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General polygonal line tilings and their matching complexes

Margaret Bayer [a,*], Marija Jelić Milutinović [b], Julianne Vega [c]

[a] Department of Mathematics, University of Kansas, Lawrence, KS, USA
[b] Faculty of Mathematics, University of Belgrade, Belgrade, Serbia
[c] Department of Mathematics, Maret School, Washington, DC, USA

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ABSTRACT

A (general) polygonal line tiling is a graph formed by a string of cycles, each intersecting the previous at an edge, no three intersecting. In 2022, Matsushita proved the matching complex of a certain type of polygonal line tiling with even cycles is homotopy equivalent to a wedge of spheres. In this paper, we extend Matsushita's work to include a larger family of graphs and carry out a closer analysis of lines of triangles and pentagons, where the Fibonacci numbers arise.

1. Introduction

For a finite simple graph G, the matching complex $\mathcal{M}(G)$ is the simplicial complex on the set of edges with faces given by matchings in the graph, where a matching is a set of edges no two of which share a vertex. The topology of matching complexes has been the subject of much research over the years. Chessboard complexes, which are the matching complexes of complete bipartite graphs, have been studied by many authors, including Athanasiadis [2], Björner, et al. [5], Jojić [11], Shareshian and Wachs [17], and Ziegler [19]. See Wachs [18] for a survey. Other matching complexes that have been studied include those for paths and cycles (Kozlov [13]) and trees (Marietti and Testa [14] and Jelić Milutinović et al. [10]). Most relevant to this paper is the study of matching complexes of grid graphs (Braun and Hough [6] and Matsushita [15]), polygonal line tilings (Matsushita [16]), and honeycomb graphs (Jelić Milutinović et al. [10]). We are particularly interested in graphs whose matching complexes are contractible or homotopy equivalent to a wedge of spheres. These are not all graphs; for example, in [5] it is shown that the matching complex of the complete bipartite graph $K_{3,4}$ is a torus.

Other papers take different approaches to the study of matching complexes. Bayer, et al. [3] start with the topology of the matching complex and identify the graphs that produce it. The current authors [4] define the *perfect matching complex*, the subcomplex of the matching complex with facets corresponding to perfect matchings, and study this complex for honeycomb graphs.

Note that other simplicial complexes associated with graphs have been studied from a topological viewpoint. See, in particular, Jonsson's book [12]. We will see that tools developed for the study of independence complexes of graphs by Adamaszek [1] and Engström [8] play an important role in our work.

In this paper we will focus on graphs that are formed from lines of polygons. This expands on the work of Matsushita [16], who studied matching complexes of the graphs formed by lines of $2n$-gons, intersecting at parallel edges. Our main

* Corresponding author.
E-mail addresses: bayer@ku.edu (M. Bayer), marijaj@matf.bg.ac.rs (M. Jelić Milutinović), jvega@maret.org (J. Vega).

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result is that any line of polygons, allowing different size polygons in the line (as long as each has at least four edges), has matching complex that is contractible or homotopy equivalent to a wedge of spheres. We also consider lines of triangles, where we can specify the dimensions and numbers of spheres in the wedge. In the case of pentagonal line tilings we give the explicit homotopy type (involving Fibonacci numbers).

2. Overview

We introduce the definitions and propositions that we use throughout the remainder of the article. Let G be a finite simple graph.

Definition 1. A *matching* of a graph G is a set of edges of G, no two of which share a vertex. The *matching complex* of a graph G is the simplicial complex $\mathcal{M}(G)$ with vertex set E, the set of edges of G, and faces the subsets $\sigma \subseteq E$ that form matchings of G.

Definition 2. An *independent set* of a graph G is a set of vertices of G, no two of which form an edge. The *independence complex* of a graph G is the simplicial complex $\mathcal{I}(G)$ with vertex set V, the set of vertices of G, and faces the subsets $\sigma \subseteq V$ that form independent sets of G.

Definition 3. The *line graph* $L(G)$ of a graph G is the graph with vertex set the set of edges of G and edge set the set of pairs of edges of G that share a vertex.

The following statement follows directly from the definitions.

Proposition 4. *The matching complex of G is the independence complex of $L(G)$.*

It is not true that every independence complex is a matching complex. For example, consider the complex with facets $\{a, b, c\}$ and $\{d\}$. This is the independence complex of $K_{1,3}$, but it is not the matching complex of any graph, since such a graph would have three independent edges and one edge that intersects all three of them.

Proposition 4 enables us to translate theorems about independence complexes to theorems about matching complexes.

Define the (open) edge neighborhood $EN_G(e)$ of an edge e in the graph G to be the set of edges adjacent to e, and the closed edge neighborhood of e to be $EN_G[e] = EN_G(e) \cup \{e\}$. (When the graph G is clear from context we write $EN(e)$ and $EN[e]$, respectively.)

For a simplex σ in a simplicial complex K the link of σ is

$$\mathrm{lk}(\sigma, K) = \{\tau \in K \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in K\}$$

and the (face) deletion of $\sigma \in K$ is

$$\mathrm{del}(\sigma, K) = \{\tau \in K \mid \sigma \not\subseteq \tau\}.$$

For a graph G and edge $e \in E(G)$, denote the corresponding vertex in $\mathcal{M}(G)$ as $\bar{e}$. Then $\mathrm{lk}(\bar{e}, \mathcal{M}(G)) = \mathcal{M}(G \setminus EN_G[e])$. Since for a vertex v of K the sequence $\mathrm{lk}(v, K) \to \mathrm{del}(v, K) \to K$ is a cofiber sequence (see [1, Section 2] for the definition of cofiber sequence and further details), we have the following result.

Proposition 5 *(Adamaszek [1], Proposition 3.1). The sequence*

$$\mathcal{M}(G \setminus EN[e]) \hookrightarrow \mathcal{M}(G \setminus \{e\}) \hookrightarrow \mathcal{M}(G)$$

is a cofiber sequence. If the inclusion $\mathcal{M}(G \setminus EN[e]) \hookrightarrow \mathcal{M}(G \setminus \{e\})$ is null-homotopic, then there is a homotopy equivalence $\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{e\}) \vee \Sigma\mathcal{M}(G \setminus EN[e])$.

