# Dynamic Behavior of 2D Thermoelastic Solid Continua Using Mathematical Model Derived Based on Non-classical Continuum Mechanics with Internal Rotations 

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#### Abstract

This thesis considers dynamic behavior of non-classical thermoelastic solid continua. The mathematical model consists of the conservation and balance laws of non-classical continuum mechanics that incorporates additional physics of internal rotations arising due to deformation gradient tensor. We consider plane stress behavior with small deformation, small strain physics only. Galerkin Method with Weak Form (GM/WF) in space is considered to construct a space-time decoupled finite element formulation giving rise to ordinary differential equations (ODEs) in time containing mass matrix, stiffness matrix due to classical as well as non-classical physics and acceleration and displacement associated with nodal degrees of freedom. This formulation is utilized to: (i) study natural undamped modes of vibration (ii) study transient dynamic response by time integrating the ODEs in time (iii) study the transient dynamic response by transforming the ODEs in time to modal basis using eigenvectors of the undamped natural modes. The ODEs in modal basis are used to construct transient dynamic response by time integrating them as well as by considering their analytical solutions. The solutions of the model problem obtained using the mathematical model based on non-classical continuum mechanics with internal rotations are presented and are compared with those obtained using the mathematical model based on classical continuum mechanics to demonstrate the influence of new physics due to internal rotations on the dynamic response of solid continua.


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## Chapter 1

## Introduction, Literature Review and Scope of Work

### 1.1 Introduction

In Lagrangian description of deforming solid continua, the Jacobian of deformation $\boldsymbol{J}$ (or deformation gradient tensor) is a fundamental quantity of the measure of the deformation physics. In general, $\boldsymbol{J}$ varies at material points. Polar decomposition of $\boldsymbol{J}$ into stretch (left or right) and pure rotation tensors shows that if $\boldsymbol{J}$ varies between material points so does the rotation tensor. Alternatively, decomposition of $\boldsymbol{J}$ into symmetric and skew symmetric tensor also confirms varying rotations between material points defined by skew symmetric part of $\boldsymbol{J}$. The rotation measures either from polar decomposition or from skew symmetric point of $\boldsymbol{J}$ are not considered in classical continuum theories, but are intrinsic part of the actual deformation physics in all deforming solid continua in which $\boldsymbol{J}$ is the fundamental measure of deformation physics.

In classical continuum theories only stretch tensor, stretch rate tensor or alternatively strain tensor and strain rate tensor contribute to energy storage and dissipation mechanisms. In the theory considered here we consider the resistance to varying rotations and rotation rates by the continua resulting in the existence of moment tensor. The moment tensor, rotations and their rates result in
additional mechanisms of energy storage and dissipation (new physics) that is completely ignored in the classical continuum theories. The theory considered here is a non-classical continuum theory based on internal rotations due to $\boldsymbol{J}$ for thermoelastic solid continua. This physics exists in all deforming solid continua. This theory should not be confused with micropolar theories [1-11] that often incorporate the scales smaller than continuum scale and are designed for this purpose. In short, the continuum theory considered here uses standard strain and stress measures as used in classical continuum mechanics but additionally incorporates the deformation physics associated with internal rotations due to $\boldsymbol{J}$. In the following we present a brief literature review of some published works related to the work presented in this thesis.

### 1.2 Literature Review

The non-classical continuum theories in the published literature appear under various categories: micropolar theories, couple stress theories, non-local theories etc. A comprehensive treatment of micropolar theories can be found in the works by Eringen [1-9]. The concept of couple stresses is presented by Koiter [10]. Balance laws for micromorphic materials are presented in reference [11]. The micropolar theories consider micro deformation due to micro constituents in the continuum. In references [12-14] by Reddy et al. and reference [15] by Zang et al. nonlocal theories are presented for bending, buckling and vibration of beams, with nanocarbon tubes and bending of plates. The nonlocal effects are primarily incorporated based on the work presented by Eringen [6] in which definition of a nonlocal stress tensor is introduced through integral relationship using the product of macroscopic stress tensor and a distance kernel representing the nonlocal effects. The continuum theory for solid continua considered and presented in this thesis is strictly local and non-micropolar. The concept of couple stress was introduced by Voigt in 1881 by assuming a couple or moment per unit area on the oblique plane of the deformed tetrahedron in addition to the stress or force per unit area. Since the introduction of this concept many published works have appeared. We cite some recent works, most of which are related to micropolar cou-
ple stress theories. Authors in reference [16] report experimental study of micropolar and couple stress elasticity of compact bones in bending. Conservation integrals in couple stress elasticity are reported in reference [17]. A microstructure dependent Timoshenko beam model based on modified couple stress theories is reported by Ma et al. [18]. Further account of couple stress theories in conjunction with beams can be found in references [19-21]. Treatment of rotation gradient dependent strain energy and its specialization to von Kármán plates and beams can be found in reference [22]. Other accounts of micropolar elasticity and Cosserat modeling of cellular solids can be found in references [23-25]. We remark that in references [16-25], Lagrangian description is used for solid matter, however the mathematical descriptions are purely derived using strain energy density functional or principle of virtual work. These energy methods work well for elastic solids in which mechanical deformation is small and is reversible. Extension of these works to thermoviscoelastic solids with and without memory is not possible. In such materials the thermal field and mechanical deformation are coupled due to the fact that the rate of work results in rate of entropy production. In references [26-37] various aspects of the kinematics of micropolar theories, couple stress theories, etc. are discussed and presented including some applications to plates and shells.

If the varying rotations and their rates result in energy storage and dissipation, then their energy conjugate moment must exist in the deforming matter. This necessitates the existence of moment (per unit area) on the oblique plane of the deformed tetrahedron. Thus, at the onset, we consider average force per unit area and displacements, and average moment per unit area and the rotations associated with the deformed tetrahedron. The work presented here follows a strictly thermodynamic approach as presented in [38-44] (discussed in the following). We consider: (i) Conservation of mass and present rationale for not considering conservation of inertia as a conservation law [45] (ii) Balance of linear momenta (iii) Balance of angular momenta (iv) Balance of moments of moments (or couples) (v) First law of thermodynamics and (vi) Second law of thermodynamics in Lagrangian description in which stress and strain rate, moment and symmetric part of the rotation gradient rates are rate of work conjugate pairs.

In references [38,39], authors presented conservation and balance laws for thermoelastic solid continua based on non-classical continuum mechanics (NCCM) incorporating internal rotations due to the deformation gradient tensor. This work considers small deformation, small strain physics in Lagrangian description. Authors in reference [40] presented constitutive theory for non-classical thermoelastic solids of $[38,39]$ based on conjugate pairs in the entropy inequality and representation theorem. Model problem studies and comparison with classical continuum mechanics (CCM) were also presented in reference [40] for BVPs. The non-classical continuum theory of references [38-40] was extended in reference [41] for finite deformation using first Piola-Kirchhoff stress tensor. In reference [42], the authors further extended the work in reference [41] for finite deformation and finite strain using second Piola-Kirchhoff stress tensor and rate of Green's strain tensor as rate of work conjugate pair. The non-classical continuum theory presented in references $[38,39]$ based on internal rotations was extended in reference [43] to include internal rotations as well as Cosserat rotations. The authors of reference [43] extended the work based on internal and Cosserat rotations for solid continua to fluent continua in reference [44].

### 1.3 Scope of Work

In this thesis, we consider conservation and balance laws of NCCM of reference $[38,39]$ and the constitutive theory of reference [40] for small deformation, small strain physics to study dynamic response of solids (IVPs). For the sake of simplicity, the mathematical model and the model problem study are only considered in $\mathbb{R}^{2}$. The mathematical model consisting of balance linear momenta is cast purely in terms of displacements. Finite element formulation is presented and applied to study model problem. The balance of linear momenta equations in displacements are used to construct finite element formulation using space-time decoupled approach based on Galerkin method with weak form (GM/WF) in space. The resulting ODEs in time are used to study natural undamped vibrations and transient dynamic response. Influence of the internal rotation physics is illustrated using a model problem.

## Chapter 2

## Mathematical Model

### 2.1 Conservation and Balance Laws

The mathematical model for non-classical continuum mechanics incorporating internal rotations due to the deformation gradient tensor $\boldsymbol{J}$ also consists of conservation of mass, balance of linear momenta, balance of angular momenta, first and second laws of thermodynamics and the constitutive theories for the constitutive variables depending upon the physics. These are similar to classical continuum mechanics in appearance but there are some modifications and differences. First, we note that due to consideration of additional physics of rotations and conjugate moment tensor in this non-classical theory, the conservation and balance laws of classical mechanics are not adequate to ensure equilibrium of the deforming matter. Yang et al. [16] and Surana et al. [45] have shown that in the non-classical continuum theories additional balance law of "balance of moment of moments is needed." The authors have discussed the necessity of the balance law and Surana et al. [45] have presented a rate derivation (necessary for a balance law) of the new balance law and have show that due to this balance law the Cauchy moment tensor is symmetric. Due to the new physics of rotations the classical continuum mechanics balance laws also need modifications [38]. In the non-classical continuum theory considered here, the Cauchy stress is not symmetric, balance of angular momenta establishes the relationship between the gradients of the Cauchy moment
tensor and the antisymmetric components of the nonsymmetric stress tensor. The energy equation and the entropy inequality establish rate of work conjugate pairs. The entropy inequality expressed in Helmholtz free energy facilitates derivation of the constitutive theories once a specific deformation physics is chosen. In the following we present conservation and balance laws and linear constitutive theories for thermoelastic solid continua based on small deformation, small strain assumption. For this deformation physics the distinction between and covariant and contravariant measures disappears [46]. Thus, stress tensor is simply non-symmetric Cauchy stress tensor $\boldsymbol{\sigma}$ and strain tensor is linear strain tensor $\boldsymbol{\varepsilon}$, a linear function of displacement gradients. We have the following conservation and balance laws in Lagrangian description [38]: conservation of mass (CM), balance of linear moment (BLM), balance of angular momenta (BAM) balance of moment of moments (BMM), first law of thermodynamics (FLT), the second law of thermodynamics (SLT), and the linear constitutive theories.

Conservation and balance laws:

$$
\begin{align*}
& \rho_{0}=|J| \rho(\boldsymbol{x}, t)  \tag{CM}\\
& \rho_{0} \frac{D v_{i}}{D t}-\rho_{0} F_{i}^{b}-\frac{\partial \sigma_{j i}}{\partial x_{j}}=0  \tag{BLM}\\
& m_{l k, l}+\epsilon_{i j k} \sigma_{i j}=0  \tag{BAM}\\
& \epsilon_{i j k} m_{i j}=0  \tag{2.4}\\
& \rho_{0} \frac{D e}{D t}+\frac{\partial q_{i}}{\partial x_{i}}-\operatorname{tr}\left(\left[{ }_{s} \sigma\right][\dot{\varepsilon}]\right)-\operatorname{tr}\left([m]\left[{ }_{s}^{[\Theta} \dot{J}\right]\right)=0  \tag{2.5}\\
& \rho_{0}\left(\frac{D \phi}{D t}+\eta \frac{D \theta}{D t}\right)+\frac{q_{i} g_{i}}{\theta}-\operatorname{tr}\left(\left[{ }_{s} \sigma\right][\dot{\varepsilon}]\right)-\operatorname{tr}\left([m]\left[{ }_{s}^{i} \Theta \dot{J}\right]\right) \leq 0
\end{align*}
$$

and

$$
\boldsymbol{\sigma}={ }_{s} \boldsymbol{\sigma}+{ }_{a} \boldsymbol{\sigma}
$$

In which $\boldsymbol{v}$ is velocity vector, $\boldsymbol{F}^{b}$ is body force vector per unit mass, $\boldsymbol{\sigma}$ is Cauchy stress tensor, ${ }_{s} \boldsymbol{\sigma}$ and ${ }_{a} \boldsymbol{\sigma}$ are symmetric and antisymmetric components of the Cauchy stress tensors $\boldsymbol{\sigma}, \boldsymbol{m}$ is
symmetric Cauchy moment tensor, $\boldsymbol{\epsilon}$ is permutation tensor, e is internal energy density, $\rho_{0}$ is density in the reference configuration, $\boldsymbol{q}$ is heat tensor, $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ are linear strain and strain rate tensors, ${ }_{i} \boldsymbol{\Theta}$ are internal rotations (due to (2.14)), ${ }^{\ominus} \boldsymbol{J}$ is rotation gradient tensor, ${ }^{i}{ }_{s}^{\Theta} \boldsymbol{J}$ is symmetric part of ${ }^{\ominus} \Theta \boldsymbol{J}, \phi$ is Helmholtz free energy density, $\eta$ is entropy density, $\theta$ is absolute temperature, $\boldsymbol{g}$ is temperature gradient tensor. The letter ' $e$ ' used here to represent internal energy density is also used to represent a typical finite element ' $e$ ' with domain $\bar{\Omega}_{x}^{e}$. The deformation gradient tensor $\boldsymbol{J}$ or Jacobian of deformation is defined as

$$
\begin{equation*}
[J]=\left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right]=[I]+\left[{ }^{d} J\right] \tag{2.7}
\end{equation*}
$$

in which the displacement gradient tensor $\left[\begin{array}{l}d \\ \left.{ }_{a} J\right]\end{array}\right.$ is given by

$$
\begin{equation*}
\left[{ }^{d} J\right]=\left[\frac{\partial\{u\}}{\partial\{x\}}\right] \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[{ }^{d} J\right]=\left[{ }_{s}^{d} J\right]+\left[{ }_{a}^{d} J\right]=[\varepsilon]+\left[{ }_{a}^{d} J\right] \tag{2.9}
\end{equation*}
$$

in which ${ }_{a}^{d} J$ contains rotations, about $o x_{i}$ axes. The internal rotations ${ }_{i} \boldsymbol{\Theta}$ are defined in the following

$$
\begin{gather*}
{\left[{ }_{a}^{d} J\right]=\frac{1}{2}\left[\begin{array}{ccc}
0 & { }_{a}^{d} J_{12} & { }_{a}^{d} J_{13} \\
-{ }_{a}^{d} J_{12} & 0 & { }_{a}^{d} J_{23} \\
-{ }_{a}^{d} J_{13} & -{ }_{a}^{d} J_{23} & 0
\end{array}\right]}  \tag{2.10}\\
{ }_{a}^{d} J_{12}=\left(\frac{\partial u_{1}}{\partial x_{2}}-\frac{\partial u_{2}}{\partial x_{1}}\right) ; \quad{ }_{a}^{d} J_{13}=\left(-\frac{\partial u_{3}}{\partial x_{1}}+\frac{\partial u_{1}}{\partial x_{3}}\right) ; \quad{ }_{a}^{d} J_{23}=\left(\frac{\partial u_{2}}{\partial x_{3}}-\frac{\partial u_{3}}{\partial x_{2}}\right) \tag{2.11}
\end{gather*}
$$

Alternatively (2.11) can be derived as

$$
\begin{equation*}
\boldsymbol{\nabla} \times \boldsymbol{u}=\boldsymbol{e}_{i} \times \boldsymbol{e}_{j} \frac{\partial u_{j}}{\partial x_{i}}=\epsilon_{i j k} \boldsymbol{e}_{k} \frac{\partial u_{j}}{\partial x_{i}} \tag{2.12}
\end{equation*}
$$

$$
\begin{gather*}
\boldsymbol{\nabla} \times \boldsymbol{u}=\boldsymbol{e}_{1}\left(\frac{\partial u_{3}}{\partial x_{2}}-\frac{\partial u_{2}}{\partial x_{3}}\right)+\boldsymbol{e}_{2}\left(\frac{\partial u_{1}}{\partial x_{3}}-\frac{\partial u_{3}}{\partial x_{1}}\right)+\boldsymbol{e}_{3}\left(\frac{\partial u_{2}}{\partial x_{1}}-\frac{\partial u_{1}}{\partial x_{2}}\right)  \tag{2.13}\\
\boldsymbol{\nabla} \times \boldsymbol{u}=\boldsymbol{e}_{1}\left({ }_{i} \Theta_{1}\right)+\boldsymbol{e}_{2}\left({ }_{i} \Theta_{2}\right)+\boldsymbol{e}_{3}\left({ }_{i} \Theta_{3}\right) \tag{2.14}
\end{gather*}
$$

