On the Existence and Stability of Normalized Ground States of the Kawahara, Fourth Order NLS and the Ostrovsky Equations

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Abstract

In this dissertation we show the existence and stability of the normalized ground states for the Kawahara, fourth order nonlinear Schrödinger (NLS) and the generalized Ostrovsky equations. One of the starting points in our investigation were numerical stability results by S. Levandosky in [32], [31] which agree with our rigorous stability results. We show existence of the waves using variational techniques together with the concentration compactness argument. On the level of construction, we encounter certain obstacles in the form of new Gagliardo–Nirenberg–Sobolev type inequalities, which impose restrictions on the parameter space. We show stability utilizing spectral theory developed in the recent work by Z.Lin and C.Zeng in [35].

For the Kawahara model, our results provide a significant extension in the parameter space of the current rigorous results. In fact, our results rigorously establish the spectral stability for all acceptable values of the parameters.

For the fourth order NLS models, we improve upon recent results on stability of, very special, explicit solutions in the one dimensional case. Our multidimensional results for the fourth order NLS equations seem to be the first of its kind. Of particular interest is a new paradigm that we discover herein. Namely, all else being equal, the form of the second order derivatives (mixed second order derivatives vs pure Laplacian) has implications on the range of the existence and stability of the normalized waves.

For the Ostrovsky equation, we show that all normalized waves we construct are spectrally stable. We also establish decay rates for the waves, extending the results in the paper by P. Zhang and Y. Liu [51].

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Contents

List of Figures

Chapter 1

Introduction

In this chapter we introduce some notations, standard inequalities, lemmas and the instability index counting theory used in subsequent chapters.

1.1 Function spaces and GNS inequalities

The L^p spaces are defined via

$$\|f\|_{L^p} = \left(\int |f(x)|^p dx \right)^{1/p}.$$

We will need some Fourier analysis basics. Fourier transform and its inverse in chapters 2 and 3 (for convenience reasons) are defined via

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx, \;\; f(x) = \int_{\mathbf{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

and

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-ix\xi} dx, \;\; f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(\xi) e^{ix\xi} d\xi,$$

respectively.

For an integer k, the classical Sobolev spaces $W^{k,p}$ are taken to be the closure of Schwartz functions in the norm $\|f\|_{W^{k,p}} = \|f\|_{L^p} + \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p}$. For a non-integer s, one may introduce a norm[1] as follows

$$\|f\|_{W^{s,p}} := \|(1-\Delta)^{s/2} f\|_{L^p}.$$

We also use the common notation H^s for $p=2$:

$$\|f\|_{H^s} = \left(\int_{\mathbf{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{1/2},$$

and the homogeneous versions $\dot{H}^s$ defined via the semi-norms

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbf{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}.$$

Recall the sharp Sobolev inequality $\|f\|_{L^q(\mathbf{R}^d)} \leq C_{s,p}\|f\|_{W^{s,p}(\mathbf{R}^d)}$, where $n\left(\frac{1}{p}-\frac{1}{q}\right) = s$. In addition, we shall make use of the Gagliardo-Nirenberg-Sobolev (GNS) inequality, which combines the Sobolev estimate with the well-known log-convexity of the complex interpolation functor $\|f\|_{[X_0,X_1]_\theta} \leq \|f\|_{X_0}^{1-\theta}\|f\|_{X_1}^{\theta}$. For example, the following estimate proves useful in the sequel

$$\|u\|_{L^q(\mathbf{R}^d)} \leq C_{q,d}\|\Delta u\|_{L^2}^{\frac{d}{2}(\frac{1}{2}-\frac{1}{q})}\|u\|_{L^2}^{1-\frac{d}{2}(\frac{1}{2}-\frac{1}{q})}, \tag{1.1.1}$$

whenever $q \in (2,\infty)$ for $d = 1,2,3,4$, and $2 < q < \frac{2d}{d-4}$ for $d \geq 5$.

We record the formula for the Green function of $(-\Delta+1)^{-1}$, that is $\hat{Q}(\xi) = (1+4\pi^2|\xi|^2)^{-1}$ (see [13], p. 418),

$$Q(x) = (2\sqrt{\pi})^{-n}\int_0^\infty e^{-(t+\frac{|x|^2}{4t})}\frac{dt}{t^{n/2}}. \tag{1.1.2}$$

Note that $Q > 0$, radial and radially decreasing. Also, $\|Q\|_{L^1(\mathbf{R}^n)} = \int_{\mathbf{R}^n} Q(x)dx = \hat{Q}(0) = 1$, but note that $Q(0) = +\infty$ for $n \geq 2$. In fact, there are the following classical estimates for it, p. 418,

[1] Which is equivalent to the standard one for an integer s.

[13],

$$|Q(x)| \leq Ce^{-|x|}, \quad |x| > 1, \tag{1.1.3}$$

$$Q(x) \sim \begin{cases} |x|^{2-n} + O(1), & n \geq 3, \\ \ln(\frac{1}{|x|}) + O(1), & n = 2, \end{cases} \quad |x| < 1. \tag{1.1.4}$$

In particular, $Q \in L^q(\mathbf{R}^n)$, whenever $q < \frac{n}{n-2}$ (or $q < \infty$, when $n = 2$).

1.2 Instability index counting theory

In this section, we present the instability index count theory , as developed in [19], [20], [44] (see also the book [21]), and, more recently, in [22], [35], which will be useful for our stability arguments later. We will only consider appropriate representative corollaries, which serve our purposes.

First, consider the following eigenvalue problem in the form

$$\mathscr{J}\mathscr{L}f = \lambda f, \tag{1.2.1}$$

where $\mathscr{J}$ is assumed to be bounded, invertible and skew-symmetric ($\mathscr{J}^* = -\mathscr{J}$), while $(\mathscr{L}, D(\mathscr{L}))$ is self-adjoint ($\mathscr{L}^* = \mathscr{L}$), with finite dimensional kernel $Ker[\mathscr{L}] : dim(Ker[\mathscr{L}]) < \infty$. In addition, the Morse index, $n(\mathscr{L})$ (that is the number of negative eigenvalues of $\mathscr{L}$), is assumed to be finite. Regarding the skew-symmetric part, we need to assume that $\mathscr{J}^{-1} : Ker[\mathscr{L}] \to Ker[\mathscr{L}]^{\perp}$.

Let k_r denote the number of real instabilities of (1.2.1) (i.e. the number of positive eigenvalues of $\mathscr{J}\mathscr{L}$), whereas k_c be the number of quadruplets of eigenvalues with non-zero real and imaginary parts. Finally, let k_i^- be the number of pairs of purely imaginary eigenvalues with negative

Krein-signature[2]. Introduce the matrix D as follows. Let $Ker[\mathscr{L}] = \{\phi_1, \dots, \phi_n\}$, then

$$D_{ij} := \langle \mathscr{L}^{-1}[\mathscr{J}^{-1}\phi_i], \mathscr{J}^{-1}\phi_j \rangle. \tag{1.2.2}$$

Note that the last formula makes sense, since $\mathscr{J}^{-1}\phi_i \in Ker[\mathscr{L}]^\perp$ and hence $\mathscr{L}^{-1}[\mathscr{J}^{-1}\phi_i]$ is well-defined. The index counting theorem[3], see Theorem 1, [20], states that if $det(D) \neq 0$, then

$$k_r + 2k_c + 2k_i^- = n(\mathscr{L}) - n(D). \tag{1.2.3}$$

In particular, if $n(\mathscr{L}) = n(D)$, we can conclude that all the terms on the left hand side of (1.2.3) are zero, so spectral stability holds true.

Second, mostly following the theory developed in [35], in the specific case when the self-adjoint operator is $\mathscr{J} = \partial_x$, we consider the following eigenvalue problem

$$\partial_x \mathscr{L} z = \mu z. \tag{1.2.4}$$

We require the following - there is a Hilbert space $\mathscr{X}$ over the reals, so that:

- $\mathscr{L} : \mathscr{X} \to \mathscr{X}^*$ is a bounded and symmetric operator, in the sense that $(u,v) \to \langle \mathscr{L}u, v \rangle$ is a bounded symmetric form on $\mathscr{X} \times \mathscr{X}$;
- $dim(Ker[\mathscr{L}]) < \infty$ and, moreover, there is an $\mathscr{L}$ invariant decomposition

$$\mathscr{X} = \mathscr{X}_- \oplus Ker[\mathscr{L}] \oplus \mathscr{X}_+, dim(\mathscr{X}_-) < \infty,$$

  so that for some $\delta > 0$, $\mathscr{L}_-|_{\mathscr{X}_-} \leq -\delta$ and $\mathscr{L}_+|_{\mathscr{X}_+} \geq \delta$. That is, for every $u_\pm \in \mathscr{X}_\pm$, there is $\langle \mathscr{L}u_-, u_- \rangle \leq -\delta \|u_-\|^2_{\mathscr{X}_-}$ and $\langle \mathscr{L}u_+, u_+ \rangle \geq \delta \|u_+\|^2_{\mathscr{X}_+}$.

Introduce the Morse index $n^-(\mathscr{L}) := dim(\mathscr{X}_-)$, which is equivalent to the number of negative

[2]The precise definition of those is provided for example in [19], [35]. For us, this is irrelevant, in our application, we will indeed have $k_i^- = 0$.

[3]See also Theorem 5.2.11 in [21] for the case $n(\mathscr{L}) = n(D)$.

eigenvalues of the operator $\mathscr{L}$, counted with their respective multiplicities. Consider the generalized eigenspace $E_0 = \{u \in \mathscr{X} : (\partial_x \mathscr{L})^k u = 0, k = 1, 2, \ldots\}$. Clearly $Ker[\mathscr{L}] \subset E_0$, so consider the complement in E_0 of $Ker[\mathscr{L}]$. That is, $E_0 = Ker[\mathscr{L}] \oplus \tilde{E}_0$. Let

$$k_0^{\leq 0} := \max\{dim(Z) : Z \text{ subspace of } \tilde{E}_0 : \langle \mathscr{L} z, z \rangle < 0, z \in Z\}.$$

Theorem 2.3, [35] asserts that[4] the number of solutions of (1.2.4), $k_{unstable}$, is estimated by

$$k_{unstable} \leq n^-(\mathscr{L}) - k_0^{\leq 0}(\mathscr{L}). \tag{1.2.5}$$

In particular, and this is what we use in this work, if $n^-(\mathscr{L}) = 1$ and $k_0^{\leq 0}(\mathscr{L}) \geq 1$, the problem (1.2.4) is spectrally stable.

Let us now derive the so-called Vakhitov-Kolokolov criteria for the stability[5]. Assume that Ψ is sufficiently smooth, $\Psi' \in Ker[\mathscr{L}]$ and, in addition, assume $\Psi \perp Ker[\mathscr{L}]$. In this case, we can identify $Q := \mathscr{L}^{-1}[\Psi]$ as an element of $Ker[(\partial_x \mathscr{L})^2] \setminus Ker[(\partial_x \mathscr{L})] \subset \tilde{E}_0$. Indeed, $\partial_x \mathscr{L} Q = \Psi'$, while $(\partial_x \mathscr{L})^2 Q = \partial_x \mathscr{L} \Psi' = \partial_x \mathscr{L} \Psi' = 0$. Now, if $\langle \mathscr{L} Q, Q \rangle < 0$, we can conclude that $k_0^{\leq 0}(\mathscr{L}) \geq 1$, which together with $n^-(\mathscr{L}) = 1$ would imply stability by (1.2.5). On the other hand,

$$\langle \mathscr{L} Q, Q \rangle = \langle \mathscr{L} \mathscr{L}^{-1} \Psi, \mathscr{L}^{-1} \Psi \rangle = \langle \mathscr{L}^{-1} \Psi, \Psi \rangle.$$

1.3 General spectral stability lemma

Lemma 1. *Let $\mathscr{H}$ be a self-adjoint operator on a Hilbert space H, so that $\mathscr{H}|_{\{\xi_0\}^\perp} \geq 0$ for some vector $\xi_0 : \xi_0 \perp Ker[\mathscr{H}], \|\xi_0\| = 1$. Assume that $\langle \mathscr{H} \xi_0, \xi_0 \rangle \leq 0$. Then,*

$$\langle \mathscr{H}^{-1} \xi_0, \xi_0 \rangle < 0.$$

[4] Theorem 2.3, [35] is actually much more general, but we state this corollary, as it is enough for us.

[5] Although the original criteria and his derivation was done, strictly speaking, in the NLS context, it introduces an important quantity, which turns out to be relevant in wide class of Hamiltonian stability problems.

Remark: Note that the condition $\xi_0 \perp Ker[\mathscr{H}]$ guarantees that $\mathscr{H}^{-1}\xi_0$ is well-defined.

Proof. Due to the self-adjointness, $\mathscr{H}^{-1} : Ker[\mathscr{H}]^\perp \to Ker[\mathscr{H}]^\perp$. Consider

$$\eta_0 := \mathscr{H}^{-1}\xi_0 - \langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle \xi_0 \perp \xi_0.$$

Note that $\langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle$ is a real. Thus,

$$\begin{aligned}
0 &\leq \langle \mathscr{H}\eta_0, \eta_0 \rangle \\
&= \langle \mathscr{H}[\mathscr{H}^{-1}\xi_0 - \langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle \xi_0], \mathscr{H}^{-1}\xi_0 - \langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle \xi_0 \rangle \\
&= \langle \xi_0 - \langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle \mathscr{H}\xi_0], \mathscr{H}^{-1}\xi_0 - \langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle \xi_0 \rangle \\
&= -\langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle + \langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle^2 \langle \mathscr{H}\xi_0, \xi_0 \rangle \\
&\leq -\langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle.
\end{aligned}$$

Thus, $\langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle \leq 0$. In fact, equality is also impossible. We argue by contradiction. Assume that $\langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle = 0$. By the assumptions, for every $\eta : \eta_0 \perp Ker[\mathscr{H}], \eta \perp \xi_0$, we have that $\langle \mathscr{H}\eta, \eta \rangle > 0$. Applying this to $\eta := \mathscr{H}^{-1}\xi_0 \in Ker[\mathscr{H}]^\perp$ and $\eta \perp \xi_0$, we conclude the contradictory statement $0 < \langle \mathscr{H}\eta, \eta \rangle = \langle \xi_0, \mathscr{H}^{-1}\xi_0 \rangle = 0$, whence we obtain that $\langle \mathscr{H}^{-1}\xi_0, \xi_0 \rangle < 0$. $\square$

1.4 Sampling a $W^{1,1}$ function

We have the following elementary lemma, which may be of independent interest.

Lemma 2. *Let $N > 1$ be an integer and $f : \mathbf{R} \to \mathbf{R}$, $f \in W^{1,1}(\mathbf{R})$. Then*

$$\sum_{n=-\infty}^{\infty} \int_{n\varepsilon}^{n\varepsilon + \frac{\varepsilon}{N}} f(x)dx = \frac{1}{N}\int_{\mathbf{R}} f(x)dx + O(\varepsilon)$$

as $\varepsilon \to 0+$. *More precisely,*

$$\left| \sum_{n=-\infty}^{\infty} \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx - \frac{1}{N}\int_{\mathbf{R}} f(x)dx \right| \leq \frac{\varepsilon}{N}\int_{\mathbf{R}} |f'(y)|dy.$$

Proof. We are going to prove the lemma for a smooth function. The statement for a $W^{1,1}(\mathbf{R})$ function can then be proven by passing to the limit. Let us split each interval $[n\varepsilon + \frac{\varepsilon}{N}, (n+1)\varepsilon)$ into $N-1$ equal intervals and compare one of them with the integral over the interval $[n\varepsilon, n\varepsilon + \frac{\varepsilon}{N})$. We have

$$\begin{aligned}
\left| \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx - \int_{n\varepsilon+\frac{m\varepsilon}{N}}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} f(x)dx \right| &= \left| \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x) - f\left(x + \frac{m\varepsilon}{N}\right) dx \right| \\
&\leq \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} \int_{x}^{x+\frac{m\varepsilon}{N}} |f'(y)|dydx \\
&\leq \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} \int_{n\varepsilon}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} |f'(y)|dydx \\
&\leq \frac{\varepsilon}{N}\int_{n\varepsilon}^{(n+1)\varepsilon} |f'(y)|dy
\end{aligned} \tag{1.4.1}$$

for all $m = 1, \ldots, N-1$.

Now, using this last estimate, we get, after adding and subtracting $\sum_{m=1}^{N-1} \int_{n\varepsilon+\frac{m\varepsilon}{N}}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} f(x)dx$,

$$\begin{aligned}
N\sum_{n=-\infty}^{\infty} \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx &= \sum_{n=-\infty}^{\infty} \left(\int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx + \sum_{m=1}^{N-1} \int_{n\varepsilon+\frac{m\varepsilon}{N}}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} f(x)dx \right) \\
&\quad + \sum_{n=-\infty}^{\infty} \left((N-1)\int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx - \sum_{m=1}^{N-1} \int_{n\varepsilon+\frac{m\varepsilon}{N}}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} f(x)dx \right) \\
&= \sum_{n=-\infty}^{\infty} \int_{n\varepsilon}^{(n+1)\varepsilon} f(x)dx + \sum_{n=-\infty}^{\infty}\sum_{m=1}^{N-1} \left(\int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx - \int_{n\varepsilon+\frac{m\varepsilon}{N}}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} f(x)dx \right) \\
&= \int_{\mathbf{R}} f(x)dx + \sum_{n=-\infty}^{\infty}\sum_{m=1}^{N-1} \left(\int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx - \int_{n\varepsilon+\frac{m\varepsilon}{N}}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} f(x)dx \right).
\end{aligned}$$

Rearranging the terms and using the estimate (1.4.1), implies

$$\begin{aligned}\left| N \sum_{n=-\infty}^{\infty} \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx - \int_{\mathbf{R}} f(x)dx \right| &\leq \sum_{n=-\infty}^{\infty} \sum_{m=1}^{N-1} \left| \int_{n\varepsilon}^{n\varepsilon+\frac{\varepsilon}{N}} f(x)dx - \int_{n\varepsilon+\frac{m\varepsilon}{N}}^{n\varepsilon+\frac{(m+1)\varepsilon}{N}} f(x)dx \right| \\ &\leq \varepsilon \sum_{n=-\infty}^{\infty} \int_{n\varepsilon}^{(n+1)\varepsilon} |f'(y)|dy \\ &= \varepsilon \int_{\mathbf{R}} |f'(y)|dy.\end{aligned}$$

Dividing by N yields the claim. □

Chapter 2

The Kawahara and Fourth Order NLS Equations

2.1 Introduction

We consider several dispersive equations in one and multiple space dimensions. Our main motivating example will be the (generalized) Kawahara equation, which is a fifth order KdV equation, allowing for the third order dispersion effects as well. Namely,

$$u_t + u_{xxxxx} + bu_{xxx} - (|u|^{p-1}u)_x = 0, x \in \mathbf{R}, t \geq 0, p > 1. \tag{2.1.1}$$

This is a model that appears in the study of plasma and capillary waves, where the third order dispersion is considered to be weak. In fact, Kawahara studied the quadratic case[1] [28] and he argued that the inclusion of a fifth order derivative is necessary for capillary-gravity waves, for values of the Bond number close to a critical one. Craig and Groves, [7] offered some further generalizations. Kichenassamy and Olver, [29] have studied the cases where explicit waves exist, see also Hunter-Scheurle, [16] for the existence of solitary waves.

Another model, which is important in the applications is the non-linear Schrödinger equation with fourth order dispersion. We consider two versions of it, which will turn out to be qualitatively different, from a the point of view of the stability of their standing waves. Namely,

$$iu_t + \Delta^2 u + \varepsilon(\langle \vec{b}, \nabla \rangle)^2 u - |u|^{p-1}u = 0, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^d, \tag{2.1.2}$$

$$iu_t + \Delta^2 u + b\Delta u - |u|^{p-1}u = 0, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^d, \tag{2.1.3}$$

[1] Where the nonlinearity is in the form $(u^2)_x$, slightly different than ours.

where $d \geq 1$, $p > 1, \varepsilon = \pm 1$. These have been much studied, both in the NLS as well as Klein-Gordon context, since the early 90's, see for example [1, 2].

For both models, we will be interested in the existence of solitons, and the corresponding close to soliton dynamics, in particular spectral stability. For the Kawahara, the relevant objects are traveling waves, in the form $u(x,t) = \Phi(x + \omega t)$, where Φ is dying off at infinity. These satisfy the profile equation of the form

$$\Phi'''' + b\Phi'' + \omega\Phi - |\Phi|^{p-1}\Phi = 0. \tag{2.1.4}$$

Similarly, standing wave solutions in the form $u = e^{-i\omega t}\Phi$, $\omega > 0$, with real-valued Φ for the fourth order NLS (2.1.2) and (2.1.3) solve the elliptic profile equations

$$\Delta^2\Phi + \varepsilon(\langle \vec{b}, \nabla \rangle)^2\Phi + \omega\Phi - |\Phi|^{p-1}\Phi = 0, \tag{2.1.5}$$

$$\Delta^2\Phi + b\Delta\Phi + \omega\Phi - |\Phi|^{p-1}\Phi = 0. \tag{2.1.6}$$

Constructing solutions to (2.1.4) and, more generally, (2.1.6)-(2.1.5) is not straightforward. In fact, it depends on the parameter p, the sign of the parameter b, as well as the dimension $d \geq 1$. Here, it is worth noting the works of Albert, [1] and Andrade-Cristofani-Natali, [2] in which the authors have mostly studied stability of some explicitly available solutions in one spatial dimension.

We proceed differently, by means of variational methods. More specifically, we employ the constrained minimization method, which minimizes the total energy with respect to a fixed particle number, or L^2 mass. In addition to being the most physically relevant, the waves constructed this way (which we refer to as normalized waves) have good stability properties[2].

This brings us to the second important goal of the chapter. Namely, we wish to examine the stability of waves arising as solutions of (2.1.4), (2.1.5) and (2.1.6). Our constructions will not yield explicit waves[3]. Thus, we need to decide, whenever possible, about their stability, based on

[2] This is probably the reason why these waves are considered the most "physical" in the first place.
[3] Although some do exist, for very specific values of the parameter b and $d = 1$, more on this below.

their construction and properties.

2.1.1 Previous results

The Kawahara model. We would like to review the history of the problem for the existence and stability of the traveling waves, by concentrating mostly on some recent results in the last twenty years or so, which we feel are the most relevant in relation to our results. We would like to draw an important point that since uniqueness results are generally lacking[4], it is hard to compare results about waves obtained by different methods, as they may differ in shape and stability properties.

In [17] and [23] the authors have shown that certain waves of depression (i.e. $b < 0$) are stable. In [23], the author establishes an important Vakhitov-Kolokolov type criteria for certain waves, but it appears that it is hard to verify outside of a few explicit examples. In [5], Bridges and Derks have studied a Kawahara model with a more general nonlinearity. They have employed the methods of Evans functions to locate the point spectrum (and hence the stability) of the corresponding linearizations. The results of their work are mostly computationally aided.

Levandosky, [30] has studied the problem for existence of such waves via an energy-momentum type argument and compensated compactness. Groves, [13] has shown the existence of multi-bump solitary waves for certain homogeneous nonlinearities. Haragus-Lombardi-Scheel, [15] have considered spatially periodic solutions and solitary waves, which are asymptotic to them at infinity. They showed spectral stability for such small amplitude solutions. We should also mention the work [2], in which the authors consider the orbital stability for explicit *periodic* solutions of the Kawahara problem, subjected to a quadratic nonlinearity.

The paper of Angulo, [3] gives some sufficient conditions for instability of such waves, both for the cases $b > 0$ and $b < 0$. Levandosky, [32] nicely summarizes the results in the literature[5] and offers rigorous analysis for stability/instability close to bifurcation points. Furthermore, his paper provides a useful, numerically aided, classification of solitary waves of the Kawahara model,

[4] For both, minimizers of the constrained variational problem and solutions of the PDE.

[5] He considers more general non-linearities containing powers of derivatives as well.

based on the type of non-linearity (i.e. the power p) and the parameters of the problem b, ω - the exhaustive tables on p. 164, [32] provided a good starting point for our investigation. We should mention that the waves considered in [32] are produced as the constrained minimizers of the following variational problem

$$\begin{cases} J_\omega[u] = \int_{\mathbf{R}^d} |\Delta u(x)|^2 - b|\nabla u(x)|^2 + \omega u^2(x) dx \to \min, \\ \int_{\mathbf{R}^d} |u(x)|^{p+1} dx = 1. \end{cases} \tag{2.1.7}$$

We take a different approach below. Namely, by constructing the normalized waves, i.e., minimizing energy constrained on the L^2 norm, in a physically relevant fashion (see Section 2.3.1). An important point we would like to make is that the procedure outlined by (2.1.7) provides waves for a considerably wider range of p than the ones produced in Section 2.3.1. Specifically, the minimizers of (2.1.7) exist for $p \in (1, p_{\max})$, with $p_{\max}(d) = \begin{cases} \infty, & d = 1,2,3,4, \\ 1 + \frac{8}{d-4}, & d \geq 5, \end{cases}$ whereas, the normalized waves constructed herein are only available for $p \in (1, 1 + \frac{8}{d})$.

Fourth order NLS models. The fourth order Schrödinger equation was introduced in [27] and [26], where it has an important role in modeling the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Moreover, the equation was also used in nonlinear fiber optics and the theory of optical solitons in gyrotropic media. The problem for the existence and the stability of the waves arising in (2.1.5) has been the subject of investigation of a few recent papers, the results of which we summarize below.

For the case of $d = 1, p = 3$ (and in fact only for the special value of $\varepsilon = -1, b = 1$ and $\omega = \frac{4}{25}$), the elliptic problem (2.1.4) (or equivalently (2.1.5)) was considered by Albert, [1] in relation to soliton solutions of related approximate water wave models. The explicit soliton, $\Phi_0(x) = \sqrt{\frac{3}{10}} sech^2\left(\frac{x}{\sqrt{20}}\right)$, was studied in detail. Important properties of the corresponding linearized operators were established. These properties allowed Natali and Pastor, [41] to establish the orbital stability of this wave (see also [8] for alternative approach and extensions to Klein-Gordon solitons). One of the central difficulties that the authors faced is that this solution is only available explicitly for an isolated value of $\omega = \frac{4}{25}$. This precludes one from differentiating with respect to the parameter ω as is customary in these types of arguments. Additionally, the problem for stability of the equation (2.1.2) in $d = 1$, $\varepsilon = -1, b = 1$ and general p were addressed in the works [24] and [25]. The numerically generated waves were shown to exists for every $p > 1$, but stable for only $p \in (1,5)$. Further (mostly numerical) investigations regarding this model are available in the papers [26], [27].

Finally, it is important to discuss the recent work [4], as it has significant overlap with ours. In it, the authors have studied (2.1.3) in great detail, including the stability of their waves. They have constructed the waves in a similar manner, in fact the existence part of our Theorem 5 is similar in nature, although more details on radial symmetry, the zero set and exponential decay rates of the waves are derived as well. In addition, they discuss some cases, in which they can show non-degeneracy, i.e. $Ker[\mathscr{L}_+] = span[\nabla\phi]$. This is verified in two cases:

- for any dimension $d \geq 1$, but with $b < 0$ and $|b|$ sufficiently large,
- the one dimensional case, $d = 1$, but with $b < 0$, $b^2 > 4\omega$.

Concerning stability of the waves, the authors of [4] do not establish stability for any given example. On the other hand, they show that orbital stability holds, once one can verify non-degeneracy and the index condition $\langle \mathscr{L}_+^{-1}\phi, \phi\rangle < 0$. The non-degeneracy was already discussed, while the verification of $\langle \mathscr{L}_+^{-1}\phi, \phi\rangle < 0$ is left as an open problem in [4]. This last condition however is essentially equivalent, modulo some easy to establish technical assumptions, to the spectral stability, see Corollary 1 below.

In this work, we actually do establish $\langle \mathscr{L}_+^{-1}\phi_\lambda, \phi_\lambda\rangle < 0$ for all waves produced in Theorems 1, 4, 5, thus answering the open problem in [4]. Our results rigorously establish spectral stability for all waves constructed therein - in all dimensions, for all values of b, positive or negative, large or small. This combined with the results of [4] would also provide orbital stability for all normalized waves enjoying the non-degeneracy property.

2.1.2 Main results: Kawahara waves

It is easy to informally summarize our results - all normalized waves, whenever they exists, turn out to be spectrally stable. Our hope is that the approach here will shed further light on this interesting phenomena in a much more general setting. As we have alluded to above, the main focus will be the Kawahara problem, (2.1.1), for both positive and negative values of b.

In order to construct solutions to the elliptic problem (2.1.4), we shall work with the following variational problem

$$\begin{cases} I[\phi] = \frac{1}{2}\int_{\mathbf{R}}[|\phi''(x)|^2 - b|\phi'(x|^2]dx - \frac{1}{p+1}\int_{\mathbf{R}}|\phi(x)|^{p+1}dx \to \min, \\ \int_{\mathbf{R}}\phi^2(x)dx = \lambda, \end{cases} \tag{2.1.8}$$

where one could take ϕ in the Schwartz class, in order to make $I[\phi]$ meaningful. Introduce the scalar function

$$m_b(\lambda) = \inf_{\phi\in H^2(\mathbf{R}), \|\phi\|_2^2=\lambda} I[\phi],$$

which will play a prominent role in the subsequent investigation. Let us say that it is not *a priori* clear whether the problem (2.1.8) is well-posed (i.e. $m_b(\lambda) > -\infty$) for all λ. We have the following existence result.

Theorem 1. *(Existence of the normalized Kawahara traveling waves)*
Let $p \in (1,9), \lambda > 0, b \in \mathbf{R}$ *satisfy one of the following*

1. $1 < p < 5, \lambda > 0$,

2. *For* $5 \le p < 9$ *and all sufficiently large*[6] λ.

