Chaos Expansion of Heat Equations With White Noise Potentials

Yaozhong HU
Department of Mathematics, University of Kansas
405 Snow Hall, Lawrence, KS 66045-2142
Email: hu@math.ukans.edu

Abstract

The asymptotic behavior as $t\to\infty$ of the solution to the following stochastic heat equations

$$\frac{\partial u_t}{\partial t}=\frac{1}{2}\sum_{i=1}^d\frac{\partial^2 u_t}{\partial x_i^2}+w\diamond u_t\,,\quad 0<t<\infty\,,\quad x\in\mathbb{R}^d\,,\qquad u_0(x)=1$$

is investigated, where w is a *space-time white noise* or a *space white noise*. The use of $\diamond$ means that the stochastic integral of Itô (Skorohod) type is considered. When $d=1$, the exact $\mathcal{L}_2$ Lyapunov exponents of the solution are studied. When the noise is space white and when $d<4$ it is shown that the solution is in some "flat" $\mathcal{L}_2$ distribution spaces. The Lyapunov exponents of the solution in these spaces are also estimated. The exact rate of convergence of the solution by its first finite chaos terms are also obtained.

1 Introduction

The Schrödinger operators with random potential have been widely studied (see [3], [4] and the references therein).

Let $(\Omega,\mathcal{F},P)$ be a probability space. The expectation on this probability space is denoted by $\mathbb{E}$. The set of all square integrable random variables on $(\Omega,\mathcal{F},P)$ is denoted by $\mathcal{L}_2:=L^2(\Omega,\mathcal{F},P)$. Let $\mathbb{R}^d$ be the d dimensional Euclidean space. Let $w_t(x)$ be a (generalized) Gaussian random field with parameters $t\in\mathbb{R}_+$ and $x\in\mathbb{R}^d$, *i.e.* $w_t(x)$ is a (generalized)

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Gaussian random variable on $(\Omega, \mathcal{F}, P)$ for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$. If $w_t(x)$ satisfies formally the following equation:

$$\mathbb{E}\left(\int_{\mathbb{R}_+\times\mathbb{R}^d} f(s,x)w_s(x)dsdx \int_{\mathbb{R}_+\times\mathbb{R}^d} g(s,x)w_s(x)dsdx\right) = \int_{\mathbb{R}_+\times\mathbb{R}^d} f(s,x)g(s,x)dsdx$$

for all $f, g \in L^2(\mathbb{R}_+ \times \mathbb{R}^d, dsdx)$, then $w_t(x)$ is called a *space-time white noise.* If $w(x)$ is a (generalized) Gaussian random field with parameters $x \in \mathbb{R}^d$ satisfying that for all $f, g \in L^2(\mathbb{R}^d, dx)$,

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(x)w(x)dx \int_{\mathbb{R}^d} g(x)w(x)dx\right) = \int_{\mathbb{R}^d} f(x)g(x)dx\,,$$

then $w(x)$ is called a *space white noise.* The rigorous definitions of various kinds of white noise are discussed in many references (see for instance [12] and [17]). In those books, the stochastic integrals of Itô type, the multiple stochastic integrals of Itô type, and the chaos expansions are discussed (see also [22], [21], [16]). These concepts will be used freely. However, in this paper only these two types of noise will be discussed and are denoted generically by a single w.

Denote $\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(x)$. The following stochastic partial differential equation

$$\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t + w \diamond u_t \tag{1.1}$$

is called stochastic heat equation with white noise potential, where $\diamond$ means that the stochastic integrals involved in (1.1) is of Itô (Skorohod) type. For instance, $\int u(s,y) \diamond w_s(y)dsdy$ is the same as $\int u(z)W(dz)$ in [22], where $z = (s,y)$. We assume that $u_0(x) = 1$ in this introduction.

When w is space-time white, this equation was studied by several authors, for example, in [20] , [21], [22], and the the references therein. It is known that when $d = 1$ the solution exists for all $t > 0$. In this case the exact $\mathcal{L}_2$ Lyapunov exponent of the solution is investigated and it is shown that

$$\lim_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}\ |u_t(x)|^2\right) = 1/4\,.$$

The main topics of this paper is stochastic heat equation with space white noise potential. Till the end of this section we assume that the noise is space white unless stated otherwise. When $d = 1$, it is known [28] that the solution is in $\mathcal{L}^2$

$$0 < \liminf_{t\to\infty} \frac{1}{t^3} \log \mathbb{E}\ |u_t(x)|^2 \le \limsup_{t\to\infty} \frac{1}{t^3} \log \mathbb{E}\ |u_t(x)|^2 < \infty\,.$$

The above upper bound and lower bounds will be improved. Moreover, my approach allows to find the exact rate of convergence of the solution by finite chaos expansion. Let $u_t(x) =$

$\sum_{n=0}^{\infty} I_n(f_n(t,x))$ be the chaos expansion of the solution and let $u_t^N(x) = \sum_{n=0}^{N} I_n(f_n(t,x))$. Then roughly speaking, $\mathbb{E}\ |u_t(x) - u_t^N(x)|^2$ is of the order $N^{-N/2}$ as $N \to \infty$. This is a surprisingly high rate of convergence. The same rate of convergence of $u_t^N(x)$ to $u_t(x)$ holds when the noise is space-time white. Professor Shiga communicated to me that he obtained limit $\limsup_{t\to\infty} \frac{1}{t^3} \log \mathbb{E}\ \log |u_t(x)|^p$ for all $p \geq 1$ if the initial condition is bounded, nonnegative, continuous (and is not identically zero). However, his paper has not been written yet. I am not sure if his approach can be applied to obtaining more precise rate estimate.

Since the noise is not white in time, we expect nicer property of the solution. In fact this is the case. We will show that the solution is more "regular". When $d = 2$, it is known that the solution is not in L^2 when the noise is space-time white. However, when the noise is space-white, the following interesting phenomena is shown: the solution to (1.1) exists in $\mathcal{L}^2$ when $t < 2$ and is not in $\mathcal{L}_2$ when $t > 2\pi$. Nevertheless, the solution is shown to be in some nicer distribution spaces (see the explanation below).

The main tool of this paper is the Itô-Wiener chaos expansion, which is also used in [12], [13], [14], [15], [16], [18], [19], [21], [22] and the references therein.

It is noted that if $u_t(x) = \sum_{n=0}^{\infty} I_n(f_n(t,x))$ is the formal chaos expansion of $u_t(x)$ (the solution to (1.1) when the noise is space white), then when $d < 4$, each chaos $I_n(f_n(t,x))$ is in $\mathcal{L}_2$. Hence, the stochastic heat equation with space white noise potential is more regular than people presumed. This phenomenon seems to be very encouraging: our living space has 3 dimensions. What a magnificent coincidence!

We introduce a new type of Hilbert space formally defined by

$$\mathcal{S}_\gamma := \{F = \sum_{n=0}^{\infty} F_n\ ; \quad \sum_{n=0}^{\infty} (n!)^\gamma \mathbb{E}\ |F_n|^2 < \infty\}\ ,$$

where $\gamma \in \mathbb{R}$ and F_n is the n-th chaos of F. It is clear that these spaces are subspaces of the distribution space introduced in [12]. Since we do not introduce weights as in [21], we shall call these spaces "flat" $\mathcal{L}_2$ type of distribution (or test) spaces. An element in $\mathcal{S}_\gamma$ has the following property: each of its chaos is in $\mathcal{L}_2$. It is shown that when $d < 4$ and when the noise is space white, there is a γ_0 such that

1) $u_t(x)$ is in $\mathcal{S}_\gamma$ for all $\gamma < \gamma_0$, $t \in (0,\infty)$ and $x \in \mathbb{R}^d$;
2) $u_t(x)$ is not in $\mathcal{S}_\gamma$ for all $\gamma > \gamma_0$, $t \in (0,\infty)$ and $x \in \mathbb{R}^d$;
3) $u_t(x)$ is in $\mathcal{S}_{\gamma_0}$ for small t and $u_t(x)$ is not in $\mathcal{S}_{\gamma_0}$ for large t.

Therefore, the regularity of the solution with respect to the noise is completely determined from the point of view of chaos expansion. Moreover, the Lyapunov exponents of the solution in $\mathcal{S}_\gamma$ for all $\gamma < \gamma_0$ are also estimated.

When the noise is space-time white and $d \geq 2$, or when the noise is space white and $d \geq 4$, none of the chaos of the solution to (1.1) is in $\mathcal{L}_2$. A renormalization procedure is proposed. It is an extension of those dealt with in [15].

The results obtained are extended to general elliptic operators by a comparison argument.

