H-CLOSED EXTENSIONS AS PARA-UNIFORM COMPLETIONS

by

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1. Historical introduction. A Hausdorff topological space is called H-closed (Hausdorff closed), or absolutely closed, if it is closed in every Hausdorff space in which it is embedded. The concept of H-closure was introduced by Alexandroff and Urysohn [AU] in 1924. An H-closed space is called an H-closed extension of each of its dense subspaces. In 1930, Tychonoff [Ty] showed that every Hausdorff space can be embedded, but not necessarily as a dense subspace, in an H-closed space. Then Stone [St, Thm. 52, p. 435] in 1937 showed that every Hausdorff space has an H-closed extension. Katětov [Kl] outlined a method for constructing an H-closed extension of an arbitrary Hausdorff space in 1940; the extension so constructed is now called the Katětov extension.

The Katětov extension was shown by Katětov [Kl] to have properties analogous to some of the properties of the Stone-Čech compactification of an arbitrary completely regular Hausdorff space. Fomin described [Fo] other H-closed extensions for arbitrary Hausdorff spaces, in 1940. Alexandroff remarked [A2] in 1960 that no method of systematically determining the H-closed extensions of a Hausdorff space had been found. This problem is simplified somewhat by considering equivalence classes which are implicit in some of Banaschewski's work and in that of Iliadis and Fomin.

In 1964, Banaschewski [Ba] showed that any T_0-extension of a T_0-space (X,τ) can be realized as a topologization of a collection of open filters on (X,τ). At the same time, he defined simple extensions and gave an alternative definition of strict extensions, which had been introduced by Stone [St]. He noted that the Katětov extension is a simple extension, but did not mention Fomin's work, or the fact that the Fomin extensions are strict extensions, although Katětov [K2] in 1947 showed that the Fomin extensions are strict. Banaschewski also showed [Ba] that every extension has an associated strict extension and an associated simple extension, both unique up to isomorphism. Iliadis and Fomin [IF] showed that every extension has associated strict and simple extensions and also that these are H-closed if and only if the original extension is. The class of all H-closed extensions of a given Hausdorff space is partitioned by the relation of having isomorphic associated strict, or simple, extensions. Thus a major step in determining the H-closed extensions of a given Hausdorff space would be the determination of the isomorphism classes of strict, or simple, H-closed extensions.

Since H-closure is a generalization of the concept of compact Hausdorff, and compact Hausdorff extensions are always strict and can be obtained as completions of totally bounded uniformities, it seems reasonable to investigate whether or not a generalization of the concept of uniformity will yield the strict H-closed extensions. This is the approach pursued in this paper. It was shown by Porter and Votaw [PV] that, in general, there are not enough uniform-like structures on a set to yield all isomorphism classes of strict H-closed extensions of a given one of its Hausdorff topologies. But a rather large class of such extensions are obtained as completions of structures called para-uniformities, which are moderate generalizations of the entourage concept of uniformities. Collections of such structures are then used to obtain a representative from each of the remaining isomorphism classes of strict H-closed extensions, in Chapter 4 of this paper. Para-uniformities are defined and investigated in Chapter 2. Para-uniform completions are studied in Chapter 3.

For a given Hausdorff space, the isomorphism classes of strict H-closed extensions which can be obtained as para-uniform completions are characterized as those which have relatively completely regular outgrowth, a phrase introduced in this paper. These include all those with relatively zero-dimensional outgrowth, a class studied by Flachsmeyer [Fl]. It is shown here that those strict H-closed extensions with relatively zero-dimensional outgrowth can be obtained as the completion of a totally bounded para-uniformity with a transitive basis, and that those with relatively completely regular outgrowth can be obtained as the completion of a totally bounded para-uniformity.

At least three other methods of obtaining a representative from each isomorphism class of strict H-closed extensions of a given Hausdorff space may be found in the literature. Stone's Boolean algebraic methods give all T_0-extensions of an arbitrary T_0-space [St, Thm. 41, p. 425], but no method of distinguishing the H-closed extensions is given. Ivanov [Iv] in 1966 used systems of open covers and the "stars" of open ultrafilters in the covers to define a collection of filters. This collection is then topologized so as to create a filter extension, in the manner described by Alexandroff in 1939 [A1]. This is also the method of generating filter extensions mentioned above in connection with Banaschewski [Ba]. Porter and Votaw remarked [PV] that the necessary collection of filters could be obtained simply by taking a maximal separated family of open filters. The method of the present paper will perhaps provide more insight into the structure of the given space by providing additional structure for the extension.

A more extensive bibliography of H-closed spaces and related concepts can be found in Berri, Porter and Stephenson [BPS]. The reader is referred to Kelley [Ke] or Bourbaki [B1] for a development of uniform space theory.

2. Basic preliminaries. Some preliminary notation and definitions are given in this section. The reader is assumed to be familiar with the concepts and notation of elementary set theory and general topology, including lattices and filters (see, e.g., Kelley [Ke] or Bourbaki [B1], [B2]). When (X,τ) is said to be a topological space, it is assumed that τ is a topology on the set X. If $x \in X$, then $\tau\langle x\rangle$ denotes the set

$\{G \in \tau | x \in G\}$; and, for $A \subseteq X$, $\tau_A\langle x \rangle$ denotes the set $\{G \cap A | G \in \tau\langle x \rangle\}$. The closure of A relative to the topology τ may be denoted by $cl_\tau(A)$, although it will be denoted by $\overline{A}$ unless there is a likelihood of confusion as to which topology is meant. When A and B are considered as sets, A-B denotes set difference. The set of positive integers is denoted by $\underline{\underline{N}}$, and the real line by $\underline{\underline{R}}$.

A list of notational definitions is given below. Let X be a set, $x \in X$, $A \subseteq X$ and $U,V \subseteq X \times X$.

(1.1) $U[x] \quad \{y \in X | (x,y) \in U\}$.

(1.2) $U[A] \quad \cup \{U[a] | a \in A\}$; $U[\emptyset] \quad \emptyset$.

(1.3) $U^{-1} \quad \{(v,y) \in X \times X | (y,v) \in U\}$.

(1.4) $U \bullet V \quad \{(v,y) \in X \times X | (v,w) \in V,\ (w,y) \in U$, for some $w \in X\}$.

(1.5) $\Delta(A) \quad \{(a,a) \in X \times X | a \in A\}$; $\Delta \quad \Delta(X)$.

(1.6) $U^0 \quad \Delta(U^{-1}[X])$.

(1.7) $U^n \quad U \bullet U^{n-1}$, for $n \in \underline{\underline{N}}$.

Some set theoretic properties related to the above notational definitions are given next. Most of these are listed in Murdeshwar and Naimpally [MN] and are included here for the convenience of the reader. The proofs are omitted. Let X be a set, Λ and Ω index sets, and $U,V,W,V_\alpha \subseteq X \times X$ for $\alpha \in \Lambda$, and let $A,B,C,A_\beta \subseteq X$ for $\beta \in \Omega$. A subset $D \subseteq X \times X$ is called <u>symmetric</u> if $D \quad D^{-1}$.

(1.8) U,V,X may be chosen so that $U \bullet V \neq V \bullet U$.

(1.9) $U \bullet (V \bullet W) \quad (U \bullet V) \bullet W$.

(1.10) $U \bullet \Delta \quad U \quad \Delta \bullet U$; $U \quad U \bullet U^0$; $U \quad \Delta(U[X]) \bullet U$.

(1.11) $U \cdot (\bigcup_{\alpha \varepsilon \Lambda} V_\alpha) = \bigcup_{\alpha \varepsilon \Lambda} (U \cdot V_\alpha)$; $(\bigcup_{\alpha \varepsilon \Lambda} V_\alpha) \cdot U \quad \bigcup_{\alpha \varepsilon \Lambda} (V_\alpha \cdot U)$.

(1.12) $U \cdot (\bigcap_{\alpha \varepsilon \Lambda} V_\alpha) \subseteq \bigcap_{\alpha \varepsilon \Lambda} (U \cdot V_\alpha)$; $(\bigcap_{\alpha \varepsilon \Lambda} V_\alpha) \cdot U \subseteq \bigcap_{\alpha \varepsilon \Lambda} (V_\alpha \cdot U)$.

(1.13) $U \subseteq V$ implies $U \cdot W \subseteq V \cdot W$ and $W \cdot U \subseteq W \cdot V$.

(1.14) $U^0 \subseteq V$ implies $U \subseteq U \cdot V$; $\Delta(U[X]) \subseteq V$ implies $U \subseteq V \cdot U$.

(1.15) $(U^{-1})^{-1} \quad U$.

(1.16) $(U \cdot V)^{-1} \quad V^{-1} \cdot U^{-1}$.

(1.17) $U \cdot U^{-1}$, $U^{-1} \cdot U$, $U \cup U^{-1}$, $U \cap U^{-1}$ are symmetric.

(1.18) $(\bigcup_{\alpha \varepsilon \Lambda} V_\alpha)^{-1} \quad \bigcup_{\alpha \varepsilon \Lambda} V_\alpha^{-1}$; $(\bigcap_{\alpha \varepsilon \Lambda} V_\alpha)^{-1} \quad \bigcap_{\alpha \varepsilon \Lambda} V_\alpha^{-1}$.

(1.19) $(A \times B)^{-1} \quad B \times A$.

(1.20) $A \subseteq B$ implies $\Delta(B)[A] \quad A$.

(1.21) $((X \times X) - U)^{-1} \quad (X \times X) - U^{-1}$.

(1.22) $U \subseteq V$ implies $U[A] \subseteq V[A]$.

(1.23) $U[x] \subseteq V[x]$ for all $x \varepsilon X$ implies $U \subseteq V$.

(1.24) $(\bigcup_{\alpha \varepsilon \Lambda} V_\alpha)[A] \quad \bigcup_{\alpha \varepsilon \Lambda} (V_\alpha[A])$.

(1.25) $(\bigcap_{\alpha \varepsilon \Lambda} V_\alpha)[A] \subseteq \bigcap_{\alpha \varepsilon \Lambda} (V_\alpha[A])$; $(\bigcap_{\alpha \varepsilon \Lambda} V_\alpha)[x] \quad \bigcap_{\alpha \varepsilon \Lambda} (V_\alpha[x])$ for all $x \varepsilon X$.

(1.26) $U[\bigcup_{\beta \varepsilon \Omega} A_\beta] \quad \bigcup_{\beta \varepsilon \Omega} U[A_\beta]$; $U[\bigcap_{\beta \varepsilon \Omega} A_\beta] \subseteq \bigcap_{\beta \varepsilon \Omega} U[A_\beta]$.

(1.27) $(U \cdot V)[A] \quad U[V[A]]$.

(1.28) $(A \times B)[C] \quad B$ if $C \cap A \neq \emptyset$; $(A \times B)[C] \quad \emptyset$ if $C \cap A = \emptyset$.

(1.29) $U^0 \subseteq U$ implies $U^n \subseteq U^{n+1}$ for all $n \varepsilon \underline{\underline{N}}$.

(1.30) $U^0 \quad V^0$ if and only if $U^{-1}[X] \quad V^{-1}[X]$.

(1.31) $U^0 \subseteq U$, $V^0 \subseteq V$ imply $(U \cap V)^0 \quad U^0 \cap V^0$.

(1.32) $U^0 \subseteq U$ implies $U^{-1}[X] \subseteq U[X]$.

(1.33) $U^0 \quad \emptyset$ if and only if $U = \emptyset$.

(1.34) $U^0[X] \quad U^{-1}[X]$.

Additional properties, related to functions, will now be listed with proofs omitted. Let X, Y, H, K be sets; let $f \quad X \to Y$, $g \quad H \to K$ be functions; let $A, B \subseteq X$ and $C, D \subseteq Y$; and let $U, U_1, U_2 \subseteq X \times X$ and $V, V_1, V_2 \subseteq Y \times Y$. For functions $r \quad D_1 \to D_2$ and $s \quad E_1 \to E_2$, define the function

$$(r \times s) \quad D_1 \times E_1 \to D_2 \times E_2$$

by $(r \times s)(d,e) \quad (r(d), s(e))$ for all $d \in D_1$, $e \in E_1$.

(1.35) If gf exists, so does $(g \times g)(f \times f)$, and $(gf) \times (gf) = (g \times g)(f \times f)$.

(1.36) If f is a bijection, so is $(f \times f)$, and $(f \times f)^{-1} = (f^{-1}) \times (f^{-1})$.

(1.37) $(f \times f)(A \times B) \quad f(A) \times f(B)$.

(1.38) $(f \times f)^{-1}(C \times D) \quad f^{-1}(C) \times f^{-1}(D)$.

(1.39) $f(U[x]) \subseteq ((f \times f)(U))[f(x)]$; equality holds if f is injective.

(1.40) $f^{-1}(V[f(x)]) \quad ((f \times f)^{-1}(V))[x]$.

(1.41) $(f \;\; f)(U^{-1}) \quad ((f \times f)(U))^{-1}$.

(1.42) $(f \times f)^{-1}(V^{-1}) \quad ((f \times f)^{-1}(V))^{-1}$.

(1.43) $(f \times f)(U_1 \cdot U_2) \subseteq ((f \times f)(U_1)) \cdot ((f \times f)(U_2))$; equality holds if f is injective.

(1.44) $(f \times f)^{-1}(V_1 \cdot V_2) \supseteq ((f \times f)^{-1}(V_1)) \cdot ((f \times f)^{-1}(V_2))$; equality holds if f is surjective.

(1.45) $V^0 \subseteq V$ implies $(f^{-1}(V^{-1}))[X] \quad f^{-1}(V^{-1}[Y])$.

The properties listed in this section will usually be used without specific mention.

CHAPTER 2

PARA-UNIFORM SPACES

1. Para-uniformities. Para-uniform spaces are defined and some of their properties established in this section. A para-uniformity on a set X will be seen to induce, in a natural way, a topology on X. Further, any topology can be induced in this manner by a para-uniformity. The collection of para-uniformities on a set X form a complete lattice when partially ordered by set inclusion. Initial and final para-uniformities will be investigated in section 2, and in section 3 some relations between topological extensions and para-uniformities will be derived.

The symbol X will always denote a nonempty set.

Definition 2.1. Let $\mathcal{U}$ be a collection of nonempty subsets of $X \times X$ which satisfies conditions (U1) through (U5).

(U1) $X \times X \in \mathcal{U}$.

(U2) $U \in \mathcal{U}$ implies $U^0 \subseteq U$.

(U3) $U \in \mathcal{U}$ implies $U \cap U^{-1} \in \mathcal{U}$.

(U4) $U, V \in \mathcal{U}$ and $U \cap V \neq \emptyset$ imply there exists a $W \in \mathcal{U}$ such that $W^2 \subseteq U \cap V$ and $W^0 \quad (U \cap V)^0$.

(U5) $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$ with $U^0 \quad V^0$ imply $V \in \mathcal{U}$.

Then $\mathcal{U}$ is a para-uniformity on X, and $(X, \mathcal{U})$ is a para-uniform space. An element of $\mathcal{U}$ is called an entourage.

Note that if (U2) is strengthened to require that $\Delta \subseteq U$ for every $U \in \mathcal{U}$, then the para-uniformity $\mathcal{U}$ is a uniformity.

Of course, in this case, some of the requirements included in (U1) through (U5) are redundant, but this shows the conditions are consistent and the collection of para-uniformities on a set X is nontrivial, in general. Further examples of para-uniformities will be given later.

For a para-uniformity $\mathcal{U}$ on X and $x \in X$, let

$$\mathcal{U}(x) \quad \{U[x] \mid U \in \mathcal{U}\} - \{\emptyset\}.$$

This notation is convenient in showing that $\mathcal{U}$ induces a topology on X.

<u>Proposition 2.2</u>. Let $(X, \mathcal{U})$ be a para-uniform space. Then the set $\{\mathcal{U}(x) \mid x \in X\}$ is a neighborhood system on X.

<u>Proof</u>. $X \times X \in \mathcal{U}$ implies $X \in \mathcal{U}(x)$ for every $x \in X$. Now let $x \in X$ and let $G \in \mathcal{U}(x)$. Then there is a $U \in \mathcal{U}$, such that $\emptyset \neq U[x] \quad G$. Then $x \in U[x]$, by (U2). Let $U[x] \subseteq A \subseteq X$ and let $V \quad (\{x\} \times A) \cup U$. Then $V \in \mathcal{U}$, by (U5), and $V[x] \quad A$, so $A \in \mathcal{U}(x)$. Now let $H \in \mathcal{U}(x)$ and let $W \in \mathcal{U}$ such that $\emptyset \neq W[x] \quad H$. Then $(x,x) \in U \cap W$, so $U \cap W \in \mathcal{U}$, by (U4) and (U5). But also, $x \in (U \cap W)[x] \quad U[x] \cap W[x]$, so $G \cap H \in \mathcal{U}(x)$. Finally, there is a symmetric $Q \in \mathcal{U}$ such that $Q^2 \subseteq U$ and $Q^0 \quad U^0$, by (U3) and (U4) (recall that Q symmetric means $Q \quad Q^{-1}$). But then $x \in Q[x]$, so $Q[x] \in \mathcal{U}(x)$. Since Q is symmetric, $y \in Q[y]$, for $y \in Q[x]$, by (U2). Thus, for $y \in Q[x]$, $y \in Q[y] \subseteq Q[Q[x]] \quad Q^2[x] \subseteq U[x] \quad G$. Thus $G \in \mathcal{U}(y)$ for every $y \in Q[x]$. Hence $\{\mathcal{U}(x) \mid x \in X\}$ is a neighborhood system on X.||

The topology on X defined by the neighborhood system $\{\mathcal{U}(x) \mid x \in X\}$, where $(X, \mathcal{U})$ is a para-uniform space, will be denoted by $\tau(\mathcal{U})$. Note that $G \in \tau(\mathcal{U})$ if and only if $x \in G$ implies there is a $U \in \mathcal{U}$ such that $x \in U[x] \subseteq G$. Note that if $\mathcal{U}, \mathcal{V}$ are para-uniformities on X with $\mathcal{U} \subseteq \mathcal{V}$, then $\tau(\mathcal{U}) \subseteq \tau(\mathcal{V})$.

<u>Definition 2.3</u>. Let $(X, \mathcal{U})$ be a para-uniform space. Then $\mathcal{U}$ is said to be <u>compatible</u> with $\tau(\mathcal{U})$. If $(X, \tau(\mathcal{U}))$ is Hausdorff, then $\mathcal{U}$ will be called a <u>separated</u>, or Hausdorff, <u>para-uniformity</u> and $(X, \mathcal{U})$ a <u>separated</u>, or Hausdorff, <u>para-uniform space</u>.

<u>Proposition 2.4</u>. Let $(X, \mathcal{U})$ be a para-uniform space, let $U \in \mathcal{U}$ and let $A, B \subseteq X$.

(a) $U^{-1}[X] \in \tau(\mathcal{U})$.

(b) If U is symmetric and A is dense in X, then

$$U[A] \quad U[X] \in \tau(\mathcal{U}).$$

(c) If U is symmetric, $A \cup B$ is dense in X and $U[A] \subseteq D \subseteq U[X]$, then D is a neighborhood of $D - (U[B])$.

<u>Proof</u>. (a): $x \in U^{-1}[X]$ implies $x \in (U \cap U^{-1})[x] \subseteq U^{-1}[X]$, by (U2), and $(U \cap U^{-1})[x] \in \mathcal{U}(x)$, by (U3).

(b): Since $A \subseteq X$, $U[A] \subseteq U[X]$. Now let $x \in U[X]$. Then $x \in U[x]$, since $U^0 \subseteq U$, and $A \cap U[x] \neq \emptyset$, since A is dense in X. Let $a \in A \cap U[x]$. Then $(x,a) \in U$, so $(a,x) \in U$, since U is symmetric. But then $x \in U[a] \subseteq U[A]$. Hence $U[X] \quad U[A]$. Since U is symmetric, $U[X] \quad U^{-1}[X]$, and $U^{-1}[X] \in \tau(\mathcal{U})$, by (a) above.

(c): Let $d \in D-(U[B])$ and let $V \in \mathcal{U}$ such that V is symmetric, $V^2 \subseteq U$ and $V^0 \quad U^0$. Since $d \in U[X] \quad V[X]$, $d \in V[d]$.

Now suppose $y \varepsilon V[d] - D$. Then $y \varepsilon V[B]$, since

$$V[d] \subseteq V[X] \quad V[A] \cup V[B],$$

by (b) above and (1.26), and $V[A] \subseteq U[A] \subseteq D$. But then $(b,y) \varepsilon V$, for some $b \varepsilon B$. Hence $(b,d) \varepsilon V^2 \subseteq U$, since V is symmetric and $(d,y) \varepsilon V$. Therefore, $d \varepsilon U[B]$, a contradiction. Hence, $V[d] \subseteq D$, so D is a neighborhood of $D - (U[B])$.||

It is convenient to consider collections with certain properties which generate, in a specified manner, unique para-uniformities. This is the subject of the next few definitions and propositions.

<u>Proposition 2.5</u>: Let $\mathcal{B}$ be a collection of nonempty subsets of $X \times X$ which satisfies (U2), (U4) of Definition 2.1 and (B3) below.

(B3) $B \varepsilon \mathcal{B}$ implies there is $D \varepsilon \mathcal{B}$ such that $D \subseteq B \cap B^{-1}$ and $D^0 \quad B^0$.

