OPERATORS IN REPRODUCING KERNEL SPACES

by

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INTRODUCTION

A problem in the field of differential and integral equations most often consists of analyzing a linear operator defined on some linear class of functions. In many other domains of mathematics problems of this type are also common. With such a problem there is usually associated a quadratic norm, having some natural connection with the problem, and this norm will then give the class of functions the character of an incomplete Hilbert space. In analyzing such an operator the general theory of linear operators in a Hilbert space has become a tool of ever increasing importance and ever widening scope.

Quite aside from the operator in question the space itself is usually investigated by analyzing the values of the functions. For this reason it is very useful if, for a fixed point, the value of a function at that point depends continuously on the function. Indeed, this is the case for the classes corresponding to all self adjoint ordinary differential equations and many partial differential equations, when the appropriate norm is considered. These spaces, where the evaluation map is continuous, are precisely the spaces considered by N. Aronszajn in [1,2] [1)] --- the functional Hilbert spaces having a reproducing kernel. Just as we learn much in these applications from the abstract theory of linear operators in a Hilbert space, we may learn even more from a theory of linear operators in a functional Hilbert space with a reproducing kernel.

As yet there have been few developments along this line, where the existence of a reproducing kernel plays an essential role.

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1) The numbers in brackets refer to the bibliography at the end.

N. Aronszajn [2] considered an interesting representation of a bounded operator L in terms of a kernel $\Lambda(x,y)$, where the scalar product of f with Λ, for each point y, gives the value of Lf at this point. The kernel Λ thus plays a role similar to that of the kernel for an integral operator. It will be our aim in the following sections to develop further a general theory of operators (possibly unbounded) represented by kernels Λ. In a recent paper A. Devinatz [1] has investigated some of the properties of such operators, and any of our considerations which operlap his will be mentioned in a footnote.

We have divided our investigation into three parts. In the first four sections of Part I we are concerned with some elementary properties of transformations in a Hilbert space, including the notion of a "subnormal" operator --- which corresponds to the restriction of a normal operator in the same fashion as a symmetric operator corresponds to the restriction of a self adjoint operator. The last section of Part I deals with the problem of finding semi-bounded self adjoint extensions of a semi-bounded symmetric operator. In this section we characterize the class of self adjoint extensions having the same semi-bound as the original operator, we give a necessary and sufficient condition for such an extension to be unique, and we discuss the other semi-bounded self adjoint extensions, including the extension of K. Friedrichs [1].

In Part II we are concerned with representable operators in an r.k. space, i.e. we have a functional Hilbert space F with a reproducing kernel K and we consider operators L represented by kernels $\Lambda(x,y)$ by the equation $Lf(y) = (f, \Lambda_y)$, where $\Lambda_y(x) = \Lambda(x,y)$. In section 2 we introduce the notions of a maximal operator L_M and a minimal operator L_m corresponding to a given kernel Λ, and we show that $\Lambda_y \in F$ and

$\Lambda_y^* \in F$, where $\Lambda^*(x,y) = \overline{\Lambda(y,x)}$, are necessary and sufficient conditions for L_M and L_m both to exist. The sum and product of representable operators is discussed in section 3, and the notion of a "subnormal" kernel is introduced in section 5 --- corresponding to the notion of a subnormal operator introduced in Part I. In section 6 symmetric kernels are analyzed, the deficiency indeces of L_m are characterized in terms of Λ, and a positive matrix Λ is shown to correspond to a positive operator L_m. Finally section 8 is concerned with the analogy between Carleman integral operators and representable operators in an r.k. space.

In Part III we consider a fixed maximal operator L in an r.k. space F. We consider the natural extension of L to the whole space F by putting $L_1 f(y) = (f, \Lambda_y)$ for each $f \in F$. As a natural r.k. space containing both the class F and the class $L_1(F)$ we take $F_1 = F + L_1(F)$, where the new reproducing kernel is $K_1(x,y) = K(x,y) + (\Lambda_y, \Lambda_x)$. The space F_1 is shown to be isomorphic to the graph of L^* (the adjoint of L), and many properties of L_1 (in the space F_1) are characterized in terms of properties of L. It is found that the extension L_1 is sometimes bounded in F_1 even though L is not bounded in F. The properties of symmetry and subnormality are seen in sections 2 and 3 to be carried over from the operator L to the extended operator L_1. Section 4 contains a rather detailed analysis of the relation between the spectrum of L and the spectrum of L_1. Finally, section 5 continues the discussion of Carleman integral operators.

We have included many examples throughout, which serve to illustrate that the situations discussed in the text may actually be attained, and sometimes to show that the theorems stated are the best possible under the given assumptions.

To a large extent our basic notation and terminology is explained in sections 1 and 2 of Part I, and in section 1 of Part II. For an explanation of the minimal inverse $T^{\ominus 1}$ and the minimal adjoint $T^{\circledast}$ we refer the reader to section 3 of Part I. Whatever notions needed beyond these will be explained in the course of the development.

When referring to formula (3)(or Theorem 3) of section 2, Part I, we shall in other sections write (I.2.3)(or Theorem I.2.3) and in the same section write simply (3)(or Theorem 3).

PART I: ABSTRACT SPACES

1. An abstract Hilbert space. Throughout this and the succeeding sections we shall assume that the reader is familiar with the general theory of Hilbert spaces as can be found in Nagy [1], Stone [1], and partially in Aronszajn [3]. Our terminology and notation will be essentially that in common to Aronszajn, Nagy, and Stone, and is listed below for convenience.

By a Hilbert space $\mathcal{H}$ we shall mean a complex Hilbert space that is not necessarily separable, but whose dimension may be any finite or transfinite cardinal number. As in Nagy [1] the "dimension" of a space is taken to be the smallest number of elements in a set whose linear combinations are dense in the space, and this number is also the number of elements in every maximal orthonormal system (called a "basis").

Unless otherwise specified the topology used will be the "strong" or norm topology, with the notation $f_n \to f$ meaning that the sequence f_n converges strongly to f. We shall also write $f_n \rightharpoonup f$, or $f = \text{w.} \lim f_n$, when f_n converges weakly to f.

A "subspace" of $\mathcal{H}$ will not necessarily be closed, and will be the "linear manifold" of Nagy [1] and Stone [1]. The symbol "(0)" will denote the subspace consisting only of the zero vector. If a subspace is closed, it will be called a "closed subspace" (= "subspace" of Nagy [1] and Stone [1]).

For any two sets A and B, AB will denote the "intersection" of A and B, and $\bar{A}$ will denote the "closure" of A. Also the inclusion $A \subset B$ will not exclude the possibility that A = B. (Our "$A \subset B$" is the "$A \subseteq B$" of Nagy [1] and Stone [1].)

For any two subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1 \dot{+} \mathcal{L}_2$ will denote their "vectorial sum" and $\mathcal{H} \ominus \mathcal{L}_1 = \mathcal{H} \ominus \overline{\mathcal{L}_1}$ will denote the "orthogonal complement" of $\mathcal{L}_1$ (in $\mathcal{H}$).

The "dimension" of a subspace is the dimension of its closure, and the "rank" of a subspace is the dimension of its orthogonal complement.

For any collection of vectors f_α, $\alpha \in A$, the notation $[f_\alpha]$ shall stand for the smallest closed subspace containing all of the f_α for $\alpha \in A$.

If $\{\mathcal{H}_\alpha\}$, $\alpha \in A$, is a collection of Hilbert spaces, then $\mathcal{H} = \sum_{\alpha \in A}^{\perp} \mathcal{H}_\alpha$ will denote the "combinatorial orthogonal sum" (the "Cartesian product" of Nagy [1]) of the spaces $\mathcal{H}_\alpha$. For two such spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we shall also write $\mathcal{H} = \mathcal{H}_1 \pm \mathcal{H}_2$.

2. Transformations. We shall distinguish between a "transformation" from one Hilbert space into another and an "operator", which will always be a transformation from a space into itself. A "functional" will be a transformation whose range lies in the one-dimensional Hilbert space of complex numbers.

If T is a transformation, then $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$, and $\mathcal{G}(T)$ will denote the "domain", "range", "nullspace", and "graph", respectively, of T. When we are dealing with only one transformation T we may write simply $\mathcal{D}$, $\mathcal{R}$, $\mathcal{N}$, and $\mathcal{G}$ instead of $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$, and $\mathcal{G}(T)$. If T has an adjoint transformation T^*, then we may also write $\mathcal{D}^*$, $\mathcal{R}^*$, and $\mathcal{N}^*$ for $\mathcal{D}(T^*)$, $\mathcal{R}(T^*)$, and $\mathcal{N}(T^*)$ respectively. Further, if we have other transformations T_1, T_2, T', etc., we may write simply $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}'$, etc., instead of $\mathcal{D}(T_1)$, $\mathcal{D}(T_2)$, $\mathcal{D}(T')$, etc.

We shall also make the distinction between a transformation "from" $\mathcal{H}$ (when $\mathcal{D} \subset \mathcal{H}$) and a transformation "of" $\mathcal{H}$ (when $\mathcal{D} = \mathcal{H}$), and between a transformation "<u>into</u>" $\mathcal{H}$ (when $\mathcal{R} \subset \mathcal{H}$) and a transformation "<u>onto</u>" $\mathcal{H}$ (when $\mathcal{R} = \mathcal{H}$).

If we are dealing with two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and two operators L_1 and L_2, one in each space, then as long as no confusion will arise we will use the same symbol "*" to denote the adjoint of each. That is, we shall use the notation L_1^*, L_2^*, $\mathcal{D}_1^*$, etc. rather than L_1^{*1}, L_2^{*2}, $\mathcal{D}_1^{*1}$, etc. Similarly if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two subspaces, $\mathcal{L}_i \subset \mathcal{H}_i$, then "$\ominus$" and "—" will be used for each as in $\mathcal{H}_1 \ominus \mathcal{L}_1$, $\mathcal{H}_2 \ominus \mathcal{L}_2$, $\overline{\mathcal{L}_1}$, and $\overline{\mathcal{L}_2}$, as long as it is obvious which norms we are referring to.

Let T be a transformation from $\mathcal{H}$ into $\mathcal{H}'$. If T_1 is another such transformation, then $T_1 \subset T$ will mean that T_1 is a "<u>restriction</u>" of T, or T is an "<u>extension</u>" of T_1, and will not exclude the possibility that $T_1 = T$. If $\mathcal{L} \subset \mathcal{H}$ then $T_{\mathcal{L}}$ will denote the restriction of T to the domain $\mathcal{L}\mathcal{D}$.

If T_1 is another transformation from $\mathcal{H}$ into $\mathcal{H}'$, then $T \dotplus T_1$ is the obvious transformation whose domain is $\mathcal{D}\mathcal{D}_1$, i.e. it is defined wherever $Tf + T_1 f$ makes sense. If T_1 is an operator, then $T_1 - \lambda = T_1 - \lambda I$, where I is the identity operator.

If T' is a transformation from $\mathcal{H}'$ into $\mathcal{H}''$, then $T'T$ denotes the obvious transformation from $\mathcal{H}$ into $\mathcal{H}''$ whose domain is the set of elements $f \in \mathcal{H}$ such that $T'(Tf)$ makes sense.

If T is one-to-one, i.e. $Tf = g$ has a unique solution f for each $g \in \mathcal{R}$, then T has an inverse transformation, denoted by T^{-1}, with domain $\mathcal{R} \subset \mathcal{H}'$ and range $\mathcal{D} \subset \mathcal{H}$.

The graph $\mathcal{J}(T)$ is a subset of the space $\mathcal{H} \dotplus \mathcal{H}'$. A transformation is "<u>closed</u>" if its graph is closed (in the space $\mathcal{H} \dotplus \mathcal{H}'$). If T has a closed extension, then its "closure" will be the smallest closed extension. If T has no closed extension, then we say that the closure of T does not exist.

The transformation T is "continuous" if $x_n \epsilon \mathfrak{D}$ and $x_n \to x \epsilon \mathfrak{D}$ implies that $Tx_n \to Tx$. A continuous transformation need not be defined on the whole space.

The transformation T is "linear" if $\mathfrak{D}$ is a subspace and if $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$ for all f, g $\epsilon \mathfrak{D}$ and arbitrary complex numbers α and β. A "bounded" linear transformation T will not necessarily be defined on the whole space. Its domain may be any subspace of $\mathcal{H}$, and its bound will be denoted by $\|T\|$.

An important theorem is the closed graph theorem, which says that a closed linear transformation with a closed domain must be bounded. It is easy to prove the two converses, that a closed bounded transformation has a closed domain, and that a bounded transformation with closed domain must be closed.

For linear operators T in the space $\mathcal{H}$, with dense domains, we also have the notions of T being "symmetric", "self adjoint", or "positive", where positive does not imply self adjoint, but only that $(Tf,f) \geq 0$ for all $f \epsilon \mathfrak{D}$ (as in Nagy [1], p. 35).

Finally, R_λ will denote the "resolvent operator" of T, and is equal to $(\lambda - T)^{-1}$ on the resolvent set of T.

3. Two lemmas on transformations. A useful criterion for determining whether two subspaces have the same dimension is given by the following lemma.

Lemma 1. A one-to-one closed linear transformation T, from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{H}'$, preserves dimensions, i.e.

<u>dim $\mathfrak{D} = \dim \mathfrak{R}$</u>. <u>More generally, if a linear transformation T has a closure $\tilde{T}$, then $\dim \mathfrak{D} = \dim \tilde{\mathfrak{X}} + \dim \mathfrak{R}$</u>.

<u>Remark</u>: It is easily seen that this lemma is no longer true if the transformation has no closure. For example we can map a Hamel basis in a separable space onto an orthonormal basis in a higher dimensional space and extend it by linearity to a one-to-one linear transformation. But this transformation will not be closed.

<u>Proof</u>. The second statement follows from the first. In fact $\mathfrak{D} \subset \tilde{\mathfrak{D}} \subset \overline{\mathfrak{D}}$, $\mathfrak{R} \subset \tilde{\mathfrak{R}} \subset \overline{\mathfrak{R}}$, and the first statement may be applied to $T_{\mathfrak{D}_0}$, where $\mathfrak{D}_0 = \tilde{\mathfrak{D}}(\mathcal{H} \ominus \tilde{\mathfrak{X}})$.

To prove the first statement we suppose first that T is bounded. We may also assume that $\mathcal{H} = \mathfrak{D}$ and $\mathcal{H}' = \overline{\mathfrak{R}}$. Clearly T maps any dense set in $\mathcal{H}$ into a dense set in $\mathfrak{R}$, and therefore $\dim \mathfrak{R} \leq \dim \mathfrak{D}$.

On the other hand, any dense set S in $\mathfrak{R}$ is dense in $\mathcal{H}'$ and therefore $T^*(S)$ is dense in $\mathfrak{R}^*$, i.e. $\dim \mathfrak{R}^* \leq \dim \mathcal{H}'$. But $\mathfrak{R}^*$ is dense in $\mathcal{H}$ since T is one-to-one, and consequently $\dim \mathfrak{D} = \dim \mathcal{H} = \dim \mathfrak{R}^* \leq \dim \mathcal{H}' = \dim \mathfrak{R}$. Thus the lemma is established for a bounded transformation.

In the general case we look at the graph $\mathcal{J}(T)$ and the projections P and P$'$ projecting the entire graph space $\mathcal{H} \pm \mathcal{H}'$ (cf. I.2) onto the first and second components, $\mathcal{H}$ and $\mathcal{H}'$, respectively. The transformations $P_{\mathcal{J}}$ and $P'_{\mathcal{J}}$ are closed, one-to-one, and bounded, so that we may apply the lemma as far as we have proved it. This gives us $\dim \mathfrak{D} = \dim \mathcal{J} = \dim \mathfrak{R}$ and completes the proof of the lemma.

We recall (cf. I.2) that the domain of a linear transformation T is not presumed to be dense, and therefore its adjoint transformation

T^* may not exist. However, we may always define a "<u>minimal adjoint</u>" $T^{\circledast}$ as follows.

If T is a linear transformation from $\mathcal{H}$ into $\mathcal{H}'$, we may also consider it as a transformation from the Hilbert space $\overline{\mathcal{D}}$ into $\mathcal{H}'$. In the latter sense T has an adjoint transformation (from $\mathcal{H}'$ into $\overline{\mathcal{D}}$), which we shall denote by $T^{\circledast}$ and call the "<u>minimal adjoint</u>" of T. Obviously $T^{\circledast}$ coincides with T^* whenever T^* exists. If T_1 denotes the linear extension of T with domain $\mathcal{D} \dot{+} (\mathcal{H} \ominus \overline{\mathcal{D}})$ and with $\mathcal{H} \ominus \overline{\mathcal{D}} \subset \mathcal{N}(T_1)$, then we also have $T^{\circledast} = T_1^*$.

This transformation $T^{\circledast}$ is clearly independent of the space $\mathcal{H}$ containing $\overline{\mathcal{D}}$, but it does depend on the space $\mathcal{H}'$ containing $\overline{\mathcal{R}}$. The dependence upon $\mathcal{H}'$, however, is only to the extent of determining the nullspace of $T^{\circledast}$, which is $\mathcal{H}' \ominus \overline{\mathcal{R}}$. The operation "$\circledast$" thus carries an arbitrary <u>closed</u> linear transformation (from $\mathcal{H}$ into $\mathcal{H}'$) into a closed linear transformation (from $\mathcal{H}'$ into $\mathcal{H}$) having a dense domain (in $\mathcal{H}'$). Further, when T has a closure, $T^{\circledast\circledast} = T^{\circledast *}$ is the closure of the extension T_1 of T defined above.

The minimal adjoint $T^{\circledast}$ may also be defined directly in terms of the scalar product or in terms of the graph of T. If we break a convention (in this paragraph only) and consider the adjoint T^* as a multi-valued transformation, then it will always exist. It will have the same domain as the single-valued transformation $T^{\circledast}$, and $T^{\circledast}f$ will be the element in the class T^*f having the smallest norm. From these remarks it is clear in what sense the minimal adjoint is minimal.

Another useful concept is that of the "<u>minimal inverse</u>"[1] $T^{\ominus 1}$ of a closed linear transformation. If T is a closed linear transformation,

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1) The theory of partial inverses, and in particular of the

from $\mathcal{H}$ into $\mathcal{H}'$, and $\mathcal{D}_0 = \mathcal{D}(\mathcal{H} \ominus \mathcal{N})$, then $T_{\mathcal{D}_0}$ is a one-to-one closed linear transformation with range $\mathcal{R} = \mathcal{R}(T)$. The minimal inverse $T^{\ominus 1}$ may then be defined as the linear extension of $(T_{\mathcal{D}_0})^{-1}$ to the domain $\mathcal{R} \dotplus (\mathcal{H}' \ominus \overline{\mathcal{R}})$ and having the nullspace $\mathcal{H}' \ominus \overline{\mathcal{R}}$. Clearly $T^{\ominus 1}$ always has a dense domain. It should be noted that when T has an inverse, T^{-1} will be only a restriction of $T^{\ominus 1}$. If T_1 denotes the linear extension of T with domain $\mathcal{D} \dotplus (\mathcal{H} \ominus \overline{\mathcal{D}})$ and with $\mathcal{H} \ominus \overline{\mathcal{D}} \subset \mathcal{N}(T_1)$, then we also have $T^{\ominus 1} = T_1^{\ominus 1}$.

In terms of the above notions we may now state a useful lemma, analogous to the well-known property that $T^{*-1} = T^{-1*}$ (whenever T^*, T^{-1}, and either T^{*-1} or T^{-1*} exist).

<u>Lemma 2. Let T be a closed linear transformation from $\mathcal{H}$ into $\mathcal{H}'$. Then $T^{\circledast \ominus 1} = T^{\ominus 1 *}$ and has $\mathcal{N}(T)$ for its nullspace.</u>

<u>Proof</u>. From the above definitions and the remarks following them, we have the general relations (for a closed T)

$$\mathcal{N}^{\circledast} = \mathcal{N}^{\ominus 1} = \mathcal{H}' \ominus \overline{\mathcal{R}},$$

and

$$\overline{\mathcal{R}^{\circledast}} = \overline{\mathcal{R}^{\ominus 1}} = \mathcal{H} \ominus \mathcal{N}.$$

Applying the first two relations to the transformations $T^{\ominus 1}$ and $T^{\circledast}$, and using the last two relations, we obtain $\mathcal{N}^{\ominus 1 *} = \mathcal{N}^{\circledast \ominus 1} = \mathcal{N}$. Thus $T^{\circledast \ominus 1}$ and $T^{\ominus 1 \circledast}$ are closed linear transformations having a common nullspace $\mathcal{N}$. To complete the proof we must now show that $(T^{\circledast \ominus 1})_{\mathcal{H} \ominus \mathcal{N}} = (T^{\ominus 1 *})_{\mathcal{H} \ominus \mathcal{N}}$.

Let $\mathcal{D}_0 = \mathcal{D}(\mathcal{H} \ominus \mathcal{N})$, $T_0 = T_{\mathcal{D}_0}$, and consider T_0 as a transformation from $\overline{\mathcal{D}}_0$ into $\overline{\mathcal{R}}$. Then T_0^*, T_0^{-1}, and T_0^{-1*} exist so that

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minimal inverse, of a closed linear transformation was introduced by Hans Hamburger. The author became familiar with these notions from a series of lectures by Hans Hamburger in the summer of 1950 at Oklahoma A. and M. College, Stillwater, Oklahoma.

$T_o^{*-1} = T_o^{-1*}$. From the remarks following the definitions we obtain $T_o^* = (T^{\circledast})_{\overline{\mathcal{R}}}$ and $T_o^{-1} = (T^{\ominus 1})_{\overline{\mathcal{R}}}$. By repeating this procedure we then obtain $T_o^{*-1} = (((T^{\circledast})_{\overline{\mathcal{R}}})^{\ominus 1})_{\overline{\mathcal{D}}_o} = (T^{\circledast\ominus 1})_{\overline{\mathcal{D}}_o}$ and $T_o^{-1*} = (((T^{\ominus 1})_{\overline{\mathcal{R}}})^{\circledast})_{\overline{\mathcal{D}}_o}$ $= (T_o^{\ominus 1\circledast})_{\overline{\mathcal{D}}_o}$. Since $T_o^{*-1} = T_o^{-1*}$ it then follows that $(T^{\circledast\ominus 1})_{\overline{\mathcal{D}}_o} = (T^{\ominus 1\circledast})_{\overline{\mathcal{D}}_o}$ and the proof is complete.

4. Subnormal operators. We know that two operators T_1 and T_2 are equal only if they have the same domain $\mathcal{D}_1 = \mathcal{D}_2$ and assign the same value $T_1 f = T_2 f$ for every element $f \in \mathcal{D}_1$. Correspondingly we define the concept of two operators T_1 and T_2 being "metrically equivalent"[1] if $\mathcal{D}_1 = \mathcal{D}_2$ and $\|T_1 f\| = \|T_2 f\|$ for all $f \in \mathcal{D}_1$. We may also say that T_1 and T_2 are "metrically equivalent on $\mathcal{L}$" if $\mathcal{L} \subset \mathcal{D}_1 \mathcal{D}_2$ and the restrictions $T_{1_{\mathcal{L}}}$ and $T_{2_{\mathcal{L}}}$ are metrically equivalent. We may now give the following definitions.

An operator T is said to be "normal" if it is closed, linear, has a dense domain, and $T^*T = TT^*$. An equivalent definition is as follows. An operator T is said to be "normal" if it is linear, has a dense domain, and if T and T^* are metrically equivalent.[2] The equivalence proof of these two definitions can be found in Nagy [1] (p.33) and will not be reproduced here.

We note immediately that every self adjoint operator is normal, but that a symmetric operator will not in general be normal since $\mathcal{D}$ and $\mathcal{D}^*$ will be different. We also recall that any restriction (with

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1) The "metrisch gleich" of Nagy [1].

2) Both definitions are from V. Neumann [2]. Further equivalent definitions can be found in V. Neumann [1,3] and Stone [1].

dense domain) of a self adjoint operator will be symmetric. With these remarks in mind we introduce the following definition.

An operator T is said to be "subnormal" if it is linear, has a dense domain $\mathcal{D} \subset \mathcal{D}^*$, and if T and T^* are metrically equivalent on $\mathcal{D}$.

Clearly it is a generalization of both symmetric and normal operators, and it is easily seen that the closure of a subnormal operator is again subnormal.[1)]

For closed operators we may rephrase the definition in the following manner, analogous to the first definition of a normal operator.

Theorem 1. Let T be a closed linear operator with a dense domain. Then necessary and sufficient conditions for T to be subnormal are $\mathcal{D} \subset \mathcal{D}^*$ and $T^*T = T_1T_1^*$, where $T_1 = ((T^*)_{\mathcal{D}})^*$.

Proof. Suppose first that T is subnormal and closed. Then $(T^*)_{\mathcal{D}}$ is closed, since it is metrically equivalent to T, and $T_1^* = (T^*)_{\mathcal{D}}$. Hence for $f \in \mathcal{D}(T^*T)$ and $g \in \mathcal{D}$ we have $(T^*Tf,g) = (Tf,Tg) = (T^*f,T^*g)$ $= (T_1^*f,T_1^*g) = (T_1T_1^*f,g)$, the last equality being true because $(T_1^*f,T_1^*g) = (T^*Tf,g)$ shows that $T_1^*f \in \mathcal{D}_1$. Thus we have shown that $T^*T \subset T_1T_1^*$.

Further, for $f \in \mathcal{D}(T_1T_1^*)$ and $g \in \mathcal{D}$ we have $(Tf,Tg) = (T^*f,T^*g)$ $= (T_1^*f,T_1^*g) = (T_1T_1^*f,g)$, which shows that $Tf \in \mathcal{D}^*$ and consequently that $T^*T = T_1T_1^*$.

On the other hand if T is a closed linear operator with dense domain $\mathcal{D} \subset \mathcal{D}^*$ and $T^*T = T_1T_1^*$, then for every $f \in \mathcal{D}(T^*T) = \mathcal{D}(T_1T_1^*)$ we have (Tf,Tf) $= (T^*Tf,f) = (T_1T_1^*f,f) = (T_1^*f,T_1^*f) = (T^*f,T^*f)$. Finally for any $f \in \mathcal{D}$ we can find a sequence $\{f_n\} \subset \mathcal{D}(T^*T)$ so that $f_n \to f$ and $Tf_n \to Tf$

1) We reserve the problem of finding normal extensions of a subnormal operator to be considered in another paper.

(cf. Nagy [1], p.30). From what we have already proved we have $\|T^*f_m - T^*f_n\| = \|Tf_m - Tf_n\|$ and therefore T^*f_n is convergent. Consequently $f \in \mathcal{D}^*$, $T^*f_n \to T^*f$ (since T^* is closed), and $\|Tf\| = \lim\|Tf_n\| = \lim\|T^*f_n\| = \|T^*f\|$, which completes the proof of the theorem.

