FERMI NEUTRINO THEORY

by

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FERMI NEUTRINO THEORY

INTRODUCTION

We recognize the radio-active β-decay as a transformation of a nucleus of charge Z into a new nucleus of charge $Z \pm 1$ with the simultaueous emission of a negative or a positive electron. Early investigations of such processes have lead to difficulties in the theoretical treatment of the β-emission. These difficulties are principally based on the fact that if we confine our attention to the observable radiations emitted from the radioactive sources[1], there is no conservation of mechanical integrals of motion. This may be readily shown by the energy equations for the process.

If we call the energy of the initial nucleus E_i and the energy of the final nucleus E_f , (fig.1.) then the energy set free in each process in which E_i is transformed into E_f with the simultaneous emission of a positive or a negative electron is $\Delta E = E_i - E_f$

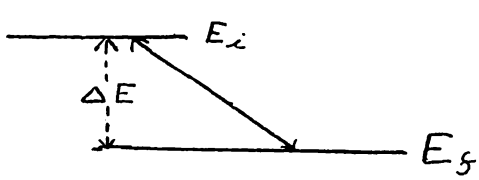


fig. 1

The energy of the emitted electron is not as one would expect equal to the well defined amount ΔE but varies between fairly wide limits in a continuous spectrum of the type shown in Fig. 2.

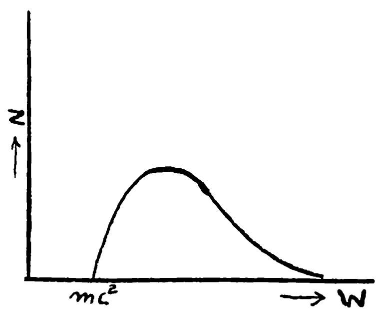


Fig. 2

In order to maintain the conservation laws then we must in some way account for the energy difference between ΔE and the energy W of the emitted electron. If we call this energy difference W' we may write for our energy balance $\Delta E = W + W'$ (1a)

and similarly for any other mechanical quantity A obeying a conservation law $A = A \ (+) \ A'$ (1b)

where $(+)$ includes the possibility of vector quantities, and symmetry properties of the eigenfunctions.

No definite information regarding the fate of the primed quantities is known at present. It may be lost, which would be a contradiction of conservation laws, or it may be described by the production of a second particle possessing properties necessary for the maintenance of the conservation laws. The treatment of the β-decay as a double process, founded upon the foregoing evidence, has been common to all theoretical attempts to account for the experimental evidence. Beck and Sitte[2] have tried to connect the β-emission with the well established phenomenon of the electron pair production according to the Dirac Theory of the electron. Following this hypothesis they have been lead to assume that one of the two particles is subsequently to be treated as captured by the nucleus without conservation of mechanical integrals. Their analysis of the empirical data however, indicated that another assumption is required in addition to the introduction of the second particle. The purpose of this assumption was to exclude the emission of slow electrons as particles of lowest angular momentum.

Describing (1a and 1b) by the production of a particle of extremely high penetrating power, Fermi[3] has proposed a formalism somewhat similar to that of Beck and Sitte in which the second particle (neutrino) has the spin momentum $\frac{1}{2}\, h/2\pi$, zero charge and Fermi

Statistics. An additional assumption is that the mass of the neutrino be zero[4]. Beck and Sitte[5] immediately criticised Fermi's treatment because of the lack of an assumption similar to that which they had found indispensable in accounting for the experimental data*. Fermi stated[14] however that while experimental results might be questionable for the low emission energies, the neutrino hypothesis would fit the experimental curves upon the inclusion of higher values of angular momentum transfer.

A more precise calculation of the Fermi integrals was therefore desirable, and it is the purpose of this paper to present a convenient method for the evaluation of the Fermi integrals. The treatment permits the immediate calculation of the emission curves for any assumption on the angular momentum balance. The results confirm the cited criticism[5] of Fermi's paper.

*) Recent investigations by Alichanjan, Alichanov, and Dzelepov (Zeits für Physik, 93, 350, 1935) indicate, however, that in the case of heavy nuclei a large number of slow electrons (s - electrons) may be emitted. This would agree with Fermi's curve as well as with Beck and Sitte's curves for s - electrons. Viz. Alichanjan, Alichanov, Dzelepov. Sov. Phys. 11, 204 (1937).

In addition to that, they may be of some interest for the discussion of the β -spectra of elements of extremely high life period (K,Rb) for which large changes of angular momentum have been suggested.

This evaluation of the Fermi integrals for the radioactive β -decay will not only allow one to discuss the Fermi theory in the light of experimental evidence, but also allows one to calculate certain terms in the expression for the cross section for inverse processes. Such expressions being thus evaluated will allow for an estimate of the numerical value of the cross section of a neutrino for processes inverse to the radio-active β -decay. These results add further agreement to the second section discussion of the discrepancies of the Fermi theory of the β -decay.

Evaluation of the Fermi Integrals

I wish now to give a treatment of the Fermi theory of the β-decay. I shall first develop a method of calculating the Fermi integrals.

In order that the treatment does not lead to the production of a pair of differently charged particles we must assume that the processes involved in the β-decay take place entirely within nuclear dimensions. We shall of course suppose that the laws of conservation of mechanical quantities hold and in addition that the emission probability of electrons and neutrinos is proportional to the respective density functions. The latter, it is to be recalled, is valid for all wave emission processes known to date[15]. If we deny this assumption[16] we must of necessity say that the β-decay is a process distinct from all known wave emissions and therefore definite physical arguments concerning its type of interaction function will need to be formulated. For the purpose of examining the Fermi theory then we will make the three assumptions in the following treatment.

1.) The laws of conservation of mechanical quantities hold.
2.) The emission probability of electrons and neutrinos is proportional to the respective density functions.

3.) Processes involved in the β-decay take place entirely within nuclear dimensions.

The Fermi theory[3] then gives us for the probability of β-emission an absolute square of an invariant expression which can be written

$$J_{(w)} = C\left| \int \left\{ U A^0 V^* + \sum U \alpha_i A^i V^* \right\} d\tau \right|^2 \qquad 16.$$

where U is the eigenfunction of the neutron in the initial nucleus and V the eigenfunction of the proton in the final nucleus. α_i Dirac matrices.

In equation (16) the Dirac theory is assumed for both particles. This is certainly incompatible with the experimental evidence of Stern and Rabi[11]. It will be shown however (P.P. 19-20) that the discussion is valid for any relativistic expression of the type (16).
A^0 , A^1 , A^2 , A^3 , are the components of a four vector which depends upon the eigenfunctions of the emitted neutrino (ψ') and the eigenfunctions of the emitted electron (ψ). This four vector is defined by

$$A^0 = \psi \delta \psi' \qquad \delta = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} \qquad 17.$$
$$A^i = -\psi \delta \alpha_i \psi'$$

corresponding to the emission of an electron and a neutrino in the positive part of its energy spectrum.

ψ is here understood to refer to an electron emitted with the energy W (including m_0c^2). ψ' represents the neutrino of energy W' where evidently $W + W' = \Delta E$ if ΔE is the energy liberated by the transformed nucleus. $\mathcal{U}$ and V are assumed to be given. $\mathcal{U}$ refers in particular to a given angular momentum of the initial nuclear state. V refers to the angular momentum of the final state. Bessel functions extended over the region of nuclear dimensions R will be used for the radial dependence of $\mathcal{U}$ and V

Now as has been pointed out before, ψ and ψ' (in polar coordinates) refer to well defined angular momenta of the emitted particles. We are interested in the total number of emitted particles without regard to their angular momenta. Assuming then that emission probabilities in which different angular momenta are involved may be regarded as independent processes; (16) should be evaluated for every possible pair of solutions ψ_κ and $\psi_{\kappa'}$ (limited by conservation laws) and the sum over all of these possible transitions is then to be taken.

Equation (16) then becomes

$$J(w) = C \sum_{KmK'm'} \left| \int \left(\mathcal{U}_{\lambda n} A^{0}_{KmK'm'} V^{*}_{\lambda' n'} + \sum_{i=1}^{3} \mathcal{U}_{\lambda n} A^{i}_{KmK'm'} \alpha_i V^{*}_{\lambda n} \right) d\tau \right|^2 \qquad 18.$$

where the summation is to be extended over all possible values and combinations of K, m, K', and m'. K, m refering to the electron quantum numbers and K', m' to the quantum numbers of the neutrinos.

Regarding the nucleus as a spherical system the involved eigenfunctions will be of the form

Neutron	$\mathcal{U}^{\alpha}_{m} = R_1 W^{\alpha}_{\lambda n}$	$\mathcal{U}^{\gamma}_{m} = R_2 W^{\gamma}_{\lambda n}$
	$\mathcal{U}^{\beta}_{m} = R_1 W^{\beta}_{\lambda n}$	$\mathcal{U}^{\delta}_{m} = R_2 W^{\delta}_{\lambda n}$
Proton	$V^{\alpha}_{n} = S_1 W^{\alpha}_{\lambda' n'}$	$V^{\gamma}_{n} = S_2 W^{\gamma}_{\lambda' n'}$
	$V^{\beta}_{m} = S_1 W^{\beta}_{\lambda' n'}$	$V^{\delta}_{n} = S_2 W^{\delta}_{\lambda' n'}$ 19.
Electron	$\Psi^{\alpha}_{K} = X_K W^{\alpha}_{Km}$	$\Psi^{\gamma}_{K} = \Phi_K W^{\gamma}_{Km}$
	$\Psi^{\beta}_{K} = X_K W^{\beta}_{Km}$	$\Psi^{\delta}_{K} = \Phi_K W^{\delta}_{Km}$
Neutrino	$\Psi'^{\alpha}_{K'} = \Xi_{K'} W^{\alpha}_{K'm'}$	$\Psi'^{\gamma}_{K'} = \Phi_{K'} W^{\gamma}_{K'm'}$
	$\Psi'^{\beta}_{K'} = \Xi_{K'} W^{\beta}_{K'm'}$	$\Psi'^{\delta}_{K'} = \Phi_{K'} W^{\delta}_{K'm'}$

where R, S, Ξ, Φ, X, and Φ are radial functions and ω^{ρ} $(\rho = \alpha, \beta, \gamma, \delta)$ spherical harmonics of the type used in the Dirac theory of the electron[6]. Substituting the values of $\mathcal{U}$ and V into (18) we obtain

$$J(w) = C \sum_{KmK'm'} \Bigg| \iint A^0 \Big\{ R_1 S_1^* (\omega^{\alpha}_{\lambda m} \omega^{\alpha *}_{\lambda' m'} + \omega^{\beta}_{\lambda m} \omega^{\beta *}_{\lambda' m'}) + R_2 S_2^* (\omega^{\gamma}_{\lambda m} \omega^{\gamma *}_{\lambda' m'} + \omega^{\delta}_{\lambda m} \omega^{\delta *}_{\lambda' m'})$$

$$+ A^1 \Big[R_1 S_2^* (\omega^{\alpha}_{\lambda m} \omega^{\delta *}_{\lambda' m'} + \omega^{\beta}_{\lambda m} \omega^{\gamma *}_{\lambda' m'}) + R_2 S_1^* (\omega^{\gamma}_{\lambda m} \omega^{\beta *}_{\lambda' m'} + \omega^{\delta}_{\lambda m} \omega^{\alpha *}_{\lambda' m'}) \Big]$$

$$+ A^2 \Big[R_1 S_2^* (-i\, \omega^{\alpha}_{\lambda m} \omega^{\delta *}_{\lambda' m'} + i\, \omega^{\beta}_{\lambda m} \omega^{\gamma *}_{\lambda' m'}) + R_2 S_1^* (-i\, \omega^{\gamma}_{\lambda m} \omega^{\beta *}_{\lambda' m'} + i\, \omega^{\delta}_{\lambda m} \omega^{\alpha *}_{\lambda' m'}) \Big] \tag{20.}$$

$$+ A^3 \Big[R_1 S_2^* (\omega^{\alpha}_{\lambda m} \omega^{\gamma *}_{\lambda' m'} - \omega^{\beta}_{\lambda m} \omega^{\delta *}_{\lambda' m'}) + R_2 S_1^* (\omega^{\gamma}_{\lambda m} \omega^{\alpha *}_{\lambda' m'} - \omega^{\delta}_{\lambda m} \omega^{\beta *}_{\lambda' m'}) \Big] \Bigg|^2$$

also from (17) we have

$$A^0 = -\psi^{\alpha} \psi'^{\beta} + \psi^{\beta} \psi'^{\alpha} + \psi^{\gamma} \psi'^{\delta} - \psi^{\delta} \psi'^{\gamma}$$

$$A^1 = \psi^{\alpha} \psi'^{\gamma} - \psi^{\beta} \psi'^{\delta} - \psi^{\gamma} \psi'^{\alpha} + \psi^{\delta} \psi'^{\beta}$$

$$A^2 = i\psi^{\alpha} \psi'^{\gamma} + i\psi^{\beta} \psi'^{\delta} - i\psi^{\gamma} \psi'^{\alpha} - i\psi^{\delta} \psi'^{\beta} \tag{20a.}$$

$$A^3 = -\psi^{\alpha} \psi'^{\delta} - \psi^{\beta} \psi'^{\gamma} + \psi^{\gamma} \psi'^{\beta} + \psi^{\delta} \psi'^{\alpha}$$

which were obtained by using the values

$$\delta\alpha_1 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \qquad \delta\alpha_2 = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix} \qquad \text{20b.}$$

$$\delta\alpha_3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

Expanding (20) by use of the equalities (19) and (20a) we obtain

$$J(w) = c\sum_{KmK'm'} \Big| \iint (X_K \Xi_{K'} a + \Phi_K \Phi_{K'} b)\Big[R_1 S_1^*(\omega^{\alpha}_{\lambda m}\omega^{\alpha *}_{\lambda' m'} + \omega^{\beta}_{\lambda m}\omega^{\beta *}_{\lambda' m'}) + R_2 S_2^*(\omega^{\gamma}_{\lambda m}\omega^{\gamma *}_{\lambda' m'} + \omega^{\delta}_{\lambda m}\omega^{\delta *}_{\lambda' m'})\Big]$$

$$+ (X_K \Phi_{K'} c + \Phi_K \Xi_K d)\Big[R_1 S_2^*(\omega^{\alpha}_{\lambda m}\omega^{\gamma *}_{\lambda' m'} + \omega^{\beta}_{\lambda m}\omega^{\delta *}_{\lambda' m'}) + R_2 S_1^*(\omega^{\gamma}_{\lambda m}\omega^{\alpha *}_{\lambda' m'} + \omega^{\delta}_{\lambda m}\omega^{\beta *}_{\lambda' m'})\Big]$$

$$- (X_K \Phi_{K'} e + \Phi_K \Xi_{K'} f)\Big[R_1 S_2^*(-\omega^{\alpha}_{\lambda m}\omega^{\delta *}_{\lambda' m'} + \omega^{\beta}_{\lambda m}\omega^{\gamma *}_{\lambda' m'}) + R_2 S_1^*(\omega^{\gamma}_{\lambda m}\omega^{\beta *}_{\lambda' m'} + \omega^{\delta}_{\lambda m}\omega^{\alpha *}_{\lambda' m'})\Big] \qquad \text{21.}$$

$$+ (X_K \Phi_{K'} g + \Phi_K \Xi_{K'} h)\Big[R_1 S_2^*(\omega^{\alpha}_{\lambda m}\omega^{\delta *}_{\lambda' m'} - \omega^{\beta}_{\lambda m}\omega^{\gamma *}_{\lambda' m'}) + R_2 S_1^*(\omega^{\gamma}_{\lambda m}\omega^{\alpha *}_{\lambda' m'} - \omega^{\delta}_{\lambda m}\omega^{\beta *}_{\lambda' m'})\Big]\Big|^2$$

where

$$a = (-\omega^{\alpha}_{Km}\omega^{\beta}_{K'm'} + \omega^{\beta}_{Km}\omega^{\alpha}_{K'm'}) \qquad b = (\omega^{\gamma}_{Km}\omega^{\delta}_{K'm'} - \omega^{\delta}_{Km}\omega^{\gamma}_{K'm'})$$

$$c = (\omega^{\alpha}_{Km}\omega^{\gamma}_{K'm'} - \omega^{\beta}_{Km}\omega^{\delta}_{K'm'}) \qquad d = (-\omega^{\gamma}_{Km}\omega^{\alpha}_{K'm'} + \omega^{\delta}_{Km}\omega^{\beta}_{K'm'})$$

$$e = (\omega^{\alpha}_{Km}\omega^{\gamma}_{K'm'} + \omega^{\beta}_{Km}\omega^{\delta}_{K'm'}) \qquad f = (-\omega^{\gamma}_{Km}\omega^{\alpha}_{K'm'} - \omega^{\delta}_{Km}\omega^{\beta}_{K'm'}) \qquad \text{21b.}$$

$$g = (-\omega^{\alpha}_{Km}\omega^{\delta}_{K'm'} - \omega^{\beta}_{Km}\omega^{\gamma}_{K'm'}) \qquad h = (\omega^{\gamma}_{Km}\omega^{\beta}_{K'm'} + \omega^{\delta}_{Km}\omega^{\alpha}_{K'm'})$$

Equation (21) is very involved and cumbersome to handle; We will now show how several simplifications can be effected which will reduce (21) to a simple form. Using the relationships

$$\omega^{\alpha}_{\lambda n} = \omega^{\gamma}_{\lambda n}\cos\theta + \omega^{\delta}_{\lambda n}\sin\theta\, e^{-i\phi}$$

$$\omega^{\beta}_{\lambda n} = \omega^{\gamma}_{\lambda n}\sin\theta\, e^{-i\phi} - \omega^{\delta}_{\lambda n}\cos\theta \qquad \text{23a.}$$

it is readily shown by substitution that

$$\omega^{\alpha}_{\lambda n}\omega^{\alpha *}_{\lambda' n'} + \omega^{\beta}_{\lambda n}\omega^{\beta *}_{\lambda' n'} = \omega^{\gamma}_{\lambda n}\omega^{\gamma *}_{\lambda' n'} + \omega^{\delta}_{\lambda n}\omega^{\delta *}_{\lambda' n'}$$

The functions $\Xi_{\kappa'}$ and $\Phi_{\kappa'}$ are to be expressed by Bessel functions. For a neutrino of zero mass these reduce to

$$\Xi_{\kappa'} = \sqrt{\tfrac{1}{2}}\, X_{\kappa'-1} \quad \text{and} \quad \Phi_{\kappa'} = -i\sqrt{\tfrac{1}{2}}\, X_{-\kappa'} \quad \text{for } \kappa' < 0 \qquad \text{23b.}$$

$$\Xi_{\kappa'} = \sqrt{\tfrac{1}{2}}\, X_{+\kappa'} \quad \text{and} \quad \Phi_{\kappa'} = +i\sqrt{\tfrac{1}{2}}\, X_{\kappa'-1} \quad \text{for } \kappa' > 0 \qquad \text{23c.}$$

where

$$X_{\ell} = \sqrt{\frac{2}{\pi}}\; \frac{k\,(k\cdot r)^{\ell}}{1\cdot 3\cdot 5\cdots(2\ell+1)}$$

For small radius then (inside the nucleus)

$$\kappa' < 0, \quad \Xi_{\kappa'} \gg \Phi_{\kappa'}$$

$\Phi_{\kappa'}$ can be disregarded compared to $\Xi_{\kappa'}$

and also

$$\kappa' > 0, \quad \Xi_{\kappa'} \ll \Phi_{\kappa'}$$

$\Xi_{\kappa'}$ can be disregarded compared to $\Phi_{\kappa'}$

Integrating (21) over the angular dependence, since R_1, R_2, S_1, S_2 vanish beyond R equal to the radius of the nucleus and will yield what are essentially constants when integrated over the nucleus, we may write equation (21) in the simplified form

$$\kappa' < 0 \quad J_{(W)} = C' \sum_{\kappa m \kappa' m'} \left| \int \Xi_{\kappa'} (A X_\kappa + B \Phi_\kappa) \, d\tau \right|^2 \qquad \text{22a.}$$

$$\kappa' > 0 \quad J_{(W)} = C' \sum_{\kappa m \kappa' m'} \left| \int \Phi_{\kappa'} (A' \Phi_\kappa + B' X_\kappa) \, d\tau \right|^2 \qquad \text{22b.}$$

where C' is to designate the change in the constant.

B and B' are to designate the fact that for given transitions, integrations over the angular functions yield constants. This will be more evident in equation (43) where these terms are discussed in detail. For heavy nuclei $\left(\frac{Z}{137} = 0.6\right)$

$$X_\kappa \sim 3\, \Phi_\kappa \qquad \text{for} \quad \kappa < 0$$

22c.

$$X_\kappa \sim \frac{1}{3}\, \Phi_\kappa \qquad \text{for} \quad \kappa > 0$$

On the other hand we know that

$$\frac{B}{A} \sim \frac{v}{c} \sim \frac{1}{10}$$

and therefore we may neglect the second terms ($B\Phi_\kappa$ resp $B' X_\kappa$) in (22a) and (22b) if

$$\kappa' < 0\,,\ \kappa < 0 \qquad \text{or} \qquad \kappa' > 0\,,\ \kappa > 0$$

If however

$$\kappa' < 0\,,\ \kappa > 0 \qquad \text{or} \qquad \kappa' > 0\,,\ \kappa < 0$$

both terms become of the same order of magnitude and an error of order one can be expected if we take only the first terms of (22a) and (22b) into account. The evaluation of only the first parts of (22a) and (22b) immediately raises the question of whether or not one might affect the shape of the emission curve of $J(w)$ by such omission.

