Homological Properties of Structures in Commutative Algebra and Algebraic Combinatorics

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Abstract

The purpose of this work is to understand homological properties of structures appearing in commutative algebra and algebraic combinatorics, objects such as commutative rings and associated structures, such as ideals and modules, or simplicial complexes. In particular, we study vanishing conditions for Ext and Tor in connection with homological dimensions of the modules involved, the representation theory of maximal Cohen-Macaulay modules, and various homological properties of simplicial complexes though the lens of combinatorial commutative algebra. Specifically, we study when a Cohen-Macaulay local ring has only trivial vanishings of Ext or Tor, and provide sufficient numerical criterion under which these condition are satisfied. We apply these results to establish new cases of the famous Auslander-Reiten conjecture; other conditions on Ext and Tor are also explored in connection with this conjecture. We also study the connection between classifically studied representation types of the category of maximal Cohen-Macaulay modules of a Cohen-Macaulay local ring and newly introduced representation types which study those maximal Cohen-Macaulay modules that are not locally free on the punctured spectrum. We provide a classification theorem in dimension 1, and discuss partial results and obstacles in higher dimension. We also explore combinatorial constructions such as the nerve complex of a simplicial complex, and introduce the new notion higher nerve complexes. We explore their connection with order complexes of posets, in particular the face poset of a simplicial complex, and we prove that the depth and h-vector of the Stanley-Reisner ring of a simplicial complex can be computed in a nice way from the reduced homologies of these higher nerve complexes. We expand upon our study of these notions by studying balanced simplicial complexes, and using this abstraction we prove that, while one cannot characterize which of Serre's conditions (S_ℓ) are satisfied by a simplicial complex via the reduced homologies of higher nerve complexes, one can pin it down to one of two possible values. We also provide a depth formula for arbitrary balanced simplicial complexes and consider total Euler characteristics of links; using the latter, we provide some applications to the study of Gorenstein* complexes. Finally, we introduce the notion of minimal Cohen-Macaulay simplicial complexes and provide some necessary and sufficient conditions for this property. We conclude by showing that many recently introduced counterexamples to longstanding conjectures in the literature are minimal Cohen-Macaulay.

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Contents

Chapter 1

Introduction

Commutative algebra studies commutative rings and attendant structures such as ideals and modules. It arises naturally from the study of familar mathematical objects such as polynomials and power series', and, in particular, is motivated by our innate desire to solve equations involving these objects. For instance, the set of common zeroes of a collection of polynomials forms a geometric object called a variety and understanding this variety geometrically is tantamount to understanding algebraic properties of the polynomials involved. For example, the variety defined by the polynomial $y - x^2$ is the familiar parabola $y = x^2$, which is smooth, while the variety defined by $y^2 - x^3$ has a singularity, a cusp, at the origin $(0,0)$. Commutative algebra provides a machinery for understanding such singular points; given any point P on a variety, we can attach a ring of germs of certain functions at P. This allows us to translate questions about the geometry of a variety near a point to questions about the algebraic structure of a commutative ring. In particular, this gives rise to several algebraic conditions and invariants which serve to detect and measure singularities.

In the 1970's, some of these conditions and invariants associated to commutative rings were found to have powerful connections to algebraic combinatorics, in particular to combinatorial objects such as simplicial complexes or polytopes. Indeed, given a simplicial complex Δ on the vertices $[n] := \{1, 2, \ldots, n\}$ and a field k, we can naturally associate to Δ an ideal $I_\Delta \subseteq k[x_1, \ldots, x_n]$. We can then associate to Δ the ring $k[\Delta] := k[x_1, \ldots, x_n]/I_\Delta$. The correspondence is functorial in nature, and so, in particular, Δ is isomorphic to Γ as simplicial complexes if and only if $k[\Delta]$ and $k[\Gamma]$ are isomorphic as rings. In particular, this means that any algebraic property of $k[\Delta]$ translates to a combinatorial, and sometimes even topological, property of Δ. This correspondence is frequently called the Stanley-Reisner correspondence after Stanley and Reisner's groundbreaking independent work which led to a proof of the upper bound theorem for simplicial spheres. Of particular import, Reisner's well-known criterion for Cohen-Macaulayness plays a guiding role.

1.0.1 Overview

In this section I will overview the main results of this document which can be divided into three main themes: homological properties of modules over commutative rings to which the Chapters 3 and 4 are devoted, Cohen-Macaulay representation theory to which Chapter 5 is devoted, and combinatorial commutative algebra which is represented by Chapters 6, 7, and 8. Chapter 2 provides the necessary background in commutative algebra and algebraic combinatorics. The results of this document can be largely found in the following papers written in part by the author: [HL18, DEL19, DDD$^+$19, LMn19, KLT20, DDL20].

Chapter 3 concerns work motivated by the following open conjecture in homological algebra:

Conjecture 1.0.1 (The Auslander-Reiten Conjecture [AR75])**.** Suppose $(R,\mathfrak{m},k)$ is a Noetherian local ring. If M is a finitely generated R-module such that $\operatorname{Ext}^i_R(M,M)=\operatorname{Ext}^i_R(M,R)=0$ for every $i>0$, then M is free.

Our approach in Chapter 3 is to establish new results on the Auslander-Reiten conjecture for Cohen-Macaulay (CM) local rings by studying the following stronger condition:

Definition 1.0.1. We say a ring R satisfies *trivial vanishing* if, for any finitely generated R-modules M and N, $\operatorname{Tor}^R_i(M,N)=0$ for $i\gg 0$ implies that either M or N has finite projective dimension.

This condition was considered by Huneke and Wiegand [HW97] and independently by Miller [Mil98] who prove it holds for hypersurface rings. Jorgensen later extended this result to show that any Golod ring satisfies trivial vanishing [Jor99], while Şega showed that trivial vanishing fails for complete intersections of embedding codimension at least 2 [Ş03]. Modules over Golod rings have the fastest growth of Betti numbers while those over complete intersections have the slowest [Avr10]. Motivated by these results, we study growth rates of Betti numbers in general to establish a numerical criterion for any CM local ring to satisfy trivial vanishing. In particular our condition involves 3 numerical invariants of R:

1. The embedding codimension $\operatorname{codim} R = \mu_R(\mathfrak{m}) - \dim R$.

2. The Hilbert-Samuel multiplicity $e(R) := \lim_{n\to\infty} \dfrac{(\dim R)!\, l_R(R/\mathfrak{m}^n)}{n^{\dim R}}$ of R.

3. The *generalized Loewy length* of R, i.e., the value $\ell\ell(R) := \max_J \min\{i \mid \mathfrak{m}^i \subseteq J\}$, where J ranges over the minimal reductions of $\mathfrak{m}$. This definition is designed for reduction to the Artinian case, where $\ell\ell(R)$ is nothing but the smallest i for which $\mathfrak{m}^i = 0$.

Theorem 1.0.2 (Theorem 8.3)**.** Let R be a Cohen-Macaulay (CM) local ring and set $c = \operatorname{codim} R$ and $\ell = \ell\ell(R)$. Suppose

$$e(R) < \frac{4c+2\ell-1-\sqrt{8c+4\ell-3}}{2}.$$

Then R satisfies trivial vanishing.

Applying this result, we characterize trivial vanishing for CM rings of small multiplicity or codimension [LMn19, Theorem B]. Examples of [Ş03] and [Jc04] show that these results are sharp.

As a consequence of our work on trivial vanishing, we are able to verify the Auslander-Reiten conjecture in some new cases.

Theorem 1.0.3 (Theorem 3.5.3)**.** Let R be a CM local ring. Assume R satisfies one of the following conditions.

(1) $e(R) \leq \frac{7}{4}\operatorname{codim}(R) + 1$.

(2) $e(R) \leq \operatorname{codim}(R) + 6$ and R is Gorenstein.

Then the Auslander-Reiten conjecture holds for R. In particular, the conjecture holds if $e(R) \leq 8$, or if $e(R) \leq 11$ and R is Gorenstein.

In Chapter 4 we consider other homological conditions that can be imposed upon the modules in question. For instance, Vasconcelos proved that if R is Gorenstein and M is MCM, then $\operatorname{End}_R(M)$ is free if and only if M is free [Vas68]. On the other hand, Ulrich, Hanes-Huneke, and Jorgensen-Leuschke consider modules with a large number of generators, and prove the vanishing of $\operatorname{Ext}^i_R(M,R)$ for certain values of i is enough to conclude R is Gorenstein [Ulr84, HH05, JL07]. Several results on the Auslander-Reiten conjecture also have this flavor [HL04, HcV04, GT17].

In Chapter 4, we introduce two categories with the aim of unifying and extending these results.

Definition 1.0.4. We let $\operatorname{mod} R$ denote the category of finitely-generated R-modules, and we let $\operatorname{Deep}(R)$ denote the full subcategory of $\operatorname{mod} R$ with objects $\{X \mid \operatorname{depth} X \geqslant \operatorname{depth} R\}$. We then define the following two full subcategories of $\operatorname{mod} R$.

1. $\Omega \operatorname{Deep}(R) = \{M \mid \exists\ 0 \to M \to R^n \to X \to 0$ exact for some $n \in \mathbb{N}$ and $X \in \operatorname{Deep}(R)\}$,

2. $\operatorname{DF}(R) = \{M \mid \exists\ 0 \to R \to M^n \to X \to 0$ exact for some $n \in \mathbb{N}$ and $X \in \operatorname{Deep}(R)\}$.

We use these categories to consider two general questions whose answers provide generalizations of the aforementioned results:

Question 1.0.2. When does $\operatorname{Hom}_R(M,N)$ have a free summand?"

Towards this question, we prove the following:

Theorem 1.0.5 (Theorem 4.3.4)**.** Suppose that $\operatorname{depth} M \geq t$ and $N \in \Omega \operatorname{Deep}(R)$. Assume that $\operatorname{Hom}_R(M,N) \in \operatorname{DF}(R)$ and $\operatorname{Ext}_R^{1 \leqslant i \leqslant t-1}(M,N) = 0$. Then N has a free summand.

This allows us to generalize the previously mentioned results of Vasconcelos and Huneke-Leuschke. When $\operatorname{Hom}_R(M,N)$ is actually free rather than only in $\operatorname{DF}(R)$, one can often apply Theorem 4.1.1 to conclude than N is actually free. In some situations, one can even conclude that M is free. For instance, we prove that if R and M satisfy Serre's condition (S_2) and $\operatorname{Hom}_R(M,R)$ is free, then M is free (Theorem 4.3.7).

The second question we consider is the following:

Question 1.0.3. When is $\mathrm{Hom}_R(M,N) \cong N^r$ for some r?.

For this question, we have the following main technical result:

Theorem 1.0.6 (Theorem 4.4.3)**.** Assume that $\mathrm{depth}(N) = t$, $\mathrm{depth}(M) \geqslant t$, $\mathrm{Ass}(N) = \mathrm{Min}(N)$, and for some $s \geq t$, $\mathrm{Ext}_R^{1\leqslant i\leqslant s}(M,N) = 0$. If $\mathrm{Hom}(M,N) \cong N^r$ for some $r \in \mathbb{N}$, then $M/IM \cong (R/I)^r$ for $I = \mathrm{Ann}(N)$.

Furthermore, if one of the following holds:

(1) N is faithful.

(2) $\mathrm{Ass}(R) \subseteq \mathrm{Ass}(N)$ and $s > 0$.

then $M \cong R^r$.

As a consequence of Theorem 4.1.2, we obtain a new case of the Auslander-Reiten conjecture (Corollary 4.4.7).

Chapter 5 focuses on Cohen-Macaulay representation theory; this theory studies the structure of the category of maximal Cohen-Macaulay modules over a Cohen-Macaulay local ring R, denoted $\mathrm{CM}R$. A starting point of this theory is the observation that R is regular if and only there's only a single indecomposable object in $\mathrm{CM}R$, namely R itself. This observation shows that the number of indecomposable objects in $\mathrm{CM}R$ can be used to measure singularities of R, and leads one to consider various representation types of $\mathrm{CM}R$ for this purpose.

Definition 1.0.7. We say R has finite (resp. countable) CM-representation type if there are only finitely (resp. countably) many indecomposable objects in $\mathrm{CM}R$.

A celebrated theorem of Huneke-Leuchske shows that rings of finite CM-representation type have isolated singularities [HL02]. On the other hand, Herzog showed that Gorenstein rings of finite CM-type must be hypersurfaces [Her78]; these hypersurfaces were later classified, in case R is complete and k is algebraically closed, see [LW12, Chapter 9] for an overview of this work.

Rings of countable CM-type are not as well-understood as those of finite CM-type, but as long as R is complete or k is uncountable, we still have $\dim \operatorname{Sing} R \leq 1$ when R has countable CM-type [HL03, Tak07].

The current focus of this theory lies in following folklore conjectures:

Conjecture 1.0.4.

(1) Suppose k is uncountable. If R has countable CM-type and is an isolated singularity, then R has finite CM-type.

(2) If R is Gorenstein and has countable CM-type, then R is a hypersurface.

(3) If R has countable CM-type, then $R_{\mathfrak{p}}$ has finite CM-type for all $\mathfrak{p} \neq \mathfrak{m}$.

In Chapter 5, we aim to progress these conjectures by introducing new representation types and exploring their connection with countable CM-type.

Definition 1.0.8. We say R has finite (resp. countable) CM_{+}-representation type R admits only finitely (resp. countably) many nonisomorphic indecomposable modules that are not locally free on $\operatorname{Spec} R - \{\mathfrak{m}\}$.

We conjecture a deep connection between finite CM_{+}-type and countable CM-type:

Conjecture 1.0.5 (conj11)**.** Suppose R is Gorenstein and is not an isolated singularity. Then R has countable CM-type if and only if R has finite CM_{+}-type.

One piece of evidence for Conjecture 1.0.5 is given by Araya-Iima-Takahashi, who prove, under mild technical hypotheses, that hypersurfaces of countable CM-type have finite CM_{+}-type [AIT12]. Chapter 5 provides further evidence for Conjecture 1.0.5; for instance, we show that rings of finite CM_{+}-type have the same singularities as those of countable CM-type.

The aim of Conjecture 1.0.5 is to replace the study of infinitely many objects with that of only finitely many. In practice, this makes problems, such as those of Conjecture 1.0.4 much more tractable. For instance, we can prove a corresponding version of Conjecture 1.0.4 (3) for rings of finite CM_+-type (Theorem 5.4.3). Thus, if Conjecture 1.0.5 can be proven, then we would immediately have Conjecture 1.0.4 (3), and other results of Chapter 5 would lead to progress on (1) and (2) in small dimension (see Corollary 5.5.6 and Theorem 5.7.8).

The case where we best understand conjecture 1.0.5 is the dimension 1 case, where we provide a classification theorem for Gorenstein non-isolated singularities of finite CM_+-type.

Theorem 1.0.9 (Theorem 5.6.1)**.** Let R be a homomorphic image of a regular local ring. Suppose that R does not have an isolated singularity but is Gorenstein. If $\dim R = 1$, then the following are equivalent.

(1) The ring R has finite CM_+-representation type.

(2) There exist a regular local ring S and a regular system of parameters x, y such that R is isomorphic to $S/(x^2)$ or $S/(x^2y)$.

When either of these two conditions holds, the ring R has countable CM-representation type.

The proof of this result relies on examining how representation types pass to and from quotient rings or birational extensions, the latter of which can be thought of as understanding some essential features of integral closure in this setting. An advantage of these methods is that they are characteristic free, a counterpoint to the classical approaches towards finite/countable CM-type. Since we have a classification of hypersurfaces of countable CM-type for complete hypersurfaces with some residue field and characteristic hypotheses, Theorem 1.0.9 implies Conjecture 1.0.5 holds true in this case.

We discuss the difficulties of the higher dimension case and we present a number of partial results, especially for the dimension 2 case, which still seems to boil down to a sufficient understanding of integral closure.

Chapter 6 switches focus to combinatorial commutative algebra and deals with nerve complexes:

Definition 1.0.10. Let $A = \{A_1, A_2, \ldots, A_r\}$ be a family of sets. Consider

$$N(A) := \{F \subseteq [r] \colon \cap_{i \in F} A_i \neq \emptyset\}.$$

This simplicial complex is the *nerve complex* of A.

One interesting case occurs when A is the list of facets of a simplicial complex Δ; in this case we denote the Nerve complex by $N(\Delta)$. The famous Borsuk Nerve Theorem shows that Δ and $N(\Delta)$ have same homotopy type but, despite this, more refined algebraic and combinatorial information of Δ is not captured by $N(\Delta)$ [Bor48]. For instance, the bowtie complex, whose facets are, $[1,2,3],[3,4,5]$, which is not CM, has the same nerve complex as the complex with facets $[1,2,3],[2,3,4]$, which is CM. In Chapter 6, we consider more general notions of these Nerve complexes aimed at capturing deeper homological information.

Definition 1.0.11. Let $A = \{A_1, A_2, \ldots, A_r\}$ be the set of facets of a simplicial complex Δ. Define

$$N_i(\Delta) := \{F \subseteq [r] \colon |\cap_{j \in F} A_j| \geq i\}.$$

We call this simplicial complex the i^{th} *nerve complex* of Δ, and we refer to the $N_i(\Delta)$ as the *higher nerve complexes* of Δ. We note that $N_1(\Delta)$ is the usual nerve complex $N(\Delta)$.

We provide an extension of the Borsuk Nerve Theorem for higher nerve complexes.

Theorem 1.0.12 (Proposition 6.3.3)**.** Let P denote the face poset of Δ. For any j, let $[\Delta]_{>j}$ denote the order complex of $P_{>j}$. Then $[\Delta]_{>j}$ is homotopy equivalent to $N_{j+1}(\Delta)$.

Using this result, and some new results about reduced homologies of links and certain induced subcomplexes, we show that the higher nerve complexes of Δ do indeed capture more refined information about Δ. Specifically, we prove the following:

Theorem 1.0.13 ([DDD$^+$19, Theorem 6.5.1 and Corollary 6.6.1])**.** We have

1. $\operatorname{depth}(k[\Delta]) = \inf\{i + j \colon \widetilde{H}_i(N_j(\Delta)) \neq 0\}$.

2. For $k \geq 1$ we have $h_k(\Delta) = (-1)^{k-1} \sum_{j\geq 1} \binom{d-j}{k-1} \widetilde{\chi}(N_j(\Delta))$ where $h_k(\Delta)$ denotes the kth entry of the h-vector of $k[\Delta]$.

In Chapter 7, we explore another point of view that can be taken on the work in the preceding Chapter; we study the $[\Delta]_{>j}$, whose homologies, as Theorem 1.0.12 shows, capture the same information as those of the higher nerve complexes of Δ. It turns out that the relevant structure of the $[\Delta]_{>j}$ can be studied abstractly to great effect.

Definition 1.0.14. A simplicial complex Δ is balanced if the vertices of Δ can be colored in such a way that no face of Δ contains more than one vertex of a given color.

Balanced simplicial complexes have come up in a variety of applications; see e.g. [BFS87, BVT13, CV13, JKV19]. The order complex $\mathscr{O}(P)$ of any finite poset P has a balanced structure, which serves as a motivating example. In particular, the $[\Delta]_{>j}$ are always balanced. It's common to study balanced simplicial complexes by their rank selected subcomplexes; the S-rank selected subcomplex Δ_S is the induced subcomplex of Δ on the vertices with colors in S. The so-called rank selection theorems of Stanley and Munkres show that homological properties often pass from Δ to Δ_S. Specifically, rank selected subcomplexes of balanced CM complexes remain CM [Sta79], and, removing a single rank (i.e. color) from Δ cannot drop the depth by more than 1 [Mun84b].

As Serre's condition (S_ℓ) generalizes the Cohen-Macaulay property, it is natural to consider if there is any extension of Stanley's theorem on rank selection of CM complexes to (S_ℓ). In Chapter 7 we prove this is indeed the case.

Theorem 1.0.15 (Theorem 7.3.2)**.** Let Δ be a balanced simplicial complex. If Δ satisfies Serre's condition (S_ℓ), then so does every rank selected subcomplex of Δ.

In the case $\Delta = \mathscr{O}(P)$ for a finite poset P, $\mathscr{O}(P_{>j})$ is the subcomplex of Δ with the bottom $j+1$ ranks removed. For this case, one can nearly characterize (S_ℓ) with the vanishing of reduced homologies of the $\mathscr{O}(P_{>j})$ (Theorems 7.3.3 and 7.3.4). However, we provide examples (Examples 7.6.4 and 7.6.5) that show Serre conditions cannot be completely characterized in this manner.

In general, equality need not hold in Munkres' theorem on rank selection and depth; though the depth cannot drop by more than 1, it may stay the same or increase wildly. However, we prove that one can often find a rank which, when removed, drops the depth by exactly 1 (Proposition 7.4.1). Using this result, we provide a formula for $\operatorname{depth} k[\Delta]$, when Δ is balanced, in terms of homologies of rank selected subcomplexes (Theorem 7.4.3).

Recently, some new gluing constructions of Duval-Goeckner-Klivans-Martin have seen counterexamples emerge to several long standing conjectures in algebraic combinatorics [DGKM16, JKV19, DG18]. In Chapter 8, noticing some common structure among these examples, we consider introduce and consider complexes which have some minimality condition with respect to the Cohen-Macaulay property.

Definition 1.0.16. Let Δ be a simplicial complex of dimension $d-1$ with facet list $\{F_1,\ldots,F_e\}$, and denote by Δ_{F_i} the subcomplex of Δ with facet list $\{F_1,\ldots,F_{i-1},F_{i+1},\ldots,F_e\}$. We say Δ is *minimal CM* if Δ is CM but Δ_{F_i} is not CM for any i.

If Γ is a subcomplex of Δ generated by facets of Δ, we say Δ is *shelled over* Γ if the relative complex (Δ,Γ) is shellable (see [Sta96, Chapter III. 7] or [Sta87, Section 5]). This provides an extension of the more commonly studied notion of a shellable complex, since shellable complexes are merely those complexes that are shelled over the empty complex $\varnothing$.

In Chapter 8, we prove that every CM simplicial complex is shelled over a minimal CM complex (Theorem 8.3.1). Thus the study of CM complexes reduces, in a large sense, to the study of minimal ones. We provide various necessary and sufficient conditions for a complex to be minimal CM (Theorem 8.1.2); for instance, a minimal CM complex must be acyclic, and a ball is minimal CM if and only if it is strongly non-shellable in the sense of [Zie98].

We use these conditions to show that many important examples in the literature end up being minimal CM. For instance, the non-partitionable CM complex C_3 of [DGKM16], the balanced non-partitionable CM complex C_3 of [JKV19], and the complex Ω_3 in [DG18], a 2-fold acyclic complex with no decomposition into rank 2 boolean intervals, are all minimal CM.

Chapter 2

Background

2.1 A Historical Note

Commutative algebra traces it's roots in the study of polynomial equations; various efforts had been simultaneously made in the 19th century at studying abstract properties of polynomials and various properties of algebraic integers. For instance, Lagrange's theorem, a well-known theorem from group theory, was proven by Lagrange, not in the general form it is known in today, but as a theorem about the number of different polynomials that appear when one permutes the variables of a polynomial in n variables all $n!$ different ways [Lag70]. Similar abstractions were made by Dedekind who introduced the notion of an ideal as a subset of a set of numbers, composed of algebraic integers that satisfy polynomial equations with integer coefficients [DD18]. A great leap forward was made by Hilbert who coined the term Ring, and who proved several important properties of polynomial rings over fields including: Hilbert's basis theorem which says that such a polynomial ring is Noetherian, Hilbert's syzygy theorem which says that such a polynomial ring is regular, and Hilbert's Nullstellensatz which provides the key connection between the algebra of polynomial rings and the geometry of varieties [Hil90]. But it was Emmy Noether who gave the modern axiomatic definiton of a ring and developed the foundations of commutative ring theory.

Critically, Noether realized the importance of chain conditions such as what is now known as the Noetherian condition, and she studied and proved the existence of primary decompositions for ideals in Noetherian rings [Noe21]. Around the same time, Macaulay had proven the unmixedness theorem for polynomial rings which would be later followed by Cohen's proof of the same for power series rings [Mac94, Coh46]. Both of these results would pave the way for the introduction of what are now called Cohen-Macaulay rings by Samuel and Zariski, just in time for the so-called homological revolution of the 1950's [ZS75].

The seminal text "Homological Algebra" of Cartan and Eilenberg in 1956 provided a complete redirection, unifying various, at the time, disparate homology theories though their use of dervied functors [CE56]. Coupled with Grothendieck's reinvention of algebraic geometry around the same time, these would pave the way for homological methods in commutative algebra. The Auslander-Buchsbaum-Serre theorem, proved independently by Auslander-Buchsbaum and Serre in 1956, was instrumental in showing that localizations of regular rings are regular, and did a great deal to illustrate the utility of homological methods [AB57, Ser56]; the statement of the localization problem does not require homological algebra, but the first and standard proof is deeply homological in nature. Soon after, Gorenstein rings were introduced by Grothendieck, and Bass's highly influential paper "On the Unbiquity of Gorenstein Rings" lived up to its title and demonstrated their ubiquity as well as their homological nature [Har67, Bas63]. From here, the so-called homological conjectures have spurred forth a great deal of research, including, for instance, prime characteristic methods and tight closure theory [HH89], the use of Chern classes in local algebra [Rob89], and the recent use of perfectoid techniques and almost mathematics to attack these questions in mixed characteristic [And18a, And18b].

In the 1970's Hochster and Stanley independently, and for different purposes, began to study what Stanley originally called face rings of simplicial complexes. The upper bound conjecture had recently been proven for simplicial polytopes by McMullen, but the upper bound conjecture for simplicial spheres was an important open combinatorial problem of the day. Stanley had the ingenious observation that this conjecture would follow if one could show that the face ring of a simplicial sphere must be Cohen-Macaulay, but he did not know how to prove this [Sta75]. Hochster gave the problem to his student Gerald Reisner who used characteristic p methods and reduction to characteristic p to establish a combinatorial criterion for a face ring to be Cohen-Macaulay [Rei76]; a consequence of this criterion was that face rings of simplicial spheres are Cohen-Macaulay, and so these algebraic methods had succeeded in solving an inherently combinatorial problem, thus beginning a period of deep interplay between commutative algebra and algebraic combinatorics. For a wonderful and unique overview on the history of the upper bound theorem see [Sta14].

2.2 Background in commutative algebra

In this section we discuss some neccessary background on commutative algebra; [BH93] serves as a suitable reference for much of what appears here. Let $(R,\mathfrak{m},k)$ be a local ring, that is, R is a Noetherian ring having a unique maximal ideal $\mathfrak{m}$ and $k := R/\mathfrak{m}$ is the so-called residue field of R; e.g. R is a formal power series ring over a field or over the p-adic integers, or is a homomorphic image thereof. Let M be a finitely generated R-module with minimal generating set $m_1,\ldots,m_{r_0}$; so the minimal number of generators $\mu_R(M)$ of M is r_0. Letting $\{e_i\}$ denote the standard basis of the free module R^{r_0}, we have a surjective R-linear map $p : R^{r_0} \to M$ given by mapping $e_i \mapsto m_i$. We may repeat this process, replacing M by $\operatorname{Ker} p$, and then continue this inductively to obtain a minimal free resolution

$$F_\bullet : \cdots \to R^{r_n} \xrightarrow{A_n} R^{r_{n-1}} \to \cdots \to R^{r_1} \xrightarrow{A_1} R^{r_0} \xrightarrow{p} M \to 0$$

of M. This resolution provides a measure of how badly the module M fails to be free; one numerical measure of this is the projective dimension $\operatorname{pd}_R M$ of M i.e., the length of $F_\bullet$. Applying $\operatorname{Hom}_R(-,N)$ or $-\otimes_R N$ to a free resolution of M and taking (co)homology of the resulting complex at the ith spot produces $\operatorname{Ext}^i_R(M,N)$ or $\operatorname{Tor}^R_i(M,N)$, respectively. In particular, one can show that $\operatorname{pd}_R M = \max\{i \mid \operatorname{Ext}^i_R(M,k) \neq 0\} = \max\{i \mid \operatorname{Tor}^i_R(M,k) \neq 0\}$. We recall the following definition:

Definition 2.2.1. A local ring $(R,\mathfrak{m},k)$ is said to be regular if the minimal number of generators $\mu_R(\mathfrak{m})$ of $\mathfrak{m}$ is equal to $\dim R$, the Krull dimension of the ring R.

This definition is motivated by algebraic geometry where the local ring $\mathscr{O}_{P,Y}$ of a variety V at a point P is regular if and only if V is nonsingular at P.

The following famous theorem of Auslander-Buchsbaum and Serre shows that the inherently geometric condtion of regularity can be characterized homologically:

Theorem 2.2.2 (The Auslander-Buchsbaum-Serre Theorem [Ser56, AB57])**.** Let $(R, \mathfrak{m}, k)$ be a local ring. The following are equivalent:

1. R is regular.

2. $\operatorname{pd}_R M < \infty$ for every R-module M.

3. $\operatorname{pd}_R k < \infty$.

Expanding on the Auslander-Buchsbaum-Serre theorem, various conditions and invariants defined through properties of Ext or Tor are used to codify and measure singularities. To explore some of these, we need the following definitions:

Definition 2.2.3. A sequence of elements $\underline{x} = x_1, \ldots, x_n$ in R is said to be regular on an R-module M if we have the following:

1. For each i, x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$.

2. $\underline{x}M \neq M$.

   When R is local and M is finitely generated, then latter condition is equivalent to $\underline{x} \subseteq \mathfrak{m}$, i.e., that none of the x_i are units.

A standard example of a regular sequence (on the ring itself) is the variables $x_1, \ldots, x_n$ in a polynomial ring $R = k[x_1, \ldots, x_n]$ or power series ring $R = k[[x_1, \ldots, x_n]]$; in fact, this example serves as a strong motivation for the introduction of regular sequences and, in a manner that can be made precise, regular sequences behave much like variables in polynomial or power series rings. It turns out if R is local, or we are in an appropriate graded setting, then every permnutation of a regular sequence is regular, and further, every regular sequence can be extended to one of maximal length. The length of a maximal regular sequence is of particular import:

Proposition 2.2.1. If $(R, \mathfrak{m}, k)$ is a local ring and M a finitely generated R-module, then the length of every maximal regular sequence on M is the same and is called the depth of M; it can be computed as follows:

$$\operatorname{depth}_R M = \inf\{i \mid \operatorname{Ext}^i_R(k,M) \neq 0\}.$$

As the above Proposition shows, the depth is a homological invariant and can be thought of as the homological size of the module involved. On the other hand, the Krull dimension represents the topological size of the module. In general we have $\operatorname{depth}_R M \leq \dim_R M$ and things are much nicer when these values coincide; in this case, the geometry of R provides a reasonable representation of what's happening algebraically:

Definition 2.2.4. An R-module M is said to be Cohen-Macaulay (CM) if $\operatorname{depth} M = \dim M$. We say M is maximal Cohen-Macaulay (MCM) if $\operatorname{depth} M = \dim R$. The ring R is said to be CM if it is CM as a module over itself, that is, if $\operatorname{depth} R = \dim R$.

To paraphrase Mel Hochster, life is really worth living in a Cohen-Macaulay ring [Hoc78]. Cohen-Macaulayness has many advantages, for instance, one can “cut down" by a maximal regular sequence to get to a ring or module of dimension 0 where many problems become much more tractable. This is a common approach and will be exploited many times during the body of this text.

We also consider two refinements of the CM condition. As Proposition 2.2.1 shows, for a CM local ring R, the first nonvanishing $\operatorname{Ext}^i_R(k,R)$ occurs when $i = \operatorname{depth} R$. It turns out R has a special structure when this Ext module is as small as possible:

Definition 2.2.5. If M is a finitely generated R-module we set $r(M) = \dim_k \operatorname{Ext}^{\operatorname{depth}_R M}(k,M)$ and call it the (CM) type of M. We say R is Gorenstein if R is CM and $r(R) = 1$.

Gorenstein rings are nice from multiple angles; indeed, Hyman Bass's original paper on the topic expresses many characterization for the Gorenstein property [Bas63]. For instance, they are exactly the rings which have finite injective dimension over themselves, and the R-dual functor $(-) = \operatorname{Hom}_R(-,R)$ is especially well-behaved over a Gorenstein ring.

Finally we discuss complete intersections. As long is R is complete, the Cohen structure theorem shows that R can be written as the homomorphic image of a regular local ring. As we tacitly implied earlier, going modulo a regular sequence behaves well with respect to many properties, and, in particular, we should expect modding out a regular sequence in a regular local ring should maintain a particularly nice set of properties. This is indeed the case, and inspires the following class of rings, whose name hints at the underlying geometric theme:

Definition 2.2.6. A local ring $(R,\mathfrak{m},k)$ is said to be a complete intersection if $\hat{R} \cong S/I$for a regular local ring S and an ideal I in S generated by a regular sequence on S.

Complete intersections are as good as it gets short of being regular, and, as one might expect, we have the familiar chain of implications:

$$\text{Regular} \Rightarrow \text{Complete Intersection} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay}.$$

Entire theories are devlpoed for complete intersections which have been difficult to nearly impossible to adapt to more generality; for example the theory of support varieties developed by Avramov-Buchweitz [AB00].

2.3 Background in algebraic combinatorics

In this section we survey some needed background in algebraic combinatorics; a suitable reference for most of the material is [Sta96].

Definition 2.3.1. A simplicial complex Δ on the set S is a collection of subsets of S such that that is closed under inclusion, that is to say, if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

The elements of Δ are called faces of Δ and faces which are maximal under inclusion in Δ are called facets. The dimension of a face $\sigma \in \Delta$ is defined to be $\dim \sigma = |\sigma| - 1$ and the dimension of Δ is defined to be $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$. We say Δ is pure if every facet of Δ has the same dimension. It's common to take $S = \{1, \ldots, n\}$ for some n; we let $V(\Delta) = \{i \in S \mid \{i\} \in \Delta\}$ denote the vertex set of Δ, and it also common to assume $V(\Delta) = S$. Throughout this document, we will always assume the set S is finite.

Simplicial complexes arise naturally from several areas of mathematics including topology and commutative algebra. For instance, the proof of the classification theorem for compact surfaces depends crucially on the fact that every compact surface admits a triangulation [Mun84a]. Any simplicial complex Δ has an associated topological space $||\Delta||$, called the geometric realization of Δ, constructed by taking $||\Delta||$ to be the subset of $[0,1]^S$ consisting of all functions f satisfying the following two conditions:

1. $\{s \in S \mid f(s) > 0\} \in \Delta$
2. $\sum\limits_{s \in S} f(s) = 1.$

We then give $||\Delta||$ the subspace topology inherited from $[0,1]^S$. In particular, we can think about homologies/reduced homologies associated to Δ though the lens of its geometric realization. Many properties and invariants only depend on the homeomorphism class of $||\Delta||$, we refer to such properties as topological properties of Δ. Many properties and invariants we care about from the algebraic front will turn out to topological.

One significant class of examples of simplicial complexes comes from posets; if P is a finite poset, we let $\mathscr{O}(P)$ be the simplicial complex with vertex set P whose faces consist of all chains in P. Then $\mathscr{O}(P)$ is a simplicial complex and, in particular, this construction allows one to think of posets topologically. If P is the collection of nonempty faces of a simplicial complex Δ under inclusion, then $\mathscr{O}(P)$ is called the barycentric subdivision of Δ and is denoted $\mathrm{sd}(\Delta)$. We have the following (see e.g. [Gib10]):

Theorem 2.3.2. If Δ is a simplicial complex then $||\Delta||$ is homeomorphic to $||\,\mathrm{sd}(\Delta)||$.

Thus, in studying topological properties of Δ, we may replace Δ by $\mathrm{sd}(\Delta)$, which has additional structure. This approach will be used numerous times in the body of this text, especially in Chapter 7.

Finally, we discuss the Stanley-Reisner correspondence. Given a simplicial complex Δ with $|V(\Delta)| = n$ and a field k, we let I_Δ be the ideal of $k[x_1, \ldots, x_n]$ generated by the nonfaces of Δ, that is to say the ideal generated by monomials of the form $\prod_{i\in F} x_i$ where $F \notin \Delta$. On the other hand, if I is a squarefree monomial ideal in $k[x_1,\ldots,x_n]$, then I defines a simplicial complex by taking $\Delta = \{\sigma \subseteq [n] \mid \prod_{i\in\sigma} x_i \notin I\}$. If Δ is a simplicial complex we set $k[\Delta] = k[x_1,\ldots,x_n]/I_\Delta$ and call it the Stanley-Reisner ring of Δ over k. The Stanley-Reisner ring of Δ over k is determined, up to isomorphism, by the isomorphism class of Δ; in fact, the correspondence is functorial in nature, and it turns out that the Krull dimension of $k[\Delta]$ is always equal $\dim\Delta+1$. In particular, this correspondence allows us to associate any algebraic property of $k[\Delta]$ to a combinatorial property of Δ and vice versa. Among the most important properties in this theory remains Cohen-Macaulayness. To explore combinatorial ramifications of this property we recall the following:

Definition 2.3.3. If Δ is a simplicial complex on the vertex set S and $\sigma \subseteq S$, then the link of σ in Δ is the simplicial complex

$$\mathrm{lk}_\Delta(\sigma) = \{T \in \Delta \mid \sigma \cup T \in \Delta \text{ and } \sigma \cap T = \varnothing\}.$$

Theorem 2.3.4 (Reisner's Criterion [Rei76])**.** Let Δ be a simplicial complex and k a field. Then the following are equivalent:

1. $k[\Delta]$ is Cohen-Macaulay.

2. $\widetilde{H}_{i-1}(lk_\Delta(\sigma);k) = 0$ for all i and $\sigma \in \Delta$ such that $i+|\sigma|-1 < \dim\Delta$.

This result was critical in the proof of the upper bound theorem for simplicial sphere, as it in particular implies that any simplicial sphere is CM.

Reisner's criterion was greatly extended by Hochster's unplublished formula for the bigraded Hilbert series of local cohomology modules of Stanley-Reisner rings (see [BH93, Theorem 5.3.8]). One immediate consequence of Hochster's formula is the following extension of Reisner's result:

Proposition 2.3.1. We have that $\operatorname{depth} k[\Delta] \geq \ell$ if and only if $\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(\sigma);k) = 0$ for i and $\sigma \in \Delta$ such that $i+|\sigma| < \ell$.

Work of Munkres modifies these characterizations slightly to show that Cohen-Maculay is a topological property and the depth is a topological invariant of Δ [Mun84b].

Chapter 3

Extremal growth of Betti numbers and trivial vanishing of (co)homology

3.1 Introduction

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay (CM) local ring. In this chapter we consider the following conditions on the vanishing of (co)homology.

1. For any finitely generated R-modules M and N, $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$ implies that either M or N has finite projective dimension.

2. For any finitely generated R-modules M and N, $\operatorname{Ext}_R^i(M,N) = 0$ for $i \gg 0$ implies that either M has finite projective dimension or N has finite injective dimension.

Rings satisfying (1) have the least possible flexibility for asymptotic vanishing of homology, and those satisfying (2) have the least flexibility for asymptotic vanishing of cohomology. While (1) always implies (2), these conditions are equivalent under mild assumptions, e.g., if R has a canonical module (Theorem 3.3.2). Following Jorgensen and Şega [Jc04], we say R satisfies *trivial vanishing* if (1), and thus (2), holds for R.

In the past few decades, these rigidity conditions have gained much attention, in particular in connection with the following long-standing conjecture.

Conjecture 3.1.1 (Auslander-Reiten [AR75])**.** Let R be a Noetherian ring and M a finitely generated R-module. If $\operatorname{Ext}_R^i(M, M \oplus R) = 0$ for every $i > 0$, then M is projective.

Recently, much research activity has been centered on proving this conjecture. However, up to date, it remains open (see for example [HL04],[HcV04],[CIST18],[NSW17],[Lin17a], and see [CH10, Appendix A] for a survey on the topic).

The study of trivial vanishing was pioneered by Huneke and Wiegand [HW97] and independently by Miller [Mil98], who show this condition holds for hypersurface rings. This was later extended by Jorgensen to show that any Golod ring satisfies trivial vanishing [Jor99]. On the other hand, Şega showed that trivial vanishing fails if $\operatorname{codim}(R) \geqslant 2$ and the completion $\hat{R}$ has embedded deformations (i.e., $\hat{R} \cong Q/(\underline{f})$ for some local ring $(Q, \mathfrak{p})$ and Q-regular sequence $\underline{f} \subseteq \mathfrak{p}^2$) [Ş03]. In particular, any complete intersection of codimension larger than one cannot satisfy trivial vanishing, a result originally obtained by Avramov and Buchweitz using the theory of support varieties [AB00].

To summarize these results, we have a good understanding of the rigid behavior of Ext and Tor over rings whose modules have Betti numbers of extremal growth; modules over Golod rings have the fastest growth of Betti numbers while those over complete intersections have the slowest [Avr10, 5.3.2 and 8.1.2]. A unification of these settings is given by the *generalized Golod rings* of which both are examples; see [Avr94] or Subsection 3.2.1 for the definition. While the behavior of Betti numbers over generalized Golod rings is more subtle, they still possess a wealth of structure. In particular, any module M over a generalized Golod ring has a rational Poincaré series and these rational expressions can be made to share a common denominator [Avr94, 1.5]. By studying growth rates of Betti numbers in general, we establish a sufficient numerical condition for any CM local ring to satisfy trivial vanishing, and we provide a refined version when R is generalized Golod. Using this result, we are able to establish the Auslander-Reiten conjecture in a number of new cases.

In order to describe our main results, we need to introduce some preliminary notation. We denote by $e_R(M)$ (or simply $e(R)$ when $M = R$) the *(Hilbert-Samuel) multiplicity* of the R-module M, and we write $\mu(M)$ for the minimal number of generators of M. We let $\operatorname{codim} R := \mu(\mathfrak{m}) - \dim R$ denote the *codimension* of R. In our proofs, we may assume R has an infinite residue field

(See Proposition 3.3.5 and Remark 3.2.4). Then we can define the *Loewy length* $\ell\ell(R)$ as the maximum among $\min\{i \mid \mathfrak{m}^i \subseteq J\}$ where J ranges over the minimal reductions of $\mathfrak{m}$ (see Definition 3.2.5). We now present our first theorem; see Theorems 8.3 and 3.4.3.

Theorem A. Let R be a CM local ring and set $c = \operatorname{codim}(R)$ and $\ell = \ell\ell(R)$. Assume one of the following conditions holds.

(1) $e(R) < \frac{4c+2\ell-1-\sqrt{8c+4\ell-3}}{2}$.

(2) R is generalized Golod and $e(R) \leqslant 2c+\ell-4$.

Then R satisfies trivial vanishing.

The authors in [Jc04] provide examples of rings R that do not satisfy trivial vanishing for which the completion has no embedded deformations. The first example is a Gorenstein ring with $e(R) = 12$ and $\operatorname{codim}(R) = 5$. From [Ş03], we know this example is minimal with respect to codimension, as no such Gorenstein ring exists with $\operatorname{codim}(R) \leqslant 4$. The second example from [Jc04] has $e(R) = 8$ and $\operatorname{codim}(R) = 4$. In our next theorem, we show that both examples are minimal with respect to codimension and multiplicity; see Propositions 3.4.8 and 3.4.10.

Theorem B. Let R be a CM local ring with $\operatorname{codim}(R) \neq 1$ and assume it satisfies one of the following conditions.

(1) $\operatorname{codim}(R) \leqslant 3$.

(2) $e(R) \leqslant 7$.

(3) $e(R) \leqslant 11$ and R is Gorenstein.

Then R satisfies trivial vanishing if and only if the completion $\hat{R}$ has no embedded deformations.

We note that the assumption $\operatorname{codim}(R) \neq 1$ is necessary by [HW97], and we also remark that a key point to Theorem B is that CM rings of small codimension and multiplicity tend to be generalized Golod (see Example 3.2.8). The only case covered by Theorem B where we do not know R is generalized Golod is the case where Artinian reductions of R have h-vector $(1,4,2)$.

As a consequence of our work, we are able to verify the Auslander-Reiten conjecture in some new cases; see Theorem 3.5.3 and Corollary 3.5.4.

Theorem C. Let R be a CM local ring. Assume R satisfies one of the following conditions.

(1) $e(R) \leqslant \frac{7}{4}\operatorname{codim}(R) + 1$.

(2) $e(R) \leqslant \operatorname{codim}(R) + 6$ and R is Gorenstein.

Then the Auslander-Reiten conjecture (Conjecture 3.1.1) holds for R. In particular, the conjecture holds if $e(R) \leqslant 8$, or if $e(R) \leqslant 11$ and R is Gorenstein.

In several other cases we show that a stronger condition is satisfied, namely, the *uniform Auslander condition* which implies Conjecture 3.1.1 by [CH10]; see Corollary 3.5.1.

In [Ulr84], Ulrich provides conditions on a finitely generated R-module M so that the vanishing of $\operatorname{Ext}_R^i(M,R) = 0$ for $1 \leqslant i \leqslant \dim(R)$ forces R to be Gorenstein. In this sense, M can be used a test module for the Gorenstein property. In [JL07] and [HH05], some variations on this result are included. In the last part of the chapter, we expand upon these results by proving the following; see Theorem 3.6.8. We recall that the ring R is *generically Gorenstein* if $R_{\mathfrak{p}}$ is Gorenstein for every $\mathfrak{p} \in \operatorname{Ass}(R)$.

Theorem D. Let R be a generically Gorenstein CM local ring that has a canonical module. Assume there exists a Maximal Cohen-Macaulay R-module M with $e_R(M) \leqslant 2\mu(M)$ and such that $\operatorname{Ext}_R^i(M,R) = 0$ for $1 \leqslant i \leqslant \dim(R) + 1$. Then R is Gorenstein.

We now describe the layout of this chapter. In Section 3.2, we set the notation that is used throughout the chapter. We also discuss preliminary results and definitions that are necessary in our proofs. In Section 3.3, we give the definition of trivial vanishing and consider its behavior under change of rings, specifically under local maps of finite flat dimension and hyperplane sections. In Section 3.4, we include our main results; here we prove Theorem A (Theorems 8.3 and 3.4.3) and Theorem B (Propositions 3.4.8 and 3.4.10). Section 3.5 includes consequences for the uniform Auslander condition and the Auslander-Reiten conjecture. In particular, this section includes the proof of Theorem C (Theorem 3.5.3). The last section, Section 3.6, includes the proof of Theorem D (Theorem 3.6.8) and related results.

3.2 Notation and preliminary results

Throughout this chapter we assume all rings are Noetherian and all modules are finitely generated. Let $(R,\mathfrak{m},k)$ be a local ring of dimension d and M be an R-module. We denote by $\beta_i^R(M) = \dim_k \operatorname{Tor}_i^R(k,M)$ the *ith Betti number* of M, $\mu(M) = \beta_0^R(M)$, the minimal number of generators of M, and $P_M^R(t) = \sum_{i=0}^{\infty} \beta_i^R(M)t^n$ the *Poincaré series* of M over R. We also write $\Omega_i^R(M)$ for the *ith syzygy* of M and $r(R) = \dim_k \operatorname{Ext}_R^{\operatorname{depth} R}(k,M)$ for its *type*. We write $\hat{R}$ for the $\mathfrak{m}$-adic completion of R.

If M has dimension s, we denote by

$$e_R(M) = \lim_{n\to\infty} \frac{s!\lambda(M/\mathfrak{m}^n)}{n^s}$$

the *Hilbert-Samuel multiplicity* of M, where $\lambda(N)$ denotes the length of the R-module N. If $M = R$, we simply write $e(R)$. The following invariant has appeared in different forms and has played an important role in several results in the literature (see for example [HH05],[HcV04], [JL07],[Ulr84]). We provide the following notation and name in order to simplify some of the statements.

Definition 3.2.1. Let R be a local ring and M a non-zero R-module. We define the *Ulrich index* of M, denoted by $\mathfrak{u}_R(M)$, as the ratio

$$\mathfrak{u}_R(M) := \frac{e_R(M)}{\mu(M)}.$$

Remark 3.2.2. We note that when M is CM, we always have the inequality $e_R(M) \geqslant \mu(M)$, and therefore $\mathfrak{u}_R(M) \geqslant 1$. The Maximal Cohen-Macaulay (MCM) R-modules such that $\mathfrak{u}_R(M) = 1$ are the so-called *Ulrich modules*, therefore the Ulrich index provides a measure of how far a module is from being Ulrich.

The next lemma is the content of [JL07, 2.1] stated in our terminology (see also [JL08, 2.1]). We remark that, although the original version of part (1) assumes N is MCM, its proof does not actually requires this condition. If R has a canonical module ω_R, we write

$$M^\vee := \operatorname{Hom}_R(M, \omega_R)$$

for its canonical dual. In the following statement we use the convention $\beta^R_{-1}(N) = 0$.

Lemma 3.2.3 ([JL07, 2.1], [JL08, 2.1])**.** Let R be a CM local ring of dimension d. Let M and N be R-modules and assume M is CM. Fix $n \in \mathbb{N}$ and assume one of the following conditions holds

(1) $n \geqslant \dim M$ and $\operatorname{Tor}^R_i(M,N) = 0$ for every $n - \dim M \leqslant i \leqslant n$, or

(2) $n \geqslant 0$, R has a canonical module ω_R, N is MCM, and $\operatorname{Ext}^i_R\left(M, N^\vee\right) = 0$ for every $n + d - \dim M \leqslant i \leqslant n + d$.

Then $\beta^R_n(N) \leqslant (\mathfrak{u}_R(M) - 1)\beta^R_{n-1}(N)$.

The *codimension* of $(R,\mathfrak{m})$ is defined as $\operatorname{codim}(R) = \mu(\mathfrak{m}) - \dim(R)$. We note that, if R is Artinian, then $\operatorname{codim}(R)$ is simply $\mu(\mathfrak{m})$. In the following remark we explain the process of reduction to the Artinian case.

Remark 3.2.4. Let R be a CM local ring, it is always possible to reduce R to an Artinian ring with the same codimension and multiplicity in the following way. First, we extend R to the faithfully flat R-algebra $R[X]_{\mathfrak{m}R[X]}$ to assume R has infinite residue field. Then, we mod out R by a minimal reduction of $\mathfrak{m}$.

We recall that the *Loewy length* of an Artinian local ring R is defined as $\ell\ell(R) = \min\{i \mid \mathfrak{m}^i = 0\}$. We extend this definition to arbitrary CM local rings as follows.

Definition 3.2.5. Let R be a CM local ring. We define the *Loewy length* of R, denoted by $\ell\ell(R)$, as the maximum of the Lowey lengths of the Artinian reductions of R as in Remark 3.2.4. We remark that when R is equicharacteristic, $\ell\ell(R)$ is achieved when modding out by a general reduction of $\mathfrak{m}$ [Fou06, 5.3.3].

The following result of Gasharov and Peeva will be crucial to the proofs of the main results of this chapter. While their result is stated for Artinian rings, we include here a version of it for arbitrary CM local rings. We remark that although part (1) is not explicitly stated therein, it is included in the proof of [GP90, 2.2].