To end this introduction, let us mention some relevant results.

When w is white with respect to time t (but not white with respect to space variable x), Carmona, Molchanov, Viens, [5], [6], [7] have studied the almost sure Lyapunov exponent of the solution, *i.e.* the upper bound and lower bound of $\frac{1}{t}\log u_t(x)$ as $t \to \infty$. If (1.1) is replaced by its discrete analogue, then the moment Lyaponuv exponents were also studied in [5].

When $w_t(x)$ is space-time white and when $d \geq 2$, Nualart, Rozovskii, Zakai, Holden, Øksendal, Ubøe, Zhang, Potthoff, Våge, Watanabe [21], [22], [12], [23] studied the solution to (1.1) in some other types of generalized distribution spaces.

Piatnitski, Zheng, and Zhao [24] applied a discretization procedure to show the regularity of the solution with respect to x for a class of nonlinear stochastic heat equation.

In most of our estimates, d can also be a real number. Since it is popular to study fractional dimensions, we write $d < 4$ instead of $d \leq 3$. Throughout the paper, C denotes a generic positive constant whose value may be different in different appearances.

2 Space-Time White Noise: $d = 1$

In this section we consider the stochastic heat equation with $d = 1$. We also study the $\mathcal{L}_2$ asymptotic behavior of the solutions as t goes to infinity. Throughout the paper we assume that u_0 is deterministic and we denote

$$P_t(x) = (2\pi t)^{-d/2} e^{-\frac{|x|^2}{2t}}\,, \quad t \in \mathbb{R}_+\,,\ x \in \mathbb{R}^d\,.$$

Definition 2.1 *A measurable function $u : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is called a solution to Eq. (1.1) if $\int_0^t \int_{\mathbb{R}^d} P_{t-s}(x-y)u_s(y) \diamond w_s(y)dsdy$ exists as an element in $\mathcal{L}^2$ (or in S_γ for some $\gamma \in \mathbb{R}$) for every $x \in \mathbb{R}^d$ and $t \in (0,T]$ and the following equation is satisfied*

$$u_t(x) = P_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} P_{t-s}(x-y)u_s(y) \diamond w_s(y)dsdy\,, \quad \forall\ \ t \in \mathbb{R}_+\,,\ x \in \mathbb{R}^d\,, \tag{2.1}$$

where $\diamond$ denotes the stochastic integral of Itô type, i.e. the same integral as in [22] (see also [17], [12]).

Let us consider the following formal chaos expansion of the solution

$$u_t(x) = P_t u_0(x) + \sum_{n=1}^{\infty} I_n(f_n(t,x))\,, \tag{2.2}$$

where

$$\begin{aligned} f_n(t,x;s_1,\cdots,s_n;y_1,\cdots,y_n) \;&=\; \mathrm{Sym}\Big[\int_{\mathbb{R}^d} P_{t-s_n}(x-y_n)P_{s_n-s_{n-1}}(y_n-y_{n-1}) \\ &\qquad \cdots P_{s_2-s_1}(y_2-y_1)P_{s_1}(y_1-\tilde{y})u_0(\tilde{y})d\tilde{y}\Big] \end{aligned} \tag{2.3}$$

and

$$I_n(f_n(t,x)) = \int_{[0,t]^n\times\mathbb{R}^{nd}} f_n(t,x;s_1,\cdots,s_n;y_1,\cdots,y_n) w_{s_1}(y_1)\diamond\cdots\diamond w_{s_n}(y_n) ds_1 dy_1\cdots ds_n dy_n \tag{2.4}$$

is the multiple Itô integral with respect to the deterministic kernel $f_n(t,x)$. Here Sym denotes the symmetrization with respect to the ($d+1$ dimensional) variables (s_1,y_1), (s_2,y_2), $\cdots$, and (s_n,y_n).

In this paper we will not go into details of the existence and uniqueness of the solution. Readers are referred to [9], [21], [22], and the references therein. In this section $u_t(x)$ given by (2.2) (when $d=1$) will be discussed.

Theorem 2.1 *When $d=1$, each $I_n(f_n(t,x))$ is well-defined as an element of $\mathcal{L}_2$ and (2.2) is convergent in $\mathcal{L}_2$.*

1) If the initial value $u_0(x)$ is identically equal to 1, then the $\mathcal{L}_2$ norm of $u_t(x)$ defined by (2.2) is independent of x and the following estimate holds:

$$\lim_{t\to\infty}\frac{\log\mathbb{E}\,(|u_t(x)|^2)}{t} = \frac{1}{4}, \quad \forall\, x\in\mathbb{R}\,. \tag{2.5}$$

2) If u_0 is uniformly bounded away from ∞, i.e. there is a positive constant C such that $\sup_{x\in\mathbb{R}^d}|u_0(x)|\le C$, then

$$\limsup_{t\to\infty}\frac{\log\mathbb{E}\,(|u_t(x)|^2)}{t} \le \frac{1}{4}, \quad \forall\, x\in\mathbb{R}\,. \tag{2.6}$$

3) If u_0 is uniformly bounded away from 0, i.e. there is a positive constant C such that $\inf_{x\in\mathbb{R}^d} u_0(x)\ge C$, then

$$\liminf_{t\to\infty}\frac{\log\mathbb{E}\,(|u_t(x)|^2)}{t} \ge \frac{1}{4}, \quad \forall\, x\in\mathbb{R}\,. \tag{2.7}$$

Proof Denote $v(t,x)=\mathbb{E}\,(|u_t(x)|^2)$. Then it is easy to see that $v(t,x)$ satisfies the following (deterministic) equation:

$$v(t,x) = |P_t u_0(x)|^2 + \int_0^t\int_{\mathbb{R}} P_{t-s}^2(x-y)v(s,y)dsdy\,.$$

When the initial condition $u_0(x)\equiv 1$, $v(t,x)$ satisfies

$$v(t,x) = 1 + \int_0^t\int_{\mathbb{R}} P_{t-s}^2(x-y)v(s,y)dsdy\,.$$

By iteration, we obtain

$$v(t,x) = \sum_{n=0}^{\infty}\Theta_n\,,$$

where

$$\begin{aligned}\Theta_n &= \int_{T_n} (4\pi)^{-n/2}(t-s_n)^{-1/2}(s_n-s_{n-1})^{-1/2}\cdots(s_2-s_1)^{-1/2}ds \\ &= (4\pi)^{-n/2}\frac{\Gamma(1/2)^n}{\Gamma(n/2+1)}t^{n/2} \\ &= \frac{1}{\Gamma(n/2+1)}\left(\frac{t}{4}\right)^{n/2}\end{aligned}$$

Thus when $u_0(x)\equiv 1$,

$$\mathbb{E}\ |u_t(x)|^2 = \sum_{n=0}^{\infty}\frac{1}{\Gamma(n/2+1)}\left(\frac{t}{4}\right)^{n/2}. \tag{2.8}$$

This means that the $\mathcal{L}_2$ norm of the solution to Eq (1.1) is independent of x when the initial condition is identically 1. The summation in (2.8) is given by the Mittag-Leffler function $E_{\frac{1}{2}}(\sqrt{t/4})$, whose asymptotic behavior as $t\to\infty$ of this function is known (see [10]), *i.e.*

$$\mathbb{E}\ |u_t(x)|^2 = E_{\frac{1}{2}}(\sqrt{t/4}) = 2\exp(\frac{t}{4}) + O(\frac{1}{t}),\quad (\text{as}\quad t\to\infty).$$

This proves (2.5).

(2.6) and (2.7) can be proved in a similar way. □

Let $u_t^N(x)$ be the sum of first finite terms, $u_t^N(x) = P_t u_0(x) + \sum_{n=1}^N I_n(f_n(t,x))$. Then from (2.8) it follows that

Theorem 2.2 *(1) If there is a positive constant C such that $\sup_{x\in\mathbb{R}^d}|u_0(x)|\le C$, then there is a finite positive constant C_1, such that*

$$\sup_{0\le t\le T,x\in\mathbb{R}}\mathbb{E}\ |u_t(x)-u_t^N(x)|^2 \le C_1\frac{T^N}{4^N\Gamma(N/2+1)} \tag{2.9}$$

3) If there is a positive constant C such that $\inf_{x\in\mathbb{R}^d}u_0(x)\ge C$, then there is a finite positive constant C_2, such that

$$\inf_{x\in\mathbb{R}}\mathbb{E}\ |u_T(x)-u_T^N(x)|^2 \ge C_2\frac{T^N}{4^N\Gamma(N/2+1)} \tag{2.10}$$

Remark 1 *It is a direct consequence of (2.6) and (2.7) that if there are positive constants c and C such that $c\le u_0(x)\le C$ for all $x\in\mathbb{R}^d$, then for all $x\in\mathbb{R}^d$,*

$$\lim_{t\to\infty}\frac{\log\mathbb{E}\ (|u_t(x)|^2)}{t} = \frac{1}{4}.$$

3 Space-Time White Noise: $d \geq 2$

It is obvious that when $d \geq 2$, the formal expansion (2.2) is not convergent in $\mathcal{L}_2$ when $u_0 = 1$. In fact, one can check that the $\mathcal{L}_2$ norm of each chaos is ∞. In physics, it is usual to obtain something meaningful out of infinity by renormalization (see [11], [25], [26]). In this section we shall renormalize the solution following the idea of [15], where we dealt with the renormalization of the self-intersection local time of Brownian motions.

