_2} + \frac{\partial m_{32}}{\partial x_3}\right) = 0\\
{}^{\Theta}I_0 \rho_0 \frac{\partial^2({}_i\Theta_3)}{\partial t^2} - 2({}_a\sigma_{12}) - \left(\frac{\partial m_{13}}{\partial x_1} + \frac{\partial m_{23}}{\partial x_2} + \frac{\partial m_{33}}{\partial x_3}\right) = 0
\end{aligned}\right\} \tag{4.2}$$

BMM:

$$\left.\begin{aligned} {}^{\Theta}I_0\rho_0\left(v_2\left({}_i\omega_3\right)-v_3\left({}_i\omega_2\right)\right)-2({}_am_{23})=0 \\ {}^{\Theta}I_0\rho_0\left(v_3\left({}_i\omega_1\right)-v_1\left({}_i\omega_3\right)\right)-2({}_am_{31})=0 \\ {}^{\Theta}I_0\rho_0\left(v_1\left({}_i\omega_2\right)-v_2\left({}_i\omega_1\right)\right)-2({}_am_{12})=0 \end{aligned}\right\} \tag{4.3}$$

FLT: using $e = c_v\theta$, where c_v is constant specific heat

$$\rho_0 c_v\frac{\partial\theta}{\partial t}+\left(\frac{\partial q_1}{\partial x_1}+\frac{\partial q_2}{\partial x_2}+\frac{\partial q_3}{\partial x_3}\right)-\left({}_s^e\boldsymbol{\sigma}+{}_s^d\boldsymbol{\sigma}\right):\dot{\boldsymbol{\varepsilon}}-{}_s\boldsymbol{m}:\left({}_s^{i\Theta}\dot{\boldsymbol{J}}\right)-{}_a\boldsymbol{m}:\left({}_a^{i\Theta}\dot{\boldsymbol{J}}\right)=0 \tag{4.4}$$

$$\{{}_d\sigma\}=[D]\{\varepsilon\}+[C]\{\dot{\varepsilon}\} \text{ : voight's notation} \tag{4.5}$$

$${}_s\boldsymbol{m}=2\mu\left({}_s^{i\Theta}\boldsymbol{J}\right) \tag{4.6}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = -k\begin{Bmatrix} \partial\theta/\partial x_1 \\ \partial\theta/\partial x_2 \\ \partial\theta/\partial x_3 \end{Bmatrix} \tag{4.7}$$

$$\boldsymbol{\sigma}={}_s\boldsymbol{\sigma}+{}_a\boldsymbol{\sigma}\ ;\ {}_s\boldsymbol{\sigma}={}_s^e\boldsymbol{\sigma}+{}_s^d\boldsymbol{\sigma}\ ;\ {}_s^e\boldsymbol{\sigma}=p(\rho,\theta)\boldsymbol{\delta}\ \text{ : equation of state} \tag{4.8}$$

$$\boldsymbol{m}={}_s\boldsymbol{m}+{}_a\boldsymbol{m} \tag{4.9}$$

Equations (4.1)-(4.7) are a system of twenty five equations BLM(3), BAM(3), BMM(3), FLT(1), constitutive theories for ${}_d\boldsymbol{\sigma}(6)$, ${}_s\boldsymbol{m}(6)$, $\boldsymbol{q}(3)$ in twenty five dependent variables $\boldsymbol{u}(3)$, ${}_s\boldsymbol{m}(6)$, ${}_a\boldsymbol{m}(3)$, $\theta(1)$, ${}_s^d\boldsymbol{\sigma}(6)$, ${}_a\boldsymbol{\sigma}(3)$, $\boldsymbol{q}(3)$. Equations (4.1)-(4.7) are valid $\forall\ (\boldsymbol{x},t)\in\Omega_{xt}=\Omega_x\times\Omega_t$. We consider space-time strip [91] $\bar{\Omega}_{xt}^{(n)}=\bar{\Omega}_x\times[t_n,t_{n-1}]$. Let $\left(\bar{\Omega}_{xt}^{(n)}\right)^T=\bigcup_e\bar{\Omega}_{xt}^e$ be discretization of $\bar{\Omega}_{xt}^{(n)}$ in which $\bar{\Omega}_{xt}^{(e)}$ is p-version hierarchical space-time finite element. Let

$$\{\phi_h^e\}^T=[(u_1)_h^e,(u_2)_h^e,(u_3)_h^e,({}_s\boldsymbol{m})_h^e,({}_a\boldsymbol{m})_h^e,(\theta)_h^e,({}_s^d\boldsymbol{\sigma})_h^e,({}_a\boldsymbol{\sigma})_h^e,(\boldsymbol{q})_h^e] \tag{4.10}$$

be local appropriation of $\boldsymbol{u}$, ${}_{s}\boldsymbol{m}$, ${}_{a}\boldsymbol{m}$, θ, ${}_{s}^{d}\boldsymbol{\sigma}$, ${}_{a}\boldsymbol{\sigma}$, $\boldsymbol{q}$. If $Q(\boldsymbol{x},t)$ is a dependent variable then its space-time local approximation over $\bar{\Omega}_{xt}^{(e)}$ is given by

$$Q_h^e(\boldsymbol{x}.t) = [N(\boldsymbol{x},t)]\{\delta_Q^e\} \tag{4.11}$$

in which $[N(\boldsymbol{x},t)]$ are space-time local approximation functions and $\{\delta_Q^e\}$ are degrees of freedom. Using (4.11), we can write the following local approximation for each dependent variable (equal order, equal degree).

$$\begin{aligned}
(u_1)_h^e &= [N(\boldsymbol{x},t)]\{\delta_{u_1}^e\} \\
(u_2)_h^e &= [N(\boldsymbol{x},t)]\{\delta_{u_2}^e\} \\
(u_3)_h^e &= [N(\boldsymbol{x},t)]\{\delta_{u_3}^e\} \\
({}_{s}\boldsymbol{m})_h^e &= [N(\boldsymbol{x},t)]\{\delta_{{}_{s}\boldsymbol{m}}^e\} \\
({}_{a}\boldsymbol{m})_h^e &= [N(\boldsymbol{x},t)]\{\delta_{{}_{a}\boldsymbol{m}}^e\} \\
(\theta)_h^e &= [N(\boldsymbol{x},t)]\{\delta_{\theta}^e\} \\
({}_{s}^{d}\boldsymbol{\sigma})_h^e &= [N(\boldsymbol{x},t)]\{\delta_{{}_{s}^{d}\boldsymbol{\sigma}}^e\} \\
({}_{a}\boldsymbol{\sigma})_h^e &= [N(\boldsymbol{x},t)]\{\delta_{{}_{a}\boldsymbol{\sigma}}^e\} \\
(\boldsymbol{q})_h^e &= [N(\boldsymbol{x},t)]\{\delta_{\boldsymbol{q}}^e\}
\end{aligned} \tag{4.12}$$

We define (in the same order as in (4.10))

$$\{\delta^e\}^T = \left[\{\delta_{u_1}^e\}^T, \{\delta_{u_2}^e\}^T, \{\delta_{u_3}^e\}^T, \{\delta_{{}_{s}\boldsymbol{m}}^e\}^T, \{\delta_{{}_{a}\boldsymbol{m}}^e\}^T, \{\delta_{\theta}^e\}^T, \{\delta_{{}_{s}^{d}\boldsymbol{\sigma}}^e\}^T, \{\delta_{{}_{a}\boldsymbol{\sigma}}^e\}^T, \{\delta_{\boldsymbol{q}}^e\}^T\right] \tag{4.13}$$

in which $\{\delta^e\}$ are total degrees of freedom for element 'e' with space-time domain $\bar{\Omega}_{xt}^e$. We define $\{\delta\}$ total degrees of freedom for $\left(\bar{\Omega}_{xt}^{(n)}\right)^T$ by

$$\{\delta\} = \bigcup_e \{\delta^e\} \tag{4.14}$$

When we substitute local approximation (4.12) in (4.1)-(4.7) we obtain twenty five residual functions E_i^e ; $i = 1, 2, \ldots, 25$. If we define (4.1)-(4.7) as

$$\boldsymbol{A}\boldsymbol{\phi} - \boldsymbol{f} = 0 \;\forall\; (x,t) \in \Omega_{xt} \tag{4.15}$$

in which $\boldsymbol{A}$ is a 25×25 nonlinear differential operator matrix and $\boldsymbol{\phi}$ is a 25×1 vector of dependent variables (as in (4.10)), then

$$\boldsymbol{E} = \boldsymbol{A}\boldsymbol{\phi}_h - \boldsymbol{f} \tag{4.16}$$

in which $\{\phi_h\} = \bigcup\limits_e \{\phi_h^e\}$ and $\boldsymbol{E}$ is a 25×1 vector of residual functions corresponding to $\left(\bar{\Omega}_{xt}^{(n)}\right)^T$. The space-time residual functional $I(\boldsymbol{\phi}_h)$ over $\left(\bar{\Omega}_{xt}^{(n)}\right)^T$ can be constructed :

$$I(\boldsymbol{\phi}_h) = \sum_{i=1}^{25} (E_i, E_i)_{(\bar{\Omega}_{xt}^{(n)})^T} = \sum_e (\sum_{i=1}^{25} (E_i^e, E_i^e)_{\bar{\Omega}_{xt}^e}) \tag{4.17}$$

when $I(\boldsymbol{\phi}_h)$ is differentiable in $\boldsymbol{\phi}_h$, then $\delta I(\boldsymbol{\phi}_h)$ is unique and $\delta I(\boldsymbol{\phi}_h) = 0$ is a necessary consideration for an extremum of $I(\boldsymbol{\phi}_h)$.

$$\begin{aligned} \delta I(\boldsymbol{\phi}_h) = \sum_{i=1}^{25} 2(E_i, \delta E_i)_{(\bar{\Omega}_{xt}^{(n)})^T} = \{g(\{\delta\})\} &= \sum_e (\sum_{i=1}^{25} 2\,(E_i^e, \delta E_i^e)_{\bar{\Omega}_{xt}^e}) \\ &= \sum_e \{g(\{\delta^e\})\} = 0 \end{aligned} \tag{4.18}$$

Since $\boldsymbol{A}$ is a nonlinear space-time differential operator, $\{g\}$ in (4.18) is a nonlinear function of $\{\delta\}$, thus we must find a $\{\delta\}$ that satisfies (4.18) iteratively. We use Newton's linear method with line search [91]. Let $\{\delta\}_0$ be an assumed or starting solution in Newton's linear method, then

$$\{g(\{\delta\}_0)\} \neq 0 \tag{4.19}$$

Let $\{\Delta\delta\}$ be a change in $\{\delta\}_0$ such that

$$\{g(\{\delta\}_0 + \{\Delta\delta\})\} = 0 \tag{4.20}$$

Taylor series expansion of $\{g(\cdot)\}$ in (4.20) about $\{\delta\}_0$ and retaining only up to linear terms in $\{\Delta\delta\}$ allows us to calculate $\{\Delta\delta\}$

$$\{\Delta\delta\} = -[\delta\{g\}]^{-1}_{\{\delta\}_0}\{g(\{\delta\}_0)\} = -[\delta^2 I]^{-1}_{\{\delta\}_0}\{g(\{\delta\}_0)\} \tag{4.21}$$

Based on reference [91]

$$\delta^2 I \simeq \sum_{i=1}^{25} 2(\delta E_i, \delta E_i)_{(\bar{\Omega}^{(n)}_{xt})^T} = \sum_e 2(\sum_{i=1}^{25}(\delta E^e_i, \delta E^e_i)_{\bar{\Omega}^e_{xt}} \tag{4.22}$$

An improved solution $\{\delta\}$ is obtained using line search [91]

$$\{\delta\} = \{\delta\}_0 + \alpha^*\{\Delta\delta\} \ ; \ \alpha^* \text{ such that } I(\{\delta\}) \leq I(\{\delta\}_0) \tag{4.23}$$

We check for convergence of the Newton's linear method.
If

$$|g_i(\{\delta\})| \leq \Delta \text{ (a preset tolerance for computed zero)} \tag{4.24}$$

then $\{\delta\}$ is the converged solution. If not we set $\{\delta\}_0$ to $\{\delta\}$ and repeat (4.21)-(4.24) until converged.

Remarks

It is rather obvious that these 25 equations can be cast in reduced system by substituting ${}^{d}_{s}\boldsymbol{\sigma}$, ${}_{s}\boldsymbol{m}$, $\boldsymbol{q}$ from the constitutive theories in the balance laws. However the steps in STRF method described above do not change.

4.2.2 IVPs in Eulerian description

We consider conservation and balance laws in Eulerian description in velocities as observable quantities. In this case also, there are many alternative choices for writing the balance laws in different forms (by substituting constitutive theories), nonetheless the basic steps of STRF based on integral form are not effected. We consider the following (expanded forms can be easily obtained). We remove back subscript $^{(0)}$ for more clarity of equations.

CM

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\boldsymbol{\nabla}} \cdot (\bar{\rho}\bar{\boldsymbol{v}}) = 0 \tag{4.25}$$

BLM

$$\bar{\rho}\frac{\partial \bar{\boldsymbol{v}}}{\partial t} + \bar{\rho}\left(\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{\nabla}}\right)\bar{\boldsymbol{v}} - \bar{\rho}\bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\sigma}} = 0 \tag{4.26}$$

BAM

$${}^{\Theta}\bar{I}\bar{\rho}\left(\frac{\partial}{\partial t}({}_i\bar{\omega}_k) + \bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{\nabla}}({}_i\bar{\omega}_k)\right) - \epsilon_{ijk}\bar{\sigma}_{ij} - \bar{m}_{mk,m} = 0 \tag{4.27}$$

BMM

$$\epsilon_{jkl}{}^{\Theta}\bar{I}\bar{\rho}\bar{v}_j\left({}_i\bar{\omega}_k\right) - \epsilon_{jkl}\bar{m}_{jk} = 0 \tag{4.28}$$

FLT (using $\bar{e} = c_v\bar{\theta}$)

$$\bar{\rho}\left(\frac{\partial \bar{\theta}}{\partial t} + \bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{\nabla}}(\bar{\theta})\right) + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}_s\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - {}_s\bar{\boldsymbol{m}} : {}^{r}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}} - {}_a\bar{\boldsymbol{m}} : {}^{r}_{i}{}^{\Theta}_{a}\bar{\boldsymbol{J}} = 0 \tag{4.29}$$

and

$$\begin{aligned}
&{}^{d}_{s}\bar{\boldsymbol{\sigma}} = 2\eta\bar{\boldsymbol{D}} + \kappa\left(\mathrm{tr}\bar{\boldsymbol{D}}\right)\boldsymbol{I} \\
&{}^{e}_{s}\bar{\boldsymbol{\sigma}} = \bar{p}(\bar{\rho}, \bar{\theta})\boldsymbol{I} \;;\; \bar{p}(\bar{\rho}, \bar{\theta}) \;:\; \text{equation of state} \\
&\bar{\boldsymbol{\sigma}} = {}_s\bar{\boldsymbol{\sigma}} + {}_a\bar{\boldsymbol{\sigma}},\; {}_s\bar{\boldsymbol{\sigma}} = {}^{e}_{s}\bar{\boldsymbol{\sigma}} + {}^{d}_{s}\bar{\boldsymbol{\sigma}}
\end{aligned} \tag{4.30}$$

$${}_s\bar{\boldsymbol{m}} = 2\eta_m\left({}^{r}_{i}{}^{\Theta}_{s}\bar{\boldsymbol{J}}\right) \tag{4.31}$$

$$\bar{\boldsymbol{q}} = -k\bar{\boldsymbol{g}} \tag{4.32}$$

The mathematical model consists of 26 equations : CM(1), BLM(3), BAM(3), BMM(3), FLT(1), constitutive theories for ${}_s^d\bar{\boldsymbol{\sigma}}(6)$, ${}_s\bar{\boldsymbol{m}}(6)$, $\bar{\boldsymbol{q}}(3)$ in twenty six variables $\bar{\rho}(1)$, $\bar{\boldsymbol{v}}(3)$, ${}_s\bar{\boldsymbol{m}}(6)$, ${}_a\bar{\boldsymbol{m}}(3)$, $\bar{\theta}(1)$, ${}_s^d\bar{\boldsymbol{\sigma}}(6)$, ${}_a\bar{\boldsymbol{\sigma}}(3)$, $\bar{\boldsymbol{q}}(3)$. Equations (4.25)-(4.32) hold $\forall\ (\bar{\boldsymbol{x}}, t) \in \Omega_{\bar{x}t} = \Omega_{\bar{x}} \times \Omega_t$. We consider a space-time strip $\bar{\Omega}_{\bar{x}t}^{(n)} = \bar{\Omega}_{\bar{x}} \times [t_n, t_{n+1}]$ (see reference [91] for more details). Let $\bar{\Omega}_{\bar{x}t}^{(n)} = \bigcup\limits_e \bar{\Omega}_{\bar{x}t}^e$ in which $\bar{\Omega}_{\bar{x}t}^e$ is the space-time domain of a space-time element e. Let

$$\{\bar{\phi}_h^e\}^T = [(\bar{\rho}_h^e), (\bar{\boldsymbol{u}})_h^e, ({}_s\bar{\boldsymbol{m}})_h^e, ({}_a\bar{\boldsymbol{m}})_h^e, (\theta)_h^e, ({}_s^d\bar{\boldsymbol{\sigma}})_h^e, ({}_a\bar{\boldsymbol{\sigma}})_h^e, (\bar{\boldsymbol{q}})_h^e] \tag{4.33}$$

be local approximation of $\{\bar{\phi}\}^T = [\bar{\rho}, \bar{\boldsymbol{v}}, ({}_s\bar{\boldsymbol{m}}), ({}_a\bar{\boldsymbol{m}}), (\bar{\theta}), ({}_s^d\bar{\boldsymbol{\sigma}}), ({}_d\bar{\boldsymbol{\sigma}}), (\bar{\boldsymbol{q}})]$. If $\bar{Q}(\bar{\boldsymbol{x}}, t)$ is a dependent variable, then its space-time local approximation $\bar{Q}_h^e(\bar{\boldsymbol{x}}, t)$ over $\bar{\Omega}_{xt}^e$ can be written as

$$\bar{Q}_h^e(\bar{\boldsymbol{x}}, t) = [N(\bar{\boldsymbol{x}}, t)]\{\delta_Q^e\} \tag{4.34}$$

in which $[N(\bar{\boldsymbol{x}}, t)]$ are space-time local approximation functions and $\{\delta_Q^e\}$ are degrees of freedom. Using (4.34) we can write local approximation for each dependent variable (equal order, equal degree)

$$\begin{aligned}
\bar{\rho}_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{\bar{\rho}}\} \\
(\bar{v}_1)_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{\bar{v}_1}\} \\
(\bar{v}_2)_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{\bar{v}_2}\} \\
(\bar{v}_3)_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{\bar{v}_3}\} \\
({}_s\bar{\boldsymbol{m}})_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{{}_s\bar{\boldsymbol{m}}}\} \\
({}_a\bar{\boldsymbol{m}})_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{{}_a\bar{\boldsymbol{m}}}\} \\
(\bar{\theta})_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{\bar{\theta}}\} \\
({}_s^d\bar{\boldsymbol{\sigma}})_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{{}_s^d\bar{\boldsymbol{\sigma}}}\} \\
({}_a\bar{\boldsymbol{\sigma}})_h^e &= [N(\boldsymbol{x},t)]\{\delta^e_{{}_a\bar{\boldsymbol{\sigma}}}\} \\
(\bar{\boldsymbol{q}})_h^e &= [N(\bar{\boldsymbol{x}},t)]\{\delta^e_{\bar{\boldsymbol{q}}}\}
\end{aligned} \tag{4.35}$$

We define degrees of freedom $\{\delta^e\}$ for an element 'e' using same order as in (4.33)

$$\{\delta^e\}^T = \left[\{\delta^e_{\bar{\rho}}\}^T, \{\delta^e_{\bar{v}_1}\}^T, \{\delta^e_{\bar{v}_2}\}^T, \{\delta^e_{\bar{v}_3}\}^T, \{\delta^e_{{}_s\bar{\boldsymbol{m}}}\}^T, \{\delta^e_{{}_a\bar{\boldsymbol{m}}}\}^T, \{\delta^e_{\bar{\theta}}\}^T, \{\delta^e_{{}_s^d\bar{\boldsymbol{\sigma}}}\}^T, \{\delta^e_{{}_a\bar{\boldsymbol{\sigma}}}\}^T, \{\delta^e_{\bar{\boldsymbol{q}}}\}^T\right] \tag{4.36}$$

If $\{\delta\}$ are total degrees of freedom for $\left(\bar{\Omega}^{(n)}_{\bar{x}t}\right)^T$, then

$$\{\delta\} = \bigcup_e \{\delta^e\} \tag{4.37}$$

Let $\boldsymbol{A}\bar{\boldsymbol{\phi}} - \bar{\boldsymbol{f}} = 0 \ \forall \ (\bar{\boldsymbol{x}}, t) \ \in \ \left(\Omega^{(n)}_{\bar{x}t}\right)^T$ in which $\boldsymbol{A}$ is a 26×26 nonlinear space-time differential operator matrix and $\bar{\boldsymbol{\phi}}$ is a 26×1 vector of dependent variables ((4.33)). Then

$$\boldsymbol{E} = \boldsymbol{A}\bar{\boldsymbol{\phi}}_h - \bar{\boldsymbol{f}} \tag{4.38}$$

where $\{\bar{\phi}_h\} = \bigcup_e \bar{\phi}_h^e$ and $\boldsymbol{E}$ is a 26×1 vector of residual functions for $\left(\bar{\Omega}_{\bar{x}t}^{(n)}\right)^T$. The space-time residual functional $I(\bar{\boldsymbol{\phi}})$ over $\left(\bar{\Omega}_{\bar{x}t}^{(n)}\right)^T$ can be constructed

$$I(\bar{\boldsymbol{\phi}}_h) = \sum_{i=1}^{26} (E_i, E_i)_{(\bar{\Omega}_{\bar{x}t}^{(n)})^T} = \sum_e (\sum_{i=1}^{26} (E_i^e, E_i^e)_{\bar{\Omega}_{\bar{x}t}^e}) \tag{4.39}$$

when $I(\bar{\boldsymbol{\phi}}_h)$ is differentiable in $\bar{\boldsymbol{\phi}}_h$, then $\delta I(\bar{\boldsymbol{\phi}}_h)$ is unique and

$$\delta I(\bar{\boldsymbol{\phi}}) = 0 \tag{4.40}$$

is a necessary condition for an extremum of $I(\bar{\boldsymbol{\phi}}_h)$.

$$\begin{aligned} \delta I(\bar{\boldsymbol{\phi}}_h) = \sum_{i=1}^{26} 2(E_i, \delta E_i)_{(\bar{\Omega}_{\bar{x}t}^{(n)})^T} = \{g(\{\delta\})\} &= \sum_e (\sum_{i=1}^{26} 2\,(E_i^e, \delta E_i^e)_{\bar{\Omega}_{\bar{x}t}^e}) \\ &= \sum_e \{g(\{\delta^e\})\} = 0 \end{aligned} \tag{4.41}$$

since $\boldsymbol{A}$ is a nonlinear space-time differential operator, $\{g\}$ in (4.41) is a nonlinear function of $\{\delta\}$, thus we must find a $\{\delta\}$ that satisfies (4.41) iteratively. We use Newton's linear method with line search [91]. Let $\{\delta\}_0$ be an assumed or starting solution in Newton's linear method, then

$$\{g(\{\delta\}_0)\} \neq 0 \tag{4.42}$$

Let $\{\Delta\delta\}$ be a change in $\{\delta\}_0$ such that

$$\{g(\{\delta\}_0 + \{\Delta\delta\}) = 0 \tag{4.43}$$

Consider Taylor series expansion of $\{g(\cdot)\}$ in (4.43) about $\{\delta\}_0$ and retain only up to linear terms in $\{\Delta\delta\}$. This allows us to calculate $\{\Delta\delta\}$.

$$\{\Delta\delta\} = -[\delta\{g\}]_{\{\delta\}_0}^{-1}\{g(\{\delta\}_0)\} = -[\delta^2 I]_{\{\delta\}_0}^{-1}\{g(\{\delta\}_0)\} \tag{4.44}$$

Based on reference [91]

$$\delta^2 I \simeq \sum_{i=1}^{26} 2(\delta E_i, \delta E_i)_{(\bar{\Omega}_{\bar{x}t}^{(n)})^T} = \sum_e 2(\sum_{i=1}^{26} (\delta E_i^e, \delta E_i^e)_{\bar{\Omega}_{\bar{x}t}^e} \tag{4.45}$$

and improved solution $\{\delta\}$ is calculated using line search

$$\{\delta\} = \{\delta\}_0 + \alpha^* \{\Delta\delta\} \ ; \ \alpha^* \text{ such that } I(\{\delta\}) \leq I(\{\delta\}_0) \tag{4.46}$$

Next, we check for convergence of the Newton's linear method. If

$$|g_i(\{\delta\})| \leq \Delta \text{ (a preset tolerance for computed zero)} \tag{4.47}$$

then $\{\delta\}$ is the converged solution from Newton's linear method. Otherwise, we set $\{\delta\}_0 = \{\delta\}$ and repeat (4.44)-(4.47) until converged.

Remarks

We note that in this case also, there are alternative ways to cast equations in the mathematical model in different forms. This does influence global differentiability requirements but the basic steps in STRF are not effected.

4.3 Space-time decoupled finite element method for IVPs

In this section, we consider space-time decoupled finite element process for the mathematical models presented in sections 2.9 in Lagrangian and in section 3.9 in Eulerian description. In space-time decoupled methods, we consider a discretization in space and consider integral form of the IVPs over this discretization using GM/WF (generally) based on fundamental Lemma of calculus of variations. Space and time are decoupled in local approximations over the elements of this discretization using approximations in space and considering nodal degrees of freedom as functions of time. When the approximations are substituted in the mathematical model, after integration in space and assembly of element expressions, we obtain ODEs in time in the degrees of freedom and their time derivatives. These are integrated using direct, explicit or implicit time integration methods.

4.3.1 IVPs in Lagrangian description

Consider the mathematical model in section 4.2.1. We consider space time decoupled finite element formulation of each balance law and then subsequently put these all together as a system of ODEs in time. We consider $\boldsymbol{u}$, ${}_i\boldsymbol{\Theta}$, ${}_a\boldsymbol{m}$, ${}_a\boldsymbol{\sigma}$ and θ as dependent variables in the mathematical model consisting of CBL of CCM (in section 4.2.1). Let $\bar{\Omega}_x^T = \bigcup_e \bar{\Omega}_x^e$ be the discretization of spatial domain $\bar{\Omega}_x$ in which $\bar{\Omega}_x^e$ is the spatial domain of a p-version hierarchical finite element. If Q is a dependent variable then local approximation of Q over $\bar{\Omega}_x^e$ in space-time decoupled finite element method is given by

$$Q_h^e(\boldsymbol{x},t) = [N(\boldsymbol{x})]\{\delta_Q^e(t)\} \tag{4.48}$$

using (4.48) we can write equal order, equal degree local approximations $\boldsymbol{u}_h^e$, ${}_i\boldsymbol{\Theta}_h^e$, ${}_a\boldsymbol{m}_h^e$, ${}_a\boldsymbol{\sigma}_h^e$ and θ_h^e of $\boldsymbol{u}$, ${}_{\boldsymbol{i}}\boldsymbol{\Theta}$, ${}_a\boldsymbol{m}$, ${}_a\boldsymbol{\sigma}_h^e$ and θ over $\bar{\Omega}_x^e$. Let $\{\delta_{\boldsymbol{u}}^e(t)\}$, $\{\delta_{{}_i\boldsymbol{\Theta}}^e(t)\}$, $\{\delta_{{}_a\boldsymbol{m}}^e(t)\}$, $\{\delta_{{}_a\boldsymbol{\sigma}}^e(t)\}$, $\{\delta_\theta^e(t)\}$ be the nodal degrees of freedom associated with the local approximations. We consider each balance law.