Proposition 3.2.6 ([GP90, 2.2], [Pee98, Proposition 3]). Let R be a CM local ring of dimension d and M be an R-module. Set $c = \operatorname{codim}(R)$, $e = e(R)$, and $\ell = \ell\ell(R)$. Then for every $n > \max\{d - \operatorname{depth} M, \mu(M)\}$ we have,

(1) $\beta_n^R(M) \geqslant c\beta_{n-1}^R(M) - (e - c - \ell + 2)\beta_{n-2}^R(M)$,

(2) $\beta_n^R(M) \geqslant (2c - e + \ell - 2)\beta_{n-1}^R(M)$, and

(3) If $c \geqslant 3$ and $2c - e + \ell - 2 = 1$, there either the Betti numbers of M are eventually constant, or there exists $C > 1$ such that $\beta_n^R(M) \geqslant C\beta_{n-1}^R(M)$ for every $n \gg 0$.

We now discuss a class of rings introduced by Avramov in [Avr94].

3.2.1 Generalized Golod rings.

An *acyclic closure* of R is a DG-algebra resolution of k constructed via Tate's process of adjoining variables to kill cycles [Tat57]. The process starts with the Koszul complex $K^R = R\langle X_1 \rangle$, and, for $n \geqslant 1$, it inductively adjoins variables X_{n+1} in homological degree $n+1$ in such a way that the classes of $\partial(X_{n+1})$ minimally generate the homology $H(R\langle X_{\leqslant n}\rangle)$. Here, X_n are exterior variables if n is odd and divided powers variables if n is even. Setting $X = \bigcup_{n \geqslant 1} X_n$, the resulting acyclic closure $R\langle X \rangle$ is a minimal free resolution of k [Gul68, Sch67]. We refer the reader to [Avr10, Section 6.3] for more information.

In [Avr94], Avramov defines a local ring R to be *generalized Golod* (*of level* $\leqslant n$) if the DG-algebra $R\langle X_{\leqslant n}\rangle$ admits a *trivial Massey operation* for some $n \geqslant 1$ (see [Avr10, 5.2.1]). We notice that the classical Golod rings are precisely the generalized Golod rings of level $\leqslant 1$. One of the main motivations for introducing this class of rings is the following theorem.

Theorem 3.2.7 ([Avr10, 1.5])**.** Let R be a generalized Golod ring, then there exists a polynomial $\mathrm{Den}^R(t) \in \mathbb{Z}[t]$ such that for every R-module M there exists $p_M(t) \in \mathbb{Z}[t]$ giving

$$P_M^R(t) = \frac{p_M(t)}{\mathrm{Den}^R(t)}.$$

Moreover, when $M = k$, all the roots of $p_k(t)$ have magnitude one.

We now present some classes of rings that are generalized Golod.

Example 3.2.8. The local ring R is generalized Golod in any of the following situations:

1. R is a complete intersection [Tat57].
2. R is Golod [GL69].
3. $\mu(\mathfrak{m}) - \mathrm{depth}(R) \leqslant 3$ [AKM88, 6.4], see also [Avr89, 3.5].
4. R is Gorenstein and $\mu(\mathfrak{m}) - \mathrm{depth}(R) \leqslant 4$ [AKM88, 6.4], see also [Avr89, 3.5].
5. R is one link from a complete intersection [AKM88, 6.4].
6. R is two links from a complete intersection and R is Gorenstein [AKM88, 6.4].
7. R is almost a complete intersection of codimension four, and 2 is a unit in R [KPS94, 4.2].
8. R is presented by Huneke-Ulrich ideals of full codimension [Kus95, 5.2].
9. R is Gorenstein and $e(R) \leqslant 11$ [Gup17, 6.9].
10. R is Gorenstein, $\mathfrak{m}^4 = 0$, and $\mu(\mathfrak{m}^2) \leqslant 4$ [Gup17, 6.9].
11. R is presented by certain determinantal ideals of full codimension [AKM88, 6.5].
12. R is a CM *stretched* ring, or an *almost stretched* Gorenstein ring [CDG$^+$16, 5.4], [Gup17, 6.1].
13. Certain *compressed* Artinian rings [Rc14, 5.1], [KcV18, 7.1].

The following invariant has been studied by other authors as it measures the growth of Betti numbers of modules [Avr10, Section 4]. We give this invariant the following name and notation for clarity of our exposition.

Definition 3.2.9. Let M be an R-module of infinite projective dimension. We define the *limit ratio* of M, denoted by $\operatorname{lr}_R(M)$, as the formula

$$\operatorname{lr}_R(M) := \limsup_{n\to\infty} \frac{\beta_{n+1}^R(M)}{\beta_n^R(M)}.$$

Clearly $\operatorname{lr}_R(M) \geqslant 1$. We remark that while it is unknown if $\operatorname{lr}_R(M)$ is always finite (cf. [Avr10, 4.3.1]), this is indeed the case when R is a CM local ring [Avr10, 4.2.6]. The limit ratio is naturally related to the *curvature* of M (see paragraph before [Avr10, 4.3.6]), and to the *complexity* of M, i.e,

$$\operatorname{cx}_R(M) = \inf\{t \in \mathbb{N} \mid \text{ there exists } \beta \in \mathbb{R} \text{ such that } \beta_n^R(M) \leqslant \beta n^{t-1} \text{ for every } n \geqslant 1\}.$$

Note that, by definition, $\operatorname{cx}_R(M) = 0$ if and only if M has finite projective dimension.

In fact, in Problems 4.3.6 and 4.3.9 of [Avr10], Avramov asks whether the limit in the definition of $\operatorname{lr}_R(M)$ always exists. By a result of Sun, this limit exists for modules of infinite complexity over a generalized Golod ring.

Proposition 3.2.10 ([Sun98, Corollary])**.** Let R be a generalized Golod ring and M be an R-module. Assume $\operatorname{cx}_R(M) = \infty$. Then the limit $\lim_{n\to\infty} \dfrac{\beta_{n+1}^R(M)}{\beta_n^R(M)}$ exists, and is greater than 1.

3.2.2 MCM approximations.

The following result of Auslander and Buchweitz, and the subsequent remark, allow us to often replace arbitrary finitely generated modules for MCM modules when dealing with vanishing of Ext.

Theorem 3.2.11 ([AB89, Theorem A]). Let R be a CM local ring with canonical module ω_R and let N be an R-module. Then there exist R-modules Y and L, such that L is MCM, Y has finite injective dimension, and they fit in a short exact sequence $0 \to Y \to L \to N \to 0$.

The exact sequence from the previous lemma is commonly referred as an *MCM approximation* of N.

Remark 3.2.12. We note that, since Y in Theorem 3.2.11 has finite injective dimension, for every R-module M we have $\operatorname{Ext}^i_R(M,N) = 0$ for $i \gg 0$ if and only if $\operatorname{Ext}^i_R(M,L) = 0$ for $i \gg 0$. In particular, if $M = k$, then N has finite injective dimension if and only if L does.

We now recall another notion of complexity. The *plexity* of N is defined in terms of its *Bass numbers*, i.e., $\mu^i_R(N) = \dim_k \operatorname{Ext}^i_R(k,N)$. We have,

$$\operatorname{px}_R(N) = \inf\{t \in \mathbb{N} \mid \text{ there exists } \mu \in \mathbb{R} \text{ such that } \mu^n_R(N) \leqslant \mu n^{t-1} \text{ for every } n \geqslant 1\}.$$

For more information about these notions of complexity, see [Avr10, Section 4.2] and [Avr96]. There is a direct relation between the two complexities, we discuss this in the following remark.

Remark 3.2.13. Let R be a CM local ring of dimension d and with a canonical module ω_R. Let $0 \to Y \to L \to N \to 0$ be an MCM approximation of N (cf. Theorem 3.2.11), then

$$\mu^{i+d}_R(N) = \beta^R_i((L)^\vee) \text{ for } i > 0, \qquad \text{and then,} \qquad \operatorname{px}_R(N) = \operatorname{cx}_R((L)^\vee).$$

To see this, choose $\underline{x} = x_1,\dots,x_d$ a maximal regular sequence on R, L, and $(L)^\vee$. The claim now follows by observing that

$$\begin{aligned}\mu^{i+d}_R(N) = \mu^{i+d}_R(L) = \mu^i_{R/\underline{x}R}(L/\underline{x}L)& \\ &= \beta^{R/\underline{x}R}_i\big((L/\underline{x}L)^\vee\big) = \beta^{R/\underline{x}R}_i\big((L)^\vee/\underline{x}(L)^\vee\big) = \beta^R_i((L)^\vee)\end{aligned}$$

for every $i > 0$, where the third equality holds by Matlis duality.

3.3 Trivial vanishing

In this section we present the definition of the trivial vanishing condition for CM local rings and prove some preliminary results.

Definition 3.3.1. A CM local ring R satisfies *trivial Tor-vanishing* if, for any R-modules M and N, $\operatorname{Tor}_i^R(M,N)=0$ for $i \gg 0$ implies M or N has finite projective dimension. We say R satisfies *trivial Ext-vanishing* if $\operatorname{Ext}_R^i(M,N)=0$ for $i \gg 0$ implies M has finite projective dimension or N has finite injective dimension. If both conditions are satisfied, we simply say R satisfies *trivial vanishing*.

The relation between the two trivial vanishing conditions is explained by the following result.

Theorem 3.3.2. Let R be a CM local ring.

(1) If R satisfies trivial Tor-vanishing, then R satisfies trivial vanishing.

(2) If R has a canonical module, the two trivial vanishing conditions (Ext and Tor) are equivalent.

For the proof of this theorem we need some preparatory results.

Lemma 3.3.3. Let R be a CM local ring with canonical module ω_R. Let M and N be R-modules and assume that N, $M \otimes_R N^\vee$, and $\Omega_1^R(M) \otimes_R N^\vee$ are MCM. Then, $\operatorname{Ext}_R^1(M,N)=0$ if and only if $\operatorname{Tor}_1^R(M,N^\vee)=0$.

Proof. From the exact sequence $0 \to \Omega_R^1(M) \to R^{\mu(M)} \xrightarrow{p} M \to 0$, and the natural isomorphism $(- \otimes_R N^\vee)^\vee \cong \operatorname{Hom}_R(-,N)$, we obtain

$$(\operatorname{Ker}(p \otimes \operatorname{id}_{N^\vee}))^\vee \cong \operatorname{coker}(\operatorname{Hom}_R(p,N)).$$

If $\operatorname{Ext}_R^1(M,N)=0$, then $\operatorname{coker}(\operatorname{Hom}_R(p,N)) = \operatorname{Hom}_R(\Omega_R^1(M),N)$, so we have $\operatorname{Ker}(p \otimes_R \operatorname{id}_{N^\vee}) = \Omega_R^1(M) \otimes_R N^\vee$. Thus $\operatorname{Tor}_1^R(M,N^\vee)=0$. Similarly, if $\operatorname{Tor}_1^R(M,N^\vee)=0$, then $\operatorname{Ker}(p \otimes_R \operatorname{id}_{N^\vee}) = \Omega_R^1(N) \otimes_R N^\vee$ and so $\operatorname{coker}(\operatorname{Hom}_R(p,N)) = \operatorname{Hom}_R(\Omega_R^1(M),N)$. Thus $\operatorname{Ext}_R^1(M,N)=0$, and the proof is complete. $\square$

Lemma 3.3.4. Let R be a CM local ring of dimension d and with canonical module ω_R. Let M and N be R-modules and assume N is MCM. Then for each $n \in \mathbb{N}$ the following hold.

(1) If $\operatorname{Ext}_R^i(M,N) = 0$ for $1 \leqslant i \leqslant d$, then $M \otimes_R N^\vee$ is MCM.

(2) If $\operatorname{Ext}_R^i(M,N) = 0$ for $1 \leqslant i \leqslant d+n$, then $\operatorname{Tor}_i^R(M,N^\vee) = 0$ for $1 \leqslant i \leqslant n$.

(3) If $\operatorname{Tor}_i^R(M,N^\vee) = 0$ for $1 \leqslant i \leqslant d+n$, then $\operatorname{Ext}_R^i(M,N) = 0$ for $d+1 \leqslant i \leqslant d+n$.

Proof. The proof of (1) is included in [DEL19, 5.3]. We remark that, although M is assumed to be MCM in [DEL19], this hypothesis is not needed in the proof.

By (1), the assumption of (2) implies $\Omega_i^R(M) \otimes_R N^\vee$ is MCM for $0 \leqslant i \leqslant n$. Hence, (2) follows from applying Lemma 3.3.3.

For (3), since $N^\vee$ is MCM, the depth lemma implies $\Omega_i^R(M) \otimes N^\vee$ is MCM for $d \leqslant i \leqslant d+n$. Then the conclusion follows from Lemma 3.3.3. $\square$

We now consider how the trivial vanishing conditions behave under extensions of finite flat dimension and hyperplane sections. Many of our arguments draw inspiration from Section 2 of [CH12]. For our purposes we only consider the case where R is CM. For similar results without this hypothesis, see [AINSW20].

Proposition 3.3.5. Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local ring map with R CM.

(1) Assume S has finite flat dimension over R. If S satisfies trivial Tor-vanishing, then so does R.

(2) Assume S is flat over R. If S satisfies trivial Ext-vanishing, then so does R.

Proof. We begin with (1). Assume $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$. By replacing M and N with sufficiently high syzygies , we may assume M, N, and $M \otimes_R N$ are MCM, and that $\operatorname{Tor}_i^R(M,N) = 0$ for $i > 0$. Let $F_\bullet^M$ and $F_\bullet^N$ be minimal free resolutions of M and N, respectively. Then $F_\bullet^M \otimes_R F_\bullet^N$ is a minimal free resolution of $M \otimes_R N$. By [CFF02, 3.4(2)], we have $F_\bullet^M \otimes_R S$ and $F_\bullet^N \otimes_R S$ are minimal free S-resolutions of $M \otimes_R S$ and $N \otimes_R S$, respectively. Therefore,

$$\operatorname{Tor}_i^S(M \otimes_R S, N \otimes_R S) = H_i(F_\bullet^M \otimes_R S) \otimes_S (F_\bullet^N \otimes_R S) \cong H_i((F_\bullet^M \otimes_R F_\bullet^N) \otimes_R S) = 0,$$

where the last equality follows from [CFF02, 3.4(2)]. As S satisfies trivial vanishing, we have that either $M \otimes_R S$ or $N \otimes_R S$ has finite projective dimension over S. Since $F_\bullet^M \otimes_R S$ and $F_\bullet^N \otimes_R S$ are minimal free resolutions of these modules, it follows that either M or N has finite projective dimension over R, completing the proof.

We now prove (2). Assume S satisfies trivial Ext-vanishing and assume M and N are R-modules with $\operatorname{Ext}_R^i(M,N) = 0$ for $i \gg 0$. Since $\operatorname{Ext}_R^i(M,N) \otimes_S S \cong \operatorname{Ext}_S^i(M \otimes_R S, N \otimes_R S)$ for each i, we conclude $\operatorname{Ext}_S^i(M \otimes_R S, N \otimes_R S) = 0$ for $i \gg 0$. By assumption we must have that $M \otimes_R S$ has finite projective dimension over S or $N \otimes_R S$ has finite injective dimension over S. Therefore, the same holds over R [FT77, Corollary 1]. This concludes the proof. $\square$

The following proposition is crucial for our results as it allows us to pass to complete rings in the proofs, or even Artinian, and with infinite residue field.

Proposition 3.3.6. Let R be a CM local ring. The following conditions are equivalent

(1) R satisfies trivial Tor-vanishing.

(2) $R/\underline{x}R$ satisfies trivial Tor-vanishing for an R-regular sequence $\underline{x} \in \mathfrak{m} \setminus \mathfrak{m}^2$.

(3) $\hat{R}$ satisfies trivial Tor-vanishing.

Proof. Each condition (2)-(3) implies (1) by Proposition 3.3.5 (1). We show (1) implies both (2) and (3), starting with (1) implies (2).

Assume R satisfies trivial Tor-vanishing and let M and N be $R/\underline{x}R$-modules which satisfy $\operatorname{Tor}_i^{R/\underline{x}R}(M,N) = 0$ for $i \gg 0$. A standard change of rings spectral sequence (see [Rot09, 10.73]) induces the following long exact sequence (cf. [HJ03, (1.4)]),

$$\begin{tikzcd}
{} \arrow[r] & M \otimes_{R/\underline{x}R} N \arrow[r] & \operatorname{Tor}_1^R(M,N) \arrow[r] & \operatorname{Tor}^1_{R/\underline{x}R}(M,N) \arrow[r] & 0 \\
{} \arrow[r] & \operatorname{Tor}^1_{R/\underline{x}R}(M,N) \arrow[r] & \operatorname{Tor}_2^R(M,N) \arrow[r] & \operatorname{Tor}^2_{R/\underline{x}R}(M,N) & \\
{} \arrow[r, dashed] & \operatorname{Tor}^2_{R/\underline{x}R}(M,N) \arrow[r] & \operatorname{Tor}_3^R(M,N) \arrow[r] & \operatorname{Tor}^3_{R/\underline{x}R}(M,N) &
\end{tikzcd}$$

Since $\operatorname{Tor}_i^{R/\underline{x}R}(M,N) = 0$ for $i \gg 0$, it follows that $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$. By assumption M or N must have finite projective dimension over R. Since $\underline{x} \in \mathfrak{m} \setminus \mathfrak{m}^2$, the same holds over $R/\underline{x}R$ [Nag75, Corollary 27.5]. The conclusion follows.

We now show (1) implies (3). Let $\underline{x}$ be a maximal R-regular sequence in $\mathfrak{m} \setminus \mathfrak{m}^2$. As (1) implies (2), we know $R/\underline{x}R$ satisfies trivial Tor-vanishing. Since $R/\underline{x}R$ is Artinian, it is complete, and also $R/\underline{x}R \cong \hat{R}/\underline{x}\hat{R}$. Then $\hat{R}$ satisfies trivial Tor-vanishing by Proposition 3.3.5 (1).

□

We are now ready to prove Theorem 3.3.2.

Proof of Theorem 3.3.2. We begin with (2). By Remark 3.2.12 and by passing to sufficiently high syzygies, we may assume M and N are MCM. The result now follows from Lemma 3.3.4 (2) and (3).

We continue with (1). Assume R satisfies trivial Tor-vanishing. By Proposition 3.3.6, $\hat{R}$ satisfies trivial Tor-vanishing. Since $\hat{R}$ has a canonical module, $\hat{R}$ has trivial Ext-vanishing as well, and then so does R by Proposition 3.3.5 (2). □

Remark 3.3.7. In general, trivial vanishing need not ascend along flat local maps. Indeed, any complete equicharacteristic CM local ring $(S, \mathfrak{n})$ is a finite flat extension of a regular local ring R (Noether normalization lemma). But there exist such S that do not satisfy trivial vanishing (e.g., S is a complete complete intersection with $\operatorname{codim}(S) \geqslant 2$).

Trivial vanishing is also not preserved by modding out a regular element in $\mathfrak{m}^2$; see Theorem 3.3.8 (2).

We now summarize results in the literature that characterize the trivial vanishing condition in small codimension. We recall that R has an *embedded deformation* if there exists a a local ring $(Q, \mathfrak{p})$ such that $R \cong Q/(\underline{f})$, for some regular sequence $\underline{f} \subseteq \mathfrak{p}^2$.

Theorem 3.3.8. Let R be a CM local ring and set $c = \operatorname{codim}(R)$, then following hold.

(1) If $c \leqslant 1$, R satisfies trivial vanishing [HW97, 1.9].

(2) If $c \geqslant 2$ and $\hat{R}$ has an embedded deformation, then R does not satisfy trivial vanishing [Ş03, 4.2].

(3) If $c = 2$, then R satisfies trivial vanishing if and only if it is not a complete intersection if and only if it is a Golod ring [Sch64], [Jc04, 1.1].

(4) If $c = 3$, or $c = 4$ and R is Gorenstein, then R satisfies trivial vanishing if and only if $\hat{R}$ has no embedded deformation (Proposition 3.4.10), [Ş03, 2.1].

(5) There exists R with $c = 4$, and another Gorenstein R with $c = 5$, such that $\hat{R}$ has no embedded deformation and R does not satisfy trivial vanishing [GP90, 3.10], [Jc04, 3.3].

3.4 Main results

This section includes our main results. We present several sufficient conditions for a CM local ring to satisfy trivial vanishing.

The following is an important lemma for the proofs of our results; it allows us to relate the vanishing of Tor with the growth of the Betti numbers of the modules involved.

Lemma 3.4.1. Let R be a CM local ring and set $c = \operatorname{codim}(R)$, $e = e(R)$, and $\ell = \ell\ell(R)$. Let M and N be non-free MCM R-modules and assume $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$, then

(1)

$$(\operatorname{lr}_R(M)+1)(\operatorname{lr}_R(N)+1) \leqslant e.$$

(2) If the limits in $\operatorname{lr}_R(M)$ and $\operatorname{lr}_R(N)$ exist, we also have

$$\operatorname{lr}_R(M)\operatorname{lr}_R(N) \leqslant e - c - \ell + 2.$$

Proof. We begin with (1). By assumption, $\operatorname{Tor}^i_R(\Omega^R_j(M),N) = 0$ for $i \gg 0$ and $j \geqslant 0$. Therefore, from Lemma 3.2.3 it follows that

$$\operatorname{lr}_R(N) \leqslant \operatorname{u}_R(\Omega_j^R(M)) - 1 \text{ for every } j \geqslant 0. \tag{3.1}$$

Fix $n \in \mathbb{N}_{n>0}$ and pick $j \in \mathbb{N}$ such that $\beta_{j+1}^R(M) > (\operatorname{lr}_R(M) - 1/n)\beta_j^R(M)$. From the additivity of multiplicities, we have

$$e_R(\Omega_j^R(M)) + e_R(\Omega_{j+1}^R(M)) = e_R(R^{\beta_j^R(M)}) = e\beta_j^R(M).$$

Therefore, either

$$e_R(\Omega_j^R(M)) \leqslant \frac{e\beta_j^R(M)}{\operatorname{lr}_R(M) - 1/n + 1},$$

or,

$$e_R(\Omega_{j+1}^R(M)) \leqslant \frac{(\operatorname{lr}_R(M) - 1/n)e\beta_j^R(M)}{\operatorname{lr}_R(M) - 1/n + 1} < \frac{e\beta_{j+1}^R(M)}{\operatorname{lr}_R(M) - 1/n + 1}.$$

Hence, $\min\{\operatorname{u}_R(\Omega_j^R(M)), \operatorname{u}_R(\Omega_{j+1}^R(M))\} \leqslant \dfrac{e}{\operatorname{lr}_R(M) - 1/n + 1}$. Then from (3.1) we obtain $e \geqslant (\operatorname{lr}_R(M) - 1/n + 1)(\operatorname{lr}_R(N) + 1)$. The result of (a) now follows by taking $\lim\limits_{n\to\infty}$.

We continue with (2). We may assume R is Artinian (see Remark 3.2.4). Let $i \in \mathbb{N}$ and consider the module $\Gamma = \Omega_{i+1}^R(M) \otimes_R \Omega_{i+1}^R(N)$. By the vanishing of Tor assumption we have $\Gamma \hookrightarrow \mathfrak{m}^2(R^{\beta_i^R(M)} \otimes_R R^{\beta_i^R(N)})$. Therefore, from [GP90, 2.1], it follows that

$$(e - c - \ell + 2)\beta_i^R(M)\beta_i^R(N) \geqslant \mu(\Gamma) = \beta_{i+1}^R(M)\beta_{i+1}^R(N).$$

The conclusion now follows by dividing by $\beta_i^R(M)\beta_i^R(N)$ and taking $\lim\limits_{i\to\infty}$. $\square$

We now present the first main result of this section which gives a sufficient numerical condition for a ring to satisfy trivial vanishing.

Theorem 3.4.2. Let R be a CM local ring and set $c = \operatorname{codim}(R)$, $e = e(R)$, and $\ell = \ell\ell(R)$. If

$$e < \frac{4c + 2\ell - 1 - \sqrt{8c + 4\ell - 3}}{2}, \tag{3.2}$$

then R satisfies trivial vanishing.

Proof. By Theorem 3.3.2 it suffices to show R satisfies trivial Tor-vanishing. In order to simplify the notation, set $b := 2c+\ell-1$, then we have

$$2e < 2b+1-\sqrt{4b+1},$$

therefore $b-e > 0$. We proceed by contradiction. Assume there exists M and N of infinite projective dimension with $\mathrm{Tor}_i^R(M,N) = 0$ for $i \gg 0$. By replacing M and N with sufficiently high syzygies, we may assume M and N are MCM. From Proposition 3.2.6 (2) it follows that $\min\{\mathrm{lr}_R(M), \mathrm{lr}_R(N)\} \geqslant b-e$. Therefore, by Lemma 3.4.1 (1) we have $e \geqslant (b-e)^2$. After rearranging terms, this inequality becomes

$$e^2-(2b+1)e+b^2 \leqslant 0,$$

which implies $e \geqslant \frac{2b+1-\sqrt{4b+1}}{2}$. This contradiction proves the result. □

We are now ready to present the second main result of the section. In this theorem, we prove a stronger result than Theorem 8.3 under the extra assumption that the ring is generalized Golod (cf. [HcV04, 3.1]).

Theorem 3.4.3. Let R be a CM local ring that is also generalized Golod and set $c = \mathrm{codim}(R)$, $e = e(R)$, and $\ell = \ell\ell(R)$. Assume

$$e \leqslant 2c+\ell-3 \tag{3.3}$$

then for any R-modules M and N, we have

(1) If $\mathrm{Tor}_i^R(M,N) = 0$ for $i \gg 0$, then $\mathrm{cx}_R(M) \leqslant 1$, or, $\mathrm{cx}_R(N) \leqslant 1$.

(2) If $\mathrm{Ext}_R^i(M,N) = 0$ for $i \gg 0$, then $\mathrm{cx}_R(M) \leqslant 1$, or, $\mathrm{px}_R(N) \leqslant 1$.

(3) If, additionally, $e \leqslant 2c+\ell-4$, then R satisfies trivial vanishing.

Proof. By Proposition 3.3.6 we may assume R is complete. By Theorem 3.3.8 (1), (3) and [AB00] the conclusion holds if $c \leqslant 2$. Therefore, we can assume $c \geqslant 3$. By the assumption and Proposition 3.2.6 (2), (3), every R-module has either infinite complexity, or complexity at most one. We note that (2) follows from (1), Lemma 3.3.4 (2), and Remark 3.2.13. Then we only show the proof of (1) and (3).

We begin with (1). By replacing M and N with sufficiently high syzygies, we may assume M and N are MCM. We proceed by contradiction. Assume $\mathrm{cx}_R(M) = \mathrm{cx}_R(N) = \infty$. In order to simplify the notation, set $b := e - c - \ell + 2$. By Proposition 3.2.10 the limits in $\mathrm{lr}_R(M)$ and $\mathrm{lr}_R(N)$ exist and are larger than one. Let γ be either one of these limits. By dividing the inequality in Proposition 3.2.6 (1) by $\beta^R_{n-1}(\text{-})$ and taking $\lim\limits_{n\to\infty}$ we obtain $\gamma \geqslant c - \frac{b}{\gamma}$. Consider the polynomial $p(z) = z^2 - cz + b$ and notice $p(\gamma) \geqslant 0$. By assumption $p(z)$ has positive discriminant, $c^2 - 4b \geqslant 4(2c + \ell - 3 - e) > 0$. Hence, $p(z)$ has only real roots. Moreover, $c^2 - 4b - (c-2)^2 \geqslant 0$. Therefore, $\frac{c-\sqrt{c^2-4b}}{2} \leqslant 1$. Since $\gamma > 1$, we have $\gamma \geqslant \frac{c+\sqrt{c^2-4b}}{2}$. Therefore,

$$\mathrm{lr}_R(M)\,\mathrm{lr}_R(N) \geqslant \frac{(c+\sqrt{c^2-4b})^2}{4} > \frac{(c+\sqrt{c^2-4b})(c-\sqrt{c^2-4b})}{4} = b,$$

which contradicts Lemma 3.4.1. This finishes the proof of (1).

We continue with (3). Under the extra assumption $e \leqslant 2c + \ell - 4$, Proposition 3.2.6 (2) implies that if an R-module has finite complexity, then it has finite projective dimension. This concludes the proof. □

Remark 3.4.4. We remark that the proof of Theorem 3.4.3 requires less from R than being as in Example 3.2.8, or even generalized Golod. Indeed, it only requires that the limit $\lim\limits_{n\to\infty} \dfrac{\beta^R_{n+1}(M)}{\beta^R_n(M)}$ exist under vanishing of Tor hypothesis. This strategy was used in [HcV04, 3.1] to prove a similar result under the assumption $\mathfrak{m}^3 = 0$.

Remark 3.4.5. By Theorem 3.3.8 (2), it follows that there is no CM local ring R for which $\hat{R}$ has an embedded deformation, $\mathrm{codim}(R) \geqslant 2$, and such that it satisfies (3.2) or the assumption of Theorem 3.4.3 (3).

A CM local ring R is *stretched* if $e(R) = \operatorname{codim}(R) + \ell\ell(R) - 1$ [Sal80]. In [CDG$^+$16, 5.4] it is proved that this class of rings are generalized Golod. As a corollary of Theorem 3.4.3 we recover the following result which originally appeared in [Gup17].

Corollary 3.4.6. Let R be a stretched CM local ring such that $\operatorname{codim}(R) \geqslant 3$, then R satisfies trivial vanishing.

The following proposition provides information from vanishing of Ext and Tor over a generalized Golod rings when one knows the Poincaré series and multiplicity (cf. [Ş03, 1.5]). We recall the definition of $\operatorname{Den}^R(t)$ was given in Theorem 3.2.8.

Proposition 3.4.7. Let R be a CM local ring that is also generalized Golod and and set $c = \operatorname{codim}(R)$, $e = e(R)$, and $\ell = \ell\ell(R)$. Let $\rho = \min\{\sqrt{e} - 1, \sqrt{e - c - \ell + 2}\}$ and assume $\operatorname{Den}^R(t)$ does not have real roots in the interval $[1/\rho, 1)$, then

(1) If $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$, then either $\operatorname{cx}_R(M)$ or $\operatorname{cx}_R(N)$ is finite.

(2) If $\operatorname{Ext}_R^i(M,N) = 0$ for $i \gg 0$, then either $\operatorname{cx}_R(M)$ or $\operatorname{px}_R(N)$ is finite.

Proof. As in the proof of Theorem 3.4.3, it suffices to show (1), and also we may assume M and N are MCM. We proceed by contradiction. Assume $\operatorname{cx}_R(M) = \operatorname{cx}_R(N) = \infty$. By [Avr89, (2.3)], Theorem 3.2.7, and [Sun98, Corollary] it follows that $\min\{\operatorname{lr}_R(M), \operatorname{lr}_R(N)\} > \rho$. This contradicts Lemma 3.4.1. $\square$

We obtain the following result about trivial vanishing for rings of small multiplicity. We remark that this result is optimal in the sense that the conclusion does not hold for higher multiplicities (see Example 3.4.14). (cf. [GP90, 1.2])

Proposition 3.4.8. Let R be a CM local ring such that $\operatorname{codim}(R) \neq 1$. If $e(R) \leqslant 7$, or $e(R) \leqslant 11$ and R is Gorenstein, then R satisfies trivial vanishing if and only if $\hat{R}$ has no embedded deformation.

We first discuss when $\hat{R}$ has embedded deformations.

Remark 3.4.9. Write $\hat{R} = P/I$ where $(P, \mathfrak{n})$ is regular and $I \subseteq \mathfrak{p}^2$. Set $n = \mu(I)$. Under the assumptions of Proposition 3.4.8, one observes from the following proof that if R does not satisfy trivial vanishing, then $\operatorname{codim}(R) \leqslant 3$, or $\operatorname{codim}(R) = 4$ in the Gorenstein case. In these cases, by [Avr89, 3.6] we have that $\hat{R}$ has a embedded deformation if and only if it is a complete intersection, of type $\mathbf{H(n-1, n-2)}$ for $n \geqslant 4$ (cf. Remark 3.4.11), or of type $\mathbf{G(n-1)}$ for $n \geqslant 5$

Proof of Proposition 3.4.8. The forward direction is Theorem 3.3.8 (2); we prove the backward one. By Theorem 3.3.2 it suffices to show R satisfies trivial Tor-vanishing. Set $c = \operatorname{codim}(R)$ and $\ell = \ell\ell(R)$. By Theorem 3.3.8 (3),(4) we may further assume $c \geqslant 4$, and $c \geqslant 5$ if R is Gorenstein. Moreover, by [Jc04, 1.1] we may also assume $\ell \geqslant 3$.

If $e(R) \leqslant 7$, the result follows by applying Theorem 8.3, then we assume R is Gorenstein and $e(R) \leqslant 11$. By [Gup17, 6.9] the ring R is generalized Golod and then Theorem 3.4.3 implies that if $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$, then either M or N have complexity at most one (notice $e(R) = c+2$ if $\ell = 3$). Hence, by [GP90, 1.1], one of these modules has finite projective dimension, finishing the proof. □

The following is a general result about codimension three CM local rings. A similar result for Gorenstein local rings up to codimension four was shown by Şega in [Ş03]. Moreover, the authors of the ongoing work [AINSW20] have informed us that they show trivial vanishing holds if $\mu(\mathfrak{m}) - \operatorname{depth}(R) \leqslant 3$ (without the CM assumption) in all but some exceptional cases.

We say that a finitely generated R-module M is *periodic of period p after n steps* if there exist $p \in \mathbb{N}$ such that $\Omega_i^R(M) \cong \Omega_{i+p}^R(M)$ for every $i \geqslant n$ (cf. [GP90]).

Proposition 3.4.10. Let R be a CM local ring of dimension d and $\operatorname{codim}(R) = 3$. Then R satisfies trivial vanishing if and only if $\hat{R}$ has no embedded deformation.

Moreover, if R is not a complete intersection and M and N are R-modules with $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$, then either M or N is periodic of period two after at most $d+1$ steps.

We need the following observation prior to the proof the result

Remark 3.4.11. In [Avr89, 3.6], Avramov explains the structure of CM rings of $\operatorname{codim}(R) \leqslant 3$ and such that $\hat{R}$ has embedded deformation. Write $\hat{R} = P/I$ where $(P, \mathfrak{n})$ is regular and $I \subseteq \mathfrak{p}^2$. Indeed, $\hat{R}$ has a embedded deformation if and only if it is either a complete intersection, or of type $\mathbf{H}(\boldsymbol{n-1}, \boldsymbol{n-2})$ where $n = \mu(I) \geqslant 4$. In the latter case, I can be assumed to be of the form $J + (x)$, where J is an ideal of height two that is generated by the maximal minors of a $(n-1) \times (n-2)$ matrix with entries in $\mathfrak{n}$, and x is regular on P/J.

Proof of Proposition 3.4.10. We begin with the first statement. First we note that the forward implication follows from Theorem 3.3.8 (2), hence we may focus on the backward implication. By Proposition 3.3.6 we may assume R is complete. By Cohen Structure Theorem there exits a regular local ring $(P, \mathfrak{n}, k)$ such that $R \cong P/I$ with $I \subseteq \mathfrak{n}^2$ and by [BE77, 1.3] the minimal P-resolution of R has a DG-algebra structure. Set T to be the graded k-algebra $\operatorname{Tor}^P(R, k)$. In what follows, for a graded k-algebra B such that $B_0 = k$ and $B_i = 0$ for $i \geqslant 4$, we denote by $\mathscr{H}_B$ the vector $(\dim_k B_1, \dim_k B_2, \dim_k B_3)$.

Set $n = \mu(I)$, $\tau = r(R)$ the type of R, and note that by assumption we have $n \geqslant 4$. Then we have a minimal free S-resolution

$$0 \to S^\tau \to S^{n+\tau-1} \to S^n \to S \to R \to 0.$$

Equivalently, $\mathscr{H}_T = (n, n+\tau-1, \tau)$. By [AKM88, 2.1], there exists a graded k-algebra A and a vector space W such that T is isomorphic to the trivial extension $A \ltimes W$. We have the following possibilities for $\mathscr{H}_A$ [AKM88, 2.1]

R is of type **TE**, then $\mathscr{H}_A = (3,\ 3,\ 0)$.

R is of type **B**, then $\mathscr{H}_A = (2,\ 3,\ 1)$.

R is of type $\mathbf{G}(\boldsymbol{r})$, then $\mathscr{H}_A = (r,\ r,\ 1)$ for some $r \geqslant 2$.

R is of type $\mathbf{H}(\boldsymbol{p}, \boldsymbol{q})$ then $\mathscr{H}_A = (p+1,\ p+q,\ q)$ for some $p, q \geqslant 1$.

By comparing $\mathscr{H}_T$ and $\mathscr{H}_A$ on each of the cases above, we can immediately see that $W \neq 0$ unless R is Gorenstein of type $\mathbf{G}(\boldsymbol{n})$, or R is of type $\mathbf{H}(\boldsymbol{n-1,\tau})$. If R is not as in the latter exceptional cases, then it follows from [AINSW19, 5.3] that R satisfies trivial vanishing. If R is of type $\mathbf{G}(\boldsymbol{n})$ then R also satisfies trivial vanishing by [Ş03, 2.3]. It remains to consider the case $\mathbf{H}(\boldsymbol{n-1,\tau})$. By [CVW20, 1.1] we must have $p-1=q=\tau=n-2$. The first statement now follows by Remark 3.4.11.

We continue with the second statement. By replacing M and N with sufficiently high syzygies, we may assume M and N are non-free MCM and R has an embedded deformation. Let $(Q,\mathfrak{p})$ be a local ring such that $R \cong Q/(\underline{f})$ where $\underline{f} \subseteq \mathfrak{p}^2$ is a regular sequence. Since R is not a complete intersection, we must have $\operatorname{codim}(Q)=2$, and then Q is a Golod ring [Sch64]. Moreover, $\operatorname{Tor}_i^Q(M,N)=0$ for $i \gg 0$ [AY98, 2.6]. By Theorem 3.3.8 (3) and the Auslander-Buchsbaum formula, it follows that either M or N has projective dimension one over Q. The conclusion now follows from [Avr89, (1.6)(II)]. $\square$

3.4.1 Examples

In the remaining part of this section, we provide examples that discuss the sharpness of our results and the necessity of the conditions. As with previous results, we set $c=\operatorname{codim}(R)$, $e=e(R)$, and $\ell=\ell\ell(R)$.

We begin with the following example.

Example 3.4.12. Let $R=k[[x,y]]/(x^2,y^2)$. We note that in this case $e=4$, $c=2$, and $\ell=3$. Therefore, R satisfies equality in (3.2). However, R does not satisfy trivial vanishing by Theorem 3.3.8 (2). In fact, we may consider $M=R/xR$ and $N=R/yR$ for which we have $\operatorname{Tor}_i^R(M,N)=0$ for every $i>0$.

Example 3.4.13. Let $R_1=k[[x,y]]/(x,y)^2$, $R_2=k[[z]]/(z^2)$, and $R=R_1\otimes_k R_2$. We note that R has an embedded deformation, then this is an example of the exceptional case in Proposition 3.4.10. By Theorem 3.3.8 (2), R does not satisfy trivial vanishing. If fact, we may consider $M=R/(x,y)R$

and $N = R/zR$. We also remark that, since $e = 6$, $c = 3$, and $\ell = 3$, the inequality in (3.3) is satisfied, but not the extra condition of Theorem 3.4.3 (3). Hence, this example shows that this extra condition is necessary to guarantee trivial vanishing.

Example 3.4.14. Using [GP90, 3.1 3.4], the authors in [Jc04, 3.3] provided some examples of rings that do not satisfy trivial vanishing. The first one is a Gorenstein ring such that $e = 12$, $c = 5$, and $\ell = 4$. The second one has $e = 8$, $c = 4$, and $\ell = 3$. The completion of each these rings does not have an embedded deformation [GP90, 3.10]. In these examples the right hand side of (3.2) are approximately 9.9 and approximately 7.3, respectively.

Example 3.4.15. Let R be a Artinian Gorenstein local ring such that $c = 6$, $\ell = 4$, and $\mu(\mathfrak{m}^2) = 4$. Therefore, $e = 12$. By [Gup17, 6.9] the ring R is generalized Golod. Thus, by Theorem 3.4.3 (3), R satisfies trivial vanishing. We note that this conclusion cannot be obtained from Theorem 8.3.

We now present examples that show that the converse of our theorems do not hold.

Example 3.4.16. Assume R is a stretched CM local ring such that $c \geqslant 3$. By Corollary 3.4.6 R satisfies trivial vanishing. However, if one fixes e and c, then for $\ell \gg 0$, the inequality (3.2) does not hold. If R is Artinian Gorenstein and *almost stretched*, i.e., $\mu(\mathfrak{m}^2) \leqslant 2$, then R is generalized Golod and satisfies trivial vanishing [Gup17, 6.3]. However if R is Artinian and has h-vector $(1,5,2,2,2,2,1)$, then (3.3) fails.

Example 3.4.17. For every integer $\tau \geqslant 2$, Yoshino constructed non-Gorenstein rings that do not satisfy trivial vanishing and such that $e = 2\tau+2$, $c = \tau+1$, and $\ell = 3$ [Yos03, 4.2]. These examples show that for rings that do not satisfy trivial vanishing, the distance between e and the right hand side of (3.2) can be as large as possible, even if we assume the ring is not Gorenstein. Moreover, for $\tau = 2$, this example also shows that the extra condition of Theorem 3.4.3 (3) is necessary to conclude trivial vanishing.

3.5 The uniform Auslander condition and the Auslander-Reiten conjecture

In this section we prove that the Auslander-Reiten conjecture holds for a ring with small multiplicity with respect to its codimension. We also provide new classes of rings that satisfy the following condition.

Definition 3.5.1. The ring R satisfies the *uniform Auslander condition* (UAC) if there exists $n \in \mathbb{N}$ such that whenever $\mathrm{Ext}_R^i(M,N) = 0$ for $i \gg 0$, this vanishing holds for $i \geqslant n$.

In [CH10] the authors prove that if R satisfies the UAC then it satisfies Conjecture 3.1.1. Some classes of rings that satisfy this condition are complete intesection rings, Golod rings, CM local rings of codimension at most two, and Gorenstein rings of codimension at most four [AB00, Jc04, Ş03]. We refer the reader to [CH10, Appendix A] for a survey on the topic and to [AINSW20] for related results.

As a corollary of our results in the previous section, we are able to provide new classes of rings that satisfy UAC and hence Conjecture 3.1.1. In particular, part (3) implies that the examples constructed in [Jc04] for the failure of UAC are minimal with respect to codimension, and part (4) shows they are minimal with respect to multiplicity.

Corollary 3.5.2. Let R be a CM local ring and assume R satisfies one of the following conditions.

(1) R satisfies the inequality (3.2).

(2) R is as one of (3)-(8) in Example 3.2.8 and satisfies the inequality (3.3), or R is any generalized Golod ring and satisfies the assumption of Theorem 3.4.3 (3).

(3) $\mathrm{codim}(R) \leqslant 3$.

(4) $e(R) \leqslant 7$, or $e(R) \leqslant 11$ and R is Gorenstein.

Then R satisfies UAC and thus the Auslander-Reiten conjecture $(\text{Conjecture } 3.1.1)$ holds for R.

Proof. We can assume that R is complete [CH10, 5.5]. Let M and N be R-modules such that $\mathrm{Ext}_R^i(M,N)=0$ for every $i \gg 0$. By Remark 3.2.12 and by replacing M for a sufficiently high syzygy, we may assume M and N are MCM.

If R is as in (1), then it satisfies trivial vanishing and hence UAC.

Assume now R is as in (2). By Theorem 3.4.3, M or $N^\vee$ has finite complexity. Therefore, by

- ▷ [Avr89, 1.5] for (3)-(6),
- ▷ [KPS94, proof of 5.2] for (7), and
- ▷ [Avr89, 2.4], [Kus95, 5.2] for (8),

we have that M or $N^\vee$ has finite *complete intersection* (CI) *dimension* (see [AB00]). Therefore, R satisfies UAC by [AB00, 4.7, 4.1.5]. The second statement of (2) is clear as R satisfies trivial vanishing.

We now consider (3), we may assume $\mathrm{codim}(R)=3$ [Jc04, 1.1]. In this case Proposition 3.4.10 and its proof show that when R does not satisfy trivial vanishing, then one of M or $N^\vee$ has finite CI-dimension. Thus the conclusion follows as before.

For (4), from the proof of Proposition 3.4.8 it follows that R satisfies trivial vanishing if $\mathrm{codim}(R) \geqslant 4$, or if $\mathrm{codim}(R) \geqslant 5$ in the Gorenstein case. The result now follows from (3) and [Ş03]. $\square$

The following is the main theorem of this section.

Theorem 3.5.3. Let R be a CM local ring and set $c=\mathrm{codim}(R)$. Assume R satisfies one of the following conditions.

(1) $e(R) \leqslant \frac{7}{4}c+1$.

(2) $e(R) \leqslant c+6$ and R is Gorenstein.

Let M be an R-module such that $\mathrm{Ext}_i^R(M, M\oplus R)=0$ for $i \gg 0$. Then M has finite projective dimension, i.e., the Auslander-Reiten conjecture (Conjecture 3.1.1) holds for R.

Proof. Proceeding as in Proposition 3.3.5 (2), we may assume R is complete and k is infinite. We may also replace M by $\Omega_j^R(M)$ for any $j \in \mathbb{N}$ to assume M is MCM and $\operatorname{Ext}_i^R(M, M \oplus R) = 0$ for $i > 0$ [HL04, 1.2]. Hence by standard arguments we may assume R is Artinian (see Remark 3.2.4).

We prove (1) first. Set $\ell = \ell\ell(R)$ and $b = 2c + \ell - 1$. Notice that the right hand side of (3.2) is $f(b) := b + \frac{1}{2} - \sqrt{b + \frac{1}{4}}$, which is an increasing function for $b \geqslant 0$. By [HcV04, 4.1] the conclusion holds if $\ell \leqslant 3$. Thus we may assume $\ell \geqslant 4$ and hence $f(b) \geqslant 2c + \frac{7}{2} - \sqrt{2c + \frac{13}{4}}$. Let $g(c)$ be the right hand of the last inequality, viewed as a function of c. It suffices to show $g(c) > \frac{7}{4}c + 1$ which is equivalent to $\frac{1}{16}(c-6)^2 + \frac{3}{4} > 0$. The result follows.

We now prove (2). If $e(R) \leqslant 11$, the conclusion follows from Corollary 3.5.2 (4), and if $c \geqslant 7$, it follows from part (1) of this theorem. Hence we may assume $c = 6$ and $e(R) = 12$. By [HcV04, 4.1] and Theorem 8.3, it remains to consider when $\ell = 4$. Therefore we are in the situation of Example 3.4.15, and then R satisfies trivial vanishing. This finishes the proof. □

Corollary 3.5.4. Let R be a CM local ring such that $e(R) \leqslant 8$. Then the Auslander-Reiten conjecture holds for R.

Proof. By Corollary 3.5.2 we may assume $\operatorname{codim}(R) \geqslant 4$. The conclusion then follows from Theorem 3.5.3. □

Remark 3.5.5. We note that if $e(R) = 9$, the only CM local rings for which we do not know Conjecture 3.1.1 holds are those whose Artinian reductions have h-vectors $(1,4,3,1)$, $(1,4,2,2)$, or $(1,4,2,1,1)$. If $e(R) = 12$ and R is Gorenstein, then we do not know if this conjecture holds when these h-vectors are $(1,5,5,1)$, $(1,5,4,1,1)$, or $(1,5,3,2,1)$; the case $(1,5,2,2,1,1)$ follows from Theorem 3.4.3 or from [Gup17, 6.5].

3.6 Criteria for the Gorenstein property

In this section we expand upon several results in the literaure which provide criteria for R to be Gorenstein based on vanishing of Ext.

We begin with the following lemma. By using this result with $M = \omega_R$, we drop the generically Gorenstein condition from [JL08, 2.4] (cf. [JL07, 2.4]). This in turn allows us to provide a more general statement for [JL08, 2.4].

Lemma 3.6.1 ([CSV10, A.1])**.** Let R be a CM local ring and M be an R-module such that $e_R(M) = e(R)$. If M is not free, then $\beta_1^R(M) \geqslant \beta_0^R(M)$.

We obtain the following criterion for the Gorenstein property. Similar results have appeared in the work of several authors (see [HH05], [JL08], [Ulr84]). We recall that the notation $\mathrm{u}_R(M)$ was introduced in Definition 3.2.1.

Theorem 3.6.2. Let R be a CM local ring of dimension d and with canonical module ω_R. Let M be a CM R-module with $\mathrm{u}_R(M) < 2$. Assume $\mathrm{Ext}_R^i(M,R) = 0$ for every $d - \dim M + 1 \leqslant i \leqslant d+1$. Then R is Gorenstein.

Proof. The result follows from Lemma 3.6.1 by applying Lemma 3.2.3 (2) with $N = \omega_R$. □

In [JL07], Jorgensen and Leuschke ask the following question.

Question 3.6.3 ([JL07, 2.6])**.** If R is a CM local ring with canonical module ω_R, does $\beta_1(\omega_R) \leqslant \beta_0(\omega_R)$ imply that R is Gorenstein?

They remark that a positive answer to this question would provide improvements to their results [JL07, 2.2] and [JL07, 2.4]. We give a positive answer to Question 3.6.3 in a particular case, which is sufficient to produce the desired improvement of [JL07, 2.4], as well as [HH05, 3.4] .

We first recall the following result of Asashiba and Hoshino. In the following statement, we denote by M^* the *R-dual* $\mathrm{Hom}_R(M,R)$.

Lemma 3.6.4 ([AH94, 2.1])**.** Let M and N be R-modules. Assume M is faithful and that we have an exact sequence $0 \to N \xrightarrow{\varphi} R^2 \xrightarrow{\psi} M \to 0$. Then there exists maps α, β, and an isomorphism θ

that make the following diagram commute.

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & R^2 \arrow[r, "\psi"] \arrow[d, "\theta"'] & M \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & M^* \arrow[r, "\psi^*"] & (R^2)^* \arrow[r, "\varphi^*"] & N^* &
\end{tikzcd}
$$

We say that an R-module M has *constant rank* if there is an $r \in \mathbb{N}$ such that $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^r$ for all $\mathfrak{p} \in \operatorname{Ass}(R)$. In this case, we refer to r as the *rank* of M and denote it by $\operatorname{rank} M$.

We derive our result on Question 3.6.3 from the following more general result.

Proposition 3.6.5. Let R be a CM local ring. Assume there exists a non-free faithful MCM R-module M, such that it has constant rank and $\max\{\beta_0^R(M), \beta_1^R(M)\} \leqslant 2$. Then

(1) M is periodic of period two and is reflexive, i.e., the natural map $M \to M^{**}$ is an isomorphism.

(2) If R is not Gorenstein, then it does not satisfy trivial vanishing.