Then, the constrained minimization problem (2.1.8) *has a solution* $\phi_\lambda \in H^4(\mathbf{R}), \|\phi\|_{L^2}^2 = \lambda$ *and* $\omega = \omega(b,\lambda,\phi)$ *which satisfies*

$$\omega > \begin{cases} \frac{b^2}{4}, & \text{if } b > 0, \\ 0, & \text{if } b < 0. \end{cases} \tag{2.1.9}$$

[6] Here, for all given $p \in [5,9)$, for both $b > 0, b < 0$, there is a specific value $\lambda_{b,p}$ and we assume that $\lambda > \lambda_{b,p}$.

Moreover, ϕ_λ satisfies the Euler-Lagrange equation (2.1.4) *in a classical sense. We call such solutions ϕ_λ normalized waves.*

Concerning the stability of the waves produced in Theorem 1, we have the following results (we employ the standard definition of spectral stability, see Definition 2 in Section 2.2.3 below).

Theorem 2. *Let $\lambda > 0$ and p satisfy the requirements of Theorem 1, and ϕ_λ is any minimizer constructed therein. Then, ϕ_λ is spectrally stable, as a solution of the Kawahara equation* (2.1.1).

Remarks:

- The Lagrange multiplier ω may depend on the particular normalized wave ϕ. In particular, we can not rule out the existence of two constrained minimizers of (2.1.8), $\phi_\lambda, \tilde{\phi}_\lambda$, with $\omega(\lambda, \phi_\lambda) \neq \omega(\lambda, \tilde{\phi}_\lambda)$. This is of course related to the uniqueness problem for the minimizers of (2.1.8) (and it should be a much simpler one), but it is open at the moment.

- The results of Theorem 2 present rigorous sufficient conditions for stability of traveling waves in much wider range than previously available. In fact, our results confirm[7] the available numerical simulations by Levandosky, [32]. For example, it is quite obvious that the bifurcation point is at $p = 5$ (corresponding to the case $p = 6$ in the notations of [32]). Namely, for powers $p < 5$ all waves are stable[8], while for $p \geq 5$, some unstable waves start to appear (which are of course not normalized). For $p \geq 9$, Levandosky observed a very small set of stable waves, again, none of them normalized, but rather generated as minimizers of (2.1.7).

- The non-degeneracy, $Ker[\mathcal{L}_+] = span[\phi']$, appears to be a hard problem in the theory. In fact, an easier version would be to establish such a non-degeneracy of the kernel, if ϕ is a minimizer of (2.1.8), while a harder problem would be to do so, knowing that ϕ is just a solution to the PDE (2.1.4). In both cases, the non-degeneracy is directly relevant to the

[7]Here the usual caveat is that, since the uniqueness is not known, it is possible that the waves considered in [32] are different from ours.

[8]Except at $p = 4$ ($p = 5$ in the notations of [32]), for a small region in the parameter space, an instability is observed numerically. This could be a fluke of the computations in [32], because as we see from Theorem 1, the stable region is up to $p \leq 5$.

uniqueness of the ground state, which is an even harder open problem in the area. See [9] for the discussion about these and related issues.

We first have the following special constrained minimizers, which we call limit waves.

Proposition 1. *Let $\lambda > 0$ and p satisfy the assumptions of Theorem 1. Then, for every sequence $\delta_j \to 0$, there is a subsequence δ_{j_k}, $y_k \in \mathbf{R}$ and $\Phi_\lambda \in H^4(\mathbf{R})$, so that*

- $\lim_{k\to\infty} \|\phi_{\lambda+\delta_{j_k}}(\cdot + y_k) - \Phi_\lambda\|_{H^2(\mathbf{R})} = 0$, *in particular* $\|\Phi_\lambda\|_{L^2}^2 = \lambda$.
- Φ_λ *is a constrained minimizer for* (2.1.8)*, so in particular Theorem 1 applies to it.*

We call Φ_λ a limit wave for the Kawahara problem.

Note that if there is uniqueness for the constrained minimizers of (2.1.8), all waves are limit waves. Our next result is about the properties of the functions m, ω. This is of independent interest, as it could be helpful in future studies on the uniqueness of minimizers for such models.

Theorem 3. *The function $m_b : (0,\infty) \to \mathbf{R}$ is a negative, strictly decreasing and concave down function. In particular, m is Lipschitz continuous on bounded intervals $(a,b) \subset \mathbf{R}_+$.*

As a consequence, m has a derivative on the full measure subset $\mathscr{A}_m := \{\lambda > 0 : m'(\lambda) \text{ exists}\}$ of $\mathbf{R}_+$. For $\lambda \in \mathscr{A}_m$, there is the formula

$$m'(\lambda) = -\frac{\omega(b,\lambda,\phi_\lambda)}{2}.$$

In particular, $\omega(b,\lambda,\phi_\lambda)$ is uniquely defined (i.e. independent on the particular minimizer ϕ_λ) on the set $\mathscr{A}_m$, so we denote this a.e. defined function by $\omega_\lambda : \mathscr{A}_m \to \mathbf{R}$. For each $0 < \lambda_1 < \lambda_2$, there is the formula

$$m(\lambda_2) - m(\lambda_1) = -\frac{1}{2}\int_{\lambda_1}^{\lambda_2} \omega_\lambda \, d\lambda. \tag{2.1.10}$$

The function $\lambda \to \omega_\lambda$ is a strictly increasing function. Thus, it has a derivative on a full measure subset $\mathscr{A}_\omega := \{\lambda \in \mathscr{A}_m : \omega'(\lambda) \text{ exists}\} \subset \mathscr{A}_m$ and, in fact, there is the inequality

$$\omega'(\lambda) > \frac{p-1}{2\lambda^2}\|\phi_\lambda\|_{L^{p+1}}^{p+1} > 0. \tag{2.1.11}$$

More generally, for points $\lambda \notin \mathscr{A}_m$, there is

$$\lim_{\varepsilon\to 0+} \frac{m(\lambda+\varepsilon)-m(\lambda)}{\varepsilon} \leq -\frac{\omega(\lambda,b,\phi)}{2} \leq \lim_{\varepsilon\to 0-} \frac{m(\lambda+\varepsilon)-m(\lambda)}{\varepsilon}. \tag{2.1.12}$$

Finally, suppose $\lambda \notin \mathscr{A}_\omega$ and let Φ_λ be a limit wave at λ. Then

$$\omega_{\lambda,b,\Phi_\lambda} = \lim_{\varepsilon\to 0+} \frac{m(\lambda+\varepsilon)-m(\lambda)}{\varepsilon} \text{ or } \omega_{\lambda,b,\Phi_\lambda} = \lim_{\varepsilon\to 0-} \frac{m(\lambda+\varepsilon)-m(\lambda)}{\varepsilon}.$$

In particular, if there is uniqueness for the minimizers of (2.1.8)*, the function $\lambda \to \omega_\lambda$ is continuous.*

2.1.3 Main results: fourth order NLS waves

We start with the existence result for the models. Before we state the results for the fourth order NLS models, we need to make an obvious reduction of the equation (2.1.2). Namely, picking a rotation matrix $A \in SU(n)$, so that $\vec{b} = |\vec{b}|A\vec{e_1}$, we can clearly reduce matters (both the existence of the solutions of the profile equation (2.1.5) and its stability analysis), by the transformation $\hat{u}(\xi) \to \hat{u}(A^*\xi)$, to the consideration of the following problems:

$$iu_t + \Delta^2 u + \varepsilon|b|^2\partial_{x_1}^2 u - |u|^{p-1}u = 0, \tag{2.1.13}$$

and the associated elliptic profile equation

$$\Delta^2\phi + \varepsilon|b|^2\partial_{x_1}^2\phi + \omega\phi - |\phi|^{p-1}\phi = 0. \tag{2.1.14}$$

That is, the existence of solutions to (2.1.14) is equivalent to the existence of solutions to (2.1.5) (under the appropriate transformation) and their stability is equivalent to the stability of their counterparts. Thus, it suffices to discuss the fourth order NLS problem (2.1.13), with its solitons satisfying (2.1.14). Below are our existence and stability results.

Theorem 4. *(Stability of the normalized waves for the fourth order NLS: mixed derivatives)* *Let $d \geq 1, \varepsilon = \pm 1$. Let $p \in (1, 1+\frac{8}{d})$, $\lambda > 0$ and assume one of the following:*

1. $1 < p < 1 + \frac{8}{d+1}$ *and $\lambda > 0$;*

2. $1 + \frac{8}{d+1} \leq p < 1 + \frac{8}{d}$ *and a sufficiently large λ.*

Then, there exists $\phi_\lambda \in H^4(\mathbf{R}^d) \cap L^{p+1}(\mathbf{R}^d)$ satisfying (2.1.14), *with an appropriate $\omega(b, \lambda, \phi_\lambda)$. In addition,*

$$\omega(b, \lambda, \phi_\lambda) > \begin{cases} \frac{|b|^2}{4}, & \text{if } \varepsilon = 1, \\ 0, & \text{if } \varepsilon = -1. \end{cases} \tag{2.1.15}$$

The solution ϕ_λ is constructed as a constrained minimizer of (2.4.1), *with $\|\phi_\lambda\|_{L^2}^2 = \lambda$. The function $\lambda \to \omega_\lambda$ is increasing, in the sense that for each $0 < \lambda_1 < \lambda_2$ and any constrained minimizers, ϕ_{λ_1} ϕ_{λ_2}, we have $\omega(b, \lambda_1, \phi_{\lambda_1}) < \omega(b, \lambda_2, \phi_{\lambda_2})$.*

Finally, $e^{-i\omega_\lambda t}\phi_\lambda(x)$ is a spectrally stable solution of (2.1.13).

Despite the obvious similarities with (2.1.5), the fourth order NLS with pure Laplacian, (2.1.3) and its associated profile equation (2.1.6), turn out to be quite different - even at the level of the existence of the waves and their stability. We have the following result.

Theorem 5. *(Stability of the normalized waves for the fourth order NLS: pure Laplacian case)* *Let $d \geq 1$, $b \in \mathbf{R}$. Let $p \in (1, 1+\frac{8}{d})$, $\lambda > 0$ and assume one of the following:*

1. $1 < p < 1 + \frac{4}{d}$ *and $\lambda > 0$;*

2. $1 + \frac{4}{d} \leq p < 1 + \frac{8}{d}$ *and a sufficiently large λ.*

Then, there exists a normalized wave $\phi_\lambda \in H^4(\mathbf{R}^d) \cap L^{p+1}(\mathbf{R}^d) : \|\phi_\lambda\|^2 = \lambda$, satisfying (2.1.6), *with an appropriate $\omega = \omega(b, \lambda, \phi)$, which is increasing and satisfies* (2.1.9).

The soliton $e^{-i\omega_\lambda t}\phi_\lambda(x)$ is a spectrally stable solution of (2.1.3).

Remarks:

- The results extend the stability results of Albert, [1] for the one dimensional cubic case $p = 3$.

- The results here also extend the NLS related results of [8] (namely, stability for $p < 1 + \frac{8}{d}$ and instability otherwise), which apply to the case $b = 0$.

- Both results, Theorem 4 and 5 of course coincide for $d = 1$, but are different for $d \geq 2$. We do not have a good physical explanation as to why the range of existence and stability of standing waves for the models (2.1.13) vis a vis (2.1.3) differ. In particular, the mixed derivative model (2.1.13) seems to support all stable normalized waves in the wider range $p \in (1, 1 + \frac{8}{d+1}), \lambda > 0$, compared to $p \in (1, 1 + \frac{4}{d})$ for (2.1.3). This topic clearly merits further investigations.

2.2 Preliminaries

2.2.1 Distributional vs strong solutions of the Euler-Lagrange equation

Definition 1. *We say that $g \in H^2(\mathbf{R}^d) \cap L^{p+1}(\mathbf{R}^d)$ is a distributional solution of the equation*

$$\Delta^2 g + b\Delta g + \omega g - |g|^{p-1} g = 0, x \in \mathbf{R}^d \tag{2.2.1}$$

if the following relation holds for every $h \in H^2(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$:

$$\langle \Delta g, \Delta h \rangle + \langle b\Delta g + \omega g, h \rangle - \langle g|^{p-1} g, h \rangle = 0.$$

Proposition 2. *Let $p \in (1, 1 + \frac{8}{d})$ and b, ω be so that $b^2 - 4\omega < 0$ or $b^2 - 4\omega > 0, \omega > 0, b < 0$. Then, any weak solution g of* (2.2.1) *is in fact $g \in H^4(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d) \cap L^{1+\varepsilon}(\mathbf{R}^d)$ for any $\varepsilon > 0$. In particular, the weak solutions of* (2.2.1)*, in fact, satisfy* (2.2.1) *as L^2 functions.*

Proof. Note that by the restrictions on b, ω, we have that the operator $(\Delta^2 + b\Delta + \omega)$ is invertible on $L^2(\mathbf{R}^d)$. Let $\tilde{g} := (\Delta^2 + b\Delta + \omega)^{-1}[|g|^{p-1} g]$. From Sobolev embedding, we easily get that

$\tilde{g} \in H^\alpha(\mathbf{R}), \alpha < 4 - \frac{d(p-1)}{2(p+1)}$, since

$$\|\tilde{g}\|_{H^\alpha(\mathbf{R}^d)} \leq \||g|^{p-1}g\|_{H^{4-\alpha}(\mathbf{R}^d)} \leq C\||g|^{p-1}g\|_{L^{\frac{p+1}{p}}} \leq C\|g\|^p_{L^{p+1}}.$$

In addition, for every test function h, we have

$$\langle \Delta\tilde{g}, \Delta h\rangle + \langle b\Delta\tilde{g} + \omega\tilde{g}, h\rangle = \langle |g|^{p-1}g, h\rangle = \langle \Delta g, \Delta h\rangle + \langle b\Delta g + \omega g, h\rangle.$$

It follows that $g = \tilde{g}$ in the sense of distributions, whence $g \in H^\alpha(\mathbf{R}^d)$.

We will show that $g \in L^\infty(\mathbf{R}^d)$. In fact, denote

$$q_0 = \sup\{q : g \in L^q(\mathbf{R}^d)\}.$$

Clearly, $q_0 \geq p+1$, by assumption. We will show first that $q_0 = \infty$. Assume not. By Sobolev embedding, we have

$$\|g\|_{L^q(\mathbf{R}^d)} = \|\tilde{g}\|_{L^q(\mathbf{R}^d)} \leq C\||g|^{p-1}g\|_{L^{\frac{p+1}{p}}} \leq C\|g\|^p_{L^{p+1}} < \infty$$

as long as $\frac{1}{q} > \frac{p}{p+1} - \frac{4}{d}$. In particular, we can take q as close to ∞, as we please, and hence $q_0 = \infty$, if $d \leq 4$. So, assume $d \geq 5$. It follows that $\frac{1}{q_0} \leq \frac{p}{p+1} - \frac{4}{d}$.

Take any $q_0 < q < \infty$. We have, by Sobolev embedding

$$\|\tilde{g}\|_{L^q(\mathbf{R}^d)} \leq C\||g|^{p-1}g\|_{L^r} \leq C\|g\|^p_{L^{rp}}, \tag{2.2.2}$$

so long as $d(\frac{1}{r} - \frac{1}{q}) \leq 4$ or $\frac{1}{r} \leq \frac{4}{d} + \frac{1}{q}$. If $\frac{4}{d} + \frac{1}{q} < 1$, we take $r : \frac{1}{r} = \frac{4}{d} + \frac{1}{q}$, whereas, if we have $\frac{4}{d} + \frac{1}{q} \geq 1$, we can take $r = \frac{p+1}{p}$ and we have a contradiction right away, since the left-hand side of (2.2.2) is unbounded (by the definition of q_0), while the right-hand is bounded. For the remainder, take $r : \frac{1}{r} = \frac{4}{d} + \frac{1}{q}$.

Clearly, if $rp < q_0$, this would be a contradiction, because the left-hand side is supposed to be unbounded (by the definition of q_0), while the right-hand is clearly bounded. We claim that this is the case, under our restrictions for $p \in (1, 1+\frac{8}{d})$. We have

$$\frac{1}{r} - \frac{p}{q_0} = \frac{4}{d} + \frac{1}{q} - \frac{p}{q_0} = \frac{4}{d} - \frac{p-1}{q_0} + o(q-q_0).$$

So, if we show that $\frac{4}{d} > \frac{p-1}{q_0}$, we will have achieved the contradiction, as we can take q very close to q_0. Indeed, by the inequality for $\frac{1}{q_0}$, we have $\frac{p-1}{q_0} \leq (p-1)\left(\frac{p}{p+1} - \frac{4}{d}\right)$. Resolving the inequality

$$(p-1)\left(\frac{p}{p+1} - \frac{4}{d}\right) < \frac{4}{d},$$

leads to the solution $1 < p < 1 + \frac{8}{d-4}$, which, of course, contains the interval $(1, 1+\frac{8}{d})$. Therefore, it is true for all p in the set that we are interested in. We have reached a contradiction, with $q_0 < \infty$.

Thus, $q_0 = \infty$. This does not mean yet that $g \in L^\infty(\mathbf{R}^d)$, but this follows easily by Sobolev embedding, once we know that $g \in \cap_{2 \leq q < \infty} L^q(\mathbf{R}^d)$. Furthermore, we see that the same type of arguments imply $g \in H^5(\mathbf{R}^d)$ and that for every $p < \infty$ and for every $\varepsilon > 0$, $g \in W^{4-\varepsilon, p}(\mathbf{R}^d)$.

For our next step, we shall need a representation of the Green's function of the operator $(\Delta^2 + b\Delta + \omega)^{-1}$ as follows. We have

$$\begin{aligned}(\Delta^2 + b\Delta + \omega)^{-1} &= \left(-\Delta + \frac{-b+\sqrt{b^2-4\omega}}{2}\right)^{-1}\left(-\Delta + \frac{-b-\sqrt{b^2-4\omega}}{2}\right)^{-1} \\ &= (b^2-4\omega)^{-1/2}\left[\left(-\Delta + \frac{-b-\sqrt{b^2-4\omega}}{2}\right)^{-1} - \left(-\Delta + \frac{-b+\sqrt{b^2-4\omega}}{2}\right)^{-1}\right].\end{aligned}$$

In the case $b^2 - 4\omega > 0, \omega > 0, b < 0$, both $\frac{-b\pm\sqrt{b^2-4\omega}}{2}$ are positive numbers, so clearly the corresponding Greens function G has decay $e^{-\sqrt{\frac{-b-\sqrt{b^2-4\omega}}{2}}|x|}$, according to (1.1.3).

As far as the case $b^2 - 4\omega < 0$ is concerned, it is not hard to see, in the same way, that the

Green's function G has decay rate $e^{-k_\omega |x|}$, where

$$k_\omega := \begin{cases} \frac{\sqrt{2\sqrt{\omega}+b}}{2}, & b<0, \quad |b|<2\sqrt{w}, \\ \frac{\sqrt{2\sqrt{\omega}-b}}{2}, & b>0, \quad |b|<2\sqrt{w}. \end{cases}$$

In both cases, the Green's function enjoys exponential rate of decay.

For $p \geq 2$, we can actually conclude that $g \in L^1(\mathbf{R}^d)$ since by the Hardy-Littlewood-Sobolev inequality

$$\|\tilde{g}\|_{L^1(\mathbf{R}^d)} \leq \|G\|_{L^1(\mathbf{R}^d)} \||g|^{p-1}g\|_{L^1(\mathbf{R}^d)} \leq C\|g\|^p_{L^p(\mathbf{R}^d)} < \infty,$$

as $g \in L^2 \cap L^\infty$, in particular $g \in L^p(\mathbf{R}^d)$. For $p<2$, denote $q_0 = \inf\{q : g \in L^q(\mathbf{R}^d)\}$. Our claim is that $q_0 = 1$. Assume for a contradiction that $q_0 > 1$. We will show that for every $q > q_0$, we have that $g \in L^{\frac{q}{p}}(\mathbf{R}^d)$, which would be a contradiction with $q_0 > 1$. Indeed, by Hardy-Littlewood-Sobolev inequality

$$\|\tilde{g}\|_{L^{\frac{q}{p}}(\mathbf{R}^d)} \leq \|G\|_{L^1(\mathbf{R}^d)} \||g|^{p-1}g\|_{L^{\frac{q}{p}}(\mathbf{R}^d)} \leq C\|G\|_{L^1}\|g\|^p_{L^q(\mathbf{R}^d)}.$$

This establishes the contradiction with $q_0 > 1$, hence $g \in \cap_{1<q} L^q(\mathbf{R}^d)$.

□

2.2.2 Concavity criteria

The following result was obtained in [47].

Lemma 3. *Let $f : (a,b) \to \mathbf{R}$ be a continuous function that satisfies*

$$\limsup_{\delta \to 0} \sup_{\lambda \in (a,b)} \frac{f(\lambda+\delta)+f(\lambda-\delta)-2f(\lambda)}{\delta^2} \leq 0.$$

Then, f is concave down on (a,b).

2.2.3 Linearized problems and spectral stability

We next discuss the linearized problems and the stability of the waves. For solutions Φ of (2.1.4), we introduce the traveling wave ansatz, $u(t,x) = \Phi(x+\omega t) + v(t,x+\omega t)$. Plugging this back in (2.1.1) and ignoring all terms $O(v^2)$, we obtain the following linearized problem

$$v_t + \partial_x[\partial_x^4 + b\partial_x^2 + \omega - p|\Phi|^{p-1}]v = 0. \tag{2.2.3}$$

Denoting $\mathscr{L}_+ := \partial_x^4 + b\partial_x^2 + \omega - p|\Phi|^{p-1}$, the associated eigenvalue problem is obtained by setting $v(t,x) \to e^{-\mu t}z(x)$ in (2.2.3), which results in

$$\partial_x \mathscr{L}_+ z = \mu z. \tag{2.2.4}$$

We proceed similarly with the linearization of the NLS problem (2.1.2). Consider solutions Φ of (2.1.14) and then perturbations of the solution $u(t,x) = e^{-i\omega t}\Phi$ of (2.1.13) in the form $u = e^{-i\omega t}[\Phi + z_1 + iz_2]$. Plugging this ansatz into (2.1.2), retaining only the linear in z terms and taking real and imaginary parts leads us to the system

$$\partial_t z = \mathscr{J}\mathscr{L} z, \tag{2.2.5}$$

where

$$\mathscr{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathscr{L} := \begin{pmatrix} \mathscr{L}_+ & 0 \\ 0 & \mathscr{L}_- \end{pmatrix},$$

with

$$\begin{cases} \mathscr{L}_+ & = \Delta^2 + \varepsilon(\langle \vec{b}, \nabla \rangle)^2 + \omega - p|\Phi|^{p-1}, \\ \mathscr{L}_- & = \Delta^2 + \varepsilon(\langle \vec{b}, \nabla \rangle)^2 + \omega - |\Phi|^{p-1}. \end{cases}$$

The eigenvalue problem associated with (2.2.5) ($\vec{z} \to e^{\lambda t}\vec{z}$) takes the form

$$\mathscr{J}\mathscr{L}\vec{z} = \lambda \vec{z}. \tag{2.2.6}$$

For solutions Φ of (2.1.6), the linearized problem appears in the form

$$\partial_t z = \mathcal{J}\mathcal{L} z, \tag{2.2.7}$$

where

$$\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{L} := \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix},$$

with

$$\begin{cases} \mathcal{L}_+ & = \Delta^2 + b\Delta + \omega - p|\Phi|^{p-1}, \\ \mathcal{L}_- & = \Delta^2 + b\Delta + \omega - |\Phi|^{p-1}. \end{cases}$$

The eigenvalue problem associated with (2.2.7) ($\vec{z} \to e^{\lambda t}\vec{z}$) takes the form

$$\mathcal{J}\mathcal{L}\vec{z} = \lambda\vec{z}. \tag{2.2.8}$$

We are now ready to give the definition of the spectral stability.

Definition 2. *The Kawahara waves are stable, provided the eigenvalue problem* (2.2.4) *does not have non-trivial solutions*[9] (μ, z)*:* $\Re\mu > 0$*,* $z \in H^4(\mathbf{R})$.

The NLS waves Φ *are stable, if the eigenvalue problem* (2.2.6) *((*2.2.8*) respectively) does not have non-trivial solutions* $(\mu, \vec{z})$*:* $\Re\mu > 0$*,* $\vec{z} \in H^4(\mathbf{R}^d) \times H^4(\mathbf{R}^d)$.

We are going to use the following corollaries from the instability index counting theory presented in section 1.2.

Corollary 1. *For the spectral problems* (2.2.6) *and* (2.2.8) *stability follows, provided*

- $n(\mathcal{L}_+) = 1$*,* $\mathcal{L}_- \geq 0$*,*
- $\phi \perp Ker[\mathcal{L}_+]$*,*

[9]Note that by the Hamiltonian symmetry of the problem $\mu \to -\mu$, the existence of eigenvalues $\mu : \Re\mu < 0$ is equivalent to the existence of $\mu : \Re\mu > 0$.

- $\langle \mathscr{L}_+^{-1}\phi, \phi \rangle < 0.$

Corollary 2. *For the spectral problem* (2.2.4)*, stability follows, provided*

- $n(\mathscr{L}_+) = 1,$
- $\phi \perp Ker[\mathscr{L}_+],$
- $\langle \mathscr{L}_+^{-1}\phi, \phi \rangle < 0.$

2.2.4 Necessary conditions for the existence of solutions of (2.1.5)

We have the following Pohozaev identities.

Lemma 4. *(Pohozaev's identities) Let some smooth and decaying ϕ satisfy*

$$\Delta^2\phi + \varepsilon \sum_{j,k}^{n} b_j b_k \partial_{j,k}\phi + \omega\phi - |\phi|^{p-1}\phi = 0. \tag{2.2.9}$$

Then

$$\int_{\mathbf{R}^d} |\Delta\phi|^2 dx = \frac{d(p-1)-2(p+1)}{2(p+1)} \int_{\mathbf{R}^d} |\phi|^{p+1} dx + \omega \int_{\mathbf{R}^d} |\phi|^2 dx, \tag{2.2.10}$$

$$\varepsilon \int_{\mathbf{R}^d} |\vec{b}\cdot\nabla\phi|^2 dx = \frac{d(p-1)-4(p+1)}{2(p+1)} \int_{\mathbf{R}^d} |\phi|^{p+1} dx + 2\omega \int_{\mathbf{R}^d} |\phi|^2 dx. \tag{2.2.11}$$

Proof. Multiplying (2.2.9) by ϕ and integrating over $\mathbf{R}^d$ we get

$$\int_{\mathbf{R}^d} |\Delta\phi|^2 dx - \varepsilon \int_{\mathbf{R}^d} |\vec{b}\cdot\nabla\phi|^2 dx - \int_{\mathbf{R}^d} |\phi|^{p+1} dx + \omega \int_{\mathbf{R}^d} |\phi|^2 dx = 0.$$

Also, multiplying (2.2.9) by $x\cdot\nabla\phi$ and integrating over $\mathbf{R}^d$ we get

$$\left(2-\frac{d}{2}\right) \int_{\mathbf{R}^d} |\Delta\phi|^2 dx - \left(1-\frac{d}{2}\right) \varepsilon \int_{\mathbf{R}^d} |\vec{b}\cdot\nabla\phi|^2 dx + \frac{d}{p+1} \int_{\mathbf{R}^d} |\phi|^{p+1} dx - \omega\frac{d}{2} \int_{\mathbf{R}^d} |\phi|^2 dx = 0.$$

Let $A = \int_{\mathbf{R}^d} |\Delta\phi|^2 dx$, $B = \varepsilon \int_{\mathbf{R}^d} |\vec{b}\cdot\nabla\phi|^2 dx$, $C = \int_{\mathbf{R}^d} |\phi|^{p+1} dx$ and $D = \int_{\mathbf{R}^d} |\phi|^2 dx$. Solving for

A and B in terms of C and D we get

$$\begin{cases} A &= \frac{d(p-1)-2(p+1)}{2(p+1)}C+\omega D, \\ B &= \frac{d(p-1)-4(p+1)}{2(p+1)}C+2\omega D, \end{cases}$$

which is (2.2.10) and (2.2.11). $\square$

Corollary 3. *If $d = 1,2$, then $\omega > 0$. If $\varepsilon = -1$ and $\omega > 0$, then $p < p_{\max}$. If $\vec{b} = 0$, then $\omega > 0$ and $p < p_{\max}$.*

Proof. If $d = 1,2$, the first term on the right of (2.2.10) is negative, forcing the positivity of the second term, so $\omega > 0$. Next, from the relation (2.2.11), we see that if $\omega > 0, \varepsilon = -1$, then $\frac{d(p-1)-4(p+1)}{2(p+1)} < 0$, or $p < p_{\max}$.

If $\vec{b} = 0$, it is clear from (2.2.11) that either $\omega > 0$ and $p < p_{\max}$ or $\omega < 0$ and $p > p_{\max}$ (the second one being impossible immediately for $d = 1,2,3,4$). For $d \geq 5$, assume for a moment that $\omega < 0$ and $p > p_{\max} = \frac{d+4}{d-4}$. Let us look at (2.2.10). The second term is now negative, while for the first term, since $p > p_{\max} > \frac{d+2}{d-2}$, we also conclude its negativity. It follows that the right hand side of (2.2.10) is negative a contradiction. Thus, $\omega > 0$, $p < p_{\max}$. $\square$

As we see from the results of Corollary 3, the Pohozaev's identities are by themselves not strong enough to derive necessary conditions on ω, p that are close to the sufficient ones.