Proposition 6 *(Engström [7], Lemma 2.4). Let G be a graph that contains two different edges e and h such that $EN(e) \subset EN(h)$. Then $\mathcal{M}(G)$ collapses to $\mathcal{M}(G \setminus \{h\})$. That is, $\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\})$.*

Proposition 7 *(Adamaszek [1], Theorem 3.3). Let G be a graph that contains two different edges e and h such that $EN[e] \subset EN[h]$. Then*

$$\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h]).$$

An edge e in G is simplicial if $L(G[EN(e)])$ is a complete graph. That is, every pair of edges adjacent to e are themselves adjacent.

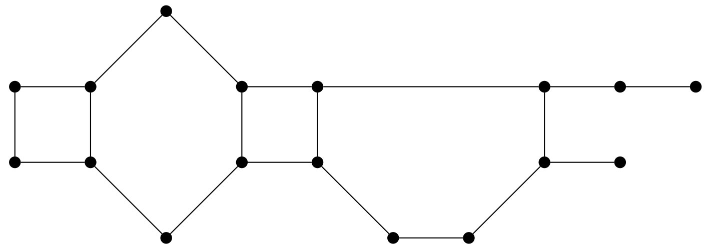

Fig. 1. Example of extended polygonal line tiling in $\mathcal{G}^{2,1}_{4,6,4,6}$.

Proposition 8 *(Engström [7], Lemma 2.5). If e is a simplicial edge in G, then there is a homotopy equivalence*

$$\mathcal{M}(G) \simeq \bigvee_{w \in EN(e)} \Sigma\mathcal{M}(G \setminus EN[w]).$$

Corollary 9. *If G is a graph, and if P is a path of length 3 that intersects G at just one endpoint, then $\mathcal{M}(G \cup P) \simeq \Sigma\mathcal{M}(G)$.*

Proposition 10 *(Engström [9], Lemma 2.2). If G is a graph with a path X of length 4 whose internal vertices are of degree two and whose end vertices are distinct, then $\mathcal{M}(G) \simeq \Sigma\mathcal{M}(G/X)$, where G/X is the contraction of X to a single edge with endpoints given by the endpoints of X.*

The resulting contraction may have parallel edges. The following proposition explains the homotopy type of the matching complex in those situations.

Proposition 11. *Let G be a graph and e an arbitrary edge in G. Consider a graph $G \cup \{x\}$ obtained by adding an edge x parallel to e (x and e have same endpoints). Then:*

$$\mathcal{M}(G \cup \{x\}) \simeq \mathcal{M}(G) \vee \Sigma\mathcal{M}(G \setminus EN_G[e]).$$

Proof. Observe that $EN_{G\cup\{x\}}[e] = EN_{G\cup\{x\}}[x]$, so by Proposition 7 we have

$$\mathcal{M}(G \cup \{x\}) \simeq \mathcal{M}(G) \vee \Sigma\mathcal{M}((G \cup \{x\}) \setminus EN_{G\cup\{x\}}[x]).$$

Then we have $(G \cup \{x\}) \setminus EN_{G\cup\{x\}}[x] = G \setminus EN_G[e]$, and the result follows. □

3. General polygonal line tilings

A (general) polygonal line tiling is a graph formed by a string of cycles, each intersecting the previous at an edge, no three intersecting. To maintain the last property, we assume all the cycles are of length at least four. We want to prove that the matching complex of such a graph is contractible or homotopy equivalent to a wedge of spheres. Our methods will require that we expand the class of graphs slightly by allowing two paths attached to adjacent vertices of the final cycle in the string. Here is the formal definition.

Definition 12. Let n be a positive integer, k and ℓ nonnegative integers, and $s_1, s_2, \ldots, s_n$ be a sequence of integers satisfying $s_j \geq 4$ for all j. Let $\mathcal{G}^{k,\ell}_{s_1,s_2,\ldots,s_n}$ be the set of graphs obtained as follows.

- For each i, $1 \leq i \leq n-1$, C_i is an s_i-cycle containing two disjoint edges $a_{i-1}b_{i-1}$ and c_id_i. C_n is an s_n-cycle containing two disjoint edges $a_{n-1}b_{n-1}$ and a_nb_n.
- T is a length k path on vertices $t_0, t_1, \ldots, t_k$ (if $k \geq 1$) and U is a length ℓ path on vertices $u_0, u_1, \ldots, u_\ell$ (if $\ell \geq 1$).
- The following pairs of vertices are identified: $a_i = c_i$ and $b_i = d_i$ for all i, $1 \leq i \leq n-1$, and $a_n = t_0$, $b_n = u_0$ (if the latter vertices exist).

Any graph in $\mathcal{G}^{k,\ell}_{s_1,s_2,\ldots,s_n}$ is called an *(extended) polygonal line tiling*. See Fig. 1.

Note: This set can contain different graphs with the same s_i, k and ℓ. This is not important for our arguments.

Theorem 13. *If G is any graph in $\mathcal{G}^{k,\ell}_{s_1,s_2,\ldots,s_n}$, then the matching complex of G is contractible or homotopy equivalent to a wedge of spheres.*

Proof. The proof is by induction on n, with an internal induction on k and ℓ.

Base case. Suppose $n = 1$. By symmetry, we can assume $k \geq \ell$. If $k = \ell = 0$, then $\mathcal{G}^{0,0}_{s_1}$ is simply the s_1-cycle, which is known to have matching complex homotopy equivalent to a sphere or a wedge of two spheres (Kozlov, [13]).

If $k = 1$, $0 \leq \ell \leq 1$, let $e = t_0t_1 = a_1t_1$ and let h be the edge of the s_1-cycle, $h = a_1b_1$. Thus $EN[e] \subset EN[h]$, and by Proposition 7,

$$\begin{aligned}\mathcal{M}(G) &\simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h]) \\ &= \mathcal{M}(P_{s_1+\ell+1}) \vee \Sigma\mathcal{M}(P_{s_1-2})\end{aligned}$$

where P_n denotes a path on n vertices.

It is known that the matching complex of a path is contractible or homotopy equivalent to a sphere [13], so in this case $\mathcal{M}(G)$ is contractible or homotopy equivalent to a wedge of spheres.

