AN INVESTIGATION OF THE THEORY OF THE DISPERSION FORMULA FOR NUCLEAR REACTIONS

by

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Submitted to the Department of Physics and Astronomy and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Master of Science.

June, 1949

403406

1-JL-49

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to Dr. Max Dresden for his constant encouragement and help, and Dr. J. D. Stranathan who made this investigation possible.

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INTRODUCTION

The purpose of this investigation is to gain insight into the theory of nuclear structure and the theory of resonances in scattering processes.

Previous to the discovery of the neutron in 1931 by Chadwick, the nucleus was commonly supposed to be made up of electrons and protons, with possibly alpha particles making up an intermediate building unit (Gamow). Among the difficulties which beset this model for the nucleus were: a) it gave an incorrect value for the spin of H^2 and N^{14}; b) it gave incorrect statistics for these same two nuclei; c) the excess magnetic moment due to the electrons, over that of the protons, is not observed; d) using the observed value of the nuclear radius (about 10^{-12} cm.), there would be required a well depth of approximately 200 Mev, plus binding energy, in order that the nucleus be stable; and e) it is impossible to construct a potential barrier which will hold the electron with the nucleus.

The discovery of the neutron, however, resolved these difficulties; and new theories for the structure of the nucleus appeared in 1932 and 1933, (1,2). The non-relativistic quantum-mechanics could now be applied with some confidence to the description of the nucleus.

Before the finding of the neutron, however, the

theory of alpha-decay was adequately described using non-relativistic quantum-mechanics, by Gamow, and by Condon and Gurney, (3,4). Both treatments are the same, and both assume a square-well of positive energy, connected with a coulomb potential outside the nucleus.

In 1932 the treatment of the alpha-decay problem was generalized by Casimir (5) to include an arbitrary potential function inside the nucleus, instead of the square-well. The significance of the treatment by Casimir, was that it showed, in general, that solutions to the time-dependent Schrodinger equation exist, which exhibit the proper behavior in the course of time, independent of any assumption of a specific potential function.

A review of the developments in the theory of nuclear structure, pertinent to the previously stated purpose, will be presented in this thesis. In Chapt. I will be given a qualitative study of the presently accepted theory of nuclear structure. The theory will be developed from its inception by Bohr to the work of Casimir.

In Chapt. II one of the most significant steps in the development of the theory will be presented, and that is the derivation of the dispersion formula. There have, at various times, been derived by different methods, many dispersion formulae. It is of interest to present several of these derivations to show the essential differences, similarities, and short-comings of each. The derivations included are those of: Kapur and Peierls,

Siegert, Wigner, and Bethe.

Since these many dispersion formulae exist, and are essentially alike, it is of significance to find experimental verification for them. Chapt. III will be devoted to examining the work done toward verifying the formulae experimentally. To this end, two essentially different phenomena are examined: the total cross-section measurements, and measurements involving the angular distribution of the reaction products of the $Li^7(p,He^4)He^4$ reactions are examined.

The impossibility of writing the wave-functions for complicated nuclei, has precluded any success in attempting to calculate, from the dispersion formulae, the various cross-sections.

The summing of the results in the form of conclusions and comments, will be presented in Chapt. IV.

In so far as was possible the notation of the original authors has been used in the formalism. Time-independent wave functions have been used with only one exception. The exception is in the work of Casimir, and no confusion should result from this one case. The notation employed in each section is defined at the beginning of that section.

CHAPTER I

-DEVELOPMENT OF THE THEORY-

(A) Qualitative Description

Early studies in the scattering of alpha-particles and interpretation of line-spectra, established the existence of the positively charged nucleus, (6). These early studies resulted in the old quantum theory and the Rutherford scattering law. The Rutherford scattering law involved only classical concepts. This law was later to give much valuable information concerning the nucleus, when the scattering deviated from that predicted, (7). The existence of strong, short-range forces is inferred from these deviations (anamolous scattering), as well as a concept of radius of the nucleus.

When the strong, short-range forces were established, much speculation and effort was put forth, to give the nucleus a "structure". From these efforts and subsequent experiments came the realization of the existence of the proton and the neutron in the nucleus, but the "nuclear structure" remained somewhat nebulous as to its description. Many experiments were devised to gain insight into the formation of the nucleus, but scattering experiments remain one of the primary sources of knowledge concerning the nucleus.

Many models were proposed to explain the great stability of the nucleus, and among these were the elect-

ron-proton model, proton-neutron, and various combinations of these. The great success of quantum-mechanics in explaining the atomic structure, and associated spectral phenomena, led some to try to develop the theory of the nucleus in a manner similar to the atom (as a whole). The method used with the atom usually involved an approximation which reduced the quantum-mechanical problem to an "equivalent" one-body problem (Hartree Method).

It was pointed out by Bohr (8), early in the study of nuclear processes, that the nuclear processes could not in any approximation be treated as a one-body problem. The existence of large nuclear forces, whose range was of the same order of magnitude as the distance between the particles within the nucleus, required the "many-body" approach. In the absence of an exact, many-body theory, this means a statistical theory. In addition to being large, the forces between nucleons exhibit a saturation character, in such a way that the interaction between one particle and the whole nucleus is of the same order of magnitude as the interaction of only a few particles inside the nucleus (~8 Mev). The energy of the interaction may be much larger than the energy of an incident particle, whereas in atomic scattering problems, the energy of interaction may usually be treated as a small perturbation.

Since the interaction between the individual particles is large, it rarely happens that a particle in-

cident upon the nucleus, in such a manner that specifically nuclear forces are involved, suffers an elastic collison. If a particle falls upon the nucleus, the probability of its passing through the nucleus with no change in energy is extremely minute because the average distance between the particles of the nucleus is of the same order of magnitude as the range of the forces. Since the forces are large, any interaction will be a large one.

Consider a particle which is incident upon the nucleus. As it approaches, it will interact strongly with a few particles and exchange part of its energy. Successive interactions with other particles will cause other energy exchanges. Eventually the energy originally possessed by the incident particle will be distributed among all of the particles of the nucleus and the incident particle. Only after a lapse of time will enough energy be concentrated, by accident, upon one particle so that it may escape from the nucleus.

The particle which escapes need not be the same one which entered the nucleus, nor even the same kind. The energy of the escaping particle need not be the same as the entering particle, even if it is the same kind. Only in the case of the same kind of escaping particle, and the state of the nucleus being unchanged, may we speak of an elastic collision.

It is evident that the time spent in the nucleus by the particle may be relatively long. It may be quite

large compared to the time necessary for the particle to traverse the nucleus without interaction. Assuming a neutron with energy 1 Mev, the necessary time to traverse an average sized nucleus ($\sim 2 \times 10^{-12}$cm.) would be $\sim 1.4 \times 10^{-21}$sec. The lifetime of the combined state (average time spent in the nucleus) for a neutron emission is $\sim 5 \times 10^{-14}$sec. Clearly this is a difference of a factor of 10^7 in the orders of magnitude. It is quite evident that this long life-time represents a relatively stable situation within the nucleus. (The factor may be made more evident if the two times are raised by a factor of 10^{21}, the comparison is then 1.4 sec. to 9.6 yrs.)

It is quite clear that any description of a nuclear process, scattering or transmutation, must include this relatively stable situation. This combined, stable state, is given the name "compound nucleus". It is also called "intermediate state", "quasi-stationary state" and a "virtual state".

The description of a nuclear collison will now logically be described as a two step process. The first step being the formation of the compound nucleus and the second being the "radioactive" decay of the compound nucleus. Symbolically the process is described as:

$A + B \rightarrow C^* \rightarrow P + Q$ (*-denotes excited state)

where the particles, A, fall on the nucleus, B, in the target, to form the compound nucleus C^*. C^* subsequently

disintegrates into the residual nucleus P and particles Q. For a given A and B there may be many different P and Q. A and B may or may not be simple particles (protons, neutrons, etc.).

An example of different end products for the same reacting components would be:

$$p + {}_3Li^7 \rightarrow {}_4Be^{8*} \rightarrow {}_2He^4 + {}_2He^4$$

and

$$p + {}_3Li^7 \rightarrow {}_4Be^{8*} \rightarrow {}_4Be^7 + {}_0n^1$$

The energy at which either of these reactions start to take place is dependent upon the Li^7 nucleus, but the relative yield is a function of the energy of the bombarding particle. Along with the two previous reactions might be given an example of the same end-products but different reacting components:

$$ {}_1H^2 + {}_3Li^6 \rightarrow {}_4Be^{8*} \longrightarrow {}_2He^4 + {}_2He^4$$

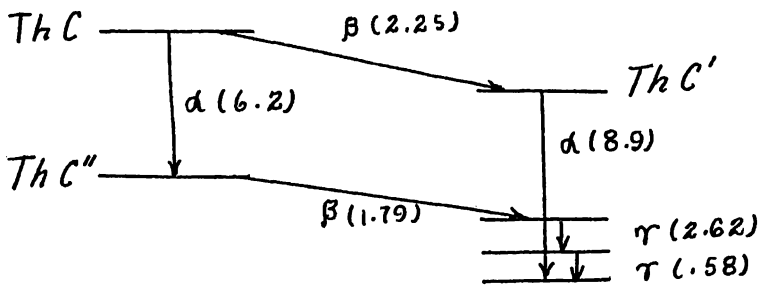
Another type of inelastic collision will take place under very special conditions, that is the simple capture process. In this type of reaction the compound nucleus does not decay, but is stable. A gamma-ray may be emitted and the process is then known as "radiative-capture". In this case the state is no longer "virtual" or "quasi-stationary".

The description of the nucleus and nuclear processes

has so far been very qualitative in character. To make the description more quantitative, the model must be given some characteristics which are amenable to the mathematical methods available. The success of the description of the atomic model again leads to the assumption of a discrete energy spectrum for the nucleus.

The assumption of the discrete spectrum is not merely a convenience, because there is strong evidence from experiments for its existence. The relatively complete monochromaticity of observed alpha and gamma rays in various decay processes was the first evidence of this discrete spectrum and is still one of the principle sources of information about the spectrum. The success of theories for beta-decay, based on the assumption of discrete levels, (and a third particle, the neutrino) lend weight to the assumption of a discrete spectrum. The agreement of the beta-decay theories with experiment cannot, in themselves, be considered too conclusive.

An example of a decay scheme, exhibiting the discrete character is illustrated below:



This decay scheme also serves to illustrate the immense complexity which is both expected and observed. The

complexity may be anticipated from the strong coupling between particles. It would be highly improbable that a single particle would be excited very much without exciting several other particles. If there were no interaction, there would be a large number of energy levels in the nucleus, corresponding to the various distributions of the excitation energy among several particles. In addition there would be levels corresponding to the excitation of a single particle. But the particles do interact and strongly; therefore the levels do not fall into classes according to the excitation of one, two, or more particles. For any arbitrary state, the excitation energy will be for a time concentrated on one particle, then perhaps another, then perhaps shared by two or more, etc. If it were possible to write the eigenfunctions of the nuclear states, it would be possible to calculate the probability of the energy being in any particular distribution.

Because of the large number of possibilities for the distributing of the energy among the particles, there will be a large number of energy levels associated with the nucleus. The levels will also, in general, be degenerate.

As a "rough picture" of the nucleus, one may imagine an aggregate of billiard balls connected by short heavy springs and exhibiting rotational degrees of freedom as well as vibrational. The balls must have the property

of being able to exchange places, interact magnetically, as well as electrically; and when a previously unbound ball approaches close enough, a new spring becomes available to bind it into the group. This "picture" must not be taken too literally, but it serves to illustrate the complexity of the problem facing nuclear physicists.

It is known that, with reasonable accuracy, the potential energy acting on a nuclear particle may be represented by a simple rectangular hole (or square-well), (9). In such a hole the energy levels of the individual particles are almost uniformly distributed from the bottom of the hole to the top (actually the density varies as $E^{\frac{1}{2}}$ where E is measured from the bottom of the hole). The depth of the hole is about 18 Mev. Since the binding energy of the most loosely bound particle is $\sim$8 Mev, the hole is normally filled up to 8 Mev from the top. There is a large number of empty levels between this energy and the top.

Now consider a state of the nucleus which has, for instance, 6 Mev more energy than the ground state. Such a state may be obtained by exciting one particle by 6 Mev and leaving the remainder unexcited. Equally well the state may be obtained by exciting two, three, or more particles. As many as ten excited particles would be possible. It is clear that the variety of ways in which a given excitation energy may be shared among the particles is very large, whereas the configuration whereby

only one particle is excited is only a single possibility. In the actual case there is interaction between the individual particles, which causes a mixing of the eigenfunctions of the various levels, therefore each level is of mixed character. The exact probability of a particular distribution of the energy could be deduced from the eigenfunctions of the nuclear state. The number of energy levels for the nucleus as a whole is seen to be very large and their spacing small, in fact very much smaller than the spacing of the levels for the individual particles would be.

It is very easy to imagine now that if the nucleus were excited by an energy greater than the dissociation energy (~8 Mev) that there exists many distributions of the energy in which no one particle is excited by as much as 8 Mev.

It may be concluded then, that apart from a disintegration (rare event), a nucleus has a series of closely spaced energy levels which have essentially the same character both above and below the dissociation energy. Furthermore, the character of these levels will be changed markedly only when the excitation energy is of the order of a few Mev <u>per particle</u> in the nucleus. For an average nucleus this amounts to excitation energies of the order of 100 Mev, a value seldom realized until recently.

The compound nucleus has provided the most char-

ecteristic feature of nuclear collisions. The compound nucleus is responsible for the phenomenon of resonance. If the energy of the incident particle is such that the total energy of the system is just equal, or nearly equal, to one of the energy levels of the compound nucleus, the probability of the formation of the compound nucleus is much greater than if the energy falls between that of the levels. This may be seen in the calculation of the quantum mechanical current through a potential barrier. The current exhibits sharp minima for those energies near the virtual states, (10).

The study of resonance phenomena is one of the most important in the study of nuclear processes. First of all, from the resonances may be deduced the spacing of the levels. The spacing, as a function of the mass number and the excitation energy will provid a check on the theories of nuclear structure.

It is clear at this point that the compound state, as described above, is not defined by that description, (11). The concept of a level should be made clear, as well as that of a "virtual" level. The usual idea is to carry over the idea of emission probability for photons from the atom to the nucleus.

Consider the excited electronic states of an atom which can emit photons of several different wave lengths by making optical transitions. This can be described quantitatively with the Einstein emission probabilities,

without quantum mechanics. It is considered that the electrons "jump" from one level to another with the simultaneous emission of a photon. It is then said that the level "width" is increased because the level is not a real, stationary level, and that the emitted lines will be "fuzzy". The argument is then justified by an appeal to the indeterminacy relation. In reality the word level has been used in two ways. As an electronic energy state, the interaction with the radiation was neglected and the atomic system pictured as an idealized mechanical unit. This usage of "level" is quite clear since the level lies in the discrete spectrum as it should. A diffuse (or "fuzzy") level on the other hand is not a level of an atomic system alone but of the atomic system plus the radiation field. This kind of level is not well-defined in the description of virtual states.

Weisskopf and Wigner (12), have shown that if an atom is put into an excited state, j, so that it is definitely in that excited state, j, at a time $t = 0$, then at later times there is an exponentially decaying probability of the atom being in that excited state, j, and a growing probability for the lower atomic states i, k, l, ..., plus photons of corresponding energies. It was shown that the band of photons had an intensity distribution of the typical resonance type:

$$\frac{(Const.)}{(E - E_{ij})^2 + (\Gamma_i + \Gamma_k + \Gamma_l)^2} \qquad where: \quad E_{ij} = E_j - E_i$$

with the peak at the emission center E_{ji}. The half width of the line (width at half-maximum intensity) $2(\Gamma_i + \Gamma_k + \Gamma_l)$ was shown to be the sum of the half-widths due to the separate emissions, ji, jk, jl, with the emission probabilities $\frac{2\Gamma_i}{\hbar}$, $\frac{2\Gamma_k}{\hbar}$, $\frac{2\Gamma_l}{\hbar}$. They also explained that for the compound system (atom plus radiation field), the energy of the initial state (no photons, atom in state, j,) is not sharp but has a width connected with $\Gamma = \Gamma_i + \Gamma_k + \Gamma_l$ by the indeterminancy relation. This usually takes the form:

$$\Delta E \cdot \Delta t \simeq \hbar/2$$

then

$$2(\Delta E) = \hbar/(\Delta t)$$

If (Δt) is taken as the smallest time available in which to make a measurement, or the mean life of the state, and if $\frac{2\Gamma_i}{\hbar}$ is the emission probability per sec, τ may be written as: $\frac{2\Gamma_i}{\hbar} = \frac{1}{\tau}$ or $\Gamma_i = \frac{\hbar}{2\tau}$. (ΔE) need only be taken as Γ_i to get $(\Delta E) = \Gamma_i = \frac{\hbar}{2\tau}$

This concept of "level" and width may be carried over to nuclear transitions, with the following observations and cautions:

a. Nuclear resonances often overlap, and coupling through emission may be expected to be frequent.

b. The direct change of one photon into another may be explained in terms of separate emissions and

absorptions. Such simplification in the nuclear case is not possible. Also direct transitions between states in the continuum are possible, in nuclear transitions.

c. The effect of a nuclear resonance is of interest outside the main resonance peak.

d. The interaction between an atom and the radiation field is always small. This is not the case in nuclei.

The total width, or the total probability of a transition, yields information concerning the probability of the concentration of energy on one particle or a group of particles (for instance: alpha-decay).

In the fore-going discussion, the conclusions are generally the result of conjectures based upon experiments and the atomic analogies. The virtual-states, or states of the nucleus capable of dissociation were inferred from knowledge gained by experiment. It is significant to discover if there is a foundation for their existence in nuclear theory. The simplest and perhaps most general investigation carried out which shows that the foundation does indeed exist, is that carried out by Casimir, (4).

Casimir considered the problem of alpha-decay in general. Potential functions were chosen, subject only to two conditions:

1) the long range potential was the observed Coulomb potential, and
2) a short-range potential was to be strong and an attraction.

Justification for the choice of potential functions, lies in experiments. The coulomb potential is justified in ordinary Rutherford Scattering. The strong, short-range attraction is implied by the large binding energies of the nuclei and the agreement of subsequent scattering experiments.

The problem of Casimir is in essence a generalization of the Gamow treatment of the theory of alpha-decay.

(B) Description by Casimir

Before describing in detail Casimir's work it is well to describe briefly the Gamow treatment. It is assumed by Gamow that the alpha-particle (already an entity) is moving in the field of the remainder of the nucleus with zero angular momentum, (l = 0). A potential function is assumed as shown if Fig I. The wave functions (time-independent) are then written for the separate regions (I-classically admissable, II-non-classical, and III-classically admissable). The wave functions are then connected by W. K. B. Approximations and the Kramers connection formula.

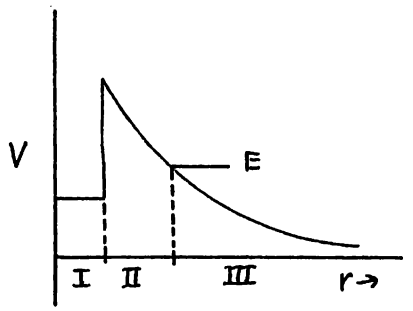


Fig. I

This procedure does not give directly the time-dependence of the wave-function since only time-independent functions were used in the connection. One now recalls the indeterminancy principle and forms a wave-packet corresponding to a group of energies very close to-gether. The behavior

of this wave-packet in the course of time now describes the emission of the alpha-particles. This treatment is satisfactory but involves the specific assumption of a square potential well of positive energy (see Fig I), and the indeterminacy of the exact energy. Consequently the construction of the wave packet is put into the problem arbitrarily, though through need.

In a generalization of this problem, the behavior of the alpha-particle should be found to arise naturally from the solution of the time-dependent Schrodinger Equation. This behavior was found by Casimir.

Casimir considered an arbitrary potential of the form shown in Fig II. The choice of potentials convexed upward is in accordance with experimental observation.

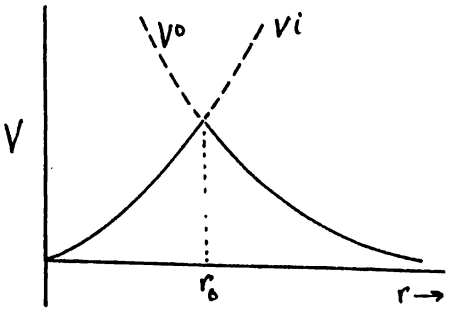


Fig II

Analytically the potential has the form:

$$V = (\delta^i V^i + \delta^o V^o) \quad \text{where:} \quad \delta^i = 1 \ (r < r_0) \quad \delta^o = 0 \qquad \delta^o = 0 \ (r > r_0) \quad \delta^o = 1$$

and the time-dependent Schrodinger Equation is:

$$(1\text{-}1) \quad -\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial r^2} + (\delta^i V^i + \delta^o V^o)\psi$$

It is assumed that V = V(r) (central symmetry) and l = 0 (no angular momentum). As in the treatment of Gamow the alpha-particle (or in general any one particle) is

assumed to be moving in the field of the nucleus as an entity. To be strictly correct there should be a factor describing the probability of formation of the alpha-particle, though we may assume it to be an independent event, and describe only the escape of the alpha-particle.

Equation (1-1) leads to two separate problems if one considers the whole of each potential. These problems are:

$$-\frac{\hbar}{i}\frac{\partial \psi^i}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2 \psi^i}{\partial r^2} + V^i \psi^i$$

(1-2a)

$$-\frac{\hbar}{i}\frac{\partial \psi^0}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2 \psi^0}{\partial r^2} + V^0 \psi^0$$

and with the following substitutions:

(1-2b) $\psi^i = b_n(t)\, u_n(r)$; $\psi^0 = k_\varepsilon(t)\, v_\varepsilon(r)$

and the definition:

$$H_0 = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r^2}$$

the equations (1-2a) are transformed into:

$$H_0 u_n(r) + V^i u_n(r) = E_n u_n(r)$$

(1-3)

$$H_0 v_\varepsilon(r) + V^0 v_\varepsilon(r) = \mathcal{E}\, v_\varepsilon(r)$$

From the assumed shapes of the potentials it is apparent that the first of the equations (1-3), has discrete eigenvalues and the second has a continuous spectrum of eigenvalues. Each of these equations supplies a complete set of orthonormal solutions. The solution to

equation (1-1) could be expanded in a series of one of the sets of solutions. The solution $\psi(r,t)$ desired however is to describe the escape of a particle from a nucleus. The particle is assumed to be in state (n) at $t = 0$, and the solution of Eq. (1-1), shall describe its behavior in the course of time. This implies a transition from a solution of the first Eq. (1-3) to a solution of the second. A combination of the solutions is therefore chosen to represent the solution of Eq. (1-1). It is assumed:

$$\psi(r,0) = u_n(r) \quad , \quad b_n(0) = 1 \tag{1-4}$$

$$\psi(r,t) = b_n(t)\, u_n(r) + \int_0^\infty k_\varepsilon(t)\, v_\varepsilon(r)\, d\varepsilon \tag{1-5}$$

It is now required to find the expansion coefficients in order to have a solution to Eq. (1-1). Since the set of solutions, v_ε , is a complete set of orthogonal functions and hence is sufficient, in itself, to represent the solution to Eq. (1-1), the relation between the coefficient b_n and the k_ε is still free, and <u>one</u> relation between them may be fixed arbitrarily. This relationship is chosen:

$$\int_0^\infty dr\, u_n(r) \int_0^\infty d\varepsilon\, k_\varepsilon(t)\, v_\varepsilon(r) = 0 \tag{1-6}$$

Putting Eq. (1-6) into Eq. (1-1) and using equations (1-3), the equations satisfied by b_n and k_ε are shown to be: (Derived in detail in Appendix I).

$$i\hbar\, \dot{b}_n(t) = b_n(t) E_n + b_n(t) V_{nn} + \int_0^\infty d\varepsilon\, k_\varepsilon(t) V_{n\varepsilon} \tag{1-7}$$

$$(1\text{-}8)\quad i\hbar\,\dot{k}_\varepsilon(t) = b_n(t)V_{\varepsilon n} + \varepsilon k_\varepsilon(t) + \int_0^\infty d\varepsilon'\, k_{\varepsilon'} V_{\varepsilon\varepsilon'} + \left[b_n(t)E_n - i\hbar\,\dot{b}_n(t)\right]\int_0^\infty dr\; u_n(r)\, v_\varepsilon^*$$

where
$$V_{nn} \equiv \int_0^\infty dr\; u_n^*(r)\;\delta^\circ(V^\circ - V^i)\; u_n(r)$$

$$V_{n\varepsilon} \equiv \int_0^\infty dr\; u_n(r)\;\delta^i(V^i - V^\circ)\; v_\varepsilon(r)$$

$$V_{\varepsilon\varepsilon'} \equiv \int_0^\infty dr\; v_\varepsilon^*\;\delta^i(V^i - V^\circ)\; v_\varepsilon$$

We may assume, that unless the energy of the alpha-particle is very near the top of the barrier, the quantities V_{nn} and $V_{\varepsilon\varepsilon'}$ are small and may be neglected. The first, V_{nn}, involves the wave-functions u_n for $r > r_o$, and they will go to zero rapidly (exponentially). The second involves only v_ε inside the barrier ($r < r_o$) and they too go to zero rapidly.

It now remains to solve equations (1-7) and (1-8), which when V_{nn} and $V_{\varepsilon\varepsilon'}$ are neglected become:

$$(1\text{-}9)\quad i\hbar\,\dot{b}_n(t) = b_n(t)E_n + \int_0^\infty d\varepsilon\; k_\varepsilon V_{n\varepsilon}$$

$$(1\text{-}10)\quad i\hbar\,\dot{k}_\varepsilon(t) = b_n(t)V_{\varepsilon n} + \varepsilon k_\varepsilon + \left[b_n(t)E_n - i\hbar\,\dot{b}_n(t)\right]\int_0^\infty dr\; u_n v_\varepsilon$$

These equations may be made simpler by putting

$$b_n(t) = a_n(t)\, e^{-\frac{i}{\hbar}E_n\cdot t} \qquad \text{and} \qquad k_\varepsilon(t) = c_\varepsilon(t)\, e^{-\frac{i}{\hbar}E_n\cdot t}$$

The equations become:

$$\dot{a}_n(t)\cdot(i\hbar) = \int_0^\infty d\varepsilon\; c_\varepsilon V_{n\varepsilon} \tag{1-11}$$

$$i\hbar\,\dot{c}_\varepsilon(t) = (\varepsilon - E_n)\,c_\varepsilon + a_n(t)\cdot V_{\varepsilon n} - i\hbar\,\dot{a}_n(t)\int_0^\infty dr\; u_n(r)\,v_\varepsilon^*(r) \tag{1-12}$$

It may be verified that the solutions of these equations are:

$$a_n(t) = e^{-\frac{\gamma t}{\hbar}} \tag{1-13}$$

$$c_\varepsilon(t) = \frac{e^{-\frac{\gamma t}{\hbar}} - e^{i\eta t/\hbar}}{\eta - i\gamma} \tag{1-14}$$

where:

$$\eta = E_n - \varepsilon$$

$$\gamma = \pi\,|V_{n\varepsilon}|^2_{\varepsilon = E_n}$$

It is apparent now that the probability of the particle being in state n, decreases in the course of time in accordance with:

$$|a_n(t)|^2 = e^{-\frac{2\gamma}{\hbar}t} \tag{1-15}$$

More important, however, is the fact that solutions to the time-dependent Schrodinger equation have been exhibited which show the exponential time-behavior and a discrete set of energy levels inside the nucleus. In

addition an indication of resonance behavior, when the energy E is varied, is indicated. This is shown rather clearly if (t) is put equal to $\hbar/(2\gamma)$ then $|C_\varepsilon|^2$ becomes:

$$|C_\varepsilon|^2 = \frac{(\text{const.}) - (2/e^2)\cos(\eta t/\hbar)}{\eta^2 + \gamma^2}$$

Now putting $\eta = (E_0 - E)$ again, it is seen that if a particle found itself in the nucleus with energy E, equal to or nearly equal to E_0, the probability of its escaping becomes greater. Also if η is large the probability is then small. This is a typical resonance behavior.

SUMMARY OF CHAPT. I

In this chapter the quelitative features of nucleon interaction are examined; with the objective being a means of arriving at a quantitative description of the nucleus.

Examples of various types of reactions are presented. The discrete character of the nuclear spectrum is illustrated, and the concept of a compound nucleus in introduced.