We believe that the necessary conditions are close to the ones required by [32] to construct solutions of the constrained minimization problem (2.1.7). Namely, we expect $p < p_{\max}$ and $\omega > \frac{b^2}{4}$ for $b > 0$, and, more generally, (2.1.9) to be necessary for the existence of localized and smooth solutions of (2.2.9) and (2.1.6). Let us show that, in fact, these follow from a natural assumption on the spectral theory for the operator $\mathscr{L}_+$. Specifically, the fact that zero cannot be an embedded eigenvalue in the continuous spectrum of $\mathscr{L}_+$. Let us note that, while for the second order Schrödinger operators $\mathscr{H} = -\Delta + V$ this is generally the case (the point spectrum does not embed into the continuous one under some decay assumptions on V), it is not the case for their fourth order counterparts, [11]. However, in physically relevant situations (and the case of $\mathscr{L}_+$ certainly merits this designation), embedded eigenvalues do not exist. If this is the case for $\mathscr{L}_+$, we, using Weyl's theorem, see that

$$\sigma_{a.c.}[\mathscr{L}_+] = \sigma_{a.c.}(\Delta^2 + b\Delta + \omega - p|\Phi|^{p-1}) = \sigma_{a.c.}(\Delta^2 + b\Delta + \omega) = \begin{cases} \omega - \frac{b^2}{4}, & b \geq 0, \\ \omega, & b < 0. \end{cases}$$

Clearly, if zero is not embedded, it must be that ω satisfies (2.1.9). If that holds, at least in the case $b < 0$, it follows from Corollary 3 that $p < p_{\max}$ as well.

2.3 Variational construction in the one dimensional case

We start with some preparatory results.

2.3.1 Variational problem: preliminary steps

We now discuss the variational problem (2.1.8). It is certainly not *a priori* clear that for a given $\lambda > 0$, such a value is finite (that is $m_b(\lambda) > -\infty$) and non-trivial (i.e. $m_b(\lambda) < 0$). In fact, in some cases, it is not finite, as we show below. Note that

$$\frac{m_b(\lambda)}{\lambda} = \inf_{\|\phi\|_2^2=1} \left\{ \frac{1}{2}\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx - \frac{\lambda^{\frac{p-1}{2}}}{p+1}\int_{\mathbf{R}} |\phi|^{p+1} dx \right\} = \inf_{\|\phi\|_2^2=1} J[\phi]. \qquad (2.3.1)$$

This is, clearly, a non-increasing function. In particular, $\frac{m_b(\lambda)}{\lambda}$ is differentiable a.e. and so is $m_b(\lambda)$. Our considerations naturally split in two case, $b > 0$ and $b < 0$. In this section, we develop criteria (based on the parameters in the problem), which address the question for finiteness and non-triviality of $m_b(\lambda)$. The next lemma shows this for $p \in (1,5)$ and, in addition, it establishes the non-finiteness of $m_b(\lambda)$ for $p > 9$.

First, we treat ***the case*** $b < 0$.

Lemma 5.

- *If* $p \in (1,5)$, $b < 0$, *then* $-\infty < m_b(\lambda) < 0$ *for all* $\lambda > 0$.

- *If $p > 9$, then $m_b(\lambda) = -\infty$ for all $\lambda > 0$.*

- *If $p = 9$, then $m_b(\lambda) = -\infty$ for all λ large enough.*

Proof. Let $\phi_\varepsilon(x) = \varepsilon^{1/2}\phi(\varepsilon x)$, where $\|\phi\|_2^2 = \lambda$. We have that

$$I[\phi_\varepsilon] = \frac{\|\phi''\|_{L^2}^2}{2}\varepsilon^4 - \frac{b\|\phi'\|_{L^2}^2}{2}\varepsilon^2 - \frac{\|\phi\|_{L^{p+1}}^{p+1}}{p+1}\varepsilon^{\frac{p-1}{2}}. \tag{2.3.2}$$

Since $0 < \frac{p-1}{2} < 2$ for $1 < p < 5$, we see that $m_b(\lambda) < 0$ in this case by choosing ε small enough. On the other hand, if $p > 9$, it is clear that $\lim_{\varepsilon\to\infty} I[\phi_\varepsilon] = -\infty$, whence $m_b(\lambda) = -\infty$.

By the GNS inequality

$$\begin{aligned} \|\phi\|_{L^{p+1}(\mathbf{R})} &\leq C_p\|\phi\|_{\dot{H}^{\frac{1}{2}-\frac{1}{p+1}}} && (2.3.3)\\ &\leq C_p\|\phi\|_{L^2}^{\frac{3}{4}+\frac{1}{2(p+1)}}\|\phi''\|_{L^2}^{\frac{1}{4}-\frac{1}{2(p+1)}}, && (2.3.4) \end{aligned}$$

we have

$$\begin{aligned} I[\phi] &= \frac{1}{2}\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |\phi|^{p+1} dx \\ &\geq \frac{1}{2}\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx - c_p\|\phi''\|_{L^2}^{\frac{p-1}{4}}\|\phi\|_{L^2}^{p+1-\frac{p-1}{4}} \\ &\geq \frac{1}{4}\|\phi''\|_{L^2}^2 - c_{p,\lambda,b}(\|\phi''\|_{L^2}^{\frac{p-1}{4}} + 1) \\ &\geq -\gamma, \end{aligned}$$

for some $\gamma > 0$ because the function $g(x) = \frac{1}{2}x^2 - c_{p,\lambda}x^{\frac{p-1}{4}}$, clearly, has a negative minimum on $[0,\infty)$ for $p \in (1,9)$. Therefore, $m_b(\lambda) \geq -\gamma > -\infty$ for $p \in (1,9)$. Letting $\varepsilon \to \infty$ in (2.3.2) shows that $m_b(\lambda) = -\infty$ for $p > 9$.

Consider now the case $p = 9$. Clearly, for large λ, $m_b(\lambda) < 0$, as it is evident from the formula (2.3.1). Assuming that $m_b(\lambda) \in (-\infty, 0)$ for some λ, let ϕ be such that $m_b(\lambda) \leq I[\phi] < \frac{m_b(\lambda)}{2}$.

Using ϕ_N as in the formula (2.3.2), we see that $\|\phi_N\|_{L^2}^2 = \lambda$, while for $N \geq 1$, we have

$$I[\phi_N] = N^4[\frac{\|\phi''\|_{L^2}^2}{2} - \frac{b\|\phi'\|_{L^2}^2}{2N^2} - \frac{\|\phi\|_{L^{10}}^{10}}{10}] \tag{2.3.5}$$

$$\leq N^4[\frac{\|\phi''\|_{L^2}^2}{2} - \frac{b\|\phi'\|_{L^2}^2}{2} - \frac{\|\phi\|_{L^{10}}^{10}}{10}] \tag{2.3.6}$$

$$\leq N^4\frac{m_b(\lambda)}{2}. \tag{2.3.7}$$

But then

$$m_b(\lambda) \leq \liminf_N I[\phi_N] = -\infty,$$

a contradiction.

□

Our next lemma shows that for $p \in [5,9]$, there is a threshold value $\lambda_p > 0$, below which $m_b(\lambda)$ is trivial.

Lemma 6. *If $b < 0$ and $p \in [5,9]$, then there exists a finite number $\lambda_p > 0$ such that for all $\lambda \leq \lambda_p$ we have $m_b(\lambda) = 0$. In addition, if $p \in [5,9)$, then for all $\lambda > \lambda_p$ we have $-\infty < m_b(\lambda) < 0$.*

Proof. Take ϕ_ε as in Lemma 5 with $\|\phi\|_2^2 = 1$. We have

$$\frac{m_b(\lambda)}{\lambda} \leq \lim_{\varepsilon \to 0} J[\phi_\varepsilon] = 0, \tag{2.3.8}$$

which implies that $m_b(\lambda) \leq 0$. Now, we are going to show that for each $p \in [5,9)$ there exists a constant $c_p > 0$ such that

$$\inf_{\phi \neq 0} \frac{\|\phi\|_2^{p-1}\left(\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx\right)}{\int_{\mathbf{R}} |\phi|^{p+1} dx} \geq c_p. \tag{2.3.9}$$

Using the GNS inequality (1.1.1), we get the following estimates for the L^{p+1} norm:

$$\|\phi\|_{p+1}^{p+1} \leq a_p \left\|\phi''\right\|_2^{\frac{p-1}{4}} \|\phi\|_2^{\frac{3p+5}{4}}$$
$$\leq a_p \left(\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx\right)^{\frac{p-1}{8}} \|\phi\|_2^{\frac{3p+5}{4}}, \tag{2.3.10}$$

and

$$\|\phi\|_{p+1}^{p+1} \leq b_p \left\|\phi'\right\|_2^{\frac{p-1}{2}} \|\phi\|_2^{\frac{3p+5}{4}}$$
$$\leq b_p \left(\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx\right)^{\frac{p-1}{4}} \|\phi\|_2^{\frac{p+3}{2}}. \tag{2.3.11}$$

Note that for $p \in [5,9]$, we have that $\frac{p-1}{8} \leq 1 \leq \frac{p-1}{4}$. Therefore, interpolating between the estimates (2.3.10) and (2.3.11) we get

$$\|\phi\|_{L^{p+1}}^{p+1} \leq c_p \|\phi\|_{L^2}^{p-1} \int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx.$$

Thus we have that for all $\phi \in H^2$ with $\|\phi\|_2^2 = 1$

$$\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx - \frac{1}{c_p} \int_{\mathbf{R}} |\phi|^{p+1} dx \geq 0.$$

This implies that for $\lambda : 0 < \lambda \leq \gamma_p = \left(\frac{p+1}{c_p}\right)^{\frac{2}{p-1}}$, $J[\phi] \geq 0$, which together with (2.3.8) implies that $m_b(\lambda) = 0$.

Observe that for a large enough λ, the quantity

$$\inf_{\|\phi\|_2^2 = 1} \left\{ \frac{1}{2} \int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx - \frac{\lambda^{\frac{p-1}{2}}}{p+1} \int_{\mathbf{R}} |\phi|^{p+1} dx \right\}$$

is strictly negative[10], so $\lambda_p < \infty$. Clearly, $\lambda_p = \sup\{\gamma > 0 : m_b(\lambda) = 0 \text{ for all } \lambda \leq \gamma\}$.

$\square$

[10]This can be seen by fixing ϕ in the infimum and taking $\lambda > \lambda(\phi)$.

Lemma 7. *Suppose $b<0$, $1<p<9$ and $-\infty<m_b(\lambda)<0$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a minimizing sequence. Then, there exists a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$ such that:*

$$\lim_{k\to\infty}\int_{\mathbf{R}}|\phi''_{n_k}(x)|^2dx=L_1,\ \lim_{k\to\infty}\int_{\mathbf{R}}|\phi'_{n_k}(x)|^2dx=L_2,\ \lim_{k\to\infty}\int_{\mathbf{R}}|\phi_{n_k}(x)|^{p+1}dx=L_3,$$

where $L_1>0$, $L_2>0$ and $L_3>0$.

Proof. We have already established in Lemma 5 that

$$I[\phi]\geq\frac{1}{4}\|\phi''\|_{L^2}^2-c_{p,\lambda,b}(\|\phi''\|_{L^2}^{\frac{p-1}{4}}+1). \tag{2.3.12}$$

Since, $\{\phi_n\}_{n=1}^{\infty}$ is minimizing, it follows that the sequence $\{\int_{\mathbf{R}}|\phi''_n(x)|^2dx\}_{n=1}^{\infty}$ is bounded. By the GNS inequality, the sequences $\{\int_{\mathbf{R}}|\phi'_n(x)|^2dx\}_{n=1}^{\infty}$ and $\int_{\mathbf{R}}|\phi_n(x)|^{p+1}dx\}_{n=1}^{\infty}$ are bounded as well. Possibly passing to a subsequence a couple of times, we get a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$ such that all of the above sequences converge(to L_1, L_2 and L_3 respectively). We claim that L_3 cannot be zero. Indeed, otherwise

$$m_b(\lambda)=\lim_{k\to\infty}[\frac{1}{2}\int_{\mathbf{R}}|\phi''_{n_k}(x)|^2dx-\frac{b}{2}\int_{\mathbf{R}}|\phi'_{n_k}(x)|^2dx]\geq 0,$$

which is a contradiction with the fact that $m_b(\lambda)<0$. By the GNS inequality, neither L_1 nor L_2 could be zero, as this would force $L_3=0$, which we have shown to be impossible.

□

In the following 3 lemmas we will be concerned with ***the case*** $b>0$**.**

Lemma 8. *If $b>0$ and $1<p<9$, then $-\infty<m_b(\lambda)<0$ for all $\lambda>0$.*

Proof. Since $0<\frac{p-1}{2}<4$, the dominant term in (2.3.2) is $\max(\varepsilon^2,\varepsilon^{\frac{p-1}{2}})$, so if we just take ε small enough, we see that $m_b(\lambda)<0$. Boundedness from below follows from (2.3.12). □

Lemma 9. *Let $1 < p < 5$, $b > 0$ and fix a constant c. Then, the inequality*

$$\|\phi\|_{L^{p+1}}^{p+1} \leq c\,\|\phi\|_{L^2}^{p-1} \left[\int_{\mathbf{R}} |\phi''(x)|^2 - b|\phi'(x)|^2 + \frac{b^2}{4}|\phi(x)|^2 dx\right] \tag{2.3.13}$$

cannot hold for all $\phi \in H^2(\mathbf{R})$.

For $p \in [5,9]$, $b > 0$, there is a $c(b,p)$, so that

$$\|\phi\|_{L^{p+1}}^{p+1} \leq c\,\|\phi\|_{L^2}^{p-1} \left[\int_{\mathbf{R}} |\phi''(x)|^2 - b|\phi'(x)|^2 + \frac{b^2}{4}|\phi(x)|^2 dx\right] \tag{2.3.14}$$

holds for all $\phi \in H^2(\mathbf{R})$.

Proof. Let $p \in [5,9]$. Write

$$\int_{\mathbf{R}} |\phi''(x)|^2 - b|\phi'(x)|^2 + \frac{b^2}{4}|\phi(x)|^2 dx = \int_{\mathbf{R}} |\hat{\phi}(\xi)|^2 \left((2\pi\xi)^2 - \frac{b}{2}\right)^2 d\xi.$$

Introducing g, so that $\hat{\phi}(\xi) := \hat{g}(2\pi\xi - \sqrt{\frac{b}{2}})$. Clearly, (2.3.14) is equivalent to the estimate

$$\|g\|_{L^{p+1}}^{p+1} \leq c\,\|g\|_{L^2}^{p-1} \int_{\mathbf{R}} |\hat{g}(\xi)|^2 |\xi|^2 |\xi - C_b|^2 d\xi \tag{2.3.15}$$

for some $C_b \neq 0$. We show (2.3.15) as follows. By Sobolev embedding and Hölder's

$$\begin{aligned}
\|g\|_{L^{p+1}} &\lesssim \|g\|_{\dot{H}^{\frac{1}{2}-\frac{1}{p+1}}} \\
&= c\left(\int_{\mathbf{R}} |\hat{g}(\xi)|^2 |\xi|^{1-\frac{2}{p+1}} d\xi\right)^{1/2} \\
&\lesssim \|g\|_{L^2}^{\frac{p-1}{p+1}} \left(\int_{\mathbf{R}} |\hat{g}(\xi)|^2 |\xi|^{\frac{p-1}{2}} d\xi\right)^{\frac{1}{p+1}}.
\end{aligned}$$

Clearly, this last estimate implies (2.3.15) as long as $2 \leq \frac{p-1}{2} \leq 4$, which is the same as $p \in [5,9]$.

Let now $p \in (1,5)$. Take a Schwartz function χ and then $\phi(x) = \chi(\varepsilon x)$. Testing (2.3.13) for this choice of ϕ leads us to $\varepsilon^{-1} \leq C\varepsilon^{-\frac{p-1}{2}}(\varepsilon^3 + \varepsilon)$. This is a contradiction as $\varepsilon \to 0+$, so (2.3.13) cannot hold. □

Lemma 10. *Suppose $b > 0, \lambda > 0$ and $1 < p < 9$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a minimizing sequence for the minimization problem $\inf_{\|\phi\|_{L^2}^2=\lambda} I[\phi]$. Assume one of the following:*

- *$p \in (1,5)$, $\lambda > 0$,*
- *$p \in [5,9)$ and for some sufficiently large $\lambda_{b,p}$, $\lambda > \lambda_{b,p}$.*

Then, there exists a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$, such that:

$$\lim_{k\to\infty} \frac{1}{2}\int_{\mathbf{R}} |\phi_{n_k}''|^2 dx = L_1, \quad \lim_{k\to\infty} \int_{\mathbf{R}} |\phi_{n_k}'|^2 dx = L_2 \text{ and } \lim_{k\to\infty} \int_{\mathbf{R}} |\phi_{n_k}|^{p+1} dx = L_3,$$

where $L_1 > 0$, $L_2 > 0$ and $L_3 > 0$.

Proof. First, by (2.3.12), the sequence $\{\int_{\mathbf{R}} |\phi_n''|^2 dx\}_{n=1}^{\infty}$ is bounded. By GNS inequality, so are the sequences $\{\int_{\mathbf{R}} |\phi_n'|^2 dx\}_{n=1}^{\infty}$ and $\{\int_{\mathbf{R}} |\phi_n|^{p+1} dx\}_{n=1}^{\infty}$. Therefore, there exists a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$ such that all three subsequences $\{\int_{\mathbf{R}} |\phi_{n_k}''|^2 dx\}_{k=1}^{\infty}$, $\{\int_{\mathbf{R}} |\phi_{n_k}'|^2 dx\}_{k=1}^{\infty}$ and $\{\int_{\mathbf{R}} |\phi_{n_k}|^{p+1} dx\}_{k=1}^{\infty}$ converge to three non-negative reals L_1, L_2, L_3 respectively.

First, suppose that $L_3 = 0$. Then, consider the following minimization problem

$$\inf_{\|\phi\|_2^2=\lambda} \frac{1}{2}\int_{\mathbf{R}} |\phi''(x)|^2 - b|\phi'(x)|^2 dx := \inf_{\|\phi\|_2^2=\lambda} \tilde{I}[\phi].$$

Observe that since $\tilde{I}[\phi] \geq I[\phi]$, we have

$$\lim_{n\to\infty} \tilde{I}[\phi_n] = \lim_{n\to\infty} I[\phi_n] = \inf_{\|\phi\|_2^2=\lambda} I[\phi] \leq \inf_{\|\phi\|_2^2=\lambda} \tilde{I}[\phi].$$

Thus, $\{\phi_n\}_{n=1}^{\infty}$ is minimizing for $\tilde{I}$ as well, and

$$\inf_{\|\phi\|_2^2=\lambda} I[\phi] = \inf_{\|\phi\|_2^2=\lambda} \tilde{I}[\phi].$$

On the other hand, $\inf_{\|\phi\|_2^2=\lambda} \tilde{I}[\phi]$ is easily seen to be $-\frac{\lambda b^2}{8}$. Indeed, for a function ϕ with $\|\phi\|_{L^2}^2 = \lambda$, we have by Plancherel's

$$2\tilde{I}[\phi]+\frac{b^2}{4}\lambda=\int_{\mathbf{R}}|\phi''(x)|^2-b|\phi'(x)|^2+\frac{b^2}{4}\phi^2(x)dx \tag{2.3.16}$$

$$=\int_{\mathbf{R}}|\hat{\phi}(\xi)|^2\left|(2\pi\xi)^2-\frac{b}{2}\right|^2d\xi\geq 0. \tag{2.3.17}$$

whence $\inf_{\|\phi\|_2^2=\lambda}\tilde{I}[\phi]\geq-\frac{\lambda b^2}{8}$. On the other hand, for any Schwartz function χ, consider

$$\hat{\phi}_\varepsilon(\xi):=\frac{\sqrt{\lambda}}{\sqrt{\varepsilon}\|\chi\|_{L^2}}\chi\left(\frac{\xi-\frac{1}{2\pi}\sqrt{\frac{b}{2}}}{\varepsilon}\right)$$

which has $\|\phi\|_{L^2}^2=\lambda$ and saturates the inequality (2.3.16) in the sense that

$$\lim_{\varepsilon\to 0+}\int_{\mathbf{R}}|\hat{\phi}_\varepsilon(\xi)|^2\left|(2\pi\xi)^2-\frac{b}{2}\right|^2d\xi\to 0.$$

Thus, $\inf_{\|\phi\|_2^2=\lambda}I[\phi]=-\frac{\lambda b^2}{8}$. So, we have that

$$-\frac{\lambda b^2}{8}=m_b(\lambda)\leq\frac{1}{2}\int_{\mathbf{R}}|\phi''(x)|^2-b|\phi'|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|\phi(x)|^{p+1}dx$$

holds for all ϕ with $\|\phi\|_2^2=\lambda$. Applying this to an arbitrary f and $\phi:=\sqrt{\lambda}\frac{f}{\|f\|_{L^2}}$, so that $\|\phi\|_{L^2}^2=\lambda$ the following inequality holds

$$\frac{\lambda^{\frac{p-1}{2}}b^{\frac{p-9}{4}}}{p+1}\int_{\mathbf{R}}|f(x)|^{p+1}dx\leq\frac{1}{2}\|f\|_2^{p-1}\left(\int_{\mathbf{R}}|f''(x)|^2-b|f'(x)|^2+\frac{b^2}{4}|f(x)|^2dx\right)$$

for all $f\neq 0$. This last inequality however contradicts Lemma 9 - for every $\lambda>0$, if $p\in(1,5)$ and for all large enough λ, if $p\in[5,9)$. Thus $L_3\neq 0$. Clearly, by Sobolev embedding $L_1>0$, $L_2>0$, otherwise L_3 must be zero, which previously lead to a contradiction.

□

The following lemma will be useful for ruling out the dichotomy option in the concentration compactness argument.

Lemma 11 (Strict sub-additivity)**.** *Let* $1 < p < 9$ *and* $\lambda > 0$*. Then for all* $\alpha \in (0, \lambda)$ *we have*

$$m_b(\lambda) < m_b(\alpha) + m_b(\lambda - \alpha). \tag{2.3.18}$$

Proof. First, suppose that $1 < p < 5$ and $b < 0$. Then

$$\begin{aligned} m_b(\lambda) &= \frac{\lambda}{\alpha} \inf_{\|\phi\|_2^2 = \alpha} \{\frac{1}{2}\int_{\mathbf{R}} |\phi''(x)|^2 - b|\phi'(x)|^2 dx - \frac{(\lambda/\alpha)^{\frac{p-1}{2}}}{p+1}\int_{\mathbf{R}} |\phi(x)|^{p+1} dx\} \\ &< \frac{\lambda}{\alpha} m_b(\alpha), \end{aligned}$$

where the last strict inequality holds because a minimizing sequence for $m_b(\alpha)$ doesn't loose $\|\phi_k\|_{p+1}$. Namely, there exists a minimizing sequence $\{\phi_k\}_{k=1}^{\infty}$, so that $\lim_k \|\phi_k\|_{p+1} > 0$. The existence of such a sequence was established in Lemmas 7 and 10. Hence the function $\lambda \to \frac{m_b(\lambda)}{\lambda}$ is a strictly decreasing function.

Assuming that $\alpha \in [\frac{\lambda}{2}, \lambda)$ (and otherwise we could just work with $\lambda - \alpha$) we get

$$\begin{aligned} m_b(\lambda) &< \frac{\lambda}{\alpha} m_b(\alpha) \\ &= m_b(\alpha) + \frac{\lambda - \alpha}{\alpha} m_b(\alpha) \\ &\le m_b(\alpha) + m_b(\lambda - \alpha), \end{aligned}$$

where we have used $\frac{m_b(\alpha)}{\alpha} \le \frac{m_b(\lambda-\alpha)}{\lambda-\alpha}$, since $\alpha \ge \lambda - \alpha$. This completes the case $p \in (1,5), b < 0$.

Let $5 \le p < 9$ and $b < 0$. Note that in this case, $m_b(\lambda)$ is zero for small λ, by Lemma 6. So, there are three possibilities:

1. $m_b(\alpha) = m_b(\lambda - \alpha) = 0$. In this case (2.3.18) trivially holds, since by assumption $m_b(\lambda) < 0$.

2. $m_b(\lambda) < 0$, but $m_b(\lambda - \alpha) = 0$. In this case we have

$$\begin{aligned} m_b(\lambda) &< \frac{\lambda}{\alpha} m_b(\alpha) \\ &= m_b(\alpha) + (\frac{\lambda}{\alpha} - 1) m_b(\alpha) \\ &< m_b(\alpha) + m_b(\lambda - \alpha). \end{aligned}$$

3. When both $m_b(\alpha), m_b(\lambda - \alpha)$ are negative, the proof is the same as in the case $1 < p < 5$ for $b < 0$.

Next, we consider the cases when $b > 0$. In this case for all $1 < p < 5$ and all $\lambda > 0$ we have that $-\infty < m_b(\lambda) < 0$. The proof is the same as in the case $b < 0, p \in (1,5)$, since we never develop the complication that $m_b(\lambda) = 0$ for any $\lambda > 0$. The case $p \in [5,9)$ and $\lambda > \lambda_{b,p}$ is similar as well. $\square$

2.3.2 Existence of the minimizer

Now, suppose

$$\begin{cases} 1 < p < 5, & \lambda > 0, \\ 5 \le p < 9, & \lambda > \lambda_{b,p}, \end{cases}$$

so that Lemma 7 and Lemma 10 hold. Let $\{\phi_k\}_{k=1}^{\infty} \subset H^2$ be a minimizing sequence, i.e.

$$\int_{\mathbf{R}} |\phi_k|^2 dx = \lambda, \qquad \lim_{k \to \infty} I[\phi_k] = m_b(\lambda).$$

Therefore, by passing to a further subsequence, by Lemma 7 and Lemma 10, we have

$$\lim_{k\to\infty} \|\phi_k''\|_2^2 = L_1 > 0, \quad \lim_{k\to\infty} \|\phi_k'\|_2^2 = L_2 > 0, \quad \lim_{k\to\infty} \|\phi_k\|_{L^{p+1}}^{p+1} = L_3 > 0.$$

Let $\rho_k = |\phi_k|^2$, so $\int \rho_k(x) dx = \lambda$. By the concentration compactness lemma of P.L.Lions, there is a subsequence (denoted again by ρ_k), so that at least one of the following is satisfied:

1. *Tightness.* There exists $y_k \in \mathbf{R}$ such that for any $\varepsilon > 0$ there exists $R(\varepsilon)$ such that for all k

$$\int_{B(y_k,R(\varepsilon))} \rho_k dx \geq \int_{\mathbf{R}} \rho_k - \varepsilon.$$

2. *Vanishing.* For every $R > 0$

$$\lim_{k\to\infty} \sup_{y\in\mathbf{R}} \int_{B(y,R)} \rho_k dx = 0.$$

3. *Dichotomy.* There exists $\alpha \in (0,\lambda)$, such that for any $\varepsilon > 0$ there exist $R, R_k \to \infty, y_k$ and k_0 such that

$$\left|\int_{B(y_k,R)} \rho_k dx - \alpha\right| < \varepsilon,\ \left|\int_{R<|x-y_k|<R_k} \rho_k dx\right| < \varepsilon,\ \left|\int_{R_k<|x-y_k|} \rho_k dx - (\lambda-\alpha)\right| < \varepsilon. \quad (2.3.19)$$

We proceed to rule out the dichotomy and vanishing alternatives, which will leave us with tightness.

Dichotomy is not an option. Assuming dichotomy, we have by (2.3.19) and $\int \rho_k(x)dx = \lambda$ that $\left|\int_{R_k<|x-y_k|} \rho_k dx - (\lambda-\alpha)\right| < 2\varepsilon$.