Proof. We begin with (1). Since M is not free, M has infinite projective dimension. Therefore, we must have $\beta_0^R(M) = \beta_1^R(M) = 2$ and $\operatorname{rank}(M) = \operatorname{rank}(\Omega_1^R(M)) = \operatorname{rank}(\Omega_2^R(M)) = 1$ by additivity of ranks. By Lemma 3.6.4, we have the following commutative diagram, where θ is an isomorphism.

$$
\begin{tikzcd}
0 \arrow[r] & \Omega_1^R(M) \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & R^2 \arrow[r, "\psi"] \arrow[d, "\theta"'] & M \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & M^* \arrow[r, "\psi^*"] & (R^2)^* \arrow[r, "\varphi^*"] & (\Omega_1^R(M))^* &
\end{tikzcd}
$$

By commutivity of the diagram, we have that α is injective and, by the snake lemma, we have that $\operatorname{Ker}\beta \cong \operatorname{coker}\alpha$. Since M is MCM and $\operatorname{Ker}\beta$ embeds into M, we have that either $\dim(\operatorname{coker}\alpha) = d$, or $\operatorname{coker}\alpha = 0$. On the other hand,

$$\operatorname{rank}(\operatorname{coker}\alpha) = \operatorname{rank}(M^*) - \operatorname{rank}(\Omega_1^R(M)) = 0.$$

Therefore, $\operatorname{coker}\alpha = 0$ and then α is an isomorphism. Hence, β is an isomorphism as well since

$$\operatorname{rank}(\operatorname{coker}\beta) = \operatorname{rank}(M) - \operatorname{rank}((\Omega_1^R(M))^*) = 0.$$

We conclude that $M \cong M^{**}$, $\operatorname{Ext}_R^1(M^*,R) = 0$, and that we have the following exact sequence $0 \to M^* \to R^2 \to M \to 0$.

Since $(\Omega_1^R(M))^* \cong M$, we have that $(\Omega_R^1(M))^*$ is faithful, and then so is $\Omega_1^R(M)$. Then we may apply Lemma 3.6.4 as above to obtain isomorphisms

$$\Omega_1^R(M) \cong (\Omega_2^R(M))^* \cong (\Omega_1^R(M))^{**}.$$

We also conclude $\operatorname{Ext}_R^1(M,R) = \operatorname{Ext}_R^1((\Omega_1^R(M))^*,R) = 0$ and that we have an exact sequence $0 \to M \to R^2 \to M^* \to 0$. Splicing this sequence with the previous sequence $0 \to M^* \to R^2 \to M \to 0$ repeatedly, we construct a resolution of M. Hence, conclusion of (1) follows.

We observe that this argument shows $\operatorname{Ext}_R^i(M,R) = 0$ for every $i > 0$ and hence (2) follows. This finishes the proof. □

Remark 3.6.6. The resolution constructed in the proof of Proposition 3.6.5 is an example of a *totally acyclic resolution of* M. That is, a complex $F_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots$ such that $M \cong \operatorname{coker}(F_1 \to F_0)$ and such that F and F^* are exact. Modules admitting such a resolution are called *totally reflexive* [AM02, 2.4].

With Proposition 3.6.5 in hand, we provide our result on Question 3.6.3. The ring R is *generically Gorenstein* if $R_{\mathfrak{p}}$ is Gorenstein for every $\mathfrak{p} \in \operatorname{Ass}(R)$, or equivalently, if ω_R has constant rank.

We recall that $r(R)$ denotes the type of R.

Corollary 3.6.7. Let R be a generically Gorenstein CM local ring such that $r(R) = 2$. Then $\beta_1^R(\omega_R) > 2$.

Proof. We proceed by contradiction. Assume $\beta_1^R(\omega_R) \leqslant 2$, then Proposition 3.6.5 (1) implies that

$(\omega_R)^*$ is MCM and we have an exact sequence $0 \to \omega_R \to R^2 \to \omega_R^* \to 0$. By canonical duality, we have the sequence $0 \to (\omega_R^*)^\vee \to \omega_R^2 \to R \to 0$ which splits. Then, by applying canonical duality again, we conclude ω_R free. The latter implies R is Gorenstein, which contradicts the assumption $r(R) = 2$. □

The following is the main result of this section. This theorem provides direct improvements to [JL07, 2.4] and [HH05, 3.4].

Theorem 3.6.8. Let R be a generically Gorenstein CM local ring of dimension d and with canonical module ω_R. Assume there exists an MCM R-module M with $\mathrm{u}_R(M) \leqslant 2$ and such that $\mathrm{Ext}^i_R(M,R) = 0$ for $1 \leqslant i \leqslant d+1$. Then R is Gorenstein.

Proof. From Lemma 3.3.4 (1) it follows that $M \otimes_R \omega_R$ is MCM. By Lemma 3.3.4 (2) we have an exact sequence

$$0 \to M \otimes_R \Omega^1_R(\omega_R) \to M \otimes_R R^{r(R)} \to M \otimes_R \omega_R \to 0 \tag{3.4}$$

whence it follows $M \otimes_R \Omega^1_R(\omega_R)$ is MCM as well. Therefore,

$$e_R(M) = e_R(M \otimes_R \omega_R) \geqslant \mu(M \otimes_R \omega_R) = \mu(M) r(R).$$

This implies $r(R) \leqslant 2$. If $r(R) = 1$, R is Gorenstein. Then we may assume $r(R) = 2$ which implies $\Omega^1_R(\omega)$ has rank 1 by (3.4). Therefore,

$$e_R(M) = e(M \otimes_R \Omega^1_R(\omega_R)) \geqslant \mu(M \otimes_R \Omega^1_R(\omega_R)) = \mu(M)\mu(\Omega^1_R(\omega_R)).$$

Hence $\mu(\Omega^1_R(\omega_R)) \leqslant 2$ which contradicts Corollary 3.6.7. This finishes the proof. □

Chapter 4

Hom and Ext, revisited

4.1 Introduction

Let R be a commutative Noetherian local ring and M,N be finitely generated R modules. The purpose of the work in this chapter is to understand a large and growing body of results which take the form: if $\mathrm{Hom}_R(M,N)$ has some nice properties and $\mathrm{Ext}_R^{1\leq i\leq n}(M,N)=0$ for some n, then M and N must be nice themselves.

For example, about 50 years ago Vasconcelos proved that if R is a Gorenstein ring of dimension 1, and M is a maximal Cohen-Macaulay (MCM) R-module such that $\mathrm{End}_R(M)$ is free, then M is free [Vas68]. Ulrich proposed tests for the Gorensteiness of R using Ext-vanishing between certain modules and R [Ulr84]. Huneke and Leuschke proved an interesting special case of the famous Auslander-Reiten conjecture. One of the main results says that if R is a normal domain of dimension d and M is a module locally free in codimension one, and if $\mathrm{Ext}_R^{1\leq j\leq d}(M,M)=\mathrm{Ext}_R^{1\leq j\leq 2d+1}(M,R)=0$, then M must be free [HL04]. These influential results have been examined and extended quite frequently, see [ACST17, GT17, Lin17b, Lin17a, CIST18] for a sample of some of this interesting recent work and the references therein. These papers all serve as the main inspiration for this chapter.

Our approach to the questions above is to first study the small dimension or depth situation. This is important since most of the proofs involve an inductive process by localization or cutting down with a regular sequence. Surprisingly, this simple-minded approach makes the problems more transparent and yields significant improvements; we can usually remove assumptions such as Cohen-Macaulayness, constant rank, $M=N$, etc., altogether. At the same time, proofs become shorter and more elementary. In fact, we do not need much preparatory material beyond graduate level commutative algebra.

We now describe this chapter in more detail. Let R be a local ring of dimension d and depth t. In Section 4.2 we define two categories of modules that are crucial for our analysis. One is called $\Omega\,\mathrm{Deep}(R)$, which consists of modules M that are a syzygy of some high-depth module. That is, such an M fits into an exact sequence $0 \to M \to F \to X \to 0$ with F free and $\operatorname{depth} X \geq t$. Somewhat dually, the second category, $\mathrm{DF}(R)$, consists of M such that there is an exact sequence $0 \to R \to M^n \to X \to 0$ with $\operatorname{depth} X \geq t$ (DF stands for “deeply faithful"). We establish a number of simple but useful results about these categories. For example, they behave well with respect to “cutting down by a general regular sequence", and any object lying in their intersection must have a free summand (we actually prove a bit more, see Theorem 4.2.8).

In Section 4.3 we study the question: when does $\operatorname{Hom}_R(M,N)$ have a free summand or is free? Our first main result is:

Theorem 4.1.1. Suppose that $\operatorname{depth} M \geq t$ and $N \in \Omega\,\mathrm{Deep}(R)$. Assume that $\operatorname{Hom}_R(M,N) \in \mathrm{DF}(R)$ and $\operatorname{Ext}_R^{1\leqslant i\leqslant t-1}(M,N) = 0$. Then N has a free summand.

This allows us to generalize both the results by Vasconcelos and Huneke-Leuschke mentioned above in Corollary 4.3.3 and Theorem 4.3.11. We also prove that if R and M satisfy Serre's condition (S_2) and $\operatorname{Hom}_R(M,R)$ is free, then M is free (see 4.3.7). The key point here is the dimension one case. Lastly, we extend a result by Goto-Tatakashi to higher rank modules (see Theorem 4.3.13).

In Section 4.4 we study when $\operatorname{Hom}_R(M,N) \cong N^r$ for some $r > 0$. Again we start with the small depth or dimension situation and build from there. Our main technical result is:

Theorem 4.1.2. Assume that $\operatorname{depth}(N) = t$, $\operatorname{depth}(M) \geqslant t$, $\operatorname{Ass}(N) = \operatorname{Min}(N)$, and for some $s \geq t$, $\operatorname{Ext}_R^{1\leqslant i\leqslant s}(M,N) = 0$. If $\operatorname{Hom}_R(M,N) \cong N^r$ for some $r \in \mathbb{N}$, then $M/IM \cong (R/I)^r$ for $I = \operatorname{Ann}(N)$.

Furthermore, if one of the following holds:

1. N is faithful.

2. $\mathrm{Ass}(R) \subseteq \mathrm{Ass}(N)$ and $s > 0$.

then $M \cong R^r$.

We give some applications, including a modest case of the Auslander-Reiten conjecture (Corollary 4.4.7).

In the last section, we address a couple of related topics: a test for Gorensteiness inspired by an old result of Ulrich (Corollary 4.5.1), and an equivalent condition for vanishing of Ext modules that slightly extends results by Huneke-Hanes, Huneke-Jorgensen and Huneke-Leuschke in [HH05, HJ03, HL04], see Corollary 4.5.4.

4.2 Two key categories

Throughout $(R,\mathfrak{m})$ is a Noetherian local ring with $\dim(R) = d$ and $\mathrm{depth}(R) = t$. In this section we define and establish basic facts about two categories of modules that play a crucial role for many of our proofs.

But first, we set some notation. We let $\mu(M)$ denote the minimal number of generators of a module M and $l(M)$ its length. We say that M is *generically free* if $M_\mathfrak{p}$ is free over $R_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}(R)$. Let (S_i) denote Serre's condition: $\mathrm{depth}\, M_\mathfrak{p} \geq \min\{i, \mathrm{ht}\,\mathfrak{p}\}$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

M is said to be *free in codimension* n if $M_\mathfrak{p}$ is free for each prime $\mathfrak{p}$ of height at most n. We say M has a *rank* if it is generically free and the rank over all $\mathfrak{p} \in \mathrm{Ass}(R)$ is constant. We use the notation $M \mid N$ to say that M is a summand of N.

For R, being (G_j) means Gorenstein in codimension j. Let $e(\underline{x}, M)$ denote the multiplicity of M with respect to a sequence $\underline{x}$ in R. Without further comment, we will often use the notation $\overline{(-)} = - \otimes_R R/\underline{x}$ when the sequence $\underline{x}$ is clear from context. In case R is Cohen-Macaulay and admits a canonical module ω_R then, for an R-module M, we set $M^\vee = \mathrm{Hom}_R(M, \omega_R)$. In case R is Cohen-Macaulay and admits a canonical module ω_R then, for an R-module M, we set $M^\vee = \mathrm{Hom}_R(M, \omega_R)$. As usual, $\mathrm{mod}(R)$ and $\mathrm{CM}(R)$ denote the category of finitely generated and maximal Cohen-Macaulay modules respectively. We let $\mathrm{Deep}(R) = \{X \mid \mathrm{depth}(X) \geqslant t\}$. If $\mathscr{C}$ is a subcategory of R-modules, we use the notation

$$\Omega\mathscr{C} := \{M \mid \exists\, 0 \to M \to R^n \to X \to 0 \text{ exact for some } X \in \mathscr{C}\}.$$

We let $\Omega^{i+1}\mathscr{C} := \Omega\Omega^i\mathscr{C}$. The first category that is important to us is $\Omega\,\mathrm{Deep}(R)$. That is:

$$\Omega\,\mathrm{Deep}(R) := \{M \mid \exists\, 0 \to M \to R^n \to X \to 0 \text{ exact for some } n \in \mathbb{N} \text{ and } X \in \mathrm{Deep}(R)\},$$

We also consider:

$$\mathrm{DF}(R) := \{M \mid \exists\ \ 0 \to R \to M^n \to X \to 0 \text{ exact for some } n \in \mathbb{N} \text{ and } X \in \mathrm{Deep}(R)\}.$$

Recall that M is said to be a *minimal syzygy* if there is an exact sequence of the form $0 \to M \to R^{\mu(X)} \to X \to 0$. We define $\Omega_{\min}\mathrm{Deep}(R) = \{M \mid \exists\, 0 \to M \to R^{\mu(X)} \to X \to 0 \text{ exact with } X \in \mathrm{Deep}(R)\}$. If $M \in \Omega_{\min}\mathrm{Deep}(R)$, we say that M is a minimal syzygy in $\Omega\,\mathrm{Deep}(R)$.

If $M \in \mathrm{Deep}(R)$, we say that M is a *deep* module, and likewise if $M \in \mathrm{DF}(R)$, we say that M is *deeply faithful*.

Remark 4.2.1.

1. R is in $\Omega\,\mathrm{Deep}(R) \cap \mathrm{DF}(R)$. Any t-syzygy module is in $\mathrm{Deep}(R)$ (in other words $\Omega^t \operatorname{mod}(R) \subseteq \mathrm{Deep}(R)$).

2. Both $\Omega\,\mathrm{Deep}(R)$ and $\mathrm{DF}(R)$ are subcategories of $\mathrm{Deep}(R)$.

3. It is clear that for any $X \in \mathrm{DF}(R)$, X is faithful, and when $\operatorname{depth}(R) = 0$ deeply faithful and faithful modules coincide.

4. One can see that
$$\Omega^{t+1}(\operatorname{mod} R) \subseteq \Omega\,\mathrm{Deep}(R).$$

5. If R is Cohen-Macaulay, then $\mathrm{Deep}(R) = \mathrm{CM}(R)$. Furthermore, if R admits a canonical module ω, then R is Gorenstein if and only if $\Omega\,\mathrm{CM}(R) = \mathrm{CM}(R)$. Indeed, if $\omega \in \Omega\,\mathrm{CM}(R)$

then there is an exact sequence $0 \to \omega \to R^n \to C \to 0$ with $C \in \mathrm{CM}(R)$. Dualizing into ω, we have an exact sequence $0 \to C^\vee \to \omega^n \to R \to 0$. This sequence splits, so R has finite injective dimension and is thus Gorenstein. For the converse, if R is Gorenstein and M is maximial Cohen-Macaulay, then M is reflexive. Then dualizing the exact sequence $0 \to \Omega^1_R(M) \to R^{\mu(M^*)} \to M^* \to 0$ gives an exact sequence $0 \to M \to R^{\mu(M^*)} \to (\Omega^1_R(M))^* \to 0$.

6. Suppose R is Cohen-Macaulay and admits a canonical module ω. Then, from canonical duality, we have

$$\Omega\,\mathrm{CM}(R) = \{X^\vee \mid X \in \mathrm{CM}(R), \exists\ \omega_R^n \twoheadrightarrow X, \text{ for some } n \in \mathbb{N}\},$$

and

$$\mathrm{DF}(R) = \{X^\vee \mid X \in \mathrm{CM}(R), \exists\ X^n \twoheadrightarrow \omega_R, \text{ for some } n \in \mathbb{N}\}.$$

7. If M is a *semi-dualizing* module (that is, if the natural map $R \to \mathrm{Hom}_R(M,M)$ is an isomorphism and $\mathrm{Ext}^{i>0}_R(M,M) = 0$), then $M \in \mathrm{DF}(R)$. See Lemma 4.3.1.

Remark 4.2.2. When the proof of a statement involves finitely many objects in $\mathrm{Deep}(R)$, one can use prime avoidance to find a regular sequence $\underline{x}$ of length at most t on all of them. In such situations we shall often say, without further comments, that $\underline{x}$ is a *general* regular sequence.

We illustrate the above remark in the following simple but useful result:

Lemma 4.2.1. Suppose $t > 0$. If $M \in \Omega\,\mathrm{Deep}(R)$ (resp. $M \in \mathrm{DF}(R)$) then, for a general regular sequence $\underline{x}$, we have $\overline{M} \in \Omega\,\mathrm{Deep}(\overline{R})$ (resp. $\overline{M} \in \mathrm{DF}(\overline{R})$).

Proof. As $M \in \Omega\,\mathrm{Deep}(R)$ (resp. $M \in \mathrm{DF}(R)$), there is an exact sequence $0 \to M \to R^n \to X \to 0$ with $X \in \mathrm{Deep}(R)$ (resp. $0 \to R \to M^n \to X \to 0$ with $X \in \mathrm{Deep}(R)$). Then, as in Remark 4.2.2, for sufficiently general $x \in \mathfrak{m}$, x is regular on R and X, and so the sequences remain exact modulo x. The result then follows from induction. $\square$

Lemma 4.2.2. Assume that $t = 0$, $0 \neq M \in \Omega\,\mathrm{Deep}(R) = \Omega\,\mathrm{mod}(R)$. The following are equivalent.

1. $R \mid M$.

2. M is faithful.

3. $\operatorname{Soc}(R) \nsubseteq \operatorname{Ann}(M)$.

4. M is not a minimal syzygy on R.

Proof. (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. For (3) $\Rightarrow$ (4), assume that M is a minimal syzygy. Consider the sequence $0 \to M \to R^n \to X \to 0$ with $n = \mu(X)$. If M is free then X has finite projective dimension, hence free, impossible. Thus M is not free, so it is a part of an infinite minimal resolution. Therefore we have $M \subseteq \mathfrak{m}R^n$ for some n, so $\operatorname{Soc}(R)M = 0$.

□

Lemma 4.2.3. Let $0 \to X \xrightarrow{i_1} Y \xrightarrow{p_1} Z \to 0$ be an exact sequence.

1. If $Y \in \Omega\operatorname{Deep}(R)$ and $Z \in \operatorname{Deep}(R)$ then $X \in \Omega\operatorname{Deep}(R)$.

2. If $X \in \operatorname{DF}(R)$ and $Z \in \operatorname{Deep}(R)$ then $Y \in \operatorname{DF}(R)$.

Proof. 1. Since $Y \in \Omega\operatorname{Deep}(R)$, there is an exact sequence of the form $0 \to Y \xrightarrow{i_2} R^n \xrightarrow{p_2} C \to 0$, where $C \in \operatorname{Deep}(R)$. Letting P be the pushout along p_1 and i_2, we have the following pushout diagram with exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "i_1"] \arrow[d, equal] & Y \arrow[r, "p_1"] \arrow[d, "i_2"] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "i_2 \circ i_1"] \arrow[d] & R^n \arrow[r] \arrow[d, "p_2"] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C \arrow[r, equal] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Since $Z, C \in \mathrm{Deep}(R)$, it follows that $P \in \mathrm{Deep}(R)$ which shows that $X \in \Omega\,\mathrm{Deep}(R)$.

2. Since $X \in \mathrm{DF}(R)$, there is an exact sequence of the form $0 \to R \xrightarrow{i_2} X^n \xrightarrow{p_2} C \to 0$ for some n. We also have the exact sequence $0 \to X^n \xrightarrow{i_1^n} Y^n \xrightarrow{p_1^n} Z^n \to 0$. Letting P be the pushout along p_2 and i_1^n, we have the following pushout diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "i_2"] \arrow[d, equal] & X^n \arrow[r, "p_2"] \arrow[d, "i_1^n"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "i_1^n \circ i_2"] \arrow[d] & Y^n \arrow[r] \arrow[d, "p_1^n"] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & Z^n \arrow[r, equal] \arrow[d] & Z^n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

But then $P \in \mathrm{Deep}(R)$ since $C, Z^n \in \mathrm{Deep}(R)$, and thus $Y \in \mathrm{DF}(R)$, as desired.

$\square$

Lemma 4.2.4. Assume that $\mathrm{depth}(R) = 0$, $M \in \Omega\,\mathrm{Deep}(R)$, and $N \in \mathrm{DF}(R)$. If there exists a surjection $M \twoheadrightarrow N$ (resp. an injection $N \hookrightarrow M$), then $R \mid M$ (resp. $R \mid M$ and $R \mid N$).

Proof. First, assume that $M \twoheadrightarrow N$. As $M \in \Omega\,\mathrm{Deep}(R)$, if $R \nmid M$ then $\mathrm{Soc}(R)M = 0$ by Lemma 4.2.2.

Since $M \twoheadrightarrow N$, $\mathrm{Soc}(R)N = 0$. But N is a nonzero faithful module which is a contradiction. Hence $R \mid M$.

Now assume that $N \hookrightarrow M$. Then M, N are both faithful and in $\Omega\,\mathrm{Deep}(R)$, so they have a free summand by Lemma 4.2.2. $\square$

Proposition 4.2.3. Suppose that $M, N \in \mathrm{Deep}(R)$ and $\mathrm{Ext}_R^{1 \leqslant i \leqslant t-1}(M, N) = 0$. Then for any general regular sequence $\underline{x}$ of length $n \leq t$ we have $\overline{\mathrm{Hom}_R(M,N)} \cong \mathrm{Hom}_{\overline{R}}(\overline{M}, \overline{N})$ when $n < t$, and

$\overline{\mathrm{Hom}_R(M,N)} \hookrightarrow \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{N})$ when $n=t$.

Proof. This is standard argument. Apply $\mathrm{Hom}_R(-,N)$ to the short exact sequence

$$0 \to M \to M \to M/xM \to 0$$

and proceed by induction using [BH93, Lemma 1.2.4]. Note this does not require $\underline{x}$ to be a regular sequence on R. □

Lemma 4.2.5. If $M \in \Omega_{\min} \mathrm{Deep}(R)$, then for a general regular sequence $\underline{x} \subseteq \mathfrak{m}$, we have $M/\underline{x}M \in \Omega_{\min} \mathrm{Deep}(R/\underline{x}R)$.

Proof. Since M is a minimal syzygy, there is an exact sequence $0 \to M \to R^{\mu(X)} \to X \to 0$ with $X \in \mathrm{Deep}(R)$. Then any regular sequence $\underline{x}$ on X suffices.

□

Lemma 4.2.6. If $M \in \Omega \mathrm{Deep}(R)$, then $R/\underline{x}R \mid M/\underline{x}M$ if and only if $R \mid M$, for a general regular sequence $\underline{x}$.

Proof. One direction is clear. If $\overline{R} \mid \overline{M}$, then $\overline{M}$ is not a minimal syzygy in $\Omega \mathrm{Deep}(\overline{R})$. Therefore M is not a minimal syzygy in $\Omega \mathrm{Deep}(R)$ which implies $R \mid M$.

□

Lemma 4.2.7. If $M \in \Omega \mathrm{Deep}(R) \cap \mathrm{DF}(R)$ then $R \mid M$.

Proof. For $t=0$ the result follows from Proposition 4.2.2. By Lemma 4.2.1 we may find a regular sequence $\underline{x}$ so that $M/\underline{x}M \in \Omega \mathrm{Deep}(R/\underline{x}R) \cap \mathrm{DF}(R/\underline{x}R)$. So $R/\underline{x}R \mid M/\underline{x}M$ which implies $R \mid M$ by Lemma 4.2.6. □

Theorem 4.2.8. Suppose that $M \in \Omega \mathrm{Deep}(R)$ and $N \in \mathrm{DF}(R)$.

1. If $M \twoheadrightarrow N$ then $R \mid M$.

2. If there is an exact sequence $0 \to N \to M \to X \to 0$ such that $X \in \mathrm{Deep}(R)$ then $R \mid M$.

Proof. We cut down using a general regular sequence and appeal to 4.2.4, 4.2.6, and 4.2.7. □

4.3 When does $\operatorname{Hom}_R(M,N)$ contain a free summand?

For this section we retain the notation of Section 4.2.

Lemma 4.3.1. Let M,N be such that and $\operatorname{Ext}_R^{1\leq i\leq t-1}(M,N)=0$.

1. Suppose $N\in\Omega\operatorname{Deep}(R)$. Then $\operatorname{Hom}_R(M,N)\in\Omega\operatorname{Deep}(R)$.

2. Suppose $\operatorname{Hom}_R(M,N)\in\operatorname{DF}(R)$ and $N\in\operatorname{Deep}(R)$. Then $N\in\operatorname{DF}(R)$.

Proof. Consider part (1). Let

$$\cdots\to F_1\to F_0\to M\to 0$$

be a minimal free resolution of M. Since $\operatorname{Ext}_R^{1\leqslant i\leqslant t-1}(M,N)=0$ we have the following exact sequence, given by applying $\operatorname{Hom}_R(-,N)$ to the resolution:

$$0\to\operatorname{Hom}_R(M,N)\to N^{l_0}\to N^{l_1}\to\cdots\to N^{l_t}\to C\to 0$$

Split this sequence into

$$0\to\operatorname{Hom}_R(M,N)\to N^{l_0}\to X\to 0$$

and

$$0\to X\to N^{l_1}\to\cdots\to N^{l_t}\to C\to 0$$

Since $N\in\Omega\operatorname{Deep}(R)$ it follows from the latter sequence that $X\in\operatorname{Deep}(R)$, and so $\operatorname{Hom}_R(M,N)\in\Omega\operatorname{Deep}(R)$, applying Lemma 4.2.3 to the former sequence.

Part (2) is proved similarly, also using Lemma 4.2.3. $\square$

Lemma 4.3.2. Suppose $t=0$ and $R\mid M^*$. Then $R\mid M$.

Proof. Take a minimal presentation $R^m\xrightarrow{A}R^n\to M\to 0$ of M. This sequence induces an exact sequence of the form

$$0\to M^*\to R^n\xrightarrow{A^T}R^m\to\operatorname{Tr}M\to 0,$$

where $\mathrm{Tr}M$ denotes the Auslander transpose of M. Setting $l = \mu(M^*)$, we have another exact sequence

$$R^l \xrightarrow{B} R^n \xrightarrow{A^T} R^m \to \mathrm{Tr}M \to 0$$

Since $R \mid M^*$, by Lemma 4.2.2, M^* is not a minimal syzygy, and so B must contain a unit. Thus there are invertible matrices P and Q so that QBP^{-1} has the block form $\begin{pmatrix} 1 & 0 \\ 0 & B' \end{pmatrix}$. But this gives rise to a chain isomorphism:

$$\begin{tikzcd}
R^l \arrow[r, "B"] \arrow[d, "P"] & R^n \arrow[r, "A^T"] \arrow[d, "Q"] & R^m \arrow[r] \arrow[d, equal] & \mathrm{Tr}M \arrow[r] \arrow[d, equal] & 0 \\
R^l \arrow[r, "QBP^{-1}"] & R^n \arrow[r, "A^TQ^{-1}"] & R^m \arrow[r] & \mathrm{Tr}M \arrow[r] & 0
\end{tikzcd}$$

Since QBP^{-1} has the form $\begin{pmatrix} 1 & 0 \\ 0 & B' \end{pmatrix}$ and since $A^TQ^{-1} \cdot QBP^{-1} = 0$, it must be that A^TQ^{-1} has a column of all 0's. Hence $(Q^T)^{-1}A$ has a row of all 0's. But this implies that $R \mid \mathrm{coker}((Q^T)^{-1}A) \cong \mathrm{coker}(A) = M$, as desired.

$\square$

Lemma 4.3.3. Suppose $\mathrm{Ext}_R^{1 \le i \le t}(M,R) = 0$. Then M^* free implies M is free.

Proof. We may suppose M is not free. Then there is part of a free resolution of M of the form $F_{t+1} \to F_t \to \cdots \to F_1 \xrightarrow{A} F_0 \to M \to 0$ where $\mathrm{im}A \subseteq \mathfrak{m}F_0$. Dualizing this sequence, we obtain, since $\mathrm{Ext}_R^{1 \le i \le t}(M,R) = 0$, an exact sequence

$$0 \to M^* \to F_0^* \xrightarrow{A^T} F_1^* \to \cdots \to F_t^* \to F_{t+1}^* \to C \to 0.$$

Split this sequence into exact sequences

$$0 \to M^* \to F_0^* \to Y \to 0$$

and

$$0 \to Y \to F_1^* \to \cdots \to F_t^* \to F_{t+1}^* \to C \to 0.$$

Then Y is a $t+1$ syzygy of C. But since M^* is free, $\operatorname{pd} C \le t$ (recall that $t = \operatorname{depth} R$). Hence Y is a free summand of F_1^*. By Lemma 1.4.7 in [BH93] there is a unit in A^T since $Y = \operatorname{im} A^T$. But, by construction, A has no unit, and we have a contradiction.

□

Remark 4.3.1. If $M \in \operatorname{Deep}(R)$, one can also derive Lemma 4.3.3 from Lemma 4.3.2 by cutting down a regular sequence and appealing to the trace map $M \otimes_R M^* \to R$, as in [Vas68].

Example 4.3.2. Suppose $t > 0$ and consider $M = R \oplus k$. Then $M^* \cong R$, and $\operatorname{Ext}_R^{1 \le i \le t-1}(M,R) = 0$, but of course M is not free. Thus, in general, one cannot reduce the Ext vanishing hypothesis to $\operatorname{Ext}_R^{1 \le i \le t-1}(M,R) = 0$.

Theorem 4.3.4. Let $M \in \operatorname{Deep}(R)$ and $N \in \Omega \operatorname{Deep}(R)$. Assume that $\operatorname{Hom}_R(M,N) \in \operatorname{DF}(R)$ and $\operatorname{Ext}_R^{1 \leqslant i \leqslant t-1}(M,N) = 0$. Then $R \mid N$.

Proof. By Lemma 4.3.1, we have $\operatorname{Hom}_R(M,N) \in \Omega \operatorname{Deep}(R) \cap \operatorname{DF}(R)$. Hence $R \mid \operatorname{Hom}_R(M,N)$ by 4.2.7. As $\overline{\operatorname{Hom}_R(M,N)} \hookrightarrow \operatorname{Hom}_{\overline{R}}(\overline{M},\overline{N})$ and $\overline{\operatorname{Hom}_R(M,N)} \in \operatorname{DF}(\overline{R})$, $\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{N}) \in \operatorname{DF}(\overline{R})$. Therefore $\overline{N} \in \operatorname{DF}(\overline{R})$. Hence $\overline{R} \mid \overline{N}$ and so $R \mid N$ by from Lemmas 4.2.2 and 4.2.5.

□

Corollary 4.3.3. Let $M \in \operatorname{Deep}(R)$ and $N \in \Omega \operatorname{Deep}(R)$. Furthermore, assume that $\operatorname{Hom}_R(M,N)$ is free and $\operatorname{Ext}_R^{1 \leqslant i \leqslant t-1}(M,N) = 0$. Then N is free.

Proof. By Theorem 4.3.4, we have that $R \mid N$. Let $N = R^n \oplus N'$ where $n > 1$. Suppose $N' \neq 0$. Since $N \in \Omega \operatorname{Deep}(R)$, $N' \in \Omega \operatorname{Deep}(R)$ by Lemma 4.2.3. Further, since $\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_R(M,R^n) \oplus \operatorname{Hom}_R(M,N')$ is free, $\operatorname{Hom}_R(M,N')$ is free and since $0 = \operatorname{Ext}_R^{1 \le i \le t-1}(M,N) \cong \operatorname{Ext}_R^{1 \le i \le t-1}(M,R^n) \oplus \operatorname{Ext}_R^{1 \le i \le t-1}(M,N')$ we have that $\operatorname{Ext}_R^{1 \le i \le t-1}(M,N') = 0$. Thus we may apply Theorem 4.3.4 to obtain that $R \mid N'$. Induction on $\mu(N)$ now shows that N is free.

□

Theorem 4.3.5. Let $M \in \mathrm{Deep}(R)$ and $N \in \Omega\,\mathrm{Deep}(R)$. Suppose $\mathrm{Hom}_R(M,N)$ is free and that $\mathrm{Ext}_R^{1\le i\le t}(M,R) = \mathrm{Ext}_R^{1\le i\le t-1}(M,N) = 0$. Then M is free.

Proof. By Corollary 4.3.3 we have that N is free. Thus $N \cong R^n$ for some n. Now we have $\mathrm{Hom}_R(M,N) \cong (M^*)^n$ is free, and thus M^* is free. Thus M is free by Lemma 4.3.3.

□

Next we address the question of when M^* is free. The most interesting case is when $\dim R \le 1$.

Lemma 4.3.6. Suppose R is Cohen-Macaulay with dimension $d \le 1$, and suppose $M \in \mathrm{CM}(R)$. Then M^* free implies M is free.

Proof. We have already obtained the result when $d = 0$ by Lemma 4.3.5. So we may suppose $d = 1$. Let x be a general regular element on R and M. Now, since $M^* \cong R^r$ for some r, we have $M_\mathfrak{p} \cong R_\mathfrak{p}^r$ for all $\mathfrak{p} \in \mathrm{Min}\, R$, from Lemma 4.3.5. Thus M has constant rank r. In particular, $M_\mathfrak{p} \cong M_\mathfrak{p}^*$ for all $\mathfrak{p} \in \mathrm{Min}\, R$. We have (see [BH93, Theorem 4.6.8])

$$e(x,M) = \sum_{p\in\mathrm{Min}\, R} e(x,R/\mathfrak{p})l(M_\mathfrak{p}) = \sum_{p\in\mathrm{Min}\, R} e(x,R/\mathfrak{p})l(M_\mathfrak{p}^*) = e(x,M^*).$$

Note that we have an exact sequence $0 \to \overline{M^*} \xrightarrow{i} \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{R})$ by Proposition 4.2.3.

Set $n = \mu(\overline{M})$. Then we have a map $\overline{R}^n \twoheadrightarrow \overline{M}$. Dualizing this gives a map $j : \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{R}) \hookrightarrow \overline{R}^n$. Since $\overline{M^*} \cong \overline{R}^r$, this gives us an exact sequence of the form

$$0 \to \overline{R}^r \xrightarrow{j\circ i} \overline{R}^n \to C \to 0.$$

But then $\mathrm{pd}\, C < \infty$ which means C is free, since $\mathrm{depth}\, \overline{R} = 0$. Thus this sequence splits, whence the map i is a split injection. Thus $\mathrm{Hom}_{\overline{R}}(\overline{M},\overline{R}) \cong \overline{R}^r \oplus L$ for some L. By Lemma 4.3.2 applied repeatedly (as Krull-Schmidt holds over $\bar{R}$ because it is Artinian), it follows that $\overline{R}^r \mid \overline{M}$. But since $l(\overline{M}) = e(x,M) = e(x,M^*) = l(\overline{M^*}) = l(\overline{R}^r)$, it must be that $\overline{M} \cong \overline{R}^r$ which implies M is free.

□

Standard arguments now allow us to show that the freeness of M^* forces that of M in general.

Theorem 4.3.7. Suppose R and M satisfy (S_2). Then M^* free implies M is free.

Proof. It suffices to show that M is reflexive. We assume $d = \dim R \geq 2$ as the small dimension case was covered by 4.3.6. Also by 4.3.6, we have that $M_{\mathfrak{p}}$ is free, in particular, reflexive, for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} \leq 1$. The natural map $M \to M^{**}$ is an isomorphism in codimension one, so is an isomorphism (or one can appeal to [BH93, Proposition 1.4.1]).

□

In the next part we extend one of the main results of [HL04]:

Theorem 4.3.8. (Huneke-Leuschke)

Suppose R is Cohen-Macaulay and is a complete intersection in codimension 1. Furthermore, assume that $\mathbb{Q} \subseteq R$. If M is an R-module that is locally free in codimension one with constant rank, $\operatorname{Ext}_R^{1 \leq i \leq d}(M,M) = 0$, and $\operatorname{Ext}_R^{1 \leq i \leq 2d+1}(M,R) = 0$, then M is free.

We start with a very well-known fact about shifting Ext modules.

Lemma 4.3.9. If $\operatorname{Ext}_R^i(M,R) = \operatorname{Ext}_R^i(\Omega M, \Omega N) = 0$ then $\operatorname{Ext}_R^i(M,N) = 0$ and if $\operatorname{Ext}_R^{i+1}(M,R) = \operatorname{Ext}_R^i(M,N) = 0$, then $\operatorname{Ext}_R^i(\Omega M, \Omega N) = 0$

Proof. The exact sequence $0 \to \Omega N \to R^n \to N \to 0$ induces an exact sequence

$$\operatorname{Ext}_R^i(M,R^n) \to \operatorname{Ext}_R^i(M,N) \to \operatorname{Ext}_R^{i+1}(M,\Omega N) \to \operatorname{Ext}_R^{i+1}(M,R)$$

from which we obtain the result.

□

Lemma 4.3.10. Suppose R is a quotient of a regular local ring and suppose R is (S_2) and (G_1) with $\mathbb{Q} \subseteq R$. Let $M \in \Omega \operatorname{Deep}(R)$ be a reflexive R-module, free in codimension 1, and suppose $\operatorname{Ext}_R^{1 \leq i \leq t-1}(M,M) = 0$. Then M is free.

Proof. We may suppose R is complete, and since M is free in codimension 1, we may also suppose $\dim R \geq 2$. We claim that M has constant rank. Take $\mathfrak{p}, \mathfrak{q} \in \operatorname{Min} R$. Since R satisfies (S_2), the Hochster-Huneke graph of R is connected (see [HH94]). This means there is a chain of minimal primes $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n = \mathfrak{q}$ such that $\operatorname{ht}(\mathfrak{p}_i + \mathfrak{p}_{i+1}) \leq 1$. Ergo, $\operatorname{rank} M_{\mathfrak{p}_i} = \operatorname{rank} M_{\mathfrak{p}_{i+1}}$ for each i, since M is free on a minimal prime of $\mathfrak{p}_i + \mathfrak{p}_{i+1}$. In particular, $\operatorname{rank} M_{\mathfrak{p}} = \operatorname{rank} M_{\mathfrak{q}}$, and so M has constant rank.

Now, we have $R \mid \operatorname{End}_R(M)$ from the trace map as explained in [HL04, Appendix]. By Theorem 4.3.4, $M = R \oplus M'$. But now M' satisfies the hypotheses again by Lemma 4.2.3, and so, proceeding inductively on the number of generators gives that M is free.

□

Theorem 4.3.11. Suppose R is a quotient of a regular local ring and satisfies (S_2) and (G_1) with $\mathbb{Q} \subseteq R$. Suppose $N \in \operatorname{mod}(R)$ such that $\operatorname{pd} N_{\mathfrak{p}} < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} \leq 1$. Set $a = \min\{t, \operatorname{depth} N\}$ and suppose $\operatorname{Ext}_R^{1 \leq i \leq t-1}(N,N) = \operatorname{Ext}_R^{1 \leq i \leq 2t+1-a}(N,R) = 0$. Then N is free.

Proof. Set $M = \Omega^{t+2-a}(N)$. Then $M \in \Omega \operatorname{Deep}(R)$ and M is reflexive. This gives us that $R \mid \operatorname{End}(M)$. By Lemma 4.3.9, we have $\operatorname{Ext}^{1 \leq i \leq t-1}(M,M) = 0$. By Lemma 4.3.10, M is free. Thus $\operatorname{pd} N \leq t+2-a$. If $\operatorname{pd} N = l$, then $\operatorname{Ext}_R^l(N,X) \neq 0$ for every finitely generated $X \neq 0$. But $t+2-a \leq 2t+1-a$ and so it must be that $l = 0$. Therefore, N is free.

□

Next we discuss and extend a result by Goto-Takahashi ([GT17, Corollary 4.3]). First, we recall their result and give a somewhat simpler proof. Note that their result does not follow directly from our previous results since, for instance, I may not be in $\Omega \operatorname{Deep}(R)$.

Theorem 4.3.12. (Goto-Takahashi)

Suppose R is CM and that I is a CM ideal of height 1. Assume that

1. $\operatorname{Hom}_R(I,I)$ is free.

2. $\operatorname{Ext}_R^{1 \leq i \leq d}(I,R) = 0$.

3. $\mathrm{Ext}_R^{1\leq i\leq d-1}(I,I)=0$.

Then I is free.

Proof. By standard reduction arguments (see [GT17, Theorem 3.3]) one can assume $\dim R=1$. By prime avoidance, there exists $a\in I$ which is not in $\mathrm{Min}(R)\cup\mathfrak{m}I$. Thus a is part of a minimal generating set for I, and we have an exact sequence of the form

$$0\to R\to I\xrightarrow{f} C\to 0.$$

Since I has height 1, it follows that R/I has finite length. Thus, localizing the exact sequence $0\to I\to R\to R/I\to 0$ at any $\mathfrak{p}\in\mathrm{Min}(R)$, we obtain that I has constant rank 1. Ergo, C has finite length. In general, if $g\in\mathfrak{m}\,\mathrm{Hom}_R(X,Y)$ then $\operatorname{im} g\in\mathfrak{m}Y$. Thus the above argument gives us that f is part of a minimal generating set for $\mathrm{Hom}_R(I,C)$. Now, since $\mathrm{Hom}_R(I,I)$ is free, it must be that $\mathrm{Hom}_R(I,I)\cong R$, since I has rank 1. Since $\mathrm{Ext}_R^1(I,R)=0$, we have the exact sequence

$$0\to\mathrm{Hom}_R(I,R)\to R\to\mathrm{Hom}_R(I,C)\to 0$$

Thus $\mathrm{Hom}_R(I,C)$ is cyclic, and $\{f\}$ a generating set. By construction, $f(a)=0$, and thus for every $g\in\mathrm{Hom}_R(I,C)$, we have $g(a)=0$. But on the other hand, $\mathrm{Hom}_R(I,\mathrm{Soc}\,C)\hookrightarrow\mathrm{Hom}_R(I,C)$ and the former is isomorphic to $\mathrm{Hom}_R(I/\mathfrak{m}I,\mathrm{Soc}(C))$. But as this is a vector space, if $C\neq 0$, we may find a map $h\in\mathrm{Hom}_R(I,C)$ so that $h(a)\in\mathrm{Soc}(C)-\{0\}$ and $h(x)=0$ for any minimal generator $x\neq a$. But this is a contradiction, and so $C=0$, whence $I\cong R$.

□

The next theorem extends the Goto-Takahashi result to modules of higher rank.

Theorem 4.3.13. Let R be a Cohen-Macaulay local normal domain. Let M be a maximal Cohen-Macaulay module such that $\mathrm{Hom}_R(M,M)$ is free and

$$\mathrm{Ext}_R^{1\leq i\leq d-1}(M,M)=\mathrm{Ext}_R^{1\leq i\leq d}(M,R)=0.$$

Then M is free.

Proof. We employ a standard argument in the theory of Brauer groups. Let $S = R^{sh}$ denote the strict Henselization of R. Then S is still a local normal domain, and it is harmless to replace R by S without affecting the assumptions and desired conclusion. Thus we assume R is Henselian with a separably closed residue field k. Let $A = \mathrm{End}_R(M)$ and set $r = \mathrm{rank}_R M$. Then as M is reflexive and A is a free module of rank r^2, A is an Azumaya algebra (see for example [CGO75], proof of Corollary 1.4). Then so is the k-algebra $B = A \otimes_R k$. Since k is separably closed, B is actually isomorphic as an algebra to $\mathrm{End}_k(k^r)$. Now as R is Hensenlian, one can lift idempotents, which shows that M splits into a direct sum of ideals. These ideals inherit all the assumptions, so by Theorem 4.3.12 they are all free, and so is M. □

Remark 4.3.4. If R^{sh} is a UFD, then our argument shows that M is free without any assumption on vanishing of Ext modules.

4.4 When is $\mathrm{Hom}_R(M,N) \cong N^r$?

In this section we try to understand the question in the title. Let t be some fixed integer. Unlike the previous sections, we don't necessarily assume $\operatorname{depth} R = t$.

Set $\upsilon_i(M) = \dim_k \mathrm{Ext}^i_R(k,M)$. We let $\mathrm{Fitt}_j(M)$ denote the j-th fitting ideal ideal of M, namely the ideal generated by $(n-j)$-minors of any presentation matrix A of M in a sequence:

$$R^m \xrightarrow{A} R^n \to M \to 0$$

We first recall a result ([GT17, Lemma 2.1]). For completeness, we provide an elementary proof that avoids spectral sequences.

Proposition 4.4.1. (Goto-Takahashi) Let M,N be such that $\operatorname{depth}(M), \operatorname{depth}(N) \geq t$. Assume that $\mathrm{Ext}_R^{1\leqslant i\leqslant t}(M,N) = 0$. Then $\upsilon_t(\mathrm{Hom}_R(M,N)) = \mu(M)\upsilon_t(N)$.

Proof. Take $F_t \to \cdots \to F_1 \to F_0 \to M \to 0$ to be part of a (possibly non-minimal) free resolution of M where $F_i \neq 0$ for each i. Note that such a resolution exists even if $\operatorname{pd} M < t$. Then a similar argument to that of Lemma 4.3.1 shows that $\operatorname{depth} \operatorname{Hom}_R(M,N) \geq t$. Now, we have $\overline{\operatorname{Hom}_R(M,N)} \cong \operatorname{Hom}_{\overline{R}}(\overline{M},\overline{N})$ for a general regular sequence of length t. Hence $\upsilon_t(\operatorname{Hom}_R(M,N)) = \dim_k \operatorname{Hom}_R(k,\overline{\operatorname{Hom}_R(M,N)}) = \dim_k \operatorname{Hom}_R(k,\operatorname{Hom}_{\overline{R}}(\overline{M},\overline{N})) = \dim_k \operatorname{Hom}_{\overline{R}}(k \otimes_R \overline{M},\overline{N}) = \mu(M) \cdot \upsilon_t(N)$.

$\square$

Theorem 4.4.1. Suppose $\dim N = 0$. Then $\operatorname{Hom}_R(M,N) \cong N^r$ if and only if $\mu(M) = r$ and $\operatorname{Fitt}_{r-1}(M)N = 0$.

Proof. First we consider the case were $\dim R = 0$

[$\Rightarrow$] Since $\operatorname{Hom}_R(M,N) \cong N^r$ we have, by Proposition 4.4.1 $\upsilon_0(\operatorname{Hom}_R(M,N)) = \upsilon_0(N^r)$ and so $\mu(M)\upsilon_0(N) = r\upsilon_0(N)$ from which we see that $\mu(M) = r$ as $\upsilon_0(N) \neq 0$. We also have $l(M \otimes_R N^\vee) = l((N^r)^\vee) = rl(N^\vee)$ by Matlis duality. Take a minimal presentation

$$R^m \xrightarrow{A} R^r \to M \to 0.$$

Tensoring with $N^\vee$ we have the exact sequence

$$(N^\vee)^m \xrightarrow{A \otimes_R id_{N^\vee}} (N^\vee)^r \to M \otimes_R N^\vee \to 0.$$

But $l((N^\vee)^r = l(M \otimes_R N^\vee)$ since $((N^\vee)^r)^\vee \cong N^r$ and $(M \otimes_R N^\vee)^\vee \cong \operatorname{Hom}_R(M,N)$. Thus $\operatorname{im}(A \otimes_R \operatorname{id}_{N^\vee}) = 0$ which implies $\operatorname{Fitt}_{r-1}(M) \subseteq \operatorname{Ann}(N^\vee) = \operatorname{Ann}(N)$.

[$\Leftarrow$] Suppose $\mu(M) = r$ and $\operatorname{Fitt}_{r-1}(M)N = 0$. Since $\mu(M) = r$, we may take a minimal presentation of M of the form

$$R^m \xrightarrow{A} R^r \to M \to 0.$$

Tensoring with $N^\vee$ we have the exact sequence

$$(N^\vee)^m \xrightarrow{A \otimes_R id_{N^\vee}} (N^\vee)^r \to M \otimes_R N^\vee \to 0.$$

Since $\mathrm{Fitt}_{r-1}(M) \subseteq \mathrm{Ann}(N) = \mathrm{Ann}(N^\vee)$, we have $\mathrm{im}(A \otimes_R \mathrm{id}_{N^\vee}) = 0$ and thus $(N^\vee)^r \cong M \otimes_R N^\vee$. That $\mathrm{Hom}_R(M,N) \cong N^r$ now follows from Matlis duality. So we have the result when $\dim R = 0$.

Now suppose $\dim R > 0$. Set $I = \mathrm{Ann}\, N$. Since N has finite length, I is $\mathfrak{m}$-primary. Set $\overline{(-)} = (-) \otimes_R R/I$. Then $\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_{\overline{R}}(\overline{M},N)$ so that $\mathrm{Hom}_{\overline{R}}(\overline{M},N) \cong N^r$. But from the dimension 0 result, this holds if and only if $\mathrm{Fitt}_{r-1}(\overline{M})N = \overline{\mathrm{Fitt}_{r-1}(M)}N = 0$ which precisely means that $\mathrm{Fitt}_{r-1}(M) \subseteq \mathrm{Ann}\, N$, as desired.

□

Corollary 4.4.2. If $\dim M = 0$ and $\mathrm{Hom}_R(M,M) \cong M^r$ then $\mathrm{Fitt}_{r-1}(M) = \mathrm{Ann}\, M$.

Proof. We always have $\mathrm{Ann}\, M \subseteq \mathrm{Fitt}_{r-1}(M)$. The result follows from combining this fact with the Theorem 4.4.1.

□

Corollary 4.4.3. Suppose $\mathrm{Hom}_R(M,N) \cong N^r$ and suppose $\mathrm{Ass}\, N = \mathrm{Min}\, N$. Then $\mathrm{Fitt}_{r-1}(M)N = 0$.

Proof. For any $\mathfrak{p} \in \mathrm{Min}(\mathrm{Fitt}_{r-1}(M)N)$, we have $\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \cong (N_\mathfrak{p})^r$.

Since $(\mathrm{Fitt}_{r-1}(M)N)_\mathfrak{p} \hookrightarrow N_\mathfrak{p}$ it follows that $N_\mathfrak{p}$ has depth 0. Thus $\mathfrak{p} \in \mathrm{Ass}(N) = \mathrm{Min}(N)$ and so $N_\mathfrak{p}$ has finite length. Theorem 4.4.1 gives us that $\mathrm{Fitt}_{r-1}(M_\mathfrak{p})N_\mathfrak{p} = (\mathrm{Fitt}_{r-1}(M))_\mathfrak{p} N_\mathfrak{p} = (\mathrm{Fitt}_{r-1}(M))N)_\mathfrak{p} = 0$. But this says that $(\mathrm{Fitt}_{r-1}(M))N)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{Fitt}_{r-1}(M)N)$ which implies $\mathrm{Fitt}_{r-1}(M)N = 0$.