5. Semi-bounded operators. There have been many investigations concerned with finding self adjoint extensions of symmetric operators. One important theorem, due to K. Friedrichs [1], is concerned with extending semi-bounded operators. It is proved here, in a slightly more general form than Friedrichs' original result,[1)] because of its close connection with Theorems 2, 3 and 4, and because of the additional light shed on the situation by the examples at the end of this section.

Theorem 1. Let L be a symmetric operator in $\mathcal{H}$ with a finite lower bound c. Then L has a well determined self adjoint extension S with the same lower bound, whose domain can be described as follows.

Let $\alpha > 0$, $H_\alpha = L - c + \alpha$, and $\|f\|_\alpha^2 = (H_\alpha f, f)$ for $f \in \mathcal{D} = \mathcal{D}(L)$. Then $\mathcal{D}(S)$ is the set of elements $f \in \mathcal{D}^*$, such that f is the strong limit of a sequence f_n in $\mathcal{D}$, with f_n a Cauchy sequence with respect to the stronger norm $\| \ \|_\alpha$. (The domain $\mathcal{D}(S)$ is independent of α.)

This theorem is an immediate consequence of the following four lemmas (letting $S = T_\alpha^{-1} + c - \alpha$). These lemmas closely parallel Friedrichs' original proof (except for the dependence upon α), and are phrased in the terminology of Theorem 1.

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1) Friedrichs considered in detail only operators with positive lower bounds. From his results it immediately follows, as a corollary, that every semi-bounded operator L possesses a self adjoint extension S with the same bound. To deduce this corollary for $L \geq c$, we must add to L a multiple, $-c + \alpha$, of the identity, with $\alpha > 0$. Friedrichs, of course, did not consider the question then arising as to whether the extension S depends on the parameter α.

Lemma 1. The completion of $\mathcal{D}$, with respect to $\| \|_\alpha$, can be realized by a unique subspace $\mathcal{D}_\alpha$ of $\mathcal{H}$ ($\mathcal{D} \subset \mathcal{D}_\alpha \subset \mathcal{H}$) with

$$(1) \qquad (f,f)_\alpha \geq \alpha(f,f) \qquad f \in \mathcal{D}_\alpha.$$

Remark: The uniqueness of $\mathcal{D}_\alpha$ follows from (1). In fact, it clearly follows that $\mathcal{D}_\alpha$ is the set of elements $f \in \mathcal{H}$, such that f is the strong limit of a sequence f_n in $\mathcal{D}$, which is a Cauchy sequence with respect to $\| \|_\alpha$. As a consequence we have the first assertion of Lemma 2 (since $\| \|_\alpha$ and $\| \|_\beta$ are equivalent on $\mathcal{D}$).

Lemma 2. The completion $\mathcal{D}_\alpha$ (except for its norm) is independent of α. We also have

$$(2) \qquad (f,g)_\alpha = (H_\alpha f,g) \qquad f \in \mathcal{D},\ g \in \mathcal{D}_\alpha.$$

Lemma 3. The equation

$$(3) \qquad (f,g) = (f,T_\alpha g)_\alpha \qquad f \in \mathcal{D}_\alpha,\ g \in \mathcal{H},$$

defines a bounded operator T_α on $\mathcal{H}$ with range $\mathcal{R}(T_\alpha)$ ($\subset \mathcal{D}_\alpha$) dense in $\mathcal{H}$.

Lemma 4. The operator T_α^{-1} exists, is a self adjoint extension of H_α, and has the same lower bound α. Further, $\mathcal{D}(T_\alpha^{-1}) = \mathcal{D}^* \mathcal{D}_\alpha$, and is consequently independent of α.

Proof of Lemma 1. From the definition of $\| \|_\alpha$, and since H_α has a lower bound α, we have

$$(4) \qquad (f,f)_\alpha = (H_\alpha f,f) \geq \alpha(f,f), \qquad f \in \mathcal{D}.$$

It follows that any α- Cauchy (i.e. with respect to $\| \|_\alpha$) sequence f_n in $\mathcal{D}$ is also a Cauchy sequence in $\mathcal{H}$, and α- equivalent sequences in $\mathcal{D}$ are equivalent sequences in $\mathcal{H}$. Consequently, every ideal element $\tilde{f}$

in the α- completion of $\mathcal{D}$ corresponds to a unique element $f \in \mathcal{H}$, which is the strong limit of each sequence in $\mathcal{D}$ determining $\tilde{f}$. To complete the proof we must now show that this linear correspondence is one-to-one; equation (1) will then follow by taking limits, and the uniqueness is covered by the remark following Lemma 1.

We suppose that f_n is an α- Cauchy sequence in $\mathcal{D}$ and that $f_n \rightarrow 0$ (with respect to $\| \ \|$). We must prove that $f_n \xrightarrow[\alpha]{} 0$. We have

$$(H_\alpha f_m - H_\alpha f_n,\ f_m - f_n) = (H_\alpha f_m,\ f_m) - (H_\alpha f_m,\ f_n) - (f_n,\ H_\alpha f_m) + (H_\alpha f_n,\ f_n),$$

and therefore

$$0 = \lim_m \lim_n (H_\alpha f_m - H_\alpha f_n,\ f_m - f_n) = \lim_m \left[(H_\alpha f_m,\ f_m) - 0 - 0 + \lim_n (H_\alpha f_n,\ f_n)\right]$$

$$= 2 \lim_n (H_\alpha f_n,\ f_n).$$

Thus $f_n \xrightarrow[\alpha]{} 0$ and the proof is complete.

Proof of Lemma 2. Let $f \in \mathcal{D}$, $g \in \mathcal{D}_\alpha$, and let g_n be a sequence in $\mathcal{D}$ α- converging strongly to g. Then $g_n \rightarrow g$ (from (1)), and (2) follows from $(f,g)_\alpha = \lim\ (f,g_n)_\alpha = \lim\ (H_\alpha f, g_n) = (H_\alpha f, g)$.

Proof of Lemma 3. For each fixed $g \in \mathcal{H}$, (f,g) is a linear functional of $f \in \mathcal{D}_\alpha$, and is bounded with respect to $\| \ \|_\alpha$ (by virtue of (1)). Consequently (3) defines a linear operator T_α of $\mathcal{H}$ into $\mathcal{D}_\alpha$. From (1) and (3) we have $0 \leq \alpha \|T_\alpha g\|^2 \leq (T_\alpha g, T_\alpha g)_\alpha = (T_\alpha g, g) \leq \|T_\alpha g\| \ \|g\|$, and therefore $0 \leq T \leq \frac{1}{\alpha}$. Finally, it follows from (3) that $\mathcal{R}(T_\alpha)$ is dense in $\mathcal{D}_\alpha$ (with respect to $\| \ \|_\alpha$), and consequently it is also dense in $\mathcal{H}$ (with respect to the weaker norm $\| \ \|$).

Proof of Lemma 4. T_α has an inverse, since $T_\alpha g = 0$ implies (cf. (3)) that g is orthogonal to the dense subspace $\mathcal{D}_\alpha$, and therefore $g = 0$. From Lemma 3 and from $0 \leq T \leq \frac{1}{\alpha}$ in the proof of Lemma 3 it follows that T_α^{-1} is self adjoint and has a lower bound $\geqslant \alpha$. It follows from (2) and (3) that

$$(5) \qquad (f,g)_\alpha = (H_\alpha f,g) = (f,H_\alpha^* g) = (f,T_\alpha H_\alpha^* g)_\alpha \qquad f \in \mathcal{D},\ g \in \mathcal{D}^* \mathcal{D}_\alpha,$$

and consequently that $T_\alpha H_\alpha^* g = g$. Therefore (since $\mathcal{D} \subset \mathcal{D}^* \mathcal{D}_\alpha$) T_α^{-1} is an extension of H_α, and T_α^{-1} has a lower bound equal to α. It also follows that $\mathcal{D}^* \mathcal{D}_\alpha \subset T_\alpha(\mathcal{H})$. Since T_α^{-1} is a self adjoint extension of H_α we have $T_\alpha(\mathcal{H}) \subset \mathcal{D}^*$, and from Lemma 3 $T_\alpha(\mathcal{H}) \subset \mathcal{D}_\alpha$. Thus $T_\alpha(\mathcal{H}) = \mathcal{D}^* \mathcal{D}_\alpha$ and the proof is complete.

Theorem 2. Let L be a symmetric operator in $\mathcal{H}$ with a finite lower bound c. Let $\alpha > 0$, $H_\alpha = L - c + \alpha$, and $\mathcal{K}_\alpha^* = \mathcal{K}(H_\alpha^*)$. Further, let $\mathcal{R}_\alpha$ be the class of all self adjoint extensions R of L with lower bound $> c - \alpha$, and let $\mathcal{U}_\alpha$ be the class of all bounded operators U on $\mathcal{K}_\alpha^*$ with the property that

$$(6) \qquad (\phi, U\phi) \geq \sup_{u \in \mathcal{D}(L)} \frac{|(u, \phi)|^2}{(H_\alpha u, u)}, \qquad \phi \in \mathcal{K}_\alpha^*.$$

Then there is a one-to-one correspondence between $\mathcal{R}_\alpha$ and $\mathcal{U}_\alpha$ given by

$$(7) \qquad (R - c + \alpha)^{-1} \phi = H_\alpha^{*(-1)} \phi + U \phi, \qquad \phi \in \mathcal{K}_\alpha^*,$$

where $H_\alpha^{*(-1)}$ is the minimal inverse of H_α^*. The extension S of Theorem 1 is characterized among all the extensions in $\mathcal{R}_\alpha$ by the property that the corresponding operator V is the smallest in the class $\mathcal{U}_\alpha$. In fact, for the operator V we have equality in (6) for all $\phi \in \mathcal{K}_\alpha^*$.

Proof. We suppose first that L (and therefore H_α) is closed. Requiring that R has a lower bound $> c - \alpha$ is the same as requiring that $R - c + \alpha$ has a positive lower bound, and again the same as requiring $R - c + \alpha$ to be positive and have a bounded inverse.

We note that $R - c + \alpha$ is a self adjoint extension of H_α and that H_α has a bounded inverse defined on the closed subspace $\mathcal{R}(H_\alpha)$. Thus an arbitrary self adjoint extension, with bounded inverse, of H_α will be completely determined by describing its inverse on $\mathfrak{X}_\alpha^* = \mathcal{H} \ominus \mathcal{R}(H_\alpha)$. If T denotes this inverse, i.e. if $H_\alpha \subset T^{-1}$ and T is bounded, then on $\mathcal{R}(H_\alpha)$ T must be the same as H_α^{-1}, and on $\mathfrak{X}_\alpha^*$, T is an arbitrary bounded operator such that $H_\alpha^* T \phi = \phi$ for all $\phi \in \mathfrak{X}_\alpha^*$.

Since H_α^{-1} is bounded, it follows that $H_\alpha^{(-1)}$ and $H_\alpha^{*(-1)} = H_\alpha^{(-1)*}$ (cf. Lemma I.3.2) are bounded operators defined on the whole space. Then the equation

(8) $$T\phi = H_\alpha^{*(-1)}\phi + U\phi, \qquad \phi \in \mathfrak{X}_\alpha^*,$$

defines a transformation U of $\mathfrak{X}_\alpha^*$ into $\mathcal{H}$. By applying H_α^* to both sides of this equation we obtain $H_\alpha^* U \phi = 0$, and consequently U is an operator on $\mathfrak{X}_\alpha^*$. Equation (8) thus gives the orthogonal decomposition of $T\phi$ into its components in $\mathcal{R}(H_\alpha)$ and in $\mathfrak{X}_\alpha^*$.

Since $T_{\mathfrak{X}_\alpha^*}$ is (so far) an arbitrary bounded transformation of $\mathfrak{X}_\alpha^*$ into $\mathcal{D}(H_\alpha^*)$ with $H_\alpha^* T \phi = \phi$, and since $H_\alpha^{*(-1)}$ is bounded, it follows from (8) that U is an arbitrary bounded operator on $\mathfrak{X}_\alpha^*$. Thus we have a one-to-one correspondence, given by (8), between the class of all self adjoint extensions T^{-1} of H_α with bounded inverse T and the class of all bounded operators U on $\mathfrak{X}_\alpha^*$. To characterize the corresponding subclasses $\mathcal{R}_\alpha$ and $\mathcal{U}_\alpha$ of the theorem we must now characterize the additional requirement that T^{-1} be positive.

An arbitrary element in the domain of T^{-1} is given by $f = \beta u + \gamma(H_\alpha^{*(-1)}\phi + U\phi)$, with $u \in \mathcal{D} = \mathcal{D}(L)$, $\phi \in \mathfrak{X}_\alpha^*$, and β and γ arbitrary complex numbers. Its transform is $T^{-1}f = \beta H_\alpha u + \gamma\phi$. Expanding the scalar product we then obtain

$$(T^{-1}f,f) = (\beta H_\alpha u + \gamma\phi,\ \beta u + \gamma H_\alpha^{*(-1)}\phi + \gamma U\phi)$$

$$= \beta\overline{\beta}\ (H_\alpha u,u) + \beta\overline{\gamma}(H_\alpha u,\ H_\alpha^{*(-1)}\phi) + \overline{\beta}\ \gamma(\phi,u) + \gamma\overline{\gamma}(\phi,U\phi)$$

$$=\beta\overline{\beta}\ (H_\alpha u,u)+\beta\overline{\gamma}(u,\phi)+\overline{\beta}\gamma(\phi,u)+\gamma\overline{\gamma}(\phi,U\phi).$$

A necessary and sufficient condition that T^{-1} be positive, i.e. that the above quadratic form in β and γ be non-negative for each u and ϕ, is that

$$|(u,\phi)|^2 \leq (H_\alpha u,u)(\phi,U\phi) \qquad u\varepsilon\mathcal{D},\ \phi\varepsilon\mathfrak{X}_\alpha^*.$$

Dividing by $(H_\alpha u,u)$ (which is positive) this condition becomes

$$(\phi,U\phi) \geq \sup_{u\,\varepsilon\,\mathcal{D}} \frac{|(u,\phi)|^2}{(H_\alpha u,u)}, \qquad \phi\varepsilon\mathfrak{X}_\alpha^*,$$

which is condition (6) of the theorem. If R is an operator in the class $\mathcal{R}_\alpha$, then $T^{-1} = R - c + \alpha$ is an operator of the type we have just analyzed, (8) is equivalent to (7), and the correspondence between $\mathcal{R}_\alpha$ and $\mathcal{U}_\alpha$ is established.

We must now prove that we have equality in (6) for the operator V corresponding to the extension S of Theorem 1. In our previous notation we have $T^{-1} = S - c + \alpha$, but in the notation of Lemmas 3 and 4 we also have $T_\alpha^{-1} = S - c + \alpha$. Thus the T of equation (8) is the same as the T_α of Lemma 3. Letting $u\varepsilon\mathcal{D}$ and $(f,g)_\alpha = (H_\alpha f,g)$ as in Theorem 1 we obtain

$$(u,\phi) = (u,\ H_\alpha^* T\phi) = (H_\alpha u,T\phi) = (u,T\phi)_\alpha$$

and

$$(H_\alpha u,u) = (u,u)_\alpha .$$

Since $\mathcal{D}$ is dense in its completion $\mathcal{D}_\alpha$ (of Lemma 1) we then have

$$(9) \qquad \sup_{u\varepsilon\mathcal{D}} \frac{|(u,\phi)|^2}{(H_\alpha u,u)} = \sup_{u\varepsilon\mathcal{D}} \frac{|(u,T\phi)_\alpha|^2}{(u,u)_\alpha} = \sup_{u\varepsilon\mathcal{D}_\alpha} \frac{|(u,T\phi)_\alpha|^2}{(u,u)_\alpha} = (T\phi,T\phi)_\alpha .$$

But from Lemma 3 and equation (8) we have

$$(T\phi,T\phi)_\alpha = (T\phi,\phi) = (H_\alpha^{*(-1)}\phi+U\phi,\phi) = (U\phi,\phi) = (\phi,U\phi),$$

and thus we have established equality in (6).

To complete the proof we must now remove the restriction that L is closed. If it is not, then our results can be applied to its closure $\tilde{L}$. We now must prove only that the right side of (6) is unchanged when we replace $\mathcal{D}$ by $\tilde{\mathcal{D}}$. From Theorem 1 we have $\mathcal{D} \subset \tilde{\mathcal{D}} \subset \mathcal{D}(S) \subset \mathcal{D}_\alpha$, and from (9) it then follows that the supremum over $\mathcal{D}$ is the same as the supremum over $\tilde{\mathcal{D}}$. This completes the proof of Theorem 2.

Theorem 3. Let L be a symmetric operator in $\mathcal{H}$ with a finite lower bound c. Let α be any positive number, and let $H_\alpha = L - c + \alpha$. Then the class $\mathcal{R}$ of all self adjoint extensions R with the same lower bound c is in a one-to-one correspondence with the class $\mathcal{U}$ of all bounded operators U on $\mathcal{X}_\alpha^* = \mathcal{X}(H_\alpha^*)$ satisfying the condition

$$(10) \qquad \sup_{u \in \mathcal{D}(L)} \frac{|(u,\phi)|^2}{(H_\alpha u,u)} \leq (U\phi,\phi) \leq \frac{1}{\alpha}(\phi,\phi) - \sup_{u \in \mathcal{D}(L)} \frac{|(u,\phi)|^2}{\frac{1}{\alpha}(H_\alpha u,H_\alpha u)-(H_\alpha u,u)}$$

for all $\phi \in \mathcal{X}_\alpha^*$. The correspondence is given by

$$(11) \qquad (R - c + \alpha)^{-1}\phi = H_\alpha^{*\ominus}\phi + U\phi, \qquad \phi \in \mathcal{X}_\alpha^*.$$

Proof. We first suppose that L (and therefore H_α) is closed. Let R be a self adjoint extension of L with the same lower bound c, and let $T = (R - c + \alpha)^{-1}$. We must now characterize the condition that $T^{-1} \geq \alpha$, or equivalently that $0 \leq T \leq \frac{1}{\alpha}$. We know from Theorem 2 that $R \in \mathcal{R}_\alpha$ characterizes the condition $0 \leq T \leq M$ for some M. We must now characterize the remaining inequality $T \leq \frac{1}{\alpha}$.

We have $R \in \mathcal{R}_\alpha$, as defined in Theorem 2, and we let U be the corresponding operator in $\mathcal{U}_\alpha$ (also defined in Theorem 2). As in the proof of Theorem 2 an arbitrary element in $\mathcal{D}(T) = \mathcal{H}$ is $g = \beta v + \gamma\phi$, where $v = H_\alpha u \in \mathcal{R}(H_\alpha)$ and $\phi \in \mathcal{X}_\alpha^*$. Also $Tg = \beta u + \gamma H_\alpha^{*\ominus}\phi + \gamma U\phi$. The condition $(Tg,g) \leq \frac{1}{\alpha}(g,g)$ then becomes $\beta\bar{\beta}(u,v) + \beta\bar{\gamma}(u,\phi) + \bar{\beta}\gamma(H_\alpha^{*\ominus}\phi,v) + \gamma\bar{\gamma}(U\phi,\phi) \leq$

$\leq \frac{\beta\overline{\beta}}{\alpha}$ $(v,v) + \frac{\gamma\overline{\gamma}}{\alpha}$ (ϕ,ϕ). Regrouping terms we obtain

$$\beta\overline{\beta}\left[\frac{1}{\alpha}(v,v) - (u,v)\right] - \beta\overline{\gamma}(u,\phi) - \overline{\beta}\gamma(\phi,u) + \gamma\overline{\gamma}\left[\frac{1}{\alpha}(\phi,\phi) - (U\phi,\phi)\right] \geqslant 0,$$

which must be true for all β and γ, and therefore is equivalent to

$$|(u,\phi)|^2 \leq \left[\frac{1}{\alpha}(v,v) - (u,v)\right]\left[\frac{1}{\alpha}(\phi,\phi) - (U\phi,\phi)\right] .$$

Dividing by $\frac{1}{\alpha}(v,v)-(u,v)$ (which is non-negative), substituting $v = H_\alpha u$, and making use of the fact that this equation must hold for all $u \in \mathcal{D}(L)$ and $\phi \in \mathcal{K}_\alpha^*$, we obtain

$$(12) \qquad \sup_{u \in \mathcal{D}(L)} \frac{|(u,\phi)|^2}{\frac{1}{\alpha}(H_\alpha u, H_\alpha u) - (u, H_\alpha u)} \leq \frac{1}{\alpha}(\phi,\phi) - (U\phi,\phi), \qquad \phi \in \mathcal{K}_\alpha^*.$$

This condition gives us an upper bound for the quadratic form $(U\phi,\phi)$, and together with the lower bound given by (6) (in Theorem 2) is the only restriction on the bounded operator U (on $\mathcal{K}_\alpha^*$). These two conditions (6) and (12) then give us the desired inequalities (10) of the theorem. Equation (11) is repeated from Theorem 2 (equation (7)).

To complete the proof we must now remove the restriction that L is closed. Since the first half of (10) is repeated from Theorem 2, we need only show that the second half of (10) is unchanged when we replace $\mathcal{D}$ by $\widetilde{\mathcal{D}}$ (where $\widetilde{L}$ is the closure of L).

For any positive quadratic form Q(f) on $\mathcal{H}$ we can form the abstract completion $\hat{\mathcal{H}}$ of $\mathcal{H}$ with respect to the pseudo-norm $Q^{\frac{1}{2}}(f)$. (Note that $\hat{\mathcal{H}}$ is not a Hilbert space, but the quotient space of equivalence classes will be a Hilbert space.) For a subspace $\mathcal{L} \subset \mathcal{H}$ we shall denote by $\hat{\mathcal{L}}$ the closure of $\mathcal{L}$ in $\hat{\mathcal{H}}$ (with respect to $Q^{\frac{1}{2}}$).

If we now let $Q(f) = \frac{1}{\alpha}(f,f) - (T_\alpha f,f) \geqslant 0$, where T_α is given by Lemma 3, we have

$$Q(H_\alpha u,\phi) = -(T_\alpha H_\alpha u,\phi) = -(u,\phi)$$

and $$Q(H_\alpha u) = \frac{1}{\alpha}(H_\alpha u, H_\alpha u) - (u, H_\alpha u),$$

where $Q(f,g)$ is the bilinear form corresponding to $Q(f)$. Thus the left side of (12) becomes

$$\sup_{v\in\mathcal{R}(H_\alpha)} \frac{|Q(v,\phi)|^2}{Q(v)} = \sup_{v\in\widehat{\mathcal{R}(H_\alpha)}} \frac{|\hat{Q}(v,\phi)|^2}{\hat{Q}(v)} = \hat{Q}(\hat{P}\phi) \tag{13}$$

where $\hat{Q}$ is the extension of Q to the completion $\hat{\mathcal{H}}$, and $\hat{P}$ is the projection of $\hat{\mathcal{H}}$ onto $\widehat{\mathcal{R}(H_\alpha)}$.

Since $Q(f) \leq \frac{1}{\alpha}(f,f)$, it follows that $\mathcal{R}(H_\alpha) \subset \overline{\mathcal{R}(H_\alpha)} \subset \widehat{\mathcal{R}(H_\alpha)}$. Consequently $\mathcal{R}(H_\alpha) \subset \mathcal{R}(\tilde{H}_\alpha) \subset \widehat{\mathcal{R}(H_\alpha)}$, and from (13) it follows that the left side of (12) is unchanged when we replace $\mathcal{D}$ by $\tilde{\mathcal{D}}$. Thus the right side of (10) is similarly unchanged, and the proof of Theorem 3 is complete.

<u>Theorem 4.</u> <u>Let L be a symmetric operator in $\mathcal{H}$ with a finite lower bound c.</u> <u>Let $\alpha > 0$, let S be the self adjoint extension of L given by Theorem 1, and let $T_\alpha = (S - c + \alpha)^{-1}$.</u> <u>Then a necessary and sufficient condition in order that a self adjoint extension of L with the same lower bound c be unique is that $\mathcal{R}(L - c + \alpha)$ be dense in $\mathcal{H}$ with respect to the quadratic pseudo-norm $[Q(f)]^{\frac{1}{2}} = \left[\frac{1}{\alpha}(f,f) - (T_\alpha f, f)\right]^{\frac{1}{2}}$.</u>

<u>Proof.</u> Theorem 3 tells us that any bounded operator U (on $\mathcal{K}_\alpha^*$) satisfying (10) will lead to a self adjoint extension R of L with the same lower bound c. Consequently a necessary and sufficient condition that such an extension R be unique is that there exist only one operator U satisfying (10). Since both the upper bound and the lower bound of $(U\phi,\phi)$ given by (10) are themselves bounded quadratic forms on $\mathcal{K}_\alpha^*$ (as shown in the proofs of Theorems 2 and 3), it follows that we may have equality at either end for an appropriate bounded operator U. Thus our necessary and sufficient condition for uniqueness is that the

two bounds given by (10) should coincide. If we let $T = (R - c + \alpha)^{-1}$, then we will have equality in (6) whenever $T = T_\alpha$. Then, using $(U\phi,\phi) = (T\phi,\phi)$, our condition for uniqueness becomes

$$\sup_{u \in \mathcal{D}} \frac{|(u,\phi)|^2}{\frac{1}{\alpha}(H_\alpha u, H_\alpha u) - (u, H_\alpha u)} = \frac{1}{\alpha}(\phi,\phi) - (T_\alpha \phi,\phi), \qquad \phi \in \mathfrak{X}_\alpha^*. \tag{14}$$

If we now let $Q(f) = \frac{1}{\alpha}(f,f) - (T_\alpha f,f)$ as in the proof of Theorem 3, this condition becomes (in view of (13))

$$\hat{Q}(\hat{P}\phi) = Q(\phi), \qquad \phi \in \mathfrak{X}_\alpha^*, \tag{15}$$

where $\hat{P}$ is again the projection of $\hat{\mathcal{H}}$ onto $\widehat{R(H_\alpha)}$. Equation (15) is obviously equivalent to $\mathfrak{X}_\alpha^* \subset \widehat{\mathcal{R}(H_\alpha)}$, or simply $\mathcal{H} = \mathfrak{X}_\alpha^* \dotplus \widehat{\mathcal{R}(H_\alpha)} \subset \widehat{\mathcal{R}(H_\alpha)}$. This completes the proof of the theorem.

