Some Results in Obstruction Theory for Projective Modules

By

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Abstract

Let A be a commutative noetherian ring and P be a finitely generated projective A-module. It is known that P can be written as $Q \oplus A$ when $rank(P) > dim(A)$. But this level of generality is far from sufficient when $rank(P) = dim(A)$. The notion of the Euler Class group was developed to address this case, and it is known, at this level of generality, that when the Euler class of P vanishes, P can be written as $Q \oplus A$.

In this dissertation, we look at overrings of polynomial rings, $B = A[X, 1/f]$ where A is a commutative noetherian ring of dimension ≥ 2 and f is a non-zero divisor of $A[X]$ so that $dim(B) = dim(A[X])$, and show, through the use of the Euler Class group, that every finitely generated projective B-module, P, with $rank(P) = dim(B)$ can be written as $Q \oplus B$.

We also prove, for a commutative noetherian ring A, which is the image of a regular ring, the equivalence of several conditions to ensure the vanishing of the entire Euler Class group over A, which will again indicate that finitely generated projective A-modules can be written as $Q \oplus A$. We also give similar results for geometrically reduced affine algebras over an infinite field.

This dissertation also looks at the history behind the development of the Euler Class group for inspiration towards future development. Several examples are also given to show some of the difficulties that come in the process of generalization.

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Chapter 1

Introduction

Projective modules arise as an object of interest in commutative algebra through the resolution of a finitely generated module over a ring. Because projective modules are direct summands of a free module, it is of interest to study just how free a projective module might be. In this dissertation, we will investigate the necessary and sufficient conditions to determine if a particular projective module may be written as the direct sum of a free module and another projective module of lesser rank.

More generally, we know from the work of Kaplansky ([K]) that projective modules over local rings are free. Subsequently, Serre conjectured that finitely generated projective modules over polynomial rings over a field were free. This conjecture was proved independently by Quillen and Suslin for finitely generated projective modules over polynomial rings over a principal ideal domain.

Serre also proved a theorem which essentially gives a ”top rank” for a finitely-generated projective module over a commutative noetherian ring. Serre's theorem states that if a commutative noetherian ring, A, has dimension d, then any projective module with rank $r > d$ can be written as $Q \oplus A^{r-d}$, where Q is a projective A-module with rank d. In other words, any projective module whose rank exceeds the dimension of the ring is always equivalent to another projective module of rank at most d direct sum with enough copies of A to boost the rank of the direct sum to r.

We also rely upon the notion of a unimodular element in a projective module. The existence of a unimodular element is tied to the existence of a surjective map from the projective module, P, to the ring A, thereby ensuring us the existence of another projective module, Q, of rank one less than P such that $P \cong Q \oplus A$.

Much work has been done towards detecting the presence of a unimodular element in a projective module. An idea of the history of the work behind the Euler Class group is given as we trace the development of an Obstruction Theory for projective modules.

Also necessary for the results in this dissertation is the concept of the Euler Class group and the Weak Euler Class group. Both of these class groups can detect the existence of a unimodular element in a projective module. A projective module contains a unimodular element precisely when its euler class or weak euler class vanishes. This, again, ensures that the projective module splits off a free summand of rank one.

Keeping Serre's theorem in mind, our efforts will focus on projective modules of "top rank", that is projective modules whose rank is the same as the dimension of the ring, since any projective module with higher rank may be written as a projective module of top rank direct sum a free module.

Among the main results given in this dissertation, proof is given that the Euler Class group of $B = A[X, 1/f]$, where A has dimension ≥ 2 and $f \in A$ is a non-zero divisor so that $dim(B) = dim(A[X])$, vanishes. As a result, any projective module of top rank over B has a unimodular element, and thus splits off a free summand of rank one.

Proof is also given to a series of equivalences regarding the vanishing of the Euler Class Group and the Weak Euler Class Group. These equivalences help to show the fundamental aspects of the vanishing of the two class groups under certain conditions.

In the last chapter, I lay out my vision for a future project which I will be undertaking beginning this fall.

Chapter 2

History and Preliminaries

2.1 Definitions

All rings are commutative noetherian and all modules are assumed to be finitely generated.

Definition 2.1.1. *An A-module, P, is said to be projective if it satisfies any one of these equivalent conditions:*

1. *Given a surjective map α between A-modules M and N, the canonical map from $Hom_A(P,M)$ to $Hom_A(P,N)$, given by composing with α, is also surjective.*
2. *Every exact sequence $0 \to M \to N \to P \to 0$ of A-modules is a split exact sequence.*
3. *There exists an A-module, Q, such that $P \oplus Q$ is free.*

It is a well known property of projective modules that they are locally free. That is to say that every projective module over a local ring is free. With this property in mind, we define the rank function for projective modules.

Definition 2.1.2. *Let A be a ring, and P be a projective A-module. Let $rank_P : spec(A) \to \mathbb{Z}$ be the function defined by $rank_P(\mathfrak{p}) =$ the rank of the free module $P_{\mathfrak{p}}$. Furthermore, if $rank_P(\mathfrak{p}) = n$ for all $\mathfrak{p} \in spec(A)$, we say that $rank(P) = n$.*

The rank function is continuous on $spec(A)$ where $\mathbb{Z}$ is endowed with the discrete topology. In other words, as long as $spec(A)$ is connected, the rank function is constant. Also, A contains a non-trivial idempotent if and only if $spec(A)$ is disconnected; therefore, the rank function is constant as long as A contains no non-trivial idempotent. For the rest of this dissertation, we will assume that A contains no such element, with the implication being that all projective A-modules have constant rank.

To formalize some of the things the dimension function does, Plumstead ([P]) defined a generalized dimension function as follows:

Definition 2.1.3 (Plumstead)**.** *Suppose A is a commutative noetherian ring. Let X be a subset of spec(A) and let $d : X \to \mathbb{Z}$ be a function.*

1. *We define a partial ordering on X by defining $\wp_1 \ll \wp_2$ if either $\wp_1 = \wp_2$ or if $\wp_1 \subseteq \wp_2$ and $d(\wp_1) > d(\wp_2)$ for $\wp_1, \wp_2 \in X$.*
2. *A function $d : X \to \mathbb{Z}$ is said to be a generalized dimension function if for any ideal I of A, $V(I) \cap X$ has only finitely many minimal elements with respect to the partial ordering $\ll$.*

Definition 2.1.4. *Let A be a ring and let P be a projective A-module of rank n. The determinant of P is defined to be the n^{th} exterior power of P, $\wedge^n(P)$. Furthermore, if $\wedge^n(P) \cong A$, we say that P has trivial determinant.*

Here we note that $\wedge^n(P)$ is a projective A-module of rank one, usually denoted as L.

Definition 2.1.5. *Let A be a ring. A row vector* $(a_1, a_2, \ldots, a_n) \in A^n$ *is called a unimodular row if there exists elements* $b_1, b_2, \ldots, b_n \in A$ *such that* $\sum a_i b_i = 1$*. The set of all unimodular rows over A of length n is denoted by* $Um_n(A)$*.*

It is clear from the definition that every free A-module has a unimodular row, namely $(1,0,0,\ldots,0)$. We note here that a unimodular row is analogous to a linear surjection from A^n to A, understanding the unimodular row to be an element of A^n which maps to 1 under the linear map given by the b_i. We can take that perspective on a unimodular row and use it to define a unimodular element in a projective module.

Definition 2.1.6. *Let A be a ring and P a projective A-module. An element* $p \in P$ *is called a unimodular element of P if there exists a map* $\alpha \in Hom_A(P,A)$ *such that* $\alpha(p) = 1$*.*

We should note here that this definition works for any A-module, not just projective A-modules. It is also useful to note that the existence of a unimodular element in a projective module gives rise to a split exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow 0,$$

where K is the kernel of α and thus $P \backsimeq K \oplus A$. In this situation, we say that P *splits off a free summand of rank one*. A is free, therefore the structure of P is contained entirely within the structure of K, which may be similarly decomposed if K contains a unimodular element. Now as long as P is not free, this process will terminate and we would have $P \backsimeq Q \oplus A^k$, where Q is a non-zero projective module which contains no unimodular element and k is a positive integer less than $rank(P)$.

The focus of the study of this dissertation is in furthering the understanding of projective modules and classifying those which split off a free summand of rank one.

For the sake of clarity, it is necessary to address the topic of basic elements.

Definition 2.1.7. *Let A be a ring and M an A-module.*

1. *An element $m \in M$ is a basic element of M at a prime ideal $\mathfrak{p}$ if m is not in $\mathfrak{p}M_{\mathfrak{p}}$.*

2. *An element m of M is a basic element of M if m is basic in M at all the prime ideals $\mathfrak{p}$ of A. We also say that m is basic in M on a subset X of $Spec(A)$ if m is basic in M at all prime ideals in X.*

As far as we are concerned, that is to say for projective modules, a basic element is the same as a unimodular element.

Lemma 2.1.8. *Let A be a ring and P a projective module. $p \in P$ is a unimodular element if and only if p is a basic element of P.*

If $p \in P$ is a unimodular element, then there exists a surjection $\phi : P \twoheadrightarrow A$ such that $\phi(p) = 1$. Thus, for any $\wp \in spec(A)$, $\phi_\wp : P_\wp \twoheadrightarrow A_\wp$ also takes p to 1. As a result, since $\phi_\wp(p) \notin \wp A_\wp$, then $p \notin \wp P_\wp$, and so p is a basic element of P.

If $p \in P$ is a basic element, then consider the short exact sequence

$$0 \longrightarrow A \longrightarrow P \longrightarrow P/pA \longrightarrow 0$$

where the map from A to P takes 1 to p. Then, localizing this sequence at any prime $\wp \in spec(A)$,

$$0 \longrightarrow A_\wp \longrightarrow P_\wp \longrightarrow (P/pA)_\wp \longrightarrow 0$$

$P_\wp$ is free, so p looks like a row $(a_1, a_2, \dots, a_r)$ with the $a_i \in A_\wp$ and $r = rank(P)$. Moreover, since p is basic in P, one of the a_i is a unit in $A_\wp$, thus the row $(a_1, a_2, \dots, a_r)$ may be completed to an invertible matrix. The first row of the matrix is $p \in P_\wp$, a basis for the image of $A_\wp$, and the other $r-1$ rows form a basis for $(P/pA)_\wp$. And so P/pA

is projective, which means that our original short exact sequence splits, resulting in a projective map from P to A, which sends p to 1. So p is a unimodular element of P.

Because of this lemma, and the fact that we are restricting ourselves to projective modules, we will be using basic element and unimodular element interchangeably.

2.2 History

It is appropriate to begin by looking at some foundational theorems in the understanding of projective modules.

Theorem 2.2.1 (Quillen, Suslin)**.** *Every projective module over a polynomial ring* $k[X_1,\ldots,X_n]$ *over a field k is free.*

This theorem was originally conjectured by Serre, and then proved independently by both Quillen and Suslin in 1976. Though using different methods, both proved Serre's conjecture by deriving the following theorem from a theorem of Horrocks.

Theorem 2.2.2 (Quillen, Suslin)**.** *Let A be a commutative noetherian ring and let R = A[X] be a polynomial ring. Suppose that P is a finitely generated projective R-module such that* P_f *is free for some monic polynomial f in R. Then P is free.*

Theorem 2.2.3 (Horrocks)**.** *Let* $(A,\mathfrak{m})$ *be a commutative noetherian local ring and let R = A[X] be a polynomial ring. Suppose P is a finitely generated projective R-module such that* P_f *is free for some monic polynomial f in R. Then P is free.*

Quillen's proof uses Horrocks' theorem to establish that $P_{\mathfrak{m}}$ is free for all maximal ideals $\mathfrak{m}$ of A, and goes on to show that P is extended from A, thereby reducing the problem to proving that $\overline{P} = P/XP$ is free. Then, using a patching diagram to find a comparable projective module, P', over $A[X^{-1}]$, he proves that P' is extended from some A-module N' which is free and also isomorphic to $\overline{P}$, thus establishing that $\overline{P}$, and therefore P, is free.

The following theorem is then derived from 2.2.2 and settles Serre's conjecture.

Theorem 2.2.4 (Quillen, Suslin)**.** *Let A be a principal ideal domain and let* $R = A[X_1,\ldots,X_n]$ *be a polynomial ring. Then any finitely generated projective R-module is free.*

This theorem is proved by induction on n by inverting all monic polynomials in $A[X_1]$ to get a projective module, P', over a polynomial ring over a principal ideal domain with fewer variables. By induction P' is free, so there exists a monic polynomial such that P_f is free and thus P is free by 2.2.2.

A classical result from Serre establishes a wide foundation for our understanding of which projective modules split off a free summand of rank one.

Theorem 2.2.5 (Serre)**.** *Let A be a ring of dimension d and let P be a finitely generated projective A-module such that rank*$(P) > d$. *Then P splits off a free summand of rank one.*

This theorem of Serre can be derived as a result of the following version of a useful theorem of Eisenbud and Evans.