Comparing (2.11) and (2.13), we note that

$$
\begin{equation*}
{ }_{a}^{d} J_{12}=\frac{1}{2}\left({ }_{i} \Theta_{3}\right) ; \quad{ }_{a}^{d} J_{13}=-\frac{1}{2}\left({ }_{i} \Theta_{2}\right) ; \quad{ }_{a}^{d} J_{23}=\frac{1}{2}\left({ }_{i} \Theta_{1}\right) \tag{2.15}
\end{equation*}
$$

The rotations ${ }_{i} \Theta_{1},{ }_{i} \Theta_{2},{ }_{i} \Theta_{3}$ defined by (2.13) are positive in the counterclockwise sense. We use this definition in the present work. Gradients of rotation angles can be easily be obtained. Let

$$
\begin{equation*}
\left\{{ }_{i} \Theta\right\}^{T}=\left[{ }_{i} \Theta_{1},{ }_{i} \Theta_{2},{ }_{i} \Theta_{3}\right] \tag{2.16}
\end{equation*}
$$

Gradients of rotations in (2.16) $\left(\left[{ }_{[ } \Theta J\right]\right)$ can be obtained using

$$
\begin{equation*}
\left.{ }^{i}{ }^{\Theta} J\right]=\left[\frac{\partial\left\{\left\{_{i} \Theta\right\}\right.}{\partial\{x\}}\right] \text { or }{ }^{\ominus} J_{j k}=\frac{\partial\left({ }_{i} \Theta_{j}\right)}{\partial x_{k}} \tag{2.17}
\end{equation*}
$$

The gradient tensor $\left[i^{\Theta} J\right]$ of rotations can be decomposed into symmetric and antisymmetric parts $\left.{ }_{[ }^{[ }{ }_{s}^{\Theta} J\right]$ and $\left[\begin{array}{c}i \\ a\end{array} \Theta J\right]$.

$$
\begin{gather*}
{\left[{ }^{\ominus} \Theta\right]=\left[{ }_{s}^{\ominus} J\right]+\left[{ }_{a}^{\Theta} J\right]}  \tag{2.18}\\
\left.\left.\left[{ }_{s}^{\Theta} J\right]=\frac{1}{2}\left(\left[{ }^{\ominus} \Theta\right]\right]+\left[{ }^{\ominus} \Theta\right]\right]^{T}\right) \\
\left.{ }_{a}^{[\Theta} J\right]=\frac{1}{2}\left(\left[{ }^{\ominus} \Theta\right]-\left[{ }^{i} \Theta J\right]^{T}\right) \tag{2.19}
\end{gather*}
$$

on the other hand polar decomposition gives

$$
\begin{equation*}
\left[{ }^{d} J\right]=\left[{ }^{d} R\right]\left[{ }^{d} S_{r}\right]=\left[{ }^{d} S_{l}\right]\left[{ }^{d} R\right] \tag{2.20}
\end{equation*}
$$

The right and left stretch tensors $\left[{ }^{d} S_{r}\right]$ and $\left[{ }^{[ } S_{l}\right]$ are symmetric and positive-definite, and $\left[{ }^{d} R\right]$ is an orthogonal rotation tensor, a rotation matrix tensor where as $\left[\begin{array}{l}d \\ a\end{array}\right]$ contains half of the rotation
angles. Deriving rotation angles from $\left[{ }^{d} R\right]$ or $[R]$ in general $\mathbb{R}^{3}$ may not be possible or unique [46]. Fortunately, there is no need for this here.

Incorporating ${ }^{d} J$ in its entirety in the derivation of the conservation and balance laws implies incorporating $\left[\begin{array}{c}d \\ { }_{s} J\end{array}\right]$, and $\left[\begin{array}{c}d \\ a_{a} J\end{array}\right]$ (i.e. rotations ${ }_{i} \Theta_{1},{ }_{i} \Theta_{2},{ }_{i} \Theta_{3}$ about the axes of a triad located at each material point). Rotations in $\left[\begin{array}{l}d \\ a^{d}\end{array}\right]$ are internal and completely defined by skew-symmetric part of $\left[{ }^{d} J\right]$ or (2.13)-(2.14).

### 2.1.1 Constitutive theories

The entropy inequality (2.6) expressed in terms of Helmholtz free energy density and the rate of work (mechanical) conjugate pairs in the entropy inequality are instrumental in determining the constitutive variables, their argument tensors as well as derivations of some constitutive theories. Choice of $\Phi, \eta,{ }_{s} \boldsymbol{\sigma}, \boldsymbol{m}$, and $\boldsymbol{q}$ as constitutive variables based on the axioms of constitutive theory $[46,47]$ and the entropy inequality as well as other balance laws is quite obvious. It is straight forward to chose the argument tensors of ${ }_{s} \boldsymbol{\sigma}, \boldsymbol{m}$ and $\boldsymbol{q}$ using the conjugate pairs in (2.6) including $\theta$ due to thermoelastic behavior.

$$
\begin{align*}
{ }_{s} \boldsymbol{\sigma} & ={ }_{s} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta)  \tag{2.21}\\
\boldsymbol{m} & =\boldsymbol{m}\left({ }_{s}^{i}{ }_{s}^{\Theta} \boldsymbol{J}, \theta\right)  \tag{2.22}\\
\boldsymbol{q} & =\boldsymbol{q}(\boldsymbol{g}, \theta) \tag{2.23}
\end{align*}
$$

The argument tensors of $\Phi$ and $\eta$ at the onset can be chosen based on principle of equipresence [46, 47]. Thus we have the following

$$
\begin{align*}
\Phi & =\Phi\left(\boldsymbol{\varepsilon},{ }_{s}^{\ominus}{ }_{s} \boldsymbol{J}, \boldsymbol{g}, \theta\right)  \tag{2.24}\\
\eta & =\eta\left(\boldsymbol{\varepsilon},{ }_{s}^{\Theta} \boldsymbol{J}, \boldsymbol{g}, \theta\right) \tag{2.25}
\end{align*}
$$

At this stage, (2.21)-(2.25) define constitutive variables and their argument tensors. In the present work we consider (2.21)-(2.23), as well as (2.24), in the derivation of the constitutive theories. Using the argument tensor of $\Phi$ in (2.24), we can write

$$
\begin{equation*}
\frac{D \Phi}{D t}=\dot{\Phi}=\frac{\partial \Phi}{\partial \varepsilon_{k l}} \dot{\varepsilon}_{k l}+\frac{\partial \Phi}{\partial\left({ }_{s}^{\Theta} J\right)_{k l}}\left(i{ }_{s}^{\Theta} \dot{J}\right)_{k l}+\frac{\partial \Phi}{\partial g_{i}} \dot{g}_{i}+\frac{\partial \Phi}{\partial \theta} \dot{\theta} \tag{2.26}
\end{equation*}
$$

Substituting (2.26) in the entropy inequality (2.6) and collecting terms we obtain the following

$$
\begin{equation*}
\left(\rho_{0} \frac{\partial \Phi}{\partial \varepsilon_{k l}}-{ }_{s} \sigma_{k l}\right) \dot{\varepsilon}_{k l}+\left(\rho_{0} \frac{\partial \Phi}{\partial\left({ }_{s}^{\Theta} J\right)_{k l}}-m_{k l}\right)\left({ }_{s}^{i}{ }_{s}^{\Theta} \dot{J}\right)_{k l}+\rho_{0}\left(\eta+\frac{\partial \Phi}{\partial \theta}\right) \dot{\theta}+\rho_{0} \frac{\partial \Phi}{\partial g_{i}} \dot{g}_{i}+\frac{q_{i} \cdot g_{i}}{\theta} \leq 0 \tag{2.27}
\end{equation*}
$$

For arbitrary but admissible choice of $\dot{\boldsymbol{\varepsilon}},{ }_{s} \Theta \boldsymbol{J}, \dot{\theta}$ and $\dot{\boldsymbol{g}}$, the entropy inequality (2.27), holds if

$$
\begin{align*}
\rho_{0} \frac{\partial \Phi}{\partial \varepsilon_{k l}}-{ }_{s} \sigma_{k l}=0 \quad & \Longrightarrow \quad{ }_{s} \sigma_{k l}=\rho_{0} \frac{\partial \Phi}{\partial \varepsilon_{k l}}  \tag{2.28}\\
\rho_{0} \frac{\partial \Phi}{\partial\left(i{ }_{s}^{\Theta} J_{k l}\right)}-m_{k l}=0 \quad & \Longrightarrow \quad m_{k l}=\rho_{0} \frac{\partial \Phi}{\partial\left({ }^{( }{ }_{s} J\right)_{k l}}  \tag{2.29}\\
\rho_{0}\left(\eta+\frac{\partial \Phi}{\partial \theta}\right)=0 \quad & \Longrightarrow \quad \eta=-\frac{\partial \Phi}{\partial \theta}  \tag{2.30}\\
\rho_{0} \frac{\partial \Phi}{\partial g_{i}}=0 & \Longrightarrow \quad \Phi \neq \Phi\left(g_{i}\right)  \tag{2.31}\\
& \frac{q_{i} g_{i}}{\theta} \leq 0 \tag{2.32}
\end{align*}
$$

and

$$
\begin{equation*}
\Phi=\Phi\left(\boldsymbol{\varepsilon},{ }_{s}^{\ominus} \boldsymbol{J}, \theta\right) \quad, \quad \eta=\eta\left(\boldsymbol{\varepsilon},{ }_{s}^{\Theta} \boldsymbol{J}, \theta\right) \tag{2.33}
\end{equation*}
$$

## Remarks

1. Equation (2.28) can be used to derive constitutive theory for ${ }_{s} \boldsymbol{\sigma}$.
2. Equation (2.29) can be used to derive constitutive theory for $\boldsymbol{m}$.
3. Equation (2.30) implies that $\eta$ is not a constitutive variable as $\eta$ is deterministic from $\partial \Phi / \partial \theta$.
4. Based on (2.31) we can conclude that $\Phi$ is not a function of $\boldsymbol{g}$.
5. Inequality in (2.32) must be satisfied by the constitutive theory for $\boldsymbol{q}$.

### 2.1.1.1 Constitutive theory for $\left[{ }_{s} \sigma\right]$ based on (2.21)

First, we consider constitutive theories for $\left[{ }_{s} \sigma\right]$ and $[m]$ based on (2.21) and (2.22) using representation theorem. Since $\left[{ }_{s} \sigma\right]$ is a symmetric tensor of rank two, the integrity based on the combined generators (symmetric tensors of rank two) of $[\varepsilon]$ and $\theta$ (symmetric tensor of rank two and tensor of rank zero) consists of tensors $[I],[\varepsilon],[\varepsilon]^{2}$, hence we can represent $\left[{ }_{s} \sigma\right]$ by a linear combination of the combined generators.

$$
\begin{equation*}
\left[{ }_{s} \sigma\right]={ }_{\alpha_{\alpha}}{ }^{0}[I]+{ }_{\sigma_{\alpha}}{ }^{1}[\varepsilon]+{ }^{\sigma} \alpha^{2}[\varepsilon]^{2} \tag{2.34}
\end{equation*}
$$

in which

$$
\begin{equation*}
{ }^{\sigma_{\alpha}}{ }^{i}={ }^{\sigma}{\underset{\sim}{Q}}^{i}\left(I_{\varepsilon}, I I_{\varepsilon}, I I I_{\varepsilon}, \theta\right) ; \quad i=0,1,2 \tag{2.35}
\end{equation*}
$$

We introduce new notations in (2.34) and (2.35) to facilitate the subsequent details of the derivation. We define $\left[{ }^{\sigma}{\underset{\sim}{G}}^{1}\right]=[\varepsilon],\left[{ }_{\sim}^{G}{ }^{2}\right]=[\varepsilon]^{2}$, i.e. $\left[{ }^{\sigma}{ }_{\sim}^{G}\right] ; i=1,2, \cdots, N(N=2)$ as the combined generators due to argument tensors $[\varepsilon]$ and $\theta$ and ${ }_{\sim}^{\sigma}{ }_{\sim}^{1}=I_{\varepsilon},{ }_{\sim}^{\sigma}{ }_{\sim}^{2}=I I_{\varepsilon},{ }_{\sim}^{\sigma}{ }^{\sigma}{ }^{3}=I I I_{\varepsilon}$, i.e. ${ }^{\sigma} \tilde{\sim}^{j} ;$ $j=1,2, \cdots, M(M=3)$ as the combined invariants of the same argument tensors.

Then, (2.34) and (2.35), can be written as

$$
\begin{align*}
& {\left[{ }_{s} \sigma\right]={ }^{\sigma}{\underset{\sim}{\alpha}}^{0}+\sum_{i=1}^{N}{ }^{\sigma_{\alpha}}{ }^{i}\left[{ }_{\sim}^{\sigma} \underline{G}^{i}\right]}  \tag{2.36}\\
& {\underset{\sim}{\alpha}}^{i}={ }^{\sigma}{\underset{\sim}{\alpha}}^{i}\left({ }_{\sim}^{\sigma} \tilde{\sim}^{j}, \theta\right) ; \quad i=1,2, \cdots, N ; \quad j=1,2, \cdots, M \tag{2.37}
\end{align*}
$$

The material coefficients in the constitutive theory for $\left[{ }_{s} \sigma\right]$ are determined by considering Taylor series expansion of ${ }^{\sigma}{\underset{\sim}{\alpha}}^{i} ; i=0,1, \cdots, N$ in ${ }_{\sim}^{\sigma}{ }_{\sim}^{j} ; j=1,2, \cdots, M$ and $\theta$ about a known configuration
$\underline{\Omega}$ and retaining only up to linear terms in ${ }^{\sigma}{ }_{\sim}^{j} ; j=1,2, \cdots, M$ and $\theta$ (for simplicity).

$$
\begin{equation*}
{ }^{\sigma} \underline{\alpha}^{i}=\left.{ }^{\sigma} \underline{\alpha}^{i}\right|_{\underline{\Omega}}+\left.\sum_{j=1}^{M} \frac{\partial^{\sigma} \tilde{\alpha}^{i}}{\partial^{\sigma} \tilde{\sim}^{j}}\right|_{\underline{\Omega}}\left({ }_{\underline{\Omega}}{ }^{j}-\left.{ }^{\sigma} \underline{I}_{\underline{\Omega}}^{j}\right|_{\underline{\Omega}}\right)+\left.\frac{\partial^{\sigma} \alpha^{i}}{\partial \theta}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right) ; \quad i=0,1, \ldots, N \tag{2.38}
\end{equation*}
$$

Substituting (2.38) in (2.34) and collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients, we can obtain

$$
\begin{align*}
& -\sum_{i=1}^{N}{ }^{\sigma}{\underset{\sim}{d}}_{i}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)\left[{ }^{\sigma} G^{G}\right]-\alpha_{t m}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.39}
\end{align*}
$$

Coefficients ${\underset{\sim}{\sigma}}_{j},{\underset{\sim}{b}}_{j},{ }_{\sim}^{\sigma} \sim_{i j},{ }_{\sim}^{d}, \alpha_{t m} ; i=1,2, \ldots, N ; j=1,2, \ldots, M$ are functions of $\left.{ }_{\sigma}{\underset{\sim}{I}}^{j}\right|_{\underline{\Omega}}$ and $\left.\theta\right|_{\underline{\Omega}}$ $j=1,2, \ldots, M$. These are the material coefficients.

## Remarks

1. This constitutive theory contains $(N=2, M=3)$ fourteen material coefficients and contains up to fifth degree terms in the components of $[\varepsilon]$, but is linear in temperature $\theta$.
2. A linear constitutive theory in which the products of ${ }^{\sigma} I^{j},\left[{ }^{\sigma} G^{G}\right]$ and $\left(\theta-\left.\theta\right|_{\Omega}\right)$ are neglected and only up to linear terms in $[\varepsilon]$ are retained is given by

$$
\begin{equation*}
[s \sigma]=\left.{\underset{\sim}{\sigma}}^{0}\right|_{\underline{\Omega}}[I]+{ }_{\sim}^{\sigma}{\underset{\sim}{1}}_{1}{\underset{\sim}{\sigma}}^{1}[I]+{ }_{\sigma_{1}}^{{\underset{b}{c}}^{[ }}\left[{ }^{\sigma} \underline{G}^{1}\right]+\left.\alpha_{t m}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.40}
\end{equation*}
$$

Using the notation ${ }_{{ }_{\sim}^{b}}^{1}$ $=\left.2 \mu\right|_{\underline{\Omega}},{ }^{\sigma}{\underset{\sim}{a}}_{1}=\left.\lambda\right|_{\underline{\Omega}}$ and using ${ }^{\sigma}{\underset{\sim}{I}}^{1}=\operatorname{tr}[\varepsilon],\left[{ }^{\sigma} G^{1}\right]=[\varepsilon]$. We can write (2.40) as

$$
\begin{equation*}
\left[{ }_{s} \sigma\right]=\left.\sigma_{\sim}^{0}\right|_{\underline{\Omega}}[I]+\left.2 \mu\right|_{\underline{\Omega}}[\varepsilon]+\left.\lambda\right|_{\underline{\Omega}}(\operatorname{tr}[\varepsilon])[I]-\left.\alpha_{t m}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.41}
\end{equation*}
$$

This is Hooke's Law in which $\mu$ and $\lambda$ are Lame's constants defined in a known configuration $\Omega$.

