Chaos Expansion of Heat Equations With White Noise Potentials

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Abstract

The asymptotic behavior as \( t \to \infty \) of the solution to the following stochastic heat equations

\[
\frac{\partial u_t}{\partial t} = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2 u_t}{\partial x_i^2} + w \circ u_t, \quad 0 < t < \infty, \quad x \in \mathbb{R}^d, \quad u_0(x) = 1
\]

is investigated, where \( w \) is a space-time white noise or a space white noise. The use of \( \circ \) means that the stochastic integral of Itô (Skorohod) type is considered. When \( d = 1 \), the exact \( \mathcal{L}_2 \) Lyapunov exponents of the solution are studied. When the noise is space white and when \( d < 4 \) it is shown that the solution is in some “flat” \( \mathcal{L}_2 \) distribution spaces. The Lyapunov exponents of the solution in these spaces are also estimated. The exact rate of convergence of the solution by its first finite chaos terms are also obtained.

1 Introduction

The Schrödinger operators with random potential have been widely studied (see [3], [4] and the references therein).

Let \((\Omega, \mathcal{F}, P)\) be a probability space. The expectation on this probability space is denoted by \( \mathbb{E} \). The set of all square integrable random variables on \((\Omega, \mathcal{F}, P)\) is denoted by \( \mathcal{L}_2 := L^2(\Omega, \mathcal{F}, P) \). Let \( \mathbb{R}^d \) be the \( d \) dimensional Euclidean space. Let \( w_t(x) \) be a (generalized) Gaussian random field with parameters \( t \in \mathbb{R}_+ \) and \( x \in \mathbb{R}^d \), i.e. \( w_t(x) \) is a (generalized)
Gaussian random variable on \((\Omega, \mathcal{F}, P)\) for \(t \in \mathbb{R}_+\) and \(x \in \mathbb{R}^d\). If \(w_t(x)\) satisfies formally the following equation:

\[
E \left( \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s, x)w_s(x)dsdx \int_{\mathbb{R}_+ \times \mathbb{R}^d} g(s, x)w_s(x)dsdx \right) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s, x)g(s, x)dsdx
\]

for all \(f, g \in L^2(\mathbb{R}_+ \times \mathbb{R}^d, dsdx)\), then \(w_t(x)\) is called a space-time white noise. If \(w(x)\) is a (generalized) Gaussian random field with parameters \(x \in \mathbb{R}^d\) satisfying that for all \(f, g \in L^2(\mathbb{R}^d, dx)\),

\[
E \left( \int_{\mathbb{R}^d} f(x)w(x)dx \int_{\mathbb{R}^d} g(x)w(x)dx \right) = \int_{\mathbb{R}^d} f(x)g(x)dx,
\]

then \(w(x)\) is called a space white noise. The rigorous definitions of various kinds of white noise are discussed in many references (see for instance [12] and [17]). In those books, the stochastic integrals of Itô type, the multiple stochastic integrals of Itô type, and the chaos expansions are discussed (see also [22], [21], [16]). These concepts will be used freely. However, in this paper only these two types of noise will be discussed and are denoted generically by a single \(w\).

Denote \(\Delta f(x) = \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}(x)\). The following stochastic partial differential equation

\[
\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t + w \circ u_t
\]

is called stochastic heat equation with white noise potential, where \(\circ\) means that the stochastic integrals involved in (1.1) is of Itô (Skorohod) type. For instance, \(\int u(s, y) \circ w_s(y)dsdy\) is the same as \(\int u(z)W(dz)\) in [22], where \(z = (s, y)\). We assume that \(u_0(x) = 1\) in this introduction.

When \(w\) is space-time white, this equation was studied by several authors, for example, in [20], [21], [22], and the the references therein. It is known that when \(d = 1\) the solution exists for all \(t > 0\). In this case the exact \(L^2\) Lyapunov exponent of the solution is investigated and it is shown that

\[
\lim_{t \to \infty} \frac{1}{t} \log \left( E |u_t(x)|^2 \right) = 1/4 .
\]

The main topics of this paper is stochastic heat equation with space white noise potential. Till the end of this section we assume that the noise is space white unless stated otherwise. When \(d = 1\), it is known [28] that the solution is in \(L^2\)

\[
0 < \liminf_{t \to \infty} \frac{1}{t} \log E |u_t(x)|^2 \leq \limsup_{t \to \infty} \frac{1}{t} \log E |u_t(x)|^2 < \infty .
\]