We can easily show, however, that in our approximation

$$\left(R \ll \lambda = \frac{h}{2\pi m c\sqrt{\left(\frac{W}{mc^2}\right)^2 - 1}} \right) \qquad \text{that} \quad \frac{X_{-\kappa}}{\Phi_\kappa} = \text{constant}$$

and therefore we may write

$$A X_\kappa + B \Phi_\kappa = \text{constant} \cdot \Phi_\kappa = \text{constant} \cdot X_{-\kappa}$$

Thus the dependence of W in every term of the sum (22a) and (22b) is not affected by the choice of the constants A and B.

$$\frac{X_{-\kappa}}{\Phi_\kappa} = \frac{(\sqrt{\kappa^2-\alpha^2} - \kappa)\sqrt{\frac{W}{mc^2}+1} - i\alpha\sqrt{\frac{W}{mc^2}-1}}{(\sqrt{\kappa^2-\alpha^2} + \kappa)\sqrt{\frac{W}{mc^2}-1} - i\alpha\sqrt{\frac{W}{mc^2}+1}}$$

$$= \frac{(\sqrt{\kappa^2-\alpha^2} - \kappa) - i\alpha q}{(\sqrt{\kappa^2-\alpha^2} + \kappa) q - i\alpha} \qquad q = \frac{\sqrt{\frac{W}{mc^2}-1}}{\sqrt{\frac{W}{mc^2}+1}}$$

$$= \frac{-i\alpha q^2(\sqrt{\kappa^2-\alpha^2}+\kappa) + i\alpha(\sqrt{\kappa^2-\alpha^2}-\kappa)}{(\sqrt{\kappa^2-\alpha^2}+\kappa)^2 q^2 + \alpha^2}$$

$$= \frac{-i\alpha}{\sqrt{\kappa^2-\alpha^2}+\kappa} \; \frac{q^2(\sqrt{\kappa^2-\alpha^2}+\kappa)^2 - (\sqrt{\kappa^2-\alpha^2}-\kappa)(\sqrt{\kappa^2-\alpha^2}+\kappa)}{(\sqrt{\kappa^2-\alpha^2}+\kappa)^2 q^2 + \alpha^2}$$

$$\frac{X_\kappa}{\Phi_\kappa} = -\frac{i\alpha}{\sqrt{\kappa^2-\alpha^2}+\kappa}$$

These considerations then allow us to evaluate only the non-relativistic parts of the integrals. For $\kappa' < 0$, $\kappa < 0$ and $\kappa' > 0$, $\kappa > 0$ equation (22) will yield $J(w)$ with an error of approximately 6%; However in case $\kappa' < 0$, $\kappa > 0$ and $\kappa' > 0$, $\kappa < 0$ the total probabilities will not be correct within a factor of the order of magnitude one. It has been shown, however, that the shape of the curves is independent of this error as $(\mathrm{Const}\,\Phi_\kappa + \mathrm{Const}\,X_\kappa)$ may be expresses as Const $\cdot\, \Phi_\kappa$ or Const $\cdot\, X_\kappa$

We wish to evaluate then the simplified integrals

$\kappa' < 0$

$$J(w) = C' \sum_{\kappa m \kappa' m'} \left| A_{\kappa m \kappa' m'} \right|^2 \cdot \left| \int \Xi_{\kappa'} X_\kappa \, r^2 \, dr \right|^2 \qquad \text{25a.}$$

and $\kappa' > 0$

$$J(w) = C' \sum_{\kappa m \kappa' m'} \left| A_{\kappa m \kappa' m'} \right|^2 \cdot \left| \int \Phi_{\kappa'} \, \varphi_\kappa \, r^2 \, dr \right|^2 \qquad \text{25b.}$$

where

$$A = \int_0^{2\pi} d\varphi \int_0^{\pi} \left(-\omega^{\alpha}_{\kappa m} \omega^{\beta}_{\kappa' m'} + \omega^{\beta}_{\kappa m} \omega^{\alpha}_{\kappa' m'} \right) \left(\omega^{\alpha}_{\lambda n} \omega^{\alpha}_{\lambda' n'} + \omega^{\beta}_{\lambda n} \omega^{\beta}_{\lambda' n'} \right) \sin\theta \, d\theta \qquad \text{26a.}$$

is the angular dependence.

Let us then calculate the integrals

$$\int_0^R \Xi_{\kappa'} X_\kappa \, r^2 dr \qquad \kappa' < 0 \qquad 26a$$

and

$$\int_0^R \Phi_{\kappa'} \varphi_\kappa \, r^2 dr \qquad \kappa' > 0 \qquad 26b.$$

The Bessel functions in (26a & 26b.) along with the normalization coefficients take the form (for a neutrino of zero mass)- free field solutions.

$$\Xi_{\kappa'} = \sqrt{\tfrac{1}{2}}\, X_{-\kappa'-1} \qquad \Phi_{\kappa'} = -i\sqrt{\tfrac{1}{2}}\, X_{\kappa'-1}$$

27a

$$X_\ell = \sqrt{\frac{2}{\pi}}\; \frac{k\,(k\cdot r)^\ell}{1\cdot 3\cdot 5 \cdots (2\ell+1)}$$

and for the electron

$$X_\kappa = N_\kappa \sqrt{\tfrac{1}{2}\left(1+\tfrac{mc^2}{W}\right)}\; (2i\,k\cdot r)^{\sqrt{\kappa^2-\alpha^2}-1}\; [b_0 + a_0]$$

$$\varphi_\kappa = N_\kappa \sqrt{\tfrac{1}{2}\left(1-\tfrac{mc^2}{W}\right)}\; (2i\,k\,r)^{\sqrt{\kappa^2-\alpha^2}-1}\; [b_0 - a_0] \qquad 27a'.$$

$$N_\kappa = \sqrt{\frac{2}{\pi}}\cdot k\cdot \epsilon^{\frac{\pi}{2kb}\frac{W}{mc^2}}\; \frac{\left|\Gamma\left(\sqrt{\kappa^2-\alpha^2}+1+\frac{i}{kb}\frac{W}{mc^2}\right)\right|}{\Gamma\left(2\sqrt{\kappa^2-\alpha^2}+1\right)}$$

where

$$a_0 = -\frac{\kappa - i/kb}{\sqrt{\kappa^2 + \frac{1}{k^2 b^2}}}$$

$$b_0 = +\frac{\sqrt{\kappa^2-\alpha^2} - \frac{i}{kb}\frac{W}{mc^2}}{\sqrt{\kappa^2 + \frac{1}{k^2 b^2}}}$$

so that

$$(b_0 + a_0) = \frac{\sqrt{\kappa^2-\alpha^2} - \kappa + \frac{i}{kb}\left(1 - \frac{W}{mc^2}\right)}{\sqrt{\kappa^2 + \frac{1}{k^2 b^2}}} \qquad \text{27b.}$$

$$(b_0 - a_0) = \frac{\sqrt{\kappa^2-\alpha^2} + \kappa - i/kb\left(1 + \frac{W}{mc^2}\right)}{\sqrt{\kappa^2 + 1/k^2 b^2}}$$

Then

$$\chi_\kappa = N_\kappa \sqrt{\tfrac{1}{2}\left(1+\tfrac{mc^2}{W}\right)}\,(2ikr)^{\sqrt{\kappa^2-\alpha^2}-1}\left(\frac{\sqrt{\kappa^2-\alpha^2} - \kappa + \frac{i}{kb}\left(1 - \frac{W}{mc^2}\right)}{\sqrt{\kappa^2 + 1/k^2 b^2}}\right) \qquad \text{27c.}$$

$$\Phi_\kappa = N_\kappa \sqrt{\tfrac{1}{2}\left(1-\tfrac{mc^2}{W}\right)}\,(2ikr)^{\sqrt{\kappa^2-\alpha^2}-1}\left(\frac{\sqrt{\kappa^2+\alpha^2} + \kappa - \frac{i}{kb}\left(1 + \frac{W}{mc^2}\right)}{\sqrt{\kappa^2 + 1/k^2 b^2}}\right)$$

In view of the relation

$$\left|\Gamma(\alpha + i\beta)\right|^2 \cong \frac{2\pi\,\beta^{2\alpha-1}}{\epsilon^{\pi\beta}} \qquad \tfrac{1}{2} < \alpha < 1$$

and

$$\Gamma(\beta) = (\beta - 1)\,\Gamma(\beta - 1) \qquad \beta > 1$$

$$\Gamma\left(\sqrt{k^2-\alpha^2} + 1 + \frac{i}{kb}\frac{W}{mc^2}\right) = \Gamma\left(\sqrt{k^2-\alpha^2} + 1 + i\beta\right) = \qquad \text{where } \beta = \frac{W}{kbmc^2}$$

$$= \left|\left(\sqrt{k^2-\alpha^2} + i\beta\right)\left(\sqrt{k^2-\alpha^2} - 1 + ib\right) \cdots \left(\sqrt{k^2-\alpha^2} - |k| + i\beta\right) \cdot \right.$$

$$\left. \cdot\, \Gamma\left(\sqrt{k^2-\alpha^2} - |k| + 1 + i\beta\right)\right| =$$

28a.

$$\doteq \sqrt{k^2-\alpha^2+\beta^2}\;\sqrt{\left(\sqrt{k^2-\alpha^2}-1\right)^2+\beta^2} \cdots\cdots \left|\Gamma\left(\sqrt{k^2-\alpha^2} - |k| + 1 + i\beta\right)\right|$$

and the last term

$$\left|\Gamma\left(\sqrt{k^2-\alpha^2} - |k| + 1 + i\beta\right)\right| \cong \sqrt{\frac{2\pi\,\beta^{2\left(\sqrt{k^2-\alpha^2}-|k|+\frac{1}{2}\right)}}{\epsilon^{\pi\beta}}} \qquad \text{28b.}$$

Substituting these values into (26a) gives

$$\int \Xi_{\kappa'} X_{\kappa} r^2 dr = \int \sqrt{\tfrac{1}{2}} \sqrt{\tfrac{2}{\pi}} \frac{k' (k' r)^{(-\kappa'-1)}}{1\cdot 3\cdot 5 \cdots (2(-\kappa'-1)+1)} \sqrt{\tfrac{2}{\pi}} \cdot k \cdot \epsilon^{\frac{\pi\beta}{2}}$$

$$\left(\sqrt{\kappa^2-\alpha^2+\beta^2} \sqrt{(\sqrt{\kappa^2-\alpha^2}-1)^2+\beta^2} \cdots\cdots\right) \frac{2\pi \beta^{2(\sqrt{\kappa^2-\alpha^2}-|\kappa|+\frac{1}{2})}}{\epsilon^{\frac{\pi\beta}{2}} \Gamma(2\sqrt{\kappa^2-\alpha^2}+1)}$$

$$\sqrt{\tfrac{1}{2}\left(1+\tfrac{mc^2}{W}\right)} (2ikr)^{\sqrt{\kappa^2-\alpha^2}-1} \left(\frac{\sqrt{\kappa^2-\alpha^2}-\kappa+\frac{i}{kb}\left(1-\frac{W}{mc^2}\right)}{\sqrt{\kappa^2-\frac{1}{k^2b^2}}}\right) r^2 dr$$

where, since

$$\sqrt{\kappa^2-\frac{1}{k^2b^2}} = \sqrt{(\kappa^2-\alpha^2)^2+\frac{1}{k^2b^2}\left(\frac{W}{mc^2}\right)^2} = \sqrt{\kappa^2-\alpha^2+\beta^2}$$

$$i^x = \epsilon^{\frac{i\pi x}{2}}$$

and

$$\sqrt{G} = \frac{\left(\sqrt{\kappa^2-\alpha^2+\beta^2} \cdots\cdots \sqrt{(\sqrt{\kappa^2-\alpha^2}-|\kappa|+1)^2+\beta^2}\right)}{\sqrt{\kappa^2-\alpha^2+\beta^2}} \qquad \kappa \neq 0$$

one obtains for $\kappa' < 0$

$$\int \Xi_{\kappa'} X_{\kappa}\, r^2 dr = \sqrt{\frac{2}{\pi}} \sqrt{1+\frac{mc^2}{W}}\; \frac{k\, k'^{|\kappa'|}}{1\cdot 3\cdot 5 \cdots (2|\kappa'|+1)} \sqrt{G}$$

$$\beta^{\sqrt{\kappa^2-\alpha^2}-|\kappa|+\frac{1}{2}} (2k)^{\sqrt{\kappa^2-\alpha^2}-1}\, e^{\frac{i\pi}{2}\left(\sqrt{\kappa^2-\alpha^2}-1\right)} \frac{R^{|\kappa'|+\sqrt{\kappa^2-\alpha^2}+1}}{\left(\sqrt{\kappa^2-\alpha^2}+|\kappa'|+1\right)} \cdot \qquad 30.$$

$$\frac{\left(\sqrt{\kappa^2-\alpha^2}-\kappa+\frac{i}{kb}\left(1-\frac{W}{mc^2}\right)\right)}{\Gamma\left(2\sqrt{\kappa^2-\alpha^2}+1\right)}$$

Combining powers of R, and using the values

$$k' = \frac{1}{\Lambda}\sqrt{\left(\frac{W'}{mc^2}\right)^2} \qquad b = \frac{137}{Z}\Lambda$$

$$k\cdot b = \frac{137}{Z}\sqrt{\left(\frac{W}{mc^2}\right)^2-1} \qquad k = \frac{1}{\Lambda}\sqrt{\left(\frac{W}{mc^2}\right)^2-1}$$

We get

$$\int \Xi_{\kappa'} X_{\kappa}\, r^2 dr = 2^{\sqrt{\kappa^2-\alpha^2}} e^{iE^*} \sqrt{G}\; k'^{|\kappa'|} (k\cdot b)^{|\kappa|-\frac{1}{2}} \left(\frac{R}{b}\right)^{\sqrt{\kappa^2-\alpha^2}} \qquad 31.$$

$$R^{1-\kappa'} \frac{\sqrt{1+\frac{mc^2}{W}} \left(\frac{W}{mc^2}\right)^{\sqrt{\kappa^2-\alpha^2}-|\kappa|+\frac{1}{2}} \left(\sqrt{\kappa^2-\alpha^2}-\kappa+\frac{i}{kb}\left(1-\frac{W}{mc^2}\right)\right)}{\sqrt{2\pi}\; \Gamma\left(2\sqrt{\kappa^2-\alpha^2}+1\right) \left(1\cdot 3\cdot 5 \cdots (2|\kappa'|+1)\right) \left(\sqrt{\kappa^2-\alpha^2}-\kappa'+1\right)}$$

where

$$E^* = \pi/2\left(\sqrt{\kappa^2-\alpha^2}-1\right)$$

Notice that the eigenfunctions has been expressed in terms of (k) in these integrals.

Since

$$dk = \frac{2\pi}{hc}\,\frac{\left(\frac{W}{mc^2}\right)}{\sqrt{\left(\frac{W}{mc^2}\right)^2-1}}\,dW \quad \text{and} \quad dk' = \frac{2\pi}{hc}\,dW' \qquad 32.$$

We have to normalize the integrals in the energy scale W instead of k The expression for $J(W)$ now becomes $\kappa'<0$

$$J(W) = K\,R^2\left(\frac{R}{\Lambda}\right)^{2\left(\sqrt{\kappa^2-\alpha^2}-\kappa'\right)}\left(\frac{Z}{137}\right)^{2\left(\sqrt{\kappa^2-\alpha^2}-|\kappa|+\frac{1}{2}\right)} G. \qquad 33.$$

$$\left(\frac{W'}{mc^2}\right)^{2(\kappa')}\left(\frac{W}{mc^2}\right)^{2\left(\sqrt{\kappa^2-\alpha^2}-|\kappa|\right)}\left\{\left(\frac{W}{mc^2}\right)^2-1\right\}^{|\kappa|-1}\left(\frac{2\pi}{hc}\right)^2\left(\frac{W}{mc^2}\right).$$

$$\left|A_{\kappa m \kappa' m'}\left(\left(\sqrt{\kappa^2-\alpha^2}-\kappa\right)\sqrt{\frac{W}{mc^2}+1}-i\frac{Z}{137}\sqrt{\frac{W}{mc^2}-1}\right)\right|^2$$

where

$$\bar{K} = \text{Const.} \frac{2^{2\sqrt{\kappa^2-\alpha^2}}}{2\pi\left[\Gamma\left(2\sqrt{\kappa^2-\alpha^2}+1\right)\right]^2\left(1\cdot 3\cdot 5 \quad (-2\kappa'+1)\right)^2\left(\sqrt{\kappa^2-\alpha^2}-\kappa'+1\right)^2}$$

and (viz. 26a)

$$A_{\kappa m \kappa' m'} = \int_0^{2\pi} d\varphi \int_0^{\pi} \left(-\omega^{\alpha}_{\kappa m}\omega^{\beta}_{\kappa' m'} + \omega^{\beta}_{\kappa m}\omega^{\alpha}_{\kappa' m'}\right) \cdot$$

$$\cdot \left(\omega^{\alpha}_{\lambda n}\omega^{\alpha *}_{\lambda' n'} + \omega^{\beta}_{\lambda n}\omega^{\beta *}_{\lambda' n'}\right) \operatorname{sen}\theta \, d\theta$$

$\left(\omega^{\alpha}_{\lambda n}\omega^{\alpha *}_{\lambda' n'} + \omega^{\beta}_{\lambda n}\omega^{\beta *}_{\lambda' n'}\right)$ may be taken as a constant for zero transfer of angular momenta and a spherical harmonic of higher order for larger values of angular momenta transfer. The integrals can be worked out explicitely for any transfer of angular momentum by the nucleus and since all those vanish which would violate the conservation of angular momentum we can readily determine those transitions which contribute to the total emission probabilities. $A_{\kappa m \kappa' m'}$ decreases in the region of the nucleus with increasing $|\kappa|$ and $|\kappa'|$ in proportion $\left(\frac{R}{\Lambda}\right)^{2|\kappa|}$ or $\left(\frac{R}{\Lambda}\right)^{2|\kappa'|}$ We thus see we have to consider but the lowest transition terms.

Before preceeding with the actual calculation it is best to note that expression (33) holds only for the case $K' < 0$ Of course in particular transfers of angular momentum the function $J(w)$ must be evaluated for $K' > 0$ (equation 26b.).

We wish to evaluate then

$$\int_0^R \bar{\Phi}_{K'} \varphi_K \, r^2 dr \qquad K' > 0$$

where of course the expressions for the neutrino eigenfunctions are changed in form from (27a) to

$K' > 0$

$$\Phi_{K'} = +i\sqrt{\tfrac{1}{2}}\, X_{K'-1} \qquad \Xi_{K'} = \sqrt{\tfrac{1}{2}}\, X_{K'} \qquad 34.$$

the electron eigenfunctions of course being unmodified. By a process exactly parallel to that of the preceeding section an expression for the integrals similar to equation (31) is found.

$$\int \bar{\Phi}_{K'} \varphi_K = +i\sqrt{\left(1-\frac{mc^2}{W}\right)}\, k^{K'} \epsilon^{iE^*} \sqrt{G}\; 2^{\sqrt{K^2-\alpha^2}} \left(\sqrt{K^2-\alpha^2} + K - \frac{i}{kb}\right) \qquad 35.$$

$$\left(1+\frac{W}{mc^2}\right)\left(\frac{W}{mc^2}\right)^{\left(\sqrt{K^2-\alpha^2} - |K| + \frac{1}{2}\right)} \left(\frac{R}{b}\right)^{\sqrt{K^2-\alpha^2}} R^{K'+1} \left(k \cdot b\right)^{|K| - \frac{1}{2}}$$

Similarily, from (32)

$$J(W) = K' R^2 \left(\frac{R}{\Lambda}\right)^{2(\sqrt{\kappa^2-\alpha^2}+\kappa')} \left(\frac{Z}{137}\right)^{2(\sqrt{\kappa^2-\alpha^2}-|\kappa|+\frac{1}{2})} G$$

$$\left(\frac{W'}{mc^2}\right)^{2(\sqrt{\kappa^2-\alpha^2}-|\kappa|)} \left\{\left(\frac{W}{mc^2}\right)^2 - 1\right\}^{|\kappa|-1} \left(\frac{2\pi}{hc}\right)^2 \left(\frac{W}{mc^2}\right). \qquad 36.$$

$$\left| i\, A_{\kappa m \kappa' m'} \left[(\sqrt{\kappa^2-\alpha^2}+\kappa)\sqrt{\frac{W}{mc^2}-1} - i\frac{Z}{137}\sqrt{1+\frac{W}{mc^2}} \right] \right|^2$$

where

$$K' = \text{Const} \frac{2^{2\sqrt{\kappa^2-\alpha^2}}}{2\pi\left[\Gamma(2\sqrt{\kappa^2-\alpha^2}+1)\right]^2 (1\cdot 3\cdot 5 \quad (2\kappa'-1))^2 (\sqrt{\kappa^2-\alpha^2}+\kappa'+1)^2}$$

In the actual calculation of the emission curves and tabulated data to follow the second non-relativistic terms in (21) were also integrated. These extra terms in the integrations give the complete $J(W)$ function as used.