Let $\psi_1, \psi_2 \in C^\infty(\mathbf{R})$, satisfying $0 \leq \psi_1, \psi_2 \leq 1$ and

$$\psi_1(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2, \end{cases}, \quad \psi_2(x) = \begin{cases} 1, & |x| \geq 1, \\ 0, & |x| \leq 1/2, \end{cases}.$$

Define $\phi_{k,1}$ and $\phi_{k,2}$ as follows:

$$\phi_{k,1}(x) = \phi_k(x)\psi_1\left(\frac{x-y_k}{R_k/5}\right), \quad \phi_{k,2}(x) = \phi_k(x)\psi_2\left(\frac{x-y_k}{R_k}\right).$$

Clearly, for k large enough we have

$$\left|\int_{\mathbf{R}} \phi_{k,1}^2(x)dx - \alpha\right| < 2\varepsilon \text{ and } \left|\int_{\mathbf{R}} \phi_{k,2}^2(x)dx - (\lambda-\alpha)\right| < 2\varepsilon.$$

In fact, by taking a sequence $\varepsilon_n \to 0$, we can find subsequence of $\phi_{k,1}, \phi_{k,2}$ (denoted again the same) and sequences $\{y_k\}_{k=1}^\infty \subset \mathbf{R}$, $\{R_k\}_{k=1}^\infty$ with $R_k \to \infty$ as $k \to \infty$, such that

$$\lim_{k\to\infty}\int_{\mathbf{R}}|\phi_{k,1}|^2dx = \alpha, \quad \lim_{k\to\infty}\int_{\mathbf{R}}\left|\phi_{k,2}\right|^2 dx = \lambda-\alpha \text{ and } \int_{R_k/5<|x-y_k|<R_k}|\phi_k|^2dx < \frac{1}{k}. \tag{2.3.20}$$

Consider $I[\phi_k]-I[\phi_{k,1}]-I[\phi_{k,2}]$. Using (2.3.20) we get

$$\begin{aligned}
& I[\phi_k]-I[\phi_{k,1}]-I[\phi_{k,2}] = \frac{1}{2}\int_{\mathbf{R}}|\phi_k''|^2-b|\phi_k'|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|\phi_k|^{p+1} \\
& -\frac{1}{2}\int_{\mathbf{R}}\left|\left(\phi_k\psi_1\left(\frac{x-y_k}{R_k/5}\right)\right)''\right|^2-b\left|\left(\phi_k\psi_1\left(\frac{x-y_k}{R_k/5}\right)\right)'\right|^2dx+\frac{1}{p+1}\int_{\mathbf{R}}\left|\left(\phi_k\psi_1\left(\frac{x-y_k}{R_k/5}\right)\right)\right|^{p+1} \\
& -\frac{1}{2}\int_{\mathbf{R}}\left|\left(\phi_k\psi_2\left(\frac{x-y_k}{R_k}\right)\right)''\right|^2-b\left|\left(\phi_k\psi_2\left(\frac{x-y_k}{R_k}\right)\right)'\right|^2dx+\frac{1}{p+1}\int_{\mathbf{R}}\left|\left(\phi_k\psi_2\left(\frac{x-y_k}{R_k}\right)\right)\right|^{p+1} \\
& =\frac{1}{2}\int_{\mathbf{R}}\left(1-\psi_1^2\left(\frac{x-y_k}{R_k/5}\right)-\psi_2^2\left(\frac{x-y_k}{R_k}\right)\right)\left[|\phi_k''(x)|^2-\frac{b}{2}|\phi_k'(x)|^2\right]dx+ \\
& +\frac{1}{p+1}\int_{\mathbf{R}}|\phi_k(x)|^{p+1}\left(\psi_1^{p+1}\left(\frac{x-y_k}{R_k/5}\right)+\psi_2^{p+1}\left(\frac{x-y_k}{R_k}\right)-1\right)dx+E_k.
\end{aligned}$$

The error term E_k, contains only terms having at least one derivative on the cutoff functions, therefore generating R_k^{-1}. At the same time, there is at most one derivative falling on the ϕ_k. So, we can estimate these terms away as follows

$$|E_k| \leq \frac{C}{R_k}\int_{R_k/5<|x|<2R_k}(|\phi_k(x)|^2+|\phi_k'(x)|^2)dx \leq \frac{C}{R_k}\|\phi_k\|_{L^2}(\|\phi_k\|_{L^2}+\|\phi_k''\|_{L^2}).$$

Since $\sup_k\|\phi_k\|_{L^2}, \sup_k\|\phi_k''\|_{L^2}<\infty$, we conclude that $\lim_k E_k = 0$. For the next term, we have the positivity relation $\int_{\mathbf{R}}\left(1-\psi_1^2\left(\frac{x-y_k}{R_k/5}\right)-\psi_2^2\left(\frac{x-y_k}{R_k}\right)\right)|\phi_k''(x)|^2dx>0$. Integration by parts yields

$$\begin{aligned}
&\int_{\mathbf{R}}\left(1-\psi_1^2\left(\frac{x-y_k}{R_k/5}\right)-\psi_2^2\left(\frac{x-y_k}{R_k}\right)\right)|\phi_k'(x)|^2dx = \\
= \; & -\int_{\mathbf{R}}\phi_k(x)\frac{d}{dx}[\left(1-\psi_1^2\left(\frac{x-y_k}{R_k/5}\right)-\psi_2^2\left(\frac{x-y_k}{R_k}\right)\right)\phi_k'(x)]dx.
\end{aligned}$$

Thus, by Hölder's

$$\left|\int_{\mathbf{R}}\left(1-\psi_1^2\left(\frac{x-y_k}{R_k/5}\right)-\psi_2^2\left(\frac{x-y_k}{R_k}\right)\right)|\phi_k'(x)|^2dx\right| \leq$$
$$\leq \quad C\|\phi_k''\|_{L^2}\|\phi_k\|_{L^2(R_k/5<|\cdot|<R_k)}+\frac{C}{R_k}\|\phi_k'\|_{L^2}\|\phi_k\|_{L^2}.$$

Note that since $R_k \to \infty$ and on the other hand $\|\phi_k\|_{H^2}$ is uniformly bounded in k, this term goes to zero, by the last estimate in (2.3.20). Finally,

$$\left|\int_{\mathbf{R}}|\phi_k(x)|^{p+1}\left(\psi_1^{p+1}\left(\frac{x-y_k}{R_k/5}\right)+\psi_2^{p+1}\left(\frac{x-y_k}{R_k}\right)-1\right)dx\right| \leq \int_{R_k/5<|x-y_k|<R_k}|\phi_k(x)|^{p+1}dx.$$

Since by GNS

$$\int_{R_k/5<|x-y_k|<R_k}|\phi_k(x)|^{p+1}dx \leq C\|\phi_k''\|_{L^2}^{\frac{p-1}{4}}\|\phi_k\|_{L^2(R_k/5<|\cdot|<R_k)}^{\frac{3p+5}{4}},$$

and $\|\phi_k''\|_{L^2}$ is uniformly bounded in k, we conclude that this term also goes to zero as $k \to \infty$.

It follows that

$$\liminf_{k\to\infty}\left[I[\phi_k]-I[\phi_{k,1}]-I[\phi_{k,2}]\right] \geq 0. \tag{2.3.21}$$

Now, let $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ be sequences such that

$$\left\|a_k\phi_{k,1}\right\|_2^2=\alpha, \quad \left\|b_k\phi_{k,2}\right\|_2^2=\lambda-\alpha.$$

Note that $a_k, b_k \to 1$. Using (2.3.21), there is β_k, $\lim_k \beta_k = 0$, so that

$$\begin{aligned}
I[\phi_k] &\geq I[\phi_{k,1}]+I[\phi_{k,2}]+\beta_k \\
&\geq I[a_k\phi_{k,1}]+I[b_k\phi_{k,2}]+\beta_k-C(|1-a_k|+|1-b_k|) \\
&\geq m_b(\alpha)+m_b(\lambda-\alpha)+\beta_k-C(|1-a_k|+|1-b_k|),
\end{aligned}$$

where we have used that $\sup_k\|\phi_k\|_{H^2}<\infty$, the estimate $|I(\phi)-I(a\phi)| \leq C(\|\phi\|_{H^2})|1-a|$ (which is a direct consequence of the definition of the functional $I[\cdot]$) and the definition of $m_b(z)$. Taking limits in k, we see that

$$m_b(\lambda) = \lim_k I[\phi_k] \geq m_b(\alpha) + m_b(\lambda - \alpha),$$

which is a contradiction with the sub-additivity of $m_b(\cdot)$ established in Lemma 11. So, dichotomy cannot occur.

Vanishing is not an option. Suppose vanishing occurs and let $\varepsilon > 0$. Let $\phi \in C^\infty$ be such that

$$\eta(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

Using GNS we have for all R and $y \in \mathbf{R}$

$$\begin{aligned} \|\phi_k\|_{L^{p+1}(B(y,R))}^{p+1} &\leq \int_{B(y,R)} |\phi_k|^{p+1} dx \\ &\leq \int_{\mathbf{R}} \left| \phi_k \eta\left(\frac{x-y}{R}\right) \right|^{p+1} dx \\ &\leq \left\| \left(\phi_k \eta\left(\frac{x-y}{R}\right)\right)'' \right\|_{L^2(\mathbf{R})}^{\frac{p-1}{4}} \|\phi_k\|_{L^2(B(y,2R))}^{\frac{3p+5}{4}} \\ &\leq C_{\eta,R} \|\phi_k\|_{L^2(B(y,2R))}^{\frac{3p+5}{4}}. \end{aligned}$$

We can cover $\mathbf{R}$ with balls of radius 2 such that every point is contained in at most 3 balls, let it be $\{B(y_j,2)\}$. Moreover, we can choose these balls so that $\{B(y_j,1)\}$ still covers $\mathbf{R}$. Choose $N \in \mathbf{N}$ so large that for all $k > N$,

$$\int_{B(y,2)} |\phi_k|^2 dx < \varepsilon,$$

for all $y \in \mathbf{R}$. We can estimate the $L^{p+1}(\mathbf{R})$ norm of ϕ_k as follows

$$\begin{aligned}
\|\phi_k\|_{L^{p+1}(\mathbf{R})}^{p+1} &\leq \sum_{j=1}^{\infty} \int_{B(y_j,1)} |\phi_k|^{p+1} dx \\
&\leq \sum_{j=1}^{\infty} C_{\eta,R} \|\phi_k\|_{L^2(B(y_j,2))}^2 \|\phi_k\|_{L^2(B(y_j,2))}^{\frac{3p-3}{4}} \\
&\leq 3C_{\eta,R} \varepsilon^{\frac{3p-3}{4}} \|\phi_k\|_{L^2(\mathbf{R})}^2 .
\end{aligned}$$

So, we get that $\|\phi_k\|_{L^{p+1}(\mathbf{R})}^{p+1} \to 0$ as $k \to \infty$ which is a contradiction. Therefore, the sequence $\rho_k = |\phi_k|^2$ is tight.

Existence of the minimizer. We have that there exists a sequence $\{y_k\}_{k=1}^{\infty}$ such that for all $\varepsilon > 0$ there exists $R(\varepsilon)$ such that

$$\int_{|x|>R(\varepsilon)} |\phi_k(y_k + x)|^2 dx < \varepsilon.$$

Define $u_k(x) := \phi_k(y_k + x)$. The sequence $\{u_k\}_{k=1}^{\infty} \subset H^2$ is bounded, therefore there exists a weakly convergent subsequence (renamed to $\{u_k\}_{k=1}^{\infty}$), say, to $u \in H^2$. By compactness criterion on $L^p(\mathbf{R}^n)$, the sequence $\{u_k\}_{k=1}^{\infty}$ has a strongly convergent subsequence in $L^2(\mathbf{R})$, say, to $\widetilde{u} \in H^2$. Since weak convergence on H^2 implies weak convergence on L^2, we have that $u = \widetilde{u}$ by uniqueness of weak limits. In addition, $\|u\|_{L^2}^2 = \lim_k \|u_k\|_{L^2}^2 = \lambda$, so u satisfies the constraint.

We also have that u_k converges to u in L^{p+1} norm. Indeed, using GNS inequality we get

$$\begin{aligned}
\|u_k - u\|_{L^{p+1}(\mathbf{R})} &\leq \left\|(u_k - u)''\right\|_{L^2(\mathbf{R})}^{\frac{p-1}{4(p+1)}} \|u_k - u\|_{L^2(\mathbf{R})}^{1-\frac{p-1}{4(p+1)}} \\
&\leq C \|u_k - u\|_{L^2(\mathbf{R})}^{1-\frac{p-1}{4(p+1)}} \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Also, since

$$\|u_k' - u'\|_{L^2}^2 \leq \|u_k'' - u''\|_{L^2} \|u_k - u\|_{L^2} \leq (\|u_k''\|_{L^2} + \|u''\|_{L^2}) \|u_k - u\|_{L^2},$$

we conclude that $\lim_k \|u_k' - u'\|_{L^2} = 0$, and, in addition, $\lim_k \int (u_k'(x))^2 dx \to \int (u'(x))^2 dx$.

Finally, by the lower semi-continuity of the L^2 norm with respect to weak convergence, we have $\liminf_k \int_{\mathbf{R}} |u_k''|^2 \geq \int_{\mathbf{R}} |u''|^2$. We conclude that

$$\liminf_k \frac{1}{2} \int_{\mathbf{R}} |u_k''|^2 - b|u_k'|^2 dx - \frac{1}{p+1} \int_{\mathbf{R}} |u_k|^{p+1} dx \geq \frac{1}{2} \int_{\mathbf{R}} |u''|^2 - b|u'|^2 dx - \frac{1}{p+1} \int_{\mathbf{R}} |u|^{p+1} dx,$$

whence we have that $m_b(\lambda) \geq I[u]$, therefore $I(u) = m_b(\lambda)$ and u is a minimizer.

2.3.3 Euler-Lagrange equation

Proposition 3. *Let $p \in (1,9), \lambda > 0$, be so that one of the following holds*

- $1 < p < 5, \lambda > 0$,
- $5 \leq p < 9, \lambda > \lambda_{b,p} > 0$.

Then, there exists a function $\omega(\lambda) > 0$, so that the minimizer of the constrained minimization problem (2.1.8) *$\phi = \phi_\lambda$, constructed in Section 2.3.2, satisfies the Euler-Lagrange equation*

$$\phi_\lambda'''' + b\phi_\lambda'' - |\phi_\lambda|^{p-1}\phi_\lambda + \omega(\lambda)\phi_\lambda = 0, \tag{2.3.22}$$

where

$$\omega(\lambda) = \frac{1}{\lambda} \int_{\mathbf{R}} b(\phi_\lambda')^2 + |\phi_\lambda|^{p+1} - (\phi_\lambda'')^2 dx.$$

In addition, $n(\mathscr{L}_+) = 1$, that is $\mathscr{L}_+$ has exactly one negative eigenvalue. In fact $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$.

Proof. We have shown that minimizers for the constrained minimization problem exist in the two cases described above, for both $b > 0$ and $b < 0$.

Consider $u_\delta = \sqrt{\lambda} \frac{\phi_\lambda + \delta h}{\|\phi_\lambda + \delta h\|}$, where h is a test function. Note that $\|u_\delta\|_{L^2}^2 = \lambda$, so it satisfies the constraint. Expanding $I[u_\delta]$ in powers of δ we obtain

$$
\begin{aligned}
I[u_\delta] = m_b(\lambda) + \delta \Bigg[& \int_{\mathbf{R}} \phi_\lambda'' h'' - b h' \phi_\lambda' - h|\phi_\lambda|^{p-1}\phi_\lambda dx \\
& + \frac{1}{\lambda} \int_{\mathbf{R}} b(\phi_\lambda')^2 + |\phi_\lambda|^{p+1} - (\phi_\lambda'')^2 dx \int_{\mathbf{R}} \phi_\lambda h dx \Bigg] \\
& + \frac{\delta^2}{2} \left[\int_{\mathbf{R}} (h'')^2 - b(h')^2 - ph^2 |\phi_\lambda|^{p-1} dx \right] \\
& + \frac{\delta^2}{\lambda} \langle h, \phi \rangle \int_{\mathbf{R}} (p+1) h |\phi|^{p-1}\phi + 2bh'\phi_\lambda' - 2h''\phi_\lambda'' dx \\
& + \frac{\delta^2}{2\lambda^2} \langle h, \phi \rangle^2 \int_{\mathbf{R}} (p+3) |\phi_\lambda|^{p+1} + 4b(\phi_\lambda')^2 - 4(\phi_\lambda'')^2 dx + \\
& + \frac{\delta^2}{2\lambda} \|h\|^2 \int_{\mathbf{R}} |\phi_\lambda|^{p+1} + b(\phi_\lambda')^2 - (\phi_\lambda'')^2 dx + O(\delta^3).
\end{aligned}
$$

Using only the first order in δ information and the fact that $I[u_\delta] \geq m_b(\lambda)$ for all $\delta \in \mathbf{R}$, we conclude that

$$\langle \phi_\lambda'''' + b\phi_\lambda'' - |\phi_\lambda|^{p-1}\phi_\lambda + \omega(\lambda)\phi_\lambda, h \rangle = 0,$$

where $\omega(\lambda) = \frac{1}{\lambda} \int_{\mathbf{R}} b(\phi_\lambda')^2 + |\phi_\lambda|^{p+1} - (\phi_\lambda'')^2 dx$. Since this is true for any test function h, we conclude that ϕ_λ is a distributional solution of the Euler-Lagrange equation (2.3.22). According to Proposition 2, this turns out to be a solution in stronger sense, in particular $\phi_\lambda \in H^4(\mathbf{R})$.

Now, using the fact that the function $g_h(\delta) := I[u_\delta]$ has a minimum at zero, we also conclude that $g_h''(0) \geq 0$. This is of course valid for all h, but in order to simplify the expression, we only look at $h : \|h\| = 1$, which are orthogonal to the wave ϕ_λ, i.e. $\langle h, \phi_\lambda \rangle = 0$. This implies that

$$\langle h'''' + bh'' + \omega(\lambda)h - p|\phi_\lambda|^{p-1}h, h \rangle \geq 0.$$

In other words, $\langle \mathscr{L}_+ h, h \rangle \geq 0$, whenever $h : \|h\| = 1, \langle h, \phi_\lambda \rangle = 0$, that is exactly the claim about $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$. In particular, this implies that the second smallest eigenvalue of $\mathscr{L}_+$ is non-negative or $n(\mathscr{L}_+) \leq 1$. On the other hand, since $\langle \mathscr{L}_+ \phi_\lambda, \phi_\lambda \rangle = -(p-1) \int |\phi_\lambda(x)|^{p+1} dx < 0$, it follows that there is a negative eigenvalue or $n(\mathscr{L}_+) = 1$. $\square$

2.3.4 Norms of the minimizers are controlled

We have the following technical proposition, which will be useful in the sequel.

Proposition 4. *Let λ, p satisfy the assumptions in Theorem 1. Then, there exists C_b, D_b, so that*

$$\int_{\mathbf{R}} |\phi_\lambda''|^2 + \int_{\mathbf{R}} |\phi_\lambda'|^2 + \int_{\mathbf{R}} |\phi_\lambda|^{p+1} \leq C_b(1+\lambda^{D_b}). \tag{2.3.23}$$

Proof. By (2.3.11), $p < 9$ and the Gagliardo-Nirenberg's inequality $\|\phi'\|_{L^2}^2 \leq C\|\phi''\|_{L^2}\|\phi\|_{L^2}$, we have that for all $\varepsilon > 0$, there is C_ε,

$$\|\phi_\lambda\|_{L^{p+1}}^{p+1} \leq \varepsilon\|\phi_\lambda''\|_{L^2}^2 + C_\varepsilon \lambda^D.$$

Thus,

$$0 \geq m(\lambda) = I[\phi_\lambda] > \frac{1}{4}\|\phi_\lambda''\|_{L^2}^2 - C\lambda^D.$$

This yields the inequality for $\|\phi_\lambda''\|_{L^2}^2$. For all the others, we use the GNS bounds and (2.3.11). □

2.4 Variational construction in higher dimensions

In this section, we follow the approach and constructions from Section 2.3. Most, if not all, of the steps go through essentially unchanged, save for the numerology, which is of course impacted by the dimension d. Thus, we will be just indicating the main points, without providing full details, where the arguments follow closely the one dimensional case. We work with

$$\begin{cases} I[\phi] = \frac{1}{2}\int_{\mathbf{R}^d} [|\Delta\phi(x)|^2 - \varepsilon|\vec{b}|^2|\partial_{x_1}\phi(x)|^2]dx - \frac{1}{p+1}\int_{\mathbf{R}^d} |\phi(x)|^{p+1}dx \to \min, \\ \int_{\mathbf{R}^d} \phi^2(x)dx = \lambda. \end{cases} \tag{2.4.1}$$

Again, we introduce

$$m_{\vec{b}}(\lambda) = \inf_{\phi\in H^2\cap L^{p+1}, \|\phi\|_2^2=\lambda} I[\phi].$$

Noting that

$$\frac{m_{\vec{b}}(\lambda)}{\lambda} = \inf_{\|\phi\|_2^2=1} \left\{ \frac{1}{2} \int_{\mathbf{R}^d} [|\Delta\phi(x)|^2 - \varepsilon|\vec{b}|^2|\partial_{x_1}\phi(x)|^2] dx - \frac{\lambda^{\frac{p-1}{2}}}{p+1} \int_{\mathbf{R}^d} |\phi(x)|^{p+1} dx \right\}, \qquad (2.4.2)$$

and hence $\lambda \to \frac{m_{\vec{b}}(\lambda)}{\lambda}$ is non-increasing, we conclude that $m_{\vec{b}}(\lambda)$ is differentiable a.e..

As before, we split our discussion in the cases $\varepsilon = 1$, $\varepsilon = -1$.

2.4.1 The case $\varepsilon = -1$

We have the following regarding $m_{\vec{b}}(\lambda)$.

Lemma 12. *Let* $\varepsilon = -1$. *then,*

- *for* $p \in (1, 1 + \frac{8}{d+1})$ *and* $\lambda > 0$, *we have that* $-\infty < m_{\vec{b}}(\lambda) < 0$,
- *for* $p \in (1, 1 + \frac{8}{d})$, $m_{\vec{b}}(\lambda) > -\infty$,
- *for* $p > 1 + \frac{8}{d}$, $m_{\vec{b},\lambda} = -\infty$ *for all* $\lambda > 0$.
- *for* $p = 1 + \frac{8}{d}$, $m_{\vec{b},\lambda} = -\infty$ *for all* λ *large enough.*

Proof. The proof goes through the same steps as in Lemma 5. Pick $\phi_\delta = \delta^{\frac{d+1}{2}} \phi(\delta^2 x_1, \delta x')$, with $\|\phi\|_{L^2}^2 = \lambda$. Clearly, $\|\phi_\delta\|_{L^2}^2 = \lambda$, while

$$I[\phi_\delta] = \frac{\delta^4 \|\Delta' \phi\|^2 + \delta^8 \|\partial_{x_1}^2 \phi\|_{L^2}^2}{2} + \frac{|\vec{b}|^2 \|\phi_{x_1}\|^2}{2} \delta^4 - \frac{\|\phi\|_{L^{p+1}}^{p+1}}{p+1} \delta^{\frac{(d+1)(p-1)}{2}}.$$

Since for δ small enough and $p < 1 + \frac{8}{d+1}$, the last term is dominant, we have $m_b(\lambda) < 0$. Similarly, using $\psi_\delta = \delta^{\frac{d}{2}} \phi(\delta x)$ we obtain

$$I[\psi_\delta] = \frac{\delta^4 \|\Delta\phi\|^2 + \delta^2 |\vec{b}|^2 \|\phi_{x_1}\|^2}{2} - \frac{\|\phi\|_{p+1}^{L^{p+1}}}{p+1} \delta^{\frac{d(p-1)}{2}},$$

and taking the limit $\delta \to \infty$ yields $m_b(\lambda) = -\infty$, for $p > 1 + \frac{8}{d}$.

Next, by GNS, we have that

$$\|\phi\|_{L^{p+1}(\mathbf{R}^d)} \leq C_p \|\phi\|_{\dot{H}^{d(\frac{1}{2}-\frac{1}{p+1})}} \leq C_p \|\phi\|_{L^2}^{1-d(\frac{1}{4}-\frac{1}{2(p+1)})} \|\Delta\phi\|_{L^2}^{d(\frac{1}{4}-\frac{1}{2(p+1)})}.$$

Thus,

$$\begin{aligned} I[\phi] &= \frac{1}{2}\int_{\mathbf{R}^d} [|\Delta\phi(x)|^2 + |\vec{b}|^2 |\partial_{x_1}\phi(x)|^2] dx - \frac{1}{p+1}\int_{\mathbf{R}^d} |\phi(x)|^{p+1} dx \\ &\geq \frac{1}{2}\int_{\mathbf{R}^d} |\Delta\phi|^2 + |\vec{b}|^2 |\partial_{x_1}\phi(x)|^2 dx - c_p \|\Delta\phi\|_{L^2}^{d\frac{p-1}{4}} \|\phi\|_{L^2}^{p+1-d\frac{p-1}{4}} \\ &\geq \frac{1}{4}\|\Delta\phi\|_{L^2}^2 - c_{p,\lambda,b}\|\Delta\phi\|_{L^2}^{d\frac{p-1}{4}} \geq -\gamma, \end{aligned}$$

where in the last inequality, we have used that $p < 1 + \frac{8}{d}$ (whence $d\frac{p-1}{4} < 2$). Therefore, $\|\Delta\phi\|_{L^2}^2$ is dominant. The fact that $m_b(\lambda) = -\infty$ for λ large enough, when $p = 1 + \frac{8}{d}$ follows in the same fashion as in Lemma 5. □

Next, we present a technical lemma.

Lemma 13. *For $1 + \frac{8}{d+1} \leq p \leq 1 + \frac{8}{d}$, the following inequality holds*

$$\|g\|_{L^{p+1}(\mathbf{R}^d)}^{p+1} \leq C_p \|g\|_{L^2}^{p-1} \int_{\mathbf{R}^d} |\Delta g|^2 + |\partial_{x_1} g|^2 dx. \tag{2.4.3}$$

For $p \in (1, 1 + \frac{8}{d+1})$, such an estimate cannot hold.

Proof. We apply the Sobolev embedding in the variables x_1 and then in $x' = (x_2, \ldots, x_d)$

$$\|g\|_{L^{p+1}(\mathbf{R}^d)} \lesssim \||\nabla_{x'}|^{(d-1)(\frac{1}{2}-\frac{1}{p+1})} |\nabla_{x_1}|^{(\frac{1}{2}-\frac{1}{p+1})} g\|_{L^2(\mathbf{R}^d)}. \tag{2.4.4}$$

Next, by Plancherel's, Hölder's inequality and Young's inequality

$$\||\nabla_{x'}|^{(d-1)(\frac{1}{2}-\frac{1}{p+1})}|\nabla_{x_1}|^{(\frac{1}{2}-\frac{1}{p+1})}g\|_{L^2(\mathbf{R}^d)} = \left(\int_{\mathbf{R}^d}|\hat{g}(\xi)|^2|\xi'|^{(d-1)(1-\frac{2}{p+1})}|\xi_1|^{1-\frac{2}{p+1}}d\xi\right)^{1/2}$$
$$\lesssim \|g\|_{L^2}^{\frac{p-1}{p+1}}\left(\int_{\mathbf{R}^d}|\hat{g}(\xi)|^2|\xi'|^{(d-1)\frac{p-1}{2}}|\xi_1|^{\frac{p-1}{2}}d\xi\right)^{\frac{1}{p+1}}$$
$$\lesssim \|g\|_{L^2}^{\frac{p-1}{p+1}}\left(\int_{\mathbf{R}^d}|\hat{g}(\xi)|^2[|\xi'|^4+|\xi_1|^{\frac{q'(p-1)}{2}}]d\xi\right)^{\frac{1}{p+1}},$$

where $q=\frac{8}{(d-1)(p-1)}$. Clearly, (2.4.3) follows, provided $2\leq\frac{q'(p-1)}{2}\leq 4$. Solving this inequality yields exactly $1+\frac{8}{d+1}\leq p\leq 1+\frac{8}{d}$.

If $p<1+\frac{8}{d+1}$, take $\phi=\chi(\varepsilon^2x_1,\varepsilon x')$ in (2.4.3). Assuming the validity of (2.4.3), we obtain that $\varepsilon^{(d+1)\frac{p-1}{2}}\leq const(\varepsilon^4+\varepsilon^8)$. This is a contradiction for $\varepsilon<<1$ and $p\in(1,1+\frac{8}{d+1})$.

□

The next two lemmas are the generalizations of Lemma 6 and Lemma 7 to higher dimensions.

Lemma 14. *If $\varepsilon=-1$ and $p\in[1+\frac{8}{d+1},1+\frac{8}{d})$, then there exists a finite number $\lambda_{\vec{b},p}>0$ such that*

- *for all $\lambda\leq\lambda_{\vec{b},p}$ we have $m_b(\lambda)=0$,*
- *for all $\lambda>\lambda_{\vec{b},p}$ we have $-\infty<m_b(\lambda)<0$.*

Proof. The inequality $m(\lambda)\leq 0$ follows in the same way as in Lemma 6. Then, by Lemma 13, we have

$$\inf_{\phi\neq 0}\frac{\|\phi\|_{L^2}^{p-1}\int_{\mathbf{R}^d}[|\Delta\phi|^2-\varepsilon|\vec{b}|^2|\phi_{x_1}|^2]dx}{\int_{\mathbf{R}^d}|\phi|^{p+1}dx}\geq c_{\vec{b},p}>0. \tag{2.4.5}$$

Thus, for all $\phi\in H^2(\mathbf{R}^d)$, we have

$$\int_{\mathbf{R}^d}[|\Delta\phi|^2-\varepsilon|\vec{b}|^2|\phi_{x_1}|^2]dx-\frac{c_{\vec{b}p}}{\lambda^{p-1}}\int_{\mathbf{R}^d}|\phi|^{p+1}dx\geq 0,$$

which by (2.4.2) implies that for $\lambda\leq\lambda_{\vec{b},p}:=\left(\frac{c_{\vec{b},p}(p+1)}{2}\right)^{\frac{2}{p-1}}$, $m_{\vec{b}}(\lambda)\geq 0$. Since we always have the opposite inequality, this implies $m_{\vec{b}}(\lambda)=0$, when λ is small enough. Note that for very large λ, the quantity in (2.4.2) is clearly negative, so this implies that $\lambda_{\vec{b},p}<\infty$. □

The next lemma is the generalization of Lemma 7 to the higher dimensional case. Its proof follows similar path and it is thus omitted.

Lemma 15. *Suppose $\varepsilon = -1$, $p \in (1, 1+\frac{8}{d})$ and $-\infty < m_b(\lambda) < 0$. That is, one of the following holds:*

- $p \in (1, 1+\frac{8}{d+1}), \lambda > 0$,
- $p \in [1+\frac{8}{d+1}, 1+\frac{8}{d})$ *and* $\lambda > \lambda_{\vec{b},p}$.

Let $\{\phi_n\}_{n=1}^{\infty}$ be a minimizing sequence for the constrained minimization problem (2.4.1). *Then, there exists a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$ such that:*

$$\lim_{k\to\infty}\int_{\mathbf{R}^d} |\Delta\phi_{n_k}(x)|^2 dx = L_1, \quad \lim_{k\to\infty}\int_{\mathbf{R}^d} |\partial_{x_1}\phi_{n_k}(x)|^2 dx = L_2, \quad \lim_{k\to\infty}\int_{\mathbf{R}^d} |\phi_{n_k}(x)|^{p+1} dx = L_3,$$

where $L_1 > 0$, $L_2 > 0$ and $L_3 > 0$.

We now turn to the case $\varepsilon = 1$.

2.4.2 The case $\varepsilon = 1$

The first observation is that for $\phi_\delta(x) = \delta^{\frac{d}{2}}\phi(\delta x)$, we have

$$I[\phi_\delta] = \delta^4 \frac{\|\Delta\phi\|_{L^2}^2}{2} - \delta^2 \frac{\|\partial_{x_1}\phi\|_{L^2}^2}{2} - \delta^{d\frac{p-1}{2}} \frac{\|\phi\|_{L^{p+1}}^{p+1}}{p+1}.$$

Clearly for $p \in (1, 1+\frac{8}{d})$ and $0 < \delta << 1$, we conclude that $m_b(\lambda) < 0$. Boundedness from below follows from the estimate

$$I[\phi] \geq \frac{1}{4}\|\Delta\phi\|_{L^2}^2 - c_{p,\lambda,b}(\|\Delta\phi\|_{L^2}^{d\frac{p-1}{4}} + 1) \geq -\gamma,$$

established earlier. Hence, we have shown the following.

Lemma 16. *If $1 < p < 1+\frac{8}{d}$ and $\varepsilon = 1$, then $-\infty < m_{\vec{b},p}(\lambda) < 0$.*

Next, we have a generalization of Lemma 9 to the case $d > 1$.