<u>Example 1</u>. We shall first give an example of a semi-bounded operator L having more than one self adjoint extension with the same (semi-) bound. We shall construct our operator in a separable Hilbert space with basis $\{g_n\}$. Any $f \in \mathcal{H}$ can be written as $f = \sum_1^\infty \alpha_n g_n$, where $\sum |\alpha_n|^2 < \infty$. We first define an operator S as follows. We let $Sf = \sum_1^\infty (n-1)\alpha_n g_n$, where $f \in \mathcal{D}(S)$ when $\sum n^2 |\alpha_n|^2 < \infty$. We note immediately that S is a self adjoint operator with lower bound 0 (since $(Sf,f) \geq 0$ and $Sg_1 = 0$).

We now let $f_1 = \sum_2^\infty \frac{1}{n-1} g_n$ and observe that $f_1 \notin \mathcal{D}(S)$. We let L^* be the linear extension of S to the domain $\mathcal{D}(S) \dotplus [f_1]$, where $L^* f_1 = 0$. This extension L^* is closed since its graph is the vectorial sum of a closed subspace and a finite dimensional subspace. Thus our notation L^* is justified, and L will be the adjoint of L^*. Letting $f, g \in \mathcal{D}(S)$ we have

$$(L^*(f+\beta f_1),g) = (Sf,g) = (f,Sg).$$

Since $L \subset S$ we then see that $g \in \mathcal{D} = \mathcal{D}(L)$ if and only if $g \in \mathcal{D}(S)$ and $Sg \perp f_1$, i.e. $f \in \mathcal{D}$ when $\sum n^2|\alpha_n|^2 < \infty$ and $\sum_2^\infty \alpha_n = 0$. From $L \subset S$ and $Lg_1 = 0$, it now follows that L has exactly the lower bound 0.

Thus far we have constructed a semi-bounded operator L and a self adjoint extension S with the same lower bound 0. (The extension S will in fact be the one given by Theorem 1.) We must now construct another self adjoint extension S' with lower bound 0.

We define $\mathcal{D}(S') = \mathcal{D} \dotplus [f_1]$, with (of course) $S' \subset L^*$. For $f \in \mathcal{D}$ we have

$$(S'(f+\beta f_1),\ f+\beta f_1) = (Lf,f) + \overline{\beta}(Lf,f_1) = (Lf,f) \geqslant 0,$$

and thus $S' \geqslant 0$. Further, $S'^* \subset L^*$, and for $f \in \mathcal{D}$, $g \in \mathcal{D}(S)$ we have

$$(S'(f+\beta f_1),\ g+\gamma f_1) = (Lf,g) + \overline{\gamma}(Lf,f_1) = (Lf,g) = (f,Sg).$$

Consequently $g + \gamma f_1 \in \mathcal{D}(S'^*)$ if and only if $Sg \perp f_1$, or $\mathcal{D}(S'^*) = \mathcal{D} \dotplus [f_1] = \mathcal{D}(S')$. Thus S' is self adjoint and the example is complete.

Remarks concerning Example 1. The essential property of the operator in Example 1 was that the range $\mathcal{R}$ was closed. Indeed for any symmetric operator L with lower bound c we may proceed in a similar fashion whenever $\mathcal{R}(L-c)$ is closed. To simplify our notation we shall suppose that c = 0 and that L is closed, and proceed as follows.

We suppose that L is a positive operator with lower bound zero, i.e. $0 = \inf_{f \in \mathcal{D}} \frac{(Lf,f)}{(f,f)}$, and we suppose further that $\mathcal{R} = \mathcal{R}(L)$ is closed. If L has an inverse L^{-1}, we would have $\|L^{-1}\| \leqslant M < \infty$ (since $\mathcal{R}$ is closed) and consequently $(Lf,f) \geqslant \frac{1}{M}(f,f)$, which is contrary to our hypothesis. Thus $\mathcal{X} \neq (0)$. Further, $\mathcal{R}$ being closed is equivalent to $\mathcal{R}^*$ being closed, and we have $\mathcal{H} = \mathcal{X} \dotplus \mathcal{R}^*$. We may now state the situation as follows. Except for its nullspace the operator L actually has a positive lower bound, i.e. the restriction $L_{\mathcal{R}^*}$ has a positive lower bound.

Finding a self adjoint extension S of L is equivalent to finding a self adjoint extension $R = S_{\mathcal{R}^*}$ of $L_{\mathcal{R}^*}$ (in the space $\mathcal{R}^*$). Moreover, the extension S will be positive whenever R is positive. In this situation we may always find one extension R with the same positive lower bound as for $L_{\mathcal{R}^*}$ (cf. Theorem 1), and we may find another positive extension R' with the nullspace $\mathcal{X}^* \ominus \mathcal{X}$ (i.e. the extension S' of L has the nullspace $\mathcal{X}^*$). We may directly construct S' as follows.

We let $\mathcal{D}(S') = \mathcal{D} \dotplus \mathcal{X}^*$, $f \in \mathcal{D}$, $f_0 \in \mathcal{X}^*$, and obtain $(S'(f+f_0), f+f_0)$ $= (Lf, f + f_0) = (Lf,f) \geqslant 0$. Further, for $g \in \mathcal{D}^*$ we have $(S'(f+f_0), g)$ $= (Lf,g) = (f, L^*g)$, so that $g \in \mathcal{D}(S'^*)$ if and only if $g \in \mathcal{D}^*$ and $L^*g \in \mathcal{H} \ominus \mathcal{X}^* = \mathcal{R}$. Thus $\mathcal{D}(S'^*) = \mathcal{D} \dotplus \mathcal{X}^*$ and S' is a positive self adjoint extension of L.

Before going further we need to connect the extensions of L and of $L_{\mathcal{R}^*}$. If we let $L_0 = L_{\mathcal{R}^*}$, then L_0 is a symmetric operator with a positive lower bound in the space $\mathcal{R}^*$. We shall now prove that $\mathcal{D}(S) = \mathcal{D}(S_0) \dotplus \mathcal{X}$, where S is the extension of L given by Theorem 1 and S_0 is the similar extension of L_0 (in the space $\mathcal{R}^*$). We shall also prove that $S(f' + f'') = S_0 f'$ when $f' \in \mathcal{D}(S_0)$ and $f'' \in \mathcal{X}$.

For any $f \in \mathcal{D}$ we may write $f = f' + f''$, where $f' \in \mathcal{D}\mathcal{R}^*$ and $f'' \in \mathcal{X}$. Similarly for $g, h \in \mathcal{H}$ we can write $g = g' + g''$ and $h = h' + h''$, where g', $h' \in \mathcal{R}^*$ and g'', $h'' \in \mathcal{X}$. We then have $(Lf,g) = (L_0 f', g')$ and (f,h) $= (f',h') + (f'',h'')$. Consequently we see that $\mathcal{D}^* = \mathcal{D}_0^* \dotplus \mathcal{X}$, and that $L^*g = L_0^* g'$ for $g' \in \mathcal{D}_0^*$. If we now prove that $\mathcal{D}_\alpha = \mathcal{D}_{0\alpha} \dotplus \mathcal{X}$, then it will follow that $\mathcal{D}(S) = \mathcal{D}(S_0) + \mathcal{X}$ as stated above.

The inclusion $\mathcal{D}_\alpha \subset \mathcal{D}_{0_\alpha} \dotplus \mathcal{X}$ is an immediate consequence of the definition of $\mathcal{D}_\alpha$ (and of $\mathcal{D}_{0_\alpha}$) given in Lemma 1. Now let $f \in \mathcal{D}_\alpha$. Then there exists a sequence f_n in $\mathcal{D}$ which is α-Cauchy and $f_n \to f$, where $(h,h)_\alpha = (Lh,h) + \alpha(h,h)$ as in Theorem 1. We can write $f_n = f_n' + f_n''$ and $f = f' + f''$, where f_n', $f' \in \mathcal{R}^*$ and f_n'', $f'' \in \mathcal{X}$. We then find that f_n' is an α-Cauchy sequence in $\mathcal{D}\mathcal{R}^* = \mathcal{D}_0$ and that $f_n' \to f'$. Consequently $f' \in \mathcal{D}_{0_\alpha}$ and we have proved that $\mathcal{D}_\alpha = \mathcal{D}_{0_\alpha} \dotplus \mathcal{X}$. Thus we have $\mathcal{D}(S) = \mathcal{D}^*\mathcal{D}_\alpha = \mathcal{D}_0^* \mathcal{D}_{0_\alpha} \dotplus \mathcal{X} = \mathcal{D}(S_0) \dotplus \mathcal{X}$.

From what we have just proved we see that extending L by Theorem 1 amounts to the same thing as extending $L_{\mathcal{R}^*}$ (in the space $\mathcal{R}^*$) by Theorem 1. As a corollary we see that the nullspace of L is not increased, i.e. $\mathcal{X}(S) = \mathcal{X}$. As a last result we shall now prove that there is always a positive self adjoint extension S'' whose nullspace is any preassigned closed subspace between $\mathcal{X}$ and $\mathcal{X}^*$.

We let $\mathcal{L}$ be any closed subspace with $\mathcal{X} \subset \mathcal{L} \subset \mathcal{X}^*$. We then define an extension $L''(L \subset L'' \subset L^*)$ by putting $\mathcal{D}(L'') = \mathcal{D} \dotplus \mathcal{L}$. For $f \in \mathcal{D}$ and $f_0 \in \mathcal{L}$ we have $(L''(f + f_0), f + f_0) = (Lf,f) \geq 0$, and thus L'' is positive. Observing that $\mathcal{R}(L'') = \mathcal{R}$ is a closed subspace and that L'' is closed, since $\mathcal{X}'' = \mathcal{L}$ is also a closed subspace, we then know that extending L'' by Theorem 1 will give us a positive self adjoint extension S'' with the same nullspace $\mathcal{L}$. This completes the remarks.

<u>Example 2</u>. If we have two semi-bounded operators L_1 and L_2 with lower bound c_1 and c_2, and with $\mathcal{D}_1\mathcal{D}_2$ dense, then their sum $L = L_1 + L_2$ will be symmetric with a lower bound $c \geq c_1 + c_2$. We may extend each of these by means of Theorem 1 to the operators S, S_1, and S_2. The operator $S_1 + S_2$ may not be self adjoint, but we may ask if it is related to the self adjoint operator S, i.e. if $S \supset S_1 + S_2$. Our next example

will show that this may not be the case, even when we have $\mathcal{D} = \mathcal{D}_1 = \mathcal{D}_2$, $c = c_1 + c_2$, and $S_1 + S_2$ self adjoint.

We let $f = \sum \alpha_n g_n$ and define

$$S_1 f = \sum_{\text{odd}} n\,\alpha_n g_n + \sum_{\text{even}} \alpha_n g_n, \quad \text{when} \quad \sum_{\text{odd}} n^2 |\alpha_n|^2 + \sum_{\text{even}} |\alpha_n|^2 < \infty,$$

and

$$S_2 f = \sum_{\text{odd}} \alpha_n g_n + \sum_{\text{even}} n\,\alpha_n g_n, \text{ when } \sum_{\text{odd}} |\alpha_n|^2 + \sum_{\text{even}} n^2 |\alpha_n|^2 < \infty.$$

The operators S_1 and S_2 are obviously self adjoint and their sum $T = S_1 + S_2$ is also self adjoint. If we now define $\mathcal{D} = \mathcal{D}_1 = \mathcal{D}_2$ by the two restrictions $\sum_1^\infty n^2 |\alpha_n|^2 < \infty$ and $\sum_2^\infty \frac{\alpha_n}{\sqrt{n+1}} = 0$, we are led to corresponding restrictions L, L_1, and L_2. If we consider the extension of T to the domain $\mathcal{D}(T) \dotplus [f_1]$, where $f_1 = \sum_2^\infty \frac{1}{(n+1)^{3/2}} g_n$ and f_1 is thrown into 0, we see that the adjoint of this extension will be L. Thus $\mathcal{D}$ is dense and the restrictions L, L_1, and L_2 are symmetric. We also observe that the lower bounds are respectively $c = 2$, $c_1 = 1$, and $c_2 = 1$.

We shall next prove that the closures of L_1 and L_2 are respectively S_1 and S_2. This will of course show that S_1 and S_2 are the extensions given by Theorem 1 and that their sum (= T) is self adjoint.

We let $f = \sum \alpha_n g_n \in \mathcal{D}_1$ and $g = \sum \beta_n g_n \in \mathcal{D}_1^*$. Then $(L_1 f, g) =$

$$\sum_{\text{odd}} n\,\alpha_n \overline{\beta}_n + \sum_{\text{even}} \alpha_n \overline{\beta}_n \text{ must be a bounded functional of } f \in \mathcal{D}.$$

Since the last term is automatically a bounded functional, the first term must be also, and we have $\sum_{\text{odd}} n^2 |\beta_n|^2 < \infty$. Thus $\mathcal{D}_1^* \subset \mathcal{D}(S_1)$, and consequently $\mathcal{D}_1^* = \mathcal{D}(S_1)$ and the closure of L_1 is equal to S_1. A similar proof will show that the closure of L_2 is S_2.

To complete the example we must now prove that $\mathcal{D}(S) \neq \mathcal{D}(T)$, where S is the extension of L given by Theorem 1. To this effect we shall find the completion $\mathcal{D}_\alpha$ (defined in Lemma 1). We let $\alpha = 2$ so that $(f,f)_\alpha = (Lf,f)$ for $f \in \mathcal{D}$. If $f = \sum \alpha_n g_n$ is an arbitrary element in $\mathcal{D}$ then $\{\beta_n\} = \{\sqrt{n+1}\,\alpha_n\}$ is an arbitrary element of ℓ_2 satisfying $\sum_1^\infty (n+1)\,|\beta_n|^2 < \infty$ and $\sum_2^\infty \frac{\beta_n}{n+1} = 0$. Completing $\mathcal{D}$ with respect to the α- norm is now equivalent to closing this subspace of ℓ_2, since $(f,f)_\alpha = \sum_1^\infty (n+1)\,|\alpha_n|^2 = \sum_1^\infty |\beta_n|^2$. The closure in ℓ_2 of this subspace is obviously the orthogonal complement of $\left[\left\{\frac{1}{n+1}\right\}_2^\infty\right]$. Consequently $\mathcal{D}_\alpha$ is characterized by the conditions $\sum_1^\infty n\,|\alpha_n|^2 < \infty$ and $\sum_2^\infty \frac{\alpha_n}{\sqrt{n+1}} = 0$. From this it is clear that elements in $\mathcal{D}(S) = \mathcal{D}^*\mathcal{D}_\alpha$ satisfy, among other conditions, $\sum_2^\infty \frac{\alpha_n}{\sqrt{n+1}} = 0$, and therefore $\mathcal{D}(S) \neq \mathcal{D}(T)$. This completes Example 2.

Example 3. For a semi-bounded operator L and for each $\alpha > 0$, Theorem 3 gives a characterization of the class of self adjoint extensions with the same (semi-) bound. The characterization is given by equation (10), and is in terms of a bounded operator U on $\mathcal{X}_\alpha^*$. We have already noted (in Theorem 2) that the minimal operator U in the class $\mathcal{U}$ leads to the Friedrichs' extension S of Theorem 1, and is thus independent of α. We may also consider the maximal operator in the class $\mathcal{U}$ and ask about the properties of the corresponding extension of L. If U_α is this maximal operator, then we are asking about the properties of the corresponding extension R_α given by (11). Contrary to the situation for the minimal operator, the extension R_α is not independent of α, and thus it does not lead us to any well-determined extension of L. This will be shown by the following example.

We let L be the operator defined in Example 1, and let S be its Friedrichs' extension, also described there. We must first find the nullspace $\mathcal{X}^*_\alpha = \mathcal{X}(L^*+\alpha)$. We let $f_1 = \sum_2^\infty \frac{1}{n-1} g_n$ as in Example 1, $f = \sum_1^\infty \alpha_n g_n$, and $g = \sum_1^\infty \beta_n g_n$. Then $f = g + \beta f_1$ is an arbitrary element in $\mathcal{D}^*$ when $\sum n^2 |\beta_n|^2 < \infty$, and we have

$$(16)\quad \begin{aligned}(L^*+\alpha)f &= (L^*+\alpha)g + \alpha\beta f_1 = Sg + \alpha g + \alpha\beta f_1 \\ &= \sum_1^\infty (n-1+\alpha)\beta_n g_n + \alpha\beta \sum_2^\infty \frac{1}{n-1} g_n .\end{aligned}$$

Thus $(L^*+\alpha)f = 0$ when $\beta_1 = 0$ and $\beta_n = \frac{-\alpha\beta}{(n-1)(n-1+\alpha)}$ for $n \geqslant 2$. Translating this condition to the α_n we obtain $\alpha_1 = 0$ and

$$\alpha_n = \frac{-\alpha\beta}{(n-1)(n-1+\alpha)} + \frac{\beta}{n-1} = \frac{\beta}{n-1+\alpha}, \qquad n \geqslant 2.$$

Consequently $\mathcal{X}^*_\alpha = [f_2]$, where $f_2 = \sum_2^\infty \frac{1}{n-1+\alpha} g_n$.

We next let $f = (L^*+\alpha)^{-1} f_2$, i.e. $f_2 = (L^*+\alpha)f$ and $(f,f_2) = 0$. Using (16) we obtain $\alpha_1 = \beta_1 = 0$, and for $n \geqslant 2$ we also obtain

$$\beta_n = \frac{1}{(n-1+\alpha)^2} - \frac{\alpha\beta}{(n-1)(n-1+\alpha)} .$$

Substituting $\alpha_n = \beta_n + \frac{\beta}{n-1}$ and the above expression into $(f,f_2) = 0$, we then obtain

$$0 = \sum_2^\infty \frac{\alpha_n}{n-1+\alpha} = \sum_2^\infty \frac{1}{(n-1+\alpha)^3} + \beta \sum_2^\infty \frac{1}{(n-1+\alpha)^2} .$$

Consequently $\beta = -\frac{C_\alpha^{(3)}}{C_\alpha^{(2)}}$, where $C_\alpha^{(p)} = \sum_1^\infty \frac{1}{(n+\alpha)^p}$. Further substitution now yields

$$\alpha_n = \beta_n + \frac{\beta}{n-1} = \frac{1}{(n-1+\alpha)^2} - \frac{C_\alpha(3)}{C_\alpha(2)} \frac{1}{n-1+\alpha} .$$

Thus we have found

$$(17) \qquad (L^* + \alpha)^{-1} f_2 = \sum_2^\infty \frac{1}{(n-1+\alpha)^2} g_n - \frac{C_\alpha(3)}{C_\alpha(2)} \sum_2^\infty \frac{1}{n-1+\alpha} g_n$$

Our next step is to construct the quadratic form $Q(f)$ given in Theorem 4. In order to do this we must calculate $T_\alpha f$, where $T_\alpha = (S+\alpha)^{-1}$. A simple calculation gives us $T_\alpha f = \sum_1^\infty \frac{\alpha_n}{n-1+\alpha} g_n$, and consequently

$$Q(f) = \frac{1}{\alpha}(f,f) - (f,T_\alpha f) = \frac{1}{\alpha} \sum_2^\infty \frac{n-1}{n-1+\alpha} |\alpha_n|^2 .$$

Thus we see that the pseudo-norm $Q^{\frac{1}{2}}$ is equivalent to the original norm on the subspace $\mathcal{H} \ominus [g_1]$, and that the Q-closure $\hat{\mathcal{L}} = \overline{\mathcal{L}} \dotplus [g_1]$ for any $\mathcal{L} \subset \mathcal{H}$. Consequently $\widehat{\mathcal{R}(L+\alpha)} = \overline{\mathcal{R}(L+\alpha)} = \mathcal{R}(L+\alpha)$, and in view of (13) our maximal operator U_α is characterized by the condition

$$(U_\alpha f_2, f_2) = \frac{1}{\alpha}(f_2,f_2) - \hat{Q}(\hat{P} f_2) = \frac{1}{\alpha}(f_2,f_2).$$

Thus we have $U_\alpha f_2 = \frac{1}{\alpha} f_2$, and $\mathcal{D}(R_\alpha) = \mathcal{D} \dotplus [h_\alpha]$, where

$$h_\alpha = (R_\alpha + \alpha)^{-1} f_2 = (L^* + \alpha)^{-1} f_2 + U_\alpha f_2$$

$$= \sum_2^\infty \frac{1}{(n-1+\alpha)^2} g_n + \left[\frac{1}{\alpha} - \frac{C_\alpha(3)}{C_\alpha(2)}\right] \sum_2^\infty \frac{1}{n-1+\alpha} g_n$$

(making use of (17)).

We have now found the domains $\mathcal{D}(R_\alpha)$, and to complete the example we must show that they are dependent upon α. To prove this we suppose the contrary, that $h_1 \in \mathcal{D}(R_\alpha)$ for every α. Then we have $h_1 = f + \gamma h_\alpha$, where $f = \sum \alpha_n g_n \in \mathcal{D}$, and f and γ each depend on α. Equating the coefficients we get $\alpha_1 = 0$, and for $n \geq 2$ we obtain

$$\alpha_n = \frac{1}{n^2} - \frac{\gamma}{(n-1+\alpha)^2} + \frac{\alpha-1}{n(n-1+\alpha)} \left[1 - \frac{C_1^{(3)}}{C_1^{(2)}}\right]$$

$$+ \frac{1}{n-1+\alpha} \left[(1 - \frac{C_1^{(3)}}{C_1^{(2)}}) - \gamma(\frac{1}{\alpha} - \frac{C_\alpha^{(3)}}{C_\alpha^{(2)}})\right].$$

Since $\sum n^2 |\alpha_n|^2 < \infty$, the last term must be zero, and

(18) $$\gamma = \frac{1 - \dfrac{C_1^{(3)}}{C_1^{(2)}}}{\dfrac{1}{\alpha} - \dfrac{C_\alpha^{(3)}}{C_\alpha^{(2)}}}$$

The final condition for $f \in \mathscr{D}$, that of $\sum_2^\infty \alpha_n = 0$, now gives us

(19) $$C_1^{(2)} - \gamma C_\alpha^{(2)} + (\alpha-1) \left[1 - \frac{C_1^{(3)}}{C_1^{(2)}}\right] C_{1,\alpha}^{(2)} = 0,$$

where $C_{1,\alpha}^{(2)} = \sum_2^\infty \frac{1}{n(n-1+\alpha)}$

We next analyze equations (18) and (19) as $\alpha \to 0$. The denominator of (18) becomes infinite so that $\gamma \to 0$, and (19) becomes

$$C_1^{(2)} - \left[1 - \frac{C_1^{(3)}}{C_1^{(2)}}\right] C_{1,o}^{(2)} = 0.$$

Since $C_{1,o}^{(2)} = 1$, we must consequently have

(20) $$C_1^{(3)} = C_1^{(2)}(1-C_1^{(2)}) = (\frac{\pi^2}{6} - 1)(2 - \frac{\pi^2}{6}).$$

A simple evaluation gives $(\frac{\pi^2}{6} - 1)(2 - \frac{\pi^2}{6}) > 0.229$ and $C_1^{(3)} < \frac{1}{8} + \frac{1}{12} < 0.209$. This contradiction completes the example.

PART II: REPRESENTABLE OPERATORS

1. Functional spaces and reproducing kernels. From this point on we shall consider a "functional Hilbert space" F, i.e. (1) a linear class F of functions f defined on a basic set $\mathcal{E}$, where addition and scalar multiplication are the usual operations on functions, and (2) a quadratic norm $\| \ \|$ defined on F giving it the character of a (complete) Hilbert space.

If $M(x,y)$ is a function defined on the product set $\mathcal{E} \times \mathcal{E}$, it will be convenient to use the notation $M^*(x,y) = \overline{M(y,x)}$, and the symbol M_y for the function whose value at the point x is $M(x,y)$, i.e. $M_y(x) = M(x,y)$.

Finally when applying an operator to the function M we shall always consider M as a function of its first variable, with the second variable acting as a parameter. That is, if $M_y \in F$ for each y and if A is an operator in F defined for each M_y, then AM will denote the function on the product set $\mathcal{E} \times \mathcal{E}$ whose value at the point x,y is $(AM)(x,y) = AM(x,y) = AM_y(x) = (AM_y)(x)$. Similarly $AM_y = (AM)_y$, $(AM)^*(x,y) = \overline{AM(y,x)} = \overline{AM_x(y)}$, but note that $AM(x,y) \neq AM^*(y,x)$ even though we may have $M(x,y) = M^*(y,x)$.

In our future considerations the functional space F (or more properly $\{F, \| \ \|\}$ to emphasize the norm we are using in the class F) will always be supposed to possess a reproducing kernel. We say that the function $K(x,y)$ defined on the product $\mathcal{E} \times \mathcal{E}$ is the "reproducing kernel" (as defined by N. Aronszajn) of the space F if (1) $K_y \in F$ for each $y \in \mathcal{E}$ and (2) $f(y) = (f, K_y)$ for each $f \in F$ and $y \in \mathcal{E}$. Such a space $\{F, \| \ \|, K\}$, i.e. a functional space with a reproducing kernel, we shall refer to simply as an "r.k. space". We shall use the results of Aronszajn [1,2] throughout, and for additional explanation of the properties of reproducing kernels and for the proofs of these properties we refer the reader to these papers.

(1) It is shown in Aronszajn [1] that every reproducing kernel is a "positive matrix" in the sense of E.H. Moore [1,2]. (He introduced the term "positive Hermitian matrix" which was shortened by N. Aronszajn to our present form.) The function $K(x,y)$ is a positive matrix if it satisfies the property

$$(1') \qquad \sum_{i,j=1}^{n} \xi_i \bar{\xi}_j \, K(y_j, y_i) \geq 0$$

for every sequence of n numbers ξ_i and n points $y_i \in \mathcal{E}$, $n = 1,2, \ldots$ It is also shown that conversely, to each positive matrix K there corresponds a unique functional space $\{F, \| \ \|\}$ for which K is the reproducing kernel.

(2) It will be convenient for us to use the symbol $\mathcal{K}$ for the (non-closed) subspace consisting of linear combinations of the functions K_y, $y \in \mathcal{E}$. For such a combination $\sum_1^n \xi_i K_{y_i}$ in $\mathcal{K}$ the norm will be given by (1'). Some of the other results (in Aronszajn [2]) that we shall use are as follows.

(3) A functional space $\{F, \| \ \|\}$ has a reproducing kernel if and only if, for each $y \in \mathcal{E}$, $f(y)$ is a bounded functional of $f \in F$.

(4) If $\{F_1, \| \ \|_1, K_1\}$ and $\{F_2, \| \ \|_2, K_2\}$ are two r.k. spaces and $F_1 \subset F_2$, $\|f\|_1 \geq \|f\|_2$ for every $f \in F_1$, then $K_1 \ll K_2$, where $M \ll N$ means that $N - M$ is a positive matrix. Conversely if $K_1 \ll K_2$ for two positive matrices, and $\{F_1, \| \ \|_1\}$ and $\{F_2, \| \ \|_2\}$ are the corresponding functional spaces respectively, then $F_1 \subset F_2$ and $\|f\|_1 \geq \|f\|_2$ for every $f \in F_1$.

