LINEARITY AND SEPARABILITY OF REGIONAL GENERAL EQUILIBRIUM INPUT/OUTPUT MODELS UNDER STATIC COBB-DOUGLAS ASSUMPTIONS

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Discussion Papers
No. 1987.1

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The author has benefitted from comments by Mohamed El-Hodiri and Bob Glass. A previous version of this paper appeared as IPPBR Economic Research Technical Note No. 35. The research reported here has been supported by the State of Kansas.

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ABSTRACT

El-Hodiri and Nourzad have pointed out that an Input-Output analysis of industrial composition can be justified by assuming a generalized Cobb-Douglas production function. Their approach is a substantial improvement over the traditional fixed-coefficients production function in several respects, which are summarized in the Introduction. This paper extends their insight by giving the Cobb-Douglas Input-Output analysis for a complete static general equilibrium model of a regional economy. If all demands are determined by Cobb-Douglas production functions or utility functions (or have unitary price elasticities), and if all supplies are constant cost (or perfectly elastic), then the following results hold:

1. All relations between dollar flows are linear. (This fact leads to an input-output analysis, with dollar flows taking the place which is occupied by quantity flows in the traditional Leontief analysis.)

2. All relationships between log-prices are linear. Consequently, the general equilibrium model can be completely linearized.

3. The price relationships are completely separable from the dollar flow relationships. In other words, economic flows measured in nominal terms are completely independent of changes in relative prices over time. (Of course, real variables do depend on relative prices, and nominal variables do depend on the general price level.) This fact provides a rigorous rationale for input/output projections which ignore relative price effects.

A comparison is then made to a similar separability result under fixed-coefficient assumptions. It is pointed out that Cobb-Douglas assumptions lead to complete linearity and separability under more general endogeneity conditions than fixed coefficient assumptions.

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1. INTRODUCTION:

El-Hodiri and Nourzad [forthcoming] have pointed out that an Input-Output analysis of industrial composition can be justified by assuming a generalized Cobb-Douglas production function. Their approach is a substantial improvement over the traditional fixed-coefficients production function used by Leontief [1951] in several respects. First, the fixed coefficients production function has been widely criticized as unrealistic, because it denies the possibility of substitution between inputs when relative prices change. The Cobb-Douglas production function, in contrast, has been very widely employed in empirical measurements, at least with respect to labor and capital inputs, and has often been found to be consistent with empirical tests. For an interregional example, see Lande [1978]. (Whether Cobb-Douglas assumptions are also empirically suited to intermediate product inputs, as assumed in this paper, remains to be seen.)

Second, the traditional fixed coefficients assumption leads to a linear input/output relationship between quantity flow variables. In practical applications of input/output analysis, quantity flow measurements are rarely available; dollar flow variables are used instead, based on an explicit or implicit assumption that relative prices will remain constant over time. In contrast, the Cobb-Douglas assumption leads directly to a linear input/output relationship between the observable dollar flow variables.

Samuelson's Non-substitution Theorem [1951] provides an alternative justification for the use of Input/Output analysis when production functions are not fixed coefficient. This approach allows for any constant returns to scale production functions and any utility functions, which is a significant generalization over the Cobb-Douglas assumption. Moreover, the linear input/output relationship applies not only to the dollar flows, but also to the quantity flows. However, in other respects, the Non-substitution approach is very restrictive: relative prices must remain constant over time, only one factor of production is allowed, and joint production is forbidden. Therefore, in all situations where the Cobb-Douglas assumption can be justified the present approach is superior.

This paper extends El-Hodiri and Farrokh's insight by giving the Cobb-Douglas Input-Output analysis for a complete static general equilibrium model of a regional economy. If all demands are determined by Cobb-Douglas production functions or utility functions (or have unitary price elasticities), and if all supplies are constant cost (or perfectly elastic), then the following results hold:

1. All relationships between dollar flows are linear. This fact leads to a general equilibrium input-output analysis, with dollar flows taking the place which would be occupied by quantity flows in the traditional Leontief analysis.

2. All relationships between the logarithms of prices are linear. Consequently, the general equilibrium model can be completely linearized.

3. The price relationships are completely separable from the dollar flow relationships. In other words, economic flows measured in nominal terms are completely independent of changes in relative prices over time. (Of course, real variables do depend on relative prices, and nominal variables do depend on the general price level.) This fact provides the first rigorous rationale for input/output projections which ignore relative price effects in an environment with changing prices.

