ON THE HAAR FUNCTIONS

by

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INTRODUCTION

In his Götingen dissertation [5], published in 1910, Alfred Haar constructed a system of orthonormal functions which are defined by, and known as Haar functions,

\[
\chi_i(t) = \begin{cases} 
1 & (0 \leq t \leq \frac{1}{2}) \\
-1 & \left(\frac{1}{2} < t \leq 1\right) \\
0 & (t = \frac{1}{2}) 
\end{cases}
\]

\[
\chi^{(l)}_{nj}(t) = \begin{cases} 
\sqrt{2^{n-1}} & (0 \leq t < \frac{1}{2^n}) \\
\sqrt{2^{n-1}} & \left(\frac{1}{2^n} < t < \frac{2}{2^n}\right) \\
-\frac{\sqrt{2^{n-1}}}{2} & (t = \frac{2}{2^n}) \\
0 & \text{everywhere else in } [0,1] 
\end{cases}
\]

\[
\chi^{(k)}_{nj}(t) = \begin{cases} 
\sqrt{2^{n-1}} & \left(\frac{2k-2}{2^n} < t < \frac{2k-1}{2^n}\right) \\
\sqrt{2^{n-1}} & \left(\frac{2k-1}{2^n} < t < \frac{2k}{2^n}\right)
\end{cases}
\]

1 Numbers in square brackets refer to the bibliography at the end of the paper.
\[
\chi_n^{(2^{n-1})}(t) = \begin{cases} 
\sqrt{2^{n-1}} / 2 & (t = \frac{2k-2}{2^n} ) \\
- \sqrt{2^{n-1}} / 2 & (t = \frac{2k}{2^n} ) \\
0 & \text{everywhere else in } [0,1] 
\end{cases}
\]

for \( k = 2, 3, \ldots, 2^{n-1}-1, \)

\[
\chi_n^{(2^{n-1})}(t) = \begin{cases} 
+ \sqrt{2^{n-1}} & (\frac{2^{n-2}}{2^n} < t < \frac{2^{n-1}}{2^n} ) \\
- \sqrt{2^{n-1}} & (\frac{2^{n-1}}{2^n} < t < 1 ) \\
\sqrt{2^{n-1}} / 2 & (t = \frac{2^{n-2}}{2^n} ) \\
0 & \text{everywhere else in } [0,1] 
\end{cases}
\]

This system of functions has a very interesting property that if a function \( f(x) \) is Lebesgue integrable, then its Fourier expansion in terms of this system converges almost everywhere to \( f(x) \). In particular, if the function \( f(x) \) is continuous everywhere then its Haar Fourier expansion converges to \( f(x) \) everywhere (here Haar Fourier expansion means the Fourier expansion in terms of Haar functions). In the same year, Faber pointed out that if a function is continuous everywhere in \([0,1]\) except at a point \( X_0 \), where it makes a finite jump,
then its Haar Fourier expansion at $x_0$ converges to $\frac{1}{2} [f(x_0^+) + f(x_0^-)]$, in case that $x_0$ is a dyadic rational, and diverges otherwise. The Haar functions did not receive the attention which they deserve. To our best knowledge, practically no paper concerning them was published in the period 1910-1922. Then appeared two papers by Walsh in 1923, in one of which [15], he introduced a new system of orthonormal functions which is closely related to the Haar system and since then known as Walsh functions.

We observe that although the Haar Fourier expansion of a continuous function represents the function, yet the Haar functions themselves are discontinuous. The question which naturally arises is whether there exists an orthonormal system of continuous functions in that the expansion of any continuous function represents the function everywhere. This has been affirmatively answered by Philip Franklin [3],

A very interesting property of Haar functions was discovered by Schauder [12] namely, if $f(x)$ is $\alpha$th summable $(\alpha \geq 1)$, so is

$$\lim_{n \to \infty} \int_0^1 |f(x) - \sum_{i=1}^{\alpha} \sum_{p=1}^{2^{n-1}} a_n (p) \lambda^n (p)(x)|^\alpha dx = 0,$$

where $\{a_n (p)\}$ are the Fourier coefficients. Concerning Schauder's result as well as the relationship between the
Haar functions and spaces of vector valued functions, Gelbaum [4] made some investigations recently. To our best knowledge, these are the only main papers published concerning Haar functions besides some occasional mention of them as counter examples; in this respect we refer to a recent paper by Rudin [10].

It is the purpose of this paper to make a systematic study of Haar functions in the traditional manner. First of all, we try to extend the class of functions which admit a uniformly convergent Haar Fourier expansion. This extension has gone not very far, our main results are contained in Theorems 1.3 and 1.4, and may be stated as follows. Let \( f(x) \) be a function continuous everywhere in \([0,1]\), except on a set \( B \) of dyadic rationals where \( f(x) \) makes a finite jump, and at the point of discontinuity \( x_0 \), \( f(x) \) is defined to be

\[
\frac{1}{2}[f(x_0^+) + f(x_0^-)]
\]

Let the derived set of \( B \) be denoted by \( B' \). Then the Haar Fourier expansion converges uniformly to \( f(x) \) if either \( B \cap B' = \emptyset \), or \( B' \) is finite. Next we take up the summability problem. By making use of a theorem of Raikov [9], we prove that there exists a set \( B \) of measure zero such that for any function \( f(x) \) being discontinuous of first kind at \( x_0 \), its Haar Fourier expansion is summable \((C,1)\) to

\[
\frac{1}{2}[f(x_0^+) + f(x_0^-)] \quad \text{if} \quad x_0 \notin B.
\]

Along the familiar track we come now to the considera-
tion of Fourier coefficients. We find that the rapidity of convergence of the sequence of Fourier coefficients does not increase when the functions considered have derivatives of higher order. This striking dissimilarity to Fourier series has been shown also by Walsh functions [2]. The problem of absolute convergence is treated in Chapter III. Here we are able to prove some theorems similar to those for Fourier series, without assuming that the function is of bounded variation.

The technique used in Chapter IV to investigate the uniqueness problem is the same one used by Fine for Walsh functions [2]. This is so, because we are naturally led to this way by analyzing the classical technique employed in Fourier case (pp. 275-278, [16]). Among other results we prove that if \( f(x) \) is Lebesgue integrable and associated the Haar Fourier expansion \( A_0 \chi(x) + \sum_{n=1}^{\infty} \sum_{p=1}^{2^{n-1}} a_n^{(p)} \chi_n^{(p)}(x) \), then the series

\[
A_0 \int_0^x \chi_0(t) dt + \sum_{n=1}^{\infty} \sum_{p=1}^{2^{n-1}} a_n^{(p)} \int_0^x \chi_n^{(p)}(t) dt
\]

is \((C, 1)\) summable to \( \int_0^x f(t) dt \). We are unable to prove a stronger result which holds for Fourier case, namely instead of \((C, 1)\) summability, the integrated series converges to \( \int_0^x f(t) dt \).

An analogy, both formal and analytic, between the Haar
Fourier expansion and an interpolating formula in terms of Haar functions has been carried out in the last chapter. Actually it is this property which makes the author interested in the orthonormal system of Haar.
Chapter I

Convergence of Haar Fourier Series.

1.1. Dirichlet kernel. Every function \( f(x) \) which is integrable in the sense of Lebesgue on \([0,1]\) will have associated with it a Haar Fourier series

\[
(1.1) \quad f(x) \sim A_0 \chi_0(x) + A_1 \chi_1(x) + \ldots + A_n \chi_n(x) + \ldots + \chi_{n(p)}(x) + \ldots ,
\]

where the coefficients are given by

\[
(1.2) \quad A_m^{(q)} = \int_0^1 f(x) \chi_m^{(q)}(x) dx \quad m = 0, 1, 2, \ldots , \quad q = 1, 2, \ldots , 2^{m-1}.
\]

We shall set

\[
(1.3) \quad S_n(x) = A_0 \chi_0(x) + \ldots + A_n \chi_n(x) + \ldots + A_n \chi_{n(p)}(x) =
\]

\[
= \int_0^1 \{ \chi_0(x) \chi_0(t) + \ldots + \chi_n(x) \chi_n(t) + \ldots + \chi_{n(p)}(x) \chi_{n(p)}(t) \} f(t) dt
\]

\[
= \int_0^1 K_n(x, t) f(t) dt,
\]
where \( K^{(p)}_{n}(x, t) = \chi_{n}(x) \chi_{n}(t) + \ldots + \chi_{n}(x) \chi_{n}(t). \) We call \( K^{(p)}_{n}(x, t) \) the Dirichlet kernel. It plays an important role in the study of convergence of Haar Fourier series, and we propose to make a detailed study.

We observe that

\[
K_{0}(x, t) = \chi_{o}(x) \chi_{o}(t) = 1 \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1.
\]

![Fig. 1. \( K_{0}(x, t) \).](image)

\[
K_{1}(x, t) = \chi_{o}(x) \chi_{o}(t) + \chi_{1}(x) \chi_{1}(t) = K_{0}(x, t) + \chi_{1}(x) \chi_{1}(t).
\]

We readily see that \( \chi_{1}(x) \chi_{1}(t) \) is defined as follows,

\[
\chi_{1}(x) \chi_{1}(t) = \begin{cases} 
1.1 = 1 & 0 \leq x < \frac{1}{2}, \quad 0 \leq t < \frac{1}{2} \\
1.(-1) = -1 & 0 \leq x < \frac{1}{2}, \quad \frac{1}{2} \leq t \leq 1 \\
(-1).1 = -1 & \frac{1}{2} \leq x \leq 1, \quad 0 \leq t < \frac{1}{2} \\
(-1).(-1) = 1 & \frac{1}{2} \leq x \leq 1, \quad \frac{1}{2} \leq t \leq 1
\end{cases}
\]

\[
\chi_{1}(\frac{1}{2}) \chi_{1}(t) = \chi_{1}(x) \chi_{1}(\frac{1}{2}) = 0, \text{ for all possible } x \text{ or possible } t.
\]
Fig. 2. \( \chi(x) \chi(t) \).

Imposing Fig. 2 upon Fig. 1, we obtain

Fig. 2'. \( K_1(x, t) \),

which defines \( K_1(x, t) \).

Now suppose that \( K_n \) \((2^{n-1})\) \((x, t)\) is defined as follows,

Fig. \((n+1)'\). \( K_n \) \((2^{n-1})\) \((x, t)\).

We observe that \( \chi_{n+1}(x) \chi_{n+1}(t) \) is defined as follows,
Fig. (n + 2)\(^1\). \(\chi_{n+1}(x)\chi_{n+1}(t)\).

Imposing Fig. (n+2)\(^1\) upon Fig. (n+1)\(^1\) we again obtain

\[
(1)
\]

which defines \(K_{n+1}(x, t)\). It is almost evident that for any positive integers \(m\) and \(q\) such that \(1 \leq q < 2^{m-1}\), we have

\[
K_m^{(q)}(x, t) \quad \text{and} \quad K_m^{(2^{m-1})}(x, t)
\]

defined as follows,
Now we are ready to prove

\textbf{Lemma 1.1} \quad \int_{0}^{1} K_{n}^{(p)}(x,t)dx = 1 \quad \text{for all } t,

\quad n = 1, 2, \ldots, \text{ and}

\quad p = 1, 2, \ldots, 2^{n-1}. 
Proof. Making use of the explicit representation of 
\(K_n^p(x,t)\) we are able to prove the lemma by direct evaluation of the integral. However this will make the proof quite lengthy, and the classical method is more elegant. We always have

\[ f(t) \sim A_0 K_0(t) + \sum_{n=1}^{\infty} \sum_{p=1}^{2^{n-1}} A_n^{(p)} K_n^{(p)}(t), \]

and \(S_n^{(p)}(t) = \int_0^1 f(x)K_n^{(p)}(x,t)dx.\)

In case that \(f(x) \equiv 1\), we have \(A_0 = 1\), \(A_m^{(q)} = 0\) for \(m = 1, 2, \ldots\), and \(q = 1, 2, \ldots, 2^{n-1}\). Thus we obtain \(S_n^{(p)}(t) \equiv 1\) and consequently the lemma,

\[ 1 = \int_0^1 K_n^{(p)}(x,t)dx. \]

1.2. Convergence of Haar Fourier series. We propose to give a new proof of the fundamental theorem that if a function \(f(x)\) is continuous everywhere in \([0, 1]\) then its Haar Fourier series converges uniformly to the function in \([0, 1]\). There are two reasons of doing this. First of all, our "new" proof is based on an "old" technique. This will bring out a closer comparison between Haar functions and
classical orthonormal systems. Secondly the new proof is more suggestive for further study.

Theorem 1.1. Let \( f(x) \) be continuous on \([0,1]\) and associated the Haar Fourier series

\[
f(x) \sim A_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{p=1}^{2^{n-1}} A_n \chi_n(x).
\]

Its partial sum is denoted by

\[
S_n^{(p)}(x) = \int_0^1 f(t)K_n^{(p)}(x,t)dt.
\]

Then \( S_n^{(p)}(x) \) converges uniformly to \( f(x) \) in \([0,1]\).

Proof. From Lemma 1.1, it follows that

\[
f(x) = \int_0^1 f(x)K_n^{(p)}(x,t)dt.
\]

Thus we observe that

\[
(1.4) \quad |f(x) - S_n^{(p)}(x)| \leq \int_0^1 |f(x) - f(t)| K_n^{(p)}(x,t)dt.
\]

From the diagrammatic representation of \( K_n^{(p)} \) we observe that for any \( x \) (fixed) there exists a closed interval \( I_n \) such that \( x \in I_n \) and \( K_n^{(p)}(x,t) \) vanishes when \( t \notin I_n \). Thus (1.4) is reduced to

\[
|f(x) - S_n^{(p)}(x)| \leq \int_{I_n} |f(x) - f(t)| K_n^{(p)}(x,t)dt.
\]
By the uniform continuity of $f(x)$, for any given $\varepsilon > 0$, we can make $n$ so large such that $|f(t) - f(x)| < \varepsilon$ whenever $x, t$ both belong to $I_n$. Thus we obtain

$$\left| f(x) - S_n^{(p)}(x) \right| \leq \varepsilon \int_{I_n} K_n^{(p)}(x, t) dt = \varepsilon.$$ 

This completes the proof.

Theorem 1.2. Let $f(x)$ be continuous everywhere in $[0,1]$ except at $X_0$ which is a point of discontinuity of first kind. Then (i) if $X_0$ is a dyadic rational, its Haar Fourier series converges to $\frac{1}{2}[f(X_0^+) + f(X_0^-)]$ at $X_0$, and uniformly to $f(x)$ in $[0, X_0) \cup (X_0, 1]$; (ii) if $X_0$ is a dyadic irrational, its Haar Fourier series diverges at $X_0$, and converges at every point of $[0, X_0) \cup (X_0, 1]$, but not necessarily uniformly.

Proof. We define the following auxiliary functions,

$$\varphi_1(x) = \begin{cases} f(x) & 0 \leq x < X_0 \\ f(X_0^-) & X_0 \leq x \leq 1 \end{cases}, \quad \varphi_2(x) = \begin{cases} f(X_0^+) & 0 \leq x \leq X_0 \\ f(x) & X_0 < x \leq 1 \end{cases}.$$ 

The so defined functions $\varphi_1(x)$ and $\varphi_2(x)$ are continuous everywhere on $[0, 1]$. By the previous theorem their Haar Fourier series converge to them respectively. We shall set

$$\lambda_n^{(p)}(X_0) = \int_0^{X_0} K_n^{(p)}(X_0, t) dt.$$

We observe that the partial sum $S_n^{(p)}(X_0)$ of the series
associated with \( f(x) \) evaluated at \( x_0 \) can be written in the following,

\[
(1.6) \quad S_n^{(p)}(x_0) = \int_{-\infty}^{1} f(t)K_n^{(p)}(x_0,t)dt = \\
\int_{-\infty}^{x_0} + \int_{x_0}^{1} f(t)K_n^{(p)}(x_0,t)dt = \\
= \int_{-\infty}^{x_0} \varphi_1(t)K_n^{(p)}(x_0,t)dt + \int_{x_0}^{1} \varphi_2(t)K_n^{(p)}(x_0,t)dt = \\
= \int_{-\infty}^{1} \varphi_1(t)K_n^{(p)}(x_0,t)dt - \int_{-\infty}^{x_0} f(x_0^-)K_n^{(p)}(x_0,t)dt + \\
+ \int_{x_0}^{1} \varphi_2(t)K_n^{(p)}(x_0,t)dt - \int_{x_0}^{1} f(x_0 +)K_n^{(p)}(x_0,t)dt = \\
= \int_{-\infty}^{1} \varphi_1(t)K_n^{(p)}(x_0,t)dt - \int_{-\infty}^{1} f(x_0^-)K_n^{(p)}(x_0,t)dt + \\
+ \int_{-\infty}^{x_0} f(x_0 -)K_n^{(p)}(x_0,t)dt + \int_{x_0}^{1} \varphi_2(t)K_n^{(p)}(x_0,t)dt \\
- \int_{x_0}^{1} f(x_0 +)K_n^{(p)}(x_0,t)dt = \\
= \int_{-\infty}^{1} [\varphi_1(t) + \varphi_2(t)]K_n^{(p)}(x_0,t)dt - f(x_0 -) + \\
+ \int_{-\infty}^{x_0} [f(x_0 -) - f(x_0 +)]K_n^{(p)}(x_0,t)dt =
\]
By the continuity (hence uniform continuity) on \([0, 1]\) of \(\mathcal{F}_1(x)\) and \(\mathcal{F}_2(x)\), \(\int_0^1 \left[ \mathcal{F}_1(t) + \mathcal{F}_2(t) \right] K_n^{(p)}(x_0, t) \, dt \) converges to \(f(x_0^-) + f(x_0^+)\). Thus the problem of convergence of \(S_n^{(p)}(X_0)\) is reduced to the study of convergence of \(\lambda_n^{(p)}(x_0)\).

