DESIGN OF GENERAL
COLD-FORMED STEEL
SHAPES

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by

Robert S. Glauz, P.E.
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ABSTRACT

     The structural design of cold-formed steel members is
a complex process. In addition to limiting the stresses
to those allowed by the material, other modes of failure
must be considered. Typically, cold-formed steel shapes
are made from relatively thin material, and therefore are
subject to local buckling. Also, since these shapes nor-
mally have an open cross-section, torsional buckling may
control.

     The American Iron and Steel Institute recently
released the 1986 Edition of the Cold-Formed Steel Design
Manual. This latest edition presented some changes to the
design procedure requiring a large amount of mathematical
computations. First, the cross-sectional properties must
be determined, including shear center and torsional war-
ping constant. Then, the capacity of the member is com-
puted using a new "unified" design approach. Since parts
of a cross-section are subject to local buckling under
compressive load, a portion of each element may be deemed
ineffective, thereby reducing the effective cross-section.
The design procedure is therefore iterative.

     This report presents the development of a software
package for the design of general cold-formed steel shapes.
The program, called CFS, is an efficient interactive sys-
tem to assist the design engineer. Any general cold-

formed steel shape may be defined, including composites of more than one shape. Concurrent graphical representation assists the user in verifying correct input. Extensive flexibility is provided for modifying the shape definition. Properties of the cross-section are computed. Allowable capacities are determined based on input boundary conditions. Load combinations are analyzed and the resulting effective properties are determined.

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1. INTRODUCTION

1.1 Background

The Marley Cooling Tower Company designs and produces the components and structure of cooling towers, which are used for cooling water in applications such as air conditioning systems, manufacturing processes, and power generation. The towers range in size from about a four foot cube to the several hundred foot tall hyperbolic towers.

At the small end of this range, the cooling tower is manufactured and shipped to the customer as a complete unit. The structure for this category of tower has been cold-formed steel for many years because of its strength, versatility, and low cost.

The structural design of cold-formed steel components can be very complicated. In addition to limiting the stresses to those allowed by the material, other modes of failure must be considered. Typically, cold-formed steel shapes are made from relatively thin material, and therefore are subject to local buckling. Also, since these shapes normally have an open cross-section, torsional buckling may control.

The American Iron and Steel Institute (AISI) has been researching the structural behavior of cold-formed steel shapes since 1939. From their research evolved a Design Manual which has generally been accepted as the standard

for the design of cold-formed steel.

The recently released 1986 Edition of the Design Manual, consists of a specification, commentary, supplemental information, illustrative examples, charts and tables, computer aids, and test procedures. As with any type of structural design, certain physical properties of the cross-sections of structural members are needed for design. Although the manual is quite thorough, including charts and tables for properties of some commonly used shapes, there are many shapes encountered for which properties are not given. It is a very time consuming task to manually calculate the properties of these uncommon shapes, especially the torsional properties such as shear center and warping constant. ____

This latest edition also presented some substantial changes in the procedure for designing structural members. The capacity of a particular shape is determined using the properties of the 'effective' cross-section. Since wide thin elements of a cross-section are subject to local buckling under compressive load, a portion of each element may be deemed ineffective, thereby reducing the effective cross-section. The effective width of each element is determined by the stress level in the element, and the stress level in each element is determined by the properties of the effective section. Therefore, an iterative process must be used to determine the capacity of a shape.

This iterative process is also very time consuming, making the total design process a very lengthy task. However it is possible to automate this process by the use of a computer. The cross-section properties (both full and effective) and the load capacity of a shape could be figured by a computer program. This would be of great assistance to the design engineers at The Marley Cooling Tower Company as well as companies in other industries which use cold-formed steel.

1.2 Objectives

The general objectives of this project were to create a computer program or programs which would compute the cross-sectional properties and load capacities of any general cold-formed steel shape, including composite sections consisting of more than one cold-formed steel shape acting together as one cross-section.

The input to the program should consist of the thickness of the part, and a description of each flat element of the cross-section (i.e., length, angle, and radius of bend between elements). For composite sections, additional input would be required to locate each part of the composite. All input should be assisted by graphical representation of the cross-section being created.

The computed properties should include: area, location of center of gravity, weight per foot, moments of

inertia about both x and y axes, product of inertia, angle of principal axes, moments of inertia about principal axes, section moduli for tension and compression about both x and y axes, radii of gyration about both x and y axes and principal axes. Additionally, other more complicated properties should be computed, such as location of shear center, St. Venant torsion constant, torsional warping constant, polar moment of inertia, polar radius of gyration, and torsional-flexural buckling constant.

Based on additional input defining the boundary conditions of the structural member (i.e., unbraced length, bending coefficients, yield strength, etc.), the following allowable capacities should be computed: tension, compression, shear, and positive and negative moments about each axis.

Once these allowable capacities have been determined, loading combinations could be analyzed using applicable interaction equations. Also the effective section properties could be computed at the given loadings for use in deflection determinations.

Each of the above segments of the program should provide user friendly input and editing capabilities. Also, a printout of the input and computed data should be made available for each segment. To ensure that these intentions were adequately fulfilled, it would be necessary that comments be received from fellow engineers at Marley

throughout the development of the program. Therefore, meetings should be set up to review the progress of the program on a regular basis.

It was anticipated that correspondence with AISI would be necessary to clarify portions of the Design Manual where their intent may not be clear. Such correspondence could be delegated to other engineers so as to provide them with greater exposure to the details of the specification.

2. DEFINITION OF CROSS-SECTION

2.1 General Shape

The cross-section of a general cold-formed steel member can be defined by a consecutive series of rectangular areas of common thickness, joined by sectors of annuli tangent to the rectangular portions and of the same thickness (see Figure 1). Therefore, the program input necessary to define a cross-section is the thickness of the part, the length and orientation of each rectangular element, and the radius of the annulus between consecutive elements.

Common shapes, such as angles and channels, are typically described using the overall length of the legs or flanges, and the overall depth of the web. Therefore,

SECTOR OF
ANNULUS

RECTANGLE
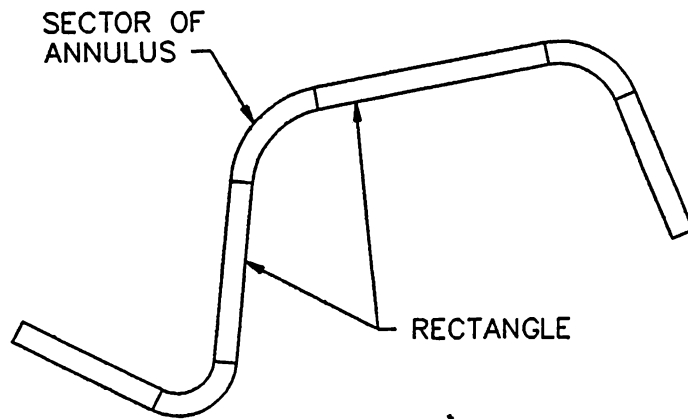
FIGURE 1.   General Cross-Section

it is appropriate for the program input of the element

length to be the overall length to the exterior inter-

section points (see Figure 2).

It is common to refer to the thickness of a part

using a name such as '10 gage'. It is also common to

state an inside bend radius for all bends between

elements, and this radius is typically constant for a

particular thickness. Therefore, it is appropriate for

the program to use an editable data file for definition of

thickness names and their corresponding thicknesses and

default bend radii. But the program should also allow

manual input of actual thickness and overide of the

default radius.
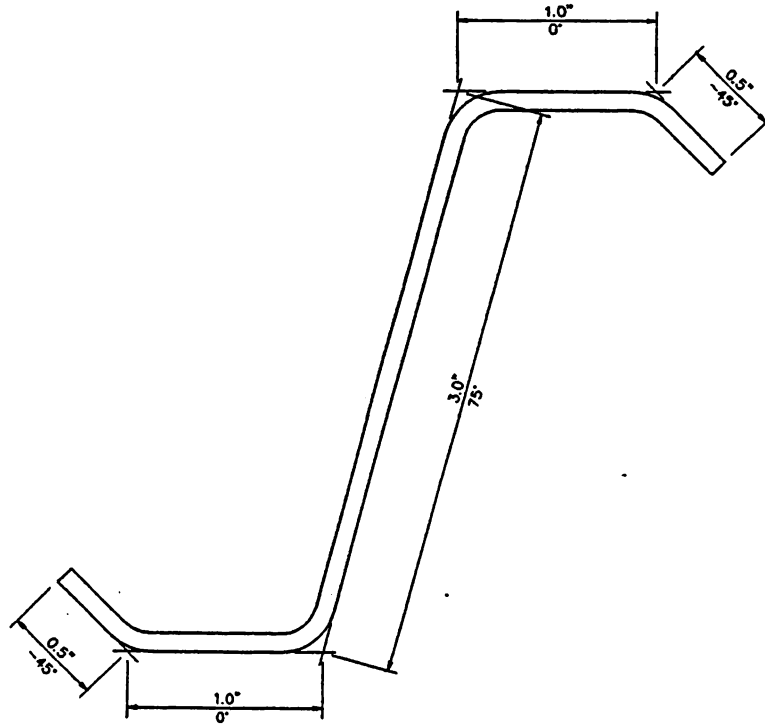
FIGURE 2.   Cross-Section Definition

The part definition is therefore minimized to
specifying the thickness, and consecutively defining the
length and angle of each element, starting at one free
edge and ending at the other.   The cross-section can then
be easily edited by changing the thickness, the radius, or
any element definition, by adding or deleting elements at
either free edge, and by rotating or mirroring the entire
cross-section.   A graphical representation of the cross-

section is displayed upon each change to the definition. This greatly assists the user in verifying the correct input.

2.2 Composite Section

An accumulation of several parts may be grouped to act as a single cross-section, called a composite section. A composite section is defined by locating predefined parts on an X-Y coordinate system. The properties and capacities of a composite section are computed in the same manner as a single part, however certain assumptions must be made regarding the connections between the parts.
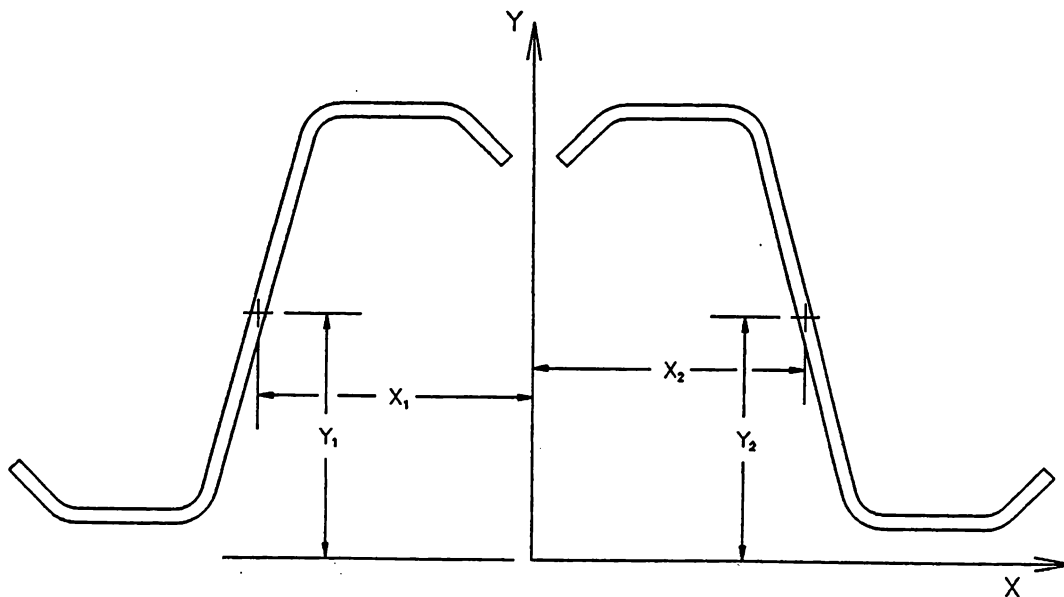

FIGURE 3. Composite Definition

3. SECTION PROPERTIES

3.1 Properties of Area

The following properties are computed by integrating
over the area of the cross-section: area, weight per
foot, location of centroid relative to the boundaries of
the part, moments of inertia about the x and y centroidal
axes, radii of gyration about x and y centroidal axes,
section moduli for both tension and compression about x
and y centroidal axes, moments of inertia about principal
axes, radii of gyration about principal axes, polar moment
of inertia about centroid, polar radius of gyration about
centroid, product of inertia, angle of major axis.

The derivation of the computations for these prop-
erties are presented in Appendix C. The integrations were
performed along the length of each element as well as
across the thickness, for both the rectangular and sector
portions of the part. Therefore, the derivations are
exact.

3.2 Torsional Properties

In addition to the above properties, the following
properties pertaining to torsional behaviour are computed:
location of shear center relative to centroid, polar
moment of inertia about shear center, polar radius of
gyration about shear center, torsional warping constant,

St. Venant torsion constant, and jx and jy used for tor-
sional-flexural buckling computations. The computation of
the shear center and torsional warping constant require
integrating along the line of action of the shear flow,
which is the centerline of the thickness of the part.
This integration was accurately performed for both the
rectangular and sector portions, unlike many references
which use simplifying assumptions.

3.3 Composites

For composites, the torsional warping constant and
St. Venant torsion constant are computed simply as the
summation of those for the individual parts. This is
correct if there is no shear interaction between parts
when subjected to torsion. However, depending on the
shear connectivity, the values of these properties could
be higher.

Also for composites, the location of the shear center
is calculated as follows:

$$X_o = \Sigma(I_x \cdot X_{sc})/\Sigma I_x - X_{cc} \quad \dots \dots \dots \dots \dots \dots (1)$$

$$Y_o = \Sigma(I_y \cdot Y_{sc})/\Sigma I_y - Y_{cc} \quad \dots \dots \dots \dots \dots \dots (2)$$

where,

(X_o, Y_o) = Coordinates of shear center of composite relative to centroid of composite

I_x , I_y = Moments of inertia of a part about its own x and y centroidal axes

(X_{sc}, Y_{sc}) = Coordinates of shear center of a part relative to origin of composite

(X_{cg}, Y_{cg}) = Coordinates of centroid of composite relative to origin of composite

This calculation also assumes no shear connectivity between parts. Based on this assumption, the calculation is correct for symmetrical composites. It is also correct if a principal axis of each part lies on one principal axis of the composite. For other composites, this is merely an approximation.

4. MEMBER CAPACITY

The AISI Cold-Formed Steel Design Manual, 1986 Edition, has two procedures in the Specification for determining member capacities. Procedure I is more general and is therefore used here. Procedure II can only be used in very restrictive circumstances.

The capacities computed are for tension, shear in the x and y directions, compression with and without torsional-flexural buckling, and positive and negative moments about the x and y axes with and without torsional-flexural buckling.

4.1 Boundary Conditions

A necessary condition for stress and buckling computations is that the cross-section must first be oriented so that the principal axes correspond to the x and y axes.

Some boundary conditions must be established before the capacities may be determined. These parameters are: unbraced length (L) and effective length factor (K) for buckling about the x and y axes and for twisting, bending coefficient (C_b) and moment coefficient (C_m) for bending about x and y axes, and yield strength (F_y) and modulus of elasticity (E) of the steel.

4.2 Tension

The allowable tensile capacity is computed simply as the full cross-sectional area times the yield strength and divided by a safety factor of 1.67. The Design Manual has provisions in the Specification to account for holes in the member. However, this is not incorporated into the program.

4.3 Shear

The AISI Specification has provisions for determining the allowable shear load for the flat portion of an element of a cross-section, which consider the shear buckling strength of the element. These calculations reflect an assumption that all of the shear is taken by the flat portion of the load carrying elements. Therefore, the computed allowable shear is overly conservative for parts which have relatively large bend radii.