This paper also gives a comparison to a similar separability result under fixed-coefficient assumptions. It is pointed out that Cobb-Douglas assumptions lead to complete linearity and separability under more general conditions than do fixed coefficient assumptions.

In the course of the exposition, a complete general equilibrium model is developed, and the operational meanings of the A-matrix and the B-matrix under Cobb-Douglas assumptions are clarified. For purposes of exposition, I abstract away from all non-linearities related to investment non-negativity and capacity constraints, and I assume there are no corner solutions. I also ignore taxes, saving, government, ordinary imports and exports, regional income transfers, and all problems of aggregation. All consumption is out of labor income.

However, so as to maintain a flavor of regional modeling, I assume that all capital services are rented from the Rest of the World (ROW) at a fixed price. These services are paid for by sales of investment goods to the ROW, with a unitary demand elasticity. A balance of payments is maintained under a fixed exchange rate.

A concluding section of the paper discusses possible extensions of the model.

2. NOTATION:

I adopt the convention that quantity flows are denoted by lower case roman variables; dollar flows by the corresponding upper case roman variables; prices by upper case P, with an appropriate subscript; and log prices by lower case p with a subscript. Therefore, for most vectors **q** we have an identity

(1) $\mathbf{Q} = \hat{\mathbf{q}}\mathbf{P_q}$,

where the overstrike "^" denotes a matrix generated by diagonalizing a vector. The main variables of the model are given in the following table. If a variable has a star, "*", by it, then the value of that variable is determined exogenously from the model. Bold face variables in the table are vectors.

variable	quantity	price	dollar flow	log(price)
Output:	x	P_x	$X = P_x\hat{x}$	p_x
Gross Investment:	i	P_i	$I^* = P_i\hat{i}$	p_i
Labor services:	l	P_l	$L = P_l\hat{l}$	p_l
Capital services:	k	P_{k^*}	$K = P_k\hat{c}$	p_{k^*}
Intermediate product input demands by sector "n"				
from sector "m":	d_{mn}	P_{xm}	$D_{mn} = P_{xm}d_{mn}$	p_{xm}
Total intermediate				
product demands:	d	P_x	$D = P_x\hat{d}$	p_x
Investment requirements by sector "n"				
from sector "m":	j_{mn}	P_{xm}	$J_{mn} = P_{xm}j_{mn}$	p_{xm}
Total investment				
requirements:	j	P_x	$J = P_j\hat{j}$	p_x
Consumption:	c	P_x	$C = P_x\hat{c}$	p_x
Total consumption:	c_o	P_o	c_oP_o	p_o

3. PRODUCTION:

Each output x_n ($n = 1,2, \ldots N$) is generated from labor services, capital services, and intermediate products according a Cobb-Douglas production function:

$$(2)\ \log x_n = \alpha_n + \lambda_n \log l_n + \kappa_n \log k_n + \sum_m A_{mn} \log d_{mn}.$$

I assume constant returns to scale. This implies

$$\lambda_n + \kappa_n + \sum_m A_{mn} = 1 \text{ for each } n, \text{ or}$$

$$(3)\ \boldsymbol{\lambda} + \boldsymbol{\kappa} + \mathbf{A}'\mathbf{1} = \mathbf{1},$$

where **1** is a column vector of 1's. I also assume that the firms are price-taking and profit-maximizing, so that the value marginal product equals marginal cost, or

$$(4)\ P_{xn}\partial x_n/\partial q = P_q,$$

where q is any input. From (2) and (4) in the case of capital services we have

$$P_{xn}x_n\kappa_n/k_n = P_k.$$

From (1) this leads to the demand for capital services:

$$(5)\ \mathbf{K} = \hat{\boldsymbol{\kappa}}\mathbf{X}.$$

This is the first of several linear equations relating the dollar flows of inputs to the dollar flows of output. Similar derivations lead to the labor demand

$$(6)\ \mathbf{L} = \hat{\boldsymbol{\lambda}}\mathbf{X},$$

and the intermediate product demand

$$(7)\ D_{mn} = A_{mn}X_n.$$

In the following, we will not be concerned with the detailed intersectoral flows, D_{nm}, but only with the total intermediate demands by commodity - say,

$$(8)\ D_m = \sum_n D_{mn}.$$

From (7) and (8) it follows that

$$(9)\ \mathbf{D} = \mathbf{AX}.$$

At the same time, perfect competition and technology determine the output price as a well-known function of input prices. This relation can be derived by substituting the appropriate versions of (1) into (5), (6), and (7) to yield relationships of the form

$$q = (\text{constant})X/P_q,$$

as well as

$$x = X/P_x.$$

These equations are then substituted into the production function (2), whereupon X drops out because of constant returns to scale (3). The result simplifies to

$$(10) \quad \log P_{xn} = \varphi_n + \lambda_n \log P_{ln} + \kappa_n \log P_{kn} + \sum_m A_{mn} \log P_{x^m}.$$