Theorem 2.2.6 (Eisenbud-Evans)**.** *Let A be a commutative noetherian ring, and let* $d : X \to \mathbb{N}$ *be a generalized dimension function on a subset X of spec(A). Let M be a finitely generated A-module. If* $\mu(M_{\wp}) > d(\wp)$ *for all* $\wp \in X$ *then M has a basic element on X.*

For the proof of Serre's theorem 2.2.5 in the commutative noetherian case, we can let X be $spec(A)$, and we will use the standard dimension function as the required generalized dimension function. Then, because a basic element in a projective module is the same as a unimodular element, we can then infer that P splits off a free summand of rank one by the discussion which followed Definition 2.1.6.

The following theorem was originally conjectured by Eisenbud and Evans in single variable form and then proved by Plumstead ([P]). It was later expanded upon in its following form as a question by Bass and subsequently proved by Bhatwadekar and Roy. It can be viewed as a generalization combining Serre's theorem 2.2.5 and Serre's conjecture 2.2.1.

Theorem 2.2.7. *Let A be a ring of dimension d, let $R = A[X_1, \ldots, X_n]$ be a polynomial ring in n variables over A, and let P be a projective R-module. If rank$(P) > d$, P splits off a free summand of rank one.*

Bhatwadekar and Roy proved this theorem in ([BR]) assuming Serre's conjecture 2.2.1. Later, Lindel gave another proof in ([L]) without assuming Serre's conjecture which implies Serre's conjecture as a result.

From the preceding theorems, we see the basis for our study. Serre's theorem establishes the dimension of the ring as a maximum rank for projective modules which cannot be written as the direct sum of a projective module of lesser rank and a free module. We will refer to such projective A-modules as these, whose rank equals the dimension of A as *projective modules of top rank*.

The following example illustrates how the maximum set by Serre's theorem is the best possible over arbitrary rings.

Example 2.2.8. *Let $A = \mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1)$, be the coordinate ring of the real 2-sphere, and let α be the surjection $\alpha : A^3 \twoheadrightarrow A$ given by $\alpha(e_1) = x$, $\alpha(e_2) = y$ and $\alpha(e_3) = z$ where x,y and z are the images of X,Y and Z in A. Now, since $\alpha(x,y,z) = 1$, (x,y,z) is a unimodular element, and then letting $P = ker(\alpha)$, we note that $P \oplus A \cong A^3$ and thus we see that P is a projective A-module of rank 2. We also note that $rank(P) = dim(A)$. However, we will see that P cannot contain a unimodular element, and therefore does not split off a free summand of rank one.*

To see this, we will begin by uniquely extending any given A-linear map from P to A to an A-linear map $\beta : A^3 \to A$ which sends the element $w = (x,y,z) \in A^3$ to 0. To this map we can associate a continuous vector field on the real 2-sphere in the following way:

Let $f_i(x,y,z) = \beta(e_i) \in A, 1 \leq i \leq 3$, and let $F_i(X,Y,Z) \in \mathbb{R}[X,Y,Z]$ be any preimages of the f_i. By reassembling the F_i into a map which sends an element (a_1,a_2,a_3) to $(F_1(a_1,a_2,a_3),F_2(a_1,a_2,a_3),F_3(a_1,a_2,a_3))$, we get a continuous map from the real 2-sphere to $\mathbb{R}^3$ which depends only on the f_i and not the choice of preimages F_i. Additionally, since $\beta(x,y,z) = 0$, the vector $(F_1(a_1,a_2,a_3),F_2(a_1,a_2,a_3),F_3(a_1,a_2,a_3))$ is perpendicular to (a_1,a_2,a_3). Now we can define a continuous vector field on the real 2-sphere as $F(a_1,a_2,a_3) = (a_1,a_2,a_3)+(F_1(a_1,a_2,a_3),F_2(a_1,a_2,a_3),F_3(a_1,a_2,a_3))$. This vector field vanishes at any points on the real 2-sphere where $F_i(a_1,a_2,a_3) = 0$ for $i = 1,2,3$. We note here that if we let J be the ideal (f_1,f_2,f_3) of A, the zeros of the vector field are given by exactly those maximal ideals, $\mathfrak{m}$, of A which contain J and such that $A/\mathfrak{m} \cong \mathbb{R}$.

Now, if P contains a unimodular element, we have a surjection from P to A which lifts uniquely to a surjection $\gamma : A^3 \twoheadrightarrow A$ such that $\gamma(w) = 0$. However, since γ is a surjection, $J = A$, and we then obtain a nowhere vanishing continuous vector field on the real 2-sphere which contradicts a well known topological result. Therefore, P may not contain a unimodular element and as such does not then split off a free summand of rank one.

So, we see that, in general, projective modules of top rank do not necessarily split off a free summand. For more examples, including projective modules over algebraically closed fields which do not split a free summand of rank one, refer to section 4.3.

It was at this point that Murthy and Mohan Kumar began their work on Chern classes in an effort to detect the presence of a unimodular element in a projective module.

Chapter 3

Chow Groups and Chern Classes

3.1 Definitions & Prerequisites

We begin with the definition of the Chow Group.

Definition 3.1.1. *Let X be an algebraic scheme and let Z(X) be the free abelian group generated by $\{\ X_i : X_i$ is an integral closed subscheme of X$\}$. We define $Z^p(X)$ as the free abelian group generated by $\{\ X_i : X_i$ is an integral closed subscheme of X with codimension p$\}$. We define $Z_p(X)$ as the free abelian group generated by $\{\ X_i : X_i$ is an integral closed subscheme of X with dimension p$\}$. We note here that*

$$Z(X) = \bigoplus_p Z^p(X) = \bigoplus_p Z_p(X)$$

We will first define the relations on Z(X) where $X = Spec(A)$ as follows: First, let $\mathfrak{p} \in Spec(A)$ have height $p-1$. For $f \in A/\mathfrak{p}$, we define

$$(f) = (\mathfrak{p}, f) = \sum_{height\,\wp = p} l_{A_\wp}(A/(\mathfrak{p}, f))\wp$$

We now define $R^p(X)$ to be the free abelian group generated by $(f/g) = (f) - (g)$, where $f, g \in A/\mathfrak{p}$ over all $f \in A/\mathfrak{p}$, for all $\mathfrak{p} \in Spec(A)$ with height p-1, and

$$R(X) = \bigoplus_p R^p(X).$$

Then the total chow group is defined $CH(X) = Z(X)/R(X)$.

Generally, for a coherent sheaf, $\mathscr{F}$, and its support, $Supp(\mathscr{F}) = \cup w_i$ where w_i are the irreducible components, we define the cycle of $\mathscr{F}$ to be

$$Z(\mathscr{F}) = \sum_i l_{A_{w_i}}(\mathscr{F}_{w_i}) w_i$$

where $A_i = \mathscr{O}_{X,w_i}$.

More specifically for a module, M, with $\mathfrak{m}_i$ being the minimal primes of ann M, we define the cycle of M to be

$$Z(M) = \sum_i l_{A_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) V(\mathfrak{m}_i).$$

We note here that if X is smooth over a field, then the chow group, $CH(X)$ has a graded ring structure([Fu]).

We also define the Grothendieck group and give a few comments about it.

Definition 3.1.2. *Let $\mathscr{C}$ be a category with short exact sequences, and define $F(\mathscr{C})$ to be the free abilian group generated by isomorphism classes of objects in $\mathscr{C}$. Let $R(\mathscr{C})$ be the subgroup of $F(\mathscr{C})$ generated by $M - M' - M''$ where $0 \to M' \to M \to M'' \to 0$ is a short exact sequence. Then the Grothendieck group is defined $K_0(\mathscr{C}) = F(\mathscr{C})/R(\mathscr{C})$.*

Remark 3.1.3. *If X is a scheme and $\mathscr{C}$ is the category of locally free sheaves, then we write $K_0(X)$ for $K_0(\mathscr{C})$.*

Similarly, if X = Spec(A) and $\mathscr{C}$ is the category of projective A-modules, then we write $K_0(A)$ for $K_0(\mathscr{C})$.

Lemma 3.1.4. *Let X be a scheme and $\mathscr{P}$ be the category of coherent sheaves with finite locally free resolution. The natural map $K_0(X) \to K_0(\mathscr{P})$ is an isomorphism.*

Similarly, let X' = spec(A) and $\mathscr{P}'$ be the category of modules with finite projective dimension. The natural map $K_0(A) \to K_0(\mathscr{P}')$ is an isomorphism.

Obviously, projective A-modules have finite projective dimension, but the inverse image of a module M with finite projective resolution:

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

is

$$[M] = \sum_{i=0}^{n} (-1)^i [P_i] \in K_0(A).$$

With this isomorphism in mind, $K_0(A)$ will represent the Grothendieck group of the category of finitely generated A-modules with finite projective dimension. Also, $K_0(X)$ will represent the Grothendieck group of the category of coherent sheaves with finite locally free resolution.

3.2 Chern Class Formalism

For general reference regarding Chern Classes, we refer the reader to Fulton & Lang ([FL]) and Fulton ([Fu]).

In [Fu], Fulton defined chern classes as maps with certain degree as follows:

Definition 3.2.1. *For a noetherian scheme, X, of dimension n, and locally free sheaf $\mathscr{E}$, we define $C^p(\mathscr{E})\cap : CH(X) \to CH(X)$ as a degree p homomorphism with the following properties:*

1. $C^0(\mathscr{E})\cap = 1$
2. $C^p(\mathscr{E})\cap = 0$ *for all* $p >$ *rank*$\mathscr{E}$
3. *If* $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ *is exact, then* $C^p(\mathscr{E})\cap = (C^p(\mathscr{E}')\cap)(C^p(\mathscr{E}'')\cap)$
4. *Although CH(X) is not a ring in general,* $\mathfrak{E}$ *has a graded ring structure if we define* $\mathfrak{E}^p =$ *degree p endomorphisms of CH(X), and* $\mathfrak{E} = \oplus \mathfrak{E}^p$.

Definition 3.2.2. *For a noetherian scheme, X, of dimension n, and locally free sheaf $\mathscr{E}$, we define the chern class polynomial to be:*

$$C_t(\mathscr{E}) = \sum_{i=0}^{n} C^i(\mathscr{E})t^i$$

Remark 3.2.3. *For a noetherian scheme, X, of dimension n, and any trivial bundle $\mathscr{F}$, $C_t(\mathscr{F}) = 1$.*

From the definition of chern class, for a general X, $CH(X)$ is not known to be a ring. However, there are cases in which $CH(X)$ has a graded ring structure. This is formalized by Fulton in [Fu]:

Lemma 3.2.4. *Let X be nonsingular over a field, then CH(X) is a graded ring. For a locally free sheaf $\mathscr{E}$, it follows that $C^i(\mathscr{E})\cap : CH(X)\to CH(X)$ is completely determined by $C^i(\mathscr{E})\cap X \in CH^i(X)$.*

There is a second case that interests us, and that is the case where X is an integral scheme:

Lemma 3.2.5. *Let X be an integral scheme with n = dim X, and let $\mathscr{E}$ be a locally free sheaf with rank $\mathscr{E} = n$. Then $C^n(\mathscr{E})\cap$ is completely determined by $C^n(\mathscr{E})\cap X \in CH^n(X)$.*

Although X being an integral scheme is not enough for $CH(X)$ to even be a ring, it does at least tell us that $CH^0(X) = \mathbb{Z}$ so that we have the result that $C^n(\mathscr{E})$ is determined by where it sends the identity. Even though this only works for the case where the degree equals the dimension of the scheme and the rank of the sheaf, this is exactly the case we're interested in.

In light of these situations, we will begin looking at these "top" chern classes (i.e. degree n endomorphisms on CH(X) where X is nonsingular or an integral scheme with dimension n) as elements of $CH^n(X)$.

Facts 3.2.6. *Note the following facts about $CH(X)$:*

1. *There is a chern class homomorphism:*

$$c_t : K_0(X)\to 1+\bigoplus_{i=1}^{n} CH^i(X)t^i$$

   which is an additive to multiplicative group homomorphism.

2. *For nonnegative integer k, $F^kK_0(X)$ will denote the subgroup of $K_0(X)$ generated by $[\mathscr{M}]$, where $\mathscr{M}$ runs through all coherent sheaves on X with the support of*

$\mathscr{M}$ having codimension at least k. For such a coherent sheaf $\mathscr{M}$, $Cycle(\mathscr{M})$ will denote the codimension k-cycle in the Chow group of X.

3. *If x is in $F^rK_0(X)$ then*

$$C^i(x) = 0; 1 \leq i < r$$

$$C^r(x) = (-1)^{r-1}(r-1)!Cycle(x)$$

4. *If $\mathscr{E}$ is a locally free sheaf of finite rank over X and there is a surjective map from $\mathscr{E}$ onto a locally complete intersection ideal sheaf $\mathscr{I}$ of height r, then*

$$C^r(\mathscr{E}) = (-1)^r Cycle(\mathscr{O}_X/\mathscr{I}).$$

Murthy gave an alternative definition of the n^{th} Chern class of a projective module which takes value in F^nK_0. This definition is not surprising when we look at the isomorphism that exists between $CH^n(X)$ and $K_0(X)$ when X is nonsingular over an algebraically closed field.