If \(X_0\) is a dyadic rational say \(\frac{i}{2^j}\) where \(i\) is odd, then

we observe that

\[
\lambda^{(1)}_{j+1} \left( \frac{2i}{2^j+1} \right) = \int_0^{2^{j+1}} K^{(1)}_{j+1} \left( \frac{2i}{2^j+1}, t \right) \, dt = \int_0^{2^{j+1}} 2^j \, dt = \frac{1}{2^j+1}.
\]
$$\lambda^{(i+1)}_{j+1} (\frac{21}{2j+1}) = \int_{0}^{2^{j+1}} K^{(i+1)}_{j+1} \left( \frac{21}{2j+1}, t \right) dt = \int_{0}^{2^{j+1}} 2^{j} dt = \frac{1}{2},$$

$$X_0 = \frac{21}{2j+1},$$

$$K^{(i+1)}_{j+1} (X, t),$$

$$\lambda^{h}_{j+k} \left( \frac{21}{2j+1} \right) = \int_{0}^{2^{j+k}} K^{(h)}_{j+k} \left( \frac{21}{2j+1}, t \right) dt = 2^{j+k-2} \frac{1}{2^{j+k-1}} = \frac{1}{2},$$

$$k>1, 2h<2^{k},$$

$$X_0 = \frac{21}{2j+k+1},$$

$$K^{(h)}_{j+k} (X, t),$$

$$\lambda^{h}_{j+k} \left( \frac{21}{2j+k+1} \right) = \int_{0}^{2^{j+k}} K^{(h)}_{j+k} \left( \frac{21}{2j+k+1}, t \right) dt = 2^{j+k-2} \frac{1}{2^{j+k-1}} = \frac{1}{2},$$

$$k>1, 2h<2^{k},$$
Thus the sequence \( \{ \lambda_{n}^{(p)}(X_{0}) \} \) converges to \( \frac{1}{2} \), and consequently the first part of (i) is proved. We call a dyadic rational interval any interval \( I_{n} : (k/2^{n}, k+1/2^{n}) \). Where \( k \) is any positive integer such that \( k+1 \leq 2^{n} \), and \( n = 1, 2, \ldots \).

Consider any point \( x \) which belongs to, say \( (X_{0}, 1] \). If \( X_{0} \) is a dyadic rational say \( 1/2^{j} \), then for any \( n \geq j \), \( f(x) \) is uniformly continuous in any \( I_{n} \). By the same reasoning employed in the previous theorem we prove the uniform convergence on \( (X_{0}, 1] \). Similarly for \( [0, X_{0}) \), thus we obtain the second part of (i).

For (ii), it is sufficient to show the divergence of \( \{ \lambda_{n}^{(p)}(X_{0}) \} \) when \( X_{0} \) is a dyadic irrational. First of all, we observe that

\[
\lambda_{n}^{(p)}(X_{0}) = \int_{0}^{X_{0}} \lambda_{n}^{(p)}(x, t)dt = \int_{0}^{X_{0}} 2^{n}dx = 2^{n}(X_{0} - \frac{pn}{2^{n}}),
\]

\[k > 1, \ 2h \geq 2i,\]
when \( x_0 < \frac{2p}{2^n} \), and \( p_n \) is the greatest positive integer such that \( \frac{p_n}{2^n} < x_0 \),

\[
\lambda_n^{(p)}(x_0) = \int_0^{x_0} K_n(x_n, t) \, dt = \int_{\frac{p_n-1}{2^{n-1}}}^{\frac{x_0}{2^{n-1}}} \frac{p_n-1}{2^{n-1}} \cdot \left( x_0 - \frac{p_n-1}{2^{n-1}} \right),
\]

when \( x_0 > \frac{2p}{2^n} \), and \( p_{n-1} \) is the greatest positive integer such that \( \frac{p_{n-1}}{2^{n-1}} < x_0 \). Thus the sequence \( \left\{ \lambda_n^{(p)}(x_0) \right\} \) for \( n = 0, 1, 2, \ldots \), \( p = 1, 2, \ldots, 2^{n-1} \) is in the following:

\[ x_0; \ldots; 2^{n-1}(x_0 - \frac{p_{n-1}}{2^{n-1}}); \ldots; 2^{n-1}(x_0 - \frac{p_{n-1}}{2^{n-1}}); \]
Thus the problem is furthermore reduced to the study of the sequence $x_0, 2(x_0 - p_1/2), \ldots, 2^n(x_0 - p_n/2^n), \ldots$. We denote $2^n(x_0 - p_n/2^n)$ by $\eta_n$.

If $x_0$ is a rational, say $x_0 = \frac{a}{b}$, where $b \neq 2^m$ for any positive integer $m$, then $2^n(x_0 - p_n/2^n) = \frac{2^n a - b p_n}{b}$.

$2^n a - b p_n$ being integers; if the sequence $\{2^n a - b p_n\}$ converges we must have, for sufficiently large $m$ and $n$,

$$2^n a - b p_n = 2^m a - b p_m.$$

This gives $a = \frac{p_n - p_m}{b} \cdot \frac{2^n - 2^m}{2^n - 2^m}$, in particular, for $n = m+1$, $a = \frac{p_m + 1 - p_m}{2^m}$. This contradicts the dyadic irrationality of $x_0$.

Thus the sequence $\{2^n a - b p_n\}$ diverges, so does $\{\lambda_n(p)(x_0)\}$.

Let $x_0$ be any irrational, first of all, we observe that

$$0 < 2^n(x_0 - p_n/2^n) < 2^n(p + 1/2^m - p_m/2^n) = 1.$$}

We also observe that $m \neq n$ implies $2^n(x_0 - p_m/2^m) \neq 2^n(x_0 - p_n/2^n)$. 

Furthermore, we see that

i) \[ p_{n+1} = 2p_n \quad \text{when} \quad \frac{p_n}{2^n} < x_0 < \frac{2p_n+1}{2^{n+1}} , \]

\[ 0 \quad \frac{p_n}{2^n} = x_0 \quad \frac{2p_n+1}{2^{n+1}} \]

\[ = \frac{2p_n}{2^{n+1}} = \frac{p_{n+1}}{2^{n+1}} \quad \text{by def. of } p_{n+1}. \]

ii) \[ p_{n+1} = 2p_n + 1 \quad \text{when} \quad \frac{2p_n+1}{2^{n+1}} < x_0 < \frac{2p_n+2}{2^{n+1}} , \]

\[ 0 \quad \frac{p_n}{2^n} = \frac{2p_n}{2^{n+1}} \quad \frac{2p_n+1}{2^{n+1}} = x_0 \quad \frac{p_{n+1}}{2^{n+1}} = \frac{2p_n+2}{2^{n+1}} \]

\[ = p_{n+1}/2^{n+1} \quad \text{(by def.)} \]

iii) \[ p_n = p_{n+1} \quad \text{when} \quad p_n = 0 \quad \text{and} \quad x_0 < \frac{1}{2^{n+1}} \]

These three properties have the following three consequences,

i) \[ |2^n x_0 - p_{n+1} - 2^n x_0 + p_n| = |2^n x_0 - p_n| = 2^n x_0 - p_n , \]
Thus we conclude that \( \eta_n \) has the following three properties,

1) \( 0 < \eta_n < 1 \), for all \( n \),

2) \( \eta_n \neq \eta_m \), when \( n \neq m \),

3) For any \( n \), we have

\[
|\eta_{n+1} - \eta_n| = \begin{cases} 
\text{either } \eta_n \\
1 - \eta_n 
\end{cases}
\]

Now we suppose that \( \eta_n \to \eta \). By 1) we have that \( 0 \leq \eta \leq 1 \).

First of all, we wish to show that \( \eta \neq 0 \) and \( \eta \neq 1 \).

\( \alpha \) Let \( \eta = 0 \), we take \( \varepsilon = \frac{1}{4} \), the assumption of convergence implies the existence of an \( N \) such that

\[
\eta_n < \frac{1}{4} \quad \text{for all } n \geq N.
\]

Thus, for \( n \geq N \), we have that \( |\eta_{n+1} - \eta_n| \neq 1 - \eta_n \) because \( 1 - \eta_n \geq 3/4 \).

Thus by condition 3) we have \( |\eta_{n+1} - \eta_n| = \eta_n \). This will imply that \( \eta_{N+1} = 2\eta_N \), consequently \( \eta_{N+p} = 2^p \eta_N \). Thus for sufficiently large \( p \), \( \eta_{N+p} > 1 \). This leads to a contradiction.

\( \beta \) Let \( \eta = 1 \), \( \varepsilon = \frac{1}{4} \). Again the convergence will imply
the existence of an \( N \) such that, for \( n \geq N \),

\[
|1 - \eta_n| < \frac{3}{4}.
\]

This will imply that \( \eta_n > \frac{3}{4} \), whenever \( n \geq N \). Thus by 3), we have \( |\eta_{n+1} - \eta_n| \neq \eta_n \) and therefore \( |\eta_{n+1} - \eta_n| = 1 - \eta_n \). Since \( \eta_{n+1} \neq 1 \), we obtain that \( \eta_{n+1} = 2\eta_n - 1 \). This recursive formula enables us to obtain \( \eta_{N+p} = 2^p(\eta_N - 1) + 1 \). Since \( \eta_N - 1 < 0 \), for sufficiently large \( p \), we will have \( \eta_{N+p} < 0 \). This leads to a contradiction.

\( \gamma \) Let \( 0 < \eta < 1 \), \( \epsilon = \eta/4 \). The convergence will assure us the existence of \( N \) such that, whenever \( n \geq N \), we have

\[
|n - n| \leq \eta/4.
\]

It follows immediately that \( 3\eta \leq 5 \eta \). If

\[
|n_{n+1} - n_n| = n_n,
\]

this will imply that either \( n_{n+1} = 0 \) or \( n_{n+1} = 2^n \eta \). Both cases are impossible. If \( |n_{n+1} - n_n| = 1 - \eta_n \),

this will imply either \( n_{n+1} = 1 \) (this is impossible), or \( n_{n+1} = 2^n - 1 \). The second case will lead us also to a contradiction as we have seen already in \( \beta \).

This proves the divergence of \( \eta_n \), consequently the first part of (ii).

Let \( x \in \text{say} \ (X_0, 1) \). There exists an integer \( N \), such that whenever \( n \geq N \), the dyadic rational interval, \( I_n \):
[k/2^n, k+1/2^n], containing x, will not contain \( x_0 \). Thus \( f(t) \) is uniformly continuous in \( I_n \). Hence

\[
\left| f(x) - S_n(x) \right| < \int_{I_n} \left| f(x) - f(t) \right| K_n(x, t) dt < \varepsilon
\]

preassigned \( \varepsilon \).

This proves the convergence of the series at any point of the interval \((x_0, 1]\). Similarly for \([0, x_0)\).

Now we suppose the uniform convergence over say \((x_0, 1]\), i.e., for any \( \varepsilon > 0 \), the existence of an \( N \) such that

\[
\left| f(x) - S_n^{(p)}(x) \right| < \varepsilon
\]

whenever \( n \geq N \) and for all \( x \in (x_0, 1] \). Let \( \frac{1}{2^n} \left| f(x_0^+) - f(x_0^-) \right| \) be the smallest integer such that \( x_0 < \frac{p_{N+1}}{2^{N+1}} \), and \( x^1 \) be any point such that \( x_0 < x^1 < \frac{N+1}{2^{N+1}} \). Then

we have \( \frac{p}{N+1} - \frac{1}{2^{N+1}} < x_0 < x^1 < \frac{p_{N+1}}{2^{N+1}} \). Hence we see readily that \( \left| f(x^1) - S_{N+1}^{(p)}(x^1) \right| \leq \varepsilon \), e.g. for the step function \( f(x) \) which makes a finite jump at a dyadic irrational.

This completes the proof.
We can generalize our theorem as follows,

**Theorem 1.3.** Let $f(x)$ be a function continuous everywhere in $[0, 1]$ except on a set $B$ of dyadic rationals where $f(x)$ makes a finite jump, and at the point of discontinuity $f(x)$ is defined to be

$$\frac{1}{2} [f(x^+) + f(x^-)].$$

Furthermore, let us denote by $B'$ the derived set of $B$. If $B \cap B' = 0$, then the Haar Fourier series converges uniformly to $f(x)$.

**Remark.** We have imposed quite few conditions on the function which we are interested. We would like to ask whether these conditions are consistent as well as independent. The affirmative answers can be shown by the following examples.

**Example 1.** Consider

$$f(x) = \begin{cases} 0 & \text{for } x = 1 \\ 1 & \text{for } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{for } \frac{1}{2} < x < \frac{3}{4} \\ \frac{1}{2^2} & \text{for } \frac{3}{4} < x < \frac{7}{8} \\ \vdots & \text{for } \vdots \\ \vdots & \text{for } \vdots 
\end{cases}$$
\[
\frac{2^n - 1}{2^{n-1}} = \frac{1}{2} \left[ \frac{1}{2^{n-1}} + \frac{1}{2^n} \right].
\]

Thus this function makes a finite jump at infinitely many dyadic rationals. \(B = \left\{ \frac{1}{2^n} \mid n = 1, 2, \ldots \right\}\), \(B^1 = \{1\}\), and \(B \cap B^1 = 0\). This shows that our conditions are consistent.

Example 2. Let

\[f(x) = \begin{cases} 
1 & (0 \leq x < \frac{1}{4}) \\
-\frac{1}{2} & (\frac{1}{2} \leq x < \frac{3}{4}) \\
\frac{1}{4} & (\frac{3}{8} \leq x < \frac{7}{16}) \\
\end{cases}\]

Evidently \(\frac{1}{2} \in B^1\) and \(\frac{3}{8} \in B\). This shows that our conditions are not dependent.

Proof of the theorem. The main idea lies on the fact that the whole unit interval can be divided into a finite number of open intervals in each of which the series is uniformly convergent to the function. Once this division is performed, the theorem follows easily.
Let \( b' \in B' \). By the assumption that \( B \cap B' = \emptyset \), \( b' \) is a point of continuity. Thus for a given \( \epsilon > 0 \), there exists an interval \( I' = (b' - \frac{1}{2^m}, b' + \frac{1}{2^m}) \) such that for every \( x_1, x_2 \in I' \) we have

\[
|f(x_1) - f(x_2)| \leq |f(x_1) - f(b')| + |f(b') - f(x_2)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Thus for sufficiently large \( n \), there exists a subinterval

\[
I = (p - \frac{1}{2^n}, p + \frac{1}{2^n})
\]

such that \( b \in I \subseteq I' \), and \( p \) is an integer. Consequently we have

\[
\left| f(x) - S_n^{(p)}(t, x) \right| \leq \int_0^1 K_n^{(p)}(x, t) \left| f(x) - f(t) \right| dt \leq \epsilon \text{ for all } x \in I.
\]

Thus to each point \( b' \in B' \), there corresponds an open subinterval \( I = (p - \frac{1}{2^n}, p + \frac{1}{2^n}) \) on which the series converges uniformly to the function. Since \( B' \) is bounded and closed, by Borel's covering theorem, a finite number of these intervals will cover \( B' \), call them \( I_1, I_2, \ldots, I_n \).
Outside these intervals, there are only a finite number of points of $B$. Because suppose that there are infinitely many, then there will be a limiting point, say $q$. But this point $q$ must belong to one of these intervals, say $I_j = (p_j - 1/2^n, p_j/2^n)$.

Let $\varepsilon = \min \left\{ \frac{1}{2^n} (p_j - q), \frac{1}{2} (q - \frac{p_j - 1}{n_j}) \right\}$, then the interval $(q - \varepsilon, q + \varepsilon) \subset I_j$, contains no points of $B - \bigcup_{j=1}^n I_j$.

This leads to a contradiction. Call these finitely many points $b_1, b_2, \ldots, b_k$. These $k$ points will divide the set $S = [0, 1] - \bigcup_{j=1}^n I_j$ into a finite number of subintervals $J_1, J_2, \ldots, J_m$ in each of which the function $f(x)$ is continuous and hence the series is uniformly convergent to the function.

Thus for any given $\varepsilon > 0$, we have

$$\left| S_n^{(p)}(x) - f(x) \right| < \varepsilon,$$

for all $x \in I_i (J_j)$, whenever $n > N_i (n_j)$, $i = 1, 2, \ldots, n$ ($j = 1, 2, \ldots, m$). Let $q_{I_i} (q_{J_j})$ be the left end point of the interval $I_i (J_j)$ we have also
whenever $n \geq NqI_i$. There are only a finite number of such $N$'s. Let $N$ be the greatest one, we have

$$\left| S_n^{(p)}(q_{I_i}) - \frac{1}{2} \left[ f(q_{I_i}^+) + f(q_{I_i}^-) \right] \right| < \varepsilon,$$

for $x \in [0, 1]$, whenever $n \geq N$.

This completes the proof.

Remark. The condition $B \cap B' = 0$ is certainly sufficient for the uniform convergence. But in case that $B'$ is infinite, whether $B \cap B' = 0$ is a necessary condition is still left open. For the convenience of the following discussion we make a definition.

Definition. By a $g$-function we understand a function $f(x)$ which is continuous everywhere except on a set $B$ of dyadic rationals where it makes a finite jump and is defined to be $\frac{1}{2} \left[ f(x^+) + f(x^-) \right]$. The set $B$ of dyadic rationals is called a $B$-set.

Theorem 1.4. Let $f(x)$ be a $g$-function and $B'$ be the derived set of the $B$-set. If $B'$ is finite, then the Haar Fourier series of $f(x)$ converges uniformly to $f(x)$. 
Proof. For simplicity we suppose that \( B^t \) contains only one point \( q \). If it is a point of continuity, nothing left for proof. If it is not a point of continuity, we consider the following auxiliary functions

\[
\varphi_1(x) = \begin{cases} 
  f(q+) & 0 \leq x \leq q \\
  f(x) & q < x \leq 1,
\end{cases}
\]

\[
\varphi_2(x) = \begin{cases} 
  f(x) & 0 \leq x < q \\
  f(q-) & q < x \leq 1.
\end{cases}
\]

Thus both \( \varphi_1(x) \) and \( \varphi_2(x) \) are continuous at \( q \), consequently their sum is also continuous at \( q \). By the theorem \( \varphi_1(x) + \varphi_2(x) \) admits a uniformly convergent Haar Fourier series. Now consider

\[
\psi(x) = \begin{cases} 
  f(q+) & 0 \leq x < q \\
  -f(q-) & q < x \leq 1 \\
  \frac{1}{2}[f(q^-) + f(q^+)] & x = q
\end{cases}
\]

\( q \), being a point of discontinuity, is a dyadic rational. Thus \( \psi(x) \) admits a Haar Fourier series which converges uniformly to the function itself. We obtain that

\[
f(x) = \psi(x) + \varphi_1(x) + \varphi_2(x).
\]

Hence the Haar Fourier series converges uniformly to \( f(x) \).
This method can certainly be extended to the case when \( B' \) is finite.

This completes the proof.

A sort of converse of the above theorems can be easily proved.

**Theorem 1.5.** Let \( S(x) = a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{p=1}^{2^{n-1}} a_n \chi_n(x) \) be a uniformly convergent Haar series, then i) \( S(x) \) is continuous at every dyadic irrational; ii) \( S(x_0^+) \) and \( S(x_0^-) \) exist, and \( S(x_0) = \frac{1}{2} [S(x_0^+) + S(x_0^-)] \).

**Proof.** The proof is exactly the same as that of the limiting function of a uniformly convergent series of continuous functions is continuous.