Let us consider the following stochastic heat equation with approximate space-time white noise

$$\frac{\partial u_t^\varepsilon}{\partial t} = \frac{1}{2}\Delta u_t^\varepsilon + w_t^\varepsilon \diamond u_t^\varepsilon\,, \tag{3.1}$$

where $u_0^\varepsilon(x) = u_0(x)$ is given and is deterministic. For simplicity, we consider the regularization only in space variable, *i.e.*

$$w_t^\varepsilon(x) = \int_{\mathbb{R}^d} P_\varepsilon(x-y) w_t(y) dy\,.$$

Since $w_t^\varepsilon(x)$ exists only in the distribution sense (there is singularity with respect to time), we need to give meaning to the solution to the above equation. Let us denote

$$P_{t,\varepsilon}(z) f(x) = \int_{\mathbb{R}^d} P_t(x-y) P_\varepsilon(y-z) f(y) dy\,.$$

Definition 3.1 *$u_t^\varepsilon(x)$ is called a solution to (3.1) if*

$$u_t^\varepsilon = P_t u_0 + \int_{\mathbb{R}^d}\int_0^t P_{t-s,\varepsilon}(z) u_s^\varepsilon \diamond w_s(z) ds dz\,, \tag{3.2}$$

where $\diamond$ means that the Itô type of stochastic integral is considered.

Let $I_n(f_n(t,x))$ be defined by (2.4) and

$$\begin{aligned}
& f_n^\varepsilon(t,x;s_1,\cdots,s_n;z_1,\cdots,z_n) \\
= \; & \mathrm{Sym}\Big[P_{t-s_n,\varepsilon}(z_n) P_{s_n-s_{n-1},\varepsilon}(z_{n-1}) \cdots P_{s_2-s_1,\varepsilon}(z_1) P_{s_1} u_0(x)\Big] \\
= \; & \mathrm{Sym}\Big[\int_{\mathbb{R}^{(n+1)d}} P_{t-s_n}(x-y_n) P_{s_n-s_{n-1}}(y_n-y_{n-1}) \cdots P_{s_2-s_1}(y_2-y_1) \\
& P_\varepsilon(y_n-z_n) \cdots P_\varepsilon(y_1-z_1) P_{s_1}(y_1-\tilde{y}) u_0(\tilde{y}) dy_1 \cdots dy_n d\tilde{y}\Big]\,,
\end{aligned} \tag{3.3}$$

where Sym is the symmetrization with respect to (s_1,z_1), $\cdots$, (s_{n-1},z_{n-1}) and (s_n,z_n). It is easy to verify that the following chaos expansion

$$u^\varepsilon(t,x) = P_t u_0(x) + \sum_{n=1}^\infty I_n(f_n^\varepsilon(t,x))\,, \tag{3.4}$$

is in $\mathcal{S}_\gamma$ for some $\gamma \in \mathbb{R}$. Moreover, $u^\varepsilon(t,x)$ is a solution to (3.1) in $\mathcal{S}_\gamma$. If we formally let $\varepsilon \to 0$ in (3.3), we obtain (2.3). Let us compute $\mathbb{E}\,|I_n(f_n(t,x))|^2$ for $u_0(x) \equiv 1$. Denote it by Θ_n (Θ_n will be different in different sections. But it is the same in each section. It will always be relevant to the $\mathcal{L}_2$ norm of the solution to stochastic heat equation.) To compute Θ_n here and later in Section 5, the following will be used.

Lemma 3.1 *Let s, t, a, b be positive numbers. Then*

$$\int_{\mathbb{R}^{2d}} P_a(x-z)P_s(x-y)P_t(z-\tilde{y})P_b(y-\tilde{y})dyd\tilde{y} = (2\pi(a+b+s+t))^{-d/2} P_c(x-z),$$

where

$$c = \frac{a(b+s+t)}{a+b+s+t}.$$

Let us return to the computation of Θ_n. Integrating with respect to $z_1, \cdots, z_n$, we obtain

$$\begin{aligned}\Theta_n &= \int P_{t-s_n}(x-y_n)\cdots P_{s_2-s_1}(y_2-y_1)P_{t-s_n}(x-\tilde{y}_n)\cdots P_{s_2-s_1}(\tilde{y}_2-\tilde{y}_1) \\ &\qquad P_{2\varepsilon}(y_n-\tilde{y}_n)\cdots P_{2\varepsilon}(y_1-\tilde{y}_1)dsdyd\tilde{y}.\end{aligned}$$

It follows from Lemma 3.1

$$\Theta_n = \int_{T_n} \prod_{i=1}^{n-1} [2\pi(2\varepsilon+\kappa_i+2s_{i+1}-2s_i)]^{-d/2} [2\pi(\kappa_n+2(t-s_n))]^{-d/2} ds, \qquad (3.5)$$

where $\kappa_1 = 2\varepsilon$ and

$$\kappa_{i+1} = \frac{2\varepsilon(\kappa_i+2s_{i+1}-2s_i)}{2\varepsilon+\kappa_i+2s_{i+1}-2s_i}, \quad i = 1, 2, \cdots, n-1.$$

It is obvious that

$$\kappa_{i+1} \geq \frac{2\varepsilon\kappa_i}{2\varepsilon+\kappa_i}.$$

From this it follows that

$$\kappa_i \geq \varepsilon/i.$$

Therefore

$$\Theta_n \leq (4\pi)^{-nd/2} \int_{T_n} \prod_{i=1}^{n-1} (\varepsilon+s_{i+1}-s_i)^{-d/2} (\varepsilon/(2n)+t-s_n)^{-d/2} ds.$$

When $d > 2$, it is easy to see that for all $t > 0$

$$\int_0^t (\kappa+t-u)^{-d/2} du \leq \frac{1}{d/2-1}\kappa^{-d/2+1}.$$

Thus we see that when $d > 2$,

$$\Theta_n \leq C^n \varepsilon^{(-d/2+1)n}. \qquad (3.6)$$

When $d=2$ and when $a<1/e$, $b>e$,

$$\int_0^t (a/b+t-u)^{-1}du \quad \le \quad -\log a+\log b+\log(t+a/b)$$

It is easy to see that when $a,b,c\ge 1$, $a+b\le 2ab$ and $a+b+c\le 3abc$. Thus when $\varepsilon<1/e$, we obtain

$$\int_0^t (\varepsilon+t-u)^{-1}du \le 2\log(e+T)(-\log\varepsilon)$$

and

$$\int_0^t (\varepsilon/(2n)+t-u)^{-1}du \le 3\log(2n)\log(e+T)(-\log\varepsilon)\,.$$

When $\varepsilon<e^{-1}$,

$$\Theta_n \le 3\log(2n)2^{n-1}\log(e+T)^n(-\log\varepsilon)^n\,.$$

Hence

$$\Theta_n \le \begin{cases} C^n\varepsilon^{-(\frac{d-2}{2})n} & \text{If } d>2 \\ \\ C^n(-\log\varepsilon)^n(\log(e+T))^n & \text{If } d=2\,. \end{cases} \tag{3.7}$$

We introduce the following generalized second quantization operator $\Gamma(\lambda,\gamma)$ as follows

$$\Gamma(\lambda,\gamma)F=\sum_{n=0}^{\infty}(n!)^{\lambda}\gamma^n F_n$$

if $F=\sum_{n=0}^{\infty}F_n$ is the Wiener-Itô's chaos expansion of F. Introduce also $\Gamma(\gamma)F=\Gamma(1,\gamma)F$.