$\kappa' < 0$

$$J_{\kappa\kappa'}(W) = K R^2 \left(\frac{R}{\Lambda}\right)^{2\left(\sqrt{\kappa^2-\alpha^2} - |\kappa'| + \frac{1}{2}\right)} G \left(\frac{W'}{mc^2}\right)^{-2\kappa'} \left(\frac{W}{mc^2}\right)^{2\left(\sqrt{\kappa^2-\alpha^2} - |\kappa|\right)}$$

$$\left\{\left(\frac{W}{mc^2}\right)^2 - 1\right\}^{|\kappa|-1} \left(\frac{2\pi}{hc}\right)^2 \left(\frac{W}{mc^2}\right) \cdot \Bigg| A_{\kappa m \kappa' m'}$$

$$\Bigg[\left(\left(\sqrt{\kappa^2-\alpha^2} - \kappa\right)\sqrt{\frac{W}{mc^2}+1} - i\frac{Z}{137}\sqrt{\frac{W}{mc^2}-1}\right) - i\left(\frac{R}{\Lambda}\right)\left(\frac{W'}{mc^2}\right)\frac{\left(\sqrt{\kappa^2-\alpha^2} - \kappa' + 1\right)}{\left(-2\kappa'+1\right)\left(\sqrt{\kappa^2-\alpha^2} - \kappa' + 2\right)}$$

$$\cdot\left(\left(\sqrt{\kappa^2-\alpha^2} + \kappa\right)\sqrt{\frac{W}{mc^2}-1} - i\frac{Z}{137}\sqrt{\frac{W}{mc^2}+1}\right)\Bigg]\Bigg|^2 \qquad \textbf{37a.}$$

$\kappa' > 0$

$$J_{\kappa\kappa'}(W) = K' R^2 \left(\frac{R}{\Lambda}\right)^{2\left(\sqrt{\kappa^2-\alpha^2} + \kappa'\right)} \left(\frac{Z}{137}\right)^{2\left(\sqrt{\kappa^2-\alpha^2} - |\kappa| + \frac{1}{2}\right)} \left(\frac{W'}{mc^2}\right)^{2\kappa'}$$

$$G \left(\frac{W}{mc^2}\right)^{2\left(\sqrt{\kappa^2-\alpha^2} - |\kappa|\right)} \left\{\left(\frac{W}{mc^2}\right)^2 - 1\right\}^{|\kappa|-1} \left(\frac{2\pi}{hc}\right)^2 \left(\frac{W}{mc^2}\right) \Bigg| A_{\kappa m \kappa' m'}$$

$$\Bigg[\frac{\left(\frac{R}{\Lambda}\right)\left(\frac{W'}{mc^2}\right)\left(\sqrt{\kappa^2-\alpha^2} + \kappa' + 1\right)\left(\left(\sqrt{\kappa^2-\alpha^2} - \kappa\right)\sqrt{\frac{W}{mc^2}+1} - i\frac{Z}{137}\sqrt{\frac{W}{mc^2}-1}\right)}{\left(2\kappa'+1\right)\left(\sqrt{\kappa^2-\alpha^2} + \kappa' + 2\right)} \qquad \textbf{37.b}$$

$$+ i\left[\left(\sqrt{\kappa^2-\alpha^2} + \kappa\right)\sqrt{\frac{W}{mc^2}-1} - i\frac{Z}{137}\sqrt{\frac{W}{mc^2}+1}\right]\Bigg|^2$$

where in both (37a) and (37b) the definitions of G and $A_{\kappa m \kappa' m'}$ are the same as in equation (33) or(and) (36). As will be shown later, the actual evaluation of the expressions (37a), (37b) confirm our conclusions regarding the terms $B \, \Phi_\kappa$ and $B \, X_\kappa$ (see equation (45b, pp 37).

The angular integrals $A_{\kappa m \kappa' m'}$ can be simplified. They are

$$A_{\kappa m \kappa' m'} = \int_0^{2\pi} d\varphi \int_0^{\pi} \left(-\omega^{\alpha}_{\kappa m}\,\omega^{\beta}_{\kappa' m'} + \omega^{\beta}_{\kappa m}\,\omega^{\alpha}_{\kappa' m'}\right)\left(\omega^{\gamma}_{\lambda n}\,\omega^{\alpha *}_{\lambda' n'} + \omega^{\beta}_{\lambda n}\,\omega^{\beta *}_{\lambda' n'}\right) \sin\theta \, d\theta \qquad 38.$$

where from the Dirac theory

$\kappa < 0$

$$\omega^{\alpha} = (-\kappa + m)\, P^{m}_{-\kappa-1}\, e^{im\varphi} \qquad \omega^{\alpha} = (\kappa - m)\, P^{m}_{\kappa}\, e^{im\varphi}$$

$$\omega^{\beta} = -P^{m+1}_{-\kappa-1}\, e^{i(m+1)\varphi} \qquad \omega^{\beta} = P^{m+1}_{\kappa}\, e^{i(m+1)\varphi} \qquad 38a.$$

$$\omega^{\gamma} = (-\kappa - m)\, P^{m}_{-\kappa}\, e^{im\varphi} \qquad \omega^{\gamma} = (\kappa + m)\, P^{m}_{\kappa-1}\, e^{im\varphi}$$

$$\omega^{\delta} = P^{m+1}_{\kappa}\, e^{i(m+1)\varphi} \qquad \omega^{\delta} = -P^{m+1}_{\kappa-1}\, e^{i(m+1)\varphi}$$

In order to discuss these integrals (38a) must be determined as functions of κ and m for the two cases $\kappa < 0$ and $\kappa > 0$ Expressions for (38a), where $m = -|\kappa|$ $|\kappa|-1$ are to be found in the appendix tables A_1 and A_2.

Since

$$P_l^m(\xi) = \frac{1}{2^l l!}\left[\sqrt{1-\xi^2}\right]^m \frac{\partial^{l+m}}{\partial \xi^{l+m}}(\xi^2-1)^l$$

For the case $\xi = \cos\theta$, P_l^m can be evaluated for the various forms found in Table A_1 and A_2. Expressions for $P_l^m(\cos\theta)$ are given in the appendix table B_1.

So that substituting the values of table B_1 into (39) the ω array is obtained in final complete form

K	m	$W^{\alpha}_{\kappa m}$	$W^{\beta}_{\kappa m}$	$W^{\gamma}_{\kappa m}$	$W^{\delta}_{\kappa m}$
$+1$	-1	$-\sqrt{\frac{1}{4\pi}}\sin\theta\, e^{-i\varphi}$	$\sqrt{\frac{1}{4\pi}}\cos\theta$	0	$-\sqrt{\frac{1}{4\pi}}$
	0	$\sqrt{\frac{1}{4\pi}}\cos\theta$	$\sqrt{\frac{1}{4\pi}}\sin\theta\, e^{i\varphi}$	$\sqrt{\frac{1}{4\pi}}$	0
$+2$	-2	$\frac{1}{2}\sqrt{3!/4\pi}\sin^2\theta\, e^{-2i\varphi}$	$-\frac{1}{2}\sqrt{3!/4\pi}\sin\theta\cos\theta\, e^{-i\varphi}$	0	$\frac{1}{2}\sqrt{3!/4\pi}\sin\theta\, e^{-i\varphi}$
	-1	$-\frac{3}{2}\sqrt{\frac{1}{2\pi}}\sin\theta\cos\theta\, e^{-i\varphi}$	$\frac{1}{2}\sqrt{\frac{1}{2\pi}}(3\cos^2\theta-1)$	$-\frac{1}{2}\sqrt{\frac{1}{2\pi}}\sin\theta\, e^{-i\varphi}$	$-\sqrt{\frac{1}{2\pi}}\cos\theta$
	0	$\sqrt{\frac{1}{8\pi}}(3\cos^2\theta-1)$	$3\sqrt{\frac{1}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}$	$2\sqrt{\frac{1}{8\pi}}\cos\theta$	$-\sqrt{\frac{1}{8\pi}}\sin\theta\, e^{i\varphi}$
	1	$3\sqrt{\frac{1}{24\pi}}\sin\theta(3\cos^2\theta-1)$	$3\sqrt{\frac{1}{24\pi}}\sin^2\theta\, e^{2i\varphi}$	$3\sqrt{\frac{1}{24\pi}}\sin\theta\, e^{i\varphi}$	0
$+3$	-3	$-\frac{1}{8}\sqrt{5!/4\pi}\sin^3\theta\, e^{-3i\varphi}$	$\frac{1}{8}\sqrt{5!/4\pi}\cos\theta\sin^2\theta\, e^{-2i\varphi}$	0	$-\frac{1}{8}\sqrt{5!/4\pi}\sin^2\theta\, e^{-2i\varphi}$
	-2	$\frac{5}{8}\sqrt{4!/4\pi}\cos\theta\sin^2\theta\, e^{-2i\varphi}$	$-\frac{1}{8}\sqrt{4!/4\pi}\sin\theta(5\cos^2\theta-1)\, e^{-2i\varphi}$	$\frac{1}{8}\sqrt{4!/4\pi}\sin^2\theta\, e^{-2i\varphi}$	$\frac{1}{2}\sqrt{4!/4\pi}\sin\theta\cos\theta\, e^{-i\varphi}$
	-1	$-\frac{1}{2}\sqrt{3!/8\pi}\sin\theta(5\cos^2\theta-1)$	$\frac{1}{2}\sqrt{3!/8\pi}(5\cos^3\theta-3\cos\theta)$	$-\sqrt{3!/8\pi}\sin\theta\cos\theta\, e^{-i\varphi}$	$-\frac{1}{2}\sqrt{3!/8\pi}(3\cos^2\theta-1)$
	0	$\frac{3}{2}\sqrt{1/12\pi}(5\cos^3\theta-3\cos\theta)$	$\frac{3}{2}\sqrt{\frac{1}{12\pi}}\sin\theta(5\cos^2\theta-1)e^{i\varphi}$	$-\frac{3}{2}\sqrt{1/12\pi}(3\cos^2\theta-1)$	$-3\sqrt{1/12\pi}\sin\theta\cos\theta\, e^{i\varphi}$
	1	$\frac{3}{2}\sqrt{\frac{1}{4!\pi}}\sin\theta(5\cos^2\theta-1)e^{i\varphi}$	$\frac{15}{2}\sqrt{\frac{1}{4!\pi}}\sin^2\theta\cos\theta\, e^{2i\varphi}$	$6\sqrt{\frac{1}{4!\pi}}\sin\theta\cos\theta\, e^{i\varphi}$	$-\frac{3}{2}\sqrt{\frac{1}{4!\pi}}\sin^2\theta\, e^{2i\varphi}$
	2	$\frac{15}{2}\sqrt{\frac{1}{5!\pi}}\sin^2\theta\cos\theta\, e^{2i\varphi}$	$\frac{15}{2}\sqrt{\frac{1}{5!\pi}}\sin^3\theta\, e^{3i\varphi}$	$\frac{15}{2}\sqrt{\frac{1}{5!\pi}}\sin^2\theta\, e^{2i\varphi}$	0

K	m	ω^{α}_{Km}	ω^{β}_{Km}	ω^{γ}_{Km}	ω^{δ}_{Km}
-1	-1	0	$-\sqrt{\frac{1}{4\pi}}$	$\sqrt{\frac{1}{4\pi}}(-\sin\theta)e^{-i\varphi}$	$\sqrt{\frac{1}{4\pi}}\cos\theta$
	0	$\sqrt{\frac{1}{4\pi}}$	0	$\sqrt{\frac{1}{4\pi}}\cos\theta$	$\sqrt{\frac{1}{4\pi}}\sin\theta\, e^{i\varphi}$
-2	-2	0	$\frac{1}{2}\sqrt{3!/4\pi}\,\sin\theta\, e^{-i\varphi}$	$\frac{1}{2}\sqrt{3!/4\pi}\,\sin^2\theta\, e^{-2i\varphi}$	$-\frac{1}{2}\sqrt{3!/4\pi}\,\sin\theta\cos\theta\, e^{-i\varphi}$
	-1	$-\frac{1}{2}\sqrt{2!/4\pi}\,\sin\theta\, e^{-i\varphi}$	$-\sqrt{2!/4\pi}\,\cos\theta$	$-\frac{3}{2}\sqrt{2!/4\pi}\,\sin\theta\cos\theta\, e^{-i\varphi}$	$\frac{1}{2}\sqrt{2!/4\pi}\,(3\cos^2\theta-1)$
	0	$2\sqrt{1/8\pi}\,\cos\theta$	$-\sqrt{1/8\pi}\,\sin\theta\, e^{i\varphi}$	$\sqrt{1/8\pi}\,(3\cos^2\theta-1)$	$3\sqrt{1/8\pi}\,\sin\theta\cos\theta\, e^{i\varphi}$
	1	$3\sqrt{1/24\pi}\,\sin\theta\, e^{i\varphi}$	0	$3\sqrt{1/24\pi}\,\sin\theta\cos\theta\, e^{i\varphi}$	$3\sqrt{1/24\pi}\,\sin^2\theta\, e^{2i\varphi}$
-3	-3	0	$-\frac{1}{8}\sqrt{5!/4\pi}\,\sin^2\theta\, e^{-2i\varphi}$	$-\frac{1}{8}\sqrt{5!/4\pi}\,\sin^3\theta\, e^{-3i\varphi}$	$\frac{1}{8}\sqrt{5!/4\pi}\,\cos\theta\,\sin^2\theta\, e^{-2i\varphi}$
	-2	$\frac{1}{8}\sqrt{4!/4\pi}\,\sin^2\theta\, e^{-2i\varphi}$	$\frac{1}{2}\sqrt{4!/2\pi}\,\sin\theta\cos\theta\, e^{-i\varphi}$	$\frac{5}{8}\sqrt{4!/4\pi}\,\cos\theta\sin^2\theta\, e^{-2i\varphi}$	$-\frac{1}{8}\sqrt{4!/4\pi}\,\sin\theta\,(5\cos^2\theta-1)e^{-i\varphi}$
	-1	$-\sqrt{3/8\pi}\,\sin\theta\cos\theta\, e^{-i\varphi}$	$-\frac{1}{2}\sqrt{3!/8\pi}\,(3\cos^2\theta-1)$	$-\frac{1}{2}\sqrt{3!/8\pi}\, e^{-i\varphi}\sin\theta(5\cos^2\theta-1)$	$\frac{1}{2}\sqrt{3!/8\pi}\,(5\cos^3\theta-3\cos\theta)$
	0	$-\frac{3}{2}\sqrt{1/12\pi}\,(3\cos^2\theta-1)$	$-3\sqrt{1/12\pi}\,\sin\theta\cos\theta\, e^{i\varphi}$	$\frac{3}{2}\sqrt{1/12\pi}\,(5\cos^3\theta-3\cos\theta)$	$\frac{3}{2}\sqrt{1/12\pi}\,\sin\theta\,(5\cos^2\theta-1)e^{i\varphi}$
	1	$12\sqrt{\frac{1}{4!\,4\pi}}\, e^{i\varphi}\sin\theta\cdot\cos\theta$	$-3\sqrt{\frac{1}{4!\,4\pi}}\,\sin^2\theta\, e^{2i\varphi}$	$3\sqrt{\frac{1}{4!\,4\pi}}\,\sin\theta\,(5\cos^2\theta-1)e^{i\varphi}$	$15\sqrt{\frac{1}{4!\,4\pi}}\,\sin^2\theta\cos\theta\, e^{2i\varphi}$
	2	$15\sqrt{\frac{1}{5!\,4\pi}}\,\sin^2\theta\, e^{2i\varphi}$	0	$15\sqrt{\frac{1}{5!\,4\pi}}\,\sin^2\theta\cos\theta\, e^{2i\varphi}$	$15\sqrt{\frac{1}{5!\,4\pi}}\,\sin^3\theta\, e^{3i\varphi}$

Each term in (41) has been multiplied by the proper normalization factor

$$N_{\kappa m} = \sqrt{\frac{1}{4\pi} \frac{(|\kappa| - m - 1)!}{(|\kappa| + m)!}}$$

which yields these values necessary in (41). See table C_1 in appendix.

All necessary quantities for the evaluation of (38) being now determined, (since (37) is complete) $J(W)$ can be determined for various transfers of angular momentum by the nucleus. It has been remarked that only those transitions for low transfer of angular momentum need be calculated. We will then calculate $J(W)$ for transfer of angular momentum 0, $h/2\pi$ and $2\, h/2\pi$, by the nucleus.

CALCULATION OF $J(W)$ FOR ZERO TRANSFER

In the calculation of the emission curves $3mc^2$ is used as the upper limit for the emission energy of the β particle*. This energy can be divided between the two emitted particles, neutrino and β -electron. We have then for zero transfer of angular momentum the following possibilities.

*) A good average value. See experimental data for Ra E

	κ	κ'	$\Sigma A^2_{\kappa m \kappa' m'}$	
$S_{\frac{1}{2}} \longleftrightarrow \sigma_{\frac{1}{2}}$	-1	-1	2	
$S_{\frac{1}{2}} \longleftrightarrow \Pi_{\frac{1}{2}}$	-1	$+1$	0	
$P_{\frac{1}{2}} \longleftrightarrow \sigma_{\frac{1}{2}}$	$+1$	-1	0	42.
$P_{\frac{1}{2}} \longleftrightarrow \Pi_{\frac{1}{2}}$	$+1$	$+1$	2	

the higher terms need not be calculated as pointed out previously*.

Since the emission probability is to be summed up for all possible combinations of κ, m, κ', m', we first consider the possible combinations of κ and κ' shown in table (42). Now the possible combinations of m and m' are limited by the relation $m = -m' - 1$. Since here $\kappa = \pm 1$ and $\kappa' = \pm 1$; the possible values are $m = 0$, $m' = 0$; and $m = 0$, $m' = 1$ The total emission probability will then be the sum for these two possibilities for the various values of κ, κ', m and m'.

*) See pp. 23

$K=-1,\ K'=-1\ ;\quad m=-1\ ,\quad m'=0$

$$\left|A_{KmK'm'}\right|^2=\left|\int_0^{2\pi}d\varphi\int_{-1}^{+1}\left(-\omega_{Km}^{\alpha}\,\omega_{K'm'}^{\beta}+\omega_{Km}^{\beta}\,\omega_{K'm'}^{\alpha}\right)d(\cos\theta)\right|^2$$

$$=\left|\int_0^{2\pi}d\varphi\int_{-1}^{+1}\left(-0+-\sqrt{\tfrac{1}{4\pi}}\sqrt{\tfrac{1}{4\pi}}\right)d(\cos\theta\right|^2 \qquad \textbf{43a.}$$

$$=\left|\int_0^{2\pi}d\varphi\left(-\tfrac{1}{2\pi}\right)\right|^2=1$$

and for $K=-1,\ K'=+1\ ;\quad m=-1\ ,\ m'=0$

$$\left|A_{KmK'm'}\right|^2=\left|\int_0^{2\pi}d\varphi\int_{-1}^{+1}-\tfrac{1}{4\pi}\cos\theta\ d(\cos\theta)\right|^2=0 \qquad \textbf{43b.}$$

For the other possibilities
$K=+1\ ,\ K'=+1\ ;\quad m=-1\ ,\ m'=0$

$$\left|A_{KmK'm'}\right|^2=\left|\int_0^{2\pi}d\varphi\int_{-1}^{+1}-\tfrac{1}{4\pi}\cos\theta\ d(\cos\theta)\right|^2=0 \qquad \textbf{43c.}$$

and for
$\kappa = +1$, $\kappa' = +1$; $m = -1$ $m' = 0$

$$\left| A_{\kappa m \kappa' m'} \right|^2 = \left| \int_0^{2\pi} d\varphi \int_{-1}^{+1} \frac{1}{4\pi} (\sin^2\theta + \cos^2\theta)\, d(\cos\theta) \right|^2 = 1 \qquad 43d.$$

For the alternative choice of the m and the m', these contributions turn out again to be zero for (43b) and (43c). In the case of (43a) and (43d) it is readily seen by reference to the tabulated values of ω, that there is a definite symmetry to the table and each corresponding inverse transition yields exactly the same value of $A_{\kappa m \kappa' m'}$. Adding these contributions to those of (43a) and (43b) the total $\sum A^2_{\kappa m \kappa' m'}$ is found for the considered transitions. See table (42). The $A_{\kappa m \kappa' m'}$ as calculated indicates that in cases two and three of (42) there would not be conservation of angular momentum. Thus only the first and fourth transition need be calculated for zero transfer of angular momentum by the nucleus.