Let \( x_0 \) be a dyadic irrational and \( \epsilon > 0 \), we wish to find a \( \delta > 0 \) such that \( |S(x_0) - S(x)| < \epsilon \) whenever \( |x - x_0| < \delta \). We write

\[
S(x) = a_0 \chi_0(x) + \sum_{n=1}^{m} \sum_{p=1}^{2} a_n \chi_n(x) + R_m(x).
\]

Thus we obtain

\[
|S(x_0) - S(x)| \leq |a_0| \left| \chi_0(x_0) - \chi_0(x) \right| + \sum_{n=1}^{m} \sum_{p=1}^{2} |a_n| \left| \chi_n(x_0) - \chi_n(x) \right| + |R_m(x_0)| + |R_m(x)|.
\]
By uniform convergence, we can choose $m$ sufficiently large so that both $R_m(x_0)$ and $R_m(x)$ are less than $\varepsilon/2$. With this $m$ there exists a $p_m$ such that $(p_m-1)/2^m < x_0 < p_m/2^m$. If we choose $\delta = \min \left\{ (p_m/2^m - x_0), (x_0 - (p_m-1)/2^m) \right\}$ then we have

$$\chi_m^{(p)}(x_0) - \chi_m^{(p)}(x) = 0 \quad \text{for} \quad n \leq m, \ \text{all} \ p, \ \text{if} \quad |x-x_0| < \delta.$$

This proves the continuity of $S(x)$ at $x_0$.

To prove the second half of the theorem, let $x_0$ be a dyadic rational, (for $x_0$ to be a dyadic irrational we already proved the continuity, and hence this assertion) we wish to prove that $S(x_0 +)$ exists, i.e. for any sequence of points $x_n$ such that $x_n > x_0$ and $x_n \to x_0$, then $\{S(x_n)\}$ converges. We observe that

$$|S(x_i) - S(x_j)| \leq \sum_{m=1}^{k} \sum_{p=1}^{2^{m-1}} A_m \left| \chi_m^{(p)}(x_i) - \chi_m^{(p)}(x_j) \right| + \left| R_k(x_i) \right| + \left| R_k(x_j) \right|.$$

By uniform convergence we can make the last two terms as small as we please. By the assumptions that $x_0$ is a dyadic rational and $x_i > x_0$, $x_j > x_0$, if, for the already fixed $k$, we make $i, j$ sufficiently large then

$$\chi_m^{(p)}(x_i) - \chi_m^{(p)}(x_j) = 0 \quad \text{for} \quad m = 1, 2, \ldots, k.$$

This proves the existence of $S(x_0 +)$, similarly for $S(x_0 -)$. 
It remains to prove that \( S(X_0) = \frac{1}{2} [S(X_0^+) + S(X_0^-)] \).

To do this, let \( \{X_j\} \) be any sequence of points such that \( X_j > X_0 \) and \( X_j \to X_0 \), and \( \{X_j\} \) be one such that \( X_0 > X_j \) and \( X_j \to X_0 \). We observe that

\[
A_0 \chi_0(X_0) + \sum_{m=1}^{\infty} \sum_{p=1}^{2^m} A_m \chi_m(X_0) = \frac{1}{2} \left\{ A_0 \chi_0(X_1) + \sum_{m=1}^{\infty} \sum_{p=1}^{2^m} A_m \chi_m(X_j) + \right.
\]

\[+ A_0 \chi_0(X_j) + \sum_{m=1}^{\infty} \sum_{p=1}^{2^m} A_m \chi_m(X_j) \right\} \leq \frac{q}{2} \sum_{m=1}^{2^m} A_m \chi_m(X_j). \]

\[
\left| \chi_m(X_0) - \frac{1}{2} \left[ \chi_m(X_1) + \chi_m(X_j) \right] \right| + \left| Rq(X_0) \right| +
\]

\[+ \frac{3}{2} \left| Rq(X_1) \right| + \frac{3}{2} \left| Rq(X_j) \right|. \]

By choosing \( q \) sufficiently large, we can make the last three terms as small as we please. With the fixed \( q \), let \( X_0 = \frac{a}{2^b} \).

If \( X_1 \) and \( X_j \) are sufficiently close to \( X_0 \), then \( \chi_n(X_0) = \chi_n(X_1) = \chi_n(X_j) \) for \( n < b \), hence \( \chi_n(X_0) = \chi_n(X_1) + \chi_n(X_j) \) is zero. \( \chi_b(X_0) = 0, \chi_b(X_1) = -\sqrt{2-1}, \chi_b(X_j) = \sqrt{2-1} \), hence we have also \( \chi_b(X_0) = \).
\[-\frac{1}{2} \left\{ \chi_{b}^{(q)}(x_1) + \chi_{b}^{(q)}(x_j) \right\} = 0. \] For \( b < n \), \( \chi_{n}^{(q)}(x_0) = -\frac{1}{2} \sqrt{2}^{n-1} \), \( \chi_{n}^{(q)}(x_1) = \frac{1}{2} \sqrt{2}^{n-1} \), \( \chi_{n}^{(q)}(x_1) = 0 \), \( \chi_{n}^{(q+1)}(x_1) = \sqrt{2}^{n-1} \), \( \chi_{n}^{(q)}(x_j) = -\sqrt{2}^{n-1} \), \( \chi_{n}^{(q+1)}(x_j) = 0 \), thus we have \( \chi_{n}^{(q)}(x_0) = \frac{1}{2} \left\{ \chi_{n}^{(q)}(x_1) + \chi_{n}^{(q)}(x_j) \right\} = 0 \), \( \chi_{n}^{(q+1)}(x_1) = \frac{1}{2} \left\{ \chi_{n}^{(q+1)}(x_1) + \chi_{n}^{(q+1)}(x_j) \right\} \).

Thus the first term is dropped. This proves that

\[ \mathcal{S}(x_0) = \frac{1}{2} \left[ \mathcal{S}(x_0^+) + \mathcal{S}(x_0^-) \right]. \]

This completes the proof.

1.3. A criterion of continuity of Haar Fourier series.

We have already seen that not only continuous function but also \( g \)-function with some restrictions about \( B \)-set admits a uniformly convergent Haar Fourier series. Now the question is, given a uniformly convergent Haar series, how to tell if its limiting function is continuous.

We recall Wiener's theorem on functions of bounded variation. Let \( f \) be a function of bounded variation, \( a_n, b_n \) its Fourier coefficients, and \( \int_{n}^{2} = a_n^2 + b_n^2 \), then a
necessary and sufficient condition that \( f \) be continuous is that \( a_n = (\int_1 + 2\int_2 + \ldots + n\int_n) / n \to 0 \). For Haar series any theorem such as this is certainly false. Consider a Haar series which contains only a finite number of terms, the coefficients go to zero as fast as we like, yet it represents a discontinuous function.

Theorem 1.6 Let \( \{ f_n \} \) be a sequence of functions defined on \([0, 1]\). \( f_n \) is continuous everywhere on \([0, 1]\) except on a set \( B_n \) (denumerable or not). If \( \{ f_n \} \) converges uniformly to a function \( f(x) \), and \( \lim_{n} \inf B_n = 0 \), then \( f(x) \) is continuous everywhere.

Proof. Let \( x_0 \in [0, 1] \), we observe that

\[
|f(x_0 + h) - f(x_0)| \leq |f(x_0 + h) - f_n(x_0 + h)| + |f_n(x_0 + h) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.
\]

By uniform convergence we can choose \( n \) sufficiently large so that the first and last term on the right are as small as we please. If \( x_0 \notin \bigcup_{n=1}^{\infty} B_n \), then \( f_n(x) \) (for all \( n \)) is continuous at \( x_0 \), hence the mid-term can be made as small as we please by choosing \( h \) small enough. If \( x_0 \) belongs to \( B_n \) for certain \( n \)'s, then by the assumption that \( \lim \inf B_n = 0 \), we are able
to choose a sufficiently large \( n \) such that \( f_m(x), m > n, \) is continuous at \( X_0. \) Hence a sufficiently small \( h \) will make the mid-term sufficiently small.

This completes the proof.

Remark. We see readily that \( \lim \inf B_n = 0 \) is not a necessary condition from the following example.

\[
\begin{cases}
\frac{1}{n} & 0 \leq x < \frac{1}{2} \\
-\frac{1}{n} & \frac{1}{2} < x \leq 1 \\
0 & x = \frac{1}{2}
\end{cases}
\]

It is evident that \( f_n(x) \) converge uniformly to zero, \( f(x) \) is continuous, the point \( \frac{1}{2} \) belongs to the set \( \lim \inf B_n. \)

In order to make a further study, we introduce a notation. Let \( f_n(x) \) be a function continuous everywhere except on a set \( B_n \) where it makes a finite jump. We denote by \( J[f_n(p)] \) the jump made by \( f_n \) at the point \( p, \) namely

\[
|f_n(p^+) - f_n(p^-)|.
\]

Theorem 1.7. Let \( \{f_n(x)\} \) be a sequence of functions such that \( f_n(x) \) is continuous everywhere except on a set \( B_n \) where it makes a finite jump. When \( x \in B_n \) we define \( f(x) \) to be any value \( k_n \) such that \( \min(f_n(x^+), f_n(x^-)) < k_n < \max(f_n(x^+), f_n(x^-)) \),
\( f_n(x) \) converge uniformly to \( f(x) \). Then \( f(x) \) is continuous at a point \( p \) if and only if \( J[f_n(p)] \to 0 \) as \( n \to \infty \).

Proof. Suppose that \( J[f_n(p)] \to 0 \). We have always that

\[
|f(p) - f(p+\delta')| \leq |f(p) - f_n(p)| + |f_n(p) - f_n(p+\delta')| + |f_n(p+\delta') - f_n(p+\delta')|.
\]

For any given \( \varepsilon > 0 \), by the uniform convergence, there exists an \( N \) such that, for \( n \geq N \), the first and last term are less than \( \varepsilon/4 \) respectively. By the assumption that \( J[f_n(p)] \to 0 \), there exists an \( N' \) such that, for \( n \geq N' \), \( J[f_n(p)] = |f_n(p+\delta) - f_n(p-\delta)| < \varepsilon/4 \). Now let \( M = \max\{N, N'\} \), with this \( M \), we can choose a \( \delta' \) so small that \( |f_n(p+\delta') - f(p+\delta')| < \varepsilon/4 \) whenever \( 0 \leq \delta' \leq \delta \). Thus we obtain

\[
|f_n(p+\delta') - f(p+\delta')| \leq |f_n(p+\delta') - f_n(p)| + |f_n(p) - f(p+\delta')| \leq J[f_n(p)] + \varepsilon/4 < \varepsilon/2,
\]

whenever \( 0 \leq \delta' \leq \delta \). This proves the continuity of \( f(x) \) at \( p \).

Conversely we suppose that \( f(x) \) is continuous at \( p \), we wish to prove that \( J[f_n(p)] \to 0 \). We have

\[
J[f_n(p)] = |f_n(p+\delta) - f_n(p-\delta)| \leq |f_n(p+\delta) - f_n(p)| + |f_n(p) - f_n(p+\delta)| + |f_n(p+\delta) - f(p+\delta)| + |f(p+\delta) - f(p-\delta)| + |f(p-\delta) - f_n(p-\delta)| + |f_n(p-\delta) - f_n(p-\delta)|.
\]
By the uniform convergence, for any \( \epsilon > 0 \), there exists \( N \) such that the second and fourth term are less than \( \epsilon / 5 \) for \( n > N \). For each \( n > N \) we can choose a \( \delta \) such that the rest terms are less than \( \epsilon / 5 \) respectively.

This completes the proof.

Theorem 1.8. Let \( S(x) = A_0 \chi(x) + \sum_{n=1}^{\infty} \sum_{p=1}^{2} A_n \chi_n(x) \) be a uniformly convergent Haar series, and \( (2q-1)/2^m \) be a dyadic rational point in \([0, 1]\). The necessary and sufficient condition that \( S(x) \) is continuous at \( (2q-1)/2^m \) is that the sequence \( \{K_p, L_p\} \) converges to zero where

\[
K_p = \sqrt{\frac{m+1}{2}} A_m + \sum_{k=0}^{p-2} \left\{ \frac{k}{2} \frac{2(2q-1)}{m+k+1} + \frac{k}{2} \frac{2(2q-1)+1}{m+k+1} \right\} \sqrt{\frac{m+k-2}{2}} - \frac{p-1}{2} \frac{2(2q-1)}{m+p} \sqrt{\frac{m+k-3}{2}},
\]

\[
L_p = \sqrt{\frac{m+1}{2}} A_m + \sum_{k=0}^{p-1} \left\{ \frac{k}{2} \frac{2(2q-1)}{m+k+1} + \frac{k}{2} \frac{2(2q-1)+1}{m+k+1} \right\} \sqrt{\frac{m+k-2}{2}}.
\]

Proof. By the previous theorem, it is sufficient to prove that \( \{K_p, L_p\} \) is the sequence of jumps made by the partial sums of our series. We observe that the partial sums...
up to the one \[ \sum_{n=0}^{m-1} \sum_{p=1}^{2^n-1} A_n \chi_n(x) + \sum_{p=1}^{q-1} A_m \chi_m(x) \]
are continuous at \( \frac{2q-1}{2^m} \). Then the function \( \chi_m(x) \) introduces the first jump of the height \( \sqrt{2^m A_m} \), this jump is maintained until the function \( \chi^{(2q-1)}_{m+1}(x) \) comes in with a new jump \(-\sqrt{2^m A_m}\). The very next function \( \chi^{(2q)}_{m+1}(x) \) introduces a third jump \( \sqrt{2^m A_m} \). Thus we can see easily that

the jump of the partial sum \( A_0 \chi(x) + \sum_{n=1}^{m-1} \sum_{i=1}^{2^n} A_n \chi_{n}(x) + \sum_{i=1}^{m+p-1} \sum_{m+p\setminus m+p} A \chi_{(2q-1)}(x) \) is \( K_p \) and the jump of the partial sum \( A_0 \chi(x) + \sum_{n=1}^{m-1} \sum_{i=1}^{2^n} A_n \chi_{n}(x) + \sum_{i=1}^{p-1} A \chi_{(2q-1)+1}(x) \) is \( L_p \). Thus the theorem follows easily from the previous one.

1.4. Degree of convergence of Haar Fourier series.

All the theorems of this section admit easy proofs and
familiar forms.

Theorem 1.9. Let \( f(x) \) be a function defined on \([0, 1]\) and satisfying a Lipschitz condition of order \( \alpha' \), i.e.

\[
|f(x_1) - f(x_2)| < \lambda |x_1 - x_2|^{\alpha'} \quad 0 < \alpha' 
\]

and \( S_n(x) \) be the partial sum of its Haar Fourier series. Then for all values of \( x \), we have

\[
|f(x) - S_n^{(p)}(x)| \leq \frac{\lambda}{n^{\alpha'}}.
\]

Proof. We have

\[
f(x) = \int_0^1 f(x) K_n(x,t) dt
\]

and

\[
S_n^{(p)}(x) = \int_0^1 f(t) K_n^{(p)}(x,t) dt.
\]

Thus we see that

\[
|f(x) - S_n^{(p)}(x)| \leq \int_0^1 |f(x) - f(t)| K_n^{(p)}(x,t) dt = \int \frac{1}{2^n} |f(x) - f(t)| K_n^{(p)}(x,t) dt,
\]

where \( i \) is such that \( \frac{i-1}{2^n} \leq x \leq \frac{i}{2^n} \). By Lipchitz condition, we have

\[
|f(x) - S_n^{(p)}(x)| \leq \lambda \int \frac{1}{2^n} |x - t|^\alpha K_n^{(p)}(x,t) dt \leq \lambda \frac{1}{2^{n\alpha'}}.
\]
This completes the proof.

Similarly, we have the following.

Theorem 1.10. If \( f(x) \) is a function with a modulus of continuity \( \omega(\delta) \), then for all values of \( x \) we have

\[
\left| f(x) - S_n(x) \right| \leq \omega\left(\frac{1}{2^n}\right).
\]

The proof can be made word for word as it has been done previously and hence is omitted.
Fourier Coefficients

In studying the Fourier expansion of a function in terms of orthonormal functions, the asymptotic properties of the coefficients or its average order often play an important role. Usually the rapidity of convergence of the coefficient gives some information about the convergence of the series. When we deal with Haar functions the first sad thing we encounter is the following.

2.1. A counter example. In the theory of Fourier series we have the celebrated theorem of Riemann-Lebesgue which states that the Fourier coefficients of an integrable function are of the order o(1). We will give an example to show that this is no longer true for Haar function.

Let

\[ f(x) = 2 \chi_2(x) + 3 \chi_3(x) + \ldots + n \chi_n^{(2^{-1})}(x) + \ldots \]

The series on the right hand side converges for every \( x \) in \([0, 1]\) and hence defines the function \( f(x) \), in particular, we observe that \( f(1) = 0 \). \( f(x) \) being the limiting function of a sequence of measurable functions is measurable, it remains to prove that it is integrable.
Let $E_1 = \bigcup_{n=2}^{\infty} \left[ \frac{2^n-4}{2^n}, \frac{2^n-3}{2^n} \right]$; $f(x) \geq 0$ when $\{f(x)\}_m = \begin{cases} f(x) & \text{when } x \in E_1 \text{ and } f(x) \leq m \\ 0 & \text{when } x \in E_1 \text{ and } f(x) > m \end{cases}$. and there exists an integer $k$ such that $k \sqrt{2} \leq m \leq (k+1) \sqrt{2}$.