Definition 3.2.7. *Let A be a geometrically reduced affine k-algebra over an infinite field k. Let $dim(A) = n$, and let P be a projective A-module of rank n and $P^* = Hom_A(P,A)$. Then the n^{th} Chern class $C^n(P)$ is*

$$C^n(P) = \sum_{i=0}^{n}(-1)^i(\wedge^i P^*) \in K_0(A).$$

Remark 3.2.8. 1. *We will use the same notation C^n for the n^{th} Chern class although it takes value in $F^nK_0(A)$ and not $CH(A)$.*

2. *In some cases, the two definitions coincide and there is no ambiguity.*

3. *It will be clear from the context which definition we are referring to.*

With this new definition in hand, we remark on a few things.

Remark 3.2.9. *Let A and P be as in the previous definition.*

1. *If $P \twoheadrightarrow I$, where I is a local complete intersection ideal of height n, then this definition of $C^n(P)$ results in*

$$C^n(P) = \sum_{i=0}^{n} (-1)^i (\wedge^i P^*) = (-1)^n (A/I) \in F^n K_0(A).$$

2. *If A is regular, k is algebraically closed, and $X = Spec(A)$, then $CH^n(X)$ is isomorphic to $F^n K_0(A)$. Moreover, the definition of Chern class as an element of $F^n K_0(A)$ coincides with the earlier definition as an element of $CH^n(X)$.*

3. $C^n(P^*) = (-1)^n C^n(P)$

4. *If $P \cong Q \oplus A$, P does not necessarily have to have rank n, then*

$$\sum_i (-1)^i (\wedge^i P) = 0.$$

The last remark follows from the fact that

$$\wedge^i P = \wedge^i (Q \oplus A) = \wedge^i Q \oplus \wedge^{i-1} Q$$

combined with the alternating sum creates an alternating sum, resulting in a zero sum.

Chapter 4

Background of Obstruction Theory

4.1 Introduction

There is a lot of similarity between vector bundles over real manifolds and projective modules over commutative rings. For instance, the similarity between the following theorems has sparked a lot of the research that is used in the work found in this dissertation.

Theorem 4.1.1. *If X is a compact real manifold of dimension n and V is a real vector bundle of rank larger than n, then V has a nowhere vanishing section.*

Correspondingly:

Theorem 4.1.2. *If A is a ring with dimension n, and P is a projective A-module with rank larger than n, then P splits off a free summand of rank one.*

These similarities find their root in the connection between the set $\Gamma(X, \mathscr{V})$ of sections of a vector bundle, $\mathscr{V}$, as a projective module over the ring of all continuous functions on X.

However, topologically, this theory is more advanced than it is algebraically. This is due, in large part, to the availability of Obstruction theory. For instance, if a vector bundle V has rank r over a smooth compact oriented manifold X, there is the Euler class $e(V)$ in $\mathbb{Z}$. This $e(V)$ is simply the number of zeros of a general section of V. In general, $e(V)$ takes values in the cohomology group $H^n(X,\mathbb{Z})$.

Theorem 4.1.3. *[MiS] Let X be a smooth compact oriented manifold with dimension n, and let V be a vector bundle of rank n. Then* $e(V) = 0$ *if and only if V has a nowhere vanishing section.*

A search for an analogue of Obstruction theory in Algebra began with the work of Mohan Kumar and Murthy ([MK, MKM, Mu]) on vector bundles over affine algebras over algebraically closed fields. Their idea was to utilize the top chern class to determine properties of vector bundles and hence projective modules. The main theorem proved in this regard is the following:

Theorem 4.1.4 (Murthy)**.** *Let X be a reduced affine variety of dimension n over an algebraically closed field k, and let V be a vector bundle of rank n. Then the top chern class* $C^n(V) = 0$ *in the Chow group if and only if V has a nowhere vanishing section.*

As a result of this theorem, then, for smooth affine algebras over an algebraically closed field and their projective modules whose rank equals the dimension of the algebra, the vanishing of the top Chern class implies the existence of a unimodular element.

4.2 Results in Obstruction Theory

In chapter 2, we gave the real 2-sphere as an example where we can have a projective module of top rank that does not split off a free summand of rank one. One might think that the projective module in that example failed to have a unimodular element perhaps due to the fact that $\mathbb{R}$ is not algebraically closed. However, Nori has a theorem in ([Mu2]) that proves the existence of indecomposable projective modules of top rank over $\mathbb{C}$. In the same paper, Murthy gives us a specific example of exactly this situation which we will provide in the next section.

Theorem 4.2.1 (Nori)**.** *Let X be a smooth projective variety of dimension n over* $\mathbb{C}$*. Suppose that* $H^n(X, \mathscr{O}_X) \neq 0$*. Let V be any smooth affine variety birational to X. Then there exist indecomposable projective A-modules rank n, where A is the coordinate ring of V.*

Because of this, we see that the base field being algebraically closed is still not enough to ensure that all projective modules of top rank will split off a free summand of rank one.

So, in the case where the base field is algebraically closed, Murthy used the vanishing of the top Chern class, $C_n(P)$ of P, a rank n projective module, to imply the existence of a unimodular element in P. In other words, if $C_n(P) = 0$, then P has a unimodular element, and therefore splits off a free summand of rank one. Meanwhile, the converse is true even if the base field is not algebraically closed: If P has a unimodular element, then $C_n(P) = 0$.

The top Chern class of P is obtained by taking a generic surjection $P \twoheadrightarrow J$ (we may take a local complete intersection ideal of height n), and then take the image of the cycle of A/J in the chow group $CH_0(A)$. It can be shown that this image in $CH_0(A)$ is independent of the generic surjection chosen. We recall that the image of the cycle of

A/J in $CH_0(A)$ is zero when J is a complete intersection, however the converse is not true in general.

Theorem 4.2.2 (Murthy). *Let A be a smooth affine domain of dimension n over an algebraically closed field k. let P be a projective A-module of rank n. If $C^n(P) = 0$, then P has a unimodular element.*

Essentially what Murthy proved is that under the conditions given, if J is a locally complete intersection ideal of A with height n such that J/J^2 is generated by n elements and $[J] = 0$, then J is generated by n elements. The proof of the theorem is completed by the following result given by Mohan Kumar in ([MK])

Theorem 4.2.3 (Mohan Kumar). *Let A be a smooth affine domain of dimension n over an algebraically closed field k. Let P be a projective A-module of rank n. Take a generic surjection $P \twoheadrightarrow J$, and if J is a complete intersection ideal of height n, then P has a unimodular element.*

So, we can see that by putting these two results together, over smooth affine domains over algebraically closed fields, projective modules, P, of top rank that have $C^n(P) = 0$ will admit a free summand of rank one.

But, as we saw with the real 2-sphere, the vanishing of the top Chern class is not sufficient to ensure the admittance of a free summand of rank one. However, the real 2-sphere is not the only non-algebraically closed case that gives us projective modules without free summands of rank one despite the vanishing of their top Chern class. The following theorem from Bhatwadekar, Das and Mandal [BDM] ensures the existence of projective modules whose top Chern class vanish yet do not split a free summand of rank one for certain even dimensional smooth affine varieties.

Theorem 4.2.4. *Let $X = Spec(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers. Let K denote the canonical module $\wedge^n(\Omega_{A/\mathbb{R}})$. Let P be a* *projective A-module of rank n and let* $\wedge^n(P) = L$*. Assume that* $C^n(P) = 0 \in CH_0(X)$*. Then P splits off a free summand of rank one in the following cases:*

1. $X(\mathbb{R})$ *has no compact connected component.*

2. *For every compact connected component C of* $X(\mathbb{R})$*,* $L_C \neq K_C$ *where* K_C *and* L_C *denote the restriction of the induced line bundles on* $X(\mathbb{R})$ *to C.*

3. *n is odd.*

Moreover, if n is even and L is a rank one projective A-module such that there exists a compact connected component C of $X(\mathbb{R})$ *with the property that* $L_C = K_C$*, then there exists a projective A-module P of rank n such that*

$$P \oplus A = (L \oplus A^{n-1}) \oplus A$$

(hence $C^n(P) = 0$*) but P does not admit a free summand of rank one.*

So, if we want to detect the existence of a unimodular element for smooth affine domains over non-algebraically closed fields, we need something more than the Chern class approach which Murthy used. In response to this problem, Nori proposed the Euler Class group for smooth affine domains over infinite perfect fields. Nori's proposal was solidified and expanded to general noetherian rings by Bhatwadekar and Sridharan.

The next theorem from Mandal and Murthy [MM] gives us a little insight into the natural transition from the top Chern class point of view towards the direction of the Euler Class Group and its lifting of surjective maps as it was eventually formalized by Bhatwadekar and Sridharan.

Theorem 4.2.5. *Let A be a reduced affine algebra of dimension* $n \geq 2$ *over an algebraically closed field k. Let P be a projective A-module of rank n and I be a local* *complete intersection ideal of height n in A. Let* $\overline{f} : P/IP \twoheadrightarrow I/I^2$ *be a surjective map. Suppose that* $F^nK_0(A)$ *has no* $(n-1)!$ *torsion. Then there exists a surjective map* $f : P \twoheadrightarrow I$ *such that* $f \otimes A/I = \overline{f}$ *if and only if* $C^n(P^*) = (A/I) \in F^nK_0(A)$.

4.3 Examples

In the last section, we gave a theorem of Nori, Theorem 4.2.1 that proves the existence of indecomposible projective modules over algebraically closed fields. Murthy gives a specific example of this situation in [Mu]. As a result, we know that it is not simply the algebraic closure of the field that causes a top rank projective module to fail to have a unimodular element.

Example 4.3.1 (Murthy)**.** *Let X be the hyper surface in $\mathbb{P}^{n+1}_{\mathbb{C}}$ defined by the equation*

$$X_0^m + X_1^m + \ldots + X_{n+1}^m = 0$$

It follows from Theorem 4.2.1, that there are indecomposable projective A-modules of rank n over

$$A = \mathbb{C}[X_1, \ldots, X_{n+1}]/(\sum_{i=1}^{n+1} X_i^m + 1)$$

for $m \geq n+2$.

We also saw in the previous section, that Murthy and Mohan Kumar used the top Chern class to detect the existence of a unimodular element in projective modules over smooth affine domains over algebraically closed fields. We again look to the real 2-sphere for an example of how the condition that the field be algebraically closed may not be relaxed.

Example 4.3.2. *Let A and P be as in example 2.2.8. Let J be the ideal (y,z) of A and define a surjection $\phi : A^3 \twoheadrightarrow J$ by sending e_1 to 0, e_2 to $-z$ and e_3 to y. ϕ induces a surjection $P \twoheadrightarrow J$ and hence $C^2(P) = 0$ because J is 2-generated, but as we have already seen, P cannot contain a unimodular element due to topological constraints.*

The example of the real 2-sphere can be expanded to a general n-dimensional sphere. The next example gives some details regarding the tangent bundle of an n-dimensional real sphere.

Example 4.3.3. *Let A_n be the coordinate ring of the n-dimensional real sphere. $A_n = \mathbb{R}[X_0, \ldots, X_n]/(\sum_{i=0}^{n} X_i^2 - 1)$, and let x_i denote the image of X_i in A_n. Then, let T be the projective module corresponding to the tangent bundle of the n-dimensional real sphere.*

$$0 \longrightarrow T \longrightarrow A_n^{n+1} \xrightarrow{\alpha} A_n \longrightarrow 0$$

where the map $\alpha : A_n^{n+1} \to A_n$ is given by $(x_0, \ldots, x_n)$.

1. *If n is even, T does not contain a unimodular element.*

2. *If n is odd, then we can find a unimodular element in T, $(-x_1, x_0, -x_3, x_2, \ldots, -x_n, x_{n-1})$. Therefore, T splits off a free summand of rank one.*

3. *The tangent bundle is trivial, i.e. T is free, in the cases $n = 1, 3, 7$.*

4. *T is not cancellative. T is stably free and so*

$$A_n^{n+1} = T \oplus A_n;$$

   but T is not free.

To see that the tangent bundle of an even dimensional real sphere does not have a unimodular element, we appeal to a proof by Milnor ([Mi]). The essential idea is that a differentiable nowhere vanishing vector field on the tangent bundle corresponds to the tangent bundle splitting off a free summand of rank one. This, of course, is equivalent to the tangent bundle containing a unimodular element.

So, if the tangent bundle has a differentiable nowhere vanishing vector field, it can be adjusted and extended to a region surrounding the sphere. A family of functions are then defined on that region which map to $\mathbb{R}^{n+1}$ where n is the (even) dimension of the sphere. A contradiction arises between the calculation of the volume of the image of the region under those maps via calculus and the calculation that comes from the geometry of the image.

We can also show that the tangent bundle does not split a free summand of rank one through algebraic methods. In Example 5.2.1 we will use Euler class methods to show exactly when T contains a unimodular element.

Now that we've thoroughly exhausted the real spherical examples, one might wonder if there are other examples. From the results of Theorem 4.2.4 in the previous section, we give some specific contexts for its use, which result in more examples of projective modules where $C^n(P) = 0$ without P containing a unimodular element.