In order to facilitate the evaluation of $J(w)$, equations (37a) and (37b), it is well to tabulate values of the various expressions for needed values of κ and κ'. In the following table

	$K=+1$	$K=+2$	$K=+3$	$K=-1$	$K=-2$	$K=-3$
$\sqrt{K^2-\alpha^2}$	.8	1.90_8	2.93_9	8	$1\,90_8$	2.93_9
$2^{2\sqrt{K^2-\alpha^2}}$	3.03	14.1_2	58.7_8	3.03	14.1_2	58.7_8
$\{1\cdot3\cdot5\cdots(2\lvert K\rvert-1)\}^2$	1	9.00	225	1	9	225
G	1	$.824+\beta^2$	$(3.76+\beta^2)(.881+\beta^2)$	1	$.824+\beta^2$	$(3.76+\beta^2)(.881+\beta^2)$
$\lvert K'\rvert=1$ $(\sqrt{K^2-\alpha^2}+\lvert K'\rvert+1)$	7.84	15.29	24.4	7.84	15.29	24.4
$\lvert K'\rvert=2$ "	14.4_4	24.1_1	35.28	14.4_4	24.1_1	35.2_8
$\lvert K'\rvert=3$ "	23.0_4	34.9_3	48.16	23.0_4	34.9_3	48.16
$\left(\frac{Z}{137}\right)^{2(\sqrt{K^2-\alpha^2}-\lvert K\rvert+\frac{1}{2})}$	.736	.663	.638	.736	.663	.638
$\left[\Gamma(2\sqrt{K^2-\alpha^2}+1)\right]^2$	2.03	338	9467	2.03	338	9467

$\Delta E = 3mc^2$; $W + W' = \Delta E$; $\Delta E - mc^2 \geq W' \geq 0$

W/mc^2	$\left(\frac{W}{mc^2}\right)^2$	$\sqrt{\frac{W}{mc^2}+1}$	$\sqrt{\frac{W}{mc^2}-1}$	$\left(\frac{W'}{mc^2}\right)^{2K'}$	$\left(\frac{W}{mc^2}\right)^{2(\sqrt{K^2-\alpha^2}-\lvert K\rvert)}$	$\left(\left(\frac{W}{mc^2}\right)^2-1\right)^{\lvert K\rvert-1}$	K
0	0	1	i	9	0	1	1
				81	0	−1	2
				729	0	1	3
0.5	.25	1.225	.7i	6.25	1.32	1	1
				39.1	1.15	−.75	2
				244	1.087	.562	3
1.0	1.0	1.41	0	4	1	1	1
				16	1	0	2
				64	1	0	3
1.5	2.25	1.581	.705	2.25	.855	1	1
				5.06	.925	1.25	2
				11.2	.953	1.56	3
2.0	4.0	1.732	1	1	.758	1	1
				1	.876	3	2
				1	.920	9	3
2.5	6.25	1.87	1.225	.25	.692	1	1
				.0625	.831	5.25	2
				.0156	.896	27.6	3
3.0	9.0	2.0	1.414	0	.644	1	1
				0	.803	8.0	2
				0	.876	64.0	3

$\alpha = 0.6 \quad \frac{Z}{137} = 0.6$

Using the results from table (44) in the equations (37a) and (37b) for the transition $S_{\frac{1}{2}} \longleftrightarrow 6_{\frac{1}{2}}$ $J(w)$ becomes

$$k' < 0 \,, \quad \frac{W}{mc^2} = 0 \qquad J(w) = 0$$

$$k = -1 \qquad k' = -1$$

$$\frac{W}{mc^2} = .5$$

$$J(w) = K R^2 \left(\frac{R}{\Lambda}\right)^{3.6} (.736)(1)(6.25)(1)(1)\left(\frac{2\pi}{hc}\right)^2 (.5)(1) \qquad \textbf{45a.}$$

$$\Sigma A^2_{kmk'm'} \left| (1.8 \times 1.22 - .6\, i\, .705) - \left(\frac{R}{\Lambda}\right) \frac{(2.5)\,(.8)(-.2)\,(.705 i)}{3 \times 3.8} \right|^2$$

$$J(w) = K R^2 \left(\frac{R}{\Lambda}\right)^{3.6} \left(\frac{2\pi}{hc}\right)^2 \Sigma A^2_{kmk'm'} \left| (2.30) \middle/ 2.6 - \left(\frac{R}{\Lambda}\right) .0247 \right|^2 \qquad \textbf{45b.}$$

where the second term is negligible as noted on pp. 13.

$$J(w) = K R^2 \left(\frac{R}{\Lambda}\right)^{3.6} \left(\frac{2\pi}{hc}\right)^2 15.55\, \Sigma A^2_{kmk'm'}$$

$$K = \frac{c}{2\pi} \quad (.2076) \qquad \textbf{45c.}$$

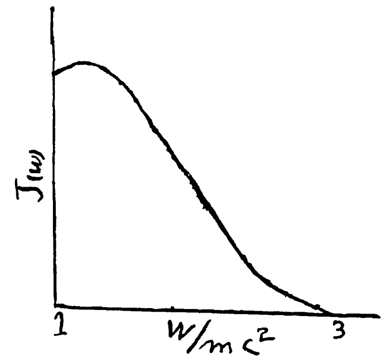
$$J(w) = \frac{c}{2\pi} R^2 \left(\frac{R}{\Lambda}\right)^{3.6} \left(\frac{2\pi}{hc}\right)^2 2.97_2\, \Sigma A^2_{kmk'm'}$$

The same procedure can be followed for the other chosen values of $\left(\frac{W}{mc^2}\right)$. The results of such calculation are given in table (46).

Transition	W/mc^2	$J(W)$	
$\kappa = -1 \quad \kappa' = -1$	0	0	
$S_{\frac{1}{2}} \longleftrightarrow \sigma_{\frac{1}{2}}$	1.0	7.92 E	
	1.2	8.12 E	
	1.5	7.28 E	
$E = \frac{c}{2\pi} R^2 \left(\frac{R}{\Lambda}\right)^{3.6} \cdot \left(\frac{2\pi}{hc}\right)^2$	2.0	4.68 E	
	2.5	1.56 E	
	3.0	0	
$\kappa = +1 \quad \kappa' = +1$	0	0	
	1.0	.878 E	
$P_{\frac{1}{2}} \longleftrightarrow \pi_{\frac{1}{2}}$	1.2	1.63_2 E	
	1.5	2.21 E	
$E = \frac{c}{2\pi} R^2 \left(\frac{R}{\Lambda}\right)^{3.6} \left(\frac{2\pi}{hc}\right)^2$	2.0	2.00 E	
	2.5	.696 E	
	3.0	0	

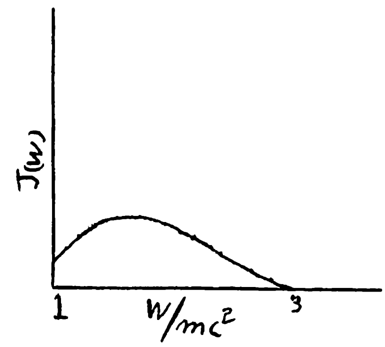
To interpret the data (46) it is plotted below.

$S_{\frac{1}{2}} \longleftrightarrow G_{\frac{1}{2}}$



(47a)

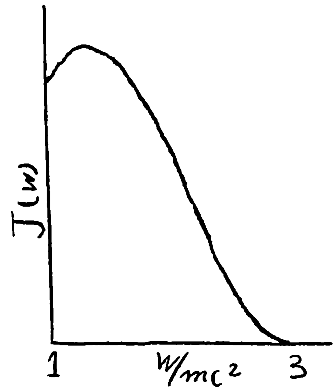
$P_{\frac{1}{2}} \longleftrightarrow \Pi_{\frac{1}{2}}$



47a'

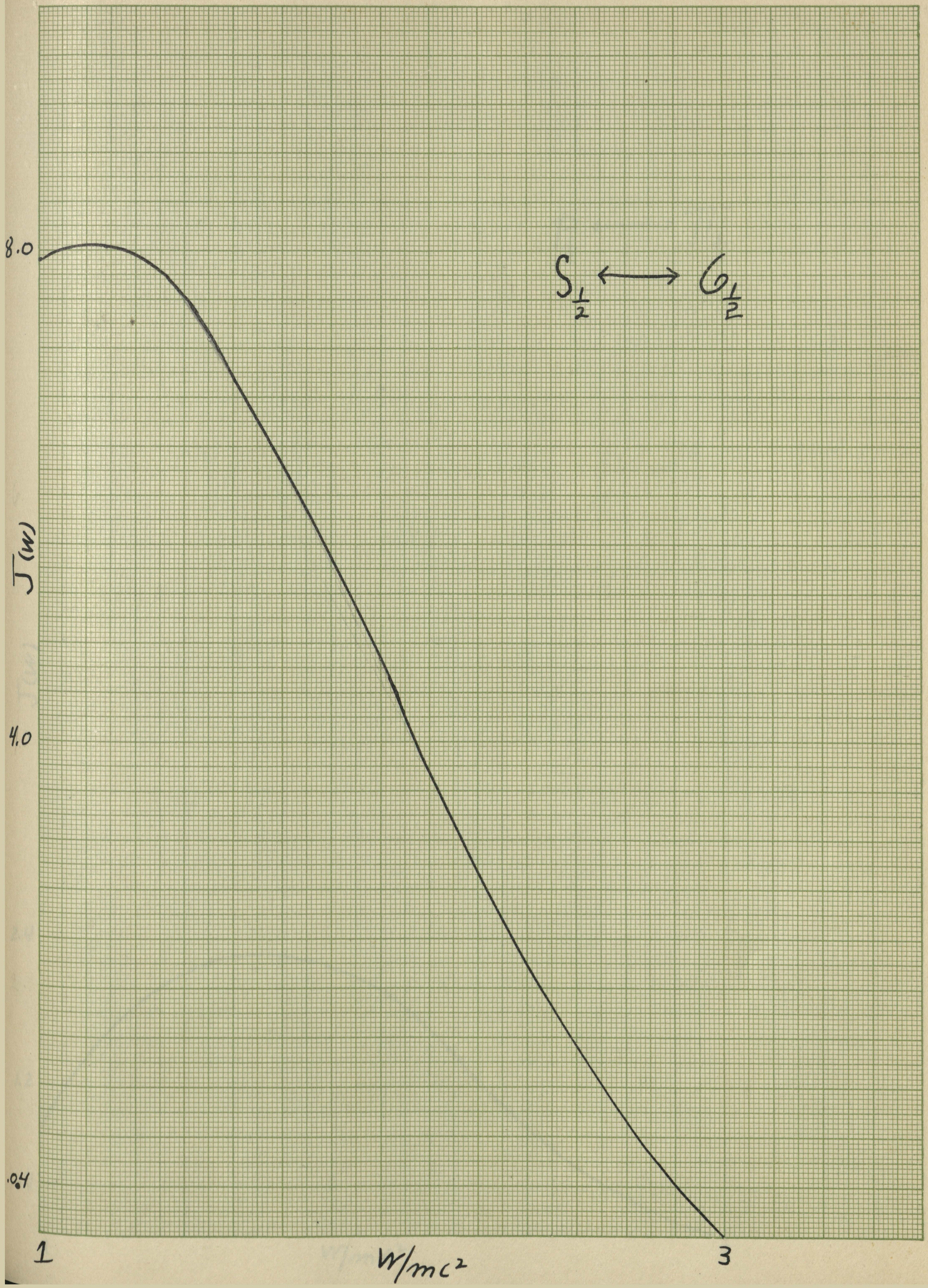
Now since by assumption the transition probabilities are independent processes the total probability of emission for zero transfer of angular momentum is given by the sum of the probabilities of emission for the $S_{\frac{1}{2}} \longleftrightarrow G_{\frac{1}{2}}$ and $P_{\frac{1}{2}} \longleftrightarrow \Pi_{\frac{1}{2}}$ transitions.

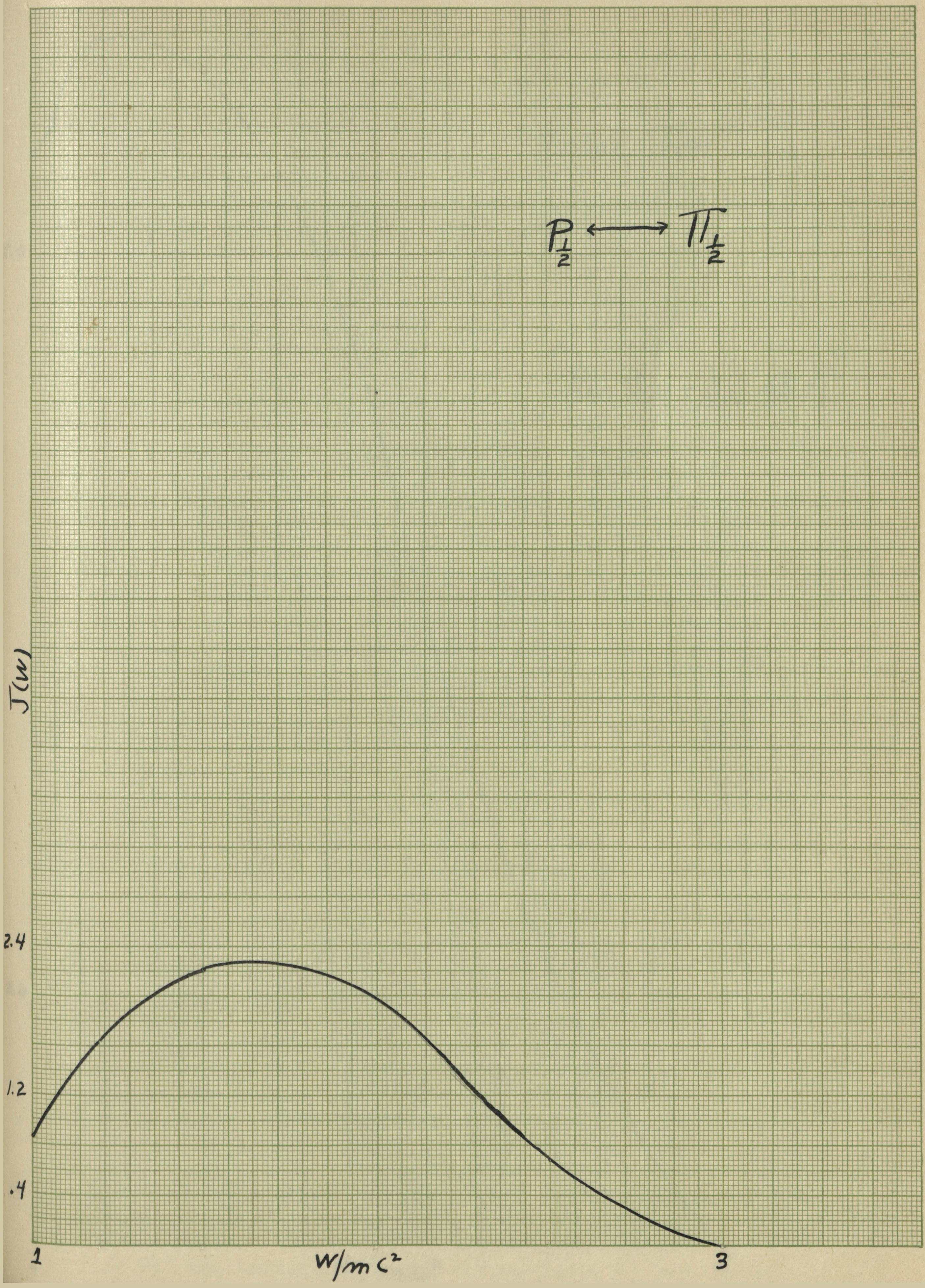
That is



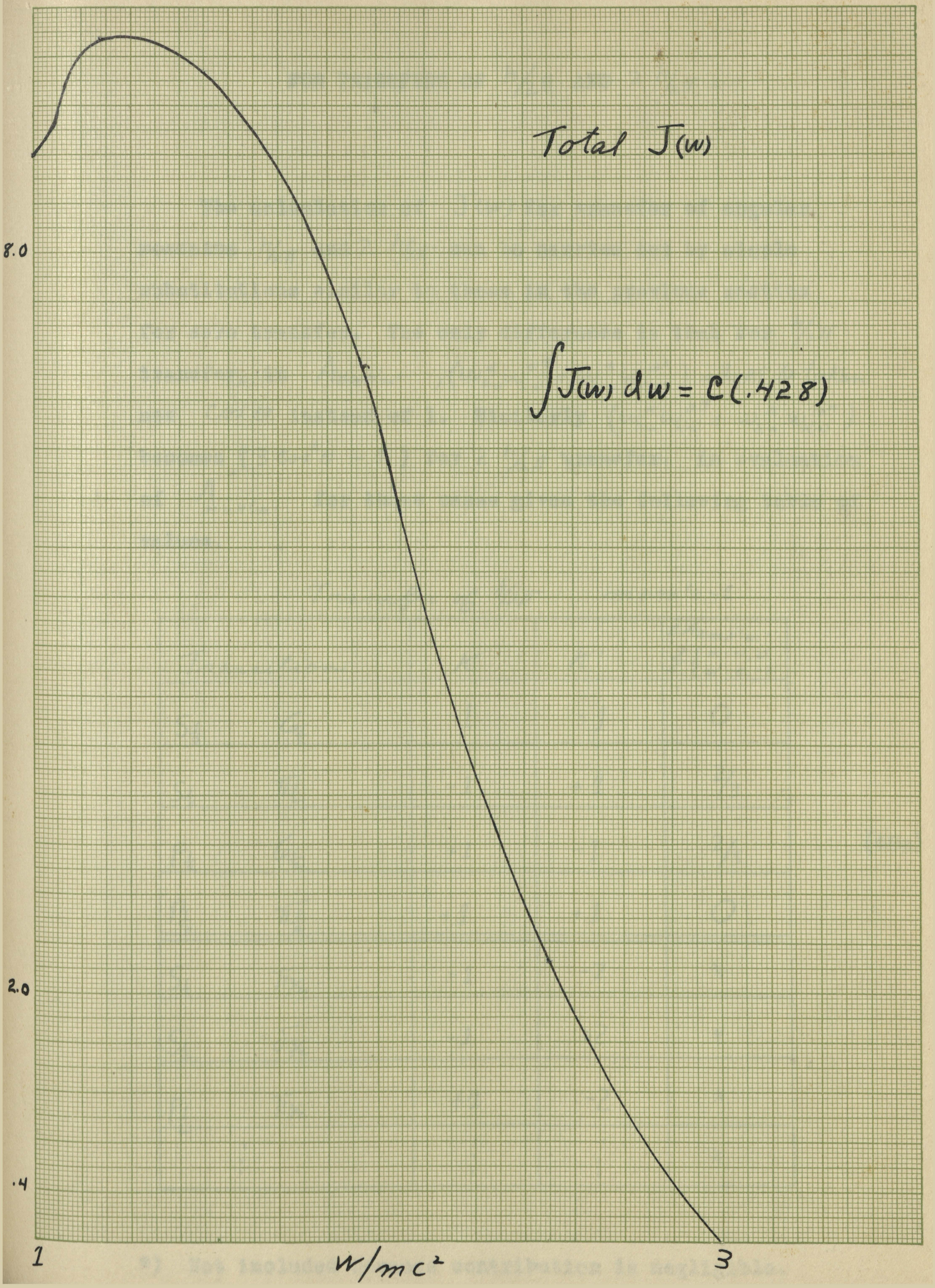
47b.

$$\int J(w)\, dw = c\ (.428)$$


8.0
4.0
.04
J(w)
$S_{\frac{1}{2}} \longleftrightarrow G_{\frac{1}{2}}$
1
W/mc²
3



$P_{\frac{1}{2}} \longleftrightarrow \Pi_{\frac{1}{2}}$
J(w)
2.4
1.2
.4
1
W/mc^2
3


Total J(w)
∫J(w) dw = C(.428)
8.0
2.0
.4
1
3
w/mc²

FOR TRANSFERS OF $h/2\pi$ AND $2\,h/2\pi$.

The calculation of $J(W)$ for transfer of angular momentum $h/2\pi$ and $2\,h/2\pi$ can be carried out by simple substitutions similar to those in the previous section for zero transfer. The only difference is that for $h/2\pi$ transfer, in $A_{\kappa m \kappa' m'}$, $(w^{\alpha}_{\lambda n} w^{\alpha *}_{\lambda' n'} + w^{\beta}_{\lambda n} w^{\beta *}_{\lambda' n'})$ shall be taken now $\cos\theta$ instead of 1. Similarly $(w^{\alpha}_{\lambda n} w^{\alpha *}_{\lambda' n'} + w^{\beta}_{\lambda n} w^{\beta *}_{\lambda' n'})$ becomes $(3\cos^2\theta - 1)$ for $2\,h/2\pi$ transfer. An evaluation of $A_{\kappa m \kappa' m'}$ for these cases gives the following table of values.

Transfer of $h/2\pi$ $\qquad m+m' = -1$

Transition		κ	κ'	$\Sigma A_{\kappa m \kappa' m'}$ $\sum \begin{cases} m=-1,0 \\ m'=0,-1 \end{cases}$
$S_{\frac{1}{2}}$	$\sigma_{\frac{1}{2}}$	-1	-1	0
$S_{\frac{1}{2}}$	$\pi_{\frac{1}{2}}$	-1	$+1$	$2/3$
$P_{\frac{1}{2}}$	$\sigma_{\frac{1}{2}}$	$+1$	-1	$2/3$
$P_{\frac{1}{2}}$	$\pi_{\frac{1}{2}}$	$+1$	$+1$	0
$S_{\frac{1}{2}}$	$\pi_{3/2}$	-1	-2	*
$S_{\frac{1}{2}}$	$\delta_{3/2}$	-1	$+2$	*
$P_{\frac{1}{2}}$	$\pi_{3/2}$	$+1$	-2	*
⋮		⋮	⋮	*

(48a)

*) Not included because contribution is negligible.

Transition	W/mc^2	Value $J(w)$
$K=+1$, $K'=-1$	1.0	.0326 E
$P_{\frac{1}{2}} \longleftrightarrow G_{\frac{1}{2}}$	1.2	.0606 E
	1.5	.0416 E
$E=\frac{c}{2\pi}R^2\left(\frac{R}{\Lambda}\right)^{3.6}\cdot\left(\frac{2\pi}{hc}\right)^2$	2.0	.0742 E
	2.5	.0300 E
	3.0	0
$K=-1$, $K'=+1$	1.0	.292 E
$S_{\frac{1}{2}} \longleftrightarrow \Pi_{\frac{1}{2}}$	1.2	.295 E
	1.5	300 E
$E=\frac{c}{2\pi}R\left(\frac{R}{\Lambda}\right)^{3.6}\cdot\left(\frac{2\pi}{hc}\right)^2$	2.0	.174 E
	2.5	.050 E
	3.0	0

48b.

Transfer of $2\,h/2\pi$				
Transition	K	K'	$\Sigma A_{KmK'm'}$	
$S_{\frac{1}{2}} \longleftrightarrow \Pi_{3/2}$	-1	-2	0	
$S_{\frac{1}{2}} \longleftrightarrow \delta_{3/2}$	-1	$+2$	$\frac{16}{25}$	$m=-1,\ m'=0\quad A=8/25$ $m=0,\ m'=-1\quad A=8/25$
$P_{\frac{1}{2}} \longleftrightarrow \Pi_{3/2}$	$+1$	-2	$16/25$	$m=-1,\ m'=0\quad A=8/25$ $m=0,\ m'=-1\quad A=8/25$
$P_{\frac{1}{2}} \longleftrightarrow \delta_{3/2}$	$+1$	$+2$	0	
$P_{3/2} \longleftrightarrow \sigma_{\frac{1}{2}}$	-2	-1	0	
$P_{3/2} \longleftrightarrow \Pi_{\frac{1}{2}}$	-2	$+1$	$16/25$	$m=-1,\ m'=0\quad A=8/25$ $m=0,\ m'=-1\quad A=8/25$
$D_{3/2} \longleftrightarrow \sigma_{\frac{1}{2}}$	$+2$	-1	$16/25$	$m=-1,\ m'=0\quad A=8/25$ $m=0,\ m'=-1\quad A=8/25$
$D_{3/2} \longleftrightarrow \Pi_{\frac{1}{2}}$	$+2$	$+1$	0	
				*

*) Higher terms are negligible.