(5) If $\{F_1, \| \ \|_1, K_1\}$ and $\{F_2, \| \ \|_2, K_2\}$ are two r.k. spaces, then $K_1 + K_2$ is the reproducing kernel for the functional space $F_1 \dotplus F_2$ with

the norm given by

(5') $$\|f\|^2 = \min\,(\|f_1\|_1^2 + \|f_2\|_2^2),$$

where the minimum is taken over all decompositions $f = f_1 + f_2$ with $f_i \varepsilon F_i$. This minimum is realized by the unique decomposition for which

(5'') $$(f_1,h)_1 = (f_2,h)_2 \text{ for all } h\varepsilon F_1F_2.$$

(6) If a linear class F becomes an r.k. space under two different norms $\|\ \|_1$ and $\|\ \|_2$, then the norms are equivalent and the corresponding reproducing kernels are equivalent, i.e. there exist positive numbers m and M such that

(6') $$m\,\|f\|_1 \leq \|f\|_2 \leq M\,\|f\|_1, \qquad f\,\varepsilon\,F,$$

and

(6'') $$\frac{1}{M^2}K_1 \ll K_2 \ll \frac{1}{m^2}K_1.$$

Of course any r.k. space $\{F, \|\ \|, K\}$, under an equivalent norm $\|\ \|_1$, will again become an r.k. space, i.e. it will possess a reproducing kernel with respect to the new norm (cf. (3)).

(7) Finally, a linear class F will be called simply an "<u>r.k.</u> <u>class</u>" if it becomes an r.k. space under some norm, and any two such norms will necessarily be equivalent. From a previous remark we now see that if F and G are two r.k. classes, then $F\dot{+}G$ is an r.k. class; it is also true that the intersection FG is an r.k. class.

2. <u>A representable operator</u> --- <u>maximal and minimal operators</u>. In this section we shall consider a fixed r.k. space $\{F, \|\ \|, K\}$ and a certain class of operators L having their domain $\mathscr{D}$ and range $\mathscr{R}$ in F. Alternatively we shall also consider the class of functions Λ on $\mathcal{E}\times\mathcal{E}$ with the property that $\Lambda_y\,\varepsilon\,F$.

For an operator L we may ask if it is represented by some Λ by the formula

(1) $$Lf(y) = (f, \Lambda_y), \qquad f \in \mathcal{D}.$$

If it is, we say that L is "<u>representable</u>", that "<u>Λ represents L</u>", or that L is "<u>represented by</u>" the kernel Λ.

This representation was analyzed in Aronszajn [2] for the case of bounded operators and in Devinatz [1] for the case of unbounded operators. We shall give here some extensions for the (possibly) unbounded operators.

We see immediately that a representable operator must have a linear character, i.e. it must have a linear extension. (It may not actually be linear because its domain may not be a subspace.) In fact, we shall see shortly (in Theorem 2) that it has closed linear extensions and maximal ones of a certain type. It is also obvious that two different operators may be represented by the same kernel (e.g. one a restriction of the other) and that two kernels Λ_1 and Λ_2 may represent the same operator L (if $\Lambda_{1_y} - \Lambda_{2_y} \perp \mathcal{D}$ for each y).

<u>Theorem 1.</u> <u>A linear operator L is representable if and only if $K_y \in \mathcal{D}(L^{\circledast})$ for each y, where $L^{\circledast}$ is the minimal adjoint of L (cf. I.3). If Λ represents L, then $\Lambda_y - L^{\circledast} K_y \perp \overline{\mathcal{D}}$ for each y.</u>[1]

<u>Proof</u>. If $K_y \in \mathcal{D}(L^*)$, then $Lf(y) = (Lf, K_y) = (f, L^{\circledast}K_y)$ and L is represented by $L^{\circledast}K$. On the other hand if Λ represents L, then $(Lf, K_y) = Lf(y) = (f, \Lambda_y)$ and $K_y \in \mathcal{D}(L^{\circledast})$. Further, $L^{\circledast}K$ also represents L and therefore $(f, \Lambda_y - L^{\circledast}K_y) = 0$ for $f \in \mathcal{D}$, or $\Lambda_y - L^{\circledast}K_y \perp \overline{\mathcal{D}}$.

<u>Theorem 2.</u> <u>If Λ is a function on $\mathcal{E} \times \mathcal{E}$ such that $\Lambda_y \in F$ for each y, then there exists a unique operator L_M represented by Λ such that every operator represented by Λ is a restriction of L_M. This</u>

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1) This theorem was proved by Devinatz [1] (Theorem 1).

<u>"maximal" (or largest) operator L_M is a closed linear operator whose domain is the collection of functions f for which $(f, \Lambda_y) \in F$, when considered as a function of y.</u>[1)]

<u>Proof</u>. If we take $\mathcal{D}_M = \mathcal{D}(L_M)$ as described in the theorem and define $L_M f(y) = (f, \Lambda_y)$, for $f \in \mathcal{D}_M$, then L_M is clearly a linear operator in the space F. Since F is an r.k. space we know that strong convergence $f_n \rightarrow f$ (or even weak convergence $f_n \rightharpoonup f$) implies pointwise convergence $f_n(x) \rightarrow f(x)$. Using this fact we see that L_M is closed. In fact, if $f_n \rightarrow f$ and $L_M f_n \rightharpoonup g$, then $L_M f_n(y) = (f_n, \Lambda_y) \rightarrow (f, \Lambda_y)$, $L_M f_n(y) \rightarrow g(y)$, and therefore $(f, \Lambda_y) = g(y) \in F$. If Λ represents an operator L, then for each $f \in \mathcal{D} = \mathcal{D}(L)$ we must have $Lf \in F$ and obviously $L \subset L_M$. This implies also that L_M is unique.

In Theorems 1 and 2 the operators in question may not have dense domains. Of course, in Theorem 1 we could have limited ourselves to the usual case of $\overline{\mathcal{D}} = F$, and then the statement would involve simply the ordinary adjoint. However, if we examine only the kernel Λ as in Theorem 2, we cannot tell immediately whether the domain $\mathcal{D}_M$ will be dense. In case it is not dense we can always replace Λ by another kernel Λ' so that Λ' represents L_M, i.e. $L_M \subset L'_M$, and so that $\mathcal{D}'_M$ is dense. For example we may take Λ'_y as the projection of Λ_y onto $\overline{\mathcal{D}}_M$, which amounts to extending L_M to be zero in the orthogonal complement of $\mathcal{D}_M$. We shall see in the next theorem that when also $\Lambda^*_y \in F$, as in the usual cases, then $\mathcal{D}_M$ is dense and the previous remarks are unnecessary.

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1) This theorem is essentially contained in Definition 1 of Devinatz [1].

From the general theory of transformations we know that a closed linear operator L always has a proper closed linear extension. In fact, we may choose any element $f_0 \notin \mathcal{D}$ and extend L to the domain $\mathcal{D} \dotplus [f_0]$ by letting $Lf_0 = g_0$, for an arbitrary $g_0 \in F$. This linear extension is closed since its graph is the vectorial sum of a closed subspace and a finite dimensional subspace.

By taking adjoints we see that the class of closed linear operators with dense domains has no minimal elements. Since there are no minimal operators in the general case, it is certainly consistent for us to define a new notion of a minimal operator under an additional restriction. We shall say that an operator L is a "<u>minimal</u>" operator if $\mathcal{K} \subset \mathcal{D}$ and L is the closure of its restriction $L_{\mathcal{K}}$ to $\mathcal{K}$ (cf. II. 1).

To each kernel Λ, with $\Lambda_y \in F$, there corresponds a maximal operator L_M. If $\mathcal{D}_M \supset \mathcal{K}$ then there also corresponds a minimal operator L_m = closure of $L_{M_{\mathcal{K}}}$. If $\mathcal{D}_M \not\supset \mathcal{K}$ then we say that the minimal operator L_m does not exist.

On the other hand, if we start with a representable operator L with dense domain, then its kernel Λ is uniquely determined, and L has a (unique) "<u>maximal</u>" representable extension which is the operator L_M corresponding to Λ. If $\mathcal{K} \subset \mathcal{D}$ then we say also that L has a "<u>minimal restriction</u>" L_m = closure of $L_{\mathcal{K}}$, and we note that an operator L may not have a minimal restriction even though the corresponding kernel Λ has a minimal operator.

<u>Theorem 3.</u> <u>If L is a closed linear operator with dense domain then</u>

(a) <u>L_m (the minimal restriction of L) exists if and only if L^* is representable.</u>

(b) <u>L is representable implies $(L_M)^* = (L^*)_m$,</u>[1)]

(c) <u>L_m exists implies $(L_m)^* = (L^*)_M$.</u>

<u>A kernel Λ, with $\Lambda_y \in F$ for each y, has a corresponding minimal operator L_m if and only if $\Lambda_y^* \in F$ for each y.</u>

Proof. (a) is a restatement of Theorem 1 for operators with dense domains. (b) follows immediately from Theorem 1 and the definitions of minimal and maximal operators. (c) follows from (b)(and (a)) by taking adjoints. To prove the last statement we notice that $\Lambda_y^*(x) = \overline{\Lambda_x(y)} = (K_y, \Lambda_x)$. It is then obvious that $\Lambda_y^* \in F$ if and only if $K_y \in \mathcal{D}_M$, which completes the proof of the theorem.

<u>Remark</u>: The conditions $\Lambda_y \in F$ and $\Lambda_y^* \in F$ in the theorem are analogous to the conditions on an infinite matrix that its rows and columns belong to ℓ_2. These conditions for a matrix are equivalent, in a general Hilbert space, to specifying a fixed basis $\{g_n\}$ and asking that the matrix represent an operator T with $\{g_n\} \subset \mathcal{D}(T)\,\mathcal{D}(T^*)$. The analogous situation in the r.k. space F is that Λ(a generalized matrix where $\mathcal{E}$ replaces the set of natural numbers) should represent an operator L with $\mathcal{K} \subset \mathcal{D}(L)\,\mathcal{D}(L^*)$. (Where the matrix element is given by $\alpha_{ij} = (Tg_j, g_i)$, the kernel is given by $\Lambda^*(x,y) = (LK_y, K_x)$, i.e. the orthonormal sequence $\{g_n\}$ is replaced by the family $\{K_y\}$.)

We may also make the obvious remark that when Λ is a kernel with $\Lambda_y \in F$, the nullspace of the maximal operator L_M is given by $F \ominus [\Lambda_y]$.[2)]

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1) Devinatz [1] did not specifically consider the notion of a minimal operator. He did, however, prove that $(L_M)^*$ is minimal in his Theorem 2, part (a).

2) Cf. section I.1. This result is essentially contained in Theorem 1 of Devinatz [1].

This follows immediately from formula (1) and the definition of $\mathcal{D}_M$. Consequently, if Λ_y^* also belongs to F, then the minimal operator L_m exists and $\overline{\mathcal{R}}_m = [\Lambda_y^*]$. This last result follows from $\overline{\mathcal{R}}_m = F \ominus \mathcal{X}((L_m)^*)$ $= F \ominus \mathcal{X}((L^*)_M) = [\Lambda_y^*]$.

Before closing these general remarks concerning a representable operator it would be worthwhile to investigate just which operators in the abstract sense may be represented in the form (1). In a fixed r.k. space we know from Theorem 1 exactly which operators are representable, but now we are asking which operators in an abstract Hilbert space are isomorphic to some representable operator in some r.k. space. The answer is given in the final theorem of this section.

Theorem 4. To each closed linear operator T in an abstract Hilbert space $\{\mathcal{H}, \| \|'\}$ there corresponds an r.k. space $\{F, \| \|, K\}$ of functions defined on a set $\mathcal{E}$ and an isomorphism φ of the abstract space $\{\mathcal{H}, \| \|'\}$ onto the functional space $\{F, \| \|\}$ with the following properties. It carries T into an operator L, by putting $\mathcal{D}(L) = \varphi(\mathcal{D}(T))$ and $L\varphi u = \varphi Tu$ for $u \in \mathcal{D}(T)$, and this operator L is representable in the r.k. space $\{F, \| \|, K\}$.

Proof. The space F may be constructed in many ways. One manner is as follows.

For the set $\mathcal{E}$ we may take any set of elements whose linear combinations are dense in $\mathcal{H}$ and which lies in the domain $\mathcal{D}(T^{\circledast})$, where $T^{\circledast}$ is the minimal adjoint of T as defined in I.3. The class F is the collection of all functions U where $U(x) = (u,x)$ for $x \in \mathcal{E}$ and u is any element in $\mathcal{H}$.

In F we define the scalar product $(U,V) = (u,v)$ where $V(x) = (v,x)$ for $x \in \mathcal{E}$. F then becomes a Hilbert space with the reproducing kernel

$K(x,y) = (y,x)$. In fact $\varphi u = U$, where $U(x) = (u,x)$ for $x \in \mathcal{E}$, defines an isomorphism of $\{\mathcal{H}, \| \|'\}$ onto $\{F, \| \|\}$. We note of course, that φ is one-to-one because the linear combinations of elements in $\mathcal{E}$ are dense in $\mathcal{H}$. That K is the reproducing kernel of the functional space F is verified by $(U,K_y) = (u,y) = U(y)$.

We have thus constructed an r.k. space $\{F, \| \|, K\}$ isomorphic by φ to the Hilbert space $\mathcal{H}$. Now we need only verify that the induced operator L in F is representable, i.e. that $K_y \in \mathcal{D}(L^{\circledast})$.

From $(LU,V) = (\varphi Tu, \varphi v) = (Tu,v)$ we obtain $\mathcal{D}(L^{\circledast}) = \varphi(\mathcal{D}(T^{\circledast}))$, and since $K_y = \varphi y$ we see that L is representable if and only if $\mathcal{E} \subset \mathcal{D}(T^{\circledast})$. Thus our construction and the proof of the theorem are complete.[1)]

We should be careful that this theorem does not make us tend to forget about the additional information given by a reproducing kernel. Every separable Hilbert space is isomorphic to the sequential r.k. space; where $\mathcal{E} = \{1,2,3,\ldots\}$, a function f on $\mathcal{E}$ is in F if and only if

$\sum_{i=1}^{\infty} |f(i)|^2 < \infty$, and $K(i,j) = \delta_{ij}$. However this does not mean that

every separable r.k. space is isomorphic to the sequential r.k. space, in the sense of preserving also the reproducing kernel. Indeed this last statement is false since among other things it would mean that all the sets $\mathcal{E}$ have the same cardinal number.

Further, if we have a nonrepresentable operator in an r.k. space it would usually do us very little good to drop the original functional character of the space as might be suggested by the theorem. The real

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1) The simple construction of this isomorphic functional space was presented by N. Aronszajn in a seminar on reproducing kernels at the University of Kansas in the Spring of 1952.

significance of the operator is usually connected directly with the functional character of the original r.k. space.

3. Sums and products of operators and kernels. In the previous section we considered the individual members of the class of representable operators in a given r.k. space F. We now consider the question of whether this class has the properties of an algebra, i.e. can we add and multiply two operators within this class (without going outside of the class).

If L_1 and L_2 are two representable operators, represented by Λ_1 and Λ_2 respectively, then clearly $L_1 + L_2$ is representable, and in fact it is represented by $\Lambda_1 + \Lambda_2$. However, the product L_1L_2 may not be representable, as will be shown in Example 1 at the end of this section.

Since we cannot multiply within the class of all representable operators, we might then ask if the situation is any better in the subclass of operators L having both a maximal extension L_M and a minimal restriction L_m, i.e. the subclass of representable operators (with dense domains and) with representable adjoints (cf. Theorem II.2.3, part (a)).

We see immediately that the sum of two operators in this latter class will again be in the class (since each is defined for $\mathcal{K}$). However, the product of two operators in this class may not belong to the class, and in fact may not even belong to the larger class of representable operators.

In Example 1 we shall construct an operator L which is representable and has a representable adjoint L^*. In addition we will have L and L^* simultaneously maximal and minimal, L^2 maximal but having no minimal restriction, and L^{*2} minimal but not representable. This example will then prove the two previous remarks concerning products of representable operators.

Instead of considering the class of representable operators L we may equally well consider the class of kernels Λ with $\Lambda_y \in F$. This amounts approximately to considering two representable operators as equivalent when they have the same kernel. It is only approximate because the same operator may have more than one kernel. In order for a representable operator L to have a unique kernel Λ we must require that $\overline{\mathcal{D}} = F$. This requirement also reduces the class of kernels Λ to include only those "regular" ones whose corresponding domain $\mathcal{D}_M$ is dense. Considering this reduced class of kernels Λ is now (exactly) the same as considering all representable operators with dense domains, where two such operators are considered to be equivalent if they have the same kernel Λ.

For two such operators L_1 and L_2 we know that the sum $L_1 + L_2$ is representable, but may not have a dense domain. This may be so even if L_1 and L_2 are both maximal, and poses a serious difficulty in trying to define addition in this class. We may, however, argue that $L_1 + L_2$ certainly has extensions in our class and we need only to pick out a well determined or natural one.

From the viewpoint of the kernels any definition of addition would be unsatisfactory unless it took Λ_1 and Λ_2 into their ordinary sum $\Lambda_1 + \Lambda_2$. This kernel will then lead us to a well determined extension of $L_1 + L_2$ and the problem is solved, <u>provided</u> this kernel $\Lambda_1 + \Lambda_2$ represents an operator with dense domain.

On the other hand, if the maximal operator corresponding to $\Lambda_1 + \Lambda_2$ does not have a dense domain, then $\Lambda_1 + \Lambda_2$ is not in our class and we reach a dead end. Looking at the adjoint operators L_i^* we see that the difficulty arises because the sum of two closed operators may have a dense domain but no closure. This unusual situation will be exhibited in Example 2.

Somewhat similar remarks can be made about the problem of defining a product of two operators in this class. One solution we might try is to define the product by choosing the extension of L_1L_2 corresponding to the kernel $L_2^* L_1^* K$, provided of course that this kernel is defined. We are in trouble, however, even when $L_2^* L_1^*$ is defined on $\mathcal{K}$. As we shall see in Example 3, we may have a closed linear operator L whose square has a dense domain but no closure. This in turn shows that $L_2^* L_1^* K$ may be defined but not in our class (even when $L_1 = L_2$).

Thus far we have given only negative information about multiplying operators. Some conditions under which the product will be representable are given by the following theorem.

Theorem 1. Let $\Lambda_{1_y} \in F$, $\Lambda_{2_y}^* \in F$, and $\Lambda(x,y) = (\Lambda_{1_y}, \Lambda_{2_x}^*)$. Suppose further that the maximal operator represented by Λ_2^* has a dense domain, and let L_{2_m} be the adjoint of this operator. If $\Lambda_y \in F$, then $L_{1_M} L_{2_m}$ is represented by Λ. If $\Lambda_y^* \in F$, then $(L_{1_M} L_{2_m})^*$ is represented by Λ^*. In order that for some operators $L_1 \subset L_{1_M}$ and $L_2 \supset L_{2_m}$, $L_1 L_2$ be defined on $\mathcal{K}$ (or representable and defined on $\mathcal{K}$) it is necessary and sufficient that $\Lambda_y^* \in F$ (or $\Lambda_y \in F$ and $\Lambda_y^* \in F$).

Proof. The operator L_{2_m}, as defined in the theorem, is a minimal operator as the notation suggests in view of Theorem II.2.3, part (b).

If $\Lambda_y \in F$ we have (from Theorem II.2.2 and Theorem II.2.1 for operators with dense domains) $\Lambda_y = (L_{2_m})^* \Lambda_{1_y} = (L_{2_m})^* (L_{1_M})^* K_y$. Since $(L_{1_M} L_{2_m})^* \supset (L_{2_m})^* (L_{1_M})^*$ it then follows that $(L_{1_M} L_{2_m})^*$ is defined on $\mathcal{K}$, and consequently that $L_{1_M} L_{2_m}$ is represented by Λ.

If $\Lambda_y^* \varepsilon F$ we have (for similar reasons) $\Lambda_y^* = L_{1_M}\Lambda_{2_y}^* = L_{1_M}L_{2_m}K_y$, and consequently $(L_{1_M}L_{2_m})^*$ is represented by Λ^*.

We shall now prove the last statement of the theorem. If some L_1L_2, as in the theorem, is defined on $\mathcal{K}$, then clearly $L_{1_M} L_{2_m}$ is also defined on $\mathcal{K}$. Since $L_{1_M} L_{2_m} K_y = \Lambda_y^*$ it follows that $\Lambda_y^* \varepsilon F$, and the necessity is proved. The sufficiency follows from what we have already proved above by putting $L_1L_2 = L_{1_M}L_{2_m}$. For the other half of the statement we suppose that some L_1L_2 is both representable and defined on $\mathcal{K}$. In this case we have $\Lambda_y^* \varepsilon F$ from the first half of the statement. L_1L_2 is representable implies that $(L_2f, \Lambda_{1_y}) = (L_1L_2f,K_y)$ is a bounded functional of $f \varepsilon \mathcal{D}(L_1L_2)$ (cf. Theorem II.2.1). Since L_1L_2 is defined on $\mathcal{K}$ we then find that (L_2f, Λ_{1_y}) is a bounded functional of $f \varepsilon \mathcal{K}$, and consequently that $\Lambda_{1_y} \varepsilon \mathcal{D}((L_{2_m})^*)$. From this it follows that $\Lambda_y = (L_{2_m})^* \Lambda_{1_y} \varepsilon F$, and the necessity is established. The sufficiency easily follows from what we have already proved by putting $L_1L_2 = L_{1_M}L_{2_m}$.

Example 1. We shall construct this example in the sequential r.k. space where $f(i)$ is a sequence in ℓ_2, $\|f\|^2 = \sum_{i=1}^{\infty} |f(i)|^2$, and $K(i,j) = e_j(i) = \delta_{ij}$. We shall construct a maximal representable operator L by specifying the values of $\Lambda_j = L^*e_j$. These values define by closure the operator L^*, and the operator L will be its adjoint. By this procedure L is automatically a maximal operator.

We let $L^*e_1 = 0$, and $L^*e_k = \frac{1}{k} e_1 + ke_k$ for $k \geq 2$. Then for $f \varepsilon \mathcal{K}$,

$(L^*f,e_1) = \sum_2^{\infty} \frac{1}{k} f(k)$, and $(L^*f,e_n) = nf(n)$ for $n \geq 2$. This shows us

that each $e_i \varepsilon \mathcal{D}$, that $Le_1 = \sum_2^{\infty} \frac{1}{k} e_k$, and that $Le_n = ne_n$ for $n \geq 2$.

From this information we know that L_m exists. Also we know that the graph of $L_{\mathcal{K}}$ is the vectorial sum of two graphs --- the graph of L on $[e_1]$ and the graph of L on $\mathcal{K}\ominus [e_1]$. Since the first of these graphs is one-dimensional and the closure of the second is well know, it follows that the domain of L_m is the set of functions f where $\{nf(n)\}$ is a sequence in ℓ_2.

Now for $f \in \mathcal{D}_m$ we have $(Lf,g) = f(1) \sum_2^\infty \frac{1}{n} \overline{g(n)} + \sum_2^\infty n\, f(n)\, \overline{g(n)}$.

The first term is clearly bounded in $f \in \mathcal{D}_m$ and the second term will be bounded if and only if $\{ng(n)\}$ is in ℓ_2. Thus $\mathcal{D}_m = \mathcal{D}((L_m)^*)$.

On the other hand for any $f \in \mathcal{D}_m$, its initial segments f_n (truncated after n coordinates) will converge to f with $L^*f_n \to L^*f$. Consequently $(L_m)^* = L^*$ is both maximal and minimal and the same holds for $L_m = L_M = L$, their common domain being $\mathcal{D} = \mathcal{D}^*$ = the set of functions f such that $\{nf(n)\}$ is in ℓ_2.

Further, the operator L^2 is not defined for e_1. Its domain $\mathcal{D}(L^2)$ is in fact the set of functions f such that $\{f(1) + n^2 f(n)\}$ is in ℓ_2, and its values are $L^2 f = \sum_2^\infty (f(1) + n^2 f(n))e_n$. It is also clear that anything orthogonal to $\mathcal{D}(L^2)$ must be zero, so that $\mathcal{D}(L^2)$ is dense and L^{2*} exists.

The operator L^{*2} has for its domain the set of functions f with $\{n^2 f(n)\} \in \ell_2$ and takes the values $L^{*2} f = (\sum_2^\infty f(n))e_1 + \sum_2^\infty n^2 f(n)\, e_n$. From this we see also that L^{*2} is minimal.

Finally we look at $(L^2 f,g) = \sum_2^\infty (f(1) + n^2 f(n))\overline{g(n)} = f(1) \sum_2^\infty \overline{g(n)}$ $\sum_2^\infty n^2 f(n)\, \overline{g(n)}$, where we have some reservations about the last equality.

If $\{g(n)\}$ is not in ℓ_1, then the last equality may not hold and we look at the previous sum. By letting $f = e_n (n > 1)$ we see that in this case $(L^2 f, g)$ is not bounded in $f \varepsilon \mathscr{D}(L^2)$. Consequently, to find the adjoint L^{2*} we can restrict ourselves to $g \varepsilon \ell_1$ and in this case the last equality holds. Then the first term is obviously bounded in $f \varepsilon \mathscr{D}(L^2)$, and the second term will be bounded if and only if $\{n^2 g(n)\}$ is in ℓ_2. Thus we see that $\mathscr{D}(L^{2*}) = \mathscr{D}(L^{*2})$ and $L^{*2} = L^{2*}$.

To sum up, we have shown that L and L^* are representable and that each of them is maximal and minimal at the same time. Further, L^2 is representable and maximal but has no minimal restriction, and L^{*2} is minimal but is not representable.

Example 2. Our second example will be that of two closed linear operators whose sum has a dense domain but no closure. Since this property has no particular bearing on r.k. spaces (and can be translated to them by means of Theorem II.2.4) we shall make the construction in an abstract separable Hilbert space $\mathscr{H}$.

Let $\{g_n\}$ be a complete orthonormal sequence in $\mathscr{H}$ and define $T_1 g_n = n^2 g_n$ and $T_2 g_n = n^2 g_n - nu$, where $u = \sum_n \gamma_n g_n$ is an arbitrary non-zero element in $\mathscr{H}$. The operator T_1 is easily seen to have a closure, and in fact its closure is positive and self adjoint. The operator $T_1 - T_2$ is defined for the sequence $\{g_n\}$ and therefore has a dense domain. We also have $\frac{1}{n} g_n \to 0$ and $(T_1 - T_2)(\frac{1}{n} g_n) = u \neq 0$ and consequently $T_1 - T_2$ has no closure. It remains to be proved that T_2 has a closure.