The above upper bound and lower bounds will be improved. Moreover, my approach allows to find the exact rate of convergence of the solution by finite chaos expansion. Let \(u_t(x) = \)
\[ \sum_{n=0}^{\infty} I_n(f_n(t,x)) \] be the chaos expansion of the solution and let \( u_i^N(x) = \sum_{n=0}^{N} I_n(f_n(t,x)) \). Then roughly speaking, \( \mathbb{E} |u_i(x) - u_i^N(x)|^2 \) is of the order \( N^{-N/2} \) as \( N \to \infty \). This is a surprisingly high rate of convergence. The same rate of convergence of \( u_i^N(x) \) to \( u_i(x) \) holds when the noise is space-time white. Professor Shiga communicated to me that he obtained limit \( \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \log |u_i(x)|^p \) for all \( p \geq 1 \) if the initial condition is bounded, nonnegative, continuous (and is not identically zero). However, his paper has not been written yet. I am not sure if his approach can be applied to obtaining more precise rate estimate.

Since the noise is not white in time, we expect nicer property of the solution. In fact this is the case. We will show that the solution is more “regular”. When \( d = 2 \), it is known that the solution is not in \( L^2 \) when the noise is space-time white. However, when the noise is space-white, the following interesting phenomena is shown: the solution to (1.1) exists in \( \mathcal{L}^2 \) when \( t < 2 \) and is not in \( \mathcal{L}_2 \) when \( t > 2\pi \). Nevertheless, the solution is shown to be in some nicer distribution spaces (see the explanation below).

The main tool of this paper is the Itô-Wiener chaos expansion, which is also used in [12], [13], [14], [15], [16], [18], [19], [21], [22] and the references therein.

It is noted that if \( u_i(x) = \sum_{n=0}^{\infty} I_n(f_n(t,x)) \) is the formal chaos expansion of \( u_i(x) \) (the solution to (1.1) when the noise is space white), then when \( d < 4 \), each chaos \( I_n(f_n(t,x)) \) is in \( \mathcal{L}_2 \). Hence, the stochastic heat equation with space white noise potential is more regular than people presumed. This phenomenon seems to be very encouraging: our living space has 3 dimensions. What a magnificent coincidence!

We introduce a new type of Hilbert space formally defined by

\[
\mathcal{S}_\gamma := \{ F = \sum_{n=0}^{\infty} F_n ; \sum_{n=0}^{\infty} (n!)^\gamma \mathbb{E} |F_n|^2 < \infty \},
\]

where \( \gamma \in \mathbb{R} \) and \( F_n \) is the \( n \)-th chaos of \( F \). It is clear that these spaces are subspaces of the distribution space introduced in [12]. Since we do not introduce weights as in [21], we shall call these spaces “flat” \( \mathcal{L}_2 \) type of distribution (or test) spaces. An element in \( \mathcal{S}_\gamma \) has the following property: each of its chaos is in \( \mathcal{L}_2 \). It is shown that when \( d < 4 \) and when the noise is space white, there is a \( \gamma_0 \) such that

1) \( u_i(x) \) is in \( \mathcal{S}_\gamma \) for all \( \gamma < \gamma_0 \), \( t \in (0, \infty) \) and \( x \in \mathbb{R}^d \);
2) \( u_i(x) \) is not in \( \mathcal{S}_\gamma \) for all \( \gamma > \gamma_0 \), \( t \in (0, \infty) \) and \( x \in \mathbb{R}^d \);
3) \( u_i(x) \) is in \( \mathcal{S}_{\gamma_0} \) for small \( t \) and \( u_i(x) \) is not in \( \mathcal{S}_{\gamma_0} \) for large \( t \).

Therefore, the regularity of the solution with respect to the noise is completely determined from the point of view of chaos expansion. Moreover, the Lyapunov exponents of the solution in \( \mathcal{S}_\gamma \) for all \( \gamma < \gamma_0 \) are also estimated.

When the noise is space-time white and \( d \geq 2 \), or when the noise is space white and \( d \geq 4 \), none of the chaos of the solution to (1.1) is in \( \mathcal{L}_2 \). A renormalization procedure is proposed. It is an extension of those dealt with in [15].