The quantitative description of the nucleus by Casimir is presented in detail. This treatment of alpha-decay by Casimir is the logical starting point for the development of the dispersion formula. It is shown that there exist solutions to the time-dependent Schrodinger equation which exhibits a discrete character and an exponentially decaying probability.

CHAPTER II

-DERIVATION OF DISPERSION FORMULAE-

The derivations to be presented are not necessarily in the chronological order of their dates of publication. The derivations are presented in the more logical order of their degree of generalization and logical extension.

(A) Kapur and Peierls (13).

The derivation by Kapur and Peierls provides the first logical extension of the work of Casimir given in Chapt. I. This derivation does, however, involve some rather stringent assumptions which, as will be shown later, may be removed by altering the formalism. This derivation attempts to make only assumptions which are experimentally satisfied.

The fundamental assumptions to be used are: a). The levels are narrow i.e. $\frac{\Gamma_n}{2} \ll \left| E_n - E_{n+1} \right|$ and b). The escape of a particle is a rare event. As pointed out in the introduction, the time spent within the nucleus is several orders of magnitude greater than the transit-time of the particle, hence the second assumption is not illogical. The approximation based on the second assumption is that if the particle is prevented from escaping at all, there will be little if any change in the state of the nucleus. This may be accomplished by suitable restriction imposed upon the wave-function somewhere outside the nucleus. Then if the wave-functions

and energy levels of the restricted system are known, they will be approximately those of the unrestricted system. It will be shown that the latter system can be obtained from the former system by a kind of perturbation calculus applied to the boundary condition rather than to the Hamiltonian of the system.

The derivation presented will be the simplest example of the method, in which only states with no angular momentum, and no dependence on spin are considered. The de Broglie wave-length of the particles incident upon the nucleus is assumed to be large compared to r_0, $(\lambda \gg r_0)$, where r_0 is the radius of a sphere about the origin which wholly contains the nuclear potential. Such a sphere may be constructed because of the short-range character of the nuclear forces. (Potential is arbitrary but short-range.)

The scattering of a single particle is considered and the wave-function $u(r) = r\psi(r)$ satisfies the radial equation:

$$\text{(A-1)} \qquad \frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + (E - V(r))u = 0$$

where: E = particle energy

$V(r) = 0$ for $r > r_0$,

for $r > r_0$ the equation reduces to:

$$\text{(A-2)} \qquad \frac{d^2u}{dr^2} + k^2u = 0$$

$$\text{(A-3)} \qquad k^2 = \frac{2m}{\hbar^2}E$$

The solution to equation (A-2) may be written as:

$$(A\text{-}4) \qquad u = \frac{I}{k} \sin kr + S e^{ikr}$$

where I represents the amplitude of the incident wave and S represents the amplitude of the scattered wave. Taking the derivative of Eq. (A-4):

$$(A\text{-}5) \qquad \frac{du}{dr} = I \cos kr + ik S e^{ikr}$$

Equations (A-3) and (A-4) may be solved for I and S, to get: (quantities evaluated at $r = r_0$).

$$(A\text{-}6) \qquad I e^{-ikr_0} = \left.\frac{du}{dr}\right|_{r_0} - ik\, u(r_0)$$

and

$$(A\text{-}7) \qquad S = u(r_0) \cos kr_0 - \frac{1}{k}\left(\frac{du}{dr}\right)_{r_0} \sin kr_0$$

It is now apparent from Eq. (A-6) that if there were no incident wave present (I = 0) that the situation would be represented by:

$$(A\text{-}8) \qquad \left.\frac{du}{dr} - ik\, u\right|_{r_0} = 0$$

The imaginary and real components must be separately zero, therefore u must be zero every where to satisfy (A-8). This, of course, is an untenable situation; however, it shows that the boundary condition (A-8) is not compatible with Eq. (A-1). This may be seen by considering that the two boundary conditions, Eq. (A-8) and the condition that

$\psi(r)$ be finite at the origin ($u(0) = 0$), are two homogeneous boundary conditions at the ends of a finite interval and therefore give a discrete spectrum of eigenvalues. The energy E of Eq. (A-1) is not an eigenvalue of this problem.

It may be supposed that the eigenvalues (W_n) of this problem are known. The solution (v_n) of

$$\text{(A-9)} \quad \frac{\hbar^2}{2M}\frac{d^2 v_n}{dr^2} + \left(W_n - V(r)\right) v_n = 0$$

are then assumed known. The eigenvalues W_n are then complex because the boundary conditions are complex. In the following considerations it is important to state that the wave vector (k) corresponding to E of Eq. (A-1) is fixed and is the same for all W_n. For each value of k, there exists a different set of solutions v_n and W_n. This has the consequence that the energy levels of the compound nucleus will be dependent upon the energy of the incident particle. As was pointed out in the introduction, this was to be expected for large energies; but for low energies, however, it can be neglected as small. This will be dealt with in detail later.

To show that the solutions of Eq. (A-9) form a complete set of orthogonal functions, Eq. (A-9) may be multiplied by $v_{n'}$, then n interchanged with n' to get a second equation. Subtracting the second from the first, and integrating with respect to r, there results:

$$(W_n - W_{n'}) \int_0^\infty v_n v_{n'}\, dr + \frac{\hbar^2}{2M}\left[v_{n'}\frac{dv_n}{dr} - v_n \frac{dv_{n'}}{dr} \right]_0^{r_0} = 0$$

The second term vanishes because of boundary condition (A-8), hence:

$$\text{(A-10)} \qquad \int_0^{r_o} v_n \, v_{n'} \, dr = 0 \qquad \text{for} \quad W_n \neq W_{n'}$$

It should be noticed Eq. (A-10) is not the ordinary orthogonality relation wherein the square of the absolute value appears under the integral sign.

With orthogonality established it follows that any function, f(r), satisfying boundary condition may be expressed as a series:

$$\text{(A-11)} \qquad f(r) = \sum_n a_n \, v_n$$

The functions u(r) do not satisfy boundary condition (A-8) but Eq. (A-6). It may be supposed, however, that an arbitrary function (χ) satisfying:

$$\text{(A-12)} \qquad \left.\frac{d\chi}{dr}\right|_{r_o} - i k \chi(r_o) = I e^{-i k r_o}$$

subtracted from u(r) such that the remainder is expressible in a series, as follows:

$$\text{(A-13)} \qquad u(r) - \chi(r) = \sum_n a_n \, v_n$$

This implies that u(r) is composed of two functions, one satisfying equation (A-6) and the other satisfying equation (A-8).

The coefficients (a_n) may be determined in equation (A-13), since it must satisfy equation (A-1).

Putting (A-13) in (A-1), and using Eq. (A-9), we have:

$$(H_o - E)u = \sum_n a_n (W_n - E) v_n + (H_o - E)\chi = 0$$

where:

$$H_o \equiv \left(\frac{\hbar^2}{2M} \frac{d^2}{dr^2} + V(r) \right)$$

Multiply now by a particular v_n, and integrate with respect to the space variable, to get:

$$\int_0^{r_o} \sum_n a_n (W_n - E) v_{n'} v_n \, dr + \int_0^{r_o} v_{n'} (H_o - E)\chi \, dr = 0$$

From the orthogonality relation (Eq. (A-10)) this becomes:

$$a_n (W_n - E) \int_0^{r_o} v_n^2 \, dr + \int_0^{r_o} v_n (H_o - E)\chi \, dr = 0$$

The second integral may be partially integrated to give:

$$-\frac{\hbar^2}{2M} \left(v_n \frac{d\chi}{dr} - \chi \frac{dv_n}{dr} \right)_{r_o} - \int_0^{r_o} dr \, \chi (H_o - E) v_n$$

apply Eq. (A-12) this becomes:

$$-\frac{\hbar^2}{2M} \left[v_n (I e^{ikr_o} + ik\chi(r_o)) - \chi(r_o) \frac{dv_n(r_o)}{dr} \right]$$

$$- \int_0^{r_o} \chi (H_o - E) v_n \, dr$$

Now using Eq. (A-8) this becomes: (Since v_n is to satisfy (A-8)).

$$-\frac{\hbar^2}{2M} \left[v_n(r_o) I e^{ikr_o} + ik v_n(r_o) \chi(r_o) - ik v_n(r_o) \chi(r_o) \right]$$

$$- (W_n - E) \int_0^{r_o} v_n \chi \, dr$$

(The second integral's change being obvious)

or finally:

$$(A\text{-}15)\quad a_n(W_n-E)\int_0^{r_0} v_n^2\,dr = \frac{\hbar^2}{2M}\,v_n(r_0)\,I\,e^{-ikr_0} - (W_n-E)\int_0^{r_0}\chi\,v_n\,dr$$

It may be assumed that the functions (v_n) are normalized in the usual way:

$$(A\text{-}16)\quad \int_0^{r_0} dr\,|v_n|^2 = 1$$

Then define:

$$(A\text{-}17)\quad \int_0^{r_0} v_n^2\,dr \equiv N_n$$

N_n is now a fixed constant, in general complex, and of modulus less than unity.

For the integral on the right, may be put:

$$(A\text{-}18)\quad b_n = \int_0^{r_0} v_n\,\chi\,dr$$

The coefficients a_n now become:

$$(A\text{-}19)\quad a_n = \frac{\hbar^2}{2M}\,\frac{v_n(r_0)\,I\,e^{-ikr_0}}{(W_n-E)\,N_n} - \frac{b_n}{N_n}$$

In principle the wave-function may now be determined from Eq. (A-7).

Inserting (A-13) into (A-7), S becomes:

$$S = \Big(\sum_n a_n v_n(r_0) + \chi(r_0)\Big)\cos kr_0 - \frac{1}{k}\Big[\frac{d}{dr}\Big(\sum_n a_n v_n + \chi\Big)\Big]_{r_0}\sin kr_0$$

Applying condition (A-8),

$$S = \Big(\sum_n a_n v_n(r_0) + \chi(r_0)\Big)\cos kr_0 - \frac{1}{k}\Big[\sum_n a_n (ik) v_n + \frac{d\chi}{dr}\Big]_{r_0}\sin kr_0$$

and condition (A-12),

$$S = \left(\sum_n a_n v_n(r_0) + \chi(r_0) \right) \cos kr_0 - \frac{1}{k} \left[\sum_n a_n (ik) v_n + I e^{-ikr_0} + ikr_0 \right] \sin kr_0$$

Combining terms (and remembering, $e^{i\theta} = (\cos\theta + i \sin\theta)$) this becomes:

$$\text{(A-20)} \quad S = \sum_n a_n v_n(r_0) e^{-ikr_0} + \chi(r_0) e^{ikr_0} - \frac{I}{k} \sin kr_0 \left(e^{-ikr_0} \right)$$

and substituting (a_n) from (A-19), (A-20) becomes:

$$S = I \left\{ \sum_n \frac{\hbar^2}{2M} \frac{e^{-2ikr_0} v_n^2(r_0)}{(W_n - E) N_n} - \frac{1}{k} \sin kr_0 \left(e^{-ikr_0} \right) \right\}$$

$$\text{(A-20a)} \quad + e^{ikr_0} \chi(r_0) - \sum_n \frac{b_n}{N_n} v_n(r_0) e^{-ikr_0}$$

The last two terms of this equation may be made arbitrarily small, by choosing $\chi(r)$ properly, requiring only that it satisfy Eq. (A-12).

Suppose:

$$\chi(r) = A e^{-\alpha(r_0 - r)} \qquad (A = \text{const.})$$

Eq. (A-12) is satisfied for

$$A = \frac{I e^{-ikr_0}}{(\alpha - ik)}$$

but since alpha is arbitrary, $\chi(r)$ may be made arbitrarily small, by allowing alpha to become very large. Thus (A-20) becomes:

$$\text{(A-21)} \quad S = I \left\{ \sum_n \frac{\hbar^2}{2M} \frac{e^{-2ikr_0} v_n^2(r_0)}{(W_n - E) N_n} - \frac{1}{k} \sin kr_0 \; e^{-ikr_0} \right\}$$

Now abbreviating:

(A-22) $\varphi_n \equiv \sqrt{\frac{\hbar^2}{2M}}\, v_n$

and putting

(A-23) $W_n = E_n - \frac{i}{2}\Gamma_n$ since W_n is complex.

$\hbar/\Gamma_n$, as was shown in the introduction represents the "mean-life" of the state (n).

Γ_n may be connected to the boundary of the eigenfunction v_n as follows: Multiply Eq. (A-9) by v_n^*, subtract the conjugate equation, and integrate over (r):

$$\frac{\hbar^2}{2M}\left\{ v_n^* \frac{d^2 v_n}{dr^2} - v_n \frac{d^2 v_n^*}{dr^2} \right\} + \left[W_n - W_n^* \right] v_n^* v_n = 0$$

Integrating:

(A-24) $$i\Gamma_n \int_0^{r_0} |v_n|^2 dr = \frac{\hbar^2}{2M}\left\{ v_n^*(r_0) \frac{dv_n(r_0)}{dr} - v_n(r_0) \frac{dv_n^*(r_0)}{dr} \right\}$$

apply boundary condition (A-8)

$$i\Gamma_n \int_0^{r_0} |v_n|^2 dr = \frac{\hbar^2}{2M}\left\{ ik\, v_n^* v_n + v_n (ik) v_n^* \right\}_{r_0}$$

or shortening, (applying normalization condition):

$$\Gamma_n = \frac{\hbar^2}{M} k \left| v_n^*(r_0)\, v_n(r_0) \right|^2$$ and from Eq. (A-22) this becomes finally:

(A-25) $$\Gamma_n = \left| \varphi_n(r_0) \right|^2$$

So far there has been no approximating assumption. It was pointed out at the beginning that there would be

little change in the system if the particle were prevented from escaping from the nucleus. This was carried out in the making of (r) small and neglecting it.

It is possible now to define the scattering cross-section in terms of this derivation. If S is the amplitude of the scattered wave and I is the amplitude of the incident wave, then the probability of the particle being scattered is represented by: $\frac{|S|^2}{|I|^2} = \left|\frac{S}{I}\right|^2$. The total cross-section is given by:

$$\sigma(E) = 4\pi \left|\frac{S}{I}\right|^2$$ (4π comes from integration over the sphere).

this becomes:

$$\text{(A-26)} \quad \sigma(E) = \frac{\pi}{k^2} \left| \frac{\sum_n |\varphi_n|^2 e^{-2ikr_0}}{(E_n - E - i/2\,\Gamma_n)} - 2\sin kr_0\, e^{-ikr_0} \right|^2$$

As was pointed out before, the φ_n's depend upon k, through (A-8), however the way it depends on k has not been established. Also the φ_n's are in general complex and hence have complex eigenvalues.

It is possible, <u>in first order</u>, to show that φ_n is real, $N_n = 1$, and $|\varphi_n|^2$ is proportional to k.

It is apparent from Eqs. (A-22), (A-16), and (A-17), that, $N_n = 1$ and φ_n is real, if $v_n(r)$ were real, and it would be proportional to k if $v_n(r_o)$ were independent of k.

$v_n(r_o)$ may be shown to be approximately independent of k if the level width (Γ_n) is sufficiently small. If $\Gamma_n = 0$, then from (A-25), $v_n(r_o)$ must vanish, and from (A-8), $v_n(r_o)$ and $dw/dr|_{r_o}$ vanish. Since (A-8) must

hold for all k, it must hold for k = 0, hence with no contradiction, the boundary condition may be written:

(A-27) $$\frac{dv_n}{dr}\Big|_{r_0} = 0$$

Condition (A-27) is real, and independent of k, hence so must be the solutions (v_n).

To decide how narrow the levels must be for this approximation the following may be considered:

If the level is sufficiently narrow, the solution v_n, satisfying (A-8) must be similar to the solutions ψ_n of Eq. (A-9) satisfying Eq. (A-27), having reel eigenvalues, E^o_n. For this case (A-9) and boundary (A-27) become:

(A-28) $$\frac{\hbar^2}{2M}\frac{d^2\psi_n}{dr^2} + \left(E^o_n - V(r)\right)\psi_n = 0$$

(A-28a) $$\frac{d\psi}{dr}\Big|_{r_0} = 0$$

The wave-function v_n may now be thought of as composed of the sum of the wave functions ψ_n plus a correction term $v_n^{(1)}$ arising from a perturbation of the boundary condition (A-8):

(A-29) $$v_n = \psi^o_n + v_n^{(1)}$$

It may be supposed that $v_n^{(1)}$ and $\mathcal{E}_n$, ($\mathcal{E}_n \equiv W_n - E^o_n$) are small in the first order.

Neglecting terms of second order, the wave-equation is:

(A-30) $(H_o - E_n^o)\, v_n^{(1)} = \epsilon_n \psi_n$

and

(A-31) $ik\, \psi_n(r_o) = \left.\dfrac{dv_n^{(1)}}{dr}\right|_{r_o}$

Expanding $v_n^{(1)}$ as was done previously for v_n in a series of the solutions ψ_m:

(A-32) $v_n^{(1)} = \sum_m a_m \psi_m + \chi(r)$

and as before, $\chi(r)$ satisfies:

(A-33) $\left.\dfrac{d\chi}{dr}\right|_{r_o} = ik\, \psi_n(r_o)$

Using Eqs. (A-8) and (A-12) as before on Eq. (A-30), and dropping terms containing $\chi(r)$:

(A-34) $a_m (E_m^o - E_n^o) = \epsilon_n \delta_{nm} + ik \dfrac{\hbar^2}{2M} \psi_m(r_o)\, \psi_n(r_o)$

For $m = n$, this equation shows that:

(A-35) $\epsilon_n \equiv \left(E_n - E_n^o - \dfrac{i}{2}\Gamma_n\right) = ik \dfrac{\hbar^2}{2M} \psi_n^2(r_o)$

and there is no shift in the energy level, in this first approximation.

For $m \neq n$, a_m becomes:

(A-36) $a_m = ik \dfrac{\hbar^2}{2M} \dfrac{\psi_m(r_o)\, \psi_n(r_o)}{E_m^o - E_n^o}$

and to maintain the normalization of v_n, $a_m \equiv 0$ for $m = n$, then from Eqs. (A-36), (A-29), and (A-32), and dropping

the (χ's), v_n becomes:

$$(A\text{-}37)\quad v_n(r_0) = \psi_n(r_0) + ik\,\psi_n(r_0)\,\frac{\hbar^2}{2M}\,{\sum_m}'\,\frac{\psi_m^2(r_0)}{E_m^0 - E_n^0}$$

(where the prime on the summation indicates that the sum is not to be extended over m = n).

Now using (A-35), it is seen that the ratio of the imaginary part of W_n, to the real parts are given by:

$$(A\text{-}38)\quad \frac{1}{2}\,{\sum_m}'\,\frac{\Gamma_m}{E_m^0 - E_n^0}$$

Also it is apparent from (A-17) that since $a_n \equiv 0$, $N_n = 1$, in first order.

The extension may now be made to the second order by putting $v_n = \left(\psi_n + v_n^{(1)} + v_n^{(2)}\right)$ where $v_n^{(1)}$ is pure imaginary, $v_n^{(2)}$ is real, and $v_n^{(1)}$ is orthogonal to ψ_n. For N_n then, (neglecting terms of $(v^{(1)} \times v^{(2)})$):

$$(A\text{-}39)\quad N_n = 1 + \int_0^{r_0} \left(v_n^{(1)}\right)^2 dr + 2\int_0^{r_0} \psi_n\, v_n^{(2)}\, dr$$

From the normalization of v_n:

$$(A\text{-}40)\quad 1 = 1 - \int_0^{r_0} \left(v_n^{(1)}\right)^2 dr + 2\int_0^{r_0} \psi_n\, v_n^{(2)}\, dr$$

Using (A-40), then:

$$N_n = 1 + 2\int_0^{r_0} \left(v_n^{(1)}\right)^2 dr = 1 + 2\,{\sum_m}'\,a_m^2$$

$$= 1 - \frac{1}{2}\,{\sum_m}'\,\frac{\Gamma_m \Gamma_n}{(E_m - E_n)^2}$$

It is now apparent that N_n differs from unity by:

$$(A\text{-}41) \qquad 1 - N_n = \frac{1}{2} \sum_m{}' \frac{\Gamma_m \Gamma_n}{(E_m - E_n)^2}$$

The necessary relations for determining how narrow the levels may be have now been derived.

The terms in (A-38) may be individually large and the sum still converge, since the signs of the terms may be both positive and negative. However, in Eq. (A-41) each term must be separately small since all terms are positive. In fact, each term in (A-41) must be less than unity. This means that $\Gamma_m < |E_m - E_n|$ for convergence. If the sum is to converge at all rapidly and thus be amenable to easy calculation, then $\Gamma_m << |E_m - E_n|$.

This feature, sharp levels, in the formula just derived, is not in keeping with the theories of nuclear structure. The conclusion one draws from this derivation is that a resonance formula is only justified if the levels are sharply defined (Γ_n is small). It was pointed out, however, in Chapt. I that the levels above and below the dissociation energy of the nucleus will not only be close together but in general will overlap because of the strong interactions encountered.

The requirement of sharp levels may however be removed as will be shown in the next section.

There is another bad feature of the foregoing derivation, and that is the definition of (r_o). The only requirement placed on (r_o) was that the potential

must vanish for $r \geqslant r_o$. Clearly r_o can be made very large or small, as we choose. If r_o is taken too large however, the second term of (A-34) becomes large and would have to be compensated for by larger terms in the summation. Also if the r_o is increased, the number of levels is in general increased. This implies that the number of energy states of the compound nucleus is dependent upon the chosen mathematical formalism and not vice-versa. While it may be true that r_o cannot (or has not) been measured precisely, it is, in a consistent theory, determined by the physical attributes of the nuclear particles and not the descriptive formalism. This last fault is also to be removed in the next section.

(B) Derivation by Siegert (14).

In this derivation by Siegert, the simple model assumed by Kapur and Peierls will again be assumed. The scattering of one particle will again be examined and it will be assumed that the particle and the nucleus have no angular momentum about their center of mass ($l = 0$).

The principle purpose of this derivation is to remove the objections, given before, to the Kapur and Peierls derivation. This is to be accomplished by changing the definition of the compound states. In the previous treatment the compound states were defined by the solutions v_n of (A-9) and the eigenvalues W_n. In this treatment the compound states will be taken as

those states for which the scattering becomes singular.

The essential difference in treatment will be that instead of expanding the wave function $u_n(r)$ in an ortho-normal set of functions, the singularities which occur for certain complex energy values will be investigated.

At a singularity the cross-section may be replaced by its singular part and a smooth function of the energy.

The physical significance of the new definition of compound state may be seen from the following considerations. The cross-section is defined (as in part A) as the ratio of the outgoing to the incoming wave. The cross-section will become singular if there is only an outgoing wave. The states characterized by these singularities will then be the "radio-active" states of the system.

To find these singular states it may be recalled that the wave function was to satisfy (radial equation):

$$(B\text{-}1)\quad \frac{\hbar^2}{2M}\frac{d^2u(r)}{dr^2} + (E - V(r))\,u(r) = 0$$

where $u(0) = 0$; $V = V(r)$ and $V(r_0) = 0$.

For $r > r_0$ the wave function becomes:

$$(B\text{-}2)\quad u(r) = \frac{I}{k}\sin kr + Se^{ikr}$$

where: $k^2 = \frac{2ME}{\hbar^2}$

M = reduced mass

The scattering cross-section is again given by:

$$\text{(B-3)} \qquad \sigma(E) = 4\pi \left| \frac{S}{I} \right|^2$$

If the ratio S/I be considered as a function of the energy including complex energies, the cross-section may be written, in terms of (A-6) and (A-7), to get:

$$\text{(B-4)} \qquad \frac{S}{I} = \frac{u(r_0)\cos kr_0 - [u'(r_0)\sin kr_0]/k}{u'(r_0) - ik\,u(r_0)}$$

(Prime indicates differentiation with respect to r).

The ratio S/I is now to be examined for singularities which arise from the vanishing denominator. The energies, W_n, for which the denominator vanishes are defined by the eigenvalue problem:

$$\text{(B-5)} \qquad \frac{\hbar^2}{2M}\frac{d^2v_n}{dr^2} + (W_n - V(r))\,v_n = 0$$

with the boundary conditions:

$$\text{(B-6)} \qquad v_n(0) = 0$$

$$\frac{dv_n}{dr} - ik_n v_n = 0 \qquad \text{for } r = r_0$$

Note that here k_n is taken instead of the k arising in (B-1). This has as a consequence, the definition of the compound states are independent of the energy of the incoming particle. It will also cause their invariance with changes in r_0.

Now multiplying (B-1) by v_n and (B-5) by (u) and

subtracting, there results:

$$\text{(B-7)}\quad \frac{\hbar^2}{2M}\left(u\frac{d^2v_n}{dr^2} - v_n\frac{d^2u}{dr^2}\right) + (W_n - E)\,u\cdot v_n = 0$$

Now integrating over (r) by parts the first term becomes:

$$\int_0^{r_0}(u v_n'' - v_n u'')\,dr = (u v_n' - v_n u')_{r_0} - \int_0^{r_0}(u' v_n' - v_n' u')\,dr$$

Partially integrating the integral on the right this becomes:

$$\int_0^{r_0}(u v_n'' - v_n u'')\,dr = (u v_n' - v_n u')_{r_0} - (u' v_n - v_n' u) + \int_0^{r_0}(u'' v_n - v_n'' u)\,dr$$

$$= 2(u v_n' - v_n u')_{r_0} - \int_0^{r_0}(u v_n'' - v_n u'')\,dr$$

Hence the first term of (B-7) reduces to:

$$\frac{\hbar^2}{2M}\int_0^{r_0}(u v_n'' - v_n u'')\,dr = (u v_n' - v_n u')_{r_0}$$

Now applying the boundary condition (B-6), the expression (B-7) when integrated becomes:

$$\text{(B-8)}\quad \frac{\hbar^2}{2M} v_n(r_0)\left(i k_n u(r_0) - \frac{du}{dr}\Big|_{r_0}\right) + (W_n - E)\int_0^{r_0} v_n u\,dr$$

The denominator of (B-4) now becomes:

$$u'(r_0) - i k u(r_0) = \frac{W_n - E}{\left(\frac{\hbar^2}{2M} v_n(r_0)\right)}\int_0^{r_0} v_n u\,dr + i(k_n - k)u(r_0)$$

$$\text{(B-9)}\qquad = \frac{W_n - E}{\left(\frac{\hbar^2}{2M} v_n(r_0)\right)}\left\{\int_0^{r_0} v_n u\,dr + \frac{i\,v_n(r_0)\,u(r_0)}{(k_n + k)}\right\}$$

where

$$(B\text{-}10)\quad (k_n - k) = \frac{2M}{\hbar^2}\left(\frac{W_n - E}{k_n + k}\right)$$

If the eigenvalue W_n is non-degenerate then in the limit, $E \to W_n$, $u \to v_n$, and

$$(B\text{-}11)\quad u'(r_0) - iku(r_0) \longrightarrow \frac{(W_n - E)2M}{\hbar^2 v_n(r_0)}\left\{\int_0^{r_0} v_n^2\, dr + \frac{i v_n^2(r_0)}{2k_n}\right\}$$

the numerator becomes:

$$(B\text{-}12)\quad u(r_0)\cos kr_0 - \frac{u'(r_0)\sin kr_0}{k} \to v_n(r_0)\, e^{-ik_n r_0}$$

and the ratio becomes:

$$(B\text{-}13)\quad \frac{S}{I} = \frac{1}{W_n - E}\;\frac{\left(\frac{\hbar^2}{2M}\right) v_n^2(r_0)\, e^{-2ik_n r_0}}{\int_0^{r_0} v_n^2\, dr + \frac{i v_n^2(r_0)}{2k_n}} + \mathcal{F}(E)$$

Where $\mathcal{F}(E)$ is a smooth function of the energy in the domain about W_n.

The quantities $1/(W_n - E)$ and W_n, are independent of the energy of the particle, E, at this singularity, and depend only upon the model assumed for the nucleus. The W_n are also invariant with respect to changes in r_0.

To show that W_n is invariant with respect to r_0, as long as $V(r) = 0$, for $r \geq r_0$, consider a new wave-function φ_n defined by (B-5) and (B-6) when r_0 is changed into $r_1 > r_0$. It is readily seen that (B-6) is satisfied by $\varphi_n = Av_n$ and the boundary conditions are

satisfied since:

$$\psi_n(r) = A v_n(r) = A v_n(r_0) e^{i k_n (r - r_0)} \tag{B-14}$$

(for $r > r_0$), and A constant. W_n is not changed.