Transfer of $2\,h/2\pi$

Transition	W/mc^2	$J(W)$	
$K = -1 \quad K' = +2$	1.0	.0622 F	
$S_{\frac{1}{2}}$ — $\delta_{3/2}$	1.1	.0558 F	
	1.2	.0510 F	
$F = \frac{c}{2\pi} R^2 \left(\frac{R}{\Lambda}\right)^{5.6}$	1.5	.0216 F	
$\cdot \left(\frac{2\pi}{hc}\right)^2$	2.0	.0093 F	
	2.5	.00077 F	
$K = +1, \; K' = -2$	1.0	.0070 F	
$P_{\frac{1}{2}} \longleftrightarrow \Pi_{3/2}$	1.2		
	1.5	.0096 F	
$K' = .0115 \frac{c}{2\pi}$	2.0	.0040 F	
	2.5	.0004 F	
$K = -2 \quad K' = +1$	1.0	.0012 Q	
$P_{3/2} \longleftrightarrow \Pi_{\frac{1}{2}}$	1.5	.0064 Q	
$Q = \frac{c}{2\pi} R^2 \left(\frac{R}{\Lambda}\right)^{5.8}$	2.0	.0088 Q	
$\left(\frac{2\pi}{hc}\right)^2$	2.5	.0051 Q	
	2.5	.0051 Q	
$D_{3/2} \longleftrightarrow G_{\frac{1}{2}}$	1.0	.00003 Q	
	1.5	.00068 Q	
	2.0	.00156 Q	
$K' = \frac{c}{2\pi} (.00275)$	2.5	.00115 Q	
	3.0	0	

49b.

For transitions $P_{3/2} \longleftrightarrow \Pi_{\frac{1}{2}}$ and $D_{3/2} \longleftrightarrow G_{\frac{1}{2}}$ it is of interest to note that $G = c^2 + \beta^2$ and since $\beta = \frac{W}{mc^2} \frac{Z}{137} \frac{1}{\sqrt{\left(\frac{W}{mc^2}\right)^2 - 1}}$ it appears that

$\beta \rightarrow \infty$ as $\frac{W}{mc^2} \rightarrow 1$ This difficulty is

readily overcome if the term $\left\{\left(\frac{W}{mc^2}\right)^2 - 1\right\}^{|K|-1}$ is included with G This gives for $|K| = 2$.

$$G\left\{\left(\frac{W}{mc^2}\right)^2 - 1\right\} = .824\left\{\left(\frac{W}{mc^2}\right)^2 - 1\right\} + (.6)^2\left(\frac{W}{mc^2}\right)^2$$

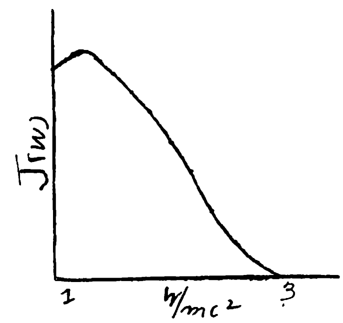
Or.

$K = \pm 2$	$\frac{W}{mc^2} = 1$	$\frac{W}{mc^2} = 1.5$	$\frac{W}{mc^2} = 2$	$\frac{W}{mc^2} = 2.5$
$G\left\{\left(\frac{W}{mc^2}\right)^2 - 1\right\}^{\lvert K\rvert - 1}$	.36	1.84	3.91	6.57

49c.

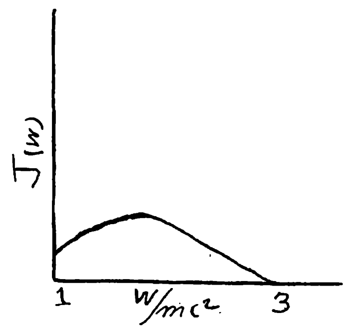
The total probability of emission of β- particles has been calculated on the Fermi theory for, respectively, zero, $h/2\pi$ and $2\, h/2\pi$ transfers of angular momentum by the nucleus. The tables (46, 48, 49) show that the necessity of evaluating the emission probability for higher transfers of angular momentum by the nucleus is unnecessary.

$S_{\frac{1}{2}} \longleftrightarrow \Pi_{1/2}$



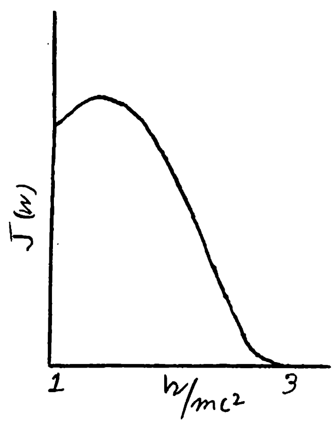
50a.

$P_{\frac{1}{2}} \longleftrightarrow G_{1/2}$



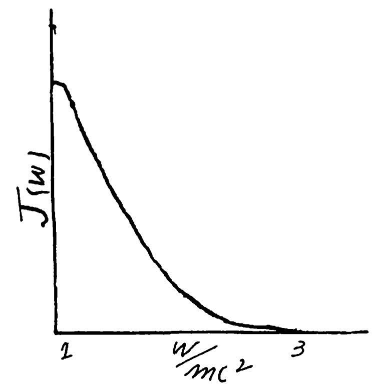
$\int J(w)\, dw$

$= C(.428)$



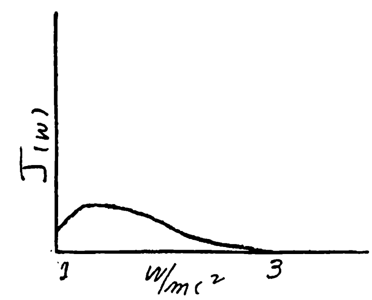
50b.

$S_{\frac{1}{2}} \longleftrightarrow \delta_{3/2}$

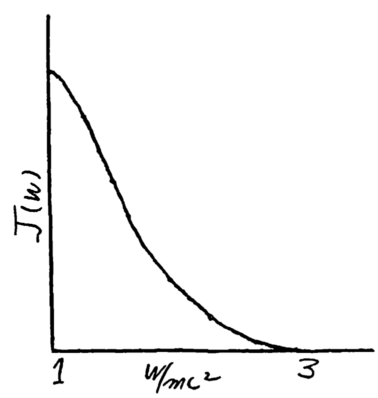


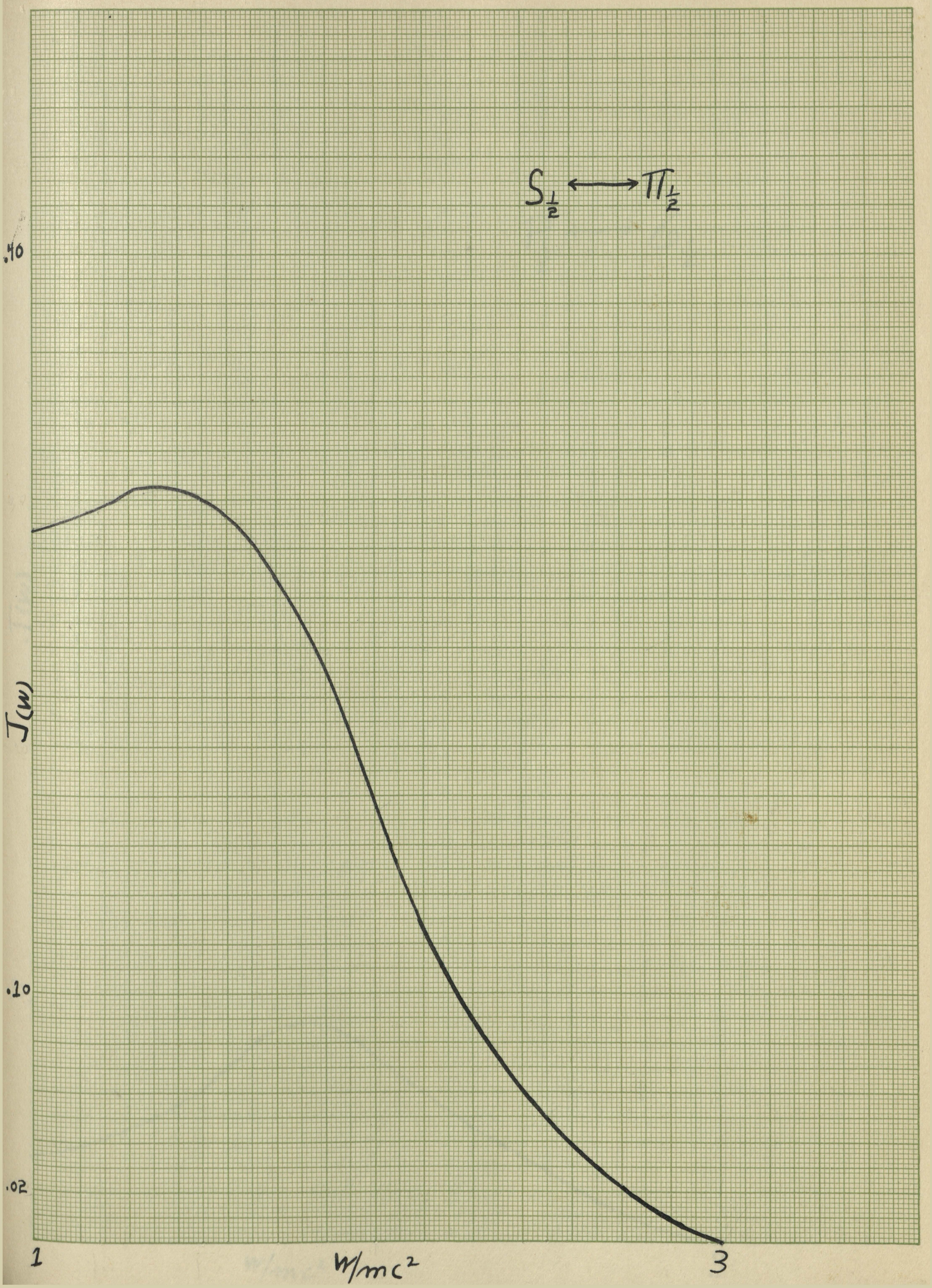
50a.

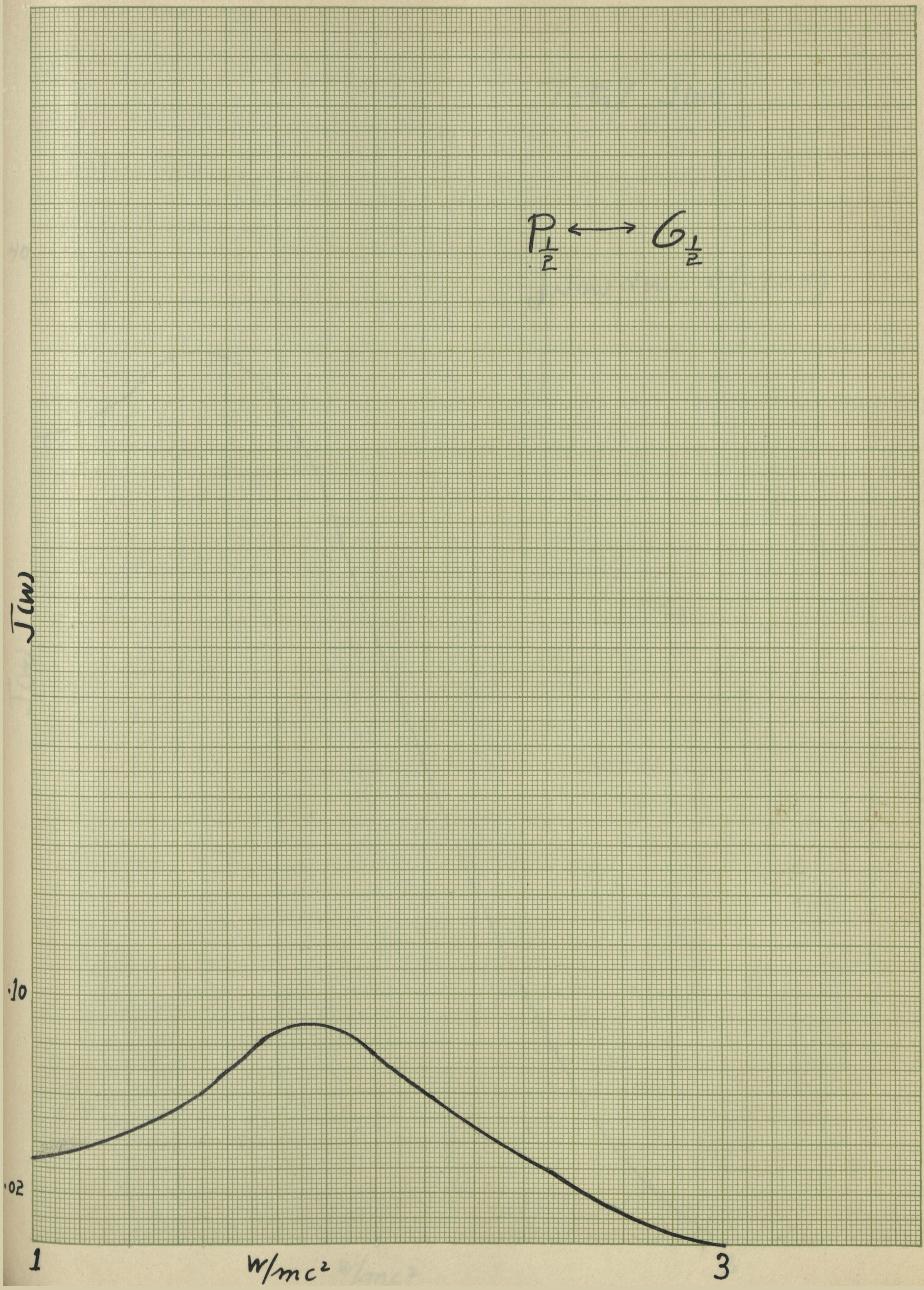
$P_{\frac{1}{2}} \longleftrightarrow \Pi_{3/2}$

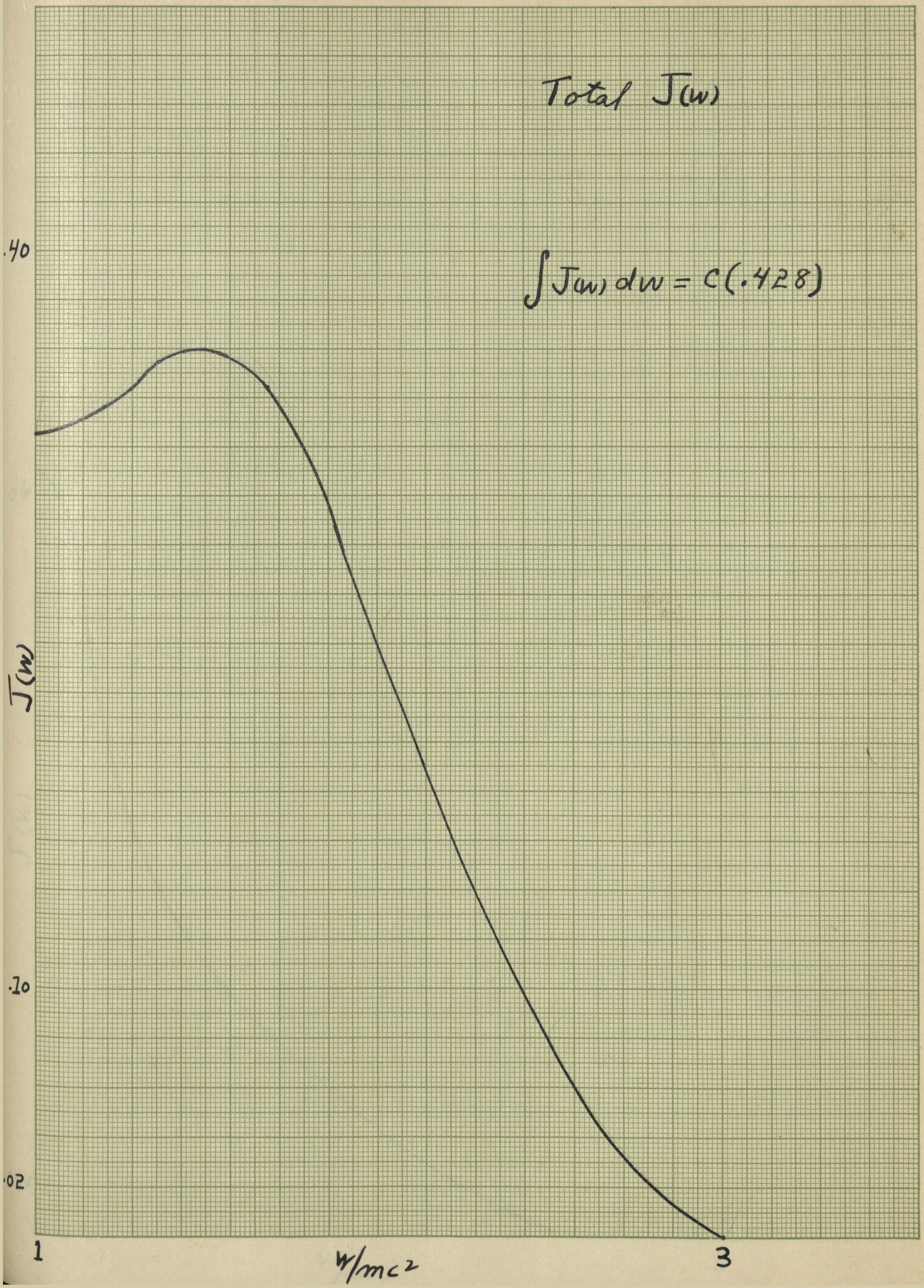


$$\int J(w)\, dw$$
$$= c\left(\frac{\Pi}{\Lambda}\right)^2 (.0665)$$

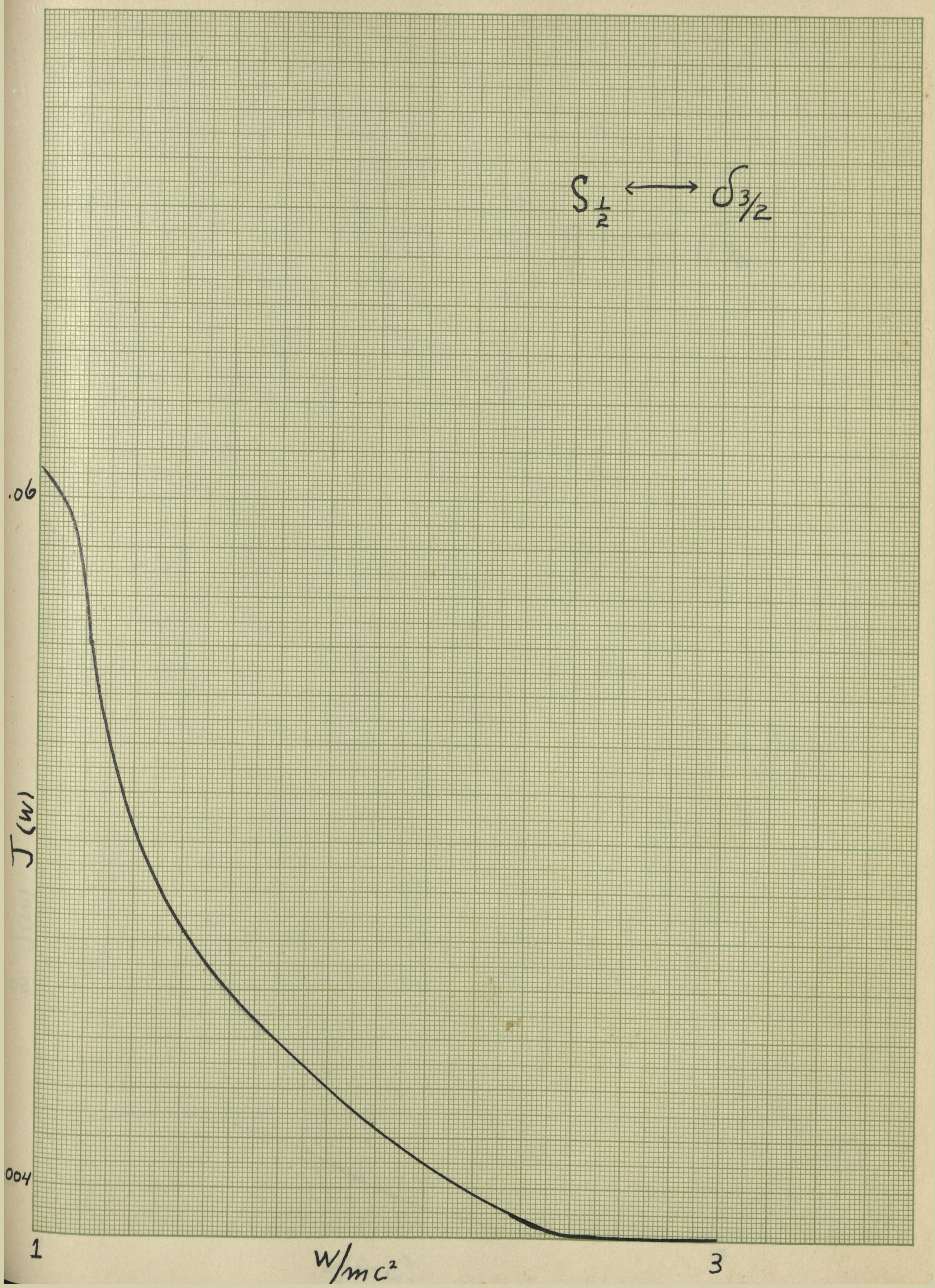



$S_{\frac{1}{2}} \longleftrightarrow \Pi_{\frac{1}{2}}$
.40
.10
.02
$J(w)$
1
W/mc^2
3