The AISI Specification also states "for a web consisting of two or more sheets, each sheet shall be considered as a separate element carrying its share of the shear force." There are no provisions for determining the distribution of the shear force to each of the elements. Since the formulas given for allowable shear in an element are no more than empirical approximations, and since exact computations for the true distribution in a general shape are very complex, a simplifying approximation is made for this distribution. The program distributes the shear force in the following manner:

$$V_i = t_i h_i v \cos \alpha_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where,

V_i = Shear load in element i

t_i = Thickness of element i

h_i = Flat length of element i

α_i = angle between direction of load and direction of element i

v = shear flow in cross-section (assumed to be constant)

The total shear is then

$$V_T = \Sigma\ V_i\ \cos\ \alpha_i = \Sigma\ t_i h_i\ v\ \cos^2 \alpha_i\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (4)$$

The portion of the shear load distributed to each element is determined by:

$$\frac{V_i}{V_T} = \frac{v\ t_i h_i\ \cos\ \alpha_i}{\Sigma\ v\ t_i h_i\ \cos^2 \alpha_i} = \frac{t_i h_i\ \cos\ \alpha_i}{\Sigma\ t_i h_i\ \cos^2 \alpha_i}\ \ .\ .\ .\ .\ .\ .\ (5)$$

Using this shear distribution, the allowable total shear is reached when the shear in the controlling element reaches the allowable shear for that element. In mathematical form,

$$V_{aT} = MIN \left[V_{ai} \frac{\Sigma \; t_i \, h_i \; \cos^2 \alpha_i}{t_i \, h_i \; \cos \alpha_i} \right] \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (6)$$

4.4 Compression and Bending

Computation of the allowable compression and bending loads requires an iterative process. Since these types of loadings produce compressive stresses in all or part of the section, local buckling must be considered. The AISI Specification handles local buckling by establishing an 'effective' section which is a function of the stresses in the member.

The allowable load is determined by dividing the nominal strength by the appropriate factor of safety. The nominal strength is the load at which the maximum stress computed using the properties of the effective section is equal to the critical stress. Initially, this load is computed based on the full section properties. With this applied load, the properties of the effective section are determined. Then the load is recomputed based on the effective section properties. This iterative process is repeated until the change in the applied load is negligible.

For load determination without torsional-flexural buckling, the critical stress is simply the yield stress. All stresses, whether tension or compression, must be less

than or equal to this critical stress. For load deter-
mination with torsional-flexural buckling, the critical
stress is taken as the maximum compressive stress caused
by the critical buckling load using the properties of the
full section. All compressive stresses must be less than
or equal to this critical stress. Since tension stresses
are not checked here, it is possible for this computation
to produce a higher load than that computed using yield.
If this occurs, the load is set equal to the latter.

The determination of the allowable moments requires
the use of Se which is defined as the elastic section mod-
ulus of the effective section calculated with the extreme
compression or tension fiber at F_y. The definition of the
extreme fiber is subject to interpretation. It could be
the extreme fiber of the full section or the extreme fiber
of the effective section. The program uses the extreme
fiber of the full section.

4.5 Torsional-Flexural Buckling

The computations for torsional-flexural buckling of a
general unsymmetrical shape require a closer look at buck-
ling theory than what is presented in the AISI Design
Manual. The solution to the three differential equations
of equilibrium of a column subjected to a longitudinal
load P, with biaxial eccentricities e_x and e_y, is shown
below as taken from reference 6.

$$\begin{bmatrix} (P_{y_0}-P) & 0 & -Pa_y K'_{13} \\ 0 & (P_{x_0}-P) & Pa_x K'_{23} \\ -Pa_y K'_{31} & Pa_x K'_{32} & r_0^2(P_{\phi_0}-P) \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ \phi_0 \end{bmatrix} = \begin{bmatrix} -(P^2/P_{y_0})e_x K_1 \\ -(P^2/P_{x_0})e_y K_2 \\ -P^2\left(\dfrac{a_y e_x}{P_{y_0}} - \dfrac{a_x e_y}{P_{x_0}}\right) K_3 \end{bmatrix}$$

When the determinant of the coefficients becomes zero, the deflections become infinite, i.e., buckling occurs. By setting the determinant equal to zero and expanding, the equation becomes

$$- [1 + \beta_1 e_y/\overline{r}_0^2 + \beta_2 e_x/\overline{r}_0^2 - K_{13}^2 a_y^2/\overline{r}_0^2 - K_{23}^2 a_x^2/\overline{r}_0^2]\ P^3$$

$$+ [P'_{\phi_0} + (1+\beta_1 e_y/\overline{r}_0^2 + \beta_2 e_x/\overline{r}_0^2)(P_{x_0}+P_{y_0})$$

$$- (K_{13}^2 a_y/\overline{r}_0^2)P_{x_0} - (K_{23}^2 a_x^2/\overline{r}_0^2)P_{y_0}]\ P^2$$

$$- [(P_{x_0}+P_{y_0})P'_{\phi_0} + (1+\beta_1 e_y/\overline{r}_0^2 + \beta_2 e_x/\overline{r}_0^2)P_{x_0}P_{y_0}]\ P$$

$$+ P_{x_0}P_{y_0}P'_{\phi_0} = 0 \quad . \quad . \quad . \quad . \quad (8)$$

For concentrically loaded compression members, $e_x = e_y = 0$. The resulting equation is somewhat reduced, however still remains a cubic equation. The program solves this cubic equation using the Newton-Raphson method of convergence.

The positive and negative elastic buckling moments about the x axis are determined by $M_{e_x}=Pe_y$ as e_y approaches positive or negative infinity and P approaches zero, while $e_x=0$. By reformulating equation 8 as a cubic equation in Pe_y, the coefficient of the cubed term drops out as e_y approaches infinity. The resulting quadratic

equation reduces to

$$M_{ex} = -(P_{y,e}/K_{13}^2)(\beta_1/2)\left[1 \pm \sqrt{1 + \left(\frac{\overline{r}_o{}^2}{(\beta_1/2)^2}\right)\left(\frac{P'_{\phi e}}{P_{y,e}/K_{13}^2}\right)}\right] \quad . \quad . \quad (9)$$

Using the terms defined in the AISI Design Manual,

$$M_{ex} = -C_{bx}A\sigma_{ey} [j_y \pm \sqrt{j_y + r_o^2(\sigma_t/\sigma_{ey})}] \quad . \quad . \quad . \quad . \quad (10)$$

Similarly, for bending about the y axis,

$$M_{ey} = -C_{by}A\sigma_{ex} [j_x \pm \sqrt{j_x + r_o^2(\sigma_t/\sigma_{ex})}] \quad . \quad . \quad . \quad . \quad (11)$$

4.6 Effective Section

The AISI Specification has provisions for determining the effective width of each flat element of a cross-section subjected to compressive stress. Computations for the effective width are divided into three categories of element types: unstiffened elements, partially stiffened elements, and webs or fully stiffened elements. There are also provisions for stiffened elements with circular holes, but the program does not provide for specifying holes in the member.

The computed effective width of an element is dependent on several factors: the element type, the magnitude of the compressive stress, the way the compressive stress varies along the element, the width to

thickness ratio (w/t) of the element, and the modulus of elasticity.

The elements at the free edges of a part are unstiffened elements. If the compressive stress is not uniform throughout the width of the element, the program uses the maximum compressive stress in the element for computing the effective width. If the free edge of the element is in compression but the other edge is in tension, then the unstiffened width is considered as only the portion in compression. If the free edge is in tension but the other edge is in compression, then the element is considered as a web or stiffened element with a stress gradient.

The elements adjacent to unstiffened elements are considered partially stiffened elements with edge stiffeners, unless the edge stiffener is adequate to fully stiffen the element. The stress used for this computation is the maximum compressive stress in the element. If the element is partially in compression and partially in tension, it is considered as a web or stiffened element with a stress gradient.

The program assumes that all elements from the third element to the third to last element are fully stiffened. This holds true for almost all cases. However, the AISI Specification has provisions for uniformly compressed elements with an intermediate stiffener, which are more limiting than the provisions for stiffened elements. The

program does not use these computations because it does
not recognize elements which constitute intermediate
stiffeners.

If a part has only three elements, the middle element
is considered as fully stiffened. This situation is not
clearly defined in the AISI Specification, and is there-
fore subject to interpretation. In some cases this may
result in unconservative results.

There are no provisions in the AISI Specification for
determining the plate buckling coefficient (k) for par-
tially stiffened elements when the ratio of stiffener
depth to element width (D/w) exceeds 0.8. The program
conservatively uses the plate buckling coefficient for
unstiffened elements when this situation arises.

For composites, the program makes no provisions for
any connections beween parts which may provide inter-
mediate stiffening of elements.

5. COMBINED LOADINGS

Given a set of combined loads acting on the member
simultaneously, the resulting effect can be analyzed.
These loads may consist of any or all of the following:
compression or tension, positive or negative moment about
the x axis, positive or negative moment about the y axis,
shear in the y direction, and shear in the x direction.

5.1 Interaction Formulas

The acceptability of the magnitudes of combined bending and shear is determined by an interaction formula as defined in the AISI Specification Equation C3.3-1.

$$\text{Eq C3.3-1:} \quad (M/M_a)^2 + (V/V_a)^2 \leq 1.0 \quad \ldots \ldots \quad (12)$$

For situations with biaxial bending, the moments are handled separately. Bending about the x axis is combined with shear in the y direction, whereas bending about the y axis is combined with shear in the x direction.

Combined axial and bending loads are checked using interaction formulas C5-1 and C5-2 from the AISI Specification. C5-1 is used for locations between brace points. C5-2 is used for locations at brace points.

$$\text{Eq C5-1:} \quad \frac{P}{P_a} + \frac{C_{ax}M_x}{M_{ax}\alpha_x} + \frac{C_{ay}M_y}{M_{ay}\alpha_y} \leq 1.0 \quad \ldots \ldots \quad (13)$$

$$\text{Eq C5-2:} \quad \frac{P}{P_{ao}} + \frac{M_x}{M_{axo}} + \frac{M_y}{M_{ayo}} \leq 1.0 \quad \ldots \ldots \quad (14)$$

5.2 Effective Section

Load combinations that produce compressive stresses may cause a reduction in the properties of the cross-section. This reduction is computed in a manner similar

to that used for the determination of the member capacities.

The effective section is computed based on stresses caused by the applied loads over the full section. Then the effective section is recomputed based on stresses caused by the same applied loads over the previously computed effective section. This process is repeated until the change in the properties of the effective section are negligible.

The properties of effective area and moments of inertia about centroidal x and y axes can be used for deflection determination.

6. COMPUTER PROGRAM

6.1 General

A computer program was written to accomplish the objectives set forth. The program was written for IBM compatible personal computers. An advanced version of compiled Basic, called Turbo Basic by Borland International, was used as the programming language. This was chosen because of its speed, versatility, and graphics capabilities.

The program is completely documented with a well-structured program listing, program commentary, variable list, supporting diagrams, and a user's manual.

To inquire about obtaining a copy of the program and
supporting documents, you may write to:

> Robert S. Glauz
>
> 6507 Widmer
>
> Shawnee, KS 66216

The following sections of this chapter describe the
program and its usage.

6.2 Installation

Hardware Requirements:
- IBM-PC, XT, AT, or Compatible Computer
- 256K Memory (minimum)
- Color/Graphics Adapter

Recommended Additional Hardware:
- Dot Matrix Printer
- Math Coprocessor

Files on CFS disk:

CFS.EXE	Program
CFS.CFG	Configuration
CFS.THK	Standard Thicknesses
CFS.DRV	Printer Drivers

All files should be copied onto the disk from which
the program will be run, whether it be another floppy disk
or a hard disk. Do not use the original disk for running
the program. It should remain write protected so as to
preserve the original form of the files.

The sections on Configuration and Customization
provide further information on installing the program for
your system.

Running CFS:

To start the program simply type CFS and press Enter.
However, be aware of the following situations:

- If CFS.EXE is not in the current drive and
  directory then the appropriate path must precede
  the CFS command.
- If .CFS.CFG and CFS.THK are not in the current
  drive and directory then the program will
  terminate.
- If CFS.DRV is not in the current drive and
  directory then you will not be able to change
  the printer type.
- If the configuration file specifies a default
  drive or directory that does not exist then the
  program will terminate.

6.3 Part Design

The cross-section of any general cold-formed steel shape of uniform thickness can be defined by describing the geometry of each straight portion of the cross-section. As can be seen in Figure 4, the required information about each element is the length of the element (measured to the external intersection points), the angle of the element, and the internal bend radius between it and the adjacent element.
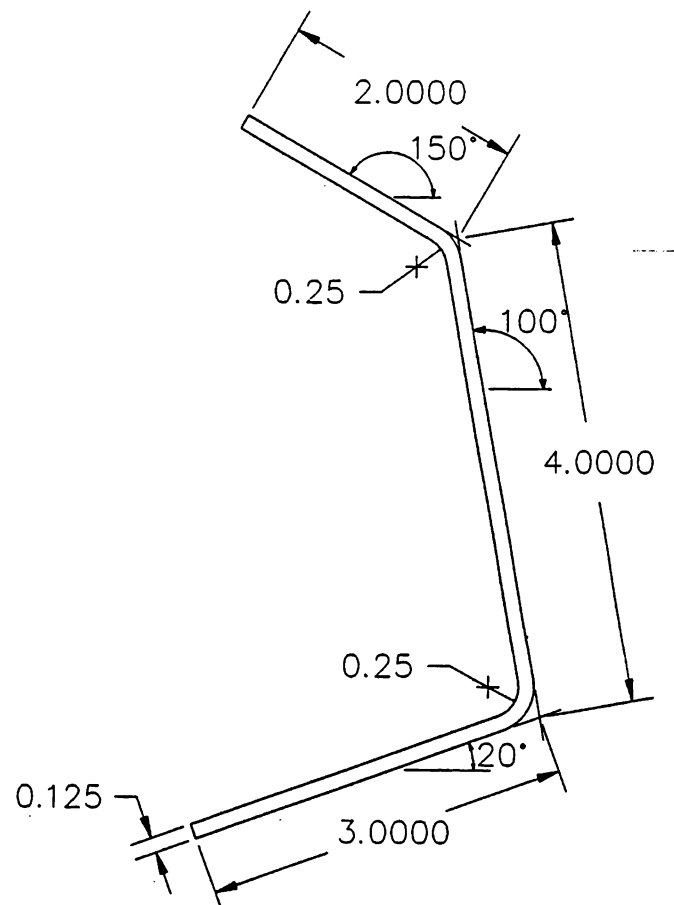

FIGURE 4.   Part Definition

The creation of the cross-section therefore starts at
one edge defining the first element, and then consecutive-
ly defining all subsequent elements. A part may consist
of up to 9 elements.

The screen display for creating a part is shown in
Figure 5. All of the commands are displayed in the lower
left window. The small window to the right displays the
orientation angle reference. The window above displays
the description of the current part, and the window above
that is used for prompts and input. The large window on
the right displays the current part, except when it is
needed for the temporary display of properties or design
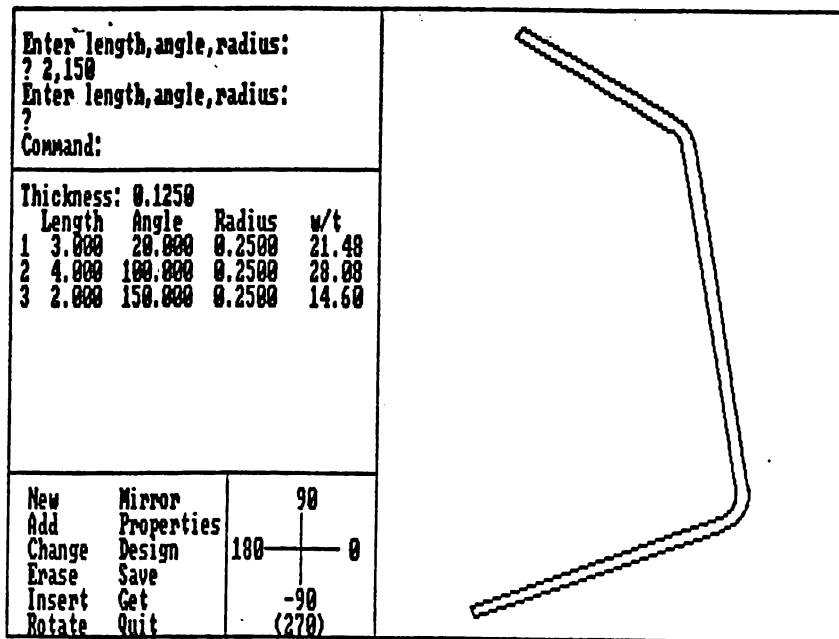data.

FIGURE 5.   Part Creation Screen

When this screen is displayed the command prompt will appear. One of the twelve commands must be entered at this point by pressing the key corresponding to the first letter of the command. Do not press Enter after pressing the letter key. The following is a description of each command.

6.3.1 New

This command will eliminate the current part (if any) and prompt you for the thickness (inches) of a new part. You have the choice of entering either the name of standard thickness or entering the actual thickness. Names of standard thicknesses are defined in the CFS.THK file (see section on Customization). Each name corresponds to a thickness and standard bend radius. If you do not enter the name of a standard thickness then the standard bend radius is equal to the thickness entered. The thickness must be between 0 and 1.

The standard radius is stored as the bend radius of the first element as a default radius. This radius may be over-written when defining the first element.