□

Remark 4.4.4. We remark that the converse to Corollary 4.4.3 does not hold. Indeed, if M has constant rank r, then $\mathrm{Fitt}_{r-1}(M) = 0$, while $\mathrm{Hom}_R(M,N)$ need not be isomorphic to N^r. To be more explicit, one could take R with $\mathrm{depth}\, R = 1$ and let $M = \mathfrak{m}$ and $N = R$.

Lemma 4.4.2. Let M,N,P be nonzero $R-$modules such that $\mathrm{Ass}(P) \subseteq \mathrm{Ass}(N) = \mathrm{Min}(N)$. Suppose that $\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(P,N)$ and $\mathrm{Ext}^1_R(M,N) = 0$. Then if $P \twoheadrightarrow M$, $P \cong M$.

Proof. First assume that $\dim(N) = 0$. From the exact sequence $0 \to X \to P \to M \to 0$ we have

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(P,N) \to \mathrm{Hom}_R(X,N) \to 0.$$

By assumption $l(\mathrm{Hom}_R(X,N)) = 0$. Therefore $X = 0$.

If $\dim(N) > 0$, for any $\mathfrak{p} \in \mathrm{Min}(N) = \mathrm{Ass}(N)$, $X_\mathfrak{p} = 0$. So $\mathrm{Ass}(\mathrm{Hom}_R(X,N)) = \mathrm{Supp}(X) \cap \mathrm{Ass}(N) = \emptyset$, and thus $\mathrm{Hom}_R(X,N) = 0$.

Now, if $X \neq 0$, take $\mathfrak{q} \in \mathrm{Min}(X)$. Then $\mathfrak{q} \in \mathrm{Ass}(P) \subseteq \mathrm{Ass}(N)$. But then $\mathrm{Hom}_{R_\mathfrak{q}}(X_\mathfrak{q}, N_\mathfrak{q}) \neq 0$, a contradiction. $\square$

Theorem 4.4.3. Assume that $\mathrm{depth}(M) \geqslant t$, $\mathrm{depth}(N) = t$, $\mathrm{Ass}(N) = \mathrm{Min}(N)$, and for some $s \geq t$, $\mathrm{Ext}_R^{1 \leqslant i \leqslant s}(M,N) = 0$. If $\mathrm{Hom}_R(M,N) \cong N^r$ for some $r \in \mathbb{N}$, then $M/IM \cong (R/I)^r$ for $I = \mathrm{Ann}(N)$.

Furthermore, if one of the following holds:

1. N is faithful.

2. $\mathrm{Ass}(R) \subseteq \mathrm{Ass}(N)$ and $s > 0$.

then $M \cong R^r$.

Proof. By Proposition 4.4.1, $v_t(N^r) = \mu(M) \cdot v_t(N)$. Hence $\mu(M) = r$. Since $\mathrm{Ass}(N) = \mathrm{Min}(N)$, Corollary 4.4.3 tells us that $\mathrm{Fitt}_{r-1}(M) \subseteq I$. Since M/IM is still r-generated over R/I, it must be a free R/I module of rank r.

For the furthermore statements, if $I = 0$ then $M \cong R^r$. Assume the second set of conditions. By Lemma 4.4.2, we have $M \cong R^r$. $\square$

Corollary 4.4.5. Let $R \to S$ be a finite local homomorphism of local rings. Assume that S is regular of dimension t, $\mathrm{depth} M \geq t$, and $\mathrm{Ext}_R^{1 \leq i \leq t}(M,S) = 0$. If one of the following holds

1. S is faithful as an R-module.

2. $t > 0$ and $\operatorname{Ass}_R R \subseteq \operatorname{Ass}_R S$.

Then M is free.

Proof. Since $\operatorname{Ext}_R^{1 \le i \le t}(M,S) = 0$, we have $\operatorname{Hom}_R(M,S) \in \operatorname{CM}(S)$. But, since S is regular, we have $\operatorname{Hom}_R(M,S) \cong S^l$ for some $l \in \mathbb{N}$. Hence M is free, by Theorem 4.4.3. $\square$

The following example shows that the conditions of 4.4.3 and 4.4.5 are needed.

Example 4.4.6. Let $R = k[\![x,y]\!]/(xy)$, let $S = R/(x)$ and $M = R/(x)$ as in Corollary 4.4.5. Then $\operatorname{Ext}_R^1(M,S) = 0$ but of course M is not free.

It is worth noting that our results in this section can also be viewed as modest confirmation of the Auslander-Reiten conjecture. For example Theorem 4.4.3 gives:

Corollary 4.4.7. Let $M = R/I$ and $\operatorname{depth} M = t$. Assume that $\operatorname{Ass}(R) \subseteq \operatorname{Ass}(M) = \operatorname{Min}(M)$. Then M is free if $\operatorname{Ext}_R^{1 \le i \le \max\{1,t\}}(M,M) = 0$.

Proof. Obviously $\operatorname{Hom}_R(M,M) \cong M$, so we can apply Theorem 4.4.3. $\square$

4.5 Some other applications

In this section we treat some similar problems that have appeared in the literature. The first one involves tests for Gorensteiness, in the spirit of [Ulr84]. Throughout this section we assume R is a Cohen-Macaulay local ring with $\dim R = d$ and with canonical module ω.

Corollary 4.5.1. Suppose $\operatorname{Ext}_R^{1 \le i \le d}(M,R) = 0$ and M is (S_2). If $M^\vee \cong M^*$ then R is Gorenstein.

Proof. Since M is Cohen-Macaulay in codimension 1, the natural map $M \to M^{\vee\vee}$ is an isomorphism in codimension 1, thus an isomorphism. Since $\operatorname{Ext}_R^{1 \le i \le d}(M,R) = 0$ it follows, as in the proof of Lemma 4.3.1 that $M^* \in \operatorname{CM}(R)$ and so $(M^*)^\vee \cong M^{\vee\vee} \cong M \in \operatorname{CM}(R)$.

By assumption and Proposition 4.4.1 we have

$$\upsilon_d(M^\vee) = \upsilon_d(M^*) = \mu(M)\upsilon_d(R).$$

Since $\upsilon_d(M^\vee) = \mu(M)$ (one can appeal to Proposition 4.4.1 again), R has type one, and so is Gorenstein. $\square$

Remark 4.5.2. The above was inspired by Theorem 2.1 of [Ulr84]. The situation there is as follows. Let $R \to S$ be a finite extension with $\dim S = \dim R$ and S is Cohen-Macaulay, local, and factorial. Under mild conditions, $\operatorname{Hom}_R(S,R)$ is isomorphic as an S module to a rank one reflexive ideal of S, thus $\operatorname{Hom}_R(S,R) \cong S$. Also $\operatorname{Hom}_R(S,\omega_R) \cong \omega_S \cong S$. One can now appeal to 4.5.1, with $M = S$ to give an Ext-vanishing test for the Gorensteiness of R.

For completeness, we give the following, which extends [HH05, Lemma 2.1], [Jor09, Theorem 2.7], and [HJ03, Theorem 5.9].

Lemma 4.5.1. Let $M,N \in \mathrm{CM}(R)$ Consider the conditions:

1. $\operatorname{Ext}_R^{1\le i\le d}(M,N) = 0$.

2. $M \otimes_R N^\vee$ is in $\mathrm{CM}(R)$.

Then $(1) \Rightarrow (2)$. If $\operatorname{Ext}_R^{1\le i\le d}(M,N)$ have finite length then $(2) \Rightarrow (1)$.

Proof. Suppose $\underline{x}$ is a regular sequence. Set $e(M) := e(\underline{x},M)$. First we have $e(\operatorname{Hom}_R(M,N)) = e(M \otimes_R N^\vee)$ by the Associativity formula [BH93, Theorem 4.7.8]. Then, since $\operatorname{Hom}_R(M,N) \in \mathrm{CM}(R)$, we have

$$e(\operatorname{Hom}_R(M,N)) = l(\overline{\operatorname{Hom}_R(M,N)}) = l(\operatorname{Hom}_R(\overline{M},\overline{N})) = l(\overline{M} \otimes_{\overline{R}} \overline{N}^\vee).$$

But this says $e(M \otimes_R N^\vee) = l(\overline{M} \otimes_R \overline{N}^\vee) = l(\overline{M \otimes_R N^\vee})$ from which we deduce that $M \otimes_R N^\vee$ is MCM.

For the converse, first consider the case where where $\dim R = 1$. Let x be R-regular and $x\operatorname{Ext}_R^1(M,N) = 0$. The short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ induces the exact sequence

$$0 \to \operatorname{Hom}_R(M,N) \xrightarrow{x} \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M/xM,N/xN) \to \operatorname{Ext}_R^1(M,N) \to 0.$$

In this case, it suffices then, to show that $\operatorname{Hom}_R(M,N)/x\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_R(M/xM,N/xN)$. Since $M\otimes_R N^\vee$ is MCM, we see that $\dfrac{(M\otimes_R N^\vee)^\vee}{x(M\otimes_R N^\vee)^\vee} \cong \operatorname{Hom}_R(\dfrac{M\otimes_R N^\vee}{x(M\otimes_R N^\vee)}, \dfrac{\omega}{x\omega})$. But since $(M\otimes_R N^\vee)^\vee \cong \operatorname{Hom}_R(M,N)$, this gives us that $\operatorname{Hom}_R(M,N)/x\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_R(M/xM,N/xN)$ so that $\operatorname{Ext}^1_R(M,N)=0$.

Now suppose $\dim R > 1$ and choose a regular $x \in \bigcap_{1\le i\le d} \operatorname{Ann}(\operatorname{Ext}^i_R(M,N))$. Then the long exact sequence in Ext, coming from the short exact sequence $0\to M \xrightarrow{x} M \to M/xM \to 0$, decomposes into short exact sequences

$$0 \to \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^{i+1}_R(M/xM,N) \to \operatorname{Ext}^{i+1}_R(M,N) \to 0$$

for each $1\le i\le d-1$. Further, we have $\operatorname{Ext}^{i+1}_R(M/xM,N) \cong \operatorname{Ext}^i_{R/xR}(M/xM,N/xN)$ for each $1\le i\le d-1$. Thus it suffices to show $\operatorname{Ext}^i_{R/xR}(M/xM,N/xN)=0$ for each $1\le i\le d-1$. But since $M\otimes_R N^\vee$ MCM implies $M/xM\otimes_{R/xR}(N/xN)^\vee$ is MCM over R/xR, this follows from induction, and we're done.

□

To explore the previous Theorem a bit more, we make the following definition. A pair of modules $M,N\in \operatorname{CM}(R)$ is called *tight* if $\operatorname{Ext}^{1\le i\le d}_R(M,N)=0$ forces $\operatorname{Ext}^i_R(M,N)=0$ for all $i>0$.

Remark 4.5.3. A pair (M,N) of maximal Cohen-Macaulay modules is tight in any of the following situations:

1. M has projective dimension or N has finite injective dimension.

2. M is locally free in codimension one, M^* is maximal Cohen-Macaulay and $N=(M^*)^\vee$ (when R is Gorenstein the last two conditions simply mean $N=M$) [ACST17, Theorem 1.4].

3. M has finite complete intersection dimension and the complexity of M is at most $d-1$ [CD11, Theorem 1.2].

Corollary 4.5.4. Suppose R is Cohen-Macaulay with canonical module ω and let $M,N\in\operatorname{CM}(R)$ such that for all $\mathfrak{p}\in\operatorname{Spec}(R)-\{\mathfrak{m}\}$, the pair $(M_\mathfrak{p},N_\mathfrak{p})$ is tight. Then the following are equivalent:

1. $\operatorname{Ext}_R^{1\leq i\leq d}(M,N)=0$.

2. $M\otimes_R N^\vee \in \mathrm{CM}(R)$.

Proof. Assume (2). Let $\mathfrak{p}$ be a non-maximal prime. By induction on $\dim R$, $\operatorname{Ext}_R^{1\leq i\leq \operatorname{ht}\mathfrak{p}}(M,N)_\mathfrak{p} = 0$. By assumption on tightness of the pair $(M_\mathfrak{p},N_\mathfrak{p})$, $\operatorname{Ext}_R^{1\leq i\leq d}(M,N)_\mathfrak{p}=0$. So the modules $\operatorname{Ext}_R^{1\leq i\leq d}(M,N)$ have finite length, and we are done. $\square$

Chapter 5

Maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum

5.1 Introduction

Cohen-Macaulay representation theory has been studied widely and deeply for more than four decades. The theorems of Herzog [Her78] in the 1970s and of Buchweitz, Greuel and Schreyer [BGS87] in the 1980s are recognized as some of the most crucial results in this long history of Cohen-Macaulay representation theory. Both are concerned with Cohen–Macaulay local rings of finite/countable CM-representation type, that is, Cohen-Macaulay local rings possessing finitely/countably many nonisomorphic indecomposable maximal Cohen–Macaulay modules. Herzog proved that quotient singularities of dimension two have finite CM-representation type and that Gorenstein local rings of finite CM-representation type are hypersurfaces. Buchweitz, Greuel and Schreyer proved that the local hypersurfaces of finite (resp. countable) CM-representation type are precisely the local hypersurfaces of type (A_n) with $n \geq 1$, (D_n) with $n \geq 4$, and (E_n) with $n = 6,7,8$ (resp. (A_∞) and (D_∞)).

At the beginning of this century, Huneke and Leuschke [HL02] proved that Cohen-Macaulay local rings of finite CM-representation type have isolated singularities. However, there are ample examples of Cohen-Macaulay local rings not having isolated singularities, including the local hypersurfaces of type (A_∞) and (D_∞) appearing above. Cohen–Macaulay representation theory for non-isolated singularities has been studied by many authors so far; see [AKM17, BD17, HN94, IW14] for instance. It should be remarked that a Cohen-Macaulay local ring with a non-isolated singularity always admits maximal Cohen–Macaulay modules that are *not* locally free on the punctured spectrum. Focusing on these modules, Araya, Iima and Takahashi [AIT12] found out that the local hypersurfaces of type (A_∞) and (D_∞) have *finite* CM_+*-representation type*, that is, there exist only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules that are *not* locally free on the punctured spectrum.

In this chapter, we investigate Cohen-Macaulay local rings of finite CM_+-representation type from various viewpoints. Our basic landmark is the following conjecture, which includes the converse of the result of Araya, Iima and Takahashi stated above. We shall give positive results to this conjecture.

Conjecture 5.1.1. Let R be a complete local Gorenstein ring of dimension d not having an isolated singularity. Then the following two conditions are equivalent.

(1) The ring R has finite CM_+-representation type.

(2) The ring R has countable CM-representation type.

Combining the result of Buchweitz, Greuel and Schreyer, this conjecture says that, when R is a hypersurface having an uncountable algebraically closed coefficient field of characteristic not 2, condition (2) is equivalent to R being an (A_∞) or (D_∞) singularity. In this setting, the implication $(2) \Rightarrow (1)$ holds by [AIT12, Proposition 2.1].

From now on, we state our main results and the organization of this chapter. Section 5.2 is devoted to a couple of preliminary definitions and lemmas, while Section 5.3 presents some conjectures and questions on finite/countable CM-representation type. Our results are stated in the later sections. In what follows, let R be a Cohen–Macaulay local ring.

In Section 5.4, we consider the (Zariski-)closedness and (Krull) dimension of the singular locus $\operatorname{Sing} R$ of R in connection with the works of Huneke and Leuschke [HL02, HL03]. As we state above, they proved in [HL02] that if R has finite CM-representation type, then it has an isolated singularity, i.e., $\operatorname{Sing} R$ has dimension at most zero. Also, they showed in [HL03] that if R is complete or has uncountable residue field, and has countable CM-representation type, then

$\operatorname{Sing} R$ has dimension at most one. In relation to these results, we prove the following theorem, whose second assertion extends the result of Huneke and Leuschke [HL03] from countable CM-representation type to countable CM_+-representation type (i.e., having infinitely but countably many nonisomorphic indecomposable maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum).

Theorem 5.1.2 (Theorem 5.4.1 and Corollary 5.4.2)**.** Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring.

(1) Suppose that R has finite CM_+-representation type. Then the singular locus $\operatorname{Sing} R$ is a finite set. Equivalently, it is a closed subset of $\operatorname{Spec} R$ with dimension at most one.

(2) Suppose that R has countable CM_+-representation type. Then the set $\operatorname{Sing} R$ is at most countable. It has dimension at most one if R is either complete or k is uncountable.

Furthermore, Huneke and Leuschke [HL03] proved that if R admits a canonical module and has countable CM-representation type, then the localization $R_{\mathfrak{p}}$ at each prime ideal $\mathfrak{p}$ of R has at most countable CM-representation type as well. We prove a result on finite CM_+-representation type in the same context.

Theorem 5.1.3 (Theorem 5.4.3)**.** Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with a canonical module. Suppose that R has finite CM_+-representation type. Then $R_{\mathfrak{p}}$ has finite CM-representation type for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. In particular, $R_{\mathfrak{p}}$ has finite CM_+-representation type for all $\mathfrak{p} \in \operatorname{Spec} R$.

In Section 5.5 we provide various necessary conditions for a given Cohen–Macaulay local ring to have finite CM_+-representation type.

Theorem 5.1.4 (Theorem 5.5.2)**.** Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an ideal of R such that R/I is maximal Cohen–Macaulay over R. Then R has infinite CM_+-representation type in each of the following cases.

(1) The ring R/I has infinite CM_+-representation type.

(2) The set $\mathrm{V}(I)$ is contained in $\mathrm{V}(0:I)$, and either R/I has infinite CM-representation type or $d \geq 2$.

(3) The ideal $I+(0:I)$ is not $\mathfrak{m}$-primary, R/I has infinite CM-representation type, and R/I is either Gorenstein, a domain, or analytically unramified with $d=1$.

This theorem may look technical, but it actually gives rise to a lot of restrictions which having finite CM_+-representation type produces, and is used in the later sections. One concrete example where Theorem 5.5.2 applies is when $I=(x)$ and $(0:x)=(x)$; see Corollary 5.5.5. Here we introduce one of the applications of the above theorem. Denote by $\mathrm{CM}(R)$ the category of maximal Cohen–Macaulay R-modules, and by $\mathrm{D_{sg}}(R)$ the singularity category of R.

Theorem 5.1.5 (Theorem 5.5.4)**.** Let R be a Cohen–Macaulay local ring of dimension $d>0$. Let I be an ideal of R with $\mathrm{V}(I) \subseteq \mathrm{V}(0:I)$ such that R/I is maximal Cohen–Macaulay over R. Suppose that R has finite CM_+-representation type. Then one must have $d=1$. If $I^n=0$ for some integer $n>0$, then $\mathrm{CM}(R)$ has dimension at most $n-1$ in the sense of [DT15]. If R is Gorenstein, then R is a hypersurface and $\mathrm{D_{sg}}(R)$ has dimension at most $n-1$ in the sense of [Rou08].

There are folklore conjectures that a Gorenstein local ring of countable CM-representation type is a hypersurface, and that, for a Cohen–Macaulay local ring R of countable CM-representation type, $\mathrm{CM}(R)$ has dimension at most one. The above theorem gives partial answers to the variants of these folklore conjectures for finite CM_+-representation type.

In Section 5.6, we prove the following, which characterizes the Gorenstein rings or finite CM_+-representation type not having an isolated singularity in the dimension 1 case. This theorem has the consequence of answering Conjecture 5.1.1 in the affirmative when R has an uncountable algebraically closed coefficient field of characteristic not equal to 2.

Theorem 5.1.6 (Theorem 5.6.1)**.** Let R be a homomorphic image of a regular local ring. Suppose that R does not have an isolated singularity but is Gorenstein. If $\dim R = 1$, the following are equivalent.

(1) The ring R has finite CM_+-representation type.

(2) There exist a regular local ring S and a regular system of parameters x, y such that R is isomorphic to $S/(x^2)$ or $S/(x^2y)$.

When either of these two conditions holds, the ring R has countable CM-representation type.

In Section 5.7, we explore the higher-dimensional case, that is, we try to understand the Cohen–Macaulay local rings R of finite CM_+-representation type in the case where $\dim R \geq 2$. We prove the following two results in this section.

Theorem 5.1.7 (Corollary 5.7.7)**.** Let R be a complete local hypersurface of dimension $d \geq 2$ which is not an integral domain. Suppose that R has finite CM_+-representation type. Then one has $d = 2$, and there exist a regular local ring S and elements $x, y \in S$ with $R \cong S/(xy)$ such that $S/(x)$ and $S/(y)$ have finite CM-representation type and $S/(x, y)$ is an integral domain of dimension 1.

Theorem 5.1.8 (Corollaries 5.7.9 and 5.7.10)**.** Let R be a 2-dimensional non-normal Cohen–Macaulay complete local domain. Suppose that R has finite CM_+-representation type. Then the integral closure $\overline{R}$ of R has finite CM-representation type. If R is Gorenstein, then R is a hypersurface.

The former theorem gives a strong restriction of the structure of a hypersurface of finite CM_+-representation type which is not an integral domain. The latter theorem supports the conjecture that a Gorenstein local ring of finite CM_+-representation type is a hypersurface. Note that, under the assumption of the theorem plus the assumption that R is equicharacteristic zero, the integral closure $\overline{R}$ is a quotient surface singularity by the theorem of Auslander [Aus86] and Esnault [Esn85].

5.2 Preliminaries

This section is devoted to stating our conventions, and to recalling the definitions of the notions which repeatedly appear in this chapter.

Convention 5.2.1. Throughout this chapter, unless otherwise specified, we adopt the following convention. Rings are commutative and noetherian, and modules are finitely generated. Subcategories are full and strict (i.e., closed under isomorphism). Subscripts and superscripts are often omitted unless there is a risk of confusion. An identity matrix of suitable size is denoted by E.

Definition 5.2.2. Let R be a ring.

(1) An R-module M is *maximal Cohen–Macaulay* if the inequality $\operatorname{depth} M_{\mathfrak{p}} \geq \dim R_{\mathfrak{p}}$ holds for all $\mathfrak{p} \in \operatorname{Spec} R$. Hence, by definition, the zero module is maximal Cohen–Macaulay.

(2) We denote by $\operatorname{mod} R$ the category of (finitely generated) R-modules, and by $\operatorname{CM}(R)$ the subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay R-modules. For a subcategory $\mathscr{X}$ of $\operatorname{mod} R$, we denote by $\operatorname{ind} \mathscr{X}$ the set of isomorphism classes of indecomposable R-modules in $\mathscr{X}$, and by $\operatorname{add}_R \mathscr{X}$ the *additive closure* of $\mathscr{X}$, that is, the subcategory of $\operatorname{mod} R$ consisting of direct summands of finite direct sums of objects in $\mathscr{X}$.

(3) A subset S of $\operatorname{Spec} R$ is called *specialization-closed* if $\mathrm{V}(\mathfrak{p}) \subseteq S$ for all $\mathfrak{p} \in S$. This is equivalent to saying that S is a union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.

(4) Let S be a subset of $\operatorname{Spec} R$. Then it is easy to see that

$$\sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in S\} \geq \sup\{n \geq 0 \mid \text{there exists a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } S\},$$

and the equality holds if S is specialization-closed. The *(Krull) dimension* of a specialization-closed subset S of $\operatorname{Spec} R$ is defined as this common number and denoted by $\dim S$.

(5) The *singular locus* of R, denoted by $\operatorname{Sing} R$, is by definition the set of prime ideals $\mathfrak{p}$ of R such that $R_{\mathfrak{p}}$ is not a regular local ring. It is clear that $\operatorname{Sing} R$ is a specialization-closed subset of $\operatorname{Spec} R$. If R is excellent, then by definition $\operatorname{Sing} R$ is a closed subset of $\operatorname{Spec} R$ in the Zariski topology.

(6) For an $m \times n$ matrix A over R, we denote by $\operatorname{Cok}_R A$ the cokernel of the map $R^{\oplus n} \to R^{\oplus m}$ given by $x \mapsto Ax$, and by $\mathrm{I}_s(A)$ the ideal of R generated by all the s-minors of A.

(7) For an R-module M, we denote by $\operatorname{Fitt}_r(M)$ is the rth *Fitting invariant* of M, that is, we have $\operatorname{Fitt}_r(M) = \mathrm{I}_{m-r}(A)$ if there exists an exact sequence $R^{\oplus n} \xrightarrow{A} R^{\oplus m} \to M \to 0$.

Definition 5.2.3. Let $(R, \mathfrak{m}, k)$ be a local ring.

(1) For an R-module M, we denote by $\nu_R(M)$ the minimal number of generators of M, that is, $\nu_R(M) = \dim_k(M \otimes_R k)$.

(2) Let M an R-module and $n \geq 0$ an integer. We denote by $\Omega_R^n M$ (or simply $\Omega^n M$) the n-th *syzygy* of M, i.e., the image of the n-th differential map in the minimal free resolution of M. This is uniquely determined up to isomorphism.

(3) We denote by $\operatorname{edim} R$ the embedding dimension of R, and by $\operatorname{codepth} R$ the codepth of R, i.e., $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$. We say that R is a *hypersurface* if $\operatorname{codepth} R \leq 1$.

(4) An R-module M is called *periodic* if $\Omega^e M \cong M$ for some $e > 0$.

(5) The *complexity* of an R-module M, denoted by $\operatorname{cx}_R M$, is defined as the infimum of nonnegative integers n such that there exists a real number r satisfying the inequality $\beta_i^R(M) \leq ri^{n-1}$ for $i \gg 0$, where $\beta_i^R(M)$ stands for the ith Betti number of M.

(6) The *Loewy length* of R is defined by $\ell\ell(R) = \inf\{n \geq 0 \mid \mathfrak{m}^n = 0\}$. Note that $\ell\ell(R) < \infty$ if and only if R is artinian.

Definition 5.2.4. Let R be a local ring.

(1) For a subcategory $\mathscr{X}$ of $\operatorname{mod} R$ we denote by $[\mathscr{X}]$ the smallest subcategory of $\operatorname{mod} R$ containing R and $\mathscr{X}$ that is closed under finite direct sums, direct summands and syzygies, i.e., $[\mathscr{X}] = \operatorname{add}_R(\{R\} \cup \{\Omega^i X \mid i \geq 0, X \in \mathscr{X}\})$. When $\mathscr{X}$ consists of a single object X, we simply denote it by $[X]$.

(2) For subcategories $\mathscr{X}, \mathscr{Y}$ of $\operatorname{mod} R$ we denote by $\mathscr{X} \circ \mathscr{Y}$ the subcategory of $\operatorname{mod} R$ consisting of objects M which fit into an exact sequence $0 \to X \to M \to Y \to 0$ in $\operatorname{mod} R$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. We set $\mathscr{X} \bullet \mathscr{Y} = [[\mathscr{X}] \circ [\mathscr{Y}]]$.

(3) Let $\mathscr{C}$ be a subcategory of $\operatorname{mod} R$. Put

$$[\mathscr{C}]_r = \begin{cases} \{0\} & (r = 0), \\ [\mathscr{C}] & (r = 1), \\ [\mathscr{C}]_{r-1} \bullet \mathscr{C} = [[\mathscr{C}]_{r-1} \circ [\mathscr{C}]] & (r \geq 2). \end{cases}$$

If $\mathscr{C}$ consists of a single object C, then we simply denote it by $[C]_r$.

(4) Let $\mathscr{X}$ be a subcategory of $\operatorname{mod} R$. We define the *dimension* of $\mathscr{X}$, denoted by $\dim \mathscr{X}$, as the infimum of the integers $n \geq 0$ such that $\mathscr{X} = [G]_{n+1}$ for some $G \in \mathscr{X}$.

5.3 Conjectures and questions

In this section, we present several conjectures and questions which we deal with in later sections. First of all, let us give several definitions of representation types, including that of finite CM_+-representation type, which is the main subject of this chapter.

Definition 5.3.1. Let R be a Cohen–Macaulay ring. By $\mathrm{CM}_0(R)$ we denote the subcategory of $\mathrm{CM}(R)$ consisting of modules that are locally free on the punctured spectrum of R, and set

$$\mathrm{CM}_+(R) := \mathrm{CM}(R) \setminus \mathrm{CM}_0(R).^1$$

For each $\mathsf{X} \in \{\mathrm{CM}, \mathrm{CM}_0, \mathrm{CM}_+\}$ we say that R has *finite* (resp. *countable*) X-representation type if there exist only finitely (resp. countably) many isomorphism classes of indecomposable modules in $\mathsf{X}(R)$. We say that R has *infinite* (resp. *uncountable*) X-representation type if R does not have

[1]The index 0 (resp. +) in $\mathrm{CM}_0(R)$ (resp. $\mathrm{CM}_+(R)$) means that it consists of modules whose nonfree loci have zero (resp. positive) dimension.

finite (resp. countable) X-representation type. Also, R is said to have *bounded* X-representation type if there exists an upper bound of the multiplicities of indecomposable modules in $\mathsf{X}(R)$, and said to have *unbounded* X-representation type if R does not have bounded X-representation type.

Let R be a complete local hypersurface with uncountable algebraically closed coefficient field of characteristic not two. Buchweitz, Greuel and Schreyer [BGS87, Theorem B] (see also [LW12, Theorem 14.16]) prove that R has countable CM-representation type if and only if it is either an (A_∞)-singularity or a (D_∞)-singularity. Moreover, when this is the case, they give a complete classification of the indecomposable maximal Cohen–Macaulay R-modules. Using this result, Araya, Iima and Takahashi [AIT12, Theorem 1.1 and Corollary 1.3] prove the following theorem (see [DT15, Proposition 3.5(3)]), which provides examples of a Cohen–Macaulay local ring of finite CM_+-representation type.

Theorem 5.3.2 (Araya–Iima–Takahashi)**.** Let R be a complete local hypersurface with uncountable algebraically closed coefficient field of characteristic not two. If R has countable CM-representation type, then the following statements hold.

(1) The ring R has finite CM_+-representation type.

(2) There is an inequality $\dim \mathrm{CM}(R) \leq 1$.

By definition, there is a strong connection between finite CM_+-representation type and finite CM-representation type. The first assertion of Theorem 5.3.2 suggests to us that finite CM_+-representation type should also be closely related to countable CM-representation type. Several conjectures have been presented so far concerning finite/countable CM-representation type, and we set the following proposal.

Proposal 5.3.3. One should consider the conjectures on finite/countable CM-representation type for finite CM_+-representation type.

There has been a folklore conjecture on countable CM-representation type probably since the 1980s. Recently, this conjecture has been studied by Stone [Sto14].

Conjecture 5.3.4. A Gorenstein local ring R of countable CM-representation type is a hypersurface.

This conjecture holds true if R has finite CM-representation type; see [Yos90, Theorem (8.15)]. Also, the conjecture holds if R is a complete intersection with algebraically closed uncountable residue field; see [AI07, Existence Theorem 7.8]. The following example shows that the assumption in the conjecture that R is Gorenstein is necessary.

Example 5.3.5. Let $S = \mathbb{C}[\![x,y,z]\!]/(xy)$. Then S is an (A_∞)-singularity of dimension 2, and has countable CM-representation type by [BGS87, Theorem B]. Let R be the second Veronese subring of S, that is, $R = \mathbb{C}[\![x^2,xy,xz,y^2,yz,z^2]\!] \subseteq S$. Then R is a Cohen–Macaulay non-Gorenstein local ring of dimension 2. We claim that R has countable CM-representation type. Indeed, let $N_1, N_2, \dots$ be the non-isomorphic indecomposable maximal Cohen–Macaulay S-modules. Let M be an indecomposable maximal Cohen–Macaulay R-module. Then $N = \mathrm{Hom}_R(S,M)$ is a maximal Cohen–Macaulay S-module, and one can write $N \cong N_{a_1}^{\oplus b_1} \oplus \cdots \oplus N_{a_t}^{\oplus b_t}$. Since R is a direct summand of S, the module M is a direct summand of N, and hence it is a direct summand of N_{a_i} for some i. The claim follows from this.

Combining Conjecture 5.3.4 with Proposal 5.3.3 gives rise to the following question.

Question 5.3.6. Let R be a Gorenstein local ring which is not an isolated singularity. Suppose that R has finite CM_+-representation type. Then is R a hypersurface?

Here, the assumption that R is not an isolated singularity is necessary. Indeed, if R is an isolated singularity, then $\#\mathsf{ind}\,\mathrm{CM}_+(R) = 0 < \infty$. We shall give answers to Question 5.3.6 in Sections 5.5 and 5.7.

Theorem 5.3.2(2) leads us to the following conjecture.

Conjecture 5.3.7. Let R be a Cohen–Macaulay local ring R of countable CM-representation type. Then there is an inequality $\dim \mathrm{CM}(R) \leq 1$.

This conjecture holds true if R has finite CM-representation type; see [DT15, Proposition 3.7(1)]. Let R be a Gorenstein local ring. Then the stable category $\underline{\mathrm{CM}}(R)$ of $\mathrm{CM}(R)$ is a triangulated category, and one can consider the *(Rouquier) dimension* $\dim \underline{\mathrm{CM}}(R)$ of $\underline{\mathrm{CM}}(R)$; we refer the reader to [Rou08] for the details. One has $\dim \underline{\mathrm{CM}}(R) \leq \dim \mathrm{CM}(R)$ with equality if R is a hypersurface; see [DT15, Proposition 3.5]. There seems to be a folklore conjecture asserting that every (noncommutative) selfinjective algebra Λ of tame representation type satisfies the inequality $\dim(\underline{\mathrm{mod}}\,\Lambda) \leq 1$. So Conjecture 5.3.7 is thought of as a Cohen–Macaulay version of this folklore conjecture. Combining Conjecture 5.3.7 with Proposal 5.3.3 leads us to the following question.

Question 5.3.8. Let R be a Cohen–Macaulay local ring of finite CM_+-representation type. Then does one have $\dim \mathrm{CM}(R) \leq 1$?

We shall give partial answers to this question in Section 5.5.

Huneke and Leuschke ([HL03, Theorem 1.3]) prove the following theorem, which solves a conjecture of Schreyer [Sch87, Conjecture 7.2.3] presented in the 1980s.

Theorem 5.3.9 (Huneke–Leuschke)**.** Let $(R, \mathfrak{m}, k)$ be an excellent Cohen–Macaulay local ring. Assume that R is complete or k is uncountable. If R has countable CM-representation type, then $\dim \operatorname{Sing} R \leq 1$.

Indeed, the assumption that R is excellent is unnecessary; see [Tak07, Theorem 2.4]. This result naturally makes us have the following question.

Question 5.3.10. Let R be a Cohen–Macaulay local ring. If R has finite CM_+-representation type, then does $\operatorname{Sing} R$ have dimension at most one?

We shall give a complete answer to this question in the next Section 5.4. In fact, we can even prove a stronger statement.

5.4 The closedness and dimension of the singular locus

In this section, we discuss the structure of the singular locus of a Cohen–Macaulay local ring of finite CM_+-representation type. First, we consider what the finiteness of the singular locus means.

Lemma 5.4.1. Let R be a local ring with maximal ideal $\mathfrak{m}$. The following are equivalent.

(1) $\operatorname{Sing} R$ is a finite set.

(2) $\operatorname{Sing} R$ is a closed subset of $\operatorname{Spec} R$ in the Zariski topology, and has dimension at most one.

Proof. (2) $\Longrightarrow$ (1): We find an ideal I of R such that $\operatorname{Sing} R = \mathrm{V}(I)$. As $\operatorname{Sing} R$ has dimension at most one, so does the local ring R/I. Hence $\operatorname{Spec} R/I = \operatorname{Min} R/I \cup \{\mathfrak{m}/I\}$, and this is a finite set.

(1) $\Longrightarrow$ (2): Write $\operatorname{Sing} R = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. As $\operatorname{Sing} R$ is specialization-closed, it coincides with the finite union $\mathrm{V}(\mathfrak{p}_1) \cup \cdots \cup \mathrm{V}(\mathfrak{p}_n)$ of closed subsets of $\operatorname{Spec} R$. Hence $\operatorname{Sing} R$ is closed.

To show the other assertion, we claim (or recall) that a local ring R of dimension at least two possesses infinitely many prime ideals of height one. Indeed, for any $x \in \mathfrak{m}$ we have $\operatorname{ht}(x) \leq 1$ by Krull's principal ideal theorem, that is, (x) is contained in some prime ideal $\mathfrak{p}$ with $\operatorname{ht} \mathfrak{p} \leq 1$. This argument shows that $\mathfrak{m} = \bigcup_{\mathfrak{p} \in \operatorname{Spec} R, \operatorname{ht} \mathfrak{p} \leq 1} \mathfrak{p}$. Now suppose that there exist only finitely many prime ideals of R having height one. Then, since the number of the minimal primes is finite, so is the number of prime ideals of height at most one. Therefore the above union is finite, and by prime avoidance $\mathfrak{m}$ is contained in some $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} \leq 1$. This implies $\dim R \leq 1$, which is a contradiction. Thus the claim follows.

Now, assume that $\operatorname{Sing} R$ has dimension at least 2. Then $\dim R/\mathfrak{p} \geq 2$ for some $\mathfrak{p} \in \operatorname{Sing} R$. The above claim shows that the ring $R/\mathfrak{p}$ has infinitely many prime ideals of height one, which have the form $\mathfrak{q}/\mathfrak{p}$ with $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$. Then $\mathfrak{q}$ is also in $\operatorname{Sing} R$, and hence $\operatorname{Sing} R$ contains infinitely many prime ideals. This contradiction shows that the dimension of $\operatorname{Sing} R$ is at most 1. $\square$

The following theorem clarifies a close relationship between finite (resp. countable) CM_+-representation type and finiteness (resp. countablity) of the singular locus.

Theorem 5.4.1. Suppose R is a Cohen–Macaulay local ring of finite (resp. countable) CM_+-representation type. Then $\operatorname{Sing} R$ is a finite (resp. countable) set.

Proof. We first consider the case where R has finite CM_+-representation type. Write $\operatorname{ind} \mathrm{CM}_+(R) = \{G_1, \dots, G_t\}$, and pick $\mathfrak{p} \in \operatorname{Sing} R \setminus \{\mathfrak{m}\}$. Set $C = \Omega_R^d(R/\mathfrak{p})$. We claim that $\mathfrak{p} = \operatorname{ann}_R \operatorname{Tor}_1^R(C, C)$.

Indeed, $\operatorname{Tor}_1^R(C,C)$ is isomorphic to $T := \operatorname{Tor}_{1+2d}^R(R/\mathfrak{p}, R/\mathfrak{p})$, which is killed by $\mathfrak{p}$. Hence $\mathfrak{p}$ is contained in the annihilator. Also, $T_\mathfrak{p}$ is isomorphic to $\operatorname{Tor}_{1+2d}^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$, which does not vanish as $\mathfrak{p}$ belongs to the singular locus. Hence $\mathfrak{p}$ is in the support of T, and contains the annihilator. Now the claim follows.

Note that $C_\mathfrak{p}$ is stably isomorphic to $\Omega_{R_\mathfrak{p}}^d(\kappa(\mathfrak{p}))$, which is not $R_\mathfrak{p}$-free since $R_\mathfrak{p}$ is singular. This means that C belongs to $\mathrm{CM}_+(R)$, and we get an isomorphism $C \cong G_{l_1}^{\oplus a_1} \oplus \cdots \oplus G_{l_s}^{\oplus a_s} \oplus H$ with $s \geq 1$ and $1 \leq l_1 < \cdots < l_s \leq t$ and $a_1, \dots, a_s \geq 1$ and $H \in \mathrm{CM}_0(R)$. It is easy to see that

$$\mathfrak{p} = \left(\textstyle\bigcap_{1 \leq i,j \leq s} \operatorname{ann}_R \operatorname{Tor}_1^R(G_{l_i}, G_{l_j})\right) \cap \operatorname{ann}_R \operatorname{Tor}_1^R(H, M)$$

for some R-module M. Since a prime ideal is irreducible in general, $\mathfrak{p}$ coincides with one of the annihilators in the right-hand side. The module H is locally free on the punctured spectrum, and $\operatorname{ann}_R \operatorname{Tor}_1^R(H,M)$ contains a power of $\mathfrak{m}$. As $\mathfrak{p}$ is a nonmaximal prime ideal, it cannot coincide with $\operatorname{ann}_R \operatorname{Tor}_1^R(H,M)$. We thus have $\mathfrak{p} = \operatorname{ann}_R \operatorname{Tor}_1^R(G_{l_p}, G_{l_q})$ for some p, q. This shows that we have only finitely many such prime ideals $\mathfrak{p}$. Consequently, $\operatorname{Sing} R \setminus \{\mathfrak{m}\}$ is a finite set, and so is $\operatorname{Sing} R$.

We can analogously deal with the case where R has countable CM_+-representation type. In this case, we can write $\operatorname{ind} \mathrm{CM}_+(R) = \{G_1, G_2, G_3, \dots\}$, and for each $\mathfrak{p} \in \operatorname{Sing} R \setminus \{\mathfrak{m}\}$ there exist p, q such that $\mathfrak{p} = \operatorname{ann}_R \operatorname{Tor}_1^R(G_{l_p}, G_{l_q})$. $\square$

Theorem 5.4.1 yields the following corollary, which gives a complete answer to Question 5.3.10. We should remark that the second assertion of the corollary highly refines Theorem 5.3.9 due to Huneke and Leuschke.

Corollary 5.4.2. Let R be a Cohen–Macaulay local ring.

(1) If R has finite CM_+-representation type, then $\operatorname{Sing} R$ is closed and has dimension at most one.

(2) Suppose that R has countable CM_+-representation type.

  (a) If k is uncountable, then $\operatorname{Sing} R$ has dimension at most one.

  (b) If R is complete, then $\operatorname{Sing} R$ is closed and has dimension at most one.

Proof. (1) The assertion follows from Theorem 5.4.1 and Lemma 5.4.1.

(2) Theorem 5.4.1 implies that $\operatorname{Sing} R$ is a countable set. Note that $\operatorname{Sing} R$ is specialization-closed. If R is complete or k is uncountable, then we can apply [Tak07, Lemma 2.2] to deduce that $\dim R/\mathfrak{p} \leq 1$ for all $\mathfrak{p} \in \operatorname{Sing} R$. In case R is complete, $\operatorname{Sing} R$ is closed as well since R is excellent. □

Next we investigate the relationship of finite CM_+-representation type with localization of the base ring at a prime ideal. In particular, we prove the following theorem, which says that finite CM_+-representation type implies finite CM-representation type on the punctured spectrum. This especially shows that finite CM_+-representation type localizes, which should be compared with the result of Huneke and Leuschke [HL03, Theorem 2.1] asserting that countable CM-representation type localizes under the same assumption as in this theorem. This is also connected with the conjecture that a Cohen–Macaulay local ring with an isolated singularity having countable CM-representation type has finite CM-representation type [HL03, Page 3006].

Theorem 5.4.3. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with a canonical module ω. Suppose that R has finite CM_+-representation type. Then $R_\mathfrak{p}$ has finite CM-representation type for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

Proof. Assume that there exists a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ such that $R_\mathfrak{p}$ has infinite CM-representation type. Then the set $\operatorname{ind} \mathrm{CM}(R_\mathfrak{p}) \setminus \{\omega_\mathfrak{p}\}$ is infinite, and we may choose $\mathscr{N} = \{N_1, N_2, N_3, \dots\}$ to be an infinite subset.

Fix a module $N \in \mathscr{N}$. Then we can choose an R-module L such that $N \cong L_\mathfrak{p}$. Take a *maximal Cohen–Macaulay approximation* of L over R, that is, a short exact sequence

$$\sigma : 0 \to Y \to X \to L \to 0$$

of R-modules such that X is maximal Cohen–Macaulay and Y has finite injective dimension; see [AB89, Theorem 1.1]. Localization gives an exact sequence $\sigma_\mathfrak{p} : 0 \to Y_\mathfrak{p} \to X_\mathfrak{p} \to N \to 0$. As N is maximal Cohen–Macaulay, $Y_\mathfrak{p}$ is a maximal Cohen–Macaulay $R_\mathfrak{p}$-module of finite injective dimension. It follows from [BH93, Exercise 3.3.28(a)] that $Y_{\mathfrak{p}} \cong \omega_{\mathfrak{p}}^{\oplus n}$ for some $n \geq 0$. The exact sequence $\sigma_{\mathfrak{p}}$ splits, and we get an isomorphism $X_{\mathfrak{p}} \cong N \oplus \omega_{\mathfrak{p}}^{\oplus n}$. Note that $\omega_{\mathfrak{p}}$ is an indecomposable $R_{\mathfrak{p}}$-module.

Let $X = X_1 \oplus \cdots \oplus X_m$ be a decomposition of X into indecomposable R-modules. Then there is an isomorphism $(X_1)_{\mathfrak{p}} \oplus \cdots \oplus (X_m)_{\mathfrak{p}} \cong N \oplus \omega_{\mathfrak{p}}^{\oplus n}$. For each i write $(X_i)_{\mathfrak{p}} = Z_i \oplus \omega_{\mathfrak{p}}^{\oplus l_i}$ with $l_i \geq 0$ an integer and Z_i not containing $\omega_{\mathfrak{p}}$ as a direct summand; then Z_i is a maximal Cohen–Macaulay $R_{\mathfrak{p}}$-module. We get an isomorphism

$$Z_1 \oplus \cdots \oplus Z_m \oplus \omega_{\mathfrak{p}}^{\oplus(l_1+\cdots+l_m)} \cong N \oplus \omega_{\mathfrak{p}}^{\oplus n}.$$

Since $\operatorname{End}_{R_{\mathfrak{p}}}(\omega_{\mathfrak{p}}) \cong R_{\mathfrak{p}}$ is a local ring, the module $Z_1 \oplus \cdots \oplus Z_m$ does not contain $\omega_{\mathfrak{p}}$ as a direct summand by [LW12, Lemma 1.2], while N is an indecomposable $R_{\mathfrak{p}}$-module with $N \ncong \omega_{\mathfrak{p}}$. Further, [LW12, Lemma 2.1] also implies that $Z_1 \oplus \cdots \oplus Z_m \cong N$ and $l_1 + \cdots + l_m = n$, so we may assume that $Z_1 \cong N$ and $Z_2 = \cdots = Z_m = 0$. We thus have that $(X_1)_{\mathfrak{p}} \cong N \oplus \omega_{\mathfrak{p}}^{\oplus l_1}$.

Suppose that $(X_1)_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free. Then so are N and $\omega_{\mathfrak{p}}$, and we have $N \cong R_{\mathfrak{p}} \cong \omega_{\mathfrak{p}}$, which contradicts the choice of N. Hence $(X_1)_{\mathfrak{p}}$ is not $R_{\mathfrak{p}}$-free, which implies that $X_1 \in \mathrm{CM}_+(R)$.

Thus we have shown that for each integer $i \geq 1$ there exist an integer $n_i \geq 0$ and a module $C_i \in \operatorname{ind}\mathrm{CM}_+(R)$ such that $(C_i)_{\mathfrak{p}} \cong N_i \oplus \omega_{\mathfrak{p}}^{\oplus n_i}$. Assume that $C_i \cong C_j$ for some $i \neq j$. Then $N_i \oplus \omega_{\mathfrak{p}}^{\oplus n_i} \cong N_j \oplus \omega_{\mathfrak{p}}^{\oplus n_j}$, and, appealing again to [LW12, Lemma 1.2], we see that $N_i \cong N_j$ (and $n_i = n_j$), contrary to the choice of $\mathscr{N}$. Hence $C_i \ncong C_j$ for all $i \neq j$, and we conclude that R has infinite CM_+-representation type. This contradiction completes the proof of the theorem. □

Remark 5.4.4. In Corollary 5.4.2(1) we proved that the singular locus of a Cohen–Macaulay local ring of finite CM_+-representation type has dimension at most one. As an application of Theorem 5.4.3, we can get another proof of this statement under the assumption that R admits a canonical module.

Let R be a d-dimensional Cohen–Macaulay local ring with a canonical module, and suppose that R has finite CM_+-representation type. Then $R_{\mathfrak{p}}$ has finite CM-representation type for all nonmaximal prime ideals $\mathfrak{p}$ of R by Theorem 5.4.3. In particular, $R_\mathfrak{p}$ has an isolated singularity for all such $\mathfrak{p}$ by [HL02, Corollary 2]. This implies that $R_\mathfrak{q}$ is a regular local ring in codimension $d-2$, and therefore $\dim \operatorname{Sing} R \leq 1$.

5.5 Necessary conditions for finite CM_+-representation type

In this section, we explore necessary conditions for a Cohen–Macaulay local ring to have finite CM_+-representation type. For this purpose we begin with stating and showing a couple of lemmas.

Lemma 5.5.1. Let R be a local ring.

(1) The subcategory of $\operatorname{mod} R$ consisting of periodic modules is closed under finite direct sums: if the R-modules $M_1,\dots,M_n$ are periodic, then so is $M_1 \oplus \cdots \oplus M_n$.

(2) Let $0 \to M_1 \to \cdots \to M_n \to 0$ be an exact sequence in $\operatorname{mod} R$. Let $r \geq 0$ and $1 \leq t \leq n$ be integers. If $\operatorname{cx}_R(M_i) \leq r$ for all $1 \leq i \leq n$ with $i \neq t$, then $\operatorname{cx}_R(M_t) \leq r$.

Proof. (1) is obvious so we need only show (2), and it suffices to show the statement when $n = 3$. Suppose that M_2, M_3 have complexity at most r. Then we find $p, q \in \mathbb{R}_{>0}$ such that $\beta_j^R(M_2) \leq p j^{r-1}$ and $\beta_j^R(M_3) \leq q j^{r-1}$ for $j \gg 0$. The induced exact sequence $\operatorname{Tor}_{j+1}^R(M_3,k) \to \operatorname{Tor}_j^R(M_1,k) \to \operatorname{Tor}_j^R(M_2,k)$ shows that $\beta_j^R(M_1) \leq \beta_j^R(M_2) + \beta_{j+1}^R(M_3) \leq (p+qr)j^{r-1}$ for $j \gg 0$. Therefore we obtain $\operatorname{cx}_R(M_3) \leq r$. The other cases are handled similarly. $\square$

The subcategory $\mathrm{CM}_+(R)$ of $\operatorname{mod} R$ is stable under syzygies.

Lemma 5.5.2. Let R be a local ring. Let $0 \to N \to F \to M \to 0$ be an exact sequence in $\operatorname{mod} R$ such that F is free and M is maximal Cohen–Macaulay. Then M belongs to $\mathrm{CM}_+(R)$ if and only if so does N.