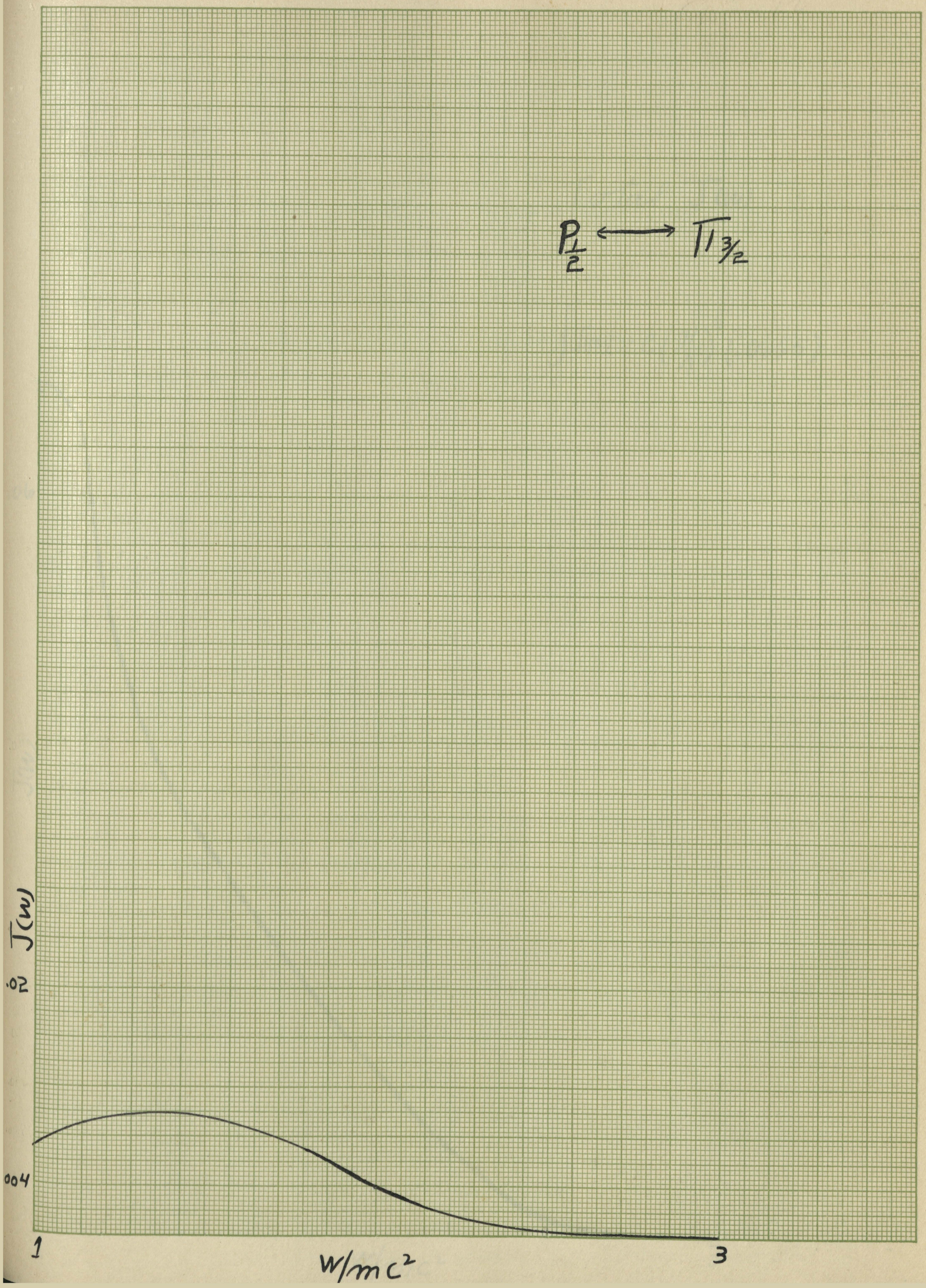

$P_{\frac{1}{2}} \longleftrightarrow G_{\frac{1}{2}}$
$\bar{J}(w)$
·10
·02
1
W/mc^2
3

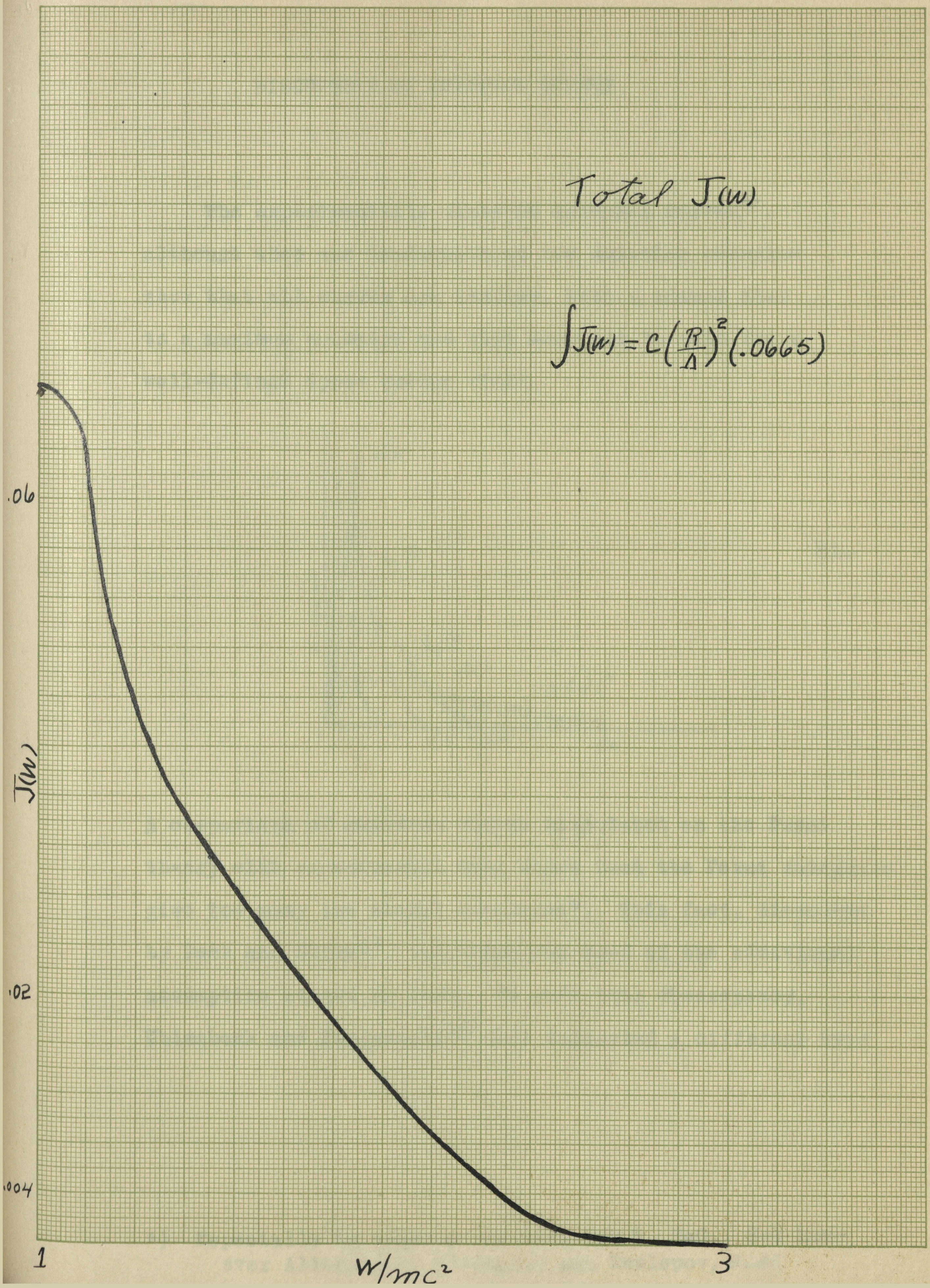


Total J(w)
∫ J(w) dw = C(.428)
.40
.10
.02
J(w)
1
w/mc²
3



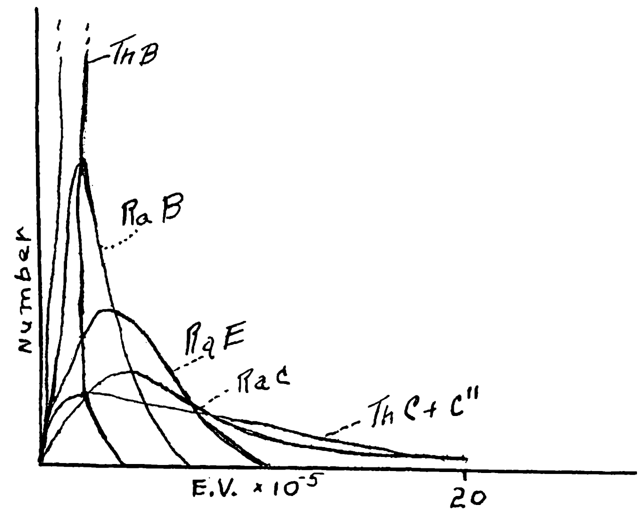
$S_{\frac{1}{2}} \longleftrightarrow \delta_{3/2}$
.06
J(W)
004
1
W/mc²
3


$P_{\frac{1}{2}} \longleftrightarrow \Pi_{3/2}$
$J(w)$
.02
.004
1
3
W/mc^2


Total J(w)
$\int J(w) = C\left(\frac{R}{\Delta}\right)^2 (.0665)$
.06
J(w)
.02
.004
1
3
W/mc²

DISCUSSION OF EMISSION CURVES

The experimentally observed emission curves[17], although they are inadequate at low emission energies show that all curves are similar, with a common rise to a maximum emission and then decrease to zero for a well-defined upper energy limit.



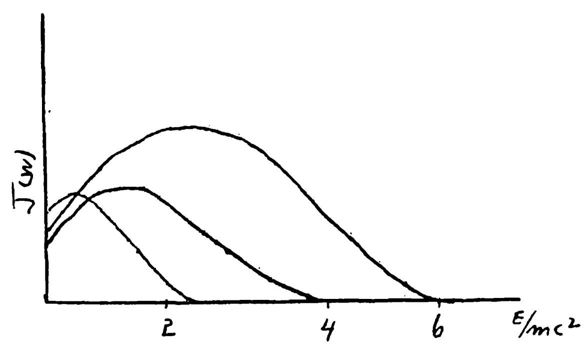
51.

A comparison of emission curves calculated on the Fermi theory with experimental data shows that the Fermi integrals give too many low energy electrons*. This fact, discussed by Beck and Sitte[13], indicates the need of the additional assumption stated by them. To meet this discrepancy, Uhlenbeck and Konopinski[16] have suggested a different type

*) Especially in experiments with RaE + RaC. See however Alichanjan, Alichanov and Dzelepov, Bibl.

of interaction from Fermi's. They consider the Hamiltonian function to contain not only the eigenfunctions of the electron and the neutrino but also their derivatives with respect to the coordinates. The theoretical curves thus obtained agree closely with the experimental data. However it would seem that one should consider this as a provisional expression for the electron-neutrino field for models similar to that they propose are not in common use. The U-K type of interaction is used to a large extent in current literature.

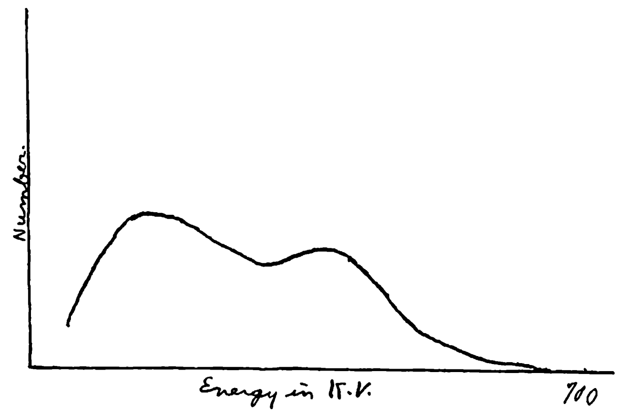
The curves calculated by (50) agree favorable with Fermis estimated curves.



52.

It is evident however that the agreement with experimental evidence is not improved upon the inclusion of the higher order transitions by having the maximum of the curves shifted to higher emission energies. The calculations for transfer of $h/2\pi$ yield a total probability which is of the order of 1/30 of that for zero transfer. It is not of the order $\left(R/\Lambda\right)^2$ smaller as Fermi had estimated; in fact the ratio is independent of the ratio $\left(R/\Lambda\right)^2$ Since $\frac{1}{T} \cong \int J(w)\,dw$ we cannot reconcile the emission curves with the Sargent curves. The emission probability for the transfer is 1/30 that for zero emission while from the Sargent curves the factor is 1/1000 for $\Delta E \doteq 3mc^2$. The question arises whether this discrepancy can be explained by saying that the first Sargent curve represents the sum of these two which we have not been able, experimentally, to differentiate between and the second curve represents a transfer of say $2\,h/2\pi$ This is hardly possible, but admitting this as a possibility for the moment consider the ratio of the total probability of emission for the transfer of $2\,h/2\pi$ to that for zero transfer. This ratio we see to be less than $1/300{,}000$; which of course is even worse than for the case of the $h/2\pi$ transfer. It does seem then that the calculated emission characteristics can neither be reconsiled with experimental data nor do they become better reconciled upon the inclusion of higher ordered transitions, as it was thought the case might be.

The large ratio in the total emission probability of the $2\,h/2\pi$ transfer to zero transfer suggests the plausibility of explaining the weak radioactivity of potassium and rubidium as beind due to a transition of large angular momentum transfer. Experimentally two groups are found in the β -spectrum of potassium.



53.

Sketch(21)

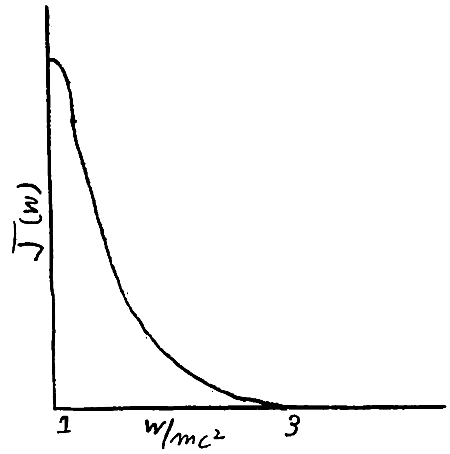
A calculation of the emission integrals for transfer of angular momentum $4\,h/2\pi$ gives

Transfer of $4\,h/2\pi$

W/mc^2	1.0	1.5	2.0	2.5
$S_{\frac12} \leftrightarrow \delta_{7/2}$	9.18 c'	1.6 c'	.099 c'	.00021 c'
$P_{\frac12} \leftrightarrow \Phi_{9/2}$	2.21 c'	.71 c'	.0576 c'	.00017 c'
$P_{3/2} \leftrightarrow \Phi_{5/2}$	.278 c'	1.2 c'	.387 c'	.0143 c'
$D_{3/2} \leftrightarrow \delta_{5/2}$	.00624 c'	.315 c'	.157 c'	.050 c'
$D_{5/2} \leftrightarrow \delta_{3/2}$	.00033 c'	.228 c'	.271 c'	.0715 c'
$F_{5/2} \leftrightarrow \Pi_{3/2}$	.0000007 c'	.077 c'	.392 c'	.0220 c'
$F_{7/2} \leftrightarrow \Pi_{1/2}$	.0000002 c'	.0031 c'	.296 c'	.596 c'
$G_{7/2} \leftrightarrow G_{\frac12}$	0	.0063 c'	.099 c'	.349 c'
C' an arbitrary constant ~ 10^{-4}				

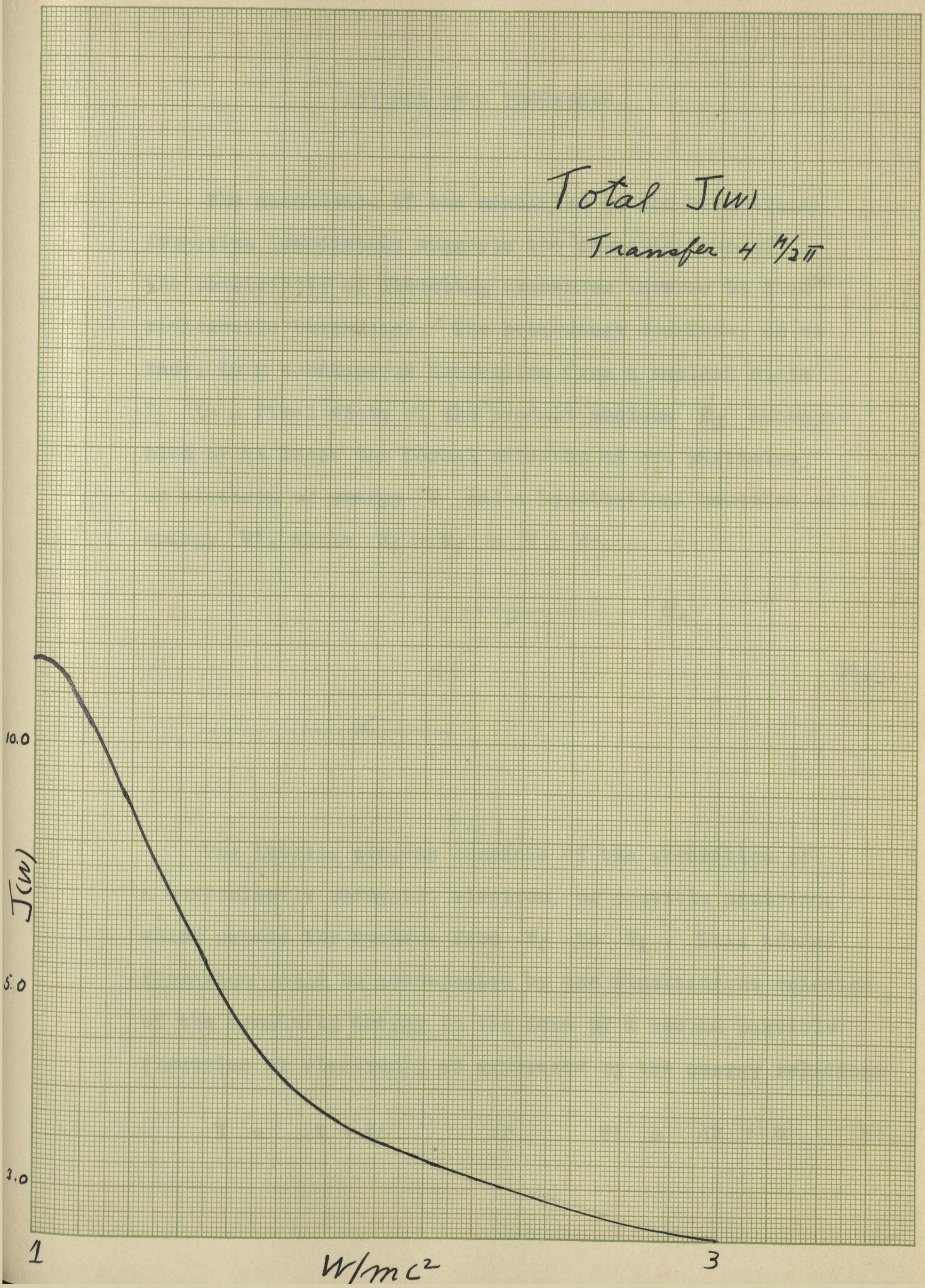
53a'.

The total $J(W)$ for $4\ h/2\pi$ transfer is



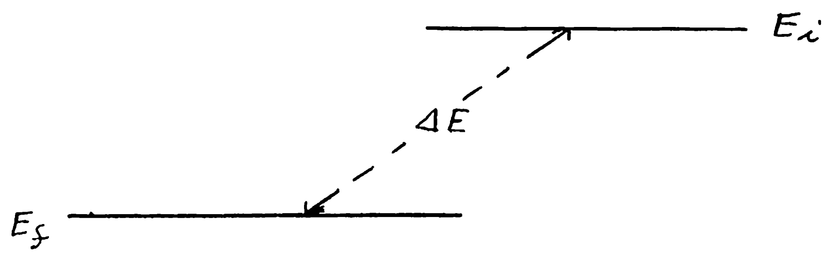
53b'

Comparing (53) with the emission curves for transfer $4\ h/2\pi$ we see that the theoretical curves as calculated do not indicate any such groups; and therefore such an explanation of the potassium and rubidium radioactivity is hardly justified.


Total J(w)
Transfer 4 $\hbar/2\pi$
10.0
5.0
1.0
J(w)
1
3
W/mc²

INVERSE BETA PROCESSES.

The discussion of the mechanism of the beta transformation immediately suggests the investigation of the possibility of detecting processes inverse to this radioactive beta-decay. The beta-decay consists, as we know, in a spontaneous transition from a nuclear state E_i to a final state of the product nucleus E_f accompanied by at least the formal emission of two particles; on electron of energy W and a hypothetical neutrino of energy W', where $E_i - E_f = W + W'$.



54.

fig 1

The inverse process consists of the absorption of a sufficiently energetic electron, or possible neutrino, which raises the nucleus from E_f to E_i This transformation would be accompanied by the simultaneous emission of the remaining energy in the form of a second particle (neutrino or electron), as expressed by the energy relations

$$W = \Delta E + W' \qquad \text{or} \qquad W' = \Delta E + W \qquad 55.$$

The second process, evidently, will occur only if such a particle as the neutrino really exists and does not merely represent a formal means of accounting for the lost energy amount. Both processes are rather closely related so that we can confine our attention here only to the absorption of a fast impinging electron under emission of the hypothetical second particle. Treating by the well known perturbation method (See appendix) we obtain the formula

$$\lambda = \frac{4\pi^2}{h} / c /^2 \left| V^*_{W, W+\Delta E} \right|^2 \qquad 56a.$$

where

$$V^*_{W, W+\Delta E} = \sum_{\rho} \int_0^R \Psi_W^{\rho *} V^* \Psi^{\rho}_{W+\Delta E} \, r^2 \, dr \int d\Omega \qquad 56b.$$

now the perturbation energy V^* is to be taken as a slowly varying quantity inside R This treatment of β-processes is essentially equivalent with the treatment given by Fermi and used in the above calculations. It differs from the latter only by replacing the eigenfunctions of the neutrino by the corresponding anti-neutrino. It has been shown by Konopinshi and Uhlenbeck, that both methods are mathematically identical if the mass of the neutrino is assumed to be zero.