If $k = 2$, $0 \leq \ell \leq 2$, let $e = t_1t_2$ and let $h = t_0t_1 = a_1t_1$. Thus $EN[e] \subset EN[h]$, and by Proposition 7

$$\begin{aligned}\mathcal{M}(G) &\simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h]) \\ &= \mathcal{M}(H \sqcup P_2)) \vee \Sigma\mathcal{M}(P_{s_1-1+\ell})\end{aligned}$$

where H is the s_1-cycle with an ℓ-path attached to the cycle at an endpoint. Since $\mathcal{M}(P_2)$ is a single vertex, and the matching complex of a disjoint union is the join of the matching complexes of the components, $\mathcal{M}(H \sqcup P_2)$ is contractible. So $\mathcal{M}(G)$ is contractible or homotopy equivalent to a wedge of spheres.

Now, inductively, assume that if both k and ℓ are at most $m \geq 2$, then the matching complex of $G \in \mathcal{G}^{k,\ell}_{s_1}$ is homotopy equivalent to a wedge of spheres. Let $\ell \leq k = m + 1$ and let $G \in \mathcal{G}^{m+1,\ell}_{s_1}$. Let $e = t_mt_{m+1}$ and $h = t_{m-1}t_m$. Thus $EN[e] \subset EN[h]$, and by Proposition 7,

$$\begin{aligned}\mathcal{M}(G) &\simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h]) \\ &= \mathcal{M}(H \sqcup P_2)) \vee \Sigma\mathcal{M}(J)\end{aligned}$$

where H is the disjoint union of a graph in $\mathcal{G}^{m-1,\ell}_{s_1}$ and P_2, and hence has contractible matching complex, and $J \in \mathcal{G}^{m-2,\ell}_{s_1}$. If $\ell \leq m$, then by the induction assumption, $\mathcal{M}(J)$ is homotopy equivalent to a wedge of spheres. If $\ell = m + 1$, we repeat the argument with the roles of k and ℓ reversed, and reduce to the suspension of a matching complex for a graph in $\mathcal{G}^{m-2,m-2}_{s_1}$. So, again, by induction the matching complex of G is homotopy equivalent to a wedge of spheres.

This completes the base case, $n = 1$.

Now we assume the result for extended polygonal line tilings with fewer than n basic cycles, $n \geq 2$, and let $G \in \mathcal{G}^{k,\ell}_{s_1,s_2,\ldots,s_n}$. As in the base case, we consider different values of k and ℓ, assuming $k \geq \ell$.

Assume $k = \ell = 0$; then we need to consider separate cases based on the size of s_n.

Consider $s_n = 4$. Let $e = a_nb_n$ and $h = a_{n-1}b_{n-1}$. Then $EN(e) \subset EN(h)$, by Proposition 6, $\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\})$. The graph $G \setminus \{h\}$ is in the set $\mathcal{G}^{0,0}_{s_1,s_2,\ldots,s_{n-2},s_{n-1}+2}$, so by the induction assumption, the matching complex of G is homotopy equivalent to a wedge of spheres.

Now consider $s_n = 5$. The 5-cycle minus the edge $a_{n-1}b_{n-1}$ forms a path of length four with internal vertices of degree 2. Then, by Proposition 10, $\mathcal{M}(G)$ is homotopy equivalent to the suspension of the matching complex of the (multi)graph H obtained by shrinking the 5-cycle to a 2-cycle (pair of parallel edges). Let e and h be those parallel edges in H. Then $EN[e] = EN[h]$, and by Proposition 10 and Proposition 11 we obtain:

$$\begin{aligned}\mathcal{M}(G) &\simeq \Sigma\mathcal{M}(H) \simeq \Sigma\left(\mathcal{M}(H \setminus \{h\}) \vee \Sigma\mathcal{M}(H \setminus EN[h])\right) \\ &\simeq \Sigma\left(\mathcal{M}(H \setminus \{h\})\right) \vee \Sigma^2\mathcal{M}(H \setminus EN[h]).\end{aligned}$$

Here $H \setminus \{h\} \in \mathcal{G}^{0,0}_{s_1,s_2,\ldots,s_{n-1}}$ and $H \setminus EN[h] \in \mathcal{G}^{k',\ell'}_{s_1,s_2,\ldots,s_{n-2}}$ for some k' and ℓ' with $k' + \ell' = s_{n-1} - 4$. (If $n = 2$, $H \setminus EN[h]$ is a path of length $s_{n-1} - 3$.) By the induction assumption, the matching complex of each is homotopy equivalent to a wedge of spheres, and hence so is the matching complex of G.

We now consider $s_n = 6$. The 6-cycle minus the edge $a_{n-1}b_{n-1}$ contains a path of length four with internal vertices of degree 2. So by Proposition 10 $\mathcal{M}(G)$ is homotopy equivalent to the suspension of the matching complex of the graph H obtained by shrinking the 6-cycle to a 3-cycle. Let $h = a_{n-1}b_{n-1}$ and let e be one of the other edges of the 3-cycle in H. Then $EN[e] \subset EN[h]$, and by Proposition 7, $\mathcal{M}(G) \simeq \Sigma\mathcal{M}(H) \simeq \Sigma\left(\mathcal{M}(H \setminus \{h\})\right) \vee \Sigma^2\mathcal{M}(H \setminus EN[h])$. Here $H \setminus \{h\} \in \mathcal{G}^{0,0}_{s_1,s_2,\ldots,s_{n-1}+1}$ and $H \setminus EN[h] \in \mathcal{G}^{k',\ell'}_{s_1,s_2,\ldots,s_{n-2}}$ for some k' and ℓ' with $k' + \ell' = s_{n-1} - 4$. (Again, if $n = 2$ $H \setminus EN[h]$ is a path of length $s_{n-1} - 3$.) By the induction assumption, the matching complex of each is contractible or homotopy equivalent to a wedge of spheres, and hence so is the matching complex of G.