Example 4.3.4. *Let $X = Spec(A)$ be an affine open subvariety of the projective 2-space $\mathbb{P}^2(\mathbb{R})$ which is the complement of $V(X^2+Y^2+Z^2)$. $X(\mathbb{R}) = X$ is connected and compact, so by Theorem 4.2.4, there exists a projective A-module, P, of rank 2 such that $L := \wedge^2(P) = \wedge^2(\Omega_{A/\mathbb{R}})$ and $P \oplus A = (L \oplus A) \oplus A$ so that $C^n(P) = 0$ and yet P does not split off a free summand of rank one.*

We may extend this example to a similar affine open subvariety of any even dimensional projective space.

Remark 4.3.5. *Let $X = Spec(A)$ be an affine open subvariety of the projective 2k-space $\mathbb{P}^{2k}(\mathbb{R})$ which is the complement of $V(x_0^2 + x_1^2 + \ldots + x_{2k}^2)$. The results of Example 4.3.4 are valid in this case as well. There exists P, a projective A-module of rank 2k whose top Chern class vanishes, yet does not split off a free summand.*

And so we see that over non-algebraically closed fields, the top chern class is not a sufficient indicator of the presence of a unimodular element. We continue our discussion in the next chapter by following the work of Bhatwadekar and Sridharan in formalizing the algebraic structure for the Euler Class group.

Chapter 5

Euler Class Groups

5.1 Euler Class Group

S. M. Bhatwadekar and Raja Sridharan formalized the definitions of both the Euler Class Group and the Weak Euler Class Group as follows in ([BRS2]). For the following definitions, let A be a Noetherian ring with $dim(A) = n \geq 2$, P be a projective A-module of rank n and let L be a rank one projective A-module isomorphic to $\wedge^n(P)$.

Definition 5.1.1. *Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let α and β be two surjections from $L/JL \oplus (A/J)^{n-1}$ to J/J^2. We say that α and β are related if there exists an automorphism σ of $L/JL \oplus (A/J)^{n-1}$ of determinant 1 such that $\alpha\sigma = \beta$. If we define $F = L \oplus A^{n-1}$:*

$$\begin{tikzcd} F/JF \arrow[r, two heads, "\beta"] \arrow[d, "\sigma", "\wr"'] & J/J^2 \\ F/JF \arrow[ur, two heads, "\alpha"'] & \end{tikzcd}$$

This defines an equivalence relation on the set of surjections from $L/JL \oplus (A/J)^{n-1}$ to J/J^2. We will let $[\alpha]$ denote the equivalence class of α, and we will call $[\alpha]$ a local L-orientation of J.

Definition 5.1.2. *A local L-orientation $[\alpha]$ of J is called a global L-orientation of J if α can be lifted to a surjection from $L \oplus A^{n-1}$ to J.*

$$\begin{tikzcd}
L \oplus A^{n-1} \arrow[r, two heads] \arrow[d] & J \arrow[d] \\
L \oplus A^{n-1}/J(L \oplus A^{n-1}) \arrow[r, two heads] & J/J^2
\end{tikzcd}$$

Here we note that since $dim(A/J) = 0$, if any surjection α from $L/JL \oplus (A/J)^{n-1}$ to J/J^2 can be lifted, so may any β equivalent to α under the equivalence relation defined above. Also, by abuse of notation, we will simply use α to refer to $[\alpha]$, the equivalence class of α.

Definition 5.1.3. *Let $\mathfrak{m}$ be a maximal ideal of height n, and N be an $\mathfrak{m}$-primary ideal. Let ω_N be a local L-orientation of N. Then let G be the abelian group generated by all the possible pairs of primary ideals of height n and all corresponding local L-orientations, (N, ω_N).*

Given a pair (J, ω_J) where J has height n and ω_J an L-orientation, we associate (J, ω_J) in G as follows:

Let

$$J = N_1 \cap N_2 \cap \cdots \cap N_k$$

be an irredundant primary decomposition with each N_i as an $\mathfrak{m}_i$-primary ideal, and let ω_J be a local L-orientation of J. Now ω_J induces local L-orientations ω_{N_i} of the N_i for each i. We will denote

$$(J, \omega_J) = \sum_{i=1}^{k} (N_i, \omega_{N_i})$$

.

Now let H be the subgroup of G generated by the pairs (J, ω_J) where J has height n and ω_J is a global L-orientation of J.

We define the Euler Class group of A with respect to L as the quotient G / H, and we denote it by E(A, L). Specifically, we denote the Euler Class group of A with respect to A as E(A) rather than E(A,A).

If P is a projective A-module of rank n, we know from a theorem of Eisenbud-Evans that for most linear maps $\alpha : P \to A$, the image of P in A, call it J, has the property that $height(J) = n = dim(A)$. We also know, from a theorem of Sridharan, in such a situation if $\mu(J) = n$ and the minimal generators are lifts of the generators of J/J^2 which come from α and a generator of $\wedge^n(P)$, then P splits off a free direct summand of rank 1. It is precisely this obstruction which is detected by the vanishing of the euler class of a projective module.

We now define the Euler class of P, a projective module of top rank with $L = \wedge^n P$.

Definition 5.1.4. *Let P be a projective A-module of rank n with determinant L, and let χ be an isomorphism from $\wedge^n(L \oplus A^{n-1})$ to $\wedge^n P$. We call χ an $L-orientation of P$. We associate the pair (P, χ) to an element of E(A, L), $e(P, \chi)$, by the following method:*

Take a generic surjection, $\lambda : P \twoheadrightarrow J_0$, and let bar denote reduction modulo J_0. There is an induced surjection $\overline{\lambda} : P/J_0P \twoheadrightarrow J_0/J_0^2$. We now choose an isomorphism $\overline{\gamma} : L/J_0L \oplus (A/J_0)^{n-1} \twoheadrightarrow P/J_0P$ so that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Now let

$$\omega_{J_0} = \overline{\lambda}\overline{\gamma} : L/J_0L \oplus (A/J_0)^{n-1} \twoheadrightarrow J_0/J_0^2.$$

Note that ω_{J_0} is a local L-orientation of J_0. To the pair (P,χ) we will associate $e(P,\chi)$, defined to be $e(P,\chi) = (J_0, \omega_{J_0}) \in E(A,L)$.

$$\begin{tikzcd}
P \arrow[r, two heads, "\lambda"] \arrow[d, two heads] & J_0 \arrow[d, two heads] \\
P/J_0P \arrow[r, two heads, "\overline{\lambda}"] & J_0/J_0^2 \\
L/J_0L \oplus (A/J_0)^{n-1} \arrow[u, "\overline{\gamma}", "\wr"'] \arrow[ur, two heads, "\omega_{J_0}"] &
\end{tikzcd}$$

It remains to be seen that $e(P,\chi)$ is well defined, but we need the following before we can establish that fact.

Lemma 5.1.5 (Bhatwadekar-Sridharan, [BRS1]). *Let $\lambda : P \twoheadrightarrow J_0$ and $\mu : P \twoheadrightarrow J_1$ be surjections, where J_0 and J_1 are height n ideals of A. Then there exists an ideal I of $A[T]$ with height n along with a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ such that $I(0) = J_0$ and $I(1) = J_1$, $\alpha(0) = \lambda$ and $\alpha(1) = \mu$.*

The proof of this lemma is fairly straightforward. After tensoring λ and μ with $A[T]$, we construct α by taking $T\lambda(T) + (1-T)\mu(T)$ and adjusting it by an appropriate element from $P[T]^*$ to get I to have height n.

Proposition 5.1.6 (Bhatwadekar-Sridharan, [BRS1]). *Let $(n-1)!$ be invertible in A, and let $F = L \oplus A^{n-1}$ and $\chi : \wedge^n(F) \to \wedge^n(P)$ be an isomorphism. Suppose that $\alpha(T) : P[T] \twoheadrightarrow I$ is a surjection, where I is a height n ideal of A[T]. Then there exists a homomorphism $\phi : F \to P$, a height n ideal K of A which is comaximal with $I \cap A$ and a surjection $\rho(T) : F[T] \twoheadrightarrow I \cap KA[T]$ such that:*

1. *$\wedge^n(\phi) = u\chi$ where $u = 1$ modulo $I \cap A$*
2. *$(\alpha(0)\phi)(F) = I(0) \cap K$*

3. $\alpha(T)\phi(T)\otimes A[T]/I = \rho(T)\otimes A[T]/I$

4. $\rho(0)\otimes A/K = \rho(1)\otimes A/K$

So, to show that $e(P,\chi)$ is well defined, take another generic surjection $\mu : P \twoheadrightarrow J_1$. By the previous lemma, there exists a surjection $\alpha[T] : P[T] \twoheadrightarrow I$ where I is a height n ideal of A[T], $\alpha(0) = \lambda$ and $\alpha(1) = \mu$. There are then induced local orientations $\omega(0)$ and $\omega(1)$ of J_0 and J_1 respectively. And the proposition essentially tells us that there exists an ideal K of A with height n along with a local orientation ω_K so that $(J_0,\omega(0))+(K,\omega_K) = 0 = (J_1,\omega(1))+(K,\omega_K)$ in $E(A,L)$, and thus $(J_0,\omega(0)) = (J_1,\omega(1))$. As a result, we see that $e(P,\chi)$ is indeed well defined under the added condition that $(n-1)!$ is invertible in A.

It is worth noting here that the invertibility of $(n-1)!$ in A is a requirement only for the well-definedness of the association which takes a pair, a projective module of top rank and an isomorphism between its determinant and L, and associates it to an element of the Euler Class group. As we will see later, this assumption may be dropped when proving that the Euler Class group is identically zero.

The following example shows that $E(A,L)$ varies with L.

Example 5.1.7. *Let X = spec(A) be an open affine subariety of projective 2-space* $\mathbb{P}^2(\mathbb{R})$ *which is the complement of* $V(X^2+Y^2+Z^2)$*. Then* $E(A) = \mathbb{Z}/2$*. However, for the case when L is the canonical module of A over* $\mathbb{R}$*, then* $E(A,L) = \mathbb{Z}$*.*

By the definition of the Euler class group, we know that for $(J,\omega_J) \in E(A,L)$, if ω_J is a global orientation of J then (J,ω_J) is zero in $E(A,L)$. The converse, though non-trivial, is also true.

Theorem 5.1.8 (Bhatwadekar & R. Sridharan [BRS2])**.** *Let A be a commutative noetherian ring with dimension* $n \geq 2$*. Let L be a rank 1 projective A-module. Let* $J \subset A$

be an ideal with height n such that J/J^2 is generated by n elements, and let ω_J : $L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ be a local L-orientation of J. Suppose that the image of (J, ω_J) is zero in E(A,L). Then ω_J is a global L-orientation of J.

The previous theorem by S. Bhatwadekar & R. Sridharan settles the original conjecture from Nori, though it is clearer if we look at the following corollaries.

Corollary 5.1.9 (Bhatwadekar & R. Sridharan [BRS2])**.** *Let A be a ring of dimension $n \geq 2$. Let P be a projective A-module of rank n with determinant L and χ be an L-orientation of P. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let ω_J be a local L-orientation of J. Suppose that $e(P, \chi) = (J, \omega_J)$ in E(A,L). Then there exists a surjection $\alpha : P \twoheadrightarrow J$ such that (J, ω_J) is obtained from (α, χ).*

Corollary 5.1.10 (Bhatwadekar & R. Sridharan [BRS2])**.** *Let A be a ring of dimension $n \geq 2$. Let P be a projective A-module of rank n with determinant L and χ be an L-orientation of P. Then $e(P, \chi) = 0$ if and only if P has a unimodular element. In particular, if the determinant of P is trivial and P has a unimodular element, then P maps onto any ideal of height n generated by n elements.*

5.2 The Real n-Sphere

As promised, we will now use the methods of the Euler Class group to prove the absence of a unimodular element in the tangent bundle for any even-dimensional real sphere.

Example 5.2.1. *Let A_n be the coordinate ring of the n-dimensional real sphere. $A_n = \mathbb{R}[X_0, \ldots, X_n]/(\sum_{i=0}^{n} X_i^2 - 1)$, and let x_i denote the image of X_i in A_n. Then, let T be the projective module corresponding to the tangent bundle of the n-dimensional real sphere.*

$$0 \longrightarrow T \longrightarrow A_n^{n+1} \xrightarrow{\alpha} A_n \longrightarrow 0$$

where the map $\alpha : A_n^{n+1} \to A_n$ is given by $(x_0, \ldots, x_n)$. Then if n is even, T does not contain a unimodular element.

Proof: Let $\mathbb{R}(S^n) = S^{-1}A_n$ where S is the multiplicative set of all $f \in A_n$ that do not pass through any real point of $Spec(A_n)$. Now we have the following:

1. Since all line bundles are trivial, we have only one Euler class group $E(A_n, A_n)$ to be denoted by $E(A_n)$. Similarly, we have only one weak Euler class group $E_0(A_n)$.