BLM: If we substitute $\boldsymbol{\sigma} \; =_s \boldsymbol{\sigma} +_a \boldsymbol{\sigma}$, ${}_s\boldsymbol{\sigma} \; = {}_s^e\boldsymbol{\sigma} + {}_s^d\boldsymbol{\sigma}$ and the constitutive theory for ${}_s^d\boldsymbol{\sigma}$ ((4.5)) in BLM and multiply each equation by $\beta_1 = \delta(u_1)_h^e$, $\beta_2 = \delta(u_2)_h^e$, and $\beta_3 = \delta(u_3)_h^e$, integrate over $\bar{\Omega}_x^e$ and perform integration by parts once for the terms associated with the constitutive theory for ${}_s^d\boldsymbol{\sigma}$, then we can obtain the following for (4.1)

$$[{}^1M^e]\{\ddot{\delta}_u^e\} + [{}^1C^e]\{\dot{\delta}_u^e\} + [{}^1K_u^e]\{\delta_u^e\} + [{}^1K_{{}_a\sigma}^e]\{\delta_{{}_a\sigma}^e\} - \{{}^1f_u^e\} - \{{}^1P_u^e\} \tag{4.49}$$

BAM: We use $\boldsymbol{m} \; =_s \boldsymbol{m} +_a \boldsymbol{m}$ and substitute constitutive theory for ${}_s\boldsymbol{m}$ in (4.2), multiply by $\beta_4 = \delta({}_i\Theta_1)$, $\beta_5 = \delta({}_i\Theta_2)$ and $\beta_6 = \delta({}_i\Theta_3)$, integrate over $\bar{\Omega}_x^e$, perform integration by parts once for the terms containing second derivative of ${}_i\Theta$ with respect to x_i to obtain

$$[{}^2M^e]\{\ddot{\delta}^e_\Theta\} + [{}^2K^e_{{}_a\sigma}]\{\delta^e_{{}_a\sigma}\} + [{}^2K^e_\Theta]\{\delta^e_\Theta\} + [{}^2K^e_{{}_am}]\{\delta^e_{{}_am}\} - \{{}^2f^e_\Theta\} - \{{}^2P^e_\Theta\} \tag{4.50}$$

BMM: We multiply (4.3) by $\beta_7 = \delta({}_am_{23})^e_h$, $\beta_8 = \delta({}_am_{31})^e_h$, $\beta_9 = \delta({}_am_{12})^e_h$, integrate over $\bar{\Omega}^e_x$ to obtain

$$[{}^3K^e_\Theta]\{\delta^e_\Theta\} + [{}^3K^e_{{}_am}]\{\delta^e_{{}_am}\} \tag{4.51}$$

FLT: We multiply (4.4) by $\beta_{10} = \delta\Theta^e_h$, substitute constitutive theory for $\boldsymbol{q}$, and integrate over $\bar{\Omega}^e_x$. We transfer one order of differentiation to β_{10} from the terms containing gradients of Θ using integration by parts to obtain

$$[{}^4C_\Theta]\{\dot{\delta}^e_\Theta\} + [{}^4H^e]\{\delta^e_\theta\} + [{}^4C^e_u]\{\dot{\delta}^e_u\} + [{}^4C^e_\Theta]\{\dot{\delta}^e_\Theta\} - \{{}^4f^e_\theta\} - \{{}^4P^e_\theta\} \tag{4.52}$$

The last set of equations are defined in terms of ${}_i\boldsymbol{\Theta} = \boldsymbol{\nabla} \times \boldsymbol{u}$. Multiply by $\beta_{11} = \delta_i\Theta_1$, $\beta_{12} = \delta_i\Theta_2$, $\beta_{13} = \delta_i\Theta_3$ and integrate over $\bar{\Omega}^e_x$ to obtain

$$[{}^5K^e_\Theta]\{\delta^e_\Theta\} + [{}^5K^e_u]\{\delta^e_u\} \tag{4.53}$$

Assembly of (4.49)-(4.53) for $(\bar{\Omega}_x)^T$ yields the following ODEs in time.

$$[{}^1M]\{\ddot{\delta}_u\} + [{}^1C]\{\dot{\delta}_u\} + [{}^1K_u]\{\delta_u\} + [{}^1K_{{}_a\sigma}]\{\delta_{{}_a\sigma}\} = \{{}^1f_u\} + \{{}^1P_u\} \tag{4.54}$$

$$[{}^2M]\{\ddot{\delta}_\Theta\} + [{}^2k_{{}_a\sigma}]\{\delta_{{}_a\sigma}\} + [{}^2K_\Theta]\{\delta_\Theta\} + [{}^2K_{{}_am}]\{\delta_{{}_am}\} = \{{}^2f_\Theta\} + \{{}^2P_\Theta\} \tag{4.55}$$

$$[{}^3K_\Theta]\{\delta_\Theta\} + [{}^3K_{{}_am}]\{\delta_{{}_am}\} = 0 \tag{4.56}$$

$$[{}^4C_\Theta]\{\dot{\delta}_\Theta\} + [{}^4H]\{\delta_\theta\} + [{}^4C_u]\{\dot{\delta}_u\} + [{}^4C_\Theta]\{\dot{\delta}_\Theta\} = \{{}^4f_\theta\} + \{{}^4P_\theta\} \tag{4.57}$$

$$[{}^5K_\Theta]\{\delta_\Theta\} + [{}^5K_u]\{\delta_u\} = 0 \tag{4.58}$$

In which

$$\{\delta_u\} = \bigcup_e \{\delta_u^e\} \quad ; \quad \{\dot{\delta}_u\} = \bigcup_e \{\dot{\delta}_u^e\} \quad ; \quad \dots \quad ; \quad etc \tag{4.59}$$

Equations (4.54)-(4.59) can be arranged as follows

$$\begin{aligned}
&\begin{bmatrix} [^1M] & [0] & [0] & [0] & [0] \\ [0] & [^2M] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{\delta}_u\} \\ \{\ddot{\delta}_\Theta\} \\ \{\ddot{\delta}_{am}\} \\ \{\ddot{\delta}_{a\sigma}\} \\ \{\ddot{\delta}_\theta\} \end{Bmatrix} + \begin{bmatrix} [^1C] & [0] & [0] & [0] & [0] \\ [0] & [^2M] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] \\ [^4C_u] & [^4C_\Theta] & [0] & [0] & [^4C_\theta] \\ [0] & [0] & [0] & [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\dot{\delta}_u\} \\ \{\dot{\delta}_\Theta\} \\ \{\dot{\delta}_{am}\} \\ \{\dot{\delta}_{a\sigma}\} \\ \{\dot{\delta}_\theta\} \end{Bmatrix} \\
&+ \begin{bmatrix} [^1K_u] & [0] & [0] & [^1K_{a\sigma}] & [0] \\ [0] & [^2K_\Theta] & [^2K_{am}] & [^2K_{a\sigma}] & [0] \\ [0] & [^3K_\Theta] & [^3K_{am}] & [0] & [0] \\ [0] & [0] & [0] & [0] & [^4H] \\ [^5K_u] & [^5K_\Theta] & [0] & [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\delta_u\} \\ \{\delta_\Theta\} \\ \{\delta_{am}\} \\ \{\delta_{a\sigma}\} \\ \{\delta_\theta\} \end{Bmatrix} = \{f\} + \{P\}
\end{aligned} \tag{4.60}$$

This is a system of second order ODEs in degrees of freedom $\{\delta\}$. Determination of a complete initial solutions using ICs is critical. We present details in the following.

(a) $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ are known ICs hence $\{\delta_u\}$ and $\{\dot{\delta}_u\}$ are known

(b) All stresses and moments are zero (system at rest) other than those used to define BCs.

(c) FLT is first order ODE in θ hence θ i.e., $\{\delta_\theta\}$ are known ICs.

(d) From (4.54) we can determine $\{\ddot{\delta}_u\}$

(e) From (4.55) we can determine $\{\ddot{\delta}_\Theta\}$ as $\{\delta_\Theta\}$ and $\{\dot{\delta}_\Theta\}$ are known (zero due to initial conditions on $\boldsymbol{\Theta}$ and $\dot{\boldsymbol{\Theta}}$).

Thus, all degrees of freedoms and their time derivatives are known at the commencement of evolution $(t_0 = 0)$.

Remarks

(1) We can write (4.60) as

$$[\underset{\sim}{M}]\{\ddot{\delta}\} + [\underset{\sim}{C}]\{\dot{\delta}\} + [\underset{\sim}{K}]\{\delta\} = \{P\} + \{F\} \tag{4.61}$$

(2) Addition of $[\underset{\sim}{M}]$, $[\underset{\sim}{C}]$ and $[\underset{\sim}{K}]$ leaves zero on the diagonal, hence in the time integration methods pivoting is needed when solving linear equations.

(3) An alternative to (2) is to rearrange degrees of freedom in $\{\delta\}$, hence $\{\dot{\delta}\}$ and $\{\ddot{\delta}\}$, such that diagonals of $[\underset{\sim}{M}] + [\underset{\sim}{C}] + [\underset{\sim}{K}]$ are not zero, thus avoiding use of pivoting.

(4) Equations (4.61) are a system of second order nonlinear ODEs in time. These can be integrated in time with iterative methods to achieve converged solution for each increment of time.

(5) Newmark's linear acceleration method with Newton's linear method for obtaining converged solution for each time step can be used for time integration of (4.60).

4.3.2 IVPs in Eulerian description

Consider the mathematical model in section 4.2.2. Following the procedure similar to the one described in section 4.3.1 we can obtain the following system of first order ODEs in time for $\bar{\Omega}^T_{\bar{x}} = \bigcup_e \bar{\Omega}^e_{\bar{x}}$.

CM

$$[^1M_{\bar{\rho}}]\{\dot{\delta}_{\bar{\rho}}\} + [^1K_{\bar{v}}]\{\delta_{\bar{v}}\} = 0 \tag{4.62}$$

BLM

$$[^2M_{\bar{v}}]\{\dot{\delta}_{\bar{v}}\} + [^2K_{\bar{v}}]\{\delta_{\bar{v}}\} + [^2K_{_a\bar{\sigma}}]\{\delta_{_a\bar{\sigma}}\} = \{^2f\} + \{^2P\} \tag{4.63}$$

BAM

$$[^3M_{\bar{\omega}}]\{\dot{\delta}_{\bar{\omega}}\} + [^3K_{\bar{\omega}}]\{\delta_{\bar{\omega}}\} + [^3K_{{}_a\bar{\sigma}}]\{\delta_{{}_a\bar{\sigma}}\} + [^3K_{{}_a\bar{m}}]\{\delta_{{}_a\bar{m}}\} = \{^3f\} + \{^3P\} \tag{4.64}$$

BMM

$$[^4K_{\bar{\omega}}]\{\delta_{\bar{\omega}}\} + [^4K_{{}_a\bar{m}}]\{\delta_{{}_a\bar{m}}\} = 0 \tag{4.65}$$

FLT

$$[^5C]\{\dot{\delta}_{\bar{\theta}}\} + [^5H]\{\delta_{\bar{\theta}}\} + [^5K_{\bar{v}}]\{\delta_{\bar{v}}\} + [^5K_{{}_a\bar{m}}]\{\delta_{{}_a\bar{m}}\} = 0 \tag{4.66}$$

and

$$[^6K_{\bar{\omega}}]\{\delta_{\bar{\omega}}\} + [^6K_{\bar{v}}]\{\delta_{\bar{v}}\} = 0 \tag{4.67}$$

using

$$\{\delta\}^T = \left[\{\delta_{\bar{\rho}}\}^T, \{\delta_{\bar{\boldsymbol{v}}}\}^T, \{\delta_{\bar{\boldsymbol{\omega}}}\}^T, \{\delta_{{}_a\bar{\boldsymbol{m}}}\}^T, \{\delta_{{}_a\bar{\boldsymbol{\sigma}}}\}^T, \{\delta_{\bar{\theta}}\}^T\right] \tag{4.68}$$

using (4.68), (4.62)-(4.67) can be written as

$$[\underset{\sim}{C}]\{\dot{\delta}\} + [\underset{\sim}{K}]\{\delta\} = \{f\} + \{P\}$$

This is a system of first order nonlinear ODEs in time. Determination of total solution for all degrees of freedom and their derivatives using ICs follows a procedure similar to one described for IVPs in Lagrangian description except that in this case the ODEs are first order (see reference [91]). These can be integrated using implicit time integration methods. Newmark method based on linear $\{\dot{\delta}\}$ for $t \in [t_n, t_{n+\Delta t}]$ with newton's linear method for each linear step is a possible time integration method.

Chapter 5

Model problems

In this section, we consider model problems for micropolar solid and fluid media. Model problems are intentionally kept simple so that significant features of micropolar aspects in the presence of rotational inertial physics can be demonstrated.

5.1 Translational and rotational waves in micropolar solids

The objective of this study is to demonstrate existence of pure rotational waves in micropolar solids with rotational inertial physics. Just like in classical continuum mechanics (CCM) we can show existence, propagation, reflection and interaction of translational (stress) waves, in case of micropolar solids with rotational inertial physics we can additionally show existence, propagation, reflection and interaction of rotational (moment) waves. That is, in micropolar solids with rotational inertial physics, translational and rotational waves coexist. In the following study we consider purely one dimensional case of BLM and BAM. In the studies presented here, we show that BLM permits translational waves where as BAM with rotational inertial physics permits pure rotational waves.

Mathematical models

One dimensional form of BLM and BAM with rotational inertial physics (equations (2.118) and (2.119) from complete mathematical model (2.118)-(2.125),(2.127)) for small strain small deformation and the associated constitutive theories for thermoviscoelastic (without memory) micropolar solid with dissipation mechanism due to rate of strain (CCM) and due to rate of symmetric part of the rotation gradient tensor are given by (in the absence of body forces, body moments, initial stress and equilibrium Cauchy stress, thus $\sigma_{11} = {}_s\sigma_{11} = {}_s^d\sigma_{11}$ and using x for x_1)

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial({}_s^d\sigma_{11})}{\partial x} = 0 \tag{5.1}$$

$$ {}^{\Theta}I_0 \rho_0 \frac{\partial^2 ({}_i\Theta_1)}{\partial t^2} - \frac{\partial({}_s m_{11})}{\partial x} = 0 \tag{5.2}$$

$$ {}_s^d\sigma_{11} = (2\mu + \lambda)\frac{\partial u_1}{\partial x} + (2\eta + \kappa)\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x}\right) \tag{5.3}$$

$$ {}_s m_{11} = (\alpha_1)\frac{\partial({}_i\Theta_1)}{\partial x} + (\alpha_2)\frac{\partial}{\partial t}\left(\frac{\partial({}_i\Theta_1)}{\partial x}\right) \qquad \alpha_1 = \mu_m \quad , \quad \alpha_2 = \eta_m \tag{5.4}$$

Substituting (5.3) in (5.1) and (5.4) in (5.2) we obtain

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} - (2\mu + \lambda)\frac{\partial^2 u_1}{\partial x^2} - (2\eta + \kappa)\frac{\partial}{\partial t}\left(\frac{\partial^2 u_1}{\partial x^2}\right) = 0 \tag{5.5}$$

$$ {}^{\Theta}I_0 \rho_0 \frac{\partial^2 ({}_i\Theta_1)}{\partial t^2} - (\alpha_1)\frac{\partial^2 ({}_i\Theta_1)}{\partial x^2} - (\alpha_2)\frac{\partial}{\partial t}\left(\frac{\partial^2 ({}_i\Theta_1)}{\partial x^2}\right) = 0 \tag{5.6}$$

for 1D case and for incompressible physics $2\mu + \lambda = E$, modulus of elasticity, as Poison's ratio $\nu = 0$ and $\kappa = 0$, hence (5.6) reduces to (but (5.6) remains same).

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} - E\frac{\partial^2 u_1}{\partial x^2} - 2\eta\frac{\partial}{\partial t}\left(\frac{\partial^2 u_1}{\partial x^2}\right) = 0 \tag{5.7}$$

$$ {}^{\Theta}I_0 \rho_0 \frac{\partial^2 ({}_i\Theta_1)}{\partial t^2} - (\alpha_1)\frac{\partial^2 ({}_i\Theta_1)}{\partial x^2} - (\alpha_2)\frac{\partial}{\partial t}\left(\frac{\partial^2 ({}_i\Theta_1)}{\partial x^2}\right) = 0 \tag{5.8}$$

Remarks

1. Equation (5.7) describes translational or ${}_s^d\sigma_{11}$ stress wave in a viscous elastic medium. In the absence of the last term in (5.7), (5.7) represents translational waves in inviscid elastic medium. Dissipation (last term in (5.7)) results in amplitude decay and base elongation.

2. Equation (5.8) is analogous to equation (5.7) and can be obtained b replacing u_1, ρ_0, E, and 2η in (5.7) by ${}_i\Theta_1$, ${}^{\Theta}I_0\rho_0$, α_1 and α_2. Thus, (5.8) describes rotational waves i.e., moment $({}_sm_{11})$ wave. The third term in (5.8) is due to micropolar dissipation physics which would result in amplitude decay and base elongation of ${}_sm_{11}$ rotational wave.

3. We note the lack of coupling between (5.7) and (5.8) due to zero antisymmetric part of the deviatoric Cauchy stress. This is not the case in 2D and 3D applications.

4. Presence of (5.7) and (5.8) confirm coexistence of translational and rotational waves in micropolar elastic solids.

Dimensionless form of the mathematical model

We first nondimensionalize (5.7) and (5.8) before presenting their solutions. We write (5.7) and (5.8) with hat ($\wedge$) on all quantities indicating that all quantities have their usual dimensions (units)

$$\hat{\rho}_0 \frac{\partial^2 \hat{u}_1}{\partial \hat{t}^2} - \hat{E}\frac{\partial^2 \hat{u}_1}{\partial \hat{x}^2} - 2\hat{\eta}\frac{\partial}{\partial \hat{t}}\left(\frac{\partial^2 \hat{u}_1}{\partial \hat{x}^2}\right) = 0 \tag{5.9}$$

$${}^{\Theta}\hat{I}_0\hat{\rho}_0 \frac{\partial^2 ({}_i\hat{\Theta}_1)}{\partial \hat{t}^2} - (\hat{\alpha}_1)\frac{\partial^2 ({}_i\hat{\Theta}_1)}{\partial \hat{x}^2} - (\hat{\alpha}_2)\frac{\partial}{\partial \hat{t}}\left(\frac{\partial^2 ({}_i\hat{\Theta}_1)}{\partial \hat{x}^2}\right) = 0 \tag{5.10}$$

We note that in general the speed of translational wave is different than the speed of rotational wave. However, since in this case (5.9) and (5.10) are decoupled, it is possible to nondimensionalize (5.9) and (5.10) such that in the dimensionless domain the speed of prorogation of both translation and rotational waves is unity. Obviously in 2D and 3D cases it is not possible to do so. We present details in the following.

BLM: Model TW1

Choosing L_0, η_0, $(\rho_0)_{ref}$, t_0, $v_0 = L_0/t_0 = \sqrt{\frac{E_0}{\rho_0}}$ and $E_0 = (\rho_0)_{ref}v_0^2$ as reference quantities in which v_0 is the speed of translational wave (speed of sound in CCM).

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} - E\frac{E_0}{(\rho_0)_{ref} v_0^2}\frac{\partial^2 u_1}{\partial x^2} - \frac{2\eta}{Re}\frac{\partial}{\partial t}\left(\frac{\partial^2 u_1}{\partial x^2}\right) = 0 \tag{5.11}$$

where $1/Re = \frac{\eta_0}{L_0(\rho_0)_{ref} v_0}$, Reynolds number. If we choose $(\rho_0)_{ref} = \hat{\rho}_0$, $E_0 = \hat{E} = (\rho_0)_{ref} v_0^2$ and $C_d = \frac{2\eta}{Re}$ as damping coefficient, then (5.11) can be reduced to

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} - c_d\frac{\partial}{\partial t}\left(\frac{\partial^2 u_1}{\partial x^2}\right) = 0 \tag{5.12}$$

In (5.12), the dimensionless speed of sound (based on reference quantities) in an inviscid medium is unity. We use (5.12) in the numerical studies.

BAM: Using speed of rotational waves as reference velocity : Model RW1

In this case, when nondimensionalizing BAM we choose speed of rotational wave as reference velocity. We also choose some of the same reference quantities as in BLM except we choose $({}^{\Theta}I_0)_{ref}$, $(\alpha_1)_0$, $(\alpha_2)_0$, $\omega_0 = \sqrt{(\alpha_1)_0/({}^{\Theta}I_0)_{ref}(\rho_0)_{ref}}$ and $t_0 = L_0/\omega_0$, $(\alpha_1)_0 = ({}^{\Theta}I_0)_{ref}(\rho_0)_{ref}\omega_0^2$, then (5.10) can be written as

$${}^{\Theta}I_0\rho_0 \frac{\partial^2({}_i\Theta_1)}{\partial t^2} - (\alpha_1)\frac{(\alpha_1)_0}{({}^{\Theta}I_0)_{ref}(\rho_0)_{ref}\omega_0^2}\frac{\partial^2({}_i\Theta_1)}{\partial x^2} - (\alpha_2)\frac{1}{Re^*}\frac{\partial}{\partial t}\left(\frac{\partial^2({}_i\Theta_1)}{\partial x^2}\right) = 0 \tag{5.13}$$

If $({}^{\Theta}I_0)_{ref} = {}^{\Theta}\hat{I}_0$, $(\rho_0)_{ref} = (\hat{\rho}_0)$, then (5.13) can be reduced to

$$\frac{\partial^2({}_i\Theta_1)}{\partial t^2} - \frac{\partial^2({}_i\Theta_1)}{\partial x^2} - \frac{(\alpha_2)}{Re^*}\frac{\partial}{\partial t}\left(\frac{\partial^2({}_i\Theta_1)}{\partial x^2}\right) = 0 \tag{5.14}$$

in which $Re^* = \frac{({}^{\Theta}I_0)_{ref}(\rho_0)_{ref}\omega_0 L_0}{(\alpha_2)_0}$ is the Reynolds number associated with dissipation due to micropolar physics. If we define $c_d^* = \frac{\alpha_2}{Re^*}$ as the dissipation coefficient due to micropolar physics, then (5.14) can be written as

$$\frac{\partial^2({}_i\Theta_1)}{\partial t^2} - \frac{\partial^2({}_i\Theta_1)}{\partial x^2} - c_d^* \frac{\partial}{\partial t}\left(\frac{\partial^2({}_i\Theta_1)}{\partial x^2}\right) = 0 \tag{5.15}$$

Equation (5.12) and (5.15) are the simplified dimensionless forms of BLM and BAM describing translational and rotational waves in the presence of dissipation mechanisms. In the numerical studies we use equations (5.12) and (5.15).

BAM: Using translation wave speed as reference velocity : Model RW2

In this case we choose

$$\left.\begin{aligned} v_0 &= \sqrt{\frac{E_0}{(\rho_0)_{\text{ref}}}} \\ t_0 &= \frac{L_0}{v_0} \end{aligned}\right\} \tag{5.16}$$

using these reference values we can nondimensionalize (5.10) as follows.

$$\begin{aligned} {}^{\Theta}I_0 \rho_0 \frac{\partial^2({}_i\Theta_1)}{\partial t^2} - (\alpha_1)\frac{(\alpha_1)_0}{({}^{\Theta}I_0)_{ref}(\rho_0)_{ref}v_0^2}\frac{\partial^2({}_i\Theta_1)}{\partial x^2} \\ - (\alpha_2)\frac{(\alpha_2)_0}{({}^{\Theta}I_0)_{ref}(\rho_0)_{ref}L_0 v_0}\frac{\partial}{\partial t}\left(\frac{\partial^2({}_i\Theta_1)}{\partial x^2}\right) = 0 \end{aligned} \tag{5.17}$$

If we choose $({}^{\Theta}I_0)_{ref} = {}^{\Theta}\hat{I}_0$, $(\rho_0)_{ref} = \hat{\rho}_0$ and define

$$\alpha_1^* = \frac{(\alpha_1)_{0/2}}{({}^{\Theta}I_0)_{ref}(\rho_0)_{ref}v_0^2} \quad , \quad c_d^* = \frac{\alpha_2}{Re^*} \quad , \quad 1/Re^* = \frac{(\alpha_2)_0}{{}^{\Theta}I_0)_{ref}(\rho_0)_{ref}L_0 v_0}$$

then (5.17) can be reduced to

$$\frac{\partial^2({}_i\Theta_1)}{\partial t^2} - \alpha_1^*\frac{\partial^2({}_i\Theta_1)}{\partial x^2} - c_d^* \frac{\partial}{\partial t}\left(\frac{\partial^2({}_i\Theta_1)}{\partial x^2}\right) = 0 \tag{5.18}$$

In this case the speed of dimensionless rotational wave is $\sqrt{\alpha_1^*}$ which may not be unity. We use

(5.18) in the numerical studies.

Remarks

1. We consider (5.12), (5.15) and (5.18) for presenting numerical studies.

2. Numerical studies are presented using space-time coupled finite element method based on space-time residual functional for a space-time strip with time marching [91] in which space-time local approximation for a space-time element are p-version hierarchical with higher order global differentiability in space and time.

3. We recast (5.12), (5.15) and (5.18) as a system of first order PDEs for convenience of defining BCs and ICs.

First order system of PDEs

Translational (or stress) waves (Model TW1)

We recast (5.12) as follows

$$\left.\begin{aligned}
&\frac{\partial v_1}{\partial t} - \frac{\partial \left({}_s^d\sigma_{11}\right)}{\partial x} = 0 \\
&{}_s^d\sigma_{11} - \frac{\partial u_1}{\partial x} - c_d \frac{\partial v_1}{\partial x} = 0 \\
&v_1 - \frac{\partial u_1}{\partial t} = 0
\end{aligned}\right\} \tag{5.19}$$

Rotational (or moment) waves (Model RW1)

We recast (5.15) as

$$\left.\begin{aligned}
&\frac{\partial\left({}_i\omega_1\right)}{\partial t}-\frac{\partial({}_sm_{11})}{\partial x}=0\\
&{}_sm_{11}-\frac{\partial({}_i\Theta_1)}{\partial x}-c_d^*\frac{\partial({}_i\omega_1)}{\partial x}=0\\
&{}_i\omega_1-\frac{\partial({}_i\Theta_1)}{\partial t}=0
\end{aligned}\right\}\tag{5.20}$$

Rotational (or moment) waves (Model RW2)

$$\left.\begin{aligned}
&\frac{\partial({}_i\omega_1)}{\partial t}-\frac{\partial({}_sm_{11})}{\partial x}\\
&{}_sm_{11}-\alpha_1^*\frac{\partial({}_i\Theta_1)}{\partial x}-c_d^*\frac{\partial({}_i\omega_1)}{\partial x}=0\\
&{}_i\omega_1-\frac{\partial({}_i\Theta_1)}{\partial t}=0
\end{aligned}\right\}\tag{5.21}$$

Numerical studies

Evolutions are computed for models TW1, RW1 and RW2 using space-time coupled finite element method for a space-time strip with time marching [91]. The space-time local approximation for a space-time element is p-version hierarchical with higher order global differentiability in space and time. Discretizations, p-levels and the minimally confirming spaces are chosen such that for each space-time strip the space-time residual functional for the discretization is O(10^{-8}) or lower ensuring that PDEs are satisfied accurately by the computed solution. In the present studies we choose a uniform mesh of 30, nine node p-version hierarchical finite elements with p-levels of 9 in space and time for all elements of the space-time strip. We choose local approximations in space and time of class C^{11}.

A time increment of $\Delta t = 0.1$ is used in all computations. The solution is computed for the first space-time strip $(0 \leq t \leq \Delta t)$ and then time marched to obtain evolution for desired value of τ (final value of time). Since all evolutions considered here are smooth, choice of solutions of class C^{11} in space and time suffices. For this choice of the order of the approximation space, the space-time integrals for the discretization of the space-time strip are Riemann. The 30 element uniform discretization with p-levels of 9 for space and time and $k = 2$ for both space and time, yield space-time residual functional values for each space-time strip of the order $O(10^{-8})$ or lower during the entire evolution, confirming good accuracy of the computed solution. In the following, we present computed evolutions for: translation wave using Model TW1 and rotations wave using Models TR1 and TR2.

Figure 1 shows schematic, space-time domain $\bar{\Omega}_{xt}$, discretization $\bar{\Omega}_{xt}^T = \bigcup_e \bar{\Omega}_{xt}^{(e)}$ of the space time domain into space-time strips, discretization $(\bar{\Omega}_{xt}^{(1)})^T = \bigcup_e \bar{\Omega}_{xt}^{(e)}$ of the first space-time strip $\bar{\Omega}_{xt}^{(1)}$ and the details of stress ${}_s^d\sigma_{11}$ boundary conditions (for TW1) applied to the last space-time element face at $x = 1$. In the numerical studies, we choose negative σ_0 (compressive) for the applied pulse at $x = 1$. The details of applying BC at $x = L$ for mathematical models RW1 and RW2 are similar.

Translational wave : Model TW1

We consider mathematical model TW1. Evolution is computed for 30 time steps for the undamped case, $c_d = 0$ using space-time coupled finite element formulation based on space-time residual functional. Since in this case, the speed of wave propagation is one, in ten time steps with $\Delta t = 0.1$, the applied pulse at $x = 1$ reaches the impermeable boundary at $x = 0$. Due to zero damping, we expect the applied pulse shape to be preserved during evolution. Figure 2(a) shows a plot of ${}_s^d\sigma_{11}$ versus x for various values of time. Time steps $2, 6, 10$ show the incident compressive pulse entering the spatial domain at $x = L$ and propagating without amplitude decay or base elongation confirming the absence of numerical dispersion. At the 10^{th} time step the compressive pulse is precisely at $x = 0$ as expected. Reflection of the compressive incident pulse at the impermeable boundary (at $x = 0$) results in reflected compressive pulse at the 11^{th} time step with double the peak value. Upon further evolution the reflected compressive pulse recovers its peak and base at the 12^{th} time step (not shown). At the 14^{th} time step, we see reflected pulse with the same amplitude and base as the original incident pulse propagating toward the right end. At the 20^{th} time step the reflected compressive pulse reaches the free boundary at $x = L = 1$ and reflects as a tensile pulse of the same amplitude and base as the original incident pulse propagating towards the boundary at $x = 0$ (time steps 23 and 28 in figure 2(a)).