After entering the thickness, the Add command is automatically invoked.

6.3.2 Add

This allows you to continue adding elements to the current part. You are prompted for the element length, angle, and radius. The length must be between 0 and 100 inches. The angle must be between -180° and +360°. The radius must be between 0 and 10 inches.

The radius of the first element has no effect on the cross-section, but it defines the default radius for the rest of the elements. If you do not enter a radius (or you enter 0 radius) then the default remains as defined when the thickness was specified.

The radius for all other elements represents the radius of bend between the element being entered and the preceding element. If you do not enter a radius (or you enter 0 radius) then the default radius will automatically be used.

If an entered element is in the same or opposite direction as the preceding element, then the element is invalid. If the length of an entered element is too short to accomodate the bend radius, then the element is invalid.

After entering the element data, the Add command is automatically repeated for successive input of the elements of the cross-section. The Add command is terminated by entering a length of 0 (or simply pressing Enter).

6.3.3 Change

The Change command allows you to change the thickness, radius, or element description.

When changing the thickness, all of the radii are automatically changed to standard radius for the thickness given. If any element length is too short to accomodate the change in thickness, then the thickness change is invalid.

When changing the radius, all of the radii change to the new value. The radius must be between 0 and 10 inches. If any element length is too short to accomodate the change in radius, then the radius change is invalid.

To change an element, first specify the element number. Then re-enter the description for that element. If the length is too short then the change is invalid.

6.3.4 Erase

You may erase either the first element or the last element by specifying "F" or "L". If the first element is erased then the radius of the new first element is set to the previous default radius.

6.3.5 Insert

The Insert command is used to add an element to the beginning of the part (a new first element). In doing so, the radius you define becomes the new default radius.

6.3.6 Rotate

This command allows you to rotate the entire cross-section by any specified angle. A positive angle will rotate the part counter-clockwise. All of the element angles are adjusted accordingly.

6.3.7 Mirror

This command allows you to reverse the cross-section (as if you were to look at the member from the opposite end). The angle of the mirror is required to specify how
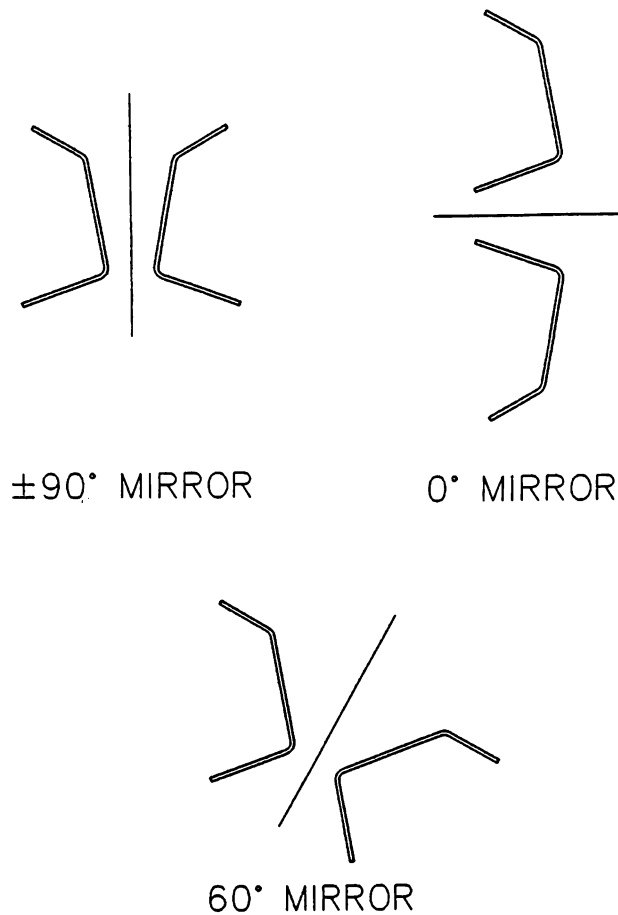
±90° MIRROR   0° MIRROR

60° MIRROR

FIGURE 6.   Mirror Examples

to reverse the part. Figure 6 illustrates that a vertical mirror (±90° angle) will flip the image left to right, and a horizontal mirror (0° or 180° angle) will flip the image top to bottom. Any angle of mirror is allowed. All of the element angles are adjusted accordingly.

6.3.8 Properties.

See explanation in section 6.5 Properties.

6.3.9 Design

See explanation in section 6.6 Design.

6.3.10 Save

This command allows you to save the definition of the current part in a file. You are prompted for a file name. If the part has previously been saved then the name will be displayed in brackets. You may keep this name by simply pressing Enter. After entering a new name, the default drive and directory is checked to see if a file already exists under that name. If so then you may not use that name. After a valid name is entered, the part is saved in the current drive and directory under the name specified, with a .PRT extension.

6.3.11 Get

This command allows you to retrieve a previously saved part. A window will display all possible names of parts to retrieve from the current drive and directory. You are then prompted for the file name to retrieve. If you simply press Enter then the Get command is cancelled. Otherwise you must enter one of the names displayed. The current part (if any) will be eliminated and the new part will be displayed.

6.3.12 Quit

This command returns you to the Main Menu. This and other commands which eliminate the current part from memory (New and Get) will check first to see if the current part is saved. If not, you will be prompted with the question "Are you sure?". Simply answer "Y" or "N".

6.4 Composite Design

A composite is a cross-section of more than one cold-formed steel part which act together as one member. A composite is defined by inserting saved parts on to an X-Y coordinate system which you define. You must provide an orientation angle and an X and Y coordinate for each part of a composite. A composite may consist of up to 9 parts. The screen display is very similar to that for creating a part.
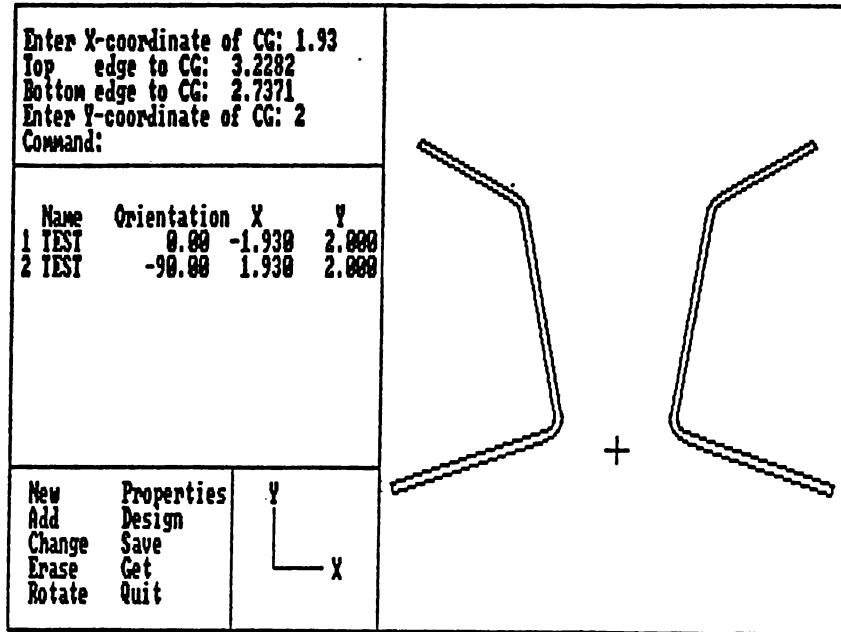
FIGURE 7.   Composite Creation Screen

When this screen is displayed the command prompt will appear.   One of the ten commands must be entered at this point by pressing the key corresponding to the first letter of the command.   Do not press Enter after pressing the letter key.   The following is a description of each command.

6.4.1 New

This command simply clears the current composite.   If it has not been saved, you will be prompted with the question "Are you sure?".   Simply answer "Y" or "N".

6.4.2 Add

This command is used to add a part to the cross-section. A window will display all possible names of parts to retrieve from the current drive and directory. You are then prompted for the file name to retrieve. A small image of the part is displayed in the coordinate reference window.

You are first asked for the orientation angle of the part. This is an opportunity to rotate or mirror the part before locating it. Entering a positive angle will rotate the part. Entering a negative angle will mirror the part. Entering 0 (or just Enter) will leave the orientation unchanged.

The horizontal location of the centroid relative to the left and right edges of the part are displayed and you are prompted for the X coordinate of the centroid. A positive X coordinate locates the centroid of the part to the right of the origin.

The vertical location of the centroid relative to the top and bottom edges of the part are displayed and you are prompted for the Y coordinate of the centroid. A positive Y coordinate locates the centroid of the part above the origin.

6.4.3 Change

This command allows you to change the orientation or location of an existing part of the current composite. You are first prompted for the number of the part to change. Then you must provide the new orientation and location just as you did with the Add command.

6.4.4 Erase

This command allows you to remove a part from the current composite. You are prompted for the number of the part to Erase.

6.4:5 Rotate

The Rotate command will rotate the entire composite about the origin by the specified angle (counter-clockwise is positive) and will adjust the orientation angle and location coordinates accordingly.

6.4.6 Properties.

See explanation in section 6.5 Properties.

6.4.7 Design

See explanation in section 6.6 Design.

6.4.8 Save

This command allows you to save the definition of the
current composite in a file. You are prompted for a file
name. If the composite has previously been saved then the
name will be displayed in brackets. You may keep this
name by simply pressing Enter. After entering a new name,
the default drive and directory is checked to see if a
file already exists under that name. If so then you may
not use that name. After a valid name is entered, the
composite is saved in the current drive and directory
under the name specified, with a .CMP extension.

6.4.9 Get

This command allows you to retrieve a previously
saved composite. A window will display all possible names
of composites to retrieve from the current drive and
directory. You are then prompted for the file name to
retrieve. If you simply press Enter then the Get command
is cancelled. Otherwise you must enter one of the names
displayed. The current composite (if any) will be
eliminated and the new composite will be displayed.

The files for each of the parts of a composite must
still exist or the composite cannot be retrieved. If a
part was revised and re-saved after the composite was
saved then a warning message will be displayed that the
part has changed.

6.4.10 Quit

This command returns you to the Main Menu. This and other commands which eliminate the current composite from memory (New and Get) will check first to see if the current composite is saved. If not, you will be prompted with the question "Are you sure?". Simply answer "Y" or "N".

6.5 Properties

When the Properties command is invoked from either the Part creation screen or Composite creation screen, the properties of the full cross-section will be computed. When done calculating, the properties are displayed in the large window.

The properties are defined below:

Area   Cross-sectional area (in^2)

Wt/Ft   The weight per foot of the member in pounds, based on a density of 0.2833 lb/in^3 (3.4 lb/ft per in^2)

Sx(t)   Section Modulus for extreme top fiber (in^3)

Sx(b)   Section Modulus for extreme bottom fiber (in^3)

Sy(l)   Section Modulus for extreme left fiber (in^3)

Sy(r)   Section Modulus for extreme right fiber (in^3)

Ix   Moment of Inertia about X centroidal axis (in^4)

Iy   Moment of Inertia about Y centroidal axis (in^4)

I1	Moment of Inertia about major axis (in^4)
I2	Moment of Inertia about minor axis (in^4)
Ic	Polar Moment of Inertia about centroid (in^4)
Io	Polar Moment of Inertia about shear center (in^4)
rx	Radius of Gyration about X centroidal axis (in)
ry	Radius of Gyration about Y centroidal axis (in)
r1	Radius of Gyration about major axis (in)
r2	Radius of Gyration about minor axis (in)
rc	Polar Radius of Gyration about centroid (in)
ro	Polar Radius of Gyration about shear center (in)
Ixy	Product of Inertia about X and Y centroidal axes
Alpha	Angle of major axis relative to positive X axis, in degrees
Cw	Torsional Warping Constant (in^6)
J	St. Venant Torsion Constant (in^4)
Xo	X coordinate of shear center relative to centroid
Yo	Y coordinate of shear center relative to centroid
jx	$[\int x^3 dA + \int xy^2 dA]/(2Iy) - Xo$ (in)
jy	$[\int y^3 dA + \int yx^2 dA]/(2Ix) - Yo$ (in)
	Used for torsional-flexural buckling computations (AISI Specification C3.1.2)

All of the above properties, except the torsional properties, are exact computations for the defined cross-section, regardless of the relative thickness. The torsional properties (Cw, J, Xo, Yo) are based on assumptions

for relatively thin elements. For almost all cold-formed

steel applications, these properties will be highly

accurate. Unlike many tables and formulas for typical

shapes, the radius portions of the part are considered as

arc sections, not treated as square corners. However, the

greater the relative thickness of the part, the greater

the error in these properties.

For composites, Cw and J are computed simply as the

summation of the Cw and J for the individual parts. This

is correct if there is no shear interaction between parts

of the cross-section. However, depending on the shear

connectivity, the values of these properties could be

higher.

After the properties are displayed, you are asked

whether to print the output. Simply press "Y" or "N".

The printed output consists of a heading, the

definition of the cross-section, a small graphic image,

and the properties. The heading consists of the name of

the part or composite, a code number, and the date. The

name will only appear if the cross-section was saved (see

Save command). The code number is a randomly selected

four digit number which changes each time a change is made

to the part. It is therefore a means of identifying the

version of a part on the printout. Other types of print-

outs which refer to the same part will also contain the

same four digit number. The date is set when your

computer is turned on. You should make sure that it is
the correct date.

The printed graphic image of the part will be created
if you selected one of the dot matrix printers when con-
figuring (see section on Configuration). The graphic
image is suppressed by selecting the "None" option.

After printing, the cross-section is displayed again.
Since the properties are now known, the center of gravity
and shear center locations are displayed on the cross-
section. The center of gravity is marked by two perpen-
dicular lines, one long and one short. The long line
represents the major axis and the short line represents
the minor axis. The shear center is marked by a cross and
circle.

6.6 Design

When the Design command is invoked from either the
Part creation screen or the Composite creation screen, the
properties of the cross-section are calculated (if not
already calculated). Then the orientation of the section
is checked to make sure that the X and Y axes are the
principal axes. If not, then a message is displayed
telling you to rotate the section to the principal axes.
This must be done so that the stress and buckling
computations will be valid.

If the section is a composite, then you are given an opportunity to change the values of Cw and J to reflect the true behaviour of the section. The calculated value is displayed in brackets as the default. To use this value simply press Enter. You may enter any number, as long as it is higher than the calculated value.

Next, a set of design parameters must be established, which define the boundary conditions. These parameters are:

L   Unbraced length (inches)

K   Effective length factor (AISI C3.1, C4, & C5)

Cb   Bending coefficient (AISI C3.1.2)

Cm   Moment coefficient (AISI C5)

Fy   Yield strength (psi or ksi)

E   Modulus of Elasticity (psi or ksi)

The default values will be displayed. By using the arrow keys, you can move the pointer to any parameter which needs to be changed. Pressing the Enter key will allow you to change the parameter at the pointer location. You are prompted for a new value, and if acceptable it is assigned to that parameter, as well as the remaining parameters displayed to the right of the one changed. By pressing the F1 key, the most recent changes will be cancelled and the values of the parameters will be restored.

By pressing the F2 key, the parameters will be set to the default values (as defined by the configuration). When you are finished making changes, press the Esc key.

The program then calculates the capacities of the cross-section. When done calculating, the allowable loads are displayed. They are defined as follows:

Ta	Allowable Tension
Pao	Allowable Compression disregarding column buckling
Pa	Allowable Compression
Vay	Allowable Shear in the Y direction
Vax	Allowable Shear in the X direction
Maxo	Allowable Moment about X axis disregarding lateral buckling
Max	Allowable Moment about X axis
Mayo	Allowable Moment about Y axis disregarding lateral buckling
May	Allowable Moment about Y axis

A positive moment about the X axis causes compression above the X axis. A positive moment about the Y axis causes compression to the right of the Y axis.

The units associated with these allowable loads are consistent with the units used in defining the parameters. If Fy and E are in psi then the allowable loads are in lbs and in-lbs. If Fy and E are in ksi then the allowable

loads are in kips and in-kips.

These allowables are computed using Procedure I as set forth in the AISI Cold-Formed Steel Design Manual, 1986 Edition. The following assumptions are made in computing the loads:

1. Section B2.2 has provisions for uniformly compressed stiffened elements with circular holes. The program computes loads assuming that there are no holes in the member. A message is displayed every time the allowables are displayed stating that holes in the member will reduce the allowable loads.