Finally, consider $s_n \geq 7$. The s_n-cycle minus the edge $a_{n-1}b_{n-1}$ contains a path of length four with internal vertices of degree 2. Then by Proposition 10 $\mathcal{M}(G)$ is homotopy equivalent to the suspension of the matching complex of the graph H obtained by shrinking the s_n-cycle to an $(s_n - 3)$-cycle. This process can be repeated until the cycle shrinks to a cycle of

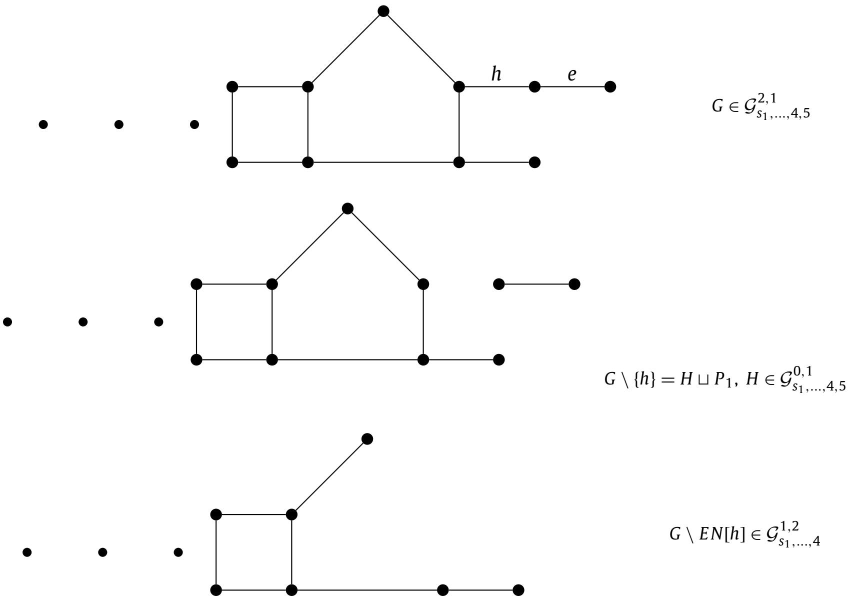

Fig. 2. Example of reduction for $s_n = 5$, $k = 2$.

length at most 6. So by the above cases, the matching complex of G is contractible or homotopy equivalent to a wedge of spheres.

This completes the $k = \ell = 0$ case for n.

Now assume $k = 1$, $0 \le \ell \le 1$. Let $e = t_0t_1 = a_nt_1$ and let h be the edge of the s_n-cycle $h = a_nb_n$. Thus $EN[e] \subset EN[h]$, and by Proposition 7, $\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h])$. Here $G \setminus \{h\} \in \mathcal{G}^{k',\ell'}_{s_1,s_2,\dots,s_{n-1}}$, with $k' + \ell' = s_n - 1 + \ell$, and $G \setminus EN[h] \in \mathcal{G}^{k'',\ell''}_{s_1,s_2,\dots,s_{n-1}}$, with $k'' + \ell'' = s_n - 4$. By the induction assumption, the matching complex of each is contractible or homotopy equivalent to a wedge of spheres, and hence so is the matching complex of G.

Next assume $k = 2$, $0 \le \ell \le 2$. (For an example see Fig. 2.) Let $e = t_1t_2$ and let $h = t_0t_1 = a_nt_1$. Thus $EN[e] \subset EN[h]$, and by Proposition 7

$$\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h]).$$

Here $G \setminus \{h\} = H \sqcup P_2$, where $H \in \mathcal{G}^{0,\ell}_{s_1,s_2,\dots,s_n}$. Since $\mathcal{M}(P_2)$ is a single vertex, and the matching complex of a disjoint union is the join of the matching complexes of the components, $\mathcal{M}(H \sqcup P_2)$ is contractible. Also, $G \setminus EN[h] \in \mathcal{G}^{k',\ell'}_{s_1,s_2,\dots,s_{n-1}}$, with $k' + \ell' = s_n - 3 + \ell$. By the induction assumption, the matching complex of each is contractible or homotopy equivalent to a wedge of spheres, and hence so is the matching complex of G.

Now, inductively, assume that if both k and ℓ are at most $m \ge 2$, then the matching complex of $G \in \mathcal{G}^{k,\ell}_{s_1,s_2,\dots,s_n}$ is contractible or homotopy equivalent to a wedge of spheres. Let $\ell \le k = m + 1$ and let $G \in \mathcal{G}^{m+1,\ell}_{s_1,s_2,\dots,s_n}$ Let $e = t_mt_{m+1}$ and $h = t_{m-1}t_m$. Thus $EN[e] \subset EN[h]$, and by Proposition 7,

$$\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h]).$$

Here $G \setminus \{h\}$ is the disjoint union of a graph in $\mathcal{G}^{m-1,\ell}_{s_1,s_2,\dots,s_n}$ and P_2, and hence has contractible matching complex, and $\Sigma\mathcal{M}(G \setminus EN[h]) \in \mathcal{G}^{m-2,\ell}_{s_1,s_2,\dots,s_n}$. If $\ell \le m$, then by the induction assumption, $\mathcal{M}(G \setminus EN[h])$ is homotopy equivalent to a wedge of spheres. If $\ell = m + 1$, we repeat the argument with the roles of k and ℓ reversed, and reduce to the suspension of a matching complex for a graph $\mathcal{G}^{m-2,m-2}_{s_1,s_2,\dots,s_n}$. So, again, by induction the matching complex of G is contractible or homotopy equivalent to a wedge of spheres.

This completes the induction on n and hence the proof. $\square$

4. Line tilings by triangles

In the last section, we restricted the cycles in the tilings to be of length four or greater, to avoid three cycles intersecting at a point. Now we look at the special case of a line of triangles.

Definition 14. Let t be a positive integer. A *regular triangular line tiling* is a graph $P_{3,t}$ with vertex set V and edge set E as follows:

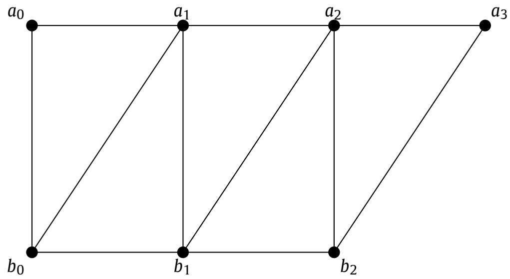

Fig. 3. $P_{3,5}$.

- $V = \{a_i \mid 0 \le i \le \lceil t/2 \rceil\} \cup \{b_i \mid 0 \le i \le \lfloor t/2 \rfloor\}$
- $E = \{a_i a_{i+1} \mid 0 \le i \le \lfloor (t-1)/2 \rfloor\} \cup \{b_i b_{i+1} \mid 0 \le i \le \lfloor (t-2)/2 \rfloor\} \cup \{a_{i+1} b_i \mid 0 \le i \le \lfloor (t-1)/2 \rfloor\} \cup \{a_i b_i \mid 0 \le i \le \lfloor t/2 \rfloor\}$.

We extend this definition to $t = 0$, where it gives a single edge $a_0 b_0$.

See Fig. 3.