(φ_n is a function of the constants introduced earlier, but that doesn't concern us here.) I restate this in matrix terms as the linear relationship

$$(11) \quad \mathbf{p_x} = \boldsymbol{\varphi} + \hat{\lambda}\mathbf{p_l} + \hat{\kappa}\mathbf{p_k} + \mathbf{A}'\mathbf{p_x}.$$

The linear relations (5), (6), (7), and (11) completely describe the behavior of the production industries under the stated assumptions. Production behavior is therefore summarized by (5), (6), (9) and (11).

4. INVESTMENT:

The production of investment goods **i** is assumed to proceed by accretion of the required production goods **j** from several industries, just as in Leontief models. However, unlike Leontief models, substitution is allowed between the different input goods, with a substitution elasticity of unity. The result is a sort of secondary production function for a capital composite by sector, parallel to the primary production function except there is no additional value added (i.e., no additional capital or labor inputs). The resulting production function for investment goods is

$$(12)\ \log i_n = \beta_n + \sum_m B_{mn} \log j_{mn}.$$

I assume constant returns to scale, so that

$$(13)\ \mathbf{B'1} = \mathbf{1}.$$

By reasoning parallel to that used in the primary production case, we arrive at linear relations for investment commodity requirements and prices under perfect competition:

$$(14)\ J_{mn} = B_{mn} I_n.$$

$$(15)\ J_m = \sum_n J_{mn}.$$

$$(16)\ \mathbf{J} = \mathbf{BI}.$$

$$(17)\ \mathbf{p_i} = \tau + \mathbf{B'p_x},$$

where the constant τ depends on the other constants. I assume that the investment demands $\mathbf{i}$ are given exogenously in the ROW with unitary price elasticities:

$$i_m = I_m{}^*/P_m.$$

It follows that

(18) $\mathbf{I} = \mathbf{I}^*$, where $\mathbf{I}^*$ is exogenous. Hence

$$(19)\ \mathbf{J} = \mathbf{BI}^*.$$

Investment behavior can then be summarized by the linear equations (17) and (19).

5. CONSUMPTION DEMAND AND LABOR SUPPLY:

I assume that there is a single household in the economy which maximizes a utility function U. In the interior of some region, U is given by

(20) $U = c_0 - \lambda \sum_n l_n$, where

(21) $c_0 = \prod_n c_n^{\gamma_n}$.

U is subject to the budget constraint

$\mathbf{P_x}'\mathbf{c} = \mathbf{P_l}'\mathbf{l}$.

I assume that

(22) $\sum_n \gamma_n = 1$. Note that $\sum_n l_n$ is the total labor supply, and the wage bill $\mathbf{P_l}'\mathbf{l} = Y$ is the total income. (I assume that all capital income is exported from the region, and there are no interregional transfer payments.)

The consumption portion of U is the familiar Cobb-Douglas utility function. Maximization leads to the usual consumption demands

(23) $c_n = \gamma_n \mathbf{P_l}'\mathbf{l}/P_{xn}$, or

(24) $C = \gamma Y$. From the dollar flow identity (1), Y can be restated as

(25) $Y = \mathbf{L}'\mathbf{1}$, where $\mathbf{1}$ is a vector of 1's.

It will be convenient in the following to define a consumption price index P_0:

(26) $P_0 = \exp(\upsilon)\prod_n Px_n^{\gamma_n}$; or stated in terms of log prices,

(27) $p_0 = \upsilon + \gamma'\mathbf{p_x}$.