To show that $1/(W_n - E)$ is not changed it is sufficient to show that the first term on the right of (B-13) is unchanged, since the total scattering is not changed and F(E) is independent of the W_n. In this respect it becomes sufficient to show that the factor multiplying $1/(W_n - E)$ does not change.

In terms of r_1 this factor is:

$$\frac{\psi_n^2(r_1)\, e^{-2ik_n r_1}}{\int_0^{r_1} \psi_n^2(r)\, dr + i \dfrac{\psi_n^2(r_1)}{2k_n}} \tag{B-15}$$

$$= \frac{v_n^2(r_0)\, e^{2ik_n(r_1 - r_0)}\, e^{-2ik_n r_1}}{\int_0^{r_0} v_n^2(r)\, dr + v_n^2(r_0) \dfrac{e^{2ik_n(r_1 - r_0)} - 1}{2ik_n} + i v_n^2(r_0) \dfrac{e^{2ik_n(r_1 - r_0)}}{2k_n}}$$

Obviously this reduces to (B-13), and it is concluded that $1/(W_n - E)$ does not change. It has now been verified that the definition of resonance scattering as the singular part of the scattering and the "potential scattering" as the regular part, has removed the dependence upon r_0 from the scattering for $r > r_0$. r_0 may now be chosen as the observed nuclear radius and be made consistent with the theory of nuclear

structure, without fear of difficulty when chosen a bit too large.

Equation (B-13) may be expressed in a slightly different form. Using Eq. (B-5) and its complex conjugate, and following the same method used to derive (B-8), there results:

$$\text{(B-16)} \qquad \frac{\hbar^2}{2M}\left|v_n(r_0)\right|^2 = \frac{\Gamma_n}{k_n + k_n^*}\int_0^{r_0}|v_n|^2\,dr$$

(where Γ_n is defined by $W_n = (E_n - \frac{i}{2}\Gamma_n)$: E_n and Γ_n being real).

For sufficiently small Γ_n, (as shown by Kapur and Peierls) the wave-function $v_n(r)$ may be made real by multiplying by a constant, A, ($A = \alpha + i\beta$) and ($A^2 = 1$). This makes $Av_n(r)$ real in the region where it contributes to the integral: $\left\{\int_0^{r_0}|v_n(r)|^2\,dr\right\}$

Thus in this limit A is fixed by:

$$\text{(B-17)} \qquad \int_0^{r_0} A^2\,(v_n(r))^2\,dr = \int_0^{r_0}|v_n(r)|^2\,dr$$

It is apparent then that:

$$\text{(B-18)} \qquad A\,v_n(r) = |v_n(r_0)|\,e^{i\left(k_n r_0 + \frac{\delta_n}{2}\right)}$$

where δ_n is a phase-factor determined by the properties of the compound

state (n), and independent of r_0. Thus Eq. (B-13) becomes:

$$\text{(B-19)}\quad \frac{S}{I} = \frac{1}{W_n - E} \cdot \frac{\left(\frac{\hbar^2}{2M}\right) |V_n(r_0)|^2 e^{i\delta_n}}{\int_0^{r_0} |V_n(r_0)|^2 dr + \frac{i}{2k_n} |V_n(r_0)|^2 e^{2i(k_n r_0 +}$$

or using (B-16) this becomes:

$$\text{(B-20)}\quad \frac{S}{I} = \frac{\Gamma_n e^{i\delta_n}}{W_n - E} \cdot \frac{1}{k_n + k_n^* + \frac{i M}{\hbar^2 k_n} e^{2i(k_n r_0 + \frac{\delta_n}{2})}}$$

Now as $\Gamma_n \to 0$, k_n becomes real and the last term in the denominator of (B-20) negligible, and hence in this limit:

$$\text{(B-21)}\quad \frac{S}{I} = \frac{\Gamma_n}{E_n - E - \frac{i}{2}\Gamma_n} \cdot \frac{e^{i\delta_n}}{2k_n} + \mathcal{F}(E)$$

the sum of the singularities with small Γ_n may be subtracted from (S/I) and the difference will be the function, F(E) or, from (B-21):

$$\text{(B-22)}\quad \mathcal{F}(E) = \frac{S}{I} - \sum_n{}' \frac{\Gamma_n}{E_n - E - \frac{i}{2}\Gamma_n} \cdot \frac{e^{i\delta_n}}{2k_n}$$

(The prime on the summation indicates that the sum is to be taken only over those levels for which Γ_n is small.) The resulting function, F(E), is a smooth function of the energy and is to be called the "potential scattering" term though in general it will contain contributions from the broad levels, as well. The requirement that the sum be extended only over narrow levels is to insure its convergence, just as in the treatment by Kapur and Peierls.

The scattering cross section is now given by:

$$(B\text{-}23)\quad \sigma(E) = 4\pi\left|\frac{S}{I}\right| = 4\pi\left|\sum_n{}' \frac{\Gamma_n}{(E_n - E - \frac{i}{2}\Gamma_n)} \cdot \frac{e^{i\delta_n}}{2k_n} + \mathcal{F}(E)\right|^2$$

It may be observed that the existence of broad levels overlapping each other, has in no way effected the derivation. They do, however, provide a limit in the computing of the cross section.

(B-23) can easily be shown to reduce to the ordinary formulations of the one level formula, derived by perturbation methods. The usual assumptions are that the levels are sharp, F(E) is small, and the cross section is to be examined in the region of a particular level. The sum reduces to one term. Clearing the fraction and squaring, there occur resonance terms, and interference terms. As $E \longrightarrow E_n$, however, only the resonance terms are of significance, and the cross section is given by the following relationship:

$$\sigma(E) = \frac{\pi}{k^2}\left|\frac{\Gamma_n}{(E_n - E - \frac{i\Gamma_n}{2})}\right|^2$$

(B-24)

$$= \frac{\pi}{k^2} \cdot \frac{\Gamma_n^2}{(E_n - E)^2 + \frac{1}{4}\Gamma_n^2}$$

Nothing so far has been said about resonance reactions, however both of the foregoing derivations may be generalized to allow different kinds of reactions to occur.

(C) Derivation by Wigner (15).

In this more general method for deriving the dispersion formula, the resonance scattering is presented first, to serve as a model for the generalization to include resonance reactions.

The following assumptions about the system are made and the consequences of each noted:

(a) If the particles are all close together, forming a compound nucleus, the wave function of that compound nucleus is, apart from a constant, independent of the mode of formation of that compound nucleus. This means that it for instance the $_5B^{11}$ nucleus is formed by the following reactions:

$$_3Li^7 + {}_2He^4 \longrightarrow {}_5B^{11}$$
$$_4Be^9 + {}_1H^2 \longrightarrow {}_5B^{11}$$
$$_5B^{10} + {}_0n^1 \longrightarrow {}_5B^{11}$$
$$_5B^{11} + \gamma \longrightarrow {}_5B^{11}$$
etc,

then the wave functions of the $_5B^{11}$ nucleus is the same for all of the reactions. From the preceding derivation it is recognized that this assumption restricts one to considering only those energies near a resonance peak. A subsequent derivation by E. P. Wigner, (16), has been presented in which the region between the resonance peaks are considered.

(b) The amplitude of the wave function of the compound nucleus will be very large compared to the amplitude of the wave function representing other configurations of

the particles. This can be recognized as the assumption of sharp resonances, or narrow widths, and consequently a long lifetime for the compound state.

(c) It will be assumed that the plane wave which describes the collision will be decomposed into spherical waves (a usual description) and the one with zero angular momentum is responsible for the scattering. This is a good approximation for low energies, but the resulting expression is not expected to hold for high energies because of assumptions (a) and (b).

The method used, is that of stationary states (as was used previously) with constant incident flux and constant outgoing flux. The wave function representing the system when the particles are far apart, may then be written:

$$\text{(C-1)} \qquad \varphi_E = \frac{1}{\sqrt{4\pi}\, u_E r}\left(e^{-ikr} - U_E e^{ikr} \right) \psi(i)$$

Where: $u = \sqrt{v}$, and v is the relative velocity of the particle and nucleus; M is the reduced mass of the system.

k is the wave-number $\left(k = \frac{Mv}{\hbar} \right)$

$\psi(i)$ is the normalized, real, wave function of the internal coordinates of the colliding particles.

The first term represents the decomposed plane wave of the incident beam, and the second term represents the scattered wave.

Since both waves are normalized to unit flux, the conservation of particles demands that $|U_E| = 1$.

The factor in the brackets may be written as: $(-2ie^{i\delta})\sin(kr + \delta)$ and $U_E = e^{2i\delta}$ where δ is now the phase-shift.

U_E depends upon the energy, but $\Psi(i)$ is independent of E

When the compound nucleus is formed the wave function, according to the first assumption may be written:

(C-2) $\varphi_E = \alpha_E \Psi$ where Ψ is the normalized wave function of all coordinates of all particles.

and Ψ is independent of E.

α_E is independent of the space variables but is a function of E, such that $|\alpha_E|^2$ has the dimensions of time and may be interpreted as the lifetime of the compound state.

The existence of a sphere about the nucleus, completely containing the nucleus, is now assumed. The radius is chosen as small as possible consistent with with the requirement that φ_E has the form of Eq. (C-1) on the surface of the sphere.

Now if φ_1 represents a state with relative energy E_1, and φ_2 corresponds to E_2, then:

(C-3) $H\varphi_1 = E_1\varphi_1$

(C-3a) $H\varphi_2 = E_2\varphi_2$

Multiplying Eqs. (C-3) by φ_2^* and the conjugate

of equation (C-3a) by ψ_1, and subtracting, there results:

$$(\psi_2^* H \psi_1 - \psi_1 H \psi_2^*) = (E_1 - E_2)(\psi_1 \psi_2^*)$$

$$\{ \psi_1 \psi_2^* = \psi_2^* \psi_1 \}$$

This may be integrated over the sphere:

$$\int_V dv \left(-\frac{\hbar^2}{2M}\right)\left[\psi_2^*(\Delta_1 + V(r))\psi_1 - \psi_1(\Delta_2 + V(r))\psi_2^*\right]$$

$$= (E_1 - E_2)\int_V \psi_1 \psi_2^*$$

The potential terms drop out, and there is left:

$$(C\text{-}4)\quad -\frac{\hbar^2}{2M}\int_V dv\left[\psi_2^* \Delta_1 \psi_1 - \psi_1 \Delta_2 \psi_2^*\right] = (E_1 - E_2)\int_V dv\, \psi_1 \psi_2^*$$

The straightforward substitution of (C-1) with appropiate indices, into (C-4), and using the symmetrical Green's Theorem, yields:

$$(C\text{-}5)\quad -\frac{\hbar^2}{2M}\int ds\,(\psi_2^* \operatorname{grad} \psi_1 - \psi_1 \operatorname{grad} \psi_2^*)_n$$

$$= (E_1 - E_2)\int dV\, \psi_1 \psi_2^*$$

The index, n, indicates the normal component.

This appears to contradict Eq. (C-2), but the sphere was chosen so that Eq. (C-1) was applicable to the sphere. Eq. (C-5) actually corresponds to an approximation of one higher order than Eq. (C-2). This can be shown by perturbation calculus.

It may be assumed that the wave-function ψ_E corresponding to energy E, is known, with the exception of

the quantites U_E. It is required to find the cross-section as a function of energy. This will be done by relating the cross-section to the energy through the quantities U_E. The exact theory of scattering (7, pg 210) shows that the total cross-section for a scattering process to be given by:

$$\sigma(E) = \sum_l \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$$

with the assumption that ($l = 0$) this becomes

$$\sigma(E) = \frac{4\pi}{k^2}\sin^2\delta_0$$

The above treatment, however, assumes that the scattering center is a solid sphere and does not relate the scattering to the internal structure. The treatment will still be exact if this dependence upon the internal structure could be established through the phases. Eq. (C-5) shows how this connection will be established.

Let Eq. (C-1) be substituted into the left hand side of Eq. (C-5) and Eq. (C-2) into the right hand side. Then neglecting in the sense of assumption (b) the region where Eq. (C-2) does not hold, one gets: (performing the integration).

(C-6)
$$\frac{-\hbar^2}{2Mu_1u_2}\left(e^{ik_2a} - U_2^* e^{ik_2a}\right)(ik_1)\left(-e^{-ik_1a} - U_1 e^{ik_1a}\right)$$
$$\frac{+\hbar^2}{2Mu_1u_2}\left(e^{-ik_1a} - U_1 e^{ik_1a}\right)(ik_2)\left(e^{ik_2a} + U_2^* e^{-ik_2a}\right)$$
$$= (E_1 - E_2)\alpha_1\alpha_2^*$$

Where (a) is the radius of the sphere. Now if (ka) << 1, (remembering $k = Mu^2/\hbar$) Eq. (C-6) becomes:

$$(C\text{-}7)\quad \frac{u_1}{u_2}(1+U_1)(1\ U_2^*) + (1+U_1)(1+U_2^*)\frac{u_2}{u_1} = \frac{2i}{\hbar}(E_1-E_2)\,\alpha_1\alpha_2^*$$

If ka is not small (C-7) is still true for:

$$(C\text{-}7a)\quad U_1 = \bar{U}_1 e^{-2ik_1a}\ ;\ U_2^* = \bar{U}_2^* e^{2ik_2a}$$

$$\alpha_1 = \bar{\alpha}_1 e^{-ik_1a}\ ;\ \alpha_2^* = \bar{\alpha}_2^* e^{ik_2a}$$

It is evident that putting $E_1 = E_2$, hence $v_1 = v_2$, and $U_1 = U_2$, yields: (from Eq. (C-7))

$$(C\text{-}8)\quad |U_1 U_1^*| = |U_1|^2 = 1$$

and the conservation of particles is still satisfied.

The condition that the wave-function of the compound nucleus be real in the vicinity of a resonance level may be obtained by considering the following: Since the wave functions of the nucleus must conform to assumption (a), the conjugate complex of φ_E must be, apart from a constant, equal to φ_E. The only constant which changes in (C-1) in going to φ_E^* is the U_E, since $\Psi(1)$ is real. The constant then which changes $\alpha_E \Psi$ to $\alpha_E^* \Psi^*$ is $(-U_E^*)$. Hence: $\alpha_E^* \Psi = -U_E^* \alpha_E \Psi$
and this shows that Ψ may be chosen real, and as a consequence,

$$(C\text{-}9)\quad \alpha_E^* = -U_E^* \alpha_E\ ;\ \alpha_E = -U_E \alpha_E^*$$

This equation holds for the barred quantities in (C-7) as well as for the unbarred ones.

The equations (C-7) through (C-9) represent an $(\infty)^2$ of equations for the double infinity of quantities α_E and U_E. These equations may be solved by making the following substitution:

$$\text{(C-10)}\quad \bar{U} = \frac{1+i\nu\bar{S}}{1-i\nu\bar{S}} \quad \text{or} \quad \nu\bar{S} = i\,\frac{1-\bar{U}}{1+\bar{U}}$$

where S is real. The physical interpretations of S is that, if multiplied by the velocity, it should give the tangent of the phase shift. This is equivalent to the trigonometric substitution,

$$\tan\frac{\vartheta}{2} = \sqrt{\frac{1+\cos\vartheta}{1-\cos\vartheta}} \quad ; \quad \nu\bar{S} \simeq \tan\delta = i\sqrt{\frac{1+e^{2i\delta}}{1-e^{2i\delta}}}$$

The substitutions for the barred quantities shows that the trigonometric interpretation will be only approximate depending on (ka) being small.

For the quantity $\sin^2\delta$, one now has:

$$\text{(C-11)}\quad \sin^2\delta = \frac{1}{4}\left(2 - e^{2i\delta} - e^{-2i\delta}\right) = \frac{1}{4}\left(2 - \bar{U}e^{-2ika} - \bar{U}^* e^{2ika}\right)$$

or putting in the substitution for $(\bar{U})$

$$\sin^2\delta = \frac{1}{4}\left[2 - \frac{1+i\nu\bar{S}}{1-i\nu\bar{S}}e^{-2ika} - \frac{1-i\nu\bar{S}}{1+i\nu\bar{S}}e^{+2ika}\right]$$

$$\text{(C-12)}\qquad = \frac{(\nu\bar{S}\cos ka - \sin ka)^2}{1+\nu^2\bar{S}^2}$$

or if ka << 1, Eq. (C-12) becomes:

(C-13) $$\sin^2\delta = \frac{(vS)^2}{1+v^2S^2}$$

where $S = \bar{S}\cos ka - \frac{1}{v}\sin ka$

or putting $k = Mv/\hbar$ and $\sin ka \simeq ka$

(C-14) $$S \sim \bar{S} - \frac{Ma}{\hbar}$$

If Eq. (C-7) be divided by $\{-iu_1(1+\bar{U}_1)(1+\bar{U}_2^*)u_2\}$ one obtains:

(C-15) $$-\bar{S}_2 + \bar{S}_1 = \frac{2}{\hbar}(E_1 - E_2)\frac{\bar{\alpha}_1}{u_1(1+\bar{U}_1)}\quad\frac{\bar{\alpha}_2^*}{u_2(1+\bar{U}_2^*)}$$

Since the S's are real, the right hand side must be real, hence if:

(C-16) $$\frac{\bar{\alpha}_1}{u_1(1+\bar{U}_1)} = b_1\omega$$ (b_1 is real and ω is the phase factor)

it follows that $\omega = \pm i$, $+i$ may be chosen since b_1 may be allowed to be negative, and ω is independent of energy. Putting Eq. (C-16) into Eq. (C-15):

(C-18) $$\bar{S}_1 - \bar{S}_2 = \frac{-2}{\hbar}(E_2 - E_1)b_1 b_2$$

or for other pairs of energies E_1, E_3: E_2, E_3:

(C-18a) $$\bar{S}_2 - \bar{S}_3 = \frac{2}{\hbar}(E_2 - E_3)\, b_2 b_3$$

(C-18b) $$\bar{S}_3 - \bar{S}_1 = \frac{2}{\hbar}(E_3 - E_1)\, b_3 b_1$$

adding these, one finds:

$$(C\text{-}19)\ (E_2-E_1)\,b_1 b_2 + (E_3-E_2)\,b_2 b_3 + (E_1-E_3)\,b_1 b_3 = 0$$

or obviously:

$$(C\text{-}19a)\quad b_1 = \frac{(E_2-E_3)\,b_2 b_3}{E_1(b_1-b_2) + E_2 b_2 - E_3 b_3}$$

It may be easily verified that if the energies are taken at equal intervals (and they may, since they are essentially a continuum) then the b's are multiples of a common value (b_0). To show this put $E_2 - E_1 = \Delta$ and

$$b_n = \frac{\frac{1}{2}\hbar b_0}{E_n - E_0} = \frac{1}{2}\,\frac{\hbar b_0}{n\Delta}$$

putting this into Eq. (C-19), it becomes after obvious cancellations:

$$(C\text{-}19b)\quad \frac{1}{\Delta} + \frac{1}{\Delta} + \frac{2}{-\Delta} = 0 \qquad \left\{(E_3-E_1) = -(E_1-E_3)\right\}$$

Now writing for b_1:

$$b_1 = \frac{\frac{1}{2}\hbar b_0}{E_1 - E_0}$$

in which b_0 and E_0 are independent of the energy. Putting this into Eq. (C-18) one gets:

$$\bar{S}_1 - \bar{S}_2 = \frac{1}{2}\hbar b_0^2\,\frac{(E_1-E_2)}{(E_1-E_0)(E_2-E_0)}$$

$$(C\text{-}20)\qquad = \frac{1}{2}\hbar b_0^2\left(\frac{-1}{E_1-E_0} + \frac{1}{E_2-E_0}\right)$$

Now writing this equation for all pairs of energy

in the order of increasing index and adding them, it is apparent that all except two cancel, hence:

$$(C\text{-}21)\quad \bar{S}_1 - \bar{S}_\infty = \frac{-\frac{1}{2}\hbar b_0^2}{E_1 - E_0}$$

where S_∞ is independent of E_1. Using now the definition of S, Eq. (C-10) becomes:

$$(C\text{-}22)\quad \bar{U}_1 = \frac{(1 + i v_1 S_\infty)(E_1 - E_0) - \frac{1}{2} i \hbar b_0^2 v_1}{(1 - i v_1 S_\infty)(E_1 - E_0) + \frac{1}{2} i \hbar b_0^2 v_1}$$

and with Eq. (C-16) this becomes:

$$(C\text{-}23)\quad \bar{\alpha}_1 = \frac{i \hbar b_0 \bar{U}_1}{(1 - i v_1 S_\infty)(E_1 - E_0) + \frac{1}{2} \hbar b_0^2 v_1}$$

If, however, $ka << 1$ then S differs from $\bar{S}$ only by $Ma/\hbar$, but $Ma/\hbar$ may be absorbed into the S_∞ and henceforth use the unbarred quantities, in Eq. (C-21).

Recalling that the cross-section was given by $(4\pi/k^2)\sin^2\delta$, in terms of U, this is:

$$\sigma(E) = \frac{\pi}{k^2}(2 - U - U^*) = \frac{4\pi}{k^2}\frac{(vS)^2}{1 + v^2 S^2}$$

Since Eq. (C-22) retains its form even when multiplied by e^{-2ika}. Putting $k^2 = M^2 v^2/\hbar^2$ into the last relation above, it may be written as:

$$\sigma(E) = \frac{4\pi(\hbar S/M)^2}{1 + v^2 S^2}$$

or putting $S = \bar{S}$ into

the last relation from Eq. (C-21) the cross-section becomes:

$$(C\text{-}24)\quad \sigma(E) = \frac{4\pi\hbar^2}{M^2} \cdot \frac{\left[S_\infty (E_1 - E_0) - \frac{1}{2}\hbar b_0^2\right]}{\left[S_\infty (E_1 - E_0) - \frac{1}{2}\hbar b_0^2\right]^2 + (E_1 - E_0)^2}$$

The usual dispersion formula is obtained from Eq. (C-24) if one puts $S_\infty = 0$ and $\Gamma \equiv \hbar b_0^2 v$,

$$(C\text{-}25)\quad \sigma(E) = \frac{\pi}{k^2} \frac{\Gamma^2}{\frac{1}{4}\Gamma^2 + (E - E_0)^2}$$

This is the usual "one-level" formula.

Equation (C-24) may be written in a different way, which has the appearance of the resonance formula even if $S_\infty \neq 0$, putting: $\Gamma = b_0^2 \hbar v / (1 + v^2 S_\infty^2)$

$$(C\text{-}26)\quad \sigma(E) = \frac{\pi}{k^2(1 + v^2 S_\infty^2)} \left\{ 4 S_\infty^2 v^2 + \frac{\Gamma^2 - 4\Gamma S_\infty v (E - E_0)}{\frac{1}{4}\Gamma^2 + (E - E_0 - \frac{1}{2}\Gamma S_\infty v)^2} \right\}$$

where the first term may be called the "potential scattering", and the second contains the sum of the resonance scattering and terms due to interference between potential and resonance scattering.

Although it has not been explicitly stated, the formula derived is for a single energy level, E_0. If more resonances are present, the same formula will apply to them, only with a new E_0. The region of validity for this relation (except when other resonances are present

and close by) extends approximately one Mev on either side of the resonance.

It is usually assumed that the scattering at infinite energy is zero, and as a consequence the S_∞ is put to zero, (corresponds to zero phase shift) but it has been shown by Wigner in a report to the Uranium Project (unavailable) that Eq. (C-24) is in general correct.

The interpretation of the quantity (b_o) is rather evident from the definition of (Γ) leading to Eq. (C-25). Since Γ is $\hbar$ divided by the mean life (τ) of the compound state, $1/b^2$ may be interpreted as the average distance the escaping particle could cover during the life-time, $\ell = v\tau = (v\hbar/\Gamma) = 1/b^2$ Putting in experimentally observed Γ's for (n, γ) reactions, it is found that $1/b^2$ is $\sim 10^6$ greater than the nuclear radius. This agrees well with the discussed lifetime in Chapt. I.

(D) Resonance Reactions (16).

The case of resonance reaction is now to be discussed. The generalization allows the possibility of more than one method of decay. It will be assumed again that no spin is involved, the angular momentum is zero. Assumptions (a) and (b) remain, but assumption (c) will need modification. The asymptotic wave-function will in general contain both incoming and outgoing waves, corresponding to the several possible modes of formation and disintegration for instance, the reactions of $Li^7 + He^4 \rightarrow B^{11}$ and $B^{11} \rightarrow Li^7 + {}_2He^4$. These different

possibilities will be distinguished by the indices j,l,...

It may be assumed that one can prepare the system so that it contains only one kind of incoming wave, say the reaction envolving a pair "1" as the reacting components. The state of the system may then be specified by the wave-equation:

$$\text{(D-1)}\quad \psi_l = \sum_j \frac{1}{\sqrt{4\pi}\,(r_j u_j)} \left(e^{-i k_j r_j} \delta_{lj} - U_{lj}\, e^{i k_j r_j} \right) \psi_j(i_j)$$

where

$\psi_j(ij)$-is the product of the real, normalized wave-functions of the two reacting particles, which correspond to disintegration into the pair (j).

r_j-is the distance between the particles which produce the pair (j).

u_j-is the square root of their velocity.

The first term in the equation corresponds to the approaching particles, the δ_{lj} indicates they are of the type (l) only (Kronecker δ). (For $j = l$, the system reduces to the foregoing resonance scattering.)

The U_{lj} are the elements of the "collision matrix" introduced by J. A. Wheeler, or the Heisenberg S-Matrix. (17,18,19). It is a set of (n^2) constants, if the number of possible disintegrations is (n). The quantity $|U_{lj}|^2$ is the probability of a reaction, yielding the pair (j) if the pair (l) collide.

The form of ψ_j is assumed to be as in the case

of scattering:

(D-2) $$\varphi_j = \alpha_j \Psi$$

subject to assumption (a).

The energy dependent quantities u_j, k_j, $U_{\ell j}$, and α_j will be given a number index to correspond to the different relative energies E_1, E_2, ...

The Matrix, $||U_{ij}||$, will be denoted by U, and when it corresponds to energy E_1, it will be denoted by U_1.

The quantities α_j will be considered a vector of (n) components and denoted by $[\alpha]$. The vector, $[\alpha]$, will also be given energy subscripts in the manner of U_1.

The quantities u_1, u_2, ... will be defined as a real diagonal matrix and will be denoted u_1, u_2, ..., etc.

The equation corresponding to Eq. (C-5) may be derived first by writing:

(D-3) $$H\varphi_{1m} = E_1 \varphi_{1m}$$
$$H\varphi_{2\ell} = E_2 \varphi_{2\ell}$$

The significance of the indices is obvious from the foregoing discussion.

In a manner entirely analogous to the derivation of Eq. (C-5), the first of Eq.'s (D-3) is multiplied by $\varphi^*_{2\ell}$ and the second by the conjugate of φ_{1j}, the equations subtracted and integrated over the sphere to get:

(D-4) $$-\frac{\hbar^2}{2M}\int dS\,(\varphi^*_{2\ell}\,\nabla\varphi_{1m} - \varphi_{1m}\,\nabla\varphi^*_{2\ell})_n = (E_1 - E_2)\int dV\,\varphi_{1m}\varphi^*_{2\ell}$$

Eq. (D-1) is now substituted into the left-hand side and Eq. (D-2) into the right hand side, and the equivalent of Eq. (C-5) is given after the integration by:

$$\sum_j \frac{-\hbar^2}{2M_j u_{1j} u_{2j}} \left(e^{ik_{2j}a_j} \delta_{\ell j} - U_{2\ell j}^{*} e^{-ik_{2j}a_j} \right)(ik_{1j}) \times$$

$$\left(-e^{-ik_{1j}a_j} \delta_{mj} - U_{1mj} e^{ik_{1j}a_j}\right) + \sum_j \left(\frac{\hbar^2}{2M_j u_{1j} u_{2j}}\right) \times$$

(D-5)
$$(ik_{2j})\left(e^{-ik_{1j}a_j} \delta_{mj} - U_{1mj} e^{ik_{1j}a_j}\right) \times$$

$$\left(e^{ik_{2j}a_j} \delta_{\ell j} + U_{2\ell j}^{*} e^{-ik_{2j}a_j}\right)$$

$$= (E_1 - E_2)\, \alpha_{1m}\, \alpha_{2\ell}^{*}$$

With the assumption that (ka) is small this becomes:

$$\sum_j i\hbar \left(\delta_{mj} - U_{1mj}\right) u_{1j} u_{2j}^{-1} \left(\delta_{\ell j} - U_{2\ell j}\right) + i\hbar \left(\delta_{mj} - U_{1mj}\right) \times$$

(D-6)
$$u_{1j}^{-1} u_{2j} \left(\delta_{\ell j} + U_{2\ell j}^{*}\right) = 2 (E_1 - E_2)\, \alpha_{1m}\, \alpha_{2\ell}^{*}$$

or in the matrix notation adopted above:

$$i\hbar \left(1 + U_1\right) a_1 a_2^{-1} \left(1 - U_2^{\dagger}\right) + i\hbar \left(1 - U_1\right) a_1^{-1} a_2 \left(1 + U_2^{\dagger}\right)$$

(D-7)
$$= 2 (E_1 - E_2)\, (A_{12})$$

where $(A_{12})_{m\ell} = \alpha_{1m} \alpha_{2\ell}^{*}$ and $1_{m\ell} = \delta_{m\ell}$

Eq. (D-7) corresponds to Eq. (C-7) and involves the same approximations.