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The results obtained are extended to general elliptic operators by a comparison argument. To end this introduction, let us mention some relevant results.

When \( w \) is white with respect to time \( t \) (but not white with respect to space variable \( x \)), Carmona, Molchanov, Viens, [5], [6], [7] have studied the almost sure Lyapunov exponent of the solution, i.e. the upper bound and lower bound of \( \frac{1}{t} \log w_t(x) \) as \( t \to \infty \). If (1.1) is replaced by its discrete analogue, then the moment Lyapunov exponents were also studied in [5].

When \( w_t(x) \) is space-time white and when \( d \geq 2 \), Nualart, Rozovskii, Zakai, Holden, Øksendal, Uböe, Zhang, Potthoff, Våge, Watanabe [21], [22], [12], [23] studied the solution to (1.1) in some other types of generalized distribution spaces.

Platnitski, Zheng, and Zhao [24] applied a discretization procedure to show the regularity of the solution with respect to \( x \) for a class of nonlinear stochastic heat equation.

In most of our estimates, \( d \) can also be a real number. Since it is popular to study fractional dimensions, we write \( d < 4 \) instead of \( d \leq 3 \). Throughout the paper, \( C \) denotes a generic positive constant whose value may be different in different appearances.

## 2 Space-Time White Noise: \( d = 1 \)

In this section we consider the stochastic heat equation with \( d = 1 \). We also study the \( \mathcal{L}_2 \) asymptotic behavior of the solutions as \( t \) goes to infinity. Throughout the paper we assume that \( u_0 \) is deterministic and we denote

\[
P_t(x) = (2\pi t)^{-d/2}e^{-\frac{|x|^2}{2t}}, \quad t \in \mathbb{R}_+ , \ x \in \mathbb{R}^d.
\]

**Definition 2.1** A measurable function \( u : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R} \) is called a solution to Eq. (1.1) if \( \int_0^T \int_{\mathbb{R}^d} P_{t-s}(x-y)u_s(y) \circ w_s(y)dsdy \) exists as an element in \( \mathcal{L}_2 \) (or in \( S_\gamma \) for some \( \gamma \in \mathbb{R} \)) for every \( x \in \mathbb{R}^d \) and \( t \in (0, T] \) and the following equation is satisfied

\[
u_t(x) = P_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} P_{t-s}(x-y)u_s(y) \circ w_s(y)dsdy, \quad \forall \ t \in \mathbb{R}_+, \ x \in \mathbb{R}^d,
\]

(2.1)

where \( \circ \) denotes the stochastic integral of Itô type, i.e. the same integral as in [22] (see also [17], [12]).

Let us consider the following formal chaos expansion of the solution

\[
u_t(x) = P_t u_0(x) + \sum_{n=1}^{\infty} I_n(f_n(t, x)), \quad (2.2)
\]

where

\[
f_n(t, x; s_1, \cdots, s_n; y_1, \cdots, y_n) = \text{Sym} \left[ \int_{\mathbb{R}^d} P_{s_n-s_{n-1}}(y_n-y_{n-1}) \cdots P_{s_2-s_1}(y_2-y_1) P_1(y_1-\bar{y})u_0(\bar{y})d\bar{y} \right]
\]

(2.3)
and

\[ I_n(f_n(t, x)) = \int_{[0, t]^{n+1} \times \mathbb{R}^d} f_n(t, x; s_1, \ldots, s_n; y_1, \ldots, y_n) w_{s_1}(y_1) \cdots w_{s_n}(y_n) ds_1 dy_1 \cdots ds_n dy_n \]

(2.4)

is the multiple Itô integral with respect to the deterministic kernel \( f_n(t, x) \). Here Sym denotes the symmetrization with respect to the \((d + 1)\) dimensional variables \((s_1, y_1), (s_2, y_2), \ldots, (s_n, y_n)\).  

In this paper we will not go into details of the existence and uniqueness of the solution. Readers are referred to [9], [21], [22], and the references therein. In this section \( u_t(x) \) given by (2.2) (when \( d = 1 \)) will be discussed.

**Theorem 2.1** When \( d = 1 \), each \( I_n(f_n(t, x)) \) is well-defined as an element of \( \mathcal{L}_2 \) and (2.2) is convergent in \( \mathcal{L}_2 \).