Theorem 15. *Let $P_{3,t}$ be a regular triangular line tiling. Then,*

$$\mathcal{M}(P_{3,0}) \simeq *, \quad \mathcal{M}(P_{3,1}) \simeq \mathcal{M}(P_{3,2}) \simeq \bigvee_2 S^0$$

$$\mathcal{M}(P_{3,3}) \simeq S^1, \quad \mathcal{M}(P_{3,4}) \simeq \bigvee_5 S^1$$

and for $t \ge 5$,

$$\mathcal{M}(P_{3,t}) \simeq \Sigma\mathcal{M}(P_{3,t-3}) \vee \Sigma\mathcal{M}(P_{3,t-3}) \vee \Sigma^2\mathcal{M}(P_{3,t-5}).$$

Thus $\mathcal{M}(P_{3,t})$ is contractible or homotopy equivalent to a wedge of spheres for all $t \ge 1$.

Proof. The homotopy types for $t \le 4$ are straightforward. See Appendix. Now assume $t \ge 5$.

Since $EN(a_0b_0) \subset EN(a_1b_1)$ in $P_{3,t}$, Proposition 6 gives us $\mathcal{M}(P_{3,t}) \simeq \mathcal{M}(P_{3,t} \setminus \{a_1b_1\})$.

In $\mathcal{M}(P_{3,t} \setminus \{a_1b_1\})$ we see $EN[a_0b_0] \subset EN[a_1b_0]$. Hence by Proposition 7,

$$\mathcal{M}(P_{3,t} \setminus \{a_1b_1\}) \simeq \mathcal{M}(P_{3,t} \setminus \{a_1b_1, a_1b_0\}) \vee \Sigma\mathcal{M}((P_{3,t} \setminus \{a_1b_1\}) \setminus EN[a_1b_0]).$$

Since $(P_{3,t} \setminus \{a_1b_1\}) \setminus EN[a_1b_0]$ is isomorphic to $P_{3,t-3}$,

$$\mathcal{M}(P_{3,t} \setminus \{a_1b_1\}) \simeq \mathcal{M}(P_{3,t} \setminus \{a_1b_1, a_1b_0\}) \vee \Sigma\mathcal{M}(P_{3,t-3}).$$

We now turn our attention to $\mathcal{M}(P_{3,t} \setminus \{a_1b_1, a_1b_0\})$. In $P_{3,t} \setminus \{a_1b_1, a_1b_0\}$ the vertices a_2, a_1, a_0, b_0, and b_1 form an induced path of length 4, call it X. Contracting path X we obtain a graph isomorphic to $P_{3,t-3}$ with an additional double edge x (with same vertices as edge a_2b_1); call this graph $P'_{3,t-3} = P_{3,t-3} \cup \{x\}$. Then by Proposition 10,

$$\mathcal{M}(P_{3,t} \setminus \{a_1b_1, a_1b_0\}) \simeq \Sigma\mathcal{M}(P'_{3,t-3}).$$

Further, we apply Proposition 11 and obtain

$$\mathcal{M}(P'_{3,t-3}) \simeq \mathcal{M}(P_{3,t-3}) \vee \Sigma\mathcal{M}(P_{3,t-3} \setminus EN_{P_{3,t-3}}[x]).$$

The graph $P_{3,t-3} \setminus EN_{P_{3,t-3}}[x]$ is isomorphic to $P_{3,t-5}$. Together the homotopy equivalences obtained imply

$$\mathcal{M}(P_{3,t}) \simeq \Sigma\mathcal{M}(P_{3,t-3}) \vee \Sigma\mathcal{M}(P_{3,t-3}) \vee \Sigma^2\mathcal{M}(P_{3,t-5}).$$

Hence $\mathcal{M}(P_{3,t})$ is homotopy equivalent to a wedge of spheres, or contractible for all $t \ge 1$. $\square$

Corollary 16. *Let $s(t,d)$ be the number of spheres of dimension d in the wedge that is the homotopy type of $\mathcal{M}(P_{3,t})$. Then for $t \ge 7$ and $d \ge 2$, $s(t,d) = 2s(t-3,d-1) + s(t-5,d-2)$ and*

$$\sum_{t \ge 2, d \ge 0} s(t,d) x^t y^d = \frac{2x^2 + x^3y + 5x^4y + 2x^6y^2}{1 - 2x^3y - x^5y^2}.$$

Proof. Let $q(x,y)=\sum_{\substack{t\ge 2\\ d\ge 0}} s(t,d)x^t y^d$. From Theorem 15, we get $s(t,d)$ for $t\le 6$ or $d\le 2$, and the recursion $\mathcal{M}(P_{3,t})\simeq \bigvee_2 \Sigma\mathcal{M}(P_{3,t-3})\vee \Sigma^2\mathcal{M}(P_{3,t-5})$ gives $s(t,d)=2s(t-3,d-1)+s(t-5,d-2)$ for $t\ge 7$, $d\ge 2$.

Thus

$$\begin{aligned}
q(x,y) &= 2x^2+x^3y+5x^4y+4x^5y+4x^6y^2+\sum_{\substack{t\ge 7\\ d\ge 2}} s(t,d)x^t y^d\\
&= 2x^2+x^3y+5x^4y+4x^5y+4x^6y^2\\
&+\sum_{\substack{t\ge 7\\ d\ge 2}} 2s(t-3,d-1)x^t y^d+\sum_{\substack{t\ge 7\\ d\ge 2}} s(t-5,d-2)x^t y^d\\
&= 2x^2+x^3y+5x^4y+4x^5y+4x^6y^2\\
&+\sum_{\substack{t\ge 4\\ d\ge 1}} 2s(t,d)x^{t+3}y^{d+1}+\sum_{\substack{t\ge 2\\ d\ge 0}} s(t,d)x^{t+5}y^{d+2}\\
&= 2x^2+x^3y+5x^4y+4x^5y+4x^6y^2\\
&+\sum_{\substack{t\ge 2\\ d\ge 0}} 2s(t,d)x^{t+3}y^{d+1}-4x^5y-2x^6y^2+\sum_{\substack{t\ge 2\\ d\ge 0}} s(t,d)x^{t+5}y^{d+2}.
\end{aligned}$$

So

$$q(x,y)(1-2x^3y-x^5y^2)=2x^2+x^3y+5x^4y+2x^6y^2. \quad \square$$

Theorem 17. *For $t\ge 2$, let D_t be the set of dimensions of the spheres occurring in the wedge of spheres that gives the homotopy type of $\mathcal{M}(P_{3,t})$. Let $I_t=\left[\left\lfloor \frac{t}{3}\right\rfloor, \frac{2t-f(t)}{5}\right]$, where*

$$f(t)=\begin{cases} 5 & \text{if } t\equiv 0 \pmod 5\\ 2 & \text{if } t\equiv 1 \pmod 5\\ 4 & \text{if } t\equiv 2 \pmod 5\\ 1 & \text{if } t\equiv 3 \pmod 5\\ 3 & \text{if } t\equiv 4 \pmod 5\end{cases}.$$

Then $D_t=I_t$.