2. By [BRS3, Theorem 4.13] (or see [BDM]),

$$E(\mathbb{R}(S^n)) = \mathbb{Z}$$

3. Also, by [BRS3, Theorem 4.10], we have

$$E_0(\mathbb{R}(S^n)) = \mathbb{Z}/(2).$$

4. We have by [BRS3, Theorem 5.5] or [BDM, Theorem 2.10], $E_0(A_n) \xrightarrow{\sim} CH_0(A_n)$.

5. From [BDM, Theorem 4.29], we have that φ is an isomorphism in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & E^{\mathbb{C}}(A_n) \arrow[r] \arrow[d, "\varphi", "\wr"'] & E(A_n) \arrow[r] \arrow[d, "\Theta"] & E(\mathbb{R}(S^n)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & CH(\mathbb{C}) \arrow[r] & CH_0(A_n) \arrow[r] & CH_0(\mathbb{R}(S^n)) \arrow[r] & 0
\end{tikzcd}$$

   where $CH(\mathbb{C})$ is the kernel of the map between the Chow groups, and likewise $E^{\mathbb{C}}(A_n)$ is the kernel of the map between the Euler Class groups. We note here that $E^{\mathbb{C}}(A_n)$ is generated by complex points.

6. Since complex points in A_n are complete intersections (see [MS, Lemma 4.2]), $CH(\mathbb{C}) = E^{\mathbb{C}}(A_n) = 0$ and so the above diagram reduces to

$$\begin{tikzcd}
E(A_n) \arrow[r, "\sim"] \arrow[d, "\Theta"] & E(\mathbb{R}(S^n)) = \mathbb{Z} \arrow[d] \\
CH_0(A_n) \arrow[r, "\sim"] & CH_0(\mathbb{R}(S^n)) = \mathbb{Z}/(2).
\end{tikzcd}$$

7. Let $m_0 = (x_0 - 1, x_1, \ldots, x_n), m_1 = (x_0 + 1, x_1, \ldots, x_n) \in Spec(A_n)$ then m_0 corresponds to the real point $(1, 0, \ldots, 0) \in S^n$ and m_1 corresponds to the real point $(-1, 0, \ldots, 0) \in S^n$.

8. We also have $m_0 \cap m_1 = (x_1, \ldots, x_n), \quad m_0 = (x_1, \ldots, x_n) + m_0^2, \quad m_1 = (x_1, \ldots, x_n) + m_1^2$.

9. Write $F = A_n^n$ and $J = m_0 \cap m_1 = (x_1, \ldots, x_n)$. Let

$$\alpha : F \twoheadrightarrow J \quad where\ for \quad i \geq 1 \qquad \alpha(e_i) = x_i.$$

10. Then, α induces the local orientation;

$$\omega : F/JF \to J/J^2 \qquad where \qquad \omega(e_i) = image(x_i)$$

and for $j = 0, 1$ let

$$\omega_j : F/m_jF \to m_j/m_j^2 \qquad where \qquad \omega_j(e_i) = image(x_i).$$

11. We will consider $(m_0, \omega_0) = 1$ as the generator of $E(A_n) = \mathbb{Z}$.

12. Since (J, ω) is global, it follows

$$(m_0, \omega_0) + (m_1, \omega_1) = (J, \omega) = 0.$$

Hence

$$(m_1, \omega_1) = -(m_0, \omega_0) = -1.$$

13. Since $E(A_n) = E(S^n)$, we can apply [BDM, Lemma 4.2] and we have

$$(m_1, \omega_1) + (m_1, -\omega_1) = 0$$

therefore

$$(m_0, \omega_0) + (m_1, -\omega_1) = 2(m_0, \omega_0) = 2.$$

14. Let D be the diagonal map $D = (-x_0, 1, \dots, 1) : F/JF \to F/JF$, then D is an automorphism and $det(D) = image(-x_0)$.

15. Now, let $\eta = \omega D : F/JF \to J/J^2$. In fact,

$$\eta(e_1) = image(-x_0 x_1) \qquad and \qquad \eta(e_i) = image(x_i) \quad \forall i > 1.$$

16. Note, $D = (-1, 0, \ldots, 0) \quad mod \quad m_0$ and $D = (+1, 0, \ldots, 0) \quad mod \quad m_1$

17. Since, ω_i are reduction of ω modulo m_i we have

$$(J, \eta) = (m_0, -\omega_0) + (m_1, \omega_1) = -[(m_0, \omega_0) + (m_1, -\omega_1)] = -2.$$

18. Now we appply [BRS2, Lemma 5.1], to $\alpha : F \twoheadrightarrow J$, with $a = b = image(-x_0)$. We have the following:

    (a) Define T by the exact sequence

    $$0 \to T \to A_n \oplus F = A^{n+1} \stackrel{\Phi}{\to} A_n \to 0$$

    where

    $$\Phi = -(x_0, x_1, \ldots, x_n) = (b, -\alpha).$$

    (b) We have (J, ω) is obtained from $(\alpha, \chi_0 = Id_{A_n})$.

    (c) By [BRS2, Lemma 5.1], T has an orientation $\chi : A_n \to \wedge^n T$ such that

    $$e(T, \chi) = (J, image(-x_0)^{n-1}\omega) = (m_0, (-1)^{n-1}\omega_0) + (m_1, \omega_1).$$

    (d) If n is **EVEN**, we have

    $$e(T, \chi) = (m_0, -\omega_0) + (m_1, \omega_1) = -2.$$

And if, n is **ODD**, we have

$$e(T,\chi) = (m_0,\omega_0) + (m_1,\omega_1) = 0.$$

(e) Note that $T = kernel(\Phi) \equiv kernel(-\Phi)$ is the tangent bundle. Therefore, if n is **EVEN**, the tangent bundle does not have a unimodular element.

19. If n is **EVEN**, the tangent bundle T over S^n does not have a unimodular element.

5.3 Weak Euler Class Group

We also are interested in a particular quotient of the Euler Class group, $E_0(A,L)$, the weak Euler Class group, the details of which we set forth now. We will again be letting A be a noetherian ring with $dim(A) = n \geq 2$, P be a projective A-module of rank n and L be a rank one projective A-module isomorphic to $\wedge^n(P)$

Definition 5.3.1. *Let G be the free abelian group generated by the set of ideals $\mathscr{I}$ with the following properties:*

1. *$\mathscr{I}/\mathscr{I}^2$ is generated by n elements.*
2. *$\mathscr{I}$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ with height n.*

For any ideal J of A for which J/J^2 is generated by n elements, take

$$J = N_1 \cap N_2 \cap \cdots \cap N_k,$$

an irredundant primary decomposition of J, with each N_i being $\mathfrak{m}_i$-primary and each $\mathfrak{m}_i$ a distinct maximal ideal of height n. Then we denote the element

$$(J) = \sum_{i=1}^{k} (N_i)$$

from G.

Now let H be the subgroup of G generated by the elements (J) where J is an ideal of A with height n such that there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$.

We now define the weak Euler Class group of A with respect to L as $E_0(A,L) = G/H$; and by an abuse of notation, (J) will now represent the image of (J) in $E_0(A,L)$ instead of $(J) \in G$.

Now we need to define an association between projective modules of top rank and elements of the weak Euler class group.

Definition 5.3.2. *Take a generic surjection, $\lambda : P \twoheadrightarrow J_0$, P_0 a height n ideal of A. We define $e_0(P) = (J_0)$ in $E_0(A,L)$ as the weak Euler Class of P.*

Once again, we need to show that $e_0(P)$ is well defined. If we take a second generic surjection $\mu : P \twoheadrightarrow J_1$, then by 5.1.5 there exists a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ with $\alpha(0) = \lambda$ and $\alpha(1) = \mu$. Then by 5.1.6 we have $(J_0) = (J_1)$ in $E_0(A,L)$ and thus we see that $e_0(P)$ is indeed well defined.

We note here that there is a canonical surjective homomorphism $E(A,L) \twoheadrightarrow E_0(A,L)$ which simply drops the orientations. Again, along the same lines as the Euler class group, if $L = A$, we write $E_0(A)$ instead of $E_0(A,A)$. Also along the same lines, we note that it is necessary for $(n-1)!$ to be invertible in A for $e_0(P)$ to be well defined, yet we may drop this assumption when proving that the weak Euler class is identically zero.

However, a fundamental difference between $E(A,L)$ and $E_0(A,L)$ is established by the following theorem. We saw in Example 5.1.7 that $E(A,L)$ varies with L, but we will see that $E_0(A,L)$ does not. But before we give the theorem, a few lemmas are necessary.

Lemma 5.3.3. *Let A be a noetherian ring of dimension 2 and $J \subset A$ be an ideal of height 2 such that $J = (f,g) + J^2$. Let L be a projective A-module of rank one. Then, there exists a projective A-module P of rank 2 having determinant isomorphic to L and a surjection from P to J.*

Proposition 5.3.4. *Let A be a noetherian ring of dimension n and P, P_1 projective A-modules of rank n such that $[P] = [P_1]$ in $K_0(A)$. Then, there exists an ideal $J \subset A$ of height $\geq n$ such that J is a surjective image of both P and P_1.*

Theorem 5.3.5. *The groups $E_0(A,L)$ and $E_0(A)$ are canonically isomorphic.*

Proof: We show that the map $\alpha : E_0(A) \to E_0(A,L)$, sending (J) in $E_0(A)$ to $E_0(A,L)$, is well defined; and the same for the map $\beta : E_0(A,L) \to E_0(A)$, which sends (J) in $E_0(A,L)$ to (J) in $E_0(A,L)$ will not be shown, but follows in a similar fashion to α. It then follows immediately that the two maps are isomorphisms and are inverses of each other.

Let $J \subset A$ be an ideal of height n and generated by n elements. This means that $(J) = 0$ in $E_0(A)$, and we show then that $(J) = 0$ in $E_0(A,L)$. Let $J = (a_1, \ldots, a_n)$, and without loss of generality, we may assume that the ideal $J_1 = (a_3, \ldots, a_n)$ is a height $n-2$ ideal of A. We will denote reduction modulo J_1 by an overline, $\overline{A} = A/J_1$.

The ring $\overline{A}$ has dimension 2. Now, since $\overline{J}$ is generated by two elements, specifically $\overline{a_1}$ and $\overline{a_2}$, it follows from Lemma 5.3.3 that there exists a projective $\overline{A}$-module P_0 of rank 2 with determinant $\overline{L}$ and a surjection from P_0 to $\overline{J}$. Since $\overline{J}$ is generated by two elements, $[P_0] = [\overline{L} \oplus \overline{A}]$ in $K_0(\overline{A})$. Then by the previous proposition, there exists an ideal J_0 of A which contains J_1 such that $\overline{J_0}$ has height ≥ 2 and is also the surjective image of both P_0 and $\overline{L} \oplus \overline{A}$. If $\overline{J_0} = \overline{A}$ then P_0 is isomorphic to $\overline{L} \oplus \overline{A}$ and since $\overline{J}$ is a surjective image of P_0, we have that J is also then a surjective image of $L \oplus A^{n-1}$and therefore $(J) = 0$ in $E_0(A,L)$. As such, we may continue under the assumption that $\overline{J_0}$ has height 2. Since there are surjections from P_0 to both $\overline{J}$ and $\overline{J_0}$, 5.1.5 implies the existence of an ideal $I \subset A[T]$ containing $J_1A[T]$ and also a surjection from $P_0[T]$ to $\overline{I}$ where the height of $\overline{I}$ is 2 and $\overline{I(0)} = \overline{J}$ and $\overline{I(1)} = \overline{J_0}$. Then by applying 5.1.6 we see that there exists an ideal $K \subset A$, containing J_1 and also a surjection from $\overline{L[T]} \oplus \overline{A[T]}$ to $\overline{I} \cap \overline{KA[T]}$ where:

1. $\overline{K} \subset \overline{A}$ is an ideal of height ≥ 2 such that $\overline{K}/\overline{K}^2$ is generated by two elements.

2. $\overline{I} + \overline{KA[T]} = \overline{A[T]}$. Thus, $I \cap KA[T]$ is a surjective image of $L[T] \oplus A[T]^{n-1}$.

By restricting to $T = 0$ and $T = 1$, we see that the ideals $J \cap K$ and $J_0 \cap K$ are both surjective images of $L \oplus A^{n-1}$. As a result, $(J) = (J_0)$ in $E_0(A,L)$. Not only that, but since $\overline{J_0}$ is a surjective image of $\overline{L} \oplus \overline{A}$, it follows that J_0 is a surjective image of $L \oplus A^{n-1}$. Therefore $(J) = 0 = (J_0)$ in $E_0(A,L)$ and so we see that α is well defined. The proof that β is well defined is similar, and as mentioned earlier, with α and β both well defined, it immediately follows that they are isomorphisms and inverses of each other, yielding the result that $E_0(A)$ and $E_0(A,L)$ are isomorphic.

With this in mind, we may redefine the association between a projective A-module and an element of the weak Euler class group. For a commutative noetherian ring of dimension n, A, and P a rank n projective A-module with determinant L, we still take a generic surjection $P \twoheadrightarrow J$. The association we defined earlier, $e_0(P)$, which links P to (J) in $E_0(A,L)$ is well defined, as is the isomorphism α, which takes (J) in $E_0(A,L)$ to (J) in $E_0(A)$, so we may combine the two, P and $\alpha(J)$ so that the association between these is now an association of a projective A-module and an element of $E_0(A)$ instead of $E_0(A,L)$. By an abuse of notation we will now mean $(J) = \alpha(J) \in E_0(A)$ when we write $e_0(P)$, rather than $(J) \in E_0(A,L)$ as before.