2. Certain assumptions are made in determining what each element type is according to the location and stress of the element. Figure 8 illustrates some of these assumptions.

    a. The first and last element of each part are considered as unstiffened, unless the free edge is in tension.

    b. The second and second to last element are considered as partially stiffened, if it is entirely in compression. The stress used for this computation is the maximum compressive stress in the element.

c. Any element which is partially in compression and
    partially in tension is considered as a web or
    stiffened element with a stress gradiant.

d. All elements between the third and third to last
    element are considered as fully stiffened.
    Therefore, an element adjacent to an intermediate
    stiffener will be considered stiffened, even
    though section B4 considers it a partially
    stiffened element.

e. If a part has only three elements, then the middle
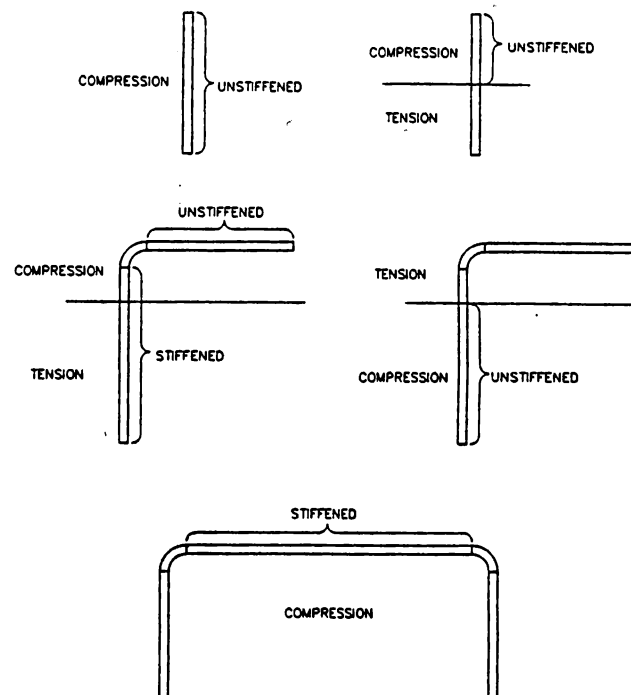   element is considered as fully stiffened.

FIGURE 8.   Element Types

f. AISI has no provisions for determining k for partially stiffened elements when D/w exceeds 0.8. The program conservatively uses k=0.43 and displays a messages when this situation occurs.

g. For composites, no provisions are made for any connections between parts which may provide intermediate stiffening of elements.

3. The determination of the allowable moments requires the use of Se which is defined as the elastic section modulus of the effective section calculated with the extreme compression or tension fiber at Fy. The definition of the extreme fiber is subject to inter-pretation. It could be the extreme fiber of the full section or the extreme fiber of the effective section. In most cases they are the same, however in cases where they are not, the program uses the extreme fiber of the full section.

4. Section C3.2 has provisions for shear strength for both unreinforced webs and webs with transverse stiffeners. The program assumes unreinforced webs.

5. The calculations for allowable shear strength reflect the assumption that all of the shear is taken by the

flat portion of the web. Therefore, for parts which
have relative large radius portions, the allowable
shear will be unrealistically low.

6. There is a statement "For a web consisting of two or
more sheets, each sheet shall be considered as a
separate element carrying its share of the shear
force." Since no provisions are given for determining
the distribution of the shear force between the
elements, the shear in each element is assumed to be
proportional to the length of the element times the
cosine of the angle between the direction of the
element and the direction of the load.

After the allowable loads are displayed, you are asked
whether to print the output. Simply press "Y" or "N".

The printed output consists of a heading, the design
parameters, the allowable loads, and any applicable mes-
sages. The heading consists of the name of the part or
composite, the code number, and the date. The name will
only appear if the cross-section was saved (see Save
command).

After printing, you are asked whether to compute
interactions. If you select "Yes" then you are prompted
for values of P, Mx, My, Vy, and Vx. You may simply press
Enter to indicate a load of zero. The loads may be

positive or negative. A positive axial load P represents
compression whereas as a negative load represents tension.
Positive and negative moments are as defined above for the
allowables. The program will accept loads greater than
the allowable loads but not greater than the ultimate
loads (without safety factors).

Based on the input loads, the properties of the
effective section are computed, which may be used for
deflection calculations. The properties A, Ix, and Iy are
displayed. Interaction equations C5-1, C5-2, and C3.3-1
are computed and displayed. You are then asked whether to
print the output. After printing, you may compute more
interactions. Up to three interactions may be printed per
page. Each printed interaction is numbered to provide
page sequencing if more than one page of interactions is
printed. After printing, the cross-section is displayed
again.

6.7 Configuration

6.7.1 Drive and Directory

You may specify a different drive and/or directory
than the original drive and directory (when starting
program) for which data may be saved and retrieved. The
original drive and directory is displayed (referred to as
default program path). The current data path is also

displayed.   Then you must enter a new data path, or press

Enter to use the default program path.

6.7.2 Printer Type

The type of printer must be specified only if a

printed graphic image of the current cross-section is

desired.   This occurs when printing the definition and

properties of the cross-section.   All other printed output

may be sent to any line printer.

A list of dot matrix printers is displayed for you to

select from.   The currently selected printer is displayed

in brackets.   To leave the selection unchanged, simply

press Enter.   Selection number 0 ("None") is used to

supress the printed graphic image.   The last selection

("Other") can be used for a dot matrix printer not listed,

but you must be able to provide the printer escape se-

quences to perform the appropriate printer tasks.   See

also the section on Customization.

6.7.3 Design Parameters

The parameters used in the Design part of the program

may be assigned initial default values.   Each parameter is

displayed with the current value in brackets.   This is

your opportunity to change any value by entering a new

value at the prompt.   To leave the current value unchanged

simply press Enter.

6.7.4 Restore Configuration

This selection will restore the values of all the configuration data back to the initial values when the program was started. This is done by simply reloading the CFS.CFG file.

6.7.5 Save Configuration

Any of the above changes that you make to the configuration will remain in effect until you exit the program. The next time the program is started, it will use the original configuration, as defined by the CFS.CFG file. If you want the current configuration to be used the next time the program is run, then you must save the configuration. The CFS.CFG file will then be re-written.

6.8 Customization

6.8.1 Standard Thicknesses

The file CFS.THK stores information relating to standard thicknesses. You may customize this file to include the standard thicknesses that you use most often.

Each line in the file contains three items separated by commas. The first item is a name describing the thickness (e.g., "10" for 10 gage). The name must be enclosed in double quotes. The second item is the actual thickness associated with that name. It must be a number between

zero and one. The third item is the standard bend radius
used for that thickness. It must be a number between zero
and ten.

You may edit this file using most any text editor.
Note that the first line in the file is simply a heading.
You may have up to 40 lines of thickness definitions.

6.8.2 Printer Definition

If you are using a dot matrix printer which is not
displayed on the list when configuring, you may still use
it by selecting "Other" and providing additional infor-
mation about the printer. You may also add your printer
to the list by editing the CFS.DRV file which contains
printer driver information for each of the listed print-
ers. There may be as many as 20 printers listed in this
file. Each line of the file contains six items separated
by commas.

The first item is the name of the printer. The
second item is the number of dots per inch printed
horizontally in graphics mode. The third item is the
number of dots per inch printed vertically in graphics
mode.

The fourth item is a sequence of ASCII characters
which when sent to the printer will set the line spacing
equal to the distance for 8 vertical dots.

The fifth item is a sequence of ASCII characters

which when sent to the printer will set the printer to graphics mode. Typically, two characters are required at the end of this sequence to specify the number of columns to print in graphics mode, however these are supplied by the program. Therefore, do not include them.

The sixth item is a sequence of ASCII characters which when sent to the printer will set the line spacing back to the standard spacing (usually 1/6 inches).

7. PROJECT MANAGEMENT

The project to develop the proposed computer program consisted of much more than merely writing a program. Due to deadlines involved, a planning stage was necessary to outline the project and develop a schedule. This outline included the various tasks required to complete the project, which were: proposal, research, computer programming, program documentation, user's manual, and report.

7.1 Planning

The planning began by identifying the objectives of the project and the tasks required to complete it. Additionally, the assignment of responsibilities and time required to complete each task was established. The result of this planning was summarized in the project proposal.

Due to the nature of the project, much of the work was intended to be performed by the author. The Marley Cooling Tower Company allowed some company time to be spent on the project, including some time spent by other engineers in the division. However, many other company activities had priority over this project. It was known that any work which could not be completed according to schedule must be finished outside of company time by the author.

It was intended that the extent of the work performed by other engineers would be primarily to review work performed by the author, provide recommendations for the program development, and test the completed program.

All of these factors were important in the development of the project schedule. The project had a completion deadline of April 8, 1988. This was required for the completion of degree requirements by the end of the Spring 1988 semester. The project schedule was developed according to this deadline.

7.2 Research

The 1986 Edition of the Cold-Formed Steel Design Manual presented substantial changes in the design procedure for cold-formed steel members. Much of the changes were based on recent research and more theoretical approaches to design. As a result, a detailed review of

the Design Manual and its supporting material was required.

The determination of section properties, which are crucial to any type of structural design, required the derivation of mathematical formulas. Also, the derivation of critical buckling loads was necessary, as the Design Manual had only formulas for specific types of members.

Although the original schedule had specific deadlines for this research, some of the research extended beyond these deadlines and continued throughout the programming stage. This was due to unexpected findings in some of the research material which required more research and derivations than originally anticipated.

Also included in the category of research was the determination of the needs of the users of the program. Several engineers at Marley were consulted about this, however it was difficult to obtain recommendations about a program that did not yet exist. Also, they had little motivation to devote time to generating ideas. It was difficult to develop the entire program concept before the program was written.

As a result, a rough concept of the program structure was developed by the author to begin the actual programming. As the program was developed, the potential users began to see the usefulness of the program. This created more interest and the motivation to provide assistance and recommendations.

7.3 Computer Programming

The primary parts of the program are the input and editing capabilities, the graphics display, the section properties, the member capacities, load combinations, the printed output, and the disk input/output. Since the derivation of section properties was performed near the beginning of the project, the programming began with the computation of the properties.

After the programming for section properties, graphics display, and data input was completed, the engineers began to experiment with the partial program. Many good recommendations resulted from their efforts, and changes were made accordingly. These changes had some impact on the schedule. Time was spent rewriting parts of the program instead of proceeding with the next segment. Also, some of the changes altered the way the remainder of the program was intended to be done.

After completion, the program was used many times by several engineers. The original project schedule did not include the task of testing and debugging the program. It was intended that each segment of the program be debugged as it was developed. This was done, however there should have been some time allotted for testing the completed program. Some minor errors were found and corrected.

7.4 Documentation

Documentation is a very important part of any widely used computer program. Although it can be very time-consuming, documentation provides a future reference of all of the thought processes that went into the development of the program. This will save time in making corrections or additions to the program.

The documentation for this program consisted of the program listing, commentary, variable list, and supporting diagrams. This material was reviewed by other engineers to ensure accuracy and completeness.

7.5 User's Manual

The User's Manual serves several purposes. It breifly describes the application of the program. It outlines the computer hardware requirements. It explains how to install and initiate the program. It describes each command and each part of the program. It states any assumptions made by the program in determining section properties or member capacities.

Since this document is probably the only one that the end user will have, it is important that it be comprehensive and easy to understand. Therefore, it was also reviewed by other engineers. The most valuable comments and suggestions came from those unfamiliar with the program. Unclear statements were identified quickly.

7.6 Lessons Learned

The most valuable management lesson learned from this project was that it's difficult to obtain suggestions from co-workers about something that does not yet exist. It is easy to find fault with an existing product, but it takes an exceptionally creative person to make recommendations about a product prior to its creation. Motivation is also a factor. When the program was first usable, the engineers recognized the benefit to them and therefore became motivated to suggest improvements.

7.7 Schedule

A comparison of the final schedule to the original schedule is shown on the following page. In general the project went according to schedule. There was a period of time, around the Christmas holiday, that progress fell behind schedule. There was no allowance made in the original schedule for the disruption of work activities during the holiday season. However, shortly afterward, things were back on schedule due to an extra effort made in January to catch up.

PROJECT SCHEDULE

	ORIGINAL SCHEDULE	ACTUAL COMPLETION
A. INVESTIGATION		
1. Mathematical Formulas.	11-20-87	11-10-87
2. AISI Code Review.	1-08-88	1-02-88 *
3. User Needs.	1-29-88	1-08-88
B. PROGRAMMING		
1. Program Outline.	11-20-87	11-13-87 **
2. General Properties.	11-27-87	11-18-87
3. Torsional Properties.	12- 4-87	11-20-87
4. Graphics.	12-11-87	11-23-87
5. Input/Output.	12-18-87	11-23-87
6. Effective Section Prop.	1-01-88	1-04-88
7. Buckling Computations.	1-15-88	1-04-88
8. Load Combinations.	1-22-88	1-08-88
9. Program Documentation.	2-12-88	1-22-88
C. USER'S MANUAL	3-04-88	2-19-88
D. REPORT	4-01-88	3-10-88

   *   Primary review completed by 1-02-88; some follow-up
       work not completed until 1-22-88.

   ** Initial program outline completed by 11-13-87; however,
       changes were made as program was developed.

8. SUMMARY

An efficient interactive software package, called CFS, has been developed for the design of general cold-formed steel shapes according to the recently released 1986 Edition of the Cold-Formed Steel Design Manual by the American Iron and Steel Institute. Any general cold-formed steel shape may be defined, including composites of more than one shape. Concurrent graphical representation assists the user in verifying correct input. Extensive flexibility is provided for modifying the shape definition. Properties of the cross-section are computed. Allowable capacities are determined based on input boundary conditions. Load combinations are analyzed and the resulting effective properties are determined.

APPENDIX A - References

1. American Iron and Steel Institute, Cold-Formed Steel
   Design Manual, 1983 Edition, (Part I - Specification,
   1980 Edition, Part II - Commentary, Part III -
   Supplementary Information, Part IV - Illustrative
   Examples, Part V - Charts and Tables), 1000 16th
   Street N.W., Washington, D.C.

2. American Iron and Steel Institute, Cold-Formed Steel
   Design Manual, 1986 Edition, 1st Printing of March
   1987, (Part I - Specification, Part II - Commentary,
   Part III - Supplementary Information, Part IV -
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   Part VI - Computer Aids, Part VII - Test Procedures),
   1000 16th Street N.W., Washington, D.C.

3. Beyer, W. H., Editor, "Standard Mathematical Tables,"
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APPENDIX B - Notation

Composite Shear Center (Section 3.3):

(X_o, Y_o) = Coordinates of shear center of composite relative to centroid of composite

I_x , I_y = Moments of inertia of a part about its own x and y centroidal axes

(X_{sc}, Y_{sc}) = Coordinates of shear center of a part relative to origin of composite

(X_{cg}, Y_{cg}) = Coordinates of centroid of composite relative to origin of composite

Shear Load Distribution (Section 4.3):

V_i = Shear load in element i

t_i = Thickness of element i

h_i = Flat length of element i

α_i = angle between direction of load and direction of element i

v = shear flow in cross-section (assumed to be constant)

Torsional-Flexural Buckling (Section 4.5):

A = Cross-sectional area

a_y = $y_o - e_y$

a_x = $x_o - e_x$

C_w = Torsional warping constant

E = Modulus of Elasticity

e_x = Eccentricity of load in x direction

e_y = Eccentricity of load in y direction

G = Shear modulus of elasticity

$I_\bullet$ = Polar moment of inertia about shear center

I_x = Moment of inertia about x centroidal axis

I_y = Moment of inertia about y centroidal axis

J = St. Venant torsion constant

j_x = $\beta_2/2$

j_y = $\beta_1/2$

K_{13}^2 = $K'_{13}K'_{31}$

K_{23}^2 = $K'_{23}K'_{32}$

K_{ij} = constants reflecting boundary conditions

P = Axial load; positive when compressive

$P_{y\bullet}$ = $K_{11}EI_y \ \pi^2/l^2$

$P_{x\bullet}$ = $K_{22}EI_x \ \pi^2/l^2$

$P_{\theta\bullet}$ = $(K_{33}EC_w \ \pi^2/l^2 + GJ)/r_\bullet^2$

$P'_{\theta\bullet}$ = $(K_{33}EC_w \ \pi^2/l^2 + GJ)/\bar{r}_\bullet^2$

$r_\bullet^2$ = $\beta_1 e_y + \beta_2 e_x + \bar{r}_\bullet^2$

$\bar{r}_\bullet$ = Polar radius of gyration about shear center

$r_\bullet$ = $\bar{r}_\bullet$ for equations 10 and 11

β_1 = $[\int y^3 \ dA + \int yx^2 \ dA]/I_x - 2y_\bullet$

β_2 = $[\int x^3 \ dA + \int xy^2 \ dA]/I_y - 2x_\bullet$

$\sigma_{\bullet x}$ = $(P_{x\bullet}/K_{23}^2)/A$

$\sigma_{\bullet y}$ = $(P_{y\bullet}/K_{13}^2)/A$

σ_t = $P'_{\theta\bullet}/A$

Interaction Formulas (Section 5.1):