Restricting our attention to such processes only, in which the nucleus does not transfer any angular momentum to the particles involved in the β -process, V^* becomes independent of the angular variables and can therefore be taken before the integral as approximating a constant.

$$V^*_{W,W+\Delta E} = \sum_\rho V^* \int d\Omega \int_0^R \psi_W^\rho \psi_{W+\Delta E}^\rho r^2 dr \qquad 57.$$

C being proportional to the number of particles in the plane wave having differing values of angular momentum can be evaluated by comparing the eigenfunction of the plane wave with the expression for an electron in a central field. For a central field and lowest angular momentum ($s_{\frac{1}{2}}$, $p_{\frac{1}{2}}$)

$$C = e^{i\delta}\sqrt{\pi/2}\;\frac{1}{k} \qquad \text{normalized in k} \qquad 58.$$

where $e^{i\delta}$ is the phase.

Normalizing in W

$$C = e^{i\delta}\sqrt{\pi/2}\cdot\frac{1}{k}\;\frac{1}{\sqrt{\frac{2\pi mc}{h}\,\frac{\frac{W}{mc^2}\,\frac{1}{mc^2}}{\sqrt{\left(\frac{W}{mc^2}\right)^2-1}}}} \qquad 59.$$

So that equation (56) becomes.

$$\lambda = \frac{4\pi^2}{h} \frac{V}{2} \frac{1}{k^2} \frac{hc}{2\pi} \frac{\sqrt{\left(\frac{W}{mc^2}\right)^2-1}}{\left(\frac{W}{mc^2}\right)} \left| V^*_{W,W+\Delta E} \right|^2 \qquad 60.$$

Dividing both sides of (60) by v (velocity), we obtain ($\frac{\lambda}{v}$) on the left which will be the cross section. To show briefly that ($\frac{\lambda}{v}$) has units cm^2 ($\frac{\lambda}{v} \overset{d}{\sim} cm^2$) consider

$$\frac{\lambda}{v} = \frac{\pi^2}{\frac{v}{c} k^2} \frac{\sqrt{\left(\frac{W}{mc^2}\right)^2-1}}{\left(\frac{W}{mc^2}\right)} \left| V^*_{W,W+\Delta E} \right|^2 \qquad 61.$$

It is readily seen that v/c and (W/mc^2) are dimensionless and since $\int dW \int |\psi_W|^2 d\tau = 1$

$$\psi_W \overset{d}{\sim} \frac{1}{\sqrt{cm^3}} \frac{1}{\sqrt{W}} \qquad 61b.$$

or

$$V^{*}_{W,W+\Delta E} \overset{d}{\sim} \frac{1}{\sqrt{cm^3}}\,\frac{1}{\sqrt{W}}\cdot W \cdot \frac{1}{\sqrt{cm^3}}\cdot\frac{1}{\sqrt{W}}\cdot cm^3 \overset{d}{\sim} 1 \qquad 61c.$$

Thus $V^{*}_{W,W+\Delta E}$ is also dimensionless. This leaves only k which has the dimensions 1/cm, so that

$$\frac{\lambda}{v} \overset{d}{\sim} cm^2 \qquad 61d.$$

For convenience let Q $=$ λ/v, the cross section of the considered inverse process. Then

$$Q = \frac{\pi^2}{\frac{v}{c}k^2}\sqrt{1-\left(\frac{mc^2}{W}\right)^2}\,\left|V^{*}_{W,W+\Delta E}\right|^2 \qquad 62.$$

It is in the evaluation of the expression $\left|V^{*}_{W,W+\Delta E}\right|^2$ of (62) that the preceding discussion of the Fermi theory is very useful. As already shown the major portion of the probability is yielded by the lowest cases of angular momentum transfer, so that the calculation of $\left|V^{*}_{W,W+\Delta E}\right|^2$ can be simplified. Substituting eigenfunctions into the expression (61b) for $\left|V^{*}_{W,W+\Delta E}\right|$

$$V^{*}_{W,W+\Delta E} \cong \bar{V}^{*} \iint \Xi^{*}_{W}\,\Xi_{W+\Delta E}\left(\omega^{\alpha *}_{W}\,\omega^{\alpha}_{W+\Delta E} + \omega^{\beta *}_{W}\,\omega^{\beta}_{W+\Delta E}\right) d\tau$$

$$+\ \Phi^{*}_{W}\,\Phi_{W+\Delta E}\left(\omega^{\delta *}_{W}\,\omega^{\delta}_{W+\Delta E} + \omega^{\gamma *}_{W}\,\omega^{\gamma}_{W+\Delta E}\right) d\tau \qquad 63.$$

where the notation is that of the Fermi theory. Since*

$$\Xi_{W} = \sqrt{\frac{2\pi}{\hbar c}\,\frac{\frac{W}{mc^2}}{\sqrt{\left(\frac{W}{mc^2}\right)^2-1}}}\;\Xi_{k}\ ,\qquad \Phi_{W} = \sqrt{\frac{2\pi}{\hbar c}\,\frac{\frac{W}{mc^2}}{\sqrt{\left(\frac{W}{mc^2}\right)^2-1}}}\;\Phi_{k} \qquad 64$$

written in full becomes

$$\varphi = \frac{\pi^2}{\frac{v}{c}k^2}\,\frac{\sqrt{\left(\frac{W}{mc^2}\right)^2-1}}{\frac{W}{mc^2}} \Big/ V^{*}\int\left(-\omega^{\alpha}_{W}\omega^{\beta}_{W+\Delta E} + \omega^{\beta}_{W}\,\omega^{\alpha}_{W+\Delta E}\right) \qquad 65.$$

$$\left(\omega^{\alpha *}_{\lambda n}\omega^{\alpha}_{\lambda' n'} + \omega^{\beta *}_{\lambda n}\omega^{\beta}_{\lambda' n'}\right)\sin\theta\, d\theta \int\left(\Xi^{*}_{k}\,\Xi_{k} + \Phi^{*}_{k}\,\Phi_{k}\right) r^2\, dr$$

*) By the method used in developing (23) it can be readily shown $\left(\omega^{\alpha *}_{W}\omega^{\alpha}_{W+\Delta E} + \omega^{\beta *}_{W}\omega^{\beta}_{W+\Delta E}\right) = \left(\omega^{\delta}_{W}\omega^{\delta *}_{W+\Delta E} + \omega^{\gamma}_{W}\omega^{\gamma *}_{W+\Delta E}\right)$

We find the value of Q by substituting our values of Ξ_K and Φ_K for the two cases $(K<0, K>0)$ and performing the integration. We have however to get an estimate of the ∇^* before performing this integration.

Evaluation of V^* in (65)

λ is given by

$$\lambda = c \int_{mc^2}^{\Delta E} J(w)\, dw \qquad \text{66a.}$$

in the treatment given above and by (13 & 14)

$$\lambda = \frac{4\pi^2}{h} \int_{mc^2}^{\Delta E} dw \left| V^*_{w,\, w+\Delta E} \right|^2 \qquad \text{66b.}$$

according to the method now adopted.

Considering only the lowest angular momentum transfer ($K = K' = \pm 1$), as the angular integration yields one (see equations (43)), we will have from (63)

$$\lambda = \frac{4\pi^2}{h} \bar{V}^* \int_{mc^2}^{\Delta E} \left| \int_0^R \left(\Xi_W \Xi_{W+\Delta E} + \Phi_W \Phi_{W+\Delta E}\, r^2\, dr \right| \right.^2 \qquad \text{67a.}$$

and from (66a) for ($K = K' = \pm 1$)

$$\lambda = Const \int_{mc^2}^{\Delta E} dw \left| \int_0^R \left(\Xi_{K'} X_K + \Phi_{K'} \varphi_K \right) r^2\, dr \right|^2 \qquad \text{67b.}$$

which is similar to (67a) and

$$\int_{mc^2}^{\Delta E} dW \left| \int_0^R \left(\Xi_{\kappa'} X_\kappa + \Phi_{\kappa'} \varphi_\kappa \right) r^2 dr \right|^2$$

can be readily calculated from the emission curves for zero transfer of angular momentum by the nucleus (50a). We see then from (67a), that in (67b)

$$Const = \frac{4\pi^2}{h} \overline{V}^{*2} \qquad 68.$$

By (50a) the total probability of transition is

$$area = 5.85 \times \frac{c}{2\pi} R^2 \left(\frac{R}{\Lambda}\right)^{3.6} mc^2 \left(\frac{2\pi}{hc}\right)^2 \qquad 68a.$$

The curve coordinates being (W/mc^2) necessitates the introduction of the mc^2 term. Putting this value into (66a)

$$\lambda = \frac{4\pi^2}{h} \overline{V}^{*2} mc^2 \, 5.85 \frac{R^2}{2\pi} \left(\frac{R}{\Lambda}\right)^{3.6} \left(\frac{2\pi}{hc}\right)^2 \qquad 68b.$$

but since $\lambda \sim 5 \times 10^{-4}$

$$5 \times 10^{-4} = V^{*2} \frac{4\pi^2}{h} \left(5.85 \left(\frac{2\pi}{hc}\right)^2 \cdot \frac{R^2}{2\pi} \left(\frac{R}{\Lambda}\right)^{3.6} mc^2 \right) \qquad 69 .$$

or

$$\overline{V}^{*2} = 5\times10^{-4}\left(\frac{h}{2\pi}\right)^3\left(\frac{\Lambda}{R}\right)^{3.6}\frac{c}{5.85\,(R^2)\,m_1} \qquad \text{69b.}$$

Evaluation of Cross Section.

Returning now to equation (65) we may write in our value for $\overline{V}^{*2}$ Recalling that the second part of the integral has been shown to be negligible*.

$$Q = \frac{\pi^2}{\frac{v}{c}k^2}\sqrt{1-\left(\frac{mc^2}{w}\right)^2}\;\overline{V}^{*2}\left|\int_0^R\left(\underline{\overline{H}}^{*}_{w}\;\underline{\overline{H}}_{w+\Delta E}\right)r^2dr\right|^2 \qquad \text{70.}$$

Now normalized in (w) for ($\kappa = \pm 1$)

$$\underline{\overline{H}} = \sqrt{\frac{1}{2}\left(1+\frac{mc^2}{w}\right)}\;X_0\sqrt{\frac{2\pi}{hc}\frac{w/mc^2}{\sqrt{\left(\frac{w}{mc^2}\right)^2-1}}} \qquad \text{71.}$$

or

$$Q = \frac{\pi^2}{\frac{v}{c}k^2}\sqrt{1-\left(\frac{mc^2}{w}\right)^2}\;\overline{V}^{*2}\left|\left(\sqrt{\frac{1}{2}\left(1+\frac{mc^2}{w}\right)\frac{2\pi}{hc}}\sqrt{\frac{2}{\pi\sqrt{1-\left(\frac{mc^2}{w}\right)^2}}}\right)^2(k\cdot k')\,\frac{R^3}{3}\right|^2 \qquad \text{72.}$$

*) Development similar to that of (67a).

Using the approximations

$$W > mc^2 \qquad \text{or} \qquad v/c \rightarrow 1$$
$$W' > mc^2 \qquad \qquad v'/c \rightarrow 1$$

$$\varphi = \frac{\pi^2}{k^2} \overline{V}^{+2} \left| \frac{2}{(hc)^2} k \cdot k' \frac{R^3}{3} \right|^2 \qquad \text{73a.}$$

$$\varphi = \left(\frac{2\pi}{hc}\right)^2 \overline{V}^{+2} \frac{R^6}{9} k'^2 \qquad \text{73b.}$$

or putting in the value of $\overline{V}^{+2}$ from (69b)

$$\varphi = \left(\frac{2\pi}{hc}\right)^2 \cdot \lambda \left(\frac{h}{2\pi}\right)^3 \left(\frac{\Lambda}{R}\right)^{3.6} \frac{c}{5.85\, R^2 m} k'^2 \frac{R^6}{9} \qquad \text{73c.}$$

for purposes of calculation assume

$$\Lambda \simeq 4 \times 10^{-11} \; ; \; R \simeq 10^{-12}$$

$$\varphi = \frac{5 \times 10^{-4}}{3 \times 10^{10} \times 62.65} \left(4 \times 10^{-11}\right)^{2.6} \left(10^{-12}\right)^{.4} \left(\frac{W'}{mc^2}\right)^2 \qquad \text{75.}$$

where 1 is neglected compared to $(W'/mc^2)^2$

Hence

$$Q \cong 46 \times 10^{-50} \left(\frac{W'}{mc^2}\right)^2 \qquad 76a.$$

If the limiting value $W'/mc^2 \sim 100$ is used

$$Q \cong 5 \times 10^{-45} \, \overline{cm}^2 \qquad 76b.$$

Let us then calculate how far a neutrino must penetrate lead before such a process would be expected to happen. If N be the number of lead molecules per cubic centimeter of apth, L represent the length of path necessary for absorption and Q the cross section then

$$N \cdot L \cdot Q = 1$$

or $$L = 1/QN \sim \frac{1}{5 \times 10^{-45} \times 3.5 \times 10^{22}}$$

$$L \cong 5.7 \times 10^{16} \text{ Km. lead.} \qquad 77.$$

which of course is another means of stating the impossibility of deciding the question of the existence of a neutrino by direct observation.

The calculated cross section is seen to be affected but little by any assumptions upon (R), but is primarily a small quantity because of $\overline{V}^*$ being small. This quantity $\overline{V}^*$, however, depends to a large extent on the special assumptions which have been found to be indispensable for a consistent theory of the beta decay[18]. Beck and Sitte have estimated $\overline{V}^{*}$ to be of the order 10^{-16} ergs, Fermi's theory herein yields a similar value $\sim 10^{-14}$ ergs. If, however, higher spacial derivatives are included in the interaction function responsible for the beta decay, this value would be considerably increased. The value obtained

$$\varphi \cong 10^{-44}\ \mathrm{cm}^2 \qquad 78.$$

is about the order of magnitude previously-estimated by Bethe and Peierls[19].

Using Uhlenbeck and Konopinski's interaction terms the value of (78) would be considerably increased, by a factor of the order 137^2. This would, however, hardly increase the possibility to find processes of the indicated type experimentally.

A second reason, which would lead to an increase of the cross section intimated in (78) is suggested by Bohr's theory of the structure of atomic nuclei (N. Bohr, Nature 137 (344) 1936). According to this theory, a heavy nucleus possesses a very large number of excited energy levels. Thus we are lead to assume that the nucleus can be excited to any one of these levels by a beta process and we have to take the sum over all these possibilities in order to obtain the total cross section.

So far very little is known about the dependence of the probabilities of beta processes on the nuclear transitions involved. We can, however, obtain a very rough estimate on the influence of the higher excited states, if we take into account, that according to (75) only those transitions have considerable probabilities, in which neutrinos of considerable final energies W' are involved. Assuming that only transitions exciting the nucleus to about $20.mc^2 = 10.10^6$ e.V. play a considerable role and assuming in this region a mean distance of about 100 e.v. between two levels* we find that (78) has to be multiplied by a factor not exceeding about 10^5.

Recently Heisenberg and Weizsacker[20] have pointed out that the phenomenon of the beta decay could account for the interaction between proton and neutron by exchange of charges if ∇^* is taken to be of the order $137mc^2$. It may be worth while to discuss the inverse beta-process in this case as the Heisenberg-Weizsacker assumption leads to cross sections observable in the cosmic rays.

*) This value isobtained as an average value between the level distance of order 10^5 e.v. for the lowest levels and of order 10 e.v. for highly excited nuclei.

Introducing $\overline{V}^{*} \sim 137mc^2$ into say (73b) Q becomes of the order

$$Q \leq a^2 \qquad 79.$$

where a denotes the radius of the electron e^2/mc^2. Further; the decay constant λ for a spontaneous beta-decay of the energy $\Delta E \doteq 137mc^2$ tends to a value

$$\lambda \doteq c/a \qquad 80.$$

and thus makes the beta decay transformation a very frequent phenomenon. Since the decay constants for smaller values of ΔE are very small compared to (80) we would have to assume that the decay constant increases very rapidly with ΔE. It may be observed, however, that Sargent's empirical curves show no such a rapid change.

Beta Absorption of Electrons in Cosmic Rays.

(75) indicates that the absorption cross section increases rapidly with the energy of the incident particles. The best chance to detect processes of this type is therefore with electrons of energy of about $137mc^2$. These occur quite frequently in cosmic rays. Owing to the fairly large extrapolation from known beta processes and considering our lack of knowledge of the additional assumptions on the mechanism of the beta-decay, a more exact upper

limit for the cross section in question than that given in (79) cannot be obtained at present. In the following discussion therefore, it is assumed that $\varphi \leq a^2$.

Comparing the assumed value of Q with the cross section of other individual processes this phenomenon would be about 50 times less frequent than radiative collisions in heavy materials; they would however, predominate in light nuclei (e.g. in air). Assuming the most favorable case, $Q = a^2$, the average path of an electron of energy $137mc^2$ (for the considered process) would be about 10^5 cm. of air. The cosmic rays should therefore contain a considerable number of neutrinos.

Possibly the best way to search for beta-absorptions of high energy electrons would be by means of cloud chamber photographs of showers. One readily concludes that the neutrino emission to be expected during the beta-absorption would not be confined to small angles and thus would give rise to recoil tracks of the nucleus corresponding to an energy of several mc^2. These recoil tracks should be easily recognized in the photographs. Assuming a Wilson Cloud Chamber of 50 cm. diameter and an average of 20 high speed electrons per shower, one beta-absorption should be found in every 100 showers photographs. It is not impossible that such phenomena be observed, even if the cross section Q is somewhat smaller than assumed above.

SUMMARY

The foregoing discussion of the Fermi theory based upon the simplified method developed for the evaluation of the probability integrals has shown distinct disagreement with experimental evidence. The calculations give curves similar to those obtained by Fermi. These curves, however, do not become better upon the inclusion of higher ordered transitions nor do they agree with experimental evidence when applied to the radioactivity of potassium. The discussion supports the criticism by Beck and Sitte which cites the indispensability of an additional* assumption necessary to prevent the emission of slow electrons.

When applied to the solution of the cross section for processes inverse to the beta-decay; cross sections are calculated which are found to depend to a large extent upon the assumptions regarding the mechanism of the β - decay. The cross sections calculated however are much to small to suggest the possibility of detecting the inverse β -processes. If one is to agree with Bethe and Peierls in assuming ∇^* to be approximately $137mc^2$, then for electrons of energy $137mc^2$ one might well expect to find β absorption processes.

Since electrons of this energy are to be found in cosmic ray shower photographs, an investigation of these shower photographs could furnish us with valuable information regarding the mechanism of the beta transformation processes.

The author wishes to express his indebtedness to Dr. G. Beck under whose direction this work was done.

APPENDIX

PERTURBATION THEORY

A perturbation treatment of the Dirac equation with the object in mind of determining the probability of a transition due to the perturbation potential, yields probability equations similar to those suggested by Fermi[7]. Using Fermi's expression for the four vector potential, the energy distribution of the β electrons for any assumption on the angular momentum balance can be calculated according to the Fermi theory. This evaluation yields pertinent information regarding the agreement of the Fermi theory and experimental evidence.

A further extension of the perturbation theory allows one to calculate the probably cross section for the Fermi neutrinos[8], which cross section of course gives the probable chance of detecting the particle assumed by Fermi in order to conserve both energy and momentum[9].

Consider the effect of a perturbation of the Dirac equation describing a system by an energy term $q_o(rt)$.