Lemma 17. *Let $p : 1 < p < 1 + \frac{8}{d+1}$, $\varepsilon = 1, \kappa \neq 0$ and fix a constant c. Then, the inequality*

$$\|\phi\|_{L^{p+1}}^{p+1} \leq c\|\phi\|_{L^2}^{p-1} \left[\int_{\mathbf{R}^d} |\Delta\phi(x)|^2 - 2\kappa^2 |\partial_{x_1}\phi(x)|^2 + \kappa^4 |\phi(x)|^2 dx \right] \tag{2.4.6}$$

cannot hold for all $\phi \in H^2(\mathbf{R}^d)$.

For $p \in [1 + \frac{8}{d+1}, 1 + \frac{8}{d}]$ and $\varepsilon = 1$, there is a $c = c(\kappa, d)$, so that

$$\|\phi\|_{L^{p+1}}^{p+1} \leq c\|\phi\|_{L^2}^{p-1} \left[\int_{\mathbf{R}^d} |\Delta\phi(x)|^2 - 2\kappa^2 |\partial_{x_1}\phi(x)|^2 + \kappa^4 |\phi(x)|^2 dx \right] \tag{2.4.7}$$

holds for all $\phi \in H^2(\mathbf{R}^d)$.

Proof. Note that to prove (2.4.6) it is enough to prove a stronger inequality

$$\|\phi\|_{L^{p+1}}^{p+1} \leq c\|\phi\|_{L^2}^{p-1} \left[\int_{\mathbf{R}^d} |\hat{\phi}(\xi)|^2 \left(|\xi'|^4 + (\xi_1^2 - \kappa^2)^2 \right) d\xi \right].$$

Thus, one introduces a function $g : \hat{g}(\xi_1 - \kappa, \xi') = \phi(\xi)$, so that (2.4.6) is now equivalent to

$$\|g\|_{L^{p+1}}^{p+1} \leq c\|g\|_{L^2}^{p-1} \left[\int_{\mathbf{R}^d} |\hat{g}(\xi)|^2 \left(|\xi'|^4 + |\xi_1|^2 |\xi_1 + 2\kappa|^2 \right) d\xi \right]. \tag{2.4.8}$$

According to the estimate in Lemma 13, we have, with $q = \frac{8}{(d-1)(p-1)}$,

$$\|\phi\|_{L^{p+1}} \leq \|g\|_{L^2}^{\frac{p-1}{p+1}} \left(\int_{\mathbf{R}^d} |\hat{g}(\xi)|^2 \left(|\xi'|^4 + |\xi_1|^{\frac{q'(p-1)}{2}} \right) d\xi \right)^{\frac{1}{p+1}}.$$

Again, this implies (2.4.8), provided $2 \leq \frac{q'(p-1)}{2} \leq 4$ or $1 + \frac{8}{d+1} \leq p \leq 1 + \frac{8}{d}$. The contradiction in the case $1 < p < 1 + \frac{8}{d+1}$ is obtained in the same way as in the proof of Lemma 13.

□

Our next lemma is a generalization of Lemma 10. Its proof follows verbatim the proof of Lemma 10, where one needs to make some adjustments to account for the dimension.

Lemma 18. *Suppose $\varepsilon = 1$, $\lambda > 0$ and $1 < p < 1 + \frac{8}{d}$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a minimizing sequence*[11] *for the constrained minimization problem* (2.4.1). *In addition, assume one of the following:*

- *$p \in (1, 1 + \frac{8}{d+1})$, $\lambda > 0$,*
- *$p \in [1 + \frac{8}{d+1}, 1 + \frac{8}{d})$ and λ is sufficiently large.*

Then, there exists a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$, such that:

$$\lim_{k\to\infty} \frac{1}{2} \int_{\mathbf{R}^d} |\Delta\phi_{n_k}(x)|^2 = L_1, \quad \lim_{k\to\infty} \int_{\mathbf{R}^d} |\partial_{x_1}\phi_{n_k}(x)|^2 = L_2 \text{ and } \lim_{k\to\infty} \int_{\mathbf{R}^d} |\phi_{n_k}|^{p+1} dx = L_3,$$

where $L_1 > 0$, $L_2 > 0$ and $L_3 > 0$.

2.4.3 Existence of minimizers

Before we go ahead with the existence of minimizers, we need analogs of Lemma 11 and Lemma 23. Their proofs in the higher dimensional case goes in an identical manner.

Lemma 19. *Suppose the assumptions of Lemma 15 and Lemma 18 on λ, p and d hold. Then $\lambda \to m_{\vec{b},p}(\lambda)$ is strictly sub-additive. That is, for every $\alpha \in (0, \lambda)$,*

$$m_{\vec{b},p}(\lambda) < m_{\vec{b},p}(\alpha) + m_{\vec{b},p}(\lambda - \alpha).$$

In addition, $\lambda \to m_{\vec{b},p}(\lambda)$ is twice differentiable a.e..

With the basic results in place, we can now proceed to establish the existence of the minimizers of (2.4.1). Supposing

$$\begin{cases} 1 < p < 1 + \frac{8}{d+1}, & \lambda > 0, \\ 1 + \frac{8}{d+1} \le p < 1 + \frac{8}{d}, & \lambda > \lambda_{b,p}, \end{cases}$$

we take a minimizing sequence $\{\phi_k\}_{k=1}^{\infty} \subset H^2(\mathbf{R}^d)$, with $I[\phi_k] \to m_{\vec{b},p}(\lambda)$ as $k \to \infty$. Possibly passing to a subsequence, using either Lemma 15 for $\varepsilon = -1$ or Lemma 18 for $\varepsilon = 1$, we can

[11] According to Lemma 16, $m(\lambda)$ is well-defined, hence such a sequence always exists.

assume that[12]

$$\lim_{k\to\infty}\frac{1}{2}\int_{\mathbf{R}^d}|\Delta\phi_k(x)|^2=L_1,\quad \lim_{k\to\infty}\int_{\mathbf{R}^d}|\partial_{x_1}\phi_k(x)|^2=L_2 \text{ and } \lim_{k\to\infty}\int_{\mathbf{R}^d}|\phi_k|^{p+1}dx=L_3,$$

where $L_1>0$, $L_2>0$ and $L_3>0$. The next task is to show that this sequence does not split nor vanish. The absence of splitting is established in the same way as the first part of Section 2.3.2.

Next, we rule out vanishing. The proof presented in Section 2.3.2 works for $d=1,2,3,4$, but breaks down in $d\geq 5$, so let us present another one that works in all dimensions. More concretely, for all $R>0$ and $y\in\mathbf{R}^d$ and a cutoff function η introduced in Section 2.3.2, we have by the GNS inequality

$$\begin{aligned}\|\phi_k\|_{L^{p+1}(B(y,R))}^{p+1} &\leq \int_{\mathbf{R}^d}|\phi_k(x)\eta\left(\frac{|x-y|}{R}\right)|^{p+1}dx\lesssim\|\phi_k\eta_R\|^{p+1}_{\dot{H}^{d\left(\frac{1}{2}-\frac{1}{p+1}\right)}}\\ &\lesssim \|\Delta[\phi_k\eta_R]\|_{L^2}^{(p+1)\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|\phi_k\eta_R\|_{L^2}^{(p+1)-(p+1)\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)}.\end{aligned}$$

Since $p<1+\frac{8}{d}$, it follows that $(p+1)\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)<2$. In addition $\|\phi_k\eta_R\|_{L^2}\leq\|\phi_k\|_{L^2(B(y,2R)}$, whence

$$\|\phi_k\|_{L^{p+1}(B(y,R))}^{p+1}\leq C_{R,\eta}\|\phi_k\|^2_{H^2(B(y,2R))}\|\phi_k\|^{p-1}_{L^2(B(y,2R))}.$$

So, if we assume that vanishing occurs, then for every $\varepsilon>0$, we will be able to cover $\mathbf{R}^d$ with balls of radius 1, say $B(y_j,1)$, so that $\int_{B(y_j,3)}|\phi_k(x)|^2dx<\varepsilon$. Then,

$$\begin{aligned}\|\phi_k\|_{L^{p+1}(\mathbf{R}^d)}^{p+1} &\leq \sum_{j=1}^{\infty}\int_{B(y_j,1)}|\phi_k|^{p+1}dx\leq\sum_{j=1}^{\infty}C_{\eta,R}\|\phi_k\|^2_{H^2(B(y_j,2))}\|\phi_k\|^{p-1}_{L^2(B(y_j,2))}\\ &\leq 10C_{\eta,R}\varepsilon^{\frac{p-1}{2}}\|\phi_k\|^2_{H^2(\mathbf{R}^d)}.\end{aligned}$$

Clearly, since $\|\phi_k\|_{H^2(\mathbf{R}^d)}$ is uniformly bounded in k, we conclude that $\|\phi_k\|_{L^{p+1}}\to 0$, which is in a contradiction with $\lim_k\int_{\mathbf{R}^d}|\phi_k|^{p+1}dx=L_3>0$.

From here, it follows that the sequence $\rho_k=|\phi_k(x)|^2$ is tight and the existence of the minimizer

[12] For conciseness, we use ϕ_k instead of ϕ_{n_k}.

is done as in Section 2.3.2.

The Euler-Lagrange equation, together with the appropriate properties of the linearized operators is done similar to Proposition 3.

Proposition 5. *Let $p \in (1, 1+\frac{8}{d}), \lambda > 0$, be so that one of the following holds:*

- $1 < p < 1+\frac{8}{d+1}, \lambda > 0$,
- $1+\frac{8}{d+1} \le p < 1+\frac{8}{d}, \lambda > \lambda_{b,p} > 0$.

Then, there exists a function $\omega(\lambda) > 0$, so that the minimizer of the constrained minimization problem (2.4.1)*, $\phi = \phi_\lambda$, satisfies the Euler-Lagrange equation*

$$\Delta^2 \phi_\lambda + \varepsilon |\vec{b}|^2 \partial_{x_1}^2 \phi_\lambda - |\phi_\lambda|^{p-1}\phi_\lambda + \omega(\lambda)\phi_\lambda = 0. \tag{2.4.9}$$

In addition, $n(\mathscr{L}_+) = 1$, that is $\mathscr{L}_+$ has exactly one negative eigenvalue. Finally, $\mathscr{L}_- \ge 0$, with a simple eigenfunction at zero, i.e. $Ker[\mathscr{L}_-] = span[\phi_\lambda]$.

As we mentioned above, the proof goes along the lines of Proposition 3. The only new element are the statements about $\mathscr{L}_-$, which we now prove.

Note that by direct inspection, $\mathscr{L}_-[\phi_\lambda] = 0$, by (2.4.9), so zero is an eigenvalue. Assuming that there is a negative eigenvalue, say $\mathscr{L}_-[\psi] = -\sigma^2\psi, \|\psi\| = 1$, we clearly would have $\psi \perp \phi_\lambda$. In addition, since[13] $\mathscr{L}_+ < \mathscr{L}_-$,

$$\langle \mathscr{L}_+\psi, \psi\rangle < \langle \mathscr{L}_-\psi, \psi\rangle = -\sigma^2,$$
$$\langle \mathscr{L}_+\phi_\lambda, \phi_\lambda\rangle < 0.$$

But then $\mathscr{L}_+|_{span\{\psi,\phi_\lambda\}} < 0$, and $dim(span\{\psi,\phi_\lambda\}) = 2$. This would force $n(\mathscr{L}_+) \ge 2$, a contradiction. Thus, $\mathscr{L}_- \ge 0$. Finally, 0 is a simple eigenvalue of $\mathscr{L}_-$ along the same line of reasoning.

[13]This is an obvious statement, once we realize that ϕ_λ cannot vanish on an interval. Indeed, otherwise, since it solves the fourth order equation (2.4.9), it follows that ϕ_λ is trivial, which it is not.

Indeed, take $\psi : \mathscr{L}_- \psi = 0, \psi \perp \phi_\lambda$. Again, we conclude $\mathscr{L}_+|_{span\{\psi,\phi_\lambda\}} < 0$, which leads to a contradiction.

2.4.4 Discussion of the proof of Theorem 5: existence of the waves

We do not provide an extensive review of the existence claims in Theorem 5, as this would be repetitious, but we would like to make a few notable points. In particular, we would like to clarify the range of indices in p. More concretely, we have the following analogue of Lemmas 13.

Lemma 20. *For* $1 + \frac{4}{d} \leq p < 1 + \frac{8}{d}$,

$$\|g\|_{L^{p+1}(\mathbf{R}^d)}^{p+1} \leq C_p \|g\|_{L^2}^{p-1} \int_{\mathbf{R}^d} |\Delta g|^2 + |\nabla g|^2 dx. \tag{2.4.10}$$

For $p \in (1, 1 + \frac{4}{d})$*, such an estimate cannot hold.*

The proof proceeds in a similar fashion, so we omit it. A combination of arguments in the flavor of the proofs for Lemma 12 and Lemma 14 leads us to the following variant of Lemma 14 and Lemma 15.

Lemma 21. *If* $b < 0$ *and* $p \in [1 + \frac{4}{d}, 1 + \frac{8}{d})$*, then there exists a finite number* $\lambda_{b,p} > 0$ *so that*

- *for all* $\lambda \leq \lambda_{b,p}$ *we have* $m_b(\lambda) = 0$*,*
- *for all* $\lambda > \lambda_p$ *we have* $-\infty < m_b(\lambda) < 0$*.*

In addition, assume that $-\infty < m_b(\lambda) < 0$*, that is one of the following holds:*

- $p \in (1, 1 + \frac{4}{d}), \lambda > 0$*,*
- $p \in [1 + \frac{4}{d}, 1 + \frac{8}{d})$ *and* $\lambda > \lambda_{b,p}$*.*

Let $\{\phi_n\}_{n=1}^{\infty}$ *be a minimizing sequence for the constrained minimization problem*

$$\begin{cases} I[\phi] = \frac{1}{2} \int_{\mathbf{R}} |\Delta \phi(x)|^2 - b|\nabla \phi(x|^2 dx - \frac{1}{p+1} \int_{\mathbf{R}} |\phi(x)|^{p+1} dx \to \min, \\ \int_{\mathbf{R}} \phi^2(x) dx = \lambda, \end{cases} \tag{2.4.11}$$

then there exists a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$ *such that:*

$$\lim_{k\to\infty}\int_{\mathbf{R}^d}|\Delta\phi_{n_k}(x)|^2dx = L_1, \quad \lim_{k\to\infty}\int_{\mathbf{R}^d}|\nabla\phi_{n_k}(x)|^2dx = L_2, \quad \lim_{k\to\infty}\int_{\mathbf{R}^d}|\phi_{n_k}(x)|^{p+1}dx = L_3,$$

where $L_1 > 0$*,* $L_2 > 0$ *and* $L_3 > 0$*.*

With these tools at hand, the existence of the waves follows in the same manner as before, so we omit the details.

2.5 Stability of the normalized waves

Interestingly, the proof of the spectral stability proceeds by a common argument, both for the Kawahara and the fourth order NLS case. From corollaries 1 and 2, it suffices to show that $n(\mathscr{L}_+) = 1$, $\mathscr{L}_- \geq 0$, $\phi_\lambda \perp Ker[\mathscr{L}_+]$ and to verify that $\langle \mathscr{L}_+^{-1}\phi_\lambda, \phi_\lambda\rangle < 0$. Indeed, the condition $n(\mathscr{L}_+) = 1$ was already verified as part of the variational construction, see Proposition 3 and 5. Similarly, $\mathscr{L}_- \geq 0$ was verified in the higher dimensional case in Proposition 5.

First, we show the weak non-degeneracy.

Lemma 22. *For each constrained minimizer* ϕ_λ*, we have that* $\phi_\lambda \perp Ker[\mathscr{L}_+]$*.*

Proof. Take any element of $Ker[\mathscr{L}_+]$, say $\Psi : \|\Psi\|_{L^2} = 1$. We need to show $\langle\Psi, \phi_\lambda\rangle = 0$. To this end, consider $\Psi - \|\phi_\lambda\|^{-2}\langle\Psi, \phi_\lambda\rangle\phi_\lambda \perp \phi_\lambda$. Recall that due to the construction $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$. We have

$$0 \leq \langle \mathscr{L}_+[\Psi - \|\phi_\lambda\|^{-2}\langle\Psi, \phi_\lambda\rangle\phi_\lambda\lambda], \Psi - \|\phi_\lambda\|^{-2}\langle\Psi, \phi_\lambda\rangle\phi_\lambda\rangle = \|\phi_\lambda\|^{-4}\langle\Psi, \phi_\lambda\rangle^2\langle\mathscr{L}_+\phi_\lambda, \phi_\lambda\rangle \leq 0,$$

where we have used that $\langle\mathscr{L}_+\phi_\lambda, \phi_\lambda\rangle = -(p-1)\int|\phi_\lambda|^{p+1} < 0$. The only way the last chains of inequalities is non-contradictory, is if $\langle\Psi, \phi_\lambda\rangle = 0$, which is the claim. $\square$

Apply Lemma 1 to the vector $\xi_0 := \phi_\lambda$ and the operator $\mathscr{H} := \mathscr{L}_+$. Recall that the construction of ϕ_λ involved the property $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$. By Lemma 22, we have that $\phi_\lambda \perp Ker[\mathscr{L}_+]$. Finally, $\langle\mathscr{L}_+\phi_\lambda, \phi_\lambda\rangle < 0$ was used repeatedly. Thus, we conclude that $\langle\mathscr{L}_+^{-1}\phi_\lambda, \phi_\lambda\rangle < 0$.

These arguments establish rigorously the spectral stability of the waves for the Kawahara and the fourth order problems, i.e., Theorem 2 and the stability claims in Theorems 4 and 5.

2.6 Additional properties of the function m: Proof of Theorem 3

In this section, we prove Theorem 3. It is worth mentioning that identical results hold for the multidimensional case as well, but it would be repetitious to prove it separately, so we just restrict our attention to the one dimensional case.

Interestingly, a number of properties has already been established and utilized already for the purposes of the variational construction. For example, we have shown that for λ, p satisfying Theorem 1, we have that $-\infty < m(\lambda) < 0$, see Lemma 5, Lemma 6 for $b < 0$ and Lemma 8 for the case $b > 0$. In Lemma 11, we have established the strict sub-additivity of m, see (2.3.18), $m(\lambda) < m(\alpha) + m(\lambda - \alpha)$, whenever $0 < \alpha < \lambda$. This, together with the fact that $m(\lambda - \alpha) < 0$, implies that m is strictly decreasing. As a strictly decreasing function, m is differentiable at all, but possibly countably many points. It also admits left and right derivatives at each point in $(0, \infty)$.

The remaining claims in Theorem 3 will be proved in a sequence of lemmas.

2.6.1 m is Lipschitz continuous

We start with the following lemma.

Lemma 23. *The function $\lambda \to m_b(\lambda)$ is a Lipschitz continuous function. Moreover, m is twice differentiable a.e. in $(0, \infty)$.*

Proof. The simple proof is based on the representation formula (2.3.1). According to it, set

$$g(\mu) = \inf_{\|\phi\|_2^2 = 1} \left\{ \frac{1}{2} \int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx - \frac{\mu}{p+1} \int_{\mathbf{R}} |\phi|^{p+1} dx \right\},$$

so that $g\left(\lambda^{\frac{p-1}{2}}\right) = \frac{m_b(\lambda)}{\lambda}$. Clearly, the properties of $\lambda \to m_b(\lambda)$ listed in the statement follow from the concavity of the function g, which we are about to prove. So, it suffices to prove that g is concave down.

To this end, denote $\tilde{J}_\mu[\phi] := \frac{1}{2}\int_{\mathbf{R}} |\phi''|^2 - b|\phi'|^2 dx - \frac{\mu}{p+1}\int_{\mathbf{R}} |\phi|^{p+1} dx$. Clearly, for every $a \in (0,1), \mu_1, \mu_2 > 0$, we have

$$\tilde{J}_{a\mu_1+(1-a)\mu_2}[\phi] = a\tilde{J}_{\mu_1}[\phi] + (1-a)\tilde{J}_{\mu_2}[\phi].$$

Hence, taking $\inf_{\|\phi\|_{L^2}=1}$ on both sides

$$\begin{aligned} g(a\mu_1 + (1-a)\mu_2) &= \inf_{\|\phi\|_{L^2}=1} \tilde{J}_{a\mu_1+(1-a)\mu_2}[\phi] \\ &\geq a \inf_{\|\phi\|_{L^2}=1} \tilde{J}_{\mu_1} + (1-a) \inf_{\|\phi\|_{L^2}=1} \tilde{J}_{\mu_2} \\ &= ag(\mu_1) + (1-a)g(\mu_2). \end{aligned}$$

Hence, the function g is concave down, whence twice differentiable a.e..

□

Our next result concerns the derivative of m, whenever it exists.

2.6.2 Computing the derivative of m

Lemma 24. *On the set $\mathscr{A}_m$, $m'(\lambda) = -\frac{\omega(\lambda)}{2}$. Moreover, m is concave down, and there is the inequality,* (2.1.12) *for $\lambda \notin \mathscr{A}_m$. That is,*

$$m'(\lambda+) \leq -\frac{\omega(\lambda, \phi_\lambda)}{2} \leq m'(\lambda-). \tag{2.6.1}$$

In particular, the function $\lambda \to \omega(\lambda, \phi)$ is non-decreasing, in the sense that for every $0 < \lambda_1 < \lambda_2 < \infty$ and for every $\phi_{\lambda_1}, \phi_{\lambda_2}$, we have the inequality

$$\omega(\lambda_1, \phi_{\lambda_1}) \leq \omega(\lambda_2, \phi_{\lambda_2}).$$

Proof. According to Lemma 23, the function m is continuous and differentiable at all but countably many points, at which left and right derivatives still exists. It also has a second derivative a.e. We can now compute the derivative $m'(\lambda)$, whenever it exists. To that end, consider $\phi_\lambda + \varepsilon h$, for any λ. We have, for a fixed test function h,

$$\|\phi_\lambda + \varepsilon h\|_2^2 = \lambda + 2\varepsilon\langle\phi_\lambda, h\rangle + \varepsilon^2\|h\|_2^2,$$

whence according to the definition of $m_b(\cdot)$,

$$I[\phi_\lambda + \varepsilon h] \geq m_b(\|\phi_\lambda + \varepsilon h\|_2^2) = m_b(\lambda + 2\varepsilon\langle\phi_\lambda, h\rangle + \varepsilon^2\|h\|_2^2). \tag{2.6.2}$$

Expanding $I[\phi_\lambda + \varepsilon h]$ in powers of ε yields

$$\begin{aligned}
I[\phi_\lambda + \varepsilon h] &= m_b(\lambda) + \varepsilon\langle\phi_\lambda'''' + b\phi_\lambda'' - |\phi_\lambda|^{p-1}\phi_\lambda, h\rangle + \\
&\quad + \frac{\varepsilon^2}{2}\left[\int_{\mathbf{R}} |h''(x)|^2 - b|h'(x)|^2 - p|\phi_\lambda(x)|^{p-1}\phi_\lambda h dx\right] + O(\varepsilon^3) \\
&= m_b(\lambda) - \varepsilon\omega(\lambda)\langle\phi_\lambda, h\rangle + \frac{\varepsilon^2}{2}\langle(\mathscr{L}_+ - \omega(\lambda))h, h\rangle + O(\varepsilon^3).
\end{aligned}$$

where $\mathscr{L}_+ := \partial_x^4 + b\partial_x^2 - p|\phi_\lambda|^{p-1} + \omega_\lambda$. Take $h = \phi_\lambda$. From (2.6.2) it follows that $m_b(\lambda) - \varepsilon\omega(\lambda)\lambda + O(\varepsilon^2) \geq m_b(\lambda + 2\lambda\varepsilon + O(\varepsilon^2))$, or

$$\frac{m_b(\lambda + 2\lambda\varepsilon + O(\varepsilon^2)) - m_b(\lambda)}{2\lambda} \leq -\varepsilon\frac{\omega(\lambda)}{2} + O(\varepsilon^2).$$

This gives two inequalities. For $\varepsilon > 0$, we obtain

$$\frac{m_b(\lambda + 2\lambda\varepsilon + O(\varepsilon^2)) - m_b(\lambda)}{2\lambda\varepsilon} \leq -\frac{\omega(\lambda)}{2} + O(\varepsilon), \tag{2.6.3}$$

while for $\varepsilon < 0$, we obtain

$$\frac{m_b(\lambda + 2\lambda\varepsilon + O(\varepsilon^2)) - m_b(\lambda)}{2\lambda\varepsilon} \geq -\frac{\omega(\lambda)}{2} + O(\varepsilon). \tag{2.6.4}$$

Taking into account the fact that m_b is Lipschitz, we can write

$$m_b(\lambda + 2\lambda\varepsilon + O(\varepsilon^2)) = m_b(\lambda + 2\lambda\varepsilon) + O(\varepsilon^2).$$

Setting $\varepsilon \to -\varepsilon$ in (2.6.4), we obtain the double inequality for all $\varepsilon > 0$

$$\frac{m_b(\lambda + 2\lambda\varepsilon) - m_b(\lambda)}{2\lambda\varepsilon} + O(\varepsilon) \leq -\frac{\omega(\lambda)}{2} \leq \frac{m_b(\lambda - 2\lambda\varepsilon) - m_b(\lambda)}{-2\lambda\varepsilon} + O(\varepsilon). \tag{2.6.5}$$

Form here, we deduce that if m has a derivative at λ, then clearly $m'(\lambda) = -\frac{\omega(\lambda)}{2}$. Even when m does not have a derivative, i.e. $\lambda \notin \mathscr{A}_m$, we can still take limits in (2.6.5) and conclude that

$$m'(\lambda+) \leq -\frac{\omega(\lambda)}{2} \leq m'(\lambda-).$$

Finally, we derive the concavity of m, but we need to involve the terms $O(\varepsilon^2)$ in (2.6.3), (2.6.4) in our analysis. To this end,

$$\langle (\mathscr{L}_+ - \omega(\lambda))\phi_\lambda, \phi_\lambda \rangle = -\omega(\lambda)\lambda - (p-1)\int |\phi_\lambda|^{p+1} dx,$$

since ϕ_λ satisfies (2.1.4). Thus, we have the inequalities for all $\varepsilon > 0$

$$\frac{m(\lambda + 2\lambda\varepsilon + \lambda\varepsilon^2) - m(\lambda)}{2\lambda\varepsilon} \leq -\frac{\omega(\lambda)}{2} - \frac{\varepsilon}{4\lambda}[\omega(\lambda)\lambda + (p-1)\int |\phi_\lambda|^{p+1}] + O(\varepsilon^2), \tag{2.6.6}$$

$$\frac{m(\lambda - 2\lambda\varepsilon + \lambda\varepsilon^2) - m(\lambda)}{2\lambda\varepsilon} \leq \frac{\omega(\lambda)}{2} - \frac{\varepsilon}{4\lambda}[\omega(\lambda)\lambda + (p-1)\int |\phi_\lambda|^{p+1}] + O(\varepsilon^2). \tag{2.6.7}$$

Setting $\delta = 2\lambda\varepsilon + \lambda\varepsilon^2$ in (2.6.6) and $\delta = 2\lambda\varepsilon - \lambda\varepsilon^2$ in (2.6.7), we can rewrite the previous two relations in the form

$$\frac{m(\lambda+\delta)-m(\lambda)}{\delta} \le \left[-\frac{\omega(\lambda)}{2} - \frac{\delta}{8\lambda^2}[\omega(\lambda)\lambda + (p-1)\int|\phi_\lambda|^{p+1}]\right](1-\frac{\delta}{4\lambda}) + O(\delta^2),$$
$$\frac{m(\lambda-\delta)-m(\lambda)}{\delta} \le \left[\frac{\omega(\lambda)}{2} - \frac{\delta}{8\lambda^2}[\omega(\lambda)\lambda + (p-1)\int|\phi_\lambda|^{p+1}]\right](1+\frac{\delta}{4\lambda}) + O(\delta^2).$$

Here, it is important to observe that the terms $O(\delta^2)$ are bounded by $C\delta^2\int|\phi_\lambda|^{p+1}$. Adding the last two inequalities results in

$$\frac{m(\lambda+\delta)+m(\lambda-\delta)-2m(\lambda)}{\delta^2} \le -\frac{(p-1)}{4\lambda^2}\int|\phi_\lambda|^{p+1} + C\delta\int|\phi_\lambda|^{p+1}. \tag{2.6.8}$$

This immediately implies that whenever $\omega'(\lambda)$ exists, we have the inequality

$$-\frac{\omega'(\lambda)}{2} = \lim_{\delta\to 0+}\frac{m(\lambda+\delta)+m(\lambda-\delta)-2m(\lambda)}{\delta^2} \le -\frac{(p-1)}{4\lambda^2}\int|\phi_\lambda|^{p+1},$$

which implies the estimate (2.1.11).

Now, for each interval $(a,b)\subset \mathbf{R}_+$, we have

$$\limsup_{\delta\to 0+}\sup_{\lambda\in(a,b)}\frac{m(\lambda+\delta)+m(\lambda-\delta)-2m(\lambda)}{\delta^2} \le 0,$$

provided, we can show that $\sup_{\lambda\in(a,b)}\int|\phi_\lambda|^{p+1} \le C_{a,b}$. We can then apply Lemma 3 to the continuous function m to conclude the concavity of m. The bound for $\sup_{\lambda\in(a,b)}\int|\phi_\lambda|^{p+1}dx$ in terms of the function $m(\lambda)$ is contained in (2.3.23).