Proof. Note that all the modules N, F, M are maximal Cohen–Macaulay. Hence the assertion is equivalent to saying that M belongs to $\mathrm{CM}_0(R)$ if and only if so does N. The "if" part follows from the fact that $\mathrm{CM}_0(R)$ is stable under syzygies. To show the "only if" part, assume that N

is in $\mathrm{CM}_0(R)$. Let $\mathfrak{p}$ be a nonmaximal prime ideal of R. Then $N_\mathfrak{p}$ is $R_\mathfrak{p}$-free, and we see that the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ has projective dimension at most 1. Note that $M_\mathfrak{p}$ is maximal Cohen–Macaulay over $R_\mathfrak{p}$. The Auslander–Buchsbaum formula implies that $M_\mathfrak{p}$ is free. Hence M is in $\mathrm{CM}_0(R)$. $\square$

We state some containments among indecomposable maximal Cohen–Macaulay modules over Cohen–Macaulay local rings, one of which is a homomorphic image of the other.

Proposition 5.5.1. Let R be a Cohen–Macaulay local ring of dimension d. Let I be an ideal of R such that R/I is a maximal Cohen–Macaulay R-module. Then the following statements hold.

(1) $\operatorname{ind}\mathrm{CM}(R/I)$ is contained in $\operatorname{ind}\mathrm{CM}(R)$.

(2) $\operatorname{ind}\mathrm{CM}_+(R/I)$ is contained in $\operatorname{ind}\mathrm{CM}_+(R)$.

(3) $\operatorname{ind}\mathrm{CM}(R/I)$ is contained in $\operatorname{ind}\mathrm{CM}_+(R)$, if $\mathrm{V}(I) \subseteq \mathrm{V}(0:I)$.

Proof. Let M be an indecomposable maximal Cohen–Macaulay R/I-module. The definition of indecomposability says $M \neq 0$. The equalities $\operatorname{depth} M = \dim R/I = \dim R$ imply M is a maximal Cohen–Macaulay R-module. It is directly checked that M is indecomposable as an R-module. Now (1) follows.

Let $\mathfrak{p}$ be a prime ideal of R such that $M_\mathfrak{p} \cong (R_\mathfrak{p})^{\oplus n}$ for some $n \geq 0$. If $n = 0$, then $M_\mathfrak{p} = 0$. If $n > 0$, then $IR_\mathfrak{p} = 0$ since $IM = 0$, and hence $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus n} = (R/I)_\mathfrak{p}^{\oplus n}$.

Let us consider the case where M is in $\mathrm{CM}_+(R/I)$. Then there is a prime ideal $\mathfrak{q}$ of R with $I \subseteq \mathfrak{q} \neq \mathfrak{m}$ such that $M_\mathfrak{q}$ is not $(R/I)_\mathfrak{q}$-free. Letting $\mathfrak{p} := \mathfrak{q}$ in the above argument, we observe that $M_\mathfrak{q}$ is not $R_\mathfrak{q}$-free (note that the zero module is free). Thus M is in $\mathrm{CM}_+(R)$, and (2) follows.

Next we consider the case where M is in $\mathrm{CM}_0(R)$. As $\dim M = \dim R/I = d > 0$, there is a nonmaximal prime ideal $\mathfrak{r}$ of R such that $M_\mathfrak{r} \neq 0$. Letting $\mathfrak{p} := \mathfrak{r}$ in the above argument, we have $IR_\mathfrak{r} = 0$. Hence $\mathfrak{r}$ is not in the support of the R-module I, which is equivalent to saying that $\mathfrak{r}$ does not contain $(0:I)$. On the other hand, $\mathfrak{r}$ is in the support of the R-module M, which implies that $\mathfrak{r}$ contains I. Thus $\mathrm{V}(I)$ is not contained in $\mathrm{V}(0:I)$. We now observe that (3) holds. $\square$

The lemma below says finite CM-representation type is equivalent to finite CM_0-representation type.

Lemma 5.5.3. Let R be a Cohen–Macaulay local ring. If R has infinite CM-representation type, then R has infinite CM_0-representation type.

Proof. Suppose that R has finite CM_0-representation type. Then by [DT15, Corollary 1.2] it is an isolated singularity. Hence $\mathrm{CM}(R) = \mathrm{CM}_0(R)$, and we have $\mathsf{ind}\,\mathrm{CM}(R/I) = \mathsf{ind}\,\mathrm{CM}_0(R/I)$, which is a finite set. This contradicts the assumption that R has infinite CM-representation type. $\square$

Now we can prove the first main result of this section, which gives various necessary conditions for a Cohen–Macaulay local ring to have finite CM_+-representation type.

Theorem 5.5.2. Let R be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an ideal of R, and assume that R/I is a maximal Cohen–Macaulay R-module. Then R has infinite CM_+-representation type in each of the following cases.

(1) R/I has infinite CM_+-representation type.

(2) $\mathrm{V}(I) \subseteq \mathrm{V}(0 : I)$ and

(a) R/I has infinite CM-representation type, or

(b) $d \geq 2$.

(3) $\mathrm{ht}(I + (0 : I)) < d$, R/I has infinite CM-representation type, and

(a) R/I is a Gorenstein ring, or

(b) R/I is a domain, or

(c) $d = 1$ and R/I is analytically unramified, or

(d) $d = 1$, k is infinite, and R/I is equicharacteristic and reduced.

Proof. (1)&(2a) These assertions immediately follow from (2) and (3) of Proposition 5.5.1, respectively.

(2b) In view of (2a), we may assume that R/I has finite CM-representation type. It follows from [HL02, Corollary 2] that R/I is an isolated singularity. As $d \geq 2$, the ring R/I is a (normal) domain. Hence $\mathfrak{p} := I$ is a prime ideal of R. As $\dim R/\mathfrak{p} = d$, the prime ideal $\mathfrak{p}$ is minimal. The assumption $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{V}(0 : \mathfrak{p})$ implies $(0 :_R \mathfrak{p}) \subseteq \mathfrak{p}$. Localizing this inclusion at $\mathfrak{p}$, we get an inclusion $(0 :_{R_\mathfrak{p}} \mathfrak{p}R_\mathfrak{p}) \subseteq \mathfrak{p}R_\mathfrak{p}$, which particularly says that $R_\mathfrak{p}$ is not a field. Therefore $\mathfrak{p}$ belongs to $\operatorname{Sing} R$.

Suppose that R has finite CM_+-representation type. Then Corollary 5.4.2(1) implies that $\operatorname{Sing} R$ has dimension at most one. In particular, we obtain $d = \dim R/\mathfrak{p} \leq 1$, which is a contradiction. Consequently, R has infinite CM_+-representation type.

(3) We find a nonmaximal prime ideal $\mathfrak{p}$ of R that contains the ideal $I + (0 : I)$. Then, as $\mathfrak{p}$ contains I, the prime ideal $\mathfrak{p}/I$ of R/I is defined, which is not maximal. Also, since $\mathfrak{p}$ contains $(0 : I)$ as well, we see that $IR_\mathfrak{p}$ is a nonzero proper ideal of $R_\mathfrak{p}$.

We establish several claims.

Claim 5.5.4. Let $M \in \operatorname{ind} \mathrm{CM}_0(R/I)$ with $M_\mathfrak{p} \neq 0$. Then $M \in \operatorname{ind} \mathrm{CM}_+(R)$.

Proof of Claim. Proposition 5.5.1(1) implies $M \in \operatorname{ind} \mathrm{CM}(R)$. There exists an integer $n \geq 0$ such that

$$M_\mathfrak{p} = M_{\mathfrak{p}/I} \cong (R/I)^{\oplus n}_{\mathfrak{p}/I} = (R/I)^{\oplus n}_\mathfrak{p} = (R_\mathfrak{p}/IR_\mathfrak{p})^{\oplus n}.$$

Since $M_\mathfrak{p}$ is nonzero, we have to have $n > 0$. Since $IR_\mathfrak{p}$ is a nonzero proper ideal of $R_\mathfrak{p}$, we have that $M_\mathfrak{p}$ is not a free $R_\mathfrak{p}$-module. We now conclude that M belongs to $\operatorname{ind} \mathrm{CM}_+(R)$. □

Claim 5.5.5. When R/I is Gorenstein, for each $M \in \operatorname{ind} \mathrm{CM}_0(R/I)$, either M or $\Omega_{R/I} M$ is in $\operatorname{ind} \mathrm{CM}_+(R)$.

Proof of Claim. If $M_\mathfrak{p} \neq 0$, then $M \in \operatorname{ind} \mathrm{CM}_+(R)$ by Claim 5.5.4. Assume $M_\mathfrak{p} = 0$. There is an exact sequence $0 \to N \to (R/I)^{\oplus n} \to M \to 0$, where we set $N := \Omega_{R/I} M$ and $n := \nu_{R/I}(M) > 0$. Localization at $\mathfrak{p}$ gives an isomorphism $N_\mathfrak{p} \cong (R_\mathfrak{p}/IR_\mathfrak{p})^{\oplus n}$. As $n > 0$ and $IR_\mathfrak{p}$ is a proper ideal, the module $N_\mathfrak{p}$ is nonzero. Since R/I is Gorenstein, we apply Lemma 5.5.2 and [Yos90, Lemma (8.17)] to see that N belongs to $\operatorname{ind} \mathrm{CM}_0(R/I)$. Using Claim 5.5.4 again, we obtain $N \in \operatorname{ind} \mathrm{CM}_+(R)$. □

Claim 5.5.6. There is an inclusion

$$\{M \in \operatorname{ind} \mathrm{CM}_0(R/I) \mid M \text{ has a rank as an } R/I\text{-module}\} \subseteq \operatorname{ind} \mathrm{CM}_+(R).$$

Proof of Claim. Take M from the left-hand side. Since the R/I-module M is maximal Cohen–Macaulay, its annihilator has grade 0. Hence M has positive rank, and we see that $\operatorname{Supp}_{R/I} M = \operatorname{Spec} R/I$. Therefore $M_{\mathfrak{p}} = M_{\mathfrak{p}/I}$ is nonzero. It follows from Claim 5.5.4 that M belongs to $\operatorname{ind} \mathrm{CM}_+(R)$. □

(3a) Suppose that R has finite CM_+-representation type, namely, $\operatorname{ind} \mathrm{CM}_+(R)$ is a finite set. Lemma 5.5.3 guarantees that the set $\operatorname{ind} \mathrm{CM}_0(R/I)$ is infinite, and hence the set difference

$$\mathscr{S} := \operatorname{ind} \mathrm{CM}_0(R/I) \setminus \operatorname{ind} \mathrm{CM}_+(R)$$

is infinite as well. Thus we can choose a (countably) infinite subset $\{M_1, M_2, M_3, \dots\}$ of $\mathscr{S}$. By Claim 5.5.5 we see that $\Omega_{R/I} M_i$ belongs to $\operatorname{ind} \mathrm{CM}_+(R)$ for all i. Note that $\Omega_{R/I} M_i \ncong \Omega_{R/I} M_j$ for all distinct i, j since R/I is Gorenstein and M_i, M_j are maximal Cohen–Macaulay over R/I. It follows that $\operatorname{ind} \mathrm{CM}_+(R)$ is an infinite set, which is a contradiction. Thus R has infinite CM_+-representation type.

(3b) Since R/I is a domain, every R/I-module has rank. Claim 5.5.6 implies that $\operatorname{ind} \mathrm{CM}_0(R/I)$ is contained in $\operatorname{ind} \mathrm{CM}_+(R)$, while $\operatorname{ind} \mathrm{CM}_0(R/I)$ is an infinite set by Lemma 5.5.3. It follows that R has infinite CM_+-representation type.

(3c) Note that $\mathrm{CM}(R/I) = \mathrm{CM}_0(R/I)$. Since R/I is analytically unramified and has infinite CM-representation type, it follows from [LW12, Theorem 4.10] that the left-hand side of the inclusion in Claim 5.5.6 is infinite, and so is the right-hand side $\operatorname{ind} \mathrm{CM}_+(R)$, that is, R has infinite CM_+-represenation type.

(3d) Since k is infinite and R/I is equicharacteristic, we can apply [LW12, Theorem 17.10] to deduce that if R/I has unbounded CM-representation type, then the left-hand side of the inclusion in Claim 5.5.6 is infinite (as R/I is reduced), and we are done. Hence we may assume that R/I has bounded CM-representation type. By [LW12, Theorems 10.1 and 17.10] the completion $\widehat{R/I}$ has infinite and bounded CM-representation type. According to [LW12, Theorem 17.9], the ring $\widehat{R/I}$ is isomorphic to one of the following three rings.

$$k[\![x,y]\!]/(x^2), \quad k[\![x,y]\!]/(x^2y), \quad k[\![x,y,z]\!]/(yz, x^2-xz, xz-z^2).$$

The indecomposable maximal Cohen–Macaulay modules over these rings are classified; one can find complete lists of those modules in [BGS87, Propositions 4.1 and 4.2] and [LW12, Example 14.23]. We can check by hand that each of these rings has an infinite family of nonisomorphic indecomposable maximal Cohen–Macaulay modules of rank 1. This family of modules is extended from a family of R/I-modules by [LW05, Corollary 2.2], and these are nonisomorphic indecomposable maximal Cohen–Macaulay R/I-modules of rank 1. Again, the left-hand side of the inclusion in Claim 5.5.6 is infinite, and the proof is completed. $\square$

Two irreducible elements p,q of an integral domain R are said to be *distinct* if $pR \neq qR$. Applying our Theorem 5.5.2, we can obtain the following corollary, which is a basis in the next Section 5.6 to obtain a stronger result (Theorem 5.6.1).

Corollary 5.5.3. Let $(S,\mathfrak{n})$ be a regular local ring of dimension two. Take an element $0 \neq f \in \mathfrak{n}$ and set $R = S/(f)$. Suppose that R is not an isolated singularity (equivaently, is not reduced) but has finite CM_+-representation type. Then f has one of the following forms:

$$f = \begin{cases} p^2qr & \text{with } p,q,r \text{ distinct irreducibles and } S/(pqr) \text{ has finite CM-representation type,} \\ p^2q & \text{with } p,q \text{ distinct irreducibles and } S/(pq) \text{ has finite CM-representation type,} \\ p^2 & \text{with } p \text{ irreducible and } S/(p) \text{ has finite CM-representation type.} \end{cases}$$

Proof. As S is factorial, we can write $f = p_1^{a_1} \cdots p_n^{a_n}$, where $p_1, \dots, p_n$ are distinct irreducible elements and $n, a_1, \dots, a_n$ are positive integers. If $a_1 = \cdots = a_n = 1$, then R is reduced, and hence it is an isolated singularity, which is a contradiction. Thus we may assume $a_1 \geq 2$.

Put $x := p_1 \cdots p_n \in R$. We have

$$(x) + (0 : x) = (p_1 \cdots p_n, p_1^{a_1 - 1} p_2^{a_2 - 1} \cdots p_n^{a_n - 1}) \subseteq (p_1),$$

and hence $\operatorname{ht}((x) + (0 : x)) = 0 < 1$. Taking advantage of Theorem 5.5.2(3a), we observe that $R/(x)$ has finite CM-representation type. Also, $R/(x) = S/(p_1 \cdots p_n)$ has multiplicity at least n. By [LW12, Theorem 4.2 and Proposition 4.3] we see that $n \leq 3$.

Assume either $a_1 \geq 3$ or $a_l \geq 2$ for some $l \geq 2$, say $l = 2$. Then put $x := p_1^2 p_2 \cdots p_n \in R$. We have

$$(x) + (0 : x) = (p_1^2 p_2 \cdots p_n, p_1^{a_1 - 2} p_2^{a_2 - 1} \cdots p_n^{a_n - 1}) \subseteq \begin{cases} (p_1) & (\text{if } a_1 \geq 3), \\ (p_2) & (\text{if } a_2 \geq 2) \end{cases}$$

and hence $\operatorname{ht}((x) + (0 : x)) = 0 < 1$. The ring $R/(x) = S/(p_1^2 p_2 \cdots p_n)$ is not reduced, so it is not an isolated singularity. By [HL02, Corollary 2], it has infinite CM-representation type. Theorem 5.5.2(3a) implies that R has infinite CM_+-representation type, which is a contradiction. Thus $a_1 = 2$ and $a_2 = \cdots = a_n = 1$.

Getting together all the above arguments completes the proof of the corollary. $\square$

To give applications of Theorem 5.5.2, we establish a lemma.

Lemma 5.5.7. Let R be a Gorenstein local ring of finite CM_+-representation type. Then for all $M \in \operatorname{ind} \mathrm{CM}_+(R)$ one has $\operatorname{cx}_R M = 1$.

Proof. As R is Gorenstein, $\Omega^i M \in \operatorname{ind} \mathrm{CM}_+(R)$ for all $i \geq 0$ by Lemma 5.5.2 and [Yos90, Lemma 8.17]. Since $\operatorname{ind} \mathrm{CM}_+(R)$ is a finite set, $\Omega^t M$ is periodic for some $t \geq 0$. Hence M has complexity at most one. As M is in $\mathrm{CM}_+(R)$, it has to have infinite projective dimension. Thus the complexity of M is equal to one. $\square$

Let R be a ring. We denote by $\mathrm{D}_{\mathrm{sg}}(R)$ the *singularity category* of R, that is, the Verdier quotient of the bounded derived category of finitely generated R-modules by perfect complexes. For an R-module M, we denote by $\mathrm{NF}_R(M)$ the *nonfree locus* of M, that is, the set of prime ideals $\mathfrak{p}$ of R

such that $M_\mathfrak{p}$ is nonfree as an $R_\mathfrak{p}$-module. Now we prove the following result by using Theorem 5.5.2.

Theorem 5.5.4. Let R be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an ideal of R with $\mathrm{V}(I) \subseteq \mathrm{V}(0 : I)$, and assume that R/I is a maximal Cohen–Macaulay R-module. Suppose that R has finite CM_+-representation type. Then:

(1) One has $d = 1$.

(2) If $I^n = 0$, then $\dim \mathrm{CM}(R) \leq n-1$.

(3) If R is Gorenstein, then R is a hypersurface and $\dim \mathrm{D_{sg}}(R) \leq n-1$.

Proof. (1) This is a direct consequence of Theorem 5.5.2(2b).

(2) It follows from Theorem 5.5.2(2a) that R/I has finite CM-representation type. Hence there exists a maximal Cohen–Macaulay R/I-module G such that $\mathrm{CM}(R/I) = \mathrm{add}_{R/I}\, G$. Take any maximal Cohen–Macaulay R-module M and put $M_0 := M$. For each integer $0 \leq i \leq n-1$ we have an exact sequence $0 \to (0 :_{M_i} I) \xrightarrow{f_i} M_i \to M_{i+1} \to 0$, where f_i is the inclusion map.

Let us show that for all $0 \leq i \leq n-1$ the R-module M_i is maximal Cohen–Macaulay and annihilated by I^{n-i}. We use induction on i. It clearly holds in the case $i = 0$, so let $i \geq 1$. Applying the functor $\mathrm{Hom}_R(-, M_{i-1})$ to the natural exact sequence $0 \to I \to R \to R/I \to 0$ induces an exact sequence $0 \to (0 :_{M_{i-1}} I) \xrightarrow{f_{i-1}} M_{i-1} \to \mathrm{Hom}_R(I, M_{i-1})$, and hence M_i is identified with a submodule of $\mathrm{Hom}_R(I, M_{i-1})$. The induction hypothesis implies that M_{i-1} is maximal Cohen–Macaulay and $I^{n-i-1}M_{i-1} = 0$. Then $\mathrm{Hom}_R(I, M_{i-1})$ has positive depth (see [BH93, Exercise 1.4.19]), and so does M_i. Since $d = 1$ by (1), the R-module M_i is maximal Cohen–Macaulay. Also, $I^{n-i}M_{i-1}$ is contained in $(0 :_{M_{i-1}} I)$, which implies that I^{n-i} annihilates $M_{i-1}/(0 :_{M_{i-1}} I) = M_i$.

Thus, for each $0 \leq i \leq n-1$ the submodule $(0 :_{M_i} I)$ of M_i is also maximal Cohen–Macaulay (as $d = 1$ again). Since it is killed by I, it is a maximal Cohen–Macaulay R/I-module. Therefore $(0 :_{M_i} I)$ belongs to $\mathrm{add}_R\, G = [G]_1$ for all $0 \leq i \leq n-1$. Using that fact that $M_0 = M$ and $M_n = 0$, we easily observe that M belongs to $[G]_n$. It is concluded that $\mathrm{CM}(R) = [G]_n$, which means that $\dim \mathrm{CM}(R) \leq n-1$.

(3) We claim that the R-module R/I has complexity at most one. Indeed, we have

$$\mathrm{NF}_R(R/I) = \mathrm{V}(I + (0:I)) = \mathrm{V}(I) \cap \mathrm{V}(0:I) = \mathrm{V}(I),$$

where the first equality follows from [Tak10, Proposition 1.15(4)]. As I is not $\mathfrak{m}$-primary, $\mathrm{NF}_R(R/I)$ contains a nonmaximal prime ideal of R. Hence R/I is in $\mathrm{CM}_+(R)$. Since R/I is a local ring, it is an indecomposable R-module, and therefore $R/I \in \operatorname{ind}\mathrm{CM}_+(R)$. It is seen from Lemma 5.5.7 that R/I has complexity at most one as an R-module. Now the claim follows.

Let X be an indecomposable R/I-module which is a direct summand of $C := \Omega^d_{R/I}k$. Proposition 5.5.1(3) implies that X belongs to $\operatorname{ind}\mathrm{CM}_+(R)$. As in the proof of the first claim, $\Omega^i_R X$ belongs to $\operatorname{ind}\mathrm{CM}_+(R)$ for all $i \geq 0$, and $\Omega^n_R X$ is periodic for some $n \geq 0$. Therefore, we find an integer $m \geq 0$ such that $\Omega^m_R C$ is periodic; see Lemma 6.7. This implies that C has complexity at most one. There is an exact sequence

$$0 \to C \to (R/I)^{\oplus r_{m-1}} \to \cdots \to (R/I)^{\oplus r_2} \to (R/I)^{\oplus r_1} \to R/I \to k \to 0.$$

As $\mathrm{cx}_R C \leq 1$ and $\mathrm{cx}_R(R/I) \leq 1$, we get $\mathrm{cx}_R k \leq 1$. By [Avr10, Theorem 8.1.2] the ring R is a hypersurface. The last assertion follows from [Buc87, Theorem 4.4.1] and [DT15, Proposition 3.5(3)]. $\square$

The above theorem gives rise to the two corollaries below. Note that the theorem and the two corollaries all give answers to Questions 5.3.6 and 5.3.8.

Corollary 5.5.5. Let R be a Cohen–Macaulay local ring of dimension $d > 0$ possessing an element $x \in R$ with $(0:x) = (x)$. Suppose that R has finite CM_+-representation type. Then $d = 1$ and $\dim \mathrm{CM}(R) \leq 1$. If R is Gorenstein, then R is a hypersurface and $\dim \mathrm{D_{sg}}(R) \leq 1$.

Proof. We have $x^2 = 0$. The sequence $\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots$ is exact, which implies that $R/(x)$ is a maximal Cohen–Macaulay R-module. The assertions follow from Theorem 5.5.4. $\square$

Corollary 5.5.6. Let R be a Gorenstein non-reduced local ring of dimension one. If R has finite CM_+-representation type, then R is a hypersurface.

Proof. Since R does not have an isolated singularity, $\operatorname{Sing} R$ contains a nonmaximal prime ideal $\mathfrak{p}$. It is easy to see that $(R/\mathfrak{p})_\mathfrak{p} = \kappa(\mathfrak{p})$ is not $R_\mathfrak{p}$-free, and we also have $\mathrm{V}(\mathfrak{p}) = \{\mathfrak{p}, \mathfrak{m}\} \subseteq \operatorname{Supp}_R(\mathfrak{p}) = \mathrm{V}(0 : \mathfrak{p})$ as $\mathfrak{p}R_\mathfrak{p} \neq 0$. Lemma 5.5.7 implies that the R-module $R/\mathfrak{p}$ has complexity at most 1, and the local ring R is a hypersurface by virtue of Theorem 5.5.4(3). $\square$

5.6 The one-dimensional hypersurfaces of finite CM_+-representation type

The purpose of this section is to prove the following theorem.

Theorem 5.6.1. Let R be a homomorphic image of a regular local ring. Suppose that R does not have an isolated singularity but is Gorenstein. If $\dim R = 1$, then the following are equivalent.

(1) The ring R has finite CM_+-representation type.

(2) There exist a regular local ring S and a regular system of parameters x, y such that R is isomorphic to $S/(x^2)$ or $S/(x^2y)$.

When either of these two conditions holds, the ring R has countable CM-representation type.

In fact, the last assertion and the implication (2) $\Rightarrow$ (1) follow from [BGS87, Propositions 4.1 and 4.2] and [AIT12, Proposition 2.1], respectively. The implication (1) $\Rightarrow$ (2) is an immediate consequence of the combination of Corollaries 5.5.3, 5.5.6 with Theorems 5.6.2, 5.6.4, 5.6.5 shown in this section. Note by Theorem 5.3.2 that the above theorem guarantees that under the assumption that R is a complete Gorenstein local ring of dimension one, Question 5.3.8 has an affirmative answer.

We establish three subsections, whose purposes are to prove Theorems 5.6.2, 5.6.4 and 5.6.5, respectively.

5.6.1 The hypersurface $S/(p^2)$

For a ring A we denote by $\mathrm{NZD}(A)$ the set of non-zerodivisors of A, and by $\mathrm{Q}(A)$ the total quotient ring $A_{\mathrm{NZD}(A)}$ of A. A ring extension $A \subseteq B$ is called *birational* if $B \subseteq \mathrm{Q}(A)$.

Lemma 5.6.1. Let $A \subseteq B$ be a birational extension. Let M be a B-module which is torsion-free as an A-module. If M is indecomposable as a B-module, then M is indecomposable as an A-module as well.

Proof. From the proof of [LW12, Proposition 4.14], we have $\mathrm{End}_A(M) = \mathrm{End}_B(M)$. The claim then follows from from [LW12, Proposition 1.1]. $\square$

Let A be a ring and M an A-module. We denote by $\underline{\mathrm{End}}_A(M)$ the quotient of $\mathrm{End}_A(M)$ by the endomorphisms factoring through projective A-modules. For a flat A-algebra B one has $\underline{\mathrm{End}}_A(M) \otimes_A B \cong \underline{\mathrm{End}}_B(M \otimes_A B)$; this can be shown by using [Yos90, Lemma 3.9].

Lemma 5.6.2. Let $A \subseteq B$ be a finite birational extension of 1-dimensional Cohen–Macaulay local rings. Then $\mathsf{ind}\,\mathrm{CM}_+(B)$ is contained in $\mathsf{ind}\,\mathrm{CM}_+(A)$.

Proof. Let $M \in \mathsf{ind}\,\mathrm{CM}_+(B)$. Then $\mathrm{depth}_A M = \mathrm{depth}_B M > 0$, which shows that M is maximal Cohen–Macaulay as an A-module. Lemma 5.6.1 implies $M \in \mathsf{ind}\,\mathrm{CM}(A)$. Set $Q = \mathrm{Q}(A) = \mathrm{Q}(B)$. Applying the functor $Q \otimes_A -$ to the inclusions $A \subseteq B \subseteq Q$ yields $B \otimes_A Q = Q$. Hence we have

$$M \otimes_B Q = M \otimes_B (B \otimes_A Q) = M \otimes_A Q, \quad \underline{\mathrm{End}}_A(M) \otimes_A Q \cong \underline{\mathrm{End}}_Q(M \otimes_A Q) \cong \underline{\mathrm{End}}_Q(M \otimes_B Q).$$

Since M is in $\mathrm{CM}_+(B)$, there is a minimal prime P of B such that M_P is not B_P-free. Note that $M_P = (M \otimes_B Q) \otimes_Q Q_P$ and $Q_P = B_P$. Hence $M \otimes_B Q$ is not Q-projective, and we obtain $\underline{\mathrm{End}}_Q(M \otimes_B Q) \neq 0$. Therefore $\underline{\mathrm{End}}_A(M) \otimes_A Q$ is nonzero, which means that the A-module $\underline{\mathrm{End}}_A(M)$ is not torsion. Thus $\mathrm{Supp}_A(\underline{\mathrm{End}}_A(M))$ contains a minimal prime of A, which implies that M belongs to $\mathrm{CM}_+(A)$. Consequently, we obtain $M \in \mathsf{ind}\,\mathrm{CM}_+(A)$, and the lemma follows. $\square$

The following lemma is a consequence of [Yos90, Corollary 7.6], which is used not only now but also later.

Lemma 5.6.3. Let $(S, \mathfrak{n})$ be a regular local ring and $x \in \mathfrak{n}$, and set $R = S/(x)$. Then

$$\{M \in \mathrm{CM}(R) \mid M \text{ is cyclic}\}/_{\cong} = \{R/yR \mid y \in S \text{ with } x \in yS\}/_{\cong}.$$

In particular, there exist only finitely many nonisomorphic indecomposable cyclic maximal Cohen–Macaulay R-modules.

Now we can achieve the purpose of this subsection.

Theorem 5.6.2. Let $(S, \mathfrak{n})$ be a regular local ring of dimension two, and let $p \in \mathfrak{n}^2$ be an irreducible element. Then $R = S/(p^2)$ has infinite CM_+-representation type.

Proof. Take any element $t \in \mathfrak{n}$ that is regular on R. We consider the S-algebra $T = S[z]/(tz - p, z^2)$, where z is an indeterminate over S. We establish two claims.

Claim 5.6.4. The ring T is a local complete intersection of dimension 1 and codimension 2 with t being a system of parameters.

Proof of Claim. It is clear that $T = S[[z]]/(tz - p, z^2)$, which shows that T is a local ring, and $\dim T = \dim S[[z]] - \mathrm{ht}(tz - p, z^2) \geq 3 - 2 = 1$ by Krull's Hauptidealsatz. We note that $T/tT = S[[z]]/(t, p, z^2) = (S/(t, p))[[z]]/(z^2)$. As $S/(t, p)$ is artinian, so is T/tT. Hence $\dim T = 1$ and t is a system of parameters of T, and thus T is a complete intersection (the equalities $\dim S[[z]] = 3$ and $\dim T = 1$ imply $\mathrm{ht}(tz - p, z^2) = 2$, whence $tz - p, z^2$ is a regular sequence). As $(tz - p, z^2) \subseteq \mathfrak{n}^2$, the local ring T has codimension 2. $\square$

Claim 5.6.5. The ring R is naturally embedded in T, and this embedding is a finite birational extension.

Proof of Claim. Let $\phi : S \to T$ be the natural map and put $I = \operatorname{Ker} \phi$. As $p^2 = t^2 z^2 = 0$ in T, we have $(p^2) \subseteq I$. Hence the map ϕ factors as $S \twoheadrightarrow R \twoheadrightarrow S/I \hookrightarrow T$. It is seen that T is an R-module generated by $1, z$ and S/I is an R-submodule of T. Since T has positive depth by Claim 5.6.4, so does S/I. Thus S/I is a maximal Cohen–Macaulay cyclic module over the hypersurface R, and Lemma 5.6.3

implies that I coincides with either (p) or (p^2). If $I=(p)$, then $T=T/pT=S[z]/(tz,p,z^2)$, which contradicts the fact following from Claim 5.6.4 that t is T-regular. We get $I=(p^2)$, which means the map $R\to T$ is injective.

Let C be the cokernel of the injection $R\hookrightarrow T$. Then C is generated by z as an R-module. Note that $tz=p=0$ in C. Hence C is a torsion R-module, which means $C\otimes_R \mathrm{Q}(R)=0$. Thus $(\mathrm{Q}(R)\to T\otimes_R \mathrm{Q}(R))=(R\hookrightarrow T)\otimes_R \mathrm{Q}(R)$ is an isomorphism, while the natural map $T\to T\otimes_R \mathrm{Q}(R)$ is injective as T is maximal Cohen–Macaulay over R by Claim 5.6.4. Thus the embedding $R\hookrightarrow T$ is birational. $\square$

By Claim 5.6.4, the ring T is a complete intersection, which implies that the element z^2 is regular on the ring $S[z]/(tz-p)$ and so is z. It is easy to check that $(0:_T z)=zT$. Claim 5.6.4 also guarantees that T is not a hypersurface. It follows from Corollary 5.5.5 that T has infinite CM_+-representation type. Combining Claim 5.6.5 with Lemma 5.6.2, we obtain the inclusion $\operatorname{ind}\mathrm{CM}_+(T)\subseteq \operatorname{ind}\mathrm{CM}_+(R)$. We now conclude that R has infinite CM_+-representation type, and the proof of the theorem is completed. $\square$

5.6.2 The hypersurface $S/(p^2qr)$

Setup 5.6.3. Throughout this subsection, let $(S,\mathfrak{n})$ be a 2-dimensional regular local ring and p,q,r pairwise distinct irreducible elements of S. Let $R=S/(p^2qr)$ be a local hypersurface of dimension 1. Setting $\mathfrak{p}=pR$, $\mathfrak{q}=qR$, $\mathfrak{r}=rR$ and $\mathfrak{m}=\mathfrak{n}R$, one has $\operatorname{Spec}R=\{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{m}\}$. For each $i\geq 1$ we define matrices

$$A_i=\begin{pmatrix} p & 0 & r^i\\ 0 & pq & p\\ 0 & 0 & pr\end{pmatrix},\qquad B_i=\begin{pmatrix} pqr & 0 & -qr^i\\ 0 & pr & -p\\ 0 & 0 & pq\end{pmatrix}$$

over S. Put $M_i=\operatorname{Cok}_S A_i$ and $N_i=\operatorname{Cok}_S B_i$.

Lemma 5.6.6. (1) For every $i\geq 1$ it holds that $M_i,N_i\in\mathrm{CM}_+(R)$, $\Omega_R M_i=N_i$ and $\Omega_R N_i=M_i$.

(2) For all positive integers $i\neq j$, one has $M_i\not\cong M_j$ and $N_i\not\cong N_j$ as R-modules.

Proof. (1) It is clear that $A_iB_i = B_iA_i = p^2qrE$. Hence A_i, B_i give a matrix factorization of p^2qr over S, and we have $M_i, N_i \in \mathrm{CM}(R)$, $\Omega_R M_i = N_i$ and $\Omega_R N_i = M_i$; see [Yos90, Chapter 7]. Note that q, r are units and $p^2 = 0$ in $R_{\mathfrak{p}} = S_{(p)}/p^2S_{(p)}$. There are isomorphisms

$$(M_i)_{\mathfrak{p}} \cong \mathrm{Cok}\begin{pmatrix} p & 0 & r^i \\ 0 & p & p \\ 0 & 0 & p \end{pmatrix} \cong \mathrm{Cok}\begin{pmatrix} p & 0 & 1 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \cong \mathrm{Cok}\begin{pmatrix} 0 & 0 & 1 \\ 0 & p & 0 \\ -p^2 & 0 & p \end{pmatrix} \cong \mathrm{Cok}\begin{pmatrix} 0 & 0 & 1 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \mathrm{Cok}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \cong R_{\mathfrak{p}} \oplus \kappa(\mathfrak{p}),$$

where all the cokernels are over $R_{\mathfrak{p}}$. Therefore $M_i \in \mathrm{CM}_+(R)$, and we get $N_i \in \mathrm{CM}_+(R)$ by Lemma 5.5.2.

(2) Suppose that there is an R-isomorphism $M_i \cong M_j$. It then holds that $\mathrm{Fitt}_2(M_i) = \mathrm{Fitt}_2(M_j)$, which means $(p, r^i)R = (p, r^j)R$. This implies that $(\overline{r}^i) = (\overline{r}^j)$ in the integral domain $R/\mathfrak{p} = S/(p)$. Since $\overline{r} \neq \overline{0}$ in this ring, we get $i = j$. If $N_i \cong N_j$, then $M_i \cong \Omega_R N_i \cong \Omega_R N_j \cong M_j$ by (1), and we get $i = j$. $\square$

Lemma 5.6.7. There is an equality

$$\{M \in \mathrm{CM}_+(R) \mid M \text{ is cyclic}\}/_{\cong} = \{R/(p), R/(pq), R/(pr), R/(pqr)\}/_{\cong}.$$

Proof. Let M be a cyclic R-module with $M \in \mathrm{CM}_+(R)$. It follows from Lemma 5.6.3 that M is isomorphic to R/fR for some element $f \in S$ which divides p^2qr in S. The localizations $R_{\mathfrak{q}}, R_{\mathfrak{r}}$ are fields, and hence $M_{\mathfrak{p}}$ is not $R_{\mathfrak{p}}$-free. As $p^2 = 0$ in $R_{\mathfrak{p}} = S_{(p)}/p^2S_{(p)}$, it is observed that $f \in pS \setminus p^2S$. Thus $f \in \{p, pq, pr, pqr\}$. Conversely, for any $g \in \{p, pq, pr, pqr\}$ we have $(R/gR)_{\mathfrak{p}} \cong \kappa(\mathfrak{p})$ and get $R/gR \in \mathrm{CM}_+(R)$. $\square$

Lemma 5.6.8. Let $i \geq 1$ be an integer. Then neither $\mathrm{Cok}_{S/(pq)}\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}$ nor $\mathrm{Cok}_{S/(pr)}\begin{pmatrix} p \\ qr^i \end{pmatrix}$ contains $S/(p)$ as a direct summand.

Proof. (1) Set $T = S/(pq)$ and $C = \mathrm{Cok}_T\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}$. Consider the sequence

$$T^{\oplus 2} \xleftarrow{\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}} T^{\oplus 2} \xleftarrow{\begin{pmatrix} q \\ 0 \end{pmatrix}} T$$

of homomorphisms of free T-modules. Clearly, this is a complex. Let $\binom{a}{b} \in T^{\oplus 2}$ be such that $\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}\binom{a}{b} = \binom{0}{0}$. In S we have $pa + r^i b = pqc$ and $pb = pqd$ for some $c, d \in S$, and get $b = qd$ and $pa + r^i qd = pqc$. Hence $pa \in qS \in \operatorname{Spec} S$ and $a \in qS$; we find $e \in S$ with $a = qe$. Then $pqe + r^i qd = pqc$, and $pe + r^i d = pc$. Therefore $r^i d \in pS \in \operatorname{Spec} S$, and $d \in pS$; we find $f \in S$ with $d = pf$ and get $b = qpf$. In $T^{\oplus 2}$ we have $\binom{a}{b} = \binom{qe}{pqf} = \binom{qe}{0} = \binom{q}{0}(e)$. It follows that the above sequence is exact, and the sequence

$$\cdots \xrightarrow{p} T \xrightarrow{q} T \xrightarrow{p} T \xrightarrow{\binom{q}{0}} T^{\oplus 2} \xrightarrow{\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}} T^{\oplus 2} \to C \to 0$$

gives a minimal free resolution of the T-module C.

Now, assume that $S/(p) = T/pT$ is a direct summand of C. Then $C \cong T/pT \oplus T/I$ for some ideal I of T. There are equalities of Betti numbers

$$2 = \beta_1^T(C) = \beta_1^T(T/pT \oplus T/I) = \beta_1^T(T/pT) + \beta_1^T(T/I) = 1 + \beta_1^T(T/I),$$

and we get $\beta_1^T(T/I) = 1$. This means I is a nonzero proper principal ideal of T; we write $I = gT$ where g is a nonzero nonunit of T. The uniqueness of a minimal free resolution yields a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "p"] & T \arrow[r, "q"] \arrow[d, "u_3", "\cong"'] & T \arrow[r, "p"] \arrow[d, "u_2", "\cong"'] & T \arrow[r, "\binom{q}{0}"] \arrow[d, "u_1", "\cong"'] & T^{\oplus 2} \arrow[r, "\begin{pmatrix} p & 0 \\ 0 & g \end{pmatrix}"] \arrow[d, "\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} =: t", "\cong"'] & T^{\oplus 2} \arrow[r] \arrow[d, "\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} =: s", "\cong"'] & C \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r, "p"'] & T \arrow[r, "q"'] & T \arrow[r, "p"'] & T \arrow[r, "\binom{q}{0}"'] & T^{\oplus 2} \arrow[r, "\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}"'] & T^{\oplus 2} \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

whose vertical maps are isomorphisms. As s, t are isomorphisms, their determinants $s_1 s_4 - s_2 s_3$ and $t_1 t_4 - t_2 t_3$ are units of T. The commutativity of the diagram shows $s_3 p = p t_3$ and $t_3 q = 0$ in T, which imply $s_3 - t_3 \in (0 :_T p) = qT$ and $t_3 \in (0 :_T q) = pT$. Hence s_3 is a nonunit of T, and therefore s_1, s_4 are units of T. Again from the commutativity of the diagram we get $s_4 g = p t_4$ and

$s_2 g = pt_2 + r^i t_4$ in T, which give $p(s_2 s_4^{-1} t_4 - t_2) = r^i t_4$. Hence $r^i t_4 \in pT \in \operatorname{Spec} T$ and $t_4 \in pT$. We now get $t_1 t_4 - t_2 t_3$ is in pT, which contradicts the fact that it is a unit of T. Consequently, $S/(p)$ is not a direct summand of C.

(2) Put $T = S/(pr)$ and $C = \operatorname{Cok}_T \binom{p}{qr^i}$. We have $\operatorname{Spec} T = \{pT, rT, \mathfrak{n}T\}$. Since $(p, qr^i)T$ is not contained in pT or rT, it is $\mathfrak{n}T$-primary and has positive grade. Hence the sequence

$$0 \to T \xrightarrow{\binom{p}{qr^i}} T^{\oplus 2} \to C \to 0$$

is exact, which gives a minimal free resolution of the T-module C. This implies $\operatorname{pd}_R C = 1$.

Suppose that $S/(p) = T/pT$ is a direct summand of C. Then T/pT has projective dimension at most one, which contradicts the fact that its minimal free resolution is $\cdots \xrightarrow{p} T \xrightarrow{q} T \xrightarrow{p} T \to T/pT \to 0$. It follows that $S/(p)$ is not a direct summand of C. $\square$

Lemma 5.6.9. (1) The ring $S/(p,q)$ is artinian, and hence the number $\ell\ell(S/(p,q))$ is finite.

(2) Let $n \geq \ell\ell(S/(p,q))$ be a positive integer.

- (i) If $X \in \mathrm{CM}_+(R)$ is a cyclic direct summand of M_n, then X is isomorphic to $R/(pqr)$.
- (ii) If $Y \in \mathrm{CM}_+(R)$ is a cyclic direct summand of N_n, then Y is isomorphic to $R/(pqr)$.

Proof. (1) The factoriality of S shows that pS is a prime ideal of S. As $pS \neq qS$, we have $\operatorname{ht}(p,q)S > \operatorname{ht} pS = 1$. Since S has dimension two, the ideal $(p,q)S$ is $\mathfrak{n}$-primary. Thus $S/(p,q)S$ is an artinian ring.

(2i) There is an R-module Z such that $M_n \cong X \oplus Z$. According to Lemma 5.6.7, it holds that $X \cong R/(f)$ for some $f \in \{p, pq, pr, pqr\}$. There are isomorphisms

$$R/(f,r) \oplus Z/rZ \cong M_n/rM_n \cong \operatorname{Cok}_{R/(r)} \begin{pmatrix} p & 0 & 0 \\ 0 & pq & p \\ 0 & 0 & 0 \end{pmatrix} \cong \operatorname{Cok}_{R/(r)} \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \cong (R/(p,r))^{\oplus 2} \oplus R/(r).$$

Taking the completions and using the Krull–Schmidt property and [Eis95, Exercise 7.5], we observe that the ideal $(f,r)R$ coincides with either $(p,r)R$ or rR. Hence $f \neq pq$. Similarly, there are isomorphisms

$$R/(f,q)\oplus Z/qZ \cong M_n/qM_n \cong \operatorname{Cok}_{R/(q)}\left(\begin{smallmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & pr \end{smallmatrix}\right) \cong \operatorname{Cok}_{R/(q)}\left(\begin{smallmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{smallmatrix}\right) \cong R/(q)\oplus \operatorname{Cok}_{R/(q)}\left(\begin{smallmatrix} p & r^n \\ 0 & p \end{smallmatrix}\right).$$

The assumption $n \geq \ell\ell(S/(p,q))$ implies $r^n \in \mathfrak{n}^n \subseteq (p,q)S$. We observe from this that $\operatorname{Cok}_{R/(q)}\left(\begin{smallmatrix} p & r^n \\ 0 & p \end{smallmatrix}\right)$ is isomorphic to $\operatorname{Cok}_{R/(q)}\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, and obtain an isomorphism $R/(f,q)\oplus Z/qZ \cong R/(q)\oplus (R/(p,q))^{\oplus 2}$. It follows that $(f,q)R$ coincides with either qR or $(p,q)R$, which implies $f \neq pr$. Finally, consider the isomorphisms

$$R/(f,pq)\oplus Z/pqZ \cong M_n/pqM_n \cong \operatorname{Cok}_{R/(pq)}\left(\begin{smallmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & pr \end{smallmatrix}\right)$$

$$\cong \operatorname{Cok}_{R/(pq)}\left(\begin{smallmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{smallmatrix}\right) \cong R/(pq)\oplus \operatorname{Cok}_{R/(pq)}\left(\begin{smallmatrix} p & r^n \\ 0 & p \end{smallmatrix}\right).$$

If $f = p$, then $R/(f,pq) = R/(p)$ and we see that this is a direct summand of $\operatorname{Cok}_{R/(pq)}\left(\begin{smallmatrix} p & r^n \\ 0 & p \end{smallmatrix}\right)$, which contradicts Lemma 5.6.8. Thus $f \neq p$, and we conclude that $f = pqr$.

(2ii) We go along the same lines as the proof of (2i). We have $N_n \cong Y \oplus Z$ for some $Z \in \operatorname{mod} R$, and get $Y \cong R/(f)$ for some $f \in \{p, pq, pr, pqr\}$ by Lemma 5.6.7. The isomorphisms

$$R/(f,r)\oplus Z/rZ \cong N_n/rN_n \cong \operatorname{Cok}_{R/(r)}\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & pq \end{smallmatrix}\right) \cong \operatorname{Cok}_{R/(r)}\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{smallmatrix}\right) \cong R/(p,r)\oplus (R/(r))^{\oplus 2},$$

$$R/(f,q)\oplus Z/qZ \cong N_n/qN_n \cong \operatorname{Cok}_{R/(q)}\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & pr & p \\ 0 & 0 & 0 \end{smallmatrix}\right) \cong \operatorname{Cok}_{R/(q)}\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{smallmatrix}\right) \cong R/(p,q)\oplus (R/(q))^{\oplus 2}$$

show that (f,q) (resp. (f,r)) coincides with either (p,q) or (q) (resp. either (p,r) or (r)), which implies $f \neq pq, pr$. We also have isomorphisms

$$R/(f,pr)\oplus Z/prZ \cong N_n/prN_n \cong \operatorname{Cok}_{R/(pr)}\left(\begin{smallmatrix} 0 & 0 & qr^n \\ 0 & 0 & p \\ 0 & 0 & pq \end{smallmatrix}\right)$$

$$\cong \operatorname{Cok}_{R/(pr)}\left(\begin{smallmatrix} 0 & 0 & qr^n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{smallmatrix}\right) \cong \operatorname{Cok}_{R/(pr)}\left(\begin{smallmatrix} p \\ qr^n \end{smallmatrix}\right) \oplus R/(pr).$$

Using Lemma 5.6.8, we see that $f \neq p$, and obtain $f = pqr$. $\square$

The purpose of this subsection is now fulfilled.

Theorem 5.6.4. Let S be a regular local ring of dimension two. Let p,q,r be distinct irreducible elements of S. Then $R = S/(p^2qr)$ has infinite CM_+-representation type.

Proof. We assume that R has finite CM_+-representation type, and derive a contradiction. It follows from Lemma 5.6.6(1) that there exists an integer $a \geq 1$ such that both M_i and N_i are decomposable for all $i \geq a$; we write $M_i \cong X_i \oplus Y_i$ for some R-modules X_i, Y_i with $\nu(X_i) = 1$ and $\nu(Y_i) = 2$. In view of Lemmas 5.6.3 and 5.6.6(2), we see that there exists an integer $b \geq a$ such that Y_h is indecomposable for all $h \geq b$ and that $Y_i \ncong Y_j$ for all $i, j \geq b$ with $i \neq j$. Then, we have to have $Y_i \in \mathrm{CM}_0(R)$ for all $i \geq b$, and hence $X_i \in \mathrm{CM}_+(R)$ for all $i \geq b$ (by Lemma 5.6.6(1)). Putting $c = \max\{b, \ell\ell(S/(p,q))\}$ and applying Lemma 5.6.9(2i), we obtain that X_i is isomorphic to $R/(pqr)$ for all $i \geq c$. There are isomorphisms

$$N_i \cong \Omega_R M_i \cong \Omega_R X_i \oplus \Omega_R Y_i \cong \Omega_R(R/(pqr)) \oplus \Omega_R Y_i \cong R/(p) \oplus \Omega_R Y_i,$$

where the first isomorphism follows from Lemma 5.6.6(1). Since $R/(p)$ is in $\mathrm{CM}_+(R)$, it follows from Lemma 5.6.9(2ii) that $R/(p) \cong R/(pqr)$, which is absurd. $\square$

5.6.3 The hypersurface $S/(p^2q)$

The goal of this subsection is to prove the following theorem.

Theorem 5.6.5. Let $(S, \mathfrak{n})$ be a 2-dimensional regular local ring. Let p,q be distinct irreducible elements of S. Suppose that $R = S/(p^2q)$ has finite CM_+-representation type. Then $p, q \notin \mathfrak{n}^2$.

Note that the rings R and R/p^2R are local hypersurfaces of dimension one. If $p \in \mathfrak{n}^2$, then $R/p^2R = S/(p^2)$ has infinite CM_+-representation type by Theorem 5.6.2, and so does R by Theorem 5.5.2(1), which contradicts the assumption of the theorem. Hence $p \notin \mathfrak{n}^2$, and p is a member of a regular system of parameters of S. Thus we establish the following setting.

Setup 5.6.6. Throughout the remainder of this subsection, let $(S,\mathfrak{n})$ be a regular local ring of dimension two. Let x,y be a regular system of parameters of S, namely, $\mathfrak{n}=(x,y)$. Let $h\in\mathfrak{n}^2$ be an irreducible element, and write $h=x^2s+xyt+y^2u$ with $s,t,u\in S$. Let $R=S/(x^2h)$ be a local hypersurface of dimension one. One has $\operatorname{Spec}R=\{\mathfrak{p},\mathfrak{q},\mathfrak{m}\}$, where we set $\mathfrak{p}=xR$, $\mathfrak{q}=hR$ and $\mathfrak{m}=\mathfrak{n}R$. For each integer $i\ge 1$ we define matrices

$$A_i=\begin{pmatrix} x & 0 & y^i \\ 0 & xy & x \\ 0 & xh & 0 \end{pmatrix}, \qquad B_i=\begin{pmatrix} xh & -y^ih & y^{i+1} \\ 0 & 0 & x \\ 0 & xh & -xy \end{pmatrix}$$

over S. We put $M_i=\operatorname{Cok}_S A_i$ and $N_i=\operatorname{Cok}_S B_i$.

In what follows, we argue along similar lines as in the previous subsection.

Lemma 5.6.10 (cf. Lemma 5.6.6)**.** (1) Let $i\ge 1$ be an integer. The modules M_i and N_i belong to $\mathrm{CM}_+(R)$, and it holds that $\Omega_R M_i=N_i$ and $\Omega_R N_i=M_i$.