To prove that T_2 has a closure we let $f_n = \sum_k \alpha_k^{(n)} g_k$, where $\alpha_k^{(n)} = 0$ for $k > m_n$, and suppose that $f_n \to 0$, $T_2 f_n \to g = \sum_k \beta_k g_k$.

Then

$$T_2 f_n = \sum_k k^2 \alpha_k^{(n)} g_k - (\sum_k k\,\alpha_k^{(n)})(\sum_i \gamma_i g_i)$$

$$= \sum_k \Big[k^2 \alpha_k^{(n)} - \gamma_k (\sum_i i\,\alpha_i^{(n)})\Big]\ g_k,$$

$\lim_n (k^2 \alpha_k^{(n)}) = 0$, and by equating the coordinates of g to the limit of the coordinates of $T_2 f_n$ we obtain $\beta_k = -c\,\gamma_k$, where $c = \lim_n \sum_k k\,\alpha_k^{(n)}$.

Consequently $(\sum_k k\,\alpha_k^{(n)})u$ and $\sum_k k^2 \alpha_k^{(n)} g_k$ each converge separately and their limits must therefore be cu and 0 respectively. Thus $\sum_k k^4 |\alpha_k^{(n)}|^2 \to 0$ and therefore $c = \lim_n (\sum_k (\frac{1}{k}) k^2 \alpha_k^{(n)}) = 0$ (since $\{\frac{1}{k}\}$ is a sequence in ℓ_2 and $\{k^2 \alpha_k^{(n)}\}$ converges to zero in ℓ_2). Consequently g = 0 and we have proved that T_2 has a closure.

<u>Example</u> 3. Our third example is that of a closed linear operator T whose square has a dense domain but no closure. Since this property has no particular bearing on r.k. spaces we shall again make the construction in an abstract separable Hilbert space $\mathcal{H}$.

Let $\{g_n\}$ be a complete orthonormal sequence in $\mathcal{H}$ and define $Tg_n = n(n+1)g_{n+1}$ for n even and $Tg_n = \frac{1}{n}\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{2k}\,g_{2k}$ for n odd.

Then $\frac{1}{2n}\,g_{2n} \to 0$, while $T^2(\frac{1}{2n}g_{2n}) = T((2n+1)g_{2n+1}) = \sum_{k=1}^{n} \frac{1}{2k}\,g_{2k} \to y \neq 0$.

Thus T^2 has no closure and our proof will be completed by showing that T itself does have a closure.

To prove this we suppose that $x^{(n)} = \sum_{k=1}^{m_n} \alpha_k^{(n)} g_k \to 0$ and $Tx^{(n)} \to y$, and we must show that y = 0. Applying T to $x^{(n)}$, with the convention that $\alpha_k^{(n)} = 0$ for $k > m_n$, and rearranging the terms we get

$$Tx^{(n)} = \sum_{k \text{ even}}^{\infty} \alpha_k^{(n)} k(k+1) g_{k+1} + \sum_{k \text{ odd}}^{\infty} \alpha_k^{(n)} \frac{1}{k} \sum_{i \text{ even}}^{k-1} \frac{1}{i} g_i$$

$$= \sum_{k=1}^{\infty} \beta_k^{(n)} g_k,$$

where $\beta_k^{(n)} = k(k-1)\,\alpha_{k-1}^{(n)}$ for k odd and $\beta_k^{(n)} = \frac{1}{k} \sum_{\substack{i \text{ odd} \\ i > k}} \frac{1}{i} \alpha_i^{(n)}$ for k even.

Letting $y = \sum_1^{\infty} \beta_k g_k$ we have $\beta_k = \lim_n k(k-1) \alpha_{k-1}^{(n)} = 0$ for k odd.

When k is even we obtain

$$\left|\beta_k^{(n)} - \beta_{k+2}^{(n)}\right| = \left|\frac{1}{k(k+1)} \alpha_{k+1}^{(n)} + \left(\frac{1}{k} - \frac{1}{k+2}\right) \sum_{\substack{i \text{ odd} \\ i > k+2}} \frac{1}{i} \alpha_i^{(n)}\right|$$

$$\leq \frac{1}{k(k+1)} \left|\alpha_{k+1}^{(n)}\right| + \left(\frac{1}{k} - \frac{1}{k+2}\right) \left(\sum_{i=1}^{\infty} \frac{1}{i^2}\right)^{\frac{1}{2}} \|x^{(n)}\| \ .$$

Consequently, since $\|x^{(n)}\| \to 0$, we have $\beta_k - \beta_{k+2} = \lim_n (\beta_k^{(n)} - \beta_{k+2}^{(n)}) = 0$.

Thus $\beta_2 = \beta_4 = \beta_6 = \ldots$, which is possible only when they are all zero, and we have proved that $y = 0$. This completes the proof that T has a closure.

4. Limits of operators and expansions of kernels. In Aronszajn [2] (pp. 374-375) there is a theorem concerning an expansion of the kernel $\Lambda(x,y)$ as an infinite series and another theorem concerning the weak limit of a sequence of operators. Extending these theorems to unbounded operators they become slightly weaker and are as follows.

Theorem 1. Let L_n be a sequence of operators with dense domains, represented by the kernels Λ_n. If $Tf = w.\lim L_n^* f$ for f $\mathcal{K}$,

$\mathcal{D}(T) = \mathcal{K}$, then T^* is represented by the kernel $\Lambda(x,y) = \lim \Lambda_n(x,y)$.

Proof. $\Lambda = TK$ certainly represents the operator T^*. Further, for each x and y, $\Lambda_n(x,y) = (L_n^* K_y, K_x) \to (TK_y, K_x) = \Lambda(x,y)$ and the proof is complete.

Theorem 2. Let L_n be a sequence of operators with dense domains, represented by the kernels Λ_n. If $Lf = w. \lim L_n f$ for all $f \in \mathcal{D}(L)$, $\overline{\mathcal{D}(L)} = F$, and $\|\Lambda_{n_y}\|$ is a bounded sequence for each y, then L is represented by the kernel $\Lambda(x,y) = \lim \Lambda_n(x,y)$.

Proof. We have $(f, \Lambda_{n_y}) = (f, L_n^* K_y) = (L_n f, K_y) \to (Lf, K_y)$ for each $f \in \mathcal{D}(L)$. It follows, since $\mathcal{D}(L)$ is dense and $\|\Lambda_{n_y}\|$ is a bounded sequence, that $L_n^* K_y = \Lambda_{n_y}$ converges weakly. By taking linear combinations we see that the hypothesis of Theorem 1 is satisfied with $T = (L^*)_{\mathcal{K}}$. Since $L \subset T^*$, it then follows that L is represented by Λ.

Theorem 3. If L and L^* are representable and $\{g_n\}$ is a complete orthonormal sequence in F, $\{g_n\} \subset \mathcal{D}\mathcal{D}^*$, then

$$\Lambda(x,y) = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \alpha_{mn}\, g_m(x)\, \overline{g_n(y)} \right] = \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \alpha_{mn}\, g_m(x)\, \overline{g_n(y)} \right],$$

where $\alpha_{mn} = (g_n, Lg_m) = (L^* g_n, g_m)$. If L is bounded then the above expansion is convergent in the sense of $\lim_{p,q \to \infty} \sum_{m=1}^{p} \sum_{n=1}^{q}$. It is absolutely convergent if L has a finite norm.

Proof. The last two statements are simply restatements of what is proved in Aronszajn [2]. In the general situation Λ_y may be expanded as

$$\Lambda_y = \sum_{m=1}^{\infty} (\Lambda_y, g_m) g_m \quad \text{and} \quad (\Lambda_y, g_m) = (L^* K_y, g_m) = (K_y, Lg_m) = \overline{Lg_m(y)}.$$

Then $Lg_m = \sum_{n=1}^{\infty} \overline{\alpha_{mn}}\, g_n$, with the α_{mn} above, in the sense of strong convergence, which certainly implies $Lg_m(y) = \sum_{n=1}^{\infty} \overline{\alpha_{mn}}\, g_n(y)$ as a pointwise limit. Substitution then yields the first expansion in the theorem.

Similarly we can expand $\Lambda_x^* = \sum_{n=1}^{\infty} (\Lambda_x^*, g_n) g_n$, where $(\Lambda_x^*, g_n) = (LK_x, g_n) = (K_x, L^* g_n) = \overline{L^* g_n(x)}$. Further, $L^* g_n = \sum_{m=1}^{\infty} \alpha_{mn}\, g_m$ and consequently $L^* g_n(x) = \sum_{m=1}^{\infty} \alpha_{mn}\, g_m(x)$. Substitution into $\Lambda(x,y) = \overline{\Lambda_x^*(y)}$ now yields the second expansion in the theorem, and the proof is complete.

We should remark that $\sum_{m=1}^{p} \sum_{n=1}^{q} \alpha_{mn}\, g_m(x)\, \overline{g_n(y)}$

$= (\sum_{n=1}^{q} \overline{g_n(y)}\, g_n,\ L \sum_{m=1}^{p} \overline{g_m(x)}\, g_m) = (P_q K_y, LP_p K_x)$ as for the proof in Aronszajn [2], where P_n is the projection onto $[g_1, g_2, \dots g_n]$. However, when L is unbounded we have no quarantee that LP_pK_x will converge as $p \to \infty$. If we happen to know that LP_pK_x and $L^*P_pK_x$ are each bounded uniformly in p (for any fixed point x) then we may conclude that the convergence in Theorem 3 is in the sense of

$\lim_{p,q \to \infty} \sum_{m=1}^{p} \sum_{n=1}^{q}$, since we would have

$$|(P_qK_y, LP_pK_x) - (P_{q'}K_y, LP_{p'}K_x)| \leq |(P_qK_y - P_{q'}K_y, LP_pK_x)| + |(L^*P_{q'}K_y, P_pK_x - P_{p'}K_x)| \to 0.$$

5. Subnormal operators and kernels. In section II.2 we considered some of the relations between a representable operator L and its kernel Λ. In the present section we wish to characterize the kernels Λ which represent subnormal operators. More precisely we shall characterize the kernels Λ which represent a subnormal operator defined on $\mathcal{K}$.

We shall say that a kernel Λ is a "<u>subnormal kernel</u>" if $\Lambda_y \varepsilon F$, $\Lambda_y^* \varepsilon F$, and

(1) $$(\Lambda_y, \Lambda_x) = (\Lambda_y^*, \Lambda_x^*).$$

The composed kernels (Λ_y, Λ_x) and $(\Lambda_y^*, \Lambda_x^*)$, in view of Theorem II.3.1 (and Theorem II.2.3), correspond to the operators $L_M(L_M)^*$ and $(L_m)^*L_m$. Thus we see a direct analogy between formula (1) and the formula $T^*T = T_1T_1^*$ in Theorem I.4.1. The exact relation is given by the following theorem.

<u>Theorem 1</u>. <u>Let $\Lambda(x,y)$ be a kernel with $\Lambda_y \varepsilon F$ for each y. Let L_M be the corresponding maximal operator and L_m the corresponding minimal operator (if it exists). Then the following statements are equivalent.</u>

(a) <u>$(L_M)^*$ exists and is a subnormal operator.</u>

(b) <u>L_m exists and is a subnormal operator.</u>

(c) <u>Λ is a subnormal kernel.</u>

<u>Proof</u>. First, if $(L_M)^*$ exists and is subnormal, then $\mathcal{K} \subset \mathcal{D}((L_M)^*) \subset \mathcal{D}(L_M)$ and L_m clearly exists. Also $\|L_m u\| = \|L_M u\| = \|(L_M)^* u\| = \|(L_m)^* u\|$ for any $u \varepsilon \mathcal{K}$. Thus $L_{m_{\mathcal{K}}}$, and consequently its closure L_m, is subnormal.

Second, if L_m exists and is subnormal, then $\Lambda = (L_M)^*K = (L_m)^*K$, $\Lambda^* = L_mK$, and consequently $(\Lambda_y, \Lambda_x) = ((L_m)^*K_y, (L_m)^*K_x) = (L_mK_y, L_mK_x) = (\Lambda_y^*, \Lambda_x^*)$.

Finally, if Λ is subnormal then $\mathcal{K} \subset \mathcal{D}_M$ and $(L_M)^*$ exists (cf. Theorem II.2.3). Further, if $u = \sum_{i=1}^{n} \alpha_i K_{y_i}$ is an arbitrary element of $\mathcal{K}$, then $\|L_M u\|^2 = \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (L_M K_{y_i}, L_M K_{y_j}) = \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (\Lambda_{y_i}^*, \Lambda_{y_j}^*)$

$$= \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (\Lambda_{y_i}, \Lambda_{y_j}) = \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j ((L_M)^* K_{y_i}, (L_M)^* K_{y_j}) = \|(L_M)^* u\|^2.$$

Consequently $((L_M)^*)_{\mathcal{K}}$ is subnormal, and therefore its closure $(L_M)^*$ (cf. Theorem II.2.3.b) is also subnormal. This completes the proof.

6. Symmetric operators and kernels. In this section we shall relate the symmetry of operators with the symmetry of their kernels. We shall also give some further information concerning self adjoint operators and positive operators.

By a symmetric kernel (sometimes called "Hermitian symmetric") we shall mean a kernel Λ with the property that $\Lambda(x,y) = \Lambda^*(x,y)$. It would be convenient if every symmetric kernel, with $\Lambda_y \in F$, represented a symmetric operator, and this is indeed the case. It would also be convenient if every representable symmetric operator had a symmetric kernel. This, however, will not always happen (e.g. when L^* is minimal and L is not self adjoint).

These remarks are immediate consequences of the following theorem.

Theorem 1. Let $\Lambda(x,y)$ be a kernel with $\Lambda_y \in F$ for each y. Let L_M be the corresponding maximal operator and L_m the corresponding minimal operator (if it exists). Then the following statements are equivalent.

(a) $(L_M)^*$ exists and is symmetric

(b) L_m exists and is symmetric (and therefore $= (L_M)^*$)

(c) Λ is symmetric

Proof. First, if $(L_M)^*$ exists and is symmetric, then $\mathcal{K} \subset \mathcal{D}((L_M)^*) \subset \mathcal{D}(L_M)$ and L_m exists. From $L_m \subset L_M$ and $(L_M)^* \subset L_M$ it follows that L_m and $(L_M)^*$ agree on the subspace $\mathcal{K}$. Since L_m and $(L_M)^*$ are each minimal operators (cf. Theorem II.2.3.b) it then follows that $L_m = (L_M)^*$, and L_m is symmetric.

Second, if L_m exists and is symmetric, then $\Lambda = (L_m)^*K = L_mK$ and $\Lambda(x,y) = (L_mK_y,K_x) = (K_y,L_mK_x) = \overline{(L_mK_x,K_y)} = \Lambda^*(x,y)$.

Finally, if Λ is symmetric, then $\Lambda_y^* \in F$ and $\mathcal{K} \subset \mathcal{D}_M$ (cf. Theorem II.2.3). Thus $(L_M)^*$ exists. Further, $(L_M)^*K = \Lambda = \Lambda^* = L_MK$ and by taking linear combinations we obtain $((L_M)^*)_{\mathcal{K}} = (L_M)_{\mathcal{K}}$. Since $(L_M)^*$ is the closure of $((L_M)^*)_{\mathcal{K}}$ (cf. Theorem II.2.3.b) it follows that $(L_M)^* \subset L_M$, and the proof is complete.

Theorem 2. Let $\Lambda(x,y)$ be a symmetric kernel with $\Lambda_y \in F$ for each y, and let L_M and L_m be the corresponding maximal and minimal operators. Then the deficiency indeces (p,q) of the symmetric operator L_m are given by

$$p = \text{rank } [\Lambda_y + iK_y]^{1)},$$

$$q = \text{rank } [\Lambda_y - iK_y] .$$

Proof. If V is the Cayley transform corresponding to L_m, then $\mathcal{D}(V) = (L_m + i)(\mathcal{D}_m)$ and $\mathcal{R}(V) = (L_m - i)(\mathcal{D}_m)$. Since $\mathcal{K} \subset \mathcal{D}_m$ we certainly have $\{\Lambda_y + iK_y\} \subset \mathcal{D}(V)$ and $\{\Lambda_y - iK_y\} \subset \mathcal{R}(V)$. On the other hand the sets $\{\Lambda_y + iK_y\}$ and $\{\Lambda_y - iK_y\}$ are dense in $\mathcal{D}(V)$ and $\mathcal{R}(V)$ respectively, because L_m is the closure of its restriction to $\mathcal{K}$. The result of the theorem them follows immediately.

Theorem 3. Let $\Lambda(x,y)$ be a symmetric kernel with $\Lambda_y \in F$ for each y, and let L_M and L_m be the corresponding maximal and minimal operators. Then the following statements are all equivalent.[2)]

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1) Cf. section I.1.

2) Devinatz [1] proved the equivalence of (c) and (e) in his Theorem 2, part (a), except that only half of condition (e) appears, due to a misprint.

(a) $L_m = L_M$

(b) L_m is self adjoint

(c) L_M is self adjoint

(d) L_M is symmetric

(e) $[\Lambda_y + iK_y] = [\Lambda_y - iK_y] = F$ [1)]

Proof. From Theorem 1 we have $L_m = (L_M)^*$ is symmetric. Together with Theorem 2 this gives us the equivalence of (a), (b), (c), (d), and (e).

Theorem 4. Let $\Lambda(x,y)$ be a symmetric kernel with $\Lambda_y \varepsilon F$ for each y, and let L_M and L_m be the corresponding maximal and minimal operators. Then the conditions

(a) $(L_M)^*$ is positive,

(b) L_m is positive,

and (c) $\Lambda \gg 0$

are mutually equivalent.

Proof. $L_m = (L_M)^*$ from Theorem 1 and the proof will be completed by showing the equivalence of (b) and (c). If L_m is positive then

$$\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j \Lambda(y_j,y_i) = \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (L_m K_{y_i}, K_{y_j})$$

$$= (L_m \sum_{i=1}^{n} \alpha_i K_{y_i}, \sum_{i=1}^{n} \alpha_i K_{y_i}) \geqslant 0.$$

On the other hand, if $\Lambda \gg 0$, then $(L_m u,u) \geqslant 0$ for all $u \varepsilon \mathcal{K}$ (by the above expansion). Hence $L_{m_{\mathcal{K}}}$, and consequently its closure L_m, is a positive operator.

1) Cf. section I.1.

N. Aronszajn [2] has a theorem characterizing the symmetric kernels which represent bounded operators with lower bound c and upper bound C. It is a trivial matter to extend this theorem to the semi-bounded case for the last result of this section.

Theorem 5. If Λ is an arbitrary symmetric kernel with $\Lambda_y \in F$ for each y, then Λ represents a minimal operator L_m. A necessary and sufficient condition in order that $c(f,f) \leq (L_m f,f)$ or $(L_m f,f) \leq C(f,f)$, for all $f \in \mathcal{D}_m$, is that $cK \ll \Lambda$ or $\Lambda \ll CK$ respectively.[1]

Proof. The necessity follows as in Aronszajn [2] because $L_m - c$ (or $C - L_m$) is a positive operator represented by the kernel $\Lambda - cK$ (or $CK - \Lambda$).

For the sufficiency we suppose that $\Lambda - cK$ (or $CK - \Lambda$) is a positive matrix. It then has a corresponding minimal operator $L_m - c$ (or $C - L_m$) which is positive. From this it immediately follows that $(L_m f,f) \geq c(f,f)$ (or that $(L_m f,f) \leq C(f,f)$) and the proof is complete.

Note that for a semi-bounded kernel we had to include the hypothesis that $\Lambda_y \in F$. When we have both inequalities $cK \ll \Lambda \ll CK$ (as in Aronszajn [2]) we do not need the additional hypothesis as it follows from these inequalities. However, with only one inequality we have no guarantee that the functions Λ_y are in our space, and the additional hypothesis is necessary.

7. Altering nonrepresentable operators. We shall consider now a closed linear operator L with a dense domain in an r.k. space $\{F, \| \; \|, K, \mathcal{E}\}$. We have explicitly referred here to the basic set $\mathcal{E}$ where our functions are defined, because this set will be of importance

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1) This theorem is contained in Theorem 2, part (b), of Devinatz [1].

in our discussion. We know (from Theorem II.2.1) that the condition for L to be representable is that $\mathcal{K} \subset \mathcal{D}^*$. We now consider the case when L is not representable, and we ask the question "Why does it fail to be representable?"

Our first and the most obvious answer to this question is that L is defined on too large a domain $\mathcal{D}$. If we restrict L to a smaller domain $\mathcal{D}' \subset \mathcal{D}$, then we will have $\mathcal{D}'^* \supset \mathcal{D}^*$ and this may give us the desired inclusion $\mathcal{K} \subset \mathcal{D}'^*$. This will certainly be the case whenever $\mathcal{D}'$ is finite dimensional (and therefore L' is bounded). However, if we restrict the operator L to too small a domain $\mathcal{D}'$ we may have lost a great deal of its essential character, and this procedure would not be at all satisfactory. It is important, therefore, to restrict the operator as little as possible, and in particular we would like to have $\mathcal{D}'$ dense in F.

The question of whether there exists a representable restriction of L with a dense domain is the same as the question of whether there exists a closed extension of L^* whose domain contains $\mathcal{K}$. If $K_y \notin \mathcal{D}^*$ for only a finite number of points $y_1, y_2, \ldots y_n$, then it is always possible to extend L^* to a closed operator T having the domain $\mathcal{D}^* \dotplus \mathcal{K} = \mathcal{D}^* \dotplus [K_{y_1}, K_{y_2}, \ldots K_{y_n}]$. A representable restriction $L' \subset L$ will then be given by $L' = T^*$.

On the other hand, if there are an infinite number of points y for which $K_y \notin \mathcal{D}^*$, then there may not exist a closed extension of L^* defined on $\mathcal{K}$. Consequently, in this case there may not exist any representable restriction of L having a dense domain.

As another possibility, we may of course ignore the functional character of the Hilbert space F. Then we can always map the Hilbert space $\{F, \| \|\}$ isomorphically onto an r.k. space $\{F', \| \|', K', \mathcal{E}'\}$ where the operator L', induced by L, will be representable (cf. Theorem II.2.4). In most applications this procedure is of no value, however, since the functional character of the original space F is usually quite important.

Although we should not ignore completely the functional character of the space F, we may still try to alter the space slightly --- retaining most of its original character. Let us analyze what happens to the space $\{F, \| \|, K, \mathcal{E}\}$ when we restrict all of the functions to a set $\mathcal{E}' \subset \mathcal{E}$. One thing that may happen is that two different functions in F may have the same restriction to $\mathcal{E}'$. In this event we have somehow "lost" part of the functions in F, and the class F' of restrictions is not as closely related to the class F as we might desire. We are in fact led to an r.k. space $\{F', \| \|', K', \mathcal{E}'\}$, where $K'(x,y)$ is the restriction of $K(x,y)$ to $\mathcal{E}' \times \mathcal{E}'$ and $\|f'\|' = \min \|f\|$, where the minimum is taken over all functions $f \in F$ whose restriction if f' (cf. section 5, Aronszajn [2]). When two different functions in F have the same restriction to $\mathcal{E}'$, some non-zero function (their difference) will have a restriction to $\mathcal{E}'$ that is identically zero. This shows us that the norm of a function $f \in F$ may be different than the norm of its restriction $f' \in F'$, i.e. $\|f\| > \|f'\|'$. Consequently we would have little connection between the Hilbert spaces $\{F, \| \|\}$ and $\{F', \| \|'\}$, and in this case the r.k. space $\{F', \| \|', K', \mathcal{E}'\}$ would be of little use.

If, however, the set $\mathcal{E}'$ is large enough so that different functions in F always have different restrictions, then the situation is much better. In this case the r.k. space $\{F', \| \|', K', \mathcal{E}'\}$ is very closely related to the original r.k. space $\{F, \| \|, K, \mathcal{E}\}$. In fact we have a canonical isomorphism between $\{F, \| \|\}$ and $\{F', \| \|'\}$ which carries the functions K_y into their restrictions K_y' whenever $y \in \mathcal{E}'$. In this case an operator L in the space F induces an operator L' in the space F', where $\mathcal{D}$ and $\mathcal{D}'$ are essentially the same, i.e. $\mathcal{D}'$ is the set of restrictions f' of functions $f \in \mathcal{D}$. Similarly $\mathcal{D}'^*$ is the set of restrictions f' of functions $f \in \mathcal{D}^*$.

We are now in a position to give a second answer to the question of what causes an operator L to be nonrepresentable. We may well consider that the basic set $\mathcal{E}$ is too large. For any set $\mathcal{E}' \subset \mathcal{E}$, with the property that each non-zero function in F has a non-zero restriction to $\mathcal{E}'$, we can form the r.k. space $\{F', \| \|', K_y', \mathcal{E}'\}$ mentioned above. The induced operator L' will now be representable if and only if $\mathcal{K}' \subset \mathcal{D}'^*$.

We remarked above that the adjoint domains $\mathcal{D}^*$ and $\mathcal{D}'^*$ are essentially the same. However, the class $\mathcal{K}'$ is essentially smaller than the class $\mathcal{K}$, so that it may happen that L' is now representable. In fact L' will be representable precisely when $\mathcal{E}'$ does not contain any of the points y for which $K_y \notin \mathcal{D}^*$. Thus we may formulate a necessary and sufficient condition under which this procedure will give us a representable operator L'. We let $\mathcal{E}_0 \subset \mathcal{E}$ be the set of points y for which $K_y \in \mathcal{D}^*$. Then the condition is that every non-zero function in F have a non-zero restriction to $\mathcal{E}_0$, i.e. that $\mathcal{K}_0$ (the linear combinations of K_y with $y \in \mathcal{E}_0$) be dense in F. In this case $\mathcal{E}' = \mathcal{E}_0$ will lead to a representable operator L', which is essentially the same as the original operator L. This procedure will be applied to the Carleman integral operators in the next section.

As the final consideration of this section we shall analyze what happens when we introduce a new norm on the functional class F. We would like the class F, with the new norm, to again be an r.k. space, and from paragraph 6 of section II.1 this means that the new norm must be equivalent to the old norm. Thus we proceed as follows.

Let $\{F, \| \|, K\}$ be an r.k. space and let $\| \|'$ be an equivalent norm on F, i.e. $m \|f\| \leq \|f\|' \leq M \|f\|$ (with $m > 0$). For a fixed $g \in F$ the scalar product $(f,g)'$ is a linear functional of $f \in F$, bounded with respect to the original norm. It therefore defines a linear operator T by putting $(f,g)' = (f,Tg)$ for $f,g \in F$. The new norm is now expressible as $\|f\|'^2 = (f,Tf)$, and we have the inequalities

(1) $$m^2(f,f) \leq (f,Tf) \leq M^2(f,f), \qquad f \in F.$$

From this it follows that T is a positive, bounded operator with a bounded inverse (and $\|T\| \leq M^2$, $\|T^{-1}\| \leq \frac{1}{m^2}$). Since T^{-1} is closed and bounded we know that $\mathcal{R}(T)$ is closed, and from (1) it follows that $\mathcal{R}(T)$ is dense. Consequently T^{-1} has the same properties as T --- it is a positive, bounded linear operator defined on the whole space and with a bounded inverse.