In the next study for TW1, we consider dimensionless damping coefficient $c_d = 0.002$. Figure 2(b) shows evolution of ${}_s^d\sigma_{11}$ versus x for time steps 2, 6, 10; 11, 14, 18; 23 and 28 (same as in figure 2(a) for the undamped case). In the incident pulse, amplitude decays and the base elongates during propagation (time steps 2, 6, 10). During reflection (time step 11), the magnitude of the pulse increases but recovers upon further evolution (time step 14). The reflected pulse reaches the free boundary and reflection from the free boundary results in a tensile pulse that propagates to the left. We observe the amplitude decay is more pronounced in the propagating incident pulse. The reflected pulses also shows some amplitude decay, but the base elongation in the reflected pulse is more pronounced. All evolutions are smooth i.e., free of oscillations. Integrated sum of squares of the space-time residual functional are of the order of $O(10^{-8})$ or lower for each space-time strip ensuring that the computed solution is accurate. Choice of $k = 2$ (order of the approximation space in space and time) is minimally conforming for the first order system of PDEs, hence all space-time integrals for $\left(\bar{\Omega}_{xt}^{(i)}\right)^T$, discretization of space-time strip $\bar{\Omega}_{xt}^{(i)}$ are Riemann. The space-time residual functional of the order $O(10^{-8})$ ensures that the PDEs are satisfied in point wise sense over each space-time strip discretization $\bar{\Omega}_{xt}^{(i)}$, hence good time accuracy of the evolution is ensured.

Rotational wave : Model RW1

We consider mathematical model RW1 defined by equations (5.20). In this case, the equations are nondimensionalized using the speed of rotational wave, hence the dimensionless speed of the rotational wave is one. In the first study, we apply moment pulse i.e., ${}_{s}m_{11}$ pulse of time duration $2\Delta t$ (similar to ${}^{d}_{s}\sigma_{11}$ shown in figure 1(e)). We choose p-levels of 9 in space and time, $\Delta t = 0.1$ and order k of the approximation space in space and time is chosen to be 2, which is minimally confirming for a first order system of PDEs. We choose peak values of ${}_{s}m_{11} = -1$. Evolution is computed using space-time strip with time marching, Integral form in space-time finite element process is based on space-time residual functional. Figure 3(a) shows the evolution for $c_d^* = 0$ (undamped case). Incident moment pulse propagates towards the impermeable boundary without amplitude decay and base elongation. Upon reflection at $x = 0$, the peak amplitude doubles during reflection but the pulse recovers to the original shape in the 12^{th} time steps and continues to propagate toward the free boundary. Reflection from the free boundary results in tensile moment pulse of the same shape as original incident compressive moment pulse that continues to propagate toward the impermeable boundary at $x = 0$.

Figure 3(b) shows the propagation, reflection and propagation upon reflection of the same compressive ${}_{s}m_{11}$ pulse as in figure 3(a) but in the presence of dissipation. We choose dimensionless dissipation coefficient $c_d* = 0.002$. From figure 3(b) we observe almost the same behavior of the moment pulse as that of ${}^{d}_{s}\sigma_{11}$ pulse in figure 2(b). Presence of dissipation resulting in continued amplitude decay and base elongation during the propagation. Amplitude decay is more pronounced for the incident pulse and the base elongation is more significant in the reflected pulse.

Rotational Wave : Model RW2

In this study, we consider mathematical model RW3, equations (5.21). In this mathematical model rotational wave speed is $\sqrt{\alpha_1^*}$ and the dissipation is controlled by the dimensionless dissipation coefficient c_d^*. We apply a moment $({}_{s}m_{11})$ pulse of duration $2\Delta t$ at the boundary $x = L = 1$.

In the first study, we choose $\Delta t = 0.1$, $\alpha_1^* = 2.25$ and $c_d^* = 0$ (undamped case). The wave speed is $\sqrt{2.25} = 1.5$, thus the pulse would reach $x = 0$ boundary in $(1/1.5)/0.1 = 6.67$ time steps. Figure 4(a) show propagation and reflection of the moment pulse. Figure 4(b) show propagation and reflection of the moment pulse for $\alpha_1^* = 2.25$ i.e., at the wave speed of 1.5 when $c_d^* = 0.002$. Diminished amplitudes and elongated base of the pulse is clearly observed during evolution. This is similar to RW1 $(c_d^* = 0.002)$. In the next study, we apply the same moment pulse as in the previous study but choose $\alpha_1^* = 0.49$ and $c_d^* = 0$, hence rotational wave speed of $\sqrt{0.49} = 0.7$, thus the moment pulse will reach the boundary at $x = 0$ in $(1/0.7)/0.1 = 14.285$ time steps. Figure 5(a) and 5(b) show propagation and reflection of the moment wave for $c_d^* = 0$ and $c_d^* = 0.002$. The undamped pulse $(c_d^* = 0)$ maintains its base and amplitude during the entire evolution and when $c_d^* = 0.002$ progressive amplitude decay and base elongation of the pulse is observed for both values of α_1^*. We remark that when the rotational wave speed is one, we are able to precisely locate the pulse at the impermeable boundary as the time to reach the impermeable boundary is integer multiple of Δt. In this case, we see perfect reflection of the ${}_sm_{11}$ pulse at the 11^{th} time step and the peak value of ${}_sm_{11}$ doubles, same as in the case of the translational stress wave. However, when rotational wave speed is not an integer multiple of Δt, precise arrival of ${}_sm_{11}$ pulse at $x = 0$ cannot be simulated as elapsed time is always an integer multiple of Δt. For this reason, we are not able to observe 2 (${}_sm_{11}$) in the peak values during the reflection process of the undamped case (figures 4(a) and 5(a)) when the rotational speed is not one.

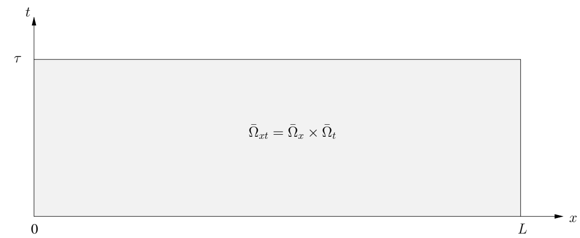
Remarks

1. In the model problem studies presented here the translational and rotational waves are decoupled. In $\mathbf{R}^2$ and $\mathbf{R}^3$ this may not be the case. Since the stress tensor is a function of $[{}_s^dJ]$ and the moment tensor is a function of $[{}_a^dJ]$, both stress and moment tensor waves are dependent on gradients of displacement. Thus, in $\mathbf{R}^2$ and $\mathbf{R}^3$ we expect interaction between the two waves. That is, as shown here, rotational waves depend upon $\hat{\alpha}_1$ and ${}^{\Theta}\bar{I}$ which in turn influences $[{}^dJ]$, thus $[{}_s^dJ]$, hence influencing translational waves. The model problem

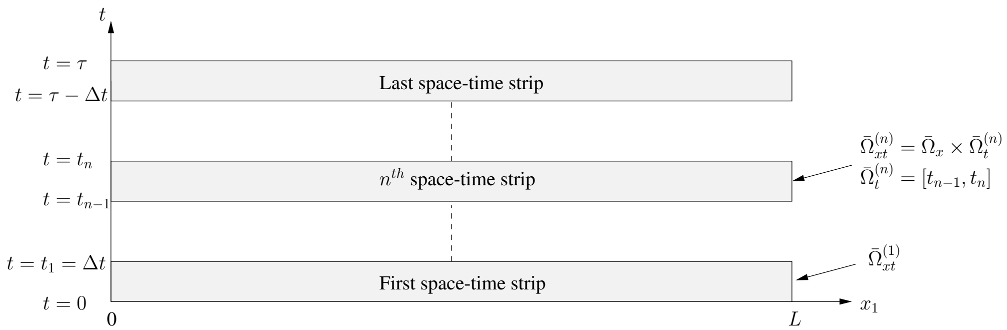
studies in $\mathbf{R}^2$ and $\mathbf{R}^3$ are needed to illustrate this physics.

2. The medium with and without microconstituents is always considered isotropic and homogeneous. Thus, in this micropolar theory there is absence of wave dispersion in the micropolar medium as it is considered to be isotropic and homogeneous. This micropolar theory can only simulate wave propagation physics of translational and rotational waves and their interaction (when it exists) in $\mathbf{R}^2$ and $\mathbf{R}^3$ (future work).

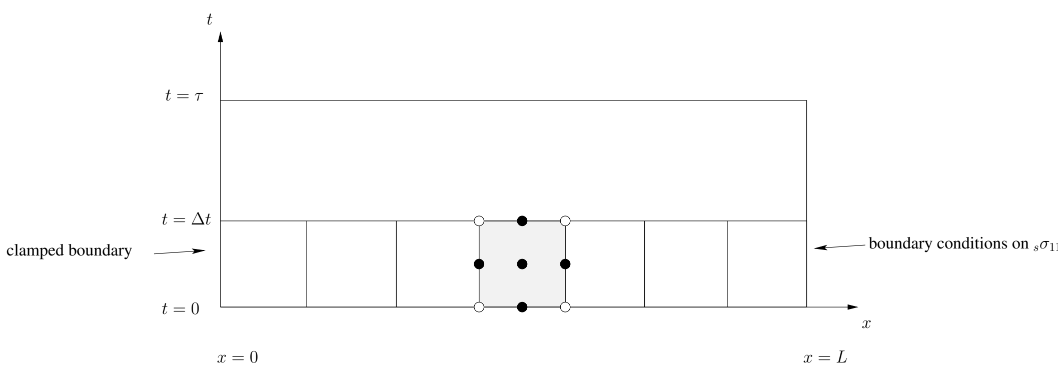
(a) schematic of spatial domain



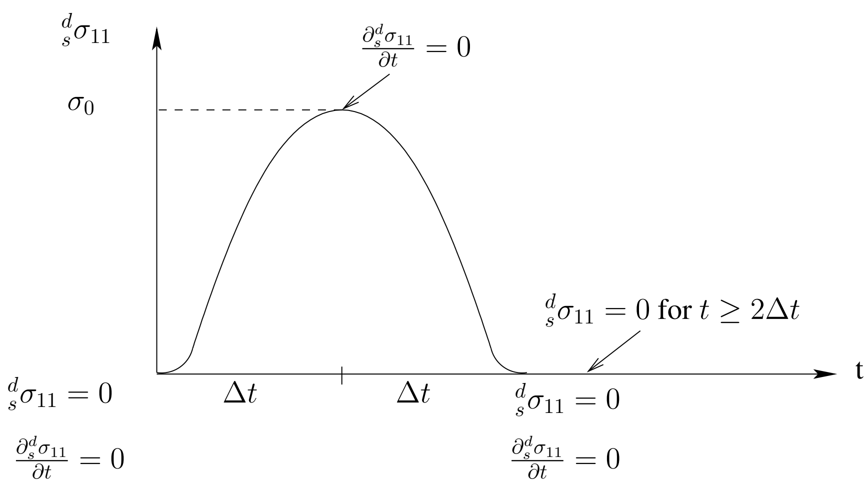
(b) Space time domain $\bar{\Omega}_{xt}$



(c) Discretization $\bar{\Omega}_{xt}^T = \bigcup_i \bar{\Omega}_{xt}^{(i)}$ in space-time strips

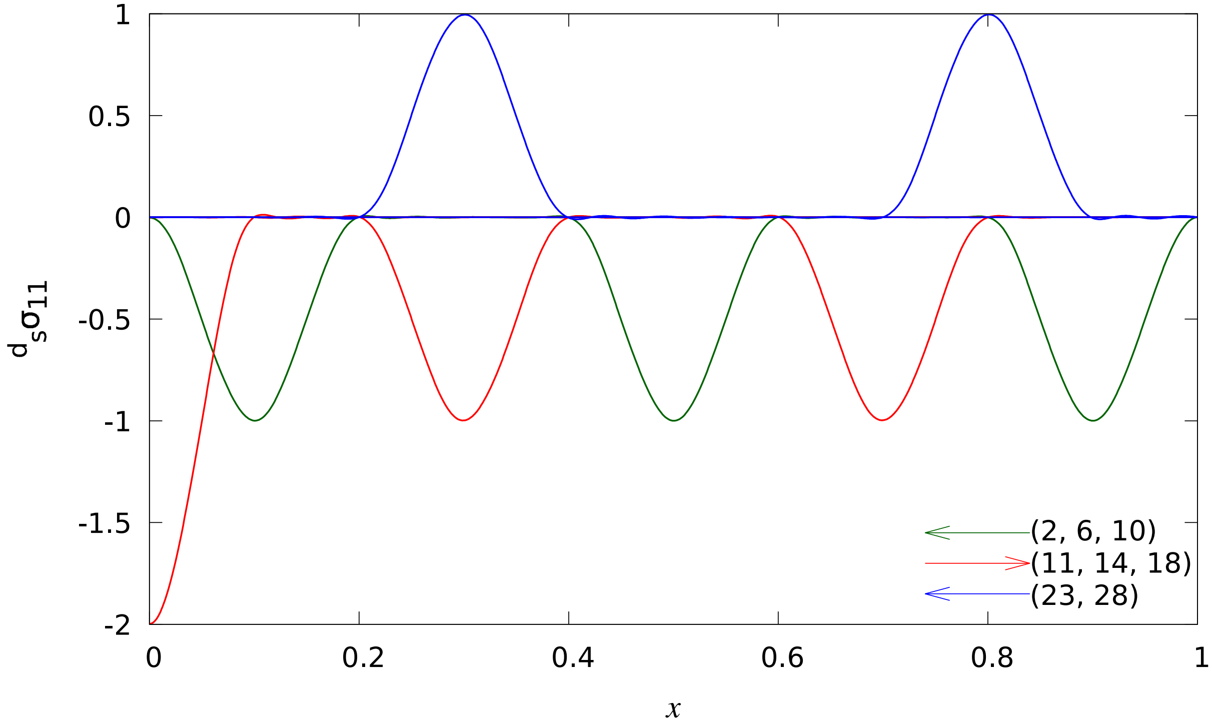


(d) Discretization $\left(\bar{\Omega}_{xt}^{(1)}\right)^T = \bigcup_e \bar{\Omega}_{xt}^{(e)}$ of the first space-time strip $\bar{\Omega}_{xt}^{(1)}$

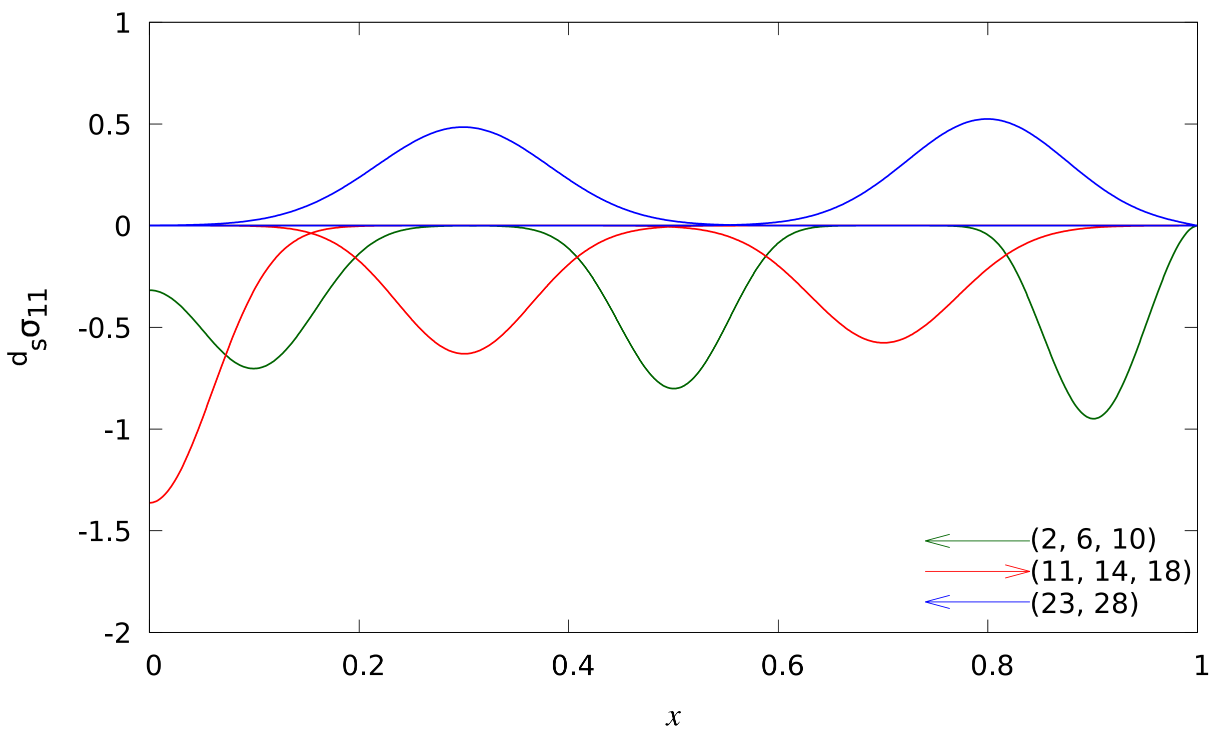


(e) ${}^d_s\sigma_{11}(t)$ as a function of time for $0 \leq t \leq 2\Delta t$ and for $t \geq 2\Delta t$

Figure 1: Spatial domain, Space-time domain, Discretization into space-time strips, First space-time strip, Boundary conditions

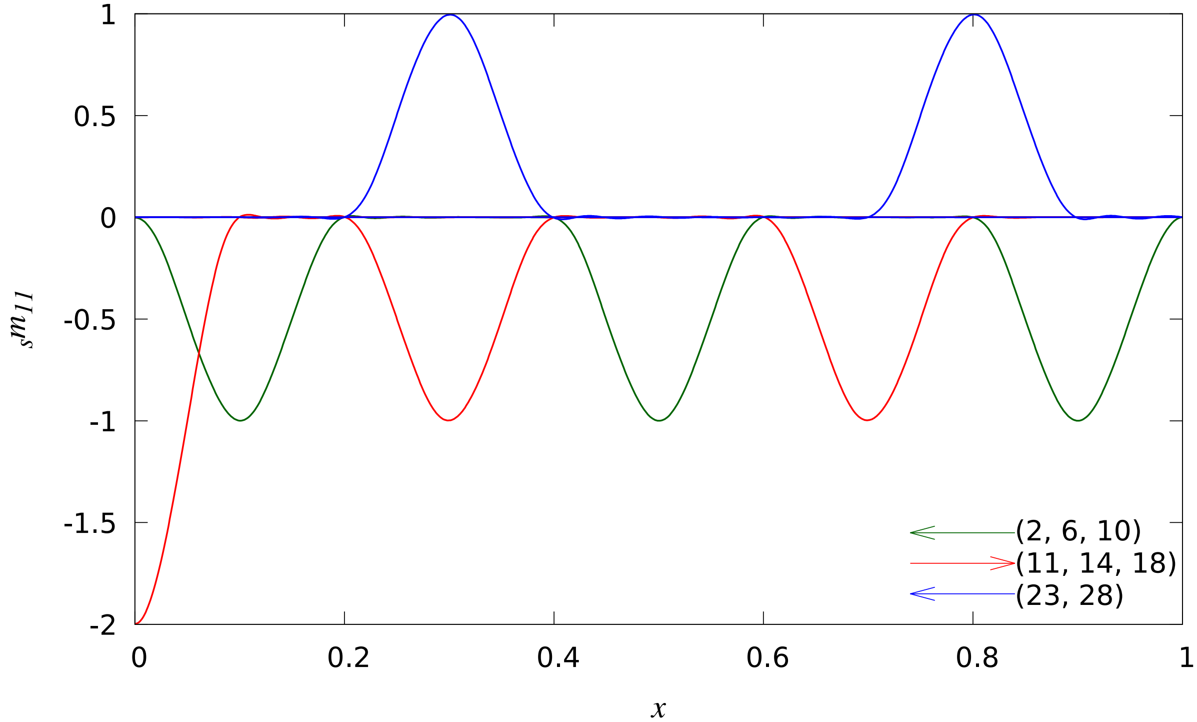


(a) Undamped case ($c_d = 0$)

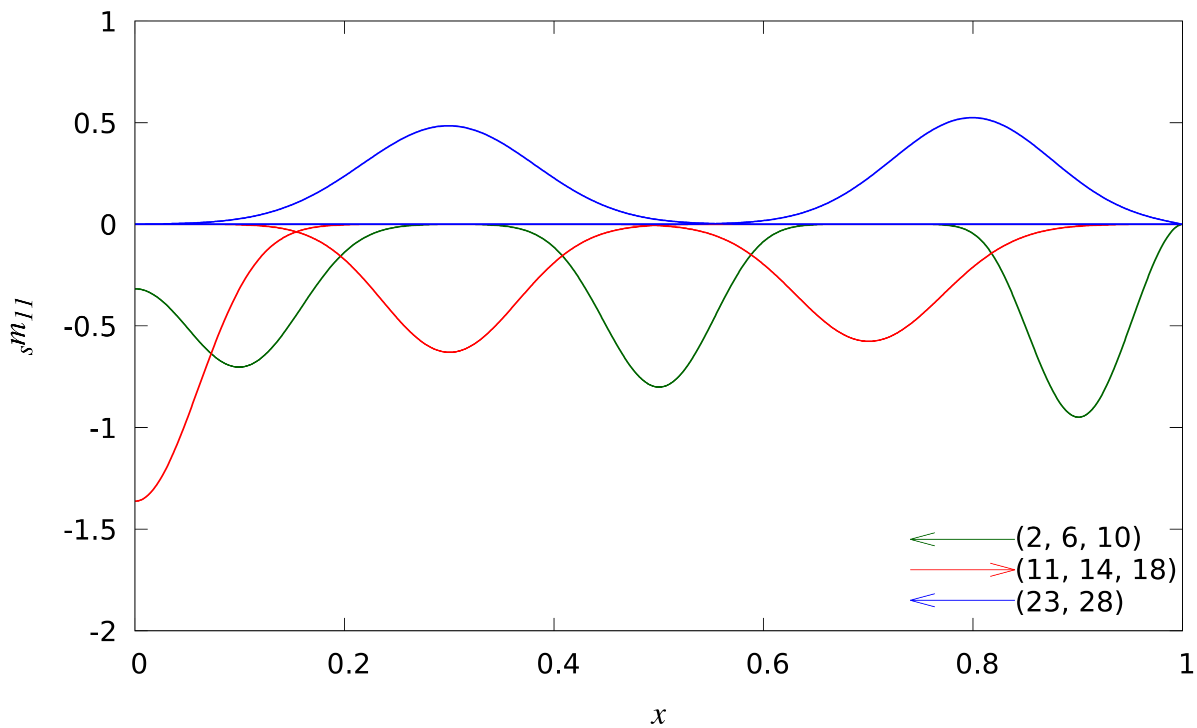


(b) Damped case ($c_d = 0.002$)

Figure 2: Evolution of deviatoric stress wave : ${}^d_s\sigma_{11}$ versus position x (Model TW1)

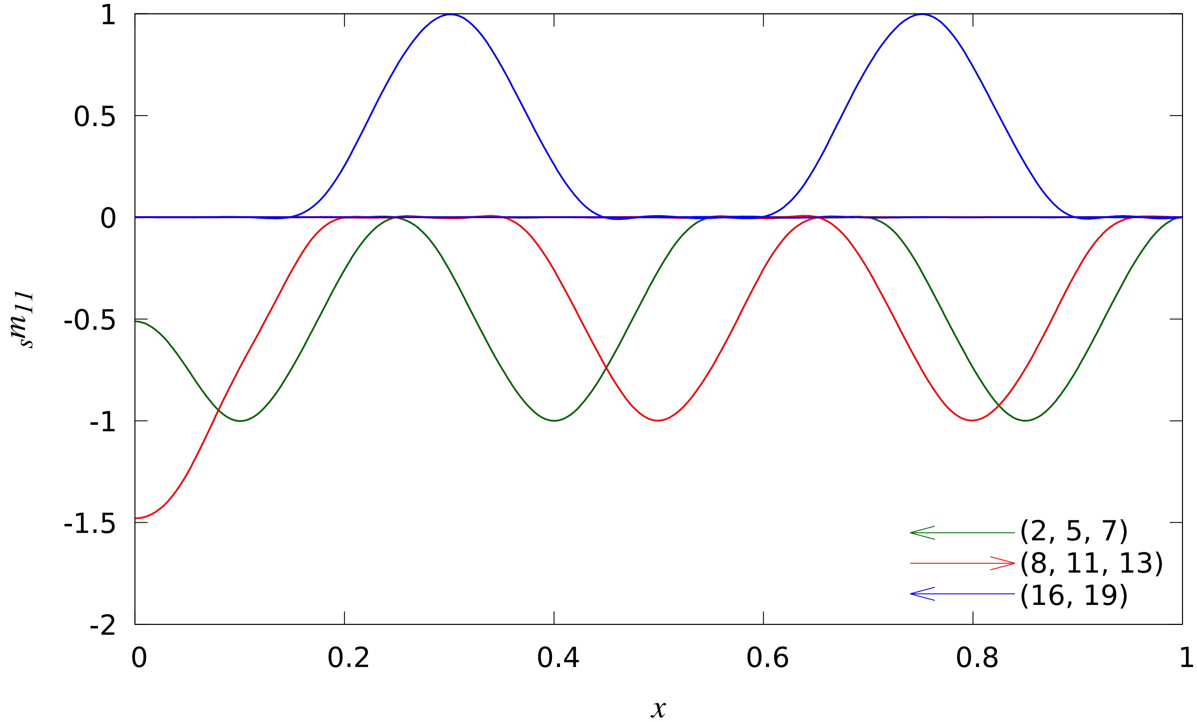


(a) Undamped case ($c_d^* = 0$)

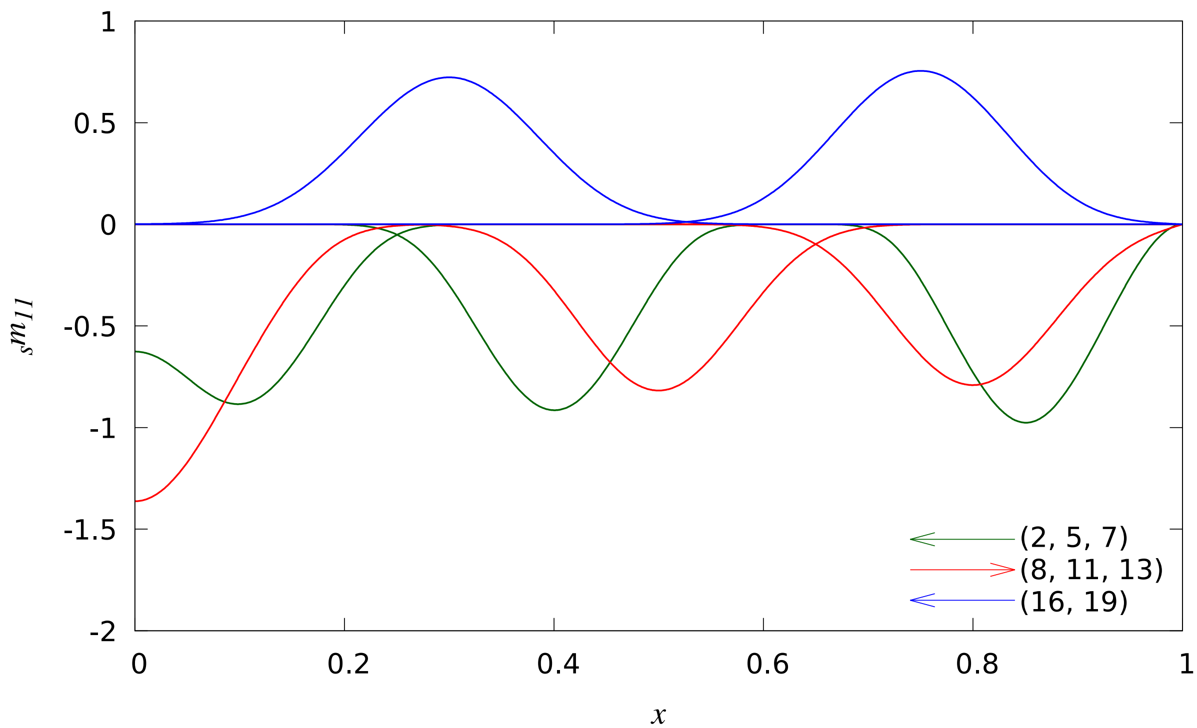


(b) Damped case ($c_d^* = 0.002$)

Figure 3: Evolution of Cauchy Moment wave : $_sm_{11}$ versus position x (Model RW1)