(2) Let $i,j\ge 1$ be integers with $i\ne j$. One then have $M_i\ncong M_j$ and $N_i\ncong N_j$ as R-modules.

Proof. (1) We have $A_iB_i=B_iA_i=x^2hE$. The matrices A_i,B_i give a matrix factorization of x^2h over S. We have that M_i,N_i are maximal Cohen–Macaulay R-modules with $\Omega_R M_i=N_i$ and $\Omega_R N_i=M_i$. Note that y,h are units and $x^2=0$ in $R_\mathfrak{p}=S_{(x)}/x^2S_{(x)}$. We have

$$\begin{aligned}(M_i)_\mathfrak{p} &\cong \operatorname{Cok}_{R_\mathfrak{p}}\begin{pmatrix} x & 0 & y^i \\ 0 & xy & x \\ 0 & x & 0 \end{pmatrix} \cong \operatorname{Cok}_{R_\mathfrak{p}}\begin{pmatrix} x & 0 & y^i \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \cong \operatorname{Cok}_{R_\mathfrak{p}}\begin{pmatrix} x & 0 & 1 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \cong \operatorname{Cok}_{R_\mathfrak{p}}\begin{pmatrix} 0 & 0 & 1 \\ -x^2 & 0 & x \\ 0 & x & 0 \end{pmatrix}\\ &= \operatorname{Cok}_{R_\mathfrak{p}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \cong \operatorname{Cok}_{R_\mathfrak{p}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix} \cong \operatorname{Cok}_{R_\mathfrak{p}}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \cong R_\mathfrak{p}\oplus\kappa(\mathfrak{p}),\end{aligned}$$

which shows that $M_i\in\mathrm{CM}_+(R)$, and Lemma 5.5.2 implies $N_i\in\mathrm{CM}_+(R)$ as well.

(2) If $M_i\cong M_j$, then $(x,y^i)R=\operatorname{Fitt}_2(M_i)=\operatorname{Fitt}_2(M_j)=(x,y^j)R$, and $(\bar{y}^i)=(\bar{y}^j)$ in the discrete valuation ring $R/xR=S/(x)$ with $\bar{y}$ a uniformizer, which implies $i=j$. As N_i,N_j are the first syzygies of M_i,M_j by (1), we see that if $N_i\cong N_j$, then $i=j$. $\square$

Lemma 5.6.11 (cf. Lemma 5.6.7)**.** It holds that $\{M\in\mathrm{CM}_+(R)\mid M \text{ is cyclic}\}/_{\cong}=\{R/(x),R/(xh)\}/_{\cong}$.

Proof. It is easy to see that neither $(R/(x))_{\mathfrak{p}}$ nor $(R/(xh))_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free. Let $M \in \mathrm{CM}_{+}(R)$ be cyclic. As $R_{\mathfrak{q}}$ is a field, $M_{\mathfrak{p}}$ is not $R_{\mathfrak{p}}$-free. Using Lemma 5.6.3, we get $M \cong R/fR$ for some $f \in S$ with $f \mid x^2h$, $x \mid f$ and $x^2 \nmid f$. Hence, either $f = x$ or $f = xh$ holds. $\square$

Lemma 5.6.12 (cf. Lemma 5.6.9)**.** Let $i \geq 1$ be an integer. Let C be a cyclic R-module with $C \in \mathrm{CM}_{+}(R)$. If C is a direct summand of either M_i or N_i, then C is isomorphic to $R/(xh)$.

Proof. (1) First, consider the case where C is a direct summand of M_i. Assume that C is not isomorphic to $R/(xh)$. Then $C \cong R/(x)$ by Lemma 5.6.11. Application of the functor $- \otimes_R R/(xy)$ shows that $C/xyC = C \cong R/(x)$ is a direct summand of

$$M_i/xyM_i = \mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & 0 & y^i \\ 0 & 0 & x \\ 0 & xh & 0 \end{pmatrix} \cong \mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & y^i & 0 \\ 0 & x & 0 \\ 0 & 0 & xh \end{pmatrix} \cong \mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix} \oplus R/(xy, xh).$$

As $(x) \neq (xy, xh)$, we have $R/(x) \ncong R/(xy, xh)$ and hence $R/(x)$ is a direct summand of $\mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix}$. Note that $R/(xy) = S/(x^2h, xy) = S/(x^2(x^2s + xyt + y^2u), xy) = S/(x^4s, xy)$. Put $T := R/(xy, x^4) = S/(x^4, xy)$. Applying the functor $- \otimes_R R/(x^4)$, we see that $T/(x) = R/(x)$ is a direct summand of $L := \mathrm{Cok}_T \begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix}$. Write $L = T/(x) \oplus D$ with $D \in \mathrm{mod}\, T$. It is easy to verify that the sequence

$$0 \leftarrow L \leftarrow T^{\oplus 2} \xleftarrow{\begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix}} T^{\oplus 2} \xleftarrow{\begin{pmatrix} y & x^3 & 0 \\ 0 & 0 & x^3 \end{pmatrix}} T^{\oplus 3}$$

is exact, and we observe $D \cong T/(v)$ for some $v \in T$. Uniqueness of a minimal free resolution gives rise to a commutative diagram

$$\begin{tikzcd}
0 & L \arrow[l] \arrow[d, "\cong"'] & T^{\oplus 2} \arrow[l] \arrow[d, "\cong"', "\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}"] & T^{\oplus 2} \arrow[l, "\begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix}"'] \arrow[d, "\cong"', "\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}"] & T^{\oplus 3} \arrow[l, "\begin{pmatrix} y & x^3 & 0 \\ 0 & 0 & x^3 \end{pmatrix}"'] \\
0 & T/(x) \oplus T/(v) \arrow[l] & T^{\oplus 2} \arrow[l] & T^{\oplus 2} \arrow[l, "\begin{pmatrix} x & 0 \\ 0 & v \end{pmatrix}"] &
\end{tikzcd}$$

with vertical maps being isomorphisms. The elements $a_1a_4 - a_2a_3$ and $b_1b_4 - b_2b_3$ are units of T. We have $a_1y^i + a_2x = xb_2$ and $a_1x = xb_1$ in T. Hence $a_1y^i \in (x) \in \mathrm{Spec}\, T$, which implies $a_1 \in (x)$.

Also, $a_1 - b_1 \in (0:x) = (x^3, y)$, which implies $b_1 \in (x,y)$. It follows that a_2, a_3, b_2, b_3 are units of T. The equality $a_3 x = v b_3$ implies that $(x) = (v)$ in T. We obtain isomorphisms

$$T/(x^3,y) \oplus T/(x^3) \cong \operatorname{Cok}_T \left(\begin{smallmatrix} y & x^3 & 0 \\ 0 & 0 & x^3 \end{smallmatrix}\right) \cong \Omega_T L \cong (x) \oplus (v) \cong (x)^{\oplus 2} \cong (T/(x^3,y))^{\oplus 2}.$$

It follows that $(x^3) = (x^3, y)$ in T, which is a contradiction. Consequently, C is isomorphic to $R/(xh)$.

(2) Next we consider the case where C is a direct summand of N_i. The proof is analogous to that of (1). Again, assume $C \not\cong R/(xh)$. Then $C \cong R/(x)$ by Lemma 5.6.11. Set $T := R/(xh) = S/(xh)$. Applying $- \otimes_R T$, we see that $R/(x) = T/(x)$ is a direct summand of

$$N_i/xhN_i = \operatorname{Cok}_T \left(\begin{smallmatrix} 0 & -y^i h & y^{i+1} \\ 0 & 0 & x \\ 0 & 0 & -xy \end{smallmatrix}\right) \cong \operatorname{Cok}_T \left(\begin{smallmatrix} 0 & -y^i h & y^{i+1} \\ 0 & 0 & x \\ 0 & 0 & 0 \end{smallmatrix}\right) \cong T \oplus \operatorname{Cok}_T \left(\begin{smallmatrix} y^i h & y^{i+1} \\ 0 & x \end{smallmatrix}\right),$$

which implies that $T/(x)$ is a direct summand of $L := \operatorname{Cok}_T \left(\begin{smallmatrix} y^i h & y^{i+1} \\ 0 & x \end{smallmatrix}\right)$. There are an isomorphism $L \cong T/(x) \oplus T/(v)$ with $v \in T$ and a commutative diagram:

$$\begin{tikzcd}
0 & L \arrow[l] \arrow[d, "\cong"'] & T^{\oplus 2} \arrow[l] \arrow[d, "\cong"', "\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)"] & T^{\oplus 2} \arrow[l, "\left(\begin{smallmatrix} y^i h & y^{i+1} \\ 0 & x \end{smallmatrix}\right)"'] \arrow[d, "\cong"', "\left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)"] \\
0 & T/(x) \oplus T/(v) \arrow[l] & T^{\oplus 2} \arrow[l] & T^{\oplus 2} \arrow[l, "\left(\begin{smallmatrix} x & 0 \\ 0 & v \end{smallmatrix}\right)"]
\end{tikzcd}$$

Note that $\operatorname{Spec} T = \{(x), (h), \mathfrak{n}T\}$. We have $(h) \ni a_1 y^i h = x b_1 \in (x)$, which implies $a_1 \in (x)$ and $b_1 \in (h)$. As $a_1 a_4 - a_2 a_3$ and $b_1 b_4 - b_2 b_3$ are units, so are a_2, a_3, b_2, b_3. The equalities $a_3 y^i h = v b_3$ and $a_3 y^{i+1} + a_4 x = v b_4$ imply $a_3 y^i (b_3^{-1} h b_4 - y) = a_4 x \in (x)$, which gives $b_3^{-1} h b_4 - y \in (x)$. Hence $y \in (x,h) = (x, x^2 s + xyt + y^2 u) = (x, y^2 u)$ in T, which is a contradiction. Thus $C \cong R/(xh)$. □

Lemma 5.6.13 (cf. Theorem 5.6.4)**.** The ring R has infinite CM_+-representation type.

Proof. Assume contrarily that R has finite CM_+-representation type. Then, by (1) and (2) of Lemma 5.6.10, there exists an integer $a \geq 1$ such that M_i is decomposable for all integers $i \geq a$.

Suppose that for some $i \geq 1$ the module M_i has a cyclic direct summand $C \in \mathrm{CM}_+(R)$. Then C is isomorphic to $R/(xh)$ by Lemma 5.6.12, and $\Omega_R C = R/(x)$ is a direct summand of $\Omega_R M_i = N_i$ by Lemma 5.6.10(1). Applying Lemma 5.6.12 again, we have to have $R/(x) \cong R/(xh)$, which is a contradiction.

Thus M_i has no cyclic direct summand belonging to $\mathrm{CM}_+(R)$ for all $i \geq 1$. This means that for every $i \geq a$ the R-module M_i has an indecomposable direct summand $Y_i \in \mathrm{CM}_+(R)$ with $\nu(Y_i) = 2$. This, in turn, contradicts the assumption that R has finite CM_+-representation type. □

Now the purpose of this subsection is readily accomplished:

Proof of Theorem 5.6.5. The theorem is an immediate consequence of Lemma 5.6.13 and what we state just after the theorem. □

5.7 On the higher-dimensional case

In this section, we explore the higher-dimensional case: we consider Cohen–Macaulay local rings R with $\dim R \geq 2$ and having finite CM_+-representation type. In particular, we give various results supporting Conjecture 5.1.1. We begin with presenting an example by using a result obtained in Section 4.

Example 5.7.1. Let S be a regular local ring with a regular system of parameters x, y, z. Then $R = S/(xyz)$ has infinite CM_+-representation type.

Proof. Let $I = (xy)$ be an ideal of R. Then $(0 : I) = (z)$ in R, and $\mathrm{ht}(I + (0 : I)) = \mathrm{ht}(xy, z) = 1 < 2 = \dim R$. The ring $R/I = S/(xy)$ is a 2-dimensional hypersurface which does not have an isolated singularity. We see by [HL02, Corollary 2] that R/I has infinite CM-representation type. It follows from Theorem 5.5.2(3a) that R has infinite CM_+-representation type. □

Remark 5.7.2. We remark that the indecomposables in CM_0(R) for $R = k[[x, y, z]]/(xyz)$ have been classified by Burban and Drozd (see [BD17, Theorem 8.6]).

We consider constructing from a given hypersurface of infinite CM_+-representation type another hypersurface of infinite CM_+-representation type. For this we establish the following lemma, which provides a version of Knörrer's periodicity theorem for $\mathrm{CM}_+(R)$.

Lemma 5.7.1. Let $(S,\mathfrak{n})$ be a regular local ring, and let $f,g \in S$. Let $R = S/(f)$ and $R^\sharp = S[[x]]/(f + x^2g)$ be hypersurfaces with x an indeterminate over S. Then the following statements hold.

(1) There is an additive functor

$$\Phi : \mathrm{CM}_+(R) \to \mathrm{CM}_+(R^\sharp), \qquad \mathrm{Cok}(A,B) \mapsto \mathrm{Cok}\left(\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right), \left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right)\right).$$

(2) Let $M \in \operatorname{ind}\mathrm{CM}_+(R)$ and put $N = \Phi(M)$. Then one has either $N \in \operatorname{ind}\mathrm{CM}_+(R^\sharp)$ or $N \cong X \oplus Y$ for some $X,Y \in \operatorname{ind}\mathrm{CM}_+(R^\sharp)$.

Proof. (1) It holds that $\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right)\left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right) = \left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right)\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right) = (f + x^2g)E$. If $(V,W) : (A,B) \to (A',B')$ is a morphism of matrix factorizations of f over S, then $\left(\left(\begin{smallmatrix} V & 0 \\ 0 & W \end{smallmatrix}\right), \left(\begin{smallmatrix} W & 0 \\ 0 & V \end{smallmatrix}\right)\right) : \left(\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right), \left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right)\right) \to \left(\left(\begin{smallmatrix} A' & -xE \\ xgE & B' \end{smallmatrix}\right), \left(\begin{smallmatrix} B' & xE \\ -xgE & A' \end{smallmatrix}\right)\right)$ is a morphism of matrix factorizations of $f + x^2g$ over $S[[x]]$. We observe that Φ defines an additive functor from $\mathrm{CM}(R)$ to $\mathrm{CM}(R^\sharp)$.

Fix $M \in \mathrm{CM}_+(R)$. Let (A,B) be the corresponding matrix factorization. Set $N = \mathrm{Cok}_{S[[x]]}\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right)$. There is a nonmaximal prime ideal $\mathfrak{p}$ of S such that $M_\mathfrak{p}$ is not $R_\mathfrak{p}$-free. Put $\mathfrak{q} = \mathfrak{p}S[[x]] + xS[[x]]$. We see that $\mathfrak{q}$ is a nonmaximal prime ideal of $S[[x]]$. Suppose that $N_\mathfrak{q} \cong (R^\sharp)_\mathfrak{q}^{\oplus n}$ for some n. Then

$$R_\mathfrak{p}^{\oplus n} \cong ((R^\sharp/xR^\sharp)^{\oplus n})_\mathfrak{q} \cong N_\mathfrak{q}/xN_\mathfrak{q} \cong \mathrm{Cok}_{S[[x]]_\mathfrak{q}}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \cong \mathrm{Cok}_{S_\mathfrak{p}} A \oplus \mathrm{Cok}_{S_\mathfrak{p}} B \cong M_\mathfrak{p} \oplus (\Omega_R M)_\mathfrak{p},$$

which implies that $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free, a contradiction. Therefore $N_\mathfrak{q}$ is not $(R^\sharp)_\mathfrak{q}$-free, and we obtain $N \in \mathrm{CM}_+(R^\sharp)$. Thus Φ induces an additive functor from $\mathrm{CM}_+(R)$ to $\mathrm{CM}_+(R^\sharp)$.

(2) Let (A,B) be the matrix factorization which gives M. Then $N = \mathrm{Cok}_{S[[x]]}\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right)$. Suppose that N is decomposable. Then $N \cong X \oplus Y$ for some nonzero modules $X,Y \in \mathrm{CM}(R^\sharp)$. It holds that

$$X/xX \oplus Y/xY \cong N/xN \cong \mathrm{Cok}_S\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \cong \mathrm{Cok}_S A \oplus \mathrm{Cok}_S B \cong M \oplus \Omega_R M.$$

Since R is Gorenstein, not only M but also $\Omega_R M$ is indecomposable; see [Yos90, Lemma 8.17]. Nakayama's lemma guarantees that X/xX and Y/xY are nonzero, and both X and Y have to be indecomposable. We may assume that $M \cong X/xX$ and $\Omega_R M \cong Y/xY$. Take a nonmaximal prime ideal $\mathfrak{p}$ of S such that $M_{\mathfrak{p}}$ is not $R_{\mathfrak{p}}$-free. Then $\mathfrak{q} := \mathfrak{p}S[[x]] + xS[[x]]$ is a nonmaximal prime ideal of $S[[x]]$ as in the proof of (1). We easily see that the $R_{\mathfrak{p}}$-module $(\Omega_R M)_{\mathfrak{p}}$ is not free. Now it follows that neither $X_{\mathfrak{q}}$ nor $Y_{\mathfrak{q}}$ is free over $(R^\sharp)_{\mathfrak{q}}$, which shows that $X, Y \in \mathrm{CM}_+(R^\sharp)$. □

Infinite CM_+-representation type ascends from R to $R^\sharp$.

Proposition 5.7.3. Let $(S, \mathfrak{n})$ be a regular local ring and $f, g \in S$. Let $R = S/(f)$ and $R^\sharp = S[[x]]/(f + x^2 g)$ be hypersurfaces with x an indeterminate. If R has infinite CM_+-representation type, then so does $R^\sharp$.

Proof. Pick any $M_1 \in \operatorname{ind}\mathrm{CM}_+(R)$. The set $\operatorname{ind}\mathrm{CM}_+(R) \setminus \{M_1, \Omega M_1\}$ is infinite, and we pick any M_2 in this set. The set $\operatorname{ind}\mathrm{CM}_+(R) \setminus \{M_1, \Omega M_1, M_2, \Omega M_2\}$ is infinite, and we pick any M_3 in it. Iterating this procedure, we obtain modules $M_1, M_2, M_3, \dots$ in $\operatorname{ind}\mathrm{CM}_+(R)$ such that $M_i \ncong M_j$ and $M_i \cong \Omega M_j$ for all $i \neq j$. We put $N_i = \Phi M_i$ for each i, where Φ is the functor defined in Lemma 5.7.1. Then by the lemma N_i is either in $\operatorname{ind}\mathrm{CM}_+(R^\sharp)$ or isomorphic to $X_i \oplus Y_i$ for some $X_i, Y_i \in \operatorname{ind}\mathrm{CM}_+(R^\sharp)$.

Assume $N_i \cong N_j$ for some $i \neq j$. Then, as we saw in the proof of the lemma, there are isomorphisms $M_i \oplus \Omega M_i \cong N_i/xN_i \cong N_j/xN_j \cong M_j \oplus \Omega M_j$ and the modules $M_i, \Omega M_i, M_j, \Omega M_j$ are indecomposable. This contradicts the choice of these modules. Hence we have $N_i \ncong N_j$ for all $i \neq j$.

Suppose that there are only a finite number, say n, of indecomposable modules in $\mathrm{CM}_+(R)$. Then it is seen that the set $\{N_1, N_2, N_3, \dots\}/_{\cong}$ has cardinality at most $n + \binom{n+1}{2}$, which is a contradiction. We now conclude that $R^\sharp$ has infinite CM_+-representation type, and the proof of the proposition is completed. □

Here is an application of Proposition 5.7.3.

Corollary 5.7.4. Let R be a 2-dimensional complete local hypersurface with algebraically closed residue field k of characteristic 0 and not having an isolated singularity. Suppose that R has multiplicity at most 2. If R has finite CM_+-representation type, then $R \cong k[[x,y,z]]/(f)$ with $f = x^2 + y^2$ or $f = x^2 + y^2 z$, and hence R has countable CM-representation type.

Proof. If $\mathrm{e}(R) = 1$, then R is regular, which contradicts the assumption that R does not have an isolated singularity. Hence $\mathrm{e}(R) = 2$, and the combination of Cohen's structure theorem and the Weierstrass preparation theorem shows $R \cong k[[x,y,z]]/(x^2 + g)$ for some $g \in k[[y,z]]$; see [Yos90, Proof of Theorem 8.8]. It follows from Proposition 5.7.3 that the 1-dimensional hypersurface $S := k[[y,z]]/(g)$ has finite CM_+-representation type. By virtue of Theorem 5.6.1, we obtain $g = y^2$ or $g = y^2 z$ after changing variables (i.e., after applying a k-algebra automorphism of $k[[y,z]]$). We observe that R is isomorphic to either $k[[x,y,z]]/(x^2 + y^2)$ or $k[[x,y,z]]/(x^2 + y^2 z)$. It follows from [LW12, Propositions 14.17 and 14.19] that R has countable CM-representation type. $\square$

Proposition 5.7.3 can provide a lot of examples of hypersurfaces of infinite CM_+-representation type of higher dimension. The following example is not covered by this proposition or any other general result given in this chapter.

Example 5.7.5. Let S be a regular local ring with a regular system of parameters x, y, z. Let

$$f = x^n + x^2 y a + y^2 b$$

be an irreducible element of S with $n \geq 4$ and $a, b \in S$. Then the hypersurface $R = S/(f)$ has infinite CM_+-representation type.

Proof. Putting $g = x^2 a + yb$, we have $f = x^n + yg$. For each integer $i \geq 0$ we define a pair of matrices $A_i = \begin{pmatrix} x^2 & xz^i \\ 0 & -x^2 \end{pmatrix}$ and $B_i = \begin{pmatrix} x^{n-2} & x^{n-3}z^i \\ 0 & -x^{n-2} \end{pmatrix}$, which gives a matrix factorization of x^n over S and $S/(y)$. Define another pair of matrices $A_i^\sharp = \begin{pmatrix} A_i & -yE \\ gE & B_i \end{pmatrix}$ and $B_i^\sharp = \begin{pmatrix} B_i & yE \\ -gE & A_i \end{pmatrix}$. These form a matrix factorization of f over S, and hence $M_i := \mathrm{Cok}_S(A_i^\sharp)$ is a maximal Cohen–Macaulay R-module.

There are equalities

$$\mathrm{Fitt}_3^S(M_i) = \mathrm{I}_1(A_i^\sharp) = (x^2, xz^i, x^{n-2}, x^{n-3}z^i, y, g)S = (x^2, xz^i, y)S$$

of ideals of S, where we use $n \geq 4$.

Suppose that $M_i \cong M_j$ for some $i < j$. Then $(x^2, xz^i, y)S = (x^2, xz^j, y)S$ and $(x^2, xz^i)\overline{S} = (x^2, xz^j)\overline{S}$, where $\overline{S} := S/(y)$ is a regular local ring having the regular system of parameters x, z. Hence $z^i \in (x, z^j)\overline{S}$ and $z^i \in z^j\widetilde{S}$ where $\widetilde{S} := \overline{S}/x\overline{S}$ is a discrete valuation ring with z a uniformizer. This gives a contradiction, and we see that $M_i \not\cong M_j$ for all $i \neq j$.

Let $\mathfrak{p} = (x, y)S \in \operatorname{Spec} S$, and fix an integer $i \geq 0$. Note that all the entries of A_i, B_i are in $\mathfrak{p}$ since $n \geq 4$. It follows from [Yos90, Remark 7.5] that the $R_\mathfrak{p}$-module $(M_i)_\mathfrak{p}$ does not have a nonzero free summand. Since f is assumed to be irreducible, R is an integral domain. Hence each nonzero direct summand X of the maximal Cohen–Macaulay R-module M_i has positive rank, and hence has full support. Therefore $X_\mathfrak{p} \neq 0$, and thus all the indecomposable direct summands of M_i belong to $\mathsf{ind}\,\mathrm{CM}_+(R)$. Since all the M_i are generated by four elements, it is observed that $\mathsf{ind}\,\mathrm{CM}_+(R)$ is an infinite set. $\square$

To prove our next result, we prepare a lemma on unique factorization domains.

Lemma 5.7.2. Let R be a Cohen–Macaulay factorial local ring with $\dim R \geq 3$. Let I be an ideal of R generated by two elements. Then $\operatorname{depth} R/I > 0$.

Proof. We write $I = (x, y)R$ and put $g = \gcd\{x, y\}$. Then $x = gx'$ and $y = gy'$ for some $x', y' \in R$, and we set $I' = (x', y')R$. There is an exact sequence $0 \to R/I' \xrightarrow{g} R/I \to R/gR \to 0$ of R-modules. As R is Cohen–Macaulay, we have $\operatorname{depth} R \geq 3$ and $\operatorname{ht} I' = \operatorname{grade} I'$. Since R is a domain and $g \neq 0$, we have $\operatorname{depth} R/gR = \operatorname{depth} R - 1 \geq 2$. If $\operatorname{ht} I' = 1$, then I' is contained in a principal prime ideal, which contradicts the fact that x', y' are coprime. Hence $\operatorname{ht} I' = 2$, and the sequence x', y' is R-regular. It follows that $\operatorname{depth} R/I' = \operatorname{depth} R - 2 \geq 1$, and the depth lemma implies $\operatorname{depth} R/I \geq 1$. $\square$

Now we can prove the following theorem, which provides the shape of a hypersurface of infinite CM_+-representation type.

Theorem 5.7.6. Let $(S,\mathfrak{n})$ be a regular local ring and $x,y \in \mathfrak{n}$. Suppose that the ideal (x,y) of S is neither prime nor $\mathfrak{n}$-primary. Then $R = S/(xy)$ has infinite CM_+-representation type.

Proof. Lemma 5.7.2 guarantees that there exists an $S/(x,y)$-regular element $a \in \mathfrak{n}$. Take a minimal prime $\mathfrak{p}$ of (x,y). Since (x,y) is not prime, we can choose an element $b \in \mathfrak{p} \setminus (x,y)$. Set $z_n = a^n b$ for each n. The matrices $A_n = \left(\begin{smallmatrix} x & z_n \\ 0 & -y \end{smallmatrix}\right)$ and $B_n = \left(\begin{smallmatrix} y & z_n \\ 0 & -x \end{smallmatrix}\right)$ with $n \geq 1$ form a matrix factorization of xy over S, and $M_n = \mathrm{Cok}_S A_n$ is a maximal Cohen–Macaulay R-module. Put $I_n := \mathrm{I}_1(A_n) = (x,y,z^n) \subseteq S$. Since the I_n are pairwise distinct, the M_n are pairwise nonisomorphic. If M_n is decomposable, it decomposes into two cyclic R-modules, while Lemma 5.6.3 says that there are only finitely many such cyclic modules up to isomorphism. Thus we find infinitely many n such that M_n is indecomposable. Since (x,y,z_n) is contained in $\mathfrak{p}$, each $(M_n)_{\mathfrak{p}}$ has no nonzero free summand by [Yos90, (7.5.1)]. In particular, we have $M_n \in \mathrm{CM}_+(R)$. Now it is seen that R has infinite CM_+-representation type. $\square$

Applying the above theorem, we can obtain a couple of restrictions for a hypersurface of dimension at least 2 which is not an integral domain but has finite CM_+-representation type.

Corollary 5.7.7. Let R be a complete local hypersurface of dimension $d \geq 2$ which is not a domain. Suppose that R has finite CM_+-representation type. Then one has $d = 2$, and there exist a complete regular local ring S of dimension 3 and elements $x,y \in S$ satisfying the following conditions.

(1) R is isomorphic to $S/(xy)$.

(2) $S/(x)$ and $S/(y)$ have finite CM-representation type.

(3) $S/(x,y)$ is a domain of dimension 1.

Proof. Corollary 5.4.2(1) says that R satisfies Serre's condition $(\mathbb{R}_{d-2})$. Suppose $d \geq 3$. Then R satisfies $(\mathbb{R}_1)$, and hence it is normal. In particular, R is a domain, contrary to our assumption. Therefore, we have to have $d = 2$. Cohen's structure theorem yields $R \cong S/fS$ for some 3-dimensional complete regular local ring $(S,\mathfrak{n})$ and $f \in \mathfrak{n} \setminus \mathfrak{n}^2$. As R is not a domain, there are elements $x,y \in S$ with $f = xy$. Since $\dim S = 3$, the ideal $(x,y)S$ is not $\mathfrak{n}$-primary. Hence

$\dim S/(x,y)S = 1$, and $S/(x,y)S$ is a domain by Theorem 5.7.6. We have $\dim R = \dim R/xR = 2$, $(0 :_R x) = yR$ and $\mathrm{ht}(xR + (0 :_R x)) < 2$. It follows from Theorem 5.5.2(3a) that S/xS has finite CM-representation type, and similarly so does S/yS. $\square$

Proposition 5.7.3 gives an ascent property of infinite CM_+-representation type. Now we presents a descent property of infinite CM_+-representation type.

Theorem 5.7.8. Let $\phi : (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a finite local homomorphism of Cohen–Macaulay local rings of dimension d such that S is a domain. Set $\mathfrak{p} = \mathrm{Ker}\,\phi$ and assume the following.

(a) The induced embedding $R/\mathfrak{p} \hookrightarrow S$ is birational.

(b) There exists $\mathfrak{q} \in \mathrm{V}(\mathfrak{p}) \setminus \{\mathfrak{m}\}$ such that $R_\mathfrak{q}$ is not a direct summand of $S_\mathfrak{q}$.

If S has infinite CM-representation type, then R has infinite CM_+-representation type.

Proof. We prove the theorem by establishing several claims.

Claim 5.7.3. Let $X \neq 0$ be an R-submodule of a maximal Cohen–Macaulay S-module M. Then $X_\mathfrak{q} \neq 0$.

Proof of Claim. Assume $X_\mathfrak{q} = 0$. Then there exists an element $s \in \mathrm{ann}_R X$ such that $s \notin \mathfrak{q}$. As $\mathfrak{p} \subseteq \mathfrak{q}$, we have $s \notin \mathfrak{p}$, which means $\phi(s) \neq 0$. Choose a nonzero element $x \in X$. Since s annihilates X, we have $0 = s \cdot x = \phi(s)x$ in M. This contradicts the fact that M is torsion-free over the domain S. $\square$

Claim 5.7.4. Let $M \in \mathrm{CM}_0(S)$. Let X be an indecomposable R-module which is a direct summand of M. Then $X \in \mathsf{ind}\,\mathrm{CM}_+(R)$.

Proof of Claim. As $\mathrm{depth}_R M = \mathrm{depth}_S M \geq d$, we have $M \in \mathrm{CM}(R)$ and hence $X \in \mathsf{ind}\,\mathrm{CM}(R)$. To show the claim, it suffices to verify that $X_\mathfrak{q}$ is not $R_\mathfrak{q}$-free.

Take an exact sequence $\sigma : 0 \to \Omega_S M \to S^{\oplus n} \to M \to 0$. Since M belongs to $\mathrm{CM}_0(S)$, the S-module $E := \mathrm{Ext}^1_S(M, \Omega_S M)$ has finite length. The induced field extension $k \hookrightarrow l$ is finite because so is the homomorphism ϕ, and hence E also has finite length as an R-module. As $\mathfrak{q}$ is a nonmaximal prime ideal of R, we have $0 = E_\mathfrak{q} = \mathrm{Ext}^1_{S_\mathfrak{q}}(M_\mathfrak{q}, (\Omega_S M)_\mathfrak{q})$, and the exact sequence

$\sigma_{\mathfrak{q}} : 0 \to (\Omega_S M)_{\mathfrak{q}} \to S_{\mathfrak{q}}^{\oplus n} \to M_{\mathfrak{q}} \to 0$ corresponds to an element in this Ext module. Hence $\sigma_{\mathfrak{q}}$ has to split, and $M_{\mathfrak{q}}$ is a direct summand of $S_{\mathfrak{q}}^{\oplus n}$ as an $S_{\mathfrak{q}}$-module. (Note that $S_{\mathfrak{q}}$ is not necessarily a local ring.) The $R_{\mathfrak{q}}$-module $X_{\mathfrak{q}}$ is a direct summand of $M_{\mathfrak{q}}$, and is nonzero by Claim 5.7.3.

Suppose that $X_{\mathfrak{q}}$ is $R_{\mathfrak{q}}$-free. Then $R_{\mathfrak{q}}$ is a direct summand of $S_{\mathfrak{q}}^{\oplus n}$ in $\operatorname{mod} R_{\mathfrak{q}}$. As $R_{\mathfrak{q}}$ is a local ring, $R_{\mathfrak{q}}$ is a direct summand of $S_{\mathfrak{q}}$. This contradicts the assumption of the theorem, and thus $X_{\mathfrak{q}}$ is not $R_{\mathfrak{q}}$-free. $\square$

Claim 5.7.5. One has the inclusion $\operatorname{ind} \mathrm{CM}_0(S) \subseteq \operatorname{ind} \mathrm{CM}_+(R)$.

Proof of Claim. Take $M \in \operatorname{ind} \mathrm{CM}_0(S)$. Lemma 5.6.1 implies that M is indecomposable as an $R/\mathfrak{p}$-module, and it is indecomposable as an R-module. Taking $X := M$ in Claim 5.7.4, we have $M \in \operatorname{ind} \mathrm{CM}_+(R)$. $\square$

It follows from Lemma 5.5.3 that S has infinite CM_0-representation type. Claim 5.7.5 implies that R has infinite CM_+-representation type, and the proof of the theorem is completed. $\square$

We obtain an application of the above theorem, which gives an answer to Question 5.3.6. For a ring R we denote by $\overline{R}$ the integral closure of R. Recall that a typical example of a henselian Nagata ring is a complete local ring.

Corollary 5.7.9. Let R be a 2-dimensional henselian Nagata Cohen–Macaulay non-normal local ring. Suppose that R has finite CM_+-representation type. Then the following statements hold.

(1) There exists a minimal prime $\mathfrak{p}$ of R such that the integral closure $\overline{R/\mathfrak{p}}$ has finite CM-representation type. In particular, if R is a domain, then $\overline{R}$ has finite CM-representation type.

(2) If R is Gorenstein, then R is a hypersurface.

Proof. By Corollary 5.4.2(1) the singular locus of R has dimension at most one, so that R satisfies Serre's condition $(\mathbb{R}_0)$. As R is Cohen–Macaulay, it is reduced. Let $S = \overline{R}$ be the integral closure of R. We have a decomposition $S = \overline{R/\mathfrak{p}_1} \oplus \cdots \oplus \overline{R/\mathfrak{p}_n}$ as R-modules, where $\operatorname{Min} R = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ (see [HS06, Corollary 2.1.13]). Since R is Nagata, the extension $R \subseteq S$ is finite. The ring S is normal and has dimension two, so it is Cohen–Macaulay.

We claim that if $\mathfrak{p}$ is a nonmaximal prime ideal of R such that $S_\mathfrak{p}$ is $R_\mathfrak{p}$-free, then $R_\mathfrak{p}$ is a regular local ring. In fact, if $\operatorname{ht}\mathfrak{p} = 0$, then $R_\mathfrak{p}$ is a field. Let $\operatorname{ht}\mathfrak{p} = 1$. The induced map $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, and we find a prime ideal P of S such that $P \cap R = \mathfrak{p}$. We easily see $\operatorname{ht} P = 1$. As S is normal, S_P is regular. The induced map $R_\mathfrak{p} \to S_P$ factors as $R_\mathfrak{p} \xrightarrow{a} S_\mathfrak{p} \xrightarrow{b} S_P$, where a is a finite free extension, and b is flat since $S_P = (S_\mathfrak{p})_{PS_\mathfrak{p}}$. Hence $R_\mathfrak{p} \to S_P$ is a flat local homomorphism. As S_P is regular, so is $R_\mathfrak{p}$.

Since R does not have an isolated singularity, there exists a nonmaximal prime ideal $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is not regular. The claim implies that $S_\mathfrak{p}$ is not $R_\mathfrak{p}$-free, whence $S \in \mathrm{CM}_+(R)$. There exists an integer $1 \le l \le n$ such that $T := \overline{R/\mathfrak{p}_l}$ belongs to $\mathrm{CM}_+(R)$.

Put $\mathfrak{p} := \mathfrak{p}_l \in \operatorname{Min} R$. The ring $R/\mathfrak{p}$ is also Nagata, and the extension $R/\mathfrak{p} \subseteq T$ is finite and birational. The ring T is a 2-dimensional henselian normal local domain, whence it is a Cohen–Macaulay. Choose a nonmaximal prime ideal $\mathfrak{q}$ of R such that $T_\mathfrak{q}$ is not $R_\mathfrak{q}$-free. If $\mathfrak{p}$ is not contained in $\mathfrak{q}$, then $(R/\mathfrak{p})_\mathfrak{q} = \kappa(\mathfrak{p})_\mathfrak{q} = 0$ and $T_\mathfrak{q} = 0$, which particularly says that $T_\mathfrak{q}$ is $R_\mathfrak{q}$-free, a contradiction. Hence $\mathfrak{p} \subseteq \mathfrak{q}$.

Suppose that $R_\mathfrak{q}$ is a direct summand of $T_\mathfrak{q}$. Then there is an isomorphism $T_\mathfrak{q} \cong R_\mathfrak{q} \oplus X$ of $R_\mathfrak{q}$-modules. Since $T_\mathfrak{q}$ is annihilated by $\mathfrak{p}$, so is $R_\mathfrak{q}$. We have ring extensions $R_\mathfrak{q} = (R/\mathfrak{p})_\mathfrak{q} \subseteq T_\mathfrak{q} \subseteq \kappa(\mathfrak{p})$, which especially says that $R_\mathfrak{q}$ is a domain and that $T_\mathfrak{q}$ has rank one as an $R_\mathfrak{q}$-module. Hence the $R_\mathfrak{q}$-module X has rank zero, and it is easy to see that $X = 0$. We get $T_\mathfrak{q} \cong R_\mathfrak{q}$, which contradicts the choice of $\mathfrak{q}$. Consequently, $T_\mathfrak{q}$ does not have a direct summand isomorphic to $R_\mathfrak{q}$.

Now, application of Theorem 5.7.8 proves the assertion (1). To show (2), we consider the T-module $U = \Omega_T^2(T/\mathfrak{m}T)$. Fix any nonzero direct summand X of U or T in $\operatorname{mod} R$. Note that $T = \overline{R/\mathfrak{p}}$ is a torsion-free module over $R/\mathfrak{p}$. Since U is a submodule of a nonzero free T-module, U is also torsion-free over $R/\mathfrak{p}$, and so is X. We easily see from this that $X_\mathfrak{q} \neq 0$. The module $X_\mathfrak{q}$ is a direct summand of $U_\mathfrak{q} \cong T_\mathfrak{q}^{\oplus \operatorname{edim} R - 1}$. As $R_\mathfrak{q}$ is not a direct summand of $T_\mathfrak{q}$, it is not a direct summand of $X_\mathfrak{q}$. In particular, X belongs to $\mathrm{CM}_+(R)$. Thus, all the indecomposable direct summands of U and of T in $\operatorname{mod} R$ belong to $\operatorname{ind}\mathrm{CM}_+(R)$, and it follows from Lemma 5.5.7 that they have complexity at most one. Hence U and T have complexity at most one over R, and so does $T/\mathfrak{m}T$. We obtain $\operatorname{cx}_R k \leq 1$, and R is a hypersurface by [Avr10, Theorem 8.1.2]. $\square$

The above result yields a strong restriction for finite CM_+-representation type in dimension two.

Corollary 5.7.10. Let R be a 2-dimensional non-normal Gorenstein complete local ring. If R has finite CM_+-representation type, then the integral closure $\overline{R}$ has finite CM-representation type.

Proof. If R is a domain, then the assertion follows from Corollary 5.7.9(1). Hence let us assume that R is not a domain. By Corollary 5.7.9(2) the ring R is a hypersurface. We can apply Corollary 5.7.7 to see that there exists a 3-dimensional regular local ring S and elements $x, y \in S$ such that R is isomorphic to $S/(xy)$ and $S/(x), S/(y)$ have finite CM-representation type. Note by [HL02, Corollary 2] that $S/(x), S/(y)$ are normal. As in the beginning of the proof of Corollary 5.7.9, the ring R is reduced. Hence $(x) \neq (y)$, and we have an isomorphism $\overline{R} \cong \overline{S/(x)} \times \overline{S/(y)} = S/(x) \times S/(y)$; see [HS06, Corollary 2.1.13]. There is a natural category equivalence $\operatorname{mod} \overline{R} \cong \operatorname{mod} S/(x) \times \operatorname{mod} S/(y)$, which induces a category equivalence $\mathrm{CM}(\overline{R}) \cong \mathrm{CM}(S/(x)) \times \mathrm{CM}(S/(y))$. It is observed from this that $\overline{R}$ has finite CM-representation type. $\square$

The converse of Corollary 5.7.10 does not necessarily hold, as the following example says.

Example 5.7.11. Let $R = k[[x,y,z]]/(x^4 - y^3 z)$ be a quotient of the formal power series ring $k[[x,y,z]]$ over a field k. Then R is a 2-dimensional complete non-normal local hypersurface. The assignment $x \mapsto s^3 t$, $y \mapsto s^4$, $z \mapsto t^4$ gives an isomorphism from R to the subring $S = k[[s^4, s^3 t, t^4]]$ of the formal power series ring $T = k[[s,t]]$. The integral closure of S is the fourth Veronese subring $k[[s^4, s^3 t, s^2 t^2, s t^3, t^4]]$ of T, which has finite CM-representation type by [LW12, Theorem 6.3]. Hence $\overline{R}$ has finite CM-representation type. However, as $x^4 - y^3 z = x^4 + x^2 y \cdot 0 + y^2(-yz)$, the ring R does not have finite CM_+-representation type by Example 5.7.5.

Remark 5.7.12. The integral closure has to actually be regular (under the assumptions of Corollary 5.7.10) provided that our conjecture that countable CM-representation type is equivalent to finite CM_+-representation type holds true in this setting.

Chapter 6

Higher Nerves of simplicial complexes

6.1 Introduction

The nerve complex has been an important object of study in algebraic combinatorics [Bas03, Bjö03, Bor48, CJS15, Grü70, KM05, LSVJ11, PUV16]. We remind the reader of its definition:

Definition 6.1.1. Let $A = \{A_1, A_2, \dots, A_r\}$ be a family of sets. Consider

$$N(A) := \{F \subseteq [r] \colon \cap_{i \in F} A_i \neq \emptyset\}.$$

This simplicial complex is the *nerve complex* of A.

Of special interest is the case where A is the set of facets of a simplicial complex Δ; in this case, one sets $N(\Delta) := N(A)$. We propose a natural extension of this notion.

Definition 6.1.2. Let $A = \{A_1, A_2, \dots, A_r\}$ be the set of facets of a simplicial complex Δ. Define

$$N_i(\Delta) := \{F \subseteq [r] \colon |\cap_{j \in F} A_j| \geq i\}.$$

We call this simplicial complex the i^{th} *nerve complex* of Δ and we refer to the $N_i(\Delta)$ and the *higher nerve complexes* of Δ.

When $i = 1$, this definition recovers $N(\Delta)$.

The Nerve Theorem of Borsuk [Bor48] gives that $N(\Delta)$ and Δ have the same homologies. We now explain how the higher nerves relate to the original complex in a more subtle manner. Namely, their homologies determine important algebraic and combinatorial properties of Δ. We summarize our main quantitative results below.

Theorem 6.1.1 (Main Theorem)**.** Let k be a field, let Δ be a simplicial complex of dimension $d-1$, and let $k[\Delta]$ be the associated Stanley-Reisner ring. Let $\widetilde{H}_i$ denote ith reduced simplicial homology with coefficients in k, and let χ denote Euler characteristic. Then:

1. $\widetilde{H}_i(N_j(\Delta)) = 0$ for $i+j > d$ and $1 \le j \le d$ (see Corollary 6.3.4).

2. $\operatorname{depth}(k[\Delta]) = \inf\{i+j \colon \widetilde{H}_i(N_j(\Delta)) \neq 0\}$ (see Theorem 6.5.2).

3. For $i \ge 0$, $f_i(\Delta) = \displaystyle\sum_{j=i+1}^{d} \binom{j-1}{i} \chi(N_j(\Delta))$ (see Theorem 6.6.1).

In short, the numbers $b_{ij} = \dim \widetilde{H}_i(N_j(\Delta))$ for $0 \le i \le d-j$ and $1 \le j \le d$ can be presented in a nice table which determine both the depth and the f-vector (and thus also the h-vector) of Δ. We provide an explicit example below.

Example 6.1.3. Consider the simplicial complex Δ with facets

$$\{F_1 := ABCD, F_2 := BCDE, F_3 := DEFG, F_4 := DFGH\}.$$

The following are geometric realizations of the complex and its higher nerves:

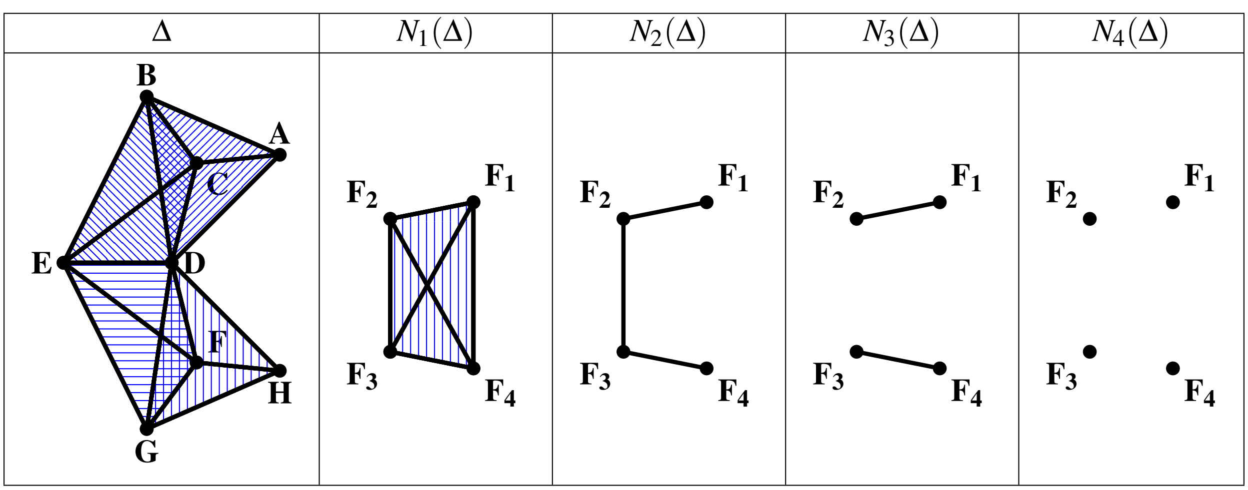

Figure 6.1: Nerves of Δ

	$\widetilde{H}_0$	$\widetilde{H}_1$	$\widetilde{H}_2$	χ
N_1	0	0	0	1
N_2	0	0	0	1
N_3	1	0	0	2
N_4	3	0	0	4

Figure 6.2: Nerve Homologies

Using our main theorem and the Table 2, $\operatorname{depth} k[\Delta] = 3$ and $f(\Delta) = (1, 8, 17, 14, 4)$.

There are consequences to our main results. For instance, we provide a formula to compute the regularity of any monomial ideal, not necessarily square-free, in Theorem 6.7.1. Other algebraic properties such as Serre's condition (S_r) can also be detected from the nerve table; this is a topic that will be explored in Chapter 7.

Remark 6.1.4. Though we will not consider it in this document, one can also define higher nerves in a more general setting. Let A be a collection of subsets of a topological space X. Define $N_i(A) := \{F \subseteq [r] : \dim \cap_{j \in F} A_j \geq i\}$, where dim represents Krull dimension. In this setting, special interest is given to the case where X is a Noetherian algebraic scheme; in this case, one sets $N_i(X) := N_i(A)$, where A is the collection of irreducible components of X. In particular, if $X = \operatorname{Spec} R$ for a local ring R, then the $N_i(X)$ provide a natural generalization of the Lyubeznik complex of R (see [Lyu07, Theorem 1.1] for the definition). If, instead, $X = \operatorname{Spec} R$ for R a Stanley-Reisner ring of a simplicial complex Δ, then the complex defined in this remark coincides with that of Definition 6.1.2, via the Stanley-Reisner correspondence. Our results in the Stanley-Reisner case raise some intriguing questions about higher nerve complexes of local schemes that can be viewed as extensions of results by Hartshorne and Katzman-Lyubeznik-Zhang ([Har62, KLZ16]).

We now briefly describe the structure of this chapter. In Section 8.2, we cover combinatorial background and fix the notation we will use throughout the chapter. In Section 6.3, we recall and prove certain basic facts about depth and connectivity of a complex, which motivate our results and will be used in our proofs. We provide a strengthened version of the classical Nerve Theorem that suits our purpose in Proposition 6.3.3. This proposition is a critical component of parts (1) and (2) of our main theorem. We conclude this section by proving part (1) of our main theorem. In Section 6.4, we provide several lemmata, the main technical tools of most of our proofs. Section 8.3 is devoted to the proof of the second part of our main theorem. Section 6.6 gives the proof of the third part of our main theorem and provides a formula for the h-vector in terms of homologies of higher nerves in Corollary 6.6.1. Section 6.7 applies our main theorem to give a formula for computing the Castelnuovo-Mumford regularity of any monomial ideal.

6.2 Notation and definitions

In this section we introduce the notation we will use throughout this chapter. Unless otherwise stated, we fix the field k and let $\widetilde{H}_i$ denote ith reduced simplicial or singular homology, whichever is appropriate, always with coefficients in k.

We will use $V(\Delta)$ to represent the vertex set of a simplicial complex Δ; we will use V instead of $V(\Delta)$ when the choice of Δ is clear; we also set $n := |V(\Delta)|$ and $S := k[x_1, \ldots, x_n]$. We denote a subcomplex of Δ induced on the vertex set W as $\Delta|_W := \{F \in \Delta \colon F \subseteq W\}$.

Given a subset $T \subseteq V(\Delta)$, we may define the star, the anti-star, and the link of T, denoted $\mathrm{st}_\Delta(T)$, $\mathrm{ast}_\Delta(T)$, and $\mathrm{lk}_\Delta(T)$, respectively, as follows:

$$
\begin{aligned}
\mathrm{st}_\Delta T &:= \{G \in \Delta \colon T \cup G \in \Delta\} \\
\mathrm{ast}_\Delta T &:= \{G \in \Delta \colon T \cap G = \varnothing\} = \Delta|_{V \setminus T} \\
\mathrm{lk}_\Delta T &:= \{G \in \Delta \colon T \cup G \in \Delta \text{ and } T \cap G = \varnothing\} = \mathrm{st}_\Delta T \cap \mathrm{ast}_\Delta T
\end{aligned}
$$

The star and link of T are the void complex exactly when $T \notin \Delta$, and the link of T is the irrelevant complex $\{\varnothing\}$ exactly when T is a facet. On the other hand, the anti-star of any $T \subsetneq V(\Delta)$ is nonempty.

We call $\Delta^{(k)} := \{\sigma \in \Delta \colon |\sigma| \leq k+1\}$ the *k-skeleton of* Δ.