By (3.7), it follows

Theorem 3.2 *1) Let $d>2$ and let u_0 be bounded on $\mathbb{R}^d$. Then for all $\lambda<0$, $\Gamma\left(\lambda,\varepsilon^{\frac{d-2}{4}}\right)u_t^\varepsilon$ is a uniformly bounded sequence in $\mathcal{L}^2$ with respect to $\varepsilon\in(0,e^{-1}]$, $t\in(0,\infty)$ and $x\in\mathbb{R}^d$.*

2) Let $d=2$ and let u_0 be bounded on $\mathbb{R}^d$. Then for all $\lambda<0$, $T\in(0,\infty)$ and $x\in\mathbb{R}^d$, $\Gamma\left(\lambda,\sqrt{-\log\varepsilon}\right)u_t^\varepsilon$ is a uniformly bounded sequence in $\mathcal{L}^2$ with respect to $\varepsilon\in(0,e^{-1}]$, $t\in(0,T]$ and $x\in\mathbb{R}^d$.

It is known that when $d\ge 2$, $u_t(x)$ is in general not a square integrable random variable. Thus we expect that when $\varepsilon\to 0$, u_t^ε is not bounded in $\mathcal{L}_2$. In fact a more precise result is stated.

Theorem 3.3 *Let the initial condition $f(x)\ge c$, where $c\in(0,\infty)$ is a constant. Then*

1) When $d>2$, for all continuous function $\gamma(\varepsilon)$ on $(0,\infty)$ with $\varepsilon^{d/4-1/2}=o(\gamma(\varepsilon))$ as $\varepsilon\to 0$,

$$\lim_{\varepsilon\to 0}\inf_{x\in\mathbb{R}^d}\mathbb{E}\left|\Gamma\left(\lambda,\gamma(\varepsilon)\right)u_t^\varepsilon(x)\right|^2=\infty \tag{3.8}$$

for all $\lambda\in\mathbb{R}$, $t>0$.

2) When $d=2$, for all continuous function $\gamma(\varepsilon)$ on $(0,\infty)$ with $\sqrt{\frac{1}{-\log\varepsilon}} = o(\gamma(\varepsilon))$ as $\varepsilon\to 0$,

$$\lim_{\varepsilon\to 0}\inf_{x\in\mathbb{R}^d}\mathbb{E}\,\left|\Gamma\left(\lambda,\gamma(\varepsilon)\right)u_t^\varepsilon(x)\right|^2=\infty \tag{3.9}$$

for all $\lambda\in\mathbb{R}$, $t>0$.

Proof It suffices to show this theorem for $u_0\equiv 1$. Let us consider the first chaos of $u_t^\varepsilon(x)$:

$$I_1^\varepsilon=\int_{\mathbb{R}^d} f(t,x;s,z)\diamond w_s(z)dsdz\,,$$

where

$$f(t,x;s,z)=\int_0^t P_{t-s}(x-y)P_\varepsilon(y-z)dyds=\int_0^t P_{t-s+\varepsilon}(x-z)ds\,.$$

By definition of $\Gamma(\gamma(\varepsilon))$ and the orthogonality of different chaos, we have that

$$\mathbb{E}\,|\Gamma(\gamma(\varepsilon))u_t^\varepsilon(x)|^2\geq\gamma(\varepsilon)^2\mathbb{E}\,|I_1(f_1(t,x))|^2\,.$$

It is easy to check that

$$\begin{aligned}\mathbb{E}\,|I_1(f(t,x))|^2 &= (2\pi)^{-d/2}\int_0^t(2t-2s+2\varepsilon)^{-d/2}ds\\ &= \begin{cases}(-d/2+1)^{-1}\left[(2\varepsilon)^{-d/2+1}-(2t+2\varepsilon)^{-d/2+1}\right] & \text{when } d>4\\ \log\frac{(t+\varepsilon)^2}{\varepsilon(2t+\varepsilon)} & \text{when } d=4\,.\end{cases}\end{aligned}$$

Thus the theorem follows. □

4 Space White Noise: $d<4$

From now on let $w=w(x)\,,x\in\mathbb{R}^d$ be a space white noise. Consider the following stochastic heat equation with space white noise potential:

$$\frac{\partial u_t(x)}{\partial t}=\frac{1}{2}\Delta u_t(x)+w(x)\diamond u_t(x)\,, \tag{4.1}$$

where the initial condition u_0 is a given deterministic function. Consider the following formal expansion of the solution

$$u_t(x)=P_tf(x)+\sum_{n=1}^\infty I_n(f_n(t,x))\,, \tag{4.2}$$

where

$$\begin{aligned}f_n(t,x;y_1,\cdots,y_n) &= \int_{T_n}\int_{\mathbb{R}^d}P_{t-s_n}(x-y_n)P_{s_n-s_{n-1}}(y_n-y_{n-1})\cdots\\ &\qquad P_{s_2-s_1}(y_2-y_1)P_{s_1}(y_1-\tilde{y})f(\tilde{y})d\tilde{y}ds\end{aligned} \tag{4.3}$$

and

$$I_n(f_n(t,x))=\int_{\mathbb{R}^{nd}}f_n(t,x;y_1,\cdots,y_n)w(y_1)\diamond\cdots\diamond w(y_n)dy_1\cdots dy_n\,. \tag{4.4}$$

Theorem 4.1 *When $d < 4$, each $I_n(f_n(t,x))$ is well-defined as an element in $\mathcal{L}^2$.*

1) Let $d < 2$ and let $\alpha = 1 - d/4$. If $|u_0(x)| \le C < \infty$, then

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log\left\{ \sup_{x\in\mathbb{R}^d} \mathbb{E}\, |u_t(x)|^2 \right\} \le (U(\alpha))^{1/(2\alpha-1)} \,, \tag{4.5}$$

where

$$U(\alpha) := \frac{(2\alpha-1)^{2\alpha-1}}{(4\pi)^{d/2}\alpha^{2\alpha}} \Gamma(\alpha)^2 \,.$$

If $u_0(x) \ge C > 0$, then

$$\liminf_{t\to\infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log\left\{ \inf_{x\in\mathbb{R}^d} \mathbb{E}\, |u_t(x)|^2 \right\} \ge (L(\alpha))^{1/(2\alpha-1)} \,, \tag{4.6}$$

where

$$L(\alpha) := \frac{(2\alpha-1)^{2\alpha-1}\Gamma(2\alpha)}{(2\pi)^{d/2}(2\alpha)^{2\alpha}} \,.$$

2) Let $d = 2$. If $|u_0(x)| \le C < \infty$, then when $t < 2$, $u_t(x)$ is defined as an element in $\mathcal{L}^2$ for all $x \in \mathbb{R}^d$. If $u_0(x) \ge C > 0$, then when $t > 2\pi$,

$$\mathbb{E}\, |u_t(x)|^2 = \infty \,, \quad \forall\, x \in \mathbb{R}^d \,. \tag{4.7}$$

3) When $d > 2$, $\mathbb{E}\, |u_t(x)|^2 = \infty$ for all $t > 0$ and $x \in \mathbb{R}^d$ if $u_0(x) \ge C > 0$ for some constant $C > 0$.

Remark 2 *When $d = 1$, the upper bound $U(1/2) \approx 0.1381$ and $L(1/2) \approx 0.0145$. In [28], the upper is given by $1/4$ and lower bound is $1/(81\pi e^3) =\approx 0.0002$.*

Remark 3 *To get a better idea about the difference between the two bounds let us divide the above upper bound by the lower bound ((4.5) and (4.6)) and denote this number by* Ratio. *Thus*

$$\text{Ratio} = \left(\frac{2^{2\alpha}\Gamma(\alpha)^2}{2^{d/2}\Gamma(2\alpha)} \right)^{1/(2\alpha-1)} \,.$$

When $d = 1$, we have $\alpha = 3/4$, $2\alpha - 1 = 1/2$. Hence

$$\text{Ratio} = \left(\frac{2\Gamma(3/4)^2}{\Gamma(3/2)} \right)^2 \,.$$

From [1], $\Gamma(3/4) \approx 1.2254$, $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \sqrt{\pi}/2 \approx 0.8862$. *Thus* Ratio ≈ 11.4844.