Then we have

\[
\int_{E_1} f(x) \, dx = \lim_{m \to \infty} \int_{E_1} \{f(x)\}_m \, dx = \lim_{k \to \infty} \int_{\bigcup_{n=2}^{\infty} \left[ \frac{2^n-4}{2^n}, \frac{2^n-3}{2^n} \right]} f(x) \, dx
\]

\[
= \lim_{k \to \infty} \sum_{n=2}^{2} \frac{2^n-3}{2^n} \frac{2^n-1}{2^n} \, dx = \lim_{k \to \infty} \sum_{n=2}^{k} \frac{1}{2^n} \frac{1}{2^n} < +\infty.
\]

Similarly we have, when $f(x) < 0$, $x \in E_2 = \bigcup_{n=2}^{\infty} \left( \frac{2^n-3}{2^n}, \frac{2^n-2}{2^n} \right)$,

\[
\int_{E_2} f(x) \, dx = - \lim_{k \to \infty} \sum_{n=2}^{k} \frac{1}{2^n} \sqrt{2^n} \sqrt{2^n} \to \text{converges}.
\]

Thus $f(x)$ is integrable. Suppose that $A_n$'s are the Fourier coefficients. Then
\[ A_n^{(p)} = \int_0^1 f(x) \chi_n^{(p)}(x) \, dx = \int_0^{2p-2} \frac{2p}{2^n} f(x) \chi_n^{(p)}(x) \, dx = \]

\[ = \int_0^{\frac{2^{n-2}}{2^n}} n \left[ \chi_n^{(2^n-1)}(x) \right]^2 \, dx = 2^{n-1} \cdot n \frac{1}{2^{n-1}} = n, \]

if \( p = 2^{n-1} - 1, \)

because in this interval \( f(x) = n \chi_n^{(2^n-1)}(x), \)

\[ A_n^{(k)} = 0, \quad \text{otherwise by the orthogonality}. \]

Thus the series in (2.1) is the Haar Fourier series of its limiting function, with unbounded coefficients. This shows that there is no analogy of Riemann Lebesgue theorem for Haar functions. The interesting point is that \( f(x) \) is so defined that after the integrability is established no term by term integration is involved, the series reduced to one term only in the specified interval where the coefficient is evaluated. This example also serves to show that the everywhere convergence of Haar Fourier series does not guarantee the boundedness of its coefficients.
2.2. **Integrated Haar functions.** The trigonometric functions or polynomials have a strong point which may be called closure under integration. We often make a good use of it, for instance, in the theory of Fourier series, when we estimate the asymptotic properties of Fourier coefficients or degree of convergence of the series of a function having smooth derivatives up to a certain order. After we integrate Haar functions once we obtain functions which are continuous, uniformly bounded, but they are no longer step functions and even not orthogonal. This is not surprising because they are all non-negative. However, they still resemble Haar functions in one respect, namely they vanish everywhere except in those subintervals where the original Haar functions assume values different from zero. But if we integrate twice, we lose this property too.

Let

\[ L_n(x) = \int_0^x \chi_n(t) \, dt = \begin{cases} 
0 & 0 \leq x \leq \frac{2n-2}{2^n} \\
\sqrt{\frac{n-1}{2^n}}(x - \frac{2p-2}{2^n}) & \frac{2p-2}{2^n} \leq x \leq \frac{2p-1}{2^n} \\
\sqrt{\frac{n-1}{2^n}} \left( \frac{2p}{2^n} - x \right) & \frac{2p-1}{2^n} \leq x \leq \frac{2p}{2^n} \\
0 & \frac{2p}{2^n} \leq x \leq 1
\end{cases} \]
$L_n^{(p)}(x)$ is continuous and hence admits a uniformly convergent Haar Fourier series.

$$L_n^{(p)}(x) = \frac{\sqrt{2^{n-1}}}{2^n} + \sum_{m=1}^{n-1} p_m(\pm 1) \frac{\sqrt{2^{m-2}}}{2^n} \chi_m^{(q'm)}(x) +$$

$$(2.2) + \sum_{m=n+1}^{\infty} \left\{ -\frac{\sqrt{2^{m+n-2}}}{2^{2m}} \frac{k}{2^{p-2}} \frac{k-1}{q=2^{k_p-(2k-1)}} \chi_m^{(q)}(x) +

+ \frac{\sqrt{2^{m-2}}}{2^{2m}} \frac{k}{2^{p-2}} \frac{2^{2k_p-(2k-1)-1}}{q=2^{k_p-(2k-1)}} \chi_m^{(q)}(x) \right\},$$

where $k = m-n$, $q'm$ is such that $\frac{2^{m-2}}{2^m} \leq \frac{2p-2}{2^n} < \frac{2p}{2^n} \leq \frac{2^{q'm}}{2^m}$,
and $p_m(\pm 1) = \begin{cases} 1 & \text{if } \frac{2q_m-2}{2^m} \leq \frac{2p-2}{2n} < \frac{2p}{2n} \leq \frac{2q_m-1}{2^m} \\ -1 & \text{if } \frac{2q_m-1}{2^m} \leq \frac{2p-2}{2n} < \frac{2p}{2n} \leq \frac{2q_m}{2^m} \end{cases}$.

The verification involves a straightforward calculation and hence is omitted.

Let

$K_n(x) = \int_0^x L_n(t) dt = \begin{cases} 0 & 0 \leq x \leq \frac{2p-2}{2^n} \\ \frac{\sqrt{n-1}}{2} \left( x - \frac{2p-2}{2^n} \right)^2 & \frac{2p-2}{2^n} \leq x \leq \frac{2p-1}{2^n} \\ \frac{\sqrt{n-1}}{2} \left[ \frac{2}{2^{2n}} - (x - \frac{2p}{2^n})^2 \right] & \frac{2p-1}{2^n} \leq x \leq \frac{2p}{2^n} \\ \frac{\sqrt{n-1}}{2^{2n}} & \frac{2p}{2^n} \leq x \leq 1 \end{cases}$.
These functions $K_n(x)$ vanish only when $0 \leq x \leq 2^n$. This gives some difficulty which we will see in Chapter IV. Of course, these continuous functions admit a uniformly convergent Haar Fourier expansion. It is a little more complicated and of less use.

2.3. Fourier coefficients. In this section we shall mainly discuss the asymptotic properties of the Fourier coefficients of different class of functions. We shall assume $f(x)$ to be integrable and associated with its Haar Fourier series,

$$f(x) \sim A_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{p=1}^{2^n-1} A_n^{(p)} \chi_n^{(p)}(x).$$

By $A_n^{(p)} = O(n)$ and $A_n = o(n)$, we shall mean that

$$|A_n^{(p)}| < A/n$$

when $n$ is sufficiently large and for all $p$ in the former, and that for any given $\epsilon > 0$, there exists an $N$ such that $|A_n^{(p)}| < \epsilon/n$ for all $n \geq N$ and all $p$ in the latter case.

Theorem 2.1. If $f(x)$ is bounded and measurable then $A_n^{(p)} = 0 \left(\frac{1}{n}\right)$. 

\[\]
Proof. \[ |A_n^{(p)}| = \left| \int_{\frac{2p-2}{2^n}}^{\frac{2p}{2^n}} f(x) \chi_n^{(p)}(x) \, dx \right| \leq M \sqrt{2^{n-1}} \frac{2}{2^n} = \sqrt{2} M \frac{1}{2^n} \]

where \( M \) is such that \( |f(x)| \leq M \).

This completes the proof.

Theorem 2.2. If \( f(x) \) is a function with a modulus of continuity \( \omega(\delta) \), then

\[ |A_n^{(p)}| \leq \sqrt{\frac{1}{2^n} \omega\left(\frac{1}{2^{n+1}}\right) \frac{1}{2^n}} \quad \text{for} \quad p = 1, 2, \ldots, 2^{n-1}. \]

Proof. \[ |A_n^{(p)}| = \left| \int_{\frac{2p-2}{2^n}}^{\frac{2p}{2^n}} f(x) \chi_n^{(p)}(x) \, dx \right| = \]

\[ \left| \int_{\frac{2p-2}{2^n}}^{\frac{2p}{2^n}} f(x) \sqrt{2^{n-1}} \, dx \right| = \]

\[ \left| \int_{\frac{2p-2}{2^n}}^{\frac{2p}{2^n}} f(x) \sqrt{2^{n-1}} \, dx \right| = \left| \int_{\frac{2p-2}{2^n}}^{\frac{2p}{2^n}} \left\{ f(x) - f(x + \frac{1}{2^n}) \right\} \sqrt{2^{n-1}} \, dx \right| \]

\[ \leq \sqrt{2^{n-1}} \int_{\frac{2p-2}{2^n}}^{\frac{2p}{2^n}} \omega\left(\frac{1}{2^n}\right) \, dx = \sqrt{\frac{2^n}{2}} \frac{1}{2^n} \omega\left(\frac{1}{2^n}\right) = \]
This completes the proof.

It follows easily the following corollary which does not have an analogy in the theory of Fourier series.

Corollary. If \( f(x) \) is continuous then \( a_n^{(p)} = o\left(\frac{1}{2^{n^p}}\right) \). This is so because \( f(x) \) is continuous if and only if \( \omega(\delta) \to 0 \) as \( \delta \to 0 \). We remark that except for the constant function, the coefficients of a continuous function can not go to zero too rapid. After we prove one more theorem, we will make some general discussions.

Theorem 2.3. If \( f(x) \in \text{Lip } \alpha, \quad 0 < \alpha \leq 1 \), then

\[
a_n^{(p)} = O\left(\frac{1}{2^{(\alpha+\frac{1}{2})n}}\right).
\]

Proof. From the assumption, we have

\[
\left| f(x_1) - f(x_2) \right| \leq M \left| x_1 - x_2 \right|^\alpha
\]

where \( M \) is an absolute constant. We have

\[
\left| a_n^{(p)} \right| \leq \sqrt{2^{n-1}} \int_{\frac{2^{p-1}}{2^n}}^{\frac{2^p}{2^n}} \left| f(x) - f(x + \frac{1}{2^n}) \right| dx \\
= \frac{M}{\sqrt{2}} \left(\frac{1}{2^{(\alpha+\frac{1}{2})n}}\right).
\]
This completes the proof.

For \( \alpha = 1 \) the following example will indicate that the above theorem is the best possible.

Let \( f(x) = x \). We have

\[
  a_n^{(p)} = \int \frac{2^{n-1}}{2^n} x \sqrt{2^n} \, dx - \int \frac{2^n}{2^n} x \sqrt{2^{n-1}} \, dx = -\frac{1}{\sqrt{2}} \frac{1}{2^{3n/2}}.
\]

Thus

\[
  a_n^{(p)} \neq o\left(\frac{1}{2^{3n/2}}\right).
\]

Let \( f(x) \) be a function having a derivative \( f'(x) \in \text{Lip} 1 \), and

\[
  f(x) \sim a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{p=1}^{n-1} a_n^{(p)} \chi_n^{(p)}(x).
\]

We wish to estimate the rapidity of convergence of its coefficients. By partial integration, we have

\[
  a_n^{(p)} = \int \frac{2^n}{2^n} f(x) \chi_n^{(p)}(x) \, dx = f(x) L_n^{(p)}(x) \left| \frac{2^n}{2^n} \right| - \int \frac{2^{n-2}}{2^n} f'(x) L_n^{(p)}(x) \, dx.
\]

The first term vanishes. \( L_n^{(p)}(x) \), being continuous, admits a uniformly convergent Haar Fourier series which we have
already evaluated in (2.2). Substituting in the above we obtain that

\[
\alpha_n = \frac{1}{2n} \int_{\frac{2n}{2}}^{\frac{2n}{2}} f'(x) \chi_{n}(x) \, dx = \frac{1}{2n} \int_{\frac{2n}{2}}^{\frac{2n}{2}} \chi_{n} - \sum_{m=1}^{n-1} \chi_{m} \chi_{m}(x) + \\
\sum_{m=n+1}^{\infty} \sqrt{2^{m-n-2}} \left\{ \sum_{q=2^{k-1}}^{2^{k-1}} \chi_{m}(x) + \sum_{q=2^{k-1}}^{2^{k-1}} \chi_{m}(x) \right\} \, dx.
\]

Recalling the definitions of \( p_m(\pm 1) \) and \( q_m \), we observe that

\[
p_m(\pm 1) \chi_{m}(x) = \sqrt{2^{m-1}} \quad \text{for} \quad \frac{2n}{2} < x < \frac{2n}{2}.
\]

Thus by uniform convergence and consequently term by term integration, we have

\[
\alpha_n = -\int_{\frac{2n}{2}}^{\frac{2n}{2}} f'(x) \, dx \left\{ \sqrt{2^{m-n-1}} + \sum_{m=1}^{n-1} \sqrt{2^{m-n-3}} \right\} - \\
\int_{\frac{2n}{2}}^{\frac{2n}{2}} \sum_{m=n+1}^{\infty} \sqrt{2^{m-n-2}} \left[ \sum_{q=2^{k-1}}^{2^{k-1}} \chi_{m}(x) + \sum_{q=2^{k-1}}^{2^{k-1}} \chi_{m}(x) \right] \, dx.
\]

(2.3)

\[
\alpha_n = -\sum_{m=n+1}^{\infty} \sqrt{2^{m-n-2}} \left[ \sum_{q=2^{k-1}}^{2^{k-1}} b_m^{(q)} + \sum_{q=2^{k-1}}^{2^{k-1}} b_m^{(q)} \right],
\]

where \( b_m \)'s are the Fourier coefficients of \( f'(x) \). Since \( f'(x) \) is assumed to be of Lip 1, we have \( b_m^{(q)} = O(1/2^{m-n}) \).

Thus for sufficiently large \( n \), we have
Thus we have that \[ \left| \sum_{m=n+1}^{\infty} \frac{1}{2^m} \right| = o \left( \frac{1}{2^{3n/2}} \right). \] Furthermore \[ \int_{2^{k-1}2^n}^{2^k2^n} f(x) \, dx = o \left( \frac{1}{2^{3n/2}} \right) \], i.e., for \( \epsilon > 0 \)

\[ \exists \ N \ \ni \ \left| \int_{2^{k-1}2^n}^{2^k2^n} f(x) \, dx \right| < \epsilon \quad \text{whenever } n \geq N, \text{ and all possible } p. \]

This implies that \( f'(x) \) vanishes at every dyadic rationals.

Since \( f'(x) \in \text{Lip 1} \), hence is continuous, this makes \( f'(x) \equiv 0 \).
or \( f(x) = \text{constant} \). This establishes

Theorem 2.4. The only function having a first derivative \( f' \in \text{Lip} \) and the Fourier coefficients of order \( o(1/2^{3n/2}) \) is the constant function.

This shows a remarkable difference between Haar functions and the trigonometric functions or Legendre polynomials when the rapidity of convergence of coefficients increases for functions having smooth derivative of higher order.

Although we do not have the Riemann Lebesgue theorem, yet we have the following which is a sharper result than in Fourier series, when \( p \geq 2 \).

Theorem 2.5. Let \( f(x) \in L^p \), \( 1 < p \) then

\[
a_n^{(q)} = o\left( \frac{1}{2^{\frac{n-2}{2} n}} \right).
\]

Proof. Applying Holder's inequality, we obtain that

\[
|a_n^{(q)}| = \left| \int_{\frac{2q}{2^n}}^{\frac{2q}{2^n}} f(x) \chi_n^{(q)}(x) \, dx \right| \leq \left\{ \int_{\frac{2q-2}{2^n}}^{\frac{2q}{2^n}} |f(x)|^\frac{1}{p} \, dx \right\} \left\{ \int_{\frac{2q-2}{2^n}}^{\frac{2q}{2^n}} |\chi_n^{(q)}(x)|^\frac{n}{p-1} \, dx \right\}^\frac{p-1}{p} \]
2.4. A second example. We conclude this chapter by giving another example which is almost exactly like the one given at the beginning. But it serves different purpose.

Let

\[ f(x) = \sqrt{2} \chi_2^{(1)}(x) + \sqrt{2} \chi_3^{(3)}(x) + \cdots + \sqrt{2^{n-1}} \chi_n^{(2^{n-1})}(x) + \cdots. \]

Let \( E_1 = \bigcup_{n=2}^\infty \left[ \frac{2^{n-4}}{2^n}, \frac{2^{n-3}}{2^n} \right] \) where \( f(x) \geq 0 \).

For any \( m > 0 \) there exists an integer \( k \) such that \( 2^{k-1} \leq m \leq 2^k \).

Then we have

\[
\int_{E_1} f(x) \, dx = \lim_{m \to \infty} \int_{E_1} f(x) \, dx = \lim_{k \to \infty} \int_{E_1} f(x) \, dx = \lim_{k \to \infty} \int_{E_1} \left[ \frac{2^{n-4}}{2^n}, \frac{2^{n-3}}{2^n} \right] f(x) \, dx = \\
= \lim_{k \to \infty} \sum_{n=2}^k \int_{\left[ \frac{2^{n-4}}{2^n}, \frac{2^{n-3}}{2^n} \right]} \sqrt{2^{n-1}} \chi_n^{(2^{n-1})}(x) \, dx = 
\]

This first term is \( o(1) \), hence the result.
Similarly \( \int_{E_2} f(x)dx \) diverges where \( E_2 \) is the set on which \( f(x) < 0 \) is. Thus \( f(x) \) is not integrable. This shows that there exists an everywhere convergent Haar trigonometrical series which is not a Haar Fourier series of its sum. It is not hard to see it is not a Haar Fourier series of any integrable function. Suppose we deny, i.e. there exists a Lebesgue integrable function \( \varphi(x) \) such that

\[
\varphi(x) \sim \sqrt{2} \left[ \chi_{1}^{(1)}(x) + \sqrt{2} \chi_{3}^{(3)}(x) + \cdots + \sqrt{2^{n-1}} \chi_{n}^{(2^{n-1})}(x) + \cdots \right]
\]

Then by the Theorem of Haar, the series will converge to \( \varphi(x) \) almost everywhere. Thus \( \varphi(x) \) is equal to \( f(x) \) almost everywhere. But this is impossible.

Remark. The interesting point of the function \( f(x) = \sqrt{2} \chi_{1}^{(1)}(x) + \sqrt{2} \chi_{3}^{(3)}(x) + \cdots + \sqrt{2^{n-1}} \chi_{n}^{(2^{n-1})}(x) + \cdots \) lies on the fact that the operators \( \int_{a}^{1} f(x) \chi_{n}^{(p)}(x) \, dx \) which give the Fourier coefficients are defined for \( f(x) \), for \( n = 1, 2, \ldots; p = 1, 2, \ldots, 2^{n-1} \). But it is not defined for \( A_{0} = \int_{a}^{1} f(x) \chi_{0}(x) \, dx = \int_{a}^{1} f(x) \, dx \) !
Chapter III

The Absolute Convergence and Summability of Haar Series

The investigation carried out in the last chapter has made for us the preparation to discuss the absolute convergence of both Haar Fourier and Haar trigonometrical series. In some classical theorems of orthonormal system concerning the asymptotic property of coefficients and the convergence property of the series, the uniform boundedness of the system plays an essential role. Failing to satisfy this assumption by Haar functions costs us some pretty theorems.

Continuing to study the Fourier expansion of functions having a finite jump at a dyadic irrational leads us to a number -- theoretic problem. However we are able to draw some conclusions by applying a theorem of Raikov [9] who proved it by using a method of Khinchin [7]. The original idea was due to G. D. Birkhoff.

3.1. The absolute convergence of Haar Fourier series.

Theorem 3.1. If $f \in \text{Lip } \alpha$, $0 < \alpha \leq 1$, then its Haar Fourier series converges absolutely.