1) If the initial value \( u_0(x) \) is identically equal to 1, then the \( \mathcal{L}_2 \) norm of \( u_t(x) \) defined by (2.2) is independent of \( x \) and the following estimate holds:

\[ \lim_{t \to \infty} \frac{\log \mathbb{E} (|u_t(x)|^2)}{t} = \frac{1}{4}, \quad \forall \ x \in \mathbb{R}. \]  

(2.5)

2) If \( u_0 \) is uniformly bounded away from \( \infty \), i.e. there is a positive constant \( C \) such that \( \sup_{x \in \mathbb{R}} |u_0(x)| \leq C \), then

\[ \limsup_{t \to \infty} \frac{\log \mathbb{E} (|u_t(x)|^2)}{t} \leq \frac{1}{4}, \quad \forall \ x \in \mathbb{R}. \]  

(2.6)

3) If \( u_0 \) is uniformly bounded away from 0, i.e. there is a positive constant \( C \) such that \( \inf_{x \in \mathbb{R}} u_0(x) \geq C \), then

\[ \liminf_{t \to \infty} \frac{\log \mathbb{E} (|u_t(x)|^2)}{t} \geq \frac{1}{4}, \quad \forall \ x \in \mathbb{R}. \]  

(2.7)

**Proof** Denote \( v(t, x) = \mathbb{E} (|u_t(x)|^2) \). Then it is easy to see that \( v(t, x) \) satisfies the following (deterministic) equation:

\[ v(t, x) = |P_t u_0(x)|^2 + \int_0^t \int_{\mathbb{R}} P_{t-s}^2(x - y) v(s, y) dy ds. \]

When the initial condition \( u_0(x) \equiv 1 \), \( v(t, x) \) satisfies

\[ v(t, x) = 1 + \int_0^t \int_{\mathbb{R}} P_{t-s}^2(x - y) v(s, y) dy ds. \]

By iteration, we obtain

\[ v(t, x) = \sum_{n=0}^{\infty} \Theta_n, \]

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where

\[ \Theta_n = \int_{T_n} (4\pi)^{-n/2} (t - s_n)^{-1/2} (s_n - s_{n-1})^{-1/2} \cdots (s_2 - s_1)^{-1/2} ds \]

\[ = (4\pi)^{-n/2} \frac{\Gamma(1/2)^n}{\Gamma(n/2 + 1)} t^{n/2} \]

\[ = \frac{1}{\Gamma(n/2 + 1)} \left( \frac{t}{4} \right)^{n/2} \]

Thus when \( u_0(x) \equiv 1 \),

\[ \mathbb{E} |u_t(x)|^2 = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n/2 + 1)} \left( \frac{t}{4} \right)^{n/2}. \] (2.8)

This means that the \( L_2 \) norm of the solution to Eq (1.1) is independent of \( x \) when the initial condition is identically 1. The summation in (2.8) is given by the Mittag-Leffler function \( E_{1/2}(\sqrt{t/4}) \), whose asymptotic behavior as \( t \to \infty \) of this function is known (see [10]), i.e.

\[ \mathbb{E} |u_t(x)|^2 = E_{1/2}(\sqrt{t/4}) = 2 \exp \left( \frac{t}{4} \right) + O\left( \frac{1}{t} \right), \ \text{(as} \ t \to \infty). \]

This proves (2.5).

(2.6) and (2.7) can be proved in a similar way. \( \square \)

Let \( u_t^N(x) \) be the sum of first finite terms, \( u_t^N(x) = P_t u_0(x) + \sum_{n=1}^N I_n(f_n(t,x)) \). Then from (2.8) it follows that

**Theorem 2.2** (1) If there is a positive constant \( C \) such that \( \sup_{x \in \mathbb{R}^d} |u_0(x)| \leq C \), then there is a finite positive constant \( C_1 \), such that

\[ \sup_{0 \leq t \leq T, x \in \mathbb{R}} \mathbb{E} |u_t(x) - u_t^N(x)|^2 \leq C_1 \frac{T^N}{4^N \Gamma(N/2 + 1)} \] (2.9)

3) If there is a positive constant \( C \) such that \( \inf_{x \in \mathbb{R}^d} u_0(x) \geq C \), then there is a finite positive constant \( C_2 \), such that

\[ \inf_{x \in \mathbb{R}} \mathbb{E} |u_T(x) - u_T^N(x)|^2 \geq C_2 \frac{T^N}{4^N \Gamma(N/2 + 1)} \] (2.10)