### 2.1.1.2 Constitutive theory for $[m]$ based on (2.22)

This derivation is exactly the same as for $\left[{ }_{s} \sigma\right]$. Based on representation theorem, we begin with

$$
\begin{align*}
& {[m]={\underset{\sim}{\alpha}}^{m}[I]+\sum_{i=1}^{N}{ }^{m}{\underset{\sim}{Q}}^{i}\left[{ }^{m} G^{i}\right]}  \tag{2.42}\\
& { }^{m}{\underset{\sim}{\alpha}}^{i}={ }^{m}{\underset{\sim}{\alpha}}^{i}\left({ }^{m} I_{\sim}^{j}, \theta\right) ; \quad i=1,2, \ldots, N ; \quad j=1,2, \ldots, M \tag{2.43}
\end{align*}
$$

in which

$$
\begin{align*}
& {\left[{ }^{m} \underline{G}^{1}\right]=\left[{ }_{s}^{\Theta} J\right], \quad\left[{ }^{m} \underline{G}^{2}\right]=\left[{ }_{s}{ }_{s} J\right]^{2} ; \quad\left[{ }^{m} \underline{G}^{i}\right] ; i=1,2, \ldots, N ; \quad N=2}  \tag{2.44}\\
& { }_{\sim}^{m}{ }_{\sim}^{1}=I_{\left(i_{s}^{\Theta} J\right)},{ }_{\sim}^{T}{\underset{\sim}{2}}^{2}=I I_{\left(i{ }_{s}^{\Theta} J\right)},{ }^{m}{\underset{\sim}{I}}^{3}=I I I_{\left({ }_{i}{ }_{s} J\right)} ; \quad\left[{ }_{\sim}^{m}{\underset{\sim}{I}}^{j}\right] ; j=1,2, \ldots, M ; \quad M=3
\end{align*}
$$

Material coefficients in (2.42) are determined using Taylor series expansion of ${ }^{m}{ }_{\sim}{ }^{i} ; i=0,1,2$ in ${ }_{\sim}^{m}{ }_{\sim}^{j} ; j=1,2, \ldots, M$ and $\theta$ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${ }^{m} \tilde{\sim}^{j} ; j=1,2, \ldots, M$ and $\theta$ (for simplicity).

Substituting (2.45) in (2.42) and collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients, we can write

$$
\begin{align*}
& {[m]=\underset{\sim}{m}[I]+\sum_{j=1}^{M}{\underset{\sim}{m}}_{\sim}^{a}{ }_{j}^{m}{\underset{\sim}{I}}^{j}[I]+\sum_{i=1}^{N}{ }_{\sim}^{m}{\underset{\sim}{i}}\left[{ }^{m}{\underset{\sim}{G}}^{i}\right]+\sum_{i=1}^{N} \sum_{j=1}^{M}{ }^{m}{\underset{\sim}{C}}_{i j}{ }^{m}{\underset{\sim}{I}}^{j}\left[{ }^{m}{\underset{\sim}{G}}^{i}\right]}  \tag{2.46}\\
& -\sum_{i=1}^{N}{ }^{m} d_{i}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)\left[{ }^{m} \underline{G}^{i}\right]-{ }^{m} \alpha_{t m}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I]
\end{align*}
$$

Coefficients ${ }^{m}{\underset{\sim}{j}}_{j},{ }_{\sim}^{m},{ }_{\sim}^{m}{\underset{\sim}{C}}_{i j},{ }^{m} \underset{\sim}{d},{ }^{m} \alpha_{t m}$ are material coefficients that can be functions of $\left.{ }^{m}{\underset{\sim}{I}}^{j}\right|_{\underline{\Omega}}$ and $\left.\theta\right|_{\underline{\Omega}}$.

## Remarks

1. This constitutive theory also contains $(N=2, M=3)$ fourteen material coefficients and contains up to fifth degree terms in the components of $\left[{ }_{s}^{\Theta} J\right]$, but is linear in $\theta$.
2. A linear constitutive theory in which the products of ${ }^{m} \widetilde{\sim}^{j},\left[{ }^{m}{\underset{\sim}{G}}^{i}\right]$ and $\left(\theta-\left.\theta\right|_{\Omega}\right)$ are neglected and only up to linear terms in $\left[{ }_{s}^{\mathrm{\Theta}} \mathrm{O}\right]$ are retained is given by

$$
\begin{equation*}
\left.[m]=\left.{\underset{\sim}{m}}^{0}\right|_{\underline{\Omega}}[I]+{ }_{\underline{m}}^{a}{ }_{\sim}^{m} I_{\sim}^{1}[I]+{ }_{\sim}^{m}{\underset{\sim}{b}}^{[i}{ }_{s}^{\Theta} J\right]-\left.{ }^{m} \alpha_{t m}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.47}
\end{equation*}
$$

Using the notation ${ }^{m}{\underset{\sim}{a}}_{1}=\underset{\sim}{\mu}$ and noting that ${ }^{m} I^{1}=\operatorname{tr}\left[{ }_{s}^{\Theta} J\right]=0$, we can write the following form (2.47)

$$
\begin{equation*}
[m]=\left.{\underset{\sim}{m}}^{0}\right|_{\underline{\Omega}}[I]+\underset{\sim}{\mu}\left[{ }_{s}^{i}{ }_{s}^{\Theta} J\right]-\left.{ }^{m} \alpha_{t m}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.48}
\end{equation*}
$$

A further simplified theory in which the first and the last term in (2.48) are neglected is given by

$$
\begin{equation*}
[m]=\underset{\sim}{\mu}\left[{ }_{s}^{\Theta} J\right] \tag{2.49}
\end{equation*}
$$

in which the material coefficient $\underset{\sim}{\mu}$ can be dependent on the invariants of $\left[{ }_{s}^{i} J\right]$ and $\theta$ in a known configuration $\underline{\Omega}$, i.e.

$$
\begin{equation*}
\underset{\sim}{\mu}=\underset{\sim}{\mu}\left(\left.I_{\left(i_{s}^{\ominus} J\right)}\right|_{\underline{\Omega}},\left.I I_{\left(i_{s}^{\Theta} J\right)}\right|_{\underline{\Omega}},\left.I I I_{\left(i_{s}^{\Theta} J\right)}\right|_{\underline{\Omega}},\left.\theta\right|_{\underline{\Omega}}\right) \tag{2.50}
\end{equation*}
$$

### 2.1.1.3 Constitutive theory for $\left[{ }_{s} \sigma\right]$ based on (2.28) and (2.33)

Substituting (2.33) into (2.28) we obtain

$$
\begin{equation*}
{ }_{s} \boldsymbol{\sigma}=\rho_{0} \frac{\partial \Phi\left(\boldsymbol{\varepsilon},{ }_{r}^{\Theta}{ }_{s} \boldsymbol{J}, \theta\right)}{\partial \boldsymbol{\varepsilon}} \tag{2.51}
\end{equation*}
$$

Due to frame invariance requirement, $\Phi$ must be a function of the invariants of $\boldsymbol{\varepsilon},{ }_{s}^{\ominus} \boldsymbol{J}$ as well as the invariants containing both $\boldsymbol{\varepsilon}$ and ${ }_{s}^{i}{ }_{s} \boldsymbol{J}$. Let ${ }_{\sim}^{\sigma} \tilde{\sim}^{j} ; j=1,2, \ldots, M$ be the combined invariants between


$$
\begin{equation*}
{ }_{s} \sigma_{l k}=\rho_{0} \sum_{j=1}^{M} \frac{\partial \Phi}{\partial^{\sigma}{\underset{\sim}{j}}_{j}^{j}} \frac{\partial^{\sigma}{\underset{\sim}{j}}^{j}}{\partial \varepsilon_{l k}} \tag{2.52}
\end{equation*}
$$

Using ${ }^{\sigma}{ }_{\sim}^{I} j ; j=1,2, \ldots, M$, we can obtain the following from (2.52)

$$
\begin{equation*}
\left[{ }_{s} \sigma\right]={ }^{\sigma} \alpha^{0}[I]+\sum_{i=1}^{N}{ }^{\sigma_{\alpha}}{ }_{\sim}\left[{ }^{\sigma} G^{G}\right] \tag{2.53}
\end{equation*}
$$

where $\left[{ }^{m} G^{i}\right]$ are the generators, symmetric tensors of rank two and

$$
\begin{equation*}
{\underset{\sim}{\alpha}}^{\sigma^{i}}={ }^{\sigma}{\underset{\sim}{\alpha}}^{i}\left({ }_{\sim}^{\sigma}{\underset{\sim}{j}}^{j}, I_{\left(i_{s}^{\Theta} J\right)}, I I_{\left(i_{s}^{\ominus} J\right)}, I I I_{\left(i_{s}^{\Theta} J\right)}, \theta\right) ; \quad i=0,1,2, \ldots, N ; \quad j=1,2, \ldots, M \tag{2.54}
\end{equation*}
$$

The material coefficients in (2.54) are determined by expanding ${ }^{\sigma}{ }_{\alpha}{ }_{\sim}^{i} ; i=0,1, \ldots, N$ in Taylor series in ${ }_{\sim}^{\sigma}{ }_{\sim}^{j} ; j=1,2, \ldots, M$ and $\theta$ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${ }^{\sigma}{ }_{\sim}^{j} ; j=1,2, \ldots, M$ and $\theta$ (for simplicity).

$$
\begin{equation*}
{\underset{\sim}{\alpha}}^{\sigma}=\left.{ }^{\sigma}{\underset{\sim}{\alpha}}^{i}\right|_{\underline{\Omega}}+\left.\sum_{j=1}^{M} \frac{\partial^{\sigma} \widetilde{\alpha}^{i}}{\partial^{\sigma} \tilde{\sim}^{j}}\right|_{\underline{\Omega}}\left({ }_{\sim}^{\sigma} \tilde{\sim}^{j}-\left.{ }^{\sigma}{\underset{\sim}{I}}^{j}\right|_{\underline{\Omega}}\right)+\left.\frac{\partial^{\sigma} \tilde{\alpha}^{i}}{\partial \theta}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right) ; \quad i=0,1, \ldots, N \tag{2.55}
\end{equation*}
$$

Substituting (2.55) in (2.53) and collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients, we obtain

$$
\begin{align*}
{\left[{ }_{s} \sigma\right]=} & \left.\underline{\sigma}^{0}\right|_{\underline{\Omega}}[I]+\sum_{j=1}^{M}{ }^{\sigma} \underline{a}_{j}{ }^{\sigma}{ }_{\sim}{ }^{j}[I]+\sum_{i=1}^{N}{ }^{\sigma_{b}} \underline{\underline{~}}_{i}\left({ }^{\sigma} \underline{G}^{i}\right]+\sum_{i=1}^{N} \sum_{j=1}^{M}{ }^{\sigma}{\underline{c_{i}}}_{i j}{ }^{\sigma} \tilde{I}^{j}\left[{ }^{\sigma} \underline{G}^{j}\right] \\
& -\sum_{i=1}^{N}{ }^{\sigma} \underline{d}_{i}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)\left[{ }^{\sigma} \underline{G}^{i}\right]-\underline{\alpha}_{t m}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.56}
\end{align*}
$$

In which ${ }^{\sigma} \underline{a}_{j},{ }^{\sigma} \underline{b}_{i},{ }^{\sigma} \underline{C}_{i j},{ }^{\sigma} \underline{d}_{i}$ and $\underline{\alpha}_{t m} ; i=1,2, \ldots, N ; j=1,2, \ldots, M$ are material coefficients that can be functions of $\left.{ }_{\sim}^{\sigma} \tilde{\sim}^{j}\right|_{\underline{\Omega}} ; j=1,2, \ldots, M, I_{\left(i{ }_{s}^{\Theta} J\right)}| |_{\underline{\Omega}},\left.I I_{\left(i_{s}^{\Theta} J\right)}\right|_{\underline{\Omega}},\left.I I I_{\left(i_{s}^{\Theta} J\right)}\right|_{\underline{\Omega}},\left.\theta\right|_{\underline{\Omega}}$.

## Remarks

1. This constitutive theory contains $(M+2 N+M N+1)$ material coefficients.
2. A linear constitutive theory in $[\varepsilon]$ in which all product terms of ${ }^{\sigma}{\underset{\sim}{I}}^{j},\left[{ }^{\sigma} G^{G}\right]$ and $\left(\theta-\left.\theta\right|_{\Omega}\right)$ are neglected is given by

$$
\begin{equation*}
\left[{ }_{s} \sigma\right]=\left.\underline{\sigma}^{0}\right|_{\underline{\Omega}}[I]+{ }^{\sigma} \underline{a}_{1}(\operatorname{tr}[\varepsilon])[I]+\underline{b}_{1}[\varepsilon]-\underline{\alpha}_{t m}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.57}
\end{equation*}
$$

Using the notation ${ }^{\sigma} \underline{a}_{1}=\lambda \underline{\Omega},{ }^{\sigma} \underline{b}_{1}=\left.2 \mu\right|_{\underline{\Omega}}$ we can write (2.57) as

$$
\begin{equation*}
\left[{ }_{s} \sigma\right]=\left.\underline{\sigma}^{0}\right|_{\underline{\Omega}}[I]+\left.\lambda\right|_{\underline{\Omega}}(\operatorname{tr}[\varepsilon])[I]+\left.2 \mu\right|_{\underline{\Omega}}[\varepsilon]-\underline{\alpha}_{t m}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.58}
\end{equation*}
$$

where $\mu$ and $\lambda$ (Lame's constants) that can be functions of $\left.{ }_{\sim}^{\sigma} \tilde{\sim}_{\underline{\Omega}}^{j}\right|_{\underline{\Omega}} ; j=1,2, \ldots, M,\left.I_{\left(i_{s}^{\Theta} J\right)}\right|_{\underline{\Omega}}$, $\left.I I_{\left(i_{s}^{\Theta} J\right)}\right|_{\underline{\underline{\Omega}}},\left.I I I_{\left(i_{s}^{\Theta} J\right)}\right|_{\underline{\Omega}},\left.\theta\right|_{\underline{\Omega}}$.