Proof. From Theorem 2.3, we have $a_n^{(h)} = O\left(\frac{1}{2^{(\alpha+\frac{1}{2})n}}\right)$. Thus for any $x$, we have
if \( x \) is a dyadic irrational, and

\[
|a_0| + \sum_{n=1}^{\infty} |a_n^{(n)}| \sqrt{2^{-n}} \leq |a_0| + \sum_{n=1}^{\infty} M(n) \left( \frac{1}{2^n} \right)^{\frac{n}{2^{n+1}}} = \\
= |a_0| + \sqrt{2^{-1}} M \sum_{n=1}^{\infty} \left( \frac{1}{2^n} \right)^n \leq +\infty,
\]

if \( x = q/2^n \) where \( q \) is odd.

This completes the proof.

The above theorem is a better result than the corresponding one for Fourier series due to S. Bernstein which requires that \( \alpha > \frac{1}{2} \), and it is not true for \( \alpha = \frac{1}{2} \). However if the function is not only Lipschitzian but also of bounded variation, then the theorem is true under the same assumption.

Theorem 3.2. If \( f(x) \in \text{Lip} \alpha \), \( 0 < \alpha \leq 1 \) and \( \beta > \frac{1}{\alpha + \frac{1}{2}} \), then

\[
|a_0| + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} |a_n^{(\nu)}|^\beta < +\infty.
\]

Proof. We have \( |a_n^{(\nu)}| < M(n) \left( \frac{1}{2^n} \right)^{\alpha + \frac{1}{2}} \) for say \( n > N \).

Raising both sides to the \( \beta \)th power we obtain

\[
|a_n^{(\nu)}|^\beta < M^{\beta} \left( \frac{1}{2^n} \right)^{\alpha + \frac{1}{2} \beta} = M \left( \frac{1}{2^n} \right)^{1+\varepsilon} \varepsilon > 0.
\]
Thus we have

\[ |a_0| + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} |a_n^{(k)}|^\beta \leq |a_0| + \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} |a_n^{(k)}|^\beta + \sum_{n=N+1}^{\infty} \sum_{k=1}^{2^{n-1}} M \left( \frac{r}{2^n} \right)^{1+\epsilon} = \]

\[ = |a_0| + \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} |a_n^{(k)}|^\beta + \sum_{n=N+1}^{\infty} \frac{M}{2^n} \sum_{k=1}^{2^{n-1}} \left( \frac{r}{2^n} \right)^n < +\infty. \]

This completes the proof.

For continuous functions in general, we are unable to say anything concerning its absolute convergence because our result in the corollary to Theorem 2.2 is not sharp enough. For continuous function with a bounded derivative, its absolute convergence will follow immediately from Theorem 3.1. However if we consider \( F(x) \) absolutely continuous and \( F'(x) = f(x) \in L^P, p > 1 \), then we can say something.

Let

\[
F(x) \sim a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_n^{(k)} \chi_n^{(k)}(x),
\]

\[
f(x) \sim b_0 \chi_0(x) + \sum_{m=1}^{\infty} \sum_{g=1}^{2^{m-1}} b_m^{(g)} \chi_m^{(g)}(x).
\]

Similar to the calculation we made in (2.3) Ch. II, we have

\[
\alpha_n^{(k)} = -\frac{1}{\sqrt{2^3}} \frac{1}{2^{n/2}} \int_{2^{n/2}}^{2^n} f(x) dx - \sum_{m=n+1}^{\infty} \frac{1}{2^{m+n/2}} \left[ \sum_{q=2^{k-2k-1}}^{2^k - 2^{k-1}} b_m^{(q)} + \sum_{q=2^{k-2k-1}}^{2^k} b_m^{(g)} \right].
\]
From Theorem 2.5 we have \( b_{m}^{(g)} = o \left( \frac{l}{2^{p-2} \pi m} \right) \). Thus for sufficiently large \( n \),

\[
\left| \sum_{m=n+1}^{\infty} \right| \leq \sum_{m=n+1}^{\infty} \sqrt{2^{n-2}} \frac{1}{2^{m}} \left[ 2^{m-n} \frac{1}{2^{m-n}} \right] =
\]

\[
= \frac{1}{\sqrt{2}^{n-2}} \sum_{m=n+1}^{\infty} \frac{1}{2^{m-n}} \leq \frac{\varepsilon}{\sqrt{2}^{n-2}} \left( \frac{1}{2^{p-1} \pi m} \right)^{n}.
\]

By a theorem of the theory of integration, if \( f(x) \in L^{p}, \ p > 1 \),

\[
\int_{\frac{2^{p-2}}{2^{n}}}^{\frac{2^{p-1}}{2^{n}}} f(t) dt = o \left( \frac{1}{2^{p-1} \pi n} \right).
\]

Thus we obtain, for sufficiently large \( n \),

\[
\sqrt{2^{n}} \left| a_{n}^{(g)} \right| \leq M \left( \frac{1}{2^{p-1} \pi n} \right)^{n}.
\]

Since \( p > 1 \), \( \frac{l}{2^{p-1} \pi m} < 1 \), we establish

Theorem 3.3. If \( F(x) \) is absolutely continuous, \( F'(x) = f(x) \in L^{p}, \ 1 < p \), then the Haar Fourier series of \( F(x) \) converges absolutely.

Theorem 3.4. Let \( f(x) \) be a measurable convex function defined on \([0,1]\). Let us denote by \( D^{+} f(x) \) and \( D^{-} F(x) \) the upper right derivate and upper left derivate of \( f(x) \) respec-
tively. If both $D^+f(0)$ and $D^-f(1)$ are finite, then the Haar Fourier expansion of $f(x)$ converges absolutely.

**Proof.** For any $n$, we calculate the difference between two successive coefficients,

$$a_n^{(k+1)} - a_n^{(k)} = \left\{ \int_{2^{k-2}n}^{2^{k+1}n} f(x)\,dx - \int_{2^{k-1}n}^{2^{k+1}n} f(x)\,dx \right\} \sqrt{2^{-n-1}} =$$

$$= \sqrt{2^{-n-1}} \left\{ \int_{2^{k-1}n}^{2^{k+1}n} f(x)\,dx - \int_{2^{k-2}n}^{2^{k+1}n} f(x)\,dx \right\}$$

Now let $g(x) = k_1 x + k_2$ be a linear function such that it passes the points $(\frac{2^{k-1}n}{2^n}, f(\frac{2^{k-1}n}{2^n}))$ and $(\frac{2^{k+1}n}{2^n}, f(\frac{2^{k+1}n}{2^n}))$.

Since $f(x)$ is convex, we have

$$\begin{cases} f(x) \leq g(x) & \text{for } x \in \left[ \frac{2^{k-1}n}{2^n}, \frac{2^{k+1}n}{2^n} \right] \\ f(x) \geq g(x) & \text{everywhere else} \end{cases}$$

This assertion can be seen very easily. Let $0 \leq X_0 < \frac{2^{k-1}n}{2^n}$, and assume $f(X_0) < g(X_0)$. Let $E_0$ be the set of points $x$ such that $X_0 < x < \frac{2^{k-1}n}{2^n}$ and $f(x) < g(x)$. If $E_1 = \left[ X_0, \frac{2^{k-1}n}{2^n} \right]$ then take any point $x_1$ from $E_1$ and $x_2$ from $\left( \frac{2^{k+1}n}{2^n}, \frac{2^{k+1}n}{2^n} \right)$ join them by a chord. We observe that at $\frac{2^{k-1}n}{2^n}$, the point


on the curve lies above this chord. This contradicts the convexity. $E_1$ cannot be void, because convexity together with measurability implies continuity. The same continuity argument rules out the possibility of $E_1$ to be a proper subset of $[X_0, \frac{2^n-1}{2^n}]$. Making use of this assertion, we obtain

$$a_{(n+1)}^{(n)} - a_n^{(n)} \leq \left\{ \int_{\frac{2^n}{2^n}}^{\frac{2^n+1}{2^n}} g(x) \sqrt{2^{n-1}} \, dx \right\} - \left\{ \int_{\frac{2^n}{2^n}}^{\frac{2^n+2}{2^n}} g(x) \sqrt{2^{n-1}} \, dx \right\} =$$

$$= \sqrt{2^{n-1}} \left\{ (k_1 \frac{2^n}{2^n} + k_2) \frac{2^n}{2^n} - (k_1 \frac{4^n}{2^n} + k_2) \frac{1}{2^n} - (k_1 \frac{4^n+1}{2^n} + k_2) \frac{1}{2^n} \right\} = 0.$$

Thus we see that $a_{(n+1)}^{(n)} \leq a_n^{(n)}$ or $a_n^{(2^{n-1})} \leq a_n^{(2^{n-1})} \leq \ldots \leq a_n^{(1)}$.

Let $K_1 = x + K_2$ and $C_1 = x + C_2$ be respectively the chords passing through the points $(0, f(0))$ and $(\frac{1}{2^n}, f(\frac{1}{2^n}))$; and $(\frac{2^n}{2^n}, f(\frac{2^n}{2^n}))$ and $(1, f(1))$. We observe that

$$a_n^{(1)} = \int_{0}^{\frac{1}{2^n}} f(x) \sqrt{2^{n-1}} \, dx - \int_{\frac{2^n}{2^n}}^{\frac{2^n}{2^n}} f(x) \sqrt{2^{n-1}} \, dx \leq \int_{0}^{\frac{1}{2^n}} (K_1^{(n)} x + K_2^{(n)} \sqrt{2^{n-1}}) \, dx -$$

$$- \int_{\frac{1}{2^n}}^{\frac{2^n}{2^n}} (K_1^{(n)} x + K_2^{(n)} \sqrt{2^{n-1}}) \, dx = - K_1^{(n)} \frac{\sqrt{2^{n-1}}}{2^{2n}}.$$
and
\[ a^{(n)}_n = \int \frac{2^{n-1}}{2^n} f(x) \sqrt{2^{n-1}} dx - \int \frac{2^{n-1}}{2^n} f(x) \sqrt{2^{n-1}} dx \geq \int \frac{2^{n-1}}{2^n} (C_1 \chi + C_2) \sqrt{2^{n-1}} dx - \int \frac{2^{n-1}}{2^n} (C_1 \chi + C_2) \sqrt{2^{n-1}} dx = -C_1 \frac{\sqrt{2^{n-1}}}{2^{2n}}. \]

Finally we have, for a fixed \( n \), and all \( p \)
\[ |a^{(n)}_n| \leq \max \left( \left| K^{(n)}_1 \sqrt{2^{n-1}} \right|, \left| C^{(n)}_2 \sqrt{2^{n-1}} \right| \right). \]

Recalling that \( |K^{(n)}_1| = \left| \frac{f(\frac{p}{2^n}) - f(0)}{2^n} \right| = O(1) \) by our assumption, similarly for \( C_2 \), we obtain finally, for any \( x \).

\[ |a_0| + \sum_{n=1}^{\infty} |a^{(n)}_n| \sqrt{2^{n-1}} \leq |a_0| + \sum_{n=1}^{N} |a^{(n)}_n| \sqrt{2^{n-1}} + \sum_{n=1}^{N} M \frac{\sqrt{2^{n-1}}}{2^{2n}} \sqrt{2^{n-1}} < +\infty. \]

This completes the proof.

The Wiener's theorem for Fourier series, which states that if to every point \( x_0 \) in the interval \([0, 2\pi]\) corresponds a neighborhood \( I_{x_0} \) of \( x_0 \) and a function \( g(x) = g_{x_0}(x) \) such that (i) the Fourier series of \( g(x) \) converges absolutely, and (ii) \( g(x) = f(x) \) in \( I_{x_0} \), then the Fourier series of \( f \) converges absolutely, becomes almost trivial for Haar functions. If we assume the same assumption, let \( x_0 \) be any point in \([0, 1]\), we wish to prove that

\[ |a_0| + \sum_{n=1}^{\infty} |a^{(n)}_n| \sqrt{2^{n-1}} < +\infty, \]
where \( p \) depends on \( x_0 \) as usual. By our assumption there corresponds an interval \( I_{x_0} \) and \( g_{x_0}(x) \). For sufficiently large \( n \) we will have the implication that \( \frac{2^{p-2}}{2^n} \leq x_0 \leq \frac{2^p}{2^n} \) implies \( \left[ \frac{2^{p-2}}{2^n}, \frac{2^p}{2^n} \right] \subset I_{x_0} \). Since in \( I_{x_0} \), \( g_{x_0}(x) = f(x) \), 
for sufficiently large \( n \), \( b_n \) the Fourier coefficient of \( g_{x_0}(x) \) will be equal to \( a_n^{(p)} \) for that particular \( p \) such that \( \frac{2^{p-2}}{2^n} \leq x_0 \leq \frac{2^p}{2^n} \). Again by the assumption \( g_{x_0}(x) \) converges absolutely we have

\[
|a_0| + \sum_{n=1}^{\infty} |a_n^{(p)}| \sqrt{2^{-n-1}} = |a_0| + \sum_{n=1}^{N} |a_n^{(p)}| \sqrt{2^{-n-1}} + \sum_{n=N+1}^{\infty} |b_n^{(p)}| \sqrt{2^{-n-1}} < +\infty.
\]

This establishes

**Theorem 3.5.** If to every point \( x_0 \) in \([0,1]\) corresponds a neighborhood \( I_{x_0} \) of \( x_0 \) and a function \( g(x) = g_{x_0}(x) \) such that (i) the Fourier Haar series of \( g(x) \) converges absolutely, and (ii) \( g(x) = f(x) \) in \( I_{x_0} \) then the Fourier Haar series of \( f(x) \) converges absolutely.

If we take a closer look at the above theorem we will find that the assumptions almost state the conclusion only in a different wording. The absolute convergence of the Haar Fourier series of a function at a point \( x_0 \) is rather a local property. It depends mainly on the asymptotic property of the coefficients \( a_n^{(p)} \) where \( p \) is such that \( \frac{2^{p-2}}{2^n} \leq x_0 \leq \frac{2^p}{2^n} \).
For sufficiently large \( n \), \( a_n^{(p)} \) depends only on the behavior of \( f(x) \) in the neighborhood of \( X_0 \). This has been seen repeatedly from the only technique we used in proving the theorems of this section. Actually we may give one main theorem which states that the Haar Fourier series converges absolutely at \( X_0 \) if and only if the coefficients \( a_n^{(p)} = 0(\frac{1}{2^{(\frac{1}{2}+\epsilon)n}}) \) for \( \epsilon > 0 \). Then prove separately that the coefficients for different classes of functions satisfy this condition.

3.2. The absolute convergence of Haar trigonometric series. In trigonometric series the convergence of \( \sum_{n=1}^{\infty} |A_n| + |B_n| \) trivially implies the absolute convergence of the series

\[
\frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ A_n \cos n\chi + B_n \sin n\chi \right\}.
\]

This is not necessarily true for Haar trigonometric series because \( \chi_n^{(p)}(x) \)'s are not uniformly bounded. There is a converse theorem due to Lusin and Denjoy which states that if the trigonometric series converges absolutely in a set of positive measure then the series of its coefficients converges absolutely.

This is obviously not true in our case too because the absolute convergence of a Haar trigonometric series over, say a subinterval \( I \) does not give us any information of the coefficients \( A_n^{(p)} \) where \( p \) is such that \( \left[ \frac{2^{n-2}}{2^n}, \frac{2^n}{2^n} \right] \cap I = 0 \).
This immediately suggests that in studying the relation between the absolute convergence of a Haar trigonometric series over a set $E$ and the absolute convergence of the series of its coefficients, we need to specify some relations between the coefficients for the same $n$, and $E$ has to be everywhere dense. Thus we obtain

**Theorem 3.6.** Let $a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n-1}} a_n(x) \chi_n(x)$ be a Haar trigonometric series converging absolutely on a set $E$ which is everywhere dense in $[0,1]$ and in particular contains the points $0$ and $1$. Furthermore for every $n$ we have

$$|a_{n(1)}| \geq |a_{n(2)}| \geq \ldots \geq |a_{n(2^{n-1})}| \quad \text{(or} \quad |a_{n(1)}| \leq |a_{n(2)}| \leq \ldots \leq |a_{n(2^{n-1})}| \text{)}.$$ 

Then the series converges absolutely in $[0,1]$.

**Proof.** Let $x$ be any point in $[0,1]$. It is sufficient to assume that $x \notin E$. By the everywhere density of $E$ there exists an $x_0 \in E$ such that $x_0 < x$. Let $p_n$ be such that

$$\frac{2^{n-2}}{2^n} \leq x_0 \leq \frac{2^{n+1}}{2^n}.$$ 

Then by the assumption of absolute convergence over $E$, we have

$$|a_0| + \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n-1}} |a_n(x)\chi_n(x)| = |a_0| + \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n-1}} |a_n(x)\chi_n(x_0)| < +\infty.$$ 

Let $p_n'$ be such that $\frac{2^{n-2}}{2^n} \leq x_0 \leq \frac{2^{n+1}}{2^n}$. Since $x_0 < x$, we have $p_n \leq p_n'$. This in turn implies that $|a_{n(1)}| \geq |a_{n(2)}| \geq \ldots \geq |a_{n(2^{n-1})}|$. 

Hence we have

\[ |a_o| + \sum_{n=1}^{\infty} \sum_{p=1}^{2^n-1} |a_n^{(p)} \chi_n^{(p)}(x)| = |a_o| + \sum_{n=1}^{\infty} |a_n^{(p_n)} \chi_n^{(p_n)}(x)| \leq \]

\[ \leq |a_o| + \sum_{n=1}^{\infty} |a_n^{(p_n)} \chi_n^{(p_n)}(x)| \leq +\infty. \]

This completes the proof.

We recall Fatou's theorem that if the series

\[ a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos n x + \cdots, \quad |a_1| \geq |a_2| \geq \cdots \]

is absolutely convergent at a point \( x_0 \), then \( |a_1| + |a_2| + \cdots < \infty \) hence the series absolutely converges everywhere. Our theorem is not so sharp as this. An easy corollary of our theorem is that under the same assumption of monotonicity of coefficients, if the series converges absolutely at one point \( x \), different from zero or one as the case may be, then the series is absolutely convergent on a set of positive measure, namely \([0,x]\) or \([x,1]\).

### 3.3. Haar Fourier expansion of a certain function.

In Chapter I we already noticed that the Haar Fourier series of a function having discontinuity of first kind may converge if the finite jump is made at a dyadic rational. This is not surprising because the Haar functions are defined in such a way that they themselves make a finite jump at these points. We also learned from the theory of Fourier series that the
Fourier series of a function of bounded variation converges at every point \( x \) to the value \( \frac{1}{2} [f(x^+) + f(x^-)] \). These two situations show a striking difference between the above mentioned two systems of functions. In this section we propose to study the simplest function having a discontinuity of first kind. Let

\[
f_t(x) = \begin{cases} 
1 & 0 \leq x \leq t \\
0 & t < x \leq 1
\end{cases}
\]

where \( t \) is a dyadic irrational. This is a good function of bounded variation. But we know that its Haar Fourier series diverges at \( x = t \). This naturally suggests the study of its summability. We have

**Theorem 3.7.** If \( t \) is a rational then the Haar Fourier series for \( f_t(x) \) is \( (C,1) \) summable at \( t \) to a positive rational \( R \) smaller than one.