Let $\mathcal{U}(\mathcal{B}) \quad \{X \times X\} \cup (\{U \subseteq X \times X \mid$ for some $B \varepsilon \mathcal{B}$, $B \subseteq U$ and $B^0 \quad U^0\} \quad \{\emptyset\})$. Then $\mathcal{U}(\mathcal{B})$ is the smallest para-uniformity on X which contains $\mathcal{B}$.

<u>Proof</u>. Clearly, $\mathcal{B} \subseteq \mathcal{U}(\mathcal{B})$, $U \neq \emptyset$ for all $U \varepsilon \mathcal{U}(\mathcal{B})$, and $\mathcal{U}(\mathcal{B})$ satisfies (U1). Let $U \varepsilon \mathcal{U}(\mathcal{B})$. Then there is a $B \varepsilon \mathcal{B}$ such that $B \subseteq U$ and $B^0 \quad U^0$. But then $U^0 \quad B^0 \subseteq B \subseteq U$. Thus $\mathcal{U}(\mathcal{B})$ satisfies (U2). But there is also a $D \varepsilon \mathcal{B}$ such that $D \subseteq B \cap B^{-1}$ and $D^0 \quad B^0$, so $D \subseteq U \cap U^{-1}$ and $D^0 \quad U^0 \quad (U \cap U^{-1})^0$. Therefore, $U \cap U^{-1} \varepsilon \mathcal{U}(\mathcal{B})$, so $\mathcal{U}(\mathcal{B})$ satisfies (U3).

Now let $U,V \varepsilon \mathcal{U}(\mathcal{B})$ with $U \cap V \neq \emptyset$, and let $B,C \varepsilon \mathcal{B}$ with $B \subseteq U$, $C \subseteq V$, $B^0 \quad U^0$ and $C^0 \quad V^0$. Then $B \cap C \neq \emptyset$, so let $D \varepsilon \mathcal{B}$

such that $D^2 \subseteq B \cap C$ and $D^0 \quad (B \cap C)^0$. Then $D^2 \subseteq U \cap V$ and $D^0 \quad (B \cap C)^0 \quad B^0 \cap C^0 \quad U^0 \cap V^0 \quad (U \cap V)^0$. Thus $\mathcal{U}(\mathcal{B})$ satisfies (U4). And $\mathcal{U}(\mathcal{B})$ satisfies (U5) by definition of $\mathcal{U}(\mathcal{B})$. Therefore, $\mathcal{U}(\mathcal{B})$ is a para-uniformity on X which contains $\mathcal{B}$.

On the other hand, if $\mathcal{V}$ is a para-uniformity on X such that $\mathcal{B} \subseteq \mathcal{V}$, then $\mathcal{U}(\mathcal{B}) \subseteq \mathcal{V}$, since $\mathcal{V}$ satisfies (U1) and (U5) of Definition 2.1. Hence $\mathcal{U}(\mathcal{B})$ is the smallest para-uniformity on X which contains $\mathcal{B}$.||

<u>Definition 2.6</u>. Let $\mathcal{B}$ be a collection of nonempty subsets of X×X which satisfies (U2), (U4) of Definition 2.1 and (B3) of Proposition 2.5. Then $\mathcal{B}$ is a <u>para-uniform basis</u> on X and is a basis for $\mathcal{U}(\mathcal{B})$ as defined in Proposition 2.5.

<u>Proposition 2.7</u>. Let $\mathcal{S}$ be a collection of nonempty subsets of X×X which satisfies condition (U2) of Definition 2.1 and (S4) below.

(S4) $S \in \mathcal{S}$ implies there exists a $T \in \mathcal{S}$ such that

$$T^2 \subseteq S \cap S^{-1} \text{ and } T^0 \quad S^0.$$

Let $\mathcal{B}(\mathcal{S}) \quad \{\cap \mathcal{T} \mid \mathcal{T} \text{ is a finite subset of } \mathcal{S}\} \quad \{\emptyset\}$, where $\cap \emptyset \quad X{\times}X$. Then $\mathcal{B}(\mathcal{S})$ is a para-uniform basis on X and $\mathcal{U}(\mathcal{B}(\mathcal{S}))$ is the smallest para-uniformity on X which contains $\mathcal{S}$.

<u>Proof</u>. Clearly, $\mathcal{B}(\mathcal{S})$ is a collection of nonempty subsets of X and $\mathcal{S} \subseteq \mathcal{B}(\mathcal{S})$. Now let $\mathcal{T}$ be a finite subset of $\mathcal{S}$ such that $\cap \mathcal{T} \neq \emptyset$. Then, by finite induction on (1.31),

$$(\cap \mathcal{T})^0 \quad \cap \{T^0 \mid T \in \mathcal{T}\} \subseteq \cap \mathcal{T}.$$

Thus $\mathcal{B}(\mathcal{S})$ satisfies (U2) of Definition 2.1. For each $T \in \mathcal{T}$, let $F(T) \in \mathcal{S}$ such that $F(T)^2 \subseteq T \cap T^{-1}$ and $F(T)^0 \quad T^0$. Let

$\mathcal{F} = \{F(T) \mid T \in \mathcal{T}\}$. Then $\mathcal{F}$ is a finite subset of $\mathcal{S}$ and $(\cap \mathcal{F})^0 \quad (\cap \mathcal{T})^0$, so $(\cap \mathcal{F}) \neq \emptyset$. And

$$(\cap \mathcal{F}) \subseteq \cap \{T \cap T^{-1} \mid T \in \mathcal{T}\} \quad (\cap \mathcal{T}) \subseteq (\cap \mathcal{T})^{-1}.$$

Thus $\mathcal{B}(\mathcal{S})$ satisfies (B3) of Proposition 2.5.

Now let $\mathcal{F}, \mathcal{T}$ be finite subsets of $\mathcal{S}$ such that $B \cap C \neq \emptyset$, where $B \quad \cap \mathcal{F}$, $C \quad \cap \mathcal{T}$. For each $F \in \mathcal{F} \cup \mathcal{T}$, let $A(F) \in \mathcal{S}$ such that $A(F)^2 \subseteq F$ and $A(F)^0 \quad F^0$ Let $\mathcal{A} = \{A(F) \mid F \in \mathcal{F} \cup \mathcal{T}\}$ and let $A \quad \cap \mathcal{A}$. Then $\mathcal{A}$ is a finite subset of $\mathcal{S}$ and $A^0 \quad (B \cap C)^0 \neq \emptyset$, so $A \neq \emptyset$. Moreover,

$$A^2 \subseteq \cap \{A(F)^2 \mid F \in \mathcal{F} \cup \mathcal{T}\} \subseteq B \cap C,$$

so $\mathcal{B}(\mathcal{S})$ satisfies (U4) of Definition 2.1. Thus $\mathcal{B}(\mathcal{S})$ is a para-uniform basis on X and $\mathcal{S} \subseteq \mathcal{U}(\mathcal{B}(\mathcal{S}))$.

If $\mathcal{V}$ is a para-uniformity on X such that $\mathcal{S} \subseteq \mathcal{V}$, then $\mathcal{B}(\mathcal{S}) \subseteq \mathcal{V}$, since $\mathcal{V}$ satisfies (U4) and (U5) of Definition 2.1. Hence $\mathcal{U}(\mathcal{B}(\mathcal{S}))$ is the smallest para-uniformity on X which contains $\mathcal{S}$, by Proposition 2.5.||

<u>Definition 2.8</u>. Let $\mathcal{S}$ be a collection of nonempty subsets of X×X which satisfies (U2) of Definition 2.1 and (S4) of Proposition 2.7. Then $\mathcal{S}$ is a <u>para-uniform</u> <u>subbasis</u> on X and is a subbasis for $\mathcal{B}(\mathcal{S})$ as defined in Proposition 2.7 and for $\mathcal{U}(\mathcal{B}(\mathcal{S}))$. The para-uniformity $\mathcal{U}(\mathcal{B}(\mathcal{S}))$ will usually be denoted by $\mathcal{U}(\mathcal{S})$.

With these tools, some examples of para-uniformities will now be given. It will be shown that every topology on X is induced by a para-uniformity.

Theorem 2.9. Let (X,τ) be a topological space and let σ be a collection of nonempty subsets of X. Let

$$\mathcal{S} \quad \{A\times A|A \varepsilon \sigma\} \text{ and } \Delta(\sigma) \quad \{\Delta(A)|A \varepsilon \sigma\}.$$

(a) $\mathcal{S}$ and $\Delta(\sigma)$ are para-uniform subbases on X.

(b) If $\cap\beta \varepsilon \sigma$ for every finite subset $\beta \subseteq \sigma$ with $\cap\beta \neq \emptyset$, then $\mathcal{S}$ and $\Delta(\sigma)$ are para-uniform bases on X.

(c) If $\sigma \subseteq \tau$, then $\tau(\mathcal{U}(\mathcal{S})) \subseteq \tau$. Also $\tau(\mathcal{U}(\mathcal{S})) \quad \tau$ if and only if σ is a subbasis for τ.

Proof. (a): Let $A \varepsilon \sigma$. Then $(A\times A)^0 \quad \Delta(A) \quad \Delta(A)^0$, so $\mathcal{S}$ and $\Delta(\sigma)$ satisfy (U2). But $(A\times A)^{-1} \quad A\times A = (A\times A)^2$ and $\Delta(A)^{-1} \quad \Delta(A) \quad \Delta(A)^2$, so $\mathcal{S}$ and $\Delta(\sigma)$ satisfy (S4).

(b): Assume that $\cap\beta \varepsilon \sigma$ for every finite subset $\beta \subseteq \sigma$ with $\cap\beta \neq \emptyset$. Let β be a finite subset of σ such that $\cap\{A\times A|A \varepsilon \beta\} \neq \emptyset$, or equivalently $\cap\{\Delta(A)|A \varepsilon \beta\} \neq \emptyset$. Then $\cap\beta \neq \emptyset$, so $\cap\beta \varepsilon \sigma$, and $(\cap\beta)\times(\cap\beta) \quad \cap\{A\times A|A \varepsilon \beta\}$, $\Delta(\cap\beta) = \cap\{\Delta(A)|A \varepsilon \beta\}$. Thus $\mathcal{B}(\mathcal{S}) \quad \mathcal{S}$, so $\mathcal{S}$ and $\Delta(\sigma)$ are para-uniform bases on X.

(c): Assume $\sigma \subseteq \tau$ and let $x \varepsilon G \varepsilon \tau(\mathcal{U}(\mathcal{S}))$. Then there is a finite subset $\beta \subseteq \sigma$ such that $x \varepsilon (\cap\{A\times A|A \varepsilon \beta\})[x] \subseteq G$. But then $x \varepsilon A$ for all $A \varepsilon \beta$, so $(\cap\{A\times A|A \varepsilon \beta\})[x] = \cap\beta$. Thus $G \varepsilon \tau$, since $\cap\beta \varepsilon \tau$. Hence $\tau(\mathcal{U}(\mathcal{S})) \subseteq \tau$. Now assume σ is a subbasis for τ and let $y \varepsilon H \varepsilon \tau$. Then there is a finite subset $\gamma \subseteq \sigma$ such that $y \varepsilon \cap\gamma \subseteq H$. But then

$$y \varepsilon \cap\gamma \quad (\cap\{D\times D|D \varepsilon \gamma\})[y],$$

so $H \varepsilon \tau(\mathcal{U}(\mathcal{S}))$. Thus $\tau \quad \tau(\mathcal{U}(\mathcal{S}))$, if σ is a subbasis for τ.

Conversely, assume that $\tau \quad \tau(\mathcal{U}(\mathcal{S}))$ and let $x \in G \in \tau$. Then there is a finite subset $\beta \subseteq \sigma$ such that

$$x \in \cap\beta \quad (\cap\{A\times A \mid A \in \beta\})[x] \subseteq G.$$

Hence σ is a subbasis for τ.||

Note that $\{A\times A \mid A \in \sigma\} - \{\emptyset\}$ need not be a para-uniform basis when σ is a basis for τ, since there may be $A,B \in \sigma$ with $\emptyset \neq A \cap B \notin \sigma$. The following useful proposition is related to this idea.

<u>Proposition 2.10</u>. Let $(X,\mathcal{U})$ be a para-uniform space, let $\mathcal{B}$ be a basis for $\mathcal{U}$, and let $\mathcal{S}$ be a subbasis for $\mathcal{U}$. For each $x \in X$, let $\mathcal{B}(x) \quad \{X\} \cup (\{B[x] \mid B \in \mathcal{B}\} - \{\emptyset\})$ and let $\mathcal{S}(x) \quad \{S[x] \mid S \in \mathcal{S}\} \quad \{\emptyset\}$. Then $\mathcal{B}(x)$ is a $\tau(\mathcal{U})$-neighborhood base at x and $\mathcal{S}(x)$ is a $\tau(\mathcal{U})$-neighborhood subbase at x.

<u>Proof</u>. If $U \in \mathcal{U}$ with $U \neq X\times X$ and $x \in U[x]$, then there is a $B \in \mathcal{B}$ and a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $B \subseteq U$, $B^0 \quad U^0$, $\cap\mathcal{F} \subseteq U$ and $(\cap\mathcal{F})^0 \quad U^0$. Then $x \in B[x] \subseteq U[x]$ and $\cap\{F[x] \mid F \in \mathcal{F}\} \quad (\cap\mathcal{F})[x] \subseteq U[x]$. Thus $\mathcal{B}(x)$ is a $\tau(\mathcal{U})$-neighborhood base at x and $\mathcal{S}(x)$ is a $\tau(\mathcal{U})$-neighborhood subbase at x.||

<u>Remark</u>. $\{X\times X\}$ is a para-uniformity on the set X which is contained in every para-uniformity on X. $\{X\times X\}$ induces the indiscrete topology on X and will be called the <u>indiscrete para-uniformity</u> on X. According to Theorem 2.9(c), the set $\{\Delta(A) \mid \emptyset \neq A \subseteq X\}$ generates a para-uniformity $\mathcal{V}$ on X which induces the discrete topology. Moreover, it is not difficult to see that every para-uniformity on X is contained in $\mathcal{V}$, so it will be called the <u>discrete para-uniformity</u> on X.

The question arises rather naturally as to the possibility of numerous distinct para-uniformities compatible with a given topology. Or, put another way, which topological spaces have a unique compatible para-uniformity. This question is answered in the next proposition.

<u>Proposition 2.11</u>. Let (X,τ) be a topological space. Then τ has a unique compatible para-uniformity if and only if every nonempty element of τ is dense in X and τ is the only basis for τ which contains $\emptyset$ and X and is closed under finite intersections.

<u>Proof</u>. First, assume there is a $G \in \tau$ such that $\emptyset \neq G$ and $\overline{G} \neq X$. Then there is an $H \in \tau$ such that $\emptyset \neq H$ and $G \cap H = \emptyset$. But then the set $\{T\times T \mid \emptyset \neq T \in \tau\}$ is a basis for a para-uniformity $\mathcal{U}$ compatible with τ, by Theorem 2.9(b),(c). If $\beta \subseteq \tau$ such that

$$\cap\{T\times T \mid T \in \beta\} \subseteq (G\times G) \cup (H\times H) \text{ and}$$

$$(\cap\{T\times T \mid T \in \beta\})^0 \quad ((G\times G) \cup (H\times H))^0, \text{ then}$$

$$(G \cup H) \times (G \cup H) \subseteq \cap\{T\times T \mid T \in \beta\}$$

Thus $(G\times G) \cup (H\times H) \notin \mathcal{U}$. But $(G\times G) \cup (H\times H)$ is symmetric and $((G\times G) \cup (H\times H))^2 \quad (G\times G) \cup (H\times H)$, hence

$$\{(G\times G) \cup (H\times H)\} \cup \{T\times T \mid \emptyset \neq T \in \tau\}$$

forms a subbasis for a para-uniformity compatible with τ but distinct from $\mathcal{U}$.

Now assume that σ is a finitely multiplicative basis for τ such that $\emptyset, X \in \sigma$ and $\sigma \neq \tau$. Let $G \in \tau - \sigma$ and let $\mathcal{U}$ be the para-uniformity on X generated by the set $\{A\times A \mid \emptyset \neq A \in \sigma\}$

as basis (this is a basis by Theorem 2.9(b)). Then $\tau \quad \tau(\mathcal{U})$, by Theorem 2.9(c), and $G \times G \notin \mathcal{U}$. But the set $\{T \times T \mid \emptyset \neq T \in \tau\}$ generates a para-uniformity compatible with τ and containing $G \times G$.

Conversely, assume that every nonempty element of τ is dense in X and that $\mathcal{U}$ is a para-uniformity compatible with τ and τ is the only finitely multiplicative basis for τ containing $\emptyset$ and X. Let $U \in \mathcal{U}$ with $U \quad U^{-1}$ and $x \in X$ such that $x \in U[x]$. Then there is a symmetric $V \in \mathcal{U}$ such that $V^4 \subseteq U$ and $V^0 \quad U^0$. Let $y \in U[X]$. Then $y \in V[y]$, so $V[y] \cap V[x] \neq \emptyset$, since every nonempty open set is dense in X. But then $y \in V^2[x]$. Thus $U[X] \subseteq V^2[x] \subseteq U[x] \subseteq U[X]$. Hence $U[X] \times U[X] \quad V^2[x] \times V^2[x] \subseteq V^4 \subseteq U$, so $U \quad U[X] \times U[X]$. Therefore, the set $\{U^{-1}[X] \mid U \in \mathcal{U}\}$ forms a finitely multiplicative basis for τ, so $\tau - \{\emptyset\} \quad \{U^{-1}[X] \mid U \in \mathcal{U}\}$. Thus $\mathcal{U}$ is generated by $\{T \times T \mid \emptyset \neq T \in \tau\}$, so τ has a unique compatible para-uniformity.||

The next proposition provides further examples of para-uniformities on a set, although no compatibility with any given topology is implied.

<u>Proposition 2.12</u>. Let $\mathcal{B}$ be a para-uniform subbasis on X and let $\mathcal{A}$ be a collection of nonempty subsets of X. For each $A \subseteq X$ and $B \in \mathcal{B}$, let $B(A) \quad B \cap (A \times A)$.

(a) The set $\{B(A) \mid B \in \mathcal{B},\ A \in \mathcal{A}\} - \{\emptyset\}$ is a para-uniform subbasis on X.

(b) If $\mathcal{A}$ is finitely multiplicative and $\mathcal{B}$ is a para-uniform basis, then $\{B(A) \mid B \in \mathcal{B}, A \in \mathcal{A}\} - \{\emptyset\}$ is a para-uniform basis on X.

Proof. (a): Let $B \in \mathcal{B}$, $A \in \mathcal{A}$. Then $B(A)^0 \quad B^0 \cap \Delta(A) \subseteq B(A)$. Also, there is a $D \in \mathcal{B}$ such that $D^2 \subseteq B \cap B^{-1}$ and $D^0 \quad B^0$. Then $D(A)^2 \subseteq D^2(A) \subseteq B(A) \cap B(A)^{-1}$, $D(A)^0 \quad B(A)^0$, and $D(A) \neq \emptyset$ if and only if $B(A) \neq 0$. Thus (U2) and (S4) are satisfied by $\{B(A) | B \in \mathcal{B}, A \in \mathcal{A}\} - \{\emptyset\}$, hence it is a para-uniform subbasis on X, by Proposition 2.7.

(b): Assume $\mathcal{A}$ is finitely multiplicative and $\mathcal{B}$ is a para-uniform basis on X. Let $B, D \in \mathcal{B}$, $A, C \in \mathcal{A}$ with $B(A) \cap D(C) \neq \emptyset$. There is an $E \in \mathcal{B}$ such that $E \subseteq B \cap B^{-1}$ and $E^0 \quad B^0$. Then $E(A) \neq \emptyset$, $E(A) \subseteq B(A) \cap B(A)^{-1}$ and $E(A)^0 \quad B(A)^0$. Also, there is an $F \in \mathcal{B}$ such that $F^2 \subseteq B \cap D$ and $F^0 \quad (B \cap D)^0$. Then $A \cap C \in \mathcal{A}$, $F(A \cap C)^2 \subseteq F^2(A \cap C) \subseteq B(A) \cap D(C)$ and $F(A \cap C)^0 \quad F^0 \cap \Delta(A) \cap \Delta(C) \quad B \cap D \cap \Delta(A \cap C) \quad (B(A) \cap D(C))^0$. Thus (U2), (B3), and (U4) are satisfied by $\{B(A) | B \in \mathcal{B}, A \in \mathcal{A}\} - \{\emptyset\}$, so this is a para uniform basis on X, by Proposition 2.5.||

Recall that a partially ordered   is a complete lattice if every nonempty subset has an infimum and a supremum. The question naturally arises as to whether the collection of para-uniformities on a given set   forms a complete lattice, as do the collections uniformities and topologies. An affirmative answer is given by the next theorem.

Theorem 2.13. The collection of para-uniformities on a set X, partially ordered by set inclusion, forms a complete lattice.