### 2.1.1.4 Constitutive theory for $[m]$ based on (2.29) and (2.33)

Substituting (2.33) into (2.29) we obtain

$$
\begin{equation*}
\boldsymbol{m}=\rho_{0} \frac{\partial \Phi\left(\boldsymbol{\varepsilon},{ }_{5}^{\ominus}{ }_{5}^{\Theta} \boldsymbol{J}, \theta\right)}{\partial\left({ }_{s}^{\Theta}{ }_{s} \boldsymbol{J}\right)} \tag{2.59}
\end{equation*}
$$

Again, due to frame invariance requirement, $\Phi$ must be a function of the invariants of $\boldsymbol{\varepsilon},{ }_{s}^{\Theta} \boldsymbol{J}$ as well as the invariants containing both $\boldsymbol{\varepsilon}$ and ${ }_{s}{ }_{s}^{\Theta} \boldsymbol{J}$. Let ${ }^{m}{ }_{\sim}^{I} j ; j=1,2, \ldots, M$ be the combined invariants between $\boldsymbol{\varepsilon}$ and ${ }_{s}^{i}{ }_{s}^{\boldsymbol{J}} \boldsymbol{J}$ excluding the invariants of $\boldsymbol{\varepsilon}\left(\right.$ i.e. $\left.I_{\varepsilon}, I I_{\varepsilon}, I I I_{\varepsilon}\right)$, then

$$
\begin{equation*}
m_{l k}=\rho_{0} \sum_{j=1}^{M} \frac{\partial \Phi}{\partial^{m}{\underset{\sim}{I}}^{j}} \frac{\partial^{m}{\underset{\sim}{I}}^{j}}{\partial\left({ }_{\stackrel{\Theta}{\Theta} J}^{s}\right)_{l k}} \tag{2.60}
\end{equation*}
$$

Using invariants ${ }^{m}{ }_{\sim}^{j} ; j=1,2, \ldots, M$ we obtain the following form of (2.60)

$$
\begin{equation*}
[m]={ }^{m}{\underset{\sim}{\alpha}}^{0}[I]+\sum_{i=1}^{N}{ }^{m}{\underset{\sim}{\alpha}}^{i}\left[{ }^{m} \underline{G}^{i}\right] \tag{2.61}
\end{equation*}
$$

where $\left[{ }^{m} G^{i}\right] ; i=1,2, \ldots, N$ are generators, symmetric tensors of rank two and

$$
\begin{equation*}
{ }^{m_{\sim}}{ }^{i}={ }^{m}{\underset{\sim}{a}}^{i}\left({\underset{\sim}{\tau}}_{\sim}^{j}, I_{\varepsilon}, I I_{\varepsilon}, I I I_{\varepsilon}, \theta\right) ; \quad i=1,2, \ldots, N ; \quad j=1,2, \ldots, M \tag{2.62}
\end{equation*}
$$

The material coefficients in (2.61) are determined by expanding ${ }^{m}{ }_{\sim}{ }^{i} ; i=0,1, \ldots, N$ in Taylor series in ${ }^{m}{ }_{\sim}^{j} ; j=1,2, \ldots, M$ and $\theta$ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${ }^{m}{ }_{\sim}^{I} j ; j=1,2, \ldots, M$ and $\theta$ (for simplicity).

$$
\begin{equation*}
{ }^{m}{\underset{\sim}{a}}^{i}=\left.{ }^{m}{\underset{\sim}{\alpha}}^{i}\right|_{\underline{\Omega}}+\left.\sum_{j=1}^{M} \frac{\partial^{m} \widetilde{\sim}^{i}}{\partial^{m} \underline{\sim}_{\underline{I}}^{j}}\right|_{\underline{\Omega}}\left({ }^{m}{\underset{\sim}{1}}^{j}-\left.{ }^{m}{\underset{\sim}{I}}^{j}\right|_{\underline{\Omega}}\right)+\left.\frac{\partial^{m}{\underset{\sim}{\alpha}}^{i}}{\partial \theta}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right) ; \quad i=0,1, \ldots, N \tag{2.63}
\end{equation*}
$$

Substituting (2.61) in (2.63) and collecting coefficients of the terms defined in the current configuration and introducing new notations for the coefficients, we obtain

$$
\begin{align*}
{[m]=} & \left.\left.\underline{m}^{0}\right|_{\underline{\Omega}}[I]+\sum_{j=1}^{M}{ }^{m} \underline{a}_{j}{ }^{m}{\underset{\sim}{I}}^{j}[I]+\sum_{i=1}^{N}{ }^{m} \underline{b}_{i} i^{m}{\underset{\sim}{G}}^{i}\right]+\sum_{i=1}^{N} \sum_{j=1}^{M}{ }^{m}{\underline{c_{i j}}}^{m}{\underset{\sim}{I}}^{j}\left[{ }^{m}{\underset{\sim}{G}}^{i}\right]  \tag{2.64}\\
& -\sum_{i=1}^{N}{ }^{m} \underline{d}_{i}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)\left[{ }^{m} \underline{G}^{i}\right]-\underline{\alpha}_{t m}(\theta-\theta \mid \underline{\Omega})[I]
\end{align*}
$$

in which ${ }^{m} \underline{a}_{i},{ }^{m} \underline{b}_{j},{ }^{m} \underline{c}_{i j},{ }^{m} \underline{d}_{i}$ and $\underline{\alpha}_{t m} ; i=1,2, \ldots, N ; j=1,2, \ldots, M$ are material coefficients that can be functions of ${ }^{m}{\underset{\sim}{j}}^{j} ; j=1,2, \ldots, M,\left.I_{\varepsilon}\right|_{\underline{\Omega}},\left.I I_{\varepsilon}\right|_{\underline{\Omega}},\left.I I I_{\varepsilon}\right|_{\underline{\Omega}}$ and $\left.\theta\right|_{\underline{\Omega}}$.

## Remarks

1. This constitutive theory contains $(M+2 N+M N+1)$ material coefficients
2. A linear constitutive theory in $\left[{ }_{s}^{\Theta} J\right]$ in which all product terms of ${ }^{m} \tilde{\sim}^{j},\left[{ }^{m} G^{i}\right],\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)$ are neglected is given by

$$
\begin{equation*}
[m]=\left.\underline{m}^{0}\right|_{\underline{\Omega}}[I]+{ }_{\underline{m}}^{\underline{a}_{1}}\left(\operatorname{tr}\left[{ }_{s}{ }_{s}^{\Theta} J\right]\right)[I]+{ }^{m} \underline{b}_{1}\left[\left[_{s}^{i} J\right]-\underline{\alpha}_{t m}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I]\right. \tag{2.65}
\end{equation*}
$$

Using ${ }^{m} \underline{a}_{1}=\left.\underset{\sim}{\lambda}\right|_{\underline{\Omega}}[I],{ }^{m} \underline{b}_{1}=\left.\underset{\sim}{\mu}\right|_{\underline{\Omega}}$ we can write (2.65) as

$$
\begin{equation*}
[m]=\left.\underline{m}^{0}\right|_{\underline{\Omega}}[I]+\left.\underset{\sim}{\lambda}\right|_{\underline{\Omega}}\left(\operatorname{tr}\left[{ }_{s}^{[i} J\right]\right)[I]+\left.\underline{\mu}_{\underline{\Omega}_{\underline{\Omega}}}\right|_{[i}\left[{ }_{s}^{\Theta} J\right]-\underline{\alpha}_{t m}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)[I] \tag{2.66}
\end{equation*}
$$

In the work presented in this thesis, we consider the following simplified forms of the constitutive theory for ${ }_{s} \boldsymbol{\sigma}$ and $\boldsymbol{m}$ (noting that $\operatorname{tr}\left[{ }_{s}{ }_{s}^{\Theta} J\right]=0$ )

$$
\begin{equation*}
\left[{ }_{s} \sigma\right]=2 \mu[\varepsilon]+\lambda(\operatorname{tr}[\varepsilon])[I] \tag{2.67}
\end{equation*}
$$

and

$$
\begin{equation*}
[m]=\underset{\sim}{\mu}\left[{ }_{s}^{i}{ }_{s}^{\Theta} J\right] \tag{2.68}
\end{equation*}
$$

The material coefficients $\mu, \lambda$ and $\underset{\sim}{\mu}$ can be functions of invariants (shown earlier) in a known
configuration $\underline{\Omega}$.

### 2.1.1.5 Constitutive theory for $\{q\}$

We consider $\boldsymbol{q}=\boldsymbol{q}(\boldsymbol{g}, \theta)$ and use representation theorem [48-64]. The combined generators of the argument tensors $\boldsymbol{g}$ and $\theta$ that are tensors of rank one is just $\boldsymbol{g}$ and the combined invariant is $\boldsymbol{g} \cdot \boldsymbol{g}\left(\right.$ or $\left.^{q} I\right)$. Thus, we can write [46]

$$
\begin{equation*}
\{q\}=-{ }^{q} \alpha\{q\} \tag{2.69}
\end{equation*}
$$

in which ${ }^{q} \underset{\sim}{\sim}={ }_{\sim}^{q}\left({ }^{q} I, \theta\right)$.

Material coefficients in the constitutive theory for $\boldsymbol{q}$ given by (2.69) are obtained by considering Taylor series expansion of ${ }^{q} \underset{\sim}{\alpha}$ in ${ }^{q} I$ and $\theta$ about a known configuration $\underline{\Omega}$ and retaining up to linear terms in ${ }^{q} I$ and $\theta$ (for simplicity).

$$
\begin{equation*}
{\underset{\sim}{q}}_{\underline{q}}={ }^{q} \underline{\sim}_{\underline{\Omega}}+\left.\frac{\partial^{q} \alpha}{\partial{ }^{q} I}\right|_{\underline{\Omega}}\left({ }^{q} I-\left.{ }^{q} I\right|_{\underline{\Omega}}\right)+\left.\frac{\partial^{q} \alpha}{\partial \theta}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right) \tag{2.70}
\end{equation*}
$$

Substituting (2.70) in (2.69) and collecting coefficients gives

$$
\begin{equation*}
\{q\}=-\left.k_{1}\right|_{\underline{\Omega}}\{g\}-\left.k_{2}\right|_{\underline{\Omega}}\left(\{g\}^{T}\{g\}\right)\{g\}-\left.k_{3}\right|_{\underline{\Omega}}\left(\theta-\left.\theta\right|_{\underline{\Omega}}\right)\{g\} \tag{2.71}
\end{equation*}
$$

This constitutive theory is based on integrity, hence uses complete basis. From (2.70) we can derive a linear constitutive theory for $\{q\}$.

$$
\begin{equation*}
\{q\}=-\left.k_{1}\right|_{\underline{\Omega}}\{g\} \tag{2.72}
\end{equation*}
$$

The material coefficients $k_{1}, k_{2}$, and $k_{3}$ are in a known configuration $\underline{\Omega}$ and can be functions of ${ }^{q} I$, i.e. $\{g\}^{T}\{g\}$ and temperature $\theta$ in $\underline{\Omega}$.

### 2.1.2 Mathematical Model in $\mathbb{R}^{2}$

We present expanded form of equations constituting the mathematical model in $\mathbb{R}^{2}$ for isothermal case. We consider the plane stress case. First, we note that in this case we have only one rotation ${ }_{i} \Theta_{3}\left({ }_{i} \Theta_{1},{ }_{i} \Theta_{2}\right.$ are zero) given by

$$
\begin{align*}
& { }_{i} \Theta_{3}=\left(\frac{\partial u_{1}}{\partial x_{2}}-\frac{\partial u_{2}}{\partial x_{1}}\right)  \tag{2.73}\\
& {\left[{ }_{i} \Theta J\right]=\left[\frac{\partial\left\{{ }_{i} \Theta\right\}}{\partial\{x\}}\right]=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{\partial\left(i_{i} \Theta_{3}\right)}{\partial x_{1}} & \frac{\partial\left(i_{i} \Theta_{3}\right)}{\partial x_{2}} & 0
\end{array}\right]}  \tag{2.74}\\
& {\left[{ }_{i}{ }_{s} J\right]=\left[\begin{array}{ccc}
0 & 0 & \frac{1}{2} \frac{\partial\left(\Theta_{i} \Theta_{3}\right.}{\partial x_{1}} \\
0 & 0 & \frac{1}{2} \frac{\partial\left(i \Theta_{3}\right)}{\partial x_{2}} \\
\frac{1}{2} \frac{\partial\left(\Theta_{i} \Theta_{3}\right)}{\partial x_{1}} & \frac{1}{2} \frac{\partial\left(i \Theta_{3}\right)}{\partial x_{2}} & 0
\end{array}\right]} \tag{2.75}
\end{align*}
$$

Using (2.68) and (2.75) we obtain the following constitutive theories for $m_{13}$ and $m_{23}$ (others are zero)

$$
\begin{align*}
& m_{31}=m_{13}=\underset{\sim}{\mu} \frac{1}{2} \frac{\partial\left({ }_{i} \Theta_{3}\right)}{\partial x_{1}}=\frac{\mu}{2}\left(\frac{\partial^{2} u_{1}}{\partial x_{1} \partial x_{2}}-\frac{\partial^{2} u_{2}}{\partial x_{1}^{2}}\right) \\
& m_{32}=m_{23}=\underset{\sim}{\mu} \frac{1}{2} \frac{\partial\left({ }_{i} \Theta_{3}\right)}{\partial x_{2}}=\frac{\mu}{2}\left(\frac{\partial^{2} u_{1}}{\partial x_{2}^{2}}-\frac{\partial^{2} u_{2}}{\partial x_{1} \partial x_{2}}\right) \tag{2.76}
\end{align*}
$$

The complete mathematical model in $\mathbb{R}^{2}$ in the dimensionless form is given in the following

$$
\begin{align*}
& \rho_{0} \frac{\partial^{2} u_{1}}{\partial t^{2}}-\rho_{0} F_{1}^{b}-\frac{\partial\left({ }_{s} \sigma_{11}\right)}{\partial x_{1}}-\frac{\partial\left({ }_{s} \sigma_{21}\right)}{\partial x_{2}}-\frac{\partial\left({ }_{a} \sigma_{21}\right)}{\partial x_{2}}=0  \tag{2.77}\\
& \rho_{0} \frac{\partial^{2} u_{2}}{\partial t^{2}}-\rho_{0} F_{2}^{b}-\frac{\partial\left({ }_{s} \sigma_{12}\right)}{\partial x_{1}}-\frac{\partial\left({ }_{s} \sigma_{22}\right)}{\partial x_{2}}-\frac{\partial\left({ }_{a} \sigma_{12}\right)}{\partial x_{1}}=0 \tag{2.78}
\end{align*}
$$

$$
\begin{gather*}
\frac{\partial m_{13}}{\partial x_{1}}+\frac{\partial m_{23}}{\partial x_{2}}+2\left({ }_{a} \sigma_{12}\right)=0 ;  \tag{2.79}\\
{ }_{a} \sigma_{21}=-{ }_{a} \sigma_{12} \\
{ }_{s} \sigma_{11}=D_{11} \frac{\partial u_{1}}{\partial x_{1}}+D_{12} \frac{\partial u_{2}}{\partial x_{2}} \\
{ }_{s} \sigma_{22}=D_{21} \frac{\partial u_{1}}{\partial x_{1}}+D_{22} \frac{\partial u_{2}}{\partial x_{2}}  \tag{2.80}\\
m_{13}=\frac{\mu}{2} \frac{\sigma_{12}}{2}={ }_{s} \sigma_{21}=D_{33}\left(\frac{\partial u_{2}}{\partial x_{1}}+\frac{\partial u_{1}}{\partial x_{2}}\right) \\
E_{0}\left(\frac{\partial^{2} u_{1}}{\partial x_{1} \partial x_{2}}-\frac{\partial^{2} u_{2}}{\partial x_{1}^{2}}\right), \quad m_{23}=\frac{\mu}{2} \frac{E_{0}}{m_{0} L_{0}}\left(\frac{\partial^{2} u_{1}}{\partial x_{2}^{2}}-\frac{\partial^{2} u_{2}}{\partial x_{1} \partial x_{2}}\right)  \tag{2.81}\\
D_{11}=D_{22}=\frac{E}{1-\nu^{2}}, \quad D_{12}=D_{21}=\frac{\nu E}{1-\nu^{2}}, \quad D_{33}=G=\frac{E}{2(1+\nu)} \tag{2.82}
\end{gather*}
$$

The dimensionless form of the equations are obtained by first writing all equations with hat ( $\wedge$ ) on all quantities and variables indicating that they have their usual units or dimensions in terms of length $(\hat{L})$, force $(\hat{F})$ and time $(\hat{t})$. If we choose $L_{0}, F_{0}$ and $t_{0}$ as reference length, force and time, then the dimensionless length, force and time are defined as

$$
\begin{equation*}
L=\frac{\hat{L}}{L_{0}}, \quad F=\frac{\hat{F}}{F_{0}}, \quad t=\frac{\hat{t}}{t_{0}} \tag{2.83}
\end{equation*}
$$

If we let $\hat{E}=E E_{0}, E_{0}=\frac{F_{0}}{L_{0}^{2}}, \hat{x}=x L_{0}, \hat{m}=m m_{0}, m_{0}=\tau_{0} L_{0}=E_{0} L_{0}, \tau_{0}=E_{0}, \hat{\mu}=\underset{\sim}{\mu} E_{0}$, and $v_{0}=\sqrt{\frac{E_{0}}{8}}$, reference speed of sound, then $t_{0}=\frac{L_{0}}{v_{0}}$. In this case, $\frac{E_{0}}{m_{0} L_{0}}$ is in fact unity. Equations (2.77)-(2.81) are eight partial differential equations in eight variables, $u_{1}, u_{2},{ }_{\text {s }} \sigma_{11},{ }_{s} \sigma_{22},{ }_{s} \sigma_{12},{ }_{a} \sigma_{12}$, $m_{13}, m_{23}$, and ${ }_{i} \Theta_{3}$. It is straight forward to show that if we substitute (2.80) in (2.79), then (2.79) in (2.77) and (2.78), and (2.81) in (2.77) and (2.78), then we can reduce the complete mathematical model into partial differential equations (2.77) and (2.78) in displacements $u_{1}$ and $u_{2}$ given below.