Lastly, in order to show that $\omega(\lambda)$ is increasing, we observe that for any $\lambda_1<\lambda_2$, by (2.6.1) and the concavity of the function m (so m' is non-increasing),

$$\omega(\lambda_1,\phi_{\lambda_1}) \le -2m'(\lambda_1+) \le -2m'(\lambda_2-) \le \omega(\lambda_2,\phi_{\lambda_2}).$$

$\square$

2.6.3 Proof of Proposition 1

For the justification of the limit waves, we argue as in Section 2.3.2. More specifically, consider the sequence $\phi_{\lambda+\delta_j}$ of constrained minimizers. For it, we have $\|\phi_{\lambda+\delta_j}\|_{L^2}^2 = \lambda + \delta_j \to \lambda$, while from the continuity of $\lambda \to m_b(\lambda)$, we have $I[\phi_{\lambda+\delta_j}] = m(\lambda+\delta_j) \to m(\lambda)$. It follows that $\tilde{\phi}_j := \frac{\phi_{\lambda+\delta_j}}{\sqrt{\lambda+\delta_j}}$, have $\|\tilde{\phi}_j\|_{L^2}^2 = \lambda$ and $\lim_j I[\tilde{\phi}_j] = m(\lambda)$. Thus, $\tilde{\phi}_j$ is a minimizing sequence. By the arguments deployed early for the existence of the minimizers for (2.1.8), there is a subsequence j_k and $y_k \in \mathbf{R}$, $\Phi_\lambda \in H^2$, so that $\lim_k \|\tilde{\phi}_{j_k}(\cdot + y_k) - \Phi_\lambda\|_{H^2(\mathbf{R})} = 0$ and Φ_λ is a minimizer of (2.1.8), since

$$m(\lambda) = \lim_k m(\lambda + \delta_{j_k}) = \lim_k I[\tilde{\phi}_{j_k}] = I[\Phi_\lambda], \|\Phi_\lambda\|^2 = \lim_k \|\tilde{\phi}_{j_k}\|^2 = \lambda.$$

2.6.4 The range of $\lambda \to \omega_\lambda$

Our next lemma establishes the range of $\lambda \to \omega(\lambda, \phi)$.

Lemma 25. *For λ, p satisfying Theorem 1, the function $\lambda \to \omega_\lambda$ is satisfies the inequalities in* (2.1.9).

Remark: Note that our results do not imply that the range of the function ω covers the whole interval described in (2.1.9), since we cannot rule out discontinuities.

Proof. Since ω_λ is non-decreasing, by Lemma 24, we have that for every $\lambda > 0$,

$$\omega_\lambda \geq \limsup_{\varepsilon \to 0+} \omega(\varepsilon) \geq -2 \liminf_{\varepsilon \to 0+} m_b'(\varepsilon) = -2 \liminf_{\varepsilon \to 0+} \frac{m(\varepsilon)}{\varepsilon}.$$

In fact, we will show that $\lim_{\varepsilon \to 0+} \frac{m(\varepsilon)}{\varepsilon}$ exists and we will be able to compute it, which will then yield (2.1.9). By formula (2.3.1) and the construction of the infimum there, it is clear that for all

$\lambda \in (0,1)$,

$$\frac{m(\lambda)}{\lambda} = \inf_{\|\phi\|_{L^2}=1} J_\lambda[\phi] \;=\; \inf_{\|\phi\|_2^2=1}\left\{\frac{1}{2}\int_{\mathbf{R}}|\phi''|^2 - b|\phi'|^2 dx - \frac{\lambda^{\frac{p-1}{2}}}{p+1}\int_{\mathbf{R}}|\phi|^{p+1}dx\right\}$$
$$= \lim_k \frac{1}{2}\int_{\mathbf{R}}|\phi''_{k,\lambda}|^2 - b|\phi'_{k,\lambda}|^2 dx - \frac{\lambda^{\frac{p-1}{2}}}{p+1}\int_{\mathbf{R}}|\phi_{k,\lambda}|^{p+1}dx$$

for some minimizing sequence $\phi_{k,\lambda} : \|\phi_{k,\lambda}\|_{L^2} = 1$. Similar to our previous calculations, for k large enough

$$0 > J[\phi_k] \geq \frac{1}{4}\|\phi''_k\|_{L^2}^2 - c_{p,\lambda,b}(\|\phi''\|_{L^2}^{\frac{p-1}{4}+1} \geq -\gamma,$$

for some absolute constant γ. It follows that we have an upper bound on $\limsup_k \|\phi''_k\|_{L^2} \leq C$, which is independent on $\lambda \in (0,1)$. Thus, by GNS

$$\|\phi_{k,\lambda}\|_{L^{p+1}} \leq \|\phi''_{k,\lambda}\|_{L^2}^{\frac{p-1}{4(p+1)}}\|\phi_{k,\lambda}\|_{L^2}^{\frac{3p+5}{4(p+1)}} \leq C,$$

independent on $\lambda \in (0,1)$. Hence

$$\inf_{\|\phi\|_2^2=1}\left\{\frac{1}{2}\int_{\mathbf{R}}|\phi''|^2 - b|\phi'|^2 dx\right\} - C\lambda^{\frac{p-1}{2}} \leq \frac{m(\lambda)}{\lambda} \leq \inf_{\|\phi\|_2^2=1}\left\{\frac{1}{2}\int_{\mathbf{R}}|\phi''|^2 - b|\phi'|^2 dx\right\}.$$

It follows that

$$\lim_{\lambda\to 0+}\frac{m(\lambda)}{\lambda} = \inf_{\|\phi\|_2^2=1}\left\{\frac{1}{2}\int_{\mathbf{R}}|\phi''|^2 - b|\phi'|^2 dx\right\} = \inf_{\|\phi\|_2^2=1}\left\{\frac{1}{2}\int_{\mathbf{R}}|\hat{\phi}(\xi)|^2[(2\pi|\xi|)^2 - \frac{b}{2}]^2 dx\right\} - \frac{b^2}{8}.$$

The consideration now splits into two cases: $b \geq 0$ and $b < 0$. If $b \geq 0$, we clearly have (try $\phi_\delta : \widehat{\phi_\delta}(\xi) = \delta^{-1/2}\hat{\chi}\left(\frac{2\pi\xi - \sqrt{\frac{b}{2}}}{\delta}\right)$ for $\delta << 1$)

$$\inf_{\|\phi\|_2^2=1}\left\{\frac{1}{2}\int_{\mathbf{R}}|\hat{\phi}(\xi)|^2[(2\pi|\xi|)^2 - \frac{b}{2}]^2 dx\right\} = 0,$$

whereas for $b < 0$, we have (try $\phi_\delta : \widehat{\phi_\delta}(\xi) = \delta^{-1/2}\hat{\chi}(\delta^{-1}\xi)$ for $\delta << 1$)

$$\inf_{\|\phi\|_2^2=1} \left\{ \frac{1}{2} \int_{\mathbf{R}} |\hat{\phi}(\xi)|^2 [(2\pi|\xi|)^2 - \frac{b}{2}]^2 dx \right\} = \frac{b^2}{8}.$$

Thus, we have shown for every $\lambda > 0$,

$$\omega(\lambda) \geq -2 \lim_{\lambda \to 0+} \frac{m(\lambda)}{\lambda} = \begin{cases} \frac{b^2}{4}, & b \geq 0, \\ 0, & b < 0. \end{cases}$$

$\square$

Chapter 3

The Ostrovsky Equation

3.1 Introduction and main results

The Ostrovsky model, which is ubiquitous in the modern water waves theory, is given by,

$$(u_t - u_{xxx} - (u^2)_x)_x = u. \tag{3.1.1}$$

The related, generalized Ostrovsky/Vakhnenko/short pulse equation is the corresponding equation with a cubic nonlinearity

$$(u_t - u_{xxx} - (u^3)_x)_x = u. \tag{3.1.2}$$

These models have attracted a lot of attention in the last thirty years, as models of water waves under the action of a Coriolis force, [42, 43, 12], as well as the amplitude of a "short" pulse in an optical fiber, [46]. We shall be interested in the dynamics of a family of problems, which contain these two. More specifically, we consider the following generalized Ostrovsky models

$$(u_t - u_{xxx} - (|u|^p)_x)_x = u, \tag{3.1.3}$$

$$\left(u_t - u_{xxx} - (|u|^{p-1}u)_x\right)_x = u. \tag{3.1.4}$$

Clearly, (3.1.3), in the case $p = 2$ is nothing but (3.1.1), while (3.1.4), for $p = 3$ is (3.1.2). Let us comment on the seemingly more general form of the equations that appear in other publications,

$$(u_t - \beta u_{xxx} - \sigma(|u|^p)_x)_x = \gamma u, \quad (x,t) \in \mathbf{R} \times \mathbf{R}. \tag{3.1.5}$$

Using the scaling transformations $t \to at, x \to bx, u \to cu$, we obtain the equivalent problem

$$\left(u_t - \frac{\beta b^3}{a} u_{xxx} - \frac{\sigma b |c|^{p-2} c}{a} (|u|^p)_x\right)_x = \frac{\gamma}{ab} u,$$

which means that by choosing a, b, c appropriately, we may scale all the coefficients to plus or minus one, just as in (3.1.3). In addition, by a judicious choice of the signs of a, b, c, one concludes that all systems in the form (3.1.5) reduce to

$$(u_t - \varepsilon_1 sgn(\beta) u_{xxx} - \varepsilon_2 (|u|^p)_x)_x = sgn(\gamma) \varepsilon_1 u$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. In this work, we stick to the case[1] $sgn(\beta) = sgn(\gamma)$. In this case, an appropriate further rescale leads us to

$$(u_t - u_{xxx} - \varepsilon (|u|^p)_x)_x = u. \tag{3.1.6}$$

Thus, our model, (3.1.3), covers the cases for which $\varepsilon = 1$.

Let us record another, mostly equivalent formulation of (3.1.3) and (3.1.4). Using $u = v_x$ in (3.1.3) and integrating once (by tacitly assuming that v, v_x vanishes at $\pm\infty$), we get

$$\begin{aligned} (v_t - v_{xxx} - (|v_x|^p))_x &= v, \\ (v_t - v_{xxx} - (|v_x|^{p-1} v_x))_x &= v. \end{aligned} \tag{3.1.7}$$

Regarding local and global well-posedness for these models, most of the theory has been developed for standard quadratic and cubic models (3.1.1), [14, 46, 48, 49, 36, 45]. Extensive further references to earlier works can be found in [48, 49]. Break up in finite time is shown in various situations in [38].

The main purpose of this chapter is the study of traveling wave solutions, namely functions in the form $\phi(x - \omega t)$. More specifically, plugging in this ansatz in (3.1.7) turns it into the profile

[1] It is well-known that solitary waves do not exists in the case when $sgn(\beta) \neq sgn(\gamma)$.

equations

$$\begin{aligned} \phi'''' + \omega\phi'' + \phi + (|\phi'|^p)' = 0, \\ \phi'''' + \omega\phi'' + \phi + (|\phi'|^{p-1}\phi')' = 0. \end{aligned} \tag{3.1.8}$$

These are fourth order nonlinear ODE's, for which there is no very well-developed theory. In particular, for non-integer values of p, existence has been proved by variational methods, [33, 34, 31], so that (3.1.8) is an Euler-Lagrange equation for these constrained minimizers. Uniqueness, which is well-known to be a hard problem (even for second order problems of this type), is only known in the case $p = 2$. This is the main result of [51], where it is shown that localized solutions are unique, together with some asymptotic decay properties of ϕ and its derivatives. Note that the result obtained there rely heavily on the quadratic nonlinearity as well as the precise structure of the equation. We provide an independent analysis of the elliptic profile equations (3.1.8) and we also compute the spatial exponential rate of decay, which we believe to be sharp, see Proposition 7 below.

Our approach to (3.1.8) is variational, but rather different than the works [31, 33, 34]. More precisely, Levandosky and Liu construct their waves as minimizers of energy, subject to a fixed L^{p+1} norm. This method allows for a construction of waves for any power of $p > 1$. As shown therein, some of these waves, for large enough p, are spectrally unstable. On the other hand, our goal is to construct the so-called normalized waves - that is, we construct the waves to minimize energy, by keeping their L^2 norm fixed. As we show later in the chapter, see Theorem 6, this imposes restrictions on p, but the result is that all of these waves are necessarily spectrally stable. We state our results below, starting with the existence, and then proceeding onto the stability.

3.1.1 Existence of the normalized waves

Let us first introduce the functionals that we work with, namely

$$I[u]=\frac{1}{2}\int_{\mathbf{R}}|u_{xx}|^2+|u|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|u_x|^pu_xdx,$$
$$J[v]=\frac{1}{2}\int_{\mathbf{R}}|v_{xx}|^2+|v|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|v_x|^{p+1}dx,$$

and their variants

$$\mathscr{I}[u]=\frac{1}{2}\int_{\mathbf{R}}|u_x|^2+|\partial_x^{-1}u|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|u|^pudx,$$
$$\mathscr{J}[v]=\frac{1}{2}\int_{\mathbf{R}}|v_x|^2+|\partial_x^{-1}v|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|v|^{p+1}dx.$$

Note $I[u]=\mathscr{I}[u_x]$ and $J[v]=\mathscr{J}[v_x]$. For every $\lambda>0$, we consider the variational problems

$$\begin{cases} I[u]=\frac{1}{2}\int_{\mathbf{R}}|u_{xx}|^2+|u|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|u_x|^pu_xdx\to\min, \\ \int_{\mathbf{R}}|u_x|^2dx=\lambda, \end{cases} \tag{3.1.9}$$

$$\begin{cases} \mathscr{I}[u]=\frac{1}{2}\int_{\mathbf{R}}|u_x|^2+|\partial_x^{-1}u|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|u|^pudx\to\min, \\ u\in\dot{H}^{-1},\ \ \int_{\mathbf{R}}|u|^2dx=\lambda, \end{cases} \tag{3.1.10}$$

and

$$\begin{cases} J[v]=\frac{1}{2}\int_{\mathbf{R}}|v_{xx}|^2+|v|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|v_x|^{p+1}dx\to\min, \\ \int_{\mathbf{R}}|v_x|^2dx=\lambda, \end{cases} \tag{3.1.11}$$

$$\begin{cases} \mathscr{J}[v]=\frac{1}{2}\int_{\mathbf{R}}|v_x|^2+|\partial_x^{-1}v|^2dx-\frac{1}{p+1}\int_{\mathbf{R}}|v|^{p+1}dx\to\min, \\ v\in\dot{H}^{-1},\ \ \int_{\mathbf{R}}|v|^2dx=\lambda. \end{cases} \tag{3.1.12}$$

Our existence results are as follows.

Theorem 6 (Existence of solitary waves)**.** *Let $\lambda > 0$. Then,*

- *For $1 < p < 3$, the constrained minimization problems* (3.1.9) *and* (3.1.10) *have solutions φ_λ and ϕ_λ. In addition, $\phi_\lambda \in H^2 \cap \dot{H}^{-2}(\mathbf{R}), \varphi_\lambda \in H^4(\mathbf{R}) : \varphi'_\lambda = \phi_\lambda$ and they satisfy, for some $\omega \in (-\infty, 2)$,*

$$\partial_x^2 \phi_\lambda + \partial_x^{-2} \phi_\lambda + \omega \phi_\lambda + |\phi_\lambda|^p = 0,$$
$$\varphi''''_\lambda + \omega \varphi''_\lambda + \varphi_\lambda + (|\varphi'_\lambda|^p)' = 0,$$

respectively. The waves $\varphi_\lambda, \phi_\lambda$ are exponentially decaying, together with their derivatives, in fact

$$|\varphi_\lambda(x)| + |\phi_\lambda(x)| + |\phi'_\lambda(x)| \leq Ce^{-\kappa_\omega |x|}, k_\omega := \begin{cases} \frac{\sqrt{2-\omega}}{2} & \omega \in (-2, 2), \\ \sqrt{\frac{-\omega - \sqrt{\omega^2 - 4}}{2}} & \omega < -2. \end{cases} \tag{3.1.13}$$

- *For $1 < p < 5$, the minimization problems* (3.1.11) *and* (3.1.12) *have constrained minimizers $\varphi_\lambda \in H^4(\mathbf{R}), \phi_\lambda \in H^2 \cap \dot{H}^{-2}(\mathbf{R}) : \varphi'_\lambda = \phi_\lambda$, which satisfy, for some $\omega \in (-\infty, 2)$,*

$$\partial_x^2 \phi_\lambda + \partial_x^{-2} \phi_\lambda + \omega \phi_\lambda + |\phi_\lambda|^{p-1} \phi_\lambda = 0,$$
$$\varphi''''_\lambda + \omega \varphi''_\lambda + \varphi_\lambda + (|\varphi'_\lambda|^{p-1} \varphi'_\lambda)' = 0.$$

  The waves $\varphi_\lambda, \phi_\lambda$ satisfy (3.1.13).

Below are some remarks:

- The waves $\varphi_\lambda, \phi_\lambda$, which initially satisfy the Euler-Lagrange equation in a distributional sense, are actually smoother solutions, see Proposition 6 below.
- The waves satisfy the decay bounds (3.1.13) hold whenever one has a weak solution of (3.1.8), see Proposition 7. This result matches the results in Zhang-Liu, [51], see Lemma

3.2 and Remark 3.1, p. 824, for the case $\omega < -2$. For the case $\omega \in (-2,2)$, the new bound (3.1.13) provides the sharp rate of decay for the solitary waves.

Let us again point out that in [33, 34, 31], the authors have constructed traveling waves for values of p beyond the range of Theorem 6, due to the use of an alternative variational approach.

3.1.2 Stability results

We start by describing in detail the state of the art, regarding the stability of the Ostrovsky waves. For the reduced Ostrovsky case, that is the model without the dispersive term u_{xxx} and with quadratic or cubic non-linearities, much is known, as the model is completely integrable. A full description of its periodic waves as well as their stability can be found in the recent papers, [10, 18].

For the full Ostrovsky model under consideration, Liu and Ohta, [39] and by a slightly different method, Liu, [37] have established the orbital stability for the classical Ostrovsky's equation (i.e. $p = 2$) for large speeds. Another, set stability result, sometimes referred to as weak orbital stability, is given in [40]. In the works, [33], [34], Levandosky and Liu have constructed the waves for the generalized problems and they have shown that their orbital stability is reduced to the convexity of certain scalar functions, a la Grillakis-Shatah-Strauss. In [31], Levandosky obtained rigorously the orbital stability of the waves near some bifurcation points. In addition, he has launched an impressive numerical study, which was our main motivation for this work.

In order to state our stability results, we need to introduce the linearized operators as well. Namely, for a traveling wave ϕ, solving either one of the elliptic equations in (3.1.8), set $u(t,x) = \phi(x-\omega t) + v(t,x-\omega t)$ into (3.1.7). After ignoring $O(v^2)$ terms we get

$$(v_x)_t - v_{xxxx} - \omega v_{xx} - v - p(|\phi'|^{p-2}\phi' v_x)_x = 0. \tag{3.1.14}$$

Setting the stability ansatz $v(t,x) = e^{t\mu}z(x)$ in (3.1.14), we obtain the eigenvalue problem in the form

$$L_+ z = \mu \partial_x z, \;\; L_+ = \partial_{xxxx} + \omega\partial_{xx} + 1 + p\partial_x(|\phi'|^{p-2}\phi'\partial_x(\cdot)). \tag{3.1.15}$$

Clearly, L_+, $D(L_+) = H^4(\mathbf{R})$ is unbounded, but self-adjoint operator on $L^2(\mathbf{R})$. Spectral instability here is understood as the existence of a non-trivial pair $(\mu, z) : \Re\mu > 0, z \neq 0, z \in D(L_+)$, so that (3.1.15) is satisfied. Spectral stability means non-existence of such pairs.

The eigenvalue problem (3.1.15) is a non-standard one, although problems in this form were recently considered in the literature. An equivalent formulation, which is technically more convenient for our approach is the following: write $L_+ = -\partial_x \mathscr{L}_+ \partial_x$, where

$$\mathscr{L}_+ = -\partial_x^2 - \omega - \partial_x^{-2} - p|\phi'|^{p-2}\phi'$$

and $D(\mathscr{L}_+) = H^2(\mathbf{R}) \cap \dot{H}^{-2}(\mathbf{R})$. In terms of the new operator

$$\mu z_x = -\partial_x \mathscr{L}_+ z_x.$$

Since the function spaces require vanishing at both infinities, this is equivalent to $\mu z = -\mathscr{L}_+ \partial_x z$ or $-\mu$ is an eigenvalue for $-\mathscr{L}_+ \partial_x$. Equivalently, $-\mu$ is an eigenvalue for the adjoint $-\partial_x \mathscr{L}_+$ or

$$\partial_x \mathscr{L}_+ z = \mu z. \tag{3.1.16}$$

Thus, the spectral stability of the traveling wave $\phi(x - \omega t)$ is equivalent to the non-solvability of (3.1.16). Our main result is the following.

Theorem 7. *Let $1 < p < 3$ and $\lambda > 0$. Then, the constrained minimizers ϕ_λ for* (3.1.9) *is spectrally stable. That is* (3.1.16) *does not have non-trivial solutions $(\mu, z) : \Re\mu > 0, z \neq 0$.*

For $1 < p < 5$ and $\lambda > 0$, let ϕ_λ be a constrained minimizer for (3.1.12)*. Then, ϕ_λ is spectrally stable.*

From the instability index counting theory presented in Section 1.2 it follows that the corollary below is enough to prove spectral stability.

Corollary 4. *Suppose that the wave ϕ_λ satisfies*

1. $n^-(\mathscr{L}_+) = 1$,

2. *the wave* ϕ_λ *is weakly non-degenerate, i.e.* $\phi_\lambda \perp Ker[\mathscr{L}_+]$,

3. $\langle \mathscr{L}_+^{-1}\phi_\lambda, \phi_\lambda \rangle < 0$.

Then, the wave is strongly spectrally stable, in the sense that the eigenvalue problem (3.1.16) *does not have non-trivial solutions, and in fact* $\sigma(\partial_x \mathscr{L}_+) \subset i\mathbf{R}$.

3.2 Preliminaries

3.2.1 Weak solutions and bootstrapping regularity

In our considerations, we will need to rely, at least initially, on a weak solution formulations of certain elliptic PDE's, specifically (3.1.8). More concretely,

Definition 3. *We say that* $g \in H^2(\mathbf{R})$ *is a weak solution of the equation*

$$g'''' + \omega g'' + g + (F(g'))' = 0, \tag{3.2.1}$$

if the non-linearity satisfies $F(f') \in L^2$, *whenever* $f \in H^2$ *and for every* $h \in H^2$, *we have the relation* $\langle g'', h'' \rangle + \langle \omega g'' + g, h \rangle - \langle F(g'), h' \rangle = 0$.

A simple observation is that if g is a weak solution of (3.2.1), in the sense of Definition 3, then we can bootstrap its smoothness, namely $g \in H^3(\mathbf{R})$. Indeed, since the operator $\partial_x^4 + 1$ is invertible on $L^2(\mathbf{R})$, introduce $\tilde{g} := (\partial_x^4 + 1)^{-1}[-\omega g'' + \partial_x(F(g'))] \in L^2(\mathbf{R})$. Of course, this is the *formal* solution of (3.2.1), which should mean that $\tilde{g} = g$, which we will prove momentarily. Before that, let us observe that due to the smoothing nature of $(\partial_x^4 + 1)^{-1} : L^2 \to H^4$, we can immediately see that[2] $\tilde{g} \in H^3(\mathbf{R})$. Now, for every test function h, we have that[3]

$$\langle (1+\partial_x^4)g, h\rangle = -\omega \langle g'', h\rangle - \langle F(g'), h'\rangle = \langle (1+\partial_x^4)\tilde{g}, h\rangle.$$

It follows that $\langle g, (1+\partial_x^4)h\rangle = \langle \tilde{g}, (1+\partial_x^4)h\rangle$ for all h, whence $g = \tilde{g}$. In particular, we have shown the extra regularity $g \in H^3(\mathbf{R})$. One can immediately bootstrap this to $g \in H^4(\mathbf{R})$ by taking into account the representation $g = (\partial_x^4+1)^{-1}[-\omega g'' + \partial_x(F(g'))]$, if $\partial_x F(g') \in L^2$. This is the case for the profile equations (3.1.8). Thus, we have shown:

Proposition 6. *The weak solution g of* (3.2.1) *is in fact $g \in H^3(\mathbf{R})$. For non-linearities in the form $F(z) = |z|^p, |z|^{p-1}z$, this can be further improved to $g \in H^4(\mathbf{R})$, whence the weak solutions of* (3.2.1) *in fact satisfy* (3.2.1) *as L^2 functions.*

Due to this result, we will henceforth not make the distinction between weak and strong(er) solutions of our profile equations.

3.2.2 Exponential decay of the waves and eigenfunctions

In this section, we show that the solutions to the elliptic profile equations (3.1.8) have exponential decay at $\pm\infty$, and in fact we are able to compute explicitly the leading order terms. Similar result holds for any element in the kernels of the linearized operators $\mathscr{L}_+, L_+$. The precise result is as follows.

Proposition 7. *Let $\phi \in H^4$ solves either of the fourth order profile equations* (3.1.8)*, with $\omega < 2$. Then, ϕ, ϕ' both have exponential decay at $\pm\infty$ and in fact,*

$$|\phi(x)| + |\phi'(x)| \leq Ce^{-k_\omega |x|}, k_{\omega_\lambda} := \begin{cases} \frac{\sqrt{2-\omega}}{2}, & \omega \in (-2,2), \\ \sqrt{\frac{-\omega-\sqrt{\omega^2-4}}{2}}, & \omega < -2. \end{cases} \tag{3.2.2}$$

[2]This can be improved further to $\tilde{g}H^4$, once we impose the mild extra smoothness assumption $F(g) \in H^1$, which will not be necessary for our purposes.

[3]This is understood as pairing between an element of the distribution space H^{-2} and H^2.

In addition, every eigenfunction of $L_+\Psi = 0$ *has the same exponential decay. Similarly, let* $\phi \in H^2 \cap \dot{H}^{-2}$ *solve, for* $\omega < 2$*,*

$$\partial_x^2 \phi + \partial_x^{-2}\phi + \omega\phi + |\phi_\lambda|^p = 0.$$

Then, ϕ *has the same exponential decay as in* (3.2.2)*, together with the eigenfunctions corresponding to zero eigenvalues for* $\mathscr{L}_+$*.*

Proof. We work with the fourth order waves, namely the solutions of (3.1.8). Since $\omega < 2$, we have that $\xi^4 - \omega\xi^2 + 1 > 0$, for every $\xi \in \mathbf{R}$, and, therefore, $(\partial_x^4 + \omega\partial_x^2 + 1)^{-1}$ is a bounded operator on L^2, so we have the representation

$$\begin{aligned} \phi &= -(\partial_x^4 + \omega\partial_x^2 + 1)^{-1}[|\phi'|^{p-1}\phi'], \\ \phi &= -(\partial_x^4 + \omega\partial_x^2 + 1)^{-1}[|\phi'|^p]. \end{aligned}$$

Take a derivative in this last equation and denote $g := \phi'$, so

$$g = -\partial_x(\partial_x^4 + \omega\partial_x^2 + 1)^{-1}[|g|^{p-1}g], \tag{3.2.3}$$

$$g = -\partial_x(\partial_x^4 + \omega\partial_x^2 + 1)^{-1}[|g|^p]. \tag{3.2.4}$$

Clearly, it is enough to show the desired exponential decay for g, whence since ϕ vanishes at $\pm\infty$, one can conclude from the representations $\phi(x) = -\int_x^\infty \phi'(y)dy = \int_{-\infty}^x \phi'(y)dy$, that ϕ vanishes at the same rate at $\pm\infty$. Let $V(x) := |g(x)|^{p-1}$ or $V(x) := |g(x)|^{p-1}sgn(g(x))$, depending on whether we consider (3.2.3) or (3.2.4). Either way,

$$g = -\partial_x(\partial_x^4 + \omega\partial_x^2 + 1)^{-1}[Vg], \tag{3.2.5}$$

where $\lim_{|x|\to\infty} V(x) = 0$, since $g \in H^4 \subset C_0(\mathbf{R})$.

Let us now comment on the operator $(\partial_x^4+\omega\partial_x^2+1)^{-1}$. Clearly

$$(\partial_x^4+\omega\partial_x^2+1)^{-1}f(x)=\int_{-\infty}^{\infty}G_\omega(x-y)f(y)dy,$$

where $\widehat{G}_\omega(\xi)=\frac{1}{\xi^4-\omega\xi^2+1}$. Note that since the roots of the bi-quadratic equation $\kappa^4-\omega_\lambda\kappa^2+1=0$ are given by

$$k^2=\frac{\omega\pm\sqrt{\omega^2-4}}{2}, \tag{3.2.6}$$

from the formula $(\widehat{\xi^2+k^2})^{-1}=\frac{\pi}{k}e^{-k|\cdot|}$, it follows that G is a linear combination of two exponential functions. In fact, taking into account $\omega<2$, after some elementary analysis, we conclude that the solutions of (3.2.6), have $\Re k=\frac{\sqrt{2-\omega}}{2}$ when $\omega\in(-2,2)$ and $\Re k=\sqrt{\frac{-\omega-\sqrt{\omega^2-4}}{2}}$ for $\omega<-2$. It follows that

$$|G_\omega(x)|+|G'_\omega(x)|\leq C_\omega e^{-k_\omega|x|}, k_\omega:=\begin{cases}\frac{\sqrt{2-\omega}}{2}, & \omega\in(-2,2),\\ \sqrt{\frac{-\omega-\sqrt{\omega^2-4}}{2}}, & \omega<-2.\end{cases}$$

We are now ready to analyze (3.2.5). To this end, let $\varepsilon=\varepsilon_\omega>0$, to be selected momentarily. Let N be so large that $|V(x)|<\varepsilon$, so long as $|x|>N$. We now rewrite (3.2.5) as

$$g(x)+\int_{|y|>N}G'_\omega(x-y)V(y)g(y)dy=-\int_{|y|\leq N}G'_\omega(x-y)V(y)g(y)dy. \tag{3.2.7}$$

We can view this as an integral equation in $X_N\in L^\infty(|\cdot|>N)$, with

$$\mathscr{G}g(x)=\chi_{|x|>N}\int_{|y|>N}G'_\omega(x-y)V(y)g(y)dy,$$

acting in a bounded way on X_N. In fact, for every m, $0\leq m\leq k_\omega$ and every $g\in Y_m$, i.e.,

$$\|g\|_{Y_m}:=\sup_{|x|>N}|g(x)|e^{m|x|}<\infty,$$

we have

$$\begin{aligned}|\mathscr{G}g(x)| &\le C_\omega \varepsilon \|g\|_{Y_m} \int_{|y|>N} e^{-k_\omega|x-y|} e^{-m|y|} dy \\ &\le C_\omega \varepsilon \|g\|_{Y_m} \int_{-\infty}^{\infty} e^{-k_\omega|x-y|} e^{-m|y|} dy \\ &\le \varepsilon D_\omega \|g\|_{Y_m} e^{-m|x|},\end{aligned}$$

where we have used the fact that for $0 < a < b$, $\int_{-\infty}^{\infty} e^{-a|y|} e^{-b|x-y|} dy \le C_b e^{-a|x|}$. Hence $\mathscr{G} : Y_m \to Y_m$ with $\|\mathscr{G}\|_{B(Y_m)} \le \varepsilon D_\omega$. Thus, select $\varepsilon(\omega) : \varepsilon D_\omega = \frac{1}{2}$.