Proof. Let $\{\mathcal{U}_\alpha | \alpha \in A\}$ be a nonempty collection of para-uniformities on X and let $\mathcal{S} = \cup \{\mathcal{U}_\alpha | \alpha \in A\}$. Then $\mathcal{S}$ is a para-uniform subbasis on X and $\mathcal{U}_\alpha \subseteq \mathcal{U}(\mathcal{S})$ for every $\alpha \in A$. Thus $\mathcal{U}(\mathcal{S})\ \ \sup\{\mathcal{U}_\alpha | \alpha \in A\}$, by Proposition 2.7.

Now let $\mathcal{T}$ be the set of all para-uniformities on X which are subsets of the set $\cap\{\mathcal{U}_\alpha | \alpha \in A\}$. Note that $\{X\times X\} \in \mathcal{T}$. Then $\sup\mathcal{T} \subseteq \cap\{\mathcal{U}_\alpha | \alpha \in A\}$ and, by definition of $\mathcal{T}$, if $\mathcal{U}$ is a para-uniformity on X and $\mathcal{U} \subseteq \cap\{\mathcal{U}_\alpha | \alpha \in A\}$, then $\mathcal{U} \in \mathcal{T}$. Thus $\sup\mathcal{T}\ \ \inf\{\mathcal{U}_\alpha | \alpha \in A\}$. Therefore, the collection of para-uniformities on X is a complete lattice.||

<u>Proposition 2.14</u>. Let $\{\mathcal{U}_\alpha | \alpha \in A\}$ be a nonempty collection of para-uniformities on X. If $\mathcal{U}_\alpha$ is a uniformity for each $\alpha \in A$, then $\sup\{\mathcal{U}_\alpha | \alpha \in A\}$ and $\inf\{\mathcal{U}_\alpha | \alpha \in A\}$ are uniformities.

<u>Proof</u>. Assume that $\mathcal{U}_\alpha$ is a uniformity for each $\alpha \in A$. Then $U \in \cup\{\mathcal{U}_\alpha | \alpha \in A\}$ implies that $\Delta\ \ \Delta(X)\ \ U^0 \subseteq U$. Therefore $\Delta \subseteq V$ for all $V \in \sup\{\mathcal{U}_\alpha | \alpha \in A\}$, and, since $\inf\{\mathcal{U}_\alpha | \alpha \in A\} \subseteq \mathcal{U}_\alpha$, $\Delta \subseteq V$ for all $V \in \inf\{\mathcal{U}_\alpha | \alpha \in A\}$. Thus $\sup\{\mathcal{U}_\alpha | \alpha \in A\}$ and $\inf\{\mathcal{U}_\alpha | \alpha \in A\}$ are uniformities.||

The following theorem will be useful later.

<u>Theorem 2.15</u>. Let $(X,\mathcal{U})$ be a para-uniform space and let τ be the product topology $\tau(\mathcal{U})\times\tau(\mathcal{U})$ on $X\times X$. Then $\tau \cap \mathcal{U}$ contains a symmetric basis for $\mathcal{U}$, that is, a basis consisting of symmetric elements.

<u>Proof</u>. Let $U \in \mathcal{U}$. Then there is a symmetric $V \in \mathcal{U}$ such that $V^3 \subseteq U \cap U^{-1}$ and $V^0\ \ U^0$. Let $(x,y) \in V$ and let $G\ \ \mathrm{int}_\tau(U)$. Then $V[x]\times V[y] \subseteq V^3 \subseteq U$, so $V \subseteq G \subseteq U$. Then, for $B\ \ G \cap G^{-1}$, $V \subseteq B \subseteq U$ and B is symmetric. Now $V \subseteq B \subseteq U$ implies $B^0\ \ U^0$, so $B \in \mathcal{U}$. But $G^{-1} \in \tau$, so $B \in \tau \cap \mathcal{U}$.

Let $\mathcal{B}$ be the set of symmetric elements of $\tau \cap \mathcal{U}$. Then $\mathcal{B}$ satisfies (U2) and (B3). Let $B, C \in \mathcal{B}$ such that $B \cap C \neq \emptyset$. Then there is $A \in \mathcal{U}$ such that $A^2 \subseteq B \cap C$ and $A^0 \quad (B \cap C)^0$. But then, as shown above, there is a $D \in \mathcal{B}$ such that $D \subseteq A$ and $D^0 \quad A^0$. Then $D^2 \subseteq B \cap C$ and $D^0 \quad (B \cap C)^0$, so $\mathcal{B}$ satisfies (U4). Thus $\mathcal{B}$ is a para-uniform basis on X. With the results of the preceding paragraph, it is easy to see that $\mathcal{U} = \mathcal{U}(\mathcal{B})$.||

2. <u>Initial and final para-uniformities</u>. Let $f\colon X \to Y$ be a function, where X and Y are sets. For any subset U of $Y \times Y$, let $f^{-1}(U)$ denote the set $(f \times f)^{-1}(U)$, i.e.,

$$f^{-1}(U) \quad \{(x,y) \in X \times X \mid (f(x), f(y)) \in U\}.$$

If $\mathcal{U}$ is a collection of subsets of $Y \times Y$, $f^{-1}(\mathcal{U})$ will denote the set $\{f^{-1}(U) \mid U \in \mathcal{U}\}$. Note that $(f^{-1}(U))^{-1} \quad f^{-1}(U^{-1})$ for $U \subseteq Y \times Y$. It will be seen below that functions induce initial and final para-uniformities just as they do uniformities and topologies.

<u>Proposition 2.16</u>. Let $f\colon X \to Y$ be a function and let $\mathcal{B}$ be a para-uniform subbasis on Y. Then $f^{-1}(\mathcal{B}) - \{\emptyset\}$ is a para-uniform subbasis on X and $\mathcal{U}(f^{-1}(\mathcal{B})) \quad \mathcal{U}(f^{-1}(\mathcal{U}(\mathcal{B})))$. If $\mathcal{B}$ is a para-uniform basis on Y, then $f^{-1}(\mathcal{B}) - \{\emptyset\}$ is a para-uniform basis on X.

<u>Proof</u>. Let $B \in \mathcal{B}$. Then $(f^{-1}(B))^0 \quad f^{-1}(B^0) \subseteq f^{-1}(B)$, and there is a $T \in \mathcal{B}$ such that $T^2 \subseteq B \cap B^{-1}$ and $T^0 \quad B^0$. Now if $f^{-1}(B) \neq \emptyset$, then $f^{-1}(B^0) \neq \emptyset$, so $f^{-1}(T) \neq \emptyset$, and $(f^{-1}(T))^2 \subseteq f^{-1}(T^2) \subseteq f^{-1}(B) \cap f^{-1}(B^{-1})$. Thus $f^{-1}(\mathcal{B}) - \{\emptyset\}$ satisfies (U2) and (S4), hence is a para-uniform subbasis on X.

Since $\mathscr{B} \subseteq \mathscr{U}(\mathscr{B})$, it is clear that $\mathscr{U}(f^{-1}(\mathscr{B})) \subseteq \mathscr{U}(f^{-1}(\mathscr{U}(\mathscr{B})))$. But if $\mathscr{F}$ is a finite subset of $\mathscr{B}$ such that $\cap\mathscr{F} \neq \emptyset$, then $f^{-1}(\cap\mathscr{F}) = \cap\{f^{-1}(F) \mid F \in \mathscr{F}\}$. Therefore $\mathscr{U}(f^{-1}(\mathscr{U}(\mathscr{B}))) \subseteq \mathscr{U}(f^{-1}(\mathscr{B}))$, hence $\mathscr{U}(f^{-1}(\mathscr{B})) = \mathscr{U}(f^{-1}(\mathscr{U}(\mathscr{B})))$.

Now assume that $\mathscr{B}$ is a para-uniform basis on Y and let $B, C \in \mathscr{B}$ such that $f^{-1}(B) \cap f^{-1}(C) \neq \emptyset$. Then $B \cap C \neq \emptyset$, so there is a $D \in \mathscr{B}$ such that $D^2 \subseteq B \cap C$ and $D^0 = (B \cap C)^0$. Then $(f^{-1}(D))^2 \subseteq f^{-1}(D^2) \subseteq f^{-1}(B) \cap f^{-1}(C)$, $(f^{-1}(D))^0 = (f^{-1}(B \cap C))^0 = (f^{-1}(B) \cap f^{-1}(C))^0$, and $f^{-1}(D) \neq \emptyset$. So $f^{-1}(\mathscr{B}) - \{\emptyset\}$ satisfies (U4), in addition to (U2) and (S4). But (S4) implies (B3), so $f^{-1}(\mathscr{B}) - \{\emptyset\}$ is a para-uniform basis on X.||

Note that if $\mathscr{U}$ is a para-uniformity on Y, then $f^{-1}(\mathscr{U})$ is finitely multiplicative and satisfies (U1) and (U3) of Definition 2.1.

<u>Theorem 2.17</u>. Let $\{\mathscr{U}_\alpha \mid \alpha \in A\}$ be a collection of para-uniformities on X and let $\tau_\alpha = \tau(\mathscr{U}_\alpha)$ for each $\alpha \in A$. Then

$$\tau(\sup\{\mathscr{U}_\alpha \mid \alpha \in A\}) = \sup\{\tau_\alpha \mid \alpha \in A\} \text{ and}$$
$$\tau(\inf\{\mathscr{U}_\alpha \mid \alpha \in A\}) \subseteq \inf\{\tau_\alpha \mid \alpha \in A\}.$$

<u>Proof</u>. Let $\mathscr{U} = \sup\{\mathscr{U}_\alpha \mid \alpha \in A\}$ and let $\tau = \sup\{\tau_\alpha \mid \alpha \in A\}$. Now $\mathscr{U}_\alpha \subseteq \mathscr{U}$ implies $\tau_\alpha \subseteq \tau(\mathscr{U})$, so $\tau \subseteq \tau(\mathscr{U})$. But $x \in G \in \tau(\mathscr{U})$ implies there is a $U_\alpha \in \mathscr{U}_\alpha$ for each $\alpha \in A$, such that $x \in \cap\{U_\alpha[x] \mid \alpha \in A\} \subseteq G$, where $U_\alpha = X \times X$ for all but finitely many $\alpha \in A$. Then $x \in \mathrm{int}_\alpha(U_\alpha[x]) \in \tau_\alpha$ for all $\alpha \in A$. Hence $G \in \tau$. Thus $\tau = \tau(\mathscr{U})$.

Now $\inf\{\tau_\alpha \mid \alpha \in A\} = \cap\{\tau_\alpha \mid \alpha \in A\}$ and $\inf\{\mathscr{U}_\alpha \mid \alpha \in A\} \subseteq \mathscr{U}_\alpha$, for all $\alpha \in A$. Thus $\tau(\inf\{\mathscr{U}_\alpha \mid \alpha \in A\}) \subseteq \inf\{\tau_\alpha \mid \alpha \in A\}$.||

The following example shows that equality in the preceding theorem does not hold in the case of infimums, even for uniformities.

Example 2.18. Let X be the real line and let $\mathcal{U}$ be the usual uniformity on X. For each finite subset $A \subseteq X$, let $V(A) \quad \Delta(A) \cup ((X-A)\times(X-A))$. Then the set

$$\{V(A) \mid A \text{ is a finite subset of } X\}$$

is a basis for a uniformity $\mathcal{V}$ on X and $\tau(\mathcal{V})$ is the discrete topology on X. Thus $\inf\{\tau(\mathcal{U}), \tau(\mathcal{V})\} = \tau(\mathcal{U})$, the usual topology on X. But $\inf\{\mathcal{U}, \mathcal{V}\} \quad \{X\times X\}$. Thus

$$\tau(\inf\{\mathcal{U}, \mathcal{V}\}) \quad \{\emptyset, X\} \neq \tau(\mathcal{U}) \quad \inf\{\tau(\mathcal{U}), \tau(\mathcal{V})\}.$$

In studying functions, a continuity concept is of value. This is now introduced.

Definition 2.19. Let $f\colon X \to Y$ be a function, let $\mathcal{U}$ be a para-uniformity on X and let $\mathcal{V}$ be a para-uniformity on Y. Then $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is para-uniformly continuous if $(f^{-1}(\mathcal{V}) \quad \{\emptyset\}) \subseteq \mathcal{U}$.

Note that the composite of para-uniformly continuous functions is para-uniformly continuous.

A corollary of Proposition 2.16 is the following proposition, stated without proof.

Proposition 2.20. Let $\mathcal{U}$ be a para-uniformity on X, let $\mathcal{V}$ be a para-uniformity on Y with subbasis $\mathcal{B}$, and let $f\colon X \to Y$ be a function. Then $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is para-uniformly continuous if and only if $(f^{-1}(\mathcal{B}) \quad \{\emptyset\}) \subseteq \mathcal{U}$.

Proposition 2.21. Let $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a para-uniformly continuous function, let τ $\tau(\mathcal{U})$ and let σ $\tau(\mathcal{V})$. Then $f\colon (X,\tau) \to (Y,\sigma)$ is continuous.

Proof. If $V \varepsilon \mathcal{V}$ such that $V[f(x)] \neq \emptyset$, then $f^{-1}(V) \neq \emptyset$ and $x \varepsilon (f^{-1}(V))[x]$ with $f((f^{-1}(V))[x]) \subseteq V[f(x)]$. Hence $f\colon (X,\tau) \to (Y,\sigma)$ is continuous.||

It is not true that continuous implies para-uniformly continuous. To see this let X be a set containing at least three points, let $\mathcal{V}$ be the discrete para-uniformity on X and let $\mathcal{U}$ be the para-uniformity with the set $\{\{(x,x)\} | x \varepsilon X\}$ as basis. Then $\tau(\mathcal{U})$ $\tau(\mathcal{V})$, so the identity function $f\colon (X,\tau(\mathcal{U})) \to (X,\tau(\mathcal{V}))$ is continuous, but $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is not para-uniformly continuous, since $\{(x,x),(y,y)\} \varepsilon f^{-1}(\mathcal{V}) - \mathcal{U}$, for $x,y \varepsilon X$ with $x \neq y$.

Remark. Let $(X,\mathcal{U}),(Y,\mathcal{V})$ be para-uniform spaces, let $\mathcal{W}$ be the discrete para-uniformity on X, let $\mathcal{J}$ be the indiscrete para-uniformity on Y, and let $f\colon X \to Y$ be a function. Then $f\colon (X,\mathcal{U}) \to (Y,\mathcal{J})$ and $(X,\mathcal{W}) \to (Y,\mathcal{V})$ are para-uniformly continuous.

Definition 2.22. Let $\{(X_\alpha,\mathcal{U}_\alpha) | \alpha \varepsilon A\}$ be a collection of para-uniform spaces and let $f_\alpha\colon X_\alpha \to X$ and $g_\alpha\colon X \to X_\alpha$ be functions, for each $\alpha \varepsilon A$. Let T be the set of all para-uniformities $\mathcal{U}$ on X such that $f_\alpha\colon (X_\alpha,\mathcal{U}_\alpha) \to (X,\mathcal{U})$ is para-uniformly continuous for each $\alpha \varepsilon A$. Then the final para-uniformity on X induced by $\{f_\alpha\colon (X_\alpha,\mathcal{U}_\alpha) \to X | \alpha \varepsilon A\}$ is the para-uniformity sup T. Let S be the set of all para-uniformities $\mathcal{V}$ on X such that

$g_\alpha: (X,\mathscr{V}) \to (X_\alpha,\mathscr{U}_\alpha)$ is para-uniformly continuous for each $\alpha \in A$. Then the <u>initial para-uniformity</u> on X <u>induced by</u>

$$\{g_\alpha: X \to (X_\alpha,\mathscr{U}_\alpha) \mid \alpha \in A\}$$

is the para-uniformity inf S.

<u>Proposition 2.23</u>. Let $\{(X_\alpha,\mathscr{U}_\alpha) \mid \alpha \in A\}$ be a collection of para-uniform spaces, let $\mathscr{U}$ be the final para-uniformity on X induced by $\{f_\alpha:(X_\alpha,\mathscr{U}_\alpha) \to X \mid \alpha \in A\}$ and let $\mathscr{V}$ be the initial para-uniformity on X induced by $\{g_\alpha: X \to (X_\alpha,\mathscr{U}_\alpha) \mid \alpha \in A\}$. Then $f_\alpha: (X_\alpha,\mathscr{U}_\alpha) \to (X,\mathscr{U})$ and $g_\alpha: (X,\mathscr{V}) \to (X_\alpha,\mathscr{U}_\alpha)$ are para-uniformly continuous, for each $\alpha \in A$. Moreover, $\mathscr{V}$ is generated by the set $(\cup\{g_\alpha^{-1}(\mathscr{U}_\alpha) \mid \alpha \in A\})$ $\{\emptyset\}$ as subbasis and $\tau(\mathscr{V})$ is the initial topology on X induced by $\{g_\alpha: X \to (X_\alpha,\tau(\mathscr{U}_\alpha)) \mid \alpha \in A\}$.

<u>Proof</u>. First, let $U \in \mathscr{U}$ and let $\alpha \in A$ such that $f_\alpha^{-1}(U) \neq \emptyset$. Then there are $W_i \in \mathscr{U}$, $1 \leq i \leq n$, such that $f_\alpha^{-1}(W_i) \in \mathscr{U}_\alpha$, $\cap\{W_i \mid 1 \leq i \leq n\} \subseteq U$ and $(\cap\{W_i \mid 1 \leq i \leq n\})^0 = U^0$, from the definition of $\mathscr{U}$. But then

$$\cap\{f_\alpha^{-1}(W_i) \mid 1 \leq i \leq n\} \subseteq f_\alpha^{-1}(U) \quad \text{and}$$
$$(\cap\{f_\alpha^{-1}(W_i) \mid 1 \leq i \leq n\})^0 \quad (f_\alpha^{-1}(U))^0,$$

hence $f_\alpha^{-1}(U) \in \mathscr{U}_\alpha$. Thus $f_\alpha: (X_\alpha,\mathscr{U}_\alpha) \to (X,\mathscr{U})$ is para-uniformly continuous.

Now $g_\alpha^{-1}(\mathscr{U}_\alpha)$ $\{\emptyset\}$ is a para-uniform basis on X, for each $\alpha \in A$, by Proposition 2.16, so $(\cup\{g_\alpha^{-1}(\mathscr{U}_\alpha) \mid \alpha \in A\}) - \{\emptyset\}$ is a subbasis for a para-uniformity $\mathscr{W}$ on X. Clearly, $\mathscr{V} \subseteq \mathscr{W}$, by definition of $\mathscr{V}$. On the other hand, $\mathscr{W}$ is a subset of every para-uniformity $\mathscr{T}$ on X such that $g_\alpha: (X,\mathscr{T}) \to (X_\alpha,\mathscr{U}_\alpha)$ is para-uniformly continuous for each $\alpha \in A$. Hence $\mathscr{W} \subseteq \mathscr{V}$, so $\mathscr{W} = \mathscr{V}$.

Thus $g_\alpha\colon (X,\mathcal{V}) \to (X_\alpha,\mathcal{U}_\alpha)$ is para-uniformly continuous for each $\alpha \in A$ and $\mathcal{V}$ has the set $(\cup\{g_\alpha^{-1}(\mathcal{U}_\alpha) \mid \alpha \in A\})\quad \{\emptyset\}$ as subbasis.

For each $\alpha \in A$, let $\tau_\alpha = \tau(\mathcal{U}_\alpha)$, and let τ be the initial topology on X induced by $\{g_\alpha\colon X \to (X_\alpha,\tau_\alpha) \mid \alpha \in A\}$, i.e., τ is the smallest topology σ on X such that $g_\alpha\colon (X,\sigma) \to (X_\alpha,\tau_\alpha)$ is continuous for each $\alpha \in A$. Then $\tau \subseteq \tau(\mathcal{V})$, by the preceding paragraph and Proposition 2.21. Now let $x \in G \in \tau(\mathcal{V})$. Then there are $U_\alpha \in \mathcal{U}_\alpha$, $\alpha \in A$, such that $U_\alpha \quad X_\alpha \times X_\alpha$ for all but finitely many $\alpha \in A$, and $x \in \cap\{(g_\alpha^{-1}(U_\alpha))[x] \mid \alpha \in A\} \subseteq G$. But then $x \in \cap\{g_\alpha^{-1}\,\mathrm{int}_\alpha(U_\alpha[f_\alpha(x)]) \mid \alpha \in A\} \subseteq G$, so $G \in \tau$. Thus $\tau(\mathcal{V}) \subseteq \tau$, hence $\tau \quad \tau(\mathcal{V})$.||

By appropriately applying the identity functions to the uniform spaces of Example 2.18, it is easy to see that the final topology need not be that induced by the final para-uniformity. This fact was also shown by Himmelberg [Hi].

<u>Theorem 2.24</u>. Let $\{(X_\alpha,\mathcal{U}_\alpha) \mid \alpha \in A\}$ be a collection of para-uniform spaces. The final para-uniformity $\mathcal{U}$ on X induced by $\{f_\alpha\colon (X_\alpha,\mathcal{U}_\alpha) \to X \mid \alpha \quad A$ is characterized by the property that if $(Y,\mathcal{V})$ is any para-uniform space and $g\colon X \to Y$ is a function, then $g\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is para-uniformly continuous if and only if $gf_\alpha\colon (X_\alpha,\mathcal{U}_\alpha) \to (Y,\mathcal{V})$ is para-uniformly continuous for each $\alpha \in A$.