$$
\begin{align*}
& \rho_{0} \frac{\partial^{2} u_{1}}{\partial t^{2}}-\rho_{0} F_{1}^{b}-A_{11}\left(u_{1}, u_{2}\right)-A_{12}\left(u_{1}, u_{2}\right)=0  \tag{2.84}\\
& \rho_{0} \frac{\partial^{2} u_{2}}{\partial t^{2}}-\rho_{0} F_{2}^{b}-A_{21}\left(u_{1}, u_{2}\right)-A_{22}\left(u_{1}, u_{2}\right)=0
\end{align*}
$$

$$
\begin{align*}
& A_{11}\left(u_{1}, u_{2}\right)=\left(\frac{\partial}{\partial x_{1}}\left(D_{11} \frac{\partial u_{1}}{\partial x_{1}}\right)+\frac{\partial}{\partial x_{2}}\left(D_{33} \frac{\partial u_{1}}{\partial x_{2}}\right)\right)+\left(\frac{\partial}{\partial x_{1}}\left(D_{12} \frac{\partial u_{2}}{\partial x_{2}}\right)+\frac{\partial}{\partial x_{2}}\left(D_{33} \frac{\partial u_{2}}{\partial x_{1}}\right)\right) \\
& A_{21}\left(u_{1}, u_{2}\right)=\left(\frac{\partial}{\partial x_{1}}\left(D_{33} \frac{\partial u_{1}}{\partial x_{2}}\right)+\frac{\partial}{\partial x_{2}}\left(D_{21} \frac{\partial u_{1}}{\partial x_{1}}\right)\right)+\left(\frac{\partial}{\partial x_{1}}\left(D_{33} \frac{\partial u_{2}}{\partial x_{1}}\right)+\frac{\partial}{\partial x_{2}}\left(D_{22} \frac{\partial u_{2}}{\partial x_{2}}\right)\right)  \tag{2.85}\\
& A_{12}\left(u_{1}, u_{2}\right)=\frac{\mu}{4}\left(\frac{\partial^{4} u_{1}}{\partial x_{1}^{2} \partial x_{2}^{2}}+\frac{\partial^{4} u_{1}}{\partial x_{2}^{4}}\right)-\frac{\mu}{4}\left(\frac{\partial^{4} u_{2}}{\partial x_{1}^{3} \partial x_{2}}+\frac{\partial^{4} u_{2}}{\partial x_{2}^{3} \partial x_{1}}\right)  \tag{2.86}\\
& A_{22}\left(u_{1}, u_{2}\right)=-\frac{\mu}{4}\left(\frac{\partial^{4} u_{1}}{\partial x_{1}^{3} \partial x_{2}}+\frac{\partial^{4} u_{1}}{\partial x_{1} \partial x_{2}^{3}}\right)+\frac{\mu}{4}\left(\frac{\partial^{4} u_{2}}{\partial x_{1}^{4}}+\frac{\partial^{4} u_{2}}{\partial x_{1}^{2} \partial x_{2}^{2}}\right)
\end{align*}
$$

## Remarks

1. We note that the complete mathematical model now consists of two fourth order partial differential equations in displacements $u_{1}$ and $u_{2}$ resulting from balance of linear momenta.
2. $A_{11}\left(u_{1}, u_{2}\right)$ and $A_{21}\left(u_{1}, u_{2}\right)$ are due to classical continuum mechanics. These contain only up to second order derivatives of displacements $u_{1}$ and $u_{2}$.
3. $A_{12}\left(u_{1}, u_{2}\right)$ and $A_{22}\left(u_{1}, u_{2}\right)$ are due to internal rotations (non-classical continuum mechanics). These contain up to fourth order derivatives of displacements $u_{1}$ and $u_{2}$.
4. Equation (2.84) are ideally suited for finite element formulation based on space-time decoupled method using Galerkin method with weak form (STDGM/WF) in space. The resulting finite element formulation can be used to study: (i) natural modes of vibrations (ii) undamped transient response using mode superposition method or by using direct integration of ordinary differential equations in time.

## Chapter 3

## Finite Element Method Considerations and

## Formulation

### 3.1 Considerations in the Finite Element Formulation

For thermoelastic non-classical solid continua with internal rotations in which the mechanical deformation is reversible we consider the following.
(i) Undamped natural vibrations for which the BLM equations (2.84) in $u_{1}$ and $u_{2}$ are ideally suited. We consider the finite element method using space-time decoupled Galerkin method with weak form in space yielding mass and stiffness matrices. Eigenpairs using this finite element formulation based on classical continuum mechanics and non-classical continuum mechanics are calculated using subspace iteration method or Householder-QR method and are compared.
(ii) Undamped transient dynamic response is calculated for the mathematical model based on classical continuum mechanics and non-classical continuum mechanics using normal mode synthesis (or mode superposition) techniques utilizing the normal modes determined in (i).
(iii) Transient dynamic response using space-time coupled finite element method based on space-
time residual functionals (space-time least squares method) can also be determined. The space-time integral form in this approach is space-time variationally consistent (STVC), hence the computations remain unconditionally stable during the computation of the entire evaluation. Time response can be computed using a space-time strip with time marching. In this approach, the mathematical model of evolution in $\mathbb{R}^{2}$ naturally yields a space-time finite element requiring $3 D$ finite elements. This work is not presented in this thesis.
(iv) ODEs in time resulting from space-time decoupled method using GM/WF in space are also integrated using Wilson's $\theta$ method to obtain transient dynamic response. This solution is compared with the one obtained in (ii).
(v) Since the ODEs in time when transformed in modal basis are decoupled, and each ODE has an analytical solution, time response is also calculated using superposition of the analytical solution of each ODE in time and compared with the time response calculated in (ii) and (iv).

### 3.2 Finite Element Formulations

For thermoelastic solid continua, the space-time differential operators are not self-adjoint [65], hence space-time Galerkin method (STGM), space-time Petrov Galerkin method (STPGM), spacetime weighted residual method (STWRM) and space-time Galerkin method with weak form (STGM/WF) all yield space-time integral forms that are space-time variationally inconsistent [65], hence can not ensure unconditionally stable computations. Only space-time integral forms based on space-time residual functional (space-time least squares method or process STLSP) is space-time variationally consistent, hence ensures unconditionally stable computations for all possible choices of dimensionless parameter of the model problem and computational parameters $(h, p, k)$. In STLSP we have more than one choice of the specific form of the equations in the mathematical model [65].

In the space-time coupled methods using STLSP [65] accurate time response computations can be performed of the mathematical model. However, studies of normal modes of vibration
using the mathematical model based on classical as well as non-classical continuum mechanics requires mass and stiffness information which is only possible in space-time decoupled finite element formulations based on Galerkin method with weak form in space. We consider this approach. For space-time decoupled GM/WF in space, the mathematical model in terms of BLM expressed in terms of $u_{1}$ and $u_{2}$ is highly meritorious, but is cumbersome to work with. In this thesis, we present an alternative procedure that is also based on space-time decoupled approach using GM/WF in space that has a simpler derivation but eventually yields same integral form as using BLM in $u_{1}$ and $u_{2}$.

### 3.2.1 Space-time Decoupled Finite Element Formulation using GM/WF in Space

We begin with the balance of linear momenta in $u_{1}, u_{2},{ }_{s} \boldsymbol{\sigma}$ and ${ }_{a} \boldsymbol{\sigma}$ (equations (2.84)) $\forall(x, t)$ $\in \Omega_{x t}=\Omega_{x} \times \Omega_{t}$,

$$
\begin{align*}
& A_{1}\left(u_{1}, u_{2}\right)=\rho_{0} \frac{\partial^{2} u_{1}}{\partial t^{2}}-\rho_{0} F_{1}^{b}-A_{11}\left(u_{1}, u_{2}\right)-A_{12}\left(u_{1}, u_{2}\right)=0  \tag{3.1}\\
& A_{2}\left(u_{1}, u_{2}\right)=\rho_{0} \frac{\partial^{2} u_{2}}{\partial t^{2}}-\rho_{0} F_{2}^{b}-A_{21}\left(u_{1}, u_{2}\right)-A_{22}\left(u_{1}, u_{2}\right)=0 \tag{3.2}
\end{align*}
$$

in which

$$
\begin{array}{ll}
A_{11}\left(u_{1}, u_{2}\right)=\frac{\partial\left({ }_{s} \sigma_{11}\right)}{\partial x_{1}}+\frac{\partial\left({ }_{s} \sigma_{21}\right)}{\partial x_{2}} ; & A_{12}\left(u_{1}, u_{2}\right)=\frac{\partial\left({ }_{a} \sigma_{21}\right)}{\partial x_{2}}  \tag{3.3}\\
A_{21}\left(u_{1}, u_{2}\right)=\frac{\partial\left({ }_{s} \sigma_{12}\right)}{\partial x_{1}}+\frac{\partial\left({ }_{s} \sigma_{22}\right)}{\partial x_{2}} ; & A_{22}\left(u_{1}, u_{2}\right)=\frac{\partial\left({ }_{a} \sigma_{12}\right)}{\partial x_{2}}
\end{array}
$$

Let $\bar{\Omega}_{x}^{T}=\bigcup_{e} \bar{\Omega}_{x}^{e}$ be discretization of spatial domain $\bar{\Omega}_{x}$ in which $\bar{\Omega}_{x}^{e}$ is a typical finite element $e$. We consider nine node p-version hierarchical finite element $\bar{\Omega}_{x}^{e}[65,66]$. The local approximation over $\bar{\Omega}_{x}^{e}$ are higher order and higher degree in the local approximation space $V_{h} \subset H^{k, p}\left(\bar{\Omega}_{x}^{e}\right) . \bar{\Omega}_{x}^{e}$ is mapped into a two unit square in natural coordinate space $\xi, \eta$ [65]. We construct integral form of (3.1) and (3.2) over $\bar{\Omega}_{x}^{e}$ using fundamental lemma of calculus of variations [65]. Let $\left(u_{1}\right)_{h}^{e}$ and
$\left(u_{2}\right)_{h}^{e}$ be approximation of $u_{1}$ and $u_{2}$ over $\bar{\Omega}_{x}^{e}$, then we can write

$$
\begin{align*}
& \left(u_{1}\right)_{h}^{e}=\sum_{i=1}^{n^{u_{1}}} N_{i}^{u_{1}}(\xi, \eta) \delta_{i}^{u_{1}}(t) \\
& \left(u_{2}\right)_{h}^{e}=\sum_{i=1}^{n^{u_{2}}} N_{i}^{u_{2}}(\xi, \eta) \delta_{i}^{u_{2}}(t) \tag{3.4}
\end{align*}
$$

in which $N_{i}^{u_{1}}(\xi, \eta)$ and $N_{i}^{u_{2}}(\xi, \eta)$ are local approximation functions of spatial natural coordinates $\xi, \eta$, and $\delta_{i}^{u_{1}}(t)$ and $\delta_{i}^{u_{2}}(t)$ are corresponding nodal degrees of freedom that are functions of time only. Let $w_{1}$ and $w_{2}$ be test functions such that $w_{1}=\delta\left(u_{1}\right)_{h}^{e}$ and $w_{2}=\delta\left(u_{2}\right)_{h}^{e}$, then

$$
\begin{array}{ll}
w_{1}=N_{j}^{u_{1}}(\xi, \eta) ; \quad j=1,2, \ldots, n^{u_{1}}  \tag{3.5}\\
w_{2}=N_{j}^{u_{2}}(\xi, \eta) ; \quad j=1,2, \ldots, n^{u_{2}}
\end{array}
$$

Consider scalar products $\left(A_{1}\left(u_{1}, u_{2}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}$ and $\left(A_{1}\left(u_{1}, u_{2}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}$

$$
\begin{align*}
\left(A_{1}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}= & \left(\rho_{0} \frac{\partial^{2}\left(u_{1}\right)_{h}^{e}}{\partial t^{2}}, w_{1}\right)_{\bar{\Omega}_{x}^{e}}-\left(\rho_{0} F_{1}^{b}, w_{1}\right)_{\bar{\Omega}_{x}^{e}}  \tag{3.6}\\
& -\left(A_{11}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}-\left(A_{12}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}} \\
\left(A_{2}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}= & \left(\rho_{0} \frac{\partial^{2}\left(u_{2}\right)_{h}^{e}}{\partial t^{2}}, w_{2}\right)_{\bar{\Omega}_{x}^{e}}-\left(\rho_{0} F_{2}^{b}, w_{2}\right)_{\bar{\Omega}_{x}^{e}}  \tag{3.7}\\
& -\left(A_{21}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}-\left(A_{22}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{align*}
$$

First consider

$$
\begin{align*}
& \left(A_{11}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}=\left(\frac{\partial\left({ }_{s} \sigma_{11}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left({ }_{s} \sigma_{21}\right)_{h}^{e}}{\partial x_{2}}, w_{1}\right)_{\bar{\Omega}_{x}^{e}}  \tag{3.8}\\
& \left(A_{21}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}=\left(\frac{\partial\left({ }_{s} \sigma_{12}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left({ }_{s} \sigma_{22}\right)_{h}^{e}}{\partial x_{2}}, w_{2}\right)_{\bar{\Omega}_{x}^{e}} \tag{3.9}
\end{align*}
$$

We transfer one order of differentiation from stress terms to $w_{1}$ and $w_{2}$ in (3.8) and (3.9).

$$
\begin{align*}
\left(A_{11}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}= & -\int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial w_{1}}{\partial x_{1}}\left({ }_{s} \sigma_{11}\right)_{h}^{e}+\frac{\partial w_{1}}{\partial x_{2}}\left({ }_{s} \sigma_{21}\right)_{h}^{e}\right) d \Omega_{x}  \tag{3.10}\\
& +\oint_{\Gamma^{e}} w_{1}\left(\left({ }_{s} \sigma_{11}\right)_{h}^{e} n_{x_{1}}+\left({ }_{s} \sigma_{21}\right)_{h}^{e} n_{x_{2}}\right) d \Gamma \\
\left(A_{21}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}= & -\int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial w_{2}}{\partial x_{1}}\left({ }_{s} \sigma_{12}\right)_{h}^{e}+\frac{\partial w_{2}}{\partial x_{2}}\left({ }_{s} \sigma_{22}\right)_{h}^{e}\right) d \Omega_{x}  \tag{3.11}\\
& +\oint_{\Gamma^{e}} w_{2}\left(\left({ }_{s} \sigma_{12}\right)_{h}^{e} n_{x_{1}}+\left({ }_{s} \sigma_{22}\right)_{h}^{e} n_{x_{2}}\right) d \Gamma
\end{align*}
$$

$\left(u_{1}\right)_{h}^{e}$ and $\left(u_{2}\right)_{h}^{e}$ are primary variables and their coefficients are secondary variables. Let $t_{x_{1}}^{c}$ and ${ }^{c} t_{x_{2}}$ be secondary variables defined by

$$
\begin{align*}
& { }^{c} t_{x_{1}}=\left({ }_{s} \sigma_{11}\right)_{h}^{e} n_{x_{1}}+\left({ }_{s} \sigma_{21}\right){ }_{h}^{e} n_{x_{2}}  \tag{3.12}\\
& { }^{c} t_{x_{2}}=\left({ }_{s} \sigma_{12}\right)_{h}^{e} n_{x_{1}}+\left({ }_{s} \sigma_{22}\right)_{h}^{e} n_{x_{2}} \tag{3.13}
\end{align*}
$$

We can now write (3.8) and (3.9) as follows

$$
\begin{align*}
& \left(A_{11}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}={ }^{c} B_{11}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right)+{ }^{c} l_{11}^{e}\left(w_{1}\right)  \tag{3.14}\\
& \left(A_{21}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}={ }^{c} B_{21}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right)+{ }^{c} l_{21}^{e}\left(w_{2}\right) \tag{3.15}
\end{align*}
$$

where

$$
\begin{align*}
{ }^{c} B_{11}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right) & =-\int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial w_{1}}{\partial x_{1}}\left({ }_{s} \sigma_{11}\right)_{h}^{e}+\frac{\partial w_{1}}{\partial x_{2}}\left({ }_{s} \sigma_{21}\right)_{h}^{e}\right) d \Omega_{x}  \tag{3.16}\\
{ }^{c} l_{11}^{e}\left(w_{1}\right) & =\oint_{\Gamma^{e}} w_{1}{ }^{c} t_{x_{1}} d \Gamma  \tag{3.17}\\
{ }^{c} B_{21}^{e}\left(\left(u_{1}\right) e_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right) & =-\int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial w_{2}}{\partial x_{1}}\left({ }_{s} \sigma_{12}\right)_{h}^{e}+\frac{\partial w_{2}}{\partial x_{2}}\left({ }_{s} \sigma_{22}\right)_{h}^{e}\right) d \Omega_{x}  \tag{3.18}\\
{ }^{c} l_{21}^{e}\left(w_{2}\right) & =\oint_{\Gamma^{e}} w_{2}{ }^{c} t_{x_{2}} d \Gamma \tag{3.19}
\end{align*}
$$