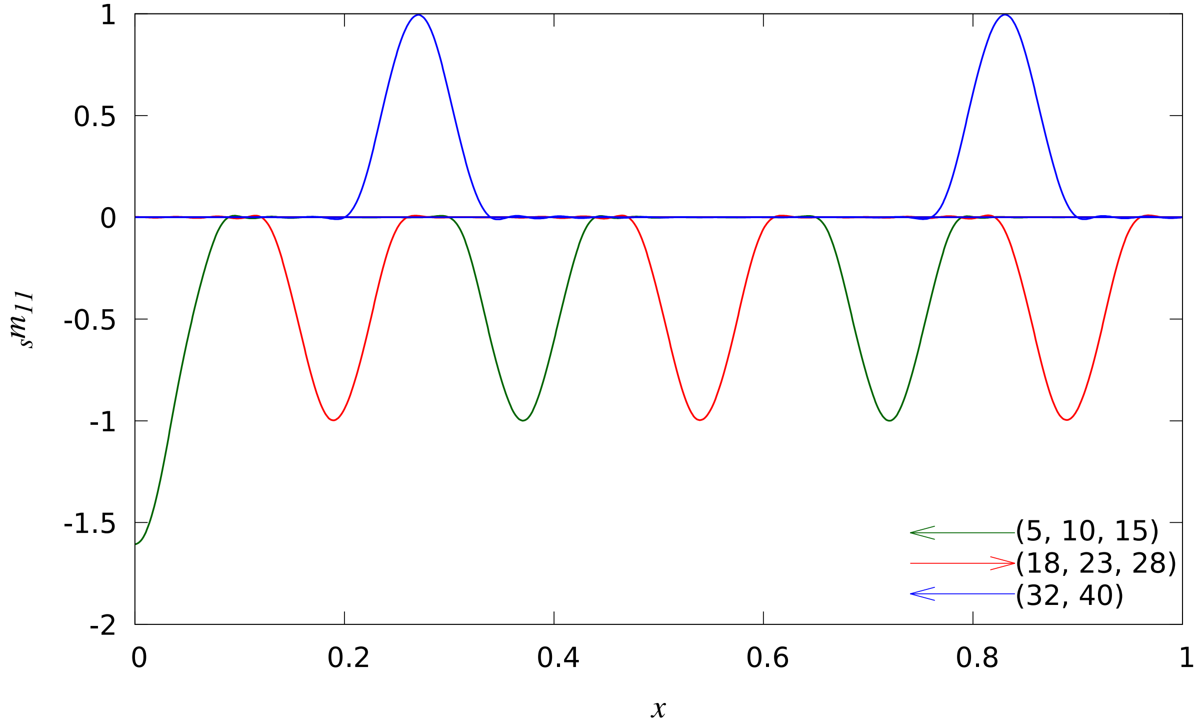


(a) Undamped case ($c_d^* = 0$)

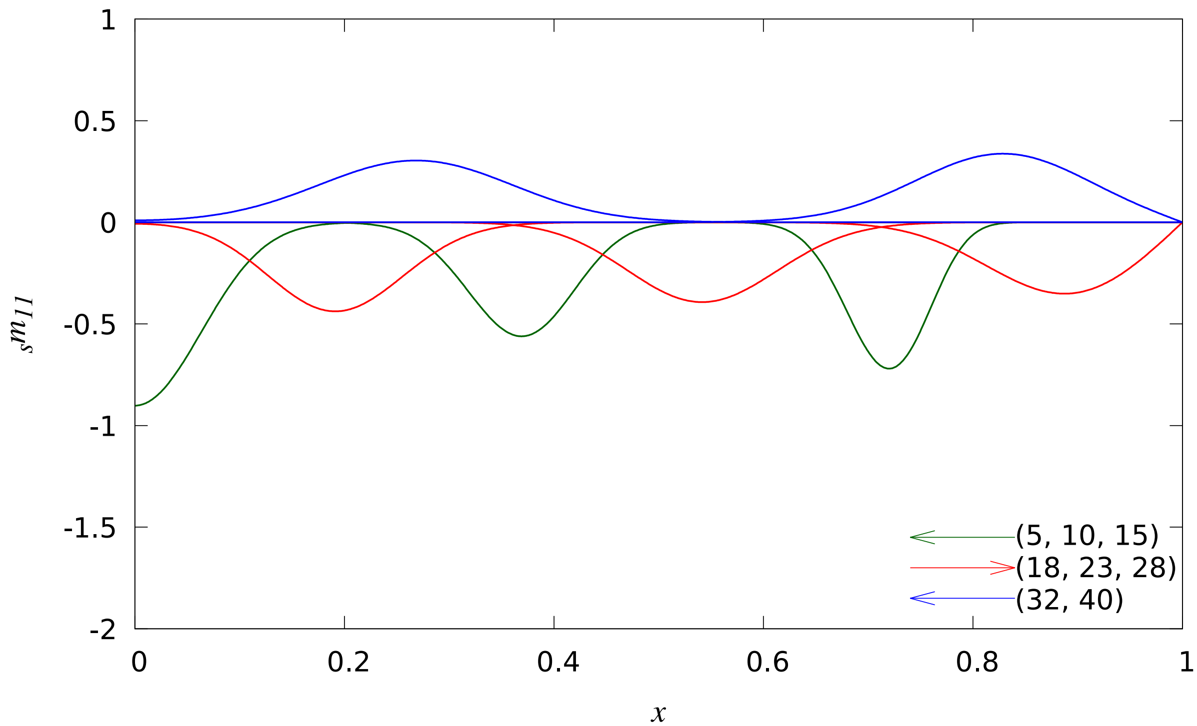


(b) Damped case ($c_d^* = 0.002$)

Figure 4: Evolution of Cauchy Moment wave : $_s m_{11}$ versus position x (Model RW2 : $\alpha^* = 2.25$)



(a) Undamped case ($c_d^* = 0$)



(b) Damped case ($c_d^* = 0.002$)

Figure 5: Evolution of Cauchy Moment wave : $_sm_{11}$ versus position x (Model RW2 : $\alpha^* = 0.49$)

5.2 Micropolar viscous fluids : Rotational inertial physics

The objective of the model problem studies presented in this section is to demonstrate the influence of rotational inertial physics in micropolar fluids in which NCCT is based on internal rotation rate physics.

5.2.1 Rotational waves in fluids?

It is well known that in classical or micropolar viscous fluids, translational deviatoric stress waves can not exist due to the absence of elasticity. In this study, we only consider balance of angular momenta in 1D and associated constitutive theories to investigate the influence of rotational inertial physics.

Mathematical model

The one dimensional form of BAM and the constitutive theory for the Cauchy moment tensor are given by (using $\bar{x}$ for $\bar{x}_1$).

$$ {}^{\Theta}\bar{I}\bar{\rho}\frac{\partial\left({}_{i}\bar{\omega}_1\right)}{\partial t}-\frac{\partial\left({}_{s}\bar{m}_{11}\right)}{\partial\bar{x}}=0 \tag{5.22} $$

$$ {}_{s}\bar{m}_{11}=\bar{\alpha}_1\left(\frac{1}{2}\frac{\partial\left({}_{i}\bar{\omega}_1\right)}{\partial\bar{x}}\right) \tag{5.23} $$

Substituting (5.23) in (5.22)

$$ {}^{\Theta}\bar{I}\bar{\rho}\frac{\partial\left({}_{i}\bar{\omega}_1\right)}{\partial t}-\frac{\bar{\alpha}_1}{2}\left(\frac{\partial^2\left({}_{i}\bar{\omega}_1\right)}{\partial\bar{x}^2}\right)=0 \tag{5.24} $$

Remarks

1. We clearly see that this equation (5.24) will not permit a moment $({}_{s}\bar{m}_{11})$ wave as it is not a wave equation due to the absence of $\frac{\partial^2({}_{i}\bar{\omega}_1)}{\partial t^2}$ in place of $\frac{\partial({}_{i}\bar{\omega}_1)}{\partial t}$.

2. However it is interesting to study its solution for varying ${}^{\Theta}\bar{I}$ values.

Dimensionless form of the mathematical model

Recasting (5.24) with hat ($\wedge$) on all quantities indicating that they have their usual dimensions (or units)

$$ {}^{\Theta}\hat{\bar{I}}\hat{\bar{\rho}}\frac{\partial\left({}_{i}\hat{\bar{\omega}}_{1}\right)}{\partial\hat{t}} - \frac{\hat{\bar{\alpha}}_{1}}{2}\left(\frac{\partial^{2}\left({}_{i}\hat{\bar{\omega}}_{1}\right)}{\partial\hat{\bar{x}}^{2}}\right) = 0 \tag{5.25} $$

and using

$$ \left.\begin{array}{l} {}^{\Theta}\bar{I} = {}^{\Theta}\hat{\bar{I}}/{}^{\Theta}I_{0} \quad , \quad \bar{\rho} = \hat{\bar{\rho}}/\rho_{0} \quad , \quad \bar{x} = \hat{\bar{x}}/L_{0} \\ {}_{i}\bar{\omega}_{1} = {}_{i}\hat{\bar{\omega}}_{1}/\omega_{0} \quad , \quad t_{0} = L_{0}/\omega_{0} \quad , \\ \text{and} \quad \omega_{0} = \sqrt{(\alpha_{1})_{0}/{}^{\Theta}I_{0}\rho_{0}} \quad , \quad \bar{\alpha}_{1} = \hat{\bar{\alpha}}_{1}/(\alpha_{1})_{0} \end{array}\right\} \tag{5.26} $$

We can write (5.25) as

$$ {}^{\Theta}\bar{I}\bar{\rho}\frac{\partial\left({}_{i}\bar{\omega}_{1}\right)}{\partial t} - \left(\frac{\bar{\alpha}_{1}}{2}\right)\frac{(\alpha_{1})_{0}}{L_{0}{}^{\Theta}I_{0}\rho_{0}}\left(\frac{\partial^{2}\left({}_{i}\bar{\omega}_{1}\right)}{\partial\bar{x}^{2}}\right) = 0 \tag{5.27} $$

If we choose ${}^{\Theta}I_{0} = {}^{\Theta}\hat{\bar{I}}$, $\rho_{0} = \hat{\bar{\rho}}$ and $(\alpha_{1})_{0} = \hat{\bar{\alpha}}_{1}$ then (5.27) can be reduced to

$$ \frac{\partial\left({}_{i}\bar{\omega}_{1}\right)}{\partial t} - \bar{c}_{2}\left(\frac{\partial^{2}\left({}_{i}\bar{\omega}_{1}\right)}{\partial\bar{x}^{2}}\right) = 0 \tag{5.28} $$

in which $\bar{c}_{2} = \left(\hat{\bar{\alpha}}_{1}\right)/\left(2L_{0}{}^{\Theta}\hat{\bar{I}}\hat{\bar{\rho}}\right)$, dimensionless dissipation coefficient. PDE (5.28) can be cast as a system of two first order PDEs by using moment ${}_{s}\bar{m}_{11}$ as auxiliary variable and we have

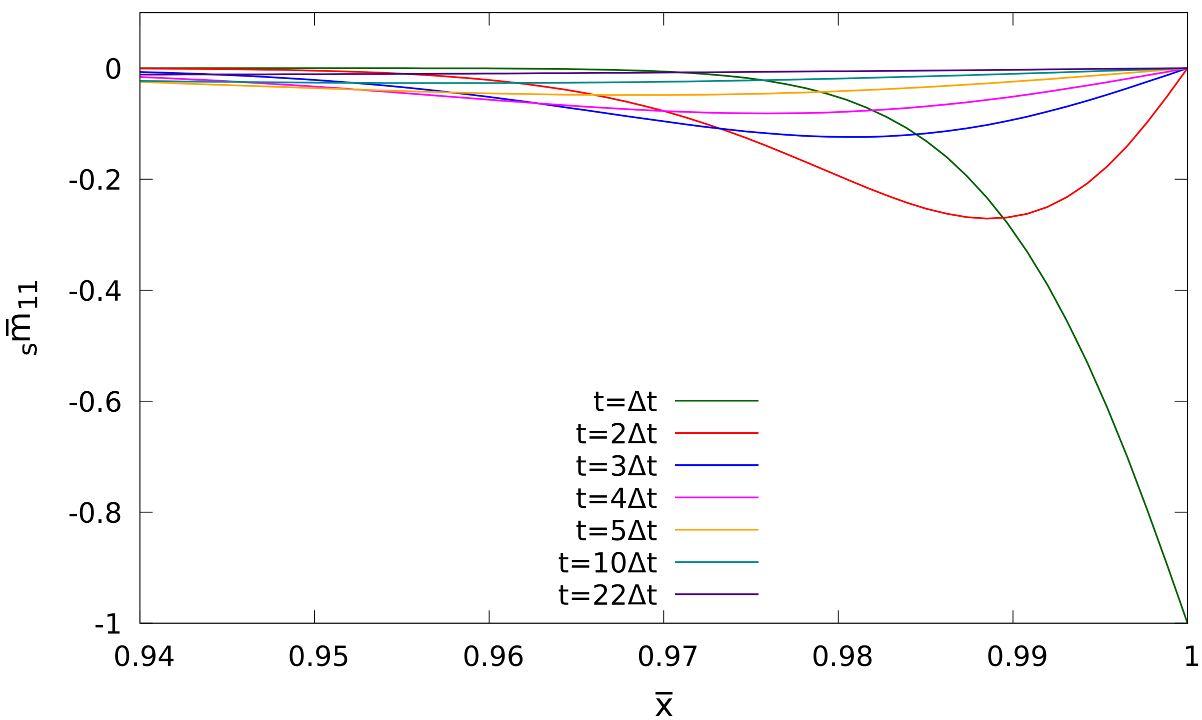
$$ \left.\begin{array}{l} \dfrac{\partial\left({}_{i}\bar{\omega}_{1}\right)}{\partial t} - \dfrac{\partial\left({}_{s}\bar{m}_{11}\right)}{\partial\bar{x}} = 0 \\ {}_{s}\bar{m}_{11} = \bar{c}_{2}\dfrac{\partial({}_{i}\bar{\omega}_{1})}{\partial\bar{x}} \end{array}\right\} \tag{5.29} $$

PDEs in (5.29) are helpful in defining BCs and ICs, hence are used in the computations.

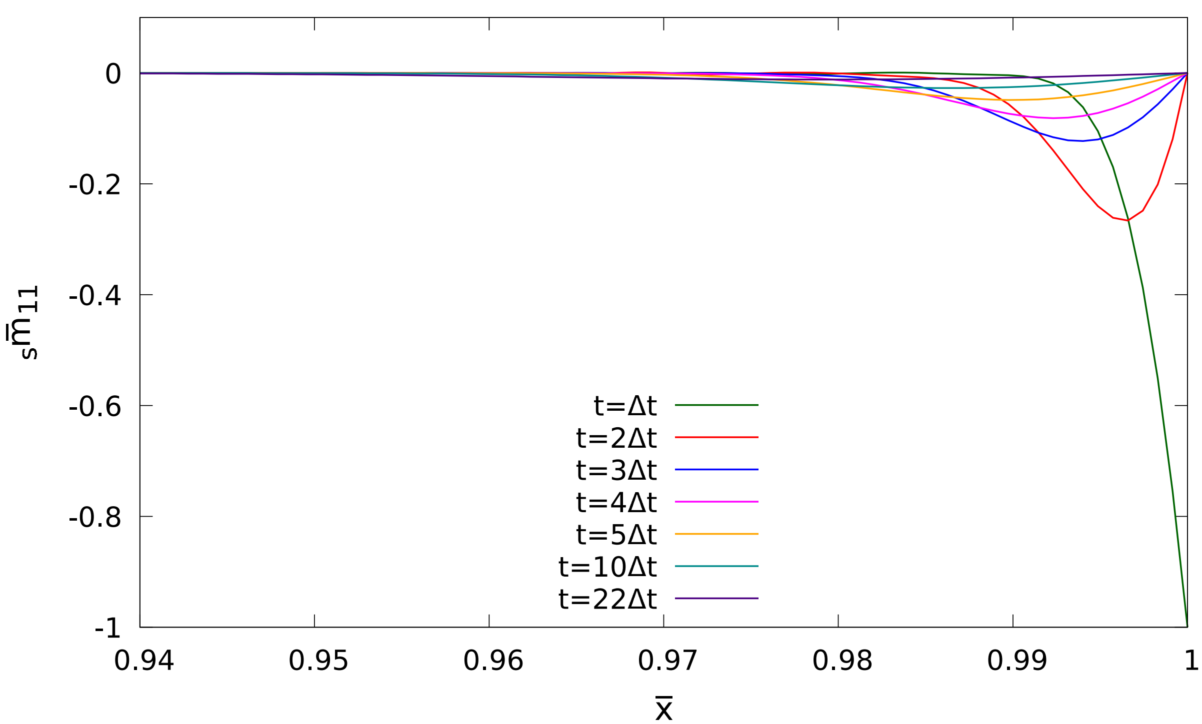
Numerical Studies

We consider mathematical model (5.28) or (5.29). It is evident (5.28) is not a wave equation as a wave equation must have second order spatial and second order time derivatives of the dependent variable. Thus, (5.28) can not describe a wave, which is not a surprise, because classical as well as micropolar fluids do not possess elasticity, hence do not have stiffness, thus they cannot support deviatoric Cauchy stress wave or Cauchy moment wave.

Equation (5.28) is in fact a time dependent diffusion equation in which $\bar{c}_2 \propto \frac{1}{{}^{\Theta}\bar{I}}$ is the diffusion coefficient. Thus low values of ${}^{\Theta}\bar{I}$ correspond to high values of $\bar{c}_2$ and vice versa. Details of the schematic, space-time strips and discretization of a space-time strip remain the same as shown in figure 1. We apply a negative ${}_s\bar{m}_{11}$ moment pulse of duration $2\Delta t$ on the boundary at $\bar{x} = 1$ such that ${}_s\bar{m}_{11} \in [0,-1]\ \forall\ t \in [0,\Delta t]$; ${}_s\bar{m}_{11} \in [-1,0]\ \forall\ t \in [\Delta t, 2\Delta t]$ and ${}_s\bar{m}_{11} = 0$ for $t \geq 2\Delta t$ (similar to BC shown in figure 1(e)). We choose a discretization of 30 nine node p-version hierarchical space-time elements with p-levels of nine in space and time. Evolution is computed for 30 time steps using $\Delta t = 0.1$ with local approximation of class C^{11} in space and time, We choose two values of $\bar{c}_2 = 0.0001, 0.001$. Evolutions for different values of time for the two choices of $\bar{c}_2$ are shown in figure 6(a) and (b). For both values of $\bar{c}_2$, the applied pulse progressively diffuses as time elapses. We observe significantly high diffusion of the applied moment pulse for $\bar{c}_2 = 0.001$ compared to $\bar{c}_2 = 0.0001$ as expected. We clearly observe lack of existence and propagation of rotational or moment waves due to lack of elasticity, demonstrating that rotational waves can not exist in micropolar fluids with rotational inertial physics.



(a) ${}_s\bar{m}_{11}$ versus $\bar{x}$ ($\bar{c}_2 = 0.001$)

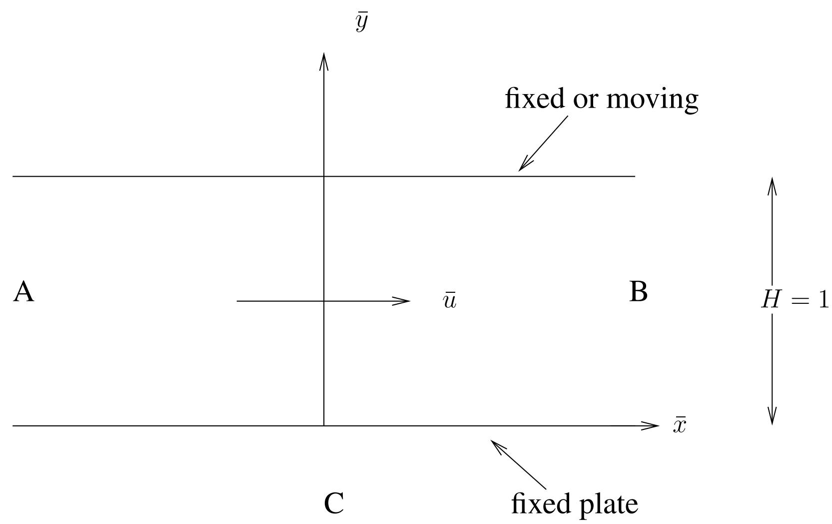


(b) ${}_s\bar{m}_{11}$ versus $\bar{x}$ ($\bar{c}_2 = 0.0001$)

Figure 6: Cauchy Moment ${}_s\bar{m}_{11}$ versus position $\bar{x}$

5.2.2 Developing pressure driven flow between parallel plates and developing Couette flow

In this study, we consider developing pressure driven flow between parallel plates and developing Couette flow in which the mathematical models are based on: CCM, NCCM without rotational inertial physics and NCCM with rotational inertial physics. Only the micropolar non-classical continuum theory based on internal rotation rates is considered [81]. Figure 7 shows a schematic of the dimensionless configuration of parallel plates.



(a) schematic of spatial domain of parallel plates

Figure 7: Spatial domain for pressure driven and Couette flow

Away from the the ends A and B, the physics of the flow is purely one dimensional (in the $\bar{y}$ direction) velocity $\bar{u}$ is a dependent variable, velocity $\bar{v}$ in the $\bar{y}$ direction is zero regardless of the type of mathematical model hence, Cauchy shear stress is the only non-zero stress (details are given in the following). We present details of three mathematical models (as mentioned above): based on CCM, NCCM without rotational inertial physics and NCCM with rotational inertial physics. We nondimensionalize these mathematical models using the following reference quantities and dimensionless quantities.

$$\left.\begin{array}{l}\bar{x}=\hat{\bar{x}}/L_0 \quad , \quad \bar{y}=\hat{\bar{y}}/L_0 \quad , \quad \bar{\eta}=\hat{\bar{\eta}}/\eta_0 \quad , \quad \bar{\alpha}=\hat{\bar{\alpha}}/\alpha_0 \\ \bar{m}_{23}^{(0)}=\hat{\bar{m}}_{23}^{(0)}/m_0 \quad , \quad {}_s^d\bar{\sigma}_{21}^{(0)}={}_s^d\hat{\bar{\sigma}}_{21}^{(0)}/\tau_0 \quad , \quad {}_a\bar{\sigma}_{21}^{(0)}={}_a\hat{\bar{\sigma}}_{21}^{(0)}/\tau_0 \\ m_0=\tau_0 L_0 \quad , \quad \tau_0=\rho_0 v_0^2=p_0 \text{ (characteristic kinetic energy)} \\ \bar{u}=\hat{\bar{u}}/v_0 \quad , \quad \bar{v}=\hat{\bar{v}}/v_0 \quad , \quad t_0=L_0/v_0 \\ {}_i\bar{\omega}_3={}_i\hat{\bar{\omega}}_3/\omega_0 \quad , \quad \omega_0=t_0 \quad , \quad {}^\Theta\bar{I}={}^\Theta\hat{\bar{I}}/{}^\Theta I_0\end{array}\right\} \tag{5.30}$$

Mathematical model based on CCM : Model A

Balance of linear momenta in the $\bar{x}$ direction and the constitutive theory for deviatoric Cauchy stress ${}_d\bar{\sigma}_{12}^{(0)}={}_d\bar{\sigma}_{21}^{(0)}$ are the only two equations needed in this case and are given in the following (with usual dimensions for all quantities) for incompressible classical fluid.

$$\left.\begin{array}{l}\hat{\bar{\rho}}\dfrac{\partial\hat{\bar{u}}}{\partial\hat{t}}+\dfrac{\partial\hat{\bar{p}}}{\partial\hat{\bar{x}}}-\dfrac{\partial\left({}_s^d\hat{\bar{\sigma}}_{21}^{(0)}\right)}{\partial\hat{\bar{y}}}=0 \\ {}_s^d\hat{\bar{\sigma}}_{21}^{(0)}=\hat{\bar{\eta}}\dfrac{\partial\hat{\bar{u}}}{\partial\hat{\bar{y}}}\end{array}\right\} \tag{5.31}$$

we can nondimensionalize (5.31) using (5.30) and if we choose $\rho_0=\hat{\bar{\rho}}$, $\eta_0=\hat{\bar{\eta}}$, then the dimensionless form of (5.31) can be written as

$$\left.\begin{array}{l}\dfrac{\partial\bar{u}}{\partial t}+\dfrac{\partial\bar{p}}{\partial\bar{x}}-\dfrac{\partial\left({}_s^d\bar{\sigma}_{21}^{(0)}\right)}{\partial\bar{y}}=0 \\ {}_s^d\bar{\sigma}_{21}^{(0)}=\dfrac{1}{Re}\dfrac{\partial\bar{u}}{\partial\bar{y}}\end{array}\right\} \tag{5.32}$$

In case of pressure driven flow, $\frac{\partial\bar{p}}{\partial\bar{x}}$ is given. For non-pressure driven Couette flow, $\frac{\partial\bar{p}}{\partial\bar{x}}=0$. In (5.32), Re is Reynolds number and is given by (based on reference quantities) $Re=\frac{L_0\rho_0 v_0}{\eta_0}$. Equation (5.32) is a first order system of two PDEs in two dependent variables $\bar{u}$ and ${}_s^d\bar{\sigma}_{21}^{(0)}$. In this case, flow characteristics depend upon dimensionless parameter Re.

Micropolar NCCT without rotational inertial physics : Model B

In this case, the mathematical model consists of BLM in $\bar{x}$ direction, balance of angular momenta about the $\bar{z}$ direction, constitutive theory for symmetric part of deviatoric Cauchy shear stress ${}_s^d\bar{\sigma}_{12}^{(0)} = {}_s^d\bar{\sigma}_{21}^{(0)}$, constitutive theory for symmetric Cauchy moment ${}_s\bar{m}_{23}$ and definition of angular rotation rate ${}_i\bar{\omega}_3$ i.e., rotation rate about $\bar{z}$. We can write the following (with their usual dimensions) using the CBL and the constitutive theories given for $\mathbb{R}^3$ by equations (3.139)-(3.150), in the absence of body forces and body moments.

$$\left.\begin{aligned}
&\hat{\bar{\rho}}\frac{\partial \hat{\bar{u}}}{\partial \hat{t}} + \frac{\partial \hat{\bar{p}}}{\partial \bar{x}} - \frac{\partial \left({}_s^d\hat{\bar{\sigma}}_{21}^{(0)}\right)}{\partial \hat{\bar{y}}} - \frac{\partial \left({}_a^d\hat{\bar{\sigma}}_{21}^{(0)}\right)}{\partial \hat{\bar{y}}} = 0 \\
&\frac{\partial \left({}_s\hat{\bar{m}}_{23}^{(0)}\right)}{\partial \hat{\bar{y}}} + 2\left({}_a^d\hat{\bar{\sigma}}_{21}^{(0)}\right) = 0 \\
&{}_s^d\hat{\bar{\sigma}}_{12}^{(0)} = {}_s^d\hat{\bar{\sigma}}_{21}^{(0)} = \hat{\bar{\eta}}\frac{\partial \hat{\bar{u}}}{\partial \hat{\bar{y}}} \\
&{}_s\hat{\bar{m}}_{23}^{(0)} = \frac{\hat{\bar{\alpha}}}{2}\frac{\partial \left({}_i\hat{\bar{\omega}}_3\right)}{\partial \bar{y}} \\
&{}_i\hat{\bar{\omega}}_3 = -\frac{1}{2}\frac{\partial \hat{\bar{u}}}{\partial \hat{\bar{y}}}
\end{aligned}\right\} \tag{5.33}$$

We can nondimensionalize (5.33) using (5.30), and if we choose ${}^{\Theta}I_0 = {}^{\Theta}\hat{\bar{I}}$, $\rho_0 = \hat{\bar{\rho}}$, $\eta_0 = \hat{\bar{\eta}}$, then

(5.33) can be written as

$$\left.\begin{aligned}
&\bar{\rho}\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{p}}{\partial \bar{x}}-\frac{\partial\left({}_{s}^{d}\bar{\sigma}_{21}^{(0)}\right)}{\partial \bar{y}}-\frac{\partial\left({}_{a}^{d}\bar{\sigma}_{21}^{(0)}\right)}{\partial \bar{y}}=0\\
&\frac{\partial\left({}_{s}\bar{m}_{23}^{(0)}\right)}{\partial \bar{y}}+2\left({}_{a}^{d}\bar{\sigma}_{21}^{(0)}\right)=0\\
&{}_{s}^{d}\bar{\sigma}_{12}^{(0)}={}_{s}^{d}\bar{\sigma}_{21}^{(0)}=\frac{1}{Re}\frac{\partial \bar{u}}{\partial \bar{y}}\\
&{}_{s}\bar{m}_{23}^{(0)}=\frac{\bar{\alpha}}{2}\frac{\partial\left({}_{i}\bar{\omega}_{3}\right)}{\partial \bar{y}}\\
&{}_{i}\bar{\omega}_{3}=-\frac{1}{2}\frac{\partial \bar{u}}{\partial \bar{y}}
\end{aligned}\right\} \tag{5.34}$$

in which $\bar{\alpha}=\left(\hat{\bar{\alpha}}/\tau_0 L_0^2 t_0\right)$, dimensionless material coefficient for micropolar non-classical physics. Equations (5.34) is a system of five first order PDEs in five dependent variables: $\bar{u}$, ${}_{s}^{d}\bar{\sigma}_{21}^{(0)}$, ${}_{a}^{d}\bar{\sigma}_{21}^{(0)}$, ${}_{s}\bar{m}_{23}^{(0)}$ and ${}_{i}\bar{\omega}_{3}$. Dimensionless parameters Re and $\bar{\alpha}$ control classical physics and micropolar non-classical physics.