In the particular case $m = 0$, we can use the von Neumann series to resolve (3.2.7)

$$g = \sum_{k=0}^{\infty} (-1)^{k+1} \mathscr{G}^k \left[\int_{|y|\le N} G'_\omega(\cdot - y) V(y) g(y) dy \right]. \tag{3.2.8}$$

Using the representation (3.2.8), the fact that $|\int_{|y|\le N} G'_\omega(x-y)V(y)g(y)dy| \le C_\omega e^{-k_\omega|x|}$, and by the mapping properties of $\mathscr{G}$, we conclude that $g \in Y_{k_\omega}$. That is, $\sup_{|x|>N} |g(x)| \le Ce^{-k_\omega|x|}$, which by the boundedness of g can be extended to $\sup_x |g(x)| \le Ce^{-k_\omega|x|}$.

Regarding the eigenfunctions, we employ the same strategy, namely if $L_+\Psi = 0$, this means that for $g = \Psi'$,

$$g(x) = -p \int_{-\infty}^{\infty} G'_\omega(x-y)[V(y)g(y)]dy, \tag{3.2.9}$$

with the same V as above. Due to the fact that $|V| \sim |\phi'(x)|^{p-1}$ has exponential decay now, clearly, (3.2.9) can be bootstrapped to produce decay for g matching the decay of G'_ω, that is $e^{-k_\omega|x|}$. Finally, $\Psi(x) = -\int_x^\infty g(y)dy = \int_{-\infty}^x g(y)dy$ recovers the same exponential decay for Ψ as for g. □

3.2.3 Pohozaev identities

Lemma 26. *Suppose $\phi \in H^2(\mathbf{R})$ is a weak solution of*

$$\phi'''' + \omega\phi'' + \phi + \partial_x(|\phi'|^{p-1}\phi') = 0. \tag{3.2.10}$$

More concretely, for every test function $h \in H^2(\mathbf{R})$,

$$\langle \phi'', h'' \rangle + \omega \langle \phi'', h \rangle - \langle |\phi'|^{p-1}\phi', \partial_x h \rangle = 0.$$

Then, the following identities hold

$$\begin{aligned} \int_{\mathbf{R}} |\phi''|^2 \, dx &= \int_{\mathbf{R}} |\phi|^2 \, dx + \frac{p-1}{2(p+1)} \int |\phi'|^{p+1} \, dx, \\ \omega \int |\phi'|^2 \, dx &= 2 \int_{\mathbf{R}} |\phi|^2 \, dx - \frac{p+3}{2(p+1)} \int |\phi'|^{p+1} \, dx. \end{aligned} \tag{3.2.11}$$

Similarly, suppose $\phi \in H^2(\mathbf{R})$ is a weak solution of

$$\phi'''' + \omega \phi'' + \phi + \partial_x(|\phi'|^p) = 0, \tag{3.2.12}$$

then

$$\begin{aligned} \int_{\mathbf{R}} |\phi''|^2 \, dx &= \int_{\mathbf{R}} |\phi|^2 \, dx + \frac{p-1}{2(p+1)} \int |\phi'|^p \, \phi' dx, \\ \omega \int |\phi'|^2 \, dx &= 2 \int_{\mathbf{R}} |\phi|^2 \, dx - \frac{p+3}{2(p+1)} \int |\phi'|^p \, \phi' dx. \end{aligned} \tag{3.2.13}$$

Proof. Multiplying (3.2.10) by ϕ and integrating over $\mathbf{R}$ we get

$$\int_{\mathbf{R}} |\phi''|^2 \, dx - \omega \int |\phi'|^2 \, dx + \int_{\mathbf{R}} |\phi|^2 \, dx - \int |\phi'|^{p+1} \, dx = 0. \tag{3.2.14}$$

Now, multiplying (3.2.10) by $x\phi'$ (recall that according to Proposition 7 this function has exponential decay) and integrating over $\mathbf{R}$ we get

$$\frac{3}{2} \int_{\mathbf{R}} |\phi''|^2 \, dx - \frac{\omega}{2} \int_{\mathbf{R}} |\phi'|^2 \, dx - \frac{1}{2} \int_{\mathbf{R}} |\phi|^2 \, dx - \frac{p}{p+1} \int_{\mathbf{R}} |\phi'|^{p+1} \, dx = 0. \tag{3.2.15}$$

Solving (3.2.14) and (3.2.15) for $\int_{\mathbf{R}} |\phi''|^2 \, dx$ and $\omega \int |\phi'|^2 \, dx$ we get (3.2.11). Finally, the proof of (3.2.13) follows similar path.

$\square$

An easy corollary of Lemma 26 is the following lemma.

Lemma 27. *Suppose $\phi \in H^1(\mathbf{R}) \cap \dot{H}^{-1}(\mathbf{R})$ is a weak solution of*

$$\phi'' + \partial_x^{-2}\phi + \omega\phi + |\phi|^{p-1}\phi = 0. \tag{3.2.16}$$

Then, the following identities hold

$$\begin{aligned} \int_{\mathbf{R}} |\phi'|^2\,dx &= \int_{\mathbf{R}} \left|\partial_x^{-1}\phi\right|^2 dx + \frac{p-1}{2(p+1)} \int |\phi|^{p+1}\,dx, \\ \omega \int |\phi|^2\,dx &= 2\int_{\mathbf{R}} \left|\partial_x^{-1}\phi\right|^2 dx - \frac{p+3}{2(p+1)} \int |\phi|^{p+1}\,dx. \end{aligned} \tag{3.2.17}$$

Similarly, suppose $\phi \in H^1(\mathbf{R}) \cap \dot{H}^{-1}(\mathbf{R})$ is a weak solution of

$$\phi'' + \partial_x^{-2}\phi + \omega\phi + \partial_x(|\phi|^p) = 0, \tag{3.2.18}$$

then

$$\begin{aligned} \int_{\mathbf{R}} |\phi'|^2\,dx &= \int_{\mathbf{R}} \left|\partial_x^{-1}\phi\right|^2 dx + \frac{p-1}{2(p+1)} \int |\phi|^p\,\phi dx, \\ \omega \int |\phi|^2\,dx &= 2\int_{\mathbf{R}} \left|\partial_x^{-1}\phi\right|^2 dx - \frac{p+3}{2(p+1)} \int |\phi|^p\,\phi dx. \end{aligned} \tag{3.2.19}$$

Proof. Just apply Lemma 26 to the function g, where $\phi = g'$. Note that $g \in H^2$ solves (3.2.10) or (3.2.12). $\square$

3.3 Variational construction

In this section, we provide the variational construction of the waves. It turns out that for some aspects of the construction, it is more beneficial to look at the following alternative $\mathscr{I}$, $\mathscr{J}$ defined in the beginning. Introduce the following functions, which are the corresponding infima, if they exists, of the constrained minimization problems

$$m_I(\lambda) = \inf_{u\in H^2, \|u_x\|^2=\lambda} \left\{ \frac{1}{2}\int_{\mathbf{R}} |u_{xx}|^2 + |u|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |u_x|^p u_x dx \right\}, \tag{3.3.1}$$

$$m_J(\lambda) = \inf_{v\in H^2, \|v_x\|^2=\lambda} \left\{ \frac{1}{2}\int_{\mathbf{R}} |v_{xx}|^2 + |v|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |v_x|^{p+1} dx \right\}, \tag{3.3.2}$$

$$m_{\mathscr{I}}(\lambda) = \inf_{U\in H^1\cap \dot{H}^{-1}, \|U\|^2=\lambda} \left\{ \frac{1}{2}\int_{\mathbf{R}} |U_x|^2 + |\partial_x^{-1} U|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |U|^p U dx \right\}, \tag{3.3.3}$$

$$m_{\mathscr{J}}(\lambda) = \inf_{V\in H^1\cap \dot{H}^{-1}, \|V\|^2=\lambda} \left\{ \frac{1}{2}\int_{\mathbf{R}} |V_x|^2 + |\partial_x^{-1} V|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |V|^{p+1} dx \right\}. \tag{3.3.4}$$

We have the following sequence of lemmas, that establishes some important properties of the functionals and the m functions.

3.3.1 The variational problems are well-posed and equivalent

Regarding well-posedness, we have the following result.

Lemma 28. *For $1 < p < 5$, $m_I, m_J > -\infty$. That is, the problems* (3.1.9) *and* (3.1.11) *are well-posed.*

Proof. Indeed, it is simple to see that $0 \geq m_I \geq m_J$. From the GNS inequality,

$$\begin{aligned} \|u_x\|_{L^{p+1}}^{p+1} &\leq C\|u_x\|_{\dot{H}^{\frac{p-1}{2}}}^{p+1} \\ &\leq C\,\|u_x\|_2^{\frac{p+3}{2}}\,\|u_{xx}\|_2^{\frac{p-1}{2}}, \end{aligned}$$

we have

$$I[u] \geq J[u] \geq \frac{1}{2}\int_{\mathbf{R}} |u_{xx}|^2 dx - C\,\|u_x\|_2^{\frac{p+3}{2}}\,\|u_{xx}\|_2^{\frac{p-1}{2}}.$$

Clearly, if $p \in (1,5)$, $\frac{p-1}{2} < 2$, so we can use Young's inequality and absorb $\|u_{xx}\|_{L^2}^{\frac{p-1}{2}}$. Thus, we get a bound

$$I[u] \geq J[u] \geq C_\lambda.$$

$\square$

The next result is about the equivalence of $m_I, m_{\mathscr{I}}$, and $m_J, m_{\mathscr{J}}$ respectively.

Lemma 29. *For $1 < p < 5$ we have that $m_I(\lambda) = m_{\mathscr{I}}(\lambda)$ and $m_J(\lambda) = m_{\mathscr{J}}(\lambda)$. Moreover, if φ_λ is a minimizer for $m_I(\lambda)$ ($m_J(\lambda)$ respectively), then $\phi_\lambda = \varphi'_\lambda$ is a minimizer for $m_{\mathscr{I}}(\lambda)(m_{\mathscr{J}}(\lambda)$ respectively).*

Proof. On one hand, let ϕ be a compactly supported function such that there exists a $\delta > 0$, so that $\widehat{\phi}(\xi) = 0$ for all $|\xi| < \delta$ and $\|\phi\|^2 = \lambda$. Note that for such functions, $\partial_x^{-1}\phi$ is well-defined.

Denote the set of all such ϕ as A_λ, noting that A_λ is dense in H^1. For such a ϕ

$$\mathscr{I}[\phi] = I[\partial_x^{-1}\phi] \geq m_I(\lambda). \tag{3.3.5}$$

Taking the infimum over all $\phi \in A_\lambda$ gives us $m_{\mathscr{I}}(\lambda) \geq m_I(\lambda)$. On the other hand,

$$m_{\mathscr{I}}(\lambda) = \inf_{u \in A_\lambda} \mathscr{I}[u] \leq \inf_{u \in A_\lambda, u = v_x \in H^2} \mathscr{I}[u] = m_I(\lambda).$$

So, $m_I(\lambda) = m_{\mathscr{I}}(\lambda)$. Now, suppose φ_λ is a minimizer for (3.3.1), then, clearly, for $\phi_\lambda := \varphi'_\lambda$ we have $I[\varphi_\lambda] = \mathscr{I}[\phi_\lambda]$. $\square$

3.3.2 Minimizing sequences produce non-trivial limits

Now that we know that the minimization problems for m_I and $m_{\mathscr{I}}$ are equivalent, suppose $\{u_k\}_{k=1}^\infty$ is a minimizing for $\mathscr{I}$, subject to the constraint $\|u\|_{L^2}^2 = \lambda$. That is

$$\lim_{k\to\infty} \mathscr{I}[u_k] = m_I, \quad \|u_k\|_{L^2}^2 = \lambda \tag{3.3.6}$$

(similarly for J). Clearly, there exists a subsequence, renamed to $\{u_k\}_{k=1}^\infty$, such that

$$\lim_{k\to\infty} \int_{\mathbf{R}} |\partial_x u_k|^2 dx = I_1, \quad \lim_{k\to\infty} \int_{\mathbf{R}} |\partial_x^{-1} u_k|^2 dx = I_2, \quad \lim_{k\to\infty} \int_{\mathbf{R}} |u_k|^p u_k dx = I_3, \tag{3.3.7}$$

or

$$\lim_{k\to\infty}\int_{\mathbf{R}}|\partial_x u_k|^2dx=J_1,\quad \lim_{k\to\infty}\int_{\mathbf{R}}|\partial_x^{-1}u_k|^2dx=J_2,\quad \lim_{k\to\infty}\int_{\mathbf{R}}|u_k|^{p+1}dx=J_3, \tag{3.3.8}$$

for $\mathscr{J}$. We have the following key lemma, that shows that such minimizing sequences can not possibly be trivially converging to zero.

Lemma 30. *For any minimizing sequence satisfying* (3.3.7)*(*(3.3.8) *respectively), we have*

i) $J_3>0$ *for* $1<p<5$*,*

ii) $I_3>0$ *for* $1<p<3$.

Proof. First of all, clearly, $I_3\geq 0$ and $J_3\geq 0$. Let $\lambda>0$. We need to show that the strict inequality holds in both cases. We treat them separately.

Proof of $J_3>0$**.** Suppose for contradiction that $J_3=0$. Then we can estimate the infimum explicitly

$$\begin{aligned} m_{\mathscr{J}}(\lambda) &= \inf_{\|u\|_2^2=\lambda}\{\frac{1}{2}\int_{\mathbf{R}}|u_x|^2+|\partial_x^{-1}u|^2dx\} \\ &= \inf_{\|u\|_2^2=\lambda}\{\frac{1}{2}\int_{\mathbf{R}}\frac{(\xi^2-1)^2}{\xi^2}|\widehat{u}(\xi)|^2d\xi+\int_{\mathbf{R}}|\widehat{u}(\xi)|^2d\xi\} \\ &\geq \lambda. \end{aligned} \tag{3.3.9}$$

In fact, there is an equality above, as it suffices to take a function, whose Fourier transform is highly localized around, say, $\xi=1$. The point is that this infimum is actually strictly smaller than λ, which would give us the contradiction sought in this case.

To see this, let χ_1 be a Schwartz function, whose Fourier transform $\widehat{\chi_1}$ is an even bump C^∞ function, supported in the interval $(-\frac{1}{100},\frac{1}{100})$. Consider then $\chi:=\chi_1^2$, so that $\widehat{\chi}=\widehat{\chi_1}*\widehat{\chi_1}$. It has essentially the same properties as χ_1, except it, is in addition, a positive function. That is, $\chi\geq 0$ and *supp* $\widehat{\chi}\subset(-\frac{1}{50},\frac{1}{50})$. Multiplication by a constant will help us to achieve $\|\chi\|_2^2=\lambda/2$, which we assume henceforth.

Next, consider the function

$$\widehat{v_{J,\varepsilon}}(\xi)=\frac{1}{\sqrt{\varepsilon}}\left(\widehat{\chi}(\frac{\xi-1}{\varepsilon})+\widehat{\chi}(\frac{\xi+1}{\varepsilon})\right).$$

By the support properties of χ and $\|\chi\|_2^2 = \lambda/2$, it is clear that for small ε, $\|v_{J,\varepsilon}\|_{L^2}^2 = \lambda$. Since $\widehat{\chi}$ is even, we have that the function

$$v_{J,\varepsilon}(x) = \sqrt{\varepsilon}\chi(\varepsilon x)(e^{ix} + e^{-ix}) = 2\sqrt{\varepsilon}\chi(\varepsilon x)\cos(x)$$

is real. Next, using the fact that $\widehat{\chi}(\frac{\xi-1}{\varepsilon})$ and $\widehat{\chi}(\frac{\xi+1}{\varepsilon})$ have disjoint supports and change of variables, we obtain

$$\begin{aligned} \frac{1}{2}\int_{\mathbf{R}} \frac{(\xi^2-1)^2}{\xi^2}|\widehat{v_{J,\varepsilon}}(\xi)|^2 d\xi &= \frac{1}{2}\int_{\mathbf{R}} \frac{(\varepsilon\xi)^2(\varepsilon\xi+2)^2}{(\varepsilon\xi+1)^2}|\widehat{\chi}(\xi)|^2 d\xi \\ &+ \frac{1}{2}\int_{\mathbf{R}} \frac{(\varepsilon\xi-2)^2(\varepsilon\xi)^2}{(\varepsilon\xi-1)^2}|\widehat{\chi}(\xi)|^2 d\xi \\ &= O(\varepsilon^2). \end{aligned} \tag{3.3.10}$$

Note that the denominators above are never problematic, as they vanish away from the support of $\widehat{\chi}$. On the other hand, using lemma 2 and the non-negativity of χ, we get

$$\begin{aligned} \int_{\mathbf{R}} |v_{J,\varepsilon}(x)|^{p+1} dx &= 2^{p+1}\varepsilon^{\frac{p-1}{2}} \int_{\mathbf{R}} \chi^{p+1}(x)\left|\cos\left(\frac{x}{\varepsilon}\right)\right|^{p+1} dx \\ &\geq 2^{p+1}\varepsilon^{\frac{p-1}{2}} \frac{\sqrt{2}}{2} \sum_{n=-\infty}^{\infty} \int_{\varepsilon(2\pi n)}^{\varepsilon(2\pi n+\pi/4)} \chi^{p+1}(x) dx \\ &\geq C\varepsilon^{\frac{p-1}{2}} \int \chi^{p+1}(x)dx + O(\varepsilon^{\frac{p+1}{2}}). \end{aligned} \tag{3.3.11}$$

Combining (3.3.10) and (3.3.11), we obtain

$$\mathscr{J}[v_{J,\varepsilon}] = O(\varepsilon^2) + \lambda - C\varepsilon^{\frac{p-1}{2}},$$

which implies that for $p < 5$, $m_{\mathscr{J}}(\lambda) < \lambda$, and this is a contradiction with (3.3.9). Thus, $J_3 > 0$.

Proof of $I_3 > 0$. The considerations in this case are considerably more involved.

Similarly to (3.3.9), we first establish that $m_{\mathscr{I}} \geq \lambda$ in this case. There is a slight twist that the quantity $\int_{\mathbf{R}} |u|^p u dx$ is not necessarily non-negative anymore. However, since the other two quantities in the definition of $\mathscr{I}$ are positive definite, we can (by switching $u \to -u$ if necessary) assume that the infimum is taken over u, with the property $\int_{\mathbf{R}} |u|^p u dx \geq 0$. This will give a better (i.e. smaller or equal) $m_{\mathscr{I}}$, which is what needs to happen anyway as $m_{\mathscr{I}}$ is the infimum. Then, it is clear that

$$m_{\mathscr{I}}(\lambda) \leq \inf_{\|u\|_2^2=\lambda} \{\frac{1}{2}\int_{\mathbf{R}} |u_x|^2 + |\partial_x^{-1} u|^2 dx\}.$$

On the other hand, our assumption that $I_3 = 0$, means that the opposite inequality also holds true as

$$\begin{aligned} m_{\mathscr{I}} &= \lim_k \left(\frac{1}{2}\int_{\mathbf{R}} |\partial_x u_k|^2 + |\partial_x^{-1} u_k|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |u_k|^p u_k dx\right) \\ &\geq \inf_{\|u\|_2^2=\lambda} \frac{1}{2}\int_{\mathbf{R}} |u_x|^2 + |\partial_x^{-1} u|^2 dx. \end{aligned}$$

This means, in particular that $m_{\mathscr{I}} \geq \lambda$, as we have argued before. We will show that this is contradictory. To that end, consider

$$\widehat{v_{I,\varepsilon}}(\xi) = \frac{1}{\sqrt{\varepsilon}}\left(\widehat{\chi}(\frac{\xi-1}{\varepsilon}) + \widehat{\chi}(\frac{\xi+1}{\varepsilon}) + \varepsilon^{\alpha}\left(\widehat{\chi}(\frac{\xi-2}{\varepsilon}) + \widehat{\chi}(\frac{\xi+2}{\varepsilon})\right)\right),$$

with χ as before and $\max(\frac{p-1}{2}, \frac{2}{p+1}) < \alpha < 1$. This is possible, due to the assumption $1 < p < 3$. Note that $\alpha > \frac{2}{p+1} > \frac{1}{2}$, due to the same assumption. Then the function

$$\begin{aligned} v_{I,\varepsilon}(x) &= \sqrt{\varepsilon}\chi(\varepsilon x)(e^{ix} + e^{-ix} + \varepsilon^{\alpha}(e^{2ix} + e^{-2ix})) \\ &= 2\sqrt{\varepsilon}\chi(\varepsilon x)(\cos(x) + \varepsilon^{\alpha}\cos(2x)) \end{aligned}$$

is real and even. Similarly to (3.3.10), we get

$$\int_{\mathbf{R}} \frac{(\xi^2-1)^2}{\xi^2} |\widehat{v_{I,\varepsilon}}(\xi)|^2 d\xi = O(\varepsilon^{2\alpha})$$

for all ε small enough. Indeed, all terms in $v_{I,\varepsilon}$ have disjoint support on the Fourier side, due to the properties of χ. However, the dominant terms, due to the choice of α, are those with ε^{α} in front of it, whence the bound $O(\varepsilon^{2\alpha})$.

Now, we are going to show that

$$\int_{\mathbf{R}} |v_{I,\varepsilon}(x)|^p v_{I,\varepsilon}(x)dx \geq C\varepsilon^{\frac{p-1}{2}+\alpha}, \tag{3.3.12}$$

which will finish the proof of the lemma, since $\frac{p-1}{2}+\alpha < 2\alpha$.

To this end, let $\gamma > 0$ be such that $(p+1)(\frac{1}{2}-\gamma) = 1$ (or $\gamma := \frac{p-1}{2(p+1)} \in (0, \frac{1}{2})$) and split the integral as follows

$$\begin{aligned}
&\varepsilon^{\frac{p+1}{2}} \int_{\mathbf{R}} \chi^{p+1}(\varepsilon x)(\cos(x)+\varepsilon^\alpha \cos(2x))|\cos(x)+\varepsilon^\alpha \cos(2x)|^p dx \\
&= \varepsilon^{\frac{p+1}{2}} \int_{|\cos(x)|\leq \varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)(\cos(x)+\varepsilon^\alpha \cos(2x))|\cos(x)+\varepsilon^\alpha \cos(2x)|^p dx \\
&+ \varepsilon^{\frac{p+1}{2}} \int_{|\cos(x)|> \varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)(\cos(x)+\varepsilon^\alpha \cos(2x))|\cos(x)+\varepsilon^\alpha \cos(2x)|^p dx
\end{aligned}$$

For the first term we have

$$\begin{aligned}
&\left| \varepsilon^{\frac{p+1}{2}} \int_{|\cos(x)|\leq \varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)(\cos(x)+\varepsilon^\alpha \cos(2x))|\cos(x)+\varepsilon^\alpha \cos(2x)|^p dx \right| \\
&\leq \varepsilon^{\frac{p+1}{2}} \int_{\mathbf{R}} \chi^{p+1}(\varepsilon x)\cdot \varepsilon^{\frac{1}{2}-\gamma}\cdot \varepsilon^{p(\frac{1}{2}-\gamma)} dx \leq C\varepsilon^{\frac{p-1}{2}+(p+1)(\frac{1}{2}-\gamma)} = C\varepsilon^{\frac{p+1}{2}}.
\end{aligned}$$

as $\varepsilon \to 0+$.

Next, we show that the second term is bounded below by $C\varepsilon^{\frac{p-1}{2}+\alpha}$, and hence is dominant. In order to prepare the calculation, note that for $x : |\cos(x)| > \varepsilon^{1/2-\gamma}$, and $\varepsilon << 1$,

$$\begin{aligned}
|\cos(x)+\varepsilon^\alpha \cos(2x)|^p &= |\cos(x)|^p \left(1+2\varepsilon^\alpha \frac{\cos(2x)}{\cos(x)} + \varepsilon^{2\alpha}\frac{\cos^2(2x)}{\cos^2(x)}\right)^{p/2} \\
&= |\cos(x)|^p \left(1+p\varepsilon^\alpha \frac{\cos(2x)}{\cos(x)}\right) + O(\varepsilon^{2\alpha+2\gamma-1}),
\end{aligned}$$

where in this calculation, we have implicitly used that $\alpha > \frac{1}{2}, \gamma > 0$. Thus, using this expansion, we obtain

$$
\begin{aligned}
& \varepsilon^{\frac{p+1}{2}} \int\limits_{|\cos(x)|>\varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)(\cos(x)+\varepsilon^{\alpha}\cos(2x))|\cos(x)+\varepsilon^{\alpha}\cos(2x)|^{p}dx \\
= \; & \varepsilon^{\frac{p+1}{2}} \int\limits_{|\cos(x)|>\varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)(\cos(x)+\varepsilon^{\alpha}\cos(2x))|\cos(x)|^{p}\left(1+p\varepsilon^{\alpha}\frac{\cos(2x)}{\cos(x)}\right)dx \\
+ \; & O(\varepsilon^{\frac{p-1}{2}+(2\alpha+2\gamma-1)}) \\
= \; & \varepsilon^{\frac{p+1}{2}} \int\limits_{|\cos(x)|>\varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)\cos(x)|\cos(x)|^{p}dx \\
+ \; & \varepsilon^{\alpha+\frac{p+1}{2}}(p+1)\left(\int_{|\cos(x)|>\varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^{p}dx\right)+O(\varepsilon^{\frac{p-1}{2}+(2\alpha+2\gamma-1)}) \\
= \; & \varepsilon^{\frac{p+1}{2}}(K+\varepsilon^{\alpha}(p+1)Q)+O(\varepsilon^{\frac{p-1}{2}+(2\alpha+2\gamma-1)})+O(\varepsilon^{\frac{p-1}{2}+2\alpha}),
\end{aligned}
$$

where we have introduced two quantities K and Q. Clearly, since $2\gamma < 1$, it follows that the term $O(\varepsilon^{\frac{p-1}{2}+(2\alpha+2\gamma-1)})$ is dominant over $O(\varepsilon^{\frac{p-1}{2}+2\alpha})$. We claim that $K = O(1)$, whereas $Q \geq C\varepsilon^{-1}$. This implies (3.3.12) and the proof of Lemma 30 will be complete.

First, let us deal with K,

$$
\begin{aligned}
K &= \int_{|\cos(x)|>\varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)\cos(x)|\cos(x)|^{p}dx \\
&= \int_{\mathbf{R}} \chi^{p+1}(\varepsilon x)\cos(x)|\cos(x)|^{p}dx + O(1).
\end{aligned}
$$

The change of variables $y = \pi/2 - x$ yields

$$
\int_{\mathbf{R}} \chi^{p+1}(\varepsilon x)\cos(x)|\cos(x)|^{p}dx = -\int_{\mathbf{R}} \chi^{p+1}(\varepsilon(\pi/2-y))\sin(y)|\sin(y)|^{p}dx.
$$

Observe, however, that for $F(u) := \int_0^u (z-z^2)^{p/2}dz$, $0 < u < 1$, we have

$$
\sin(y)|\sin(y)|^{p} = 2^{p+1}\partial_y[F(\sin^2(y/2))],
$$

and hence integrating by parts yields

$$\int_{\mathbf{R}} \chi^{p+1}(\varepsilon x)\cos(x)|\cos(x)|^p dx = -2^{p+1}\varepsilon \int_{\mathbf{R}} \frac{\partial}{\partial y}(\chi^{p+1})(\varepsilon(\pi/2-y))F(\sin^2(y/2))dy$$
$$= O(1),$$

since F is a continuous function.

Now, we prove the claim about Q. Similarly to K, we can write

$$\begin{aligned} Q &= \int_{|\cos(x)|>\varepsilon^{1/2-\gamma}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx \\ &= \int_{\mathbf{R}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx + O(\varepsilon^{p(\frac{1}{2}-\gamma)-1}). \end{aligned}$$

Noting that $p(\frac{1}{2}-\gamma)-1 > -1$, it suffices to show that the first term is bounded from below by $C\varepsilon^{-1}$.