Putting $E_1 = E_2$, it follows that $a_1 = a_2$; $U_1 = U_2$; and Eq. (D-7) becomes:

(D-8)
$$\left(1 + U_1\right)\left(1 - U_1^{\dagger}\right) + \left(1 - U_1\right)\left(1 + U_1^{\dagger}\right) = 2 - 2 U_1 U_1^{\dagger} = 0$$

It follows then that

$$U_1 U_1^{\dagger} = 1 \qquad (1) \text{ is a unit matrix, and}$$

therefore the matrices U_1 are unitary.

If Eq. (D-1) and Eq. (D-2) represent a solution then so must their conjugate imaginary. In the conjugate imaginary the second term in the bracket represents the incoming waves, hence:

$$\text{(D-9)} \qquad \varphi_\ell^* = -\sum_j U_{\ell j}^* \varphi_j$$

Since the incoming wave on the right above are the same as on the left, the two sides must be equal. It follows that the outgoing wave corresponding to the pair (m), is also the same, or:

$$\frac{1}{\sqrt{4\pi}\,(r_m u_m)} e^{i k_m r_m} \delta_{m\ell} = -\sum_j U_{\ell j}^* \frac{U_{jm}}{\sqrt{4\pi}\,(r_m u_m)} e^{i k_m r_m}$$

and for this to be so, it must follow that:

$$\delta_{m\ell} = \sum_j U_{\ell j}^* U_{jm} \quad \text{or} \quad U^* U = 1$$

With this relation and the previous one, $U U^{\dagger} = U^{\dagger} U = 1$ it follows that $U^* = U$ or the matrices U are symmetric.

The realitycondition then follows in the same manner, for: $\alpha_\ell^* \Psi^* = -\sum_j U_{\ell j}^* \alpha_j \Psi$

and consequently:

$$\text{(D-10)} \qquad \alpha_\ell^* = -\sum_j U_{\ell j}^* \alpha_j \quad \text{and} \quad \alpha_\ell = -\sum_j U_{\ell j} \alpha_j^*$$

or in matrix notation,

(D-10a) $[\alpha_i^*] = -U_i^* [\alpha_i]$ and $[\alpha_i] = -U_i [\alpha_i^*]$

The following table (ref. 20), will define the relationships between some of the special types of matrices.

Relation	Name of A	Relationship of Elements
$A = \tilde{A}$	Symmetric	$A_{pq} = A_{qp}$
$A = \tilde{A}^{-1}$	Orthogonal	Satisfies: $\sum_s A_{ps} A_{qs} = \sum_s A_{sp} A_{sq} = \delta_{pq}$
$A = A^*$	Real	$A_{pq} = A_{pq}^*$
$A = -A^*$	Pure Imaginary	$A_{pq} = iB_{pq}$ (B_{pq} is real)
$A = A^\dagger$	Hermitean	$A_{pq} = A_{qp}^*$
$A = (A^\dagger)^{-1}$	Unitary	Satisfies: $\sum_s A_{ps}^* A_{qs} = \sum_s A_{sp}^* A_{sq} = \delta_{pq}$

$\tilde{A}$ ≡ transposed matrix of A (rows and columns interchanged).

To solve Eqs. (D-7) through (D-10a) it is again necessary to substitute for the quantities U_i, as in the case of resonance scattering; therefore the quantities S_i' are defined by:

(D-11) $U_i = \dfrac{1 + i\alpha_i S_i' \alpha_i}{1 - i\alpha_i S_i' \alpha_i}$ and $\alpha_i S_i' \alpha_i = i \dfrac{1 - U_i}{1 + U_i}$

The quantities S' are matrices as were the U's and have the same number of rows and columns. Each row and column corresponds to a particular mode of disin-

tegration of the compound nucleus. The matrices S' are symmetric, hermitean, and unitary because the matrices U are also. They are therefore real symmetric matrices and:

(D-12) $$S_1' = S_1'^{*} = \tilde{S}_1' = S_1'^{\dagger}$$

Using the transformation Eq. (D-11) in Eq. (D-7), it becomes:

$$i\hbar\left[1 + \frac{(1+iU_1S_1'U_1)}{(1-iU_1S_1'U_1)}\right]U_1U_2^{-1}\left[1 + \frac{(1+iU_2S_2'U_2)}{(1-iU_2S_2'U_2)}\right]$$

$$+i\hbar\left[1 - \frac{(1+iU_1S_1'U_1)}{(1-iU_1S_1'U_1)}\right]U_1^{-1}U_2\left[1 + \frac{(1+iU_2S_2'U_2)}{(1-iU_2S_2'U_2)}\right]$$

$$= 2(E_1 - E_2)(A_{12})$$

Multiplying on the left by $\{U_1^{-1}(1-iU_1S_1'U_1)\}$ and from the right by $\{(1+iU_2S_2'U_2)\,U_2^{-1}\}$ (in matrix algebra this is equivalent to the division which leads to Eq. (C-15), this becomes:

$$-S_2' + S_1' = \frac{1}{2\hbar}(E_1 - E_2)\,U_1^{-1}(1-iU_1S_1'U_1)(A_{12})(1+iU_2S_1'U_2)\,U_2^{-1}$$

(D-13)

$$= \frac{2}{\hbar}(E_1 - E_2)(B_{12})$$

where (B_{12}) is defined by the last part of the Eq. (D-13).

from the definition of (A_{12}), Eq. (D-6):

$$(B_{12})_{m\ell} = \beta_{1\ell}\beta_{2m}^{*} \qquad \text{where:}$$

(D-14)

$$\beta_{1m} = \frac{1}{2}\sum_{K} U_{1m}^{-1}(1-iU_1S_1'U_1)_{mK}\,\alpha_{1K}$$

and the vector $[\beta]$ is thus defined in matrix notation as:

(D-15) $$[\beta_1] = \frac{1}{2}U_1^{-1}(1-iU_1S_1'U_1)[\alpha_1]$$

Since $\mathcal{S}_1'$ and $\mathcal{S}_2'$ are real it is apparent that (B_{12}) is real (from Eq. (D-13)).

Since $\mathcal{S}_1'$ and $\mathcal{S}_2'$ are symmetric (B_{12}) must be symmetric. It follows then that:

$$\beta_{1m}\,\beta_{2\ell}^{*} = \beta_{1\ell}\,\beta_{2m}^{*} \quad \text{or} \quad \frac{\beta_{1m}}{\beta_{1\ell}} = \frac{\beta_{2m}^{*}}{\beta_{2\ell}^{*}}$$

and it is apparent then that the vectors $[\beta]$ are multiples of a common vector, or that:

(D-16) $$[\beta_1] = b_1\,\omega\,[\beta_0]$$

where b_1 is a real number depending on the energy, and ω and $[\beta_0]$ are independent of the energy. ω has the absolute value of unity, the vector $[\beta_0]$ is real, and it may be assumed normalized, such that:

$$[\beta_1\,\beta_0] = \sum_j \beta_j^2 = 1$$

where (β_j^2) occurs because $[\beta]$ is real.

Eq. (D-10a) may be written in terms of the matrices $\mathcal{S}'$, in the following manner:

(D-17) $$(1 + i\pi_1 \mathcal{S}_1' \pi_1)[\alpha_1^{*}] = -(1 - i\pi_1 \mathcal{S}_1' \pi_1)[\alpha_1]$$

and comparing the coefficients with Eq. (D-15) one sees that: $\pi_1 [\beta_1^{*}] = -\pi_1 [\beta_1]$ or from Eq. (D-16) $[\beta_1^{*}] = b_1\,\omega\,[\beta_0^{*}] = [\beta_1] = b_1\,\omega\,[\beta_0]$, but the vector is real hence $\omega = \pm i$, and it will be assumed the $\omega = i$. Using Eq. (D-16) and (D-15) it follows that:

(D-18) $$[\alpha_1] = 2i\,b_1\,(1 - i\pi_1 \mathcal{S}_1' \pi_1)^{-1}\,\pi_1\,[\beta_0]$$

also for Eq. (D-14):

$$(B_{12})_{ml} = \beta_{1m}\beta_{2l}^{*} = b_1[\beta_0]_m\, b_2[\beta_0]_l$$

or putting

$$[\beta_0]_m[\beta_0]_l = (B_0)_{ml}$$

it follows that

(D-19) $$(B_{12}) = b_1 b_2 (B_0)$$

also since (B_0) has the property that when multiplied by any vector the direction of that vector is changed to the direction of the vector $[\alpha]$, (in this case), and likewise the operation on $[\alpha]$ is an idempotent and has the property that

$$(B_0) = (B_0)^2 = (B_0)^3 = \cdots$$

It is apparent that Eq. (D-13) must have the form:

(D-20) $$S_1' = S_\infty' - c_1(B_0)$$ This follows from the properties of $(B_{12}) = b_1 b_2 (B_0)$, in the same manner as in the case of resonance scattering. c_1 is an unknown real function of the energy and is a real symmetric matrix independent of E_1.

Putting Eq. (D-20) into Eq. (D-13), it becomes:

(D-21) $$-c_1 + c_2 = \frac{2}{\hbar}(E_1 - E_2)\, b_1 b_2$$

This equation may be handled in an exactly similar manner as Eq. (C-15) to get relations analogous to Eq. (C-20), and from this comes the result:

(D-22) $$c_1 = \frac{\frac{1}{2}\hbar\, b_0^2}{(E_1 - E_0)} + C_\infty$$

Absorbin the constant as before, δ'_∞ is redefined as being decreased by $[c_\infty (B_0)]$, and as before c_∞ may be put equal to zero without loss of generality and as a result:

$$\delta'_1 = \delta'_\infty - \frac{\frac{1}{2}\hbar b_0^2}{(E_1 - E_0)} (B_0) \tag{D-23}$$

It is now apparent that δ' becomes equal to δ'_∞ when $E_1 \to \infty$, hence δ'_∞ gives the phase shift for bombarding particles of infinite energy.

Using $b_1 = (\frac{1}{2}\hbar b_0^2)/(E_1 - E_0)$ (a consequence of Eq. (D-22) and Eq. (D-18)), the vector $[\alpha_1]$ becomes:

$$[\alpha_1] = \frac{i\hbar b_0}{E_1 - E_0} \left(1 - i\pi_1 \delta'_1 \pi_1\right)^{-1} \pi_1 [\beta_0] \tag{D-24}$$

Where E_0 and b_0 are arbitrary real numbers, and $[\beta_0]$ is an arbitrary real vector of length one, and δ'_∞ is an arbitrary real, symmetric matrix, and putting these into the expression for U, together with $(B_0)_{jl} = [\beta_j][\beta_l]$; $(B_0) = (B_0)^2 = (B_0)^3 = \ldots$.

$$U_1 = \left(1 + i\pi_1 \delta'_\infty \pi_1 - \frac{\frac{1}{2}\hbar b_0^2}{(E_1 - E_0)} \pi_1 (B_0) \pi_1\right) \times \left(1 - i\pi_1 \delta'_\infty \pi_1 + \frac{\frac{1}{2}\hbar b_0^2}{(E_1 - E_0)} \pi_1 (B_0) \pi_1\right)^{-1} \tag{D-25}$$

It is to be noted that the ordinary division is not defined in operations with matrices (except by a constant), but instead, an inverse matrix (A^{-1}) is defined and the fivision process is replaced by an operation ($A^{-1}B$), and this operation does not in general commute,

since multiplication does not.

Clearing the fraction Eq. (D-25) becomes:

$$U_1 = [(1 + i u_1 S_\infty u_1)(E_1 - E_0) - \tfrac{1}{2}\hbar b_0^2\, u_1 (B_0) u_1]\, \times$$

(D-25a)

$$[(1 - i u_1 S_\infty u_1)(E_1 - E_0) + \tfrac{1}{2}\hbar b_0^2\, u_1 (B_0) u_1]^{-1}$$

Another way of writing this matrix, U_1 is:

$$U_1 = \frac{(1 + i u_1 S_\infty u_1)}{(1 - i u_1 S_\infty u_1)} - \frac{i\hbar b_0^2}{(E - E_0 + \frac{i}{2}\lambda)} (1 - i u_1 S_\infty u_1)^{-1} u_1 (B_0) u_1 (1 - i u_1 S_\infty u_1)^{-1}$$

(D-25b)

where

$$\lambda = \hbar b_0^2 \sum_{j\ell} \beta_j u_j (1 - i u_1 S_0 u_1)^{-1}_{j\ell} u_\ell \beta_\ell$$

and the vector $[\alpha]$ which is the coefficient of the wave function of the compound nucleus then shows the resonance behavior:

$$[\alpha] = \frac{i\hbar b_0}{E - E_0 + \frac{i}{2}\lambda} (1 - i u S u)^{-1} u [\beta_0]$$

The transformation of Eq. (D-25a) into Eq. (D-25b) is not presented, though is is not immediately apparent.

There are two special cases of interest in this general derivation. These with the angular momentum equal to zero (this assumption has not been used previously). Eq. (D-25) is the most general form of U satisfying Eq. (D-7) and Eq. (D-10), but it is seen that U goes over to:

$U = (1 - i u S_\infty u)(1 + i u S_\infty u)^{-1}$ for very large $(E - E_0)$

while $[\alpha]$ in this case goes to zero. Thus for very large $(E - E_0)$ the scattering does not go through the intermediate state. This imposes a condition, not

previously used, on the U. The condition is that for very large $(E - E_o)$, in order that there be scattering and/or a reaction then $U = 1$ (1 is a unit matrix.) or that U becomes a diagonal matrix.

These two requirements are examined in detail in the following pages.

In the first instance, it is required that the scattering and reaction go through the compound state, then this condition requires that U become:

$$U \Rightarrow (1 + i\sigma S'_\infty \sigma)(1 - i\sigma S'_\infty \sigma)^{-1} = 1$$

The Matrix $(U - 1)$ then becomes:

$$(D\text{-}26) \quad (U - 1) = \frac{-i\hbar b_o^2}{E - E_o + \frac{i}{2}\lambda} \sigma(B_o)\sigma$$

where:

$$\lambda = \hbar\, b_o^2 \sum_j v_j \beta_j$$

The cross-section for the transition from state (j) to state ( ), is then:

$$(D\text{-}27) \quad \sigma_{jl} = \frac{\pi}{k_j^2}\left|(U-1)_{jl}\right|^2 = \frac{\pi}{k_j^2}\frac{\hbar^2 b_o^4 (v_j \beta_j \beta_l v_l)^2}{(E - E_o)^2 + \frac{1}{4}\lambda^2}$$

or

$$\sigma_{jl} = \frac{\pi}{k_j^2} \cdot \frac{\Gamma_j \Gamma_l}{\frac{1}{4}\Gamma^2 + (E - E_o)^2}$$

where:

$$\Gamma_j = \hbar\, b_o^2 \beta_j^2 v_j \quad ; \quad \Gamma = \sum_j \Gamma_j$$

The more rigorous assumption that all scattering and reaction go through the intermediate state, then

leads to the usual resonance formula.

The second restriction, and less rigorous is that, for $(E - E_0)$ large, only the reactions need go through the compound state; this has as a consequence that U must be diagonal as was pointed out. But if U is diagonal so must S_∞ be diagonal. The elements of the diagonal S_∞ may be denoted s_j, then from Eq. (D-25b), one has:

$$(U-1)_{jl} = \frac{2i s_j v_j}{1 - i v_j s_j}\,\delta_{lj} - \frac{i\hbar b_0^2}{E - E_0 + \frac{i}{2}\lambda} \times \tag{D-28}$$

$$(1 - i s_j v_j)^{-1} u_j \beta_j \beta_l u_l (1 - i s_j v_j)^{-1}$$

where:

$$\lambda = \hbar b_0^2 \sum_j \frac{\beta_j^2 v_j}{1 - i s_j v_j}$$

The quantity λ may be separated into its real and imaginary parts, $\lambda = \Gamma + 2i\Delta$

where $\Gamma = \hbar b_0^2 \sum_j \frac{\beta_j^2 v_j}{1 + v_j^2 s_j^2}$ and $\Delta = \frac{1}{2}\hbar b_0^2 \sum_j \frac{\beta_j^2 s_j v_j^2}{1 + s_j^2 v_j^2}$

Now for $j \neq l$ the first term of Eq. (D-28) vanishes and the effective cross section of the reaction yielding pair (l) when the pair (j) react is given by:

$$\sigma_{jl} = \frac{\pi}{k_j^2}\left|(U-1)_{jl}\right|^2 = \frac{\pi}{k_j^2}\,\frac{\Gamma_j \Gamma_l}{\frac{1}{4}\Gamma^2 + (E - E_0 - \Delta)^2} \tag{D-29}$$

where: $\Gamma_j = \frac{\hbar b_0^2 \beta_j^2 v_j}{1 + s_j^2 v_j^2}$, $\Gamma = \sum_j \Gamma_j$

The two main differences between this equation and Eq. (D-27) are: a) the energy shift Δ, which can be shown to be small in any case, and b) the more complicated dependence of Γ on the energy.

It might be noted that the pure scattering cross section (or $l = j$) is given by:

$$\sigma_{jj} = \frac{\pi}{k_j^2} \left| (U-1)_{jj} \right|^2$$

(D-30)

$$= \frac{\pi v_j^2}{k_j^2 (1+v_j^2 s_j^2)} \left| 2 s_j (1 - i v_j s_j) - \frac{\hbar b_0^2 \beta_j^2}{E - E_0 - \Delta + \frac{1}{2} i \Gamma} \right|^2$$

Up to now the physical interpretation of the various quantities used in this derivation has remained rather vague, and they will now be give, since the physical situation may be more easily seen in the final result.

The quantities π/k_j^2 are rather obvious.

The quantities β_j form a dimensionless vector and they serve formally to separate the total width (Γ) into partial widths (Γ_j). In the usual (n, γ) processes the β (neutron) is practically unity while β (radiation) is very small, this in spite of the fact that Γ (radiation) is very much larger than Γ (neutron). This is merely a recitation of the facts, that while the scattering of γ-rays by nuclei is very small, the probability of a γ-ray being emitted in processes involving material particles is very large. There are exceptions, of course.

The interpretation of the (b_o) is the same as before but now the quantity $(1/\beta_j\, b_o^2)$ is to be interpreted as the distance covered by the particle during the mean life. The β_j serves to define the process.

The quantities Γ_j are as usual the partial widths and represent the probability (as explained in Chapt. I) of any one of the several energetically possible transitions. It should be clear at this point how the mean-lifetime of the example in Chapt. I was calculated, i.e., a typical Γ (neutron) was taken as being ($\sim$ .02 ev), this value being in keeping with measurements.

The quantities s_j are the diagonal elements of the matrix $\mathcal{S}_\infty$, and have the dimensions of (1/v). These quantities determine the normal scattering, as illustrated by Eqs. (D-29) and (D-30). For radiation scattering $|s_j| \to 0$ because normal scattering of radiation by nuclei is negligible. It follows that, unless E_j is very large indeed, $s_j v_j << 1$ and hence $\Delta << 1$.

Some general remarks can be made concerning this derivation, and they follow. First of all, it is noted that this derivation makes no assumption of structure of the nucleus except as is contained in the assumption (a). This, of course, is rather stringent as far as the shape of the levels in the nucleus is concerned, but it provides no method of obtaining or estimating the character of the levels (that is the spacing, total width, etc.) so that they may be checked by

experiment. Instead one makes the derivation and uses the experiment to fix the constants. In this sense, the derivation amounts to the expansion, in a series, of the wave function in the vicinity of a resonance level. If there exist two or more levels close together, the region of validity is seriously restricted. At the present time, the amount of experimental data deemed sufficiently accurate to warrant the fixing of the quantity (Γ) is almost nil.

The second remark is made in reference to the Heisenberg S-Matrix Theory. It is shown in reference (19) that the resonance reactions and scattering follow as a special case of the general theory of the S-Matrix. More will be said of this later.

The last remark concerns the generality of the derivation. The results of this derivation follow from the general asymptotic behavior of the wave function assumed to describe the system. It is to be expected then that less specific information concerning the nucleus is obtained. The approximate region of validity depends upon the interpretation of the quantities in the formula and these must be taken from experimental data, which after all must be the final criterion of any theory.

(E) Derivation by Bethe (9).

The system, composed of a particle and a nucleus, will be described in terms of the stationary states, (21). The initial state will consist of particle (P) and nucleus (A); the intermediate state will be only nucleus (C), (the compound nucleus); and the final state will be composed of final nucleus (B) and outgoing particle (Q). Obviously it has been assumed that only one type of reaction can take place, that is symbolically:

$$A + P \rightarrow C^{*} \rightarrow B + Q$$

It is further assumed that only one resonance level lies in the region of validity for this derivation.

Since the system consists of a free particle and a nucleus, the energies of the system lie in the continuum. The eigenvalues of the Hamiltonian then are a continuous spectrum. The Perturbation then will produce a change in the wave functions describing the system.

The following notation will be adopted:

Let, χ_A -be the wave function of the initial or target nucleus, assumed normalized and known.

χ_B -be the wave function of the final or residual nucleus, assumed normalized and known.

χ_P -be the wave function of the incident particle (a function of its internal coordinates only). If P is a simple particle, then $\chi_P = 1$.

χ_Q -be the wave function of the emitted particle (internal coordinates only) and as before if Q

is simple particle, $\chi_Q = 1$.

ψ_P-be the wave function of the relative motion of the incident particle and target nucleus. This is not assumed to be known or normalized. ψ_P consists of an incident plane wave and an outgoing, scattered, spherical wave.

ψ_Q-be the wave function of the relative motion of the emitted particle and the residual nucleus. This is not assumed known or normalized. This is solely an outgoing wave, whose amplitude determines the probability of the disintegration process, A + P → B + Q.

The total Hamiltonian (H) may be split in the following two ways:

(E-1) $H = H_A + H_P + T_P + U_P + V_{AP}$

and

(E-2) $H = H_B + H_Q + T_Q + U_Q + V_{BQ}$

where:

H_A--contains the kinetic energy and interaction of the particles within the nucleus (A).

H_P--same significance as H_A except for the incident particle, and if P is an elementary particle then H_P 0.

T_P--is the kinetic energy of the relative motion of A and P.

U_P--is the effective potential energy of the particle P. This will be the coulomb potential for charged particles and zero for uncharge particles, out-

side the nucleus. Inside the nucleus the potential U_P must be chosen so that the wave function ψ_P decreases in amplitude rapidly, in order that the wave function of the compound nucleus is dominant. This requires that U_P be a repulsion. The physical significance of this requirement is that, according to the "many-body" process discussed in Chapt. I, the particle (or aggregate of particles) is to lose its individuality when interacting with the nucleus. This is also convenient mathematically as it will help the perturbation-series to converge more rapidly (if it converges at all).

V_{AP}-contains the interaction-potential, not contained in U_P. It in general depends upon all the coordinates of all the particle of the incident end target nuclei (or particles).

The terms in the Eq. (E-2) have the same significance for the nuclei (or particles) (B) and (Q).

The internal wave functions χ_A, χ_B, χ_P, χ_Q satisfy the equation:

(E-3) $H_i \chi_i = W_i \chi_i$

The compound nucleus wave-function satisfies the equation:

(E-4) $H \chi_c = W_c \chi_c \qquad \left(H = \sum_i H_i\right)$

except at the boundary of the compound nucleus. χ_C then is restricted to a finite extension, inside the nuclear radius. This assumption will be recognized as being the same as used previously in the other derivations.

The total wave function may be written:

(E-5) $\Psi = \chi_A \chi_P \psi_P + \chi_B \chi_Q \psi_Q + C \chi_C$ (C = constant.)

and it is required to find the wave function Ψ, which satisfy the Schrodinger equation:

(E-6) $H\Psi = W\Psi$

Since the first two terms have extensions from zero to infinity, they are assumed to be the zero order wave function and the last term ($C \chi_C$) represents the perturbation or is the correction to the unperturbed wave function. Actually χ_C is assumed known and C is unknown. This assumes as was done before that the wave functions of the compound state are the same, apart from a constant, for all energies. The constant (C) may then depend on the energy.

The energy W, in Eq. (E-5) may be written as the sum of the energies of the components of the system:

(E-7) $W = W_A + W_P + E_P$

Where W_A and W_P are the internal energies of A and P, and E_P is the kinetic energy of their relative motion.

Equation (E-5) may be written:

(E-8) $(H-W)\Psi = 0$

then it is also true that:

$$\chi_A^* \chi_P^* (H-W)\Psi = 0$$

This may be partially integrated over the coordinates of A and P (internal), and will remain

zero since the integrand is identically zero:

(E-9) $\int \chi_A^* \chi_P^* (H - W) \Psi \, d\tau_A \, d\tau_P = 0$

In a like manner, one has:

(E-10) $\int \chi_B^* \chi_Q^* (H - W) \Psi \, d\tau_B \, d\tau_Q = 0$

and

(E-11) $\int \chi_C^* (H - W) \Psi \, d\tau_C = 0$

It is evident that when the dependence upon χ_A, χ_P, χ_B, and χ_Q are integrated out, the first two Eqs., (E-9) and (E-10), will become a function of the relative coordinates of A and P, and B and Q, only. They are differential equations of the two functions ψ_P and ψ_Q. The third Eq., (E-11), will serve to fix the constant (C).