Micropolar NCCT with rotational inertial physics : Model C

The mathematical model consists of BLM in $\bar{x}$ direction, BAM about $\bar{z}$ direction, BMM balance law, constitutive theory for symmetric part of the deviatoric Cauchy shear stress ${}_{s}^{d}\bar{\sigma}_{12}^{(0)}={}_{s}^{d}\bar{\sigma}_{21}^{(0)}$, constitutive theory for symmetric Cauchy moment ${}_{s}\bar{m}_{23}^{(0)}$, definition of angular rotation rate ${}_{i}\bar{\omega}_{3}$ and the constraint equations resulting from the entropy inequality. Using the mathematical model in $\mathbb{R}^3$ (equations (3.139)-(3.150)) and noting that in this case $\bar{v}=0$, $\bar{w}=0$, we can write the following for CBL and constitutive theories (in the absence of body forces and body moments) and the constraint equation

$$\left.\begin{aligned}
&\hat{\hat{\rho}}\frac{\partial\hat{\hat{u}}}{\partial\hat{t}}+\frac{\partial\hat{\hat{p}}}{\partial\bar{x}}-\frac{\partial\left({}_s^d\hat{\hat{\sigma}}_{21}^{(0)}\right)}{\partial\hat{\hat{y}}}-\frac{\partial\left({}_a^d\hat{\hat{\sigma}}_{21}^{(0)}\right)}{\partial\hat{\hat{y}}}=0\\
&\hat{\hat{\rho}}\left({}^{\Theta}\hat{\hat{I}}\right)\left(\frac{\partial\left({}_i\hat{\hat{\omega}}_3\right)}{\partial\hat{t}}\right)-\frac{\partial\left({}_s\hat{\hat{m}}_{23}^{(0)}\right)}{\partial\hat{\hat{y}}}-\frac{\partial\left({}_a\hat{\hat{m}}_{23}^{(0)}\right)}{\partial\hat{\hat{y}}}+2\left({}_a^d\hat{\hat{\sigma}}_{21}^{(0)}\right)=0\\
&{}_s^d\hat{\hat{\sigma}}_{12}^{(0)}={}_s^d\hat{\hat{\sigma}}_{21}^{(0)}=\hat{\hat{\eta}}\frac{\partial\hat{\hat{u}}}{\partial\hat{\hat{y}}}\\
&{}_s\hat{\hat{m}}_{23}^{(0)}=\frac{\hat{\hat{\alpha}}}{2}\frac{\partial\left({}_i\hat{\hat{\omega}}_3\right)}{\partial\bar{y}}\\
&{}_i\hat{\hat{\omega}}_3=-\frac{1}{2}\frac{\partial\hat{\hat{u}}}{\partial\hat{\hat{y}}}
\end{aligned}\right\}\tag{5.35}$$

$${}^{\Theta}\hat{\hat{I}}\hat{\hat{v}}\left({}_i\hat{\hat{\omega}}_3\right)-{}_s\hat{\hat{m}}_{23}^{(0)}-{}_a\hat{\hat{m}}_{23}^{(0)}=0\quad\text{(BMM when }\bar{v}\neq0)\tag{5.36}$$

$${}_s\hat{\hat{m}}_{23}^{(0)}\left({}_i\hat{\hat{\omega}}_3\right)=0\quad\text{(constraint equation)}\tag{5.37}$$

When $\bar{v}=0$ (which is the case here), BMM (5.36) reduces to

$${}_s\hat{\hat{m}}_{23}+{}_a\hat{\hat{m}}_{23}=0\tag{5.38}$$

Equation (5.38) suggests that ${}_a\hat{\hat{m}}_{23}^{(0)}=-{}_s\hat{\hat{m}}_{23}^{(0)}$ or the nonsymmetric moment tensor $\hat{\hat{m}}_{23}^{(0)}$ is zero, implying no micropolar physics. This of course is erroneous, hence in this case (5.38) cannot be used as part of the mathematical model. The constraint equation (5.37) implies that either ${}_s\hat{\hat{m}}_{23}^{(0)}=0$ or ${}_i\hat{\hat{\omega}}_3=0$ for all $\bar{x},t\in\bar{\Omega}_{xt}$. This also is erroneous, thus (5.36) and (5.37) cannot be considered as a part of the mathematical model, thus equations (5.35) constitutes the mathematical model. We can nondimensionalize (5.35) using (5.30), and if we choose ${}^{\Theta}I_0={}^{\Theta}\hat{\hat{I}}$, $\rho_0=\hat{\hat{\rho}}$, $\eta_0=\hat{\hat{\eta}}$, then we can obtain the following from (5.35).

$$\left.\begin{aligned}
&\bar{\rho}\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{p}}{\partial \bar{x}}-\frac{\partial\left({}_{s}^{d}\bar{\sigma}_{21}^{(0)}\right)}{\partial \bar{y}}-\frac{\partial\left({}_{a}^{d}\bar{\sigma}_{21}^{(0)}\right)}{\partial \bar{y}}=0\\
&\bar{\beta}\frac{\partial\left({}_{i}\bar{\omega}_{3}\right)}{\partial t}-\frac{\partial\left({}_{s}\bar{m}_{23}^{(0)}\right)}{\partial \bar{y}}+2\left({}_{a}^{d}\bar{\sigma}_{21}^{(0)}\right)=0\\
&{}_{s}^{d}\bar{\sigma}_{12}^{(0)}={}_{s}^{d}\bar{\sigma}_{21}^{(0)}=\frac{1}{Re}\frac{\partial \bar{u}}{\partial \bar{y}}\\
&{}_{s}\bar{m}_{23}^{(0)}=\frac{\bar{\alpha}}{2}\frac{\partial\left({}_{i}\bar{\omega}_{3}\right)}{\partial \bar{y}}\\
&{}_{i}\bar{\omega}_{3}=-\frac{1}{2}\frac{\partial \bar{u}}{\partial \bar{y}}
\end{aligned}\right\}\tag{5.39}$$

These are a system of five first order PDEs in five dependent variables: $\bar{u}$, ${}_{s}^{d}\bar{\sigma}_{21}^{(0)}$, ${}_{a}^{d}\bar{\sigma}_{21}^{(0)}$, ${}_{s}\bar{m}_{23}^{(0)}$ and ${}_{i}\bar{\omega}_{3}$. Re, $\bar{\alpha}$ and $\bar{\beta}$ control classical physics, non-classical physics without rotational inertial effects and non-classical physics with rotational inertia effects, respectively. In (5.39), $\bar{\alpha}$ and $\bar{\beta}$ are defined as $\bar{\alpha}=\left(\hat{\bar{\alpha}}/\tau_0 L_0^2 t_0\right)$ and $\bar{\beta}={}^{\Theta}I_{{}_0}/L_0^2$.

Numerical studies

We consider two model problems in this section: developing flow between parallel plates and developing Couette flow. In both model problems, we compute solutions for Model A (CCM); Model B, NCCM with micropolar physics but absence of rotational inertial physics; and Model C, micropolar NCCM with rotational inertial physics. In Model A (CCM), the only dimensionless parameter is Re which controls the flow physics. In case of Model B, Re for CCM and $\bar{\alpha}$ for micropolar physics control the flow physics. In case of Model C, Re, $\bar{\alpha}_1$ and $\bar{\beta}_1$ all three control the physics. $\bar{\beta}_1$ is associated with rotational inertial physics. For both model problem studies we present evolutions for different combinations of $\bar{\alpha}$ and $\bar{\beta}$ for a fixed Re to illustrate their influence on flow physics. Computed solutions for Model B and C are always compared with Model A.

We recall that many works published by Surana et. al [86, 88] show that in micropolar NCCT presence of micro constituents offer resistance to flow i.e., increasing value of $\bar{\alpha}$ results in diminishing flow rate. Material coefficient $\bar{\beta}$, due to rotational inertial physics, related to rotational inertial physics due to microconstituents is also expected provide further resistance to flow over and beyond $\bar{\alpha}$. In other words, we expect increasing $\bar{\alpha}$ as well as increasing $\bar{\beta}$ to provide increasing resistance to flow. Since the constitutive theories are not calibrated, we do not know actual values of $\bar{\alpha}$ and $\bar{\beta}$ for various fluids with varying microconstituents. In the model problem studies presented here, we choose $\bar{\alpha}$ and $\bar{\beta}$ such that we can demonstrate their relative influence on flow physics.

Developing flow between parallel plates

Figure 8 shows schematic, space-time strips and a uniform discretization of first space-time strip using 10 p-version hierarchical space-time finite elements. We consider solution of class C^{00} in space and time with p-level of nine in space and time, $Re = 100$ with $\Delta t = 1.0$. Flow is pressure driven with $\frac{\partial \bar{p}}{\partial \bar{x}} = -0.1$.

Case I

In the first study we choose

Model A (CCM), $Re = 100$

Model B (NCCM), $Re = 100$; $\bar{\alpha} = 0.0001, 0.001$.

Figures 9(a)-(d) show plots of the evolution of velocity $\bar{u}$ for $t = 5\Delta t, 10\Delta t, 20\Delta t$ and $40\Delta t$ for Models A and B. In case of Model A (CCM) the flow has no resistance due to microconstituents. In model B, with progressively increasing $\bar{\alpha}$ the flow resistance increases resulting in diminishing flow rate. This holds true for each value of time t. At $t = 40\Delta t$, we almost have stationary state of the developing flow. Significant reduction in flow rate with increasing $\bar{\alpha}$ is also obvious from the stationary state in figure 9(d). In CCM, rotation rates are a free field, hence they do not influence

CCT. Figure 10(a) shows evolution of ${}_i\bar{\omega}_3$ versus $\bar{y}$ for Model A (CCM). At $t = 40\Delta t$, we almost have stationary sate at which $\frac{\partial({}_i\bar{\omega}_3)}{\partial\bar{y}} = constant$. Since ${}_i\bar{\omega}_3$ is a free field, ${}_s\bar{m}_{23} = 0$, shown in figure 10(b). Figures 10(c) and (d) show evolution of ${}_i\bar{\omega}_3$ versus $\bar{y}$ and ${}_s\bar{m}_{23}$ versus $\bar{y}$ for Model B for $\bar{\alpha} = 0.001$. Comparison of these evolutions with those in figures 10(a) and(b) clearly shows the influence of microconstituents on ${}_i\bar{\omega}_3$ and ${}_s\bar{m}_{23}$ as in this case ${}_i\bar{\omega}_3$ is not a free field, hence ${}_s\bar{m}_{23}$ is no longer zero.

Case II

In this study we consider the influence of both $\bar{\alpha}$ and $\bar{\beta}$ for fixed Reynolds number. We consider the following

Model A (CCM), $Re = 100$ (same as in Case I)

Model C (NCCM with rotational inertial physics),

$Re = 100$ (same as in Case I)

$$\left.\begin{array}{l}\bar{\alpha} = 0.0001\\ \bar{\alpha} = 0.001\end{array}\right\}\bar{\beta} = 0.05, 0.9$$

Discretization, p-levels and other details remain the same as in case I. Figures 11(a)-(d) and figures 12(a)-(d) show plots of velocity $\bar{u}$ versus $\bar{y}$ for $\bar{\alpha} = 0.0001$, 0.001, $\bar{\beta} = 0.05$; and $\bar{\alpha} = 0.0001$, 0.001, $\bar{\beta} = 0.9$. As we mentioned earlier both $\bar{\alpha}$ and $\bar{\beta}$ result in resistance to the flow but we expect that the degree of resistance is not the same for $\bar{\alpha}$ and $\bar{\beta}$ with similar change in their value. When comparing figures 11(a)-(d) with figures 9(a)-(d) we note that for same values of $\bar{\alpha}$ as in figures 9(a)-(d) but with $\bar{\beta} = 0$, we observe further reduction in the flow rate when $\bar{\beta} = 0.05$. For $\bar{\beta} = 0.9$ (figures 12(a)-(d)) flow rate is further reduced compared to $\bar{\beta} = 0.05$.

Case III

In this study we consider fixed $Re = 100$, choose a fixed value of $\alpha = 0.01$ and vary $\bar{\beta}$ to study evolution of velocity $\bar{u}$. We choose $\Delta t = 1.0$ and $\bar{\beta} = 0.05$, 0.2 and 0.5. Remaining details are same as in case I. Figures 13(a)-(d) show plots of $\bar{u}$ versus $\bar{y}$ for $t = 5\Delta t$, $10\Delta t$, $20\Delta t$ and $40\Delta t$. At $t = 40\Delta t$ we almost have stationary state of the flow. As expected, larger values of $\bar{\beta}$ offer more resistance to flow and requires more time to reach stationary state and vice versa. We note that stationary state is same for all $\bar{\beta}$ values. Figures 14(a) and (b) show evolution of ${}_i\bar{\omega}_3$ versus $\bar{y}$ and ${}_s\bar{m}_{23}$ versus $\bar{y}$ (case II) for $Re = 100$, $\bar{\alpha} = 0.001$ and $\bar{\beta} = 0.9$. Comparing this with figures 10(c) and (d) ($\bar{\alpha} = 0.001$, $\bar{\beta} = 0.0$), we clearly note reduced values of ${}_i\bar{\omega}_3$ during the evolution for $\bar{\beta} = 0.9$ in figure 14(a), Cauchy moment ${}_s\bar{m}_{23}$ adjusts accordingly. This of course is due to increased resistance to flow due to $\bar{\beta}$.

Developing Couette flow

The configuration of parallel plates and other details remain the same as in figure 8. We choose $Re = 5$ in all studies. A 10 element uniform mesh with p-levels of nine in space and time and a solution of class C^{00} with $\Delta t = 0.01$ is considered. At the top plate ($\bar{y} = 1.0$) a velocity of 1.0 is applied over Δt in a continuous and differentiable manner i.e., $\bar{u} \in [0, 1]\ \forall\ t \in [0, \Delta t]$ and $u = 1.0\ \forall\ t \geq \Delta t$.

Case A

Model A (CCM) : $Re = 5$

Model B (NCCM) : $Re = 5$; $\bar{\alpha} = 0.0001, 0.001$

Figure 15(a)-(c) show evolution of $\bar{u}$ versus $\bar{y}$ at $t = 2\Delta t$, $5\Delta t$ and $10\Delta t$. For each value of time the velocity $\bar{u}$ for both $\bar{\alpha}$ values is lower than from Model A (CCM) as expected. Furthermore, higher values of $\bar{\alpha}$ yield lower values of velocity $\bar{u}$ compared to lower values of $\bar{\alpha}$ during the entire evolution, demonstrating increasing resistance to flow with increasing value of $\bar{\alpha}$ during the entire evolution.

Case B

In case B we present two studies. In the first study, we choose

Model A (CCM) : $Re = 5$

Model C (NCCM) : $Re = 5$; $\bar{\alpha} = 0.01, 0.1$; $\bar{\beta} = 0.05, 0.2$

Computations are performed using the same mesh and other details as in case A. Figure 16 (a), (b) and (c), (d) show plots of evolution of $\bar{u}$ versus $\bar{y}$ at $t = 5\Delta t$ and $t = 20\Delta t$ for $\bar{\beta} = 0.05$ and $\bar{\beta} = 0.2$ for both values of $\bar{\alpha}$ in each case. Once again, combination of higher values of $\bar{\alpha}$ and $\bar{\beta}$ result in more reduction in the velocity due to higher resistance to flow. Velocities for $\bar{\beta} = 0.2$ are lower than those for $\bar{\beta} = 0.05$ for both values of $\bar{\alpha}$.

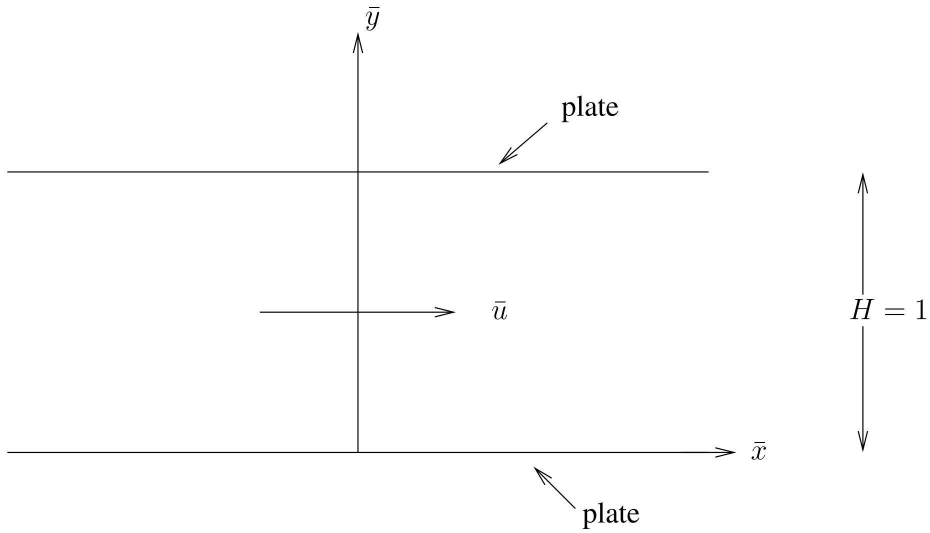
In the second study, we choose a fixed $\bar{\alpha} = 0.1$ and vary $\bar{\beta} = 0.1, 0.2, 0.3$. Figure 17 (a) and (b) shows evolution of $\bar{u}$ versus $\bar{y}$ at $t = \Delta t$ and $t = 5\Delta t$. Progressively increasing values of $\bar{\beta}$ results in progressively reduced values of velocity $\bar{u}$. Figure 18(a)-(c) show evolution of ${}_i\bar{\omega}_3$ versus $\bar{y}$ for: Model A, CCM ($Re = 5$); Model B, NCCM ($Re = 5$, $\bar{\alpha} = 0.001$) and Model C, NCCM ($Re = 5, \bar{\alpha} = 0.001$ and $\bar{\beta} = 0.05$). Corresponding ${}_s\bar{m}_{23}$ versus $\bar{y}$ graphs are shown in figures 19(a)-(c). In figures 18(a) and (b) we clearly observe the change in ${}_i\bar{\omega}_3$ free field due to micropolar physics. In figure 18(c), we observe further reduction and change in ${}_i\bar{\omega}_3$ due to $\bar{\beta} = 0.05$. Evolution of ${}_s\bar{m}_{23}$ versus $\bar{y}$ follows a trend opposite to ${}_i\bar{\omega}_3$ i.e., $\bar{\alpha} \neq 0$ introduces nonzero ${}_s\bar{m}_{23}$ and introduction of $\bar{\beta}$ further changes evolution of existing nonzero ${}_s\bar{m}_{23}$ due to only $\bar{\alpha}$.

Remarks

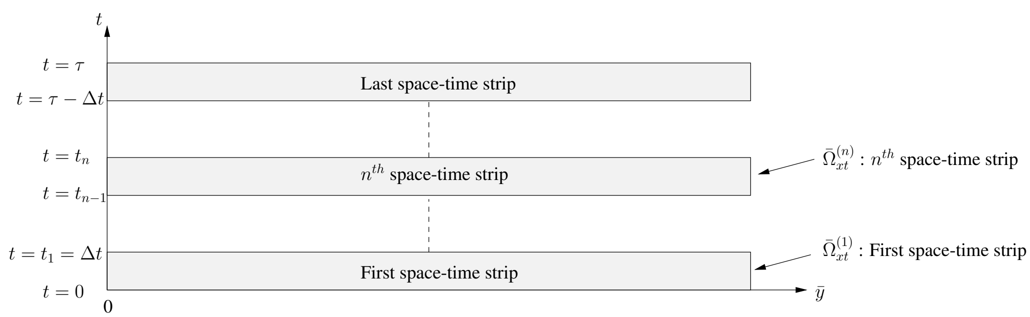
1. In this work, $\bar{\alpha}$ and $\bar{\beta}$ are two parameters related to the micropolar physics. We have seen in the model problem studies that both offer increasing resistance to flow with their increasing values.

2. Model problem studies show that resistance to flow due to $\bar{\beta}$ (${}^{\Theta}\bar{I}$ and $\bar{\rho}$ combined) is more pronounced compared to $\bar{\alpha}$.

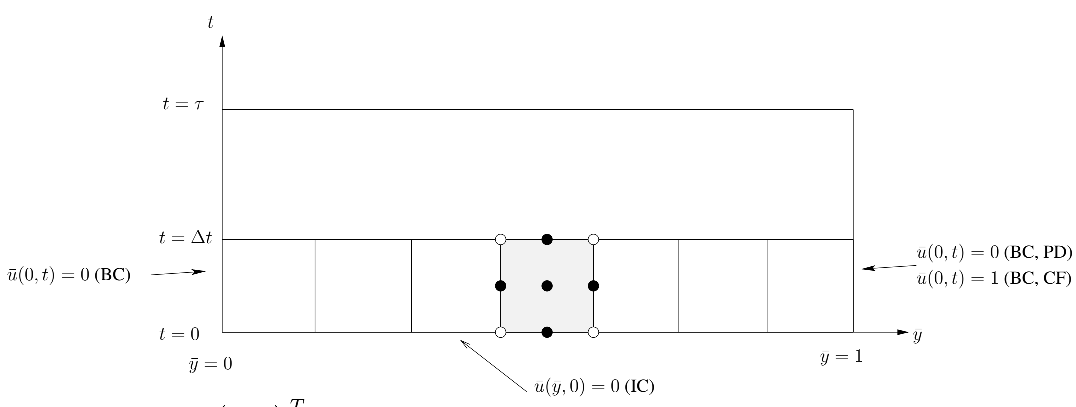
3. Parameters $\bar{\alpha}$ and ${}^{\Theta}\bar{I}$ are properties of the micropolar medium controlled by the microconstituents, but the manner in which they exert their influence on flow physics is different. The parameter $\bar{\alpha}$ is a measure of the collective resistance offered to the flow by each microconstituents. Whereas ${}^{\Theta}\bar{I}$ appears as ${}^{\Theta}\bar{I}\bar{\rho}$, suggesting that it is a volumetric or mass effect through angular acceleration. Thus ${}^{\Theta}\bar{I}\bar{\rho}$ is the collective influence of a group of microconstituents in a unit volume and perhaps provides a better explanation of why $\bar{\beta}$ is more influential than $\bar{\alpha}$.



(a) Schematic of parallel plates

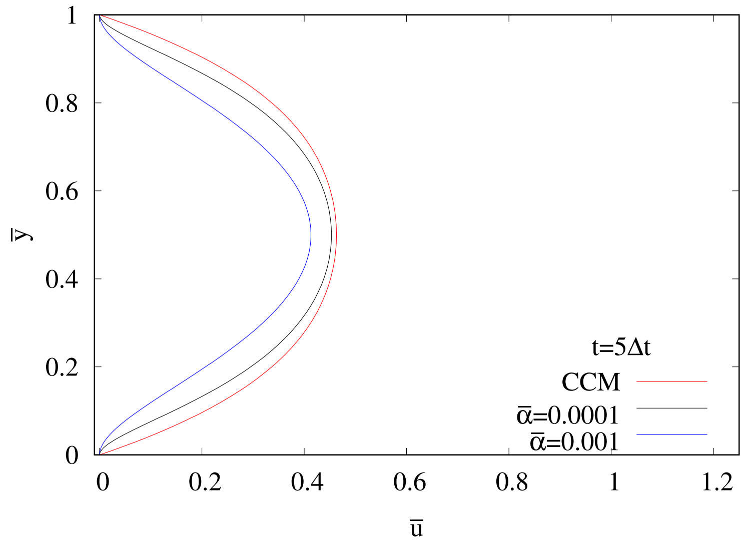


(b) Space-time strips

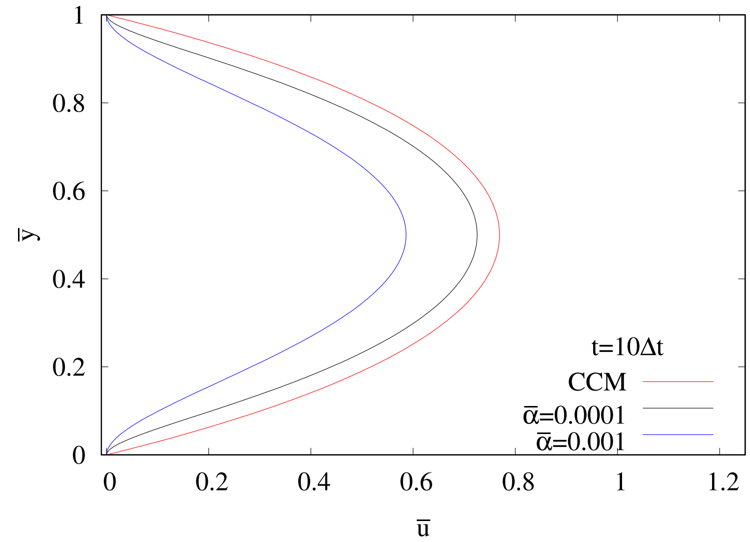


(c) Discretization $\left(\bar{\Omega}_{xt}^{(1)}\right)^T = \bigcup_e \bar{\Omega}_{xt}^{(e)}$ of the first space-time strip $\bar{\Omega}_{xt}^{(1)}$ and BCs, ICs

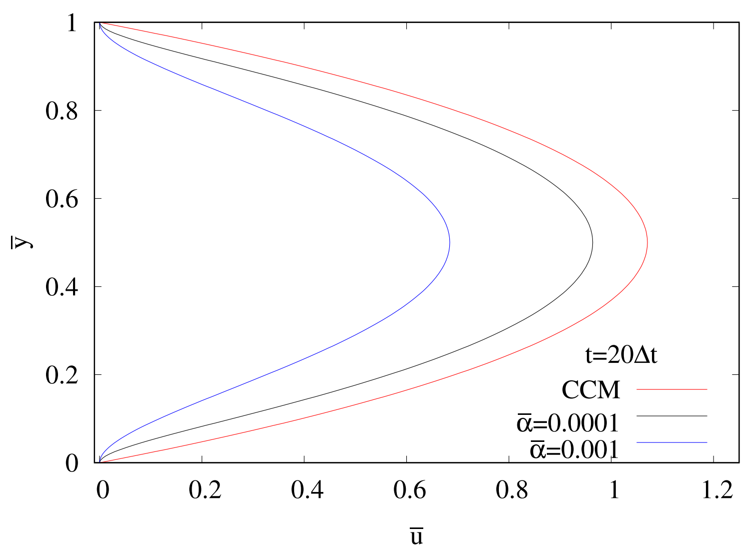
Figure 8: Developing flow between parallel plates



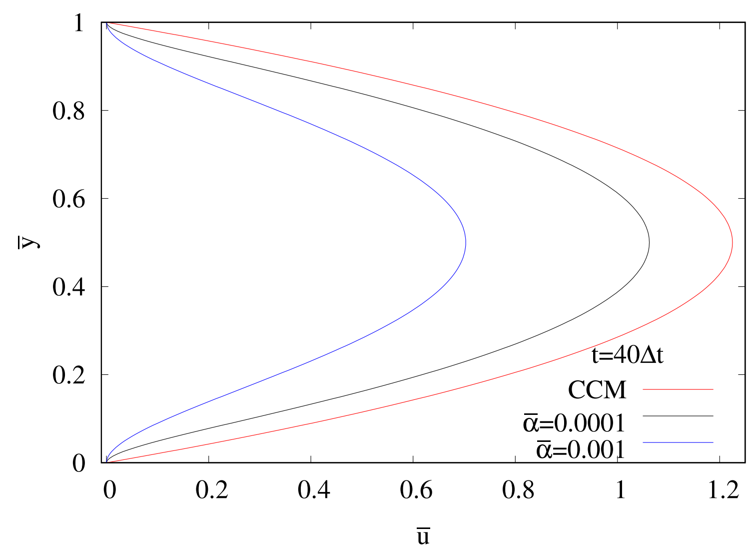
(a) $\bar{u}$ versus $\bar{y}$



(b) $\bar{u}$ versus $\bar{y}$

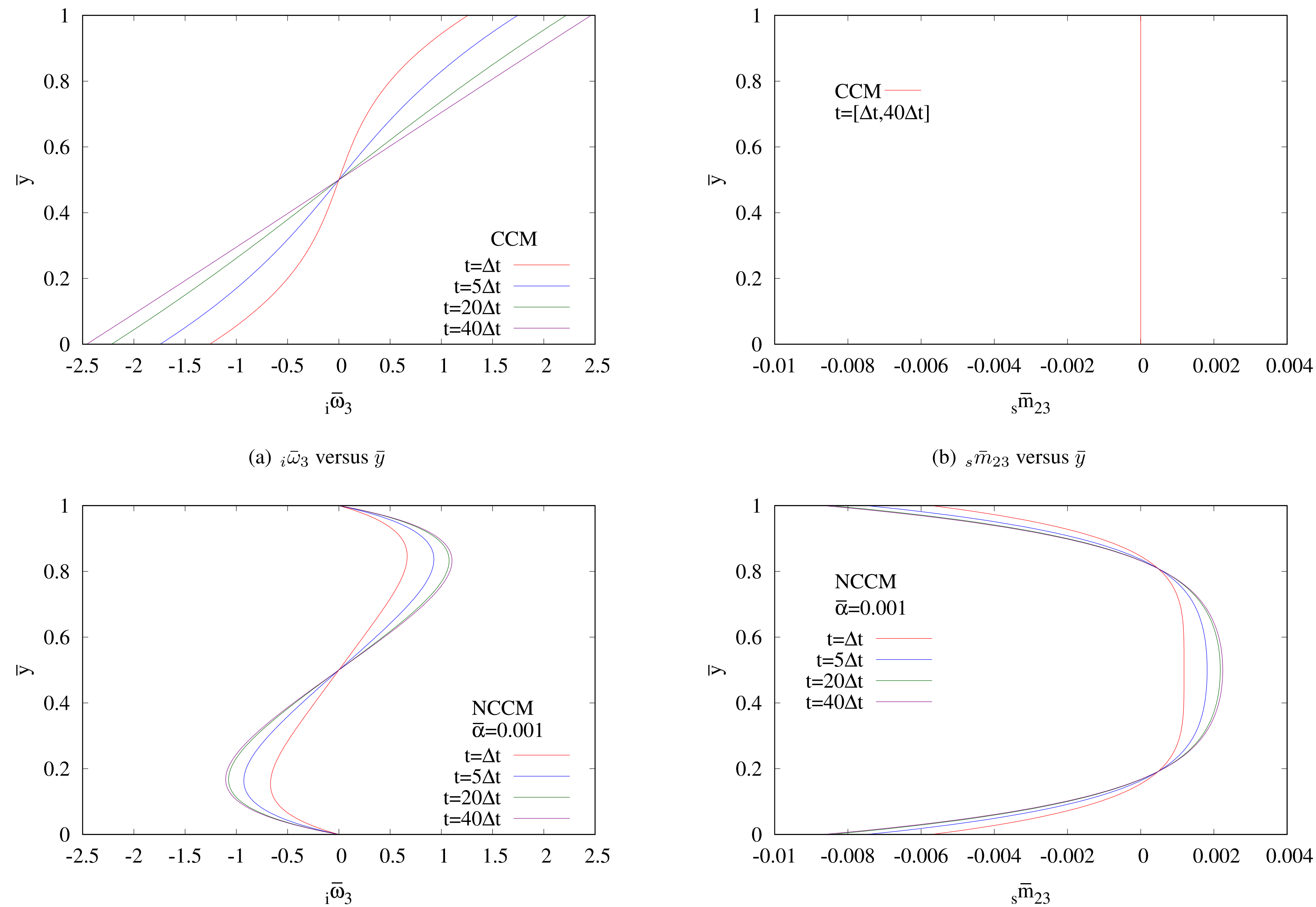


(c) $\bar{u}$ versus $\bar{y}$



(d) $\bar{u}$ versus $\bar{y}$

Figure 9: Developing flow between parallel plates : velocity $\bar{u}$ versus position $\bar{y}$: Models A and B (Case I)