Proof. Part I. $D_t\subseteq I_t$.

The proof is by induction on $t\ge 2$. The statement is true for the base cases, $2\le t\le 6$. So assume $t\ge 7$ and the statement is true for all smaller t. Theorem 15 implies that $D_t=\{r+1 \mid r\in D_{t-3}\}\cup\{r+2\mid r\in D_{t-5}\}$. We consider first the smallest integer in D_t.

$$\begin{aligned}
\min(D_t) &= \min(\min(D_{t-3})+1, \min(D_{t-5})+2)\\
&= \min\left(\left\lfloor\frac{t-3}{3}\right\rfloor+1, \left\lfloor\frac{t-5}{3}\right\rfloor+2\right)\\
&= \min\left(\left\lfloor\frac{t}{3}\right\rfloor, \left\lfloor\frac{t+1}{3}\right\rfloor\right)=\left\lfloor\frac{t}{3}\right\rfloor
\end{aligned}$$

Now for the largest integer in D_t.

$$\begin{aligned}
\max(D_t) &= \max(\max(D_{t-3})+1, \max(D_{t-5})+2)\\
&= \max\left(\frac{2t-1-f(t-3)}{5}, \frac{2t-f(t-5)}{5}\right)
\end{aligned}$$

We calculate this for each congruence class modulo 5.

- $t \equiv t-5 \equiv 0 \pmod 5$, $t-3 \equiv 2 \pmod 5$
  $\max(D_t) = \max\left(\frac{2t-5}{5}, \frac{2t-5}{5}\right) = \frac{2t-5}{5} = \frac{2t-f(t)}{5}$
- $t \equiv t-5 \equiv 1 \pmod 5$, $t-3 \equiv 3 \pmod 5$
  $\max(D_t) = \max\left(\frac{2t-2}{5}, \frac{2t-2}{5}\right) = \frac{2t-2}{5} = \frac{2t-f(t)}{5}$
- $t \equiv t-5 \equiv 2 \pmod 5$, $t-3 \equiv 4 \pmod 5$
  $\max(D_t) = \max\left(\frac{2t-4}{5}, \frac{2t-4}{5}\right) = \frac{2t-4}{5} = \frac{2t-f(t)}{5}$
- $t \equiv t-5 \equiv 3 \pmod 5$, $t-3 \equiv 0 \pmod 5$
  $\max(D_t) = \max\left(\frac{2t-6}{5}, \frac{2t-1}{5}\right) = \frac{2t-1}{5} = \frac{2t-f(t)}{5}$
- $t \equiv t-5 \equiv 4 \pmod 5$, $t-3 \equiv 1 \pmod 5$
  $\max(D_t) = \max\left(\frac{2t-3}{5}, \frac{2t-3}{5}\right) = \frac{2t-3}{5} = \frac{2t-f(t)}{5}$

So $\min(D_t) = \min(I_t)$ and $\max(D_t) = \max(I_t)$.

Part II. $I_t \subseteq D_t$.

Note that in the expansion of the rational function for $q(x, y)$ there is no subtraction. We consider one set of monomials that occur in $q(x, y)$ with positive coefficients, namely those of the form $(2x^2)(2x^3y)^\alpha(x^5y^2)^\beta = 2^{\alpha+1}x^{3\alpha+5\beta+2}y^{\alpha+2\beta}$. Each such monomial cx^ty^d represents c spheres of dimension d in the wedge of spheres for the homotopy type of $\mathcal{M}(P_{3,t})$, that is, an element d of D_t. We are not concerned with the coefficient c, which for these monomials is positive. So we will consider just the exponents. We know that for every nonnegative integers α and β, $\alpha + 2\beta \in D_{3\alpha+5\beta+2}$. Also, for every integer $t \geq 5$, there exist $\alpha \geq 0$ and $\beta \geq 0$ such that $t = 3\alpha + 5\beta + 2$. (In general, the α and β are not uniquely determined.) In what follows we sometimes assume $t \geq 14$; it is easy to check that the statement of the theorem is true for smaller t. (See Appendix.) We show that for fixed $t \geq 14$, all of these elements of D_t fill the interval I_t.

Let $A_t = \{(\alpha, \beta) \mid 3\alpha + 5\beta + 2 = t\}$ and $f(\alpha, \beta) = \alpha + 2\beta$. Note that if $(\alpha, \beta) \in A_t$ and $\alpha \geq 5$, then $(\alpha - 5, \beta + 3) \in A_t$ and $f(\alpha - 5, \beta + 3) = f(\alpha, \beta) + 1$. Using this we will produce an interval of sphere dimensions for fixed t.

Fix $t \geq 14$. The minimum of $\alpha + 2\beta$ for $(\alpha, \beta) \in A_t$ occurs when α is greatest and β is least; values are in the following table.

$t \equiv 0 \pmod 3$	$\alpha = (t-12)/3$, $\beta = 2$	$\alpha + 2\beta = t/3$
$t \equiv 1 \pmod 3$	$\alpha = (t-7)/3$, $\beta = 1$	$\alpha + 2\beta = (t-1)/3$
$t \equiv 2 \pmod 3$	$\alpha = (t-2)/3$, $\beta = 0$	$\alpha + 2\beta = (t-2)/3$

The maximum of $\alpha + 2\beta$ for $(\alpha, \beta) \in A_t$ occurs when α is least and β is greatest; values are in the following table.