On the other hand if T is any positive, bounded linear operator on the whole space and has a bounded inverse, then equation (1) follows with $M^2 = \|T\|$ and $\frac{1}{m^2} = \|T^{-1}\|$. It is then clear that (Tf,f) defines a norm equivalent to the original one.

Now let L be an operator in F with dense domain, and again let $\|f\|'^2 = (Tf,f)$ be a norm equivalent to $\| \|$. We have $(Lf,g)' = (Lf,Tg)$ and $(f,h)' = (f,Th)$ so that asking for a solution h to the equation $(Lf,g)' = (f,h)'$ is the same as asking for a solution h to $(Lf,Tg) = (f,Th)$. Thus we see that the adjoint of L with respect to the new norm is $T^{-1}L^*T$, where L^* is the adjoint with respect to the original norm.

If K is the original reproducing kernel, then $(f,TK_y') = (f,K_y')'$ $= f(y) = (f,K_y)$ shows us that the new reproducing kernel is $K' = T^{-1}K$.

Thus we see that $K_y' \varepsilon \mathcal{D}(T^{-1}L^*T) = T^{-1}(\mathcal{D}(L^*))$ if and only if $K_y \varepsilon \mathcal{D}(L^*)$, i.e. the operator L is, or is not, representable independently of the norm chosen (so long as it is an r.k. space). If L is represented by Λ with respect to the original norm, then L is represented by $\Lambda' = T^{-1}\Lambda$ with respect to the new norm (since $\Lambda' = (T^{-1}L^*T)K' = T^{-1}L^*K$).

8. Carleman integral operators. In this section we consider a σ-finite measure space (X,S,μ) (in the terminology of Halmos [1]) and a kernel $H(x,y)$ defined on $X \times X$ and measurable with respect to the product measure $\mu \times \mu$. We suppose also that H is (Hermitian) symmetric, i.e. $H(x,y) = \overline{H(y,x)}$, and that for each fixed value of x, $H(x,y)$, considered as a function of y, belongs to $\mathcal{L}_2(\mu)$.

In place of $\mathcal{L}_2(\mu)$ we shall write $\mathcal{L}_2(X)$ or simply $\mathcal{L}_2$. For a measurable set A we shall write $\mathcal{L}_2(A)$ to stand for $\mathcal{L}_2(\mu_A)$, where $\mu_A(B) = \mu(AB)$ for every $B \varepsilon S$. Finally, when we integrate over the entire space X we shall write simply $\int ()d\mu$ instead of $\int_X ()d\mu$.

The kernel H now defines a transformation T_1 by putting

$$(1) \qquad T_1 f(x) = \int H(x,y)\, f(y) d\mu_y, \qquad f \varepsilon \mathcal{L}_2.$$

The transformation T_1 may not have a range in $\mathcal{L}_2$ --- indeed $T_1(\mathcal{L}_2)$ may even contain functions not integrable over some sets of finite measure. This transformation, however, leads us to an integral operator T in $\mathcal{L}_2$, where $f \varepsilon \mathcal{D}(T)$ if $T_1 f$ is also a function in $\mathcal{L}_2$, and

$$(2) \qquad Tf(x) = \int H(x,y)\, f(y) d\mu_y, \qquad f \varepsilon \mathcal{D}(T).$$

It is also convenient to introduce the non-negative measurable function $H(x)$, defined by

$$(3) \qquad H^2(x) = \int |H(x,y)|^2\, d\mu_y.$$

Thus far we have a general kernel of Carleman type (cf. Carleman [1]). The kernels of Hilbert-Schmidt type are characterized by the additional requirement that $H(x)$ belong to $\mathcal{L}_2$.

The operator T, defined by (2), is in some sense a maximal operator in $\mathcal{L}_2$, even though the space $\mathcal{L}_2$ is not an r.k. space and this maximal sense is not precisely what we have considered before in section II.2. However, T is a closed linear operator with a symmetric adjoint and is isomorphic to a maximal operator in an r.k. space. We shall establish this by the following construction.

We let $\mathcal{E}'$ be the class of all measurable sets A such that $\mu(A) < \infty$ and $H(x) \in \mathcal{L}_2(A)$. Since the measure μ is σ- finite and since $H(x)$ is finite valued, the class $\mathcal{E}'$ is clearly a basis for the σ- ring S, in the sense that the smallest σ-ring containing $\mathcal{E}'$ is S. We now let $\mathcal{E}$ be any subclass of $\mathcal{E}'$ which is also a basis for S, and we let F be the class of indefinite integrals $\tilde{f}$ of functions f in $\mathcal{L}_2$, i.e. $\tilde{f}$ is the function on $\mathcal{E}$ defined by

$$\tilde{f}(A) = \int_A f d\mu , \qquad f \in \mathcal{L}_2,\ A \in \mathcal{E}. \tag{4}$$

For example, if X is the unit interval $(0,1)$, μ is Lebesgue measure, and $H(x)$ is bounded except in a neighborhood of $x = 1$, then for $\mathcal{E}$ we may take the subclass of intervals $(0,x)$ with $0 < x < 1$. Then $\mathcal{E}$ is in a one-to-one correspondence with the space X, and instead of $\tilde{f}(A)$ we could just as well write

$$\tilde{f}(x) = \int_0^x f d\mu .$$

In this case $\tilde{f}(x)$ is the ordinary indefinite integral of $f(x)$.

Similarly, if X is the real line, μ is Lebesgue measure, and $H^2(x)$ is integrable over finite intervals, then for $\mathcal{E}$ we may take the subclass of finite intervals with one end at the origin. Then we may again take $\tilde{f}(x)$ to be the ordinary indefinite integral of $f(x)$.

Returning to the general setting, equation (4) defines a linear transformation φ of $\mathcal{L}_2$ onto F, with f going into $\varphi f = \tilde{f}$. This transformation is also one-to-one. In fact if $\tilde{f} = \tilde{g}$, then $\int_A (f-g)d\mu = 0$ for all $A \in \mathcal{E}$, but this implies that $\int_A (f-g)d\mu = 0$ for all $A \in S$, since $\mathcal{E}$ is a basis for S, and therefore $f(x) = g(x)$ almost everywhere. If we now define a norm in F by letting $\|\tilde{f}\|^2 = \int |f|^2 d\mu$, then F clearly becomes a functional Hilbert space isomorphic to $\mathcal{L}_2$ (by the isomorphism φ).

When B is an arbitrary set in $\mathcal{E}$, and χ_B is its characteristic function, we have

$$\tilde{f}(B) = \int f\, \overline{\chi_B}\, d\mu ,$$

and we see that F has a reproducing kernel $K(A,B) = \int_A \chi_B \, d\mu = \mu(AB)$. (Note that $\chi_B \in \mathcal{L}_2$ since $\mu(B) < \infty$.) If L is the operator in F induced by T (and the isomorphism φ), then

$$L\tilde{f}(B) = (\varphi Tf)(B) = \int_B \left(\int H(x,y)\; f(y) d\mu_y \right) d\mu_x \tag{5}$$

$$= \int f(y) \left(\int_B H(x,y) d\mu_x \right) d\mu_y ,$$

for every $f \in \mathcal{D}(T)$, where we are permitted to change the order of integration because

$$\int_B \int |H(x,y) f(y)| \;\; d\mu_y \, d\mu_x \leq \int_B H(x) \left(\int |f|^2 \, d\mu \right)^{\frac{1}{2}} d\mu_x \tag{6}$$

$$\leq \left(\int |f|^2 \, d\mu \right)^{\frac{1}{2}} (\mu(B))^{\frac{1}{2}} \left(\int_B H^2(x) \; d\mu \right)^{\frac{1}{2}} < \infty$$

for every $f \varepsilon \mathcal{L}_2$ (recalling that $H(x) \varepsilon \mathcal{L}_2$ (B)). Further, $\int_B H(x,y)d\mu_x$ is in $\mathcal{L}_2$, since

$$(7) \qquad \int \left| \int_B H(x,y)d\mu_x \right|^2 d\mu_y = \int \left(\int_B H(x,y)d\mu_x \right) \overline{\left(\int_B H(z,y)d\mu_z \right)} d\mu_y$$

$$\leq \int_B \int_B \int |H(x,y) \;\; H(z,y)| \;\; d\mu_x \, d\mu_z \;\; d\mu_y$$

$$= \int_B \int_B \left(\int |H(x,y) \;\; H(z,y)| \;\; d\mu_y \right) d\mu_x \;\; d\mu_z$$

$$\leq \int_B \int_B H(x) \;\; H(z) d\mu_x \, d\mu_z < \infty .$$

Consequently, the symmetric kernel

$$(A,B) = \int_A \overline{\left(\int_B H(x,y)d\mu_x \right)} d\mu_y = \int_A \int_B H(x,y) d\mu_y \, d\mu_x$$

represents the operator L in F.

It should be remarked that the r.k. space F we have constructed is not the most natural r.k. space corresponding to $\mathcal{L}_2$. The natural r.k. space would be constructed by taking the basic set $\mathcal{E}$ equal to the class $\mathcal{E}_o$ of <u>all</u> measurable sets A with $\mu(A) < \infty$. This r.k. space is independent of the integral operator T. In this space the operator T will then induce an operator L_o, but L_o may not be representable. In order to know that L_o is representable we must be able to change the order of integration in (5) and we must have

$\int_B H(x,y)d\mu_x$ in $\mathcal{L}_2$. To insure this we have reduced the class $\mathcal{E}_o$

to the class $\mathcal{E}'$ of sets A with $H(x) \in \mathcal{L}_2(A)$, so that equations (6) and (7) then hold. This procedure of reducing the basic set in order to have a representable operator was described in the preceding section, and the situation for a Carleman integral operator is an example of where it may be applied. The condition that $\mathcal{E}$ be a basis for the σ-ring S is what insures that a non-zero function in F_0 have a non-zero restriction to $\mathcal{E}$.

PART III: EXTENSIONS OF REPRESENTABLE OPERATORS

1. General considerations. Throughout Part III we shall consider a fixed r.k. space $\{F, \| \|, K\}$ and a representable operator L. Equivalently we may consider a kernel $\Lambda(x,y)$ with $\Lambda_y \in F$, i.e. representing an operator $L \subset L_M$.

A first step in extending L would be to consider a maximal extension L_M (which is unique if $\mathcal{D}$ is dense). Next, if $\mathcal{D}_M$ is not already dense we may extend L_M to a representable operator with dense domain (e.g. replacing Λ by $P\Lambda$, where P is the projection onto $\overline{\mathcal{D}}_M$). With these remarks in mind we shall suppose in the remainder of Part III that L has already been extended to an L_M with dense domain, i.e. we are assuming that $L = L_M$ and $\mathcal{D}$ is dense. We caution the reader not to forget that the forthcoming formulas and conditions in terms of L (throughout Part III) will actually be in terms of the maximal operator L_M.

We shall now attempt to construct a representable extension L_1 of L. (Of course this means abandoning to some extent the space F, since in F, $L = L_M$ is as far as we can go.) We proceed as follows.

Examining the representation (II.2.1) we observe that (f, Λ_y) is defined for all $f \in F$, and we are immediately led to define our extension L_1, first in the whole space F, by the equation

$$L_1 f(y) = (f, \Lambda_y), \qquad f \in F. \tag{1}$$

Our problem is now to find some natural r.k. space $\{F_1, \| \|_1, K_1\}$ so that F_1 contains both F and $L_1(F)$ and, if possible, so that L_1 is representable in F_1.

This situation is similar in many respects to the case of an integral operator in $\mathcal{L}_2$ given by

$$Lf(y) = \int \Lambda^*(y,x) \quad f(x) \; dx, \tag{2}$$

with $\Lambda_y \varepsilon \mathcal{L}_2$ (i.e. a kernel of Carleman Type). When dealing with the Hilbert space $\mathcal{L}_2$ the operator is first defined for those functions f in $\mathcal{L}_2$ such that $L_1 f$ is also in $\mathcal{L}_2$. But the operator has an obvious extension to the entire space $\mathcal{L}_2$, or even to the larger class of measurable functions f where f Λ_y is an integrable function for almost all y. The Hilbert space $\mathcal{L}_2$ is of course not a proper functional space, but it can be isomorphically mapped onto an r.k. space as shown in section II.8.

Let us pause for a moment and examine just what our general situation is. We have a basic set $\mathcal{E}$ and we see no natural way of obtaining another one. We have an r.k. class F (cf. (7) of section II.1) of certain functions defined on $\mathcal{E}$, and we have another class $L_1(F)$ of certain pointwise limits of functions in F. (A function in $L_1(F)$ is the pointwise limit of some sequence in F, because each $f \varepsilon F$ is the strong limit of a sequence f_n in $\mathcal{D}$, and $Lf_n(y) = (f_n, \Lambda_y) \rightarrow (f, \Lambda_y) = L_1 f(y)$.)

The class of all functions on $\mathcal{E}$ is certainly a class containing F and $L_1(F)$, but this class is much too large since it is usually not even an r.k. class.

The most obvious of the few natural choices of an r.k. class containing both F and $L_1(F)$ is their vectorial sum $F_1 = F \dotplus L_1(F)$. In order to make F_1 into an r.k. space, and indeed to prove that F_1 is even an r.k. class, we shall first establish a norm in $L_1(F)$ making it into an r.k. space $\{L_1(F), \| \|', K'\}$. After establishing this we may take $K_1 = K + K'$ as a reproducing kernel for F_1 as explained in (5) of section II.1.

To define a norm in $L_1(F)$ we restrict L_1 to the domain $[\Lambda_y] = F \ominus \mathcal{X}$ (where $\mathcal{X} = \mathcal{X}(L)$). This establishes a one-to-one correspondence between $[\Lambda_y]$ and $L_1(F)$, and we simply transfer the norm in F by this correspondence. Clearly $L_1(F)$ then becomes a functional space $\{L_1(F), \| \|'\}$. (Note that $(L_1f, L_1g) = (f,g)$ whenever either f or g is in $[\Lambda_y]$.) This functional space also has the reproducing kernel $K'(x,y) = L_1 \Lambda(x,y) = (\Lambda_y, \Lambda_x)$, since $(L_1f, L_1\Lambda_y)' = (f, \Lambda_y) = L_1f(y)$.

As described in (5) of section II.1, $K_1 = K + L_1\Lambda$ is a reproducing kernel for F_1 with the norm

$$\|f_1\|_1^2 = \min(\|f\|^2 + \|g\|^2), \tag{3}$$

where the minimum is taken over all decompositions $f_1 = f + L_1g$ with $f \in F$ and $g \in [\Lambda_y]$. This minimum is attained for the unique couple f,g satisfying $(f,h') = (L_1g,h')'$ for all $h' \in FL_1(F)$ (cf. (5") of section II.1), i.e. satisfying $(f,Lh) = (g,h)$ for all $h \in \mathcal{D}(L)$. This condition can now be interpreted as $f \in \mathcal{D}^*$ and $g = L^*f$. Thus we see that the norm in F_1 is given by

$$\|f_1\|_1^2 = \|f\|^2 + \|L^*f\|^2, \tag{4}$$

where $f_1 = (1 + L_1L^*)f$. Further, since this representation is unique, we know that the transformation $1 + L_1L^*$ is one-to-one and that $F_1 = (1 + L_1L^*)(\mathcal{D}^*)$.

From the general theory of operators we know that $(1 + LL^*)^{-1}$ exists and is a positive bounded operator on F (i.e. of F into F) with bound ≤ 1. Since $\|f\|^2 \leq \|f\|^2 + \|L^*f\|^2 = \|(1 + L_1L^*)f\|_1^2$ for $f \in \mathcal{D}^*$, we see in addition that $(1 + L_1L^*)^{-1}$ is a bounded transformation of $\{F_1, \| \|_1\}$ into $\{F, \| \|\}$ with bound ≤ 1.

Since $\|(1 + L_1L^*)f\|_1$ is the same as the norm of $\{f, L^*f\}$ in the graph of L^*, we see that F_1 is isomorphic to the graph of L^*. From the general theory of operators we also know that L^* is the closure of its restriction to $\mathcal{D}(LL^*)$, and consequently F (considered as a subspace of F_1) is dense in F_1. Thus we have established the following theorem.

Theorem 1. Let $L = L_M$ be representable operator with dense domain in the r.k. space $\{F, \| \|, K\}$, and let Λ be the kernel representing L. Also let $F_1 = F + L_1(F)$ and $K_1 = K + L_1L^*K$, where L_1 is given by (1). Then $(1 + L_1L^*)$ has an inverse and $\{F_1, \| \|_1, K_1\}$ is an r.k. space, where $\| \|_1$ is given by either of the two equivalent formulas (3) or (4). Further, $F_1 = (1 + L_1L^*)(\mathcal{D}^*)$, F (considered as a subspace of F_1) is dense in F_1, and $(1 + L_1L^*)^{-1}$ is a bounded transformation with bound ≤ 1 of $\{F_1, \| \|_1\}$ into $\{F, \| \|\}$.

It will also be useful to have the relation

$$(5) \qquad (f, g_1)_1 = (f, g^*) \text{ for } f \in F, g_1 = (1 + L_1L^*)g^* \in F_1.$$

This formula follows easily from (4) by letting $f = (1 + LL^*)f^*$ and obtaining $(f, g_1)_1 = (f^*, g^*) + (L^*f^*, L^*g^*) = (f^*, g^*) + (LL^*f^*, g^*) = (f, g^*)$.

Thus far we have defined L_1 only on the space F, i.e. we have defined L_{1_F}. To complete the definition we now let (the entire) L_1 be the closure in the space F_1 of the operator L_{1_F}, provided of course that this closure exists.

It happens that the graph of L_{1_F}, L, and L', considered as operators in F_1, all have the same closure, where L' denotes any restriction of L with domain dense in F. This fact follows immediately from $\|L_1f\|_1^2 = \min(\| \|^2 + \| \|'^2) \leq \|0\|^2 + \|L_1f\|'^2 = \|Pf\|^2 \leq \|f\|^2$ (where P is the projection onto $[\Lambda_y]$) and from $\|f\|_1^2 = \min(\ ) \leq \|f\|^2$. To find their common adjoint operator L_1^* in F_1 we first use (5) to obtain $(Lf, g_1)_1 = (Lf, g^*)$ and $(f, h_1)_1 = (f, h^*)$, where $g_1 = (1 + L_1L^*)g^*$ and $h_1 = (1 + L_1L^*)h^*$. Then we must find for which $g^* \in \mathcal{D}^*$ the equation $(Lf, g^*) = (f, h^*)$ has a solution $h^* \in \mathcal{D}^*$ and what this solution is. The answer is now obvious: that $g^* \in \mathcal{D}(L^{*2})$ and $h^* = L^*g^*$. Consequently we obtain $\mathcal{D}_1^* = (1 + L_1L^*)(\mathcal{D}(L^{*2}))$ and $L_1^*(1 + L_1L^*)g^* = (1 + L_1L^*)L^*g^*$.

Further, the operator L has a closure in F_1 if and only if $\mathcal{D}_1^*$ is dense in F_1, i.e. if and only if L^* is the closure of its restriction to $\mathcal{D}(L^{*2})$ (using the isomorphism between F_1 and the graph of L^*). Thus we have established:

Theorem 2. In the notation of Theorem 1, if L' is any restriction of L with domain dense in F, then the graphs of L, L_{1_F}, and L', considered as operators in F_1, all have the same closure. The common adjoint operator L_1^* has domain $\mathcal{D}_1^* = (1+L_1L^*)(\mathcal{D}(L^{*2}))$ and is given by

$$L_1^*(1+L_1L^*)f = (1+L_1L^*)L^*f, \qquad f\in\mathcal{D}(L^{*2}). \tag{6}$$

The entire operator L_1, defined as the closure in F_1 of L (or L_{1_F}), exists if and only if L^* is the closure of its own restriction to $\mathcal{D}(L^{*2})$.

Even if we know that L_1 exists we still must determine whether or not it is representable. Applying Theorem II.1, the condition is that $\mathcal{K}_1 \subset \mathcal{D}_1^*$, which becomes $\mathcal{K} \subset \mathcal{D}(L^{*2})$ when referred to the space F. If L_1 is representable, then (using Theorem II.1) its kernel is $\Lambda_1 = L_1^* K_1 = (1+L_1L^*)\Lambda$.

Further, we will have $L_1 = L_{1_M}$ if and only if L_1^* is the closure of its restriction to $\mathcal{K}_1$, i.e. if and only if $\mathcal{K}_1$ is dense in the Hilbert space $\{\mathcal{D}_1^*, \|\ \|_1'\}$, where $\|\ \|_1'$ is the norm in the graph space (of L_1^*), $\|f_1\|_1'^2 = \|f_1\|_1^2 + \|L_1^*f_1\|_1^2$. Translating this condition back to the space F and the operator L^*, and again using the isomorphism between F_1 and the graph of L^*, we find that $L_1 = L_{1_M}$ if and only if $\mathcal{K}$ is dense in the Hilbert space $\{\mathcal{D}(L^{*2}), \|\ \|'\}$, where $\|f\|'^2 = \|f\|^2 + 2\|L^*f\|^2 + \|L^{*2}f\|^2$. If we wish, we may of course omit the "2" in the last expression, which gives us an equivalent norm.

In case the original operator L is bounded (in F) then it is already easy to handle in many respects. Since its domain is the whole space there is little reason to attempt any extension of the type discussed here. However, everything we have done so far certainly applies to the bounded operator L. The class F_1 is then the same as F so that all we have done is to change the norm to an equivalent one.

On the other hand it is possible for the operator L_1 to be bounded even though the operator L was not bounded. In fact L_1 being bounded is equivalent to its adjoint L_1^* being bounded, i.e. to $\mathcal{D}_1^* = F_1$. This condition is clearly equivalent to $\mathcal{D}(L^{*2}) = \mathcal{D}^*$, i.e. L_1 is bounded if and only if $\mathcal{R}^* \subset \mathcal{D}^*$.

If we ask whether this situation is really possible for an unbounded L the answer is yes, but not in the usual cases. Example 3 at the end of this section will establish the existence of such operators.

If L^* (or Λ) is subnormal, then $\mathcal{R}^* \subset \mathcal{D}^*$ is possible only if the operator L is already bounded. In fact it is sufficient to know that $\mathcal{R}^* \subset \mathcal{D}^* \subset \mathcal{D}$. From the general theory of operators we know that $\mathcal{D} \dotplus \mathcal{R}^*$ is the whole space F, and together these conditions imply that $F = \mathcal{D} + \mathcal{R}^* \subset \mathcal{D} \dotplus \mathcal{D}^* \subset \mathcal{D}$, or that L is bounded.

Summing up, we have established the following theorem.

<u>Theorem 3. In the notation of Theorems 1 and 2, the operator L (or L_1) is representable in F_1 if and only if $\mathcal{K} \subset \mathcal{D}(L^{*2})$. If L is representable in F_1 then:</u>

(a) <u>L_1 is represented by $\Lambda_1 = (1 + L_1 L^*) \Lambda$.</u>

(b) <u>$L_1 = L_{1_M}$ if and only if $\mathcal{K}$ is dense in the Hilbert space $\{\mathcal{D}(L^{*2}), \| \|'\}$, where $\|f\|'^2 = \|f\|^2 + \|L^* f\|^2 + \|L^{*2} f\|^2$.</u>

(c) <u>L_1 is bounded if and only if $\mathcal{R}^* \subset \mathcal{D}^*$.</u>

(d) <u>If $\mathcal{D}^* \subset \mathcal{D}$, then L_1 is bounded if and only if L is bounded.</u>

Before closing this section we shall make two last remarks.

Remark 1. If we had chosen some other norm in the r.k. class F_1, also making it into an r.k. space, then this new norm would be equivalent to the one we have used (cf. (6) in section II.1). Consequently the existence and the representability of L_1 would be unchanged. (The existence depends only on topological notions, and the representability with respect to an equivalent norm was discussed in section II.7.)

Remark 2. If, instead of maintaining the operator L in the space F_1 (with a new kernel), we wished to maintain the kernel Λ in the space F_1, it would represent the bounded operator $L_1(1+L_1L^*)^{-1}$. This follows from $(f_1, \Lambda_y)_1 = (f^*, \Lambda_y) = L_1f^*(y)$ (using (5)), and from $\|L_1f^*\|_1^2 \leq \|0\|^2 + \|f^*\|^2 \leq \|(1+L_1L^*)f^*\|_1^2$ (using (3) and the last statement in Theorem 1).

Example 1. The first difficulty that may arise in trying to extend L is that the operator L_1 may not exist. We shall now construct an operator so that L ($=L_M$) is representable in F, but also with the property that L has no closure in F_1.

We shall make the construction by defining $\Lambda_y = L^*K_y$. This immediately leads to the linear extension $L^*_{\mathcal{K}}$ with domain $\mathcal{K}$, and the closure of this operator will be the entire operator L^*. By this procedure the adjoint $L = L^{**}$ will automatically be a maximal representable operator in F.

We shall define our operator in the sequential r.k. space, where $\sum_1^\infty |f(n)|^2 < \infty$ and $K(i,j) = e_j(i) = \delta_{ij}$. We let $L^*e_1 = \sum_1^\infty \frac{1}{k} e_{2k}$, $L^*e_{2n} = ne_{2n+1}$, and $L^*e_{2n+1} = 0$, for $n \geq 1$. It is easily seen that the closure exists and has domain $\mathcal{D}^*$ consisting of those functions f with $nf(2n)$ a sequence in ℓ_2 (and of course $f(n)$ a sequence in ℓ_2).

We notice immediately that L^*e_1 is not in $\mathcal{D}^*$, so that L is not representable in F_1. This is not enough for our purpose, however.

For any $f \in \mathcal{D}^*$, we have $L^*f = \sum_1^\infty \frac{f(1)}{n} e_{2n} + \sum_1^\infty nf(2n)e_{2n+1}$.

It then follows that $f \in \mathcal{D}(L^{*2})$ if and only if $f \in \mathcal{D}^*$ and $f(1) = 0$. Thus $\mathcal{D}(L^{*2})$ is not even dense, and consequently L has no closure in F_1 (cf. Theorem 2).