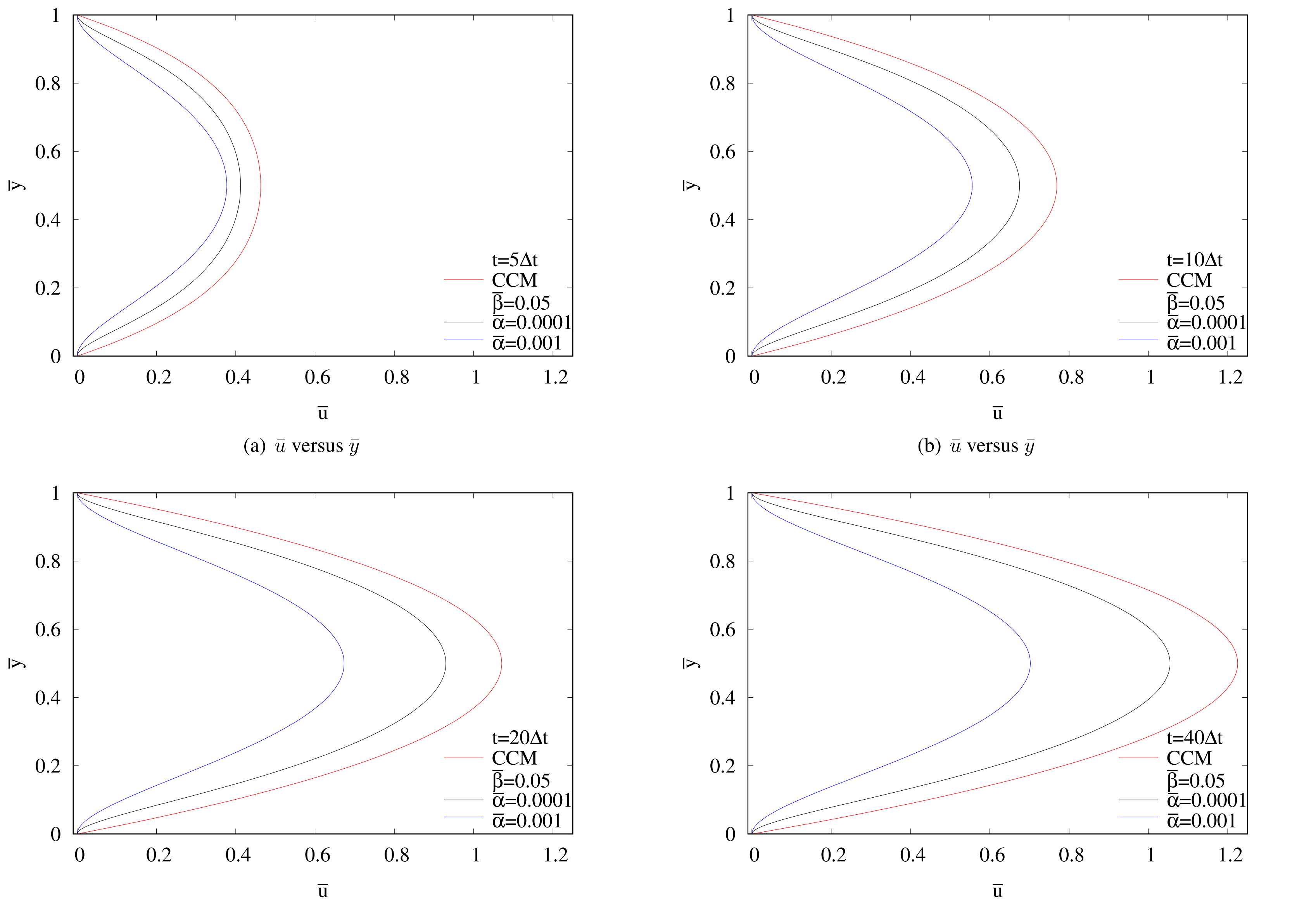
(a) $_i\bar{\omega}_3$ versus $\bar{y}$

(b) $_s\bar{m}_{23}$ versus $\bar{y}$

(c) $_i\bar{\omega}_3$ versus $\bar{y}$

(d) $_s\bar{m}_{23}$ versus $\bar{y}$

Figure 10: Developing flow between parallel plates : $_i\bar{\omega}_3$ versus $\bar{y}$ and $_s\bar{m}_{23}$ versus $\bar{y}$: Models A and B (Case I)



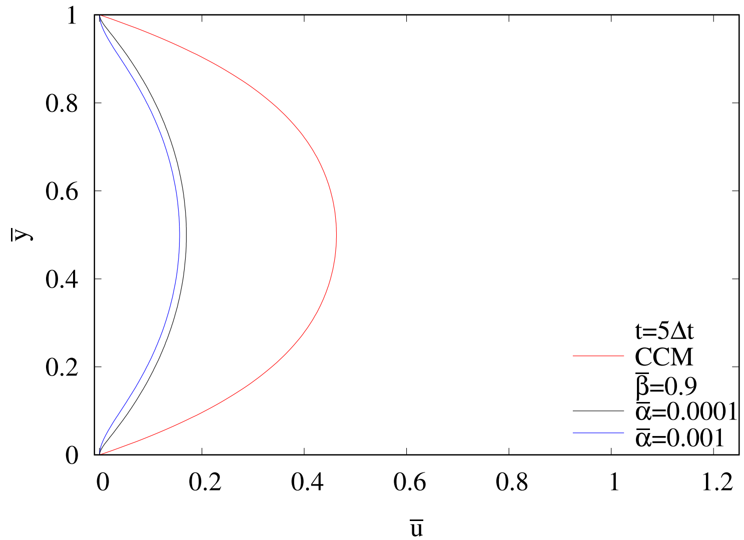
(a) $\bar{u}$ versus $\bar{y}$

(b) $\bar{u}$ versus $\bar{y}$

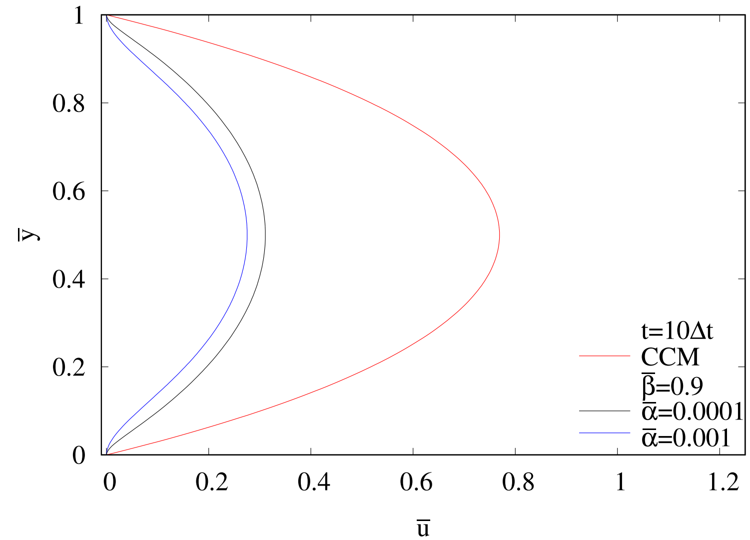
(c) $\bar{u}$ versus $\bar{y}$

(d) $\bar{u}$ versus $\bar{y}$

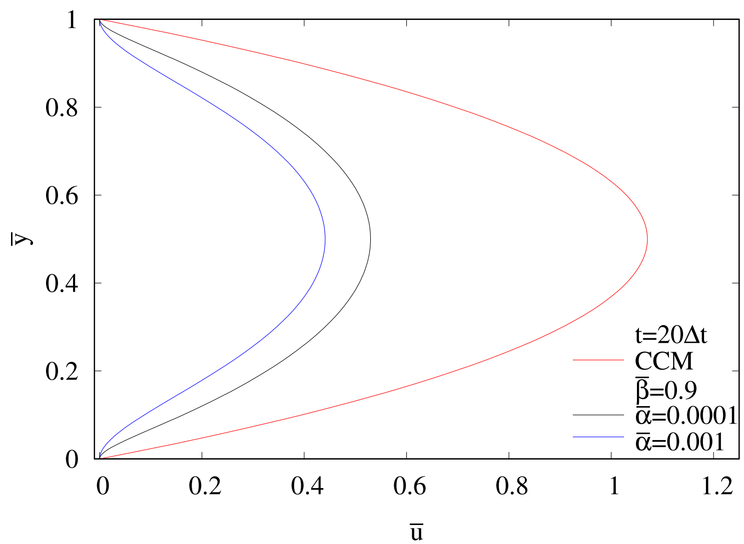
Figure 11: Developing flow between parallel plates : velocity $\bar{u}$ versus position $\bar{y}$: Models A and C (Case II), $\bar{\beta} = 0.05$



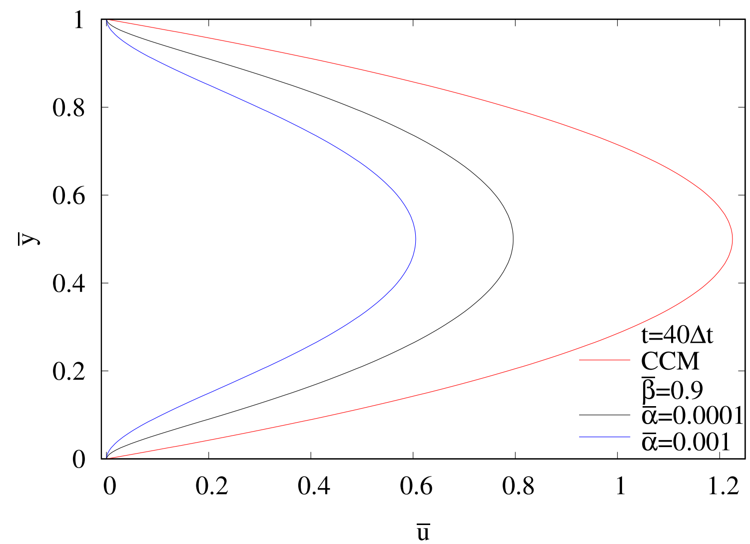
(a) $\bar{u}$ versus $\bar{y}$



(b) $\bar{u}$ versus $\bar{y}$

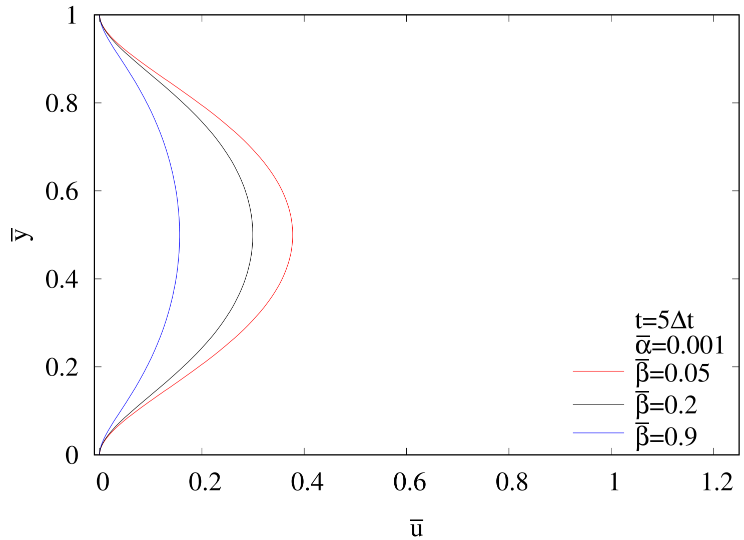


(c) $\bar{u}$ versus $\bar{y}$

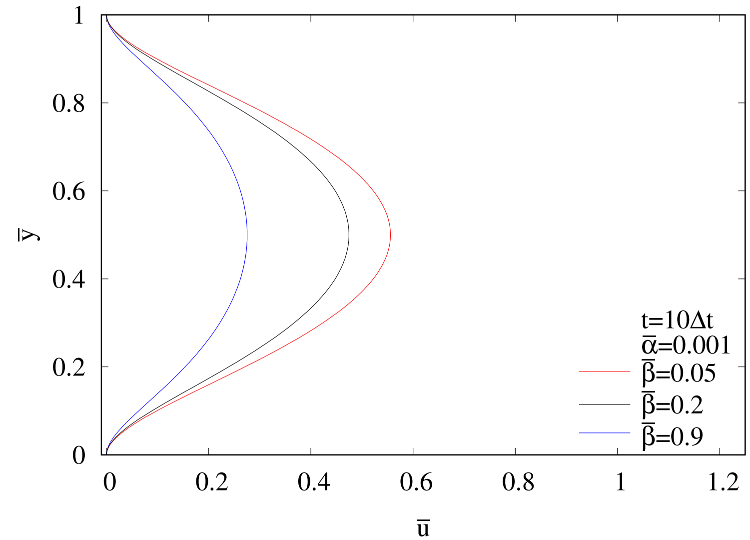


(d) $\bar{u}$ versus $\bar{y}$

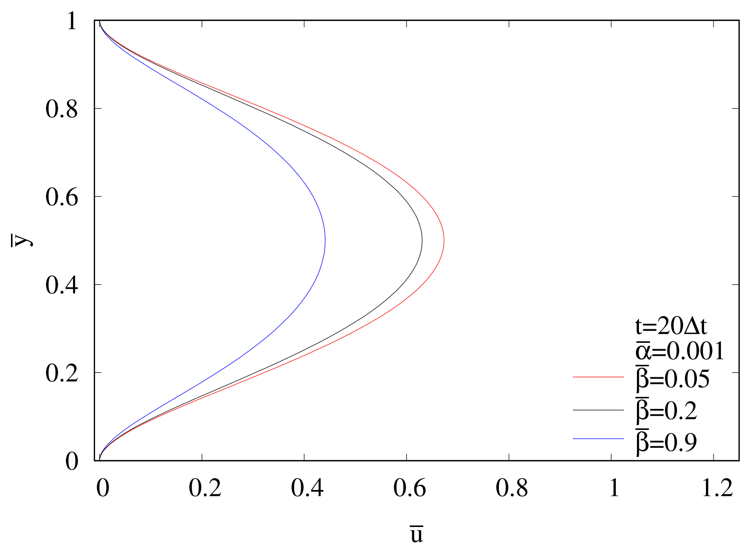
Figure 12: Developing flow between parallel plates : velocity $\bar{u}$ versus position $\bar{y}$: Models A and C (Case II), $\bar{\beta} = 0.9$



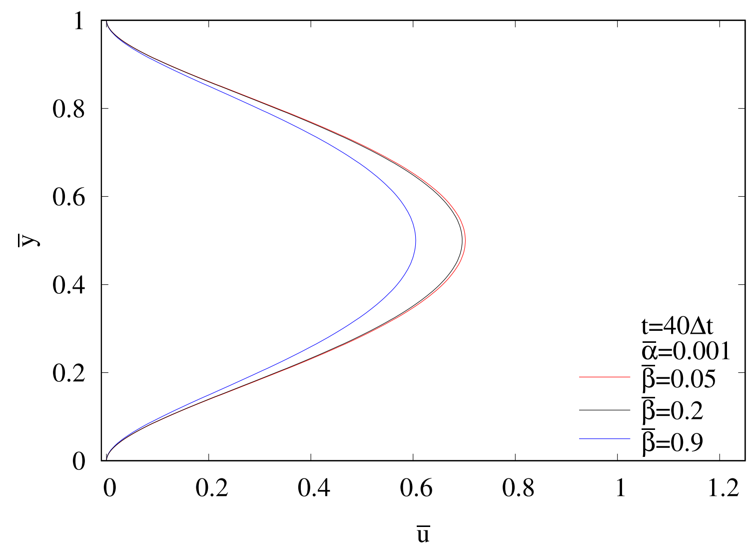
(a) $\bar{u}$ versus $\bar{y}$



(b) $\bar{u}$ versus $\bar{y}$

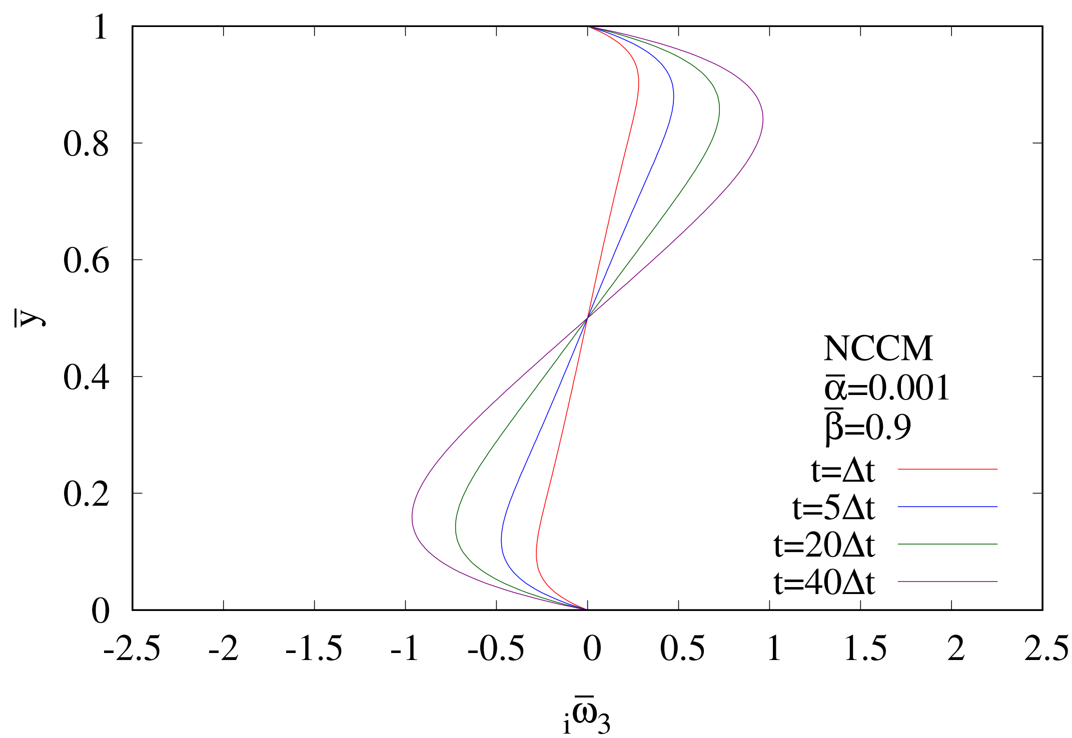


(c) $\bar{u}$ versus $\bar{y}$

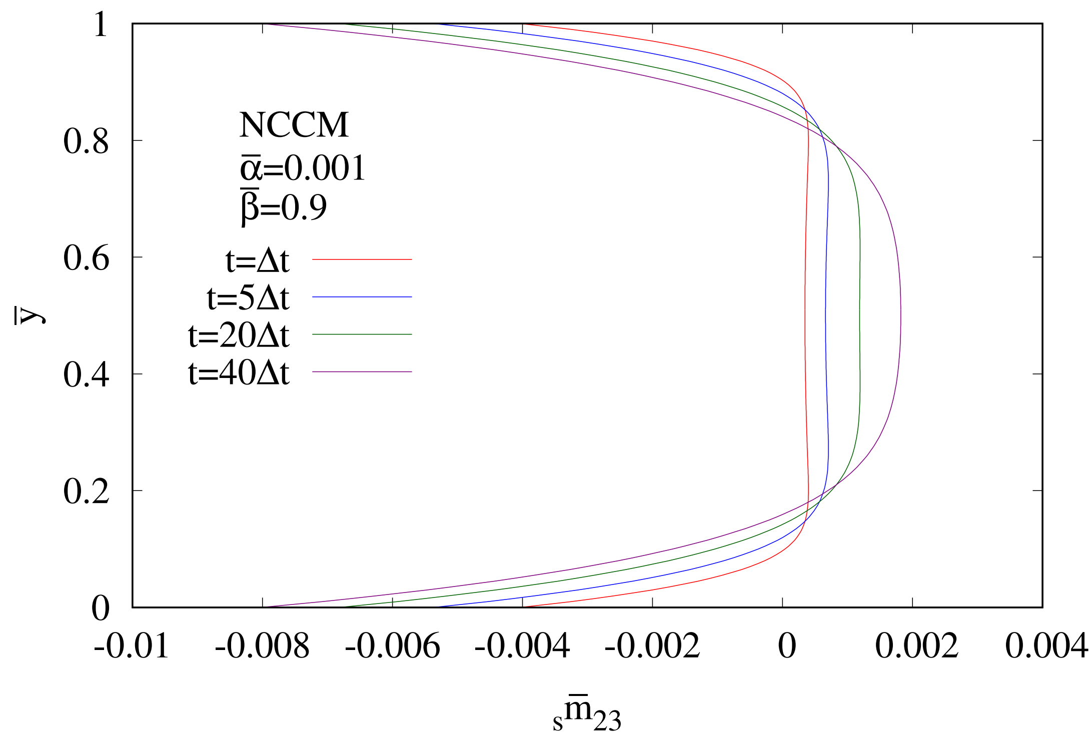


(d) $\bar{u}$ versus $\bar{y}$

Figure 13: Developing flow between parallel plates : velocity $\bar{u}$ versus position $\bar{y}$: Model C (Case III)

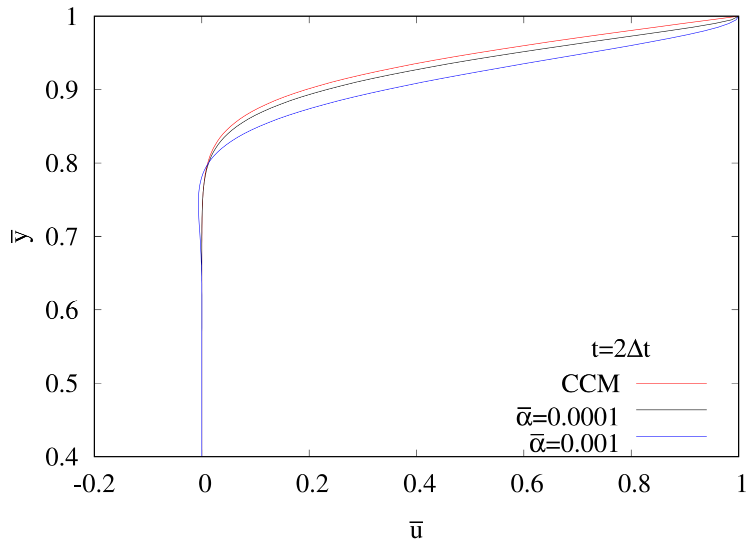


(a) ${}_i\bar{\omega}_3$ versus $\bar{y}$

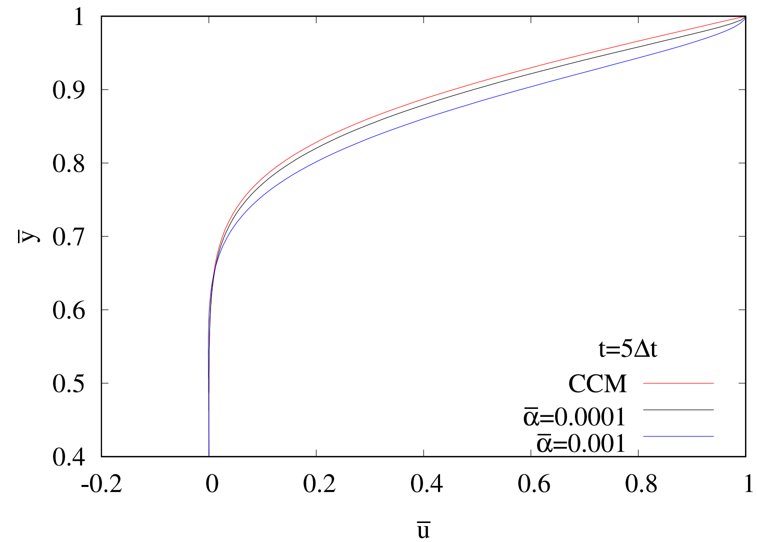


(b) ${}_s\bar{m}_{23}$ versus $\bar{y}$

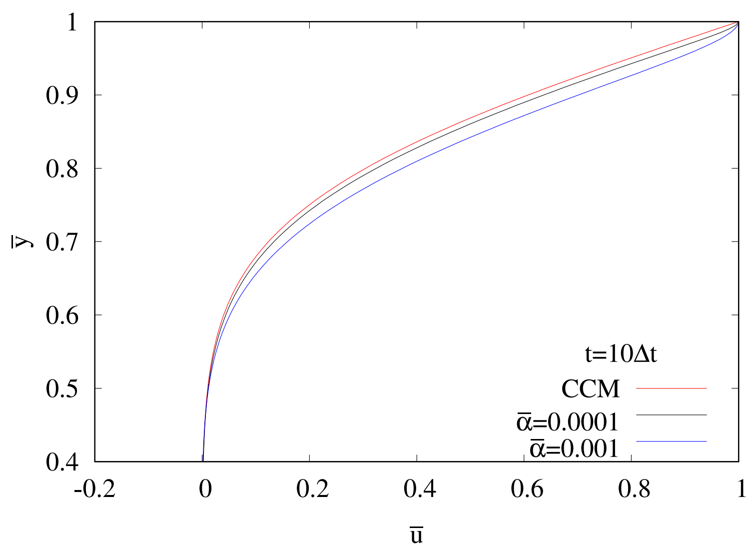
Figure 14: Developing flow between parallel plates : ${}_i\bar{\omega}_3$ and ${}_s\bar{m}_{23}$ versus $\bar{y}$: $\bar{\alpha} = 0.001$, $\bar{\beta} = 0.9$: Model B (Case II)



(a) $\bar{u}$ versus $\bar{y}$

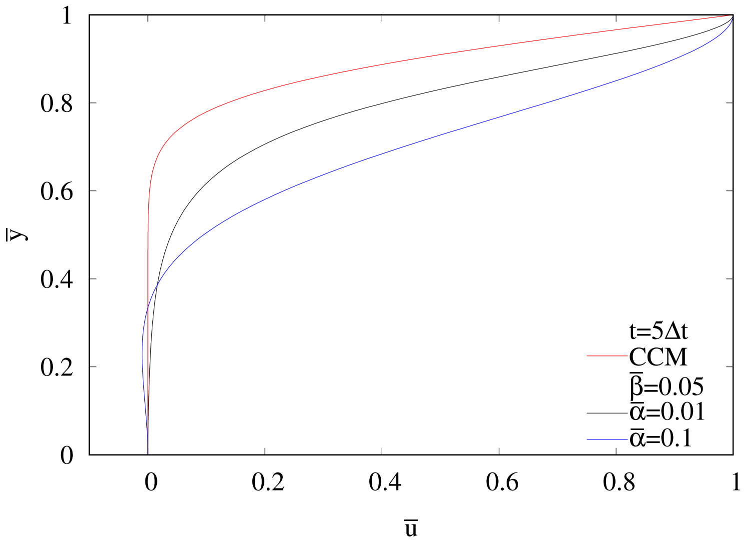


(b) $\bar{u}$ versus $\bar{y}$

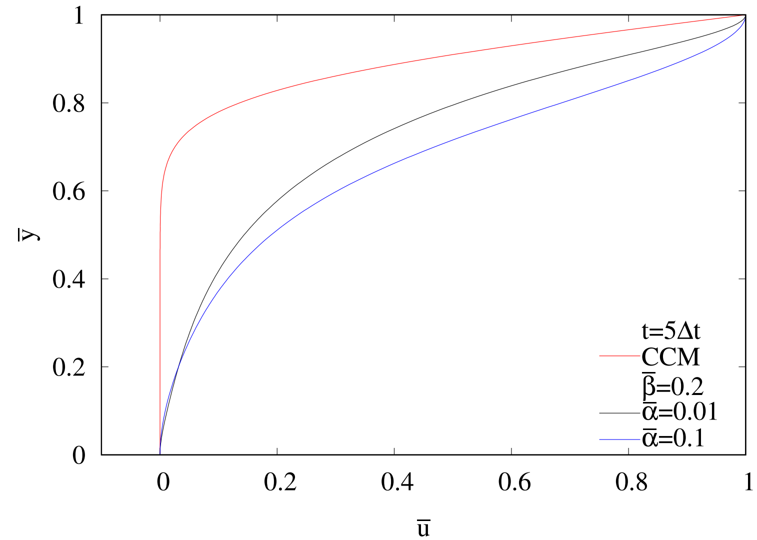


(c) $\bar{u}$ versus $\bar{y}$

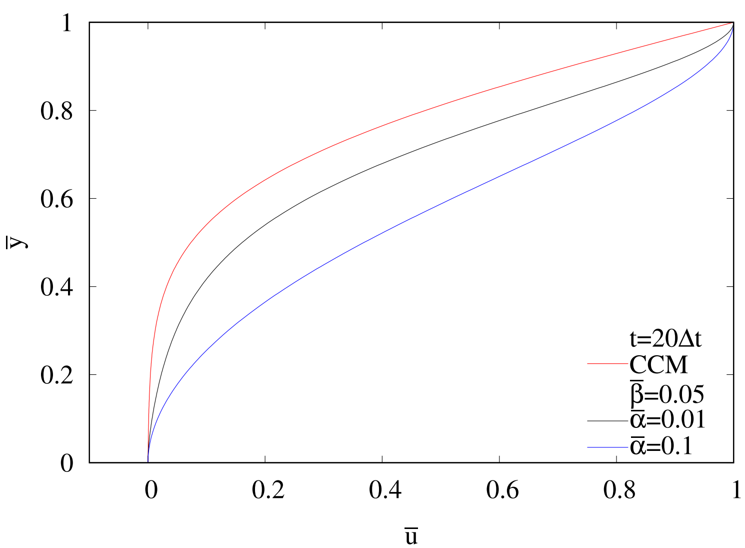
Figure 15: Developing Couette flow : velocity $\bar{u}$ versus position $\bar{y}$: Models A and B (Case A)