We have then to solve the equations

$$\left[P_0 - \frac{e}{c}(\varphi_0 + q_0)\right] + \sum_\nu \alpha_\nu \left[P_\nu - \frac{e}{c}(\Phi_\nu + q_\nu)\right] + \beta m c = 0 \qquad 1.$$

where $(q_0, q \;)$ represents the four components of the perturbing term. α_ν and β are the usual Dirac matrices.

$$\alpha_1 = \begin{vmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{vmatrix}, \quad \alpha_2 = \begin{vmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{vmatrix} \quad \alpha_3 = \begin{vmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&-1&0&0 \end{vmatrix} \quad \beta = \begin{vmatrix} -1&0&0&0 \\ 0&-1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{vmatrix} \qquad 2.$$

and $p_0 = h/2\pi \, i c \frac{\partial}{\partial t}$ with $P_\nu = \frac{h}{2\pi i} \frac{\partial}{\partial x_\nu}$

At any subsequent time after the application of the perturbation let the wave function solution be ψ^ρ
Then expanding ψ^ρ into

$$\psi^\rho = \sum_k C_k(t)\, U_k^\rho\, \epsilon^{\frac{2\pi i}{h} W_k t} + \int_{u}^{\infty} C_W(t)\, U_W^\rho\, \epsilon^{\frac{2\pi i}{h} W t}\, dW \qquad 3.$$

where $U_k^\rho \epsilon^{\frac{2\pi i}{h} W_k t}$ and $U_W^\rho \epsilon^{\frac{2\pi i}{h} W t}$ are the initial wave-functions solution before the application of the pertubation. $U_W^{\rho'}$ of course are the solutions for positive energy and are of the form of spherical harmonics

Substituting (3) into (1) we obtain

$$\frac{h}{2\pi i c}\left(\sum_k \frac{dC_k(t)}{dt} u_k^\rho \epsilon^{\frac{2\pi i}{h}W_k t} + \int_u^\infty \frac{dC_w(t)}{dt} u_w^\rho \epsilon^{\frac{2\pi i}{h}wt} dw\right) +$$

$$\frac{h}{2\pi i c}\left(\frac{2\pi i}{h}\sum_k C_k(t)\, u_k^\rho W_k \epsilon^{\frac{2\pi i}{h}W_k t} + \frac{2\pi i}{h}\int_u^\infty C_w(t)\, u_w^\rho \epsilon^{\frac{2\pi i}{h}wt} dw\right) +$$

$$-\left(\frac{e}{c}\varphi_0 \sum_k C_k(t)\, u_k^\rho \epsilon^{\frac{2\pi i}{h}W_k t} + \frac{e}{c}\varphi_0\int_u^\infty C_w(t)\, u_w^\rho \epsilon^{\frac{2\pi i}{h}wt} dw\right) + \qquad 4.$$

$$-\left(\frac{e}{c} g_0 \sum_k C_k(t)\, u_k^\rho \epsilon^{\frac{2\pi i}{h}W_k t} + \frac{e}{c} g_0 \int_u^\infty C_w(t)\, u_w^\rho \epsilon^{\frac{2\pi i}{h}wt} dw\right) +$$

$$\sum_\nu \alpha_\nu \left(P_\nu \psi^\rho - \frac{e}{c}\varphi_\nu \psi^\rho\right) - \sum_\nu \frac{e}{c}\alpha_\nu g_\nu \left(\sum_k C_k(t)\, u_k^\rho \epsilon^{\frac{2\pi i}{h}wt}\right) +$$

$$-\sum_\nu \frac{e}{c}\alpha_\nu g_\nu \int_u^\infty \frac{e}{c} C_w(t)\, u_w^\rho \epsilon^{\frac{2\pi i}{h}wt} dw + \beta m c\, \psi^\rho = 0$$

Consider the following parts of equation (4).

$$\frac{1}{c}\sum_k C_k(t)\, u_k^\rho W_k e^{\frac{2\pi i}{h}W_k t} + \frac{1}{c}\int_u^\infty w\, C_w(t)\, \epsilon^{\frac{2\pi i}{h}wt} u_w^\rho\, dw$$

$$-\frac{e}{c}\varphi_0 \sum_k C_k(t)\, u_k^\rho \epsilon^{-\frac{2\pi i}{h}W_k t} - \frac{e}{c}\varphi_0 \int_u^\infty C_w(t)\, e^{\frac{2\pi i}{h}wt} u_w^\rho\, dw$$

$$\sum_\nu \alpha_\nu \Big(P_\nu \sum_k C_k(t)\, u_k^\rho \epsilon^{\frac{2\pi i}{h}W_k t} - \sum_\nu \alpha_\nu P_\nu \int_u^\infty C_w(t)\, \epsilon^{\frac{2\pi i}{h}wt} u_w^\rho\, dw \qquad 5.$$

$$-\frac{e}{c}\varphi_\nu \left(\sum_k C_k(t)\, u_k^\rho \epsilon^{\frac{2\pi i}{h}wt}\right) - \frac{e}{c}\varphi_\nu \int_u^\infty C_w(t)\, \epsilon^{\frac{2\pi i}{h}wt} u_w^\rho\, dw$$

$$+\beta m c\left(\sum_k C_k(t)\, u_k^\rho \epsilon^{\frac{2\pi i}{h}wt}\right) + \beta m c \int_u^\infty C_w(t)\, \epsilon^{\frac{2\pi i}{h}wt} u_w^\rho\, dw$$

Summing up the two columns of (5) we obtain

$$\sum_k c_k(t)\, \epsilon^{\frac{2\pi i}{h} W t}\left(\left[\frac{W_k}{c} - \frac{e}{c}\varphi_0\right] + \sum_\nu \alpha_\nu \left(P_\nu - \frac{e}{c}\varphi_\nu\right) + \beta m c\right) u_k^\rho$$

$$+ \int_u^\infty c_W(t)\, \epsilon^{\frac{2\pi i}{h} W t}\left(\left[\frac{W}{c} - \frac{e}{c}\varphi_0\right] + \sum_\nu \alpha_\nu \left(P_\nu - \frac{e}{c}\varphi_\nu\right) + \beta m c\right) u_W^\rho\, dW$$

where $\left(\left[\frac{W}{c} - \frac{e}{c}\varphi_0\right] + \sum_\nu \alpha_\nu \left(P_\nu - \frac{e}{c}\varphi_\nu\right) + \beta m c\right) u^\rho = 0$

because of the Dirac equation. We obtain therefore

$$\sum_k \frac{h}{2\pi i c} \frac{d c_k(t)}{dt} \epsilon^{\frac{2\pi i}{h} W_k t} u_k^\rho + \int_u^\infty \frac{h}{2\pi i c} \frac{d c_W(t)}{dt} \epsilon^{\frac{2\pi i}{h} W t} u_W^\rho\, dW =$$

$$\sum_k c_k(t) \frac{e}{c} \varphi_0 \epsilon^{\frac{2\pi i}{h} W_k t} u_k^\rho + \int_u^\infty \frac{e}{c} \varphi_0\, c_W(t)\, \epsilon^{\frac{2\pi i}{h} W t} u_W^\rho\, dW \qquad 7.$$

$$+ \sum_\nu \alpha_\nu \sum_k \frac{e}{c} \varphi_0\, c_k(t)\, u_k^\rho\, \epsilon^{\frac{2\pi i}{h} W_k t} + \sum_\nu \alpha_\nu \frac{e}{c} \int_u c_W(t)\, \varphi_\nu\, \epsilon^{\frac{2\pi i}{h} W t} u_W^\rho\, dW$$

Now in Dirac's theory it is well known that

$$\sum_{\rho} \int_0^\infty u_i^{\rho *} u_k^{\rho} \, d\tau = \delta_{ik}$$ ~~for the~~ discrete spectrum 8a.

or

$$\sum_{\rho} \int_{E_1}^{E_2} dW' \int_0^\infty u_W^{\rho *} u_{W'}^{\rho} \, d\tau = 1$$ for the continuous spectrum 8b.

W lying between E_1 and E_2.

and

$$\sum_{\rho} \int_{E_1}^{E_2} dW' \int_0^\infty u_W^{\rho *} u_{W'}^{\rho} \, d\tau = 0$$ for the continuous spectrum,

W lying outside the interval E_1, E_2. We make use of these relations in discussing the equation (7). If we multiply through (7) by $u_i^{\rho *} \epsilon^{\frac{2\pi i}{h} W_i t}$ and integrate the resulting expression we can obtain, by use of condition (8) expressions for $\frac{d\, C_k(t)}{dt}$ We wish, however for the present to limit ourselves to the discussion of transitions in the continuous part of the spectrum only, omitting all coefficients C_k belonging to the discrete spectrum which are of no interest to our problem. Multiply both sides of equation (7) by $u_{W'}^{\rho} \epsilon^{-\frac{2\pi i}{h} W' t}$ where the subscript (W') is to indicate the continuous spectrum, obtaining

$$\frac{h}{2\pi i}\frac{dC_{W'}(t)}{dt} = \sum_K C_K(t)\,\epsilon^{\frac{2\pi i}{h}(W_K-W')t}\,\frac{e}{c}\,q^0_{W'K} + \int_u^\infty C_W(t)\,\epsilon^{\frac{2\pi i}{h}(W-W')t}\,\frac{e}{c}\,q^0_{W'W}\,dW$$

$$+\sum_\nu\left[\sum_K \alpha_\nu C_K(t)\,\epsilon^{\frac{2\pi i}{h}(W_K-W')t}\,\frac{e}{c}\,q^\nu_{W'K} + \alpha_\nu\int_u^\infty C_W(t)\,\epsilon^{\frac{2\pi i}{h}(W-W')t}\,\frac{e}{c}\,q^\nu_{W'W}\,dW\right] \qquad 9.$$

where we have written for brevity

$$q^a_{bc} = \sum_\rho \int u_b^{\rho *}\, q^a\, u_c^\rho\, d\tau \qquad 10.$$

Yielding an expression for $\frac{dC_{W'}(t)}{dt}$ as a function of the wavefunctions for the initial state, independent of the time and the perturbing quantity $q_{0,\nu}$.

In order to proceed with our discussion we make the assumption that the perturbation is small and of the form $e q^0 = V^* \epsilon^{\frac{2\pi i}{h}\Delta E\, t}$ Since the components q^ν will be of the order (v/c) smaller we will for convenience set $q^\nu = 0$ This omission of small terms in q^ν will not appreciable affect the transition curves for the case being considered*.

We have then

$$\frac{dC_{W'}(t)}{dt} = \frac{2\pi i}{h}\sum_K C_K(t)\,\epsilon^{\frac{2\pi i}{h}(W_K+\Delta E-W')t}\,V^*_{W',K} + \frac{2\pi i}{h}\int_u^\infty C_W(t)\,V^*_{W'W}\,\epsilon^{\frac{2\pi i}{h}(W+\Delta E-W')t}\,dW \qquad 11.$$

C_w will have a reasonable value for a small range in w, and $C_k = 0$. In the following we shall represent $\int_{W+\Delta E}^{W-\Delta E} C_W dW$ by C.

Then $\frac{dC_{W'}(t)}{dt} = \frac{2\pi i}{h} C V_{W'W}^{*} \epsilon^{\frac{2\pi i}{h}(W+\Delta E-W')t}$

or $C_{W'}(t) = \frac{C V_{W'W}^{*} \epsilon^{\frac{2\pi i}{h}(W+\Delta E-W')t}}{(W+\Delta E-W')} - K_a'$

At time $t=0$, $C_{W'}(t)$ is zero; therefore

$$K_a' = \frac{C V_{W'W}^{*}}{(W+\Delta E-W')}$$

which gives

$$C_{W'}(t) = C V_{W'W}^{*} \frac{\epsilon^{\frac{2\pi i}{h}(W+\Delta E-W')t} - 1}{(W+\Delta E-W')} \qquad 12.$$

In the expansion of the wavefunction ψ^{ρ} (see equation (3), the assumption is made that $|C_a(t)|^2$ is equal to the probability that the atom is in the state (a) at the time t. That is, $|C_{W'}(t)|^2$ will be the probability of the system being in W' state at time t. The total number of transitions is given by

*) See page 12 and 13.

$$\int_0^\infty |C_{W'}(t)|^2 dW' = |c|^2 \int_0^\infty |V^*_{WW'}|^2 \frac{2 - \left(\epsilon^{\frac{2\pi i}{h}(W+\Delta E-W')t} + \epsilon^{-\frac{2\pi i}{h}(W+\Delta E-W')t}\right)}{(W+\Delta E-W')^2} dW \qquad 13.$$

and since outside $W' \doteq W + \Delta E$ the contribution is small, $|V^*_{WW'}|^2$ can be taken from under the integral as $|V^*_{W,W+\Delta E}|^2$. Equation (13) can then be simplified by transforming it to the trigonometric form

$$\int_0^\infty |C_{W'}(t)|^2 dW' = |c|^2 |V^*_{W,W+\Delta E}|^2 \int_0^\infty \frac{4 \sin^2 \frac{\pi}{h}(W+\Delta E-W')t}{(W+\Delta E-W')^2} dW' \qquad 14.$$

but $$\int_0^\infty \frac{\sin^2 \pi/h\,(W+\Delta E-W')t}{(W+\Delta E-W')^2} dW' = \frac{t\pi}{h} \int_0^\infty \frac{\sin^2 \xi}{(\xi)^2} d\xi$$

where $$\xi = \frac{\pi}{h}(W+\Delta E-W')t$$

therefore $$\int_0^\infty |C_{W'}(t)|^2 dW' = \frac{4\pi^2}{h} |c|^2 |V^*_{W,W+\Delta E}|^2$$

Upon dividing through by the time

$$\lambda = \frac{4\pi^2}{h} |c|^2 |V^*_{W,W+\Delta E}|^2 \qquad 15.$$

where λ represents the number of transitions to state $W' \doteq W + \Delta E$ per unit time.

Formula (15) has been deduced for transitions between electron states only.

The same formula, however, can be used for transitions between two states refering to different particles obeying a wave equation of the Dirac type, e.g. electron and neutrino states if the latter particle is supposed to be described by equation (1) with $\varphi_0, \varphi_\nu = 0$, $m = 0$.

A transition of this type refers to β -processes in which the particle in the initial state, e.g. the neutrino, disappears and the particle in the final state (in our case an electron) is produced. The meaning of the matrix element $V^*_{W W'}$ is but slightly changed in this case, since we have to take in

$$V^*_{W,W'} = \int u^*_{W'} V^* u_W \, d\tau$$

for $u_{W'}$ the eigenfunction of the initial, for u_W the eigenfunction of the final particle.

Actual solution of $(a X_K + B \varphi_K) = \text{const}\, \varphi_K$

$(a X_K + B \varphi_K)$ when written in terms of eigen-functions becomes

$$\text{const}\left\{\left(\sqrt{K^2-\alpha^2}+K\right)\sqrt{\frac{W}{mc^2}-1} - i\alpha\sqrt{\frac{W}{mc^2}+1}\right. \qquad 23.$$

$$\left.-\text{const}\left\{\left(\sqrt{K^2-\alpha^2}-K\right)\sqrt{\frac{W}{mc^2}+1} - i\alpha\sqrt{\frac{W}{mc^2}-1}\right\}\right\}$$

this can be transformed to

$$\frac{\left(\sqrt{K^2-\alpha^2}+K+ic\alpha\right)}{\sqrt{K^2-\alpha^2}+K}\left\{\left(\sqrt{K^2-\alpha^2}+K\right)\sqrt{\frac{W}{mc^2}-1}\right\} - c\,\frac{\left(\sqrt{K^2-\alpha^2}-K\right)+i\alpha\left(i\alpha\sqrt{\frac{W}{mc^2}+1}\right)}{i\alpha}$$

Since for any transition the value of K and K' will be fixed,

$$\frac{\sqrt{K^2-\alpha^2}+K+ic\alpha}{\sqrt{K^2-\alpha^2}+K} = c\,\frac{\left(\sqrt{K^2-\alpha^2}-K\right)+i\alpha}{i\alpha}$$

which permits (23) to be written

$$(a X_K + B\varphi_K) = \text{const}\left\{\left(\sqrt{K^2-\alpha^2}+K\right)\sqrt{\frac{W}{mc^2}-1} - i\alpha\sqrt{\frac{W}{mc^2}+1}\right\}$$

$$= \text{const}\ \varphi_K$$

A similar method may be utilized to obtain the alternative.

TABLES

$K<0$

$K=-1$	ω^{α}_{Km}	ω^{β}_{Km}	ω^{γ}_{Km}	ω^{δ}_{Km}
$m=1$	0	$-P_0^0$	$2P_1^{-1}e^{-i\varphi}$	P_1^0
$m=0$	P_0^0	0	P_1^0	$P_1^1 e^{i\varphi}$
$K=-2$				
$m=-2$	0	$-P_1^{-1}e^{-i\varphi}$	$4P_2^{-2}e^{-2i\varphi}$	$P_2^{-1}e^{-i\varphi}$
$m=-1$	$P_1^{-1}e^{-i\varphi}$	$-P_1^0$	$3P_2^{-1}e^{-i\varphi}$	P_2^0
$m=0$	$2P_1^0$	$-P_1^1 e^{i\varphi}$	$2P_2^0$	$P_2^1 e^{i\varphi}$
$m=1$	$3P_1^1 e^{i\varphi}$	0	$P_2^1 e^{i\varphi}$	$P_2^2 e^{2i\varphi}$
$K=-3$				
$m=-3$	0	$-P_2^{-2}e^{-2i\varphi}$	$6P_3^{-3}e^{-3i\varphi}$	$P_3^{-2}e^{-2i\varphi}$
$m=-2$	$P_2^{-2}e^{-2i\varphi}$	$-P_2^{-1}e^{-i\varphi}$	$5P_3^{-2}e^{-2i\varphi}$	$P_3^{-1}e^{-i\varphi}$
$m=-1$	$2P_2^{-1}e^{-i\varphi}$	$-P_2^0$	$4P_3^{-1}e^{-i\varphi}$	P_3^0
$m=0$	$3P_2^0$	$-P_2^1 e^{i\varphi}$	$3P_3^0$	$P_3^1 e^{i\varphi}$
$m=1$	$4P_2^1 e^{i\varphi}$	$-P_2^2 e^{2i\varphi}$	$2P_3^1 e^{i\varphi}$	$P_3^2 e^{2i\varphi}$
$m=2$	$5P_2^2 e^{2i\varphi}$	0	$P_3^2 e^{2i\varphi}$	$P_3^3 e^{3i\varphi}$

Tabl A_1

$\kappa > 0$

$\kappa = 1$	$\omega^{\alpha}_{\kappa m}$	$\omega^{\beta}_{\kappa m}$	$\omega^{\gamma}_{\kappa m}$	$\omega^{\delta}_{\kappa m}$
$m=-1$	$2P_1^{-1}e^{-i\varphi}$	P_1^0	0	$-P_0^0$
$m=0$	P_1^0	$P_1^1 e^{i\varphi}$	P_0^0	0
$\kappa = +2$				
$m=-2$	$4P_2^{-2}e^{-2i\varphi}$	$P_2^{-1}e^{-i\varphi}$	0	$-P_1^{-1}e^{-i\varphi}$
$m=-1$	$3P_2^{-1}e^{-i\varphi}$	P_2^0	$P_1^{-1}e^{-i\varphi}$	$-P_1^0$
$m=0$	$2P_2^0$	$P_2^1 e^{i\varphi}$	$2P_1^0$	$-P_1^1 e^{i\varphi}$
$m=1$	$P_2^1 e^{i\varphi}$	$P_2^2 e^{2i\varphi}$	$3P_1^1 e^{i\varphi}$	0
$\kappa = +3$				
$m=-3$	$6P_3^{-3}e^{-3i\varphi}$	$P_3^{-2}e^{-2i\varphi}$	0	$-P_2^{-2}e^{-2i\varphi}$
$m=-2$	$5P_3^{-2}e^{-2i\varphi}$	$P_3^{-1}e^{-i\varphi}$	$P_2^{-2}e^{-2i\varphi}$	$-P_2^1 e^{-i\varphi}$
$m=-1$	$4P_3^{-1}e^{-i\varphi}$	P_3^0	$2P_2^{-1}e^{-i\varphi}$	$-P_2^0$
$m=0$	$3P_3^0$	$P_3^1 e^{i\varphi}$	$3P_2^0$	$-P_2^1 e^{i\varphi}$
$m=1$	$2P_3^1 e^{i\varphi}$	$P_3^2 e^{2i\varphi}$	$4P_2^1 e^{i\varphi}$	$-P_2^2 e^{2i\varphi}$
$m=2$	$P_3^2 e^{2i\varphi}$	$P_3^3 e^{3i\varphi}$	$5P_2^2 e^{2i\varphi}$	0

Tabl. A_2

$P_\ell^m =$

$\ell = 0$	$m = 0$		
	$P_0^0 = 1$		
$\ell = 1$	$m = -1$	$m = 0$	$m = +1$
	$-\frac{1}{2}\sin\theta$	$\cos\theta$	$\sin\theta$
$\ell = 2$	$m = -2$	$m = -1$	$m = 0$
	$\frac{1}{8}\sin^2\theta$	$-\frac{1}{2}\sin\theta\cos\theta$	$\frac{1}{2}(3\cos^2\theta - 1)$
	$m = 1$	$m = 2$	
	$3\sin\theta\cos\theta$	$3\sin^2\theta$	
$\ell = 3$	$m = -3$	$m = -2$	$m = -1$
	$-\frac{1}{48}\sin^3\theta$	$\frac{1}{8}\cos\theta\sin^2\theta$	$-\frac{1}{8}\sin\theta\,(5\cos^2\theta - 1)$
	$m = 0$	$m = 1$	$m = 2$
	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$	$\frac{3}{2}\sin\theta\,(5\cos^2\theta - 1)$	$15\sin^2\theta\cos\theta$
	$m = 3$		
	$15\sin^3\theta$		

Table B_1

		$N_{\kappa m}$
$\kappa = -1$	$m = -1$	$\sqrt{1/4\pi}$
	$m = 0$	$\sqrt{1/4\pi}$
$\kappa = -2$	$m = -2$	$\sqrt{1/4\pi}\ \sqrt{3!}$
	$m = -1$	$\sqrt{1/4\pi}\ \sqrt{2!}$
	$m = 0$	$\sqrt{1/4\pi}\ \sqrt{1/2!}$
	$m = 1$	$\sqrt{1/4\pi}\ \sqrt{1/3!}$
$\kappa = -3$	$m = -3$	$\sqrt{1/4\pi}\ \sqrt{5!}$
	$m = -2$	$\sqrt{1/4\pi}\ \sqrt{4!}$
	$m = -1$	$\sqrt{1/4\pi}\ \sqrt{3!/2!}$
	$m = 0$	$\sqrt{1/4\pi}\ \sqrt{2!/3!}$
	$m = 1$	$\sqrt{1/4\pi}\ \sqrt{1/4!}$
	$m = 2$	$\sqrt{1/4\pi}\ \sqrt{1/5!}$
$\kappa = +1$	$m = -1$	$\sqrt{1/4\pi}$
	$m = 0$	$\sqrt{1/4\pi}$
$\kappa = +2$	$m = -2$	$\sqrt{1/4\pi}\ \sqrt{3!}$
	$m = -1$	$\sqrt{1/4\pi}\ \sqrt{2!}$
	$m = 0$	$\sqrt{1/4\pi}\ \sqrt{1/2!}$
	$m = 1$	$\sqrt{1/4\pi}\ \sqrt{1/3!}$
$\kappa = +3$	$m = -3$	$\sqrt{1/4\pi}\ \sqrt{5!}$
	$m = -2$	$\sqrt{1/4\pi}\ \sqrt{4!}$
	$m = -1$	$\sqrt{1/4\pi}\ \sqrt{3!/2!}$
	$m = 0$	$\sqrt{1/4\pi}\ \sqrt{2!/3!}$
	$m = 1$	$\sqrt{1/4\pi}\ \sqrt{1/4!}$
	$m = 2$	$\sqrt{1/4\pi}\ \sqrt{1/5!}$

Table C_1