Proof In this section we denote

$$\Theta_n := \int_{\mathbb{R}^{nd}} \left\{ \int_{T_n} P_{t-s_n}(x-y_n) P_{s_n-s_{n-1}}(y_n - y_{n-1}) \cdots P_{s_2-s_1}(y_2-y_1) ds \right\}^2 dy \tag{4.8}$$

which is $\mathbb{E}\ (I_n(f_n(t,x)))^2/n!$ when $u_0(x)=1$. It is easy to see that

$$\mathbb{E}\ (I_n(f_n(t,x)))^2 \le n! \|f\|_\infty^2 \Theta_n\,.$$

Using the semigroup property $\int_{\mathbb{R}} P_s(x-y)P_t(y-z)dy = P_{t+s}(x-z)$, we have

$$\begin{aligned}
\Theta_n &= \int_{T_n\times T_n}\int_{\mathbb{R}^{nd}} P_{t-s_n}(x-y_n)P_{s_n-s_{n-1}}(y_n-y_{n-1})\cdots P_{s_2-s_1}(y_2-y_1) \\
&\qquad P_{t-r_n}(x-y_n)P_{r_n-r_{n-1}}(y_n-y_{n-1})\cdots P_{r_2-r_1}(y_2-y_1)dy \\
&= (2\pi)^{-nd/2}\int_{T_n\times T_n} (s_2+r_2-s_1-r_1)^{-d/2}\cdots \\
&\qquad (s_n+r_n-s_{n-1}-r_{n-1})^{-d/2}(2t-s_n-r_n)^{-d/2} ds dr\,.
\end{aligned} \tag{4.9}$$

This shows that $\mathbb{E}\ u_t(x)^2$ is independent of x if $u_0(x)=1$. We shall bound Θ_n from above and from below. Let us bound Θ_n from above first. From $2\sqrt{ab}\le a+b$, it follows that $(a+b)^{-d/2}\le 2^{-d/2}a^{-d/4}b^{-d/4}$. Thus

$$\begin{aligned}
\Theta_n &\le 2^{-nd/2}(2\pi)^{-nd/2}\int_{T_n}(s_2-s_1)^{-d/4}\cdots(s_n-s_{n-1})^{-d/4} \\
&\qquad (t-s_n)^{-d/4}(r_2-r_1)^{-d/4}\cdots(r_n-r_{n-1})^{-d/4}(t-r_n)^{-d/4}dsdr \\
&= (4\pi)^{-nd/2}\left\{\int_{T_n}(s_2-s_1)^{-d/4}\cdots(s_n-s_{n-1})^{-d/4}(t-s_n)^{-d/4}ds\right\}^2 \\
&= (4\pi)^{-nd/2}\left\{\frac{\Gamma(\alpha)^n t^{\alpha n}}{\Gamma(n\alpha+1)}\right\}^2 .
\end{aligned} \tag{4.10}$$

This shows that when $d<4$, $I_n(f_n(t,x))$ is a well-defined square integrable random variable. Throughout this paper C_x denotes a generic function of x such that there are constants ρ_1, ρ_2, $C_1>0$, and $C_2>0$ such that $C_1x^{\rho_1}\le C_x\le C_2x^{\rho_2}$ for all $x\ge 1$, where ρ_1, ρ_2, C_1, and $C_2>0$ may depend on the dimension d (*i.e.* α). C_x may be different in different appearances. From the Stirling's formula, *i.e.* there is a constant C_x such that $\Gamma(x+1)=C_x x^x e^{-x}$, it follows

$$\begin{aligned}
\frac{n!}{\Gamma(\alpha n+1)^2} &= C_n\frac{n^n e^{-n}}{(\alpha n)^{2\alpha n}e^{-2\alpha n}} \\
&= \left(\frac{(2\alpha-1)^{2\alpha-1}}{\alpha^{2\alpha}}\right)^n \frac{C_n}{((2\alpha-1)n)^{(2\alpha-1)n}\,e^{-(2\alpha-1)n}} \\
&= \left(\frac{(2\alpha-1)^{2\alpha-1}}{\alpha^{2\alpha}}\right)^n \frac{C_n}{\Gamma((2\alpha-1)n+1)}\,.
\end{aligned}$$

For all ρ and all $\varepsilon > 1$, there are constants C_1 and C_2 such that $n^\rho \le C_1 + C_2\varepsilon^n$. Thus for all $\varepsilon > 1$, there are constants C_0, C_1, C_2, and N depending on ε, dimension, and sup norm of the initial condition, such that the $\mathcal{L}_2$ norm of the solution is bounded as follows

$$\begin{aligned}
\mathbb{E}\,|u_t(x)|^2 &\le \|u_0\|_\infty^2 \sum_{n=0}^\infty n!\Theta_n \\
&= \|u_0\|_\infty^2 \sum_{n=0}^\infty \frac{n!}{\Gamma(n\alpha+1)^2}(4\pi)^{-nd/2}\Gamma(\alpha)^{2n}t^{2\alpha n} \\
&\le \sum_{n=0}^\infty \frac{C_n}{\Gamma((2\alpha-1)n+1)}\left(\frac{(2\alpha-1)^{2\alpha-1}}{\alpha^{2\alpha}}\right)^n (4\pi)^{-nd/2}\Gamma(\alpha)^{2n}t^{2\alpha n} \\
&\le C_0 + C_1 t^N + C_2\sum_{n=0}^\infty \frac{1}{\Gamma((2\alpha-1)n+1)}\left(\frac{\varepsilon(2\alpha-1)^{2\alpha-1}}{\alpha^{2\alpha}}\right)^n (4\pi)^{-nd/2}\Gamma(\alpha)^{2n}t^{2\alpha n} \\
&= C_0 + C_1 t^N + C_2 E_{2\alpha-1}\left(\frac{\varepsilon\Gamma(\alpha)^2(2\alpha-1)^{2\alpha-1}t^{2\alpha}}{(4\pi)^{d/2}\alpha^{2\alpha}}\right),
\end{aligned}$$

where $E_{2\alpha-1}$ is the Mittag-Leffler function.

Thus we see that $t \to \infty$,

$$\mathbb{E}\,|u_t(x)|^2 \le C_0 + C_1 t^N + C_2 \exp\left(\frac{\varepsilon(2\alpha-1)^{2\alpha-1}\Gamma(\alpha)^2 t^{2\alpha}}{(4\pi)^{d/2}\alpha^{2\alpha}}\right)^{1/(2\alpha-1)}.$$

Consequently we have

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log\left\{\sup_{x\in\mathbb{R}^d} \mathbb{E}\,|u_t(x)|^2\right\} \le \left(\frac{\varepsilon(2\alpha-1)^{2\alpha-1}\Gamma(\alpha)^2}{(4\pi)^{d/2}\alpha^{2\alpha}}\right)^{1/(2\alpha-1)}.$$

Since $\varepsilon > 1$ is arbitrary, we have

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log\left\{\sup_{x\in\mathbb{R}^d} \mathbb{E}\,|u_t(x)|^2\right\} \le \left(\frac{(2\alpha-1)^{2\alpha-1}\Gamma(\alpha)^2}{(4\pi)^{d/2}\alpha^{2\alpha}}\right)^{1/(2\alpha-1)},$$

proving (4.5).

Now let us bound Θ_n from below. This will be slightly more complicated. Let $\beta = 2 - d/2 = 2\alpha$. By a simple substitution,

$$\begin{aligned}
\Theta_n &= (2\pi)^{-nd/2}t^{\beta n}\int_{T_n(1)\times T_n(1)} (s_2+r_2-s_1-r_1)^{-d/2}\cdots \\
&\qquad (s_n+r_n-s_{n-1}-r_{n-1})^{-d/2}(2-s_n-r_n)^{-d/2}dsdr\,,
\end{aligned}$$

where

$$T_n(1) := \{(s_1,\cdots,s_n);\quad 0 < s_1 < \cdots < s_n < 1\}\,.$$

To bound Θ_n from below, let us make the following substitution,

$$x_1 = r_1 + s_1\,, \quad x_2 = r_2 + s_2 - r_1 - s_1\,, \quad \cdots, \quad x_n = r_n + s_n - r_{n-1} - s_{n-1}$$
$$y_1 = s_1\,, \quad y_2 = s_2 - s_1\,, \quad \cdots, \quad y_n = s_n - s_{n-1}$$

It is easy to notice that if $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ are in

$$\begin{aligned} \mathcal{Y}_n \;=\; \Big\{ &(x_1, \cdots, x_n, y_1, \cdots, y_n); \quad y_i < x_i\,, \quad x_i \ge 0\,, \\ & i = 1, 2, \cdots, n\,, \quad x_1 + x_2 + \cdots + x_n < 1 \Big\}, \end{aligned}$$

then $0 < r_1 < \cdots < r_n < 1$ and $0 < s_1 < \cdots < s_n < 1$. Namely, if $x, y \in \mathcal{Y}_n$, then $(s_1, \cdots, s_n)$ and $(r_1, \cdots, r_n)$ are in $T_n(1)$. It is also easy to check that the Jacobi determinant of the above substitution is 1. Thus

$$\begin{aligned} \Theta_n &\ge (2\pi)^{-nd/} t^{\beta n} \int_{\mathcal{Y}_n} x_2^{\beta-2} x_3^{\beta-2} \cdots x_n^{\beta-2} (2 - x_1 - x_2 - \cdots - x_n)^{\beta-2} dy_1 \cdots dy_n dx_1 \cdots dx_n \\ &\ge 2^{-d/2} (2\pi)^{-nd/2} t^{\beta n} \int_{\mathcal{Y}_n} x_2^{\beta-2} x_3^{\beta-2} \cdots x_n^{\beta-2} dy_1 \cdots dy_n dx_1 \cdots dx_n \\ &\ge 2^{-d/2} (2\pi)^{-nd/2} t^{\beta n} \int_{\substack{x_1>0, \cdots, x_n>0 \\ x_1+x_2+\cdots+x_2<1}} x_1 x_2^{\beta-1} x_3^{\beta-1} \cdots x_n^{\beta-1} dx_1 \cdots dx_n \end{aligned}$$