Next we consider $\left(A_{12}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}$ and $\left(A_{22}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}$ in (3.6) and (3.7). First we substitute ${ }_{a} \sigma_{21}$ and ${ }_{a} \sigma_{21}=-{ }_{a} \sigma_{12}$ from (2.79).

$$
\begin{gather*}
\left(A_{12}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}=-\left(\frac{\partial}{\partial x_{2}}\left(\frac{1}{2}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right)\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}  \tag{3.20}\\
\left(A_{22}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}=\left(\frac{\partial}{\partial x_{1}}\left(\frac{1}{2}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right)\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}} \tag{3.21}
\end{gather*}
$$

We transfer one order of differentiation with respect to $x_{2}$ on $w_{1}$ and one order of differentiation with respect to $x_{2}$ on $w_{2}$ in (3.20) and (3.21) respectively to obtain

$$
\begin{align*}
\left(A_{12}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}= & \frac{1}{2} \int_{\bar{\Omega}_{x}^{e}} \frac{\partial w_{1}}{\partial x_{2}}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) d \Omega_{x}  \tag{3.22}\\
& -\frac{1}{2} \oint_{\Gamma^{e}} w_{1}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) n_{x_{2}} d \Gamma \\
\left(A_{22}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}= & -\frac{1}{2} \int_{\bar{\Omega}_{x}^{e}} \frac{\partial w_{2}}{\partial x_{1}}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) d \Omega_{x}  \tag{3.23}\\
& +\frac{1}{2} \oint_{\Gamma^{e}} w_{1}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) n_{x_{1}} d \Gamma
\end{align*}
$$

We transfer all of the differentiation from $\left(m_{13}\right)_{h}^{e}$ and $\left(m_{23}\right)_{h}^{e}$ to $\frac{\partial w_{1}}{\partial x_{2}}$ and $\frac{\partial w_{2}}{\partial x_{1}}$ to obtain

$$
\begin{align*}
\left(A_{12}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}= & -\frac{1}{2} \int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial^{2} w_{1}}{\partial x_{1} \partial x_{2}}\left(m_{13}\right)_{h}^{e}+\frac{\partial^{2} w_{1}}{\partial x_{2}^{2}}\left(m_{23}\right)_{h}^{e}\right) d \Omega_{x} \\
& +\oint_{\Gamma^{e}}\left(\frac{\partial w_{1}}{\partial x_{2}}\left(\left(m_{13}\right)_{h}^{e} n_{x_{1}}+\left(m_{23}\right)_{h}^{e} n_{x_{2}}\right)\right) d \Gamma  \tag{3.24}\\
& -\oint_{\Gamma^{e}} w_{1}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) n_{x_{2}} d \Gamma
\end{align*}
$$

Primary variables are $\frac{\partial\left(u_{1} e_{h}^{e}\right.}{\partial x_{2}}$ and $\left(u_{1}\right)_{h}^{e}$ and their coefficients are secondary variables

$$
\begin{align*}
\left(A_{22}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}= & -\frac{1}{2} \int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial^{2} w_{2}}{\partial x_{1}^{2}}\left(m_{13}\right)_{h}^{e}+\frac{\partial^{2} w_{2}}{\partial x_{1} \partial x_{2}}\left(m_{23}\right)_{h}^{e}\right) d \Omega_{x} \\
& -\oint_{\Gamma^{e}}\left(\frac{\partial w_{2}}{\partial x_{1}}\left(\left(m_{13}\right)_{h}^{e} n_{x_{1}}+\left(m_{23}\right)_{h}^{e} n_{x_{2}}\right)\right) d \Gamma  \tag{3.25}\\
& +\oint_{\Gamma^{e}} w_{2}\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) n_{x_{1}} d \Gamma
\end{align*}
$$

Primary variables are $\frac{\partial\left(u_{2}\right)_{h}^{e}}{\partial x_{2}}$ and $\left(u_{2}\right)_{h}^{e}$ and their coefficients are secondary variables. We define secondary variables ${ }^{n c} t_{x_{1}},{ }^{n c} t_{x_{2}}$ and $m_{n}$ as follows

$$
\begin{align*}
{ }^{n c} t_{x_{1}} & =\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) n_{x_{2}}  \tag{3.26}\\
{ }^{n c} t_{x_{2}} & =\left(\frac{\partial\left(m_{13}\right)_{h}^{e}}{\partial x_{1}}+\frac{\partial\left(m_{23}\right)_{h}^{e}}{\partial x_{2}}\right) n_{x_{1}}  \tag{3.27}\\
m_{n} & =\left(m_{13}\right){ }_{h}^{e} n_{x_{1}}+\left(m_{23}\right){ }_{h}^{e} n_{x_{2}} \tag{3.28}
\end{align*}
$$

Now we can write (3.24) and (3.25) as follows

$$
\begin{equation*}
\left(A_{12}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}={ }^{n c} B_{12}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right)+{ }^{n c} l_{12}^{e}\left(w_{1}\right) \tag{3.29}
\end{equation*}
$$

$$
\begin{equation*}
\left(A_{22}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}={ }^{n c} B_{22}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right)+{ }^{n c} l_{22}^{e}\left(w_{2}\right) \tag{3.30}
\end{equation*}
$$

where

$$
\begin{align*}
{ }^{n c} B_{12}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right) & =-\frac{1}{2} \int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial^{2} w_{1}}{\partial x_{1} \partial x_{2}}\left(m_{13}\right)_{h}^{e}+\frac{\partial^{2} w_{1}}{\partial x_{2}^{2}}\left(m_{23}\right)_{h}^{e}\right) d \Omega_{x}  \tag{3.31}\\
{ }^{n c} l_{12}^{e}\left(w_{1}\right) & =\oint_{\Gamma^{e}} \frac{\partial w_{1}}{\partial x_{2}} m_{n} d \Gamma-\oint_{\Gamma^{e}} w_{1}{ }^{n c} t_{x_{1}} d \Gamma  \tag{3.32}\\
{ }^{n c} B_{22}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right) & =\frac{1}{2} \int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial^{2} w_{2}}{\partial x_{1}^{2}}\left(m_{13}\right)_{h}^{e}+\frac{\partial^{2} w_{2}}{\partial x_{1} \partial x_{2}}\left(m_{23}\right)_{h}^{e}\right) d \Omega_{x}  \tag{3.33}\\
{ }^{n c} l_{22}^{e}\left(w_{2}\right) & =-\oint_{\Gamma^{e}} \frac{\partial w_{2}}{\partial x_{1}} m_{n} d \Gamma+\oint_{\Gamma^{e}} w_{2}{ }^{n c} t_{x_{2}} d \Gamma \tag{3.34}
\end{align*}
$$

Using (3.14) and (3.15) and (3.29) and (3.30) in (3.6) and (3.7) we obtain the following weak form of (3.6) and (3.7)

$$
\begin{align*}
\left(A_{1}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}= & \left(\rho_{0} \frac{\partial^{2}\left(u_{1}\right)_{h}^{e}}{\partial t^{2}}, w_{1}\right)_{\bar{\Omega}_{x}^{e}}-{ }^{c} B_{11}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right)-{ }^{c} l_{11}^{e}\left(w_{1}\right)  \tag{3.35}\\
& -{ }^{n c} B_{12}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right)-{ }^{n c} l_{12}^{e}\left(w_{1}\right)-\left(\rho_{0} F_{1}^{b}, w_{1}\right)_{\bar{\Omega}_{x}^{e}} \\
\left(A_{2}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}= & \left(\rho_{0} \frac{\partial^{2}\left(u_{2}\right)_{h}^{e}}{\partial t^{2}}, w_{2}\right) \overline{\bar{\Omega}}_{x}^{e}-{ }^{c} B_{21}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right)-{ }^{c} l_{21}^{e}\left(w_{1}\right)  \tag{3.36}\\
& \left.-{ }^{n c} B_{22}^{e}\left(\left(u_{1}\right)\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right)-{ }^{n c} l_{22}^{e}\left(w_{2}\right)-\left(\rho_{0} F_{2}^{b}, w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{align*}
$$

or

$$
\begin{align*}
\left\{\begin{array}{l}
\left(A_{1}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}} \\
\left(A_{2}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{array}\right\}= & \left\{\begin{array}{l}
\left(\rho_{0} \frac{\partial^{2}\left(u_{1}\right)_{h}^{e}}{\partial t^{2}}, w_{1}\right)_{\bar{\Omega}_{x}^{e}} \\
\left(\rho_{0} \frac{\partial^{2}\left(u_{2}\right)^{e}}{\partial t^{2}}, w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{array}\right\}-\left\{\begin{array}{l}
{ }^{c} B_{11}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right) \\
\left.{ }^{c} B_{21}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)\right)_{h}^{e} ; w_{2}\right)
\end{array}\right\} \\
& -\left\{\begin{array}{l}
{ }^{n c} B_{12}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right) \\
{ }^{n c} B_{22}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right)
\end{array}\right\}-\left\{\begin{array}{l}
{ }^{c} l_{11}^{e}\left(w_{1}\right)+{ }^{n c} l_{12}^{e}\left(w_{1}\right) \\
{ }^{c} l_{21}^{e}\left(w_{2}\right)+{ }^{n c} l_{22}^{e}\left(w_{2}\right)
\end{array}\right\} \\
& -\left\{\begin{array}{l}
\left(\rho_{0} F_{1}^{b}, w_{1}\right)_{\bar{\Omega}_{x}^{e}} \\
\left(\rho_{0} F_{2}^{b}, w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{array}\right. \tag{3.37}
\end{align*}
$$

We note that

$$
\begin{align*}
\left\{\begin{array}{l}
{ }^{c} l_{11}^{e}\left(w_{1}\right)+{ }^{n c} l_{12}^{e}\left(w_{1}\right) \\
{ }^{c} l_{21}^{e}\left(w_{2}\right)+{ }^{n c} l_{22}^{e}\left(w_{2}\right)
\end{array}\right\}=\left\{\begin{array}{l}
l_{1}^{e}\left(w_{1}\right) \\
l_{2}^{e}\left(w_{2}\right)
\end{array}\right\} & =\left\{\begin{array}{l}
\oint_{\Gamma^{e}} w_{1}{ }^{c} t_{x_{1}} d \Gamma+\oint_{\Gamma^{e}} \frac{\partial w_{1}}{\partial x_{2}} m_{n} d \Gamma-\oint_{\Gamma^{e}} w_{1}{ }^{n c} t_{x_{1}} d \Gamma \\
\oint_{\Gamma^{e}} w_{2}{ }^{c} t_{x_{2}} d \Gamma+\oint_{\Gamma^{e}} \frac{\partial w_{2}}{\partial x_{1}} m_{n} d \Gamma-\oint_{\Gamma^{e}} w_{2}^{n c} t_{x_{2}} d \Gamma
\end{array}\right\}  \tag{3.38}\\
& =\left\{\begin{array}{l}
\oint_{\Gamma^{e}} w_{1}\left({ }^{c} t_{x_{1}}-{ }^{n c} t_{x_{1}}\right) d \Gamma+\oint_{\Gamma^{e}} \frac{\partial w_{1}}{\partial x_{2}} m_{n} d \Gamma \\
\oint_{\Gamma^{e}} w_{2}\left({ }^{c} t_{x_{2}}+{ }^{n c} t_{x_{2}}\right) d \Gamma+\oint_{\Gamma^{e}} \frac{\partial w_{2}}{\partial x_{1}} m_{n} d \Gamma
\end{array}\right\}
\end{align*}
$$

We substitute $\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}$ from (3.4) in the expressions for stresses and the moments given by (2.80) and (2.81). Then, we substitute (2.80), (2.81) and (3.5) in (3.16)-(3.19) and (3.31)-(3.34) to obtain the following.

$$
\begin{align*}
& \left\{\begin{array}{c}
{ }^{c} B_{11}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right) \\
{ }^{c} B_{21}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right)
\end{array}\right\}=-\left[{ }^{c} K^{e}\right]\left\{\delta^{e}(t)\right\}  \tag{3.39}\\
& \left\{\begin{array}{l}
{ }^{n c} B_{12}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{1}\right) \\
{ }^{n c} B_{22}^{e}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e} ; w_{2}\right)
\end{array}\right\}=-\left[{ }^{n c} K^{e}\right]\left\{\delta^{e}(t)\right\} \tag{3.40}
\end{align*}
$$

$$
\begin{align*}
& \left\{\begin{array}{l}
\left(\rho_{0} \frac{\partial^{2}\left(u_{1}\right)_{h}^{e}}{\partial t^{2}}, w_{1}\right)_{\bar{\Omega}_{x}^{e}} \\
\left(\rho_{0} \frac{\partial^{2}\left(u_{2}\right)_{h}^{e}}{\partial t^{2}}, w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{array}\right\}=\left[M^{e}\right]\left\{\ddot{\delta}^{e}(t)\right\}  \tag{3.41}\\
& \left\{\begin{array}{l}
\left(\rho_{0} F_{1}^{b}, w_{1}\right)_{\bar{\Omega}_{x}^{e}} \\
\left(\rho_{0} F_{2}^{b}, w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{array}\right\}=\left\{\begin{array}{l}
\left\{F_{1}^{e}\right\} \\
\left\{F_{2}^{e}\right\}
\end{array}\right\}=\left\{F^{e}\right\} \tag{3.42}
\end{align*}
$$

in which

$$
\begin{gather*}
{\left[{ }^{c} K^{e}\right]=\left[\begin{array}{cc}
{\left[{ }^{c} K_{11}^{e}\right]} & {\left[{ }^{c} K_{12}^{e}\right]} \\
{\left[{ }^{c} K_{21}^{e}\right]} & {\left[{ }^{c} K_{22}^{e}\right]}
\end{array}\right] ;\left\{\delta^{e}(t)\right\}=\left\{\begin{array}{l}
\left\{\delta^{u_{1}}(t)\right\} \\
\left\{\delta^{u_{2}}(t)\right\}
\end{array}\right\}}  \tag{3.43}\\
\left({ }^{c} K_{11}^{e}\right)_{i j}=\int_{\bar{\Omega}_{x}^{e}}\left(D_{11} \frac{\partial N_{i}^{u_{1}}}{\partial x_{1}} \frac{\partial N_{j}^{u_{1}}}{\partial x_{1}}+D_{33} \frac{\partial N_{i}^{u_{1}}}{\partial x_{2}} \frac{\partial N_{j}^{u_{1}}}{\partial x_{2}}\right) d \Omega_{x} ; \quad i, j=1,2, \ldots, n^{u_{1}}  \tag{3.44}\\
\left({ }^{c} K_{12}^{e}\right)_{i j}=\int_{\Omega_{x}^{e}}\left(D_{12} \frac{\partial N_{i}^{u_{1}}}{\partial x_{1}} \frac{\partial N_{j}^{u_{2}}}{\partial x_{2}}+D_{33} \frac{\partial N_{i}^{u_{1}}}{\partial x_{2}} \frac{\partial N_{j}^{u_{2}}}{\partial x_{1}}\right) d \Omega_{x} ; \quad i=1,2, \ldots, n^{u_{1}} ; \quad j=1,2, \ldots, n^{u_{2}}  \tag{3.45}\\
\left({ }^{c} K_{21}^{e}\right)_{i j}=\int_{\bar{\Omega}_{x}^{e}}\left(D_{33} \frac{\partial N_{i}^{u_{2}}}{\partial x_{1}} \frac{\partial N_{j}^{u_{1}}}{\partial x_{2}}+D_{21} \frac{\partial N_{i}^{u_{2}}}{\partial x_{2}} \frac{\partial N_{j}^{u_{1}}}{\partial x_{1}}\right) d \Omega_{x} ; \quad i=1,2, \ldots, n^{u_{1}} ; \quad j=1,2, \ldots, n^{u_{2}}  \tag{3.46}\\
\left({ }^{c} K_{22}^{e}\right)_{i j}=\int_{\Omega_{x}^{e}}\left(D_{33} \frac{\partial N_{i}^{u_{2}}}{\partial x_{1}} \frac{\partial N_{j}^{u_{2}}}{\partial x_{1}}+D_{22} \frac{\partial N_{i}^{u_{2}}}{\partial x_{2}} \frac{\partial N_{j}^{u_{2}}}{\partial x_{2}}\right) d \Omega_{x} ; \quad i, j=1,2, \ldots, n^{u_{2}}  \tag{3.47}\\
\left\{{ }^{c} P^{e}\right\}=\left\{\begin{array}{l}
\left\{{ }^{c} P_{1}^{e}\right\} \\
\left\{{ }^{c} P_{2}^{e}\right\}
\end{array}\right\} ; \quad\left({ }^{c} P_{1}^{e}\right)_{i}=-\int_{\Gamma^{e}} N_{i}^{c} t_{x_{1}} d \Gamma ; \quad i=1,2, \ldots, n^{u_{1}}  \tag{3.48}\\
\left({ }^{c} P_{2}^{e}\right)_{i}=-\int_{\Gamma^{e}} N_{i}^{c} t_{x_{2}} d \Gamma ; \quad i=1,2, \ldots, n^{u_{2}}
\end{gather*}
$$