C_{mx} = Coefficient for moment about x axis

C_{my} = Coefficient for moment about y axis

M = Applied moment about x or y axis

M_x = Applied moment about x axis

M_y = Applied moment about y axis

M_a = Allowable moment about x or y axis

M_{ax} = Allowable moment about x axis

M_{ay} = Allowable moment about y axis

M_{axo} = Allowable moment about x axis based on yield

M_{ayo} = Allowable moment about y axis based on yield

P = Applied axial load

P_a = Allowable axial load

P_{ao} = Allowable axial load based on yield

V = Applied shear load in y or x direction

V_a = Allowable shear load in y or x direction

α_x = Magnification factor for moment about x axis

α_y = Magnification factor for moment about y axis

Program Properties (Section 6.5):

Area   Cross-sectional area (in²)

Wt/Ft   The weight per foot of the member in pounds, based on a density of 0.2833 lb/in³ (3.4 lb/ft per in²)

Sx(t)   Section Modulus for extreme top fiber (in^3)

Sx(b)   Section Modulus for extreme bottom fiber (in^3)

Sy(l)   Section Modulus for extreme left fiber (in^3)

Sy(r)   Section Modulus for extreme right fiber (in^3)

Ix   Moment of Inertia about X centroidal axis (in^4)

Iy   Moment of Inertia about Y centroidal axis (in^4)

I1   Moment of Inertia about major axis (in^4)

I2   Moment of Inertia about minor axis (in^4)

Ic   Polar Moment of Inertia about centroid (in^4)

Io   Polar Moment of Inertia about shear center (in^4)

rx   Radius of Gyration about X centroidal axis (in)

ry   Radius of Gyration about Y centroidal axis (in)

r1   Radius of Gyration about major axis (in)

r2   Radius of Gyration about minor axis (in)

rc   Polar Radius of Gyration about centroid (in)

ro   Polar Radius of Gyration about shear center (in)

Ixy   Product of Inertia about X and Y centroidal axes

Alpha   Angle of major axis relative to positive X axis

Cw   Torsional Warping Constant (in^6)

J   St. Venant Torsion Constant (in^4)

Xo   X coordinate of shear center relative to centroid

Yo   Y coordinate of shear center relative to centroid

jx $[\int x^3 dA + \int xy^2 dA]/(2Iy) - Xo$ (in)

jy $[\int y^3 dA + \int yx^2 dA]/(2Ix) - Yo$ (in)

         Used for torsional-flexural buckling computations

   (AISI Specification C3.1.2)

Program Design (Section 6.6):

L	Unbraced length (inches)
K	Effective length factor (AISI C3.1, C4, & C5)
Cb	Bending coefficient (AISI C3.1.2)
Cm	Moment coefficient (AISI C5)
Fy	Yield strength (psi or ksi)
E	Modulus of Elasticity (psi or ksi)
Ta	Allowable Tension
Pao	Allowable Compression disregarding column buckling
Pa	Allowable Compression
Vay	Allowable Shear in the Y direction
Vax	Allowable Shear in the X direction
Maxo	Allowable Moment about X axis disregarding lateral buckling
Max	Allowable Moment about X axis
Mayo	Allowable Moment about Y axis disregarding lateral buckling
May	Allowable Moment about Y axis

APPENDIX C

DERIVATION OF SECTION PROPERTIES

Properties of Rectangular Areas
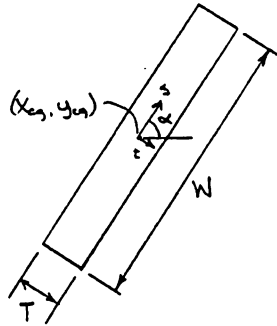
$$x = x_{cg} + s\cos\alpha + t\sin\alpha$$
$$y = y_{cg} + s\sin\alpha - t\cos\alpha$$

$$dA = dt\,ds$$

$$\int dA = \int_{-W/2}^{W/2} \int_{-T/2}^{T/2} dt\,ds$$

$$\underline{= TW}$$

$$\int x\,dA = \int_{-W/2}^{W/2} \int_{-T/2}^{T/2} (x_{cg} + s\cos\alpha + t\sin\alpha)\,dt\,ds$$

$$= \int_{-W/2}^{W/2} (Tx_{cg} + Ts\cos\alpha)\,ds$$

$$\underline{= TW x_{cg}}$$

$$\int y\,dA = \underline{TW y_{cg}}$$

$$\int x^2\,dA = \int_{-W/2}^{W/2} \int_{-T/2}^{T/2} (x_{cg} + s\cos\alpha + t\sin\alpha)^2\,dt\,ds$$

$$= \int_{-W/2}^{W/2} \int_{-T/2}^{T/2} (x_{cg}^2 + s^2\cos^2\alpha + t^2\sin^2\alpha + 2x_{cg}s\cos\alpha + 2x_{cg}t\sin\alpha + 2st\sin\alpha\cos\alpha)\,dt\,ds$$

$$= \int_{-W/2}^{W/2} (Tx_{cg}^2 + Ts^2\cos^2\alpha + \tfrac{1}{12}T^3\sin^2\alpha + 2Ts\,x_{cg}\cos\alpha)\,ds$$

$$= TW x_{cg}^2 + \tfrac{1}{12}TW^3\cos^2\alpha + \tfrac{1}{12}WT^3\sin^2\alpha$$

$$\underline{= TW x_{cg}^2 + TW(W^2\cos^2\alpha + T^2\sin^2\alpha)/12}$$

$$\int y^2\,dA = \underline{TW y_{cg}^2 + TW(W^2\sin^2\alpha + T^2\cos^2\alpha)/12}$$

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$$\int xy\,dA = \int_{-W/_2}^{W/_2} \int_{-T/_2}^{T/_2} (x_{cg} + s\cos\alpha + t\sin\alpha)(y_{cg} + s\sin\alpha - t\cos\alpha)\,dt\,ds$$

$$= \int_{-W/_2}^{W/_2} \int_{-T/_2}^{T/_2} (x_{cg}\,y_{cg} + x_{cg}\,s\sin\alpha - x_{cg}\,t\cos\alpha$$
$$+ y_{cg}\,s\cos\alpha + s^2\sin\alpha\cos\alpha - st\cos^2\alpha$$
$$+ y_{cg}\,t\sin\alpha + st\sin^2\alpha - t^2\sin\alpha\cos\alpha)\,dt\,ds$$

$$= \int_{-W/_2}^{W/_2} (T x_{cg}\,y_{cg} + T x_{cg}\,s\sin\alpha + T y_{cg}\,s\cos\alpha + T s^2\sin\alpha\cos\alpha - \tfrac{1}{12}T^2\sin\alpha\cos\alpha)\,ds$$

$$= TW x_{cg}\,y_{cg} + \tfrac{1}{12}TW^3\sin\alpha\cos\alpha - \tfrac{1}{12}WT^3\sin\alpha\cos\alpha$$

$$= TW x_{cg}\,y_{cg} + TW(W^2 - T^2)\sin\alpha\cos\alpha/12$$

$$\int x^2\,dA = \int_{-W/_2}^{W/_2} \int_{-T/_2}^{T/_2} (x_{cg} + s\cos\alpha + t\sin\alpha)^3\,dt\,ds$$

$$= \int_{-W/_2}^{W/_2} \int_{-T/_2}^{T/_2} (x_{cg}^3 + s^3\cos^3\alpha + t^3\sin^3\alpha + 3x_{cg}^2 s\cos\alpha + 3x_{cg}^2 t\sin\alpha$$
$$+ 3s^2\cos^2\alpha\,x_{cg} + 3s^2\cos^2\alpha\,t\sin\alpha + 3t^2\sin^2\alpha\,x_{cg}$$
$$+ 3t^2\sin^2\alpha\,s\cos\alpha + 6x_{cg}\,s\cos\alpha\,t\sin\alpha)\,dt\,ds$$

$$= \int_{-W/_2}^{W/_2} (T x_{cg}^3 + T s^3\cos^3\alpha + 3T x_{cg}^2 s\cos\alpha + 3T s^2 x_{cg}\cos^2\alpha + 3(\tfrac{1}{12}T^3 x_{cg}\sin^2\alpha)$$
$$+ 3(\tfrac{1}{12}T^3 s\sin^2\alpha\cos\alpha))\,ds$$

$$= TW x_{cg}^3 + 3(\tfrac{1}{12}TW^3 x_{cg}\cos^2\alpha) + 3(\tfrac{1}{12}WT^3 x_{cg}\sin^2\alpha)$$

$$= TW x_{cg}^3 + 3\left[TW(W^2\cos^2\alpha + T^2\sin^2\alpha)/12\right] x_{cg}$$

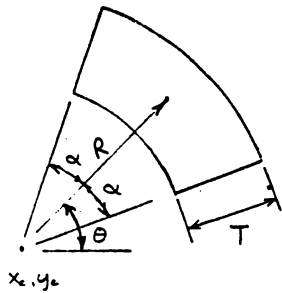$$\int y^2\,dA = TW y_{cg}^3 + 3\left[TW(W^2\sin^2\alpha + T^2\cos^2\alpha)/12\right] y_{cg}$$

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$$\int xy^2 \, dA = \int_{-W/2}^{W/2} \int_{-T/2}^{T/2} (x_{cg} + s\cos\alpha + t\sin\alpha)(y_{cg}^2 + s^2\sin^2\alpha + t^2\cos^2\alpha + 2y_{cg}s\sin\alpha$$
$$- 2y_{cg}t\cos\alpha - 2st\sin\alpha\cos\alpha) \, dt \, ds$$

$$= \int_{-W/2}^{W/2} \int_{-T/2}^{T/2} (x_{cg}y_{cg}^2 + x_{cg}s^2\sin^2\alpha + x_{cg}t^2\cos^2\alpha + 2x_{cg}y_{cg}s\sin\alpha - 2x_{cg}y_{cg}t\cos\alpha - 2x_{cg}st\sin\alpha\cos\alpha$$
$$+ y_{cg}^2 s\cos\alpha + s^3\sin^2\alpha\cos\alpha + st^2\cos^3\alpha + 2y_{cg}s^2\sin\alpha\cos\alpha - 2y_{cg}st\cos^2\alpha - 2s^2t\sin\alpha\cos^2\alpha$$
$$+ y_{cg}^2 t\sin\alpha + s^2t\sin^3\alpha + t^3\sin\alpha\cos^2\alpha + 2y_{cg}st\sin^2\alpha - 2y_{cg}t^2\sin\alpha\cos\alpha - 2st^2\sin^2\alpha\cos\alpha)$$
$$dt \, ds$$

$$= \int_{-W/2}^{W/2} (Tx_{cg}y_{cg}^2 + Tx_{cg}s^2\sin^2\alpha + \tfrac{1}{12}T^3 x_{cg}\cos^2\alpha + 2Tx_{cg}y_{cg}s\sin\alpha$$
$$+ Ty_{cg}^2 s\cos\alpha + Ts^3\sin^2\alpha\cos\alpha + \tfrac{1}{12}T^3 s\cos^3\alpha + 2Ty_{cg}s^2\sin\alpha\cos\alpha$$
$$- 2(\tfrac{1}{12}T^3 y_{cg}\sin\alpha\cos\alpha) - 2(\tfrac{1}{12}T^3 s\sin^2\alpha\cos\alpha)) \, ds$$

$$= TW x_{cg}y_{cg}^2 + \tfrac{1}{12}TW^3 x_{cg}\sin^2\alpha + \tfrac{1}{12}T^3 W x_{cg}\cos^2\alpha$$
$$+ 2(\tfrac{1}{12}TW^3 y_{cg}\sin\alpha\cos\alpha) - 2(\tfrac{1}{12}T^3 W y_{cg}\sin\alpha\cos\alpha)$$

$$= \underline{TW x_{cg}y_{cg}^2 + \tfrac{1}{12}TW(W^2\sin^2\alpha + T^2\cos^2\alpha)x_{cg} + 2(\tfrac{1}{12}TW(W^2 - T^2)\sin\alpha\cos\alpha)y_{cg}}$$

$$\int yx^2 \, dA = \underline{TW y_{cg}^2 x_{cg} + \tfrac{1}{12}TW(W^2\cos^2\alpha + T^2\sin^2\alpha)y_{cg} + 2(\tfrac{1}{12}TW(W^2 - T^2)\sin\alpha\cos\alpha)x_{cg}}$$

Properties of Sector Areas

$x = x_c + (R+t)\cos(\theta+\alpha)$
$\quad = x_c + (R+t)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$
$y = y_c + (R+t)\sin(\theta+\alpha)$
$\quad = y_c + (R+t)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$dA = (R+t)\,d\alpha\,dt$
$\quad = (R+t)\,dt\,d\alpha$

$$\int dA = \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}(R+t)\,dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha} RT\,d\alpha$$

$$= 2RT\alpha$$

$$\int x\,dA = \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}\left(x_c + (R+t)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\right)(R+t)\,dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}\left(x_c R + x_c t + (R^2 + 2Rt + t^2)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\right)dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha}\left(x_c RT + (R^2 T + \tfrac{1}{12}T^3)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\right)d\alpha$$

$$= \underline{2RT\alpha\, x_c + (R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha)}$$

$$\int y\,dA = \underline{2RT\alpha\, y_c + (R^2 T + \tfrac{1}{12}T^3)(2\sin\theta\sin\alpha)}$$

$$\int x^2 \, dA = \int_{-\alpha}^{\alpha} \int_{-T/2}^{T/2} \left(x_c + (R+t)(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \right)^2 (R+t) \, dt \, d\alpha$$

$$= \int_{-\alpha}^{\alpha} \int_{-T/2}^{T/2} \left(x_c^2 + 2x_c(R+t)(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + (R+t)^2(\cos\theta\cos\alpha - \sin\theta\sin\alpha)^2 \right)$$
$$\cdot (R+t) \, dt \, d\alpha$$

$$= \int_{-\alpha}^{\alpha} \int_{-T/2}^{T/2} \left(x_c^2(R+t) + 2x_c(R^2+2Rt+t^2)(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \right.$$
$$\left. + (R^3+3R^2t+3Rt^2+t^3)(\cos^2\theta\cos^2\alpha - 2\sin\theta\cos\theta\sin\alpha\cos\alpha + \sin^2\theta\sin^2\alpha) \right) dt \, d\alpha$$

$$= \int_{-\alpha}^{\alpha} \left(x_c^2 RT + 2x_c(R^2T + \tfrac{1}{12}T^3)(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \right.$$
$$\left. + (R^3T + 3(\tfrac{1}{12}RT^3))(\cos^2\theta\cos^2\alpha - 2\sin\theta\cos\theta\sin\alpha\cos\alpha + \sin^2\theta\sin^2\alpha) \right) d\alpha$$

$$\underline{= 2RT\alpha\, x_c^2 + 2x_c(R^2T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + (R^3T + \tfrac{1}{4}RT^3)(\alpha + (2\cos^2\theta - 1)\sin\alpha\cos\alpha)}$$

$$\int y^2 \, dA = \int_{-\alpha}^{\alpha} \int_{-T/2}^{T/2} \left(y_c + (R+t)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \right)^2 (R+t) \, dt \, d\alpha$$

$$= \int_{-\alpha}^{\alpha} \int_{-T/2}^{T/2} \left(y_c^2 + 2y_c(R+t)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) + (R+t)^2(\sin\theta\cos\alpha - \cos\theta\sin\alpha)^2 \right)$$
$$\cdot (R+t) \, dt \, d\alpha$$

$$= \int_{-\alpha}^{\alpha} \int_{-T/2}^{T/2} \left(y_c^2(R+t) + 2y_c(R^2+2Rt+t^2)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \right.$$
$$\left. + (R^3+3R^2t+3Rt^2+t^3)(\sin^2\theta\cos^2\alpha + 2\sin\theta\cos\theta\sin\alpha\cos\alpha + \cos^2\theta\sin^2\alpha) \right) dt \, d\alpha$$

$$= \int_{-\alpha}^{\alpha} \left(y_c^2 RT + 2y_c(R^2T + \tfrac{1}{12}T^3)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \right.$$
$$\left. + (R^3T + 3(\tfrac{1}{12}RT^3))(\sin^2\theta\cos^2\alpha + 2\sin\theta\cos\theta\sin\alpha\cos\alpha + \cos^2\theta\sin^2\alpha) \right) d\alpha$$

$$= 2RT\alpha\, y_c^2 + 2y_c(R^2T + \tfrac{1}{12}T^3)(2\sin\theta\sin\alpha) + (R^3T + \tfrac{1}{4}RT^3) \cdot$$
$$\cdot (\sin^2\theta\,(\alpha + \sin\alpha\cos\alpha) + \cos^2\theta(\alpha - \sin\alpha\cos\alpha))$$

$$\underline{= 2RT\alpha\, y_c^2 + 2y_c(R^2T + \tfrac{1}{12}T^3)(2\sin\theta\sin\alpha) + (R^3T + \tfrac{1}{4}RT^3) \cdot (\alpha + (2\sin^2\theta - 1)\sin\alpha\cos\alpha)}$$