**Remark 1** It is a direct consequence of (2.6) and (2.7) that if there are positive constants \( c \) and \( C \) such that \( c \leq u_0(x) \leq C \) for all \( x \in \mathbb{R}^d \), then for all \( x \in \mathbb{R}^d \),

\[ \lim_{t \to \infty} \frac{\log \mathbb{E} |u_t(x)|^2}{t} = \frac{1}{4}. \]
3 Space-Time White Noise: $d \geq 2$

It is obvious that when $d \geq 2$, the formal expansion (2.2) is not convergent in $\mathcal{L}_2$ when $u_0 = 1$. In fact, one can check that the $\mathcal{L}_2$ norm of each chaos is $\infty$. In physics, it is usual to obtain something meaningful out of infinity by renormalization (see [11], [25], [26]). In this section we shall renormalize the solution following the idea of [15], where we dealt with the renormalization of the self-intersection local time of Brownian motions.

Let us consider the following stochastic heat equation with approximate space-time white noise

$$\frac{\partial u^\varepsilon_t}{\partial t} = \frac{1}{2} \Delta u^\varepsilon_t + W^\varepsilon_t \odot u^\varepsilon_t,$$

where $u_0^\varepsilon(x) = u_0(x)$ is given and is deterministic. For simplicity, we consider the regularization only in space variable, i.e.

$$w^\varepsilon_t(x) = \int_{\mathbb{R}^d} P^\varepsilon_t(x-y)w_t(y)dy.$$

Since $w^\varepsilon_t(x)$ exists only in the distribution sense (there is singularity with respect to time), we need to give meaning to the solution to the above equation. Let us denote

$$P_{t,\varepsilon}(z)f(x) = \int_{\mathbb{R}^d} P_{t}(x-y)P_{\varepsilon}(y-z)f(y)dy.$$

**Definition 3.1** $u^\varepsilon_t(x)$ is called a solution to (3.1) if

$$u_t^\varepsilon = P_t u_0 + \int_0^t \int_{\mathbb{R}^d} P_{t-s,\varepsilon}(z)u_s^\varepsilon \odot w_s(z)dsdz,$$

where $\odot$ means that the Itô type of stochastic integral is considered.

Let $I_n(f_n(t, x))$ be defined by (2.4) and

$$f_n^\varepsilon(t, x; s_1, \ldots, s_n; z_1, \ldots, z_n)$$

$$= \text{Sym} \left[ P_{1-s_n,\varepsilon}(z_n)P_{s_n-s_{n-1},\varepsilon}(z_{n-1}) \cdots P_{s_2-s_1,\varepsilon}(z_1)P_{s_1}u_0(x) \right]$$

$$= \text{Sym} \left[ \int_{\mathbb{R}^{(n+1)d}} P_{1-s_n}(x-y_n)P_{s_n-s_{n-1}}(y_n-y_{n-1}) \cdots P_{s_2-s_1}(y_2-y_1) \right.$$

$$P_{s_1}(y_n-z_n) \cdots P_{s_1}(y_1-z_1)P_{s_1}(y_1-\tilde{y})u_0(\tilde{y})dy_1 \cdots dy_n d\tilde{y},$$

where Sym is the symmetrization with respect to $(s_1, z_1), \ldots, (s_{n-1}, z_{n-1})$ and $(s_n, z_n)$. It is easy to verify that the following chaos expansion

$$u^\varepsilon(t, x) = P_t u_0(x) + \sum_{n=1}^{\infty} I_n(f_n^\varepsilon(t, x)),$$

is in $\mathcal{S}_\gamma$ for some $\gamma \in \mathbb{R}$. Moreover, $u^\varepsilon(t, x)$ is a solution to (3.1) in $\mathcal{S}_\gamma$. If we formally let $\varepsilon \to 0$ in (3.3), we obtain (2.3). Let us compute $E |I_n(f_n(t, x))|^2$ for $u_0(x) \equiv 1$. Denote it
by $\Theta_n$ ($\Theta_n$ will be different in different sections. But it is the same in each section. It will always be relevant to the $L_2$ norm of the solution to stochastic heat equation.) To compute $\Theta_n$ here and later in Section 5, the following will be used.