$\left[{ }^{c} K^{e}\right]$ is the element stiffness matrix due to classical continuum mechanics only $\left({ }^{c} K^{e}\right)_{i j}=\left({ }^{c} K^{e}\right)_{j i}$,
i.e. $\left.{ }^{c} K^{e}\right]$ is symmetric.

$$
\left[{ }^{n c} K^{e}\right]=\left[\begin{array}{cc}
{\left[{ }^{n c} K_{11}^{e}\right]} & {\left[{ }^{n c} K_{12}^{e}\right]}  \tag{3.49}\\
{\left[{ }^{n c} K_{21}^{e}\right]} & {\left[{ }^{n c} K_{22}^{e}\right]}
\end{array}\right]
$$

and

$$
\begin{align*}
& \left({ }^{n c} K_{11}^{e}\right)_{i j}=-\frac{1}{4} \underset{\sim}{\mu} \int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial^{2} N_{i}^{u_{1}}}{\partial x_{1} \partial x_{2}} \frac{\partial^{2} N_{j}^{u_{1}}}{\partial x_{1} \partial x_{2}}+\frac{\partial^{2} N_{i}^{u_{1}}}{\partial x_{2}^{2}} \frac{\partial^{2} N_{j}^{u_{1}}}{\partial x_{2}^{2}}\right) d \Omega_{x} ; \quad i, j=1,2, \ldots, n^{u_{1}}  \tag{3.50}\\
& \left({ }^{n} K_{12}^{e}\right)_{i j}=-\frac{1}{4} \underset{\sim}{\mu} \int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial^{2} N_{i}^{u_{1}}}{\partial x_{1} \partial x_{2}} \frac{\partial^{2} N_{j}^{u_{2}}}{\partial x_{1}^{2}}+\frac{\partial^{2} N_{i}^{u_{1}}}{\partial x_{2}^{2}} \frac{\partial^{2} N_{j}^{u_{2}}}{\partial x_{1} \partial x_{2}}\right) d \Omega_{x} ; \quad i=1,2, \ldots, n^{u_{1}} ; \quad j=1,2, \ldots, n^{u_{2}} \\
& \left({ }^{n c} K_{21}^{e}\right)_{i j}=-\frac{1}{4} \underset{\sim}{\underset{\Omega_{x}}{e}} \int_{x}\left(\frac{\partial^{2} N_{i}^{u_{2}}}{\partial x_{1}^{2}} \frac{\partial^{2} N_{j}^{u_{1}}}{\partial x_{1} \partial x_{2}}+\frac{\partial^{2} N_{i}^{u_{2}}}{\partial x_{1} \partial x_{2}} \frac{\partial^{2} N_{j}^{u_{1}}}{\partial x_{2}^{2}}\right) d \Omega_{x} ; \quad i=1,2, \ldots, n^{u_{1}} ; \quad j=1,2, \ldots, n^{u_{2}}  \tag{3.51}\\
& \left({ }^{n c} K_{22}^{e}\right)_{i j}=-\frac{1}{4} \underset{\sim}{\sim} \int_{\bar{\Omega}_{x}^{e}}\left(\frac{\partial^{2} N_{i}^{u_{2}}}{\partial x_{1}^{2}} \frac{\partial^{2} N_{j}^{u_{2}}}{\partial x_{1}^{2}}+\frac{\partial^{2} N_{i}^{u_{2}}}{\partial x_{1} \partial x_{2}} \frac{\partial^{2} N_{j}^{u_{2}}}{\partial x_{1} \partial x_{2}}\right) d \Omega_{x} ; \quad i, j=1,2, \ldots, n^{u_{2}} \tag{3.53}
\end{align*}
$$

We note that $\left({ }^{n c} K^{e}\right)_{i j}=\left({ }^{n c} K^{e}\right)_{j i}$, i.e. $\left[{ }^{n c} K^{e}\right]$ is symmetric.

$$
\begin{gather*}
{\left[M^{e}\right]=\left[\begin{array}{cc}
{\left[M_{11}^{e}\right]} & {\left[M_{12}^{e}\right]} \\
{\left[M_{21}^{e}\right]} & {\left[M_{22}^{e}\right]}
\end{array}\right]}  \tag{3.54}\\
\left(M_{11}^{e}\right)_{i j}=\int_{\bar{\Omega}_{x}^{e}} \rho_{0} N_{i}^{u_{1}} N_{j}^{u_{1}} d \Omega_{x} ; \quad i, j=1,2, \ldots, n^{u_{1}} \\
\left(M_{12}^{e}\right)_{i j}=[0] ; \quad i=1,2, \ldots, n^{u_{1}} ; \quad j=1,2, \ldots, n^{u_{2}}  \tag{3.55}\\
\left(M_{21}^{e}\right)_{i j}=[0] ; \quad i=1,2, \ldots, n^{u_{2}} ; \quad j=1,2, \ldots, n^{u_{1}} \\
\left(M_{22}^{e}\right)_{i j}=\int_{\bar{\Omega}_{x}^{e}} \rho_{0} N_{i}^{u_{2}} N_{j}^{u_{2}} d \Omega_{x} ; \quad i, j=1,2, \ldots, n^{u_{2}}
\end{gather*}
$$

$$
\begin{align*}
& \left(F_{1}^{e}\right)_{i}=\int_{\bar{\Omega}_{x}^{e}} \rho_{0} F_{1}^{b} N_{i}^{u_{1}} d \bar{\Omega}_{x} ; \quad i=1,2, \ldots, n^{u_{1}} \\
& \left(F_{2}^{e}\right)_{i}=\int_{\bar{\Omega}_{x}^{e}} \rho_{0} F_{2}^{b} N_{i}^{u_{2}} d \bar{\Omega}_{x} ; \quad i=1,2, \ldots, n^{u_{2}} \tag{3.56}
\end{align*}
$$

Details of the secondary variable vector containing $l_{1}^{e}\left(w_{1}\right)$ and $l_{2}^{e}\left(w_{2}\right)$ are obtained by expanding each of the contour integrals in there over the element closed boundary $\Gamma^{e}[65,66]$. Details are straightforward. Here we simply use the following compact notation $[65,66]$.

$$
\left\{\begin{array}{l}
l_{1}^{e}\left(w_{1}\right)  \tag{3.57}\\
l_{2}^{e}\left(w_{2}\right)
\end{array}\right\}=\left\{\begin{array}{l}
\left\{P_{1}^{e}\right\} \\
\left\{P_{2}^{e}\right\}
\end{array}\right\}=\left\{P^{e}\right\}
$$

Lastly $\left\{\delta^{e}(t)\right\}$ is a vector of nodal degrees of freedom for $\left(u_{1}\right)_{h}^{e}$ and $\left(u_{2}\right)_{h}^{e}$, i.e.

$$
\left\{\delta^{e}(t)\right\}=\left\{\begin{array}{c}
\left\{\delta^{u_{1}}\right\}  \tag{3.58}\\
\left\{\delta^{u_{2}}\right\}
\end{array}\right\}
$$

in which $\left\{\delta^{u_{1}}\right\}$ and $\left\{\delta^{u_{2}}\right\}$ are nodal degrees of freedom for $\left(u_{1}\right)_{h}^{e}$ and $\left(u_{2}\right)_{h}^{e}$ for an element e with spatial domain $\bar{\Omega}_{x}^{e}$. Substituting (3.39)-(3.57) in (3.37) we obtain

$$
\left\{\begin{array}{l}
\left(A_{1}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}  \tag{3.59}\\
\left(A_{2}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{array}\right\}=\left[M^{e}\right]\left\{\ddot{\delta}^{e}(t)\right\}+\left[\left[{ }^{c} K^{e}\right]+\left[{ }^{n c} K^{e}\right]\right]\left\{\delta^{e}(t)\right\}-\left\{F^{e}\right\}-\left\{P^{e}\right\}
$$

Assembly of the element equation (3.59) follows standard procedure $[65,66]$ and we obtain

$$
\sum_{e}\left\{\begin{array}{l}
\left(A_{1}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{1}\right)_{\bar{\Omega}_{x}^{e}}  \tag{3.60}\\
\left(A_{1}\left(\left(u_{1}\right)_{h}^{e},\left(u_{2}\right)_{h}^{e}\right), w_{2}\right)_{\bar{\Omega}_{x}^{e}}
\end{array}\right\}=[M]\{\ddot{\delta}(t)\}+\left[\left[{ }^{c} K\right]+\left[{ }^{n c} K\right]\right]\{\delta(t)\}-\{F\}=\{0\}
$$

in which

$$
\begin{align*}
& {[M]=\sum_{e}\left[M^{e}\right] ; \quad\left[{ }^{c} K\right]=\sum_{e}\left[{ }^{c} K^{e}\right] ; \quad\left[{ }^{n c} K\right]=\sum_{e}\left[{ }^{n c} K^{e}\right]} \\
& \{\ddot{\delta}(t)\}=\bigcup_{e}\left\{\ddot{\delta}^{e}(t)\right\} ; \quad\{\delta(t)\}=\bigcup_{e}\left\{\delta^{e}(t)\right\}  \tag{3.61}\\
& \{P\}=\sum_{e}\left\{P^{e}\right\} \quad \text { and } \quad\{F\}=\sum_{e} \bigcup_{e}\left\{F^{e}\right\}
\end{align*}
$$

Through interelement conditions and specified nodal loads some components of $\{P\}$ are zero and the nonzero known components are absorbed in $\{F\}$. Equations (3.60) are a system of second order ODEs in time in which [ $\left.{ }^{c} K\right]$ and $\left[{ }^{n c} K\right]$ are stiffness matrices due to classical and non-classical physics and $[M]$ is the mass matrix for discretization $\bar{\Omega}_{x}^{T}$ of the spatial domain $\bar{\Omega}_{x}$.

## Chapter 4

## Solution Methods of ODEs in time

### 4.1 Methods

In this chapter, we consider solutions of ODEs in time (3.60) resulting from decoupling of space and time using GM/WF in space. We consider:
(i) Natural modes of vibrations associated with (3.60)
(ii) Transient dynamic response using normal mode synthesis and direct numerical integration of the ODEs in time.

Details of the three types of studies considered here are given in the following.

### 4.1.1 Natural Modes of Vibrations

Consider (3.60) with $[K]=\left[{ }^{c} K\right]+\left[{ }^{n c} K\right]$, total stiffness matrix for the discretization $\bar{\Omega}_{x}^{T}$.

$$
\begin{equation*}
[M]\{\ddot{\delta}(t)\}+[K]\{\delta(t)\}=\{F(t)\} \tag{4.1}
\end{equation*}
$$

For harmonic excitation

$$
\begin{equation*}
F(t)=\{\underset{\sim}{F}\} \sin (\omega t+\alpha) \tag{4.2}
\end{equation*}
$$

the response

$$
\begin{equation*}
\{\delta(t)\}=\{\Phi\} \sin (\omega t+\alpha) \tag{4.3}
\end{equation*}
$$

holds. Using (4.2) and (4.3) in (4.1) we obtain

$$
\begin{equation*}
\left[[K]-\omega^{2}[M]\right]\{\Phi(t)\}=\{\underset{\sim}{F}\} \tag{4.4}
\end{equation*}
$$

For natural vibrations $\{\underset{\sim}{F}\}=\{0\}$ and we have the following

$$
\begin{equation*}
\left[[K]-\omega^{2}[M]\right]\{\Phi\}=\{0\} \tag{4.5}
\end{equation*}
$$

eigenvalue problem with the eigenpairs $\left(\omega_{i}^{2},\{\Phi\}_{i}\right) ; i=1,2, \ldots, n$, in which $\omega_{i}$ is the $i^{\text {th }}$ natural frequency of vibrations in $(\mathrm{Hz})$ and $\{\Phi\}_{i}$ is the corresponding mode shape. These are calculated using subspace iteration method or QR-Householder method.

## Remarks

1. Since $\left[{ }^{c} K\right]+\left[{ }^{n c} K\right]$, progressively increasing non-classical physics due to progressively increasing $\underset{\sim}{\mu}$ will result in higher stiffness $[K]$. Thus when $\underset{\sim}{\mu} \neq 0$ the natural frequencies $\omega_{i}$ are always higher than the classical case $(\underset{\sim}{\mu}=0)$.
2. Since increase in stiffness results in reduced deflections, the mass normalized eigenvectors $\{\Phi\}_{i}$ are expected to exhibit progressively lower amplitude with progressively $\underset{\sim}{\mu}$ compared to classical case.
3. In the model problem study we present results for different values of $\underset{\sim}{\mu}$ and comparisons with the classical case $(\underset{\sim}{\mu}=0)$.

### 4.1.2 Time Response using Direct Numerical Integration of ODEs in Time: Wilson's $\theta$ method

We can consider direct numerical integration of ODEs in (4.1) using unconditionally stable integration methods such as: Houbolt method, Newmark method, Wilson's $\theta$ method etc. [65, 67]. Alternatively, we could also consider finite element method in time [65] with local approximation in higher order, higher degree scalar product space in time. In the present work we present model problem studies using Wilson's $\theta$ method to demonstrate the influence of non-classical physics on the evolution. A summary of Wilson's $\theta$ method is given in the following.