It can be shown that for a certain value of the constant υ, P_0 is an exact consumer's price index for aggregate consumption, defined as the indirect utility function for consumption, evaluated at constant nominal income. However, for our purposes I only need to point out that there is a value of υ such that total consumption obeys the relation

$$C'1 = P_o c_o \quad \text{at the optimum.}$$

(To prove this relationship, use equations (21), (1), (22), (23), and (24).) It follows that the linear portion of the household problem can be restated as

$$\text{MAX} \quad c_o - \sum_n l_n \qquad \text{subject to} \qquad c_o P_o = P_l{}'l.$$

Since this problem is linear, it does not lead to any definite interior solution for demands as a function of prices. Instead, the assumption of an interior solution leads to a restriction on prices

$$(28) \quad P_o = P_{ln} \quad \text{for all } n.$$

In other words, the labor supply for each type of labor is perfectly elastic at the supply price P_o. Restating this in terms of log prices, we have

$$p_o = p_{ln} \quad \text{for all } n.$$

In other words,

$$(29) \quad p_l = p_o 1.$$

Consumption demand and labor supply are then completely specified by the linear equations (24), (25), (27), and (29).

ASIDE. The perfectly elastic labor supply may seem to be the most unrealistic part of the model. In particular, the empirical measurements of the U.S. labor supply of prime-aged males are, in general, nearly consistent with a perfectly INelastic supply. However, an elastic supply might approximate the long-run experience of a small open region with high labor mobility. For a comment on extending the model to inelastic supplies, see Section 10.

6. CAPITAL SUPPLY AND BALANCE OF PAYMENTS:

Capital services are rented from the ROW at fixed prices

$P_k = P_k^*$;

or in log terms,

(30) $p_k = p_k^*$.

Since interregional payments are assumed to balance, total payments for capital services must equal total income from the sale of investment goods, so that

(31) $K'1 = I'1$.

However, I will show in the next section that the condition (31) adds no new restrictions on the model; that is, it can also be derived from the previous dollar flow equations and CRTS conditions, plus the market-clearing condition in the goods market.

7. SOLUTION OF THE DOLLAR FLOW MODEL:

The dollar flow model can now be solved in a very simple fashion. We begin with the condition that the market for produced goods clears in dollar flow terms. (The dollar flow balance equation expresses roughly the same information as does the material balance equation of Leontief models, but in different units of measurement). In the second step, all flows are restated as functions of output flows (**X**) or exogenous variables. Finally, we solve for **X**.

The dollar flow balance equation is

(32) $\mathbf{X} = \mathbf{D} + \mathbf{J} + \mathbf{C}.$

Equations (6), (9), (19), (24), and (25) lead to

(33) $\mathbf{X} = \mathbf{AX} + \mathbf{BI}^* + \gamma\lambda'\mathbf{X}.$

This leads immediately to the solution

(34) $\mathbf{X} = [\hat{1} - \mathbf{A} - \gamma\lambda']^{-1}\mathbf{BI}^*,$

where $\hat{1}$ is the identity matrix. **K**, **L**, **C**, **J**, and **D** can then be found from (5), (6), (24), (25), (19), and (9).

To check the balance of payments condition (31), use (33) to derive

$$\mathbf{X}'1 = \mathbf{X}'\mathbf{A}'1 + \mathbf{I}'\mathbf{B}'1 + \mathbf{X}'\lambda\gamma'1.$$

The CRTS conditions (3), (13), and (22) lead to

$$\mathbf{X}'1 = \mathbf{X}'[1 - \lambda - \kappa] + \mathbf{I}'1 + \mathbf{X}'\lambda, \text{ or}$$

$$0 = - \mathbf{X}'\kappa + \mathbf{I}'1.$$

The capital demand (5) then leads to the balance of payments equation (31).

8. SOLUTION OF THE PRICE MODEL:

The main equation driving the price model is the competitive price equation for primary production, equation (11). One eliminates the price of capital using the exogenous price (30). The price of labor is related back to the price of output by means of the consumption price index; that is, (27) and (29) lead to

$$p_l = \upsilon 1 + 1\gamma' p_x.$$

Then the fundamental price equation is then

(35) $p_x = \varphi + \hat{\lambda}[\upsilon 1 + 1\gamma' p_x] + \hat{\kappa} p_k^* + A' p_x,$

with the solution

(36) $p_x = [\hat{1} - 1\gamma' - A']^{-1}[\varphi + \hat{\lambda}\upsilon 1 + \hat{\kappa} p_k^*].$

The log-prices of consumption and labor can then be found from (27), and the log-price of investment goods can be found from (17).

Note that the price model is completely independent of the dollar flow model, as advertised.