<u>Proof</u>. Let $(Y,\mathcal{V})$ be a para-uniform space and let $g\colon X \to Y$ be a function. If $\quad (X,\mathcal{U}) \to (Y,\mathcal{V})$ is para-uniformly continuous, then $gf_\alpha\colon (X_\alpha,\mathcal{U}_\alpha) \to (Y,\mathcal{V})$ is para-uniformly continuous for each $\alpha \in A$. On the other hand, if $((gf_\alpha)^{-1}(\mathcal{V})\quad \{\emptyset\}) \subseteq \mathcal{U}_\alpha$ for each

$\alpha \in A$, then $(g^{-1}(\mathcal{V}) \quad \{\omega\}) \subseteq \mathcal{U}$, by definition of $\mathcal{U}$. Thus $\mathcal{U}$ has the indicated property.

Now let $\mathcal{W}$ be a para-uniformity on X with the indicated property. Let h: X→ X be the identity function. Then $f_\alpha = hf_\alpha: (X_\alpha, \mathcal{U}_\alpha) \rightarrow (X, \mathcal{U})$ is para-uniformly continuous for each $\alpha \in A$, so $h: (X, \mathcal{W}) \rightarrow (X, \mathcal{U})$ is para-uniformly continuous. Thus $\mathcal{U} \subseteq \mathcal{W}$. But $h: (X, \mathcal{W}) \rightarrow (X, \mathcal{W})$ is para-uniformly continuous, so $f_\alpha = hf_\alpha: (X_\alpha, \mathcal{U}_\alpha) \rightarrow (X, \mathcal{W})$ is para-uniformly continuous for each $\alpha \in A$, so $\mathcal{W} \subseteq \mathcal{U}$, by definition of $\mathcal{U}$. Thus $\mathcal{U} = \mathcal{W}$.||

The proof of the following theorem is similar to that of the preceding and is omitted.

<u>Theorem 2.25</u>. Let $\{(X_\alpha, \mathcal{U}_\alpha) | \alpha \in A\}$ be a collection of para-uniform spaces. The initial para-uniformity $\mathcal{U}$ on X induced by $\{f_\alpha: X \rightarrow (X_\alpha, \mathcal{U}_\alpha) | \alpha \in A\}$ is characterized by the property that if $(Y, \mathcal{V})$ is any para-uniform space, g: Y→ X is a function, then $g: (Y, \mathcal{V}) \rightarrow (X, \mathcal{U})$ is para-uniformly continuous if and only if $f_\alpha g: (Y, \mathcal{V}) \rightarrow (X_\alpha, \mathcal{U}_\alpha)$ is para-uniformly continuous for each $\alpha \in A$.

<u>Definition 2.26</u>. Let $\{(X_\alpha, \mathcal{U}_\alpha) | \alpha \in A\}$ be a collection of para-uniform spaces, let X be the set product $\Pi\{X_\alpha | \alpha \in A\}$ and let $p_\alpha: X \rightarrow X_\alpha$, $\alpha \in A$, be the projection. The <u>product para-uniformity</u>, sometimes denoted by $\Pi\{\mathcal{U}_\alpha | \alpha \in A\}$, is the initial para-uniformity on X induced by $\{p_\alpha: X \rightarrow (X_\alpha, \mathcal{U}_\alpha) | \alpha \in A\}$.

<u>Remarks</u>. 1. The product para-uniformity induces the product topology.

2. If $\mathcal{U}_\alpha$ is a uniformity, for each $\alpha \in A$, then the initial

para-uniformity on X induced by $\{f_\alpha: X \to (X_\alpha, \mathcal{U}_\alpha) | \alpha \in A\}$ is a uniformity, hence is the initial uniformity.

3. The final para-uniformity need not be a uniformity even if each $\mathcal{U}_\alpha$ is. For example, let $X = \{a,b,c\}$, $Y \quad \{e,d\}$ with $f(e) \quad a$, $f(d) \quad b$. Let $\mathcal{V} = \{Y \times Y\}$. Then the final para-uniformity on X induced by $\{f: (Y, \mathcal{V}) \to X\}$ is generated by $\{\{a,b\} \times \{a,b\}, (\{a,b\} \times \{a,b\}) \cup \{(c,c)\}\}$, hence is not a uniformity.

<u>Proposition 2.27</u>. Let $\{(X_\alpha, \mathcal{U}_\alpha) | \alpha \in A\}$ be a collection of uniform spaces and let $f_\alpha: X_\alpha \to X$ be a function, for each $\alpha \in A$, such that $X = \cup\{f_\alpha(X_\alpha) | \alpha \in A\}$ and, given any $\beta, \gamma \in A$, there is a finite subset $\{\alpha(i) | 1 \leq i \leq n\} \subseteq A$ such that $\beta \quad \alpha(1)$, $\gamma \quad \alpha(n)$ and $f_{\alpha(i)}(X_{\alpha(i)}) \cap f_{\alpha(i+1)}(X_{\alpha(i+1)}) \neq \emptyset$, for $1 \leq i < n$. Then the final para-uniformity $\mathcal{U}$ on X induced by $\{f_\alpha: (X_\alpha, \mathcal{U}_\alpha) \to X | \alpha \in A\}$ is a uniformity.

<u>Proof</u>. Let $U \in \mathcal{U}$, $x \in X$, $y \in U^0[X]$ and $\beta, \gamma \in A$ such that $y \in f_\beta(X_\beta)$ and $x \in f_\gamma(X_\gamma)$. Let $\alpha(i) \in A$, $1 \leq i \leq n$, auch that $\alpha(1) \quad \beta$, $\alpha(n) \quad \gamma$ and

$$f_{\alpha(i)}(X_{\alpha(i)}) \cap f_{\alpha(i+1)}(X_{\alpha(i+1)}) \neq \emptyset,$$

for $1 \leq i < n$. Now $y \in U^0[X] \cap f_\beta(X_\beta)$ implies that

$$\Delta(X_\beta) \subseteq f_\beta^{-1}(U),$$

since $\mathcal{U}_\beta$ is a uniformity. But then $f_\beta(X_\beta) \subseteq U^0[X]$, so there is a $v \in U^0[X] \cap f_{\alpha(2)}(X_{\alpha(2)})$. Then $f_{\alpha(2)}(X_{\alpha(2)}) \subseteq U^0[X]$ and, by induction, $f_{\alpha(n)}(X_{\alpha(n)}) \subseteq U^0[X]$, i.e., $f_\gamma(X_\gamma) \subseteq U^0[X]$. Thus $x \in U^0[X]$, hence $X \subseteq U^0[X]$. Therefore, $\Delta(X) \subseteq U$, so $\mathcal{U}$ is a uniformity.||

<u>Definition 2.28</u>. Let $(X,\mathcal{U})$ be a para-uniform space, $A \subseteq X$ and let $i\colon A \to X$ be the injection ($i(a) \quad a$ for all $a \in A$). The <u>relative para-uniformity</u> on A, denoted $\mathcal{U}_A$, is the initial para-uniformity on A induced by $\{i\colon A \to (X,\mathcal{U})\}$. $(A,\mathcal{U}_A)$ is a <u>para-uniform subspace</u> of $(X,\mathcal{U})$.

<u>Proposition 2.29</u>. Let $(X,\mathcal{U})$ be a para-uniform space and let $\emptyset \neq A \subseteq X$. Then $\mathcal{U}_A \quad \{U(A) \mid U \in \mathcal{U}\} - \{\emptyset\}$, where $U(A) \quad U \cap (A \times A)$.

<u>Proof</u>. Let $i\colon A \to X$ be the injection. Then

$$i^{-1}(\mathcal{U}) \quad \{U(A) \mid U \in \mathcal{U}\} \subseteq \mathcal{U}_A \cup \{\emptyset\}$$

and $\mathcal{U}_A$ has $i^{-1}(\mathcal{U}) \quad \{\emptyset\}$ as basis. Now let $V \in \mathcal{U}$ with $V(A) \subseteq B \subseteq A \times A$ and $V(A)^0 \quad B^0$. Then $V \subseteq V \cup B$ and $(V \cup B)^0 \quad V^0 \cup B^0 \quad V^0$, so $V \cup B \in \mathcal{U}$ and $(V \cup B)(A) \quad B$. Thus $B \in i^{-1}(\mathcal{U})$, so $\mathcal{U}_A \quad \{U(A) \mid U \in \mathcal{U}\}$.||

Note that $\tau(\mathcal{U})_A \quad \tau(\mathcal{U}_A)$, by Proposition 2.23. Also, by Proposition 2.16, if $\mathcal{S}$ is a subbasis for $\mathcal{U}$ and $\mathcal{B}$ is a basis for $\mathcal{U}$, then $\{S(A) \mid S \in \mathcal{S}\} \quad \{\emptyset\}$ is a subbasis for $\mathcal{U}_A$ and $\{B(A) \mid B \in \mathcal{B}\} \quad \{\emptyset\}$ is a basis for $\mathcal{U}_A$.

The following proposition is a corollary of Theorem 2.25 and is stated without proof.

<u>Proposition 2.30</u>. Let $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a para-uniformly continuous function and $\emptyset \neq A \subseteq X$. Then $f|_A\colon (A,\mathcal{U}_A) \to (Y,\mathcal{V})$ is para-uniformly continuous.

3. <u>Topological Extensions</u>. Some interesting relations between para-uniformities and topological extensions will be presented. A topological space (Y,σ) is an <u>extension</u> of (X,τ) if $X \subseteq Y$,

$cl_{\sigma}(X)$ Y and σ_X τ, where σ_X is the subspace topology on X induced by σ. More generally, $((Y,\sigma),f)$ is an extension of (X,τ) if $f\colon (X,\tau)\to (Y,\sigma)$ is a function such that $f\colon (X,\tau)\to (f(X),\sigma_{f(X)})$ is a homeomorphism and $cl_{\sigma}(f(X))$ Y. If $((Y,\sigma),f)$ and $((Z,\rho),g)$ are extensions of (X,τ), then a homeomorphism $h\colon (Y,\sigma)\to (Z,\rho)$ is an *isomorphism* if hf g. It is easy to see that if $((Y,\sigma),f)$ is an extension of (X,τ), then there is an extension $((Z,\rho),g)$ of (X,τ) such $X\subseteq Z$, g is the injection, and (Y,σ) is isomorphic to (Z,ρ), (see Banaschewski [Ba]). Therefore, for this paper, "(Y,σ) is an extension of (X,τ)" will always imply that $X\subseteq Y$ and $\sigma_X = \tau$.

Let (Y,σ) be an extension of (X,τ). For $y \in Y$, the set $\sigma_X\langle y\rangle$ is called the *trace filter* of y on X, and the set $\{\sigma_X\langle y\rangle \mid y \in Y\}$ is called the *filter trace* of Y on X. Banaschewski [Ba] noted that, for $G^{\#}$ $\{y \in Y \mid G \in \sigma_X\langle y\rangle\}$, the set $\{G^{\#} \mid G \in \tau\}$ is a basis for a topology $\sigma^{\#}$ on Y such that $(Y,\sigma^{\#})$ is an extension of (X,τ), $\sigma^{\#} \subseteq \sigma$, and $\sigma_X\langle y\rangle$ $\sigma^{\#}_X\langle y\rangle$ for each $y \in Y$. If σ $\sigma^{\#}$, (Y,σ) will be called a *strict extension* of (X,τ). At the same time, Banaschewski noted that the collection $\{G \cup \{y\} \mid G \in \sigma_X\langle y\rangle,\ y \in Y\}$ is a basis for a topology σ^{+} on Y such that (Y,σ^{+}) is an extension of (X,τ), $\sigma \subseteq \sigma^{+}$ and $\sigma_X\langle y\rangle$ $\sigma^{+}_X\langle y\rangle$ for each $y \in Y$. If σ σ^{+}, (Y,σ) will be called a *simple extension* of (X,τ).

If (Y,σ) is an extension of (X,τ) and $\mathcal{U}$ is a para-uniformity on Y such that σ $\tau(\mathcal{U})$, then τ $\tau(\mathcal{U}_X)$, by Proposition 2.33. The following proposition will relate $\mathcal{U}$ to σ^+ and to $\sigma^\#$ for certain $\mathcal{U}$. (Recall that $U(A) \quad U \cap (A\times A)$ for $U \in \mathcal{U}$ and $A \subseteq Y$.)

<u>Proposition 2.31</u>. Let (Y,σ) be an extension of (X,τ) and let $\mathcal{U}$ be a para-uniformity on Y compatible with σ. Then $\mathcal{B} = \{U(X \cup H) | H \subseteq Y \quad X\}$ is a basis for a para-uniformity $\mathcal{U}^+$ on Y and $\sigma^+ \quad \tau(\mathcal{U}^+)$.

<u>Proof</u>. The set $\{X \cup H | H \subseteq Y - X\}$ is a finitely multiplicative collection of nonempty subsets of Y, so $\mathcal{B}$ is a para-uniform basis on Y, by Proposition 2.12(b), since $U(X) \neq \emptyset$, for $U \in \mathcal{U}$. Let $\mathcal{U}^+ \quad \mathcal{U}(\mathcal{B})$, let $y \in Y$ and let $A \quad \{y\} \cup X$. Now let $G \in \sigma_X\langle y\rangle$. Then there is a $V \in \mathcal{U}$ such that $\emptyset \neq X \cap V[y] \subseteq G$. But then $V(A)[y] \subseteq G \cup \{y\}$, so $\sigma^+ \subseteq \tau(\mathcal{U}^+)$. Now let $U \in \mathcal{U}$ such that U is open in the product topology $\sigma\times\sigma$ on $Y\times Y$ and $y \in U[y]$. Then $U(A)[y] \quad U[y] \cap A \quad \{y\} \cup (U[y] \cap X) \in \sigma^+$ Since the open entourages in $\mathcal{U}$ form a basis for $\mathcal{U}$, by Theorem 2.15, this shows that $\tau(\mathcal{U}^+) \subseteq \sigma^+$, hence $\sigma^+ \quad \tau(\mathcal{U}^+)$.||

<u>Proposition 2.32</u>. Let (Y,σ) be an extension of (X,τ) and let $\mathcal{U}$ be a para-uniformity on Y compatible with σ such that for each $y \in Y \quad X$ and each $U \in \mathcal{U}$, there is a $G \in \sigma_X\langle y\rangle$ with $G\times G \subseteq U$. For each $U \in \mathcal{U}$, let

$U^\# \quad U(X) \cup \{(x,y) \in Y\times Y | G\times G \subseteq U \text{ for some } G \in \sigma_X\langle x\rangle \cap \sigma_X\langle y\rangle\}$.

Then $\mathcal{B} = \{U^\# | U \in \mathcal{U}\}$ is a basis for a para-uniformity $\mathcal{U}^\#$ on Y and $\sigma^\# \quad \tau(\mathcal{U}^\#)$.

<u>Proof</u>. Note that $(Y - X)\times(Y - X) \subseteq U^\#$, $(U^\#)^\circ \quad U^\circ(X) \cup (Y \quad X)$, and $U^\# \cap V^\# \quad (U \cap V)^\#$, for $U,V \in \mathcal{U}$. Also, $(U \cap U^{-1})^\# \quad U^\# \cap (U^\#)^{-1}$, hence $\mathcal{B}$ satisfies (U2) and (B3).

Now let $U,V \in \mathcal{U}$ such that $U^\# \cap V^\# \neq \emptyset$. Then $U \cap V \neq \emptyset$, so there is a symmetric $W \in \mathcal{U}$ such that $W^4 \subseteq U \cap V$ and $W^0 \quad (U \cap V)^0$. By Theorem 2.15, assume W is open in $\sigma\times\sigma$. Then $(W^\#)^0 \quad (U^\# \cap V^\#)^0$. Let $(x,y),(y,v) \in W^\#$. If $x,y,v \in X$, then $(x,v) \in W(X)^2 \subseteq (U \cap V)^\#$. If $x,y \in X$ and $v \notin X$, then there is a $G \in \sigma_X\langle y\rangle \cap \sigma_X\langle v\rangle$ such that $G\times G \subseteq W$. But then, since $(x,y) \in W$, $G \subseteq W(X)[y] \subseteq W(X)^2[x]$. Therefore, $W(X)^2[x] \in \sigma_X\langle v\rangle \cap \sigma_X\langle x\rangle$. Thus $(x,v) \in (U \cap V)^\#$, since $W(X)^2[x] \times W(X)^2[x] \subseteq W^4 \subseteq U \cap V$. Similarly, if $y,v \in X$ and $x \notin X$, $(x,v) \in (U\cap V)^\#$. So, assume that $y \notin X$. Then there are $H \in \sigma_X\langle x\rangle \cap \sigma_X\langle y\rangle$ and $K \in \sigma_X\langle y\rangle \cap \sigma_X\langle v\rangle$ such that $H \times H \subseteq W$ and $K \times K \subseteq W$. Then $H \cup K \in \sigma_X\langle x\rangle \cap \sigma_X\langle v\rangle$ and

$$(H \cup K) \times (H \cup K) \subseteq W^2 \subseteq U \cap V,$$

since $H \cap K \neq \emptyset$. Thus $(W^\#)^2 \subseteq (U \cap V)^\#$, so $\mathcal{B}$ satisfies (U4), hence is a para-uniform basis on Y.

Let $\mathcal{U}^\# \quad \mathcal{U}(\mathcal{B})$. Let $y \in G \in \sigma^\#$ and assume, without loss of generality, that $G \quad \{x \in Y | G \cap X \in \sigma_X\langle x\rangle\}$. Since $\sigma^\# \subseteq \sigma$, there is a $U \in \mathcal{U}$ such that $y \in U[y] \subseteq G$. Then there is a symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$ and $V^0 \quad U^0$. Let $x \in V^\#[y]$. If $x \in V(X)[y]$, then $x \in G$, so assume that $(y,x) \notin V(X)$. Then there is an $H \in \sigma_X\langle x\rangle \cap \sigma_X\langle y\rangle$ such that $H \times H \subseteq V$. But $H \cap V[y] \neq \emptyset$, so let $h \in H \cap V[y]$. Then

$$H \subseteq V[h] \subseteq V^2[y] \subseteq U[y] \subseteq G.$$

Thus $x \in G$, since $H \in \sigma_X\langle x\rangle$. Therefore, $G \in \tau(\mathcal{U}^\#)$, hence $\sigma^\# \subseteq \tau(\mathcal{U}^\#)$.

Now let $U \in \mathcal{U}$ and let $y \in Y$ such that $y \in U^\#[y]$.

Assume there is a $G \in \sigma_X\langle y\rangle$ such that $G \times G \subseteq U$. Then $v \in Y$ with $G \in \sigma_X\langle v\rangle$ implies that $v \in U^{\#}[y]$. Thus

$$y \in G \cup \{v \in Y \mid G \in \sigma_X\langle v\rangle\} \subseteq U^{\#}[y],$$

and $G \cup \{v \in Y \mid G \in \sigma_X\langle v\rangle\} \in \sigma^{\#}$. If $y \in Y - X$, there is such a $G \in \sigma_X\langle y\rangle$, by definition of $U^{\#}$. On the other hand, if $y \in X$, then $y \in V(X)[y] \quad X \cap V[y]$ and $V(X)[y] \times V(X)[y] \subseteq U$ for any symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$ and $V^0 \quad U^0$. Thus, in this case too, there is a $G \in \sigma_X\langle y\rangle$ with $G \times G \subseteq U$. Therefore, $U^{\#}[y]$ is a $\sigma^{\#}$-neighborhood of y, hence $\tau(\mathcal{U}^{\#}) = \sigma^{\#}$.||

CHAPTER 3

COMPLETENESS AND H-CLOSURE

In this chapter, completeness and H-closure will be introduced and some relations between them investigated. Among other results; every para-uniform space has a completion and a Hausdorff space is H-closed if and only if every compatible para-uniformity is complete.

1. Completeness. First, the concepts of Cauchy filter and completeness will be extended to para-uniform spaces.

Definition 3.1. Let $(X,\mathcal{U})$ be a para-uniform space and let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is Cauchy, or $\mathcal{U}$-Cauchy, if for each $U \in \mathcal{U}$, there is an $x \in X$ such that $U[x] \in \mathcal{F}$.

Proposition 3.2. Let $(X,\mathcal{U})$ be a para-uniform space and let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is Cauchy if and only if for every $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $F \times F \subseteq U$.

Proof. Clearly, the indicated condition implies that $\mathcal{F}$ is Cauchy, so assume that $\mathcal{F}$ is Cauchy and let $U \in \mathcal{U}$. Then there is a symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$ and there is an $x \in X$ such that $V[x] \in \mathcal{F}$. But then $V[x] \times V[x] \subseteq V^2 \subseteq U$.||

Note that if $\mathcal{F}$ is a $\mathcal{U}$-Cauchy filter then $U^0[X] \in \mathcal{F}$ for every $U \in \mathcal{U}$. Thus, if there is a $\mathcal{U}$-Cauchy filter, then $U \cap V \neq \emptyset$, for every $U,V \in \mathcal{U}$.

Proposition 3.3. Let $(X,\mathcal{U})$ be a para-uniform space and let $\mathcal{F}$ be a Cauchy filter on X. Then $\mathcal{F}$ converges to each of its adherent points, where adherence and convergence are considered relative to $\tau(\mathcal{U})$.