**Proof.** Let \( S_n^{(2^{-l})}(f; x) \) denote the partial sum of the series up to the term \( a_n^{(2^{-l})} \chi_n^{(2^{-l})}(x) \) evaluated at the point \( x \). Then we have that

\[
S_n^{(2^{-l})}(f; x) = \int_0^1 f_t(x) \chi_n^{(2^{-l})}(x, x) d\lambda = \int_{\frac{1}{2^n}}^{\frac{2^n}{2}} f_t(x) 2^n d\lambda ,
\]
where \( p' \) is such that \( \frac{p'}{2^n} < t < \frac{p'+1}{2^n} \). By the definition of \( f_t(x) \), we observe that

\[
S_n^{(\frac{2^n}{p})}(f; \frac{t}{2^n}) = \int_{\frac{p}{2^n}}^{\frac{p'+1}{2^n}} 2^n dx = 2^n(t - \frac{p}{2^n}) = 2^n t - p' = 2^n t - \lfloor 2^n t \rfloor,
\]

where \( \lfloor 2^n t \rfloor \) denotes as usual the greatest integer smaller than \( 2^n t \). The last equality can be seen very easily from the definition of \( p' \) as follows,

\[
\frac{p'}{2^n} < t < \frac{p'+1}{2^n} \Rightarrow p' < 2^n t < p' + 1 \Rightarrow \text{the required equality.}
\]

If \( t \) is a rational say \( p/q \), then we have

\[
S_n^{(\frac{2^n}{p})} = 2^n \frac{p}{q} - \lfloor 2^n \frac{p}{q} \rfloor = \frac{p_n}{q}, \quad 0 < p_n < q.
\]

The inequality \( 0 < p_n < q \) follows from the dyadic irrationality of \( t = p/q \). Thus there are only finite number of values which \( p_n \) can assume, say \( k \). We also observe that

\[
2^{n+1} t - \lfloor 2^{n+1} t \rfloor = 2 \{2^n t - \lfloor 2^n t \rfloor\} \mod 1.
\]

This is so because \( \lfloor 2^{n+1} t \rfloor = 2 \lfloor 2^n t \rfloor \mod 1 \). Thus if we start from \( S_1 \), we obtain

\[
S_1 = \frac{p_1}{q}, \quad S_2 = \frac{p_2}{q}, \ldots, S_k = \frac{p_k}{q}, \quad S_{k+1} = \frac{p_{k+1}}{q}, \ldots.
\]

For any \( n \) we can write \( n = mk + r \) where \( 0 \leq r < k \). Let \( 0 \leq r_n \) denotes the Cesaro mean we have
\[ T_n = \frac{\sum_{i=1}^{n} S_i}{n} = \frac{\sum_{i=1}^{mk} S_i + \sum_{i=mk+1}^{n} S_i}{n} = \frac{\left( \sum_{i=1}^{k} S_i \right) m}{mk + n} + \frac{\sum_{i=mk+1}^{n} S_i}{n} \].

Thus we have that \( \lim_{n \to \infty} T_n = \left( \sum_{i=1}^{k} S_i \right) / k = R \), where each \( S_i \) is a positive rational less than one. Thus \( R \) is a positive rational less than one.

This completes the proof.

A few numerical calculations led us to believe that, in most cases, \( R = \frac{1}{2} \). This conjecture can be confirmed by making use of the theorem of Raikov. To give the calculation of an exceptional case is not out of place.

Example. \( t = \frac{1}{7} \)

\[ S_1 = \frac{1}{7}, \quad S_2 = \frac{2}{7}, \quad S_3 = \frac{4}{7}, \quad S_4 = \frac{1}{7}, \ldots \]

Thus we have that \( \frac{\sum_{i=1}^{3} S_i}{3} = \frac{1}{3} \).

Whether the same theorem holds for irrationals is still left open. But it is true that for almost all irrationals the theorem holds with \( R = \frac{1}{2} \). This follows immediately from a theorem of Raikov [9] which states as follows. Let \( \varphi(x) \) be any summable periodic function with period 1 and \( a \) any natural number > 1. Let \( \varphi_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi(a^k x) \). Then for almost every \( x \), \( \varphi_n(x) \to \int_0^1 \varphi(t) \, dt \) as \( n \to \infty \).

Now we take \( \varphi(t) = t \), and \( a = 2 \), then we have
The proof of this theorem of Raikov is too lengthy to be reproduced here. However an outline of the main steps will not be amiss. We will prove the theorem by contradiction. We denote by $\varphi_n = \frac{1}{n} \sum_{k=0}^{n-1} \varphi(a^k x)$ and $I = \int_0^1 \varphi(t) \, dt$. If $\lim_n \varphi_n \neq \int_0^1 \varphi(t) \, dt$, a.e. then there exists either a number $\beta$ such that $\beta > 1$, and $\lim \sup \varphi_n(x) > \beta$ on a subset $S$ of positive measure, or a number $\alpha$, such that $\alpha < 1$ and $\lim \inf \varphi_n(x) < \alpha$ on a subset $S'$ of positive measure. Due to the symmetry we consider the first case. It can be shown that under these assumptions, the measure of $S$ is one.

We denote by $M_1$ the set of points of $S$ such that the relation, $\varphi_n > \beta$, is satisfied by $n = 1$ and larger $n$'s, but $\varphi_j < \beta$ when $j < 1$. Thus we have that $S = \sum_{k=1}^{\infty} M_k$.

Let $S_k = \sum_{k=1}^{K} M_k$, then it can be shown that $\int_{S_k} \varphi(t) \, dt \geq \beta$.

This leads to a contradiction and completes the proof.

3.4. Summability of Haar Fourier series. We recall the proof of Theorem 1.2 where the problem is reduced to the study of the convergence of the sequence $\{\eta_n = 2^n \chi_o - \mu_n = 2^n \chi_o - [2^n \chi_o]\}$. Now by Raikov's theorem, there exists a
set $S$ of measure zero such that the sequence $\{\eta_n\}$ is summable $(C,1)$ to $\frac{1}{2}$, if $x_0 \notin S$. Thus we establish

Theorem 3.8. Let $f(x)$ be a function continuous everywhere on $[0,1]$ except at $x_0$ which is a point of discontinuity of first kind. There exists a set $S$ of measure zero which does not depend on $f(x)$ such that if $x_0 \notin S$ then the Haar Fourier series of $f(x)$ is summable $(C,1)$ at $x_0$ to

$$\frac{1}{2} \left[ f(x_0^+) + f(x_0^-) \right].$$
Chapter IV

Haar Trigonometrical Series

In this chapter we deal mainly with Haar trigonometrical series, i.e.

\[ a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} a_{n,\nu} \chi_n(x) \]

where \( a_n \)'s are arbitrary real numbers not necessarily Fourier coefficients. The question which naturally arises is that under what condition a Haar trigonometrical series is a Fourier series. For instance, if \( \sum_{n=0}^{\infty} \sum_{\nu=1}^{2^{n-1}} |a_{n,\nu}|^2 \) is convergent, then the famous theorem of Riesz and Fischer will ensure us that the series (4.1) is the Fourier series of a function \( f(x) \) of the class \( L^2 \).

In the theory of trigonometrical series, it was proved by Cantor that (i) if a trigonometrical series converges everywhere to zero, then the series vanishes identically, i.e. all the coefficients are equal to zero; and proved by de la Vallie Poussin that (ii) if a trigonometrical series converges in the interval \((0,2\pi)\) to an integrable function \( f(x) \), then the series is the Fourier series of \( f(x) \). It is not hard to see that (i) follows from (ii). It is our main purpose to prove similar theorems for Haar trigonometrical series. One will notice the resemblance between the method
employed here and the Fourier method. But it resembles more a recently published and less known work of Fine who deals with Walsh functions [2].

An outline of both methods may not be amiss.

Let

\[
(4.2) \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) \rightarrow f(x),
\]

where \( f(x) \) is integrable. Integrating the series on the left side formally twice, we obtain

\[
(4.3) \quad F(x) = \frac{1}{4} a_0 x^2 - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}.
\]

This series converges uniformly and absolutely, hence

\[- \sum_{n=1}^{\infty} \frac{1}{n^2} \left( a_n \cos nx + b_n \sin nx \right) \]

is the Fourier series of \( F(x) - \frac{1}{4} a_0 x^2 \). If we are able to prove that

\[
(4.4) \quad F(x) = \int_{a}^{x} dy \int_{a}^{y} f(t) dt + Ax + B,
\]

then by integrating partially twice, we obtain that \( a_n, b_n \) are Fourier coefficients of \( f(x) \). In order to establish (4.4) we make use of the fact that if \( f(x) \) together with \( -f(x) \) is convex, then \( f(x) \) is linear. Thus it remains to
prove that \( \varphi(x) = F(x) - \int_a^x dy \int_y^x f(t) dt \) together with \( -\varphi(x) \) is convex.

This is a reason to employ the second integrated series (4.3), because if we consider

\[
L(x) = \frac{1}{2} a_0 x + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n},
\]

which is obtained by formal integration once, a classical example, \( \sum (\sin nx)/\log n \), shows that in general (4.5) need not converge everywhere, even the series on the left of (4.2) does for every \( x \). The situation is quite different for Haar functions, we will see that the first integrated series converges wherever the original series does but there exists an example that this is not true for the second integrated series. This naturally suggests us to employ only the first integration. If we wish to use the same sort of argument, we need to make use of the fact that if a function \( f(x) \) together with \( -f(x) \) is monotonically decreasing, then \( f(x) \) is constant. In order to do this we owe a lemma to Fine.


By an L-series and a K-series we mean

\[
a_0 L_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} a_n^{(\nu)} L_n^{(\nu)}(x),
\]

and

\[
a_0 K_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} a_n^{(\nu)} K_n^{(\nu)}(x),
\]
which are obtained respectively by formal integration once and twice of the series (4.1).

Theorem 4.1. If the series (4.1) converges at \( x_0 \) then so does the series (4.6) absolutely at \( x_0 \).

Proof. If \( x_0 \) is a dyadic rational, say \( p/2^{n-1} \), then

\[
L_m^{(q)}(x_0) = 0 \quad \text{for} \quad m \geq n \quad \text{and} \quad q = 1, 2, \ldots, 2^{m-1}. 
\]

Thus (4.6) reduces to a finite sum and hence the convergence follows. If \( x_0 \) is a dyadic irrational, then for any \( n \), there exists one and only one \( p_n \) such that \( \lambda_n^{(p_n)}(x_0) \neq 0 \). We observe that, for and only for this \( p_n \), \( L_n^{(p_n)}(x_0) \neq 0 \).

The convergence of

\[
a_o \lambda_n(x_0) + \sum_{n=1}^{\infty} a_n \lambda_n^{(p_n)}(x_0) \]

will imply that \( a_n^{p_n} \to 0 \). Hence converges absolutely

\[
a_o L_0(x_0) + \sum_{n=1}^{\infty} a_n L_n^{(p_n)}(x_0),
\]

where \( 0 < L_n^{(p_n)}(x_0) < \frac{1}{\sqrt{2^{n+1}}} \).

This completes the proof.

We already mentioned that there exist examples such that (4.1) converges everywhere in \((0,1)\) while its corresponding K-series does nowhere. Now we give one of them.
Example. \( \chi_0(x) + \sum_{n=1}^{\infty} 2^{2n} \chi_{n^{(1)}}(x) \).

For any \( x \) such that \( 0 < x < 1 \), there exists a smallest integer \( N \) satisfying \( 2/2^n < x \), whenever \( n \geq N \). It follows that

\[
\chi_{n^{(1)}}(x) = 0 \quad \text{for} \quad n \geq N.
\]

Thus we observe that

\[
\chi_0(x) + \sum_{n=1}^{\infty} 2^{2n} \chi_{n^{(1)}}(x) = \chi_0(x) + \sum_{n=1}^{N-1} 2^{2n} \chi_{n^{(1)}}(x),
\]

and the convergence follows. But for the same \( x \) and same \( N \)
we have

\[
K_{n^{(1)}}(x) = \frac{\sqrt{2^{n-1}}}{2^{2n}} \quad \text{for} \quad n \geq N.
\]

Thus we obtain

\[
\chi_0(x) + \sum_{n=1}^{\infty} 2^{2n} K_{n^{(1)}}(x) = \chi_0(x) + \sum_{n=1}^{N-1} 2^{2n} K_{n^{(1)}}(x) + \sum_{n=N}^{\infty} \sqrt{2^{n-1}} \to +\infty.
\]

(p)

Theorem 4.2. If \( a_n = o(1) \), then both L-series and K-series converge absolutely and uniformly.

Proof. There exists an \( N \) such that \( |a_n^{(k)}| < 1 \), whenever \( n \geq N \). For any given \( \epsilon > 0 \) then exists an \( N' \) such that

\[
\sum_{i=N}^{\infty} \frac{1}{\sqrt{2^{i+1}}} < \epsilon.
\]

Thus we obtain, for \( M = \max(N, N') \)

\[
\left| \sum_{n=M}^{M+1} \sum_{k=1}^{2^{n-1}} \alpha^{(k)}(x) L_n^{(k)}(x) \right| = \left| \sum_{n=M}^{M+1} \alpha_n^{(k)} L_n^{(k)}(x) \right| \leq \sum_{n=M}^{M+1} \sqrt{2^{n+1}} < \epsilon,
\]

for all \( x \), and all \( j \). Thus we proved the uniform and absolute
convergence of L-series, similarly we can prove for K-series.

Remark. The assumption \( a_n^{(p)} = o(1) \) can be replaced by that for any positive integer \( M \) there exists \( N \) such that

\[
|a_n^{(p)}| < M \quad \text{whenever } n \geq N.
\]

In trigonometrical series we have the theorem that if

\[
|a_0 - a_1| + |a_1 - a_2| + \cdots
\]

is convergent and \( a_k \to 0 \), then the series

\[
a_{0/2} + \sum_{k=1}^{\infty} a_k \cos kx
\]

converges uniformly in any interval \( 0 < \varepsilon \leq x \leq 2\pi - \varepsilon \). The proof is based upon the fact that \( \sum_{k=1}^{\infty} \cos kx \) has its partial sums uniformly bounded in \( 0 < \varepsilon \leq x \leq 2\pi - \varepsilon \). For Haar functions the partial sums of

\[
\sum_{n=0}^{\infty} \sum_{k=1}^{2^{n-1}} \mathcal{K}_n(x)
\]

are not uniformly bounded in any subinterval of \( [0,1] \). Hence we do not have the same theorem. But for L-series and K-series, this becomes trivial because only the assumption that \( a_n^{(p)} \to 0 \) is sufficient to make the series converge uniformly and absolutely. However we have the following corollary.

Corollary. If

\[
|a_0 - a_1| + \cdots + |a_{n-1}^{(p)} - a_n^{(p)}| + \cdots
\]

is convergent then both L-series and K-series converge absolutely and uniformly.

4.2. Integration of Haar Fourier series.

Theorem 4.3. Let
If $f(x)$ is bounded and measurable, then we have

$$\int_0^x f(t)\,dt = a_0 L_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} a_n L_n(x).$$

Proof. If $f(x)$ is measurable then its Haar Fourier series will converge to $f(x)$ almost everywhere, i.e.

$$a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} a_n \chi_n(x) \rightarrow f(x) \quad a.e.$$ 

Let

$$S_n^{(\nu)}(x) = a_0 \chi_0(x) + \sum_{m=1}^{n} \sum_{\delta=1}^{m} a_m \chi_m(x) = \int_0^1 f(t) K_{n,\nu}(x,t)\,dt.$$ 

Thus we have

$$|S_n^{(\nu)}(x)| \leq \int_0^1 |f(t)| K_{n,\nu}(x,t)\,dt \leq M \int_0^1 K_{n,\nu}(x,t)\,dt = M.$$ 

Hence by the theorem of bounded convergence, we prove our assertion.

For integrable functions we are able to prove only the following weaker form.

Theorem 4.4. Let $f(x)$ be integrable,

$$f(x) \sim a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} a_n \chi_n(x).$$ 

Then the series

$$a_0 L_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^{n-1}} a_n L_n(x)$$

is $(C,1)$ summable to $\int_0^x f(t)\,dt.$
Proof. Consider the function

\[ f_{\alpha}(x) = \begin{cases} 1 & 0 \leq x \leq \alpha \\ 0 & \alpha < x \leq 1 \end{cases}. \]

Let

\[ f_{\alpha}(x) \sim b_{0} \chi_{0}(x) + \sum_{m=1}^{\infty} \sum_{q=1}^{2^{m-1}} b_{m}^{(q)} \chi_{m}^{(q)}(x). \]

We know that the series on the right converges to \( f_{\alpha}(x) \) for \( 0 \leq x \leq \alpha \). Now we consider its Cesaro sum at any point \( x \),

\[ \sigma_{n}(f_{\alpha}; x) = (S_{0} + S_{1} + \cdots + S_{n-1})/n. \]

Since

\[ b_{m}^{(q)} = \int_{0}^{1} f_{\alpha}(x) \chi_{m}^{(q)}(x)dx = \int_{0}^{\alpha} \chi_{m}^{(q)}(x)dx = L_{m}^{(q)}(\alpha), \]

substituting

in \( \sigma_{n}(f_{\alpha}; x) \) we obtain that

\[ \sigma_{n}(f_{\alpha}; x) = \frac{L_{0}(\alpha)\chi_{0}(x) + (L_{0}(\alpha)\chi_{0}(x) + L_{1}(\alpha)\chi_{1}(x)) + \cdots + (L_{0}(\alpha)\chi_{0}(x) + \sum_{m=1}^{n-1} \sum_{q=1}^{2^{m-1}} L_{m}^{(q)}(\alpha)\chi_{m}^{(q)}(x))}{n}. \]

Multiplying both sides by \( f(x) \) and integrating term by term (we can do this simply because we have only a finite number of terms to deal with), we obtain that

\[ \int_{0}^{1} \sigma_{n}(f_{\alpha}; x) f(x) dx = \frac{a_{0}L_{0}(\alpha) + (a_{0}L_{0}(\alpha) + a_{1}L_{1}(\alpha)) + \cdots + (a_{0}L_{0}(\alpha) + \sum_{m=1}^{n-1} \sum_{q=1}^{2^{m-1}} a_{m}L_{m}(\alpha))}{n}. \]

Passing to limit, we obtain that
\[
\lim_{n \to \infty} \frac{a_0 L_0(t) + \cdots + (a_0 L_0(t) + \sum_{m=1}^{n} \frac{\sum_{q=1}^{n} a_m L_m(t)}{n})}{n} = \lim_{n \to \infty} \int_{0}^{t} \left( f_{nt} \right)(x) f(x) \, dx.
\]

We observe that, when \(0 \leq x < t\), \(\sigma_n \to 1\) as \(n \to \infty\) and when \(t < x \leq 1\), \(\sigma_n \to 0\) as \(n \to \infty\). In the case \(x = t\), from the discussion in Chapter III, we have

\[
\sigma_n (f_{nt}; x) = \frac{t + 2x - [2x] + \cdots + 2^{n-1}x - [2^{n-1}x]}{n} < 1
\]

for all \(n\).