Example 2. In order next to find an operator L having a closure L_1 in F_1 but not representable there, we go first to an abstract Hilbert space $\mathcal{H}$. There we take any unbounded closed linear operator T with dense domain $\mathcal{D} = \mathcal{D}^*$ (e.g. any unbounded normal or self adjoint T). Then T^* on $\mathcal{D}(TT^*)$ has a graph dense in the graph of T^*. Further $\mathcal{H} = \mathcal{D} \dotplus \mathcal{R}^*$, $\mathcal{D} \neq \mathcal{H}$, and therefore $\mathcal{R}^*$ cannot be contained in $\mathcal{D} = \mathcal{D}^*$, i.e. $\mathcal{D}^*$ properly contains $\mathcal{D}(TT^*)$.

Now take $\mathcal{E} = \mathcal{D}^*$ (or any set contained in $\mathcal{D}^*$ and properly containing $\mathcal{D}(TT^*)$) and construct the isomorphic r.k. space $\{F, \| \|, \mathcal{E}, K\}$ as in the proof of Theorem II.2.4. Then T induces an operator L in F which has the desired properties.

L is representable and maximal since $\mathcal{D}(LL^*) \subset \mathcal{K} \subset \mathcal{D}(L^*)$, (implied by $\mathcal{D}(TT^*) \subset \mathcal{E} \subset \mathcal{D}(T^*)$). Further, $\mathcal{D}(L) = \mathcal{D}(L^*)$ so that $\mathcal{D}(L^{*2}) = \mathcal{D}(LL^*)$, and therefore L has a closure L_1 in F_1 (since L^* is the closure of its own restriction to $\mathcal{D}(LL^*)$). Finally $\mathcal{K}$ is not in $\mathcal{D}(LL^*) = \mathcal{D}(L^{*2})$ so that L_1 is not representable (cf. Theorem 3).

Example 3. In this example we shall show that the extension L_1 may be bounded even though L was not bounded. We shall also examine more fully the extended space F_1 and operator L_1. As in Example 1 we shall use the sequential space and begin by defining L^*e_n.

We let $L^*e_{2n} = ne_{2n-1}$ and $L^*e_{2n-1} = \frac{1}{n} e_{2n}$, for $n = 1,2,3,\ldots$ The domain $\mathcal{D}^*$ is easily seen to be characterized by $nf(2n)$ being a sequence in ℓ_2 (and of course $f(2n-1)$ is in ℓ_2 so that $f \in F$). We also note that $\mathcal{D}^* = \mathcal{R}^*$ and $L^{*-1} = L^*$.

From this we know that L_1 exists (cf. Theorem 2), is representable (cf. Theorem 3), and is bounded in F_1 (cf. Theorem 3, part (c)). We also see that $\mathcal{D}$ is characterized by $nf(2n-1)$(and of course $f(2n)$) being in ℓ_2, and that $Le_{2n} = \frac{1}{n} e_{2n-1}$, $Le_{2n-1} = ne_{2n}$. Thus $\mathcal{D} = \mathcal{R}$ and $L^{-1} = L$. The kernel Λ is given by $\Lambda(i,j) = L^*e_j(i)$, or $\Lambda(i,2n)$ $= ne_{2n-1}(i)$ and $\Lambda(i,2n-1) = \frac{1}{n} e_{2n}(i)$.

The class $F_1 = F + L_1(F)$ is the class of functions f such that $f(2n-1)$ and $\frac{1}{n}f(2n)$ are in ℓ_2. The operator $(1 + L_1L^*)$ has, of course, $\mathcal{D}^*$ for its domain, and letting $f_1 = (1 + L_1L^*)f^*$ we get $f_1(2n-1)$ $= (1 + \frac{1}{n^2})f^*(2n-1)$ and $f_1(2n) = (1 + n^2)f^*(2n)$. Inverting these formulas we get $f^*(2n-1) = \frac{n^2}{n^2+1} f_1(2n-1)$ and $f^*(2n) = \frac{1}{n^2+1} f_1(2n)$. Calculating L^*f^* and substituting into $(f_1,g_1)_1 = (f^*,g^*) + (L^*f^*, L^*g^*)$

we finally obtain

$$(f_1,g_1)_1 = \sum_1^{\infty} \frac{n^2}{n^2+1} f_1(2n-1)\, \overline{g_1(2n-1)} + \sum_1^{\infty} \frac{1}{n^2+1} f_1(2n)\, \overline{g_1(2n)}.$$

When f_1 is in F we may calculate L_1f_1 and we find that $\|L_1f_1\|_1 = \|f_1\|_1$. This tells us that L_1 is a unitary operator on F_1. By computing

$$(L_1f_1,f_1)_1 = \sum_1^{\infty} \frac{n}{n^2+1} f_1(2n)\overline{f_1(2n-1)} + \sum_1^{\infty} \frac{n}{n^2+1} f_1(2n-1)\, \overline{f_1(2n)},$$

which is always real, we see that L_1 is also self adjoint, and from these two properties it follows that L_1 is of order 2 (i.e. $L_1^2 = I$). Consequently we find that L_1 is a very special and simple operator, called a symmetry, though L and L^* are not even subnormal.

Example 4. For our final example we let $L^*e_{6n+1} = e_{6n+2} + ne_{6[\frac{n}{2}]+6}$, where $[\frac{n}{2}]$ denotes the largest integer in $\frac{n}{2}$, $L^*e_{6n+2} = ne_{6n+3} + e_{6n+1}$, $L^*e_{6n+3} = 0$, $L^*e_{6n+4} = e_{6n+5} + ne_{6n+3}$, $L^*e_{6n+5} = e_{6n+4} + 2ne_{6n+6}$, and $L^*e_{6n+6} = 0$. This operator will have the property that L_1 exists and is representable, but $L_1 \neq L_{1M}$ even though by our convention, $L = L_M$.

Finding the domain $\mathcal{D}^*$ of the closure of this operator requires more care than in the previous examples. The result is that $f \in \mathcal{D}^*$ if and only if the sequences $\{f(n)\}$, $\{nf(6n+2) + nf(6n+4)\}$, and $\{2nf(6n+5) + 2nf(12n+1) + (2n+1)\ f(12n+7)\}$ are all in ℓ_2. This result may be obtained as follows.

We must construct a sequence f_k of finite combinations of the e_n so that $f_k \to f$ and L^*f_k is a Cauchy sequence. One way to construct the approximating function f_k is to truncate the infinite sequence $f(n)$ appropriately so that after applying L^* the coefficients of e_{6n+3} and of e_{6n+6} will have their "non-ℓ_2" components cancelled out. Roughly this means taking the same number of $f(6n+2)$'s as of $f(6n+4)$'s and taking twice as many $f(6n+1)$'s as $f(6n+5)$'s.

L_1 exists and is representable in F_1 because $\mathcal{D}(L^{*2})$ obviously contains the kernels $K_j = e_j$.

In order to show that $L_1 \neq L_{1M}$ we must find some function f for which $L^{*2}f$ is defined and such that it is impossible to find a sequence $\{f_k\}$ in $\mathcal{K}$ with $f_k \to f$, $L^*f_k \to L^*f$, and $L^{*2}f_k \to L^{*2}f$ (cf. Theorem 3, part (b)). Such an element is given by $f = \sum_1^\infty \frac{1}{n} e_{6n+2} - \sum_1^\infty \frac{1}{n} e_{6n+4}$. Then $L^*f = \sum_1^\infty \frac{1}{n} e_{6n+1} - \sum_1^\infty \frac{1}{n} e_{6n+5}$, and $L^{*2}f = f$.

We suppose now that there exists a sequence $\{f_k\} \subset \mathcal{K}$ with the properties that $f_k \to f$, $L^* f_k \to L^* f$, and $L^{*2} f_k \to f$. We may assume that $f_k(i) = 0$ except when $i = 6n+2$ or $i = 6n+4$, since each convergence is improved by applying such a projection. Thus

$$f_k = \sum_n f_k(6n+2) e_{6n+2} + \sum_n f_k(6n+4) e_{6n+4},$$

$$L^* f_k = \sum_n f_k(6n+2) e_{6n+1} + \sum_n f_k(6n+4) e_{6n+5}$$

$$+ \sum_n n \left[f_k(6n+2) + f_k(6n+4) \right] e_{6n+3},$$

$$L^{*2} f_k = \sum_n f_k(6n+2) e_{6n+2} + \sum_n f_k(6n+4) e_{6n+4}$$

$$\sum_n \left[2n f_k(12n+2) + (2n+1) f_k(12n+8) + 2n f_k(6n+4) \right] e_{6n+6}.$$

The three conditions $f_k \to f$, $L^* f_k \to L^* f$, and $L^{*2} f_k \to f$ now become

(i) $$\sum_n \left| f_k(6n+2) - \tfrac{1}{n} \right|^2 \to 0,$$

(ii) $$\sum_n \left| f_k(6n+4) + \tfrac{1}{n} \right|^2 \to 0,$$

(iii) $$\sum_n n^2 \left| f_k(6n+2) + f_k(6n+4) \right|^2 \to 0,$$

(iv) $$\sum_n n^2 \left| f_k(12n+2) + \frac{2n+1}{2n} f_k(12n+8) + f_k(6n+4) \right|^2 \to 0.$$

Conditions (iii) and (iv) then imply that

(v) $$\sum_n n^2 \left| f_k(12n+2) + \frac{2n+1}{2n} f_k(12n+8) - f_k(6n+2) \right|^2 \to 0.$$

We then choose an index k_o such that $|f_{k_o}(8)| > \frac{2}{3}$ and such that the sum in equation (v) is smaller than $\frac{1}{32}$. Letting $g(n) = n f_{k_o}(6n+2)$

and $\delta_n = \frac{1}{2}g(2n)+\frac{1}{2}g(2n+1)-g(n)$, we obtain $|g(1)| > \frac{2}{3}$, $\sum_n |\delta_n|^2 < \frac{1}{32}$, and $g(n) = \delta_n = 0$ for $n > p$ (where p depends upon k_o).

If we now add the equations

$$g(1) + \delta_1 = \tfrac{1}{2}g(2) + \tfrac{1}{2}g(3),$$

$$\tfrac{1}{2}g(2) + \tfrac{1}{2}\delta_2 = \tfrac{1}{4}g(4)+\tfrac{1}{4}g(5),$$

$$\tfrac{1}{2}g(3) + \tfrac{1}{2}\delta_3 = \tfrac{1}{4}g(6)+\tfrac{1}{4}g(7),$$

$$\tfrac{1}{4}g(4) + \tfrac{1}{4}\delta_4 = \tfrac{1}{8}g(8)+\tfrac{1}{8}g(9),$$

$$\cdots\cdots\cdots\cdots$$

we will obtain $g(1)+\delta_1 + \frac{1}{2}(\delta_2+\delta_3) + \frac{1}{4}(\delta_4+ \ldots + \delta_7) + \frac{1}{8}(\delta_8 + \ldots + \delta_{15}) + \ldots = 0$. Dividing this equation by 2 we finally obtain $\frac{1}{2}|g(1)| \leq |\delta_1| + \frac{1}{2}|\delta_2| + \frac{1}{3}|\delta_3| + \frac{1}{4}|\delta_4| + \ldots$ $\leq (\sum \frac{1}{n^2})^{\frac{1}{2}}(\sum|\delta_n|^2)^{\frac{1}{2}} \leq \sqrt{2}\sqrt{\frac{1}{32}} = \frac{1}{4}$, which contradicts $|g(1)| > \frac{2}{3}$.

2. Subnormal operators. In the previous section we discussed some of the properties of the extended operator L_1. A question of interest is whether the process of going from L to L_1 preserves the notion of subnormality. In this connection we have the following theorem.

Theorem 1. Let $L = L_M$ be the maximal operator represented by the subnormal kernel Λ, and let F_1 be the r.k. space of Theorem III.1.1. Also let L_1 be the closure, if it exists, of the operator L in the space F_1.

(a) $\mathcal{D}_1^* \subset F$.

(b) <u>If L_1 exists, then $L(\mathcal{D}(L^{*2}))\subset\mathcal{D}^*$ is a necessary and sufficient condition for L_1^* to be subnormal.</u>

(c) <u>$\mathcal{D}\mathcal{R}^*\subset\mathcal{D}^*$ and $L(\mathcal{D}(L^{*2}))\subset\mathcal{D}^*$ are necessary and sufficient conditions for L_1 to exist and be normal.</u>

(d) <u>If L_1 exists and is represented (in F_1) by Λ_1, then $\Lambda_y^*\varepsilon\,\mathcal{D}^*$ is a necessary and sufficient condition for Λ_1 to be subnormal.</u>

<u>Proof</u>. By our hypothesis we have Λ_y, $\Lambda_y^*\ \varepsilon\ F$ and L^* subnormal (Theorem II.5.1). Consequently, whether L_1 exists or not, we have $\mathcal{D}_1^* = (1+L_1L^*)(\mathcal{D}(L^{*2}))$ (Theorem III.1.2). From the general theory of operators we also have $F = (1+LL^*)(\mathcal{D}(LL^*))$. Since $\mathcal{D}^*\subset\mathcal{D}$, it then follows that $\mathcal{D}(L^{*2})\subset\mathcal{D}(LL^*)$ and $\mathcal{D}_1^*\subset F$. Thus (a) is established.

We shall next show that (c) will follow from (b). We have $\mathcal{D}(L^{*2}) = \mathcal{D}(LL^*)$ if and only if $\mathcal{D}\mathcal{R}^*\subset\mathcal{D}^*$, and this inclusion is therefore a necessary and sufficient condition for $\mathcal{D}_1^* = F$. Clearly L_1 will exist when $\mathcal{D}_1^* = F$ (since F is dense in F_1). Consequently necessary and sufficient conditions for L_1 to exist and be normal are that $\mathcal{D}_1^* = F$, that L_1^* be subnormal, and that L_{1_F} be already closed. Whenever $\mathcal{D}_1^* = F$ and L_1^* is subnormal, L_{1_F} will automatically be closed, since L_{1_F} is metrically equivalent to the closed operator L_1^* (cf. section I.4). Thus $\mathcal{D}\mathcal{R}^*\subset\mathcal{D}^*$ and L_1^* subnormal are necessary and sufficient conditions for L_1 to exist and be normal.

In order to prove (b) and (d) we now let $f = (1+LL^*)f^*\varepsilon\,\mathcal{D}_1^*$ and obtain $L_1^*f = (1+L_1L^*)L^*f^*$ (cf. Theorem III.1.2) and

(1) $$\|L_1^*f\|_1^2 = \|L^*f^*\|^2+\|L^{*2}f^*\|^2$$

(using formula III.1.4). Further, $L_1f = Lf^*+L_1LL^*f^*$, and applying formula III.1.3 we get

$$(2) \qquad \|L_1 f\|_1^2 \leq \|Lf^*\|^2 + \|LL^*f^*\|^2.$$

Since $LL^* = L_M(L_M)^* = (L_m)^*L_m$ (Theorem I.4.1), we have $L_1LL^*f^* = L_1(L_m)^*L_mf^*$ and $L_1f = (1+L_1(L_m)^*)Lf^*$. Because the minimum in formula III.1.3 is attained for the unique decomposition $L_1f = (1+L_1L^*)f'$, we see that equality holds in (2) precisely when $f' = Lf^*$ (and $L^*f' = LL^*f^*$), i.e. when $Lf^* \varepsilon \mathcal{B}^*$. Since L^* is subnormal, the right sides of (1) and (2) are equal, and we have established

$$(3) \qquad \|L_1 f\|_1 \leq \|L_1^* f\|_1, \qquad f \varepsilon \mathcal{B}_1^*,$$

with equality if and only if $Lf^* \varepsilon \mathcal{B}^*$, where $f = (1+L_1L^*)f^*$. Consequently when L_1 exists (and L_1^* therefore has a dense domain in F_1), a necessary and sufficient condition for L_1^* to be subnormal (in F_1) is that $L(\mathcal{B}(L^{*2})) \subset \mathcal{B}^*$. Thus we have established (b).

For (d) we now suppose that L_1 is represented (in F_1) by Λ_1. By applying the condition for equality in (3) to elements $f \varepsilon \mathcal{K}_1$, we see that $(L_1^*)_{\mathcal{K}_1}$ is subnormal if and only if $LK_y \varepsilon \mathcal{B}^*$. Therefore $\Lambda_y^* = LK_y \varepsilon \mathcal{B}^*$ is a necessary and sufficient condition for the closure $(L_1^*)_m$ to be subnormal, or for Λ_1 to be subnormal (cf. Theorem II.5.1). This completes the proof of the theorem.

3. <u>Symmetric operators</u>. In the process of extending a maximal operator L we may also analyze the notions of symmetry and self-adjointness. These notions for the operator L and for its extension L_1 are related as shown by the following theorems.

<u>Theorem 1</u>. <u>Let $L = L_M$ be the maximal operator represented by the subnormal kernel Λ, and let F_1 be the r.k. space of Theorem III.1.1. Also let L_1 be the closure, if it exists, of the operator L in the space F_1. If L_1 exists and is represented (in F_1) by Λ_1, then a necessary and sufficient condition that Λ_1 be symmetric is that Λ be symmetric.</u>

<u>Proof</u>. We suppose that L_1 exists and is represented by Λ_1. Then $\Lambda_1 = (1+L_1L^*)\Lambda$ (Theorem III.1.3, part (a)). Since L^* is subnormal, we also have $(L_1L^*\Lambda)^*(x,y) = (\Lambda_y, L^*\Lambda_x) = (L^*K_y, L^*\Lambda_x)$ $= (LK_y, L\Lambda_x) = (\Lambda_y^*, L\Lambda_x)$. If $\Lambda_y^* \varepsilon \mathcal{D}^*$, then $(L_1L^*\Lambda)^*(x,y) = (L^*\Lambda_y^*, \Lambda_x)$ $= L_1L^*\Lambda^*(x,y)$ and $\Lambda_1^* = (1+L_1L^*)\Lambda^*$. Consequently, when $\Lambda_y^* \varepsilon \mathcal{D}^*$, Λ_1 will be symmetric if and only if Λ is symmetric. Finally, when Λ is symmetric we have $\Lambda_y^* = \Lambda_y \varepsilon \mathcal{D}^*$, and when Λ_1 is symmetric we have $\Lambda_y^* \varepsilon \mathcal{D}^*$ from Theorem III.2.1, part (d). This completes the proof.

Throughout the remainder of this section we shall suppose that $L = L_M$ and $\Lambda = \Lambda^*$. In this case we have $L^* \subset L$ and $\mathcal{D}_1^* \subset F$ (cf. Theorem III.2.1, part (a)). For $f = (1+LL^*)f^* \varepsilon \mathcal{D}_1^*$ we have $L_1^* f$ $= (1+L_1L^*)L^*f^*$, and since $L^*f^* \varepsilon \mathcal{D}^*$, we also have $L_1 f = Lf^* + L_1LL^*f^*$ $= (1+L_1L^*)L^*f^*$. Thus L_1^* will be symmetric whenever L_1 exists (i.e. whenever L_1^* has a dense domain).

<u>Theorem 2</u>. <u>Let $\Lambda = \Lambda^*$ and $\Lambda_y \varepsilon F$ for each y.</u>

(a) <u>$L_m = L_M$ is a necessary and sufficient condition for L_1 to exist and be self adjoint, and in this case $\mathcal{D}_1 = F$.</u>

(b) <u>If $L_m \neq L_M$ and L_1 exists, then L_1^* is symmetric, $L_1^* \neq L_1$, and for every self adjoint restriction L_1' of L_1 we must have $\mathcal{D}_1'F$ non-dense in F.</u>

<u>If L_1 is also representable (in F_1), then either</u>

(c) <u>$L_m = L_M$ and L_1 is self adjoint, $\mathcal{D}_1 = F$. For every other self adjoint restriction L_1' of L_{1_M} we must have $\mathcal{D}_1'F$ non-dense in F.</u>

<u>or</u> (d) <u>$L_m \neq L_M$ and L_m is not symmetric in F_1. For every self adjoint restriction L_1' of L_{1_M} we must have $\mathcal{D}_1'F$ non-dense in F.</u>

<u>Proof</u>. (a). If $L_m = L_M \equiv L$, then $\mathcal{D}_1^* = (1+LL^*)(\mathcal{D}(LL^*)) = F$ and L_1 exists. It then follows that $L_{1_F} = L_1^*$ and L_1 is self adjoint with domain F. On the other hand, if L_1 exists and is self adjoint, then $(L_1+i)^{-1}$ and $(L_1-i)^{-1}$ both exist. Since $(L \pm i)$ is a restriction of $(L_1 \pm i)$ it follows that $(L+i)^{-1}$ and $(L-i)^{-1}$ both exist. But this implies that $L_m = L^*$ is self adjoint, and $L_m = L = L_M$.

(b). If $L_m \neq L_M = L$ and L_1 exists, then the fact that L_1^* is symmetric follows from the remarks preceding the theorem, and $L_1^* \neq L_1$ follows from (a). Let L_1' be a self adjoint restriction of L_1. If $\mathcal{D}_1'F$ were dense in F, then L_1' would be a self adjoint extension of L_1 (cf. Theorem III.1.2), which is impossible.

(c). Since $L_m = L_M$ the first statement follows from (a). We must now prove that for a self adjoint operator $L_1' \subset L_{1_M}$ with $\mathcal{D}_1'F$ dense in F, we must have $L_1' = L_1$. In fact, from Theorem III.1.2. it follows that L_1' is a self adjoint extension of L_1, and therefore $L_1' = L_1$.

(d) When $L_m \neq L_M$, it follows from (b) that L_1 is not symmetric. Since L_1 is the closure in F_1 of L_m, (cf. Theorem III.1.2), it follows that L_m is not symmetric in F_1. Let L_1' be any self adjoint restriction of L_{1_M}. Then as before, if $\mathcal{D}_1'F$ were dense in F, we would have $L_1' \supset L_1$. This is clearly impossible and the proof of the theorem is complete.

<u>Remark</u>: Theorems 1 and 2 give us the conditions under which L_1 will be symmetric (or self adjoint) whenever the original operator L_m (or the kernel Λ) is symmetric. However, we have seen in Example 3 of section III.1 that for some operators L_m ($= L_M$ and not even subnormal) the operator L_1 may even be a symmetry. (We shall reserve the problem of finding necessary and sufficient conditions for L_1 to be normal, self adjoint, unitary, etc, to be considered in another paper.)

When $L_m = L_M = L$ (and $\Lambda = \Lambda^*$) we have seen that the operators L and L_1 are each self adjoint in their respective spaces F and F_1. We now proceed to relate their corresponding resolutions of the identity.

Theorem 3. Let $L_m = L_M = L$ be represented in F by the kernel $\Lambda = \Lambda^*$. Then L is a self adjoint operator in F, and its closure L_1, in the extended space F_1, is a self adjoint operator in F_1 with domain F. Let $\{E_\lambda\}$ and $\{E_{1\lambda}\}$ be the resolutions of the identity corresponding to L and L_1 respectively. Then $E_\lambda \subset E_{1\lambda}$, and $E_{1\lambda}$ is the closure in F_1 of E_λ. For elements $f \in F$ the two norms, that in F and that in F_1, are given by

(1) $$\|f\|^2 = \int d(E_\lambda f, f),$$

and

(2) $$\|f\|_1^2 = \int \frac{1}{1+\lambda^2}\, d(E_\lambda f, f).$$

Further, for $f \in \mathcal{D}$ we have

(3) $$\|Lf\|^2 = \int \lambda^2\, d(E_\lambda f, f),$$

and for any $f \in F = \mathcal{D}_1$ we have

(4) $$\|L_1 f\|_1^2 = \int \frac{\lambda^2}{1+\lambda^2}\, d(E_\lambda f, f).$$

Also, for $f \in F$,

(5) $$L_1 f = \lim_{N\to\infty} \int_{-N}^{N} \lambda\, dE_\lambda f,$$

in the sense of a pointwise limit and a strong limit in the norm $\| \ \|_1$.

Proof. Using Theorem 2 and the Spectral Theorem we need only prove that $E_\lambda \subset E_{1\lambda}$ and derive (2), (4), and (5).

For $f \in F$ we let $f_\mu = (E_\mu - E_{-\mu})f$. For $g_1 = (1 + L_1 L^*)g^* \in F_1$ we let $g_{1_\nu} = (E_{1_\nu} - E_{1_{-\nu}})g_1 = (1 + L_1 L^*)g_\nu^*$. Then $f_\mu \in \mathcal{D}(L^n)$ and we have

$$\int_{-\mu+0}^{\mu} \lambda^n\, d(E_\lambda f_\mu, g_\nu^*) = \int_{-\infty}^{+\infty} \lambda^n\, d(E_\lambda f_\mu, g_\nu^*) = (L^n f_\mu, g_\nu^*)$$

$$= (L_1^n f_\mu, g_{1_\nu})_1 = \int_{-\infty}^{+\infty} \lambda^n \, d(E_{1_\lambda} f_\mu, g_{1_\nu})_1 = \int_{-\nu+0}^{\nu} \lambda^n \, d(E_{1_\lambda} f_\mu, g_{1_\nu})_1.$$

By a classical device we may now approximate by polynomials the characteristic function of the interval from $-\mu-\nu$ to λ and obtain, in the limit, $(E_\lambda f_\mu, g_\nu^*) = (E_{1_\lambda} f_\mu, g_{1_\nu})_1$. As $\nu \to \infty$ we will have $\|g_{1_\nu} - g_1\|_1 \to 0$ and also $\|g_\nu^* - g^*\| \to 0$, since $(1+L_1L^*)^{-1}$ is a bounded transformation of F_1 into F (cf. Theorem III.1.1). Consequently, in the limit we obtain $(E_\lambda f_\mu, g_1)_1 = (E_\lambda f_\mu, g^*) = (E_{1_\lambda} f_\mu, g_1)_1$. Since $g_1 \varepsilon F_1$ is arbitrary, we have $E_\lambda f_\mu = E_{1_\lambda} f_\mu$. Since the f_μ fill out a dense subspace of F (as f and μ vary), we finally obtain $E_\lambda \subset E_{1_\lambda}$

From the general Spectral Theorem we have $B = \int \frac{1}{1+\lambda^2} dE_\lambda$, where $B = (1+L^2)^{-1}$. From this and from $(f,f)_1 = (Bf,f)$ formula (2) follows immediately.

By using $(E_{1_\lambda} f,f)_1 = (BE_\lambda f,f) = (E_\lambda Bf,f)$ and the Spectral Theorem for L_1 we obtain

$$\|L_1 f\|_1^2 = \int \lambda^2 \, d(E_{1_\lambda} f,f)_1 = \int \lambda^2 \, d(E_\lambda Bf,f)$$

$$= (L^2 Bf,f) = \int \frac{\lambda^2}{1+\lambda^2} d(E_\lambda f,f).$$

Thus (4) is established.