Proof. Let $y \in X$ be an adherent point of $\mathcal{F}$; i.e., $y \in \cap\{\overline{F} | F \in \mathcal{F}\}$. Now let $U \in \mathcal{U}$ such that $y \in U[y]$ and let $V \in \mathcal{U}$ such that $V \quad V^{-1}$, $V^3 \subseteq U$ and $V^0 \quad U^0$. Then $y \in V[y]$ and there is an $x \in X$ such that $V[x] \in \mathcal{F}$. But then $V[x] \cap V[y] \neq \emptyset$, so $x \in V^2[y]$. Hence $V[x] \subseteq V^3[y] \subseteq U[y]$, so $U[y] \in \mathcal{F}$. Thus $\mathcal{F}$ converges to y.||

Note that, unlike the uniform case, a $\tau(\mathcal{U})$-convergent filter need not be $\mathcal{U}$-Cauchy. For example, if X is any set with at least two points, τ is a T_1 topology on X and $\mathcal{U}$ is generated by $\{T \times T | \emptyset \neq T \in \tau\}$, then no neighborhood filter is Cauchy.

To acquire desirable separation properties in the completions to be discussed later, it is convenient to have a method of selecting a "canonical" representative from each of certain collections of Cauchy filters. This is provided by the next theorem.

Theorem 3.4. Let $(X,\mathcal{U})$ be a para-uniform space, let $\mathcal{F}$ be a Cauchy filter on X and let

$$\mathcal{F}_m \quad \{U[F] | U \in \mathcal{U},\ F \in \mathcal{F}\}.$$

Then $\mathcal{F}_m$ is the smallest Cauchy filter contained in $\mathcal{F}$.

Proof. $X \times X \in \mathcal{U}$ and $\mathcal{F} \neq \emptyset$, so $\mathcal{F}_m \neq \emptyset$. Also, $U^0[X] \in \mathcal{F}$ for every $U \in \mathcal{U}$, so $U[F] \neq \emptyset$, for every $F \in \mathcal{F}$ and $U \in \mathcal{U}$. Let $U, V \in \mathcal{U}$ and let $A, B \in \mathcal{F}$. Then $U \cap V \in \mathcal{U}$, $A \cap B \in \mathcal{F}$ and $(U \cap V)[A \cap B] \subseteq U[A] \cap V[B]$. Thus $\mathcal{F}_m$ is a filter base. But if $U \in \mathcal{U}$, $A \in \mathcal{F}$ and $U[A] \subseteq D \subseteq X$, then $V \quad U \cup (\{a\} \times D) \in \mathcal{U}$, where $a \in U^0[X] \cap A$. Then $V[A] \quad D \in \mathcal{F}_m$, so $\mathcal{F}_m$ is a filter.

Now let $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ such that $V \quad V^{-1}$ and $V^2 \subseteq U$. Let $x \in X$ such that $V[x] \in \mathcal{F}$. Then $V^2[x] \in \mathcal{F}_m$ and $V^2[x] \subseteq U[x]$, so $U[x] \in \mathcal{F}_m$. Thus $\mathcal{F}_m$ is Cauchy.

Now let $\mathcal{G}$ be a Cauchy filter on X such that $\mathcal{G} \subseteq \mathcal{F}$ and let $U \in \mathcal{U}$, $F \in \mathcal{F}$. Let $V \in \mathcal{U}$ such that $V \quad V^{-1}$, $V^2 \subseteq U$ and $V^0 \quad U^0$, and let $x \in X$ such that $V[x] \in \mathcal{G}$. Then $V[x] \cap F \neq \emptyset$, so $V[x] \subseteq V^2[F] \subseteq U[F]$. Thus $U[F] \in \mathcal{G}$, so $\mathcal{F}_m \subseteq \mathcal{G} \subseteq \mathcal{F}$, hence $\mathcal{F}_m$ is the smallest Cauchy filter contained in $\mathcal{F}$. ||

<u>Definition 3.5</u>. Let $(X,\mathcal{U})$ be a para-uniform space and let $\mathcal{F}$ be a Cauchy filter on X. The filter $\mathcal{F}_m$ as defined in Theorem 3.4 will be called a <u>minimal Cauchy filter</u>.

<u>Definition 3.6</u>. Let (X,τ) be a topological space. A filter $\mathcal{F}$ on X is an <u>open filter</u>, or a τ-filter, if $\mathcal{F}$ has a base of open sets, i.e., $F \in \mathcal{F}$ implies there is a $B \in \mathcal{F} \cap \tau$ such that $B \subseteq F$. A filter on X which is maximal in the collection of open filters is an <u>open ultrafilter</u>, a τ-ultrafilter, or a maximal open filter.

Note that every open filter on (X,τ) is contained in an open ultrafilter.

A filter on a para-uniform space $(X,\mathcal{U})$ will be called open if it is a $\tau(\mathcal{U})$-filter.

<u>Proposition 3.7</u>. A minimal Cauchy filter $\mathcal{F}$ on the para-uniform space $(X,\mathcal{U})$ is open.

<u>Proof</u>. Let $F \in \mathcal{F}$. Then there is a $B \in \mathcal{F}$ and a $U \in \mathcal{U}$ such that $U[B] \subseteq F$, by Theorem 3.4. Let $V \in \mathcal{U}$ such that $V \quad V^{-1}$, $V^2 \subseteq U$ and $V^0 \quad U^0$. Then, for $y \in V[B]$,

$V[y] \subseteq V^2[B] \subseteq U[B]$. Hence, $V[B] \subseteq \mathrm{int}_{\tau(\mathcal{U})}(U[B]) \subseteq F$. But $V[B] \in \mathcal{F}$, so $\mathrm{int}_{\tau(\mathcal{U})}(U[B]) \in \mathcal{F}$. Thus $\mathcal{F}$ is open.||

Proposition 3.8. Let $\mathcal{F}$ be a Cauchy filter on the para-uniform space $(X,\mathcal{U})$ and let $\mathcal{G}$ be the minimal Cauchy filter contained in $\mathcal{F}$. Then $\mathcal{F}$ and $\mathcal{G}$ have the same $\tau(\mathcal{U})$-adherent points.

Proof. Since $\mathcal{G} \subseteq \mathcal{F}$, the $\tau(\mathcal{U})$-adherence of $\mathcal{F}$ is contained in that of $\mathcal{G}$. But $\mathcal{G}$, being Cauchy, converges to each of its $\tau(\mathcal{U})$-adherent points, by Proposition 3.3. Therefore, $\mathcal{F}$ converges to each $\tau(\mathcal{U})$-adherent point of $\mathcal{G}$, so $\mathcal{F}$ and $\mathcal{G}$ have the same $\tau(\mathcal{U})$-adherent points.||

Definition 3.9. A filter on a topological space (X,τ) is free if it has no τ-adherent points. A filter on a para-uniform $(X,\mathcal{U})$ is free if it is free on $(X,\tau(\mathcal{U}))$.

Thus Proposition 3.8 implies that the minimal Cauchy filter contained in a Cauchy filter $\mathcal{F}$ is free if and only if $\mathcal{F}$ is free. Note also that a Cauchy filter is free if and only if it does not converge, by Proposition 3.3.

Definition 3.10. A para-uniform space $(X,\mathcal{U})$ is complete, or $\mathcal{U}$ is complete, if every $\mathcal{U}$-Cauchy filter on X is $\tau(\mathcal{U})$-convergent. A complete para-uniform space $(Y,\mathcal{V})$ is a completion of $(X,\mathcal{U})$ if $(Y,\tau(\mathcal{V}))$ is an extension of $(X,\tau(\mathcal{U}))$ and $\mathcal{V}_X = \mathcal{U}$.

Let (X,τ) be a topological space and let $\mathcal{U}$ be the para-uniformity on X with $\{T\times T \mid \emptyset \neq T \in \tau\}$ as basis (see Theorem 2.9). If τ contains two nonempty, disjoint elements, then there are no $\mathcal{U}$-Cauchy filters on X, hence $(X,\mathcal{U})$ is complete. There are other, somewhat more unexpected, complete para-uniformities, as seen by the following example of a complete uniform space.

Let X be the set of rational numbers in $\underline{\underline{R}}$ and let $\mathcal{U}$ be the usual uniformity on X. For each $r \in \underline{\underline{R}}$, let $(x_{r,n};\ n \in \underline{\underline{N}})$ and $(y_{r,n};\ n \in \underline{\underline{N}})$ be increasing and decreasing, respectively, sequences from $\underline{\underline{R}} - X$ converging to r in $\underline{\underline{R}}$ with the usual topology. Let

$$G(r,1) \quad]-\infty, x_{r,1}[\, \cap X \quad \text{and} \quad H(r,1) \quad]y_{r,1}, +\infty[\, \cap X.$$

Then, for $1 < i \in \underline{\underline{N}}$, let

$$G(r,i) \quad]x_{r,i-1}, x_{r,i}[\, \cap X \quad \text{and} \quad H(r,i) \quad]y_{r,i}, y_{r,i-1}[\, \cap X.$$

Now let

$$V(r) \quad (\cup\{G(r,n)\times G(r,n) \mid n \in \underline{\underline{N}}\}) \cup (\cup\{H(r,n)\times H(r,n) \mid n \in \underline{\underline{N}}\}).$$

Then, for $r \in \underline{\underline{R}} - X$, $V(r)^{-1} \quad V(r) \quad V(r)^2$ and $\Delta(X) \subseteq V(r)$, so $\mathcal{U} \cup \{V(r) \mid r \in \underline{\underline{R}} - X\}$ is a subbasis for a uniformity $\mathcal{V}$ on X. Now $\tau(\mathcal{V}) \quad \tau(\mathcal{U})$ and $(X,\mathcal{V})$ is complete Thus there is a complete uniformity on the rationals with the usual topology.

<u>Proposition 3.11</u>. Let $\{(X_\alpha,\mathcal{U}_\alpha) \mid \alpha \in A\}$ be a collection of para-uniform spaces and let $(X,\mathcal{U})$ be the product para-uniform space, i.e., $X \quad \Pi\{X_\alpha \mid \alpha \in A\}$ and $\mathcal{U}$ is the product para-uniformity.

(a) If $(X_\alpha,\mathcal{U}_\alpha)$ is complete for each $\alpha \in A$, then $(X,\mathcal{U})$ is complete.

(b) If for each $\alpha \in A$ there is a $\mathcal{U}_\alpha$-Cauchy filter on X_α and if $(X,\mathcal{U})$ is complete, then $(X_\alpha,\mathcal{U}_\alpha)$ is complete for each $\alpha \in A$.

<u>Proof</u>. (a): Assume that $(X_\alpha,\mathcal{U}_\alpha)$ is complete for each $\alpha \in A$. Let $\mathcal{F}$ be a $\mathcal{U}$-Cauchy filter on X, let $\alpha \in A$ and let $U \in \mathcal{U}_\alpha$. Then $p_\alpha^{-1}(U) \in \mathcal{U}$, where p_α is the projection of X onto

X_α, so there is an F ε $\mathcal{F}$ such that $p_\alpha(F)\times p_\alpha(F) \subseteq U$. Thus, since $\{p_\alpha(B)|B \in \mathcal{F}\}$ is a filterbase on X_α, the filter generated by $\{p_\alpha(B)|B \in \mathcal{F}\}$ is $\mathcal{U}_\alpha$-Cauchy, hence converges to a point x_α in X_α. But is not difficult to verify that $\mathcal{F}$ then converges to the point $x \quad \cap\{p_\alpha^{-1}(x_\alpha)|\alpha \in A\}$ in X. Thus $(X,\mathcal{U})$ is complete.

(b) Assume that $(X,\mathcal{U})$ is complete and that there is a $\mathcal{U}_\alpha$-Cauchy filter $\mathcal{F}_\alpha$ on X_α, for each $\alpha \in A$. Let $\beta \in A$ and let $\mathcal{F}$ be a $\mathcal{U}_\beta$-Cauchy filter on X_β. Let $\mathcal{D}_\beta = \mathcal{F}$ and let $\mathcal{D}_\alpha = \mathcal{F}_\alpha$ for $\alpha \in (A - \{\beta\})$. Then the collection

$$\{\cap\{p_\gamma^{-1}(G_\gamma)|\gamma \in C\}|G_\gamma \in \mathcal{D}_\gamma,\ C \subseteq A,\ C \text{ is finite}\}$$

is a filter base on X and the filter generated by it is $\mathcal{U}$-Cauchy (recall that $\cup\{p_\alpha^{-1}(\mathcal{U}_\alpha)|\alpha \in A\}$ is a basis for $\mathcal{U}$), hence converges to a point x in X. But then $\mathcal{D}_\beta = \mathcal{F}$ converges to $p_\beta(x)$ in X_β. Thus $(X_\beta,\mathcal{U}_\beta)$ is complete.||

It will now be shown that every para-uniform space has a completion. For this purpose, let $(X,\mathcal{U})$ be a para-uniform space and let $X(\mathcal{U})$ be X union with the set of all free minimal Cauchy filters on X. For each $U \in \mathcal{U}$ and each $\mathcal{F} \in X(\mathcal{U}) - X$, let $U(\mathcal{F}) \quad \{(\mathcal{F},x)|x \in X \text{ and } F\times F \subseteq U \text{ for some } F \in \mathcal{U}(x)\cap\mathcal{F}\}$, and let $I(U,\mathcal{U}) = \{(\mathcal{F},\mathcal{D})|\mathcal{F},\mathcal{D} \in X(\mathcal{U}) \quad X \text{ and } A\times A \subseteq U \text{ for some } A \in \mathcal{F}\cap\mathcal{D}\}$. Now let $U^\# \quad U \cup (\cup\{U(\mathcal{F}) \cup U(\mathcal{F})^{-1}|\mathcal{F} \in X(\mathcal{U}) - X\}) \cup I(U,\mathcal{U})$. Note that $\mathcal{F} \in U^\#[\mathcal{F}]$ for every $\mathcal{F} \in X(\mathcal{U}) - X$.

<u>Theorem 3.12</u>. Let $(X,\mathcal{U})$ be a para-uniform space and let $X(\mathcal{U})$ and $U^\#$, for $U \in \mathcal{U}$, be defined as above. Then $\mathcal{B} \quad \{U^\#|U \in \mathcal{U}\}$ is a basis for a para-uniformity $\mathcal{U}^\#$ on $X(\mathcal{U})$,

$(X(\mathcal{U}),\mathcal{U}^{\#})$ is a completion of $(X,\mathcal{U})$ and $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is a strict extension of $(X,\tau(\mathcal{U}))$.

Proof. Note that $U^{\#} - U$ is symmetric, $U(\mathcal{F}) \quad (U \cap U^{-1})(\mathcal{F})$, $I(U,\mathcal{U}) \quad I(U \cap U^{-1},\mathcal{U})$ and $(U^{\#})^{0} \quad U^{0} \cup \Delta(X(\mathcal{U}) - X)$, for every $U \in \mathcal{U}$ and every $\mathcal{F} \in X(\mathcal{U}) - X$. The proof that $\mathcal{B}$ is a para-uniform basis on $X(\mathcal{U})$ is now essentially the same as the proof of a similar statement in Proposition 2.32 and is omitted. It is easily verified that $\mathcal{U}^{\#}_{X} = \mathcal{U}$ and that X is $\tau(\mathcal{U}^{\#})$-dense in $X(\mathcal{U})$. Therefore, $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is an extension of $(X,\tau(\mathcal{U}))$ and, by comparison with Proposition 2.32, is a strict extension thereof. Thus it only remains to show that $(X(\mathcal{U}),\mathcal{U}^{\#})$ is complete.

Let $\mathcal{F}$ be a free minimal $\mathcal{U}^{\#}$-Cauchy filter on $X(\mathcal{U})$, and let $\mathcal{F}_X \quad \{F \cap X | F \in \mathcal{F}\}$. Note that $\mathcal{F}_X$ is a $\tau(\mathcal{U})$-filter, since X is dense in $X(\mathcal{U})$ and minimal Cauchy filters are open filters. For $U \in \mathcal{U}$, there is an $F \in \mathcal{F}$ such that $F \times F \subseteq U^{\#}$. Then

$$(F \cap X) \times (F \cap X) \subseteq U^{\#}(X) \quad U,$$

so $\mathcal{F}_X$ is $\mathcal{U}$-Cauchy. But $\mathcal{F}_X$ is also a free $\tau(\mathcal{U})$-filter, since any $\tau(\mathcal{U})$-adherent point of $\mathcal{F}_X$ is a $\tau(\mathcal{U}^{\#})$-adherent point of $\mathcal{F}$. Thus there is a $\mathcal{G} \in X(\mathcal{U}) - X$ such that $\mathcal{G} \subseteq \mathcal{F}_X$. But then $A \cap V^{\#}[\mathcal{G}] \neq \emptyset$, for every $A \in \mathcal{F}$ and every $V \in \mathcal{U}$. Therefore, $\mathcal{G}$ is a $\tau(\mathcal{U}^{\#})$-adherent point of $\mathcal{F}$, a contradiction of the assumption that $\mathcal{F}$ is free. Thus there are no free minimal $\mathcal{U}^{\#}$-Cauchy filters on $X(\mathcal{U})$, hence no free $\mathcal{U}^{\#}$-Cauchy filters on $X(\mathcal{U})$. Therefore, $(X(\mathcal{U}),\mathcal{U}^{\#})$ is complete.||

Note that if $\mathcal{U}$ is a uniformity on X, then $\mathcal{U}^{\#}$ is a uniformity on $X(\mathcal{U})$.

Proposition 3.13. Let $(X,\mathcal{U})$ be a para-uniform space and let $\mathcal{F}$ be a free minimal $\mathcal{U}$-Cauchy filter on X. Then

$$\tau(\mathcal{U}^{\#})_X\langle\mathcal{F}\rangle = \mathcal{F} \cap \tau(\mathcal{U}).$$

Proof. Let $F \in \mathcal{F} \cap \tau(\mathcal{U})$. Then there is a $U \in \mathcal{U}$ and an $A \in \mathcal{F}$ such that $U[A] \subseteq F$, by Theorem 3.4. Let $y \in X \cap U^{\#}[\mathcal{F}]$. Then there is a $B \in \mathcal{F} \cap \mathcal{U}(y)$ such that $B \times B \subseteq U$. But $B \cap A \neq \emptyset$, so

$$y \in B \subseteq U[A] \subseteq F.$$

Thus $X \cap U^{\#}[\mathcal{F}] \subseteq F$, hence $F \in \tau(\mathcal{U}^{\#})_X\langle\mathcal{F}\rangle$. Therefore,

$$\mathcal{F} \cap \tau(\mathcal{U}) \subseteq \tau(\mathcal{U}^{\#})_X\langle\mathcal{F}\rangle.$$

Now let $G \in \tau(\mathcal{U}^{\#})\langle\mathcal{F}\rangle$. Then there is a $U \in \mathcal{U}$ such that $\mathcal{F} \in U^{\#}[\mathcal{F}] \subseteq G$ and there is a symmetric $V \in \mathcal{U}$ such that $V^4 \subseteq U$ and $V^0 \quad U^0$. Then there is an $x \in X$ such that $V[x] \in \mathcal{F}$. Let $y \in V[x]$. Then $x \in V[y]$, so $V[x] \subseteq V^2[y]$, hence

$$V^2[y] \in \mathcal{U}(y) \cap \mathcal{F}.$$

But $V^2[y] \times V^2[y] \subseteq V^4 \subseteq U$, hence $y \in X \cap U^{\#}[\mathcal{F}]$. Thus $V[x] \subseteq X \cap U^{\#}[\mathcal{F}] \subseteq G \cap X$, so $G \cap X \in \mathcal{F}$. Therefore,

$$\tau(\mathcal{U}^{\#})_X\langle\mathcal{F}\rangle = \mathcal{F} \cap \tau(\mathcal{U}). ||$$

Definition 3.14. Let (Y,σ) be an extension of (X,τ).

(a) The outgrowth, or remainder, of the extension (Y,σ) is the set $Y \quad X$.

(b) [Fl](Y,σ) has relatively zero-dimensional outgrowth if σ has a basis β such that $cl_{\sigma}(B) \quad B \subseteq X$ for every $B \in \beta$.

(c) (Y,σ) has relatively completely regular outgrowth if for every $y \in Y$ and every $G \in \sigma\langle y\rangle$ there is an $H \in \sigma$ and a continuous function $f\colon (H,\sigma_H) \to [0,1]$ such that $\{y\} \cup (Y-X) \subseteq H$, $f(y) \quad 0$ and $f(H-G) \subseteq \{1\}$, where the unit interval $[0,1]$ has the usual topology.

(d) (Y,σ) is <u>Hausdorff except for X</u> if for every $x \in Y$ and every $y \in Y - X$ with $x \neq y$, x and y have disjoint neighborhoods.

<u>Remarks</u>. 1. [F1] If (Y,σ) has relatively zero-dimensional outgrowth, as an extension of (X,τ), then it is a strict extension of (X,τ).

2. If (Y,σ) has relatively completely regular outgrowth, as an extension of (X,τ), then it is a strict extension. For, if $A \quad X \cap \operatorname{int}_\sigma(f^{-1}([0,\delta]))$, for $0 < \delta < 1$, and $v \in Y \quad X$ with $A \in \sigma_X\langle v\rangle$, then $v \in \operatorname{int}_\sigma(f^{-1}([0,\delta]))$, where f is a continuous function as in Definition 3.14(c). Hence σ has a basis of sets of the form $\{v \in Y \mid B \in \sigma_X\langle v\rangle\}$, where $B \in \tau$, so (Y,σ) is a strict extension of (X,τ).