It follows then

$$\Theta_n \ge 2^{-d/2} (2\pi)^{-nd/2} \frac{\Gamma(\beta)^{n-1}}{\Gamma((n-1)\beta + 3)} t^{\beta n}\,. \tag{4.11}$$

Thus when $u_0 \ge c > 0$,

$$\begin{aligned} \mathbb{E}\, |u_t(x)|^2 &\ge c^2 \sum_{n=0}^{\infty} n! \Theta_n \\ &\ge c^2 \sum_{n=0}^{\infty} (2\pi)^{-nd/2} \frac{n! \Gamma(\beta)^{n-1}}{\Gamma((n-1)\beta + 3)} t^{\beta n}\,. \end{aligned} \tag{4.12}$$

On the other hand, by the Stirling's formula, we have (using the generic notation C_x introduced earlier),

$$\begin{aligned} \frac{n!}{\Gamma(\beta n + 3 - \beta)} &= C_n \frac{n!}{\Gamma(\beta n)} \\ &= C_n \frac{n^n e^{-n}}{(\beta n)^{\beta n} e^{-\beta n}} \\ &\ge \left(\frac{(\beta-1)^{\beta-1}}{\beta^\beta} \right)^n \frac{C_n}{((\beta-1)n)^{(\beta-1)n} e^{-(\beta-1)n}} \\ &\ge \left(\frac{(\beta-1)^{\beta-1}}{\beta^\beta} \right)^n \frac{C_n}{\Gamma((\beta-1)n + 1)}\,. \end{aligned}$$

Thus for all $\varepsilon < 1$, there is an N such that when $n \geq N$,

$$\frac{n!}{\Gamma(\beta n + 3 - \beta)} \geq \left(\frac{\varepsilon(\beta-1)^{\beta-1}}{\beta^\beta}\right)^n \frac{1}{\Gamma((\beta-1)n+1)}\,.$$

Thus when $u_0 \geq c$ for some constant $c > 0$,

$$\begin{aligned}
\mathbb{E}\,|u_t(x)|^2 &\geq C_0 + C_1 t^N + C_2 \sum_{n=0}^{\infty} \frac{1}{\Gamma((\beta-1)n+1)} \left(\frac{\varepsilon t^\beta (\beta-1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2}\beta^\beta}\right)^n \\
&= C_0 + C_1 t^N + C_2 E_{\beta-1}\left(\frac{\varepsilon t^\beta (\beta-1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2}\beta^\beta}\right) \\
&\geq C_0 + C_1 t^N + C_2 \exp\left\{\left(\frac{\varepsilon t^\beta (\beta-1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2}\beta^\beta}\right)^{1/(\beta-1)}\right\}
\end{aligned}$$

Thus when $t \to \infty$, it follows

$$\liminf_{t\to\infty} \frac{1}{t^{\beta/(\beta-1)}} \log\left\{\inf_{x\in\mathbb{R}^d} \mathbb{E}\,|u_t(x)|^2\right\} \geq \left(\frac{\varepsilon(\beta-1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2}\beta^\beta}\right)^{1/(\beta-1)}.$$

Let $\varepsilon \to 1$, it follows

$$\liminf_{t\to\infty} \frac{1}{t^{\beta/(\beta-1)}} \log\left\{\inf_{x\in\mathbb{R}^d} \mathbb{E}\,|u_t(x)|^2\right\} \geq \left(\frac{(\beta-1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2}\beta^\beta}\right)^{1/(\beta-1)}.$$

Replacing β by 2α we show (4.6).

From the Stirling's formula as shown previously, it follows

$$\frac{n!}{\Gamma(n/2+1)^2} \leq C_n 2^n$$

for some constant C_n as defined earlier this section. Thus by (4.10) we have when $d = 2$ and $|u_0(x)| \leq C < \infty$,

$$\mathbb{E}\,|u_t(x)|^2 \leq C^2 \sum_{n=0}^{\infty} (4\pi)^{-n} \frac{n!\Gamma(1/2)^{2n} t^n}{\Gamma(n/2+1)^2} \leq \sum_{n=0}^{\infty} C_n \left(\frac{t}{2}\right)^n.$$

This series is convergent when $t < 2$, which proves the first part of 2) in Theorem 4.1.

On the other hand, $\beta = 1$ when $d = 2$. By (4.12), when $u_0 \geq C$ for some constant $C > 0$,

$$\mathbb{E}\,|u_t(x)|^2 \geq \sum_{n=0}^{\infty} C_n \frac{t^n}{(2\pi)^n}\,.$$

This shows that when $t > 2\pi$, $\mathbb{E}\,|u_t(x)|^2 = \infty$, proving (4.7).

It is also easy to see from the estimate that when $d>2$, $\mathbb{E}\,|u_t(x)|^2=\infty$ for all $t>0$ if $u_0\ge C$ for some constant $C>0$. This completes the proof of Theorem 4.1. □

Let

$$u_t^N(x) == P_t f(x)+\sum_{n=1}^N I_n(f_n(t,x))\,.$$

The above estimate can be used to find the exact rate of convergence of approximating $u_t(x)$ by the finite sum $u_t^N(x)$.

Theorem 4.2 *There are constants $C_1(N)$ and $C_2(N)$ of polynomial growth or polynomial decay such that*

$$\begin{aligned} C_1(N)\frac{U(\alpha)^N T^{2\alpha N}}{\Gamma((2\alpha-1)n+1)} &\le \inf_{x\in\mathbb{R}^d}\mathbb{E}\,|u_T(x)-u_T^N(x)|^2 \\ &\le \sup_{0\le t\le T,x\in\mathbb{R}^d}\mathbb{E}\,|u_t(x)-u_t^N(x)|^2\le C_2(N)\frac{U(\alpha)^N T^{2\alpha N}}{\Gamma((2\alpha-1)n+1)}\,. \end{aligned}$$

Proof Since $\mathbb{E}\,|u_T(x)-u_T^N(x)|^2\ge(N+1)!\Theta_{N+1}$. The lower bound follows easily from (4.11). On the other hand, we have

$$\begin{aligned} \mathbb{E}\,|u_t(x)-u_t^N(x)|^2 &= \sum_{n=N+1}^\infty c(n)\frac{U(\alpha)^n t^{2\alpha n}}{\Gamma((2\alpha-1)n+1)} \\ &\le C_2(N)\frac{U(\alpha)^N t^{2\alpha N}}{\Gamma((2\alpha-1)N+1)}\,, \end{aligned}$$

proving the theorem. □

Remark 4 *The above theorem implies in particular that when $d=1$, $\mathbb{E}\,|u_t(x)-u_t^N(x)|^2$ is of order $N^{-N/2}$. This is a very good approximation.*

Let us recall the distribution and test functional spaces of Meyer-Watanabe type.

Let $F\in\mathcal{L}_2$. Then F admits an Itô-Wiener chaos expansion

$$F=\sum_{n=0}^\infty F_n\,. \tag{4.13}$$

Let us denote by $\mathcal{G}_f$ the set of all finite sum of chaos, *i.e.* $\mathcal{G}_f=\{F: F=\sum_{n=0}^k F_n\}$. $\mathcal{G}_f$ is a dense subset of $\mathcal{L}_2$. We denote by D_γ the completion of $\mathcal{G}_f$ under the norm

$$|F|_\gamma^2 := \sum_{n=0}^k (n+1)^\gamma\mathbb{E}\,(F_n^2)\,,\quad \text{where}\quad F=\sum_{n=0}^k F_n\,.$$

Since the radius of convergence and the asymptotics as $t\to\infty$ of the series $\sum_{n=0}^\infty a_n n^\gamma t^n$ do not depend on γ, the asymptotics of $U_t(x)$ defined by (4.2) in the Meyer-Watanabe spaces will be the same as in $\mathcal{L}^2$. Namely, we can extend Theorem 4.1 to the following theorem.