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$$\int xy\,dA = \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}\left(x_c + (R+t)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\right)\left(y_c + (R+t)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)\right)(R+t)\,dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}\left(x_c y_c(R+t) + x_c(R^2+2Rt+t^2)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)\right.$$
$$+ y_c(R^2+2Rt+t^2)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$\left.+ (R^3+3R^2t+3Rt^2+t^3)(\sin\theta\cos\theta(\cos^2\alpha-\sin^2\alpha) + \sin\alpha\cos\alpha(\cos^2\theta - \sin^2\theta))\right)dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha}\left(x_c y_c RT + x_c(R^2 T + \tfrac{1}{12}T^3)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) + y_c(R^2 T + \tfrac{1}{12}T^3)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\right.$$
$$\left.+ (R^3 T + \tfrac{1}{4}RT^3)(\sin\theta\cos\theta(\cos^2\alpha - \sin^2\alpha) + (\cos^2\theta - \sin^2\theta)\sin\alpha\cos\alpha)\right)d\alpha$$

$$= 2RT\alpha\, x_c y_c + x_c(R^2 T + \tfrac{1}{12}T^3)(2\sin\theta\sin\alpha) + y_c(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha)$$
$$+ (R^3 T + \tfrac{1}{4}RT^3)(2\sin\theta\cos\theta\sin\alpha\cos\alpha)$$

$$\int x^3\,dA = \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}\left(x_c + (R+t)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\right)^3 (R+t)\,dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}\left(x_c^3(R+t) + 3x_c^2(R+t)^2(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + 3x_c(R+t)^3(\cos\theta\cos\alpha - \sin\theta\sin\alpha)^2\right.$$
$$\left.+ (R+t)^4(\cos\theta\cos\alpha - \sin\theta\sin\alpha)^3\right)dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha}\left(x_c^3 RT + 3x_c^2(R^2 T + \tfrac{1}{12}T^3)(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + 3x_c(R^3 T + \tfrac{1}{4}RT^3)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)^2\right.$$
$$\left.+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)(\cos^3\theta\cos^3\alpha - 3\sin\theta\cos^2\theta\sin\alpha\cos^2\alpha + 3\sin^2\theta\cos\theta\sin^2\alpha\cos\alpha - \sin^3\theta\sin^3\alpha)\right)d\alpha$$

$$= 2RT\alpha\, x_c^3 + 3x_c^2(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + 3x_c(R^3 T + \tfrac{1}{4}RT^3)(\alpha + (2\cos^2\theta-1)\sin\alpha\cos\alpha)$$
$$+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)(\tfrac{2}{3}\cos^3\theta\sin\alpha(\cos^2\alpha + 2) + \tfrac{2}{3}\sin^3\theta\cos\theta\sin^3\alpha)$$

$$\int y^3\,dA = \int_{-\alpha}^{\alpha}\int_{-T/2}^{T/2}\left(y_c + (R+t)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)\right)^3 (R+t)\,dt\,d\alpha$$

$$= \int_{-\alpha}^{\alpha}\left(y_c^3 RT + 3y_c^2(R^2 T + \tfrac{1}{12}T^3)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) + 3y_c(R^3 T + \tfrac{1}{4}RT^3)(\sin\theta\cos\alpha - \cos\theta\sin\alpha)^2\right.$$
$$\left.+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)(\sin^3\theta\cos^3\alpha + 3\sin^2\theta\cos\theta\sin\alpha\cos^2\alpha + 3\sin\theta\cos^2\theta\sin^2\alpha\cos\alpha + \cos^3\theta\sin^3\alpha)\right)d\alpha$$

$$= 2RT\alpha\, y_c^3 + 3y_c^2(R^2 T + \tfrac{1}{12}T^3)(2\sin\theta\sin\alpha) + 3y_c(R^3 T + \tfrac{1}{4}RT^3)(\alpha + (2\sin^2\theta-1)\sin\alpha\cos\alpha)$$
$$+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)(\tfrac{2}{3}\sin^3\theta\sin\alpha(\cos^2\alpha + 2) + \tfrac{2}{3}\sin\theta\cos^2\theta\sin^3\alpha)$$
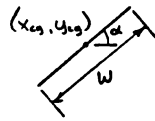
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11-10-87

$$\int xy^2 dA = \int_{-\infty}^{\alpha} \int_{-T_{1/2}}^{T_{1/2}} \left(x_c + (R+t)(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \right) \left(y_c + (R+t)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \right)^2 (R+t)\, dt\, d\alpha$$

$$\cdot \int_{-\infty}^{\alpha} \int_{-T_{1/2}}^{T_{1/2}} \left(x_c(R+t) + (R+t)^2(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \right) \left(y_c^2 + 2y_c(R+t)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \right.$$
$$\left. + (R+t)^2(\sin\theta\cos\alpha + \cos\theta\sin\alpha)^2 \right) dt\, d\alpha$$

$$= \int_{-\infty}^{\alpha} \int_{-T_{1/2}}^{T_{1/2}} \left(x_c y_c^2(R+t) + 2x_c y_c(R+t)^2(\sin\theta\cos\alpha + \cos\theta\sin\alpha) + x_c(R+t)^3(\sin\theta\cos\alpha + \cos\theta\sin\alpha)^2 \right.$$
$$+ y_c^2(R+t)^2(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + 2y_c(R+t)^3(\cos\theta\cos\alpha - \sin\theta\sin\alpha)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$\left. + (R+t)^4(\cos\theta\cos\alpha - \sin\theta\sin\alpha)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)^2 \right) dt\, d\alpha$$

$$= \int_{-\infty}^{\alpha} \left(x_c y_c^2 RT + 2x_c y_c(R^2 T + \tfrac{1}{12}T^3)(\sin\theta\cos\alpha + \cos\theta\sin\alpha) + x_c(R^3 T + \tfrac{1}{4}RT^3)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)^2 \right.$$
$$+ y_c^2(R^2 T + \tfrac{1}{12}T^3)(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + 2y_c(R^3 T + \tfrac{1}{4}RT^3)(\cos\theta\cos\alpha - \sin\theta\sin\alpha)(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)(\cos\theta\cos\alpha\sin^2\theta\cos^2\alpha + \cos\theta\cos\alpha\, 2\sin\theta\cos\theta\sin\alpha\cos\alpha$$
$$+ \cos\theta\cos\alpha\cos^2\theta\sin^2\alpha - \sin\theta\sin\alpha\sin^2\theta\cos^2\alpha - \sin\theta\sin\alpha\, 2\sin\theta\cos\theta\sin\alpha\cos\alpha$$
$$\left. - \sin\theta\sin\alpha\cos^2\theta\sin^2\alpha) \right) d\alpha$$

$$= 2RT\alpha\, x_c y_c^2 + 2x_c y_c(R^2 T + \tfrac{1}{12}T^3)(2\sin\theta\sin\alpha) + x_c(R^3 T + \tfrac{1}{4}RT^3)(\alpha + (2\sin^2\theta - 1)\sin\alpha\cos\alpha)$$
$$+ y_c^2(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + 2y_c(R^3 T + \tfrac{1}{4}RT^3)(2\sin\theta\cos\theta\sin\alpha\cos\alpha)$$
$$+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)\int_{-\infty}^{\alpha}(\sin^2\theta\cos\theta\cos^3\alpha + (2\sin\theta\cos^2\theta - \sin^3\theta)\sin\alpha\cos^2\alpha$$
$$- (2\sin^2\theta\cos\theta - \cos^3\theta)\sin^2\alpha\cos\alpha - \sin\theta\cos^2\theta\sin^3\alpha)\, d\alpha$$

$$= 2RT\alpha\, x_c y_c^2 + 2x_c y_c(R^2 T + \tfrac{1}{12}T^3)(2\sin\theta\sin\alpha) + x_c(R^3 T + \tfrac{1}{4}RT^3)(\alpha + (2\sin^2\theta - 1)\sin\alpha\cos\alpha)$$
$$+ y_c^2(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + 2y_c(R^3 T + \tfrac{1}{4}RT^3)(2\sin\theta\cos\theta\sin\alpha\cos\alpha)$$
$$\underline{+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)(\tfrac{1}{3}\sin^2\theta\cos\theta\sin\alpha(\cos^2\alpha + 2) + \tfrac{2}{3}(\cos^3\theta - 2\sin^2\theta\cos\theta)\sin^3\alpha)}$$

$$\int yx^2 dA = 2RT\alpha\, y_c x_c^2 + 2x_c y_c(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + y_c(R^2 T + \tfrac{1}{4}RT^3)(\alpha + (2\cos^2\theta - 1)\sin\alpha\cos\alpha)$$
$$+ x_c^2(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + 2x_c(R^3 T + \tfrac{1}{4}RT^3)(2\sin\theta\cos\theta\sin\alpha\cos\alpha)$$
$$+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)\int_{-\infty}^{\alpha}(\sin\theta\cos^2\theta\cos^3\alpha + (\cos^3\theta - 2\sin^2\theta\cos\theta)\sin\alpha\cos^2\alpha$$
$$+ (\sin^3\theta - 2\sin\theta\cos^2\theta)\sin^2\alpha\cos\alpha + \sin^2\theta\cos\theta\sin^3\alpha)\, d\alpha$$

$$= 2RT\alpha\, y_c x_c^2 + 2x_c y_c(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + y_c(R^2 T + \tfrac{1}{4}RT^3)(\alpha + (2\cos^2\theta - 1)\sin\alpha\cos\alpha)$$
$$+ x_c^2(R^2 T + \tfrac{1}{12}T^3)(2\cos\theta\sin\alpha) + 2x_c(R^2 T + \tfrac{1}{4}RT^3)(2\sin\theta\cos\theta\sin\alpha\cos\alpha)$$
$$\underline{+ (R^4 T + \tfrac{1}{2}R^2 T^3 + \tfrac{1}{80}T^5)(\tfrac{1}{3}\sin\theta\cos^2\theta\sin\alpha(\cos^2\alpha + 2) + \tfrac{2}{3}(\sin^3\theta - 2\sin\theta\cos^2\theta)\sin^3\alpha)}$$

Properties of Line Segments

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$$x = x_{cg} + s\cos\alpha$$
$$y = y_{cg} + s\sin\alpha$$

$$\int ds = \int_{-W/2}^{W/2} ds$$

$$= W$$

$$\int x\,ds = \int_{-W/2}^{W/2} (x_{cg} + s\cos\alpha)\,ds$$

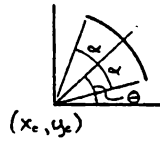$$= x_{cg} W$$

$$\int y\,ds = y_{cg} W$$

$$\int x^2\,ds = \int_{-W/2}^{W/2} (x_{cg}^2 + 2x_{cg}\,s\cos\alpha + s^2\cos^2\alpha)\,ds$$

$$= x_{cg}^2 W + \tfrac{1}{12} W^3 \cos^2\alpha$$

$$\int y^2\,ds = y_{cg}^2 W + \tfrac{1}{12} W^3 \sin^2\alpha$$

Properties of Arc Segments

Bob Glouz
11-11-87

(x_c, y_c)

$x = x_c + R\cos(\theta + \alpha)$
$\quad = x_c + R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$y = y_c + R\sin(\theta + \alpha)$
$\quad = y_c + R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$ds = R\,d\alpha$

$$\int ds = \int_{-\alpha}^{\alpha} R\,d\alpha$$

$$= 2R\alpha$$

$$\therefore \int x\,ds = \int_{-\alpha}^{\alpha} (x_c R + R^2(\cos\theta\cos\alpha - \sin\theta\sin\alpha))\,d\alpha$$

$$= 2R\alpha\, x_c + R^2(\cos\theta\sin\alpha)$$

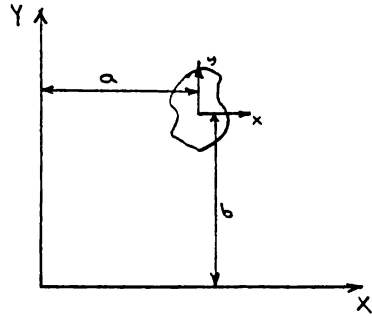$$\int y\,ds = 2R\alpha\, y_c + R^2(\sin\theta\sin\alpha)$$

$$\therefore \int x^2\,ds = \int_{-\alpha}^{\alpha} (x_c^2 R + 2x_c R^2(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + R^3(\cos^2\theta\cos^2\alpha - 2\sin\theta\cos\theta\sin\alpha\cos\alpha + \sin^2\theta\sin^2\alpha))\,d\alpha$$

$$= 2R\alpha\, x_c^2 + 2x_c R^2\cos\theta\sin\alpha + R^3(\cos^2\theta(\alpha + \sin\alpha\cos\alpha) + \sin^2\theta(\alpha - \sin\alpha\cos\alpha))$$

$$= 2R\alpha\, x_c^2 + 2x_c R^2\cos\theta\sin\alpha + R^3(\alpha + (2\cos^2\theta - 1)\sin\alpha\cos\alpha)$$

$$\int y^2\,ds = 2R\alpha\, y_c^2 + 2y_c R^2\sin\theta\sin\alpha + R^3(\alpha + (2\sin^2\theta - 1)\sin\alpha\cos\alpha)$$

Parallel - Axis Theorem

$$\int X dA = \int (x+a) dA$$
$$= \int x dA + a \int dA$$
$$= aA$$
$$\int Y dA = bA$$

$$\int X^2 dA = \int (x+a)^2 dA$$
$$= \int x^2 dA + 2a \int x dA + a^2 \int dA$$
$$= I_y' + a^2 A$$
$$\int Y^2 dA = I_x + b^2 A$$

$$\int XY dA = \int (x+a)(y+b) dA$$
$$= \int xy dA + b \int x dA + a \int y dA + ab \int dA$$
$$= I_{xy} + ab A$$

$$\int X^3 dA = \int (x+a)^3 dA$$
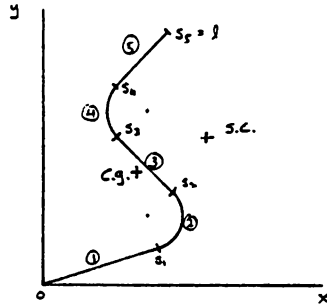$$= \int x^3 dA + 3a \int x^2 dA + 3a^2 \int x dA + a^3 \int dA$$
$$= \int x^3 dA + 3a I_y + a^3 A$$
$$\int Y^3 dA = \int y^3 dA + 3b I_x + b^3 A$$

$$\int XY^2 dA = \int (x+a)(y+b)^2 dA$$
$$= \int xy^2 dA + a \int y^2 dA + 2b \int xy dA + 2ab \int y dA + b^2 \int x dA + ab^2 \int dA$$
$$= \int xy^2 dA + a I_x + 2b I_{xy} + ab^2 A$$

$$\int YX^2 dA = \int yx^2 dA + b I_y + 2a I_{xy} + a^2 b A$$

o

Shear Center Computation

$$X_{sc} = X_{cg} + \frac{t}{I_x} \int_0^l w_e y\, ds$$

$$Y_{sc} = Y_{cg} - \frac{t}{I_y} \int_0^l w_e x\, ds$$

$$\int_0^l w_e y\, ds = \int_0^l \left(\int_0^s R_e\, ds \right) y\, ds$$

$$= \int_0^{s_1} \left(\int_0^{s_1} R_e\, ds \right) y\, ds + \int_{s_1}^{s_2} \left(\int_0^{s_1} R_e\, ds + \int_{s_1}^{s_2} R_e\, ds \right) y\, ds$$

$$+ \int_{s_2}^{s_3} \left(\int_0^{s_1} R_e\, ds + \int_{s_1}^{s_2} R_e\, ds + \int_{s_2}^{s_3} R_e\, ds \right) y\, ds$$

$$+ \int_{s_3}^{s_4} \left(\int_0^{s_1} R_e\, ds + \int_{s_1}^{s_2} R_e\, ds + \int_{s_2}^{s_3} R_e\, ds + \int_{s_3}^{s_4} R_e\, ds \right) y\, ds$$

$$+ \int_{s_4}^{s_5} \left(\int_0^{s_1} R_e\, ds + \int_{s_1}^{s_2} R_e\, ds + \int_{s_2}^{s_3} R_e\, ds + \int_{s_3}^{s_4} R_e\, ds + \int_{s_4}^{s_5} R_e\, ds \right) y\, ds$$

$$= \int_0^{s_1} \left(\int_0^{s_1} R_e\, ds \right) y\, ds + \int_{s_1}^{s_2} \left(w_{e_1} + \int_{s_1}^{s_2} R_e\, ds \right) y\, ds$$

$$+ \int_{s_2}^{s_3} \left(w_{e_1} + w_{e_2} + \int_{s_2}^{s_3} R_e\, ds \right) y\, ds + \int_{s_3}^{s_4} \left(w_{e_1} + w_{e_2} + w_{e_3} + \int_{s_3}^{s_4} R_e\, ds \right) y\, ds$$

$$+ \int_{s_4}^{s_5} \left(w_{e_1} + w_{e_2} + w_{e_3} + w_{e_4} + \int_{s_4}^{s_5} R_e\, ds \right) y\, ds$$

Likewise for $\int_0^l w_e x\, ds$.