Using the expression for Ψ, H, and W given by Eqs. (E-1), (E-7), and (E-5), Eq. (E-8) becomes:

(E-12) $$\int \chi_A^* \chi_P^* (H_A + H_P + T_P + U_P + V_{AP} - W_A - W_P - E_t) \times$$
$$(\chi_A \chi_P \psi_P + \chi_B \chi_Q \psi_Q + C \chi_C) \, d\tau_A \, d\tau_P = 0$$

It is seen immediately from Eq. (E-3) that the terms involving H_A, H_P, and W_A, W_P, will be zero so this reduces to:

(E-13) $\int \chi_A^* \chi_P^* (T_P + U_P + V_{AP} - E_P) \Psi \, d\tau_A \, d\tau_P = 0$

It is convenient to define a new operator:

(E-14) $L_P \equiv E_P - T_P - U_P$ the bracketed expression then of the Hamiltonian operator, becomes: $(V_{AP} - L_P)$, and the integral may be split into the sum of three

integrals:

$$\int \chi_A^* \chi_P^* (V_{AP} - L_P) \chi_A \chi_P \psi_P \, d\tau_A \, d\tau_P + \int \chi_A^* \chi_P^* (V_{AP} - L_P) \, c \chi_C \, d\tau_A \, d\tau_P$$

(E-15)

$$+ \int \chi_A^* \chi_P^* (V_{AP} - L_P) \chi_B \chi_Q \psi_Q \, d\tau_A d\tau_P$$

The first integral may be split again to get:

(E-15a) $$\int \chi_A^* \chi_P^* (V_{AP} - L_P) \chi_A \chi_P \psi_P \, d\tau_A \, d\tau_P - \int \chi_A^* \chi_P^* (L_P) \chi_A \chi_P \psi_P \, d\tau_A \, d\tau_P$$

Since (V_{AP}) does not contain differentiations with respect to (r_A) or (r_P), the first integral may be written:

(E-15b) $$\int |\chi_A|^2 |\chi_P|^2 V_{AP} \psi_P \, d\tau_A \, d\tau_P$$ and the second, since L_P is a function of (r_{AP}) only, yields: $(-L_P \psi_P)$. The final result is:

$$L_P \psi_P = c \int \chi_A^* \chi_P^* (V_{AP} - L_P) \chi_C \, d\tau_A \, d\tau_P$$

(E-16)

$$+ \int |\chi_A|^2 |\chi_P|^2 (V_{AP}) \psi_P \, d\tau_A \, d\tau_P$$

$$+ \int \chi_A^* \chi_P^* (V_{AP} - L_P) \chi_B \chi_Q \psi_Q \, d\tau_A d\tau_P$$

and similarly for L_Q (defined just as was L_P), one obtains:

(E-17) $$L_Q \psi_Q = c \int \chi_B^* \chi_Q^* (V_{BQ} - L_Q) \chi_C \, d\tau_B \, d\tau_Q$$

$$+ \int |\chi_B|^2 |\chi_Q|^2 (V_{BQ}) \psi_Q \, d\tau_B d\tau_Q$$

$$+ \int \chi_B^* \chi_Q^* (V_{BQ} - L_Q) \chi_A \chi_P \psi_P \, d\tau_B \, d\tau_Q$$

The third relation (E-10) may be written, using (E-1) and (E-5):

$$\int \chi_c^* (H_A + H_P + T_P + U_P + V_{AP} - W_A - W_P - E_P) \chi_A \chi_P \psi_P \, d\tau_C$$

$$+ \int \chi_c^* (H_B + H_Q + T_Q + U_Q + V_{BQ} - W_B - W_Q - E_Q) \chi_B \chi_Q \psi_Q \, d\tau_C \qquad \text{(E-18)}$$

$$+ \int \chi_c^* (H - W) C \chi_C \, d\tau_C = 0$$

The terms involving H_A, H_P, H_B, H_Q, W_A, W_P, W_B, and W_Q, will add to zero because of equation (E-3). Using the definition of the operators L_P and L_Q, Eq. (E-18) becomes:

$$\int \chi_c^* (V_{AP} - L_P) \chi_A \chi_P \psi_P \, d\tau_C$$

$$+ \int \chi_c^* (V_{BQ} - L_Q) \chi_B \chi_Q \psi_Q \, d\tau_C \qquad \text{(E-19)}$$

$$+ \int \chi_c^* (H - W) C \chi_C \, d\tau_C$$

The last term of (E-18) becomes, upon performing the indicated operation and integrating, using (E-4),

$$C \int \chi_C (W_C - W) \chi_C \, d\tau_C = C (W_C - W) \int \chi_c^* \chi_C \, d\tau_C$$

and the χ_C may be assumed normalized, hence (E-19) becomes:

$$C (W_C - W) = \int \chi_c^* (V_{AP} - L_P) \chi_A \chi_P \psi_P \, d\tau_C$$

$$+ \int \chi_c^* (V_{BQ} - L_Q) \chi_B \chi_Q \psi_Q \, d\tau_C \qquad \text{(E-20)}$$

It may be remarked, that the last term in (E-16) corresponds to transitions which do not go through the compound state, or symbolically, the transition is, $A + P \rightarrow B + Q$. This gives rise to another phenomenon, "Potential reaction", and would result in a shift of the levels of the compound state, as was seen in the pre-

ceding derivation. It was shown at that time, that this term would be small for all except extremely high energies. It will henceforth be neglected.

The second term constitutes for the first time, an actual formulation of potential scattering. Depending upon the choice of the potential U_p, this term will be small. U_p was chosen to be a rather strong repulsion within the nucleus so that the amplitude of the wave function ψ_p would decrease rapidly from the surface of the nucleus, and V_{AP} is restricted to the region of the nucleus alone. In this and the following work, this term will be neglected as small compared to the first term for the above reasons.

Similar considerations apply to Eq. (E-17).

In the light of the foregoing considerations Eq. (E-16) may now be written:

$$(E\text{-}21) \quad L_p \psi_p = c \int \chi_A^* \chi_p^* (V_{AP} - L_p) \chi_c \, d\tau_A \, d\tau_p$$

This is a differential equation which must be solved for ψ_p. There is a similar equation for ψ_Q, and the remaining unknown quantity (C) will be found from (E-20) when ψ_p and ψ_Q have been determined. In principle at least, the problem is now solved, though the solution is not immediately evident. The following discussion will be broken into two parts: a) Solutions of the particle wave equations, and b) Determining the coefficient of the compound state wave function.

To solve the inhomogeneous differential Eq. (E-21) for ψ_p, it must first be separated, in spherical polar coordinates (justified by the fact that L_p is a spherically symmetric operator). Let,

$$\text{(E-22)} \quad \psi_p = \sum_{lm} \frac{\psi_{plm}}{r_p} Y_{lm}(\vartheta, \varphi)$$

The separation is straight-forward, with proper account being taken of the inhomogeneity. The differential Eq. (E-21) is, using definition of L_p:

$$\frac{\hbar^2}{2m_p}(\Delta\psi_p) + (E_p - U_p)\psi_p = c \int \chi_A^* \chi_p^* (V_{AP} - L_p) \chi_C \, d\tau_C$$

where

$$\Delta = \frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\right\} + \frac{1}{r^2 \sin\vartheta}\left\{\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial}{\partial \vartheta}\right)\right\} + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2}{\partial \varphi^2}$$

M_p = reduced mass.

Let the inhomogeneity be denoted by ($\overline{X}$) (and the ψ_{plm}/r_p be denoted by b), and the tesseral harmonics simply by b , so that: $\sum bY = \sum_{lm} Y_{lm}(\vartheta, \varphi)\, \psi_{plm}/r_p$

Putting Eq. (E-22) into Eq. (E-21) and making use of the abbreviation, Eq. (E-21) becomes:

$$\text{(E-23)} \quad \frac{\hbar^2}{2m_p} \sum_{lm} \left\{ \frac{Y}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial b}{\partial r}\right)\right] + \frac{b}{r^2 \sin\vartheta}\left[\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial Y}{\partial \vartheta}\right)\right] \right.$$

$$\left. + \frac{b}{r^2 \sin^2\vartheta}\frac{\partial^2 Y}{\partial \varphi^2} + bY(E_p - U_p) \right\} = \overline{X}.$$

The summation may be reduced to a single term if the orthogonality properties of Y are utilized.

Multiplying by $Y^{*}_{l'm}$ and integrating over the angle ϑ one has:

$$\sum_{lm}\left\{\frac{\hbar^2}{2m_p}\int\frac{Y'^{*}Y}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial b}{\partial r}\right)\right]+\int\frac{bY'^{*}}{r^2\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)\right]\right.$$

(E-24)

$$\left.+\int\frac{bY'^{*}}{r^2\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}+\int(E_p-U_p)Y'^{*}Yb\right\}=\int\bar{X}Y'^{*}$$

but,

$$\int_{-1}^{1}dx\,Y_{l'm}(x)\,Y_{lm}(x)=\delta_{l'l}\qquad\text{(Y -normalized)}$$

therefore, Eq. (E-24) becomes:

$$\frac{\hbar^2}{2M_p}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial b}{\partial r}\right)\right]+\frac{b}{r^2}\int\frac{Y'^{*}}{\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)\right)$$

(E-25)

$$+\frac{b}{r^2}\int\frac{Y'^{*}}{\sin^2\vartheta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)+(E_p-U_p)b=\int\bar{X}Y'^{*}$$

Multiplying all terms by r^2 and dividing by (b), Eq. (E-25) becomes:

$$\frac{\hbar^2}{2M_p}\frac{1}{b}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial b}{\partial r}\right)\right]+\left\{\int\frac{Y'^{*}}{\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)\right)\right.$$

(E-26)

$$\left.+\int\frac{Y'^{*}}{\sin^2\vartheta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right\}+r^2(E_p-U_p)=\frac{r^2}{b}\int\bar{X}Y'^{*}$$

The quantity in the brackets is now a function of ϑ and φ, and separation has been achieved. Letting the separation constant be $l(l+1)$, the radial equation becomes (after multiplying by b, and dividing by r^2; and using $b=\psi_{plm}/r$, as defined):

$$\text{(E-27)}\quad\frac{\hbar^2}{2M_p}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{\psi_{plm}}{r}\right]\right)+\left(r^2(E_p-U_p)-\frac{\hbar^2}{2M_p}\frac{l(l+1)}{r^2}\right)\frac{\psi_{plm}}{r}=\int\bar{X}Y'^{*}$$

and this reduces to:

$$\frac{\hbar^2}{2M_p}\frac{d^2\psi_{p\ell m}}{dr^2} + \left(E_p - U_p - \frac{\hbar^2}{2M}\frac{\ell(\ell+1)}{r^2}\right)\psi_{p\ell m} = \int \bar{X}\, Y'^{*} r_p .$$

or finally:

$$\text{(E-28)}\quad \frac{\hbar^2}{2M}\frac{d^2\psi_{p\ell m}}{dr^2} + \left(E_p - U_p - \frac{\hbar^2}{2M_p}\frac{\ell(\ell+1)}{r^2}\right)\psi_{p\ell m} = C\, C_{p\ell m}(r_p)$$

where:

$$C_{p\ell m}(r_p) \equiv \int r_p\, Y^{*}_{p\ell m}(\vartheta_p, \varphi_p)\, \chi_A^{*}\chi_p^{*}(V_{Ap} - L_p)\, \chi_C\, d\tau_A\, d\tau_p\, d\omega_p$$

$(d\omega_p)$ is the element of solid angle $(\sin\vartheta\, d\vartheta\, d\varphi)$. Thus the inhomogeneity is a function of the relative corrdinate (r_p) only.

The general solution of this ordinary differential equation is given by an abitrary solution of Eq. (E-28) plus any multiple of the regular solution to the homogeneous equation. The inhomogeneity is dependent upon the short-range forces through (V_{AP}) and the function χ_c, hence will be appreciable only in the region within, or very close to the nucleus (C). For large (r_p) this solution must be identical to a certain solution of the homogeneous equation, and this certain solution will nor in general be the regular solution; hence the asymptotic behavior of $\psi_{p\ell m}$ is:

$$\text{(E-29)}\quad \psi_{p\ell m} \sim \alpha_{p\ell m} f_{p\ell}(r_p) + \beta_{p\ell m} \cdot g_{p\ell}(r_p)$$

where f(r) and g(r) are the regular and irregular solutions to the homogenous

equation, and satisfy:

$$\text{(E-30)}\quad \frac{\hbar^2}{2M_p}\frac{d^2}{dr^2}(f,g) + \left(E_p - U_p - \frac{\hbar^2}{2M_p}\cdot\frac{l(l+1)}{r^2}\right)(f,g) = 0$$

These solutions are well known and their asymptotic form is:

$$f \sim \sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$$
$$g \sim \cos\left(kr - \frac{l\pi}{2} + \delta_l\right) \qquad \text{where: } k^2 = \frac{2M_p E_p}{\hbar} = \frac{M^2 v^2}{\hbar^2}$$

(δ_l) is the phase shift from the free particle case caused by the potential (U_p). The normalization is taken care of by the α and β. There are three constants to be determined, hence two of them only are independent. Also since f(r) and g(r) are solutions to the homogeneous equation, they are valid for any potential U_p and for all r_p.

Equations (E-30) may be written as:

$$g\frac{d^2f}{dr^2} + \left[k^2 - 2\frac{M_p U_p}{\hbar^2} - \frac{l(l+1)}{r^2}\right]f\cdot g = 0$$

$$f\frac{d^2g}{dr^2} + \left[k^2 - 2\frac{M_p U_p}{\hbar^2} - \frac{l(l+1)}{r^2}\right]f\cdot g = 0$$

and subtracting these, one has:

$$\text{(E-31)}\quad \left[g\frac{d^2f}{dr^2} - f\frac{d^2g}{dr^2}\right] = 0 = \frac{d}{dr}\left(g\frac{df}{dr} - f\frac{dg}{dr}\right)$$

or partially integrating over variable r_p, (indefinite integral):

$$\text{(E-32)}\quad \left(g\frac{df}{dr} - f\frac{dg}{dr}\right) = \text{const.}$$

Now if the regular Equations (E-30) and (E-28) are

used, in the same manner, one has:

$$\frac{\hbar^2}{2M_p} f(r) \frac{d^2\psi_{plm}}{dr^2} + \left[E_p - U_p - \frac{\hbar^2}{2M_p} \frac{l(l+1)}{r^2}\right]\psi_{plm} \cdot f = C \cdot f \cdot C_{plm}$$

(E-33)

$$\psi_{plm} \frac{\hbar^2}{2M_p} \frac{d^2 f}{dr^2} + \left[E_p - U_p - \frac{\hbar^2}{2M_p} \frac{l(l+1)}{r^2}\right]\psi_{plm} \cdot f(r) = 0$$

Subtracting these two equations one has:

$$\left(f(r) \frac{d^2\psi_{plm}}{dr^2} - \psi_{plm} \frac{d^2 f}{dr^2}\right) = C\, C_{plm}(r)\, f(r) \left(\frac{2M_p}{\hbar^2}\right)$$

Integrating now over the variable (r_p) with the same transformation as in Eq. (E-31), this becomes:

$$\text{(E-34)} \quad \left[f(r) \frac{d\psi_{plm}}{dr} - \psi_{plm} \frac{df(r)}{dr}\right] = \frac{2C M_p}{\hbar^2} \int f(r)\, C_{plm}(r)\, dr$$

Using now: $\psi \to \alpha f + \beta g$ and $f \sim \sin kr$, $g \sim \cos kr$

the left hand side of Eq. (E-34) becomes:

$$\sin kr\,(\alpha k \cos kr - \beta k \sin kr) - (\alpha \sin kr + \beta \cos kr)(k) \cos kr$$

or

$$(\alpha k - \alpha k) \sin kr \cos kr - \beta k (\sin^2 kr + \cos^2 kr) = -\beta k$$

Hence Eq. (E-34) becomes:

$$\text{(E-35)} \quad -\beta k = \frac{2M_p}{\hbar^2} \cdot C \int_0^\infty dr\, f(r)\, C_{plm}(r)$$

The integral on the right is now extended to infinity, because the left hand side is seen to approach a constant limit ($-\beta k$) as r_p increases (since asymptotic was used) or $\lim_{r \to \infty} \left[f \frac{d\psi}{dr} - \psi \frac{df}{dr}\right] = -\beta k$

Now putting $k = (M_p v)/\hbar$ and dividing both sides of Eq. (E-35) by this expression, the constant β is

determined to be:

$$(E\text{-}36)\quad \beta_{\ell m} = \frac{-2c}{\hbar v_p}\int_0^\infty f_{p\ell}(r_p)\, Y^*_{\ell m}(\vartheta_p,\varphi_p)\,\chi^*_A\chi^*_P\,(V_{AP}-W_P)\,\chi_C\, d\tau_A\, d\tau_P\,\frac{d\Omega_P}{r_P}$$

where: $d\Omega_P = r_P^2\, dr_P\, d\omega_P$

The constant $\beta_{P\ell m}$ is therefore not arbitrary but uniquely determined by the inhomogeneous differential equation.

Eq. (E-36) may be further simplified by using the fact that (L_P) is sefl-adjoint and its contribution to the integral vanishes. To show this let:

$$(E\text{-}37)\quad F_{P\ell m} \equiv f_{P\ell m}(r)\,\frac{Y_{\ell m}(\vartheta_P,\varphi_P)}{r_P}$$

Then obviously (from the definition of L_P) $F_{P\ell m}$ satisfies the equation:

$$L_P F_{P\ell m} = 0$$

Thus $\beta_{P\ell m}$ becomes:

$$(E\text{-}38)\quad \beta_{P\ell m} = -\frac{2c}{\hbar v_P}\, V^{C\,*}_{AP\ell m}$$

where

$$(E\text{-}39)\quad V^C_{AP\ell m} = \int \chi^*_C\,(V_{AP})\,\chi_A\,\chi_P\,F_{P\ell m}\, d\tau_C$$

and $d\tau_C$ is the total volumn element $d\tau_A d\tau_P\, d\Omega_P \quad d\tau_B d\tau_Q d\Omega_Q$.

There now remains the two erbitrary constants ($\alpha_{P\ell m}$ and ζ_1) to be fixed. $\alpha_{P\ell m}$ may be fixed by considering the asymptotic behavior of the wave functions $\Psi_{P\ell m}$ and $\Psi_{Q\ell m}$. It was assumed at the beginning that

ψ_{Plm} must contain an incident plane-wave and an outgoing (scattered) spherical wave. The ψ_{Plm} and ψ_{Qlm} for $m \neq 0$ must have the form $\psi_{Plm} \sim e^{ik_P r_P}$, and $\psi_{Qlm} \sim e^{ik_Q r_Q}$ and no term behaving as e^{-ikr} except the partial waves with $m = 0$ in ψ_{Plm}, which describe the incident plane wave.

The incident plane wave may be written in the form, (normalized to unit current):

$$\psi_P^{inc} = \frac{1}{\sqrt{v_P}} e^{ikz} = \left(\frac{4\pi}{v_P}\right)^{1/2} \frac{1}{2ikr} \sum_l (2l+1)^{1/2} Y_{l0}(\vartheta) \left[e^{i\left(kr - \frac{l\pi}{2}\right)} - e^{-i\left(kr - \frac{l\pi}{2}\right)}\right] \tag{E-40}$$

and therefore for particle (P) and $m = 0$, Eq. (E-40) may be written as:

$$\psi_{Pl0} = \frac{i\pi^{1/2}(2l+1)^{1/2}}{k\, v^{1/2}} e^{-i\left(kr - \frac{l\pi}{2}\right)} + \gamma e^{ikr} \tag{E-41}$$

where γ is an arbitrary coefficient, which takes the place of α_{Plm} in this solution, howevery they are not equal. γ, as well as f and g are real.

Comparing the required asymptotic behavior of $_{Plm}$ for all (ml), ($\psi_{Plm} \sim e^{ikr}$, $m \neq 0$) and Eq. (E-41) for ($m = 0$) with the known behavior of f_{Plm} and g_{Plm} given by Eq. (E-30) one obtains:

(a) for all $m \neq 0$

$$\psi_{Plm} \sim g + if = (const)\, e^{ikr}$$

from

$$e^{ikr} = (\cos kr + i \sin kr)$$

Therefore from Eq. (E-29) it is apparent that:

$$\alpha_{\rho l m} = i\beta_{\rho l m} \qquad \text{or that:}$$

(E-42) $$\psi_{\rho l m} = \beta_{\rho l m}(g + if)$$

and as a result, the number of outgoing particles of the kind P with quantum numbers l and m, is given by:

(E-43) $$\sigma_{\rho l m} = \nu_\rho |\beta_{\rho l m}|^2$$

(b) for $m = 0$.

Putting the asymptotic behavior of $f_{\rho l}$ and $g_{\rho l}$ into the equation $\psi_{\rho l m} = \alpha_{\rho l m} f_{\rho l} + \beta_{\rho l m} g_{\rho l}$. (for large r), there results:

$$\psi_{\rho l m} = \alpha_{\rho l m} \sin(kr - \tfrac{l\pi}{2} + \delta_l) + \beta_{\rho l m} \cos(kr - \tfrac{l\pi}{2} + \delta_l)$$

$$= \alpha_{\rho l m} \left\{ \frac{e^{i(kr - \frac{l\pi}{2} + \delta_l)} - e^{-i(kr - \frac{l\pi}{2} + \delta_l)}}{2i} \right\}$$

$$+ \beta_{\rho l m} \left\{ \frac{e^{i(kr - \frac{l\pi}{2} + \delta_l)} + e^{-i(kr - \frac{l\pi}{2} + \delta_l)}}{2} \right\}$$

Now for $m \neq 0$, $\alpha_{\rho l m} = i\beta_{\rho l m}$ and for $m = 0$, one may put $\alpha_{\rho l m} - A_l = i\beta_{\rho l m}$, to take care of the partial waves of $m = 0$ which are incoming, then one has:

(E-44) $$\psi^{tot.}_{\rho l_0} = (\beta_{\rho l_0} - \tfrac{iA_l}{2}) e^{i(kr - \frac{l\pi}{2} + \delta_l)} + \tfrac{i}{2} A_l e^{-i(kr - \frac{l\pi}{2} + \delta_l)}$$

Now comparing the coefficients of e^{-ikr} in Eq. (E-41), it is evident that:

(E-45) $$A_l = \frac{2}{k}\sqrt{\frac{\pi(2l+1)}{\nu_\rho}}\, e^{i\delta_l}$$

The scattered partial wave of $m=0$ can now be obtained by subtracting Eq. (E-40) from Eq. (E-44) as ψ(tot.) $-$ ψ(incoming) $=$ ψ(outgoing), and one gets:

$$\text{(E-46)} \quad \psi^{out}_{Pl0} = \left[\beta_{Pl0}\, e^{i\delta_l} + A_l \sin\delta_l\right] e^{i(kr - l\frac{\pi}{2})}$$

The number of particles outgoing (per second) is then given by: (of the kind P_1 and $m = 0$)

$$\text{(E-47)} \quad \sigma_{Pl0} = \left|\beta_{Pl0}\, e^{i\delta_l} + A_l \sin\delta_l\right|^2 v_P$$

This term represents the interference between the "potential scattering" and the scattering by way of the compound state, ($A_1 \sin\delta_1$) representing the potential scattering.

Finally with the above considerations the wave-function $\psi^{(tot)}_{Plm}$ may be written as:

$$\text{(E-48)} \quad \psi_{Plm} = \alpha_{Plm} f_{Pl} + \beta_{Plm} \gamma_{Pl} \qquad \text{(for all r)}$$

γ_{P1} is from Eq. (E-41) and is a particular solution to the inhomogeneous equation, chosen in such a way that the last term goes over into $\beta_{Plm}\ g_{Pl}(r)$ without containing $f_{Pl}(r)$, that is for large r:

$$\beta_{Plm}\gamma_{Pl} \rightarrow \beta_{Plm} g_{Pl}$$

It is now convenient to define:

$$\text{(E-49)} \quad G_{Plm} = \frac{\gamma_{Plm}}{r_P} Y_{lm}(\vartheta,\varphi)$$

where G_{Plm} is regular everywhere.

Now the total wave-function ψ_P from Eq. (E-22)

may be written:

(E-50) $$\psi_p = \sum_{lm} \alpha_{plm} F_{pl} + \beta_{plm} G_{plm}$$

or finally:

(E-51) $$\psi_p = \sum_l A_l F_{pl_0} - \frac{2c}{\hbar v_p} \sum_{lm} V^{C\,*}_{Aplm} \left(G_{plm} + i F_{plm} \right)$$

Eq.(E-20) must now be solved for the constant (C). Using ψ_p from Eq. (E-51), (E-20) becomes:

(E-52) $$C(W - W_l) = \sum_{lm} \left\{ A_l \delta_{m0} - i \left(\frac{2c}{\hbar v_p} \right) V^{C\,*}_{Aplm} \right\} V^{C}_{Aplm}$$
$$- \left(\frac{2c}{\hbar v_p} \right) \sum_{lm} V^{C\,*}_{Aplm} V^{C}_{Aplm}$$
$$+ (\text{like terms in B and Q}).$$

Obviously: ψ_Q will have the same form as ψ_P, except for the "incoming function", since the defining equation for it is the same as ψ_P. If the derivation to this point were done over except with subscritps (B) and (Q) instead of (A) and (P), there would be in as exactly similar fashion: V^C_{BQlm}, F_{BQlm}, G_{BQlm}, β_{Qlm}, etc.

The first integral in Eq. (E-52) is just V^C_{APlm} from the definition. As was pointed out before, γ_{Pl} is real as is $f_{Pl}(r)$ and $g_{Pl}(r)$; therefore the second integral is real unless its complex character arises from the angle factors in χ_A, χ_P, χ_C, F_{Plm}, G_{Plm}. The angle dependence of V^C_{Plm} is exactly the same; therefore the second summation is real. The same applies to V^C_{BQlm} and the second sums in both A,P and B,Q may be put equal to: $\frac{1}{2} v_P \epsilon_{CP}$ and $\frac{1}{2} v_Q \epsilon_{CQ}$. These last two

quantities may be combined with W_C, to give:

(E-53) $$W_c' = W_c - \epsilon_{cp} - \epsilon_{cQ}$$

Henceforth the prime will be dropped, and W_C will be understood to be actually W_C'.

This shift of the energy level of the compound nucleus is analogour to the Dirac frequency shift in light dispersion (21, pp 203). In present considerations, the correct position of the level cannot be calculated, hence there would be no way of telling if the level had been shifted a little or had not. With these considerations in mind, Eq. (E-52) now becomes:

(E-54) $$c(W-W_c) = \sum_{lm} A_l \delta_{m0} - i V^{c}_{Apl_m}{}^{*} V^{c}_{Apl_m} \left(\frac{2c}{\hbar v_p}\right) - i \left(\frac{2c}{\hbar v_Q}\right) V^{c}_{BQl_m}{}^{*} V^{c}_{BQlm}$$

or transposing the last two terms, this becomes:

(E-55) $$c\left[(W-W_c) + i\left(\frac{2}{\hbar v_p}|V^{c}_{Apl_m}|^2 + \frac{2}{\hbar v_a}|V^{c}_{BQl_m}|^2\right)\right] = \sum_l A_l V^{c}_{Apl_m}$$

Now let:

(E-56) $$\Gamma_c = \left(\frac{2}{\hbar v_p}|V^{c}_{Apl_m}|^2 + \frac{2}{\hbar v_a}|V^{c}_{BQl_m}|^2\right)$$

and Eq. (E-55) becomes:

(E-57) $$c\left[W - W_c + i\Gamma_c\right] = \sum_l A_l V^{c}_{Apl_m}.$$

and for the constant (C):

$$(E\text{-}58) \qquad C = \frac{\sum_{l} A_l V^{c}_{APlm}}{(\omega - \omega_c + i\Gamma_c)}$$

The usual formulation of the dispersion formula may now be deduced. The normalization of the wave functions (f_{Pl}) need only be changed to "per unit energy" instead of "per unit current, per unit solid angle". Let φ_{Pl} be the wave function corresponding to (f_{Pl}) but normalized "per unit energy" (7), then:

$$(E\text{-}59) \qquad \varphi_{Pl} = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{dk}{dE}\right)^{1/2} f_{Pl}$$

and let H^{c}_{APlm} be the matrix element corresponding to V^{c}_{APlm} except with normalization per unit energy, then:

$$H^{c}_{APlm} = \left(\frac{2}{\pi\hbar v_P}\right) V^{c}_{APlm}$$

and

$$\Gamma_c = \left(\frac{4}{\hbar v_P}\right)\left(\frac{\pi\hbar v_P}{2}\right) \sum_{lm} |H^{c}_{APlm}|^2 + \left(\frac{4}{\hbar v_Q}\right)\left(\frac{\pi\hbar v_P}{2}\right) \sum_{lm} |H^{c}_{BQlm}|^2$$

or finally:

$$(E\text{-}60) \qquad \Gamma_c = 2\pi \sum_{lm} \left\{ |H^{c}_{APlm}|^2 + |H^{c}_{BQlm}|^2 \right\}$$

Also, by analogy with Eq. (E-38):

$$(E\text{-}61) \qquad \beta_{Qlm} = \frac{-2C}{\hbar v_Q} V^{c*}_{BQlm}$$

but when (C) is substituted from Eq. (E-58), and (A_1)

from Eq. (E-45), this becomes:

$$\text{(E-62)}\quad \beta_{Q\ell m} = -\left(\frac{2}{\hbar v_Q}\right)\left(\frac{2\pi}{k_p v_p}\right)^{1/2} \sum_{\ell'} \frac{(2\ell'+1)\, e^{i\delta_\ell}\, V^c_{AP\ell m}\, V^{c\,*}_{BQ\ell m}}{\omega - \omega_c + \frac{i}{2}\Gamma_c}$$

and the cross-section, for ψ_Q normalized to unit current per unit solid angle, is: $v_Q |\psi_Q|^2$, or:

$$\sigma^p_{Q\ell m} = v_Q |\psi_Q|^2 = v_Q |\beta_{P\ell m}|^2$$

(E-63)

$$= \left(\frac{4 v_Q}{\hbar^2 v_Q^2}\right)\left(\frac{4\pi}{v_p k_p^2}\right) \left| \sum_{\ell'} \frac{(2\ell'+1)\, e^{i\delta_\ell} V^c_{AP\ell' m}\, V^c_{BQ\ell m}}{\omega - \omega_c + i/2\,\Gamma_c} \right|^2$$

or changing to normalization per unit energy this becomes:

(E-64)

$$\sigma^p_{Q\ell m} = \left(\frac{4 v_Q}{\hbar^2 v_Q^2}\right)\left(\frac{4\pi}{v_p k_p^2}\right)\left(\frac{\pi\hbar v_Q}{2}\right)\left(\frac{\pi\hbar v_p}{2}\right) \left| \sum_{\ell'} \frac{(2\ell'+1)\, e^{i\delta_\ell} H^c_{AP\ell' m}\, H^{c\,*}_{BQ\ell m}}{\omega - \omega_c + \frac{i}{2}\Gamma_c} \right|^2$$

$$\text{(E-64a)}\quad = \frac{4\pi^3}{k_p^2} \left| \sum_{\ell'} \frac{(2\ell'+1)\, e^{i\delta_{\ell'}} H^c_{AP\ell m}\, H^c_{BQ\ell m}}{\omega - \omega_c + \frac{i}{2}\Gamma_c} \right|^2$$

or putting, $\lambda\!\!\!^{-} = \frac{\lambda}{2\pi} = \frac{1}{k}$

$$\text{(E-65)}\quad \sigma^p_{Q\ell m} = 4\pi^3 \lambda\!\!\!^{-2} \left| \sum_{\ell'} \frac{(2\ell'+1)\, e^{i\delta_{\ell'}} H^c_{AP\ell' m}\, H^c_{BQ\ell m}}{\omega - \omega_c + \frac{i}{2}\Gamma_c} \right|^2$$

This last relation, (E-65) is the relation which was to be shown. The scattering which will in general be present may be obtained from Eq. (E-46). Eq. (E-46) con-

tains a term due to potential scattering and one due to resonance scattering. The scattering cross section is obtained by summing Eq. (E-46) over all orbital angular momenta to obtain a modified Eq. (E-47). When this is done the cross section for a given angle (θ) is given by:

$$\text{(E-66)}\quad \sigma_p^p(\vartheta,E) = \nu_p \left| \sum_l \left(A_l \sin\delta_l + \beta_{pl_0} e^{i\delta_l} \right) Y_{l0}(\vartheta) \right|^2$$

or performing the square and integrating over all angles, (θ), this becomes:

$$\text{(E-67)}\quad \begin{aligned} \sigma_p^p(E) &= \nu_p \left| \sum_l A_l \sin\delta_l \right|^2 + \nu_p \left| \sum_l \beta_{pl_0} e^{i\delta_l} \right|^2 \\ &+ \nu_p \left\{ \sum_l \left(A_l \beta_{pl_0}^{*} e^{-i\delta_l} + A_l \beta_{pl_0} e^{i\delta_l} \right) \sin\delta_l \right\} \end{aligned}$$

This scattering formula contains three terms. The first is due only to potential scattering and is seen to agree with the exact scattering theory given by Rasetti (7). The second is the term due to resonance scattering only and is similar to that of the resonance reactions given by Eq. (E-65). The last term is the term which arises from the interference between resonance and potential scattering. Eq. (E-67) is quite similar to the last equation in the Wigner derivation, when it is assumed that ($l = 0$). The interpretation of the various quantities in the Wigner derivation must be considered when making the comparison.