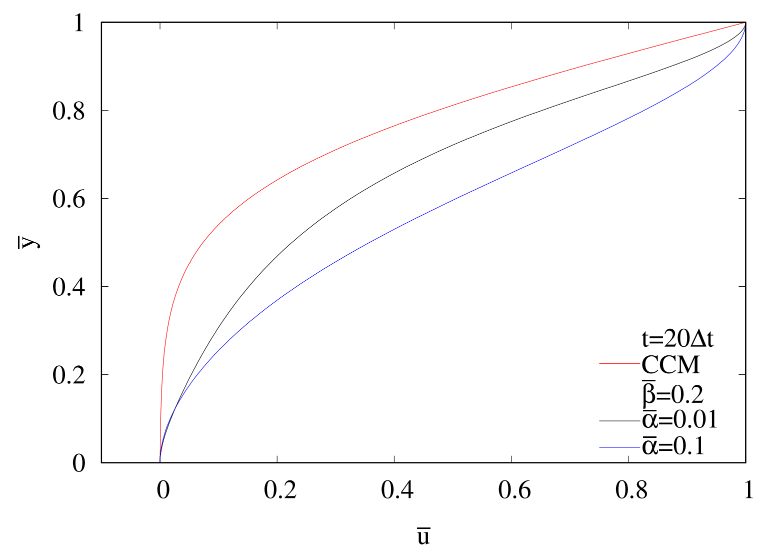
(a) $\bar{u}$ versus $\bar{y}$



(b) $\bar{u}$ versus $\bar{y}$

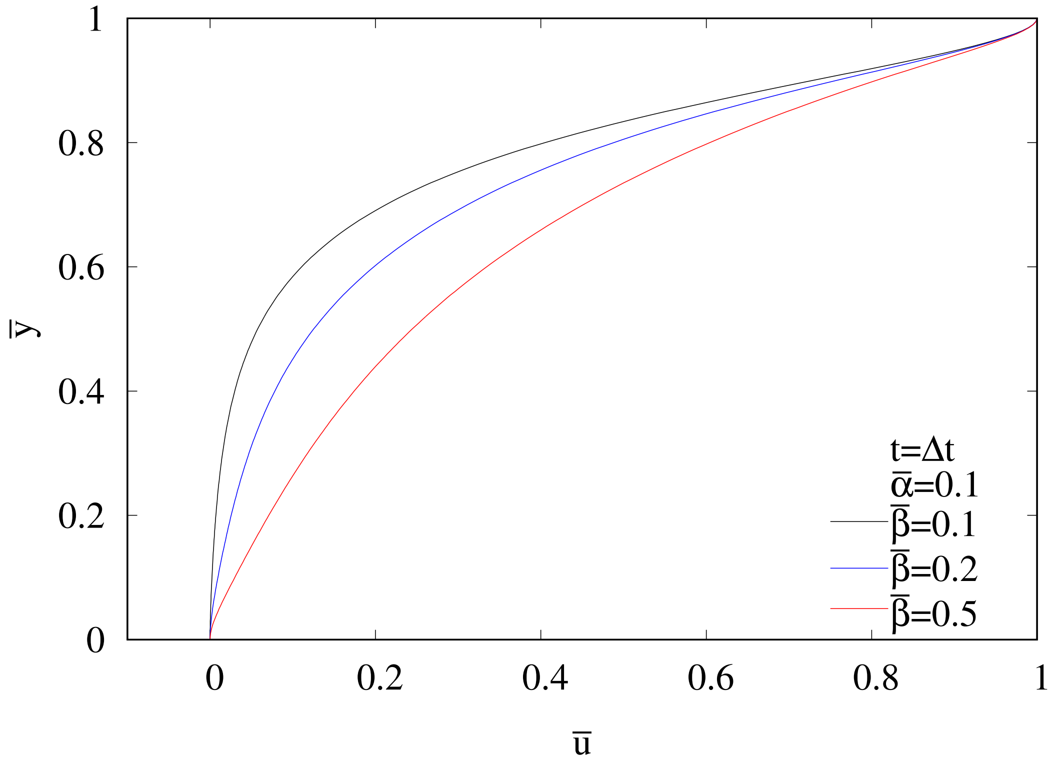


(c) $\bar{u}$ versus $\bar{y}$

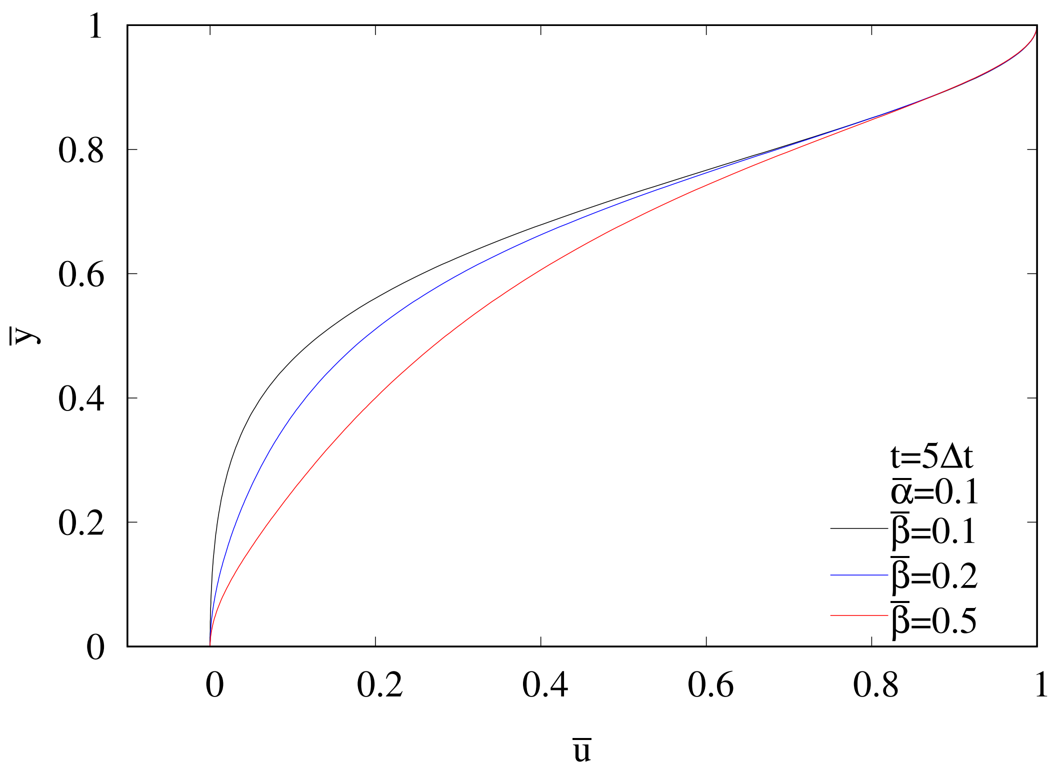


(d) $\bar{u}$ versus $\bar{y}$

Figure 16: Developing Couette flow : velocity $\bar{u}$ versus position $\bar{y}$: Models A and C (Case B)

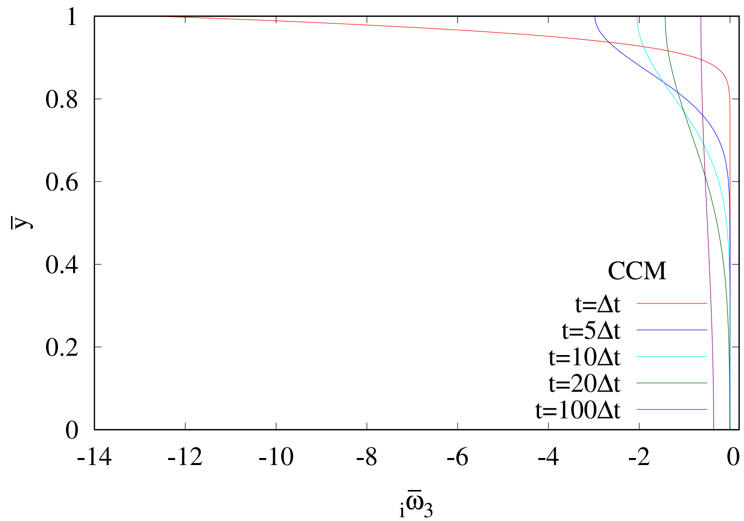


(a) $\bar{u}$ versus $\bar{y}$

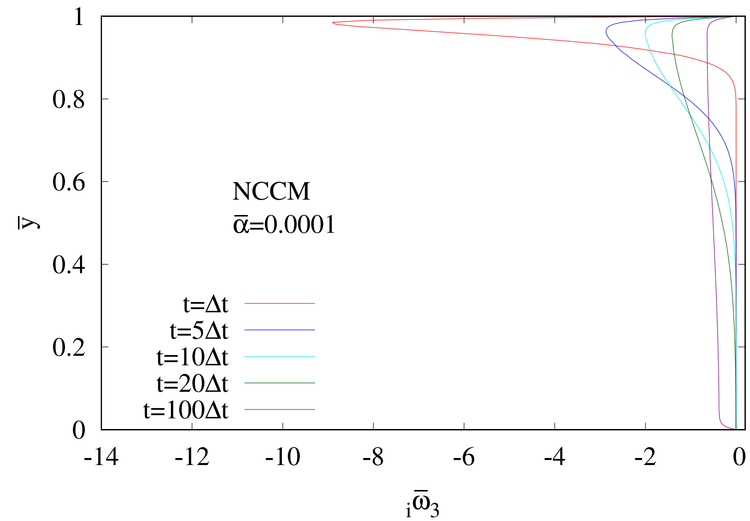


(b) $\bar{u}$ versus $\bar{y}$

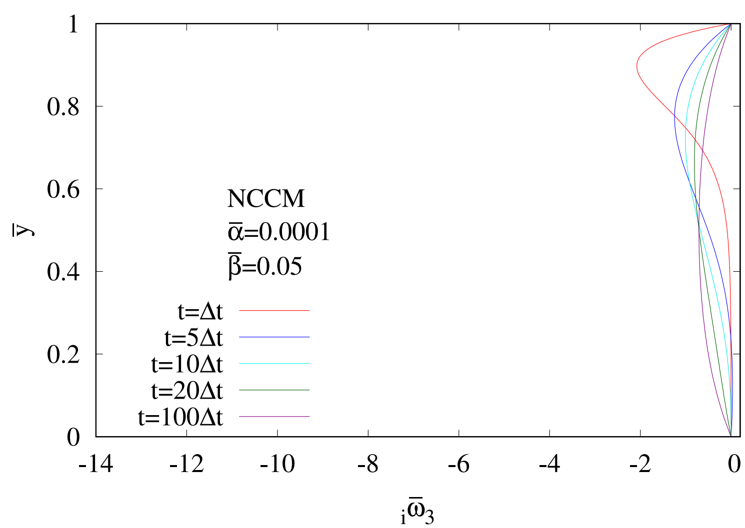
Figure 17: Developing Couette flow : velocity $\bar{u}$ versus position $\bar{y}$: (Case B, second study)



(a) $_i\bar{\omega}_3$ versus $\bar{y}$

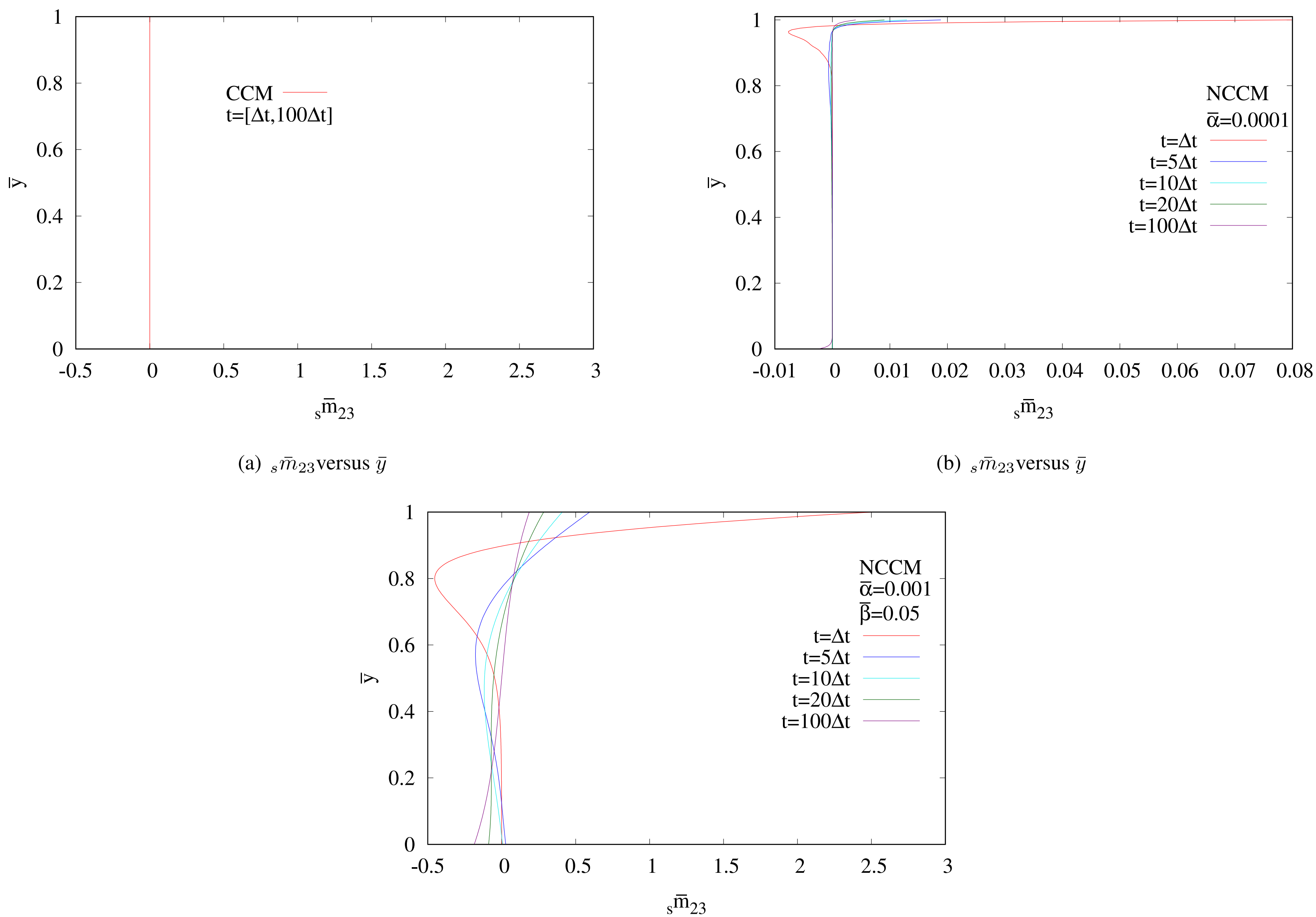


(b) $_i\bar{\omega}_3$ versus $\bar{y}$



(c) $_i\bar{\omega}_3$ versus $\bar{y}$

Figure 18: Developing Couette flow : $_i\bar{\omega}_3$ versus $\bar{y}$: Models A, B and C



(a) $_s\bar{m}_{23}$versus $\bar{y}$

(b) $_s\bar{m}_{23}$versus $\bar{y}$

(c) $_s\bar{m}_{23}$versus $\bar{y}$

Figure 19: Developing Couette flow : $_s\bar{m}_{23}$ versus $\bar{y}$: Models A, B and C

Chapter 6

Summary and conclusions

This work considers micropolar non-classical continuum theories in the presence of rotational inertial physics for solid as well as fluid media. Derivation of CBL and the constitutive theories based on entropy inequality and representation theorem are given. Model problem studies are also presented for micropolar solid and fluid media using the present micropolar NCCT with rotational inertial physics.

In the first part of this work, the conservation and balance laws of non-classical continuum mechanics [87, 95] incorporating internal rotations due to displacement gradient tensor ${}^{d}\boldsymbol{J}$ at a material point are considered for solid continua. In the evolution of a volume of matter the time dependent rotations (referred to as internal rotations arising due to ${}^{d}\boldsymbol{J}$) and their rates naturally vary between the material points, thus creating their gradients, angular velocities and angular accelerations at material points. Resistance offered by the microconstituents to these results in moments, angular momentum and rotational inertial effects. In this work, the conservation and balance laws of references [87,95] are rederived in Lagrangian description to account for the new physics due to angular velocities and angular accelerations. This work considers homogeneous, isotropic matter and small strain, small deformation physics in the presence of microconstituents.

In the second part of this work, conservation and balance laws of non-classical continuum mechanics with internal rotation rate physics [86, 88] and the constitutive theories for thermoviscous fluent continua are rederived by incorporating rotational inertia effects. In the evolution of deforming fluent continua, when the time varying rotation rates (angular velocities) and angular accelerations are resisted by the microconstituents, moments, angular momentum and angular inertial effects are realized. The work presents complete derivation of CBL and the constitutive theories in the presence of internal rotation rates due to $\bar{\boldsymbol{L}}$ and rotational inertial effects. This work considers homogeneous and isotropic thermoviscous fluent continua. We summarize the work and draw some conclusions in the following.

1. As in most non-classical continuum theories, in this work also, the Cauchy stress tensor is not symmetric.

2. In the non-classical continuum theories for solid continua [87, 95] as well as fluent continua incorporating internal rotation rates [86, 88], the Cauchy moment tensor is symmetric as a consequence of the balance of moment of moments balance law [82, 96]. In the present work, in the presence of rotational inertial physics, BMM balance law does not establish symmetry of the Cauchy moment tensor for solid as well as fluent continua, instead it yields three additional equations containing ${}_a\boldsymbol{m}$, necessary to satisfy the entropy inequality.

3. In the CBL presented here for NCCM with internal rotation rates and rotational inertial effects, BAM balance law is not just a relationship between the gradients of the Cauchy moment tensor and the skew symmetric Cauchy stress tensor, but contains additional rotational inertial effects.

4. Constitutive variables are established using SLT (in conjunction with other balance laws) and their argument tensors are determined using the conjugate pairs in the entropy inequality as well as the principle of equipresence.

5. It is shown that for micropolar solids, constitutive theories are needed only for ${}_s\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$ and $\boldsymbol{q}$ and based on Surana et. al [97] and there can not be a constitutive theory for ${}_a\boldsymbol{m}$. Thus, $\text{tr}([{}_am][{}^{i\Theta}_{\;a}\dot{J}]) = 0$ must serve as a constraint equation to unconditionally satisfy the entropy inequality. This is an additional equation in the mathematical model.

6. It is shown that for micropolar fluids, constitutive theories are needed only for ${}^{(0)}_{\;e}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}_{\;d}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}_{\;s}\bar{\boldsymbol{m}}$ and $\bar{\boldsymbol{q}}$. Based on Surana et. al [82] there cannot be a constitutive theory for ${}^{(0)}_{\;a}\bar{\boldsymbol{m}}$, thus, ${}^{(0)}_{\;a}\bar{\boldsymbol{m}} : {}^{r\Theta}_{i\;a}\dot{\bar{\boldsymbol{J}}} = 0$ must serve as a constraint equation in the mathematical model consisting of

CBL and constitutive theories to satisfy the entropy inequality for all arbitrary but admissible ${}^{r\Theta}_{i\,a}\bar{\boldsymbol{J}}$.

7. Constitutive theories for ${}_s\boldsymbol{\sigma}$ and ${}_s\boldsymbol{m}$ can be derived using representation theorem if we assume isothermal physics and equilibrium stress to be zero. These constitutive theories will be the same as those derived using representation theorem (in the absence of dissipation). The generators in the two approaches remain same, but the material coefficients do not have dependence on the same invariants. When the constitutive theories for ${}_s\boldsymbol{\sigma}$ and ${}_s\boldsymbol{m}$ are derived using Φ, the material coefficients can be functions of the invariants of $\boldsymbol{\varepsilon}$ and ${}^{i\Theta}_{s}\boldsymbol{J}$ as well as θ in a known configuration. When using representation theorem, the material coefficients for ${}_s\boldsymbol{\sigma}$ are only a function of the invariants of $\boldsymbol{\varepsilon}$ and θ in a known configuration, likewise the material coefficients for ${}_s\boldsymbol{m}$ are only a function of the invariants of ${}^{i\Theta}_{s}\boldsymbol{J}$ and θ in a known configuration.

8. Constitutive theory for ${}^{(0)}_{e}\bar{\boldsymbol{\sigma}}$, the equilibrium Cauchy stress tensor is derived using Helmholtz free energy density $\bar{\Phi}$ for compressible thermoviscous fluent continua. The constitutive theories for ${}^{(0)}_{d}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}_{s}\bar{\boldsymbol{m}}$ are derived using representation theorem.

9. Constitutive theory for $\boldsymbol{q}$ or $\bar{\boldsymbol{q}}$ is derived using representation theorem. It is shown that the constitutive theory for $\boldsymbol{q}$ or $\bar{\boldsymbol{q}}$ based on integrity is cubic in the temperature gradients.

10. BAM when compared with BLM containing the physics of translational waves clearly shows the existence of rotational waves. While translational waves are Cauchy stress waves, the rotational waves are Cauchy moment waves.

11. Since the internal rotations are due to ${}^{d}_{a}\boldsymbol{J}$ in ${}^{d}\boldsymbol{J} = {}^{d}_{s}\boldsymbol{J} + {}^{d}_{a}\boldsymbol{J}$. The stress waves are due to physics in ${}^{d}_{s}\boldsymbol{J}$ and the rotational waves are due to physics in ${}^{d}_{a}\boldsymbol{J}$, thus these two waves coexist. But, the rotational waves can not exist without the translational waves as ${}^{d}_{s}\boldsymbol{J}$ has to be present for ${}^{d}_{a}\boldsymbol{J}$ to exist.

12. The mathematical model presented in this work based on the conservation and balance laws

of NCCM incorporating internal rotations due to ${}_a^d\boldsymbol{J}$, their rates and rotational inertial physics of continua has closure.

13. We observe that since ${}^d\boldsymbol{J}$ exists in all deforming solid continua, ${}_s^d\boldsymbol{J}$ and ${}_a^d\boldsymbol{J}$ also exist in all deforming solid continua. Thus, existence of rotational waves is natural if deforming solid continua offer rotational inertial resistance.

14. Unlike micropolar solid continua, micropolar NCCT for fluids shows that in this case neither translational nor rotational waves can exist due to absence of elasticity or stiffness due to absence of strain physics in both CCT as well as NCCT. Thus, in fluent continua only pressure waves can be realized.

15. It is shown that NCCM with internal rotation rate physics also results in rate of entropy production due to ${}_s^{(0)}\bar{\boldsymbol{m}} : {}_i^r{}_s^{\Theta}\dot{\bar{\boldsymbol{J}}}$ that differs in the absence and presence of rotational inertial effects. We also have rate of entropy production due to ${}_{ds}^{(0)}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}}$. Both mechanisms of entropy production exist in compressible as well as incompressible fluent continua. In high pressure, high temperature compressible flow physics (with or without shocks) accurate determination of rate of entropy production is important as it controls shock formation, shock structure and shock relations (in general, isolated high gradient physics of dependent variables).

16. The NCCM work proposed here with internal rotation rates and rotational inertial physics may be a more realistic approach to describing the flow physics at high pressures and high temperatures that may result in severe change in state of matter that is critically influenced by the rate of entropy production.

17. We have considered the mathematical models of chapters 2 and 3 with rotational inertial physics to construct model problems (IVPs) and present their solutions using space-time coupled finite element method. The model problems are intentionally kept simple so that significant aspects of the micropolar physics due to microconstituents and in particular in-

fluence of rotational inertial physics can be demonstrated clearly.

18. The solutions of the model problems (IVPs) are obtained by using space-time coupled finite element method based on space-time residual functional for a space-time strip with time marching. The space-time local approximation are p-version hierarchical in space and time with higher order global differentiability in space and time. In this approach, with the choice of minimally confirming spaces, when the space-time residual functional for a space-time strip is $O(10^{-8})$ or lower, PDEs in the IVPs are satisfied accurately, hence the computed evolutions are almost time accurate.

19. In the present work, we consider micropolar NCCT with dissipation mechanisms due to : strain rate (CCM) and due to rate of symmetric part of internal rotational gradient tensor (micropolar non-classical dissipation). While the strain rate dissipation (CCT) is viscous, the nonviscous dissipation mechanisms in micropolar NCCT is due to microconstituents and the fluid medium.

20. From the one dimensional numerical studies presented for translational and rotational waves in micropolar solid medium using Models TW1, RW1 and RW2 we observe:

    (a) Existence and propagation of translational wave (CCM) in the absence and in the presence of strain rate dissipation. The translational wave propagates, reflects, the reflected wave propagates and reflects from the free boundary without amplitude decay and base elongation when the medium is inviscid. The same phenomenon exists in the presence of damping but with continued amplitude decay and base elongation during evolution. Amplitude decay is most pronounced for the incident wave where as base elongation is more prominent in the reflected waves.

    (b) In the case of Model RW1, in which the rotational wave speed is one, the evolution of rotational wave physics is exactly same as in Model TW1.

    (c) Model RW2 permits the choice of wave speed. Studies for wave speed faster than one and slower than one with and without dissipation are presented. In 2D and 3D applications choice of t_0 using v_0/L_0 or ${}_i\omega_0/L_0$ permits translational wave speed of one or rotational wave speed of one, but both wave speeds cannot be one. In this study, we show that wave speed different than one only influences when the propagating wave reaches the boundaries. The studies conclusively demonstrate that rotational inertial physics in micropolar solids is essential for the existence of rotational waves in micropolar solids.

(d) In the studies presented here, rotational and translational waves are decoupled but coexist. In $\mathbf{R}^2$ and $\mathbf{R}^3$ this is not the case. Since translational waves depend upon gradients of displacements in $[{}_s^dJ]$ and the rotational waves depend upon the gradients of displacements in $[{}_a^dJ]$, both $[{}_s^dJ]$ and $[{}_a^dJ]$ being due to $[J]$, we expect coupling in the two in $\mathbf{R}^2$ and $\mathbf{R}^3$.

(e) Both micropolar and non-micropolar media are assumed isotropic and homogeneous, thus this micropolar theory can not simulate wave dispersion in micropolar media due to microconstituents. This theory can only simulate the influence of micropolar physics on wave propagation physics without dispersion.

21. In the case of micropolar fluid, neither translational nor rotational waves exist. The 1D form of BAM in this case is a time dependent diffusion equation. Two numerical studies are presented for two values of diffusion coefficient. From the studies, we clearly see higher values of the diffusion coefficient result in faster diffusion of the applied pulse.

22. Numerical studies presented for pressure driven developing flow between parallel plates show:

(a) Micropolar physics with increasing $\bar{\alpha}$ $(\bar{\beta} = 0)$ results in progressively increasing resistance to flow and progressively decreasing flow rate. This holds during the entire evolution for each value of time.

(b) We have shown that $\bar{\beta}$, which controls rotational inertial physics, also offers resistance to flow. Increasing values of $\bar{\beta}$ results in decreasing flow rate.

(c) Thus, in micropolar fluids with both $\bar{\alpha}$ and $\bar{\beta}$, the resulting flow rate is decreased. Since the constitutive model is not calibrated their relative influence on the flow rate is difficult to ascertain. Based on the values of $\bar{\alpha}$ and $\bar{\beta}$ used, $\bar{\beta}$ influences flow rate more than $\bar{\alpha}$. Both $\bar{\alpha}$ and $\bar{\beta}$ offer resistance to flow but the physics of the resistance mechanisms is different in the two cases.