Finally, for any $f \varepsilon F$ we let $f_N = (E_N - E_{-N-0})f = (E_{1_N} - E_{1_{-N-0}})f$. Then $f_N \varepsilon \mathcal{D}(L)$ and $Lf_N = \int_{-N}^{N} \lambda \, dE_\lambda f$. Further, $\|L_1 f_N - L_1 f\|_1^2 \to 0$ (using the Spectral Theorem for L_1). Thus we have proved (5) in the sense of a strong limit in the norm $\| \ \|_1$. Pointwise convergence then follows from this, and the proof is complete.

We may also profit by using a resolution of the identity even in the case when $L_m \neq L_M$, as seen by the following theorem.

Theorem 4. Let $L = L_M$ be represented in F by a symmetric kernel Λ, let L_1 be given by formula (III.1.1), and let $S \subset L$ be any self adjoint restriction of L. If $\{E_\lambda\}$ is the resolution of the identity corresponding to S and if $f \in F$, then for each y $E_\lambda f(y)$ is of bounded variation in λ and

(6) $$f(y) = \int_{-\infty}^{+\infty} dE_\lambda f(y),$$

(7) $$L_1 f(y) = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda f(y).$$

Proof. For any resolution of the identity in F we have $E_\lambda f(y) = (E_\lambda f, K_y)$, and therefore $E_\lambda f(y)$ is of bounded variation in λ and (6) follows from $(f, K_y) = \int d(E_\lambda f, K_y)$.

For (7) we must use the fact that $\{E_\lambda\}$ corresponds to S. If $f \in \mathcal{R}(S)$ then $(E_\lambda S^{(-1)} f, \Lambda_y) = (S^{(-1)} E_\lambda f, \Lambda_y) = (E_\lambda f, S^{(-1)} \Lambda_y) = (E_\lambda f, K_y) = E_\lambda f(y)$ and

$$L_1 f(y) = (f, \Lambda_y) = (SS^{(-1)} f, \Lambda_y) = \int \lambda \, d(E_\lambda S^{(-1)} f, \Lambda_y) = \int \lambda \, dE_\lambda f(y).$$

For an arbitrary $f \in F$ we now define $f_\epsilon = (E_\epsilon - E_{-\epsilon-0}) f \in \mathcal{R}(S)$. Then $f_\epsilon \to f_0$, where $f - f_0 \in \dot{\mathcal{X}}(S) \subset \mathcal{X}(L)$, as $\epsilon \to 0$, and

$$L_1 f(y) = \lim L_1 f_\epsilon(y) = \lim \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty} \right) \lambda \, dE_\lambda f(y) = \int \lambda \, dE_\lambda f(y).$$

4. Spectra. The usual subdivision of the spectrum of an operator (e.g. in Stone [1]) into the point spectrum, the continuous spectrum, and the residual spectrum is a useful distinction between points of the complex plane and gives a partial description of the operator. However, it will be more useful to give a more detailed description of the spectrum as suggested by N. Aronszajn.

For a closed linear operator T with dense domain in a Hilbert space $\mathcal{H}$, we recall that the set of points λ for which $T - \lambda$ has an inverse defined on the whole of $\mathcal{H}$ is called the resolvent set of T. We also recall that the complementary set (in the complex plane) is called the spectrum of T. For every point λ in the complex plane we now define the quadruple of cardinal numbers (α, m, n, β), called the "<u>spectral value of λ</u>," as follows:

$$
(1)\qquad
\begin{aligned}
\alpha = \alpha(\lambda) &= \alpha(\lambda, T) = \begin{cases} 0 \text{ if } \mathcal{D}(T) \text{ is closed} \\ 1 \text{ otherwise.} \end{cases} \\
m = m(\lambda) &= m(\lambda, T) = \text{the dimension of } \mathcal{N}(T-\lambda). \\
n = n(\lambda) &= n(\lambda, T) = \text{the rank of } \mathcal{R}(T-\lambda). \\
\beta = \beta(\lambda) &= \beta(\lambda, T) = \begin{cases} 0 \text{ if } \mathcal{R}(T-\lambda) \text{ is closed} \\ 1 \text{ otherwise.} \end{cases}
\end{aligned}
$$

The number α can usually be omitted, since it merely signifies whether T is bounded or not, and the operator usually remains fixed throughout a discussion of the spectrum. We include it here only because we wish to compare the spectrum of an operator L with the spectrum of an extension L_1, and contrary to what we might expect α may actually change. Of course as λ varies $\alpha(\lambda)$ does not change.

The resolvent set is now the set of points having the spectral value $(\alpha, 0, 0, 0)$. The classical point spectrum corresponds to $m > 0$, the continuous spectrum to $(\alpha, 0, 0, 1)$, and the residual spectrum to $m = 0$ and $n > 0$. Instead of subdividing the spectrum only into these three sets (and possibly a multiplicity function), with the new notation we keep the continuous spectrum and split the other sets further into infinitely many smaller sets.

If (α, m, n, β) corresponds to the operator T and some point λ, then (α, n, m, β) will correspond to the adjoint operator T^* and the conjugate point $\overline{\lambda}$. That m and n are interchanged follows immediately from the relations $(T-\lambda)^* = T^* - \overline{\lambda}$ (cf. Nagy [1] ,p.29) and

$\mathcal{N}((T-\lambda)^*) = \mathcal{H} \ominus \mathcal{R}(T-\lambda)$. To prove that β is unchanged, i.e. that $\mathcal{R}(T-\lambda)$ is closed if and only if $\mathcal{R}((T-\lambda)^*)$ is closed, requires a slightly more sophisticated argument.

If we let $T' = T-\lambda$, then we may proceed as follows. A necessary and sufficient condition for $\mathcal{R}(T')$ to be closed is that the minimal inverse $T'^{\ominus 1}$ be bounded (cf. section I.3). In turn, $T'^{\ominus 1}$ will be bounded if and only if its adjoint $T'^{\ominus 1 *}$ is bounded. From Lemma 2 of section I.3 we have $T'^{\ominus 1 *} = T'^{* \ominus 1}$. Thus a necessary and sufficient condition for $\mathcal{R}(T')$ to be closed is that $\mathcal{R}(T'^*)$ is closed, and we have proved the preceding remark concerning $\mathcal{R}(T-\lambda)$ and $\mathcal{R}(T^*-\overline{\lambda})$.

Theorem 1. Let $L = L_M$ be the maximal operator represented by the kernel Λ (with $\Lambda_y \varepsilon F$), and suppose that its closure L_1 in the space F_1 exists (cf. Theorem III.1.2). Further, let (α, m, n, β) and $(\alpha_1, m_1, n_1, \beta_1)$ be the spectral value of a point λ (as defined in (1)) for the operators L and L_1 respectively. Then $\mathcal{N}(L_1^* - \overline{\lambda}) = \mathcal{N}(L^* - \overline{\lambda})$,

(a) $\alpha \geqslant \alpha_1$, (b) $m \leqslant m_1$, (c) $n = n_1$, and (d) $\beta \geqslant \beta_1$.

These relations are the best possible for each of the couples separately.

Proof. We may assume that $\lambda = 0$. In fact, if we replace L by $L-\lambda$ and construct the operator $(L-\lambda)_1$ we obtain (letting $\Lambda(T)$ be the kernel representing T): $(L-\lambda)_M = (L_M - \lambda) = L - \lambda$, $((L-\lambda)_M)^* = L^* - \overline{\lambda}$, $\Lambda(L-\lambda) = \Lambda(L) - \overline{\lambda} K$, and $(L-\lambda)_{1_F} = L_{1_F} - \lambda$. Also, for the r.k. class F_1 we get (letting $F_1(T) = F \dotplus T_1(F)$, the extended class F_1 corresponding to T) $F_1(L-\lambda) = F \dotplus (L_1 - \lambda)(F) = F + L_1(F) = F_1(L)$. The two norms in F_1, defined from L and from $L-\lambda$, will not be the same, but they will be equivalent (cf. (6) of section II.1). Since the spectral value of a point is the same under an equivalent norm, i.e. since the topology is not changed, a proof for $\lambda = 0$ will clearly suffice.

(a) simply states that L_1 is bounded if L is bounded, and this was already remarked in section III.1.

(b) is almost obvious, since $\mathfrak{N} = \mathfrak{N}(L) \subset \mathfrak{N}(L_1) = \mathfrak{N}_1$, but we must verify that the dimension $\dim \mathfrak{N}$ in the space F is the same as the dimension $\dim_1 \mathfrak{N}$ in the space F_1 (e.g. when $\mathfrak{N}$ has a transfinite dimension). The transformation carrying $f \in F$ into $f \in F_1$ is one-to-one and closed, and consequently Lemma I.3.1 tells us that these dimensions are the same.

For (c) we let $f_1 = (1 + L_1 L^*) f^*$ for an arbitrary $f_1 \in F_1$ (i.e. an arbitrary $f^* \in \mathcal{D}^*$). Then $L^* f^* = 0$ if and only if $f_1 \in \mathcal{D}_1^*$ and $L_1^* f_1 = 0$ (cf. Theorem III.1.2). Hence $\mathfrak{N}_1^* = (1 + L_1 L^*)(\mathfrak{N}^*) = \mathfrak{N}^*$ as announced prior to (a), and therefore $m(L^*) = m(L_1^*)$ or $n(L) = n(L_1)$ (again using Lemma I.3.1).

To prove (d) we use a previous remark that $\mathcal{R}$ is closed if and only if $\mathcal{R}^*$ is closed. We must prove that $\mathcal{R}$ is closed implies $\mathcal{R}_1$ is closed, and this is now equivalent to proving that $\mathcal{R}^*$ is closed implies $\mathcal{R}_1^*$ is closed. If $\mathcal{R}^*$ is closed, then $\mathcal{D}^* \mathcal{R}^*$ is complete with respect to the norm $(\|f\|^2 + \|L^* f\|^2)^{\frac{1}{2}}$, and consequently $\mathcal{R}_1^* = (1 + L_1 L^*)(\mathcal{D}^* \mathcal{R}^*)$ is closed, since $1 + L_1 L^*$ is an isomorphism of the graph of L^* onto F_1 (cf. remark preceding Theorem III.1.1).

To complete the proof of Theorem 1 we must show that separately these are the best inequalities possible. This is partially shown in Example 3 of section III.1. In this example L and L_1 have the spectral values (1,0,0,1) and (0,0,0,0), respectively, for $\lambda = 0$. This shows that (a) and (d) are the best possible. That (b) is the best possible will be shown by the examples at the end of this section.

<u>Remark</u>: When L is self adjoint we may say much more than is asserted in Theorem 1. In fact L and L_1 then have essentially the same resolution of the identity (cf. Theorem III.3.3) and it is easily verified that $(\alpha, m, n, \beta) = (\alpha_1, m_1, n_1, \beta_1)$.

<u>Theorem 2</u>. <u>Under the hypothesis of Theorem 1 we have the following. The resolvent operator $R_{1\lambda} = R_\lambda(L_1)$ exists if and only if $\overline{\mathcal{R}(L-\lambda)} = F$ and $\mathcal{D}^* \subset \mathcal{R}(L^* - \overline{\lambda})$. In particular $R_{1\lambda}$ exists whenever $R_\lambda = R_\lambda(L)$ exists, so that when passing from L to L_1 the resolvent set increases and the spectrum decreases. This increase and decrease may be proper</u>.

<u>Proof</u>. Again we may assume that $\lambda = 0$. The conditions that R_{1_0} exists are that $\mathcal{R}_1$ and $\mathcal{R}_1^*$ be dense in F_1 and one of them (and therefore both) be closed. From Theorem 1, part (c), and from $\mathcal{R}_1^* = (1 + L_1 L^*)(\mathcal{D}^* \mathcal{R}^*)$ (Theorem III.1.2), it follows that $\overline{\mathcal{R}_1} = \mathcal{R}_1^* = F$, if and only if $\overline{\mathcal{R}} = F$ and $\mathcal{D}^* \subset \mathcal{R}^*$. Also, if $\overline{\mathcal{R}} = \mathcal{R}^* = F$, i.e. if R_o exists, then R_{1_0} will exist. We now complete the proof by referring to Example 3 of section III.1, where L and L_1, for $\lambda = 0$, have the spectral values (1,0,0,1) and (0,0,0,0). Thus the increase and decrease may be proper.

In general it is not true that $m = m_1$, but from the last theorem we see that $m = m_1$ when $m = n = \beta = 0$. However this is not a necessary condition.

We have, in fact, $m = m_1$ if we weaken the condition to read $m = \beta = 0$. This is simply saying that $\mathcal{R}^* = F$ implies $\mathcal{D}^* \subset \mathcal{R}^*$, and therefore $\mathcal{R}_1^* = F_1$. An even strong result, requiring a slightly more complicated proof, is the following.

<u>Theorem 3</u>. <u>Under the hypothesis of Theorem 1, $\beta = 0$ implies $m = m_1$</u>.

Proof. We may again assume that $\lambda = 0$. From Theorem 1 we have $m \leq m_1$, and we must now prove that $m_1 \leq m$.

We have (from Theorem 1) $\beta = \beta_1 = 0$, and $\mathcal{R}^*$ and $\mathcal{R}_1^*$ are each closed. Also $m_1 = \dim \mathfrak{X}_1 = \dim(F_1 \ominus \mathcal{R}_1^*)$. The transformation $1 + L_1L^*$ is an isomorphism of $\mathcal{J}^* = \mathcal{J}(L^*)$ onto F_1 (cf. remark preceding Theorem III.1.1) and $\mathcal{R}_1^* = (1 + L_1L^*)(\mathcal{D}^* \mathcal{R}^*)$ (cf. Theorem III.1.2), so that $m_1 = \dim \mathcal{J}'$, where $\mathcal{J}' = \mathcal{J}^* \ominus \mathcal{J}(L^*_{\mathcal{D}^* \mathcal{R}^*})$.

We let P be the transformation of $\mathcal{J}'$ into F defined by $P\{f, L^*f\} = f$, and we let Q be the projection of F onto $\mathfrak{X}$. P is the restriction of a projection in the graph space $F \pm F$, and therefore QP is bounded. Also, QP has a closed domain $\mathcal{J}'$ and is thus closed. Further, QP is one-to-one, since $\{f, L^*f\} \in \mathcal{J}'$ and $QP\{f, L^*f\} = 0$ implies $f \in \mathcal{R}^*$, which means $f = 0$. Thus we may apply Lemma I.3.1 and obtain $m_1 = \dim \mathcal{J}' = \dim(QP(\mathcal{J}'))$. But $QP(\mathcal{J}') \subset \mathfrak{X}$, so that $m_1 \leq \dim \mathfrak{X} = m$, and the proof is complete.

Example 1. We shall now show that the nullspace may actually increase when the operator is extended. We shall make the construction in the sequential r.k. space by defining $\Lambda_j = L^*e_j$ (as explained in Example 1 of section III.1).

We let $L^*e_{2n-1} = ne_{2n}$ and $L^*e_{2n} = \frac{1}{n} e_{2n+1} (n \geq 1)$. Taking the linear closure we see that $\mathcal{D}^*$ is the set of functions f with nf(2n-1) (and of course f(2n)) a sequence in ℓ_2. The range $\mathcal{R}^*$ then consists of those functions in $\mathcal{D}^*$ with f(1) = 0. Thus we see that L_1 exists and is bounded (Theorem III.1.3, part (c)) and that $\alpha = 1$, $\alpha_1 = 0$.

The point of the spectrum that we wish to investigate this time is not the origin but the point $\lambda = 1$. In other words we are particularly interested in fixed points of L.

Taking the adjoint of L^* we see that $\mathcal{D}$ is characterized by the property that $nf(2n)$ is in ℓ_2. Further, $Le_1 = 0$, $Le_{2n} = ne_{2n-1}$, and $Le_{2n+1} = \frac{1}{n} e_{2n}$ for $n \geq 1$.

Letting $f \neq 0$ be a fixed point of L we must have $f(j) \neq 0$ for some j. By applying L successively we see that $f(i) \neq 0$ for all $i \leq j$, and thus we may normalize f by putting $f(1) = 1$. It then follows that $f(2) = f(3) = 1$, $f(4) = 1/2$, $f(5) = 1$, $f(6) = 1/3$, $f(7) = 1$, etc. Since $f(2n-1) = 1$ for all n, this element f is not in our space F and L has no fixed points.

However, $F_1 = F + L_1(F)$ is the set of functions f with $f(2n)$ and $\frac{1}{n}f(2n+1)$ in ℓ_2, and we see that our fixed point is in the space F_1. Therefore it is a fixed point of L_1, and for $\lambda = 1$ we have $m = 0$ and $m_1 = 1$.

In a similar manner we find that a fixed point of L^*, when $f(1) = 1$, must formally satisfy $f(2n) = n$. Therefore $1 - L^*$ is one-to-one, or $n = 0$ and (from Theorem 1) $n_1 = 0$.

We shall next show that $\beta_1 = 1$. We let $g = (1 - L_1)f$ for $f \varepsilon \mathcal{D}_1 = F_1$. Then we have

$$g(2n) = f(2n) - \tfrac{1}{n}f(2n+1),$$

$$g(2n-1) = f(2n-1) - nf(2n).$$

If $\mathcal{R}(1 - L_1) = F_1$, then there would be a solution $f \varepsilon F_1$ for $g(2n) = 0$ and $g(2n-1) = 1$. Multiplying the first equation by n and adding them we would obtain

$$1 = f(2n-1) - f(2n+1),$$

or $f(2n+1) = f(1) - n$, which is not an element in F_1. Thus $\mathcal{R}(1-L_1) \neq F_1$ and $\beta_1 = 1$ (since we already have $n_1 = 0$). Then $\beta = 1$ also (from

Theorem 1). Consequently we have established an example with the spectral values, at $\lambda = 1$, of (1,0,0,1) and (0,1,0,1) for L and L_1 respectively.

Example 2. In this example our sole aim is to exploit the result of Example 1. We first look at an auxiliary operator which will be useful.

We let $Te_1 = e_1$ and $Te_k = e_k + e_{k-1}$ for $k \geqslant 2$. This defines a bounded operator whose adjoint is given by $T^* e_k = e_k + e_{k+1}$ for $k \geqslant 1$. We see immediately that T-1 has a one-dimensional nullspace $[e_1]$ and that $\mathcal{R}(T-1) = F$. Thus for $\lambda = 1$ we have the spectral value (0,1,0,0).

We now turn to the actual construction of the operator of our interest. Let m,n, and p be arbitrary cardinal numbers. Take m copies of the sequential space, each with the operator T constructed in the last paragraph. Take n more copies of the sequential space with the operator T^*, and finally p copies of the sequential space with the operator constructed in Example 1.

Then by taking the combinatorial orthogonal sum (the "Cartesian product" of Nagy [1] --- cf. section I.1) of these $m+n+p$ spaces and operators we get an operator L in an r.k. space F. This space F consists of functions f defined on a set $\mathcal{E} = \bigcup \mathcal{E}_\alpha$, where the sets $\mathcal{E}_\alpha$ are $m+n+p$ disjoint copies of the set of natural numbers. Among all these functions the class F consists of those f with $\|f\|^2 = \sum_{x \in \mathcal{E}} |f(x)|^2 < \infty$. The domain $\mathcal{D} = \mathcal{D}(L)$ is the set of functions $f \in F$ such that the restriction f_α to $\mathcal{E}_\alpha$ is in $\mathcal{D}(L_\alpha)$ and such that $Lf = \{L_\alpha f_\alpha\}$ is in F. Further, the adjoint operator L^* is simply the combinatorial orthogonal sum of the adjoints L_α^*.

An examination of the operator $L-1$ now shows that it is unbounded, it has an m-dimensional nullspace, and its range is not closed and of rank n. Thus L has the spectral value $(1, m, n, 1)$ for $\lambda = 1$.

For the extended operator we find that F_1 is the combinatorial orthogonal sum of the spaces F_{α_1} and that L_1 is the combinatorial orthogonal sum of the operators L_{α_1}. Consequently $L_1 - 1$ is bounded (since the L_α are uniformly bounded), has an $(m+p)$- dimensional nullspace, and has a non-closed range. Consequently L_1 has the spectral value $(0, m+p, n, 1)$ for $\lambda = 1$, and the example is complete.

5. Carleman integral operators. We shall now continue our discussion of Carleman integral operators that we began in section II.8. Our notation will be that introduced in section II.8.

From equation (II.8.6) it is clear that $T_1(\mathcal{L}_2) \subset \mathcal{L}_1(B)$ for each $B \in \mathcal{E}$, and we may extend the linear transformation φ to the domain $\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)$ by the same equation (II.8.4). For any $f \in \mathcal{L}_2$ we then have

$$(1) \qquad (\varphi T_1 f)(B) = \int_B T_1 f(x) d\mu_x = \int_B \left(\int H(x,y) f(y)\, d\mu_y\right) d\mu_x$$

$$= \int f(y) \left(\int_B H(x,y)\, d\mu_x\right) d\mu_y = (\varphi f, \Lambda_B).$$

Thus we see that L is a maximal operator in the r.k. space $F = \varphi(\mathcal{L}_2)$ and agrees with our convention in sections III.1 and III.3. From (1) it is now apparent that $\varphi(\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)) = F + L_1(F) = F_1$, and that the extension L_{1_F} is the operator induced by T_1 (and by φ). We may now apply the results of section III.1 and state the following.

The linear transformation $1 + T_1 T^*$ (where T_1 is defined by (II.8.1) and T^* is the adjoint of T) is defined on $\mathcal{D}(T^*)$, has range $\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)$, and has an inverse. If in $\mathcal{L}_2$ we denote the original norm by

$\|f\|^2 = \int |f|^2 \, d\mu$, then we can introduce a new norm

(2) $$\|f\|_1^2 = ((1+T_1T^*)^{-1}f, f).$$

With respect to this new norm $\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)$ is a completion of $\mathcal{L}_2$, and the extended norm is given by

(3) $$\|f_1\|_1^2 = \|(1+T_1T^*)^{-1}f_1\|^2 + \|T^*(1+T_1T^*)^{-1}f_1\|^2$$

$$= \min\,(\|f\|^2 + \|g\|^2),$$

where the minimum is taken over all decompositions $f_1 = f + T_1 g$ with $f, g \in \mathcal{L}_2$.

In order that the operator T_1 have a self adjoint closure in the extended space $\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)$ it is necessary and sufficient that T be self adjoint in $\mathcal{L}_2$ (cf. Theorem III.3.2, part (a)). Kernels H(x,y) of this type, where the operator T is self adjoint, Carleman [1] has called kernels of Class I , all others being of Class II.[1)] Thus for Carleman kernels of Class I we may also apply the results of Theorem III.3.3 as follows.

If H(x,y) is of Class I, then the operator T is self adjoint in $\mathcal{L}_2$, and the operator T_1, with domain $\mathcal{L}_2$, is self adjoint in $\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)$ (with the norm given by (3)). If $\{E_\lambda\}$ is the resolution of the identity for T (in $\mathcal{L}_2$), then the resolution of the identity (in $\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)$) for T_1 is given by $\{E_{1\lambda}\}$, where $E_{1\lambda}$ is the closure in $\mathcal{L}_2 \dotplus T_1(\mathcal{L}_2)$ of E_λ . Then we also have

$$\|f\|^2 = \int d(E_\lambda f, f) \qquad f \in \mathcal{L}_2,$$

---

1) Carleman considered only real symmetric kernels. We have made the natural extension of his Class I and Class II to include complex (Hermitian) symmetric kernels, as did Stone [1].

$$\|f\|_1^2 = \int \frac{1}{1+\lambda^2}\, d(E_\lambda f,f) \qquad f \in \mathcal{L}_2,$$

$$\|Tf\|^2 = \int \lambda^2\, d(E_\lambda f,f) \qquad f \in \mathcal{D}(T),$$

$$\|T_1 f\|_1^2 = \int \frac{\lambda^2}{1+\lambda^2}\, d(E_\lambda f,f) \qquad f \in \mathcal{L}_2.$$

The above paragraph gives us information only about kernels of Class I. We shall now prove a final result that is true for any Carleman kernel such that T has a self adjoint restriction S (e.g. any real kernel). We then have $T^* \subset S \subset T$ and a resolution of the identity $\{E_\lambda\}$ corresponding to S. We shall make use of the fact that functions in the range of T_1 are defined everywhere (and not merely almost everywhere) by equation (II.8.1). (In fact $\mathcal{R}(T_1)$ and $\mathcal{R}(T)$ are r.k. spaces with the norms $\|T_1 f\|' = \|f\|$ and $\|Tf\|''^2 = \|f\|^2 + \|Tf\|^2$ respectively, where $f \perp \mathcal{N}(T)$.) For any function g in the range of T_1, its (well determined) values are given by $g(x) = (f,H_x)$, where f is any element of $\mathcal{L}_2$ with $T_1 f = g$.

We let f be an arbitrary element in $\mathcal{L}_2$. For $\lambda < 0$, $E_\lambda f \in \mathcal{R}(S) \subset \mathcal{R}(T)$ and $E_\lambda f(x)$ is defined everywhere. For $\lambda > 0$, $(1-E_\lambda)f \in \mathcal{R}(S) \in \mathcal{R}(T)$ and $(E_\lambda - 1)f(x)$ is defined everywhere. Hence we may consider $\Delta E_\lambda f(x)$ to be defined everywhere except for intervals (λ', λ'') containing the origin. For $\epsilon, \eta > 0$ we let $f_{\epsilon,\eta} = f - (E_{\eta-0} - E_{-\epsilon})f$. Then $f_{\epsilon,\eta} \in \mathcal{R}(S)$ and

$$(f_{\epsilon,\eta}, H_x) = (SS^{-1} f_{\epsilon,\eta}, H_x) = \int_{-\infty}^{+\infty} \lambda\, d(E_\lambda S^{-1} f_{\epsilon,\eta}, H_x)$$

$$= \left(\int_{-\infty}^{-\epsilon} + \int_{\eta}^{+\infty}\right) \lambda\, d(S^{-1} E_\lambda f_{\epsilon,\eta}, H_x)$$

$$= \left(\int_{-\infty}^{-\epsilon} + \int_{\eta}^{+\infty}\right) \lambda\, dE_\lambda f(x).$$

As $\epsilon \to 0$ and $\eta \to 0$ we have $f_{\epsilon,\eta} \to f_0$ where $f - f_0 \in \mathfrak{X}(S) \subset \mathfrak{X}(T)$. Thus $(f_{\epsilon,\eta}, H_x) \to (f_0, H_x) = (f, H_x) = T_1 f(x)$, and we have finally

$$T_1 f(x) = \lim_{\substack{\epsilon \to 0 \\ \eta \to 0}} \left(\int_{-\infty}^{-\epsilon} + \int_{\eta}^{+\infty} \right) \lambda \, dE_\lambda f(x)$$

for every point x and every $f \varepsilon \mathcal{L}_2$. The result corresponds to Theorem III.3.4 in r.k. spaces.

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