It might be remarked that this derivation depends upon essentially the same assumptions concerning the compound nucleus as the Wigner derivation. The wave-function

of the compound nucleus is independent of the energy for the region of energy considered, except for an energy dependent multiplier. The picture given for the behavior of the potential function U_p within the nucleus, is very like the assumption by Wigner that the wave function have a certain form on the surface of a finite sphere. Inside the sphere nothing is said of the behavior of the wave function except that it should be of the form $(\alpha\Psi)$. Bethe has merely presented the conditions under which this may be achieved in a practicable manner.

A principle difference in the derivations (as in the case of the first two derivations) is that the wave function of the compound nucleus is assumed to be known and therefore the energies W_c may be found theoretically, as well as the matrix elements which determine the cross section. These calculations are not possible in the formalism adopted by Wigner, and as was pointed out before, in the absence of specific assumptions about the compound nucleus, was to be expected. It might also be pointed out that even though the method of making the calculations has been outlined in this last derivation, it is still impossible at this time to write down the wave function of the compound nucleus precisely for more than two particles. There is a great deal of doubt about the particular form of the interaction between two particles, and it is usually approximated with sufficient accuracy by a square-well function.

SUMMARY OF CHAPT. II

In this chapter derivations of the dispersion formula by Kapur and Peierls, Siegert, Wigner, and Bethe are presented.

Kapur and Peierls base their method on a perturbation calculus applied to the boundary conditions rather than to the Hamiltonian of the system. A difficulty arises with the appearance of an undetermined parameter (r_o), and it is shown, in first order, that the usual dispersion formula results from the calculation and depends upon the existence of narrow levels for validity (convergence of the perturbation series). A potential scattering term also arises in the derivation.

The derivation by Seigert removes the dependence of the wave-function of the compound nucleus upon (r_o). This is done by examining the singularities of the cross section. The energies which make the cross section singular (complex eigenvalues) are then defined as the energies of the compound states. The treatment of potential scattering is, however, quite arbitrary.

The derivation by Wigner is the most general derivation presented and is based upon the "collision" matrix introduced by Wheeler. The final formulation follows from the general asymptotic behavior of the wave functions. The asymptotic behavior is related by Green's Theorem to the wave functions of the

compound nucleus. The final form is essentially the matrix of the "phase-shifts" from the free-particle case. No theoretical means is provided for the calculation of the parameters which appear in the formulation, but they may be found empirically. They could also be calculated theoretically if proper assumptions are made about the structure of the nucleus. The limitations of this derivation are similar to those of Kapur and Peierls.

This formulation is generalized to include nuclear reactions as well as scattering.

The last derivation is by Bethe. This method is a perturbation method. The assumptions concerning the nucleus are similar to those of the other derivations, (this derivation antedates all the others presented). The wave functions of the compound nucleus are assumed known and on this basis the matrix elements describing the transition probabilities are calculated.

CHAPTER III

-EXPERIMENTAL COMPARISON-

In view of the remarks following the derivations, it is quite apparent that comparisons with experiment will be rather difficult. Without specific assumptions about the wave-function of the nucleus it is impossible to start with a resonance formula and arrive at some result to be checked experimentally. The alternative is to examine the experiments and find if physicially consistent assumptions are required to secure agreement with experiment.

The two phenomena to be examined are the angular distribution and the total cross-section. The total cross-section experiments are primarily of the "scatter-in and scatter-out" type. The angular distribution measurements envolve the differential cross section measured at various angles with respect to the beam direction.

(A) Angular Distribution Analysis

The experiments to obtain angular distribution data are difficult to perform. In essence, however, the relative number of particles scattered under a particular angle are obtained as a function of the impinging particle energy. The data are then represented by a number of curves, plotted as particle energy (incident) vs. the scattering angle.

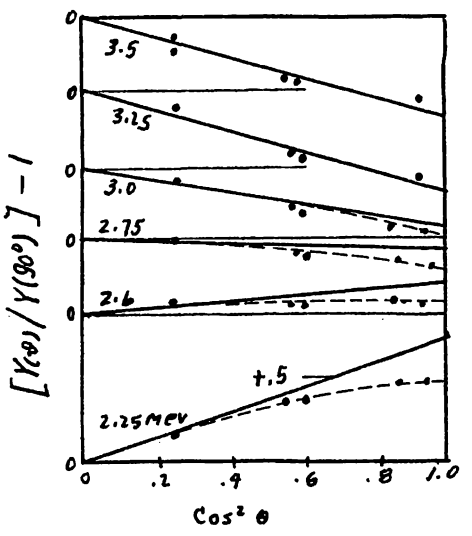


Fig II

Fig. II is a representation of typical data (22), and represents the yield at a particular angle, (θ), vs. the $\cos^2\theta$. (θ - measured in center - of - mass coordinates). The straight-lines represent a least-squares fit of the data versus the $\cos^2\theta$, only, and the dotted line represents the data plotted in the form:

$$\sigma(\theta) = A(E) + B(E)\cos^2\theta + C(E)\cos^4\theta$$

The experimental data show very good agreement with the assumed form of the equation. The coefficients, A(E), B(E), and C(E) of course vary with the energy. It is significant that the data fit a curve envolving only even powers of (cos θ), as will be seen later. The reaction chosen is the $Li^7(p,\alpha)\alpha$, because it is the only reaction which may be analyzed, without being very specific about the assumed structure of the nucleus.

The analysis will be made on the basis of the Breit-Wigner Many-level dispersion formula (23), originally given by Critchfield and Teller, (24) and extended by Inglis (25).

The Breit-Wigner Many-level formula is:

$$(F\text{-}1) \qquad \sigma(E,\vartheta) = 4\pi\lambda^2 \left| \frac{\sum_r (P/H/r)(r/H/Q)}{(E_p - E_r + \frac{i}{2}\Gamma_r)} \right|^2$$

or in the notation which has been used before:

$$\sigma(E,\vartheta) = 4\pi\lambda^2 \left| \frac{\sum_r H_p^r H_Q^{r*}}{E_p - E_r + \frac{i}{2}\Gamma_r} \right|^2$$

where:

$\sigma(E,\theta)$ - is the differential cross-section.

$(P/H/r)$ - the matrix element which measures the probability of the formation of the compound state (r).

$(r/H/Q)$ - the matrix element which measures the probability of the decay of the compound state (r) into a residual nucleus and the particle Q.

λ - the de Broglie wave-length of the incident particle, divided by 2π.

E_p - energy of the incident particle.

E_r - energy of the compound state (r).

Γ_r - the width of the state (r).

The matrix element (P/H/r) may be written:

$$(F\text{-}2) \qquad (P/H/r) = \int \psi_i H \psi_r \, d\tau$$

where:

ψ_r - is the wave function of the compound state (r).

H - is the interaction operator.

ψ_i - is the wave-function of the initial state,

which is composed of an incident particle and a target nucleus.

These elements are of the same form and significance as their analogs in the Bethe derivation of the dispersion formula.

It is clear that any results to be obtained from Eq. (F-1) must come from the assumptions about the wave-functions envolved in the matrix elements.

A wave-function which correctly describes the initial state of the system is:

$$(F\text{-}3) \quad P_m^{(S)} = \chi_m^{(S)} \sum_{l=0}^{\infty} i^l P_l(\cos\theta)(2l+1)\psi_l(kr)$$

Eq. (F-3) is clearly the product of a spin function, a plane wave, and a radial function depending upon (l). Eq. (F-3) is an eigenfunction of the orbital angular momentum and the spins separately and not the total angular momentum (j_r). The spin representation is chosen such that:

$$(F\text{-}4) \quad S = s + I$$

where:

I - is the nuclear spin.

s - is the proton spin.

S - is the vector sum of s and I.

This representation is entirely equivalent to a representation of the spins and their projections separately. This equivalence is proved as a simple theorem by J. V. Neumann (26).

In order to simplify the considerations to follow, the following notation will be adopted:

$$(l S m_l m_s /) \equiv$$

(F-5)
$$Z^{(S)}_{m_S} \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\vartheta) e^{i m_l \varphi} \psi_l(kr)$$

l - denotes orbital angular momentum in units of $\hbar$.

m - the projection of $(l\hbar)$ on the z-axis, in units of $\hbar$.

S - the total spin defined by Eq. (F-4) in units of $\hbar$.

m_S - the projection of S on the z-axis in units of $\hbar$.

When the beam direction is chosen as the z-axis, $(m_l = 0)$.

The nucleus will be concerned only with the total angular momentum (j_r) and its projection M on the z-axis; and as said before, the functions, $(lSPm_S/)$, are not eigenfunctions of (j_r). The functions $(lSOm_S/)$ may be combined with the functions $(lSm_s m'/)$ by a linear combination (with $m' = m_s - m_l$) so that they are made eigenfunctions of (j_r) and (M). This is the result of a theorem by Wigner (26,27) using the theory of groups. The theorem may be stated briefly as follows:

If the wave-function,

$$\psi^j_M = \sum_\mu (C_{\mu, M-\mu}{}^j) \psi^l_\mu \psi^S_{M-\mu}$$

is an irreducible state of the angular momentum, $j = l + S$, then the matrix,

$$\chi_{\mu'\mu} = \sum_{M,\nu} (\psi^l_{\mu'} \psi^S_\nu / \psi^j_M)(\psi^j_M / \psi^l_\mu \psi^S_\nu)$$

$$= \chi^l_j \delta_{\mu'\mu}$$

is diagonal and $X^{l}_{j} = (2j + 1)/(2\ell + 1)$ and

$$(\psi^{\ell}_{\mu}\, \psi^{s}_{m-\mu} / \psi^{j}_{m}) = C_{\mu, m-\mu}$$

and further, that

$$\sum_{m} |C_{\mu, m-\mu}|^2 = X_{\mu\mu} = X^{\ell}_{j} = \frac{2j+1}{2\ell+1}$$

In the adopted notation, the transformation coefficients $(C_{\mu}, {}^{j}_{M\ -\mu})$ will be denoted by $(\ell S0m_S/\ell S j_r M)$.

The "incoming" matrix element Eq. (F-2) may now be written:

$$(F\text{-}6) \quad (P/H/r) = \sum_{\ell=0}^{\infty} (\ell S0m/\ell S j_r m)(\ell S j_r m/H/r)$$

The numerical factors $(i^{\ell}(2\ell + 1))$ will be considered as contained in the factor $(\ell S j_r M/H/r)$. The matrix element contains an integration over the radial function as well as the angular function. The matrix element now transforms the initial eigenstate of $j\ (= j_r)$ and a given (ℓ) into an eigenstate of the resonance level (r).

Eq. (F-6) is seen to contain a summation over all values of (ℓ), and further physical considerations are necessary to make it usable.

The end-products of the reaction are two alpha-particles, which obey Einstein-Bose statistics. The E-B statistics require that the wave function of the end-products be an even function, so that a reflection, through the origin, of the coordinates, leaves the physical situation unaltered. This is just the condition

that the parity of the state be even. The parity of a system is conserved in any transition, hence the parity of the compound state must be even. Now there exists theoretical reasons for believing that the parity of the Li^7 nucleus is odd (9, pp 212), and therefore only protons having a wave function of odd parity may combine to give an initial state of even parity. Further, only those protons with odd angular momentum quantum numbers have odd parity. The summation in Eq. (F-6) is now limited to only odd values of (ℓ), or in spectroscopic notation, only the P, F,..... protons will enter into the reaction. It must be pointed out that those protons excluded by the above considerations, can and do cause a reaction, but do not give rise to the two-alpha particles as end-products.

Although the possibility of all the protons with odd angular momentum entering the nucleus and causing a reaction still exists, an examination of the relative penetrabilities of the centrifugal barrier shows that the ratio of entering F-wave protons to entering P-wave protons is of the order of 1/1000 for 500 Kev and still small at 3 Mev. The protons with higher angular momenta will contribute insignificantly within the energy range considered, (28,29).

The summation is thus reduced to only two terms, those corresponding to $\ell = 1$, and $\ell = 3$. The original analysis by Critchfield and Teller, considered only

those protons for which $\ell = 1$, but for an energy range of only 100 to 400 Kev.

A further simplification may be had by considering that if for a given value of (ℓ) there are n values of S which will combine with (ℓ) to give a certain value of the total angular momentum (j_r), then the (n) "incoming" eigenstates of (j_r), ($1Sj_r/$), may be combined into (n) orthogonal linear combinations. (n - 1) of these combinations will have zero matrix elements with H and (/r). The remaining one, $X_{\ell r}$, has a value of the matrix element ($X_{\ell r}/H/r$). This may be expressed by writing:

$$(F\text{-}7) \quad (1Sj_r/H/r) = [1Sj_r/X_{\ell r}](X_{\ell r}/H/r)$$

Eq. (F-7) arises from the fact that the "incoming" eigenstates are degenerate and the factor $[1Sj_r/X_{1r}]$ in effect, removes the degeneracy. This may be illustrated by writing down the states corresponding to $\ell = 1$, (24). To construct these states it is recalled that the projection (m) of the angular momentum of the proton must be zero. The spin of the protons is ($\frac{1}{2}$) and the nuclear spin (I) is (3/2). The spins may add either parallel or anti-parallel (assuming Russell-Saunders coupling) giving rise to S = 1,2, (from Eq. (F-4). The states (1S0m/) may now be constructed.

Let $a_{\frac{3}{2}}$, $a_{\frac{1}{2}}$, $a_{-\frac{1}{2}}$ and $a_{-\frac{3}{2}}$ be the wave functions of the target nucleus, and b_1, b_0, b_{-1} be the proton

wave functions (orbital angular momentum only). Let $s_{\frac{1}{2}}$ and $s_{-\frac{1}{2}}$ be the proton spin functions. A three-fold degenerate state and a five-fold degenerate state may be constructed; let them be called $P^{(3)}$ and $P^{(5)}$. Only the state $P^{(3)}$ will be given here, and it is:

(F-8)
$$P^{(3)} \begin{cases} \frac{1}{2}\left[a_{\frac{1}{2}} s_{\frac{1}{2}} - \sqrt{3}\, a_{\frac{3}{2}} s_{-\frac{1}{2}} \right] b_0 \\ \frac{1}{\sqrt{2}}\left[a_{-\frac{1}{2}} s_{\frac{1}{2}} - a_{\frac{1}{2}} s_{-\frac{1}{2}} \right] b_0 \\ \frac{1}{2}\left[\sqrt{3}\, a_{-\frac{3}{2}} s_{\frac{1}{2}} - a_{-\frac{1}{2}} s_{-\frac{1}{2}} \right] b_0 \end{cases}$$

The state $P^{(3)}$ is then transformed by the transformation coefficients $(l S 0 m / l S j_r m)$ into the five-fold degenerate state $X_2^{(3)}$, the lower index indicating that the state has $j_r = 2$, and the upper indicating that the state corresponds to $P^{(3)}$. Using the same notation as before,

(F-9)
$$X_2^{(3)}, \quad \frac{1}{2\sqrt{6}} \Big\{ \sqrt{3}\, a_{-\frac{3}{2}} b_1 s_{\frac{1}{2}} + 2\sqrt{2}\, a_{-\frac{1}{2}} b_0 s_{\frac{1}{2}} - a_{\frac{1}{2}} b_1 s_{-\frac{1}{2}} + a_{\frac{1}{2}} b_{-1} s_{\frac{1}{2}} - 2\sqrt{2}\, a_{\frac{1}{2}} b_0 s_{-\frac{1}{2}} - \sqrt{2}\, a_{\frac{3}{2}} b_{-1} s_{-\frac{1}{2}} \Big\}$$

$$\frac{1}{4}\left[\sqrt{6}\, a_{-\frac{3}{2}} b_0 s_{\frac{1}{2}} + 2 a_{-\frac{1}{2}} b_{-1} s_{\frac{1}{2}} - \sqrt{2}\, a_{\frac{1}{2}} b_0 s_{\frac{1}{2}} - 2 a_{\frac{1}{2}} b_{-1} s_{\frac{1}{2}} \right]$$

$$\frac{1}{2}\left[\sqrt{3}\, a_{-\frac{3}{2}} b_{-1} s_{\frac{1}{2}} - a_{-\frac{1}{2}} b_{-1} s_{-\frac{1}{2}} \right]$$

(F-9) cont. $X_2^{(3)}$,

$$\frac{1}{2}\left[a_{\frac{1}{2}}\, b_1\, s_{\frac{1}{2}} - \sqrt{3}\, a_{\frac{3}{2}}\, b_1\, s_{-\frac{1}{2}}\right]$$

$$\frac{1}{4}\left\{2\, a_{-\frac{1}{2}}\, b_1\, s_{\frac{1}{2}} + \sqrt{2}\, a_{\frac{1}{2}}\, b_0\, s_{\frac{1}{2}} - 2\, a_{\frac{1}{2}}\, b_1\, s_{-\frac{1}{2}} - \sqrt{6}\, a_{\frac{3}{2}}\, b_0\, s_{-\frac{1}{2}}\right\}$$

These functions $X_2^{(3)}$ are the functions which appear in the matrix element $(\ell Sj/\chi_{\ell r})(\chi_{\ell r}/H/r)$ and the first factor gives the value of that element.

The matrix element may be further factored if it is assumed that the radial and angle factors of the integration are independent; this results in:

(F-10) $$(\chi_{\ell r}/H/r) = \varphi_\ell(E)\, \alpha'_{\ell r}$$

where $\varphi_\ell(E)$ is real and $\alpha'_{\ell r}$ is in general complex. The energy dependence of $\varphi_\ell(E)$ is essentially that due to the penetration of the coulomb barrier.

The matrix element (P/H/r) has thus been reduced to:

(F-11) $$(P/H/r) = \sum_{\ell=1,3} (\ell 0 m/\ell S j_r m)[\ell S j_r/\chi_{\ell r}]\, \varphi_\ell(E)\, \alpha'_{\ell r}$$

The reaction is highly exoergic so that the matrix element (r/H/Q) is considered to the energy independent. The angular dependence may be found, however, from the fact that the orbital angular momentum of two alpha-particles must equal the total angular momentum of

the combined state of $j_r = L$ and $M_L = m$, (the spin of the alpha's being zero). It must also describe the fact that the observations are made at an angle (θ), and this brings into the final state wave function, the associated Legendre Polynomials. As was seen in the derivations, this final-state wave function would be an outgoing spherical wave. When observed some distance from the scattering center, however, the final state is a plane wave in the direction (θ). The angle factor is the angle part of the transformation coefficient which transforms an outgoing spherical wave into an outgoing plane wave by the following expansion, (30):

$$e^{ikR} = \sum_{l=|r}^{} i^l J_l(kr) \left\{ P_l(\cos\alpha) P_l(\cos\nu) \right.$$

(F-12)

$$\left. + 2 \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\alpha) P_l^m(\cos\nu) \cos m(\varphi-\beta) \right\}$$

where α and β are the colatitude and azimuth angles of the position vector R, and θ and φ those of the observation direction, k. Now since the two matrix elements appear as a product, the complex factors may be combined into one complex number; or α_{lr} may be taken as the product of α'_{lr} and all of Eq. (F-12) except the factor $P_L^m(\cos\theta)$.

The formula for the cross-section is now:

$$(F\text{-}13)\quad \sigma_p(E,\vartheta) = 4\pi\lambda^2 \left| \frac{\sum_r (l\,s_0 m / l s j_r m)[l s j_r / X l_r]\, \psi_l(E)\alpha_{lr} P_{j_r}(\cos\vartheta)}{E - E_r + \frac{i}{2}\Gamma_r} \right|^2$$

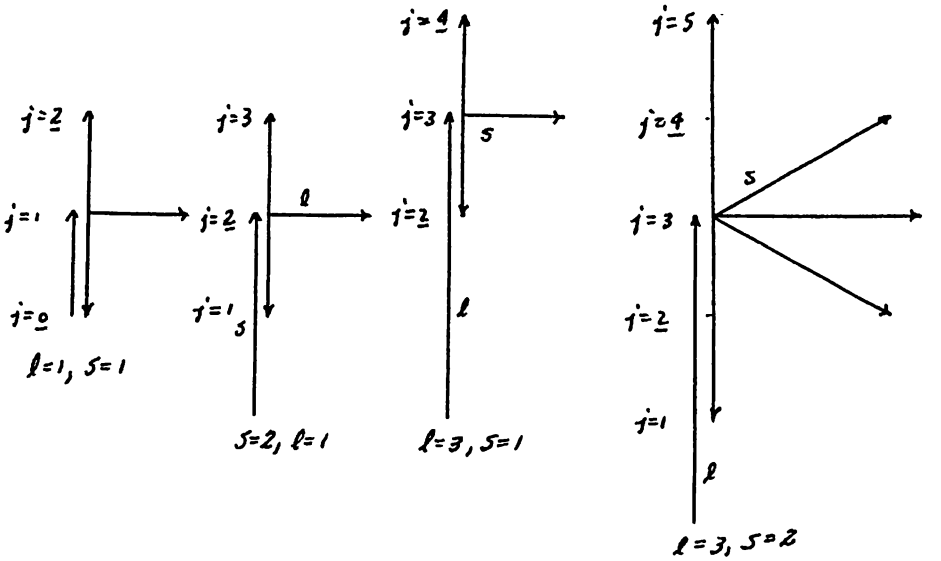
And since the beam is unpolarized, the differential cross section is:

$$(F\text{-}14)\quad \sigma(E,\vartheta) = \sum_p \sigma_p(E,\vartheta)$$

With the considerations leading to Eq. (F-13) and (F-14) the procedure in verifying the resonance formula reduces essentially to tedious manipulations in algebra.

As was mentioned before the proton spin of ($\frac{1}{2}$) combines with the nuclear spin (3/2) to give S = 1,2. The consideration of Bose statistics for the alpha-particles has led to considering only initial states with odd orbital angular momentum, of which only $l = 1,3$, need be considered. Several combinations of S and l are now possible. The Bose statistics plus partiy considerations, require only even (j_r). With $l = 1$, the spin state S leads to only two even states for (j): $j_r = 0,2$; while $l = 1$, and S = 2, leads to only $j_r = 2$. With $l = 3$ and S = 1, only $j_r = 2,4$ results, and likewise for $l = 3$, and S = 2. The following vector diagrams

illustrate this addition:



The considerations may now be limited to only those states which have even values for (j_r) or $j_r = 0,2,4$. It will be shown that the experiments may be accounted for if the $j_r = 4$ state is considered negligible.

The transformation coefficients (ℓ S0m/ℓ Sjm) may be taken from those tabulated in ref. 27, pp 76. They are as follows:

m	(110m/112m)	(120m/122m)	(320m/312m)	(320m/322m)	$(4/3)^{\frac{1}{2}}P^m(x)$
2	0	$-(2/3)^{\frac{1}{2}}$	0	$-(14)^{\frac{1}{2}}$	$(3/2)^{\frac{1}{2}}(1-x^2)$
1	$2^{-\frac{1}{2}}$	$-6^{-\frac{1}{2}}$	$7^{-\frac{1}{2}}$	$2/(14)^{\frac{1}{2}}$	$6^{\frac{1}{2}}x(1-x^2)^{\frac{1}{2}}$
0	$(2/3)^{\frac{1}{2}}$	0	$-(3/7)^{\frac{1}{2}}$	0	$(1-3x^2)$
-1	$2^{-\frac{1}{2}}$	$6^{-\frac{1}{2}}$	$7^{-\frac{1}{2}}$	$-2/(14)^{\frac{1}{2}}$	
-2	0	$(2/3)^{\frac{1}{2}}$	0	$(14)^{-\frac{1}{2}}$	

It is assumed that the compound nucleus has only two states which contribute to the reaction within the range of energies considered. The state numbered by r = 0, having $j_0 = 0$ is assumed to have a half-width (Γ_0) much greater than the range of energies considered so that the variation of its resonance denominator may be neglected. The state numbered r = 2, having $j_2 = 2$, is assumed to have a sufficiently narrow half-width (Γ_2) to account for the energy variation of the angular distribution. Now if ϵ be defined:

$\epsilon \equiv 2(E-E_2)/\Gamma_2$ Eq. (F-13) may be written:

$$\text{(F-15)}\quad \sigma(E,\vartheta) = \left[\frac{16\pi\lambda^2}{\Gamma_2^2(\epsilon^2+1)}\right] \sum_{S,M} \left| \left(\frac{\Gamma_2}{i\Gamma_0}\right)(\epsilon+i)\, H_0^\rho H_Q^0 + H_2^\rho H_Q^2 \right|^2$$

or factoring the matrix elements as was done before, this becomes:

$$\text{(F-20)}\quad \sigma(E,\vartheta) = \left[\frac{16\pi\lambda^2}{\Gamma_2^2(\epsilon^2+1)}\right] \sum_{S,M} \left| \frac{\Gamma_2}{i\Gamma_0}(\epsilon+i)\,\varphi_1 \alpha_{10}\,\delta(s,1)\,\delta_{m0} + \sum_{\ell=1,3} (\ell 50m/\ell 52m)[\ell 52/\ell 2]\,\alpha_{\ell 2}\,\varphi_\ell\, P_2^m(\cos\vartheta) \right|^2$$

or putting in the summation over (m) explicitly, this becomes:

$$\text{(F-17)}\quad \sigma(E,\vartheta) = \left[\frac{16\pi\lambda^2}{\Gamma_2^2(\epsilon^2+1)}\right] \left\{ \sum_{m=-2}^{2} \left| \sum_{\ell=1,3} (\ell 20m/\ell 22m)[\ell 22/\ell 2]\,\alpha_{\ell 2}\,\varphi_\ell \right|^2 \times \left(P_2^m(\cos\vartheta)\right)^2 + \sum_{m=\pm 1} \left| \sum_{\ell=1,3} (\ell 10m/\ell 12m)[\ell 12/\ell 2]\,\alpha_{\ell 2}\,\varphi_\ell \right|^2 \left(P_2^m\right)^2 + \left| \left(\frac{\Gamma_2 \alpha_{10}}{i\Gamma_0}(\epsilon+i)\,\varphi_1 + \sum_{\ell=1,3} (\ell 100/\ell 120)[\ell 12/\ell 2]\,\alpha_{\ell 2}\,\varphi_\ell\, P_2^0(\cos\vartheta)\right)\right|^2 \right\}$$

These summations are given in the following table:

(Angle factors only, with cos $\theta \equiv x$)

1	2	$(4/5)\sum_{m=-2}^{2}(\ell_1 20m/\ell_1 22m)$ $x(\ell_2 20m/\ell_2 22m)(P_2^m(x))^2$	$(4/5)\sum_{m=-1}^{1}(\ell_1 10m/\ell_1 12m)$ $x(\ell_2 10m/\ell 12m)(P_2^m(x))^2$
1	1	$2(1-x^2)$	$(2/3)(1-3x^2)$
1	3	$(3/7)^{\frac{1}{2}}(1-x^2)(1-5x^2)$	$-(2/7)^{\frac{1}{2}}(1-12x^2+15x^4)$
3	3	$(3/28)(1-x^2)(1+15x^2)$	$(3/7)(1-4x^2+7x^4)$

A further simplification may be had by defining the following relations between the contants:

$$\frac{\Gamma_2 \alpha_{10}}{i\Gamma_0 [112/12]\alpha_{12}} = R_0 + iI_0 \qquad \frac{[312/32]\alpha_{32}}{[112/12]\alpha_{12}} = R_2 + iI_2$$

$$\frac{[122/12]}{[112/12]} = R_1 + iI_1 \qquad \frac{[322/32]\alpha_{32}}{[112/12]\alpha_{12}} = R_3 + iI_3$$

where the R's are real and the I's are real.