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For straight line elements:

$$R_c = (x_o - x_{cg}) \sin \alpha - (y_o - y_{cg}) \cos \alpha$$

    where (x_o, y_o) is the starting point of the line (at s_o)
         α is the angle of the line in the direction
         of increasing s.

R_c is constant for any point on the line.

$$\int_{s_o}^{s} R_c \, ds = R_c (s - s_o)$$

$$y = (y_o - y_{cg}) + (s - s_o) \sin \alpha = \Delta y + (s - s_o) \sin \alpha$$

$$\int_{s_o}^{s} \left(\sum_{i}^{i-1} w_i + \int_{s_o}^{s} R_c \, ds \right) y \, ds = \int_{s_o}^{s} \left(W_c + R_c (s - s_o) \right) \left(\Delta y + (s - s_o) \sin \alpha \right) ds$$

$$= \int_{s_o}^{s} \left(W_c \Delta y + W_c \sin \alpha (s - s_o) + R_c \Delta y (s - s_o) + R_c \sin \alpha (s - s_o)^2 \right) ds$$

$$= W_c \Delta y (s - s_o) + (W_c \sin \alpha + R_c \Delta y)(s - s_o)^2 / 2 + R_c \sin \alpha (s - s_o)^3 / 3$$

$$= \underline{W_c \Delta y \, w_i + (W_c \sin \alpha + R_c \Delta y) \, w_i^2 / 2 + R_c \sin \alpha \, w_i^3 / 3}$$

Likewise,

$$\int_{s_{i-1}}^{s} \left(\sum_{i}^{i-1} w_i + \int_{s_{i-1}}^{s} R_c \, ds \right) x \, ds = \underline{W_c \Delta x \, w_i + (W_c \cos \alpha + R_c \Delta x) \, w_i^2 / 2 + R_c \cos \alpha \, w_i^3 / 3}$$

For the next element, add $\int_{s_o}^{s} R_c \, ds$ to W_c

$$W_c = W_c + R_c \, w_i$$

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For CCW arcs:

$$\alpha = \beta + \frac{\pi}{2}$$

$$\cos \beta = \sin \alpha$$
$$\sin \beta = -\cos \alpha$$
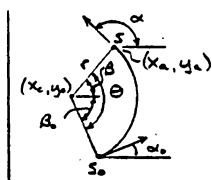
$$
\begin{aligned}
R_c &= (x_a - x_{cg}) \sin \alpha - (y_a - y_{cg}) \cos \alpha \\
&= (x_c + r\cos\beta - x_{cg}) \cos \beta + (y_c + r\sin\beta - y_{cg}) \sin \beta \\
&= \Delta x \cos\beta + \Delta y \sin\beta + r \\
&= \Delta x \sin\alpha - \Delta y \cos\alpha + r
\end{aligned}
$$

$$\alpha = \alpha_0 + \theta = \alpha_0 + \frac{s-s_0}{r}$$

$$
\begin{aligned}
R_c &= \Delta x \sin\left(\alpha_0 + \tfrac{s-s_0}{r}\right) - \Delta y \cos\left(\alpha_0 + \tfrac{s-s_0}{r}\right) + r \\
&= \Delta x \left(\sin\alpha_0 \cos\tfrac{s-s_0}{r} + \cos\alpha_0 \sin\tfrac{s-s_0}{r}\right) - \Delta y \left(\cos\alpha_0 \cos\tfrac{s-s_0}{r} - \sin\alpha_0 \sin\tfrac{s-s_0}{r}\right) + r \\
&= (\Delta x \sin\alpha_0 - \Delta y \cos\alpha_0) \cos\tfrac{s-s_0}{r} + (\Delta x \cos\alpha_0 + \Delta y \sin\alpha_0) \sin\tfrac{s-s_0}{r} + r \\
&= \qquad c_1 \cos\tfrac{s-s_0}{r} \qquad + \qquad c_2 \sin\tfrac{s-s_0}{r} \qquad + r
\end{aligned}
$$

$$
\begin{aligned}
\int_{s_0}^{s} R_c \, ds &= \int_{s_0}^{s} \left(c_1 \cos\tfrac{s-s_0}{r} + c_2 \sin\tfrac{s-s_0}{r} + r\right) ds \\
&= \left[c_1 r \sin\tfrac{s-s_0}{r} - c_2 r \cos\tfrac{s-s_0}{r} + rs \right]_{s_0}^{s} \\
&= c_1 r \sin\tfrac{s-s_0}{r} + c_2 r \left(1 - \cos\tfrac{s-s_0}{r}\right) + r^2 \left(\tfrac{s-s_0}{r}\right)
\end{aligned}
$$

$$
\begin{aligned}
y &= y_c + r\sin\beta - y_{cg} \\
&= \Delta y - r\cos\alpha \\
&= \Delta y - r\cos\left(\alpha_0 + \tfrac{s-s_0}{r}\right) \\
&= \Delta y - r\left(\cos\alpha_0 \cos\tfrac{s-s_0}{r} - \sin\alpha_0 \sin\tfrac{s-s_0}{r}\right)
\end{aligned}
$$

$$x = \Delta x + r\left(\sin\alpha_0 \cos\tfrac{s-s_0}{r} + \cos\alpha_0 \sin\tfrac{s-s_0}{r}\right)$$

For the next element, add $\int_{s_{i-1}}^{s_i} R_c \, ds$ to W_c

$$W_c = W_c + c_1 r \sin\theta_i + c_2 r (1 - \cos\theta_i) + r^2 \theta_i$$

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$$\int_{s_o}^{s}\left(W_c + \int_{s_o}^{s} R_c\,ds\right) y\,ds = \int_{s_o}^{s}\left(W_c + c_1 r \sin\tfrac{s-s_o}{r} + c_2 r\left(1-\cos\tfrac{s-s_o}{r}\right) + r^2\left(\tfrac{s-s_o}{r}\right)\right)\cdot$$
$$\left(\Delta y + r\sin\alpha_o \sin\tfrac{s-s_o}{r} - r\cos\alpha_o \cos\tfrac{s-s_o}{r}\right) ds$$

$$= \int_{s_o}^{s}\Big[W_c\,\Delta y + W_c\,r\sin\alpha_o \sin\tfrac{s-s_o}{r} - W_c\,r\cos\alpha_o \cos\tfrac{s-s_o}{r}$$
$$+ c_1 r \Delta y \sin\tfrac{s-s_o}{r} + c_1 r^2 \sin\alpha_o \sin^2\tfrac{s-s_o}{r} - c_1 r^2 \cos\alpha_o \sin\tfrac{s-s_o}{r}\cos\tfrac{s-s_o}{r}$$
$$+ c_2 r \Delta y\left(1-\cos\tfrac{s-s_o}{r}\right) + c_2 r^2 \sin\alpha_o \sin\tfrac{s-s_o}{r}\left(1-\cos\tfrac{s-s_o}{r}\right) - c_2 r^2 \cos\alpha_o \cos\tfrac{s-s_o}{r}\left(1-\cos\tfrac{s-s_o}{r}\right)$$
$$+ r^2 \Delta y\left(\tfrac{s-s_o}{r}\right) + r^3 \sin\alpha_o\left(\tfrac{s-s_o}{r}\right)\sin\tfrac{s-s_o}{r} - r^3 \cos\alpha_o\left(\tfrac{s-s_o}{r}\right)\cos\tfrac{s-s_o}{r} \Big]\,ds$$

$$= \Big[W_c\,\Delta y\,s - W_c\,r^2\sin\alpha_o \cos\tfrac{s-s_o}{r} - W_c\,r^2\cos\alpha_o \sin\tfrac{s-s_o}{r}$$
$$- c_1 r^2 \Delta y \cos\tfrac{s-s_o}{r} + c_1 r^3 \sin\alpha_o\left(\left(\tfrac{s-s_o}{r}\right)- \sin\tfrac{s-s_o}{r}\cos\tfrac{s-s_o}{r}\right)/2 - c_1 r^3 \cos\alpha_o\left(\sin^2\tfrac{s-s_o}{r}\right)/2$$
$$+ c_2 r^2 \Delta y\left(\tfrac{s-s_o}{r} - \sin\tfrac{s-s_o}{r}\right) - c_2 r^3 \sin\alpha_o \cos\tfrac{s-s_o}{r} - c_2 r^3 \sin\alpha_o\left(\sin^2\tfrac{s-s_o}{r}\right)/2$$
$$- c_2 r^3 \cos\alpha_o \sin\tfrac{s-s_o}{r} + c_2 r^3 \cos\alpha_o\left(\left(\tfrac{s-s_o}{r}\right)+ \sin\tfrac{s-s_o}{r}\cos\tfrac{s-s_o}{r}\right)/2 + r^3\Delta y\left(\tfrac{s-s_o}{r}\right)^2/2$$
$$+ r^4 \sin\alpha_o\left(\sin\tfrac{s-s_o}{r} - \left(\tfrac{s-s_o}{r}\right)\cos\tfrac{s-s_o}{r}\right) - r^4 \cos\alpha_o\left(\cos\tfrac{s-s_o}{r} + \left(\tfrac{s-s_o}{r}\right)\sin\tfrac{s-s_o}{r}\right) \Big]_{s_o}^{s}$$

$$= W_c\,r\,\Delta y\,\theta + W_c\,r^2\sin\alpha_o(1-\cos\theta) - W_c\,r^2\cos\alpha_o \sin\theta$$
$$+ c_1 r^2 \Delta y(1-\cos\theta) + c_1 r^3 \sin\alpha_o(\theta - \sin\theta\cos\theta)/2 - c_1 r^3 \cos\alpha_o(\sin^2\theta)/2$$
$$+ c_2 r^2 \Delta y(\theta - \sin\theta) + c_2 r^3 \sin\alpha_o\left((1-\cos\theta) - (\sin^2\theta)/2\right) - c_2 r^3 \cos\alpha_o\left(\sin\theta - (\theta + \sin\theta\cos\theta)/2\right)\cdot$$
$$+ r^3 \Delta y\,\theta^2/2 + r^4\sin\alpha_o\left(\sin\theta - \theta\cos\theta\right) - r^4\cos\alpha_o\left(\theta\sin\theta - (1-\cos\theta)\right)$$

$$\int_{s_o}^{s}\left(W_c + \int_{s_o}^{s} R_c\,ds\right) x\,ds = \int_{s_o}^{s}\left(W_c + c_1 r \sin\tfrac{s-s_o}{r} + c_2 r\left(1-\cos\tfrac{s-s_o}{r}\right) + r^2\left(\tfrac{s-s_o}{r}\right)\right)\cdot$$
$$\left(\Delta x + r\cos\alpha_o \sin\tfrac{s-s_o}{r} + r\sin\alpha_o \cos\tfrac{s-s_o}{r}\right) ds$$

$$= \int_{s_o}^{s}\Big[W_c\,\Delta x + W_c\,r\cos\alpha_o \sin\tfrac{s-s_o}{r} + W_c\,r\sin\alpha_o \cos\tfrac{s-s_o}{r}$$
$$+ c_1 r \Delta x \sin\tfrac{s-s_o}{r} + c_1 r^2 \cos\alpha_o \sin^2\tfrac{s-s_o}{r} + c_1 r^2 \sin\alpha_o \sin\tfrac{s-s_o}{r}\cos\tfrac{s-s_o}{r}$$
$$+ c_2 r \Delta x\left(1-\cos\tfrac{s-s_o}{r}\right) + c_2 r^2 \cos\alpha_o \sin\tfrac{s-s_o}{r}\left(1-\cos\tfrac{s-s_o}{r}\right) + c_2 r^2 \sin\alpha_o \sin\tfrac{s-s_o}{r}\left(1-\cos\tfrac{s-s_o}{r}\right)$$
$$+ r^2 \Delta x\left(\tfrac{s-s_o}{r}\right) + r^2 \cos\alpha_o\left(\tfrac{s-s_o}{r}\right)\sin\tfrac{s-s_o}{r} + r^2 \sin\alpha_o\left(\tfrac{s-s_o}{r}\right)\cos\tfrac{s-s_o}{r} \Big]\,ds$$

$$= W_c\,r\,\Delta x\,\theta + W_c\,r^2\cos\alpha_o(1-\cos\theta) + W_c\,r^2\sin\alpha_o \sin\theta$$
$$+ c_1 r^2 \Delta x(1-\cos\theta) + c_1 r^3 \cos\alpha_o(\theta - \sin\theta\cos\theta)/2 + c_1 r^3 \sin\alpha_o(\sin^2\theta)/2$$
$$+ c_2 r^2 \Delta x(\theta - \sin\theta) + c_2 r^3 \cos\alpha_o\left((1-\cos\theta) - (\sin^2\theta)/2\right) + c_2 r^3 \sin\alpha_o\left(\sin\theta - (\theta + \sin\theta\cos\theta)/2\right)$$
$$+ r^3 \Delta x\,\theta^2/2 + r^4\cos\alpha_o\left(\sin\theta - \theta\cos\theta\right) + r^4 \sin\alpha_o\left(\theta\sin\theta - (1-\cos\theta)\right)$$

For CW arcs:

$$\alpha = \beta - \frac{\pi}{2}$$

$$\cos \beta = -\sin \alpha$$
$$\sin \beta = \cos \alpha$$

$$R_c = (x_a - x_{cg}) \sin \alpha - (y_a - y_{cg}) \cos \alpha$$
$$= -(x_c + r\cos\beta - x_{cg})\cos\beta - (y_c + r\sin\beta - y_{cg})\sin\beta$$
$$= -\Delta x \cos\beta - \Delta y \sin\beta - r$$
$$= \Delta x \sin\alpha - \Delta y \cos\alpha - r$$

$$\alpha = \alpha_0 - \theta = \alpha_0 - \left(\frac{s - s_0}{r}\right)$$

$$R_c = \Delta x \sin\left(\alpha_0 - \frac{s-s_0}{r}\right) - \Delta y \cos\left(\alpha_0 - \frac{s-s_0}{r}\right) - r$$
$$= \Delta x \left(\sin\alpha_0 \cos\frac{s-s_0}{r} - \cos\alpha_0 \sin\frac{s-s_0}{r}\right) - \Delta y \left(\cos\alpha_0 \cos\frac{s-s_0}{r} + \sin\alpha_0 \sin\frac{s-s_0}{r}\right) - r$$
$$= (\Delta x \sin\alpha_0 - \Delta y \cos\alpha_0) \cos\frac{s-s_0}{r} + (\Delta x \cos\alpha_0 + \Delta y \sin\alpha_0)\left(-\sin\frac{s-s_0}{r}\right) - r$$
$$= c_1 \cos\frac{s-s_0}{r} + (-c_2)\sin\frac{s-s_0}{r} - r$$

$$\int_{s_0}^{s} R_c \, ds = \int_{s_0}^{s} \left(c_1 \cos\frac{s-s_0}{r} + (-c_2)\sin\frac{s-s_0}{r} - r\right) ds$$

$$= \left[c_1 r \sin\frac{s-s_0}{r} - (-c_2) r \cos\frac{s-s_0}{r} - rs \right]_{s_0}^{s}$$

$$= c_1 r \sin\frac{s-s_0}{r} + (-c_2) r \left(1 - \cos\frac{s-s_0}{r}\right) - r^2\left(\frac{s-s_0}{r}\right)$$

$$y = y_c + r\sin\beta - y_{cg}$$
$$= \Delta y + r\cos\alpha$$
$$= \Delta y + r\cos\left(\alpha_0 - \frac{s-s_0}{r}\right)$$
$$= \Delta y + r\left(\cos\alpha_0 \cos\frac{s-s_0}{r} + \sin\alpha_0 \sin\frac{s-s_0}{r}\right)$$

$$x = \Delta x - r\left(\sin\alpha_0 \cos\frac{s-s_0}{r} - \cos\alpha_0 \sin\frac{s-s_0}{r}\right)$$