(d) The significance of these studies is that we can conclusively see that rotational inertial physics in micropolar non-classical continuum theories for fluids offers further resistance to the motion of the fluids over and beyond rotation rates or rotation rate gradients, thus reducing velocities and flow rates.

23. Numerical studies presented for Couette flow confirms the findings reported in item (9).

24. Presence of rotational inertial physics in fluids only offers added resistance to fluid motion. It can not possibly result in rotational waves as micropolar fluids have no elasticity associated with micropolar physics. In the case of micropolar fluids, BAM is not a wave equation in rotation rate.

25. Evolution of ${}_i\bar{\omega}_3$ versus $\bar{y}$ and ${}_s\bar{m}_{23}$ versus $\bar{y}$ (figures 10(a) and (b)) for Model A (CCM) and their comparison with similar evolution when $\bar{\alpha} = 0.001$ ($\bar{\beta} = 0.0$) i.e., model B shown in figures 10(c) and (d) and those for $\bar{\alpha} = 0.001$ and $\bar{\beta} = 0.9$ in figures 14(a) and (b) clearly demonstrate how the free field ${}_i\bar{\omega}_3$ and zero ${}_s\bar{m}_{23}$ in figures 10(a) and (b) are effected by the presence of microconstituents without and with rotational inertial physics. These studies show that increasing $\bar{\alpha}$ ($\bar{\beta} = 0.0$) reduces ${}_i\bar{\omega}_3$ field due to resistance offered by the microconstituents. When both $\bar{\alpha}$ and $\bar{\beta}$ are nonzero, resistance offered by the microconstituents increases resulting in further reduction in ${}_i\bar{\omega}_3$ field. The consequences of reducing ${}_i\bar{\omega}_3$ field is increase in ${}_s\bar{m}_{23}$ as shown in the graphs.

Appendix A

Notation and Measures in Lagrangian and Eulerian Description

Notations

In the following we present notations used in this dissertation, some of which are different from what is commonly used in continuum mechanics, and provide details on the measures of rotations, rotation rates and their gradients for solid continua in Lagrangian description and Eulerian description for fluids. We only consider internal rotations, their rates and gradients at a material point, i.e., Cosserat rotations are not considered.

Following reference [97] quantities with an over-bar are quantities in the current (deformed) configuration, i.e., all quantities with over-bar are functions of coordinates $\bar{x}_i$ and time t (Eulerian description). Quantities without an over-bar are quantities referring to the reference configuration, i.e., these are functions of undeformed coordinates x_i and time t (Lagrangian description). Thus, x_i and $\bar{x}_i$ are coordinates of the same material point in the reference and current configurations, respectively, both measured in a fixed Cartesian x-frame. The configuration at time $t = t_0 = 0$, commencement of evolution, is considered as the reference configuration. Thus, x_i and $\bar{x}_i$ are coordinates of the same material point in the reference and current configurations, respectively, both measured in a fixed Cartesian x-frame. Further $\boldsymbol{x}$, $\boldsymbol{A}$, V, $\partial\boldsymbol{A}$, ∂V refer to material point coordinates (in a fixed Cartesian frame), area, volume, boundary of $\boldsymbol{A}$ and the surface bounding V, all in the reference or undeformed configuration, whereas $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{A}}$, $\bar{V}$, $\partial\bar{\boldsymbol{A}}$, $\partial\bar{V}$ are their counterparts in the current configuration. $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{x}, t)$ and $\bar{\boldsymbol{Q}} = \bar{\boldsymbol{Q}}(\bar{\boldsymbol{x}}, t)$ are Lagrangian and Eulerian descriptions of a quantity $\boldsymbol{Q}$ at a material point $\boldsymbol{x}$ in the reference configuration with its corresponding location $\bar{\boldsymbol{x}}$ in the current configuration.

Deformation Gradient, Internal Rotation and Rotation gradient Tensors

Consider the deformation gradient tensor or the Jacobian of deformation defined by $\boldsymbol{J} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial \bar{x}_j}{\partial x_i}$. The rows are the covariant base vectors, whereas in Murnaghan's notation $[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = \left[\frac{\bar{x}_1,\bar{x}_2,\bar{x}_3}{x_1,x_2,x_3}\right]$, the columns are the covariant base vectors, i.e., in this definition $[J]$ is the transpose of $\boldsymbol{J}$ in the first definition. Both definitions are obviously covariant measures in the Lagrangian description. Likewise, $\bar{\boldsymbol{J}} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial x_j}{\partial \bar{x}_i}$ and $[\bar{J}] = \left[\frac{\partial\{x\}}{\partial\{\bar{x}\}}\right] = \left[\frac{x_1,x_2,x_3}{\bar{x}_1,\bar{x}_2,\bar{x}_3}\right]$ are also deformation gradient tensors or Jacobians of deformation but they are a contravariant measure in the Eulerian description. Columns of $\bar{\boldsymbol{J}}$ are the contravariant base vectors whereas in case of $[\bar{J}]$ its rows are the contravariant base vectors, i.e., $\bar{\boldsymbol{J}}$ is transpose of $[\bar{J}]$. In this paper we only use referential description, hence we only need to consider $\boldsymbol{J}$ or $[J]$. In the remaining paper, we consider $[J]$ and $[\bar{J}]$ notation due to Murnaghan. Since the work presented in this paper only considers small strain and small deformation, the distinction between covariant and contravariant measures disappears as $\bar{x}_i \simeq x_i$, i.e., the deformed configuration is not substantially different from the undeformed configuration. For both deformation measures, det$[J]$ = det$[\bar{J}] \cong 1$.

The displacement gradient tensor is defined as

$$[{}^{d}J] = \frac{\partial\{u\}}{\partial\{x\}} = \left[\frac{u_1,u_2,u_3}{x_1,x_2,x_3}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{6.1}$$

The Cauchy stress tensor is used as a measure of stress because the deformed and undeformed tetrahedron can be treated the same for small deformation. Since $\bar{x}_i \simeq x_i$, the conservation and balance laws only need to consider $[{}^{d}J]$ instead of $[J]$. Consideration of $[{}^{d}J]$ in its entirety implies that $[{}^{d}_{s}J]$ and $[{}^{d}_{a}J]$ both are to be considered in the conservation and balance laws. The displacement gradient $[{}^dJ]$ can be written in component form as

$$ {}^dJ_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = {}^d_sJ_{ij} + {}^d_aJ_{ij} \tag{6.2} $$

in which

$$ [{}^d_aJ] = \frac{1}{2}\begin{bmatrix} 0 & -{}_i\Theta_{x3} & {}_i\Theta_{x2} \\ {}_i\Theta_{x3} & 0 & -{}_i\Theta_{x1} \\ -{}_i\Theta_{x2} & {}_i\Theta_{x1} & 0 \end{bmatrix} \tag{6.3} $$

Rotations ${}_i\Theta_{x1}$, ${}_i\Theta_{x2}$, ${}_i\Theta_{x3}$ are referred to as internal rotations at a material point and are defined by

$$ \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_i \times \boldsymbol{e}_j \frac{\partial u_j}{\partial x_i} = \epsilon_{ijk}\, \boldsymbol{e}_k\, \frac{\partial u_j}{\partial x_i} \tag{6.4} $$

$$ \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \boldsymbol{e}_2 \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \boldsymbol{e}_3 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{6.5} $$

$$ \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1 \left({}_i\Theta_{x1}\right) + \boldsymbol{e}_2 \left({}_i\Theta_{x2}\right) + \boldsymbol{e}_3 \left({}_i\Theta_{x3}\right) \tag{6.6} $$

The rotations ${}_i\Theta_{x1}$, ${}_i\Theta_{x2}$, ${}_i\Theta_{x3}$ in (6.6) are in counterclockwise sense and are assumed positive.

On the other hand using polar decomposition ($[{}^dJ] = [{}^dR][{}^dS_r] = [{}^dS_l][{}^dR]$) we can obtain the right and the left stretch tensors $[{}^dS_r]$ and $[{}^dS_l]$ that are symmetric and positive-definite and $[{}^dR]$ that is an orthogonal rotation tensor corresponding to the rotation angles defined in (6.5). Both $[{}^d_aJ]$ and $[{}^dR]$ contain the same physics as both are derived from $[{}^dJ]$ but in different forms. The terms of $[{}^d_aJ]$ are rotation angles while the rotation tensor $[{}^dR]$ is the corresponding rotation matrix or tensor. Both can be used in derivations as needed in their given form. The same holds true for $[{}^dR]$ and $[{}^d_aJ]$ derived from $[J]$. However, in general, deriving $[{}^dR]$ from $[{}^d_aJ]$ or $[{}^d_aJ]$ from $[{}^dR]$ in $\mathbb{R}^3$ may not be possible or unique. Fortunately, there is no need for this here. Incorporating $[{}^dJ]$ in its entirety in the derivation of conservation and balance laws implies incorporating both $[{}^d_sJ]$ and $[{}^d_aJ]$, i.e., rotations ${}_i\Theta_{x1}$, ${}_i\Theta_{x2}$, and ${}_i\Theta_{x3}$ about the axes of a triad located at each material point. Rotations in

$[{}^d_aJ]$ are referred to as internal rotations as they naturally arise within deforming solid continua due to $[{}^d_aJ]$ and are completely defined by the skew-symmetric part of $[{}^dJ]$ or $\boldsymbol{\nabla} \times \boldsymbol{u}$ in (6.5) and (6.6).

First and second material derivatives of the rotations $\{{}_i\Theta\}$ are given by

$$\frac{D}{Dt}\{{}_i\Theta\} = \{{}_i\dot{\Theta}\} = \{{}_i\omega\} \qquad ; \qquad \frac{D^2}{Dt^2}\{{}_i\Theta\} = \{{}_i\ddot{\Theta}\} = \{{}_ia\} \tag{6.7}$$

in which $\{{}_i\omega\}$ and $\{{}_ia\}$ are angular velocities and angular accelerations about the axes of a triad at each material point associated with internal rotations (hence the back subscript i). These are new quantities and variables introduced in this work that are not used in classical continuum theories. The axes of the triad at each material point are parallel to the fixed x-frame. If we let

$$\{{}_i\Theta\}^T = [{}_i\Theta_{x1}, {}_i\Theta_{x2}, {}_i\Theta_{x3}] \tag{6.8}$$

then, the gradients of $\{{}_i\Theta\}$ can be defined by

$$[{}^{i\Theta}J] = [\frac{\partial\{{}_i\Theta\}}{\partial\{x\}}] \qquad \text{or} \qquad {}^{i\Theta}J_{jk} = \frac{\partial({}_i\Theta_j)}{\partial x_k} \tag{6.9}$$

The rotation gradient tensor $[{}^{i\Theta}J]$ can be decomposed into symmetric and skew symmetric tensors $[{}^{i\Theta}_sJ]$ and $[{}^{i\Theta}_aJ]$.

$$\begin{aligned} [{}^{i\Theta}J] &= [{}^{i\Theta}_sJ] + [{}^{i\Theta}_aJ] \\ [{}^{i\Theta}_sJ] &= \frac{1}{2}([{}^{i\Theta}J] + [{}^{i\Theta}J]^T) \\ [{}^{i\Theta}_aJ] &= \frac{1}{2}([{}^{i\Theta}J] - [{}^{i\Theta}J]^T) \end{aligned} \tag{6.10}$$

It is well known that the rotation gradient tensor is a tensor of rank three, but a vector representation of rotations in (6.9) is helpful as it results in $[{}^{i\Theta}J]$ of rank two. When the displacement gradient tensor varies between neighboring material points, so do internal rotations $[{}^d_aJ]$ and their rates $[{}^d_a\dot{J}]$ and $[{}^d_a\ddot{J}]$. When $[{}^d_aJ]$, $[{}^d_a\dot{J}]$ and $[{}^d_a\ddot{J}]$ are resisted by deforming solid continua, then moments, angular momenta and angular inertial effects result as a consequence. Thus, on the oblique plane of the tetrahedron (defining a part of the bounding surface of the volume) or in general on the boundary of the deforming volume a resultant moment tensor can exist.

Internal rotation rates and their gradients

The velocities $\bar{\boldsymbol{v}}$ and the velocity gradients $(\bar{L}_{ij} = \frac{\partial \bar{v}_i}{\partial \bar{x}_j})$ are fundamental measures of deformation physics in fluent continua in Eulerian description, hence these in their entirety must form a basis for a complete thermodynamic framework. Decomposition of $\bar{\boldsymbol{L}}$ into symmetric tensor $\bar{\boldsymbol{D}}$ and skew-symmetric tensor $\bar{\boldsymbol{W}}$. The physics of $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{W}}$ exists in all deforming fluent continua. The currently used thermodynamic framework (classical continuum mechanics, CCM) only considers $\bar{\boldsymbol{D}}$. Hence, $\bar{\boldsymbol{W}}$ containing internal rotation rates is not considered at all. Incorporating entirety of $\bar{\boldsymbol{L}}$ in the conservation and balance laws implies that we incorporate the additional physics due to internal rotation rates in the existing thermodynamic framework for fluent continua as the physics due to $\bar{\boldsymbol{D}}$ is already present in CCM. The internal rotation rates can be visualized as the rotation rates about the axes of a triad located at a material point (a location) whose axes are parallel to the axes of the fixed Cartesian x-frame. The velocity gradient tensor $[\bar{L}]$ can be decomposed into pure rotation rate tensor $[{}^t\bar{R}]$ and the right and left stretch rates $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$. Then, $[{}^t\bar{R}]$ is orthogonal and $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$ are symmetric and positive-definite.

$$[\bar{L}] = [{}^t\bar{R}][{}^t\bar{S}_r] = [{}^t\bar{S}_l][{}^t\bar{R}] \tag{6.11}$$

Let $({}^t\lambda_i, \{\phi\}_i)$; $i = 1, 2, 3$ be the eigenpairs of $[\bar{L}]^T[\bar{L}]$ in which $\{\phi\}_i^T\{\phi\}_j = \delta_{ij}$, then

$$[\bar{L}]^T[\bar{L}] = [\bar{\Phi}][{}^t\bar{\lambda}][\bar{\Phi}]^T = [{}^t\bar{S}_r]^2 \tag{6.12}$$

The columns of $[\bar{\Phi}]$ are eigenvectors of $\{\phi\}_i$ and $[{}^t\bar{\lambda}]$ is a diagonal matrix of the eigenvalues ${}^t\lambda_i$; $i = 1, 2, 3$. If we choose

$$[{}^t\bar{S}_r] = [\bar{\Phi}]\left[\sqrt{{}^t\bar{\lambda}}\right][\bar{\Phi}]^T \tag{6.13}$$

then (6.12) holds, hence definition of $[{}^t\bar{S}_r]$ in (6.13) is valid. $[{}^t\bar{R}]$ can now be defined using (6.11).

$$[{}^t\bar{R}] = [\bar{L}][{}^t\bar{S}_r]^{-1} \tag{6.14}$$

Furthermore, using

$$[\bar{L}][\bar{L}]^T = [{}^t\bar{S}_l]^2 \tag{6.15}$$

and following a similar procedure we can establish

$$[{}^t\bar{S}_l] = [\bar{\Phi}]\left[\sqrt{{}^t\bar{\lambda}}\right][\bar{\Phi}]^T \tag{6.16}$$

$$[{}^t\bar{R}] = [{}^t\bar{S}_l]^{-1}[\bar{L}] \tag{6.17}$$

where $[{}^t\bar{R}]$ defined by (6.14) and (6.17) is unique. We note that in this approach $[{}^t\bar{R}]$ is a rotation rate transformation matrix, hence does not contain rotation angle rates. Alternatively, we can consider decomposition of $[\bar{L}]$ into symmetric ($[\bar{D}]$) and skew-symmetric ($[\bar{W}]$) tensors.

$$[\bar{L}] = \left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right] = [\bar{D}] + [\bar{W}] \tag{6.18}$$

$$[\bar{D}] = \frac{1}{2}([\bar{L}] + [\bar{L}]^T); \qquad [\bar{W}] = \frac{1}{2}([\bar{L}] - [\bar{L}]^T \tag{6.19}$$

or

$$\bar{D}_{ij} = \frac{1}{2}(\bar{v}_{i,j} + \bar{v}_{j,i}); \qquad \bar{W}_{ij} = \frac{1}{2}(\bar{v}_{i,j} - \bar{v}_{j,i}) \tag{6.20}$$

We define positive rotation rates ${}_i^r\bar{\Theta}$ using

$$\bar{\boldsymbol{\nabla}} \times \bar{\boldsymbol{v}} = \boldsymbol{e_i} \times \boldsymbol{e_j}\frac{\partial \bar{v}_j}{\partial \bar{x}_i} = \epsilon_{ijk}\boldsymbol{e_k}\frac{\partial \bar{v}_j}{\partial \bar{x}_i} \tag{6.21}$$

or

$$\bar{\boldsymbol{\nabla}} \times \bar{\boldsymbol{v}} = \boldsymbol{e_1}\left(\frac{\partial \bar{v}_3}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_3}\right) + \boldsymbol{e_2}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_1}\right) + \boldsymbol{e_3}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_1} - \frac{\partial \bar{v}_1}{\partial \bar{x}_2}\right) \tag{6.22}$$

or

$$\bar{\boldsymbol{\nabla}} \times \bar{\boldsymbol{v}} = \boldsymbol{e_1}\left({}_i^r\bar{\Theta}_{x_1}\right) + \boldsymbol{e_2}\left({}_i^r\bar{\Theta}_{x_2}\right) + \boldsymbol{e_3}\left({}_i^r\bar{\Theta}_{x_3}\right) \tag{6.23}$$

We note that

$$\left\{{}_i^r\bar{\Theta}\right\} = \{{}_i\bar{\omega}\} \quad \text{and} \quad \frac{D}{Dt}({}_i\bar{\omega}) = {}_i\dot{\bar{\omega}} = {}_i^r\dot{\bar{\Theta}} = \{{}_i\bar{a}\}$$

in which ${}_i\bar{\omega}$ are angular velocities and ${}_i\bar{a}$ are angular accelerations using the rotation rates in (6.23). We can write the expanded form of $[\bar{W}]$

$$[\bar{W}] = \begin{bmatrix} 0 & -\frac{1}{2}({}_i^r\Theta_{x_3}) & \frac{1}{2}({}_i^r\Theta_{x_2}) \\ \frac{1}{2}({}_i^r\Theta_{x_3}) & 0 & -\frac{1}{2}({}_i^r\Theta_{x_1}) \\ -\frac{1}{2}({}_i^r\Theta_{x_2}) & \frac{1}{2}({}_i^r\Theta_{x_1}) & 0 \end{bmatrix} \tag{6.24}$$

where ${}_i^r\Theta_{x_1}, {}_i^r\Theta_{x_2}, {}_i^r\Theta_{x_3}$ are rotation rates related to total 90 degree angle and are positive counter-clockwise and $[\bar{W}]$ contains half of the total rotation rate, i.e., related to half of the rate of change of 90 degree angle. Its obvious that $\bar{\boldsymbol{W}}$ is a tensor of rank two, whereas the rotation rates defined in (6.23) are clearly a tensor of rank one. In other words, rotation rates in (6.23) constitute a tensor of rank one, but the components of this tensor arranged in the form in which they appear in $[\bar{W}]$ constitute a tensor of rank two. We determine gradients of the rotation rate tensor (6.23). Let

$$\{{}_i^r\bar{\Theta}\}^T = [{}_i^r\Theta_{x_1}, {}_i^r\Theta_{x_2}, {}_i^r\Theta_{x_3}] \tag{6.25}$$

be a vector representation of (6.23), then the gradient of ${}_i^r\Theta$ can be defined by

$$\left[{}_i^{r\Theta}\bar{J}\right] = \left[\frac{\partial\{{}_i^r\Theta\}}{\partial\{\bar{x}\}}\right] \quad \text{or} \quad {}_i^{r\Theta}J_{jk} = \frac{\partial\left({}_i^r\Theta_j\right)}{\partial\bar{x}_k} \tag{6.26}$$

The gradient tensor $\left[{}_i^{r\Theta}\bar{J}\right]$ of the internal rotation rates defined by (6.26) can be decomposed into symmetric and antisymmetric tensors $\left[{}_{i\ s}^{r\Theta}\bar{J}\right]$ and $\left[{}_{i\ a}^{r\Theta}\bar{J}\right]$.

$$\left[{}_i^{r\Theta}\bar{J}\right] = \left[{}_{i\ s}^{r\Theta}\bar{J}\right] + \left[{}_{i\ a}^{r\Theta}\bar{J}\right] \tag{6.27}$$

$$
\begin{aligned}
\left[{}^{r\Theta}_{i\ s}\bar{J}\right] &= \frac{1}{2}\left(\left[{}^{r\Theta}_{i}\bar{J}\right] + \left[{}^{r\Theta}_{i}\bar{J}\right]^T\right) \\
\left[{}^{r\Theta}_{i\ a}\bar{J}\right] &= \frac{1}{2}\left(\left[{}^{r\Theta}_{i}\bar{J}\right] - \left[{}^{r\Theta}_{i}\bar{J}\right]^T\right)
\end{aligned}
\tag{6.28}
$$

when the velocity gradient tensor varies between the neighboring material points so do the internal rotation rates ${}^{r}_{i}\boldsymbol{\Theta}$ (or $[\bar{W}]$), their rates as well as their gradients and their rates. Varying ${}^{r}_{i}\boldsymbol{\Theta}$ and ${}^{r\Theta}_{i}\bar{\boldsymbol{J}}$, when resisted by deforming fluent continua, results in moments, angular momenta and angular inertial effects as a consequence. Thus, on the oblique plane of the tetrahedron defining part of $\partial\bar{V}(t)$ or defining a part of the bounding surface due to cut principle of Cauchy, resultant moment can exist.

Stress and moment tensors and choice of basis

Consider a tetrahedron in the undeformed configuration (volume V) with its oblique plane constituting a part of surface ∂V bounding V that deforms and rotates in the current configuration. Equilibrium considerations associated with conservation and balance laws require measures of stress, strain rates, etc. associated with the deformed tetrahedron. The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in the cut principle of Cauchy. Consider a tetrahedron T_1 such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other parallel to the planes of the x-frame. Upon deformation, $\underset{\sim}{V}$ and $\partial\underset{\sim}{V}$ occupy $\bar{\underset{\sim}{V}}$ and $\partial\bar{\underset{\sim}{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial\bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation) are non-orthogonal covariant base vectors $\tilde{g}_i$. The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that the assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial\bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on surface $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrium with the mating faces in volume $\bar{V}$ when the tetrahedron $\bar{T}_1$ is placed back in the volume $\bar{V}$.

Consider the deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress per unit area on plane

$\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area on plane $\bar{A}\bar{B}\bar{C}$ (henceforth referred to as moment for short), and $\bar{\boldsymbol{n}}$ be the unit exterior normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{n}}$ all have different directions when the deformation is finite. The edges of the deformation tetrahedron are covariant base vectors $\tilde{\boldsymbol{g}}_i$ that are tangent to the deformed curvilinear material lines.

$$\tilde{\boldsymbol{g}}_i = \boldsymbol{e}_k \frac{\partial \bar{x}_k}{\partial x_i} \quad ; \quad J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j} \tag{6.29}$$

Columns of $\boldsymbol{J}$ are covariant base vectors $\tilde{\boldsymbol{g}}_i$ that form non-orthogonal covariant basis. Contravariant base vectors of $\tilde{\boldsymbol{g}}^j$ are normal to the faces of the tetrahedron formed by the covariant base vectors

$$\tilde{\boldsymbol{g}}^j = \boldsymbol{e}_l \frac{\partial x_j}{\partial \bar{x}_l} \quad ; \quad \bar{J}_{ij} = \frac{\partial x_i}{\partial \bar{x}_j} \tag{6.30}$$

The rows of $\bar{\boldsymbol{J}}$ are contravariant base vectors $\tilde{\boldsymbol{g}}^j$. These form a non-orthogonal contravariant basis. Covariant and contravariant bases are reciprocal to each other [97]. If $\bar{\underset{\sim}{\boldsymbol{\sigma}}}^{(0)}$ or $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ is the contravariant stress tensor with components $\bar{\underset{\sim}{\sigma}}_{ij}^{(0)}$ or $\underset{\sim}{\sigma}_{ij}^{(0)}$ with dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$, then using dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$ or contravariant laws of transformation we can define contravariant Cauchy stress tensors $\boldsymbol{\sigma}^{(0)}$ in Lagrangian description

$$\boldsymbol{\sigma}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \, \underset{\sim}{\sigma}_{ij}^{(0)} \tag{6.31}$$

using (6.29)-(6.31), we can write

$$\boldsymbol{\sigma}^{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \, \sigma_{ij}^{(0)} \quad ; \quad \sigma_{ij}^{(0)} = J_{ik} (\underset{\sim}{\sigma}_{kl}^{(0)}) J_{jl} \tag{6.32}$$

or

$$\left[\sigma^{(0)}\right] = [J] \left[\underset{\sim}{\sigma}^{(0)}\right] [J]^T \tag{6.33}$$

where $\bar{\boldsymbol{\sigma}}^{(0)}$ is Eulerian description of $\boldsymbol{\sigma}^{(0)}$ which is obtained from (6.33) by replacing $[J]$ with $\left[\bar{J}\right]^{-1}$ and $\boldsymbol{\sigma}^{(0)}$ with $\bar{\boldsymbol{\sigma}}^{(0)}$. Since dyads of $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\boldsymbol{\sigma}^{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}^{(0)}$.

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{6.34}$$

Similarly we can define covariant Cauchy stress tensors $\boldsymbol{\sigma}_{(0)}$ or $\bar{\boldsymbol{\sigma}}_{(0)}$ and Cauchy principle between $\bar{\boldsymbol{\sigma}}_{(0)}$ and $\bar{\boldsymbol{P}}$.

$$\bar{\boldsymbol{\sigma}}_{(0)} = \tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j \left(\underset{\sim}{\sigma}_{(0)}\right)_{ij} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\bar{\sigma}_{(0)}\right)_{ij} \quad ; \quad \left(\bar{\sigma}_{(0)}\right)_{ij} = \bar{J}_{ki} \left(\underset{\sim}{\sigma}_{(0)}\right)_{kl} \bar{J}_{lj} \tag{6.35}$$

or

$$\left[\bar{\sigma}_{(0)}\right] = \left[\bar{J}\right]^T \left[\underset{\sim}{\sigma}_{(0)}\right] \left[\bar{J}\right] \tag{6.36}$$

and

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}_{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{6.37}$$

We define the contravariant and covariant Cauchy moment tensor in similar fashion and the corresponding Cauchy principle

$$\boldsymbol{m}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \, \underset{\sim}{m}_{ij}^{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \, m_{ij}^{(0)} \quad ; \quad m_{ij}^{(0)} = J_{ik} (\underset{\sim}{m}_{kl}^{(0)}) J_{jl} \tag{6.38}$$

or

$$\left[m^{(0)}\right] = [J] \left[\underset{\sim}{m}^{(0)}\right] [J]^T \quad ; \quad \left[\bar{m}^{(0)}\right] = \left[\bar{J}\right]^{-1} \left[\underset{\sim}{m}^{(0)}\right] \left[\left[\bar{J}\right]^{-1}\right]^T \tag{6.39}$$

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}^{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{6.40}$$

and

$$\bar{\boldsymbol{m}}_{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \, \left(\underset{\sim}{m}_{(0)}\right)_{ij} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \, \left(\bar{m}_{(0)}\right)_{ij} ; \left(m_{(0)}\right)_{ij} = \bar{J}_{ki} \left(\underset{\sim}{m}_{(0)}\right)_{kl} \bar{J}_{lj} \tag{6.41}$$

or

$$\left[\bar{m}_{(0)}\right] = \left[\bar{J}\right]^T \left[\underset{\sim}{m}_{(0)}\right] \left[\bar{J}\right] \quad ; \quad \left[m_{(0)}\right] = \left[[J]^{-1}\right]^T \left[\underset{\sim}{m}_{(0)}\right] [J]^{-1} \tag{6.42}$$

and

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}_{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{6.43}$$

At this state $\bar{\boldsymbol{\sigma}}^{(0)}$, $\boldsymbol{\sigma}^{(0)}$, $\bar{\boldsymbol{\sigma}}_{(0)}$, $\boldsymbol{\sigma}_{(0)}$, $\boldsymbol{m}^{(0)}$, $\bar{\boldsymbol{m}}^{(0)}$, $\bar{\boldsymbol{m}}_{(0)}$, and $\boldsymbol{m}_{(0)}$ are all nonsymmetric tensors of rank two. Thus, we note that the Cauchy stress tensors and the Cauchy moment tensors are basis dependent. It has been shown that [97] for finite strain rates the contravariant measures are meritorious. However, in deriving conservation and balance laws and the constitutive theories either measure yields a covariant mathematical model. We introduce stress measure ${}^{(0)}\bar{\boldsymbol{\sigma}}$ that could represent $\bar{\boldsymbol{\sigma}}^{(0)}$ or $\bar{\boldsymbol{\sigma}}_{(0)}$ and the moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ that could represent $\bar{\boldsymbol{m}}^{(0)}$ or $\bar{\boldsymbol{m}}_{(0)}$ depending upon our choice. We present derivation of the balance laws and constitutive theories using ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$, thus making the derivations basis independent. Basis dependent mathematical model is recoverable from the derivation by specific choice of ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$.

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