Remark 5.3.6. *Let A be a ring and L a rank one projective A-module. There are natural maps:*

$$E(A,L) \to E_0(A,L) = E_0(A) \twoheadrightarrow CH_0(A)$$

Consider the case when A is a Cohen-Macaulay ring of dimension n, and $J \subset A$ an ideal generated by an A-regular sequence of length n. Let $J = \cap_{i=1}^{k} N_i$ be an irredundant primary decomposition, where each N_i is an $\mathfrak{m}_i$-primary ideal. Then the natural map from $E_0(A)$ to $CH_0(A)$ sends (J) to $\sum_{i=1}^{k} l_{A_{\mathfrak{m}_i}}((A/N_i)_{\mathfrak{m}_i})\mathfrak{m}_i$

Moreover, the surjection from $E_0(A)$ to $CH_0(A)$ is an isomorphism in certain cases. In [Mu], we see from Corollary 3.4 that this map is an isomorphism whenever the base field is algebraically closed. Then Bhatwadekar and Sridharan, in [BRS3], show that the natural map is an isomorphism when the base field is $\mathbb{R}$ and $spec(A)$ is smooth.

Chapter 6

Vanishing

6.1 Implications of the Vanishing of the Euler Class Group

Now that we've defined the notion of the Euler class of projective modules, we investigate the link between the Euler class of a projective module and the question on the existence of a unimodular element in that projective module.

In a similar fashion to Murthy's theorems regarding the vanishing of the top Chern class corresponding to the presence of a unimodular element, we see that the vanishing of the Euler Class of a projective module indicates a unimodular element in a much broader context. This result was already stated in the chapter on the Euler Class group, in Corollary 5.1.10

Because of this result from the vanishing of the Euler class of a projective module, we are interested in cases where the entire Euler Class group vanishes. Murthy proved the following result for $F^nK_0(A)$ in [Mu]

Theorem 6.1.1. *Let A be a reduced affine k-algebra of dimension n over a field k such that k is either algebraically closed or $k = \mathbb{R}$ and the closure of $\mathbb{R}$-rational points of Spec A has dimension $\leq n-1$. Then $F^nK_0(A) = 0$ if and only if every projective A-module of rank n has a free direct summand of rank one.*

Because of Bhatwadekar and Sridharan's theorem on the vanishing of the Euler class of a projective module, we have the following results if the Euler Class group of a noetherian ring is identically zero.

Remark 6.1.2. *For the following items, let A be a ring with dimension $n \geq 2$.*

1. *Let L be a projective A-module of rank one such that $E(A,L) = 0$, then any projective A-module, P, with rank n and determinant L will split off a free summand of rank one.*

2. *If $E(A) = 0$, then every height n ideal, I, for which I/I^2 is n-generated, is a complete intersection.*

3. *If $E(A) = 0$, then for any ideal, I, for which I/I^2 is n-generated, i.e. $I/I^2 = (f_1, \ldots, f_n)$, the f_i lift to generators of I.*

4. *Let L be a projective A-module of rank one, so that $E(A,L) = 0$, then any surjection $L \oplus A^{n-1}/I(L \oplus A^{n-1}) \twoheadrightarrow I/I^2$ where I is a height n ideal of A such that I/I^2 is n-generated, lifts to a surjection $L \oplus A^{n-1} \twoheadrightarrow I$.*

Appealing to Quillen and Suslin's resolution of Serre's conjecture, Theorem 2.2.4 we have the following example:

Example 6.1.3. *Let A be a principal ideal domain, and let $R = A[X_1, \ldots, X_n]$. $ThenE(R) = 0$. Note that all rank one projective R-modules are free, so $E(R)$ is the only Euler Class group for this ring.*

Next, we give some preliminaries that lead to some results on the vanishing of the Euler Class group.

6.2 Preliminaries

First, we will recall the following patching lemma of Quillen ([Q]).

Lemma 6.2.1 (Quillen, [Q]). *Let A be a commutative ring and R be an $A-$algebra. Suppose $f \in A$ and θ be an unit in $1+TR_f[T]$, where $R[T]$ is the polynomial ring over R in the variable T. Then there is an integer k, such that for $g_1, g_2 \in A$, whenever $g_1 - g_2 \in f^kA$, there is a unit ψ in $1+fTR[T]$ such that $\psi_f(T) = \theta(g_1T)\theta(g_2T)^{-1}$.*

The following is a version of the patching lemma of Plumstead ([P]).

Lemma 6.2.2 (Plumstead, [P]). *Let A be a a commutative noetherian ring and M be an $A-$module. Suppose $As + At = 1$. Let $\alpha : M_{st} \to M_{st}$ be an isomorphism that is isotopic to the identity. Then we can find isomorphisms $\eta_1 : M_t \to M_t$ and $\eta_2 : M_s \to M_s$ such that*

- $\alpha = (\eta_2)_t(\eta_1)_s$,
- $\eta_1 \equiv Id \quad (modulo\ s)$
- $\eta_2 \equiv Id \quad (modulo\ t)$

Proof. Since α is isotopic to identity there is an isomorphism

$$\theta(T) : M_{st}[T] \xrightarrow{\sim} M_{st}[T]$$

such that $\theta(0) = Id$ and $\theta(1) = \alpha$.

Write $R = End(M_t)$. So, R is an $A-$algebra. So, θ is an unit in $1+TR_s[T]$. By 6.2.1, there is an integer $k \geq 0$ such that for $g_1, g_2 \in A$, whenever $g_1 - g_2 \in s^kA$, there is a unit ψ in $1+sTEnd(M_t)[T]$ such that $\theta(g_1T)\theta(g_2T)^{-1} = \psi_s(T)$.

Again taking $R' = End(M_s)$, we consider θ as an element in $1 + TR'_t[T]$. So, there is an integer $k \geq 0$ such that for $g_1, g_2 \in A$, whenever $g_1 - g_2 \in t^k A$, there is a unit ψ in $1 + tTEnd(M_s)[T]$ such that $\theta(g_1 T)\theta(g_2 T)^{-1} = \psi_t(T)$.

Now we can take the same integer k for both the statements above. We can write $1 = \lambda s^k + \mu t^k$. Now

$$\theta(T) = [(\theta(T)\theta(\lambda s^k T)^{-1}][\theta(\lambda s^k T)\theta(0)^{-1}].$$

Since $1 - \lambda s^k \in t^k A$, there is an unit ψ_2 in $1 + tTEnd(M_s)[T]$ such that $(\psi_2)_t(T) = \theta(T)\theta(\lambda s^k T)^{-1}$.

Similarly, since $\lambda s^k - 0 \in s^k A$ there is an unit $\psi_1 \in 1 + sTEnd(M_t)[T]$ such that $(\psi_1)_s(T) = \theta(\lambda s^k T)^{-1}\theta(0)^{-1}$. Therefore,

$$\theta(T) = (\psi_2)_t(T)(\psi_1)_s(T).$$

Substituting $T = 1$ we have $\alpha = (\eta_2)_t(\eta_1)_s$, where $\eta_i = \psi_i(1)$, for $i = 1, 2$. This completes the proof of this lemma.

The referee suggested the following version of Quillen's argument ([Q]) regarding extendibility of modules.

Proposition 6.2.3. *Let A be a noetherian commutative ring and $R = A[X]$ be the polynomial ring. Suppose N is a finitely generated $R-$module and*

$$Q = \{s \in A : N_s \approx E \otimes R_s \quad where \quad E \quad is \quad A_s - projective\}.$$

Then Q is an ideal.

Proof. Clearly, $0 \in Q$ and $as \in Q$ for all $a \in A$ and $s \in Q$. So, we need to prove that $s,t \in Q \Longrightarrow s+t \in Q$. Assume $s,t \in Q$. We will prove $s+t \in Q$. By replacing A by A_{s+t} we may assume $s+t = 1$. Write $M_1 = E_1 \otimes R_s$ and $M_2 = E_2 \otimes R_t$ where E_1 is a projective R_s-module and E_2 is a projective R_t-module. Let

$$f_1 : M_1 \xrightarrow{\sim} N_s \quad \textit{and} \quad f_2 : M_2 \xrightarrow{\sim} N_t$$

be two isomorphisms and

$$\Theta = (f_2^{-1})_s (f_1)_t : (M_1)_t \xrightarrow{\sim} (M_2)_s.$$

Let "overline" denote "*modulo X*". Then $\overline{\Theta} : (E_1)_t \xrightarrow{\sim} (E_2)_s$ is an isomorphism. Consider the following two fibre product diagrams:

$$\begin{tikzcd}
E_0 \arrow[rr, "q_2"] \arrow[d, "q_1"] & & E_2 \arrow[d] \\
E_1 \arrow[r] & (E_1)_t \arrow[r, "\overline{\Theta}"] & (E_2)_s,
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[rr, "p_2"] \arrow[d, "p_1"] & & M_2 \arrow[d] \\
M_1 \arrow[r] & (M_1)_t \arrow[r, "\Theta"] & (M_2)_s.
\end{tikzcd}$$

Clearly, E_0 is a projective $A-$module and E is a projective $R-$module. Now use standard arguments to show $E_0 \otimes R \approx E \approx N$ (see, for example, [Ma1]). This completes the proof.

6.3 Results

The following is the main theorem from [MP] on vanishing of Euler class group of polynomial rings.

Theorem 6.3.1 (Mandal-Parker)**.** *Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and $B = A[X, 1/f]$, where $f \in R$ is a non-zero divisor. Assume $dimB = dimA + 1 \geq 3$. Let $\mathscr{L}$ be rank one projective $B-$module. Then $E(B, \mathscr{L}) = 0$ and $E_0(B, \mathscr{L}) = 0$.*

Proof. Since there is a surjective map $E(B, \mathscr{L}) \to E_0(B, \mathscr{L})$, we will only prove $E(B, \mathscr{L}) = 0$. We also assume that A is reduced ([BRS2, Corollary 4.6]) and A has no non-trivial idempotent.

We will consider $\mathscr{L}$ as an invertible ideal and let $L = R \cap \mathscr{L}$. Write $n = \dim B$. We will write $\mathscr{F} = B^{n-1} \oplus \mathscr{L}$ and $F = R^{n-1} \oplus L$. Let $\mathscr{I}$ be a primary ideal of B with $height(\mathscr{I}) = n$ and let $\omega : \mathscr{F}/\mathscr{I}F \to \mathscr{I}/\mathscr{I}^2$ be a local $\mathscr{L}-$orientation of $\mathscr{I}$. We will prove that $(\mathscr{I}, \omega) = 0$ in $E(B, \mathscr{I})$.

Let

$$Q = \{s \in A : L_s \approx E \otimes R_s \quad where \quad E \quad is \quad A_s - projective\}.$$

By Proposition 6.2.3, Q is an ideal of A. Since L is an ideal and A is reduced, we have $height(Q) \geq 1$.

Let $I = \mathscr{I} \cap R$. Note that I is a primary ideal of R with $height(I) = n$. So, $m_0 = \sqrt{I}$ is a maximal ideal of height n. Write

$$\mathscr{P} = \{\wp \in Spec(R) : I \not\subseteq \wp \; and \; height(\wp) < n \;\; or \;\; Q \subseteq \wp)\}.$$

So,

$$\mathscr{P} = \{\wp \in Spec(R) : m_0 \neq \wp \; and \; height(\wp) < n \;\; or \;\; Q \subseteq \wp)\}.$$

Note that there is a generalized dimension function $d : \mathscr{P} \to \{0,1,2,\ldots\}$ such that $d(\wp) \leq n-1$ for all $\wp \in \mathscr{P}$.

Let $\gamma : \mathscr{F} \to \mathscr{I}$ be any lift of ω.

Let $\beta_0 : F \to I$ be such that $\gamma = \beta_0/f^{2k}$. Since $(\mathscr{I},\omega) = f^{2k}(\mathscr{I},\omega)$ ([BRS2, Lemma 5.4]), replacing ω by $f^{2k}\omega$, we can assume that $\gamma = \beta_0/1$.

Note that $Hom(F,R)_{\wp} = I^2 Hom(F,R)_{\wp}$ for all $\wp \in \mathscr{P}$.

So, $\beta = \beta_0 + \beta_1$ is basic in $Hom(F,R)$ on $\mathscr{P}$ for some $\beta_1 \in I^2 Hom(F,R)$. Consider the following commutative diagram

$$
\begin{tikzcd}
F \arrow[r] \arrow[d, "\beta"'] & F/IF \arrow[r, "\sim"] \arrow[d] & \mathscr{F}/\mathscr{I}F \arrow[d, "\omega"] \\
I \arrow[r] & I/I^2 \arrow[r, "\sim"] & \mathscr{I}/\mathscr{I}^2.
\end{tikzcd}
$$

Therefore, β_f is a lift of ω and also $I = image(\beta) + I^2$. So, $image(\beta) = I \cap K$ for some ideal K of R with $I + K = R$, $height(K_f) \geq n$ and $QR_f + K_f = R_f$.