$t \equiv 0 \pmod 5$	$\alpha = 1$, $\beta = (t-5)/5$	$\alpha + 2\beta = (2t-5)/5$
$t \equiv 1 \pmod 5$	$\alpha = 3$, $\beta = (t-11)/5$	$\alpha + 2\beta = (2t-7)/5$
$t \equiv 2 \pmod 5$	$\alpha = 0$, $\beta = (t-2)/5$	$\alpha + 2\beta = (2t-4)/5$
$t \equiv 3 \pmod 5$	$\alpha = 2$, $\beta = (t-8)/5$	$\alpha + 2\beta = (2t-6)/5$
$t \equiv 4 \pmod 5$	$\alpha = 4$, $\beta = (t-14)/5$	$\alpha + 2\beta = (2t-8)/5$

Note in all cases a pair in A_t produces the minimum value in the set I_t, but in some cases no pair in A_t produces the maximum value in I_t. However, the maximum value produced by a pair in A_t is at least one less than the maximum value of I_t. Since the set produced by A_t is itself an interval, missing at most one element (the top) of the interval I_t, and we know by Part I that the top element of I_t is in D_t, we conclude that $D_t = I_t$. □

5. Regular pentagonal line tilings

We consider one more particular case, pentagonal line tilings.

Definition 18. Let t be a positive integer. A *regular pentagonal line tiling* is a graph $P_{5,t}$ with vertex set V and edge set E as follows:

- $V = \{a_i \mid 0 \leq i \leq \lceil 3t/2 \rceil\} \cup \{b_i \mid 0 \leq i \leq \lfloor 3t/2 \rfloor\}$
- $E = \{a_ia_{i+1} \mid 0 \leq i \leq \lfloor (3t-1)/2 \rfloor\} \cup \{b_ib_{i+1} \mid 0 \leq i \leq \lfloor (3t-2)/2 \rfloor\} \cup \{a_{3j}b_{3j} \mid 0 \leq j \leq \lfloor t/2 \rfloor\} \cup \{a_{3j+2}b_{3j+1} \mid 0 \leq j \leq \lfloor (t-1)/2 \rfloor\}$

See Fig. 4.

Let (F_n) be the standard Fibonacci sequence, $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

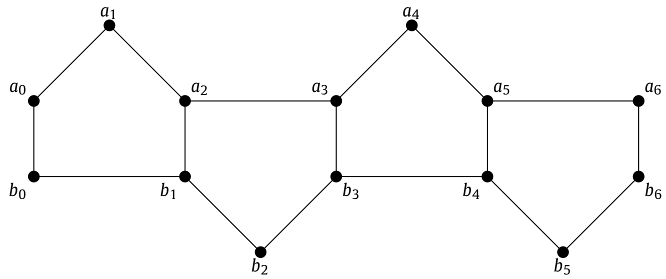

Fig. 4. Graph $P_{5,4}$.

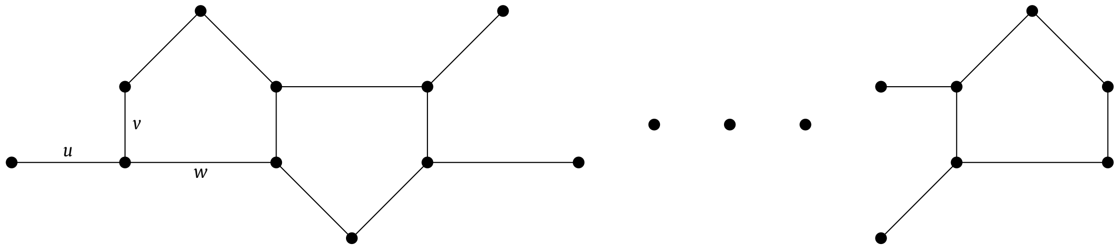

Fig. 5. Graph H_t.

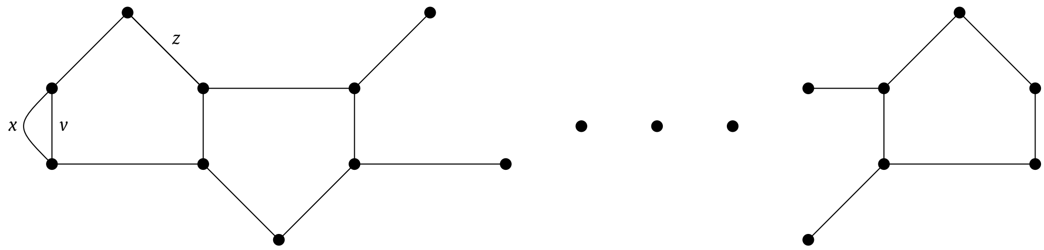

Fig. 6. Graph G'_t.

Theorem 19. *Let $P_{5,t}$ be the pentagonal line tiling with $t \geq 1$. Then* $\mathcal{M}(P_{5,t}) \simeq \bigvee_{F_{t+2}-1} S^t$.

Proof. The proof is by induction, and we work with two sequences of graphs. Let $G_t = P_{5,t}$. Let H_t be the graph obtained by appending one edge to the graph G_t, as shown in Fig. 5. We will use Proposition 10 and Proposition 11 to reduce the matching complex of G_t to the wedge of suspensions of the matching complexes of G_{t-1} and H_{t-2}.

We first find the homotopy type of the matching complex of H_t. It is straightforward to check that $\mathcal{M}(H_1) \simeq \bigvee_2 S^1$. Let u be the pendant edge and v and w its neighboring edges, as shown in Fig. 5. The edge u is a simplicial edge, because its two neighbors are neighbors of each other, so we can apply Proposition 8, and conclude that

$$\mathcal{M}(H_t) \simeq \Sigma\mathcal{M}(H_t \setminus EN[v]) \vee \Sigma\mathcal{M}(H_t \setminus EN[w]).$$

The graph $H_t \setminus EN[v]$ is isomorphic to H_{t-1}. The graph $H_t \setminus EN[w]$ is isomorphic to the graph H_{t-2} with an additional path of length 3 attached to another vertex of the first pentagon. (In the case of $t = 2$, $H_t \setminus EN[w]$ is a path of length 5.) By Corollary 9, this path can be collapsed to give $\mathcal{M}(H_t \setminus EN[w]) \simeq \Sigma\mathcal{M}(H_{t-2})$. Thus, $\mathcal{M}(H_t) \simeq \Sigma\mathcal{M}(H_{t-1}) \vee \Sigma^2\mathcal{M}(H_{t-2})$. By induction we conclude that $\mathcal{M}(H_t) \simeq \bigvee_{F_{t+2}} S^t$.