For next element, add $\int_{s_{i-1}}^{s_i} R_c \, ds$ to W_c

$$W_c = W_c + c_1 r \sin\theta_i + (-c_2) r (1 - \cos\theta_i) - r^2 \theta_i$$

$$(\theta_i \geq \emptyset)$$

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$$\int_{s_0}^{s}\left(W_c + \int_{s_0}^{s} R_c\,ds\right)\dot{y}\,ds = \int_{s_0}^{s}\left(W_c + c_1 r \sin\tfrac{s-s_0}{r} + (-c_2) r\left(1-\cos\tfrac{s-s_0}{r}\right) - r^2\left(\tfrac{s-s_0}{r}\right)\right)\cdot$$
$$\left(\Delta y + r\sin\alpha_0 \sin\tfrac{s-s_0}{r} + r\cos\alpha_0 \cos\tfrac{s-s_0}{r}\right) ds$$

$$= \int_{s_0}^{s}\Big[W_c\Delta y + W_c r \sin\alpha_0 \sin\tfrac{s-s_0}{r} + W_c r \cos\alpha_0 \cos\tfrac{s-s_0}{r}$$
$$+ c_1 r \Delta y \sin\tfrac{s-s_0}{r} + c_1 r^2 \sin\alpha_0 \sin^2\tfrac{s-s_0}{r} + c_1 r^2 \cos\alpha_0 \sin\tfrac{s-s_0}{r}\cos\tfrac{s-s_0}{r}$$
$$+(-c_2)r\Delta y\left(1-\cos\tfrac{s-s_0}{r}\right)+(-c_2)r^2\sin\alpha_0\sin\tfrac{s-s_0}{r}\left(1-\cos\tfrac{s-s_0}{r}\right)+(-c_2)r^2\cos\alpha_0\cos\tfrac{s-s_0}{r}\left(1-\cos\tfrac{s-s_0}{r}\right)$$
$$- r^2\Delta y\left(\tfrac{s-s_0}{r}\right) - r^3\sin\alpha_0\left(\tfrac{s-s_0}{r}\right)\sin\tfrac{s-s_0}{r} - r^3\cos\alpha_0\left(\tfrac{s-s_0}{r}\right)\cos\tfrac{s-s_0}{r}\Big]\,ds$$

$$= \Big[W_c\Delta y s - W_c r^2\sin\alpha_0\cos\tfrac{s-s_0}{r} + W_c r^2\cos\alpha_0\sin\tfrac{s-s_0}{r}$$
$$- c_1 r^2\Delta y\cos\tfrac{s-s_0}{r} + c_1 r^3\sin\alpha_0\left(\left(\tfrac{s-s_0}{r}\right)- \sin\tfrac{s-s_0}{r}\cos\tfrac{s-s_0}{r}\right)/2 + c_1 r^3\cos\alpha_0\left(\sin^2\tfrac{s-s_0}{r}\right)/2$$
$$+(-c_2)r^2\Delta y\left(\tfrac{s-s_0}{r} - \sin\tfrac{s-s_0}{r}\right)-(-c_2)r^3\sin\alpha_0\cos\tfrac{s-s_0}{r}-(-c_2)r^3\sin\alpha_0\left(\sin^2\tfrac{s-s_0}{r}\right)/2$$
$$+(-c_2)r^3\cos\alpha_0\sin\tfrac{s-s_0}{r}-(-c_2)r^3\cos\alpha_0\left(\left(\tfrac{s-s_0}{r}\right)+\sin\tfrac{s-s_0}{r}\cos\tfrac{s-s_0}{r}\right)/2 - r^3\Delta y\left(\tfrac{s-s_0}{r}\right)^2/2$$
$$- r^4\sin\alpha_0\left(\sin\tfrac{s-s_0}{r} - \left(\tfrac{s-s_0}{r}\right)\cos\tfrac{s-s_0}{r}\right) - r^4\cos\alpha_0\left(\cos\tfrac{s-s_0}{r} + \left(\tfrac{s-s_0}{r}\right)\sin\tfrac{s-s_0}{r}\right)\Big]_{s_0}^{s}$$

$$= W_c r \Delta y \theta + W_c r^2\sin\alpha_0(1-\cos\theta) + W_c r^2\cos\alpha_0\sin\theta$$
$$+ c_1 r^2\Delta y(1-\cos\theta) + c_1 r^3\sin\alpha_0(\theta-\sin\theta\cos\theta)/2 + c_1 r^3\cos\alpha_0(\sin^2\theta)/2$$
$$+(-c_2)r^2\Delta y(\theta-\sin\theta)+(-c_2)r^3\sin\alpha_0\left((1-\cos\theta)-(\sin^2\theta)/2\right)+(-c_2)r^3\cos\alpha_0\left(\sin\theta-(\theta+\sin\theta\cos\theta)/2\right)$$
$$- r^3\Delta y\,\theta^2/2 - r^4\sin\alpha_0(\sin\theta-\theta\cos\theta) - r^4\cos\alpha_0(\theta\sin\theta-(1-\cos\theta)) \qquad (\theta \ge 0)$$

$$\int_{s_0}^{s}\left(W_c + \int_{s_0}^{s} R_c\,ds\right)\dot{x}\,ds = \int_{s_0}^{s}\left(W_c + c_1 r \sin\tfrac{s-s_0}{r} + (-c_2) r\left(1-\cos\tfrac{s-s_0}{r}\right) - r^2\left(\tfrac{s-s_0}{r}\right)\right)\cdot$$
$$\left(\Delta x + r\cos\alpha_0 \sin\tfrac{s-s_0}{r} - r\sin\alpha_0 \cos\tfrac{s-s_0}{r}\right) ds$$

$$= \int_{s_0}^{s}\Big[W_c\Delta x + W_c r \cos\alpha_0 \sin\tfrac{s-s_0}{r} - W_c r \sin\alpha_0 \cos\tfrac{s-s_0}{r}$$
$$+ c_1 r \Delta x \sin\tfrac{s-s_0}{r} + c_1 r^2 \cos\alpha_0 \sin^2\tfrac{s-s_0}{r} - c_1 r^2 \sin\alpha_0 \sin\tfrac{s-s_0}{r}\cos\tfrac{s-s_0}{r}$$
$$+(-c_2)r\Delta x\left(1-\cos\tfrac{s-s_0}{r}\right)+(-c_2)r^2\cos\alpha_0\sin\tfrac{s-s_0}{r}\left(1-\cos\tfrac{s-s_0}{r}\right)-(-c_2)r^2\sin\alpha_0\cos\tfrac{s-s_0}{r}\left(1-\cos\tfrac{s-s_0}{r}\right)$$
$$- r^2\Delta x\left(\tfrac{s-s_0}{r}\right) - r^3\cos\alpha_0\left(\tfrac{s-s_0}{r}\right)\sin\tfrac{s-s_0}{r} + r^3\sin\alpha_0\left(\tfrac{s-s_0}{r}\right)\cos\tfrac{s-s_0}{r}\Big]\,ds$$

$$= W_c r \Delta x \theta + W_c r^2\cos\alpha_0(1-\cos\theta) - W_c r^2\sin\alpha_0\sin\theta$$
$$+ c_1 r^2\Delta x(1-\cos\theta) + c_1 r^3\cos\alpha_0(\theta-\sin\theta\cos\theta) - c_1 r^3\sin\alpha_0(\sin^2\theta)/2$$
$$+(-c_2)r^2\Delta x(\theta-\sin\theta)+(-c_2)r^3\cos\alpha_0\left((1-\cos\theta)-(\sin^2\theta)/2\right)-(-c_2)r^3\sin\alpha_0\left(\sin\theta-(\theta+\sin\theta\cos\theta)/2\right)$$
$$- r^3\Delta x\,\theta^2/2 - r^4\cos\alpha_0(\sin\theta-\theta\cos\theta) + r^4\sin\alpha_0(\theta\sin\theta-(1-\cos\theta))$$

Torsional Warping Constant C_w
Computation
(Follows shear center computation)

Bob Glauz
11-6-87

$$C_w = \int_0^l (w_o)^2 \, t \, ds \;-\; \frac{1}{A}\left[\int_0^l w_o t \, ds\right]^2$$

$$= t \int_0^l w_o^2 \, ds \;-\; \frac{t^2}{A}\left(\int_0^l w_o \, ds\right)^2$$

$$\int_0^l w_o^2 \, ds = \int_0^l \left(\int_0^s R_o \, ds\right)^2 ds$$

$$= \int_0^{s_1}\left(\int_0^{s_1} R_o \, ds\right)^2 ds + \int_{s_1}^{s_2}\left(\int_0^{s_1} R_o \, ds + \int_{s_1}^{s_2} R_o \, ds\right)^2 ds$$

$$+ \int_{s_2}^{s_3}\left(\int_0^{s_1} R_o \, ds + \int_{s_1}^{s_2} R_o \, ds + \int_{s_2}^{s_3} R_o \, ds\right)^2 ds$$

$$+ \int_{s_3}^{l}\left(\int_0^{s_1} R_o \, ds + \int_{s_1}^{s_2} R_o \, ds + \int_{s_2}^{s_3} R_o \, ds + \int_{s_3}^{l} R_o \, ds\right)^2 ds$$

$$= \int_0^{s_1}\left(\int_0^{s_1} R_o \, ds\right)^2 ds + \int_{s_1}^{s_2}\left(w_{o1} + \int_{s_1}^{s_2} R_o \, ds\right)^2 ds + \int_{s_2}^{s_3}\left(w_{o1} + w_{o2} + \int_{s_2}^{s_3} R_o \, ds\right)^2 ds$$

$$+ \int_{s_3}^{l}\left(w_{o1} + w_{o2} + w_{o3} + \int_{s_3}^{l} R_o \, ds\right)^2 ds$$

$$\int_0^l w_o \, ds = \int_0^l \left(\int_0^s R_o \, ds\right) ds$$

$$= \int_0^{s_1}\left(\int_0^{s_1} R_o \, ds\right) ds + \int_{s_1}^{s_2}\left(w_{o1} + \int_{s_1}^{s_2} R_o \, ds\right) ds + \int_{s_2}^{s_3}\left(w_{o1} + w_{o2} + \int_{s_2}^{s_3} R_o \, ds\right) ds$$

$$+ \int_{s_3}^{l}\left(w_{o1} + w_{o2} + w_{o3} + \int_{s_3}^{l} R_o \, ds\right) ds$$

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For straight line elements:

$$R_o = (x_o - x_{sc}) \sin \alpha - (y_o - y_{sc}) \cos \alpha$$

where (x_o, y_o) is the starting point of the line (at s_o)
        (x_{sc}, y_{sc}) is the shear center of the cross-section
        α is the angle of the line in the direction
               of increasing s.

R_o is constant for any point on the line

$$\int_{s_o}^{s} \left(\sum^{i-1} w_o + \int_{s_o}^{s} R_o \, ds \right) ds = \int_{s_o}^{s} (W_o + R_o(s-s_o)) \, ds$$

$$= W_o(s-s_o) + R_o(s-s_o)^2 / 2$$

$$= \underline{W_o \, w_i + R_o \, w_i^2 / 2}$$

$$\int_{s_o}^{s} \left(\sum^{i-1} w_o + \int_{s_o}^{s} R_o \, ds \right)^2 ds = \int_{s_o}^{s} (W_o + R_o(s-s_o))^2 \, ds$$

$$= \int_{s_o}^{s} (W_o^2 + 2 W_o R_o(s-s_o) + R_o^2(s-s_o)^2) \, ds$$

$$= W_o^2(s-s_o) + W_o R_o(s-s_o)^2 + R_o^2(s-s_o)^3 / 3$$

$$= \underline{W_o^2 \, w_i + W_o \, R_o \, w_i^2 + R_o^2 \, w_i^3 / 3}$$

For the next element, add $\int_{s_o}^{s} R_o \, ds$ to W_o

$$W_o = W_o + R_o \, w_i$$

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For Arcs:

$$R_o = c_1 \cos\frac{s \cdot s_o}{r} + c_2 \sin\frac{s \cdot s_o}{r} \pm r^*$$

$$c_1 = \Delta x \sin \alpha_o - \Delta y \cos \alpha_o$$
$$c_2 = \pm(\Delta x \cos \alpha_o + \Delta y \sin \alpha_o)^*$$
$$\Delta x = x_c - x_{sc}$$
$$\Delta y = y_c - y_{sc}$$
$$(x_c, y_o) = \text{arc center}$$
$$(x_{sc}, y_{sc}) = \text{shear center}$$

* Use $(+)$ for CCW arcs and $(-)$ for CW arcs

$$\int_{s_o}^{s}\left(\frac{1}{r} W_o + \int_{s_o}^{s} R_o \, ds\right) ds = \int_{s_o}^{s}\left(W_o + \int_{s_o}^{s}\left(c_1 \cos\frac{s \cdot s_o}{r} + c_2 \sin\frac{s \cdot s_o}{r} \pm r\right) ds\right) ds$$

$$= \int_{s_o}^{s}\left(W_o + c_1 r \sin\frac{s \cdot s_o}{r} + c_2 r\left(1 - \cos\frac{s \cdot s_o}{r}\right) \pm r^2\left(\frac{s \cdot s_o}{r}\right)\right) ds$$

$$= W_o r\left(\frac{s \cdot s_o}{r}\right) + c_1 r^2\left(1 - \cos\frac{s \cdot s_o}{r}\right) + c_2 r^2\left(\left(\frac{s \cdot s_o}{r}\right) - \sin\frac{s \cdot s_o}{r}\right) \pm r^3\left(\frac{s \cdot s_o}{r}\right)^2 / 2$$

$$= W_o r \theta + c_1 r^2(1 - \cos\theta) + c_2 r^2(\theta - \sin\theta) \pm r^3 \theta^2 / 2$$

$$\int_{s_o}^{s}\left(\frac{1}{r} W_o + \int_{s_o}^{s} R_o \, ds\right)^2 ds = \int_{s_o}^{s}\left[W_o + c_1 r \sin\frac{s \cdot s_o}{r} + c_2 r\left(1 - \cos\frac{s \cdot s_o}{r}\right) \pm r^2\left(\frac{s \cdot s_o}{r}\right)\right]^2 ds$$

$$= \int_{s_o}^{s}\left[W_o^2 + 2W_o c_1 r \sin\frac{s \cdot s_o}{r} + 2W_o c_2 r\left(1 - \cos\frac{s \cdot s_o}{r}\right) \pm 2W_o r^2\left(\frac{s \cdot s_o}{r}\right)\right.$$
$$+ c_1^2 r^2 \sin^2\frac{s \cdot s_o}{r} + 2c_1 c_2 r^2 \sin\frac{s \cdot s_o}{r}\left(1 - \cos\frac{s \cdot s_o}{r}\right) \pm 2c_1 r^3\left(\frac{s \cdot s_o}{r}\right)\sin\frac{s \cdot s_o}{r}$$
$$\left. + c_2^2 r^2\left(1 - 2\cos\frac{s \cdot s_o}{r} + \cos^2\frac{s \cdot s_o}{r}\right) \pm 2c_2 r^3\left(\frac{s \cdot s_o}{r}\right)\left(1 - \cos\frac{s \cdot s_o}{r}\right) + r^4\left(\frac{s \cdot s_o}{r}\right)^2\right]$$
$$ds$$

$$= W_o^2 r \theta + 2W_o c_1 r^2(1 - \cos\theta) + 2W_o c_2 r^2(\theta - \sin\theta) \pm W_o r^3 \theta^2$$
$$+ c_1^2 r^3(\theta - \sin\theta\cos\theta)/2 + 2c_1 c_2 r^3\left((1 - \cos\theta) - (\sin^2\theta)/2\right)$$
$$\pm 2c_1 r^4(\sin\theta - \theta\cos\theta) + c_2^2 r^3(\theta - 2\sin\theta + (\theta + \sin\theta\cos\theta)/2)$$
$$\pm 2c_2 r^4(\theta^2/2 - \theta\sin\theta + (1 - \cos\theta)) + r^5 \theta^3 / 3$$

For the next element, add $\int_{s_o}^{s} R_o \, ds$ to W_o.

$$W_o = W_o + c_1 r \sin\theta + c_2 r(1 - \cos\theta) \pm r^2 \theta$$