**Lemma 3.1** Let $s, t, a, b$ be positive numbers. Then

$$
\int_{\mathbb{R}^d} P_a(x - z)P_s(x - y)P_t(z - \bar{y})P_b(y - \bar{y})dyd\bar{y} = (2\pi(a + b + s + t))^{-d/2} P_c(x - z),
$$

where

$$
c = \frac{a(b + s + t)}{a + b + s + t}.
$$

Let us return to the computation of $\Theta_n$. Integrating with respect to $z_1, \cdots, z_n$, we obtain

$$
\Theta_n = \int P_{1-s_1}(x - y_1)P_{s_2}(y_2 - y_1)P_{s_3}(x - y_2)P_{s_4}(y_3 - y_2) \cdots P_{s_n}(y_n - y_{n-1})dsdy_1dy_2dy_3 \cdots dy_n.
$$

It follows from Lemma 3.1

$$
\Theta_n = \int_{T_n} \prod_{i=1}^{n-1} [2\pi(2\varepsilon + \kappa_i + 2s_i - 2s_i)]^{-d/2} [2\pi(\kappa_n + 2(t - s_n))]^{-d/2} ds, \quad (3.5)
$$

where $\kappa_1 = 2\varepsilon$ and

$$
\kappa_{i+1} = \frac{2\varepsilon(\kappa_i + 2s_i - 2s_i)}{2\varepsilon + \kappa_i + 2s_i - 2s_i}, \quad i = 1, 2, \cdots, n - 1.
$$

It is obvious that

$$
\kappa_{i+1} \geq \frac{2\varepsilon \kappa_i}{2\varepsilon + \kappa_i}.
$$

From this it follows that

$$
\kappa_i \geq \varepsilon / i.
$$

Therefore

$$
\Theta_n \leq (4\pi)^{-nd/2} \int_{T_n} \prod_{i=1}^{n-1} (\varepsilon + s_i - s_i)^{-d/2} (\varepsilon/(2n) + t - s_i)^{-d/2} ds.
$$

When $d > 2$, it is easy to see that for all $t > 0$

$$
\int_0^t (\kappa + t - u)^{-d/2} du \leq \frac{1}{d/2 - 1} \kappa^{-d/2+1}.
$$

Thus we see that when $d > 2$,

$$
\Theta_n \leq C^n \varepsilon^{-d/2+1}n. \quad (3.6)
$$
When \( d = 2 \) and when \( a < 1/e, b > e \),

\[
\int_0^t (a/b + t - u)^{-1} du \leq - \log a + \log b + \log(t + a/b)
\]

It is easy to see that when \( a, b, c \geq 1, a + b \leq 2ab \) and \( a + b + c \leq 3abc \). Thus when \( \varepsilon < 1/e \), we obtain

\[
\int_0^t (\varepsilon + t - u)^{-1} du \leq 2 \log(\varepsilon + T)(- \log \varepsilon)
\]

and

\[
\int_0^t (\varepsilon/(2n) + t - u)^{-1} du \leq 3 \log(2n) \log(\varepsilon + T)(- \log \varepsilon).
\]

When \( \varepsilon < e^{-1} \),

\[
\Theta_n \leq 3 \log(2n) 2^{n-1} \log(e + T)^n(- \log \varepsilon)^n.
\]

Hence

\[
\Theta_n \leq \begin{cases} 
  C^n \varepsilon^{-\left(\frac{d-2}{2}\right)n} & \text{if } d > 2 \\
  C^n (- \log \varepsilon)^n (\log(e + T))^n & \text{if } d = 2 \end{cases}
\]

(3.7)

We introduce the following generalized second quantization operator \( \Gamma(\lambda, \gamma) \) as follows

\[
\Gamma(\lambda, \gamma) F = \sum_{n=0}^{\infty} (n!)^\lambda \gamma^n F_n
\]

if \( F = \sum_{n=0}^{\infty} F_n \) is the Wiener-Itô’s chaos expansion of \( F \). Introduce also \( \Gamma(\gamma) F = \Gamma(1, \gamma) F \).