Now consider $G_t = P_{5,t}$. Let G'_t be the multigraph obtained from G_t by duplicating the first "vertical" edge v in G_t; denote the duplicate edge x. (See Fig. 6.) Proposition 10 gives $\mathcal{M}(G_t) = \Sigma\mathcal{M}(G'_{t-1})$. Since $G'_{t-1} = G_{t-1} \cup \{x\}$, by applying Proposition 11 we obtain:

$$\mathcal{M}(G'_{t-1}) \simeq \mathcal{M}(G_{t-1}) \vee \Sigma\mathcal{M}(G_{t-1} \setminus EN_{G_{t-1}}[v]).$$

Further, graph $G_{t-1} \setminus EN_{G_{t-1}}[v])$ is isomorphic to H_{t-2}, so

$$\mathcal{M}(G_t) \simeq \Sigma\mathcal{M}(G'_{t-1}) \simeq \Sigma\mathcal{M}(G_{t-1}) \vee \Sigma^2\mathcal{M}(H_{t-2}).$$

It is straightforward to check that $\mathcal{M}(G_1) \simeq S^1$ and $\mathcal{M}(G_2) \simeq \bigvee_2 S^2$. By induction we see that $\mathcal{M}(G_t)$ is homotopy equivalent to a wedge of t-spheres:

$$\mathcal{M}(G_t) \simeq \Sigma\mathcal{M}(G_{t-1}) \vee \Sigma^2\mathcal{M}(H_{t-2}) \simeq \Sigma(\bigvee_{F_{t+1}-1} S^{t-1}) \vee \Sigma^2(\bigvee_{F_t} S^{t-2}) \simeq \bigvee_{F_{t+1}-1+F_t} S^t.$$

So $\mathcal{M}(P_{5,t}) \simeq \bigvee_{F_{t+2}-1} S^t$. $\square$

6. Conclusion

Our focus in this paper has been on extending the set of graphs whose matching complexes are known to be contractible or homotopy equivalent to a wedge of spheres. We are interested in larger classes of graphs, particularly of planar graphs, with this property. We believe that the methods in this paper will be useful in this regard.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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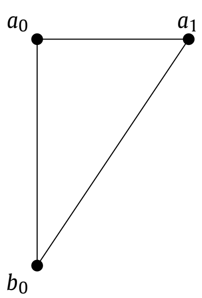
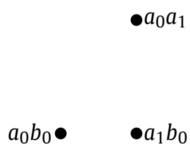
Appendix A. Matching complexes for $P_{3,t}$, $1 \le t \le 4$

$P_{3,1}$: $V = \{a_0, a_1, b_0\}$, $E = \{a_0a_1, a_1b_0, a_0b_0\}$

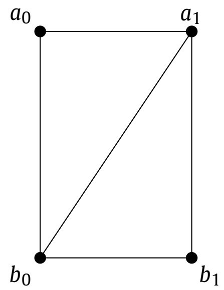


$P_{3,1}$

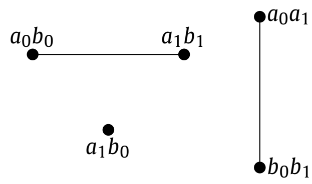


$\mathcal{M}(P_{3,1})$

$P_{3,2}$: $V = \{a_0, a_1, b_0, b_1\}$, $E = \{a_0a_1, b_0b_1, a_1b_0, a_0b_0, a_1b_1\}$

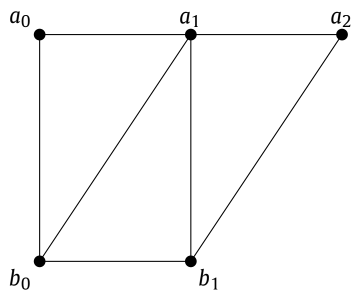


$P_{3,2}$

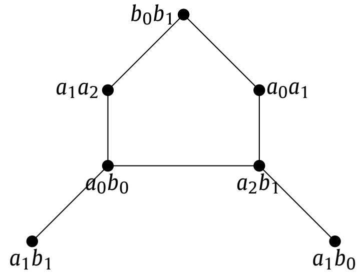


$\mathcal{M}(P_{3,2})$

$P_{3,3}$: $V = \{a_0, a_1, a_2, b_0, b_1\}$, $E = \{a_0a_1, a_1a_2, b_0b_1, a_1b_0, a_2b_1, a_0b_0, a_1b_1\}$

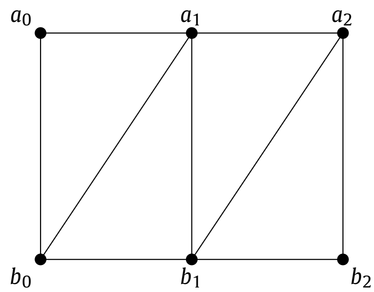


$P_{3,3}$

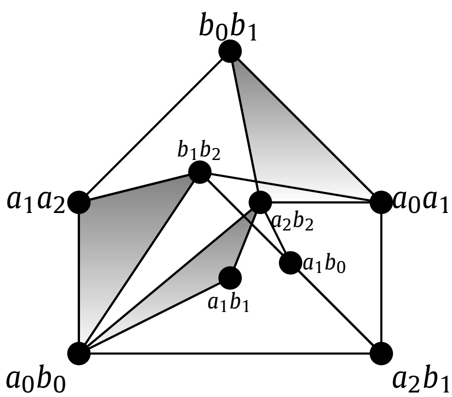


$\mathcal{M}(P_{3,3})$

$P_{3,4}$: $V = \{a_0, a_1, a_2, b_0, b_1, b_2\}$, $E = \{a_0a_1, a_1a_2, b_0b_1, b_1b_2, a_1b_0, a_2b_1, a_0b_0, a_1b_1, a_2b_2\}$,



$P_{3,4}$



$\mathcal{M}(P_{3,4})$

Appendix B. Homotopy types of $\mathcal{M}(P_{3,t})$ for small t

t	homotopy type	t	homotopy type	t	homotopy type
2	$\bigvee_{2} S^0$	6	$\bigvee_{4} S^2$	10	$\bigvee_{28} S^3$
3	S^1	7	$\bigvee_{12} S^2$	11	$\bigvee_{16} S^3 \vee \bigvee_{6} S^4$
4	$\bigvee_{5} S^1$	8	$\bigvee_{8} S^2 \vee S^3$	12	$\bigvee_{38} S^4$
5	$\bigvee_{4} S^1$	9	$\bigvee_{13} S^3$	13	$\bigvee_{64} S^4 \vee S^5$

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