Now neglecting for the moment a constant term, the cross section becomes:

$$(F\text{-}18)\quad \sigma \sim \frac{1}{\epsilon^2+1}\left\{\left[\frac{5}{2}(R_1^2+I_1^2)(1-x^2)+\frac{5}{6}(1-3x^2)\right.\right.$$

$$\left.+(R_0^2+I_0^2)(1+\epsilon^2)+\left(\frac{10}{3}\right)^{1/2}(R_0\epsilon-I_0)(1-3x^2)\right]\varphi_1^2$$

$$+\left(\frac{3}{7}\right)^{1/2}\left[\frac{5}{2}(R_1R_3+I_1I_3)(1-x^2)(1-5x^2)-\frac{5}{6}R_2(1-12x^2+15x^4)\right.$$

$$-(5)^{1/2}\left[(R_0R_2 + I_0I_2)\epsilon + R_0I_3 - R_3I_0](1-3x^2)\right]\varphi_1\varphi_3 + \left(\frac{15}{112}\right)\left[(R_3^2+I_3^2)\times (1-x^2)(1+15x^2) + 4(R_2^2+I_2^2)(1-4x^2+7x^2)\,\varphi_3^2\right]\Big\}.$$

It is convenient to rename some of these constants as:

$$a = \left(\frac{5}{2}\right)(R_1^2 + I_1^2)$$

$$b = \left(\frac{5}{2}\right)\sqrt{\frac{3}{7}}\,(R_1R_3 + I_1I_3)$$

$$c = \left(\frac{5}{\sqrt{14}}\right)R_2$$

$$d = \left(\frac{15}{7}\right)^{1/2}(R_0R_2 + I_0I_2)$$

$$e = \left(\frac{15}{112}\right)(R_3^2 + I_3^2)$$

Now writing the cross section in the form:

$$\sigma = C(\epsilon)\left\{1 + A(\epsilon)\cos^2\vartheta + B(\epsilon)\cos^4\vartheta\right\} \qquad \text{(F-19)}$$

The coefficients A(ϵ), B(ϵ), and C(ϵ) are found in terms of the previously defined constants as:

$$A(\epsilon) = \left[\frac{5}{2} - a - \sqrt{30}\,(R_0\epsilon - I_0 + 3(2b + 4c + d\epsilon + \alpha)\phi + (19e - 9\gamma)\phi^2\right]/D(\epsilon)$$

$$B(\epsilon) = \left[5(b-3c)\phi - (15e - 7\gamma)\phi^2\right]/D(\epsilon)$$

$$C(\epsilon) = \frac{\pi}{\epsilon^2+1}\left(4\pi\,[112/12]\,\frac{\alpha_{12}\varphi_1}{\sqrt{2}}\right)D(\epsilon)$$

$$D(\epsilon) = (R_0^2 + I_0^2)(\epsilon^2+1) + \frac{5}{6} + a + \frac{2\sqrt{2}}{3}(R_0\epsilon - I_0) + (b - c - d\epsilon - \alpha)\phi + (e+\gamma)\phi^2$$

where: $\phi = \dfrac{\varphi_3}{\varphi_1}$ and $\begin{cases}\alpha = \left(\frac{15}{7}\right)^{1/2}(R_0I_2 - R_2I_0) \\ \gamma = \left(\frac{15}{28}\right)(R_2^2 + I_2^2)\end{cases}$

Three degrees of approximation are now possible.

If the f-wave protons are neglected, then $\phi = 0$, because $\varphi_3 = 0$; and consequently $B(\epsilon) = 0$. The cross section becomes:

(F-21) $$\sigma(E,\vartheta) = \frac{[5/2 - a - \sqrt{30}\,(R_0\epsilon - I_0)]}{(R_0^2 + I_0^2)(\epsilon^2+1) + \frac{5}{6} + a + \frac{2\sqrt{2}}{3}(R_0\epsilon - I_0)}$$

Eq. (F-21) is seen to have one root, and that is when ($\epsilon = I_0/R_0$). This represents the extent to which the analysis was carried by Critchfield and Teller. There occur five arbitrary parameters which may be adjusted for fitting the data, and they are: E_2, Γ_2, R_0, I_0, and (a). When terms is ϕ but not ϕ^2 are taken into account, three more parameters occur: b, c, and d. When ϕ^2 is taken into account, the last parameter (e), occurs. Altogether there are nine parameters.

When $\phi \neq 0$ is considered, it is necessary to find the variation of ϕ with energy. This has been estimated from the WKB approximation to be approximately linear for a range of nuclear radii in the neighborhood of R 1.5(e^2/mc^2), and takes the form:

(F-22) $$\phi \simeq \frac{KE}{mc^2}$$ where $K \simeq .01$ for the chosen value of R.

With the energy dependence of ϕ established the terms in Eq. (F-20) may be written:

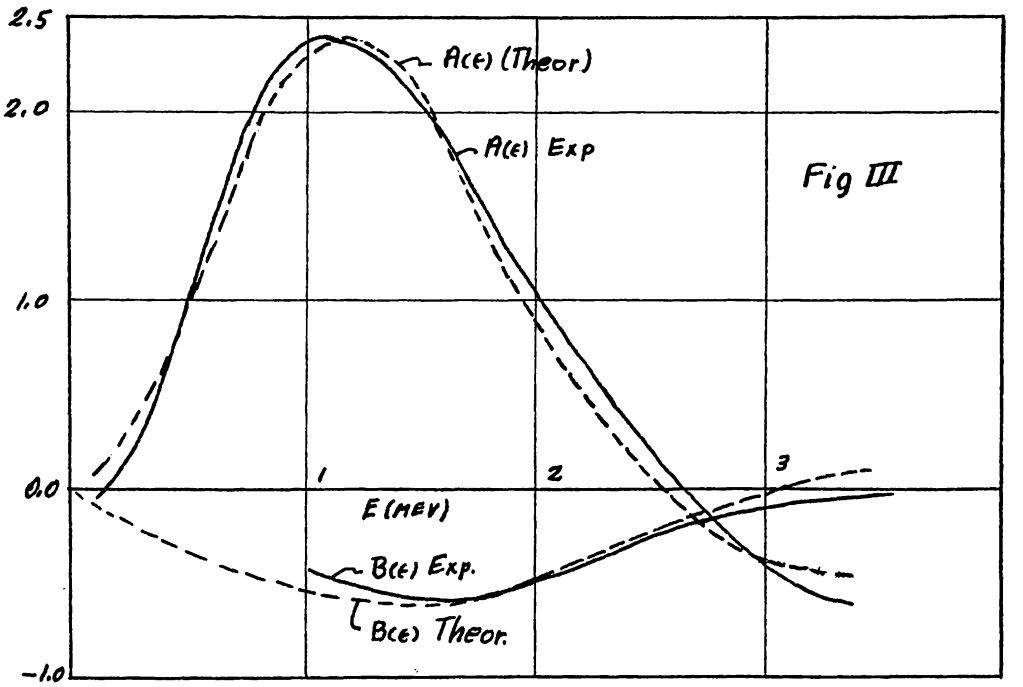
(F-23) $$A(\epsilon) = \frac{(C_0E^2 + C_1E + C_2)}{(E^2 + C_3E + C_4)}$$

and

$$B(\epsilon) = \frac{C_5 E^2 + C_6 E}{E^2 + C_3 E + C_4}$$

The constants $C_0 \ldots C_6$, being defined in terms of the parameters in Eq. (F-20), remembering that (ϵ) is the energy in terms of $2(E-E_2)/\Gamma_2$.

The fitting of the data will now only envolve the adjusting of the constants, of which there are a good many. The observed values of A(E) and B(E) together with the plot of Eq. (F-22) with appropriatly chosen constants appear in Fig III.



for: $C_0 = -1.75$, $C_1 = -2.8C_0$, $C_2 = 0.27C_0$
$C_3 = -1.91$, $C_4 = 215$, $C_5 = 0.4$, $C_6 = -1.2$.

The values of these constants are not of real significance because they cannot be calculated from the theory. It is observed that the limitations of this analysis lie in the fact that at about 7 Mev the intensity at some angles becomes negative. This is due primarily to the neglect of the higher orbital angular momenta, but for this low energy analysis the approximations are good.

It is also apparent that with the large number of constants available for fitting the curve, the fit with the data is no better than might be expected. The theoretical agreement is more significant when the appearance of two nodes in the curve for A(E) are observed experimentally. The principle conclusion which may be drawn is that the theory does not disagree with the observations.

Very useful information of a different character is derived from this analysis, however. The excitation curves for the $Li^7(d,n)Be^8$ and $Li^7(p,n)Be^7$ reactions and the excitation of lithium by scattered protons indicate that there exist four compound states within the energy range of 0 to 2.5 Mev. The effect of these reactions, however, is not observed in the $Li^7(p,\alpha)\alpha$ reaction since the curves are smooth in this range of energies. It is presumed then that these other states are states of odd parity whereas the states envolved in the $Li^7(p,\alpha)\alpha$ reaction are of even parity. The odd

parity of these other states makes interpretation of the experiments envolving these other states possible.

(B) Analysis of Total Cross Section

Total cross-sections could be gotten from the Angular Distribution curves by integrating the yield function over all angles. This would entail the angular analysis of all possible reactions at all angles, integration of each reaction yield function, and the summing of the partial cross sections. The experiments would be difficult as well as introducing several errors. The analysis of each particular reaction would, however, be possible (in principle) and much useful information gained thereby.

The usual method of making total cross-section measurements is, however, much simpler and in general not as much information is gained. The method is the "scatterer-in and scatterer-out" experiment. The experiment consists of placing a target of the material to be examined in a beam of particles. The transmission of the target is measured by counting the particles with the scatterer (or target) in the beam and then measuring the beam current without the scatterer (or "scatterer-out"). The transmission is then expressed by:

$$(G\text{-}1) \qquad T = \frac{I_i'}{I_o} = e^{-\sigma N}$$

Where: I - is the transmitted intensity.

I_o - is the beam current with scatterer out.

σ - is the total cross section.

N - is the number of target nuclei per square centimeter of target material.

This is probably clearer if expressed as:

$$(G\text{-}1a) \quad T = \frac{I_i'}{I_o} = e^{-\sigma N' x}$$

Where N' is the number of target nuclei per cubic centimeter and x is the target thickness. The Eq. (G-1) will be used in this section.

The cross-section is obviously given by:

$$(G\text{-}2) \quad \sigma(E) = \frac{1}{N} \ln \frac{I_o}{I_i} = -\frac{1}{N} \ln (T)$$

The analysis will consist of fixing the parameters in dispersion formula to fit as closely as possible the observed variation of the cross-section with the energy of the incident particle. The parameters are quite different in form, than those used in the angular analysis. There is no need to concern oneself with angle variations of the cross-section (except the usual precautions for "good" geometry), since all particles scattered out of the beam are not counted, in contradistinction to the previous analysis.

In order to make the comparison, the Breit-Wigner many-level formula must be simplified by suitable assumptions about the process to be examined. The process chosen here is the slow-neutron transmutation process. This is perhaps the simplest case to examine,

because of the simplifying assumptions allowed by the nature of the process.

The generalization of the dispersion formula will be indicated briefly, then the assumptions which allow its simplification for a particular process.

The generalized dispersion formula, which allows different initial states, as well as many combined states and many possible end-products is given by Bethe (9) as:

$$(G\text{-}3)\quad \sigma(E) = 4\pi^3\lambda^2 \left| \frac{\sum_r H_{C_r}^{AP_p} H_{BQ\,q}^{C_r}}{(W_{A_p} + W_p + E_p - W_{C_r} + \frac{i}{2}\gamma_r)} \right|^2$$

Where:

$H_{C_r}^{AP_p}$ - is the matrix element corresponding to a transition from the initial state (nucleus A plus particle P with kinetic energy E_p) to the compound nucleus in state (r).

γ_r - is the total effective width of the level (r). It is the sum of the partial widths due to various modes of disintegration, or $\gamma_r = \sum_{Q'} \gamma_{Q'}^r$ The other symbols have the same significance as before, but with the indices modified to denote different states. Among the particles Q' which may be emitted, is the particle P. The width $\gamma_{Q'}^r$ may be written:

$$\gamma_{Q'}^r = \sum_{q'} \gamma_{Q'q'}^r$$

where:

$$\gamma^{r}_{Q_\beta'} = 2\pi \left| H^{C_r}_{B'Q'q'} \right|^2$$

is that part of the width of the level of the compound state, r, due to the disintegration of C into a residual nucleus B' in state q' and a particle Q', with an energy given by the conservation law:

(G-4) $$W_{A_p} + W_p + E_p = W_{B_q} + W_Q + E_Q$$

All the quantities is Eq. (G-3) will depend upon the energy of incident particle through the conservation law. This is also true of the effective widths γ_r, and therefore γ_r is not in general the true width of the level (r). The true width (Γ_r) is obtained when the resonance energy ($E_{Pr} = W_{Cr} - W_{Ap} - W_p$) is put into Eq. (G-3) for E_p, so that:

$$\Gamma_r = \gamma_r(E_{P_r})$$

If the levels are degenerate (and in general they are) then other statistical factors must be considered. Bethe and Placzek (31) have given the form for degenerate levels as:

(G-5) $$\sigma^{P_r}_{Q_\beta} = \frac{\pi \lambda^2}{(2i+1)(2s+1)} \sum_{l_j l'_{j'} J} (2J+1) \left| \frac{\sum_r u^{C_{rJ}}_{B_p P_{lj}} u^{C_{rJ}}_{B_\beta Q_{l'j'}}}{(W_{A_p} + W_p + E_p - W_{C_{rJ}} + \frac{i}{2}\gamma_{rJ})} \right|^2$$

where i, J, and i' are the angular momenta of the initial, compound, and final states respectively.

s and s' are the spins of the incident and outgoing particles.

ℓ and ℓ' are the orbital angular momenta of the incident and outgoing particles.

j and j' are the total angular momenta of the incident and outgoing particle.

p, r, and q stand for all quantum numbers of the initial, compound, and final states other than the angular momenta.

ℓ and ℓ' extend from $0 \to \infty$.

j has all values between $|\ell - s|$ to $|\ell + s|$

j' has all values between $|\ell' - s'|$ to $|\ell' + s'|$

$u^{C_r J}_{A_p \ell_j}$ is defined by:

$$\gamma^{rJ}_{p\ell j} = \left(u^{C_r J}_{A_p \ell_j}\right)^2$$

and

$$\gamma^{rJ}_{p} = \sum_{\ell j} \gamma^{rJ}_{p\ell j}$$

If only one-level is of importance the formula reduces to:

$$(G\text{-}6)\quad \sigma^{p_p}_{Q_q} = \pi \lambda^2 \frac{(2J+1)}{(2i+1)(2s+1)} \cdot \frac{\gamma^{rJ}_{p_p}\, \gamma^{rJ}_{Q_q}}{(E_p - E_r)^2 + \frac{1}{4}\gamma_{rJ}{}^2}$$

The quantities H, u, and γ, from which Eq. (G-6) is derived depends upon the energy and the wave-functions must be normalized per unit energy. Considerations of the wave function for small (kr) (or $\lambda >>>$ nuclear

radius) show that the matrix elements depend on the wave number, k, as $[(dk/dE)^{\frac{1}{2}} k^{l-1}]$ and $(dk/dE) \sim 1/k$, hence the matrix element is proportional to $(k)^{l+\frac{1}{2}}$.

The matrix element $u^{rJ}_{P_p}$ may be written:

$$u^{rJ}_{P_p} = b^{rJ}_{P_p} k^{l+\frac{1}{2}} = b^{rJ}_{P_p} \lambda^{-(l+\frac{1}{2})}$$

or the partial width becomes:

$$\gamma^{rJ}_{P_p} = (b^{rJ}_{P_p})^2 \lambda^{-(2l+1)}$$

where (b) is a constant (independent of k).

In terms of the true width (Γ) and the value of the matrix element at resonance (U) these terms become:

$$u^{rJ}_{P_p} = U^{rJ}_{P_p} \left(\frac{\lambda P_p}{\lambda}\right)^{l+\frac{1}{2}} = U^{rJ}_{P_p} \left(\frac{E}{E_{P_p}}\right)^{\frac{1}{2}(l+\frac{1}{2})}$$

(G-7)

$$\gamma^{rJ}_{P_p} = \Gamma^{rJ}_{P_p} \left(\frac{\lambda P_p}{\lambda}\right)^{2l+1} = \Gamma^{rJ}_{P_p} \left(\frac{E}{E_{P_p}}\right)^{l+\frac{1}{2}}$$

The general form of the cross-sections is now given by:

$$\sigma^{P_p l}_{Q_q l'} = \frac{\pi}{(2s+1)(2i+1)} \lambda^{1-2l} \lambda'^{-(1+2l')}$$

(G-8)

$$\sum_{jj'J} (2J+1) \left| \frac{\sum_r b^{rJ}_{A_p P_{lj}} b^{rJ}_{B_q Q l'j'}}{(E_p - E_{c_rJ} + \frac{i}{2}\gamma_{rJ})} \right|^2$$

where according to the previous definitions:

$$b^{rJ}_{A_p P_{lj}} = U^{rJ}_{A_p P_{lj}} \lambda_{P_p}^{(l+\frac{1}{2})}$$

end, λ and λ' are the wave lengths of the incident and outgoing particles respectively.

Now the only energy dependent factors are λ, λ', and E_p it self.

In accordance with the chosen slow neutron process, the orbital angular momentum may be put to zero. The general form then reduces to:

$$(G\text{-}9)\quad \sigma_{Qq}^{Pp} = \frac{\pi}{(2i+1)(2s+1)} \lambda \sum_{l'j'J} (2J+1) \left| \frac{\sum_r U_{Rp\,Pos}^{rJ} \lambda_{p_r}^{1/2} U_{Bq\,Q\,l'j'}^{rJ}}{E_p - E_{p_r} + \frac{i}{2}\gamma_r J} \right|$$

Assuming now that only one level is important, and putting $s = \frac{1}{2}$ for neutrons Eq. (G-9) becomes:

$$(G\text{-}10)\quad \sigma_{Qq}^{No} = \frac{\pi}{2}\left(1 \mp \frac{1}{2i+1}\right) \frac{\lambda \lambda_r \Gamma_N \Gamma_{Qq}}{(E-E_r)^2 + \frac{1}{4}\gamma_r^2}$$

The (+) sign is taken if $(J = i + \frac{1}{2})$ for $l = 0$, and the (-) sign if $(J = i - \frac{1}{2})$. The γ_r contains a term due to the probability of emitting a neutron as well as other particles (Q), (from definition, Eq.(E-60)). From Eq. (G-7), it is clear that the neutron term varies as $(E^{\frac{1}{2}})$, consequently γ_r may be written:

$$\gamma_r = \Gamma_N \left(\frac{E}{E_r}\right)^{\frac{1}{2}} + \sum_{Qq} \Gamma_{Qq}$$

Neutron widths, however, may be usually neglected, for low energies, compared to all other contributions; hence they may be neglected in the following term, and $\gamma \cong \Gamma \cong \sum_{Qq} \Gamma_{Qq}$ Γ - is the true width of the level.

The total cross section is now given by:

$$(G\text{-}11)\quad \sigma_Q^N = \sum_q \sigma_{Qq}^N = \frac{\pi}{2}\left(1 \pm \frac{1}{2i+1}\right)\frac{\hbar^2}{2M(EE_r)^{1/2}}\,\frac{\Gamma_N \Gamma_Q}{(E-E_r)^2 + \frac{1}{4}\Gamma^2}$$

where $\Gamma_Q = \sum_q \Gamma_{Qq}$ = total width due to the emission of the particle Q. For low energies (say, under 10,000 volts) there is in general only one possible reaction for neutrons and a given nucleus. This may be a simple capture process or the emission of a charged particle, most often a β-decay activity is induced; and consequently in nearly all practical cases, $\Gamma_Q = \Gamma$, where Γ is the true width of the level. With these considerations the total cross section becomes:

$$(G\text{-}12)\quad \sigma_Q^N = \frac{\pi}{2}\left(1 \pm \frac{1}{2i+1}\right)\frac{\hbar^2}{2M(EE_r)^{1/2}}\,\frac{\Gamma_N \Gamma}{(E-E_r)^2 + \frac{1}{4}\Gamma^2}$$

If the energy of the incident particle now becomes equal to the resonance energy, Eq. (G-12) becomes:

$$(G\text{-}13)\quad \sigma_0 \equiv \sigma_Q^N(E_0) = \frac{\pi}{2}\left(1 \pm \frac{1}{2i+1}\right)\frac{\hbar^2}{2ME_0}\,\frac{4\Gamma_N}{\Gamma}$$

and σ_0 is the cross section at exact resonance. The cross section at any energy near the resonance may now be written:

$$(G\text{-}14)\quad \sigma_Q^N(E) = \frac{\sigma_0 \Gamma^2}{\Gamma^2 + 4(E-E_0)^2}\left(\frac{E_0}{E}\right)^{\frac{1}{2}}$$

This is the form most generally used to analyze total cross section data for thermal neutrons. This same equation is also used in a slightly different form:

$$(G\text{-}15) \qquad \sigma E^{1/2} = \frac{\sigma_0 E_0^{1/2}}{1 + \left[\frac{4(E-E_0)}{\Gamma}\right]^2}$$

There are two approaches now to the analysis of the total cross sections. One method is to calculate the transmission from the values of σ_0 and assumed values of Γ. The second is to plot the data and take the best values from the curve thus obtained.

The first method has been used by Havens, Rainwater, et. al. (32,33,34), and will be given in detail in the following paragraphs.

The method of measuring cross sections used by Havens, et. al., is to measure the transmission as a function of the neutron time-of-flight. The time of flight is measured (or provided) by a velocity spectrometer. The time of flight is inversely proportional to the velocity of the neutron and therefore inversely proportional to the square-root of the energy.

Now the transmission is given by: $T = e^{-\sigma N}$

and $\sigma(E) = \dfrac{\sigma_0 \Gamma^2 (E_0/E)^{1/2}}{4(E-E_0)^2 + \Gamma^2}$ and the variables may be changed to the dimensionless parameters:

$$x = \frac{t-t_0}{t_0}, \quad \xi = \frac{\Gamma}{2E_0} \quad \text{and} \quad \beta = N\sigma_0\xi^2$$

where (t) is the time-of-flight corresponding to energy (E), and (t_o) is the time-of-flight corresponding to (E_o).

The transmission is now given by:

$$(G\text{-}16) \quad T(x) = exp.\left\{ \frac{-B(x+1)}{\xi^2 + \left[x(x+2)/(x+1)^2\right]^2} \right\}$$

Now the transmission may be plotted for various assumed values of B and ξ. When this is done, a set of curves (illustrated in Fig IV) result, which may be compared to the experimentally observed transmission and as oppropriate set of constants derived. The curves clearly depend upon the target thickness through (B).

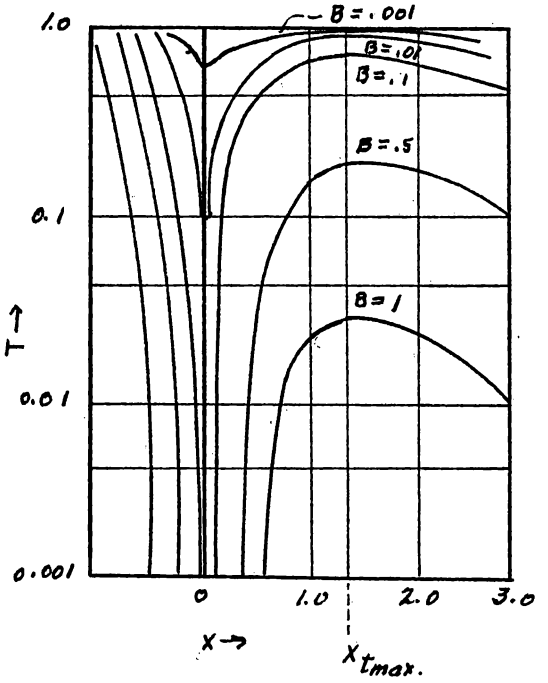


Fig. IV

Once t_o is fixed, one can very quickly evaluate the quantity ($\sigma_o \Gamma^2$) from the value of B obtained from the curve. The evaluation of Γ and σ_o separately however is a different matter. If the energy resolutions were much smaller than the width of the level (Γ), then a fair estimate of (Γ) can be made, but unfortunately it seems never to be the case. The effect of the resolution width of the instruments on (Γ) and the transmission is illustrated by the set of curves in Fig. V. The curves are calculated for different values of resolution width (Δ t) for a sample of Indium. The shape of

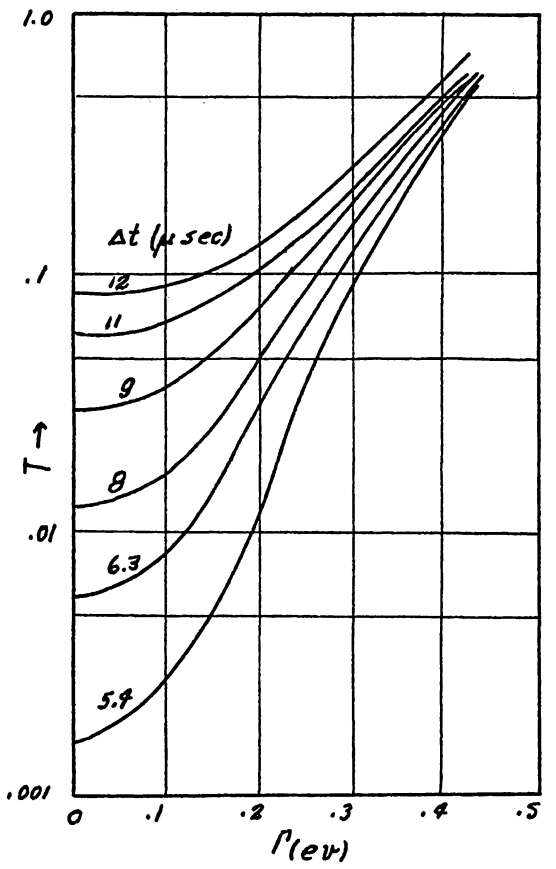


Fig. V

of these curves must also fit the data and when it can be done reasonably well an estimate of Γ is given. The final result of these measurements and calculations is illustrated with the data again gathered on Indium, Fig VI. The curves are calculated for different values of B. Curve (a) has B = .572, curve (b) has B = .676, and curve (c) has B = .889. The triangle in the corner gives the resolution as

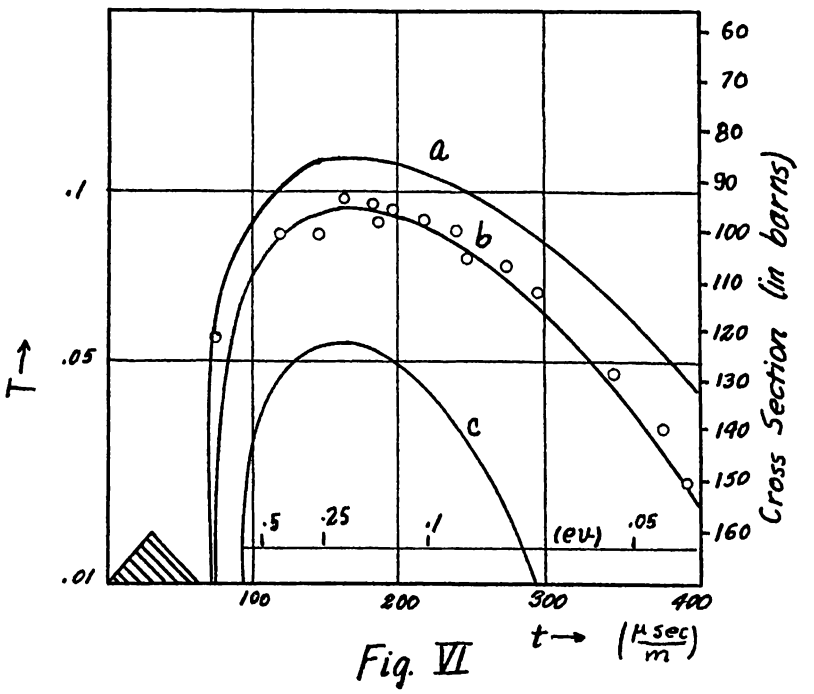


Fig. VI

a function of the time-of-flight. These data illustrated did not go as high as the resonance peak whoih occurs at about 1.44 ev. There would be more curves to the left, in the manner of Fig. IV, but these are omitted for clarity. Curve B checks very well with the data and yields the following result:

$E_o = (1.44 \pm .04)$ev. and $\sigma_o \Gamma^2 = (210 \pm 60) \times 10^{-24} cm^2 (ev)^2$

An upper limit is placed on Γ ($\simeq .2$ ev), the limit being inferred from over-all uncertainty as well as lack of sharp resolution. The best estimate of the value of Γ is .09 ev, for which $\sigma_o = 26{,}000 \times 10^{-24} (cm^2)$. These values when the resolution is considered fit the data rather well.