In Wilson's $\theta$ method with linear acceleration, the acceleration $\{\ddot{\delta}\}$ is assumed to be linear in the interval $[t, t+\theta \Delta t]$, where $\theta>1$. For this method to be unconditionally stable, $\theta \geq 1$.37, but $\theta=1.4$ is generally used. Consider

$$
\begin{equation*}
[M]\{\ddot{\delta}\}+[K]\{\delta\}-\{F\}-\{P\}=0 \tag{4.6}
\end{equation*}
$$

The method to time integrate equation (4.6) using Wilson's $\theta$ method with linear acceleration is briefly outlined below (see reference [65] for more details).
(i) When the initial conditions at time $t_{0}$ is given, say $\{\delta\}_{t_{0}}$ and $\{\dot{\delta}\}_{t_{0}}$, then the initial acceleration $\{\ddot{\delta}\}_{t_{0}}$ is obtained by solving for $\{\ddot{\delta}\}_{t_{0}}$ in equation (4.6) at time $t_{0}$.

$$
\begin{equation*}
\left\{\delta_{[2]}\right\}_{t_{0}}=[M]^{-1}\left(\{F\}_{t_{0}}+\{P\}_{t_{0}}-[K]\{\delta\}_{t_{0}}\right) \tag{4.7}
\end{equation*}
$$

(ii) Calculate $\{\delta\}_{t+\theta \Delta t}$ by solving the equation below [65].

$$
\begin{equation*}
\left[\frac{6}{(\theta \Delta t)^{2}}[M]+[K]\right]\{\delta\}_{t+\theta \Delta t}=\{f\}_{t+\theta \Delta t}+\frac{6}{(\theta \Delta t)^{2}}[M]\{\delta\}_{t}+\frac{6}{\theta \Delta t}[M]\{\dot{\delta}\}_{t}+2[M]\{\ddot{\delta}\}_{t} \tag{4.8}
\end{equation*}
$$

(iii) We calculate $\{\dot{\delta}\}_{t+\theta \Delta t}$ using the equation below.

$$
\begin{equation*}
\{\dot{\delta}\}_{t+\theta \Delta t}=\frac{3}{\theta \Delta t}\left(\{\delta\}_{t+\theta \Delta t}-\{\delta\}_{t}\right)-2\{\dot{\delta}\}_{t}-\frac{\theta \Delta t}{2}\{\ddot{\delta}\}_{t} \tag{4.9}
\end{equation*}
$$

(iv) Calculate $\{\ddot{\delta}\}_{t+\theta \Delta t}$ using the equation below.

$$
\begin{equation*}
\{\ddot{\delta}\}_{t+\theta \Delta t}=\frac{6}{(\theta \Delta t)^{2}}\left(\{\delta\}_{t+\theta \Delta t}-\{\delta\}_{t}\right)-\frac{6}{\theta \Delta t}\{\dot{\delta}\}_{t}-2\{\ddot{\delta}\}_{t} \tag{4.10}
\end{equation*}
$$

(v) Then, the solution $(\{\delta\},\{\dot{\delta}\}$, and $\{\ddot{\delta}\})$ at the next time step $t+\Delta t$ is calculated by the following equation.

$$
\begin{align*}
& \{\delta\}_{t+\Delta t}=\{\delta\}_{t}+\Delta t\{\dot{\delta}\}_{t}+\frac{(\Delta t)^{2}}{2}\{\ddot{\delta}\}_{t}+\frac{(\Delta t)^{2}}{6 \theta}\left(\{\ddot{\delta}\}_{t+\theta \Delta t}-\{\ddot{\delta}\}_{t}\right) \\
& \{\dot{\delta}\}_{t+\Delta t}=\{\dot{\delta}\}_{t}+\Delta t\{\ddot{\delta}\}_{t}+\frac{\Delta t}{2 \theta}\left(\{\ddot{\delta}\}_{t+\theta \Delta t}-\{\ddot{\delta}\}_{t}\right)  \tag{4.11}\\
& \{\ddot{\delta}\}_{t+\Delta t}=\{\ddot{\delta}\}_{t}+\frac{1}{\theta}\left(\{\ddot{\delta}\}_{t+\theta \Delta t}-\{\ddot{\delta}\}_{t}\right)
\end{align*}
$$

Steps (ii) to (v) are repeated until the desired final time is reached.
Wilson's $\theta$ method if applied to the ODEs in time before transformation to modal basis will require damping coefficients. $\theta \geq 1.37$ is a requirement for Wilson's $\theta$ method to be unconditionally stable [65, 67]. But a small enough $\Delta t$ must still be chosen for accuracy of the solution [65].

### 4.1.3 Time Response using Normal Mode Synthesis

In this method, the ODEs in time given in (4.1) are transformed to modal basis using change of basis. Let $\left(\omega_{i},\{\Phi\}_{i}\right) ; i=1,2, \ldots, n$ be the frequencies and the mode shapes obtained using (4.5) and let $\omega_{i}$ be arranged in ascending, i.e. $\omega_{1}<\omega_{2}<\cdots<\omega_{n}$. We construct a matrix $[\Phi]$ containing mass normalized eigenvectors corresponding to $\omega_{1}, \omega_{2}, \ldots, \omega_{n}$ as its columns. It is well known that the transient dynamic response of a dynamic system can be constructed using only few lower modes (m) of natural modes of vibration [65], hence the motivation for choosing only $m$ modes. Consider change of basis using

$$
\begin{equation*}
\{\delta(t)\}=[\Phi]\{x(t)\} \tag{4.12}
\end{equation*}
$$

in which $\{x(t)\}$ are the modal participation factors and $[\Phi]$ contains $m$ columns of eigenvectors. We substitute (4.12) in (4.1) and premultiply by $[\Phi]^{T}$ and noting that $\{\ddot{\delta}(t)\}=[\Phi]\{\ddot{x}\}$, we can
obtain

$$
\begin{gather*}
{[\Phi]^{T}[M][\Phi]\{\ddot{x}\}+[\Phi]^{T}[K][\Phi]\{x\}=[\Phi]\{F(t)\}}  \tag{4.13}\\
{[I]\{\ddot{x}(t)\}+\left[\omega^{2}\right]\{x\}=\{\hat{F}\}} \tag{4.14}
\end{gather*}
$$

Equation (4.14) are $m$ decoupled second order ODEs in time, thus these can be written as

$$
\begin{equation*}
\ddot{x}_{i}(t)+\omega_{i}^{2} x_{i}(t)=\hat{F}_{i} ; \quad i=1,2, \ldots, m \tag{4.15}
\end{equation*}
$$

For simple $\hat{F}_{i}$ (constant, harmonic etc.), (4.15) has analytical solution [65]. Equation (4.15) can also be integrated in time using explicit or implicit time integration methods [65]. In the present work we use analytical solutions of (4.15) [65].

## Chapter 5

## Model Problem Studies

### 5.1 Description of Solution Methods Used

The ODEs resulting from decoupling of space and time using GM/WF in space are used to study
(i) Normal modes of vibration
(ii) Time response or transient response
(a) Using normal mode synthesis
(b) Direct numerical integration using Wilson's $\theta$ method

The model problem consists of a plane strip of length 60 " width $1^{\prime \prime}$ and thickness $1^{\prime \prime}$ shown in figure 5.1 with a load of magnitude P applied at the midspan of the strip. The following reference and dimensional quantities are used to nondimensionalize the mathematical model (see reference [65]) and the model problem: $L_{0}=1 \mathrm{in}, \rho_{0}=0.289018 \mathrm{lbm} / \mathrm{in}^{3}, E_{0}=30 \times 10^{6} \mathrm{psi}, v_{0}=\sqrt{E_{0} / \rho_{0}}$, $t_{0}=L_{0} / v_{0}$ in which $L_{0}, \rho_{0}, E_{0}, v_{0}$ and $t_{0}$ are reference length, density, modulus of elasticity, velocity and time. Properties of the strip (quantities with units, indicated by hat $(\wedge)$ on them) and dimensions are given in the following (figure 5.1). $\hat{E}=30 \times 10^{6} \mathrm{psi}, \hat{\rho}=0.289018 \mathrm{lbm} / \mathrm{in}^{3}$, $\nu=0.3$. We choose $\hat{L}=60^{\prime \prime}, \hat{h} \times \hat{b}=1^{\prime \prime} \times 1^{\prime \prime}$, hence $L=60, h \times b=1 \times 1, E=1, \rho=1$.

The quantities without hat are dimensionless quantities. We choose magnitude $P=-2.96 \times 10^{-5}$ which corresponds to static deflection of -0.4 if the strip is treated as an Euler-Bernoulli beam. A uniform discretization of nine $p$-version plane stress elements is considered. We consider local approximation of class $C^{1}\left(\bar{\Omega}^{e}\right)$ is higher order scalar product space $V \subset H^{k, p}\left(\bar{\Omega}_{x}\right) ; k=2$ with $p_{\xi}=p_{\eta}=7$. Initial convergence studies performed for progressively increasing $p$-levels ( $p \geq 3$ ) confirm that for $p=7$, the completed solutions are converged. In the numerical studies we consider dimensionless $\underset{\sim}{\mu}$ of $0.001,0.01$ and 0.1 that correspond to progressively increasing influence of internal rotation physics. The $\underset{\sim}{\mu}$ value of 0.0 obviously corresponds to CCM only.


Figure 5.1: Clamped-clamped strip schematics, loading and boundary conditions

### 5.1.1 Natural modes of vibrations

Natural modes of vibration are calculated using mass matrix $[M]$ and the stiffness matrix $[K]$ corresponding to $\underset{\sim}{\mu}=0.0(\mathrm{CCM})$ and $\underset{\sim}{\mu}=0.001,0.01,0.1$. The eigenpairs are calculated using Householder QR as well as subspace iteration method. Eigenvectors are mass normalized and scaled proportionately so the their magnitude are truly relative to each other. In this thesis, we report first six eigenpairs. Figures 5.2(a)-(c) and figures 5.3(a)-(c) show plots of first six eigenvectors corresponding to the bending modes of the strip. We note that with progressively increasing $\underset{\sim}{\mu}$ the stiffness increases but the mass remains unchanged, hence for $\underset{\sim}{\mu}=0.0(\mathrm{CCM})$ we expect the lowest frequencies of vibration. With progressively increasing $\underset{\sim}{\mu}$, the natural frequencies will increase and the corresponding time period will decrease for each mode. This is confirmed by the frequencies reported in the plots in figures 5.2 and 5.3. Since for $\underset{\sim}{\mu}=0.0$ the stiffness is lowest
compared to all other values of $\mu$, hence the amplitudes of the scaled mode shapes are largest for $\underset{\sim}{\mu}=0.0$ and progressively decrease with progressively increasing value of $\underset{\sim}{\mu}$. Most dramatic illustration of progressively reducing magnitudes of the eigenvectors with progressively increasing $\underset{\sim}{\mu}$ values is seen in figure 5.3(c) (sixth eigenvector).

### 5.1.2 Transient response using normal mode synthesis

We consider load $P$ applied at time $t_{0}=0$ at the midspan of the strip and maintained for all values of time $t>0$. The ODEs in modal basis are decoupled second order system of equations that have analytical solution (used here). The time response calculations using one to three modes of vibration show that for the illustrative study considered here, use of first normal mode suffices. From the natural modes of vibration we note that the progressively increasing value of $\underset{\sim}{\mu}$ results in: (i) progressively increasing stiffness (ii) progressively increasing natural frequencies (iii) progressively decreasing time period for each mode. Thus, time response for $\underset{\sim}{\mu}>0$ will always lag the time response for $\underset{\sim}{\mu}=0$. Figures $5.4-5.6$ show time response for $\underset{\sim}{\mu}=0.001,0.01,0.1$ as well as $\underset{\sim}{\mu}=0(\mathrm{CCM})$ for $0 \leq t \leq 1.5 T$, T being the time period for the first mode when $\underset{\sim}{\mu}=0$.

Figures 5.4(a), (b), (c) show time response for $\underset{\sim}{\mu}=0$ and 0.001 . These graphs correspond to $0 \leq t \leq T / 2, T / 2 \leq t \leq T, T \leq t \leq 1.5 T$. First, we describe the motion of the strip for $\underset{\sim}{\mu}=0$. For $0 \leq t \leq T / 2$, the motion of the strip is in the negative $x_{2}$ direction, reaching maximum deflected position at $t=T / 2$ (figure 5.4(a)). For $T / 2 \leq t \leq T$, the strip motion is upward, reaching a stationary undeflected state at $t=T$. For $t \leq t \leq 1.5 T$, the strip deflects in the negative $x_{2}$ direction from its undeflected position at $t=T$ to maximum deflected position at $t=1.5 T$. Upon continued evolution, the cycle $0 \leq t \leq T$ repeats without amplitude decay. When we compare the strip deflection for $\underset{\sim}{\mu}=0.001$ in figures 5.4(a), (b) and (c) with the strip reflection for $\underset{\sim}{\mu}=0$, we note that: to arrive at a chosen deflected position for $\underset{\sim}{\mu}=0$, the evolution requires more time. The deflection of the strip for $t=T / 2, T$ and $1.5 T$ when $\underset{\sim}{\mu}=0$ is never achieved when $\underset{\sim}{\mu}=0.001$ to increased stiffness at $\underset{\sim}{\mu}=0.001$. From figures 5.5(a)-(c) and figures 5.6(a)-(c) for $\underset{\sim}{\mu}=0.01$ and 0.1, we observe similar behavior as in figures 5.4(a)-(c) but more pronounced due to increasing

(a) Mode Shape 1

(b) Mode Shape 2

(c) Mode Shape 3

Figure 5.2: Mode shapes 1-3 using QR-Householder method

(a) Mode Shape 4

(b) Mode Shape 5

(c) Mode Shape 6

Figure 5.3: Mode shapes 4-6 using QR-Householder method
stiffness with increasing $\underset{\sim}{\mu}$. Deviation between the deflected position at $t=T / 2, T$ and $1.5 T$ for $\underset{\sim}{\mu}=0$ and the maximum deflected corresponding position for $\underset{\sim}{\mu}=0.01$ and 0.1 increases as $\underset{\sim}{\mu}$ increases. Progressively increasing stiffness, increasing natural frequencies, reducing time period with progressively increasing $\underset{\sim}{\mu}$ is clearly observed in figures 5.4-5.6.

### 5.1.3 Transient dynamic response using Wilson's $\theta$ method

We consider ODEs in time resulting from the space-time decoupled GM/WF in space and integrate these using Wilson's $\theta$ method with linear acceleration. We choose an integration time step $\Delta t=T / 100$ in which $T=2 \pi / \omega_{1}, \omega_{1}$ being the first natural frequency corresponding to $\underset{\sim}{\mu}=0$. This choice of $\Delta t$ is quite conservative and works well for $\underset{\sim}{\mu}=0.001,0.01,0.1$. Figures 5.7(a)(c) through figures 5.9 show time response for $0 \leq t \leq 1.5 T$ for $\underset{\sim}{\mu}=0.001,0.01,0.1$ as well as for $\underset{\sim}{\mu}=0.0$. We observe exactly similar behavior as in figures 5.4-5.6.

We remark that response in figures 5.7-5.9 are different than in figure 5.4-5.6 due to the fact that normal mode synthesis requires large number of modes and high accuracy of the mode shapes for accurate time response. The numerical studies demonstrate the differences in time response for classical and non-classical continuum mechanics using direct integration methods.

(a) Transient response: $0 \leq t \leq 1750$

(b) Transient response: $1750 \leq t \leq 3490$

(c) Transient response: $3490 \leq t \leq 5230$

Figure 5.4: Transient response (Normal mode Synthesis) for Classical and Non-Classical

(a) Transient response: $0 \leq t \leq 1750$

(b) Transient response: $1750 \leq t \leq 3490$

(c) Transient response: $3490 \leq t \leq 5230$

Figure 5.5: Transient response (Normal mode Synthesis) for Classical and Non-Classical


(c) Transient response: $3490 \leq t \leq 5230$

Figure 5.6: Transient response (Normal mode Synthesis) for Classical and Non-Classical


Figure 5.7: Transient response (Wilson's $\theta=1.4$ Method) for Classical and Non-Classical


Figure 5.8: Transient response (Wilson's $\theta=1.4$ Method) for Classical and Non-Classical


Figure 5.9: Transient response (Wilson's $\theta=1.4$ Method) for Classical and Non-Classical

## Chapter 6

## Summary and conclusions

In this thesis, we have considered dynamic behavior of thermoelastic solids using the mathematical model based on non-classical continuum mechanics incorporating internal rotations. Mathematical model for plane stress problem in $\mathbb{R}^{2}$ is considered to present details of the explicit equations in the mathematical model including constitutive theories. The space-time differential operator in the mathematical model is linear but not symmetric. To study natural modes of vibration, mass and stiffness are essential. This necessitates that we consider space-time decoupled finite element method using GM/WF in space. It is shown that GM/WF in space results in symmetric mass matrix and the symmetric stiffness matrix that is sum of the stiffness matrices due to CCM and NCCM. The stiffness matrix due to CCM contain up to first order derivatives of the approximation functions in space whereas the stiffness matrix corresponding to the non-classical continuum mechanics contains up to second orders of the approximation functions. This necessitates that at the very least we must consider local approximations of class $C^{1}$ when considering both CCM and NCCM. For this choice of the integrals over $\bar{\Omega}_{x}^{T}$ (discretization in space) for classical mechanics are Riemann but are in Lebesgue sense for the non-classical part. We consider model problem with solutions smooth enough so that weak convergence of the solutions of lower class to the solutions of higher class are achievable. We make the following specific observations and draw some conclusions.
(i) When considering CCM the stiffness is lowest, this yields largest deflections, lowest frequencies (natural modes) and faster time response compared to NCCM with internal rotations.
(ii) Presence of internal rotation physics when resisted by the deforming solid continua results in additional energy storage, increased stiffness, higher natural frequencies and slower time response. This has been illustrated clearly in the model problem studies for: natural modes of vibration and transient dynamic response.
(iii) We remark that the internal rotation physics can not exist by itself without the presence of CCM. That is the ODEs in time in the absence of stiffness due to CCM can neither be used for natural vibration calculations nor transient dynamic response.
(iv) Since the physics of internal rotations is due to the antisymmetric part of the deformation gradient tensors $[J]$, it is always present in all deforming solid continua. In some applications, it may be more significant than the others. Varying internal rotations between neighboring material points when resisted by the deforming solid, result in moments. Thus, in the present work the moments (and Cauchy moment tensor) exist due to the presence of internal rotations. Whereas in couple stress theories assumption of the existence of moment necessitates existence of rotations.

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