Let ω' be the local $\mathscr{L}-$orientation on K_f induced by β_f. Therefore,

$$(\mathscr{I},\omega) + (K_f,\omega') = 0.$$

So, we will prove $(K_f,\omega') = 0$. Let

$$K = K_1 \cap K_2 \cap \cdots \cap K_r \cap K_{r+1} \cap \cdots \cap K_t$$

be a irredundant primary decomposition of K, where $f \notin \sqrt{K_i}$ for $i = 1,\ldots,r$ and $f \in \sqrt{K_i}$ for $i = r+1,\ldots,t$. Also note $height(K_i) = n$ for $i = 1,\ldots,r$. We have

$$(K_f,\omega') = \sum_{i=1}^{r} ((K_i)_f,\omega_i)$$

where ω_i is induced by ω'. So, we will prove $((K_i)_f, \omega_i) = 0$ for $i = 1, \ldots, r$.

Replacing (I, ω) by $((K_i)_f, \omega_i)$, we can additionally assume that $QR_f + I_f = R_f$. Since $I + Rf = R$, in fact, we have

$$QR + I = R.$$

The diagram above remains valid. By abuse of notation, the map $F/IF \to I/I^2$ will also be denoted by ω.

Write $J = I \cap A$. Since I has a monic polynomial, we have $Q + J = A$. Let $'overline'$ denote modulo I. Let $e_1, \ldots, e_{n-1}$ be the standard basis of $R^{n-1} \subseteq F$. Let $e_n = (0, \ldots, 0, l) \in F$ be such that $L/IL = (A/I)l = \mathscr{L}/\mathscr{I}L$.

Let $f_1 \in I$ be such that $\overline{f_1} = \omega(\overline{e_1})$. We can assume f_1 is a monic polynomial. We will pick $f_2 \in I$ such that $\overline{f_2} = \omega(\overline{e_2})$ and for any maximal ideal m, if $(J, f_1, f_2) \subseteq m$ then $I \subseteq m$. To do this, let $g_2 \in I$ be such that $\overline{g_2} = \omega(\overline{e_2})$. Let $m_1, \ldots, m_r, m_{r+1}, \ldots, m_l$ be the maximal ideals over (J, f_1) such that $I \not\subseteq m_i$ (that means, $m_i \neq \sqrt{I}$). Assume $g_2 \notin m_i$ for $i = 1, \ldots, r$ and $g_2 \in m_i$ for $i = r+1, \ldots, l$. Pick $\lambda \in I^2 \cap \cap_{i=1}^{r} m_i \setminus \cup_{i=r+1}^{l} m_i$. Write $f_2 = g_2 + \lambda$. Now, f_2 will satisfy this property.

For $i = 3, \ldots, n-1$ let $f_i \in I$ be any lift of $\omega(\overline{e_i})$. Let $\gamma : L \to I$ be any lift of $\omega_{|L/IL}$. (*Note, γ exists. A choice of γ could be the restriction $\beta_{|L}$ and for* $i = 3, \ldots, n-1$ *we could take $f_i = \beta(e_i)$.*)

Let $\varphi_0 : F \to I$ be given by $f_1, \ldots, f_{n-1}, \gamma$. We claim that $\varphi_0(F)_{1+J} = I_{1+J}$.

Let m be a maximal ideal of R_{1+J} such that $\varphi_0(F)_{1+J} \subseteq m$. Since $f_1 \in m$ is monic, we have $m \cap A_{1+J}$ is maximal and hence $J \subseteq m$. Therefore, $(J, f_1, f_2) \subseteq m$. Therefore, by choice, we have $I_{1+J} \subseteq m$. Since $I = \varphi_0(F) + I^2$, we have $I_{1+J} = \varphi_0(F)_{1+J}$.

So, we can pick an $s \in J$, such that the map $\varphi_1 : F_{1+s} \to I_{1+s}$ given by $f_1, \ldots, f_{n-1}, \gamma$ is surjective. Since, $Q + J = A$, we can assume that $1 + s \in Q \cap (1 + J)$.

Since $Q+J=A$, we have L_{1+J} is extended from A_{1+J} and is an invertible (projective) ideal. Since A_{1+J} is semi-local, L_{1+J} is, in fact, free. Therefore, by modifying s, we can assume that L_{1+s} is free.

Let $\varphi_2 : F_s \to I_s$ be a surjective map given by $(1,0,\ldots,0)$.

Since $F_{s(1+s)}$ is free and f_1 is monic, by a theorem of Ravi Rao ([R2]), there is elementary matrix $\alpha \in Aut(F_{s(1+s)})$ such that $(\varphi_2)_{1+s}\alpha = (\varphi_1)_s$.

Consider the following fibre product diagram:

$$\begin{tikzcd}
P \arrow[rrr, "\pi_2"] \arrow[dr, "\varphi"] \arrow[dd, "\pi_1"] & & & F_s \arrow[dr, "\varphi_2"] \arrow[dd] & \\
& I \arrow[rrr] \arrow[dd] & & & I_s \arrow[dd] \\
F_{1+s} \arrow[rr] \arrow[dr, "\varphi_1"] & & F_{s(1+s)} \arrow[r, "\alpha"] \arrow[dr, "\varphi_1"] & F_{s(1+s)} \arrow[dr, "\varphi_2"] & \\
& I_{1+s} \arrow[rr] & & I_{s(1+s)} \arrow[r, "Id"] & I_{s(1+s)}
\end{tikzcd}$$

where P is the $R-$module obtained by patching F_{1+s} and F_s via α. The map φ obtained by the properties of fibre product diagrams. Note that φ is surjective.

We will construct an isomorphism $\Psi : F \to P$ such that $(\varphi\Psi)_{1+s} \equiv \varphi_1 \quad (modulo\ \ s)$. This will mean that $(\varphi\Psi)_f$ is a surjective lift of ω.

Since α is elementary, α is isotopic to the identity. Therefore, by 6.2.2, there are isomorphisms $\eta_1 : F_{1+s} \to F_{1+s}$ and $\eta_2 : F_s \to F_s$ such that

- $\alpha = (\eta_2)_{1+s}(\eta_1)_s$,
- $\eta_1 \equiv Id \quad (modulo\ s)$,

Now let $\psi_1 = (\pi_1)_{1+s}^{-1}\eta_1^{-1} : F_{1+s} \to P_{1+s}$ and $\psi_2 = (\pi_2)_s^{-1}\eta_2 : F_s \to P_s$. Then $(\psi_1)_s = (\psi_2)_{1+s}$. So, there is an isomorphism $\Psi : F \to P$ such that $\Psi_s = \psi_2$, and $\Psi_{1+s} = \psi_1$.

Now we claim $\varphi\Psi : F \to I$ is a surjective lift of ω. We will check that $\varphi_{1+s}\Psi_{1+s} : F_{1+s} \to I_{1+s}$ is a lift of ω. We have $\varphi_{1+s}\Psi_{1+s} = \varphi_{1+s}\psi_1 = \varphi_{1+s}(\pi_1)^{-1}_{1+s}\eta_1^{-1} = \varphi_1\eta_1^{-1}$. Since, $\eta_1^{-1} \equiv Id \quad (modulo \ \ s)$ and φ_1 is a lift of ω, the claim is established.

By localizing, we have

$$\phi = (\varphi\Psi)_f : \mathscr{F} \twoheadrightarrow \mathscr{I}$$

is a surjective lift of ω. Therefore, we have $(\mathscr{I}, \omega) = 0$ in $E(B, \mathscr{L})$ and the proof is complete.

The following corollary is an immediate consequence of the above theorem 6.3.1 and theorem of Bhatwadekar and Sridharan ([BRS2], pp 199).

Corollary 6.3.2. *Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and $B = A[X, 1/f]$, where $f \in R$ is a non-zero divisor. Assume $dimB = dimA + 1 = n \geq 3$. Let L be a rank one projective $B-$module. Suppose I is an ideal in B with $height(I) = n$ and $\omega : L \oplus B^{n-1} \twoheadrightarrow I/I^2$ is a surjective map. Then ω lifts to a surjection $\varphi : L \oplus B^{n-1} \twoheadrightarrow I$.*

Now assume that $1/(n-1)! \in B$. Let P be a projective $B-$module with $rank(P) = n$ and $\det(P) = L$. Let $\chi : L \stackrel{\sim}{\to} \wedge^n P$ be an isomorphism. Let I be an ideal in B with $height(I) = n$ and $\omega : P/IP \twoheadrightarrow I/I^2$ be a surjective homomorphism. Then there is a surjective homomorphism $\varphi : P \twoheadrightarrow I$ such that (I, ω) is obtained from (φ, χ) (see [BRS2, Corollary 4.3] for clarification).

Proof. Since $E(B, L) = 0$, we have $(I, \omega) = 0$. By [BRS2, Theorem 4.2], ω lifts to a surjection $\varphi : L \oplus B^{n-1} \twoheadrightarrow I$. Note that [BRS2, Theorem 4.2] does not require that $1/(n-1)! \in B$. This completes the proof.

For the later part, since $1/(n-1)! \in B$, euler class $e(P,\chi) \in E(B,L)$ is defined. As $E(B,L) = 0$, we have $e(P,\chi) = (I,\omega) = 0$. Therefore the assertion follows from [BRS2, Corollary 4.3].

Remark 6.3.3. Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and $B = A[X, 1/f]$, where $f \in R$ is a non-zero divisor. Assume $\dim B = \dim A + 1 = n \geq 3$. It is a theorem of Ravi Rao ([R1]) that any projective $B-$module P with $rank(P) = n$, has a free direct summand. If $1/(n-1)! \in B$, then Euler classes of P are defined and hence the theorem of Rao ([R1]) follows from theorem 6.3.1.

Chapter 7

Equivalences

7.1 Preliminaries

We quote the following version of Swan's Bertini theorem from ([BRS3], pp 291).

Theorem 7.1.1. *Let A be a geometrically reduced affine ring over an infinite field and P be a projective $A-$module of rank r. Let $(\alpha, s) \in P^* \oplus A$. Then there is an element $\beta \in P^*$ such that if $I = (\alpha + s\beta)(P)$, then*

1. *Either $I_s = A_s$ or I_s is an ideal of height r such that $(A/I)_s$ is a geometrically reduced ring.*

2. *If $r < dimA$ and A_s is geometrically integral, then $(A/I)_s$ is a geometrically integral.*

3. *If A_s is smooth, then $(A/I)_s$ is smooth.*

4. *In particular, if $J = (a_1, \ldots, a_r, s)$ is an ideal of A then there exist $d_1, \ldots, d_r \in A$ such that if $J = (a_1 + sd_1, \ldots, a_r + sd_r)$, then I_s satisfies properties (1-3).*

The following is the theorem of Grothendieck ([EGA, page 158]) on the openness of the Cohen-Macaulay locus.

Theorem 7.1.2 (Grothendieck, [EGA]). *Let A be noetherian commutative ring and M be a finitely generated $A-$module. Assume that A is image of a regular ring. Define the map*

$$coDepth : Spec(A) \to \mathbb{Z} \quad given \quad by \quad coDepth(\wp) = dim(M_\wp) - depth(M_\wp).$$

Then coDepth is upper semi continuous. That means, for any $\wp_0 \in Spec(A)$ there is an open neighbourhood U of $\wp_0$ such that

$$\wp \in U \quad \Longrightarrow \quad coDepth(\wp) \quad \leq coDepth(\wp_0)$$

In fact,

$$\mathscr{U} = \{\wp \in Spec(A) : coDepth(\wp) \quad \leq \quad n\}$$

is open for all $n \in \mathbb{Z}$.

In particular, the Cohen-Macaulay locus $CM(A)$ is nonempty open in $Spec(A)$.

Proof. For the benefit of the reader, we will sketch the proof. Let B be a regular ring and $\varphi : B \to A$ be a surjective homomorphism and $\Phi : Spec(A) \to Spec(B)$ be the induced map. Consider M as a $B-$module.

It is easy to see that the map, $coDepth : Spec(A) \to \mathbb{Z}$ extends to $Spec(B)$. So, replacing A by B, we can assume that A is a regular ring.

Since A is regular, we can use Auslander-Buchsbaum formula. It follows that, for $\wp \in Spec(A)$ we have

$$coDepth(\wp) = \dim(M_\wp) - depth(M_\wp) = projDim(M_\wp)) - [height(I_\wp)]$$

where $I = ann(M)$.

It is easy to see that, for any integer $n \in \mathbb{Z}$ the sets

$$U(n) = \{\wp \in Spec(A) : projDim(M_{\wp})) \leq n\}$$

and

$$V(n) = \{\wp \in Spec(A) : height(I_{\wp}) \geq n\}$$

are open. This completes the proof.

7.2 Results

In this section we prove some results regarding vanishing of Euler class groups with respect to vanishing of certain types of cycles.

Theorem 7.2.1 (Mandal-Parker)**.** *Let A be a geometrically reduced affine algebra over an infinite field k with dim $A = n$ and L be a line bundle over $Spec(A)$. Then the following are equivalent:*

1. $E_0(A,L) = 0$.