3. If (X,τ) is Hausdorff and (Y,σ) is Hausdorff except for X, then (Y,σ) is Hausdorff.

A collection $\mathcal{B}$ of subsets of $X \times X$ is called <u>transitive</u> if $B^2 \subseteq B$ for every $B \in \mathcal{B}$.

<u>Proposition 3.15</u>. Let $(X,\mathcal{U})$ be a para-uniform space and consider $(X(\mathcal{U}),\tau(\mathcal{U}^\#))$ as an extension of $(X,\tau(\mathcal{U}))$.

(a) If $\mathcal{U}$ has a transitive basis, then $(X(\mathcal{U}),\tau(\mathcal{U}^\#))$ has relatively zero-dimensional outgrowth.

(b) $(X(\mathcal{U}),\tau(\mathcal{U}^\#))$ has relatively completely regular outgrowth.

(c) $(X(\mathcal{U}),\tau(\mathcal{U}^\#))$ is Hausdorff except for X.

<u>Proof</u>. (a): Let $U \in \mathcal{U}$ such that $U \quad U^{-1}$ and $U^2 \subseteq U$. Then $U^4 \subseteq U$, so $(U^\#)^2 \subseteq U^\#$, as was shown in the proof of Prop-

osition 2.32. Let $p \in X(\mathcal{U})$ such that $p \in U^{\#}[p]$, and let $\mathcal{F} \in X(\mathcal{U}) - X$ such that $\mathcal{F} \in cl_{\tau(\mathcal{U}^{\#})}(U^{\#}[p])$. Then, since $\mathcal{F} \in U^{\#}[\mathcal{F}]$, $U^{\#}[\mathcal{F}] \cap U^{\#}[p] \neq \emptyset$, hence $\mathcal{F} \in U^{\#}[p]$, since $U^{\#} = (U^{\#})^{-1}$! Thus $cl_{\tau(\mathcal{U}^{\#})}(U^{\#}[p]) \quad U^{\#}[p] \subseteq X$. But $U^{\#}[p] \in \tau(\mathcal{U}^{\#})$, since $q \in U^{\#}[p]$ implies $p \in U^{\#}[q] \subseteq (U^{\#})^2[p] \quad U^{\#}[p]$. Therefore, $(X(\mathcal{U}), \tau(\mathcal{U}^{\#}))$ has relatively zero-dimensional outgrowth, if $\mathcal{U}$ has a transitive basis.

(b): Let $y \in X(\mathcal{U})$ and let $G \in \tau(\mathcal{U}^{\#})\langle y\rangle$. Then there is a symmetric $U \in \mathcal{U}$ such that $y \in U^{\#}[y] \subseteq G$ and $U^{\#}[y] \in \tau(\mathcal{U}^{\#})$. Let $H \quad U^{\#}[X(\mathcal{U})]$. It is easy to verify that

$$\mathcal{V} \quad \{V(H) \mid V \in \mathcal{U}^{\#}, V^{0} \quad \Delta(H)\},$$

where $V(H) \quad V \cap (H \times H)$, is a uniformity on H and that $\tau(\mathcal{V}) \subseteq \tau(\mathcal{U}^{\#})$. But $U^{\#}[y] \in \tau(\mathcal{V})$ and $(H, \tau(\mathcal{V}))$ is completely regular [see Ke, Thm. 17, p. 188], so there is a continuous function $f: (H, \tau(\mathcal{V})) \to [0,1]$ such that $f(y) \quad 0$ and $f(H - U^{\#}[y]) \subseteq \{1\}$. Then $f: (H, \tau(\mathcal{U}^{\#})_H) \to [0,1]$ is continuous and $f(H - G) \subseteq \{1\}$, since $U^{\#}[y] \subseteq G$. Therefore, $(X(\mathcal{U}), \tau(\mathcal{U}^{\#}))$ has relatively completely regular outgrowth.

(c): Let $y \in X(\mathcal{U})$ and let $\mathcal{F} \in X(\mathcal{U}) - X$ with $y \neq \mathcal{F}$. If $y \in X$, then there is a $G \in \tau(\mathcal{U})\langle y\rangle$ and an $F \in \mathcal{F} \cap \tau(\mathcal{U})$ such that $G \cap F \quad \emptyset$, since $\mathcal{F}$ is a free filter on X. But then there is an $A \in \tau(\mathcal{U}^{\#})\langle y\rangle$ such that $G = X \cap A$ and, by Proposition 3.13, there is a $B \in \tau(\mathcal{U}^{\#})\langle\mathcal{F}\rangle$ such that $F = B \cap X$. Then $A \cap B = \emptyset$, since X is dense in $X(\mathcal{U})$.

Now assume that $y \notin X$. Since y and $\mathcal{F}$ are distinct free minimal $\mathcal{U}$-Cauchy filters on X, there is a $G \in y \cap \tau(\mathcal{U})$ and an

$F \in \mathcal{F} \cap \tau(\mathcal{U})$ such that $G \cap F \quad \emptyset$. But then there is an $A \in \tau(\mathcal{U}^{\#})\langle y\rangle$ and a $B \in \tau(\mathcal{U}^{\#})\langle\mathcal{F}\rangle$ such that $G = X \cap A$ and $F \quad X \cap B$. Then $A \cap B = \emptyset$. Thus $(X(\mathcal{U}), \tau(\mathcal{U}^{\#}))$ is Hausdorff except for X.||

An immediate consequence of Proposition 3.15(c) is the following corollary, whose proof is omitted.

<u>Corollary 3.16</u>. Let $(X, \mathcal{U})$ be a separated para-uniform space. Then $(X(\mathcal{U}), \tau(\mathcal{U}^{\#}))$ is Hausdorff.

While distinct para-uniformities on X may induce the same extension as defined in Theorem 3.12, they are related by the following proposition.

<u>Proposition 3.17</u>. Let $\{\mathcal{U}_\alpha | \alpha \in A\}$ be a collection of para-uniformities on X, each compatible with τ, such that $(X(\mathcal{U}_\alpha), \tau(\mathcal{U}_\alpha^{\#})) \quad (X(\mathcal{U}_\beta), \tau(\mathcal{U}_\beta^{\#}))$ for all $\alpha, \beta \in A$. Let $\mathcal{U} = \sup\{\mathcal{U}_\alpha | \alpha \in A\}$. Then $(X(\mathcal{U}), \tau(\mathcal{U}^{\#})) \quad (X(\mathcal{U}_\alpha), \tau(\mathcal{U}_\alpha^{\#}))$, for all $\alpha \in A$.

<u>Proof</u>. Any $\mathcal{U}$-Cauchy filter is $\mathcal{U}_\alpha$-Cauchy for each $\alpha \in A$, since $\mathcal{U}_\alpha \subseteq \mathcal{U}$. On the other hand, for $\beta \in A$, any $\mathcal{U}_\beta$-Cauchy filter is $\mathcal{U}$-Cauchy, since it is $\mathcal{U}_\alpha$-Cauchy for all $\alpha \in A$ and $\cup \{\mathcal{U}_\alpha | \alpha \in A\}$ is a subbasis for $\mathcal{U}$. (Recall that filters are closed under finite intersections and use Proposition 3.2.) But $(X(\mathcal{U}), \tau(\mathcal{U}^{\#}))$ and $(X(\mathcal{U}_\alpha), \tau(\mathcal{U}_\alpha^{\#}))$ are just the strict filter extensions of (X, τ) based on the sets of free minimal $\mathcal{U}$-Cauchy and $\mathcal{U}_\alpha$-Cauchy filters, respectively, as shown by Proposition 3.13. Since these latter sets are equal, $(X(\mathcal{U}), \tau(\mathcal{U}^{\#})) \quad (X(\mathcal{U}_\alpha), \tau(\mathcal{U}_\alpha^{\#}))$, for each $\alpha \in A$.||

Another completion can be obtained in the manner described in Proposition 2.31.

Theorem 3.18. Let $(X,\mathcal{U})$ be a para-uniform space and let $\mathcal{B} = \{V(H) \mid V \in \mathcal{U}^{\#},\ X \subseteq H \subseteq X(\mathcal{U})\}$, where $V(H) \quad V \cap (H\times H)$. Then $\mathcal{B}$ is a basis for a para-uniformity $\mathcal{U}^{+}$ on $X(\mathcal{U})$, $(X(\mathcal{U}),\mathcal{U}^{+})$ is a completion of $(X,\mathcal{U})$, $(X(\mathcal{U}),\tau(\mathcal{U}^{+}))$ is a simple extension of $(X,\tau(\mathcal{U}))$, $\tau(\mathcal{U}^{+}) \quad \tau(\mathcal{U}^{\#})^{+}$ and $\tau(\mathcal{U}^{\#}) \quad \tau(\mathcal{U}^{+})^{\#}$.

Proof. By Propositions 2.31 and 3.12, $\mathcal{B}$ is a basis for a para-uniformity on $X(\mathcal{U})$, $(X(\mathcal{U}),\ \tau(\mathcal{U}^{+}))$ is a simple extension of $(X,\tau(\mathcal{U}))$, $\tau(\mathcal{U}^{+}) \quad \tau(\mathcal{U}^{\#})^{+}$ and $\tau(\mathcal{U}^{\#}) \quad \tau(\mathcal{U}^{+})^{\#}$. Thus it only remains to prove that $(X(\mathcal{U}),\tau(\mathcal{U}^{+}))$ is a completion of $(X,\mathcal{U})$. But it is clear that $\mathcal{U}^{+}_{X} = \mathcal{U}$, from the definitions of $\mathcal{U}^{\#}$ and $\mathcal{U}^{+}$. On the other hand, if $\mathcal{G}$ is a free $\mathcal{U}^{+}$-Cauchy filter on $X(\mathcal{U})$, then $\mathcal{G}$ contains a free minimal $\mathcal{U}$-Cauchy filter $\mathcal{F}$, since $X\times X \in \mathcal{U}^{+}$. But $\{F \cup \{\mathcal{F}\} \mid F \in \mathcal{F}\}$ is a $\tau(\mathcal{U}^{+})$-neighborhood base at $\mathcal{F}$, by Proposition 3.13 and the facts that $X \in \tau(\mathcal{U}^{+})$ and $\tau(\mathcal{U}^{+}) \quad \tau(\mathcal{U}^{\#})^{+}$. Thus $\mathcal{G}$ converges to $\mathcal{F}$ in $(X(\mathcal{U}),\tau(\mathcal{U}^{+}))$, since $\mathcal{F} \subseteq \mathcal{G}$ implies $F \cup \{\mathcal{F}\} \in \mathcal{G}$ for each $F \in \mathcal{F}$ Therefore, $\mathcal{G}$ is not free, a contradiction. So there are no free $\mathcal{U}^{+}$-Cauchy filters on $X(\mathcal{U})$ and $(X(\mathcal{U}),\mathcal{U}^{+})$ is complete, hence a completion of $(X,\mathcal{U})$.||

In view of certain results in uniform space theory, it is natural to ask about the extension of functions into complete para-uniform spaces. This is the subject of the following theorem. The results are not quite as nice as in the uniform case.

A filter base is Cauchy if the filter it generates is.

Theorem 3.19. Let (Y,σ) be an extension of (X,τ), let $(Z,\mathcal{W})$ be a complete para-uniform space with $\rho \quad \tau(\mathcal{W})$, and let $f\colon (X,\tau)\to (Z,\rho)$ be a continuous function. Let $\mathcal{V}$ be a para-uniformity on Y compatible with σ and let $\mathcal{U} = \mathcal{V}_X$.

(a) If the filter base $\{f(G)\,|\,G \in \sigma_X\langle y\rangle\}$ is $\mathcal{W}$-Cauchy, for each $y \in Y - X$, then f can be extended to a continuous function $g\colon (Y,\sigma^+)\to (Z,\rho)$.

(b) If $f\colon (X,\mathcal{U})\to (Z,\mathcal{W})$ is para-uniformly continuous, $\sigma_X\langle y\rangle$ is $\mathcal{U}$-Cauchy, for each $y \in Y - X$, and $f(X)\cap W^0[Z] \neq \emptyset$, for each $W \in \mathcal{W}$, then f can be extended to a para-uniformly continuous function $g\colon (Y,(\mathcal{V}^\#)^+)\to (Z,\mathcal{W})$, where $\mathcal{V}^\#$ is defined from $\mathcal{V}$ as in Proposition 2.32 and $(\mathcal{V}^\#)^+$ is defined from $\mathcal{V}^\#$ as in Proposition 2.31.

Proof. (a): Assume the filter base $\{f(G)\,|\,G \in \sigma_X\langle y\rangle\}$ is $\mathcal{W}$-Cauchy, for each $y \in Y - X$. Then, for each $y \in Y - X$, there is some $w(y) \in Z$ such that the filter generated by $\{f(G)\,|\,G \in \sigma_X\langle y\rangle\}$ converges to $w(y)$, since $(Z,\mathcal{W})$ is complete. Define $g(y) = w(y)$, for $y \in Y - X$, and $g(x) \quad f(x)$, for $x \in X$. Then $g\colon Y\to Z$ is a function.

Now let $y \in Y$, let $H \in \rho\langle g(y)\rangle$ and let $G \quad X\cap g^{-1}(H)$. By continuity of f, $G \in \tau \subseteq \sigma^+$ Thus, if $y \in X$, then $G \in \sigma^+\langle y\rangle$ and $g(G)\subseteq H$. But, if $y \in Y - X$, then there is an $F \in \sigma_X\langle y\rangle$ such that $F\subseteq H$, by definition of $g(y)$. Then $F\subseteq G$, so $G \in \sigma_X\langle y\rangle$, hence $G\cup\{y\} \in \sigma^+\langle y\rangle$ and $g(G\cup\{y\}) \subseteq H$. Thus $g\colon (Y,\sigma^+)\to(Z,\rho)$ is continuous.

(b): Assume the indicated properties. Let $y \in Y - X$ and let $W \in \mathcal{W}$. Then $f^{-1}(W) \neq \emptyset$, so $f^{-1}(W) \in \mathcal{U}$. Therefore, there is an $H \in \sigma_X\langle y\rangle$ such that $H \times H \subseteq f^{-1}(W)$. But then $f(H) \times f(H) \subseteq W$, so the filter base $\{f(G) \mid G \in \sigma_X\langle y\rangle\}$ is $\mathcal{W}$-Cauchy. Thus f can be extended to a continuous function $g\colon (Y,\sigma^+) \to (Z,\rho)$, by (a) above.

Let $W \in \mathcal{W}$ and $U \in \mathcal{W}$ such that $U \quad U^{-1}$, $U^3 \subseteq W$ and $W^0 \quad U^0$. Then $f^{-1}(U) \in \mathcal{U}$, since $f(X) \cap U^0[Z] \neq \emptyset$, so there is a $V \in \mathcal{V}$ such that $V(X) \quad f^{-1}(U)$. Let $(x,y) \in V^{\#}((g^{-1}(U))^0[Y])$. Recall that $V^{\#} \quad V(X) \cup \{(v,w) \in Y \times Y \mid G \times G \subseteq V \text{ for some } G \in \sigma_X\langle v\rangle \cap \sigma_X\langle w\rangle\}$ and $V^{\#}((g^{-1}(U))^0[Y]) \quad V^{\#} \cap ((g^{-1}(U))^0[Y] \times (g^{-1}(U))^0[Y])$. If $(x,y) \in V(X)$, then $(g(x),g(y)) \quad (f(x),f(y)) \in U \subseteq W$. So assume that $y \in Y - X$. Then there is a $G \in \sigma_X\langle x\rangle \cap \sigma_X\langle y\rangle$ such that $G \times G \subseteq V$. But then $g(x), g(y) \in cl_\rho(f(G))$, by definition of g. Since $(x,y) \in V^{\#}((g^{-1}(U))^0[Y])$, $U[g(x)] \neq \emptyset \neq U[g(y)]$, hence there is a $p \in U[g(x)] \cap f(G)$ and a $q \in U[g(y)] \cap f(G)$. But then $(p,q) \in U$, so $(g(x),g(y)) \in U^3 \subseteq W$. Therefore, $V^{\#}((g^{-1}(U))^0[Y]) \subseteq g^{-1}(W)$. But

$$(g^{-1}(W))^0 \quad (g^{-1}(U))^0 \quad (V^{\#}((g^{-1}(U))^0[Y]))^0,$$

so $g^{-1}(W) \in (\mathcal{V}^{\#})^+$, by (U5), hence $g\colon (Y,(\mathcal{V}^{\#})^+) \to (Z,\mathcal{W})$ is a para-uniformly continuous extension of f.||

Remark. If (Z,ρ) is Hausdorff, or if (Y,σ) is Hausdorff except for X and $cl_\rho(f(X))$ is Hausdorff except for $f(X)$, in the preceding theorem, then the extension g is unique.

Corollary 3.20. Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ be para-uniform spaces, with $X \subseteq Y$, $\mathcal{U} = \mathcal{V}_X$, X dense in $(Y,\tau(\mathcal{V}))$, and $(Z,\mathcal{W})$ complete. Let $f\colon (X,\mathcal{U}) \to (Z,\mathcal{W})$ be para-uniformly continuous, with $f(X)$ dense in $(Z,\tau(\mathcal{W}))$.

(a) If $\tau(\mathcal{V})_X\langle y\rangle$ is $\mathcal{U}$-Cauchy for each $y \in Y - X$, then f can be extended to a para-uniformly continuous function $g\colon (Y,(\mathcal{V}^{\#})^{+}) \to (Z,\mathcal{W})$.

(b) f can be extended to a para-uniformly continuous function $h\colon (X(\mathcal{U}),\mathcal{U}^{+}) \to (Z,\mathcal{W})$.

Proof. (a): $f(X)$ dense in $(Z,\tau(\mathcal{W}))$ implies that $f(X) \cap W^{0}[Z] \neq \emptyset$, since $W^{0}[Z] \in \tau(\mathcal{W})$, for each $W \in \mathcal{W}$. The conclusion now follows from Theorem 3.19(b).

(b): $\mathcal{F} \in X(\mathcal{U}) \quad X$ implies $\tau(\mathcal{U}^{\#})_X\langle\mathcal{F}\rangle = \mathcal{F} \cap \tau(\mathcal{U})$, by Proposition 3.13. Hence $\tau(\mathcal{U}^{\#})_X\langle\mathcal{F}\rangle$ is $\mathcal{U}$-Cauchy. But $(\mathcal{U}^{\#})^{+} = \mathcal{U}^{+}$, by definition of $\mathcal{U}^{+}$, so the conclusion follows from (a) above.||

2. H-closure. The concept of quasi-H-closure will now be introduced and related to para-uniform completeness.

Definition 3.21. [PT] Let (X,τ) be a topological space. A subset $\beta \subseteq \tau$ whose union is dense in (X,τ) is a proximate cover of X. (X,τ) is quasi-H-closed if every open cover has a finite subcollection which is a proximate cover. A quasi-H-closed space is H-closed if it is Hausdorff.

Remark. If (Y,σ) is an extension of (X,τ), then (Y,σ) is quasi-H-closed if and only if $(Y,\sigma^{\#})$ and (Y,σ^{+}) are. [cf., IF, Lemma 8, p. 48]

It is not difficult to show that (X,τ) is quasi-H-closed if and only if every τ-filter on X has an adherent point.

<u>Definition 3.22</u>. Let A be a collection of open filters on (X,τ) and let $Y \quad X \cup A$. Then the set $\tau \cup \{F \cup \{\mathcal{F}\} | F \in \mathcal{F} \in A\}$ forms a basis for a topology σ on Y such that (Y,σ) is an extension of (X,τ) and $\sigma_X<y> \quad y$ for each $y \in Y - X$ [see Ba]. Note that $\sigma^+ \quad \sigma$, so (Y,σ) is a simple extension of (X,τ). (Y,σ) will be called the <u>simple filter extension</u> of (X,τ) <u>based on A</u>, and $(Y,\sigma^\#)$ will be called the <u>strict filter extension</u> of (X,τ) <u>based on A</u>.

It is not difficult to show that (Y,σ), in the preceding definition, is quasi-H-closed if and only if every free open ultrafilter on (X,τ) contains one of the filters in A, and $(Y,\sigma^\#)$ is Hausdorff except for X if and only if each of the filters in A is free and each open ultrafilter on (X,τ) contains at most one of the filters in A.

<u>Definition 3.23</u>. A collection M of open filters on (X,τ) is called <u>free</u> if it contains only free filters, and is called <u>separated</u> if for any two distinct $\mathcal{F},\mathcal{G} \in M$ there are disjoint elements $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Every maximal (in the set inclusion sense) free separated set of open filters on (X,τ) yields a strict, or simple, quasi-H-closed extension of (X,τ) which is Hausdorff except for X. On the other hand, any strict, or simple, quasi-H-closed extension (Y,σ) of (X,τ) which is Hausdorff except for X is isomorphic to a strict, or simple, filter extension based on a maximal free separated set of open filters on X, namely, the set $\{\sigma_X<y> | y \in Y - X\}$. Thus the problem of characterizing the strict quasi-H-closed extensions of (X,τ) which are Hausdorff except for X is the precisely the problem of characterizing the maximal free separated sets of open filters on (X,τ). But this is equivalent to characterizing those sets M of free open filters on (X,τ) such that each free τ-ultrafilter contains precisely one of the filters in M. The completions of certain para-uniform spaces provide a method for describing some of these sets and the extensions obtained from them.