Thus \(\sigma_n (f_{nt}; x)\) is uniformly bounded, hence by the general convergence theorem of Lebesgue, we have

\[
\lim_{n \to \infty} \int_{0}^{t} \sigma_n (f_{nt}; x) f(x) \, dx = \int_{0}^{t} \lim_{n \to \infty} \sigma_n (f_{nt}; x) f(x) \, dx = \int_{0}^{t} f(t) \, dx = \int_{0}^{t} f(x) \, dx.
\]

Thus we have

\[
\lim_{n \to \infty} \frac{a_0 L_0(t) + \cdots + (a_0 L_0(t) + \sum_{m=1}^{n} \frac{\sum_{q=1}^{n} a_m L_m(t)}{n})}{n} = \int_{0}^{t} f(x) \, dx.
\]

This completes the proof.

4.3. Differentiation of the L-series.

Theorem 4.4. If

\[
L(x) = a_0 L_0(x) + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} a_n eps^p L_n(t) (x)
\]

(4.9)
is absolutely continuous, and let

$$L'(x) = b_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^n-1} b_{n,\nu} \chi_n(x)$$

then $a_n(p) = b_n(p)$ for all $n = 0, 1, 2, \ldots$, and $p = 1, 2, \ldots, 2^{n-1}$.

Proof. We observe that first of all

$$L(x) = \int_0^x L'(t) \, dt.$$

Next, we calculate

$$b_n(p) = \int_{2^{p-1}}^{2^n} L'(x) \sqrt{2^{n-1}} \, dx - \int_{2^{p-1}}^{2^n} L'(x) \sqrt{2^{n-1}} \, dx$$

$$= \sqrt{2^{n-1}} \left\{ 2 \int_0^{2^{p-1}} - \int_0^{2^{p-2}} - \int_0^{2^{p-1}} L'(x) \, dx \right\}$$

$$= \sqrt{2^{n-1}} \left\{ 2 L(\frac{2^{p-1}}{2^n}) - L(\frac{2^{p-2}}{2^n}) - L(\frac{2^{p-1}}{2^n}) \right\}.$$

In computing $a_n(p)$, let

$$\chi_1 = \frac{2^{p-2}}{2^n}, \quad \chi_2 = \frac{2^{p-1}}{2^n}, \quad \chi_3 = \frac{2^{p}}{2^n}.$$

For $m > n$, by the definition of $L_m(p)$, we have

$$L_m(p) \chi_i = 0 \quad \text{for} \quad i = 1, 2, 3; \quad p = 1, 2, \ldots, 2^{m-1}.$$
For \( m < n \), there exists one and only one \( q_m \) such that

\[
\frac{2q_m - 2^{m-2}}{2^m} \leq \frac{2^{h-2}}{2^n} < \frac{2^{h-1}}{2^n} < \frac{2^h}{2^n} \leq \frac{2q_m}{2^m}.
\]

For this \( q_m \) it can be seen easily, in Fig. 4, that

\[
2 L_{m}(\frac{q_m}{2^n}) - L_{m}(\frac{2^{h-1}}{2^n}) - L_{m}(\frac{2^{h-2}}{2^n}) = 0.
\]

For other \( q \)'s we obviously have \( L_m(q_i')(x_i) = 0 \quad i = 1, 2, 3 \).

For \( m = n \), we have

\[
L_n(\frac{2^{h-1}}{2^n}) = \frac{1}{\sqrt{2^{2n+2}}} \quad , \quad L_n(x_i) = 0 \quad , \quad i = 1, 3.
\]

Thus we obtain finally

\[
2 L(\frac{2^{h-1}}{2^n}) - L(\frac{2^{h-2}}{2^n}) - L(\frac{2^{h-2}}{2^n}) = 2 \sum_{m=0}^{n-1} a_m L_m(\frac{q_m}{2^n}) + a_n(\frac{h}{2^n})^2 - \sum_{m=0}^{n-1} a_m L_m(\frac{q_m}{2^n}) - \sum_{m=0}^{n-1} a_m L_m(\frac{2^{h-2}}{2^n}) =
\]
Thus we have
\[ a_n^{(h)} = \sqrt{2^{-n-1}} \left\{ 2L \left( \frac{2^{h-1}}{2^n} \right) - L \left( \frac{2^{h-2}}{2^n} \right) - L \left( \frac{2^h}{2^n} \right) \right\} = b_n^{(h)}. \]

This completes the proof.

Remark. The absolute continuity may be replaced by either one of the following two conditions.

1) If \( L(x) \) is any function such that \( L'(x) \) exists and is finite everywhere and is integrable.

2) If \( L(x) \) is any function such that \( L'(x) \) exists everywhere and is bounded.

4.4. Uniqueness of Haar trigonometrical series. We first prove a lemma which characterize the monotonicity of a function satisfying certain conditions.

Lemma 1. (Lemma of Fine [2]). For any \( x \) in \([0,1]\) and any positive integer \( n \) there exists one and only one \( \alpha_n \) such that
\[ \alpha_n' = \frac{j_n - 1}{2^n} \leq \alpha_n = \frac{j_n - 1}{2^n} \leq \chi < \frac{j_n}{2^n} = \beta_n. \]

Let \( f(x) \) be a function defined on \([0,1]\) satisfying

1) \( \lim_{n \to \infty} f(\alpha_n') = f(x) \)
ii) \( \liminf_{n \to \infty} 2^n [f(\beta_n) - f(\alpha_n)] \leq 0 \) for all \( x \) in \( [0,1] \),

then \( f(x) \) is monotonically non-increasing.

Proof. Suppose we deny, then there exist two points \( \alpha < \beta \) such that \( f(\alpha) < f(\beta) \), say \( f(\beta) - f(\alpha) = \epsilon > 0 \). Take \( \frac{\epsilon}{4} \epsilon \), by the condition 1), there exist two positive integers \( k \) and \( i \) such that \( |f(\frac{k}{2^n}) - f(\alpha)| < \frac{\epsilon}{4} \), and \( |f(\frac{k+i}{2^n}) - f(\beta)| < \frac{\epsilon}{4} \). Thus we obtain that \( f(\frac{k}{2^n}) < f(\frac{k+i}{2^n}) \). Now we claim that there exists a positive integer \( j \), \( 0 \leq j \leq i-1 \), such that \( f(\frac{k+j}{2^n}) < f(\frac{k+j+1}{2^n}) \). Let us start with \( k/2^n \). If \( f(\frac{k}{2^n}) < f(\frac{k+1}{2^n}) \), then we are done. Otherwise, namely \( f(\frac{k+1}{2^n}) \leq f(\frac{k}{2^n}) < f(\frac{k+i}{2^n}) \), we reject \( k/2^n \) and start with \( \frac{k+1}{2^n} \).

We will reach our conclusion before we use up all the \( i \) points. Thus we have a subinterval \( I_n : (\frac{k+j}{2^n}, \frac{k+j+1}{2^n}) \) such that \( f(\frac{k+j}{2^n}) < f(\frac{k+j+1}{2^n}) \). Let us denote \( \frac{k+j}{2^n} \) and \( \frac{k+j+1}{2^n} \) by \( \alpha_n \) and \( \beta_n \) respectively, and \( 2^n [f(\beta_n) - f(\alpha_n)] = c_n > 0 \).
For any $k_1$, $0 < k_1 < 1$, we draw a straight line $L$ passing through $P$ with a slope $(1-k_1)c_n$. Obviously we have $L(\alpha_n) = f(\alpha_n)$ and $L(\beta_n) < f(\beta_n)$. Let $f(\beta_n) - L(\beta_n) = \epsilon' > 0$,

$$\alpha_{n+1} = (\alpha_n + \beta_n)/2, \hspace{1cm} \alpha_{n+2} = (\alpha_{n+1} + \beta_n)/2, \hspace{1cm} \ldots, \hspace{1cm} \alpha_{n+k} = (\alpha_{n+k-1} + \beta_n)/2, \ldots.$$  

By condition i) there exists an $\alpha_m$ such that

$$|f(\alpha_m') - f(\beta_n)| < \epsilon'. \quad \text{Evidently } \alpha_n \text{ does not satisfy this condition. There exists a smallest } t \geq 1 \text{ such that}$$

$$L(\alpha_{n+t}) < f(\alpha_{n+t}). \quad \text{Thus we obtain a sequence of subintervals}$$

$$I_n: (\alpha_n, \beta_n), \hspace{1cm} I_{n+1}: (\alpha_{n+1}, \beta_n), \hspace{1cm} \ldots, \hspace{1cm} I_{n+t-1}: (\alpha_{n+t-1}, \beta_n) \hspace{1cm} \text{and} \hspace{1cm} I_{n+t}: (\alpha_{n+t}, \alpha_{n+t}'),$$

with the property that $2^{n+t}[f(\beta_n) - f(\alpha_{n+t})] > (1-k_1)c_n$ for $s = 1, 2, \ldots, t-1$, and $2^{n+t}[f(\alpha_{n+t}) - f(\alpha_{n+t-1})] > (1-k_1)c_n$. We remark that $I_{n+t}$ is contained in the interior of $I_n$. Now starting with $(1-k_1)c_n$ and any $k_2, 0 < k_2 < 1$ and applying our method to $I_{n+t}$, we will obtain a second sequence of intervals $I_{n+t+1}: (\alpha_{n+t+1}, \alpha_{n+t}'), I_{n+t+2}: (\alpha_{n+t+2}, \alpha_{n+t}'), \ldots,$

$$I_{n+t+c-1}: (\alpha_{n+t+c-1}, \alpha_{n+t}'), \hspace{1cm} I_{n+t+c}: (\alpha_{n+t+c-1}, \alpha_{n+t+c}),$$

where $\alpha_{n+t+1} = (\alpha_{n+t-1} + \alpha_{n+t})/2, \hspace{1cm} \alpha_{n+t+2} = (\alpha_{n+t+1} + \alpha_{n+t})/2, \ldots$, with the property that $2^{n+t+d}[f(\alpha_{n+t}) - f(\alpha_{n+t+d})] > (1-k_1)(1-k_2)c_n$,.
for \( d = 1, 2, \ldots, C-1 \), and 
\[
2^{n+t+c} \left[ f(\alpha_{n+t+C}) - f(\alpha_{n+t+C-1}) \right] > (1 - k_1)(1 - k_2) c_n .
\]
Keeping going on like this with \( k_3, k_4, \ldots \), we obtain a nested sequence of intervals which will define a point \( x \) in \([0,1]\). For this point \( x \), we have

\[
\liminf_{m \to \infty} 2^m \left[ f(\beta_m) - f(\alpha_m) \right] \geq c_n \prod_{i=1}^{\infty} (1 - k_i) ,
\]
where \( \alpha_m \) and \( \beta_m \) are respectively the left and right end point of the subinterval. But we are still left free to make a choice of \( k_1 \). We certainly be able to choose them in such a way that
\[
\prod_{i=1}^{\infty} (1 - k_i) = K > 0 .
\]
But this contradicts our assumption that \( \liminf 2^m \left[ f(\beta_m) - f(\alpha_m) \right] \leq 0 \).

This completes the proof.

Lemma 2. Let
\[
L(x) = a_0 L_0(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_n \chi_m^{(k)}(x) .
\]
With the notation \( \lambda_n = \frac{k_{n-1}}{2^n} \leq x < \frac{k_n}{2^n} = \beta_n \), for a fixed \( x \), there exists a sequence of numbers \( 2^n [ L(\beta_n) - L(\alpha_n) ] \).

Let \( f(x) \) denotes formally the series
\[
\chi_0^{(k)}(x) + \sum_{m=1}^{\infty} \sum_{k=1}^{2^{m-1}} a_m \chi_m^{(k)}(x) \]
and \( S_n(f;x) \) its partial sum, then
\[ 2^n \left[ L(\beta_n) - L(\alpha_n) \right] = S_n(f; x). \]

**Proof.** We observe that

\[
2^n \left[ L(\beta_n) - L(\alpha_n) \right] = 2^n \left[ a_0 \{ L_0(\beta_n) - L_0(\alpha_n) \} + 2 \sum_{m=1}^{n-1} a_m \{ L_m(\beta_n) - L_m(\alpha_n) \} \right] \\
= 2^n a_0 \int_{\alpha_n}^{\beta_n} \chi_0(t) \, dt + 2^n \sum_{m=1}^{n-1} a_m \int_{\alpha_n}^{\beta_n} \chi_m(t) \, dt = \\
= 2^n \int_{\alpha_n}^{\beta_n} S_n(f; x) \, dx.
\]

But \( S_n(f; x) \) is constant in the interval \((\alpha_n, \beta_n)\) we obtain

\[
2^n \left[ L(\beta_n) - L(\alpha_n) \right] = S_n(f; x).
\]

This completes the proof.

**Lemma 3.** Let \( f(x) \) be any integrable function then for any \( \varepsilon > 0 \), there exist two continuous functions \( \varphi(x) \) and \( \psi(x) \) such that

i) \( |\varphi(x) - F(x)| < \varepsilon \), \( |F(x) - \psi(x)| < \varepsilon \)

where \( F(x) = \int_0^x f(t) \, dt \),

ii) at any point \( x \) where \( f(x) \neq \infty \) all derivatives of \( \varphi(x) \) exceeds \( f(x) \) and at any point \( x \) where \( f(x) \neq -\infty \) all derivatives of \( \psi(x) \) are less than \( f(x) \).
The proof of this lemma can be found in standard textbook on theory of integration, e.g. pp. 136-137, Saks, Théorie de L'intégrale, and hence is omitted.

Now we are ready to prove our main theorem.

Theorem 4.5. Let \( a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_n \chi_n(x) \) be convergent everywhere to an integrable function \( f(x) \), then

\[
\int_0^x f(t) \, dt = a_0 L_0(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_n L_n(x) + C.
\]

Proof: Let \( F(x) = \int_0^x f(t) \, dt \), for any given \( n \) let \( \phi_n(x) \) and \( \psi_n(x) \) be the continuous functions in Lemma 3 with \( \epsilon = 1/n \). By Theorem 4.1 we know that the series \( a_0 L_0(x) + \)

\[ + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_n L_n(x) \]

converges everywhere, and we denote its limiting function by \( L(x) \). We wish to prove that \( K_n(x) = L(x) - \phi_n(x) \) and \( H_n(x) = \psi_n(x) - L(x) \) are monotonically non-increasing. We observe that, first of all,

\[
\liminf_{m \to \infty} 2^m \left[ K_m(\beta_m) - K_m(\alpha_m) \right] = \liminf_{m \to \infty} 2^m \left\{ \left| L(\beta_m) - L(\alpha_m) \right| \right\} -
\]

\[ - \left\{ \phi_n(\beta_m) - \phi_n(\alpha_m) \right\} \leq \liminf_{m \to \infty} 2^m \left\{ L(\beta_m) - L(\alpha_m) \right\} -
\]

\[ - \liminf_{m \to \infty} 2^m \left\{ \phi_n(\beta_m) - \phi_n(\alpha_m) \right\} \leq f(x) - f(x) = 0,
\]

similarly we have \( \liminf_{m \to \infty} 2^m \left[ H_m(\beta_m) - H_m(\alpha_m) \right] \leq 0. \)
Given any \( x \) and any \( \varepsilon > 0 \), by the convergence of \( L(x) \) there exists an \( N' \) such that
\[
\left| \sum_{m=n'}^{\infty} \sum_{\ell=1}^{2^{m-1}} a_m^\ell l_m^\ell(x) \right| < \frac{\varepsilon}{2}
\]
owherever \( n' \geq N' \). With this \( N' \) we can make \( \lambda_n' \) sufficiently close to \( x \) so that
\[
\left| a_o \left[ L(x) - L(\lambda_n') \right] + \sum_{n=1}^{N'} \sum_{\ell=1}^{2^{n-1}} a_n^{\ell} \left( l_n^{\ell}(x) - l_n^{\ell}(\lambda_n') \right) \right| < \frac{\varepsilon}{2}
\]
owherever \( n' \geq N' \). Thus we have \( L(\lambda_n') \to L(x) \).

By continuity we have also that \( \mathcal{F}(\lambda_n') \to \mathcal{F}(x) \) and \( \varphi(\lambda_n') \to \varphi(x) \). Thus both \( K_n(x) \) and \( H_n(x) \) satisfy the two conditions of Lemma 1. Hence they are monotonically non-increasing, so are their limiting functions \( K(x) = L(x) - F(x) \) and \( H(x) = F(x) - L(x) \). Therefore \( F(x) - L(x) = \) constant \( C \), i.e.,
\[
\int_0^x f(t) \, dt = a_o L_o(x) + \sum_{n=1}^\infty \sum_{\ell=1}^{2^{n-1}} a_n^{\ell} l_n^{\ell}(x) + C.
\]

This completes the proof.

Theorem 4.6. Let \( a_o \chi_0(x) + \sum_{n=1}^\infty \sum_{\ell=1}^{2^{n-1}} a_n^{\ell} \chi_n^{\ell}(x) \) be convergent everywhere to an integrable function \( f(x) \) then
\[
a_o \chi_0(x) + \sum_{n=1}^\infty \sum_{\ell=1}^{2^{n-1}} a_n^{\ell} \chi_n^{\ell}(x)
\]
is the Fourier series of \( f(x) \).

Proof. By Theorem 4.5 we have
\[
(4.10) \quad F(x) = \int_0^x f(t) \, dt = a_o L_o(x) + \sum_{n=1}^\infty \sum_{\ell=1}^{2^{n-1}} a_n^{\ell} l_n^{\ell}(x) + C.
\]
F(x), being the integral of the integrable function f(x), is absolutely continuous. Differentiating both sides of (4.20) formally, by Theorem 4.4, we have

\[
F'(x) = f(x) \sim a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} a_n^{(k)} \chi_n^{(k)}(x).
\]

This completes the proof.