Splitting each of the intervals $[2\pi n, 2\pi(n+1))$ into eight pieces as follows

$$\begin{aligned} &\int_{\mathbf{R}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx \\ &= \left(\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{4}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx + \sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{\pi}{2}}^{2\pi n+\frac{3\pi}{4}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx\right) \\ &+ \left(\sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{3\pi}{4}}^{2\pi n+\pi} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx + \sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{\pi}{4}}^{2\pi n+\frac{\pi}{2}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx\right) \\ &+ \left(\sum_{n=-\infty}^{\infty}\int_{2\pi n+\pi}^{2\pi n+\frac{5\pi}{4}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx + \sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{3\pi}{2}}^{2\pi n+\frac{7\pi}{4}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx\right) \\ &+ \left(\sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{7\pi}{4}}^{2\pi n+2\pi} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx + \sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{5\pi}{4}}^{2\pi n+\frac{3\pi}{2}} \chi^{p+1}(\varepsilon x)\cos(2x)|\cos(x)|^p dx\right) \end{aligned}$$

and then pairing them as in Figure 3.1 yields

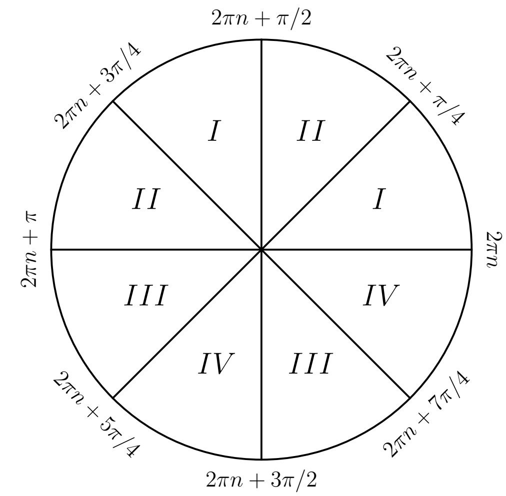

Figure 3.1

$$\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{4}} \chi^{p+1}(\varepsilon x)\cos(2x)(|\cos(x)|^p - |\sin(x)|^p)dx$$
$$+\sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{3\pi}{4}}^{2\pi n+\pi} \chi^{p+1}(\varepsilon x)\cos(2x)(|\cos(x)|^p - |\sin(x)|^p)dx$$
$$+\sum_{n=-\infty}^{\infty}\int_{2\pi n+\pi}^{2\pi n+\frac{5\pi}{4}} \chi^{p+1}(\varepsilon x)\cos(2x)(|\cos(x)|^p - |\sin(x)|^p)dx$$
$$+\sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{7\pi}{4}}^{2\pi n+2\pi} \chi^{p+1}(\varepsilon x)\cos(2x)(|\cos(x)|^p - |\sin(x)|^p)dx$$
$$+\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{4}} \left(\chi^{p+1}\left(\varepsilon(x+\frac{\pi}{2})\right) - \chi^{p+1}(\varepsilon x)\right)\cos(2x)|\sin(x)|^p dx$$
$$+\sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{3\pi}{4}}^{2\pi n+\pi} \left(\chi^{p+1}\left(\varepsilon(x-\frac{\pi}{2})\right) - \chi^{p+1}(\varepsilon x)\right)\cos(2x)|\sin(x)|^p dx$$
$$+\sum_{n=-\infty}^{\infty}\int_{2\pi n+\pi}^{2\pi n+\frac{5\pi}{4}} \left(\chi^{p+1}\left(\varepsilon(x+\frac{\pi}{2})\right) - \chi^{p+1}(\varepsilon x)\right)\cos(2x)|\sin(x)|^p dx$$
$$+\sum_{n=-\infty}^{\infty}\int_{2\pi n+\frac{7\pi}{4}}^{2\pi n+2\pi} \left(\chi^{p+1}\left(\varepsilon(x-\frac{\pi}{2})\right) - \chi^{p+1}(\varepsilon x)\right)\cos(2x)|\sin(x)|^p dx.$$

Note that the first four terms are all positive for all values of n. In addition, taking the first term

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{4}}\chi^{p+1}(\varepsilon x)\cos(2x)(|\cos(x)|^{p}-|\sin(x)|^{p})dx\\
&\geq\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{6}}\chi^{p+1}(\varepsilon x)\cos(2x)(|\cos(x)|^{p}-|\sin(x)|^{p})dx\\
&\geq c_p\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{6}}\chi^{p+1}(\varepsilon x)dx\\
&\geq\varepsilon^{-1}\sum_{n=-\infty}^{\infty}\int_{2\pi n\varepsilon}^{2\pi n\varepsilon+\frac{\pi}{6}\varepsilon}\chi^{p+1}(y)dy\\
&\geq d_p\varepsilon^{-1}\int_{\mathbf{R}}\chi^{p+1}(y)dydy+O(1),
\end{aligned}
$$

by Lemma 2.

On the other hand, for the error terms we have a bound of $O(1)$, since

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}|\int_{2\pi n}^{2\pi n+\frac{\pi}{4}}\left(\chi^{p+1}\left(\varepsilon(x+\frac{\pi}{2})\right)-\chi^{p+1}(\varepsilon x)\right)\cos(2x)|\sin(x)|^{p}dx|\\
&\leq\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{4}}\int_{\varepsilon x}^{\varepsilon(x+\frac{\pi}{2})}|(\chi^{p+1})'(y)|dydx\\
&\leq\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi n+\frac{\pi}{4}}\int_{\varepsilon 2\pi n}^{\varepsilon(2\pi n+\frac{3\pi}{4})}|(\chi^{p+1})'(y)|dydx\\
&\leq C\int_{\mathbf{R}}|(\chi^{p+1})'(x)|dx
\end{aligned}
$$

and, similarly, we estimate the three other error terms. □

We are now ready to present the main result of this section.

3.3.3 Existence of the waves

Proposition 8. *Let* $1<p<3$*, then the minimization problem* (3.3.1) *has a solution. Let* $1<p<5$*, then the minimization problem* (3.3.2) *has a solution.*

Remark: By Lemma 29, this implies the existence of solutions to (3.3.3) and (3.3.4), in the corresponding range of p. The proof of Proposition 8 is based on the method of concentrated compactness. In the compensation compactness arguments, the sub-additivity of the function $\lambda \to m(\lambda)$ plays a pivotal role. We begin with this lemma.

Lemma 31. *(Strict sub-additivity) Fix* $\lambda > 0$*.*

i) Suppose $1 < p < 3$ *. Then for all* $0 < \alpha < \lambda$ *we have that the strict sub-additivity condition holds for* m_I*, namely,*

$$m_I(\lambda) < m_I(\alpha) + m_I(\lambda - \alpha).$$

ii) Suppose $1 < p < 5$ *. Then for all* $0 < \alpha < \lambda$ *we have that the strict sub-additivity condition holds for* m_J*, namely,*

$$m_J(\lambda) < m_J(\alpha) + m_J(\lambda - \alpha).$$

Proof. The proofs of i) and ii) are identical. Let us prove i). First, we claim that the function $\frac{m_I(\lambda)}{\lambda}$ is strictly decreasing. Indeed,

$$\begin{aligned} m_I(\lambda) &= \inf_{\|u_x\|_2^2=\lambda} \{\frac{1}{2}\int_{\mathbf{R}} |u_{xx}|^2 + |u|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |u_x|^p u_x dx\} \\ &= \frac{\lambda}{\alpha} \inf_{\|u_x\|_2^2=\alpha} \{\frac{1}{2}\int_{\mathbf{R}} |u_{xx}|^2 + |u|^2 dx - \frac{(\lambda/\alpha)^{\frac{p-1}{2}}}{p+1}\int_{\mathbf{R}} |u_x|^p u_x dx \\ &< \frac{\lambda}{\alpha} m_I(\alpha), \end{aligned}$$

where the strict inequality follows from the fact that by lemma 30 there exist a minimizing sequence $\{u_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \int_{\mathbf{R}} |(u_k)_x|^p (u_k)_x dx > 0$. Finally, assuming that $\alpha \in [\lambda/2, \lambda)$ (otherwise we argue with $\lambda - \alpha$), since $\frac{m_I(\lambda)}{\lambda}$ is decreasing, we get

$$\begin{aligned} m_I(\lambda) &< \frac{\lambda}{\alpha} m_I(\alpha) \\ &= m_I(\alpha) + \frac{\lambda - \alpha}{\alpha} m_I(\alpha) \\ &\le m_I(\alpha) + m_I(\lambda - \alpha). \end{aligned}$$

□

Proof. (Proposition 8)

Define $\rho_k(x) = |\partial_x u_k|^2$. By the concentration compactness lemma at least one of the following holds:

i) *Tightness.* There exists $\{y_k\}_{k=1}^{\infty}$ such that for all $\varepsilon > 0$ there exists an $R_\varepsilon > 0$ satisfying

$$\int_{B(y_k,R_\varepsilon)} \rho_k dx \geq \int_{\mathbf{R}} \rho_k dx - \varepsilon.$$

ii) *Vanishing.* For every $R > 0$

$$\lim_{k\to\infty} \sup_{y\in R} \int_{B(y,R)} \rho_k dx = 0.$$

iii) *Dichotomy.* There exists an $\alpha \in (0,\lambda)$ such that for every $\varepsilon > 0$ there exist R, $R_k \to \infty$, y_k and k_0 such that for all $k \geq k_0$

$$\left|\int_{|x-y_k|<R} \rho_k dx - \alpha\right| < \varepsilon, \left|\int_{|x-y_k|>R_k} \rho_k dx - (\lambda-\alpha)\right| < \varepsilon, \left|\int_{R<|x-y_k|<R_k} \rho_k dx\right| < \varepsilon.$$

First, let us rule out vanishing. Suppose, it occurs. Let $0 \leq \chi \leq 1$ be a smooth bump function supported on $(-2,2)$ with $\chi \equiv 1$ on $(-1,1)$. Applying the GNS inequality we get

$$\begin{aligned}\int_{B(y,1)} |(u_k)_x|^p (u_k)_x dx &\leq \int_{\mathbf{R}} |(u_k)_x \chi(x-y)|^{p+1} dx \\ &\leq \|(u_k)_x \chi(x-y)\|_2^{\frac{p+3}{2}} \|((u_k)_x \chi(x-y))_x\|_2^{\frac{p-1}{2}} \\ &\leq C \|(u_k)_x\|_{L^2(B(y,2))}^{\frac{p+3}{2}},\end{aligned} \tag{3.3.13}$$

where in the last line, we have used that $\|u_k\|_{H^2}$ is a bounded sequence. By the assumed vanishing, choose k_0 so large that for all $k \geq k_0$

$$\int_{B(y,2)} \rho_k dx < \varepsilon,$$

for all $y \in R$. We can cover the real line with intervals $\cup_{n=0}^{\infty} B(y_n, 2)$, so that each $x \in \mathbf{R}$ belongs to at most ten intervals and $\cup_{n=0}^{\infty} B(y_n, 1)$ still covers the whole line. Using (3.3.13), we obtain

$$
\begin{aligned}
\int_{\mathbf{R}} |(u_k)_x|^p (u_k)_x dx &\leq \int_{\mathbf{R}} |(u_k)_x|^{p+1} dx \\
&\leq \sum_{n=0}^{\infty} \int_{B(y_n,1)} |(u_k)_x|^{p+1} dx \\
&\leq C\varepsilon^{\frac{p-1}{2}} \sum_{n=0}^{\infty} \|(u_k)_x\|^2_{L^2 B(y_n,2)} \\
&\leq 3C\varepsilon^{\frac{p-1}{2}} \|(u_k)_x\|^2_{L^2}
\end{aligned}
$$

which is a contradiction, for sufficiently small $\varepsilon > 0$. Indeed, recall that $\sup_k \|u_k\|_{H^2} < \infty$, while by Lemma 30, $\inf_k \int_{\mathbf{R}} |(u_k)_x|^p (u_k)_x dx > 0$. Hence, vanishing does not take place.

Next, we rule out dichotomy. For contradiction, suppose it occurs. Let $\eta_1, \eta_2 \in C^\infty(\mathbf{R})$, satisfying $0 \leq \eta_1, \eta_2 \leq 1$ and

$$
\eta_1(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2, \end{cases} \quad \eta_2(x) = \begin{cases} 1, & |x| \geq 1, \\ 0, & |x| \leq 1/2. \end{cases}
$$

Dichotomy implies that there exists a subsequence of $\{u_k\}_{k=1}^\infty$ (re-indexed to be $\{u_k\}_{k=1}^\infty$ again) and sequences $\{R_k\}_{k=1}^\infty \in \mathbf{R}$, with $\lim_{k\to\infty} R_k = \infty$ and $\{y_k\}_{k=1}^\infty \in \mathbf{R}$ such that

$$
\lim_{k\to\infty} \int_{\mathbf{R}} |(u_{k,1})_x|^2 dx = \alpha, \quad \lim_{k\to\infty} \int_{\mathbf{R}} |(u_{k,2})_x|^2 dx = \lambda - \alpha, \quad \int_{R_k/5 \leq |x-y_k| < R_k} |(u_k)_x|^2 dx \leq \frac{1}{k},
$$

where

$$
u_{k,1}(x) = u_k(x)\eta_1\left(\frac{x-y_k}{R_k/5}\right), \quad u_{k,2}(x) = u_k(x)\eta_2\left(\frac{x-y_k}{R_k}\right).
$$

Let $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ be sequences of real numbers converging to 1 such that

$$
\int_{\mathbf{R}} |(a_k u_{k,1})_x| = \alpha, \quad \int_{\mathbf{R}} |(b_k u_{k,1})_x| = \lambda - \alpha,
$$

for all k. It is easy to see that the following holds

$$I[u_k] - I[a_k u_{k,1}] - I[b_k u_{k,1}] = \frac{1}{2}\int_{\mathbf{R}} \left(1 - \eta_1^2\left(\frac{x-y_k}{R_k/5}\right) - \eta_2^2\left(\frac{x-y_k}{R_k}\right)\right)\left(|(u_k)_{xx}|^2 + |u_k|^2\right)dx$$
$$+ O\left(\frac{1}{R_k}\right) + O\left(\frac{1}{k}\right) + O\left(|1-a_k^2|\right) + O\left(|1-b_k^2|\right).$$

It follows that

$$I[u_k] \geq m_I(\alpha) + m_I(\lambda - \alpha) + \beta_k,$$

where $\beta_k \to 0$. Taking the limit as $k \to \infty$ we obtain

$$m_I(\lambda) \geq m_I(\alpha) + m_I(\lambda - \alpha),$$

which contradicts the strict sub-additivity condition shown in lemma 3.1.9. Hence dichotomy is not an option.

Finally, using tightness we show existence of a minimizer. We show it only for the I functional, but the steps for the J functional are exactly the same, if not easier. Define $v_k(x) := u_k(x - y_k)$. Since $\{v_k\}_{k=1}^{\infty}$ is bounded on $H^2(\mathbf{R})$ there exists a weakly convergent subsequence to some $v \in H^2(\mathbf{R})$, renamed to $\{v_k\}_{k=1}^{\infty}$ again. From tightness it follows that for all $\varepsilon > 0$ there exists an R_ε satisfying

$$\int_{B^c(0,R_\varepsilon)} |(v_k)_x|^2 dx < \varepsilon. \tag{3.3.14}$$

By the Rellich-Kondrachov theorem $H^1(B(0,R_\varepsilon))$ compactly embeds into $L^2(B(0,R_\varepsilon))$. So, there exists a subsequence of $\{v_k\}_{k=1}^{\infty}$ such that $(v_k)_x \to v_x$ strongly on $L^2(B(0,R_\varepsilon))$. Taking $\varepsilon = 1/n$ and letting $n \to \infty$ in (3.3.14) we can find a subsequence of $\{v_k\}_{k=1}^{\infty}$, again renamed to be the same, so that $(v_k)_x \to v_x$ strongly on $L^2(\mathbf{R})$. With this in hand, we can show that

$$\lim_{k\to\infty} \int_{\mathbf{R}} |(v_k)_x|^p (v_k)_x dx = \int_{\mathbf{R}} |v_x|^p v_x dx. \tag{3.3.15}$$

Indeed,

$$\left|\int_{\mathbf{R}} |(v_k)_x|^p (v_k)_x dx - \int_{\mathbf{R}} |v_x|^p v_x dx\right| \leq C \int_{\mathbf{R}} |(v_k)_x - v_x| (|(v_k)_x|^p + |v_x|^p) dx$$
$$\leq C \|(v_k)_x - v_x\|_2 (\|v\|_{H^2} + \|v_k\|_{H^2})^p \to 0,$$

where we have used the inequality $||x|^p x - |y|^p y| \leq C|x-y| (|x|^p + |y|^p)$ which holds for all real numbers x and y, the Cauchy-Schwartz inequality and the fact that $H^1(\mathbf{R})$ embeds into $L^\infty(\mathbf{R})$.

Finally, the lower semi-continuity of norms with respect to weak convergence and (3.3.15) imply that $m_I(\lambda) = \lim_{k\to\infty} I[v_k] \geq I[v]$, which means that $I[v] = m_I(\lambda)$ and v is the minimizer. Proposition 8 is thus proved in full. □

The next order of business is to derive the Euler-Lagrange equations.

3.3.4 The Euler-Lagrange equations - fourth order formulations

Proposition 9.

- *For $1 < p < 3$ and $\lambda > 0$ there exists a real number ω such that the minimizer of the constrained minimization problem (3.3.1) ϕ_λ satisfies the Euler-Lagrange equation*

$$\phi_\lambda'''' + \omega(\lambda)\phi_\lambda'' + \phi_\lambda + \partial_x(|\phi_\lambda'|^p) = 0, \tag{3.3.16}$$

  where $\omega = \omega(\lambda, \phi_\lambda) = \frac{1}{\lambda}\int_{\mathbf{R}}[|\phi_\lambda''|^2 + |\phi_\lambda|^2 - |\phi_\lambda'|^p \phi_\lambda']dx$.

- *For $1 < p < 5$ and $\lambda > 0$ there exists a function ω such that the minimizer of the constrained minimization problem (3.3.2) ϕ_λ satisfies the Euler-Lagrange equation*

$$\phi_\lambda'''' + \omega(\lambda)\phi_\lambda'' + \phi_\lambda + \partial_x(|\phi_\lambda'|^{p-1}\phi_\lambda') = 0, \tag{3.3.17}$$

  where $\omega = \omega(\lambda, \phi_\lambda) = \frac{1}{\lambda}\int_{\mathbf{R}} |\phi_\lambda''|^2 + |\phi_\lambda|^2 - |\phi_\lambda'|^{p+1} dx$.

Proof. Consider $u_\delta = \frac{\phi_\lambda + \delta h}{\|\phi'_\lambda + \delta h'\|}\sqrt{\lambda}$, where h is a test function. Clearly, u_δ satisfies the constraint and expanding $I[u_\delta]$ in δ we get

$$I[u_\delta] = m_I(\lambda) + \delta \left(\int_{\mathbf{R}} \phi''_\lambda h'' dx + \phi_\lambda h + |\phi'_\lambda|^p h' - \frac{1}{\lambda} (\int_{\mathbf{R}} |\phi''_\lambda|^2 + |\phi_\lambda|^2 - |\phi'_\lambda|^p \phi'_\lambda dx) \int_{\mathbf{R}} \phi'_\lambda h' dx \right) + O(\delta^2).$$

Since $I[u_\delta] \geq m_I[\lambda]$ for all $\delta \in \mathbf{R}$ we conclude that

$$\langle \phi''''_\lambda + \phi_\lambda + \omega(\lambda)\phi''_\lambda + (|\phi'_\lambda|^p)', h \rangle = 0,$$

with $\omega = \frac{1}{\lambda} \int_{\mathbf{R}} |\phi''_\lambda|^2 + |\phi_\lambda|^2 - |\phi'_\lambda|^p \phi'_\lambda dx$, holds for all h, i.e., ϕ_λ is a distributional solution of the Euler-Lagrange Equation (3.3.16). For the minimizers of (3.3.2), we proceed analogously to establish (3.3.17). □

3.3.5 The Euler-Lagrange equations - second order formulation

Proposition 10.

- *For* $1 < p < 3$*, there exists a function* $\omega(\lambda)$ *such that for all* $\lambda > 0$*, the minimizer of the constrained minimization problem* (3.3.3) ϕ_λ *satisfies the Euler-Lagrange equation*

$$\partial_x^2 \phi_\lambda + \partial_x^{-2} \phi_\lambda + \omega(\lambda)\phi_\lambda + |\phi_\lambda|^p = 0, \tag{3.3.18}$$

  where

$$\omega = \omega(\lambda, \phi_\lambda) = \frac{1}{\lambda} \int_{\mathbf{R}} [|\partial_x \phi_\lambda|^2 + |\partial_x^{-1} \phi_\lambda|^2 - |\phi_\lambda|^p \phi_\lambda] dx. \tag{3.3.19}$$

  In addition, the linearized operator $\mathscr{L}_+ := -\partial_x^2 - \partial_x^{-2} - \omega(\lambda) - p|\phi_\lambda|^{p-2}\phi_\lambda$ *satisfies* $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$*. In fact,* $\mathscr{L}_+$ *has exactly one negative eigenvalue.*

- *For* $1 < p < 5$*, there is* $\omega(\lambda)$*, such that for all* $\lambda > 0$*, the minimizer of the constrained* *minimization problem* (3.3.4) ϕ_λ *weakly satisfies the Euler-Lagrange equation*

$$\partial_x^2\phi_\lambda + \partial_x^{-2}\phi_\lambda + \omega(\lambda)\phi_\lambda + |\phi_\lambda|^{p-1}\phi_\lambda = 0, \tag{3.3.20}$$

where $\omega = \omega(\lambda, \phi_\lambda) = \frac{1}{\lambda}\int_{\mathbf{R}} |\phi_\lambda'|^2 + |\partial_x^{-1}\phi_\lambda|^2 - |\phi_\lambda|^{p+1}dx$. *The operator* $\mathscr{L}_+ = -\partial_x^2 - \partial_x^{-2} - \omega(\lambda) - p|\phi_\lambda|^{p-1}$ *has* $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$ *and it possesses exactly one negative eigenvalue.*

Proof. The derivation of the Euler-Lagrange equations is pretty similar to the one presented in the fourth order context, Proposition 9. For an arbitrary test function $h \in H^{-2} \cap H^2$ and $\delta \in \mathbf{R}$, consider $u_\delta = \sqrt{\lambda}\frac{\phi_\lambda + \delta h}{\|\phi_\lambda + \delta h\|}$. Since u_δ satisfies the constraint $\|u_\delta\|_{L^2}^2 = \lambda$, expand $I[u_\delta]$ in powers of δ. We get

$$\begin{aligned}
\mathscr{I}[u_\delta] = m_{\mathscr{I}}(\lambda) &+ \delta\bigg(\int_{\mathbf{R}} \phi_\lambda' h' dx + \int_{\mathbf{R}} \partial_x^{-1}\phi_\lambda \partial_x^{-1} h dx - \int_{\mathbf{R}} |\phi_\lambda|^p h \\
&- \frac{1}{\lambda}\int_{\mathbf{R}} |\phi_\lambda'|^2 + |\partial_x^{-1}\phi_\lambda|^2 - |\phi_\lambda|^p\phi_\lambda dx \int_{\mathbf{R}} \phi_\lambda h dx\bigg) \\
&+ \frac{\delta^2}{2}\left(\int_{\mathbf{R}} |h'|^2 + |\partial_x^{-1}h|^2 dx - p\int_{\mathbf{R}} |\phi_\lambda|^{p-2}\phi_\lambda |h|^2 dx\right) \\
&- \frac{\delta^2}{2}\frac{1}{\lambda}\left(\int_{\mathbf{R}} |\phi_\lambda'|^2 + |\partial_x^{-1}\phi_\lambda|^2 - |\phi_\lambda|^p\phi_\lambda dx\right)\int_{\mathbf{R}} |h|^2 dx \\
&+ \delta^2 \frac{\int_{\mathbf{R}} \phi_\lambda h dx}{\lambda}\left((p+1)\int_{\mathbf{R}} |\phi_\lambda|^p h dx - 2\Big(\int_{\mathbf{R}} \phi_\lambda' h' dx + \int_{\mathbf{R}} \partial_x^{-1}\phi_\lambda \partial_x^{-1} h dx\Big)\right) \\
&+ \delta^2\left(\frac{\int_{\mathbf{R}} \phi_\lambda h dx}{\lambda}\right)^2\left(2\int_{\mathbf{R}} |\phi_\lambda'|^2 + |\partial_x^{-1}\phi_\lambda|^2 dx - \frac{p+3}{2}\int |\phi_\lambda|^p\phi_\lambda dx\right) \\
&+ O(\delta^3).
\end{aligned}$$

Since $\mathscr{I}[u_\delta] \geq m_{\mathscr{I}}[\lambda]$ for all $\delta \in \mathbf{R}$, we conclude that

$$\left\langle \phi_\lambda'' + \partial_x^{-2}\phi_\lambda + \omega\phi_\lambda + |\phi_\lambda|^p, h\right\rangle = 0, \tag{3.3.21}$$

with $\omega = \frac{1}{\lambda}\int_{\mathbf{R}} |\phi_\lambda'|^2 + |\partial_x^{-1}\phi_\lambda|^2 - |\phi_\lambda|^p\phi_\lambda dx$, holds for all h. That is ϕ_λ is a distributional solution of the Euler-Lagrange Equation. According to Proposition 6, this solution is, in fact, an element of H^3 and (3.3.18) is satisfied in the sense of L^2 functions.

The fact that ϕ_λ is a minimizer also implies that the coefficient in front of δ^2 must be nonnegative. Choosing h orthogonal to ϕ_λ with $\|h\| = 1$, we conclude that

$$\left\langle -h'' - \partial_x^{-2} h - p|\phi_\lambda|^{p-2}\phi_\lambda h - \omega h, h \right\rangle \geq 0,$$

i.e., the operator $\mathscr{L}_+ = -\partial_x^2 - \partial_x^{-2} - \omega(\lambda) - p|\phi_\lambda|^{p-2}\phi_\lambda$, satisfies $\langle \mathscr{L}_+ h, h\rangle \geq 0$ for all h orthogonal to ϕ_λ with $\|h\| = 1$, which implies that it has at most one negative eigenvalue. On the other hand, recalling that $\int_{\mathbf{R}} |\phi_\lambda|^p \phi_\lambda dx > 0$, we compute

$$\langle \mathscr{L}_+ \phi_\lambda, \phi_\lambda \rangle = -(p-1)\int_{\mathbf{R}} |\phi_\lambda|^p \phi_\lambda dx < 0. \tag{3.3.22}$$

So, $\mathscr{L}_+$ has at least one negative eigenvalue. Hence it has exactly one negative eigenvalue. The second part of the proposition is proven similarly expanding $\mathscr{J}[u_\delta]$ in powers of δ. $\square$

The next corollary is a consequence of the Pohozaev's identities and the fact that our waves are minimizers[4] .

Corollary 5. *Let ϕ_λ be a minimizer for either one of* (3.3.1), (3.3.3), (3.3.2)*,* (3.3.4). *Then, for each $\lambda > 0$, $\omega < 2$.*

Proof. Let ϕ_λ be a minimizer for (3.3.1), so in particular $\|\phi_\lambda'\|_{L^2}^2 = \lambda$. Then, we have $m(\lambda) < \lambda$, as established in the proof of Lemma 30. Therefore

$$I(\phi_\lambda) = \frac{1}{2}\int_{\mathbf{R}} |\phi_\lambda''|^2 + |\phi_\lambda|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}} |\phi_\lambda'|^p \phi_\lambda' dx < \lambda = \int_{\mathbf{R}} |\phi_\lambda'|^2 dx.$$

Rearranging the terms yields

$$\frac{1}{2}\int_{\mathbf{R}} |\phi_\lambda''|^2 + |\phi_\lambda|^2 dx < \int_{\mathbf{R}} |\phi_\lambda'|^2 dx + \frac{1}{p+1}\int_{\mathbf{R}} |\phi_\lambda'|^p \phi_\lambda' dx. \tag{3.3.23}$$

[4]It is possible that the conclusions of Corollary 5 are valid, by just assuming that ϕ satisfies the elliptic profile equations, without being a constrained minimizer, but we leave this open at the present time.

Since ϕ_λ also satisfies (3.2.11), we get

$$\frac{1}{2}\int_{\mathbf{R}} |\phi_\lambda''|^2 + |\phi_\lambda|^2 dx = \frac{\omega}{2}\int_{\mathbf{R}} |\phi_\lambda'|^2 dx + \frac{1}{2}\int_{\mathbf{R}} |\phi_\lambda'|^p \phi_\lambda' dx. \tag{3.3.24}$$

Combining (3.3.23) and (3.3.24), we have that

$$\left(\frac{\omega}{2} - 1\right)\int_{\mathbf{R}} |\phi_\lambda'|^2 dx = -\frac{p-1}{2(p+1)}\int_{\mathbf{R}} |\phi_\lambda'|^p \phi_\lambda' dx.$$

Recalling again that $\int_{\mathbf{R}} |\phi_\lambda'|^p \phi_\lambda' dx > 0$, we conclude that $\omega < 2$. Similarly for the minimizers of the other three variational problems. $\square$

3.3.6 Weak non-degeneracy of the waves and the proof of Theorem 7

Our first order of business is to show that $\phi_\lambda \perp Ker[\mathscr{L}_+]$. Let us work with the second order version, for which $\mathscr{L}_+ = -\partial_x^2 - \partial_x^{-2} - \omega - p|\phi_\lambda|^{p-1}$, the other one being similar. Take any element $\Psi \in Ker[\mathscr{L}_+], \|\Psi\|_{L^2} = 1$. Note that by Proposition 10, we have that $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$. It follows that $\Psi - \lambda^{-1}\langle\Psi, \phi_\lambda\rangle\phi_\lambda \perp \phi_\lambda$, since by construction $\|\phi_\lambda\|^2 = \lambda$. Thus,

$$0 \leq \langle \mathscr{L}_+[\Psi - \lambda^{-1}\langle\Psi,\phi_\lambda\rangle\phi_\lambda], \Psi - \lambda^{-1}\langle\Psi,\phi_\lambda\rangle\phi_\lambda\rangle = \lambda^{-2}\langle\Psi,\phi_\lambda\rangle^2\langle\mathscr{L}_+\phi_\lambda,\phi_\lambda\rangle \leq 0,$$

where in the last inequality we have used (3.3.22). We conclude that $\langle\Psi,\phi_\lambda\rangle = 0$, otherwise we have a contradiction in the above chain of inequalities.

The proof of Theorem 7 consists of applying Lemma 1 to $\mathscr{H} = \mathscr{L}_+$ and $\xi_0 := \lambda^{-1/2}\phi_\lambda$. We have shown that $\mathscr{L}_+|_{\{\phi_\lambda\}^\perp} \geq 0$ and (3.3.22) establishes $\langle\mathscr{L}_+\phi_\lambda,\phi_\lambda\rangle < 0$. This verifies all the assumptions in Lemma 1, which implies $\langle\mathscr{L}_+\phi_\lambda,\phi_\lambda\rangle < 0$. Finally, corollary 4 implies the stability.

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