<u>Definition 3.24</u>. A para-uniform space $(X,\mathcal{U})$, or the para-uniformity $\mathcal{U}$, is <u>totally bounded</u> if for every $U \in \mathcal{U}$ there is a finite collection $\mathcal{A}$ of subsets of X such that $X = \cup\{\overline{A} \mid A \in \mathcal{A}\}$ and $\cup\{A\times A \mid A \in \mathcal{A}\} \subseteq U$.

Note that the above definition is equivalent to the usual definition of totally bounded in case $\mathcal{U}$ is a uniformity, since then $\overline{A} \subseteq V[A]$ for every $V \in \mathcal{U}$. Note also that it is equivalent to assume $\emptyset \notin \mathcal{A}$.

<u>Proposition 3.25</u>. Let $(X,\mathcal{U})$ be a para-uniform space. The following are equivalent.

(a) $(X,\mathcal{U})$ is totally bounded.

(b) For every $U \in \mathcal{U}$, there is a finite subset F of X such that X $\overline{U[F]}$.

(c) Every open ultrafilter on $(X,\tau(\mathcal{U}))$ is Cauchy.

<u>Proof</u>. (a) implies (b): Assume (a), let $U \in \mathcal{U}$ and let $\mathcal{A}$ be a collection of nonempty subsets of X such that $X = \cup\{\overline{A} \mid A \in \mathcal{A}\}$ and $\cup\{A\times A \mid A \in \mathcal{A}\} \subseteq U$. Let F be a finite subset of X such that $F \cap A \neq \emptyset$ for each $A \in \mathcal{A}$. Then $X = \overline{U[F]}$.

(b) implies (c): Assume (b), let $\mathcal{F}$ be an open ultrafilter on $(X,\tau(\mathcal{U}))$, and let $U \in \mathcal{U}$. There is a symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$ and $V^0 \quad U^0$, and a finite subset F of X such that $\overline{V[F]} \quad X$. Then $V[a]$ is contained in the interior of $U[a]$, for each $a \in F$, and any open ultrafilter contains one element from any finite proximate cover. Thus $U[a] \in \mathcal{F}$, for some $a \in F$, hence $\mathcal{F}$ is Cauchy.

(c) implies (a): Assume (c) and let $U \in \mathcal{U}$. There is a symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$ and $V^0 \quad U^0$. If for every finite subset F of X, $X \neq \overline{V[F]}$, then the collection

$$\{X - \overline{V[F]} \mid F \subseteq X,\ F \text{ is finite}\}$$

forms a base for an open filter on $(X,\tau(\mathcal{U}))$ which is then contained in an open ultrafilter $\mathcal{F}$ on $(X,\tau(\mathcal{U}))$. But then $\mathcal{F}$ is Cauchy, so there is an $x \in X$ such that $V[x] \in \mathcal{F}$, a contradiction, since $X - \overline{V[x]} \in \mathcal{F}$. Thus there is a finite subset F of X such that $X = \overline{V[F]}$. Then $X \quad \cup \{\overline{V[a]} \mid a \in F\}$ and $\cup \{V[a] \times V[a] \mid a \in F\} \subseteq V^2 \subseteq U.$||

<u>Corollary 3.26</u>. A topological space with a compatible totally bounded complete para-uniformity is quasi-H-closed.

<u>Proof</u>. Let $(X,\mathcal{U})$ be a totally bounded complete para-uniform space. Then every open ultrafilter on $(X,\tau(\mathcal{U}))$ is Cauchy, by Proposition 3.25, hence converges, since $\mathcal{U}$ is complete. Thus $(X,\tau(\mathcal{U}))$ is quasi-H-closed.||

<u>Corollary 3.27</u>. If $(X,\mathcal{U})$ is a totally bounded para-uniform space, then $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is quasi-H-closed.

<u>Proof</u>. Since $\mathcal{U}$ is totally bounded, so is $\mathcal{U}^{\#}$. The corollary now follows from the preceding one.||

<u>Proposition 3.28</u>. Every compatible para-uniformity on a quasi-H-closed space is complete.

<u>Proof</u>. Let $\mathcal{U}$ be a compatible para-uniformity on the quasi-H-closed space (X,τ) and let $\mathcal{F}$ be a $\mathcal{U}$-Cauchy filter on X. Then $\mathcal{F}$ contains a minimal $\mathcal{U}$-Cauchy filter $\mathcal{G}$, by Theorem 3.4. But $\mathcal{G}$ is an open filter on (X,τ), by Proposition 3.7, hence is contained in an open ultrafilter $\mathcal{H}$. Since (X,τ) is quasi-H-closed, $\mathcal{H}$ converges, hence so does $\mathcal{G}$, by Proposition 3.3. Consequently, $\mathcal{F}$ converges, so $(X,\mathcal{U})$ is complete.||

<u>Lemma 3.29</u>. Let (X,τ) be a topological space and let β be a subbasis for τ. For each $G \in \beta$, let

$$S(G) \quad (G\times G) \cup ((X - \overline{G})\times(X - \overline{G})).$$

Then the set $\mathcal{S} = \{S(G) | G \in \beta\}$ is a subbasis for a compatible totally bounded para-uniformity on (X,τ) with a transitive basis.

<u>Proof</u>. It is clear that $S(G) \neq \emptyset$,

$$S(G)^{0} \quad \Delta(G) \cup \Delta(X - \overline{G}) \subseteq S(G)$$

and $S(G)^{2} \quad S(G) \quad S(G)^{-1}$, for every $G \in \beta$. Thus $\mathcal{S}$ is a subbasis for a para-uniformity $\mathcal{U}$ on X. It is also apparent that $\tau(\mathcal{U}) = \tau$, since β is a subbasis for τ.

Now let $\mathcal{F}$ be a τ-ultrafilter on X and let $U \in \mathcal{U}$. Then there is a finite subset γ of β such that $\cap\{S(G) | G \in \gamma\} \subseteq U$ and $(\cap\{S(G) | G \in \gamma\})^{0} \quad U^{0}$. For each $G \in \gamma$, either $G \in \mathcal{F}$ or $X - \overline{G} \in \mathcal{F}$, so let $H(G) = G$ if $G \in \mathcal{F}$ and let $H(G) \quad X - \overline{G}$ if

$X - \overline{G} \in \mathcal{F}$. Then $H(G) \times H(G) \subseteq S(G)$, for each $G \in \gamma$, so

$$(\cap\{H(G)|G \in \gamma\}) \times (\cap\{H(G)|G \in \gamma\}) \subseteq U.$$

But $\cap\{H(G)|G \in \gamma\} \in \mathcal{F}$, so $\mathcal{F}$ is $\mathcal{U}$-Cauchy. Thus $(X,\mathcal{U})$ is totally bounded, by Proposition 3.25.

The set $\mathcal{B}$ $\{\cap\{S(G)|G \in \gamma\}|\gamma$ is a finite subset of $\beta\}$ is a transitive basis for $\mathcal{U}$, that is, $B \in \mathcal{B}$ implies $B^2 \subseteq B$.||

Theorem 3.30. A topological space is quasi-H-closed if and only if every compatible para-uniformity is complete.

Proof. Proposition 3.28 is one implication and the other follows from Lemma 3.29 and Corollary 3.26.||

In uniform space theory, a uniform space whose completion is compact is sometimes called precompact. For uniform spaces, this is equivalent to totally bounded, but a quasi-H-closed space may well have a compatible para-uniformity which is not totally bounded. A concept analogous to precompact is the subject of the next definition.

Definition 3.31. A para-uniform space $(X,\mathcal{U})$ is pre-H-closed if every free $\tau(\mathcal{U})$-ultrafilter is $\mathcal{U}$-Cauchy.

Proposition 3.32. Let $(X,\mathcal{U})$ be a para-uniform space. Then $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is quasi-H-closed if and only if $(X,\mathcal{U})$ is pre-H-closed.

Proof. If $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is quasi-H-closed and $\mathcal{F}$ is a free $\tau(\mathcal{U})$-ultrafilter on X, then $\mathcal{F}$ has a minimal $\mathcal{U}$-Cauchy filter $\mathcal{G}$ as an adherent point in $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$. But then $\mathcal{G} \subseteq \mathcal{F}$, by the properties of open ultrafilters and Proposition 3.13, hence $\mathcal{F}$ is $\mathcal{U}$-Cauchy. Thus $(X,\mathcal{U})$ is pre-H-closed.

Conversely, if $(X,\mathcal{U})$ is pre-H-closed, then every open ultrafilter $\mathcal{G}$ on $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ either converges or induces a free $\tau(\mathcal{U})$-ultrafilter $\mathcal{F}$ on X, which then contains a free minimal $\mathcal{U}$-Cauchy filter. But then $\mathcal{F}$ has an adherent point in $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$, hence so does $\mathcal{G}$, since X is dense in $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$. Thus every open filter on $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ has an adherent point, so $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is quasi-H-closed.||

Lemma 3.33. Let (Y,σ) be an extension of (X,τ) and let $\mathcal{V}$ be a compatible para-uniformity on Y and let $\mathcal{U}\quad\mathcal{V}_X$. If $\sigma_X\langle y\rangle$ is $\mathcal{U}$-Cauchy, then it is minimal $\mathcal{U}$-Cauchy, where $y \in Y$.

Proof. Let $G \in \sigma_X\langle y\rangle$. Then there is a $V \in \mathcal{V}$ such that $y \in V[y]$ and $X \cap V[y] \subseteq G$, and there is a $W \in \mathcal{V}$ such that $W^2 \subseteq V$, $W[y] \in \sigma\langle y\rangle$ and $W^0 \quad V^0$. Let $U \quad W(X)$. Then $U[X \cap W[y]] \subseteq X \cap W^2[y] \subseteq G$. Thus $\sigma_X\langle y\rangle$ is minimal $\mathcal{U}$-Cauchy, if it is $\mathcal{U}$-Cauchy, by Theorem 3.4.||

An extension (Y,σ) of (X,τ) is a quasi-H-closed (or H-closed) extension if it is a quasi-H-closed (or H-closed) space.

Proposition 3.34. Let (Y,σ) be a strict quasi-H-closed extension of (X,τ) which is Hausdorff except for X, and let $\mathcal{V}$ be a compatible para-uniformity on Y with $\mathcal{U} = \mathcal{V}_X$. If $\sigma_X\langle y\rangle$ is $\mathcal{U}$-Cauchy for every $y \in Y - X$, then $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is isomorphic to (Y,σ).

Proof. By Lemma 3.33, $\sigma_X\langle y\rangle$ is a minimal $\mathcal{U}$-Cauchy filter on X, for each $y \in Y - X$. Since (Y,σ) is Hausdorff except for X, $\sigma_X\langle y\rangle$ is free, for each $y \in Y - X$. Each free

τ-ultrafilter contains $\sigma_X\langle y\rangle$ for some $y \in Y - X$, since (Y,σ) is quasi-H-closed. Thus $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is a strict quasi-H-closed extension of (X,τ) which is Hausdorff except for X and has the same trace filters as (Y,σ). Therefore $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is isomorphic to (Y,σ).||

Let (X,τ) be a Hausdorff space and let $\mathcal{U}$ be the para-uniformity on X generated by the set $\{S(G)|G \in \tau\}$ as sub-basis (see Lemma 3.29). As noted in Lemma 3.29, $\mathcal{U}$ is totally bounded and has a transitive basis. Therefore, both $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ and $(X(\mathcal{U}),\tau(\mathcal{U}^{+}))$ are H-closed extensions of (X,τ), by Corollary 3.27, the remark following Definition 3.21 and Remark 3 following Definition 3.14. Also, $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ has relatively zero-dimensional outgrowth, by Proposition 3.15(a). It is easy to verify from the definitions that every free τ-ultrafilter is minimal $\mathcal{U}$-Cauchy, so the extensions $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ and $(X(\mathcal{U}),\tau(\mathcal{U}^{+}))$ are the strict and simple filter extensions based on the collection of free τ-ultra-filters on X. The latter was defined by Katětov [K1] in 1940 and the former by Fomin [Fo] in the same year. $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is now sometimes called the Fomin extension, and $(X(\mathcal{U}),\tau(\mathcal{U}^{+}))$ is commonly called the Katětov extension.

CHAPTER 4

H-CLOSED EXTENSIONS

It will be shown in this chapter that the quasi-H-closed extensions with relatively completely regular outgrowth which are Hausdorff except for X are precisely those extensions which can be obtained, up to isomorphism, as para-uniform completions $(X(\mathcal{U}),\mathcal{U}^{\#})$ for some totally bounded para-uniformity $\mathcal{U}$. A representative from each isomorphism class of strict quasi-H-closed extensions, which are Hausdorff except for X, of a given topological space (X,τ) will also be obtained in this chapter. First, some further results concerning relatively zero-dimensional outgrowth.

Let $(X,\mathcal{U})$ be a para-uniform space. The completion $(X(\mathcal{U}),\mathcal{U}^{\#})$ then yields a quasi-H-closed extension of $(X,\tau(\mathcal{U}))$ which is Hausdorff except for X and has relatively zero-dimensional outgrowth, if $\mathcal{U}$ is totally bounded and has a transitive basis, by Proposition 3.15(a), (c) and Corollary 3.27. The converse also holds, up to isomorphism, as shown in the following theorem.

Theorem 4.1. Let (Y,σ) be an extension of (X,τ). The following are equivalent.

(a) (Y,σ) is quasi-H-closed, is Hausdorff except for X, and has relatively zero-dimensional outgrowth.

(b) (Y,σ) is isomorphic to $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$, for some totally bounded para-uniformity $\mathcal{U}$ on X with a transitive basis.

Proof. That (b) implies (a) was noted above, so assume (a) and let β $\{G \in \sigma | \overline{G} - G \subseteq X\}$. Since (Y,σ) has relatively zero-dimensional outgrowth, β is a basis for τ. Moreover, β is closed under finite intersections and $G \in \beta$ implies $Y - \overline{G} \in \beta$, [cf., Fl]. The collection $\{S(G) | G \in \beta\}$, where $S(G)$ is defined as in Lemma 3.29, is a subbasis for a compatible totally bounded para-uniformity $\mathcal{V}$ on Y which has a transitive basis, by Lemma 3.29. Let $\mathcal{U} = \mathcal{V}_X$. Then $\mathcal{U}$ is a compatible totally bounded para-uniformity on X which has a transitive basis.

Let $y \in Y - X$. Then $y \in G \cup (Y - \overline{G})$, for every $G \in \sigma$ such that $\overline{G} - G \subseteq X$, hence $\sigma_X\langle y\rangle$ is $\mathcal{U}$-Cauchy. Now (Y,σ) is a strict extension of (X,τ), by Remark 1 following Definition 3.14. Hence $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is isomorphic to (Y,σ), by Proposition 3.34. Thus (a) implies (b).||

An immediate consequence of Theorem 4.1, in view of Proposition 3.15(b), is the following corollary.

Corollary 4.2. Relatively zero-dimensional outgrowth implies relatively completely regular outgrowth, for a quasi-H-closed extension of (X,τ) which is Hausdorff except for X.

Flachsmeyer [Fl] studied H-closed extensions with relatively zero-dimensional outgrowth and noted that, up to isomorphism, they could be obtained as filter extensions based on the set of maximal filters from a collection of open sets called a π-basis. (A π-basis, on (X,τ), is a basis β for τ such that $G \in \beta$ implies $X - \overline{G} \in \beta$.) Using the idea of a full

π-basis, he showed that there is a one-to-one correspondence between the full π-bases on (X,τ) and the isomorphism classes of H-closed extensions of (X,τ) with relatively zero-dimensional outgrowth, for Hausdorff (X,τ). (A <u>full</u> <u>π-basis</u> may be defined as a π-basis β with the property that $G \in \beta$ if every τ-ultrafilter containing G contains a subset of G which is an element of β.) This gives the result that there is a one-to-one correspondence between the isomorphism classes of H-closed extensions of (X,τ) with relatively zero-dimensional outgrowth and the para-uniform completions $(X(\mathcal{U}),\mathcal{U}^{\#})$, where $\mathcal{U}$ is the para-uniformity generated by $\{S(G)|G \in \beta\}$ for a full π-basis β.

Note that $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ need not have relatively zero-dimensional outgrowth, as the existence of Hausdorff compactifications of non-rimcompact Hausdorff space shows. Thus the method of para-uniform completions yields a larger class of H-closed extensions than does the method used by Flachsmeyer [Fl], since $(X(\mathcal{U}),\mathcal{U}^{\#})$ is the usual uniform completion when $\mathcal{U}$ is a totally bounded uniformity.

A characterization, up to isomorphism, of those strict quasi-H-closed extensions obtained as para-uniform completions is given in the following theorem.

<u>Theorem 4.3</u>. Let (Y,σ) be an extension of (X,τ). The following are equivalent.

(a) (Y,σ) is quasi-H-closed, is Hausdorff except for X, and has relatively completely regular outgrowth.

(b) (Y,σ) is isomorphic to $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$, for some totally bounded para-uniformity $\mathcal{U}$ on X.

<u>Proof</u>. That (b) implies (a) follows from Proposition 3.15(b), (c) and Corollary 3.27, so assume (a) holds. For each $y \in Y$ and each $G \in \sigma\langle y\rangle$, let $f(G,y)\colon H(G,y) \to [0,1]$ be a continuous function such that $f(G,y)(y) \quad 0$ and $f(G,y)(H(G,y) - G) \subseteq \{1\}$, where $Y - X \subseteq H(G,y) \in \sigma\langle y\rangle$ and $H(G,y)$ has the subspace topology induced by σ. Let $F \quad \{f(G,y) | G \in \sigma\langle y\rangle,\ y \in Y\}$, and, for each $f \in F$, let $H(f)$ be the domain of f. For $f \in F$ and $\delta > 0$, let

$V(f,\delta) \quad (Y - \overline{H(f)})\times(Y - \overline{H(f)}) \cup \{(v,w) \in Y\times Y \mid |f(v) - f(w)| < \delta\}$.

It is clear that $V(f,\delta)^{-1} \quad V(f,\delta)$, $V(f,\delta)^2 \subseteq V(f,2\delta)$ and $V(f,\delta)^0 \quad \Delta(H(f)) \cup \Delta(Y \quad \overline{H(f)}) \subseteq V(f,\delta)$. Thus the set $\{V(f,\delta) | f \in F,\ \delta > 0\}$ is a subbasis for a para-uniformity $\mathcal{V}$ on Y. Moreover, in view of the definition of $V(f,\delta)$ and the properties of the unit interval, it is not difficult to verify that $\mathcal{V}$ is totally bounded. From the definitions of $\mathcal{V}$ and F, $\sigma \subseteq \tau(\mathcal{V})$. On the other hand, for $y \in V(f,\delta)^{-1}[Y]$, $V(f,\delta)[y] \quad \{v \in Y \mid |f(y) - f(v)| < \delta\} \quad f^{-1}(]f(y) - \delta, f(y) + \delta[) \in \sigma$. Thus $\tau(\mathcal{V}) \subseteq \sigma$, so $\sigma \quad \tau(\mathcal{V})$.

Now let $\mathcal{U} = \mathcal{V}_X$. If $y \in Y - X$, $f \in F$ and $\delta > 0$, then, by definition of F, $y \in V(f,2\delta)^{-1}[Y] = H(f) \quad V(f,\delta)^{-1}[Y]$ and

$$V(f,\delta)[y]\times V(f,\delta)[y] \subseteq V(f,\delta)^2 \subseteq V(f,2\delta).$$

Therefore $\sigma_X\langle y\rangle$ is a $\mathcal{U}$-Cauchy filter on X, for each $y \in Y - X$. But (Y,σ) is a strict extension, by Remark 2 following Definition 3.14, so $(X(\mathcal{U}),\tau(\mathcal{U}^{\#}))$ is isomorphic to (Y,σ), by

Proposition 3.34.||

The question naturally arises as to whether there are any strict H-closed extensions which do not have relatively completely regular outgrowth. Clearly, there are no such Hausdorff compactifications, but there are such H-closed extensions, as shown by the following example.

Let $X = \{(n,m) | n,m,-m \in \underline{N}\}$ and let τ be the discrete topology on X. Let $p \quad (0,1)$, $q \quad (0,-1)$ and

$$Y \quad X \cup \{p,q\} \cup \{(n,0) | n \in \underline{N}\}.$$

For $n,k \in \underline{N}$, let $G(n,k) \quad \{(n,0)\} \cup \{(n,m) \in X | \ |m| > k\}$, let $G(p,k) \quad \{p\} \cup \{(j,m) \in X | j > k,\ m \in \underline{N}\}$ and let $G(q,k) \quad \{q\} \cup \{(j,m) \in X | j > k,\ -m \in \underline{N}\}$. Let σ be the topology on Y generated by

$$\{\{x\} | x \in X\} \cup \{G(n,k) | n,k \in \underline{N}\} \cup \{G(y,k) | y \in \{p,q\},\ k \in \underline{N}\}$$

as basis. Then (Y,σ) is a strict H-closed extension of (X,τ) which does not have relatively completely regular outgrowth (p and q cannot be separated by a continuous real-valued function on any neighborhood of $Y - X$). The verification of these properties is not difficult and is omitted.