2. *The cycle $(I) = 0$ in $E_0(A,L)$ for all local complete intersection ideals I with $height(I) = n$.*

3. *The cycle $(\mathfrak{m}) = 0$ in $E_0(A,L)$, for all smooth maximal ideals $m \in Spec(A)$ of height n.*

Proof:$(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are obvious. So, we prove $(3) \Rightarrow (1)$. Let I be an ideal of height n such that I/I^2 is generated by n elements. We will prove that the cycle $(I) = 0$ in $E_0(A,L)$. Write $F = L \oplus A^{n-1}$. Take any surjection from $\omega : F/IF \to I/I^2$ and lift it to $\alpha : F \to I$. Notice that α is not necessarily a surjection. However, I is generated by $image(\alpha)$ and some $s \in I^2$. By Swan's Bertini theorem 7.1.1, there we can find $\alpha' \in Hom(F,A)$ such that, with $\beta = \alpha + s\alpha'$, we have $\beta(F) = I \cap J$ for some reduced ideal J of height n with $I + J = A$.

Therefore $J = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_k$, where $\mathfrak{m}_i$ are maximal ideals. This results in $(I) + \sum_{i=1}^{k}(\mathfrak{m}_i) = (I \cap J) = 0$ in $E_0(A,L)$. Now by hypothesis, $(\mathfrak{m}_i) = 0$ for all i, yielding $(I) = 0$. This completes the proof.

The following is the Euler class group version of the above theorem.

Theorem 7.2.2 (Mandal-Parker)**.** *Let A be a geometrically reduced affine algebra over an infinite field k, with dim $A = n \geq 2$ and L be a rank one projective module. We write $F = L \oplus A^{n-1}$. Then the following are equivalent:*

1. $E(A,L) = 0$.
2. *The cycle $(I, \omega) = 0$ in $E(A,L)$ for all local complete intersection ideals I with height$(I) = n$, and local orientation $\omega : F/IF \to I/I^2$.*
3. *The cycle $(\mathfrak{m}, \omega) = 0$ in $E(A,L)$ for all for all smooth maximal ideal $\mathfrak{m} \in Spec(A)$, of height n, and local orientation $\omega : F/\mathfrak{m}F \to \mathfrak{m}/\mathfrak{m}^2$.*

Proof: We will only prove (3)$\Rightarrow$(1). Let I be an of height n and $\omega : F/IF \to I/I^2$ be a surjective map (local orientation). Now let $\alpha : F \to I$ be a lift of ω. Then $I = (\alpha(F), s)$ for some $s \in I^2$. By Swan's Bertini theorem 7.1.1 there is $\alpha' \in F^*$ such that if $\beta = \alpha + s\alpha'$ and $J = im(\beta)$ then,

1. there is a reduced ideal J' of height n so that $I + J' = A$ and $J = I \cap J'$.
2. Since J' is reduced, $J' = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$, where $\mathfrak{m}_i$ are maximal ideals with $A_{\mathfrak{m}_i}$ regular with dimension n.
3. Notice that β is a lift of ω.
4. We also have $(J, \omega_0) = 0$ in $E(A,L)$, where ω_0 is induced by β.

Therefore, in $E(A,L)$, we have $(I, \omega) + \sum_{i=1}^{k} (\mathfrak{m}_i, \omega_{\mathfrak{m}_i}) = (J, \omega_0) = 0$ where $\omega_{\mathfrak{m}_i} : F/\mathfrak{m}_i F \to \mathfrak{m}_i/\mathfrak{m}_i^2$ is the local orientation on $\mathfrak{m}_i$ induced by β. By hypothesis, each $(\mathfrak{m}_i, \omega_{\mathfrak{m}_i}) = 0$, leaving us with $(I, \omega) = 0$. This completes the proof.

The following are versions of the above theorems for rings that are images of regular rings.

Theorem 7.2.3 (Mandal-Parker)**.** *Let A be an noetherian ring with dim $A = n$. Assume that A is image of a regular ring. Let L be a projective $A-$module of rank one. Then the following are equivalent:*

1. *For all local complete intersection ideals N where N is primary with $height(N) = n$ and N/N^2 is generated by n elements, $(N) = 0$ in $E_0(A,L)$*

2. *For all local complete intersection ideals J with $height(J) = n$ and J/J^2 is generated by n elements, $(J) = 0$ in $E_0(A,L)$*

3. $E_0(A,L) = 0$.

Proof:(3⇒2) and (2⇒1) are obvious.

(1⇒3) Let $CM(A)$ denote the Cohen-Macaulay locus of $Spec(A)$. By theorem 7.1.2, $CM(A)$ is non-empty and open. So, $\mathrm{Spec}A \setminus CM(A) = V(I)$ for some ideal I. Since rings of dimension zero are Cohen-Macaulay, height of I is at least one.

Now suppose N is a primary ideal with $height(N) = n$, and N/N^2 is generated by n elements. We will prove that $(N) = 0$ in $E_0(A,L)$.

Write $F = L \oplus A^{n-1}$. There is a homomorphism $\varphi_0 : F \to N$, such that the induced map $F/NF \to N/N^2$ is surjective. So, $N = \varphi_0(F) + N^2$. Hence, $N = (\varphi_0(F), s)$ for some $s \in N^2$. Write

$$\mathscr{P} = \{\wp \in \mathrm{Spec}(A) : height(\wp) < n \quad OR \quad (I \subseteq \wp \quad and \quad N \not\subseteq \wp)\}$$

Note that $(\varphi_0, s) \in Hom(F,A) \oplus A$ basic on $\mathscr{P}$. Also there is a generalized dimension function $d : \mathscr{P} \to \mathbb{Z}$

$$d(\wp) < n = rank(F) \quad for \quad all \quad \wp \in \mathscr{P}.$$

Therefore, by theorem of Eisenbud and Evans, $\varphi = \varphi_0 + s\beta$ is basic on $\mathscr{P}$ for some $\beta \in Hom(F,A)$.

Write $J_0 = \varphi(F)$. Then $height(J_0) = n$ and $J_0 = N \cap J$ for some ideal J with $height(J) = n$ and $N + J = A$.

Suppose $\wp \in \operatorname{Spec}(A)$ and $J \subset \wp$. Then $J_0 \subseteq \wp$ and $N \not\subseteq \wp$. Therefore $I \not\subseteq \wp$. Therefore $A_\wp$ is Cohen-Macaulay. This also implies that J is locally complete intersection ideal of height n.

Looking at a primary decomposition, $J = \cap_{i=1}^{k} N_i$ with N_i primary local complete intersection for all i. Now we have $N \cap \left(\cap_{i=1}^{k} N_i\right) = J_0$. Since $J_0 = \varphi(F)$, we have $(N) + \sum_{i=1}^{k}(N_i) = (J_0) = 0$ in $E_0(A,L)$. By 1, $(N_i) = 0$ for all i. Thus $(N) = 0$. Therefore $E_0(A,L) = 0$. So, the proof of the theorem is complete.

Theorem 7.2.4 (Mandal-Parker)**.** *Let A be an noetherian ring with dim $A = n$. Assume that A is image of a regular ring. Let L be a projective $A-$module of rank one and $F = L \oplus A^{n-1}$.*

Then the following are equivalent:

1. *For all local orientations $\omega : F/NF \to N/N^2$ where N is primary local complete intersection ideal with $height(N) = n$, we have $(N, \omega) = 0$ in $E(A,L)$.*

2. *For local orientations $\omega : F/JF \to J/J^2$ where J is local complete intersection ideal with $height(J) = n$, we have $(J, \omega) = 0$ in $E(A,L)$.*

3. *$E(A,L) = 0$.*

Proof: The proof is similar to the proof of the above theorem 7.2.3. The proofs of (3⇒2) and (2⇒1) are obvious.

(1⇒3) As before, the Cohen-Macaulay locus $CM(A)$ of $Spec(A)$ is open and $\operatorname{Spec} A \setminus CM(A) = V(I)$ for some ideal I with $height(I) \geq 1$.

Now suppose N is a primary ideal with $height(N) = n$, and $\omega : F/NF \to N/N^2$ be a local orientations. We will prove that $(N, \omega) = 0$ in $E(A, L)$.

Let $\varphi_0 : F \to N$ be a lift of ω (that is not necessarily surjective). As in the proof of theorem 7.2.3, we can find $s \in N^2$, and $\varphi = \varphi_0 + s\beta$ for some $\beta \in Hom(F, A)$ such that if $J_0 = \varphi(F)$ then $J_0 = N \cap J$ where J is local complete intersection ideal of height n and $J + N = A$.

Looking at the primary decomposition

$$J = \cap_{i=1}^{k} N_i$$

of J, where N_i is primary local complete intersection for all $i = 1, \ldots, k$. For $i = 1, \ldots, k$ let $\omega_i = \varphi \otimes A/N_i$ and let $\omega_0 = \varphi \otimes A/J_0$.

Note $(J_0, \omega_0) = 0$ in $E(A, L)$ and $\varphi \otimes A/N = \omega$. Now we have

$$N \cap \left(\cap_{i=1}^{k} N_i \right) = J_0.$$

Therefore

$$(N, \omega) + \sum_{i=1}^{k} (N_i, \omega_i) = (J_0, \omega_0) = 0.$$

By 1, $(N_i, \omega_i) = 0$ for all $i = 1, \ldots, k$. Thus $(N, \omega) = 0$. Therefore $E(A, L) = 0$. This completes the proof of the theorem.

Chapter 8

Future Direction

Mathematics has proved itself to be a universal concept throughout the history of civilization. As such, it is something which I believe every person can and should grasp, as it provides a basis for understanding our mutual reality. Mathematics is not only useful philosophically, but in a very realistic sense; for along with mathematical understanding comes the ability to reason and analyze. Both of these are key abilities for success in our modern world.

When you combine the necessity of the skills given by proper mathematical study and the universality of mathematics itself, it throws the implications of a poor mathematical education into sharp relief. From one perspective mathematics may seem rigid and overly complicated; but when understood properly, the beauty of mathematics is seen in the intricacies of its simplicity. It is a proper education that reveals both the simplicity and the intricate nature of mathematics to students.

As a well-educated student of mathematics, I have come to recognize and appreciate the fluidity of the its art-like nature. Currently, the trend in mathematics education lies in textbooks. These textbooks belie the beauty of confluence in mathematics by forcing a rigid path through topics. My view is that education is best served by the motivations of the individual student and as such, the textbook approach is hardly optimal. My dream is that of a flexible curriculum where individual students may pursue their own path through a set of material and yet still have a structure that will provide an educator with the ability to assess their progress and depth of understanding.

Indeed, technology has advanced sufficiently that this dream is in no wise unreachable. Through video lessons and computerized assessments, I hope to create a system that will allow a student to both follow their own motivation through mathematical concepts and at the same time, assess their knowledge, both prior and acquired. This system will be configured to keep the student's skills developed and sharp without the dreariness of instruction by rote. Problems may be developed and assessed in ways that will catch a wide variety of common mistakes from various learning gaps and the system may then require a student to review material that covers skills in which he or she is lacking. It is also possible that this review of material could be offered with an alternate explanation, different than it was the previous time the student viewed the material.

Once the basic framework for this curriculum is established as described above, it could be additionally configured to initially give a student a basic personality test to establish their learning style. A range of exercises suiting different styles could be put into each lesson to provide students with exercises that more closely fit their own innate style of learning.

The implications of the release of such a system to public education would be far-reaching. The minimum standard in mathematical instruction could be raised quite sharply. I have personally experienced some very low standards in education. In one instance, I saw a remedial math class being taught by an art teacher who did not know that any non-zero number raised to the zero power was one, much less understand how to explain it to his struggling students. From interactions other educators as well as research data, I know that situations such as this are far too common, especially in low-income districts, where education is poorly funded, and skilled teachers are hard to find.

At its most basic, this system would still require a teacher to answer questions and help students who may be stuck on a particular concept. However, I foresee its evolution into a dynamic system which continually grows through the addition of questions and answers to address places where students get stuck. For instance, at the end of a lesson, and perhaps even at intervals during the lesson, students would have the opportunity to select from a list of commonly asked questions. After choosing a question they would be shown an explanation and returned to the list of questions. The option could even be given to choose the same question again, and the student would receive an alternate explanation if the first one did not fully resolve their difficulty. This approach would be even more helpful than sitting in a classroom with an instructor giving a lesson since often times I have noticed that students who are lost are uncertain as to what exactly they are confused about. With a selection of questions, students may be able to recognize what they couldn't give a voice to on their own.

As the body of material grows, through lessons and alternate explanations as well as questions and their responses, the instructor's role will increasingly shift from helping students with the details of the specific lessons towards helping students piece their lessons together into a larger view of how what they've learned fits together as a whole.

Additionally, if this system can be developed through grant monies, rather than through a curriculum publishing company, it could be released and expanded upon as ”open source” project, with contributions from interested users, almost in a wiki-like fashion. As an ”open source” curriculum, it would be accessible to poorly-funded schools, requiring only an initial investment in a computer lab capable of running the system. This investment would be much more useful than the purchase of a set of new textbooks, since the software system could be updated without requiring any further expenditure. Additionally, the system could be made accessible to students via the internet, allowing them to work on lessons away from school and at the same time, allowing parents to monitor their students progress.

Over the course of this next year I will be developing a very basic foundation of this system by videotaping lessons, and digitizing my lecture notes. I intend to produce something that I can present at the annual NCTM conference in an attempt to draw other teachers who would be interested in contributing, since this idea is obviously much larger than a single person could handle.

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