Theorem 4.3 *Let $d < 4$.*

1) Let $d < 2$, $\gamma \in \mathbb{R}$ and let $\alpha = 1 - d/4$. If $|u_0(x)| \le C < \infty$, then $u_t(x) \in D_\gamma$ for all $t > 0$ and

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log\left\{\sup_{x\in\mathbb{R}^d} |u_t(x)|_\gamma^2\right\} \le \left(\frac{(2\alpha-1)^{2\alpha-1}}{(4\pi)^{d/2}\alpha^{2\alpha}}\Gamma(\alpha)^2\right)^{1/(2\alpha-1)}. \tag{4.14}$$

If $u_0(x) \ge C > 0$, then

$$\liminf_{t\to\infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log\left\{\inf_{x\in\mathbb{R}^d} |u_t(x)|_\gamma^2\right\} \ge \left(\frac{(2\alpha-1)^{2\alpha-1}\Gamma(2\alpha)}{(2\pi)^{d/2}(2\alpha)^{2\alpha}}\right)^{1/(2\alpha-1)}. \tag{4.15}$$

2) Let $d = 2$. If $|u_0(x)| \le C < \infty$, then when $t < 2$, $u_t(x)$ is defined as an element in D_γ. If $u_0(x) \ge C > 0$, then when $t > 2\pi$,

$$|u_t(x)|_\gamma = \infty. \tag{4.16}$$

3) When $d > 2$, $|u_t(x)|_\gamma = \infty$ for all $t > 0$ if $u_0(x) \ge c > 0$ for some constant $c > 0$.

Now we introduce a class of "flat" $\mathcal{L}_2$ type of spaces.

Let $\mathcal{S}_\gamma$ denote the completion of $\mathcal{G}_f$ under the norm

$$|||F|||_\gamma^2 := \sum_{n=0}^{k} (n!)^\gamma \mathbb{E}\left(F_n^2\right), \quad \text{where} \quad F = \sum_{n=0}^{k} F_n.$$

These spaces are not weighted (compare these with the distribution spaces introduced in [21], [22]). For each element in this space, its n-th chaos is in $\mathcal{L}_2$.

The following theorem deals with the solutions of (4.1) in $\mathcal{S}_\gamma$.

Theorem 4.4 *Let $d < 4$. Then*

1) Let $\gamma < 2\alpha - 1 = 1 - d/2$. Then $u_t(x) \in \mathcal{S}_\gamma$ for all $t \in [0,\infty)$ and $x \in \mathbb{R}^d$. Moreover, when $|u_0(x)| \le C < \infty$,

$$\limsup_{t\to\infty} \frac{1}{t^{\frac{2\alpha}{2\alpha-1-\gamma}}} \log\left\{\sup_{x\in\mathbb{R}^d} |||u_t(x)|||_\gamma^2\right\} \le (2\alpha-1-\gamma)\left(\frac{\Gamma(\alpha)^2}{(4\pi)^{d/2}\alpha^{2\alpha}}\right)^{\frac{1}{2\alpha-1-\gamma}}. \tag{4.17}$$

When $u_0(x) \ge C > 0$,

$$\liminf_{t\to\infty} \frac{1}{t^{\frac{2\alpha}{2\alpha-1-\gamma}}} \log\left\{\inf_{x\in\mathbb{R}^d} |||u_t(x)|||_\gamma^2\right\} \ge (2\alpha-1-\gamma)\left(\frac{\Gamma(2\alpha)}{(2\pi)^{d/2}(2\alpha)^{2\alpha}}\right)^{\frac{1}{2\alpha-1-\gamma}}. \tag{4.18}$$

2) Let $\gamma = 2\alpha - 1 = 1 - d/2$. Then when $|u_0(x)| \le C < \infty$ and

$$t < \frac{\alpha(4\pi)^{d/(4\alpha)}}{\Gamma(\alpha)^{1/\alpha}},$$

the solution $u_t(x)$ exists in $\mathcal{S}_\gamma$ for all $x \in \mathbb{R}^d$. When $u_0(x) \geq C > 0$ and

$$t > \frac{2\alpha(2\pi)^{d/(4\alpha)}}{\Gamma(2\alpha)^{1/(2\alpha)}},$$

$u_t(x)$ is not an element of $\mathcal{S}_\gamma$ for any $x \in \mathbb{R}^d$.

3) Let $\gamma > 2\alpha - 1 = 1 - d/2$. If $u_0(x) \geq C > 0$, then $u_t(x)$ is not in $\mathcal{S}_\gamma$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Proof The proof is similar to that of Theorem 4.1 and is omitted. □

5 Space White Noise: $d \geq 4$

When $d \geq 4$, it is easy to see that $f_n(t,x)$ defined by (4.3) is not in $L^2(T_n^2)$ if $u_0(x) \geq C > 0$. Thus none of the chaos $I_n(f_n(t,x))$ of $u_t(x)$ defined by (4.2) is in $\mathcal{L}_2$.

Therefore we shall introduce a renormalization procedure. Let

$$w^\varepsilon(x) = \int_{\mathbb{R}^d} P_\varepsilon(x-y)w(y)dy.$$

Consider the following approximation of the stochastic heat equation

$$\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t + w^\varepsilon(x) \diamond u_t. \tag{5.1}$$

The initial condition $u_0(x)$ is given and is deterministic.

For all $\varepsilon > 0$, the solution to the above approximated equation exists in some $\mathcal{S}_\gamma$. Its Wiener chaos expansion is given by

$$u_t^\varepsilon(x) = P_t u_0(x) + \sum_{n=1}^\infty I_n(f_n^\varepsilon(t,x)), \tag{5.2}$$

where

$$\begin{aligned} f_n^\varepsilon(t,x;z_1,\cdots,z_n) &= \mathrm{Sym}\Big[\int_{T_n\times\mathbb{R}^{nd}} P_{t-s_n}(x-y_n)P_{s_n-s_{n-1}}(y_n-y_{n-1})\cdots P_{s_2-s_1}(y_2-y_1) \\ &\qquad P_\varepsilon(y_1-z_1)\cdots P_\varepsilon(y_n-z_n)P_{s_1}(y_1-\tilde{y})u_0(\tilde{y})d\tilde{y}dyds\Big] \end{aligned} \tag{5.3}$$

and

$$I_n(f_n^\varepsilon(t,x)) = \int_{\mathbb{R}^{nd}} f_n^\varepsilon(t,x;z_1,\cdots,z_n)w(z_1)\diamond\cdots\diamond w(z_n)dz_1\cdots dz_n. \tag{5.4}$$

Theorem 5.1 *1) Let $d > 4$. Let u_0 be bounded on $\mathbb{R}^d$. Then for all $\lambda < 0$, $\Gamma\left(\lambda, \sqrt{\varepsilon^{d/2-1}}\right) u_t^\varepsilon$ is a uniformly bounded sequence in $\mathcal{L}^2$ with respect to $\varepsilon \in (0, e^{-1}]$, $t \in (0,\infty)$ and $x \in \mathbb{R}^d$.*

2) Let $d = 4$. Let u_0 be bounded on $\mathbb{R}^d$. Then for all $\lambda < 0$, $T \in (0,\infty)$ and $x \in \mathbb{R}^d$, $\Gamma\left(\lambda, \sqrt{-\log\varepsilon}\right) u_t^\varepsilon$ is a uniformly bounded sequence in $\mathcal{L}^2$ with respect to $\varepsilon \in (0, e^{-1}]$, $t \in (0,T]$ and $x \in \mathbb{R}^d$.

Proof Let $\kappa_1 = 2\varepsilon$ and let

$$\kappa_{i+1} = \frac{2\varepsilon(\kappa_i + s_{i+1} - s_i + r_{i+1} - r_i)}{2\varepsilon + \kappa_i + s_{i+1} - s_i + r_{i+1} - r_i}, \quad i = 1, 2, \cdots, n-1 .$$

Similar to (3.5), we have

$$\Theta_n = \int_{T_n^2} \prod_{i=1}^{n-1} \left(2\pi(\kappa_i + 2\varepsilon + s_{i+1} - s_i + r_{i+1} - r_i)\right)^{-d/2} \left(2\pi(\kappa_n + s_{i+1} - s_i + r_{i+1} - r_i)\right)^{-d/2} ds dr .$$

Similar to (3.7),

$$\Theta_n \quad \leq \quad \begin{cases} C^n \varepsilon^{\frac{4-d}{2}n} & \text{when } d > 4 \\ \\ C^n (-\log\varepsilon)^n \log(e+T)^n & \text{when } d = 4 \end{cases}$$

The theorem then follows. □

An theorem analogous to Theorem 3.3 can be also stated and proved.