Attention is now turned to the problem of describing a representative from each isomorphism class of strict quasi-H-closed extensions which are Hausdorff except for X. First, some preliminary concepts and results will be discussed. Recall from the discussion at the end of the preceding chapter that, for a Hausdorff space (X,τ), the Katětov extension of (X,τ) can be viewed as the simple filter extension based on the set of free τ-ultrafilters on X. (For the definition of simple filter extension, see Definition 3.22.) This extension will be denoted by $(\kappa X, \kappa)$, or by κX. Iliadis and Fomin [IF] and Porter and Thomas [PT] showed that if (Y, σ) is any H-closed extension of (X, τ), there is a unique continuous surjection $f\colon (\kappa X, \kappa) \rightarrow (Y, \sigma)$ such that $f(x) \quad x$ for every $x \in X$. The function f is defined on $\kappa X - X$ by $f(\mathcal{F}) = y \in Y$ such that $\sigma_X\langle y \rangle \subseteq \mathcal{F}$. The same property holds, with the function defined in the same way, for an arbitrary topological space (X, τ), if $(\kappa X, \kappa)$ is the simple filter extension of (X, τ) based on the set of free τ-ultrafilters on X and (Y, σ) is a quasi-H-closed extension of (X, τ) which is Hausdorff except for X. If the cardinality of $f^{-1}(y)$ is greater than one for some $y \in Y - X$, where f is the indicated continuous surjection of κX onto Y, then y is called a multiple point of the extension (Y, σ).

Lemma 4.4. Let (Y, σ) be a quasi-H-closed extension of (X, τ) which is Hausdorff except for X and let $f\colon (\kappa X, \kappa) \rightarrow (Y, \sigma)$ be the continuous surjection such that $f(x) \quad x$ for all $x \in X$. Then $\tau \quad \cup(f^{-1}(y))$ is a basis for τ, for each $y \in Y - X$.

Proof. Let $y \in Y - X$ and let $x \in G \in \tau$. Then, since $\sigma_X\langle y \rangle \subseteq \cap(f^{-1}(y))$, there is a $B \in \tau\langle x \rangle$ and an $F \in \cap(f^{-1}(y))$ such that $B \cap F \quad \emptyset$. But then $B \cap G \in \tau\langle x \rangle - \cup(f^{-1}(y))$. Thus $\tau \quad \cup(f^{-1}(y))$ is a basis for τ, since it is closed under finite intersections.||

Theorem 4.5. Let (Y, σ) be a quasi-H-closed extension of (X, τ) which is Hausdorff except for X, let $f\colon (\kappa X, \kappa) \rightarrow (Y, \sigma)$

be the continuous surjection such that $f(x) \quad x$ for all $x \in X$, and let $y \in Y - X$. Then there is a totally bounded para-uniformity $\mathcal{U} = \mathcal{U}(y)$ on X which has a transitive basis and satisfies the following conditions.

(a) $\sigma_X\langle y\rangle \in X(\mathcal{U}) - X$.

(b) $X(\mathcal{U}) \quad \{\sigma_X\langle y\rangle\} \quad \kappa X - f^{-1}(y)$.

(c) The function f can be continuously factored through $(X(\mathcal{U}),\tau(\mathcal{U}^+))$.

<u>Proof</u>. By Lemma 4.4, $\tau - \cup(f^{-1}(y))$ is a basis for τ, so $\beta \quad \sigma_X\langle y\rangle \cup (\tau \quad \cup(f^{-1}(y)))$ is a subbasis for τ. Let $\mathcal{U} = \mathcal{U}(y)$ be the para-uniformity on X with $\{S(G)\,|\,G \in \beta\}$ as subbasis, where $S(G)$ is defined as in Lemma 3.29. Then $\mathcal{U}$ is totally bounded and has a transitive basis, by Lemma 3.29.

(a): Every τ-ultrafilter on X is $\mathcal{U}$-Cauchy, by Proposition 3.25. Let $\mathcal{F}$ be a τ-ultrafilter such that $f(\mathcal{F}) \quad y$, i.e., $\sigma_X\langle y\rangle \subseteq \mathcal{F}$, and let $\mathcal{G}$ be the minimal $\mathcal{U}$-Cauchy filter contained in $\mathcal{F}$. For $G \in \sigma_X\langle y\rangle$, $S(G)[G] \quad G$, so $\sigma_X\langle y\rangle \subseteq \mathcal{G}$, by Theorem 3.4. Now let $B \in \tau - \cup(f^{-1}(y))$. Then $y \notin cl_\sigma(B)$ and, therefore, $X - cl_\tau(B) \quad X \cap (Y - cl_\sigma(B)) \in \sigma_X\langle y\rangle$. Then $S(B)[x] \in \sigma_X\langle y\rangle$ for $x \in X - cl_\tau(B)$, so $\sigma_X\langle y\rangle$ is $\mathcal{U}$-Cauchy. But then $\sigma_X\langle y\rangle$ is a free minimal $\mathcal{U}$-Cauchy filter, hence is an element of $X(\mathcal{U}) \quad X$.

(b): Let $\mathcal{F} \in \kappa X - X$ such that $f(\mathcal{F}) \neq y$. Then $\mathcal{F}$ is $\mathcal{U}$-Cauchy, as noted above, so let $F \in \mathcal{F}$. Since $\mathcal{F} \notin f^{-1}(y)$, there is a $B \in \mathcal{F}$ and a $G \in \sigma_X\langle y\rangle$ such that $B \cap G = \emptyset$. Then $B \cap F \in \mathcal{F}$, $B \cap F \in \tau - \cup(f^{-1}(y))$ and $S(B \cap F)[B \cap F] \quad B \cap F \subseteq F$.

Therefore $\mathcal{F}$ is a free minimal $\mathcal{U}$-Cauchy filter on X, by Theorem 3.4, hence is an element of $X(\mathcal{U}) - X$. Thus

$$X(\mathcal{U}) \quad \{\sigma_X\langle y\rangle\} \quad \kappa X - f^{-1}(y).$$

(c): Let $g: (\kappa X, \kappa) \to (X(\mathcal{U}), \tau(\mathcal{U}^+))$ be the continuous surjection such that $g(x) \quad x$ for all $x \in X$. Define $h: (X(\mathcal{U}), \tau(\mathcal{U}^+)) \to (Y, \sigma)$ by $h(p) \quad f(p)$ for $p \in X(\mathcal{U}) - \{\sigma_X\langle y\rangle\}$, and $h(\sigma_X\langle y\rangle) \quad y$. It is easy to verify that h is continuous, and it is obvious that $hg = f$.||

<u>Definition 4.6</u>. Let C be a collection of para-uniformities on X. A filter on X is <u>C-Cauchy</u> if it is $\mathcal{U}$-Cauchy for some $\mathcal{U} \in C$. If each para-uniformity in C is compatible with the topology τ on X, then a <u>C-completion</u> of (X,τ) is an extension (Y,σ) of (X,τ) such that every C-Cauchy filter on X has a σ-adherent point in Y.

For a collection C of para-uniformities on X, each compatible with the topology τ, let M(C) be the set of free C-Cauchy filters on (X,τ). A filter $\mathcal{F}$ is said to <u>meet</u> a filter $\mathcal{G}$ if, for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $F \cap G \neq \emptyset$. Call two filters $\mathcal{F}, \mathcal{G} \in M(C)$ <u>contiguous</u> if there is a finite subset $\{\mathcal{F}_i \mid 1 \leq i \leq n\} \subseteq M(C)$ such that $\mathcal{F} = \mathcal{F}_1$, $\mathcal{G} = \mathcal{F}_n$ and $\mathcal{F}_i$ meets $\mathcal{F}_{i+1}$ for $1 \leq i < n$. It is not difficult to verify that the relation of being contiguous partitions the set M(C); the elements of this partition will be called <u>contiguity classes</u>. For $\mathcal{F} \in M(C)$, let $m(\mathcal{F})$ be the filter which is the intersection of all the filters in the contiguity class containing $\mathcal{F}$. Note that $m(\mathcal{F})$ is a τ-filter, for each $\mathcal{F} \in M(C)$, since every

C-Cauchy filter contains a C-Cauchy τ-filter.

For (X,τ), C, M(C) and $m(\mathcal{F})$ as above, let $(C(X),C(\tau))$ be the strict filter extension (see Definition 3.22) of (X,τ) based on the set $\{m(\mathcal{F}) \mid \mathcal{F} \varepsilon M(C)\}$. Note that if $\mathcal{F} \varepsilon M(C)$, then $m(\mathcal{F})$ is a $C(\tau)$-adherent point of $\mathcal{F}$ in X(C). This yields the following proposition.

<u>Proposition 4.7</u>. Let C be a collection of para-uniformities on X, each compatible with the topology τ. Then $(C(X),C(\tau))$ is a C-completion of (X,τ).

Now $(C(X),C(\tau))$ will be quasi-H-closed if and only if for each free τ-ultrafilter $\mathcal{F}$ on X there is a filter $\mathcal{G} \varepsilon M(C)$ such that $m(\mathcal{G}) \subseteq \mathcal{F}$, as is easily verified. Moreover, it is not difficult to see that $(C(X),C(\tau))$ is Hausdorff except for X if and only if the set $\{m(\mathcal{F}) \mid \mathcal{F} \varepsilon M(C)\}$ is free and separated (see Definition 3.23). Some other sufficient conditions are given in the following proposition.

<u>Proposition 4.8</u>. Let C be a collection of para-uniformities on X, each compatible with the topology τ.

(a) $(C(X),C(\tau))$ is quasi-H-closed if each free τ-ultrafilter on X is C-Cauchy.

(b) $(C(X),C(\tau))$ is Hausdorff except for X if for each $\mathcal{F} \varepsilon M(C)$ there is a finite subset A of the contiguity class containing $\mathcal{F}$ such that $m(\mathcal{F}) \quad \cap \{\mathcal{G} \mid \mathcal{G} \varepsilon A\}$.

<u>Proof</u>. (a): This follows from the preceding paragraph, since $m(\mathcal{F}) \subseteq \mathcal{F}$ for each $\mathcal{F} \varepsilon M(C)$.

(b): Assume that $\mathcal{F} \in M(C)$ implies there is a finite subset A of the contiguity class containing $\mathcal{F}$ such that $m(\mathcal{F}) \quad \cap\{\mathcal{G}|\mathcal{G} \in A\}$. Let $\mathcal{F} \in M(C)$ and let $x \in X$. Let $A \quad \{\mathcal{G}_i|1 \leq i \leq n\}$ be a finite subset of the contiguity class containing $\mathcal{F}$ such that $m(\mathcal{F}) = \cap\{\mathcal{G}|\mathcal{G} \in A\}$. By definition of M(C), for $1 \leq i \leq n$ there is a $G_i \in \mathcal{G}$ and an $H_i \in \tau\langle x\rangle$ such that $G_i \cap H_i \quad \emptyset$. Then $H \quad \cap\{H_i|1 \leq i \leq n\} \in \tau\langle x\rangle$, $G \quad \cup\{G_i|1 \leq i \leq n\} \in m(\mathcal{F})$ and $G \cap H = \emptyset$. Thus $m(\mathcal{F})$ is a free filter on (X,τ).

Now let $\mathcal{F},\mathcal{T} \in M(C)$ such that $m(\mathcal{F}) \neq m(\mathcal{T})$. Let $A \quad \{\mathcal{G}_i|1 \leq i \leq n\}$ and $B \quad \{\mathcal{S}_j|1 \leq j \leq m\}$ be finite subsets of the contiguity classes of $\mathcal{F}$ and $\mathcal{T}$, respectively, such that $m(\mathcal{F}) \quad \cap\{\mathcal{G}|\mathcal{G} \in A\}$ and $m(\mathcal{T}) \quad \cap\{\mathcal{S}|\mathcal{S} \in B\}$. Since $m(\mathcal{F}) \neq m(\mathcal{T})$, $\mathcal{G}_i$ does not meet $\mathcal{S}_j$; for $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $1 \leq j \leq m$. Then for $1 \leq i \leq n$, there is a $G_i \in \mathcal{G}_i$ and an $S_i \in \mathcal{S}_j$ such that $G_i \cap S_i \quad \emptyset$. Then $S \quad \cap\{S_i|1 \leq i \leq n\} \in \mathcal{S}_j$, $G \quad \cup\{G_i|i \leq i \leq n\} \in m(\mathcal{F})$ and $S \cap G \quad \emptyset$. Thus $\mathcal{S}_j$ does not meet $m(\mathcal{F})$, for $1 \leq j \leq m$. A similar argument now shows that $m(\mathcal{F})$ does not meet $m(\mathcal{T})$, i.e., there is an $F \in m(\mathcal{F})$ and a $T \in m(\mathcal{T})$ such that $F \cap T = \emptyset$. Thus the set $\{m(\mathcal{F})|\mathcal{F} \in M(C)\}$ has been shown to be free and separated, so $(C(X),C(\tau))$ is Hausdorff except for X.||

An immediate consequence is the following corollary.

<u>Corollary 4.9</u>. Let C be a collection of para-uniformities on X, each compatible with the topology τ.

(a) $(C(X),C(\tau))$ is quasi-H-closed if each $\mathcal{U} \in C$ is pre-H-closed (see Definition 3.31).

(b) $(C(X),C(\tau))$ is Hausdorff except for X if $m(\mathcal{F})$ is C-Cauchy for each $\mathcal{F} \in M(C)$.

The isomorphism classes of strict quasi-H-closed extensions which are Hausdorff except for X can now be described as C-completions for certain collections C.

<u>Theorem 4.10</u>. Let (Y,σ) be a quasi-H-closed extension of (X,τ) which is Hausdorff except for X. Let $C \quad \{\mathcal{U}(y) \mid y \in Y - X\}$, where $\mathcal{U}(y)$ is defined as in the proof of Theorem 4.5.

(a) $(C(X),C(\tau))$ is a quasi-H-closed extension of (X,τ) which is Hausdorff except for X.

(b) $C(X) \quad X \quad \{\sigma_X\langle y\rangle \mid y \in Y - X\}$.

(c) There is a topology ρ on $C(X)$ such that $C(\tau) \subseteq \rho \subseteq C(\tau)^+$ and $(C(X),\rho)$ is isomorphic to (Y,σ).

<u>Proof</u>. (a): By Theorem 4.5, $\mathcal{U}(y)$ is pre-H-closed, so $(C(X),C(\tau))$ is quasi-H-closed, by Corollary 4.9(a). As shown in the proof of Theorem 4.5, $\sigma_X\langle y\rangle \in M(C)$ for $y \in Y - X$. For each $\mathcal{F} \in M(C)$, there is a $y \in Y \quad X$ such that $\sigma_X\langle y\rangle$ is a subset of $\mathcal{F}$. Since (Y,σ) is Hausdorff except for X, this implies that for each $\mathcal{F} \in M(C)$, there is a $y \in Y - X$ such that $m(\mathcal{F}) \quad \sigma_X\langle y\rangle \in M(C)$. Thus $(C(X),C(\tau))$ is Hausdorff except for X, by Corollary 4.9(b).

(b): This follows from the proof of (a) above.

(c): Let $h: (C(X), C(\tau)) \to (Y, \sigma^{\#})$ be defined by $h(x) \quad x$ for $x \in X$ and $h(\sigma_X\langle y\rangle) \quad y$ for $y \in Y - X$. Then h is clearly an isomorphism, since $(C(X), C(\tau))$ and $(Y, \sigma^{\#})$ are both strict extensions of (X, τ) with the filter traces $\{\sigma_X\langle y\rangle \mid y \in Y - X\}$. But $\sigma^{\#} \subseteq \sigma \subseteq \sigma^{+}$, $C(\tau) \subseteq C(\tau)^{+}$, and, of course,

$$h: (C(X), C(\tau)^{+}) \to (Y, \sigma^{+})$$

is an isomorphism. Thus there is a topology ρ on $C(X)$ such that $C(\tau) \subseteq \rho \subseteq C(\tau)^{+}$ and $h: (C(X), \rho) \to (Y, \sigma)$ is an isomorphism.||

Note that, in the preceding theorem, if (Y, σ) is a strict extension of (X, τ), then $(C(X), C(\tau))$ is isomorphic to (Y, σ).

The results of this paper are, of course, directly applicable to H-closed extensions of Hausdorff spaces, since an extension of a Hausdorff space (X, τ) is an H-closed extension if and only if it is a quasi-H-closed extension which is Hausdorff except for X.

Problems. 1. This paper provides a method for describing the strict and simple quasi-H-closed extensions of (X, τ) which are Hausdorff except for X. An interesting question that arises asks how to describe those extension topologies lying between the strict and simple extension topologies, for a given quasi-H-closed extension.

2. As was indicated previously, there is a set of para-uniformities on (X, τ) which is, in a natural way, in one-to-one correspondence with the set of isomorphism classes of

quasi-H-closed extensions of (X,τ) which are Hausdorff except for X and have relatively zero-dimensional outgrowth. It might be useful to describe a set of para-uniformities on (X,τ) which includes the above and is, in a natural way, in one-to-one correspondence with the set of isomorphism classes of quasi-H-closed extensions of (X,τ) which are Hausdorff except for X and have relatively completely regular outgrowth.

3. This paper has pointed out the class of pre-H-closed para-uniformities as having completions which are of interest. What other classes of para-uniformities have completions which are of interest? For example, is it possible to obtain the noncompact regular-closed extensions of regular Hausdorff spaces as such completions; or the noncompact Urysohn-closed extensions of Urysohn spaces? (Regular-closed means closed in every regular Hausdorff space in which embedded. Urysohn-closed is defined similarly.)

4. The concepts of relatively zero-dimensional outgrowth and relatively completely regular outgrowth lead to the idea of relatively regular outgrowth. An extension (Y,σ) of (X,τ) has <u>relatively</u> <u>regular</u> <u>outgrowth</u> if for every $y \in Y$ and every $G \in \sigma\langle y\rangle$ there are $B,H \in \sigma\langle y\rangle$ such that $Y - X \subseteq H$ and $cl_H(B \cap H) \subseteq G$. Then, of course, a natural question is whether or not there is a structure similar to the para-uniform structure whose completions, in a natural way, yield those quasi-H-closed extensions of (X,τ) which are Hausdorff except for X and have relatively regular outgrowth.

BIBLIOGRAPHY

[A1] P. S. Alexandrov, On bicompact extensions of topological spaces, (Russian, German summary), Mat. Sb. 5 (47) (1939), 403-423.

[A2] P. Alexandroff, Some results in the theory of topological spaces, obtained within the last twenty-five years, Russian Math. Surveys 15 (1960), 23-83.

[AU] P. Alexandroff and P. Urysohn, Zur theorie der topologische Räume, Math. Ann. 92 (1924), 258-266.

[Ba] B. Banaschewski, Extensions of topological spaces, Canad. Math. Bull. 7 (1964), 1-22.

[BPS] M. P. Berri, J. R. Porter, and R. M. Stephenson, Jr., A survey of minimal topological spaces, Proc. India Topology Conference, Kanpur, 1968 (to appear).

[B1] N. Bourbaki, General Topology, Part 1. Addison-Wesley Publishing Company, Reading, Massachusetts, 1966.

[B2] N. Bourbaki, Theory of Sets. Addison-Wesley Publishing Company, Reading, Massachusetts, 1968.

[Fl] J. Flachsmeyer, Zur theorie der H-abgeschlossen Erweiterungen, Math. Z. 94 (1966), 349-381.

[Fo] S. V. Fomin, The theory of extensions of topological spaces, (Russian, German summary), Mat. Sb. 8 (1940), 285-294.

[Hi] C. J. Himmelberg, Quotient uniformities, Proc. Amer. Math. Soc. 17 (1966), 1385-1388.

[IF] S. Iliadis and S. Fomin, <u>The method of centred systems in the theory of topological spaces</u>, Russian Math. Surveys 21 (1966), 37-62.

[Iv] A. A. Ivanov, <u>Regular extensions of topological spaces</u>, Izv. Akad. Nauk BSSR Ser. Fiz. Mat. 1 (1966), 28-35.

[K1] M. Katětov, <u>Über H-abgeschlossene und bikompakte Räume</u>, Čas. Mat. Fys. 69 (1940), 36-49.

[K2] M. Katětov, <u>On H-closed extensions of topological spaces</u>, Čas. Mat. Fys. 72 (1947), 17-31.

[Ke] J. L. Kelley, <u>General Topology</u>. D. Van Nostrand Company, Inc., Princeton, New Jersey, 1955.

[MN] M. G. Murdeshwar and S. A. Naimpally, <u>Quasi-uniform Topological Spaces</u>. Noordhoff, 1966.

[PT] J. Porter and J. Thomas, <u>On H-closed and minimal Hausdorff spaces</u>, Trans. Amer. Math. Soc. 138 (1969), 159-170.

[PV] J. R. Porter and C. Votaw, <u>Partitions of the Katětov extension</u>, submitted.

[St] M. H. Stone, <u>Applications of the theory of Boolean rings to general topology</u>, Trans. Amer. Math. Soc. 41 (1937), 375-481.

[Ty] A. Tychonoff, <u>Über die topologische Erweiterung von Räumen</u>, Math. Ann. 102 (1930), 544-561.