Corollary. If \( a_0 \chi_0(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} a_n^{(k)} \chi_n^{(k)}(x) \) converges everywhere to 0, then \( a_n^{(k)} = 0 \) for \( n = 0, 1, 2, \ldots \) and \( p = 1, 2, \ldots, 2^{n-1} \).
Chapter V

Interpolation by Haar functions

In studying the interpolation of a function \( f(x) \) in terms of a sequence of functions \( \{\varphi_n(x)\} \), the first problem is to find the expression of the interpolating coefficients in terms of \( f(x) \). By this we mean to find \( n \) numbers \( A_i \)'s such that

\[
f(x) = A_1 \varphi_1(x) + A_2 \varphi_2(x) + \cdots + A_n \varphi_n(x)
\]

for \( x = x_1, x_2, \ldots, x_n \). This amounts to solving a system of \( n \) linear equations having \( n \) unknowns. We have, by means of one way or other, to show that the system does possess a solution. The following lemma might be well known, but we put in a form which is most suitable to our purpose.

5.1. **Lemma.** We recall the definition of linearly independent functions. A sequence of functions \( \{\varphi_n(x)\} \) defined on \([a, b]\) is said to be linearly independent if for any linear combination of a finite number of them, i.e.

\[
A_1 \varphi_{n_1}(x) + \cdots + A_{\xi} \varphi_{n_\xi}(x)
\]

the following holds,

\[
A_1 \varphi_{n_1}(x) + \cdots + A_{\xi} \varphi_{n_\xi}(x) \equiv 0 \Rightarrow A_1 = A_2 = \cdots = A_{\xi} = 0.
\]

From the definition it follows immediately that \( \varphi_n(x) \not\equiv 0 \) for all \( n \).
Lemma. Given a sequence of linearly independent functions \( \{ \varphi_n(x) \} \) defined on \([a, b]\), for any positive integer \( n \), there always exists a sequence of points

\[
a \leq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leq b,
\]

such that

\[
\begin{vmatrix}
\varphi_1(x_1^{(n)}) & \varphi_2(x_1^{(n)}) & \cdots & \varphi_n(x_1^{(n)}) \\
\varphi_1(x_2^{(n)}) & \varphi_2(x_2^{(n)}) & \cdots & \varphi_n(x_2^{(n)}) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1(x_n^{(n)}) & \varphi_2(x_n^{(n)}) & \cdots & \varphi_n(x_n^{(n)})
\end{vmatrix} \neq 0.
\]

Proof. For \( n = 2 \), let \( x_1^{(2)} \) be a point such that \( \varphi_1(x_1^{(2)}) \neq 0 \). Denote \( b_2 = \varphi_1(x_1^{(2)}) \neq 0 \), \( b_1 = -\varphi_2(x_1^{(2)}) \). Form the linear combination,

\[
L_2(x) = b_2 \varphi_2(x) + b_1 \varphi_1(x).
\]

From the definition of linearly independent functions, there exists at least one point, say \( x_2^{(2)} \), in \([a, b]\) such that \( L_2(x_2^{(2)}) \neq 0 \), i.e. \( \varphi_1(x_2^{(2)}) \varphi_2(x_2^{(2)}) - \varphi_2(x_1^{(2)}) \varphi_1(x_2^{(2)}) \neq 0 \).

Writing in determinant form, we have

\[
\begin{vmatrix}
\varphi_1(x_1^{(2)}) & \varphi_2(x_1^{(2)}) \\
\varphi_1(x_2^{(2)}) & \varphi_2(x_2^{(2)})
\end{vmatrix} \neq 0.
\]
Thus our assertion is true for $n = 2$.

Assume that it is true for $n$, with the points $x^{(n)}_1$, $x^{(n)}_2$, ..., $x^{(n)}_n$. Let $x^{(n+1)}_{i+1} = x^{(n)}_i$ for $i = 1, 2, ..., n$, and denote

$$b_i = \left| \begin{array}{cccc}
\varphi_{i+1}(x^{(n+1)}_2) & \cdots & \varphi_{n+1}(x^{(n+1)}_2) & \varphi_1(x^{(n+1)}_2) & \varphi_2(x^{(n+1)}_2) & \cdots & \varphi_{i-1}(x^{(n+1)}_2) \\
\varphi_{i+1}(x^{(n+1)}_3) & \cdots & \varphi_{n+1}(x^{(n+1)}_3) & \varphi_1(x^{(n+1)}_3) & \varphi_2(x^{(n+1)}_3) & \cdots & \varphi_{i-1}(x^{(n+1)}_3) \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\varphi_{i+1}(x^{(n+1)}_{n+1}) & \cdots & \varphi_{n+1}(x^{(n+1)}_{n+1}) & \varphi_1(x^{(n+1)}_{n+1}) & \varphi_2(x^{(n+1)}_{n+1}) & \cdots & \varphi_{i-1}(x^{(n+1)}_{n+1}) 
\end{array} \right|$$

In particular, by our induction hypothesis, $b_{n+1} \neq 0$. Consider

$$L^{(n+1)}(x) = b_1 \varphi_i(x) + b_2 \varphi_2(x) + \cdots + b_{n+1} \varphi_{n+1}(x).$$

By definition, there exists at least one point, say $x_1^{(n+1)}$, such that $L^{(n+1)}(x_1^{(n+1)}) \neq 0$. Writing in determinant form, we have

$$\left| \begin{array}{cccc}
\varphi_1(x^{(n+1)}_1) & \varphi_2(x^{(n+1)}_1) & \cdots & \varphi_{n+1}(x^{(n+1)}_1) \\
\varphi_1(x^{(n+1)}_2) & \varphi_2(x^{(n+1)}_2) & \cdots & \varphi_{n+1}(x^{(n+1)}_2) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1(x^{(n+1)}_{n+1}) & \varphi_2(x^{(n+1)}_{n+1}) & \cdots & \varphi_{n+1}(x^{(n+1)}_{n+1}) 
\end{array} \right| \neq 0.$$

This completes the proof.
5.2. **Fundamental formula of interpolation by Haar functions.** The above lemma, being a kind of existence theorem, does by no means guarantee that we are able to find a simple representation of the interpolating coefficients. For different sequences of functions, different techniques may be needed. However for a class of functions which possess a property called finite orthogonality, this problem becomes very simple.

**Definition.** Given a sequence of functions $\{\varphi_n(x)\}$, defined on $[a, b]$, if there exists a system of fundamental sequences of points,

$$
\begin{align*}
\begin{cases}
    a \leq x_1^{(1)} \leq b \\
    a \leq x_1^{(2)} < x_2^{(2)} \leq b \\
    \ldots \\
    a < x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} \leq b \\
\end{cases}
\end{align*}
$$

(5.1)

such that, for any positive integer $n$, the following relations hold,

$$
\sum_{i=1}^{n} \varphi_{\mu} (x_i^{(n)}) \overline{\varphi}_{\nu} (x_i^{(n)}) = \begin{cases}
    0 & \mu \neq \nu, \quad 1 \leq \mu, \nu \leq n, \\
    C_{\mu, \nu} & \mu = \nu, \quad 1 \leq \mu \leq n,
\end{cases}
$$

(5.2)

where $C_{n, q}$ is a constant depending on $n$ and $q$ only, then $\{\varphi_n(x)\}$ is said to be finite orthogonal over the fundamental sequences of points (5.1).
After we prove that the Haar functions are finite orthogonal over a certain system of sequences of points, we will have easily the interpolating formula.

Theorem 5.1. Consider the first $2^n$ Haar functions,

$$
(5.3) \quad \psi(x); \chi_{(1)}(x), \chi_{(2)}^{(1)}(x); \ldots; \chi_{(n)}^{(1)}(x), \ldots, \chi_{(n)}^{(2^{n-1})}(x),
$$

and the $2^n$ points

$$
(5.4) \quad \chi_{i} = \frac{i - \frac{1}{2^n}}{2^n}, \quad i = 0, 1, 2, \ldots, 2^n - 1.
$$

Then (5.3) is finite orthogonal over (5.4), i.e.

$$
\sum_{i=0}^{2^n-1} \chi_{m_1}^{(p_1)}(x) \chi_{m_2}^{(p_2)}(x) = 0 \quad \text{either} \quad m_1 \neq m_2, \quad \text{or} \quad p_1 \neq p_2
$$

$$
\sum_{i=0}^{2^n-1} \left[\chi_{m}^{(p)}(x)\right]^2 = 2^n \quad \text{for} \quad 0 \leq m \leq n, \quad 1 \leq p \leq 2^{n-1}.
$$

Proof. For any $\chi_{m_2}^{(p_2)}(x)$ we divide the unit interval into $2^{m_2}$ open subintervals $I_{m_2}^{(p_2)}$, $p_2 = 1, 2, \ldots, 2^{m_2}$. By definition, $\chi_{m_2}^{(p_2)}$ will be constant in each of these open subintervals. Furthermore by the choice of the $2^n$ points in (5.4), each $I_{m_2}^{(p_2)}$ will contain equal number of these points in its interior. For every $\chi_{m_1}^{(p_1)}(x)$ with $m_1 > m_2$, each $I_{m_2}^{(p_2)}$ will contain two or more subintervals of $I_{m_1}^{(p_1)}$. 

Thus we have
\[ \sum_{i=0}^{2^n-1} \chi_{m_1}(x_i) \chi_{m_2}(x_i) = \begin{cases} \sqrt{2^{m_2-1}} \sum_{i=0}^{2^n-1} \chi_{m_1}(x_i) & \text{if } m_1 \text{ is odd,} \\ \sqrt{2^{m_2-1}} \sum_{i=0}^{2^n-1} \chi_{m_1}(x_i) & \text{if } m_1 \text{ is even.} \end{cases} \]

But by the choice of \( \left\{ x_i = \frac{l}{2^{m_1}} + \frac{i}{2^n} \right\} \), we have
\[ \sum_{i=0}^{2^n-1} \chi_{m_1}(x_i) = 0. \]

Thus we have
\[ \sum_{i=0}^{2^n-1} \chi_{m_1}(x_i) \chi_{m_2}(x_i) = 0, \quad \text{if } m_1 \neq m_2. \]

In case that \( m_1 = m_2 \) and \( p_1 \neq p_2 \), for any \( x_1 \) of (5.4) either \( \chi_{m_1}(x_i) = 0 \) or \( \chi_{m_2}(x_i) = 0 \). Thus we proved the orthogonality.

We observe that, for \( \chi_{n}^{(p)}(x) \), each \( I_{n}^{(p)} \) contains exactly one of the points of (5.3), for \( \chi_{m}^{(q)}(x) \), \( m = n-k \), each \( I_{m}^{(q)} \) contains exactly \( 2^k \) points. Thus we have
\[ \sum_{i=0}^{2^n-1} \left[ \chi_{m_1}(x_i) \right]^2 = 2^{m-1} \cdot 2 \cdot 2^k = 2^{m+k} = 2^n. \]

This completes the proof.

Now we are ready to give the interpolating formula.
Let $s_n(x)$ be the interpolating sum of $f(x)$ of $n^{th}$ order, then

$$
\begin{align*}
  s_n(x) = a_0 \chi_0(x) + a_1 \chi_1(x) + \cdots + a_m \chi_m(x) + \cdots + a_{2^n} \chi_{2^n}(x),
\end{align*}
$$

where

$$
\begin{align*}
  a_m = \frac{1}{2^n} \sum_{i=0}^{2^n-1} f(x_i) \chi_m(x_i), \quad \chi_i = \frac{1}{2^n} x_i + \frac{1}{2^n}.
\end{align*}
$$

In order to justify that it is an interpolating formula, we have to verify that

$$
\begin{align*}
  s_n(x_j) = f(x_j), \quad \text{for } j = 0, 1, 2, \ldots, 2^n-1.
\end{align*}
$$

This can be done very easily. We observe that

$$
\begin{align*}
  s_n(x) &= \frac{1}{2^n} \sum_{m=0}^{n} \sum_{p=1}^{2^m-1} \sum_{i=0}^{2^m-1} f(x_i) \chi_m(x_i) \chi_m(x) \\
  &= \frac{1}{2^n} \sum_{i=0}^{2^n-1} f(x_i) \sum_{m=0}^{n} \sum_{p=1}^{2^m-1} \chi_m(x_i) \chi_m(x) \\
  &= \frac{1}{2^n} \sum_{i=0}^{2^n-1} f(x_i) K_n^{(2^n-1)}(x_i, x).
\end{align*}
$$

In particular, when $x = x_j$, we have

$$
\begin{align*}
  s_n(x_j) = \frac{1}{2^n} \sum_{i=0}^{2^n-1} f(x_i) K_n^{(2^n-1)}(x_i, x_j).
\end{align*}
$$
By the choice of \( x_1's \), we see that \( K_n(2^{-n'}) (x_i, x_j) = 0 \) when \( i \neq j \) and \( K_n(2^{-n'}) (x_j, x_j) = 2^n \). Thus finally we obtain

\[
S_n(x_j) = \frac{1}{2^n} f(x_j) 2^n = f(x_j).
\]

5.3. **Convergence and degree of convergence of interpolating sums.** We have seen already the formal analogy between the Haar Fourier expansion and the interpolation by Haar functions, the former with a Fourier coefficient,

\[
\int_0^1 f(x) \chi_m^{(n)}(x) \, dx
\]

while the latter with an interpolating coefficient \( \frac{1}{2^n} \sum f(x_i) \chi_m^{(n)}(x_i) \). It is not hard to see that the interpolating coefficient is actually a Riemann sum of the integral \( \int_0^1 f(x) \chi_m^{(n)}(x) \, dx \). Thus as \( n \to \infty \) it will approach to the integral. The analogies will be brought out more clearly when we study the analytic properties of the interpolating sum.

**Theorem 5.2.** If \( f(x) \) is continuous, then its interpolating sum \( s_n \) will converge uniformly to \( f(x) \).

**Proof.** First of all, we observe that, if \( x = j/2^n \), then

\[
S_n \left( \frac{j}{2^n} \right) = \frac{1}{2^n} \left\{ f \left( \frac{j}{2^n} - \frac{1}{2} \right) 2^{n-1} + f \left( \frac{j}{2^n} + \frac{1}{2} \right) 2^{n-1} \right\} = \frac{1}{2} \left\{ f \left( \frac{j}{2^n} - \frac{1}{2} \right) + f \left( \frac{j}{2^n} + \frac{1}{2} \right) \right\}.
\]

If \( x \) is such that \( \frac{j}{2^n} < x < \frac{j+1}{2^n} \), then \( S_n(x) = f \left( \frac{j}{2^n} + \frac{1}{2} \right) \).

Now for any given \( \epsilon > 0 \), we make \( n \) so large that
Thus if \( \frac{j}{2^n} < x < (j+1)\frac{j}{2^n} \) then we have

\[
|s_n(x) - f(x)| = |f\left(\frac{j}{2^n} + \frac{1}{2^n}\right) - f(x)| < \varepsilon.
\]

If \( x = j/2^n \), then

\[
|s_n(x) - f(x)| = \frac{1}{2} f\left(\frac{j}{2^n} - \frac{1}{2^n}\right) - \frac{1}{2} f\left(\frac{j}{2^n}\right) + \frac{1}{2} f\left(\frac{j}{2^n} + \frac{1}{2^n}\right) - \frac{1}{2} f\left(\frac{j}{2^n}\right) \\
\leq \frac{1}{2} \left| f\left(\frac{j}{2^n} - \frac{1}{2^n}\right) - f\left(\frac{j}{2^n}\right) \right| + \frac{1}{2} \left| f\left(\frac{j}{2^n} + \frac{1}{2^n}\right) - f\left(\frac{j}{2^n}\right) \right| < \varepsilon.
\]

This completes the proof.

Theorem 5.3. If \( f(x) \) is a \( g \)-function with a \( B \)-set consisting of one pt. \( x_0 = j/2^k \). Then its interpolating sum \( s_n \) will converge uniformly to \( f(x) \).

Proof. First of all we observe that

\[
|s_n\left(\frac{j}{2^k}\right) - f\left(\frac{j}{2^k}\right)| \\
\leq \frac{1}{2} f\left(\frac{j}{2^k} + \frac{1}{2^n}\right) - \frac{1}{2} f\left(\frac{j}{2^k}\right) + \frac{1}{2} f\left(\frac{j}{2^k} - \frac{1}{2^n}\right) - \frac{1}{2} f\left(\frac{j}{2^k}\right)
\]

for sufficiently large \( n \). By increasing \( n \) we can make the two terms on the right side as small as we please. Thus we have proved the convergence of \( s_n(x) \) to \( f(x) \) at \( x_0 = j/2^k \).

For \( x_0 \neq j/2^k \), \( f(x) \) is continuous, and hence the interpolating sum \( s_n \) converges uniformly to \( f(x) \) when
Similarly for $j/2^k < x \leq 1$. This completes the proof.

Theorem 5.4. If $f(x)$ satisfies a Lipschitz condition of order $\alpha$, i.e.

$$|f(x_1) - f(x_2)| < \lambda |x_1 - x_2|^\alpha,$$

then

$$|S_n(x) - f(x)| < \frac{\lambda}{2} (n+1)^{\alpha}.$$

Proof. When $j/2^n < x < j+1/2^n$, we have

$$|S_n(x) - f(x)| = |f(\frac{j}{2^n} + \frac{1}{2^{n+1}}) - f(x)| < \lambda \left| x - \left( \frac{j}{2^n} + \frac{1}{2^{n+1}} \right) \right|^\alpha < \frac{\lambda}{2} (n+1)^{\alpha}.$$

When $x = j/2^n$, we have

$$|S_n(x) - f(x)| \leq \frac{1}{2} |f(\frac{j}{2^n} + \frac{1}{2^{n+1}}) - f(x)| + \frac{1}{2} |f(\frac{j}{2^n} + \frac{1}{2^{n+1}}) - f(\frac{j}{2^n})| \leq \frac{1}{2} \lambda \frac{1}{2} (n+1)^{\alpha} + \frac{1}{2} \lambda \frac{1}{2} (n+1)^{\alpha} = \frac{\lambda}{2} (n+1)^{\alpha}.$$

This completes the proof.

Similarly we prove the following

Theorem 5.5. If $f(x)$ is continuous with a modulus of continuity $\omega(\delta)$ then

$$|S_n(x) - f(x)| < \omega\left( \frac{1}{2^{n+1}} \right).$$

Furthermore by an argument which is exactly the same as the one we carried out in proving Theorem 1.3, we establish
Theorem 5.6  If $f(x)$ is a g-function with a $B$-set $B$ such that $B' \cap B = 0$ when $B'$ is the derived set of $B$, then the interpolating sum converges uniformly to $f(x)$.
Bibliography