6 General Stochastic Heat Equations

In this section we will discuss the stochastic Heat equations when the Laplacian Δ is replaced by a general elliptic operator. For simplicity we assume that the operator is of divergence form.

Let $a_{ij}(x)$, $1 \leq i, j \leq d$ be measurable functions on $\mathbb{R}^d$ and let $A(x) = (a_{ij}(x))_{1\leq i,j\leq d}$ be symmetric positive matrices for all $x \in \mathbb{R}^d$ such that

$$\lambda I \leq A(x) \leq \mu I , \quad \forall \ x \in \mathbb{R}^d ,$$

where I is the d dimensional unit matrix; λ and μ are two positive numbers. Let $L = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ be the second order elliptic operator of divergence form with coefficient A. It is known that the semigroup associated with L is given by a kernel of the form $Q_t(x-y)$. Moreover the following Nash's estimate holds (see [8], [27])

$$C_1 P_{\mu_1 t}(x) \leq Q_t(x) \leq C_2 P_{\mu_2 t}(x) , \tag{6.1}$$

where μ_1, C_1, μ_2, and C_2 are positive constants; P_t is defined as in the beginning of Section 2.

Consider the following stochastic heat equation

$$\frac{\partial u_t}{\partial t} = L u_t + w \diamond u_t , \tag{6.2}$$

where w is a space white or space-time white noise, $u_0(x)$ is a given deterministic function.

All results obtained for Eq. (1.1) can be extended to (6.2) by (6.1). For instance, we can state the following theorem

Theorem 6.1 *(1) If $w_s(x)$ is a space-time white noise, if $d=1$ and if there is positive constant $\mu>0$ such that $|u_0(x)|\le\mu_1$ for all $x\in\mathbb{R}^d$, then*

$$\limsup_{t\to\infty}\frac{\log\mathbb{E}\,(|u_t(x)|^2)}{t}\le C\,,\quad\forall\,x\in\mathbb{R}\,. \tag{6.3}$$

If there is positive constant μ_1 such that $u_0(x)\ge\mu_1$ for all $x\in\mathbb{R}^d$, then

$$\liminf_{t\to\infty}\frac{\log\mathbb{E}\,(|u_t(x)|^2)}{t}\ge C\,,\quad\forall\,x\in\mathbb{R}\,. \tag{6.4}$$

(2) Let $w_s(x)$ be a space white noise.
i) Let $d<2$ and let $\alpha=1-d/4$. If $|u_0(x)|\le C<\infty$, then

$$\limsup_{t\to\infty}\frac{1}{t^{2\alpha/(2\alpha-1)}}\log\left\{\sup_{x\in\mathbb{R}^d}\mathbb{E}\,|u_t(x)|^2\right\}\le C\,. \tag{6.5}$$

If $u_0(x)\ge C>0$, then

$$\liminf_{t\to\infty}\frac{1}{t^{2\alpha/(2\alpha-1)}}\log\left\{\inf_{x\in\mathbb{R}^d}\mathbb{E}\,|u_t(x)|^2\right\}\ge C\,. \tag{6.6}$$

ii) Let $d=2$. If $|u_0(x)|\le C_1<\infty$, then there is a constant $\alpha>0$ when $t<\alpha$, $u_t(x)$ is defined as an element in $\mathcal{L}^2$. If $u_0(x)\ge C_2>0$, then when $t>\beta$,

$$\mathbb{E}\,|u_t(x)|^2=\infty\,. \tag{6.7}$$

iii) When $d>2$, $\mathbb{E}\,|u_t(x)|^2=\infty$ for all $t>0$ if $u_0(x)\ge C>0$ for some constant $C>0$.

Other results may be stated in a similar way.

References

[1] Abramowitz M. and Stegun, I.A. Handbook of Mathematical Functions, Dover Publications, INC., New York, 1992.

[2] Benth, F.E. On the positivity of the stochastic heat equation, Potential Analysis, 6 (1997), 127-148.

[3] Carmona, R. Random Schrödinger operators. École d'été de probabilités de Saint-Flour, XIV (1984), 1–124, Lecture Notes in Math., 1180, Springer, Berlin-New York, 1986.

[4] Carmona, R. and Lacroix, J. Spectral Theory of Random Schrödinger Operators. Probability and its Applications. Birkhuser Boston, Inc., Boston, MA, 1990.

[5] Carmona, R. A. and Molchanov, S. A. Stationary parabolic Anderson model and intermittency. Probab. Theory Related Fields 102 (1995), no. 4, 433–453.

[6] Carmona, R. A. and Viens, F. G. Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter. Stochastics Stochastics Rep. 62 (1998), no. 3-4, 251–273.

[7] Carmona, R.; Viens, F. G. and Molchanov, S. A. Sharp upper bound on the almost-sure exponential behavior of a stochastic parabolic partial differential equation. Random Oper. Stochastic Equations 4 (1996), no. 1, 43–49.

[8] Davies, E. B. Spectral theory and differential operators. Cambridge Studies in Advanced Mathematics, 42. Cambridge University Press, Cambridge, 1995.

[9] Deck, Th. and Potthoff, J. On a class of stochastic partial differential equations related to turbulent transport. Probab. Theory Related Fields 111 (1998), 101–122.

[10] Erdélyi, A., Magnus, W., Oberhettinger F. and Tricomi, F.G. High Transcendental Functions, Vol. 3, McGraw-Hill Book Company, INC. 1955.

[11] Glimm J. and Jaffe, A. Quantum physics. A functional integral point of view. Second edition. Springer-Verlag, New York-Berlin, 1987.

[12] Holden, H; Øksendal,B.; Ubøe, J. and Zhang, T.S. Stochastic Partial Differential Equations, Birkhauser, 1996;

[13] Hu, Y.Z. and Kallianpur, G. Exponential integrability and application to stochastic quantization. Appl. Math. Optim. 37 (1998), no. 3, 295–353.

[14] Hu, Y.Z. Itô-Wiener chaos expansion with exact residual and correlation, variance inequalities. J. Theoret. Probab. 10 (1997), no. 4, 835–848.

[15] Hu, Y. Z. On the self-intersection local time of Brownian motion-via chaos expansion. Publ. Mat. 40 (1996), no. 2, 337–350.

[16] Hu, Y. Z. Self-intersection local time of fractional Brownian motion-via chaos expansion. Preprint, 1998.

[17] Hida, T.; Kuo, H.-H.; Potthoff, J. and Streit, L. White noise. An infinite-dimensional calculus. Mathematics and its Applications, 253. Kluwer Academic Publishers Group, Dordrecht, 1993.

[18] Imkeller, P.; Pérez-Abreu, V. and Vives, J. Chaos expansions of double intersection local time of Brownian motion in $\mathbb{R}^d$ and renormalization. Stochastic Process. Appl. 56 (1995), no. 1, 1–34.

[19] Imkeller, P. and Yan, J.A. Multiple intersection local time of planar Brownian motion as a particular Hida distribution. J. Funct. Anal. 140 (1996), no. 1, 256–273. %bibitemKLPSW96 Kondratiev, Yu. G.; Leukert, P.; Potthoff, J.;

[20] Mueller, C. Long-time existence for the heat equation with a noise term, Prob. Theory Rel. Fields, Vol. 9 (1991), 505-517

[21] Nualart, D. and Rozovskii B. Weighted stochastic Sobolev spaces and bilinear SPDEs driven by space-time white noise, Journal of Functional Analysis, 149 (1997), 200-225,

[22] Nualart, D. and Zakai, M. Generalized Brownian functionals and the solution to a stochastic partial differential equation. J. Funct. Anal. 84 (1989), no. 2, 279- 296.

[23] Potthoff, J.; Våge, G. and Watanabe, H. Generalized solutions of linear parabolic stochastic partial differential equations. Appl. Math. Optim. 38 (1998), 95–107.

[24] Piatnitski, A.L.; Zhao, H.Z. and Zheng, W.A. Homogenization approximation to linear SPDEs in Multi-dimensions, Preprint, 1998.

[25] Rivasseau, V. From perturbative to constructive renormalization. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1991.

[26] Simon, B. The $P(\phi)_2$ Euclidean (quantum) field theory. Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1974.

[27] Stroock, D.W. Diffusion semigroups corresponding to uniformly elliptic divergence form operator. *Lect. Notes Math.* 1321, 316-347, Springer-Verlag (1988).

[28] Uemura, H. Construction of the solution of 1-dimensional heat equation with white noise potential and its asymptotic behavior. Stochastic Anal. Appl. 14 (1996), 487–506.