9. COMPARISON TO FIXED-COEFFICIENT MODELS:

It will be apparent that the dollar flow solution (34) looks very similar to a Leontief model solution with endogenous consumption and exogenous investment. However, the correct Leontief solution is in material-flows rather than dollar flows. (Note: more detailed expositions of the Leontief model and its price equations are given in Dervis, de Melo, and Robinson [1982, esp. pp. 48-52.], and Takayama [1985].)

In particular, investment must be assumed perfectly inelastic rather than unitary elastic. Consumption is proportional to labor income, so prices enter the model. It can be shown that the Leontief solution is then

(37) $x = [\hat{1} - A_L - \gamma_L \hat{\lambda}_L P_L'/(P_x'\gamma)]^{-1} B_L i^*,$

where the subscript L on the constants indicates that they are Leontief fixed-coefficient parameters, rather than Cobb-Douglas exponents.

Now it happens that if $\mathbf{A} = \mathbf{A}_L$, $\mathbf{B} = \mathbf{B}_L$, and so on, and if relative prices do not change over time, then the two equations (34) and (37) cannot be distinguished empirically, because they make identical predictions. (To demonstrate this point most simply, define quantity units in such a way that all prices equal 1, and then recover (37) from (34).) However, if these two equations are compared over time, with prices allowed to change, then the predictions will differ.

On the other hand, the usual Leontief price equations are quite different from the Cobb-Douglas price equations. In particular, under some circumstances both capital and labor prices can be taken as exogenous, say $\mathbf{P}_k{}^*$ and $\mathbf{P}_l{}^*$. Then under competitive conditions the price of produced goods is entirely determined by the prices of factor inputs according to the zero-profit condition

$$\mathbf{P}_x{}'\hat{\mathbf{x}} = \mathbf{P}_x{}'\mathbf{A}_L\hat{\mathbf{x}} + \mathbf{P}_l{}^{*\prime}\hat{\lambda}_L\hat{\mathbf{x}} + \mathbf{P}_k{}^{*\prime}\hat{\kappa}_L\hat{\mathbf{x}}.$$

As a result of CRTS, $\mathbf{x}$ drops out, leading to

$$(38) \quad \mathbf{P}_x = [\hat{1} - \mathbf{A}_L{}']^{-1}[\hat{\lambda}_L\mathbf{P}_l{}^* + \hat{\kappa}_L\mathbf{P}_k{}^*].$$

This prediction is very different from equation (35).

Under more restrictive exogeneity conditions than in the Cobb-Douglas case, there is a separability theorem for the Leontief case. In particular, if consumption is assumed exogenous and perfectly inelastic, then equation (37) is replaced by

(39) $\mathbf{x} = [\hat{1} - \mathbf{A}_L]^{-1}[c + \mathbf{B}_L i^*]$.

Note that the quantity flow equation (39) is independent of the price equation (38).

10. POSSIBLE EXTENSIONS OF THE MODEL:

The following extensions of the model are possible without changing any fundamental conclusions, but these extensions are not developed here.

1. Perfectly elastic export and Cobb-Douglas import demands can be introduced.

2. Transport, retail, and wholesale margins can be introduced by assuming that delivered goods can be produced through a Cobb-Douglas technology from inputs of undelivered goods, transport services, wholesale services, and retail services.

3. The model is easily modified to allow for exogenous taxes at fixed rates, with the proceeds exported.

4. Fixed dollar-flow interregional income transfers and Cobb-Douglas saving are also straight-forward.

The model can also be modified to allow for factors in fixed supply or outputs in fixed final demand. However, separability of the price and dollar flow models no longer holds; instead, there will be a recursive relationship. For example, with perfectly inelastic supplies, the dollar flow model can be solved independently of prices. Then the prices of the goods in fixed supply or fixed demand can be determined from their dollar flows. Finally, the remaining log-prices can be determined from a linear relationship.

Under very restrictive conditions, these ideas can be extended to intertemporal perfect foresight or Arrow-Debreu models. In the simplest case, one assumes perfect foresight, with exogenous prices and interest rates held constant (hence all prices are constant). This leads to constant capital-to-output ratios. With full capacity assumptions and exponential depreciation, one ends up with a model formally similar to a dynamic Leontief model. Consequently, the usual problems of dynamic instability and causal indeterminacy are likely to emerge [see Takayama, pp.503-506].

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