Definition 6.2.1. Let $\mathscr{F}_{>k}(\Delta)$ denote the face poset of Δ restricted to faces of Δ with cardinality strictly greater than k.

We note the face poset of Δ is $\mathscr{F}_{>-1}(\Delta)$. Furthermore $\mathscr{F}_{>(\dim\Delta+1)}(\Delta)$ is the empty poset.

Definition 6.2.2. The *order complex* of a poset P, denoted $\mathscr{O}(P)$, is the simplicial complex whose faces are all chains in P.

We will denote the geometric realization of Δ as $||\Delta||$.

Given a complex Δ, its *barycentric subdivision* may be defined as $\operatorname{sd}\Delta := \mathscr{O}(\mathscr{F}_{>0}(\Delta))$. The following is well-known (see Corollary 5.7 of [Gib10] for example).

Lemma 6.2.1. The realization $||\Delta||$ is homeomorphic to $||\operatorname{sd}\Delta||$. In particular, $\widetilde{H}_i(\Delta) = \widetilde{H}_i(\operatorname{sd}\Delta)$ for all i.

We let $\rho : \mathscr{F}_{>0}(\Delta) \to V(\operatorname{sd}\Delta)$ be the map which sends an element of $\mathscr{F}_{>0}(\Delta)$ to itself viewed as a vertex of $\operatorname{sd}\Delta$.

We will often use the following shorthand:

$$\begin{aligned}[\Delta]_{>k} &= \mathscr{O}(\mathscr{F}_{>k}(\Delta)) \\ &= \operatorname{sd}\Delta\big|_{V(\operatorname{sd}\Delta)\setminus V(\operatorname{sd}(\Delta^{(k-1)}))}\end{aligned}$$

Notice that the image of ρ may be restricted to $V([\Delta]_{>k})$ by restricting its domain to $\mathscr{F}_{>k}(\Delta)$.

A *simplicial map* $f : \Delta_1 \to \Delta_2$ is a function $f : V(\Delta_1) \to V(\Delta_2)$ so that for all $\sigma \in \Delta_1$, $f(\sigma) \in \Delta_2$. We say a simplicial map f is a *simplicial isomorphism* if f has an inverse that is a simplicial map. Note that if $f : Q \to P$ is an order-reversing or order-preserving poset map, then $f : \mathscr{O}(Q) \to \mathscr{O}(P)$ is a simplicial map.

Given a simplicial complex Δ, we also consider algebraic properties of its Stanley-Reisner ring. Readers unfamiliar with the algebraic terminology used may see [BH93] or a similar text for more background. Unless otherwise stated, we write d for $\dim k[\Delta]$, the Krull dimension of the ring $k[\Delta]$. We also use $s(\Delta)$ to mean the minimal cardinality of facets of Δ. By $\operatorname{depth} k[\Delta]$ we mean the depth of the k-algebra $k[\Delta]$; for a combinatorial characterization of $\operatorname{depth} k[\Delta]$, see Corollary 6.3.1. We say that Δ is Cohen-Macaulay whenever $k[\Delta]$ is Cohen-Macaulay, that is, whenever $\dim k[\Delta] = \operatorname{depth} k[\Delta]$.

We further note that

$$\dim k[\Delta] = \max\{|F| \colon F \text{ is a facet of } \Delta\}$$
$$\operatorname{depth} k[\Delta] = \max\{i : \Delta^{(i-1)} \text{ is Cohen-Macaulay}\} \leq s(\Delta).$$

6.3 Preparatory results

In this section, we begin by exploring what is known in the literature and use our construction to prove some immediate results. Many of these results follow as a consequence of our main theorem, but their immediacy shows that our construction is a natural one. We then prove a generalization of the Borsuk Nerve Theorem for simplicial complexes.

We now present Hochster's formula, which will be used throughout this chapter and the next. It relates the i^{th} local cohomology module of $k[\Delta]$ supported on $\mathfrak{m}$, denoted $H^i_{\mathfrak{m}}(k[\Delta])$, to the reduced homology of links of certain faces of Δ. Here $\mathfrak{m}$ is the ideal of $k[\Delta]$ generated by the residue classes of all variables in S.

Theorem 6.3.1 (Hochster [BH93])**.** Let Δ be a simplicial complex. Then the Hilbert series of the local cohomology modules of $k[\Delta]$ with respect to the fine grading is given by:

$$\operatorname{Hilb}_{H^i_{\mathfrak{m}}(k[\Delta])}(t) = \sum_{T \in \Delta} \dim_k \widetilde{H}_{i-|T|-1}(\operatorname{lk}_\Delta T) \prod_{v_j \in T} \frac{t_j^{-1}}{1-t_j^{-1}}.$$

One has $\operatorname{depth} k[\Delta] = \min\{i \colon H^i_{\mathfrak{m}}(k[\Delta]) \neq 0\}$ and $\dim k[\Delta] = \max\{i \colon H^i_{\mathfrak{m}}(k[\Delta]) \neq 0\}$, so Hochster's formula allows us to characterize depth and dimension of $k[\Delta]$ in terms of homologies of links of faces. The following is a generalization of Reisner's well known criterion for Cohen-Macaulayness.

Corollary 6.3.1. Let Δ be a simplicial complex. Then $\operatorname{depth} k[\Delta] \geq t$ if and only if $\widetilde{H}_{i-1}(\operatorname{lk}_\Delta T) = 0$ for all $T \in \Delta$ with $i + |T| < t$.

The following theorem, known as the Borsuk Nerve Theorem, is one of the main tools for working with the classical nerve complex.

Theorem 6.3.2 ([Bor48, Section 9, Corollary 2])**.** Δ and $N_1(\Delta)$ have same homotopy type. In particular, $\widetilde{H}_i(\Delta) \cong \widetilde{H}_i(N_1(\Delta))$ for all i.

Note if $\operatorname{depth} k[\Delta] \geq t$, then $\widetilde{H}_{i-1}(\Delta) = \widetilde{H}_{i-1}(N_1(\Delta)) = 0$ for $i < t$ by Corollary 6.3.1 and Corollary 6.3.2.

Following from the definition of higher nerves, we are able to quickly derive the following results.

Lemma 6.3.3. If $i \leq s(\Delta)$ and $N_i(\Delta)$ is connected, then $\Delta^{(1)}$ is an i-connected graph.

Proof. Since $N_i(\Delta)$ is connected, there is a spanning tree of $N_i(\Delta)^{(1)}$. Let S be a set of all vertices of Δ except for at most $i-1$ of them. We have that $N_1(\Delta|_S)$ is connected, since the facets of $\Delta|_S$ are a subset of the facets of Δ, and the induced spanning tree is preserved. Since connectedness is equivalent to trivial 0^{th} reduced homology and $N_1(-)$ preserves reduced homology, $\Delta|_S$ is connected. Therefore $\Delta^{(1)}$ is i-connected. □

Corollary 6.3.2. Let $t = \operatorname{depth} k[\Delta]$. Then $\Delta^{(1)}$ is a $(t-1)$-connected graph.

Proof. Since $\Delta^{(t-1)}$ is Cohen-Macaulay, the facet-ridge graph of $\Delta^{(t-1)}$ is connected by [Har62]; that is, between any pair of $(t-1)$-faces of Δ, there is a sequence of $(t-1)$-faces, so that each consecutive pair intersects in a $(t-2)$-face. Then for any pair of facets of Δ, by choosing a $(t-1)$-face for each, and finding such a sequence between them, we construct from this a sequence of facets so that each consecutive pair intersects in a $(t-2)$-face. Therefore $N_{t-1}(\Delta)$ is connected, and the result then follows from Lemma 6.3.3. □

An easy proof of Borsuk's Nerve Theorem (Theorem 6.3.2) uses the following result.

Theorem 6.3.4 ([Qui78], Proposition 1.6)**.** Let $f : \Delta \to \mathscr{O}(P)$ be a simplicial map. If for all $x \in P$ we have that $f^{-1}(P_{\geq x})$ is contractible, then f induces a homotopy equivalence between Δ and $\mathscr{O}(P)$.

This theorem also provides a proof of our generalization of the classical Nerve Theorem. This result is probably known to experts, but we could not find the statement we need, so we provide a proof.

Proposition 6.3.3 (Generalized Nerve Theorem)**.** $[\Delta]_{>j}$ is homotopy equivalent to $N_{j+1}(\Delta)$.

Proof. We use a similar approach as that of Theorem 10.6 in [Bjö95].

Let $P = \mathscr{F}_{>0}(N_{j+1}(\Delta))$ and define $f : \mathscr{F}_{>j}(\Delta) \to P$ by

$$f(\sigma) = \{F_i : \sigma \subseteq F_i \text{ facet of } \Delta\}.$$

This map is order-reversing, and it is well-defined, since $|\sigma| \geq j+1$. Therefore, $f : \mathscr{O}(\mathscr{F}_{>j}(\Delta)) \to \mathscr{O}(P)$ is a simplicial map. For any $\tau \in P$, we have that

$$f^{-1}(P_{\geq \tau}) = \bigcap_{F_i \in \tau} F_i,$$

which is a face of Δ and is thus contractible. Therefore, by Theorem 6.3.4, f induces a homotopy equivalence between $\mathscr{O}(\mathscr{F}_{>j}(\Delta))$ and $\mathscr{O}(P)$. Since $\mathscr{O}(P)$ is the barycentric subdivision of $N_{j+1}(\Delta)$, Lemma 6.2.1 says that $||\mathscr{O}(P)|| \cong ||N_{j+1}(\Delta)||$, and therefore, $\mathscr{O}(\mathscr{F}_{>j}(\Delta)) = [\Delta]_{>j}$ is homotopy equivalent to $N_{j+1}(\Delta)$. □

Notice when $j = 0$, we recover the classical Nerve Theorem.

We may now prove part (1) of our main theorem as a corollary.

Corollary 6.3.4. For a simplicial complex Δ, $\widetilde{H}_i(N_j(\Delta)) = 0$ for $i + j > d$ and $1 \leq j \leq d$.

Proof. By Proposition 6.3.3, we get

$$\widetilde{H}_i(N_j(\Delta)) = \widetilde{H}_i([\Delta]_{>j-1}).$$

But $[\Delta]_{>j-1}$ has dimension at most $d - j$ and the result follows. □

6.4 Lemmata

In this section, we introduce several lemmata that will be integral to proving our main theorem. We refer to Section 8.2 for notation.

Lemma 6.4.1. Let T be a face of Δ and $|T| = k > 0$. Then, $\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)) \cong [\mathrm{lk}_{\Delta}(T)]_{>0}$ as simplicial complexes. In particular, $\widetilde{H}_i(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T))) \cong \widetilde{H}_i(\mathrm{lk}_{\Delta}(T))$ for every i.

Proof. First note that if T is a facet then $\mathrm{lk}_{\Delta}(T) = \{\varnothing\} = [\mathrm{lk}_{\Delta}(T)]_{>0}$. But, since T is a facet, $\{\rho(T)\}$ must be a facet of $[\Delta]_{>k-1}$, since this is a chain of maximal length containing $\rho(T)$. Thus $\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)) = \{\varnothing\} = [\mathrm{lk}_{\Delta}(T)]_{>0}$, and thus we have the result if T is a facet.

Now suppose $T \in \Delta$ is not a facet and define the function $f : V([\mathrm{lk}(T)]_{>0}) \to V(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)))$ by $f(\rho(\tau)) = \rho(\tau \cup T)$. One can check that f is a simplicial isomorphism.

In particular, f induces a homeomorphism between the geometric realizations of $[\mathrm{lk}_{\Delta}(T)]_{>0}$ and $\mathrm{lk}_{[\Delta]_{k-1}}(\rho(T))$, and the result follows from Lemma 6.2.1. $\square$

Lemma 6.4.2. Suppose b is a non-isolated vertex of Δ. Then there is a Mayer-Vietoris exact sequence of the form

$$\cdots \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\mathrm{lk}_{\Delta}(b)) \to \widetilde{H}_{i-1}(\mathrm{ast}_{\Delta}(b)) \to \widetilde{H}_{i-1}(\Delta) \to \cdots$$

Proof. Notice that $\mathrm{st}_{\Delta}(b) \cup \mathrm{ast}_{\Delta}(b) = \Delta$ and $\mathrm{st}(b) \cap \mathrm{ast}_{\Delta}(b) = \mathrm{lk}_{\Delta}(b)$. Since b is non-isolated, $\mathrm{lk}_{\Delta}(b)$ is nonempty. Thus we have a Mayer-Vietoris exact sequence in reduced homology:

$$\cdots \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\mathrm{lk}_{\Delta}(b)) \to \widetilde{H}_{i-1}(\mathrm{st}_{\Delta}(b)) \oplus \widetilde{H}_{i-1}(\mathrm{ast}_{\Delta}(b)) \to \widetilde{H}_{i-1}(\Delta) \to \cdots$$

Since $\mathrm{st}_{\Delta}(T)$ is a cone, it is acyclic, and the result follows. $\square$

Lemma 6.4.3. Let T be a non-trivial, non-facet face of Δ with $|T| = k$. Let i be such that $\widetilde{H}_i([\Delta]_{>k-1}) = \widetilde{H}_{i-1}([\Delta]_{>k-1}) = 0$. Then

$$\widetilde{H}_{i-1}(\mathrm{lk}_{\Delta}(T)) \cong \widetilde{H}_{i-1}(\mathrm{ast}_{[\Delta]_{>k-1}}(\rho(T))).$$

Proof. Since T is not a facet, $\rho(T)$ is not an isolated vertex of $[\Delta]_{>k-1}$. Thus, Lemma 6.4.2 gives an exact sequence

$$\widetilde{H}_i([\Delta]_{>k-1}) \to \widetilde{H}_{i-1}(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T))) \to \widetilde{H}_{i-1}(\mathrm{ast}_{[\Delta]_{>k-1}}(\rho(T))) \to \widetilde{H}_{i-1}([\Delta]_{>k-1})$$

Since $\widetilde{H}_i([\Delta]_{>k-1}) = \widetilde{H}_{i-1}([\Delta]_{>k-1}) = 0$, the middle map in this sequence is an isomorphism, so $\widetilde{H}_{i-1}(\mathrm{ast}_{[\Delta]_{>k-1}}(\rho(T))) \cong \widetilde{H}_{i-1}(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)))$.

By Lemma 6.4.1, we have $\widetilde{H}_{i-1}(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T))) \cong \widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T))$ which gives the result. $\square$

Lemma 6.4.4. Let Δ be a simplicial complex and $J \subsetneq V = V(\Delta)$ such that $\dim(\Delta|_J) = 0$. Assume that $\widetilde{H}_{i-1}(\Delta) = \widetilde{H}_i(\Delta) = 0$. Then

$$\widetilde{H}_{i-1}(\Delta|_{V\setminus J}) \cong \bigoplus_{x\in J} \widetilde{H}_{i-1}(\Delta|_{V\setminus\{x\}}).$$

Proof. We will proceed by induction on $|J|$. When $|J| = 1$, the result is immediate. Suppose the result holds for any J of cardinality k for some $k \geq 1$, and suppose now that $|J| = k+1$. Let $x \in J$ and $J' = J \setminus \{x\}$. Suppose $\sigma \in \Delta$. If $x \in \sigma$, then $\sigma \in \Delta|_{V\setminus J'}$; otherwise if σ contained some $y \in J'$, then $\{x,y\} \in \Delta$, contradicting the fact that $\dim(\Delta|_J) = 0$. If $x \notin \sigma$, then $\sigma \in \Delta|_{V\setminus\{x\}}$. Therefore, $\Delta = \Delta|_{V\setminus J'} \cup \Delta|_{V\setminus\{x\}}$. Note that $\Delta|_{V\setminus J'} \cap \Delta|_{V\setminus\{x\}} = \Delta|_{V\setminus J} \neq \varnothing$.

We have the following Mayer-Vietoris sequence in reduced homology:

$$\cdots \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\Delta|_{V\setminus J}) \to \widetilde{H}_{i-1}(\Delta|_{V\setminus J'}) \oplus \widetilde{H}_{i-1}(\Delta|_{V\setminus\{x\}}) \to \widetilde{H}_{i-1}(\Delta) \to \cdots$$

Because $\widetilde{H}_{i-1}(\Delta) = \widetilde{H}_i(\Delta) = 0$, we have that

$$\widetilde{H}_{i-1}(\Delta|_{V\setminus J}) \cong \widetilde{H}_{i-1}(\Delta|_{V\setminus J'}) \oplus \widetilde{H}_{i-1}(\Delta|_{V\setminus\{x\}}).$$

By induction, $\widetilde{H}_{i-1}(\Delta|_{V\setminus J'}) \cong \bigoplus_{y\in J'} \widetilde{H}_{i-1}(\Delta|_{V\setminus\{y\}})$. Therefore

$$\widetilde{H}_{i-1}(\Delta|_{V\setminus J}) \cong \bigoplus_{x\in J} \widetilde{H}_{i-1}(\Delta|_{V\setminus\{x\}}).$$

$\square$

6.5 Depth and higher nerves

Theorem 6.5.1. For a fixed m, the following are equivalent:

1. $\widetilde{H}_{i-1}(N_{j+1}(\Delta)) = 0$ for all $i, j \geq 0$ such that $i+j < m$.

2. $\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ for all $i, j \geq 0$, $|T| = j$, and $i+j < m$.

Proof. We begin the proof by showing that each condition implies $m \leq s(\Delta)$ and thus we will never need to consider the case when T is a facet. Consider the first condition: if $m > s(\Delta)$, then we may take $j = s(\Delta) - 1, i = 1$. This nerve will have an isolated vertex corresponding to the facet of smallest size. The nerve will not be connected unless that facet is the only facet. However, if this facet is the only facet, then we contradict the first condition for $j = s(\Delta), i = 0$. Now consider the second condition: suppose $m > s(\Delta)$. Then take $j = s(\Delta)$, $i = 0$. Then we have a contradiction when T is a facet.

To prove equivalence, we will induct on j. Thus, let us begin by considering the case $j = 0$. The first set of equations is then $\widetilde{H}_{i-1}(N_1(\Delta)) = 0$ for all $i < m$. Using Theorem 6.3.2, we get that this statement is equivalent to $\widetilde{H}_{i-1}(\Delta) = 0$ for all $i < m$. When $j = 0$, the second set of equations is in fact $\widetilde{H}_{i-1}(\Delta) = 0$ for all $i < m$, since $|T| = 0$ implies T is the empty set. Thus we have equivalence when $j = 0$.

Now, let us take as our induction hypothesis that our theorem holds for $j = k-1$. Consider $j = k < m$. Assuming either set of equations holds, the $j = 0$ case again says that $\widetilde{H}_{i-1}(\Delta) = 0$ for all $i < m$. By Proposition 6.3.3 and the $j = k-1$ case, either set of equations yields $\widetilde{H}_{i-1}([\Delta]_{>k}) = 0$ for all $i < m-(k-1)$. Therefore, we may apply Lemma 6.4.3 for all $i < m-(k-1)-1 = m-k$. Thus, we have

$$\bigoplus_{\substack{T \in \Delta \\ |T| = k}} \widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) \cong \bigoplus_{\substack{T \in \Delta \\ |T| = k}} \widetilde{H}_{i-1}(\mathrm{ast}_{[\Delta]_{>k-1}}(\rho(T))).$$

Applying Lemma 6.4.4, we get:

$$\widetilde{H}_{i-1}([\Delta]_{>k}) \cong \bigoplus_{\substack{T\in\Delta \\ |T|=k}} \widetilde{H}_{i-1}(\operatorname{ast}_{[\Delta]_{>k-1}}(\rho(T))).$$

And by Proposition 6.3.3:

$$\widetilde{H}_{i-1}([\Delta]_{>k}) \cong \widetilde{H}_{i-1}(N_{k+1}(\Delta)).$$

Thus, we have completed the proof by induction. $\square$

Combining Corollary 6.3.1 and Theorem 6.5.1, we obtain the second part of our main theorem, Theorem 6.1.1, restated here:

Theorem 6.5.2. For a simplicial complex Δ, $\operatorname{depth}(k[\Delta]) = \inf\{i+j\colon \widetilde{H}_i(N_j(\Delta)) \neq 0\}$.

Remark 6.5.1. Since depth is a topological property ([Mun84b, Theorem 3.1]), we always have $\operatorname{depth} k[\Delta] = \operatorname{depth} k[\operatorname{sd}\Delta]$ by Lemma 6.2.1. One can apply [Hib91, Proposition 2.8] repeatedly to show that $\operatorname{depth}[\Delta]_{>j} \geq \operatorname{depth} k[\Delta] - j$ for every $j \leq d$. In particular, by Corollary 6.3.1, this implies $\widetilde{H}_i(N_j(\Delta)) = 0$ for $i < \operatorname{depth} k[\Delta] - j$. Therefore, one immediately obtains $\operatorname{depth} k[\Delta] \leq \inf\{i+j\colon \widetilde{H}_i(N_j(\Delta)) \neq 0\}$. However, the converse to [Hib91, Proposition 2.8] does not hold, even with additional hypotheses on vanishing of homology, and therefore, these methods are incapable of establishing the reverse inequality.

6.6 The f-vector and the h-vector

In this section, we prove part 3 of Theorem 6.1.1. We set $\chi(N_j(\Delta))$ to be the Euler characteristic of $N_j(\Delta)$ and $\widetilde{\chi}(N_j(\Delta))$ to be the reduced Euler characteristic of $N_j(\Delta)$. We use $f_i(\Delta)$ to indicate the i^{th} entry in the f-vector of Δ.

Theorem 6.6.1. Let $i \geq 0$,

$$f_i(\Delta) = \sum_{j=i+1}^{d} \binom{j-1}{i} \chi(N_j(\Delta))$$

We note that f_{-1} is always 1.

Proof. Before we proceed, we introduce some additional notation:

Let $f_{h,k}$ be the number of h-faces in $N_k(\Delta)$. We note that for any complex Δ, $f_{h,k}$ is 0 for large enough h and for large enough k.

If a face appears in $N_{k+1}(\Delta)$, then that face also appears in $N_k(\Delta)$. We wish to count the h-faces of $N_k(\Delta)$ which first appear in $N_k(\Delta)$. This number is given by

$$f_{h,k} - f_{h,k+1}.$$

For a collection of facets ρ let $\varphi(\rho) = \cap_{F\in\rho} F$. Note that for a given $\alpha \in \Delta$, the set of ρ such that $\alpha \subseteq \varphi(\rho)$ is a Boolean lattice. Let $y, x_1, \ldots, x_n$ be indeterminates, and let $x_\alpha = \prod_{i\in\alpha} x_i$. Then,

$$\sum_{\rho} \sum_{\alpha\subseteq\varphi(\rho)} (-1)^{|\rho|} x_\alpha y^{|\alpha|} = \sum_{\alpha\in\Delta} x_\alpha y^{|\alpha|} \sum_{\substack{\rho \\ \alpha\subseteq\varphi(\rho)}} (-1)^{|\rho|} = 0.$$

This is because for each α, the set of such ρ is Boolean, and therefore, $\sum_{\substack{\rho \\ \alpha\subseteq\varphi(\rho)}} (-1)^{|\rho|} = 0.$

Now, setting $x_i = 1$ for all i and solving for the $\rho = \emptyset$ term yields:

$$\sum_{\alpha\in\Delta} y^{|\alpha|} \;=\; -\sum_{\rho\neq\emptyset} (-1)^{|\rho|} \sum_{\alpha\subseteq\varphi(\rho)} y^{|\alpha|} \;=\; \sum_{\rho\neq\emptyset} (-1)^{|\rho|-1} \sum_{j=0}^{|\varphi(\rho)|} \binom{|\varphi(\rho)|}{j} y^j.$$

Taking the $(i+1)^{st}$ coefficient of each side yields:

$$
\begin{aligned}
f_i(\Delta) &= \sum_{\substack{\rho\neq\emptyset\\ |\varphi(\rho)|\geq i+1}} (-1)^{|\rho|-1}\binom{|\varphi(\rho)|}{i+1}\\
&= \sum_{h=0}^{\infty}\sum_{k=i+1}^{\infty}(-1)^h\binom{k}{i+1}\#\{\rho \mid |\rho|-1=h,\ \rho\in N_k(\Delta)\backslash N_{k+1}(\Delta)\}\\
&= \sum_{h=0}^{\infty}(-1)^h\sum_{k=i+1}^{\infty}\binom{k}{i+i}(f_{h,k}-f_{h,k+1})\\
&= \sum_{h=0}^{\infty}(-1)^h\sum_{k=i+1}^{\infty}(f_{h,k}-f_{h,k+1})\sum_{j=i+1}^{k}\binom{j-1}{i}\\
&= \sum_{h=0}^{\infty}(-1)^h\sum_{j=i+1}^{d}\sum_{k=j}^{\infty}\binom{j-1}{i}(f_{h,k}-f_{h,k+1})\\
&= \sum_{j=i+1}^{d}\binom{j-1}{i}\sum_{h=0}^{\infty}(-1)^h f_{h,j}\\
&= \sum_{j=i+1}^{d}\binom{j-1}{i}\chi(N_j(\Delta)).
\end{aligned}
$$

□

For the convenience of the reader, we have worked out the corresponding formula for the h-vector $(h_0=1,h_1,\ldots,h_d)$ of Δ.

Corollary 6.6.1. For $k\geq 1$ we have:

$$h_k(\Delta)=(-1)^{k-1}\sum_{j\geq 1}\binom{d-j}{k-1}\widetilde{\chi}(N_j(\Delta)).$$

We also record the following:

Corollary 6.6.2. If Δ_1 and Δ_2 are simplicial complexes with $\widetilde{H}_{i-1}(N_j(\Delta_1))\cong\widetilde{H}_{i-1}(N_j(\Delta_2))$ for all i,j, then Δ_1 and Δ_2 have identical f-vectors and h-vectors.

6.7 LCM-lattice and regularity of monomial ideals

In this section, we use our main theorem, Theorem 6.1.1, to deduce a formula for the Castelnuovo-Mumford regularity of any monomial ideal I, denoted by $\operatorname{reg}(I)$. We first fix some notation motivated by [GPW99]. Suppose $f_1,\dots,f_r$ are the minimal monomial generators of I.

Definition 6.7.1. We define the j-th LCM complex of I to be:

$$L_j(I) := \{F \subseteq [r] \colon |\operatorname{lcm}_{i\in F}(f_i)| \leq j\}.$$

Theorem 6.7.1. Let I be a monomial ideal. Then:

$$\operatorname{reg}(I) = \sup\{j-i \colon \widetilde{H}_i(L_j(I)) \neq 0\}.$$

Proof. Let $I^{pol} = (g_1,\dots,g_r)$ be the polarization of I. Then it is well-known that $\operatorname{reg}(I) = \operatorname{reg}(I^{pol})$ (see for instance [Pee11, Theorem 21.10]). From the construction of the g_i's from the f_i's, it is obvious that for any subset $F \subseteq [r]$, $\operatorname{lcm}_{i\in F}(f_i)$ and $\operatorname{lcm}_{i\in F}(g_i)$ have the same size. Thus, the problem reduces to the case when I is a square-free monomial ideal.

Now let $I^\vee$ be the Alexander dual of I. It is the Stanley-Reisner ideal of some complex Δ. We have that

$$\operatorname{reg}(I) = \operatorname{pd} S/I^\vee = n - \operatorname{depth} S/I^\vee$$

by the Eagon-Reiner theorem ([MS05, Theorem 5.59]) and the Auslander-Buchsbaum formula. We now note that each g_i is precisely the product of variables in the complement of the corresponding facet F_i of Δ. Thus $L_j(I) = N_{n-j}(\Delta)$. Putting all of these together, we have:

$$\operatorname{reg}(I) = n - \inf\{i+j \colon \widetilde{H}_i(L_{n-j}(I)) \neq 0\} = \sup\{j-i \colon \widetilde{H}_i(L_j(I)) \neq 0\}$$

as desired. □

Remark 6.7.2. Our formula above should be compared with Theorem 2.1 in [GPW99].

Chapter 7

Rank selection and depth conditions for balanced simplicial complexes

7.1 Introduction

Let k be a field, $A = k[x_1,\dots,x_n]$, and I a square-free monomial ideal in A. The Stanley-Reisner correspondence associates to $R := A/I$ a simplicial complex Δ whose topological and combinatorial properties capture the algebraic structure of R. Exploiting this correspondence has been an active line of investigation over the past few decades. Due to their combinatorial characterization ([Rei76, Theorem 1]), Stanley-Reisner rings that are Cohen-Macaulay have received particular attention. However, the Cohen-Macaulay property is quite strong in this setting, and so there has been a focus in recent years on considering weaker algebraic properties such as Serre's condition (S_ℓ) or bounds on $\operatorname{depth} R$ which still have interesting combinatorial ramifications. For instance, even (S_2) forces Δ to be pure, and (S_ℓ) implies the h-vector of R is nonnegative up to the ℓth spot [MT09]; see [PSFTY14] for a survey of related results. The main purpose of this chapter is to consider Serre's condition and the depth of Stanley-Reisner rings by studying balanced simplicial complexes.

A balanced simplicial complex Δ is a simplicial complex of dimension $d-1$, together with an ordered partition $\pi = (V_1,\dots,V_d)$ of the vertex set of Δ such that $|F \cap V_i| \leq 1$ for every $F \in \Delta$ and every i. To put it another way, the vertices of Δ are colored so that no face of Δ has more than one vertex of a given color. The motivating example of a balanced simplicial complex is the order complex $\mathscr{O}(P)$ of a finite poset P, whose vertex set is P and whose faces consist of all chains in P; we partition the vertices of $\mathscr{O}(P)$ by their height in P. When P is the face poset of a simplicial complex Δ (excluding the empty face), $\mathscr{O}(P)$ is nothing but the barycentric subdivision of Δ, and it's well known that its geometric realization is homeomorphic to that of Δ. Thus we can study topological characteristics of any simplicial complex via the combinatorial structure of a balanced simplicial complex. In particular, we may study homological properties such as the Cohen-Macaulay property and Serre's condition (S_ℓ), and numerical invariants such as depth in this manner.

Let (Δ, π) be a balanced simplicial complex of dimension $d-1$ with ordered partition $\pi = (V_1, \ldots, V_d)$, and let $k[\Delta]$ denote its Stanley-Reisner ring over the field k. If $S \subseteq [d]$, we let Δ_S be the subcomplex of Δ induced on $\bigcup_{i \in S} V_i$, and we refer to Δ_S as the S-rank selected subcomplex of Δ. It's often convenient to think about the ranks we remove rather than those we retain, and so we also set $\widetilde{\Delta}_S := \Delta_{[d]-S}$. If $S = \{i\}$ is a singleton, we abuse notation and write Δ_i or $\widetilde{\Delta}_i$, as appropriate. The so-called rank selection theorems of Stanley ([Sta79]) and Munkres ([Mun84b]) show that homological properties often pass from Δ to Δ_S. Specifically, we have the following:

Theorem 7.1.1 ([Sta79])**.** Let (Δ, π) be a balanced simplicial complex. If $k[\Delta]$ is Cohen-Macaulay, then $k[\Delta_S]$ is Cohen-Macaulay for any $S \subseteq [d]$.

Theorem 7.1.2 ([Mun84b])**.** Let (Δ, π) be a balanced simplicial complex. Then, for any $i \in [d]$, $\operatorname{depth} k[\widetilde{\Delta}_i] \geq \operatorname{depth} k[\Delta] - 1$.

As Serre's condition (S_ℓ) generalizes the Cohen-Macaulay property, it is natural to consider if there is any extension of Theorem 7.1.1 to (S_ℓ). We prove this is indeed the case.

Theorem 7.1.3. Let (Δ, π) be a balanced simplicial complex of dimension $d-1$. If $k[\Delta]$ satisfies Serre's condition (S_ℓ), then $k[\Delta_S]$ satisfies (S_ℓ) for any $S \subseteq [d]$.

If P is a finite poset, we let $P_{>j}$ be the subposet consisting of the elements of P with height greater than j. In the case $\Delta = \mathscr{O}(P)$ for a finite poset P, $\mathscr{O}(P_{>j})$ is the subcomplex of Δ with the bottom $j+1$ ranks removed. For this case, one can nearly characterize (S_ℓ) with the vanishing of reduced homologies of the $\mathscr{O}(P_{>j})$.

Theorem 7.1.4. Let P be a finite poset.

1. If $k[\mathscr{O}(P)]$ satisfies (S_ℓ), then $\widetilde{H}_{i-1}(\mathscr{O}(P_{>j});k) = 0$ whenever $i+j<d$ and $0 \le i < \ell$.

2. If P is the face poset of a simplicial complex Δ and $\widetilde{H}_{i-1}(\mathscr{O}(P_{>j});k) = 0$ whenever $i+j<d$ and $0 \le i \le \ell$, then $k[\mathscr{O}(P)]$, and thus $k[\Delta]$, satisfies (S_ℓ).

It's natural to ask whether one can fully characterize (S_ℓ) in this way i.e., whether the converse of (1) or (2) hold. We provide examples (Examples 7.6.4 and 7.6.5) that show this is not the case.

In general, equality need not hold in Theorem 7.1.2; $\operatorname{depth}\widetilde{\Delta}_i$ can be any value between $\dim\widetilde{\Delta}_i$ and $\operatorname{depth} k[\Delta]-1$. However, we prove that one can often find a rank so that equality is achieved.

Proposition 7.1.1. Let (Δ,π) be a balanced simplicial complex of dimension $d-1$, with ordered partition $\pi = (V_1,\ldots,V_d)$. If $\widetilde{H}_{\operatorname{depth} k[\Delta]-1}(\Delta) = 0$, then there is an $i \in [d]$ such that $\operatorname{depth} k[\widetilde{\Delta}_i] = \operatorname{depth} k[\Delta]-1$.

Using Proposition 7.1.1, we provide a formula for $\operatorname{depth} k[\Delta]$ (see Theorem 7.4.3).

Finally, we provide a formula for sums of reduced Euler characteristics of links. Our formula is analogous to those of [HN02, Section 2 Lemma 1 (i)] and [Swa05, Proposition 2.3].

Theorem 7.1.5. Suppose Δ is pure and let P be the face poset of Δ. Write χ for Euler characteristic and $\widetilde{\chi}$ for reduced Euler characteristic. Then

$$\sum_{\substack{T\in\Delta\\|T|=k}} \widetilde{\chi}(\operatorname{lk}_\Delta(T)) = \chi(\mathscr{O}(P_{>k})) - \chi(\mathscr{O}(P_{>k-1})).$$

We now describe the structure of this chapter. In Section 2, we set notation and provide the algebraic and combinatorial background we appeal to throughout the chapter. In Section 3, we prove Theorems 7.1.3 and 7.1.4 (see Theorems 7.3.2, 7.3.3 and 7.3.4). Section 4 contains a proof of Proposition 7.1.1 (Proposition 7.4.2) as well as a formula for $\operatorname{depth} k[\Delta]$ (Theorem 7.4.3). In Section 5, we prove Theorem 7.1.5 (Theorem 7.5.3) and provide an application to Gorenstein* complexes. The last section discusses open problems related to this work and provides examples indicating the sharpness of our results.

7.2 Background and notation

In this section we set notation and provide needed background for the chapter. Once and for all, fix the base field k. We let $\widetilde{H}_i$ denote ith simplicial or singular homology, whichever is appropriate, always taken with respect to the field k. We use χ for Euler characteristic and $\widetilde{\chi}$ for reduced Euler characteristic.

Given a simplicial complex Δ, we write $k[\Delta]$ for its Stanley-Reisner ring over k. We write $V(\Delta)$ for the vertex set of Δ, but, if Δ is clear from context, we generally write V for $V(\Delta)$ and n for $|V|$; we set $A := k[x_1,\ldots,x_n]$. We write $f_i(\Delta)$ for the number of i-dimensional faces of Δ, and $h_i(\Delta)$ for the ith entry of the h-vector of Δ; so $h_i(\Delta) = \sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}f_{j-1}(\Delta)$. We let $||\Delta||$ denote the geometric realization of Δ. We call $\Delta^{(k)} := \{\sigma \in \Delta : \dim \sigma \leq k\}$ the k-skeleton of Δ.

Given a subset $T \subseteq V(\Delta)$, we use $\Delta|_T := \{\sigma \in \Delta \mid \sigma \subseteq T\}$ for the induced subcomplex of Δ on T. We may then define the star, the anti-star, and the link of T, respectively, as follows:

$$\begin{aligned}
\mathrm{st}_\Delta T &:= \{G \in \Delta \mid T \cup G \in \Delta\} \\
\mathrm{ast}_\Delta T &:= \{G \in \Delta \mid T \cap G = \varnothing\} = \Delta|_{V-T} \\
\mathrm{lk}_\Delta T &:= \{G \in \Delta \mid T \cup G \in \Delta \text{ and } T \cap G = \varnothing\} = \mathrm{st}_\Delta T \cap \mathrm{ast}_\Delta T
\end{aligned}$$

We note that $\mathrm{st}_\Delta T$ and $\mathrm{lk}_\Delta T$ are the void complex $\varnothing$ exactly when $T \notin \Delta$, and $\mathrm{lk}_\Delta(T)$ is the irrelevant complex $\{\varnothing\}$ exactly when T is a facet of Δ. On the other hand, $\mathrm{ast}_\Delta(T)$ is nonempty as long as long as $T \neq V$. Of import, $\mathrm{st}_\Delta(T)$ is a cone over $\mathrm{lk}_\Delta(T)$ for any $T \in \Delta$, in particular is acyclic. When $T = \{v\}$, we abuse notation and write $\mathrm{st}_\Delta(v)$, $\mathrm{ast}_\Delta(v)$, and $\mathrm{lk}_\Delta(v)$.

We say that $J \subseteq V(\Delta)$ is an independent set for Δ if $\{a,b\} \notin \Delta$ for any $a,b \in J$ with $a \neq b$. Motivated by [Hib91], we say that $J \subseteq V(\Delta)$ is an excellent set for Δ if J is an independent set for Δ and $J \cap F \neq \varnothing$ for every facet $F \in \Delta$. When Δ is clear from context, we simply say that J is an independent set or that J is an excellent set, as appropriate.

The main computational tools are two exact sequences seen in the previous chapter; we record them here for the convenience of the reader:

Proposition 7.2.1 (Lemma 6.4.2)**.** Suppose b is a non-isolated vertex of Δ. Then there is a Mayer-Vietoris exact sequence of the form

$$\cdots \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\mathrm{lk}_\Delta(b)) \to \widetilde{H}_{i-1}(\mathrm{ast}_\Delta(b)) \to \widetilde{H}_{i-1}(\Delta) \to \cdots$$

Proposition 7.2.2 (Proof of Lemma 6.4.3)**.** Suppose $\{x\} \subsetneq J \subsetneq V$ and J is an independent set. Set $J' = J - \{x\}$. Then there is a Mayer-Vietoris exact sequence of the form

$$\cdots \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\mathrm{ast}_\Delta(J)) \to \widetilde{H}_{i-1}(\mathrm{ast}_\Delta(J')) \oplus \widetilde{H}_{i-1}(\mathrm{ast}_\Delta(x)) \to \widetilde{H}_{i-1}(\Delta) \to \cdots$$

We also consider algebraic properties of $k[\Delta]$; one can see [BH93] as a reference for this subject. We use $\dim k[\Delta]$ for the Krull dimension of the ring $k[\Delta]$; so $\dim \Delta = \dim k[\Delta] - 1$. We write d for $\dim k[\Delta]$ when Δ is clear from context. By $\operatorname{depth} k[\Delta]$ we mean the depth of the k-algebra $k[\Delta]$; for a combinatorial characterization of depth, see Proposition 7.2.3. We say Δ is Cohen-Macaulay whenever $k[\Delta]$ is Cohen-Macaulay, that is, if $\operatorname{depth} k[\Delta] = \dim k[\Delta]$. Recall the following:

Definition 7.2.1. A commutative Noetherian ring R satisfies *Serre's Condition*, (S_ℓ), if, for all $\mathfrak{p} \in \operatorname{Spec} R$, $\operatorname{depth} R_\mathfrak{p} \geq \min\{\ell, \dim R_\mathfrak{p}\}$.

We say Δ satisfies (S_ℓ) if $k[\Delta]$ does. Every simplicial complex satisfies (S_1), and a simplicial complex satisfies (S_d) if and only if it is Cohen-Macaulay.

The following is an immediate consequence of Hochster's formula ([BH93, Theorem 5.3.8]) and gives a useful characterization of depth for Stanley-Reisner rings in terms of reduced homologies of links:

Proposition 7.2.3. If Δ is a simplicial complex, then $\operatorname{depth} k[\Delta] \geq t$ if and only if $\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ for all $T \in \Delta$ with $i + |T| < t$.

The corresponding result for (S_ℓ) can be found in [Ter07]:

Proposition 7.2.4 ([Ter07])**.** Let Δ be a simplicial complex. Then Δ satisfies (S_ℓ) for $\ell \geq 2$ if and only if $\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ whenever $i + |T| < d$ and $0 \leq i < \ell$. In particular, (S_ℓ) complexes are pure if $\ell \geq 2$.

One can obtain similar characterizations for other algebraic properties of $k[\Delta]$. We define $\operatorname{core} V(\Delta) := \{v \in V(\Delta) \mid \mathrm{st}_\Delta(v) \neq \Delta\}$ and set $\operatorname{core}\Delta := \Delta|_{\operatorname{core} V(\Delta)}$. We say that Δ is Gorenstein if the ring $k[\Delta]$ is Gorenstein; if, in addition, $\operatorname{core}\Delta = \Delta$, we say that Δ is Gorenstein*. One has the following, see [BH93, Theorem 5.6.1]:

Theorem 7.2.2. A simplicial complex Δ is Gorenstein* if and only if

$$\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) \cong \begin{cases} k & \text{if } i = d - |T| \\ 0 & \text{if } i \neq d - |T| \end{cases}$$

Now, let P be a finite poset. If $p \in P$, we let $\mathrm{ht}(p)$ denote the length of a longest chain $p_1 \prec p_2 \prec \cdots \prec p_i = p$ and let $\mathrm{ht}\,P := \max\{\mathrm{ht}\,p \mid p \in P\}$. We denote by $P_{>j}$ the poset obtained by restricting to elements $p \in P$ so that $\mathrm{ht}\,p > j$. The *order complex* of P, denoted $\mathscr{O}(P)$, is the simplicial complex on P consisting of all chains of elements in P. Let $\mathscr{F}(\Delta)$ denote the face poset of Δ. We set $[\Delta]_{>j} := \mathscr{O}(\mathscr{F}(\Delta)_{>j})$. We note that when $j = 0$, $[\Delta]_{>0}$ is the barycentric subdivision of Δ. The following is well known (see [Gib10, Corollary 5.7], for example):

Lemma 7.2.3. The realization $||\Delta||$ is homeomorphic to $||[\Delta]_{>0}||$. In particular, $\widetilde{H}_i(\Delta) \cong \widetilde{H}_i([\Delta]_{>0})$ for all i.

We let $\rho : \Delta - \{\varnothing\} \to V([\Delta]_{>0})$ be the map which sends T to itself viewed as a vertex of $[\Delta]_{>0}$.

There are several advantages of working with $[\Delta]_{>k}$. For instance, the following result of the previous chapter:

Lemma 7.2.4 (Lemma 6.4.1)**.** Let $T \in \Delta$. Then $[\mathrm{lk}_\Delta(T)]_{>0} \cong \mathrm{lk}_{[\Delta]_{>|T|-1}}(\rho(T))$ as simplicial complexes. In particular, $\widetilde{H}_i(\mathrm{lk}_\Delta(T)) \cong \widetilde{H}_i(\mathrm{lk}_{[\Delta]_{>|T|-1}}(\rho(T)))$ for each i.

Definition 7.2.5. A balanced simplicial complex is a pair (Δ, π) satisfying:

1. Δ is $d-1$ dimensional simplicial complex on a vertex set V.

2. $\pi = (V_1, \dots, V_d)$ is an ordered partition of V.

3. For every facet $F \in \Delta$ and every $i \in [d]$, $|F \cap V_i| \leq 1$.

Balanced simplicial complexes were introduced by Stanley in [Sta79]. One can find more information on balanced simplicial complexes in [BFS87, BGS82, Gar80]; [Sta96] gives a more modern treatment of the subject. An important property of balanced simplicial complexes is that each V_i is an independent set for Δ, and, if Δ is pure, the V_i are excellent sets for Δ. If (Δ, π) is a balanced simplicial complex with $\pi = (V_1, \dots, V_d)$, and if $S \subseteq [d]$, we define the *S-rank selected subcomplex* of Δ to be the complex $\Delta_S := \Delta|_{\bigcup_{i \in S} V_i}$; for notational convenience, we also set $\widetilde{\Delta}_S = \Delta_{[d]-S}$. If (Δ, π) is a balanced simplicial complex, we often suppress the ordered partition π and simply refer to Δ as a balanced simplicial complex; in this case we always write $\pi = (V_1, \dots, V_d)$ for the corresponding ordered partition.

Now, let P be a finite poset. If we set $V_i := \{p \mid \mathrm{ht}(p) = i\}$ and $\pi = (V_1, \dots, V_{\mathrm{ht}\,P})$, then $(\mathscr{O}(P), \pi)$ is a balanced simplicial complex. In particular, this means $[\Delta]_{>j}$ is always a balanced simplicial complex for any j.

Finally, we recall the higher nerve complexes of [DDD+19]:

Definition 7.2.6. Let $\{A_1, A_2, \dots, A_r\}$ be the collection of facets of Δ. The simplicial complex

$$N_i(\Delta) := \{F \subseteq [r] : |\bigcap_{j \in F} A_j| \geq i\}$$

is called the *ith Nerve Complex* of Δ. We refer to the $N_i(\Delta)$ as the *higher Nerve Complexes* of Δ. We note that $N_0(\Delta) = 2^{[r]}$ and $N_1(\Delta)$ is the classical nerve complex of Δ.

As we will need the key properties of higher nerve complexes for this chapter, we recall the main result of the previous chapter:

Theorem 7.2.7 (Theorem 6.1.1)**.**

(1) $\widetilde{H}_{i-1}(N_{j+1}(\Delta)) = 0$ for $i+j > d$ and $1 \leq j \leq d$.

(2) $\operatorname{depth} k[\Delta] = \inf\{i+j \colon \widetilde{H}_{i-1}(N_{j+1}(\Delta)) = 0\}$.

(3) For $i \geq 0$,

$$f_i(\Delta) = \sum_{j=i}^{d-1} \binom{j}{i} \chi(N_{j+1}(\Delta)).$$

(4) $\widetilde{H}_i([\Delta]_{>k}) \cong \widetilde{H}_i(N_{k+1}(\Delta))$ for any i and any k.

7.3 Rank selection theorems for Serre's condition

In this section we prove some general statements and use them to derive Theorems 7.1.3 and 7.1.4.

Lemma 7.3.1. Suppose $J \subseteq V$ is excellent and Δ satisfies (S_ℓ). Set $\widetilde{\Delta} := \operatorname{ast}_\Delta(J)$. Then $\widetilde{\Delta}$ satisfies (S_ℓ).

Proof. We proceed by induction on ℓ. The claim is clear when $\ell = 1$, since every simplicial complex satisfies (S_1). So, suppose we know the result for all $1 \leq j \leq \ell$ and suppose Δ satisfies $(S_{\ell+1})$. Inductive hypothesis gives us that $\widetilde{\Delta}$ satisfies (S_ℓ), and we will show $\widetilde{\Delta}$ satisfies $S_{\ell+1}$; the Lemma will then follow from induction.

By Proposition 7.2.4, we have that $\widetilde{H}_{i-1}(\operatorname{lk}_\Delta(T)) = 0$ whenever $i+|T| < d$ and $0 \leq i \leq \ell$, $\widetilde{H}_{i-1}(\operatorname{lk}_{\widetilde{\Delta}}(T)) = 0$ whenever $i+|T| < d-1$ and $0 \leq i < \ell$, and it remains only to show that $\widetilde{H}_{\ell-1}(\operatorname{lk}_{\widetilde{\Delta}}(T)) = 0$ for all $T \in \widetilde{\Delta}$ with $\ell+|T| < d-1$.

Pick $T \in \widetilde{\Delta}$ such that $\ell+|T| < d-1$. Let $\sigma \supseteq T$ be a facet of Δ. Since J is excellent, there is a $b \in J \cap \sigma$, and thus $\{b\} \cup T \in \Delta$. Since $b \notin T$, this means $b \in \operatorname{lk}_\Delta(T)$. Note $T \cup \{b\}$ cannot be a facet of Δ, since this would mean $|T|+1 = d$, whilst $\ell+|T| < d-1$. Set $S = J \cap V(\operatorname{lk}_\Delta(T))$; then we have $\operatorname{lk}_{\widetilde{\Delta}}(T) = \operatorname{ast}_{\operatorname{lk}_\Delta(T)}(S)$. By Proposition 7.2.1, we have, for any $b \in S$, the exact sequence:

$$\widetilde{H}_\ell(\operatorname{ast}_{\operatorname{lk}_\Delta(T)}(b)) \xrightarrow{i_b^*} \widetilde{H}_\ell(\operatorname{lk}_\Delta(T)) \to \widetilde{H}_{\ell-1}(\operatorname{lk}_{\operatorname{lk}_\Delta(T)}(b)) \to \widetilde{H}_{\ell-1}(\operatorname{ast}_{\operatorname{lk}_\Delta(T)}(b)) \to \widetilde{H}_{\ell-1}(\operatorname{lk}_\Delta(T))$$

where i_b^* is the induced map coming from the inclusion $i_b : \operatorname{ast}_{\operatorname{lk}_\Delta(T)}(b) \hookrightarrow \operatorname{lk}_\Delta(T)$. As $\operatorname{lk}_{\operatorname{lk}_\Delta(T)}(b) =$

$\mathrm{lk}_\Delta(T\cup\{b\})$ and since $\ell+|T|<d-1$, we have $\widetilde{H}_{\ell-1}(\mathrm{lk}_{\mathrm{lk}_\Delta(T)}(b))=0$. Since $\widetilde{H}_{\ell-1}(\mathrm{lk}_\Delta(T))=0$, we obtain $\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(b))=0$ and that i_b^* is surjective, from exactness.

Now, since J is an independent set in Δ, S is an independent set in $\mathrm{lk}_\Delta(T)$. We claim that $\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(I))=0$ for any $\varnothing\subsetneq I\subseteq S$. To see this, we induct on $|I|$. Note that the claim is true when $|I|=1$, from above. Now suppose the claim is true for every I with $|I|=k$, and suppose we are given an I with $|I|=k+1$. Write $I=L\cup\{a\}$ so that $|L|=k$. By Proposition 7.2.2 we have the exact sequence

$$\begin{tikzcd}
\widetilde{H}_\ell(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(a))\oplus\widetilde{H}_\ell(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(L)) \arrow[r, "i_a^*-k^*"] & \widetilde{H}_\ell(\mathrm{lk}_\Delta(T)) \arrow[dl] \\
\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(I)) \arrow[r] & \widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(a))\oplus\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(L))
\end{tikzcd}$$

where k^* is the induced map coming from the inclusion $k:\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(L)\hookrightarrow\mathrm{lk}_\Delta(T)$.