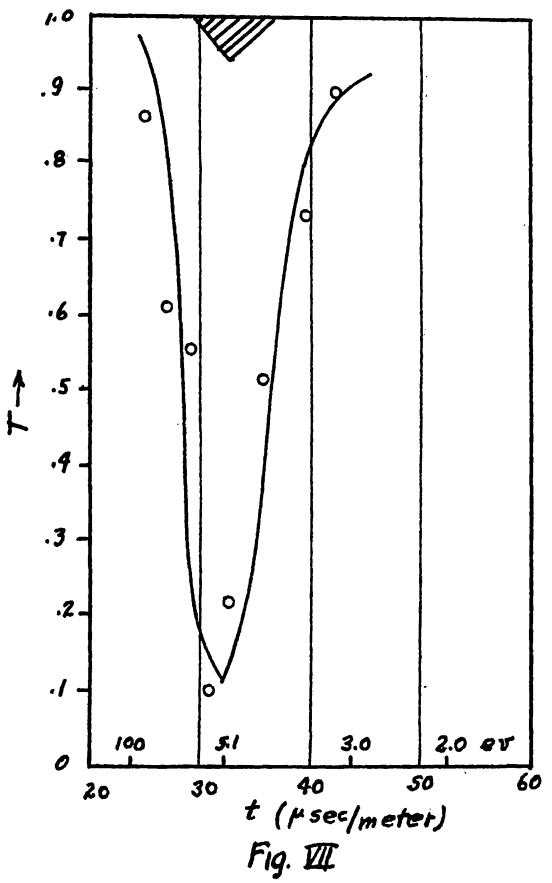


Fig. VII

A somewhat better illustration of the relative size of the resolution width is afforded by Fig. VII. This is the cross-section in the vicinity of the main peak of the Ag cross-section curve. The triangle again is the resolution width and the curve is calculated for $\Gamma = 0$. (See Fig. V). That the data fit well, is quite apparent. In some cases the existence of negative energy

levels is inferred by the shape of the curve at low energy. In the case of negative levels, the Breit-Wigner formula takes the same form as before:

$$\sigma = \sigma_0 \left[\frac{E_0}{E}\right]^{\frac{1}{2}} \frac{\Gamma^2}{\Gamma^2 + 4(E-E_0)^2}$$

where $-E_0$ is now the energy of the level. If $\Gamma \ll E_0$, then Γ^2 may be neglected in the denominator and the cross-section may be written as:

$$(G\text{-}17) \qquad \sigma = \left[\frac{\sigma_0 \Gamma^2}{4 E_0^{3/2} E^{1/2}}\right]\left[1 + \frac{E}{E_0}\right]^{-2}$$

The first term in this equation corresponds to the (1/v) law for slow neutrons.

The cross-section of Hg shows this case very well.

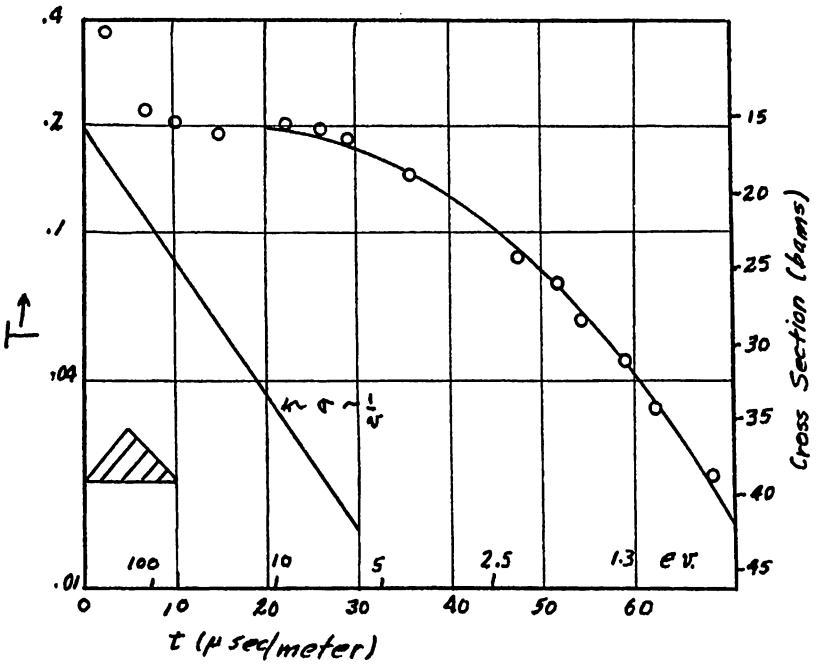


Fig. VIII

The existence of a level at $(-E_0) = -2$ ev is illustrated in Fig. VIII. This curve (Fig. VIII) was repeated

for three different target thicknesses and showed the same behavior each time. The disagreement of the theory above 10 ev indicates the existence of positive levels, and these are shown in the table at the end of this chapter.

The table at the end of the chapter lists the positions of the slow neutron levels for all those nuclei for which they are found. This table was compiled in an effort to find if any periodicities or regularities existed for periodic system. No attempt was made to ascribe the listed cross section to any particular isotope of the listed elements, since they are in general unknown. It may be argued theoretically that the cross sections should be due primarily to the odd numbered isotopes. Mass spectrographic analysis of samples before and after bombardment with slow neutrons verify this in at least one instance,(36). The plotted curve (Fig. IX) shows the variation of position, and therefore spacing, of the first two levels plotted against the nuclear charge (Z). Many levels of higher energy in the elements with $Z < 40$, have not been shown because they would not be slow-neutron processes.

Though the spacing is small in all cases shown (under 25 ev) it's variation is quite marked. There seems to be no correlation between the observed spacing and the atomic number. Data on heavier nuclei ($Z > 80$)

are nearly non-existent (not released or not available).

It would be of some significance to ascertain the strength of each level, but for reasons mentioned before, this could not be done with any certainty.

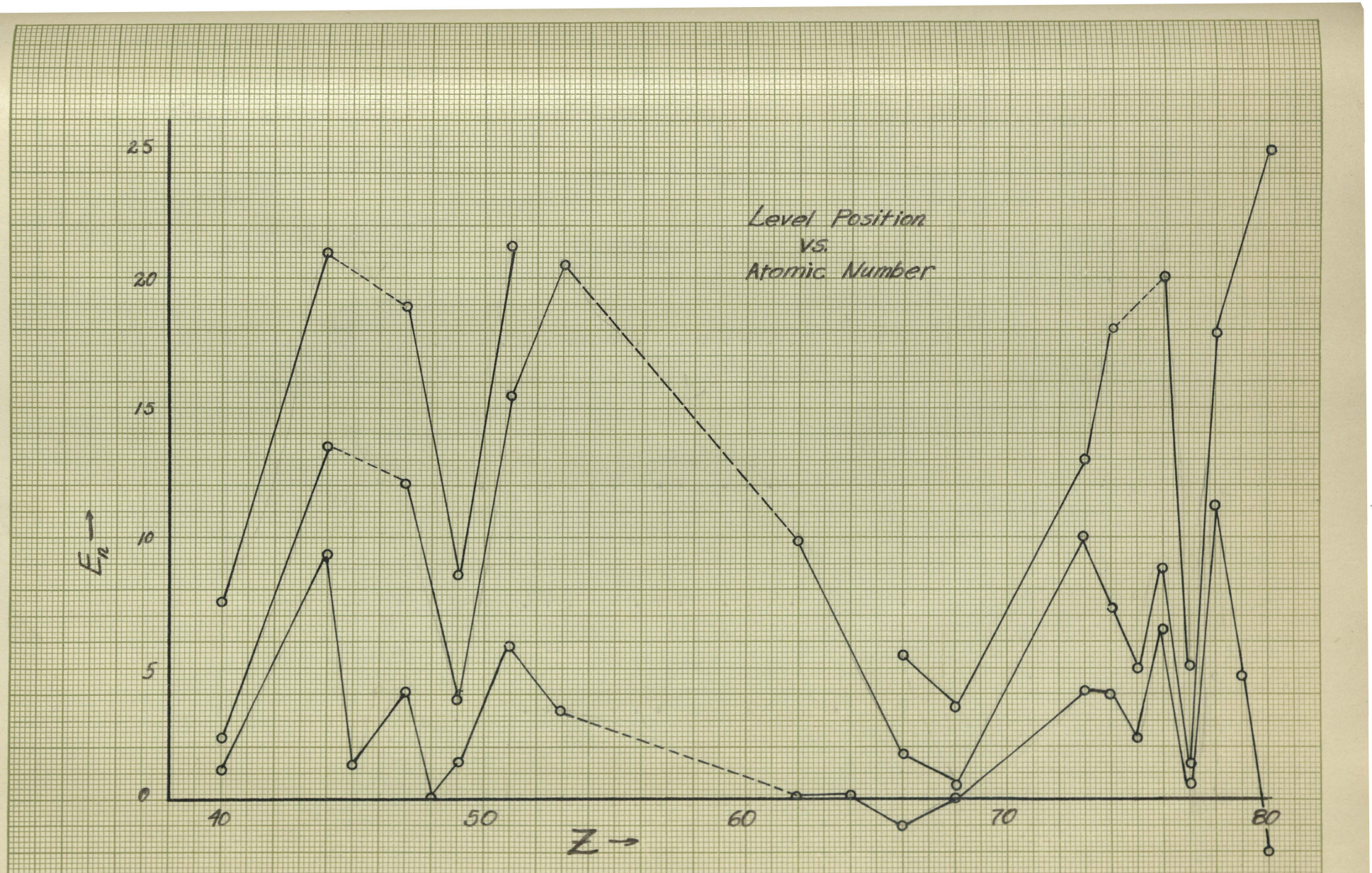
In general it might be concluded that either the experiments must become better or better techniques devised if accurate checks on the theory are to be made.

TABLE OR NEUTRON RESONANCE LEVELS (under 1000 ev)

Element	Z	E_1	E_2	E_3	E_4	E_5	E_6
C (1,3,4)	6	0.006	0.157				
Mg (1)	12	220 (approx.)					
Al (1)	13	330					
S	16	150					
Cr (1)	24	360					
Mn (1)	25	30					
Co (1)	27	115					
Zn (1,u)	30	495					
Ge (1)	32	95					
Zr (1)	40	1.09	2.3	7.6			
Ru (1,u)	44	9.4	13.5	21			
Rh (2)	45	1.28					
Ag (1)	47	5.1	12.1	19	43.5		
Cd*	48	0.196					
In (1,6)	49	1.44	3.8	8.6			
Sb (1)	51	5.9	15.5	21.2			
I (1)	53	3.3?	20.6	32.4	42.0	86.0	
Sm (2)	62	0.096	10.0	33.0	40?	55.0	
Gd (2)	64	0.031					
Dy (2)	66	-1.01	1.75	5.5			
Eu (2)	68	-0.011	0.465	3.5	9.5	22.0	40.0
Ta (1)	73	4.1	10.0	13.0	22.0	37.0	
W (1)	74	4.0	7.4	18.0	45.0	117.0	180.0
Re (7,u)	75	2.3	5.0				
Os (1)	76	6.5	8.8	20.0	28.0	42.0	84.0
Ir (2)	77	0.635	1.35	5.2	8.9	25?	
Pt (1)	78	11.5	18.0	100.0	1000.0		
Au (1,2)	79	4.8					
Hg (1)	80	-2.0	25.0 (and higher levels not resolved)				
Tl (1)	81	270.0	1100.0 (approx.)				

(1) Havens, Rainwater, Wu, Dunning, etc (32,33,34)
(2) Sturm, (35)
(3) Hanstein, P. R., 59, 489, (1941)
(4) J. Marshall, P. R., 70, 107(A), (1946)
(5) Zinn, P. R., 71, 752, (1947)
(6) McDaniel, P. R., 70, 832, (1946)
(7) Argonne lab group

(u) Unpublished
(*) (1,2,5, and others)
(?) Questionable level


Level Position
vs.
Atomic Number
25
20
15
10
5
0
E_n →
40
50
60
70
80
Z →

SUMMARY OF CHAPT. IV

In this chapter, two essentially different methods of checking the dispersion formula are presented.

The first method, is the measurement of the angular distribution of the reaction end products. It is shown that the distribution follows from the angular momentum properties of the incoming state and the compound state. The Bose statistics of the final state (two alpha-particles in the $Li^7(p,\alpha)\alpha$ reaction) make it possible to fix the angular properties of the initial state. The large number of parameters which arise in the development are sufficient to fit the experimental data. It is also shown that while the checking of the dispersion formula is not particularly significant, much valuable information of a different character is obtained.

The second method was the fitting of the total cross section data with the one level formula, modified to describe the slow neutron processes. The method was that due to W. W. Havens, in which the transmission of a sample is calculated with the use of constants obtained from the data. The fit with data was good, but not so much information was obtainable. The lack of good resolution of the instruments precludes the fixing of the level widths, or the cross section at exact resonance with precision. The positions of the levels, however, are well defined in the low lying states.

A table of the level positions of the low lying states in included.

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CHAPTER IV

-CONCLUSION-

The specific similarites and differences in the treatment of resonance processes in nuclear reactions have been pointed out at the end of each derivation. There are, however, two categorical differences which should be pointed out. These differences are primarily in the points of view.

The first approach assumes that an "equation of state" may be written for the compound nucleus as well as the initial and residual nuclei. The problem then is to quaulitatively describe the transitions from one of the initial states to the compound state and thence to the final state. The state equations are set-up for an arbitrary potential and the cross section is derived. No attempt is made to make the results insensitive to the chosen potential function. The specific difficulties in this approach are immediately manifest. The choice of a potential function is at the present time still very arbitrary although relatively good approximations may be made for low energies ($\sim$10 Mev). The most important limitation to this theory is, however, the inability to solve exactly, many body mechanical systems when the interactions are large. This limitation requires rather formidable approximation methods, as was seen in the derivation by Bethe.

In principle at least the solution by the method of Bethe, Kapur and Peierls will provide the widths of levels, the spacing, and the cross-section as a function of the energy and angle of observation.

The second point of view recognizes all the difficulties of the first method and provide a formal derivation envolving many parameters which may be interpreted in terms of the observables of the process. No theoretical means is provided for calculating the parameters, and they are to be fixed by experimental results. This is the approach due principally to Wheeler, Wigner, and Heisenberg.

Neither of the approaches is without merit; though the second approach will provide less specific information about the nucleus than the first. When the results of the more general second method are properly interpreted in terms of the experiments, they should provide much useful information for formulating the theory by means of the first method.

At the present time there exists neither sufficiently accurate nor pleniful experimental data to prove or disprove the validity of the results of either approach. This is amply illustrated by the results in Chapt. III. The best one can do is to show that in certain cases the theories do not disagree.

The validity of the first method is limited to narrow levels, even theoretically; and the second

method suffers similar limitation when attempts are made to get the same results as the first. This is not altogether unexpected since a description envolving transitions between states only makes sense when the states are reasonably well defined. In terms of the observations, this means that the time actually consumed making a transition must be small compared to the mean life of the state. The proper description in the case of broad overlapping levels is not that of a state with a given "width" but a superposition of many states. The formalism must then be altered accordingly.

During the course of this investigation it was thought that some speculations as to the future developments in the theory would be evolved. In the light of the most recent experiments at California University (Berkeley), however, speculations are not feasible at this time. Those experiments showed that the forms of the nuclear forces presently in use do not agree with experimental data at high energies. The data are not sufficient to formulate new ideas at present. The general form of the dispersion formula as advanced by Wigner would, however, be applicable inasmuch as the parameters may be taken from experiment.

APPENDIX

Derivation of Eq's. (1-7) and (1-8).

a) Equation (1-7)

Equation (1-5) is substituted into Equation (1-1), the resulting equation multiplied by u_n^* and integrated with respect to the variable (r). One has then:

$$i\hbar \int_0^\infty \dot{b}_n u_n^* u_n \, dr + i\hbar \int_0^\infty \int_0^\infty u_n^* \dot{k}_\varepsilon v_\varepsilon \, d\varepsilon \, dr = b_n \int_0^\infty u_n^* E_n u_n \, dr \qquad (I)$$

$$-b_n \int_0^\infty u_n^* V^i u_n \, dr + \int_0^\infty \int_0^\infty dr \; u_n^* k_\varepsilon \varepsilon v_\varepsilon - \int_0^\infty \int_0^\infty dr \, d\varepsilon \; u_n^* k_\varepsilon V^o v_\varepsilon$$
$$(II) \qquad (III) \qquad (IV)$$

$$+ \int_0^\infty dr \, u_n^* (\delta^i V^i + \delta^o V^o) u_n b_n + \int_0^\infty dr \; u_n^* (\delta^i V^i + \delta^o V^o) \int_0^\infty k_\varepsilon v_\varepsilon \, d\varepsilon$$
$$(V) \qquad (VI)$$

Since the function u_n is assumed normalized, the first integral on the left is simply: $(i \dot{b})$. (The dot over b indicates the time derivative).

The second integral on the left may be disposed of by writing: (using equation (1-6))

$$i\hbar \int_0^\infty \int_0^\infty u_n^* \dot{k}_\varepsilon v_\varepsilon \, d\varepsilon \, dr = i\hbar \frac{d}{dt} \left\{ \int_0^\infty dr \; u_n^* \int_0^\infty k_\varepsilon v_\varepsilon \, d\varepsilon \right\} = 0$$

Integral (I) is obviously just $(b_n E_n)$.

Integrals (II) and (V) may be combined as:

$$(II) + (V) = b_n \int_0^\infty u_n^* (\delta^i V^i + \delta^o V^o - V^i) u_n \, dr.$$

and this is equal to:

$$\int_0^\infty u_n^* (\delta^o)(V^o - V^i)\, u_n \, dr$$

and this will be abbreviated to V_{nn}.

Integral (IV) and (VI) may be combined in the same manner as (II) and (V) to give:

$$(IV) + (VI) = \int_0^\infty dr\, u_n^* \int_0^\infty (\delta^i V^i + \delta^o V^o - V^o)\, k_\varepsilon \nu_\varepsilon \, d\varepsilon$$

$$= \int_0^\infty \int_0^\infty dr\, u_n^*\, \delta^i (V^i - V^o)\, k_\varepsilon \nu_\varepsilon \, d\varepsilon \equiv \int_0^\infty d\varepsilon\, k_\varepsilon\, V_{n\varepsilon}.$$

Integral (III) may be partially integrated with respect to (ε) to give:

$$\int_0^\infty dr\, u_n^* \int_0^\infty k_\varepsilon\, \varepsilon\, \nu_\varepsilon \, d\varepsilon = \int_0^\infty dr\, u_n^* \Big[\varepsilon \int_0^\infty k_\varepsilon \nu_\varepsilon \, d\varepsilon$$

$$- \int_0^\infty dr\, u_n^* \int_0^\infty d\varepsilon \left\{ \int_0^\infty d\varepsilon'\, k_{\varepsilon'} \nu_{\varepsilon'} \right\}$$

The first integral on the right is zero from equation (1-6). The order of the integration of the second may be interchanged to give: (from Eq.(1-6))

$$- \int_0^\infty d\varepsilon \int_0^\infty dr\, u_n^* \int_0^\infty k_{\varepsilon'}\, \nu_{\varepsilon'}\, d\varepsilon' = 0$$

There remains then only Equation (1-7):

$$(1\text{-}7) \quad i\hbar\, \dot{b}_n = b_n E_n + b_n V_{nn} + \int_0^\infty d\varepsilon\, k_\varepsilon\, V_{n\varepsilon}$$

b) To derive Eq. (1-8), equation (1-5) is substituted into Eq. (1-1), the resulting equation is multiplied by $(\nu_{\varepsilon'})$ and again integrated over the

the variable (r), and one has:

$$i\hbar\,\dot{b}_n\int_0^\infty u_n\,v_{\varepsilon'}^*\,dr \;+\; i\hbar\int_0^\infty dr\int_0^\infty v_{\varepsilon'}^*\,\dot{k}_\varepsilon\,v_\varepsilon\,d\varepsilon = b_n\int_0^\infty dr\; v_{\varepsilon'}^* H_0 u_n$$

$$+\int_0^\infty dr\; v_{\varepsilon'}^*\int_0^\infty k_\varepsilon\,H_0\,v_\varepsilon\,d\varepsilon \;+\; \int_0^\infty dr\; v_{\varepsilon'}^*\,(\delta^i V^i+\delta^0 V^0)\,b_n\,u_n$$

$$+\int_0^\infty dr\; v_{\varepsilon'}^*\,(\delta^i V^i+\delta^0 V^0)\int_0^\infty k_\varepsilon\,v_\varepsilon\,d\varepsilon.$$

Replacing $H_0 u_n$ by $(E_n u_n - V^i u_n)$

and $H_0 v$ by $(\varepsilon v_\varepsilon - V^0 v_\varepsilon)$ (from Eq. (1-2)).

The right hand side becomes:

$$= b_n E_n\int_0^\infty dr\; v_{\varepsilon'}^* u_n \;-\; b_n\int_0^\infty dr\; v_{\varepsilon'}^* V^i u_n + \int_0^\infty dr\; v_{\varepsilon'}^*\int_0^\infty k_\varepsilon\,v_\varepsilon\,d\varepsilon$$

$$-\int_0^\infty dr\; v_{\varepsilon'}^*\int_0^\infty k_\varepsilon\,V^0 v_\varepsilon\,d\varepsilon \;+\; \int_0^\infty dr\; v_{\varepsilon'}^*\,(\delta^i V^i+\delta^0 V^0)\,\dot{b}_n\,u_n$$

$$+\int_0^\infty dr\; v_{\varepsilon'}^*\,(\delta^i V^i+\delta^0 V^0)\int_0^\infty k_\varepsilon\,v_\varepsilon\,d\varepsilon$$

and combining as before, this becomes:

$$= b_n E_n\int_0^\infty dr\; v_{\varepsilon'}^* u_n + b_n\int_0^\infty dr\; v_{\varepsilon'}^*\,\underbrace{(\delta^0 V^0+\delta^i V^i - V^i)}_{(I)}\,u_n$$

$$+\int_0^\infty dr\; v_{\varepsilon'}^*\,\underbrace{(\delta^i V^i+\delta^0 V^0 - V^0)}_{(II)}\int_0^\infty k_\varepsilon v_\varepsilon\,d\varepsilon \;+\; \underbrace{\int_0^\infty dr\; v_{\varepsilon'}^*\int_0^\infty k_\varepsilon\,\varepsilon\,v_\varepsilon\,d\varepsilon}_{(III)}$$

These integrals may be treated seperately as follows:

$$(I) = b_n\int_0^\infty dr\; v_{\varepsilon'}^*\,\delta^0(V^0-V^i)\,u_n \equiv b_n V_{\varepsilon' n}$$

$$(II) = \int_0^\infty dr\; v_{\varepsilon'}^*\,\delta^i(V^i-V^0)\int_0^\infty k_\varepsilon v_\varepsilon\,d\varepsilon = \int_0^\infty d\varepsilon\; k_\varepsilon\,V_{\varepsilon'\varepsilon} = \int_0^\infty d\varepsilon'\,k_{\varepsilon'}\,V_{\varepsilon\varepsilon'}$$

$$(III) = \int_0^\infty d\varepsilon\int_0^\infty dr\; v_{\varepsilon'}^*\,k_\varepsilon\,\varepsilon\,v_\varepsilon = \int_0^\infty d\varepsilon\; k_\varepsilon\,\varepsilon\,\delta(\varepsilon'-\varepsilon) = k_{\varepsilon'}\,\varepsilon'$$

Now solving the left hand side for $(i\hbar\dot{k})$ and using the above definitions, one has:

$$i\hbar\dot{k}_\varepsilon = b_n V_{\varepsilon n} + \varepsilon' k_{\varepsilon'} + \int_0^\infty d\varepsilon'\, k_{\varepsilon'} V_{\varepsilon\varepsilon'} + (b_n E_n - i\hbar\dot{b}_n)\int_0^\infty dr\, u_n v_{\varepsilon}^*$$

but this is just Eq. (1-8).

BIBLIOGRAPHY

(1) Heisenberg, Z. fur Phys., 77, 1, (1932)

(2) Majorana, Z. fur Phys., 82, 137, (1933)

(3) Gamow, Z. fur Phys., 122, 439, (1928)

(4) Condon and Gurney, Nature, 122, 439, (1928)

(5) Casimir, Physica, 7, 193, (1934)

(6) Stranathan, Particles of Modern Physics, Chpt. 11, (1934)

(7) Rassetti, Elements of Nuclear Physics, Chpt. 6, (1936)

(8) Bohr, Nature, 137, 344, (1936)

(9) Bethe, Rev. Mod. Phys., 9, 73, (1937)

(10) Frenkel, Wave Mechanics, Elementary Theory, Oxford Press

(11) Breit, Phys. Rev., 58, 506, (1940)

(12) Weisskopf and Wigner, Z. fur Phys., 63, 54, (1930)

(13) Kapur and Peierls, Proc. Roy. Soc., 166, 277, (1938)

(14) Siegert, Phys. Rev., 56, 750, (1939)

(15) Wigner, Phys. Rev., 70, 15, (1946)

(16) Wigner, Phys. Rev., 70, 606, (1946)

(17) Wheeler, Phys. Rev., 52, 1107, (1937)

(18) Møller, Nature, 158, 403, (1946)

(19) Ning Hu, Phys. Rev., 74, 131, (1948)

(20) Marganau and Murphy, Math. of Phys. and Chem. (pg 296)

(21) Dirac, Quantum Mechanics, 3rd Ed., Oxford, Chpt. 8

(22) Heydenburg, Hudson, Inglis, and Whitehead,
Phys. Rev., 74, 405, (1948) and 73, 241, (1948)

(23) Breit and Wigner, Phys. Rev., 49, 519, (1936)

(24) Critchfield and Teller, Phys. Rev., 60, 10, (1941)

(25) Inglis, Phys. Rev., 74, 21, (1938)

(26) Breit and Darling, Phys. Rev., 71, 402, (1947)

(27) Condon and Shortley, Theory of Atomic Spectra, Oxford Press, New York, (1935)

(28) Wheeler, Yost, and Breit, Phys. Rev., 49, 174, (1936)

(29) Christy and Latter, Rev. Mod. Phys., 20, 185, (1948)

(30) Whittaker and Watson, Modern Analysis, MacMillan Co., page 395, (1947)

(31) Bethe and Placzek, Phys. Rev., 51, 450, (1937)

(32) Havens and Rainwater, Phys. Rev., 70, 154, (1946)

(33) Rainwater, et. al., Phys. Rev., 73, 733, (1948)

(34) Havens, et. al., Phys. Rev., 71, 165, 174, (1947)

(35) Sturm, Phys. Rev., 71, 752, (1947)

(36) Lapp, Van Horn, and Dempster, Phys. Rev., 71, 745, (1947)