By inductive hypothesis, we have that $\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(a))\oplus\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(L))=0$. As we saw previously, i_a^* is surjective so that $i_a^*-k^*$ is as well. Thus we obtain $\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(I))=0$ from exactness. Therefore, induction gives us that $\widetilde{H}_{\ell-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T)}(S))=\widetilde{H}_{\ell-1}(\mathrm{lk}_{\widetilde{\Delta}}(T))=0$, and thus, $\widetilde{\Delta}$ satisfies $(S_{\ell+1})$.

□

Theorems 7.1.3 and 7.1.4 (1) now follow as quick consequences of Lemma 7.3.1:

Theorem 7.3.2. Let Δ be a balanced simplicial complex. If Δ satisfies (S_ℓ), then Δ_S satisfies (S_ℓ) for any $S\subseteq[d]$.

Proof. The claim is clear when $\ell=1$. When $\ell\geq 2$, Δ is pure, and the result follows by applying Lemma 7.3.1 inductively on each $i\in[d]-S$.

□

Theorem 7.3.3. If P is a finite poset satisfying (S_ℓ), then $\widetilde{H}_{i-1}(\mathscr{O}(P_{>j}))=0$ whenever $i+j<d$ and $0\leq i<\ell$. In particular, if Δ is a simplicial complex satisfying (S_ℓ), then $\widetilde{H}_{i-1}([\Delta]_{>j})=0$

whenever $i+j<d$ and $0 \leq i < \ell$.

Proof. Suppose P is (S_ℓ). By Theorem 7.3.2, $\mathscr{O}(P_{>j})$ satisfies (S_ℓ) for each $0 \leq j \leq d-1$. In particular, $\widetilde{H}_{i-1}(\mathscr{O}(P_{>j})) = 0$ for $i < d-j$ and $0 \leq i < \ell$. It only remains to remark that if Δ is a simplicial complex satisfying (S_ℓ), then, since $||\Delta|| \cong ||[\Delta]_{>0}||$ and since (S_ℓ) is a topological property ([Yan11, Theorem 4.4 (d)]), $[\Delta]_{>0}$ satisfies (S_ℓ).

□

Remarkably, Theorem 7.3.3 admits a partial converse (Theorem 7.1.4 (2)) when P is the face poset of a simplicial complex.

Theorem 7.3.4. If $\widetilde{H}_{i-1}([\Delta]_{>j}) = 0$ whenever $i+j<d$ and $0 \leq i \leq \ell$, then Δ satisfies (S_ℓ).

Proof. We follow a similar approach to that of Lemma 7.3.1; we induct on ℓ. The result is clear when $\ell = 1$. Suppose we know the result for ℓ and suppose $\widetilde{H}_{i-1}([\Delta]_{>j}) = 0$ whenever $i+j<d$ and $0 \leq i \leq \ell+1$. From induction hypothesis, we have that Δ satisfies (S_ℓ). Note that we assumed, in particular, that $\widetilde{H}_0([\Delta]_{>j}) = 0$ whenever $j < d-1$. Thus, no facet of Δ can have cardinality less than or equal to $d-1$; that is, Δ is pure. Since Δ is (S_ℓ), we have $\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ whenever $i+|T|<d$ and $0 \leq i < \ell$, and we need only show that $\widetilde{H}_{\ell-1}(\mathrm{lk}_\Delta(T)) = 0$ whenever $|T| < d-\ell$. To see this, we proceed by induction on $|T|$. When $|T| = 0$, we have $\widetilde{H}_{\ell-1}(\mathrm{lk}(T)) = \widetilde{H}_{\ell-1}(\Delta) = \widetilde{H}_{\ell-1}([\Delta]_{>0}) = 0$. Suppose $\widetilde{H}_{\ell-1}(\mathrm{lk}(T)) = 0$ whenever $j = |T| < d-\ell$, and consider $T \in \Delta$ with $j+1 = |T| < d-\ell$.

Letting $S = \{\rho(T) \mid T \in \Delta, |T| = j+1\}$ and writing $S = I \cup \{\rho(T)\}$, we have, by Proposition 7.2.2, the exact sequence

$$\widetilde{H}_{\ell-1}([\Delta]_{>j+1}) \longrightarrow \widetilde{H}_{\ell-1}(\mathrm{ast}_{[\Delta]_{>j}}(\rho(T))) \oplus \widetilde{H}_{\ell-1}(\mathrm{ast}_{[\Delta]_{>j}}(I)) \longrightarrow \widetilde{H}_{\ell-1}([\Delta]_{>j})$$

Since $\widetilde{H}_{\ell-1}([\Delta]_{>j+1}) = 0 = \widetilde{H}_{\ell-1}([\Delta]_{>j})$, we have $\widetilde{H}_{\ell-1}(\mathrm{ast}_{[\Delta]_{>j}}(\rho(T))) \oplus \widetilde{H}_{\ell-1}(\mathrm{ast}_{[\Delta]_{>j}}(I)) = 0$. In particular, $\widetilde{H}_{\ell-1}(\mathrm{ast}_{[\Delta]_{>j}}(\rho(T))) = 0$.

As Δ is pure, T is not a facet, and so $\rho(T)$ is a non-isolated vertex of $[\Delta]_{>j}$. By Proposition 7.2.1, we have the exact sequence

$$\widetilde{H}_{\ell}(\mathrm{ast}_{[\Delta]_{>j}}(\rho(T))) \to \widetilde{H}_{\ell}([\Delta]_{>j}) \to \widetilde{H}_{\ell-1}(\mathrm{lk}_{[\Delta]_{>j}}(\rho(T))) \to \widetilde{H}_{\ell-1}(\mathrm{ast}_{[\Delta]_{>j}}(\rho(T))) \to \widetilde{H}_{\ell-1}([\Delta]_{>j})$$

Since $\widetilde{H}_{\ell-1}(\mathrm{ast}_{[\Delta]_{>j}}(\rho(T))) = 0 = \widetilde{H}_{\ell}([\Delta]_{>j})$, we have $\widetilde{H}_{\ell-1}(\mathrm{lk}_{[\Delta]_{>j}}(\rho(T))) = 0 = \widetilde{H}_{\ell-1}(\mathrm{lk}(T))$, by Proposition 7.2.3. Thus, Δ satisfies $(S_{\ell+1})$, and the result follows from induction. □

Remark 7.3.1. When $\ell = 2$, the conclusion of Theorem 7.3.3 is equivalent to $\widetilde{H}_0([\Delta]_{>d-2}) = 0$, since, for a pure complex, connectivity of $[\Delta]_{>j}$ implies connectivity of $[\Delta]_{>j-1}$.

Remark 7.3.2. Since, by Theorem 7.2.7 (4), $\widetilde{H}_{i-1}([\Delta]_{>j}) \cong \widetilde{H}_{i-1}(N_{j+1}(\Delta))$ for any i and j, Theorems 7.3.3 and 7.3.4 also serve as a version of Theorem 7.2.7 (2) for (S_{ℓ}).

Remark 7.3.3. Theorems 7.3.3 and 7.3.4 show that the reduced homologies of the $[\Delta]_{>j}$ determine one of two values as the largest ℓ such that Δ satisfies (S_{ℓ}). As Examples 7.6.4 and 7.6.5 show, the reduced homologies of the $[\Delta]_{>j}$ alone cannot determine which of these values ℓ actually is. We would be quite interested to know what information can be used in tandem with the $\widetilde{H}_{i-1}([\Delta]_{>j})$ to determine this value; see Question 7.6.1.

7.4 Depth of rank selected subcomplexes

The following lemma follows from [Hib91, Proposition 2.8] and a slightly weaker version can be found in [Mun84b, Theorem 6.4]:

Lemma 7.4.1. Suppose J is an independent set. Set $\widetilde{\Delta} = \mathrm{ast}_{\Delta}(J)$. Then $\operatorname{depth}\widetilde{\Delta} \geq \operatorname{depth}\Delta - 1$.

We first provide a variation on this lemma:

Lemma 7.4.2. Let $\operatorname{depth}\Delta = \ell$ and suppose $\widetilde{H}_{\ell-1}(\Delta) = 0$. Choose $T \in \Delta$ of minimal cardinality such that $\widetilde{H}_{\ell-|T|-1}(\mathrm{lk}_{\Delta}(T)) \neq 0$ (that such a T exists follows from Proposition 7.2.3). Let J be an independent set and suppose $T = T' \cup \{b\}$ with $b \in J$. Set $\widetilde{\Delta} = \mathrm{ast}_{\Delta}(J)$. Then $\widetilde{H}_{\ell-|T'|-2}(\mathrm{lk}_{\widetilde{\Delta}}(T')) \neq 0$. In particular, $\operatorname{depth}\widetilde{\Delta} = \ell - 1$.

Proof. If T is a facet of Δ, then we have that $|T| = \ell$ by minimality, and, as $\mathrm{lk}_\Delta(T) = \mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b)$, that $\{b\}$ is a facet of $\mathrm{lk}_\Delta(T')$. By our minimality hypothesis, $\widetilde{H}_0(\mathrm{lk}_\Delta(T')) = 0$. It follows that $\mathrm{lk}_\Delta(T')$ is a simplex with facet $\{b\}$, and so $\mathrm{lk}_{\widetilde{\Delta}}(T') = \mathrm{ast}_{\mathrm{lk}_\Delta(T')}(b) = \{\varnothing\}$. Thus T' is a facet of $\widetilde{\Delta}$, and so $\widetilde{H}_{\ell-1-|T'|-1}(\mathrm{lk}_{\widetilde{\Delta}}(T')) = \widetilde{H}_{-1}(\mathrm{lk}_{\widetilde{\Delta}}(T')) \neq 0$.

Otherwise, set $S = J \cap V(\mathrm{lk}_\Delta(T'))$ and note that $\mathrm{lk}_{\widetilde{\Delta}}(T') = \mathrm{ast}_{\mathrm{lk}_\Delta(T')}(S)$. Lemma 7.2.1 gives the following exact sequence

$$\widetilde{H}_{\ell-|T|}(\mathrm{lk}_\Delta(T')) \to \widetilde{H}_{\ell-|T|-1}(\mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b)) \to \widetilde{H}_{\ell-|T|-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T')}(b)) \to \widetilde{H}_{\ell-|T|-1}(\mathrm{lk}_\Delta(T'))$$

By minimality of $|T|$ and Proposition 7.2.3, we have $\widetilde{H}_{\ell-|T|}(\mathrm{lk}_\Delta(T')) = \widetilde{H}_{\ell-|T|-1}(\mathrm{lk}_\Delta(T')) = 0$. Thus, $\widetilde{H}_{\ell-|T|-1}(\mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b)) \cong \widetilde{H}_{\ell-|T|-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T')}(b))$. But, $\mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b) = \mathrm{lk}_\Delta(T' \cup \{b\}) = \mathrm{lk}_\Delta(T)$, and so, in particular, $\widetilde{H}_{\ell-|T|-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T')}(b)) \neq 0$.

But now, [DDD+19, Lemma 4.3] gives that

$$\widetilde{H}_{i-|T|-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T')}(S)) \cong \bigoplus_{x \in S} \widetilde{H}_{i-|T|-1}(\mathrm{ast}_{\mathrm{lk}_\Delta(T')}(x)),$$

in particular, that $\mathrm{lk}_{\widetilde{\Delta}}(T')$ is nonzero. That $\operatorname{depth} \widetilde{\Delta} = \ell - 1$ now follows from Lemma 7.4.1 and Proposition 7.2.3.

□

Proposition 7.4.1. Let Δ be a balanced simplicial complex. Suppose $\widetilde{H}_{\ell-1}(\Delta) = 0$. Then there exists an i such that $\operatorname{depth} \mathrm{ast}_\Delta(V_i) = \ell - 1$.

Proof. This follows immediately from Lemma 7.4.2.

□

With these results in hand, we now provide a formula for $\operatorname{depth} \Delta$.

Theorem 7.4.3. If Δ is a balanced simplicial complex, then

$$\operatorname{depth} \Delta = \min\{i + |S| \mid \widetilde{H}_{i-1}(\widetilde{\Delta}_S) \neq 0\}.$$

Proof. That

$$\operatorname{depth}\Delta \leq \min\{i+|S| \mid \widetilde{H}_{i-1}(\widetilde{\Delta}_S) \neq 0\}$$

follows at once from Lemma 7.4.1, so we need only concern ourselves with the reverse inequality. We proceed by induction on $\operatorname{depth}\Delta$, noting that the claim is clear when $\operatorname{depth}\Delta = 0$, that is, when $\Delta = \{\varnothing\}$. Suppose $\operatorname{depth}\Delta = \ell$. The claim is clear if $\widetilde{H}_{\ell-1}(\Delta) \neq 0$, so we may suppose this is not the case. By Proposition 7.4.1, there is an i with $\operatorname{depth}\operatorname{ast}_\Delta(V_i) = \ell - 1$. From inductive hypothesis, we have $\ell - 1 = \min\{i+|S| \mid \widetilde{H}_{i-1}(\operatorname{ast}_\Delta(V_i)_{[d]-S})\}$. In particular, there is an $S \subseteq [d-1]$ with $\widetilde{H}_{\ell-|S|-2}(\operatorname{ast}_\Delta(V_i)) = \widetilde{H}_{\ell-|S\cup\{i\}|-1}(\widetilde{\Delta}_{|S\cup\{i\}|}) \neq 0$, and the result follows.

□

Corollary 7.4.2. Let P be a finite poset. For any $S \subseteq \{1, \ldots, \operatorname{ht} P\}$, let $\widetilde{P}_S$ denote the poset obtained by restricting P to elements whose height is not in S. Then

$$\operatorname{depth}\mathscr{O}(P) = \min\{i+|S| \mid \widetilde{H}_{i-1}(\mathscr{O}(\widetilde{P}_S)) \neq 0\}.$$

In particular, for any simplicial complex Δ, we can compute $\operatorname{depth}\Delta$ by taking P to be the face poset of Δ.

7.5 Euler characteristics of links and truncated posets

We now shift our attention to Theorem 7.1.5. Through this section we set $F_k = \{T \in \Delta, |T| = k\}$.

Lemma 7.5.1. Suppose Δ is pure. Then

$$\sum_{T\in F_k} f_{i-1}(\operatorname{lk}_\Delta(T)) = \binom{i+k}{k} f_{i+k-1}(\Delta).$$

Proof. Note that $\binom{i+k}{k}$ the number of $(k-1)$-dimensional faces contained in each $(i+k-1)$-dimensional face. Thus the right hand side counts each $(i+k-1)$-dimensional face exactly once for each of its subfaces of dimension $k-1$. On the other hand, $f_{i-1}(\operatorname{lk}_\Delta(T))$ counts each $(i+k-1)$-

dimensional face containing T. Thus on the left hand side, each $(i+k-1)$-dimensional face is also counted exactly once for each $(k-1)$-dimensional face it contains, and so the sides are equal.

□

As in [HN02, Section 2 Lemma 1 (i)] and [Swa05, Proposition 2.3], one can combine this with Theorem 1.3 (3) to obtain a formula for $\sum_{\substack{T\in\Delta \\ |T|=k}} h_i(\mathrm{lk}_\Delta(T))$ in terms of Euler characteristics of higher nerves. We follow a similar approach to obtain a particularly simple formula for $\sum_{\substack{T\in\Delta \\ |T|=k}} \widetilde{\chi}(\mathrm{lk}_\Delta(T))$. To do this we need the following identity:

Lemma 7.5.2. If k is a positive integer, then, for any nonnegative integer j, we have

$$\sum_{i=0}^{j}(-1)^{i+1}\binom{i+k}{k}\binom{j}{i+k-1} = \begin{cases} -1 & j=k-1 \\ 1 & j=k \\ 0 & j\neq k,k-1 \end{cases}.$$

Proof. By Pascal's identity, we have that $\sum_{i=0}^{j}(-1)^{i+1}\binom{i+k}{k}\binom{j}{i+k-1}$ equals

$$\sum_{i=0}^{j-k+1}(-1)^{i+1}\binom{i+k-1}{k-1}\binom{j}{i+k-1}+\sum_{i=0}^{j-k+1}(-1)^{i+1}\binom{i+k-1}{k}\binom{j}{i+k-1}$$

Applying the subset of a subset identity to both terms, this equals

$$\sum_{i=0}^{j-k+1}(-1)^{i+1}\binom{j}{k-1}\binom{j-k+1}{i}+\sum_{i=0}^{j-k+1}(-1)^{i+1}\binom{j}{k}\binom{j-k}{i-1}$$

$$=-\binom{j}{k-1}\sum_{i=0}^{j-k+1}(-1)^{i}\binom{j-k+1}{i}-\binom{j}{k}\sum_{i=0}^{j-k+1}(-1)^{i}\binom{j-k}{i-1}$$

$$=-\binom{j}{k-1}\sum_{i=0}^{j-k+1}(-1)^{i}\binom{j-k+1}{i}+\binom{j}{k}\sum_{i=0}^{j-k}(-1)^{i}\binom{j-k}{i}.$$

The first term is 0 unless $j=k-1$ and the second is 0 unless $j=k$. We easily check that the sum is -1 when $j=k-1$ and 1 when $j=k$, giving the result.

$\square$

Theorem 7.5.3. Suppose Δ is pure. Then

$$\sum_{\substack{T\in\Delta\\|T|=k}} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>k}) - \chi([\Delta]_{>k-1})$$

Proof. The claim is clear if $k=0$, since $[\Delta]_{>0}$ is the barycentric subdivision of Δ, and since $[\Delta]_{>-1}$ is a cone. So we suppose $k \geq 1$.

We have

$$\begin{aligned}
\sum_{T\in F_k} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) &= \sum_{i=0}^{d-k}\sum_{T\in F_k}(-1)^{i+1} f_{i-1}(\mathrm{lk}_\Delta(T)) \\
&= \sum_{i=0}^{d-k}(-1)^{i+1}\binom{i+k}{k} f_{i+k-1}(\Delta) \\
&= \sum_{i=0}^{d-k}\sum_{j=i+k-1}^{d-1}(-1)^{i+1}\binom{i+k}{k}\binom{j}{i+k-1}\chi(N_{j+1}(\Delta)) \\
&= \sum_{j=0}^{d-1}\sum_{i=0}^{j}(-1)^{i+1}\binom{i+k}{k}\binom{j}{i+k-1}\chi(N_{j+1}(\Delta)) \\
&= \chi(N_{k+1}(\Delta)) - \chi(N_k(\Delta)) \qquad \text{(By Lemma 7.5.2).}
\end{aligned}$$

The result then follows from Theorem 7.2.7 (4).

$\square$

Note that, as long as $k \neq d$, $\sum_{T\in F_k} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>k}) - \chi([\Delta]_{>k-1}) = \widetilde{\chi}([\Delta]_{>k}) - \widetilde{\chi}([\Delta]_{>k-1})$.

Corollary 7.5.1. Suppose Δ is pure. Then

$$\sum_{k=j}^{i}\sum_{T\in F_k} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>i}) - \chi([\Delta]_{>j-1}).$$

In particular,

$$\sum_{k=0}^{i}\sum_{T\in F_k} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>i}).$$

As an application, we provide a result analogous to those of sections 7.3 and 7.4 for Gorenstein* complexes.

Corollary 7.5.2. Suppose Δ is Gorenstein*. Then

$$\dim_k \widetilde{H}_{i-1}([\Delta]_{>j}) = \begin{cases} \dim_k \widetilde{H}_{j-1}(\Delta^{(j-1)}) & \text{if } i = d-j \\ 0 & \text{if } i \neq d-j. \end{cases}$$

The converse holds if $\mathrm{lk}_\Delta(T)$ is non-acyclic for each $T \in \Delta$.

Proof. By Theorem 7.2.7 (4), $\widetilde{H}_{i-1}([\Delta]_{>j}) \cong \widetilde{H}_{i-1}(N_{j+1}(\Delta))$ for any i and j. Thus, by Theorems 7.2.7 (1) and 7.2.2, both conditions imply Δ is Cohen-Macaulay, in particular, that $\Delta^{(j-1)}$ is Cohen-Macaulay for every j ([Fr0, Theorem 8]). In this case, we have

$$\dim_k \widetilde{H}_{j-1}(\Delta^{(j-1)}) = (-1)^j \widetilde{\chi}(\Delta^{(j-1)}) = \sum_{k=0}^{j} (-1)^{j-k} f_{k-1}(\Delta).$$

Suppose Δ is Gorenstein*. Then, by Theorem 7.2.2

$$\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) \cong \begin{cases} k & \text{if } i = d-j \\ 0 & \text{if } i \neq d-j \end{cases}$$

Likewise, since Δ is Cohen-Macaulay, we have $\widetilde{H}_{i-1}(N_{j+1}(\Delta)) = 0$ unless $i = d-j$ by Theorem 7.2.7. By Corollary 7.5.1 we have

$$\sum_{k=0}^{j} \sum_{T \in F_k} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \sum_{k=0}^{j} \sum_{T \in F_k} (-1)^{d-k-1}$$

$$= \sum_{k=0}^{j} (-1)^{d-k-1} f_{k-1}(\Delta) = (-1)^{d-j-1} \dim_k \widetilde{H}_{d-j-1}([\Delta]_{>j})$$

and the result follows.

Now suppose $\mathrm{lk}_\Delta(T)$ is non-acyclic for each $T \in \Delta$ and that

$$\dim_k \widetilde{H}_{i-1}([\Delta]_{>j}) = \begin{cases} \sum_{k=0}^{j} (-1)^{j-k} f_{k-1}(\Delta) & \text{if } i = d-j \\ 0 & \text{if } i \neq d-j. \end{cases}$$

.

Since Δ is Cohen-Macaulay, $\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ unless $i = d - |T|$. Now we induct on $|T|$ to show that $\widetilde{H}_{d-|T|-1}(\mathrm{lk}_\Delta(T)) \cong k$ for each T. For the case $T = \varnothing$, we have $\dim \widetilde{H}_{d-1}(\mathrm{lk}_\Delta T) = \dim \widetilde{H}_{d-1}(\Delta) = \dim \widetilde{H}_{d-1}([\Delta]_{>0}) = f_{-1}(\Delta) = 1$. Now suppose $\widetilde{H}_{d-|T|-1}(\mathrm{lk}_\Delta(T)) \cong k$ whenever $|T| < j$. Then

$$\sum_{k=0}^{j} \sum_{T \in F_k} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \widetilde{\chi}([\Delta]_{>j}) = (-1)^{d-j-1} \dim_k \widetilde{H}_{d-j-1}(N_{j+1}(\Delta)) = \sum_{k=0}^{j} (-1)^{d-k-1} f_{k-1}(\Delta)$$

Similarly,

$$\sum_{k=0}^{j-1} \sum_{T \in F_k} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \sum_{k=0}^{j-1} (-1)^{d-k-1} f_{k-1}(\Delta),$$

and thus

$$\sum_{T \in F_j} \widetilde{\chi}(\mathrm{lk}_\Delta(T)) = \sum_{T \in F_j} (-1)^{d-j-1} \dim_k \widetilde{H}_{d-j-1}(\mathrm{lk}_\Delta(T)) = (-1)^{d-j-1} f_{j-1}(\Delta).$$

Then

$$\sum_{T \in F_j} \dim_k \widetilde{H}_{d-j-1}(\mathrm{lk}_\Delta(T)) = f_{j-1}(\Delta),$$

but, since $\mathrm{lk}_\Delta(T)$ is non-acyclic for each T, we must have $\dim_k \widetilde{H}_{d-j-1}(\mathrm{lk}_\Delta(T)) = 1$ for each $T \in F_j$, by pigeonhole. The result now follows from induction.

$\square$

Remark 7.5.3. We claim the result of Corollary 7.5.2 is analogous to those of Sections 7.3 and 7.4, but this is perhaps not obvious. To see this, note that $\dim_k \widetilde{H}_{i-1}(\Delta^{(j-1)}) = \dim_k \widetilde{H}_{i-1}(P^{-1}_{>d-j})$ where P is the face poset of Δ. In essence, our result says that, when Δ is Gorenstein*, removing

j ranks from the bottom of P gives the same homologies as removing $d-j$ ranks from the top, though these homologies are in different degrees.

7.6 Open problems and examples

We say that $A \subseteq \Delta$ is independent if $\sigma \cup \tau \notin \Delta$ for all $\sigma, \tau \in A$ with $\sigma \neq \tau$. We say that A is excellent if, additionally, for every facet F of Δ, $F \supseteq \sigma$ for some (necessarily unique) $\sigma \in A$. Note that $J = \{v_1, \ldots, v_m\} \subseteq V$ is independent (resp. excellent) if and only if $\{\{v_1\}, \ldots, \{v_m\}\}$ is an independent (resp. excellent) subset of Δ. If $A \subseteq \Delta$ is independent, we set

$$\Delta_A := \Delta - \{\sigma \in \Delta \mid \sigma \supseteq \tau \text{ for some } \tau \in A\}.$$

If $A = \{\{v_1\}, \ldots, \{v_m\}\}$ where $J = \{v_1, \ldots, v_m\} \subseteq V$ is independent, then $\Delta_A = \operatorname{ast}_\Delta(J)$. Essentially the same argument as [Hib91, Proposition 2.8] shows the following extension of Lemma 7.4.2:

Proposition 7.6.1. Suppose $A \subseteq \Delta$ is independent. Then $\operatorname{depth} \Delta_A \geq \operatorname{depth} \Delta - 1$.

We conjecture a similar extension of Lemma 7.3.1.

Conjecture 7.6.2. Suppose $A \subseteq \Delta$ is excellent. If Δ satisfies (S_ℓ), then Δ_A satisfies (S_ℓ).

Remark 7.6.3. If A is independent and $\ell \geq 2$, the conclusion can only hold if A is excellent, since (S_2) complexes are pure. Similar to Proposition 7.6.1, one can modify the argument of [Hib91, Proposition 2.8] to show that Δ_A satisfies $(S_{\ell-1})$ whenever Δ satisfies (S_ℓ) and A is excellent. However, as in the proof of Theorem 7.3.3, one often needs to cut away excellent subsets inductively, and, for this purpose, $(S_{\ell-1})$ is not generally good enough; in particular, we cannot conclude anything when Δ only satisfies (S_2). A positive answer to this conjecture would allow one to extend Theorem 7.1.3 to balanced complexes of a more general type, along the lines of [Hib91, Section 3].

The following examples show the converses of Theorems 7.3.3 and 7.3.4 do not hold, even for face posets of simplicial complexes:

Example 7.6.4. Consider the complex Δ_1 with facets:

$$\{4,5,6\},\{1,5,6\},\{1,3,5\},\{2,3,6\},\{2,5,6\},\{2,4,6\}.$$

This complex is not (S_2) but has $\widetilde{H}_{i-1}([\Delta_1]_{>j}) = 0$ for all i, j with $i + j < d$ and $0 \leq i < 2$.

Example 7.6.5. Consider the complex Δ_2 with facets:

$$\{4,5,6\},\{3,5,6\},\{2,3,5\},\{2,3,4\},\{1,3,4\},\{2,4,6\}.$$

This complex is (S_2) but $\widetilde{H}_1([\Delta_2]_{>0})$ is non-trivial.

In fact, $\widetilde{H}_{i-1}([\Delta_1]_{>j}) \cong \widetilde{H}_{i-1}([\Delta_2]_{>j})$ for every i and every j. Since Δ_2 is (S_2) and Δ_1 is not, this shows that (S_2) cannot be determined in general by reduced homologies of the $[\Delta]_{>j}$. Further, Example 7.6.5 is Buchsbaum while Example 7.6.4 is not, so Buchsbaum cannot be determined either. In a similar fashion, the following example shows that Gorenstein cannot be detected in general.

Example 7.6.6. Let Γ_1 be the complex with facets

$$\{2,3,4\},\{1,3,4\},\{1,2,5\},\{2,3,5\},\{1,2,4\},\{1,3,5\}$$

and Γ_2 the complex with facets

$$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\}.$$

Then $[\Gamma_1]_{>j}$ and $[\Gamma_2]_{>j}$ have isomorphic homologies for each j, but Γ_1 is Gorenstein whilst Γ_2 is not (it is not even 2-Cohen-Macaulay).

The above discussion leads us to ask the following general question:

Question 7.6.1. In addition to the reduced homologies of the $[\Delta]_{>j}$, what information does one need to determine if a simplicial complex satisfies conditions such as (S_ℓ), Buchsbaum, or Gorenstein?

Chapter 8

Minimal Cohen-Macaulay simplicial complexes

8.1 Introduction

In this chapter, we introduce and study the notion of a minimal Cohen-Macaulay complex. Fix a field k. Let Δ be a simplicial complex. We say Δ is minimal Cohen-Macaulay (over k) if it is Cohen-Macaulay and removing any facet from the facet list of Δ results in a complex which is not Cohen-Macaulay. See Section 8.2 for precise definitions.

For the rest of the chapter we shall write CM for Cohen-Macaulay. We first observe a crucial fact.

Theorem 8.1.1. Any CM complex is shelled over a minimal CM complex. (Theorem 8.3.1)

Thus, in a strong sense, understanding CM complexes amounts to understanding the minimal ones. We support this claim by demonstrating that many interesting examples of CM complexes in combinatorics are minimal. Theorem 8.1.1 also puts shellable complexes in a broader context: they are precisely complexes shelled over the empty one. Its proof relies on a simple but somewhat surprising statement (Lemma 8.3.1), which might be of independent interest.

Below is a collection of the main technical results of this chapter which establish various necessary and sufficient conditions for a complex to be minimal CM.

Theorem 8.1.2. The following statements hold.

1. A minimal CM complex is acyclic. (Corollary 8.3.3)

2. Let Δ be CM and i-fold acyclic. If no facet of Δ contains more than $i-1$ boundary ridges, then Δ is minimal CM. (Theorem 8.3.3)

3. If Δ is a ball, then Δ is minimal CM if and only if it is strongly non-shellable in the sense of [Zie98]. (Proposition 8.3.7)

4. If Δ is minimal CM and Γ is CM, then $\Delta \star \Gamma$ is minimal CM. (Theorem 8.4.2)

In Section 8.2, we give the formal definitions, provide background, and set notation. Section 8.3 contains the proofs of Theorem 8.3.1, Theorem 8.3.3, Corollary 8.3.3 and Proposition 8.3.7. In Section 8.4, we provide many ways to build new minimal Cohen Macaulay from old ones, such as gluing (Corollary 8.4.1, Proposition 8.4.2) and taking joins (Theorem 8.4.2). In the last section, we use our results to examine many classical and recent examples of Cohen-Macaulay complexes from the literature and show that they are minimal.

8.2 Background and notation

Once and for all, fix the base field k. We let $\widetilde{H}_i$ denote the ith simplicial or singular homology, as appropriate, always with coefficients in k. We use $\widetilde{\chi}$ for reduced Euler characteristic. Throughout this chapter, we let Δ be a simplicial complex of dimension $d-1$ with facet list $\{F_1, \dots, F_e\}$, and we denote by Δ_{F_i} the subcomplex of Δ with facet list $\{F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_e\}$. We write $f_i(\Delta)$ for the number of i-dimensional faces of Δ, and $h_i(\Delta)$ for the ith entry of the h-vector of Δ; so $h_i(\Delta) = \sum_{k=0}^{i} \binom{d-k}{i-k} (-1)^{i-k} f_{k-1}(\Delta)$. In particular, we note that $h_d(\Delta) = \sum_{k=0}^{d} (-1)^{d-k} f_{k-1}(\Delta) = (-1)^{d-1} \widetilde{\chi}(\Delta)$. The depth of Δ is, by definition, the depth of the Stanley-Reisner ring $k[\Delta]$ of Δ. We say Δ is CM if $\operatorname{depth} \Delta = d$. The following consequence of Hochster's formula ([BH93, Theorem 5.3.8]) is an extension of Reisner's famous criterion for Cohen-Macaulayness ([Rei76, Theorem]) and gives a combinatorial characterization of depth.

Proposition 8.2.1. $\operatorname{depth} \Delta \geq \ell$ if and only if $\widetilde{H}_{i-1}(\operatorname{lk}_\Delta(\sigma)) = 0$ for all $\sigma \in \Delta$ such that $i + |\sigma| < \ell$.

We use $\Delta^{(i)} := \{\sigma \in \Delta \colon |\sigma| \leq i+1\}$ to denote the i-skeleton of Δ; it is well known that $\operatorname{depth} \Delta = \max\{i \mid \Delta^{(i-1)} \text{ is CM}\}$, see for example [Fr0, Theorem 8].

The following definition gives the main focus of this chapter .

Definition 8.2.1. We say Δ is **minimal CM** if Δ is CM but Δ_{F_i} is not CM for any i. In particular, we note that a simplex is minimal CM if and only if it is the void complex $\varnothing$.

The following related concept provides an extension of the notion of shellability.

Definition 8.2.2. We say Δ_F to Δ is a **shelling move** if $\langle F\rangle \cap \Delta_F$ is generated by facets of ∂F, that is to say, if $\langle F\rangle \cap \Delta_F$ is pure of codimension 1. If Γ is a subcomplex of Δ generated by facets of Δ, we say Δ is **shelled over** Γ if there exists a sequence of shelling moves taking Γ to Δ. To put it another way, Δ is shelled over Γ if the relative complex (Δ, Γ) is shellable (see [Sta96, Chapter III. 7] or [AS16, Section 4]).

We note that shellable complexes are exactly those which are shelled over $\varnothing$.

Definition 8.2.3. The *intersection face* of a facet $F \in \Delta$ is the face of Δ defined as

$$ß(F) := \{x \in F : F - \{x\} \notin \Delta_F\}.$$

If Δ_F to Δ is a shelling move, then $ß(F)$ is the intersection of the facets of $\langle F\rangle \cap \Delta_F$. Further, in this context, $ß(F)$ is the complement of the unique minimal face of $\langle F\rangle \setminus \Delta_F$, which is commonly called the of the restriction face of F; see [BW96].

Definition 8.2.4. We say that Δ is l-fold acyclic if $\mathrm{lk}_\Delta(\sigma)$ is acyclic for all $\sigma \in \Delta$ with $|\sigma| < l$.

Definition 8.2.5. We say a ridge $\sigma \in \Delta$ is a boundary ridge if σ is contained in a unique facet of Δ.

Definition 8.2.6. A *pseudomanifold* is a simplicial complex that is pure, has a connected facet-ridge graph, and every ridge is in exactly two facets.

Definition 8.2.7 (See [MT09])**.** We say Δ satisfies Serre's condition (S_ℓ) if $\ell = 1$ or if $\ell \geq 2$ and, for any $\sigma \in \Delta$, $\widetilde{H}_{i-1}(\mathrm{lk}_\Delta(\sigma)) = 0$ whenever $i + |\sigma| < d$ and $0 \leq i < \ell$. In particular, we note that any complex satisfying (S_2) is pure.

8.3 Main results

In this section we prove most of our main technical results regarding minimal CM complexes. We start with the following lemma.

Lemma 8.3.1. Suppose Δ satisfies Serre's condition (S_2). Then, for any facet $F \in \Delta$, Δ is shelled over Δ_F.

Proof. The claim is clear if Δ is a simplex, so we suppose this is not the case. Noting that Δ is pure (since it satisfies (S_2)), we want to show that $\langle F \rangle \cap \Delta_F$ is pure of dimension $d-2$. Let σ be a facet of $\langle F \rangle \cap \Delta_F$. Then $F \setminus \sigma$ is a facet of $\mathrm{lk}_\Delta(\sigma)$, and any other facet of $\mathrm{lk}_\Delta(\sigma)$ can be written in the form $G \setminus \sigma$ where $G \in \Delta_F$. If $v \in F \setminus \sigma \cap G \setminus \sigma$, then $\sigma \cup \{v\} \in \langle F \rangle \cap \Delta_F$, contradicting that σ is a facet. Thus $F \setminus \sigma \cap G \setminus \sigma = \varnothing$. Thus $F \setminus \sigma$ has trivial intersection with every other facet of $\mathrm{lk}_\Delta(\sigma)$, and so $\mathrm{lk}_\Delta(\sigma)$ is disconnected. Since Δ satisfies (S_2), this must mean that σ is $(d-2)$-dimensional. Ergo, $\langle F \rangle \cap \Delta_F$ is pure of dimension $d-2$, and so Δ is shelled over Δ_F.

□

Theorem 8.3.1. If Δ is a CM complex, then there is a minimal CM complex Γ so that Δ is shelled over Γ.

Proof. Since Δ is CM, it also satisfies (S_2). We can then apply Lemma 8.3.1 to conclude that, for every facet F of Δ, Δ is shelled over Δ_F. If none of these is CM, then Δ is minimal CM by definition. If not, we may continue this process to eventually reach a minimal one. □

Remark 8.3.2. It is not hard to see that a given CM complex can be shelled over two different minimal ones. For instance, let $\Delta = K_6^{(2)}$ be the complete two-skeleton of the simplex on 6 vertices, and let Γ be a triangulation of the projective plane on 6 vertices. Then Δ is shellable and is also shelled over Γ. That Γ is minimal CM follows from Corollary 8.3.5.

Next, we aim to prove that minimal CM complexes are acyclic. This is accomplished by showing a more general result.

Theorem 8.3.2. Suppose $\widetilde{H}_{d-1}(\Delta) \neq 0$. Then there is a facet of maximal cardinality F_i of Δ so that the following hold:

$$\dim \widetilde{H}_{i-1}(\Delta_{F_i}) = \begin{cases} \dim \widetilde{H}_{i-1}(\Delta) & \text{if } 0 \leq i < d \\ \dim \widetilde{H}_{i-1}(\Delta) - 1 & \text{if } i = d \end{cases} \tag{1}$$

$$f_{k-1}(\Delta_{F_i}) = \begin{cases} f_{k-1}(\Delta) & \text{if } 0 \leq i < d \\ f_{k-1}(\Delta) - 1 & \text{if } i = d \end{cases} \tag{2}$$

$$\operatorname{depth} \Delta = \operatorname{depth} \Delta_{F_i}. \tag{3}$$

Proof. Let $C_\bullet : 0 \to C_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$ be the associated chain complex of Δ. Choose a nonzero $\mathfrak{C} \in \widetilde{H}_{d-1}(\Delta) = \operatorname{Ker} \partial_{d-1}$. Write $\mathfrak{C} = \sum_{i=1}^{e} r_i F_i$ where, without loss of generality, $r_1 \neq 0$. Then $\partial_{d-1}(\mathfrak{C}) = \sum_{i=1}^{e} r_i \partial_{d-1}(F_i) = 0$ so $\partial(F_1) = \sum_{i=2}^{e} (-\frac{r_i}{r_1}) \partial_{d-1}(F_i)$. Thus, if we remove F_1, its boundary remains, and so it follows that

$$\Delta^{(d-2)} = \Delta_{F_1}^{(d-2)}. \tag{8.1}$$

So (1) and (2) are now immediate, and it only remains to show (3). Suppose $\sigma \in \Delta_{F_1}$. Following from (8.1), we have $(\operatorname{lk}_\Delta(\sigma))^{(\dim \operatorname{lk}_\Delta(\sigma)-1)} = (\operatorname{lk}_{\Delta_{F_1}}(\sigma))^{(\dim \operatorname{lk}_{\Delta_{F_1}}(\sigma)-1)}$ and so $\operatorname{lk}_{\Delta_{F_1}}(\sigma)$ and $\operatorname{lk}_\Delta(\sigma)$ have the same homologies except potentially in top degree. It follows that $\operatorname{depth} \Delta = \operatorname{depth} \Delta_{F_1}$. □

Corollary 8.3.3. Minimal CM complexes are acyclic.

Proof. This is an immediate consequence of Lemmas 8.3.1 and Theorem 8.3.2. □

While highly acyclic CM complexes need not be minimal CM (a complex with a shelling such that each restriction face is a vertex is $(d-2)$-acyclic but not minimal CM), one can provide some additional conditions under which they are. We accomplish this via the following Lemma:

Lemma 8.3.4. Suppose Δ is a CM complex. If F is a facet of Δ such that $\widetilde{\chi}(\mathrm{lk}_\Delta(\text{ß}(F))) = 0$, then Δ_F is not CM.

Proof. Set $j = |\text{ß}(F)|$. We first note that

$$h_{d-j}(\mathrm{lk}_\Delta(\text{ß}(F)) = \sum_{i=0}^{d-j}(-1)^{d-j-i} f_{i-1}(\mathrm{lk}_\Delta(\text{ß}(F)) = (-1)^{d-j-1}\widetilde{\chi}(\mathrm{lk}_\Delta(\text{ß}(F))) = 0.$$

Since every ridge in $F \setminus r(F)$ in $\mathrm{lk}_\Delta(\text{ß}(F))$ is contained in some other facet of $\mathrm{lk}_\Delta(\text{ß}(F))$,

$$f_{i-1}(\mathrm{lk}_{\Delta_F}(r(F))) = \begin{cases} f_{i-1}(\mathrm{lk}_\Delta(\text{ß}(F))) & \text{if } 0 \leq i < d-j \\ f_{i-1}(\mathrm{lk}_\Delta(\text{ß}(F))) - 1 & \text{if } i = d-j. \end{cases}$$

This implies

$$\begin{aligned} h_{d-j}(\mathrm{lk}_{\Delta_F}(\text{ß}(F))) &= (-1)^{d-j-1}\widetilde{\chi}(\mathrm{lk}_{\Delta_F}(\text{ß}(F))) \\ &= (-1)^{d-j-1}\widetilde{\chi}(\mathrm{lk}_\Delta(\text{ß}(F))) - 1 \\ &= 0 - 1 \\ &= -1. \end{aligned}$$

Thus $\mathrm{lk}_{\Delta_F}(\sigma)$ has a negative entry in its h-vector, and so cannot be CM. In particular, Δ_F is not CM, so Δ is minimal CM.

□

Theorem 8.3.3. If Δ is an ℓ-fold acyclic CM simplicial complex and F is a facet of Δ that contains no more than $\ell - 1$ boundary ridges, then Δ_F is not CM. In particular, if every facet of Δ contains no more than $\ell - 1$ boundary ridges, then Δ is minimal CM.

Proof. Let F be a facet of Δ. Since Δ is ℓ-fold acyclic and since $|\text{ß}(F)| \leq \ell - 1$, $\mathrm{lk}_\Delta(\text{ß}(F))$ is acyclic. The result now follows from Lemma 8.3.4.

□

Setting $\ell = 1$ immediately gives the following special case.

Corollary 8.3.5. If Δ is an acyclic CM with no boundary ridges, then Δ is minimal CM. In particular, acyclic CM psuedomanifolds are minimal CM.

We now consider the relationship between the minimal CM property and the strongly non-shellable property for balls. Strongly non-shellablility has been used quite frequently in the study of non-shellable balls (see e.g. [DK78, Hac00, Lut04a, Lut04b, Zie98]). It is defined as follows:

Definition 8.3.4. We say a ball B is strongly non-shellable if B_F is a non-ball for any facet $F \in B$.

Remark 8.3.6. A strongly non-shellable ball is often defined (as in [Zie98]) as a ball B that does not contain a free facet, i.e., a facet F such that $\langle F\rangle \cap \partial B$ is a ball of dimension $d-2$. It's easy to see this definition is equivalent to the one we provide; it follows immediately from [Zie98, Proposition 2.4 (iii)] that any ball with a free facet cannot be strongly non-shellable in our notion. On the other hand, if B_F is a ball, then, as in the proof of Proposition 8.3.7, $\langle F\rangle \cap \partial B$ is generated by the ridges not containing σ, so it must also be a ball.

Proposition 8.3.7. For a ball B, the following are equivalent:

(1) B is minimal CM.

(2) B is strongly non-shellable.

Proof. We first show (1) implies (2). Suppose B is minimal CM. Then removing any facet of B gives a complex that is not CM. This certainly can't be a ball, so B is strongly non-shellable.

We now show that (2) implies (1). Suppose B is strongly non-shellable. Let F be a facet of B. Let $\rho_1, \dots \rho_k$ be the ridges of Δ contained in both $\langle F\rangle$ and ∂B. Let $\rho_{k+1} \dots, \rho_{d+1}$ be the other ridges of Δ contained in $\langle F\rangle$. We consider two complexes, $\mathbf{I} := B_F \cap \langle F\rangle$ and $\mathbf{O} := \partial B \cap \langle F\rangle$. Every ρ_i is either F and another facet, and therefore in **I**, or only in F, and therefore in **O**. This implies that **I** and **O** partition the ρ_i. Furthermore, any facet of **I** that is not a ρ_i is contained in **B**, and vice versa.

Since **O** cannot be a ball by [Zie98, Proposition 2.4 (iii)], it has some facet which is not a ρ_i and more generally has a face which is not contained in $\langle \rho_{k+1}, \ldots, \rho_{d+1} \rangle$. Since ß$(F)$ is the unique minimal face of $\langle F \rangle$ not contained in $\langle \rho_{k+1}, \ldots, \rho_{d+1} \rangle$, ß$(F)$ must be contained in **O**. By the definition of **O**, we get that ß(F) is in the boundary of B. This means that $\mathrm{lk}_B($ß$(F))$ is a ball, and by Lemma 8.3.4, B_F is not CM. This is true for all facets of B, so B is minimal CM.

□

8.4 Building new minimal CM complexes from old ones

In this section we provide several results which show operations such as gluing or taking joins can be used to construct new examples of minimal CM complexes.

We begin with results on gluing, starting with the following corollary of Theorem 8.3.3.

Corollary 8.4.1. Suppose Δ_1 and Δ_2 are CM acyclic complexes. Set $\Gamma = \Delta_1 \cap \Delta_2$. Suppose $\partial\Delta_1, \partial\Delta_2 \subseteq \Gamma$, that $\dim(\partial\Delta_1) = \dim\Gamma = \dim(\partial\Delta_2)$, and that Γ is acyclic and CM. Then $\Delta_1 \cup \Delta_2$ is minimal CM.

Proof. The assumptions and construction ensures that any ridge of $\Delta = \Delta_1 \cup \Delta_2$ is contained in at least two facets. Thus, we can appeal to Corollary 8.3.5.

□

For our next gluing result, we need the following dual notion of minimal CM.

Definition 8.4.1. We say Δ is **strongly CM** if Δ is CM and Δ_{F_i} is CM for any i.

Proposition 8.4.2. If Δ and Γ are minimal CM of dimension $d-1$ and $\Delta \cap \Gamma$ is strongly CM of dimension $d-1$, then $\Delta \cup \Gamma$ is minimal CM.

Proof. We have an exact sequence $0 \to k[\Delta \cup \Gamma] \to k[\Delta] \oplus k[\Gamma] \to k[\Delta \cap \Gamma] \to 0$ and then $k[\Delta \cup \Gamma]$ is CM by the depth lemma. Now let F be a facet of $\Delta \cap \Gamma$. Without loss of generality, assume $F \in \Delta$. If $F \notin \Delta \cap \Gamma$, then we have the exact sequence

$$0 \to k[\Delta_F \cup \Gamma] \to k[\Delta_F] \oplus k[\Gamma] \to k[\Delta \cap \Gamma] \to 0.$$

Since $k[\Delta_F]$ is not CM, neither is $k[\Delta_F \cup \Gamma]$. Otherwise, $F \in \Delta \cap \Gamma$, and we have the exact sequence

$$0 \to k[(\Delta \cup \Gamma)_F] \to k[\Delta_F] \oplus k[\Gamma_F] \to k[(\Delta \cap \Gamma)_F] \to 0.$$

Since $\Delta \cap \Gamma$ is CM and $k[\Delta_F] \oplus k[\Gamma_F]$ is not, $k[(\Delta \cup \Gamma)_F]$ is not CM, completing the proof. $\square$

Remark 8.4.3. The proof of Proposition 8.4.2 does not actually require $\Delta \cap \Gamma$ to be strongly CM, only that $\Delta \cap \Gamma$ have dimension and depth $d-1$, and that $(\Delta \cap \Gamma)_F$ be CM for every facet $F \in \Delta \cap \Gamma$.

We end this section by showing that the join of a minimal CM complex and another (not necessarily minimal) CM complex is minimal CM.

Theorem 8.4.2. If Δ is minimal CM and Γ is CM, then $\Delta \star \Gamma$ is minimal CM.

Proof. First we note that $\Delta \star \Gamma$ is CM. Now, let F be a facet of $\Delta \star \Gamma$. We may write $F = F' \star G$ for some facets F' of Δ and G of Γ. Since $\Delta_{F'}$ is not CM, there exists $\sigma \in \Delta_{F'}$ such that $\widetilde{H}_i(\mathrm{lk}_{\Delta_{F'}}(\sigma)) \neq 0$ for some $i < \dim(\Delta) - |\sigma|$.

We now show that $L = \mathrm{lk}_{\Delta_{F'}}(\sigma)$ and $L' = \mathrm{lk}_{(\Delta\star\Gamma)_F}(\sigma \star G))$ are isomorphic posets. If $\tau \in L$, it is immediate that $\tau \star \emptyset \in L'$. Furthermore, if $\tau \star \emptyset \in L'$, then $\tau \in L$. Suppose $\tau \star \gamma \in L'$, then $\gamma \cup G \in \Gamma$ and $\gamma \cap G = \emptyset$. This implies that $\gamma = \emptyset$. So L and L' are isomorphic posets.

With this isomorphism, we see that $\widetilde{H}_i(\mathrm{lk}_{(\Delta\star\Gamma)_F}(\sigma \star G)) \neq 0$ for some $i < \dim(\Delta \star \Gamma) - |\sigma \star G|$. Thus $(\Delta \star \Gamma)_F$ is not CM by Reisner's Criterion ([Rei76, Theorem 1]), and therefore, $\Delta \star \Gamma$ is minimal CM. $\square$

8.5 Examples

In this section, we consider some notable examples of CM complexes from the literature, and show that they are minimal CM. We expect that this is far from a complete list of minimal CM examples currently published.

A large class of minimal CM complexes are those that satisfy the conditions of Corollary 8.3.5. The following complexes fall in this class:

- Triangulations of $\mathbb{RP}^{2n}$ (over k of characteristic not 2)
- The dunce hat
- Bing's House with 2 rooms [Hac99]
- The pastry [Doo18]

The following are all strongly non-shellable balls, which are minimal CM by Proposition 8.3.7.

- Rudin's Ball [Rud58]
- $B^3_{16,48}$, $B^3_{12,37,a}$, and $B^3_{12,37,b}$ [Lut04b]
- $B_{3,9,18}$ [Lut04a]
- Ziegler's Ball [Zie98]

This next class of minimal CM complexes are constructible complexes which are not themselves balls. Each of these were verified to be minimal CM by applying Theorem 8.3.3.

- The complex C_3 in [DGKM16], a non-partitionable CM complex
- The complex C_3 in [JV18], a balanced non-partitionable CM complex
- The complex Ω_3 in [DG18], a 2-fold acyclic complex with no decomposition into rank 2 boolean intervals

Each of these complexes is a counterexample to an associated conjecture in the literature; see the references for more details. They are each the result of gluing many copies of a CM complex along a CM subcomplex, a similar process to Proposition 8.4.2.

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