By (3.7), it follows

**Theorem 3.2** 1) Let \( d > 2 \) and let \( u_0 \) be bounded on \( \mathbb{R}^d \). Then for all \( \lambda < 0 \), \( \Gamma\left(\lambda, \varepsilon^{\frac{d-2}{4}}\right) u_\varepsilon \)

is a uniformly bounded sequence in \( \mathcal{L}^2 \) with respect to \( \varepsilon \in (0, e^{-1}], t \in (0, \infty) \) and \( x \in \mathbb{R}^d \).

2) Let \( d = 2 \) and let \( u_0 \) be bounded on \( \mathbb{R}^d \). Then for all \( \lambda < 0 \), \( T \in (0, \infty) \) and \( x \in \mathbb{R}^d \),

\( \Gamma\left(\lambda, \sqrt{-\log \varepsilon}\right) u_\varepsilon \) is a uniformly bounded sequence in \( \mathcal{L}^2 \) with respect to \( \varepsilon \in (0, e^{-1}], t \in (0, T] \) and \( x \in \mathbb{R}^d \).

It is known that when \( d \geq 2 \), \( u_\varepsilon(x) \) is in general not a square integrable random variable. Thus we expect that when \( \varepsilon \to 0 \), \( u_\varepsilon \) is not bounded in \( \mathcal{L}_2 \). In fact a more precise result is stated.

**Theorem 3.3** Let the initial condition \( f(x) \geq c \), where \( c \in (0, \infty) \) is a constant. Then

1) When \( d > 2 \), for all continuous function \( \gamma(\varepsilon) \) on \((0, \infty)\) with \( \varepsilon^{d/4 - 1/2} = o(\gamma(\varepsilon)) \) as \( \varepsilon \to 0 \),

\[
\liminf_{\varepsilon \to 0} \int_{x \in \mathbb{R}^d} |\Gamma(\lambda, \gamma(\varepsilon)) u_\varepsilon(x)|^2 = \infty
\]

for all \( \lambda \in \mathbb{R}, t > 0 \).
2) When $d = 2$, for all continuous function $\gamma(\varepsilon)$ on $(0, \infty)$ with $\sqrt{-\log \varepsilon} = o(\gamma(\varepsilon))$ as $\varepsilon \to 0$,
\[ \lim \inf_{\varepsilon \to 0} \mathbb{E} |\Gamma(\lambda, \gamma(\varepsilon)) u^\varepsilon_t(x)|^2 = \infty \]  
(3.9)
for all $\lambda \in \mathbb{R}$, $t > 0$.

Proof. It suffices to show this theorem for $u_0 \equiv 1$. Let us consider the first chaos of $u^\varepsilon_t(x)$:
\[ I^\varepsilon_1 = \int_{\mathbb{R}^d} f(t, x; s, z) \circ w_s(z) ds dz, \]
where
\[ f(t, x; s, z) = \int_0^t P_t-s(x - y) P_s(y - z) dy ds = \int_0^t P_t-s+x(x - z) ds. \]
By definition of $\Gamma(\gamma(\varepsilon))$ and the orthogonality of different chaos, we have that
\[ \mathbb{E} |\Gamma(\gamma(\varepsilon)) u^\varepsilon_t(x)|^2 \geq \gamma(\varepsilon)^2 \mathbb{E} |I_1(f_1(t, x))|^2. \]
It is easy to check that
\[ \mathbb{E} |I_1(f(t, x))|^2 = (2\pi)^{-d/2} \int_0^t (2t - 2s + 2\varepsilon)^{-d/2} ds \]
\[ = \begin{cases} (-d/2 + 1)^{-1} [(2\varepsilon)^{-d/2+1} - (2t + 2\varepsilon)^{-d/2+1}] & \text{when } d > 4 \\
\log \left( \frac{t+\varepsilon}{\varepsilon} \right)^2 & \text{when } d = 4. \end{cases} \]
Thus the theorem follows. \hfill \Box

4 Space White Noise: $d < 4$

From now on let $w = w(x), x \in \mathbb{R}^d$ be a space white noise. Consider the following stochastic heat equation with space white noise potential:
\[ \frac{\partial u_t(x)}{\partial t} = \frac{1}{2} \Delta u_t(x) + w(x) \circ u_t(x), \]  
(4.1)
where the initial condition $u_0$ is a given deterministic function. Consider the following formal expansion of the solution
\[ u_t(x) = P_t f(x) + \sum_{n=1}^{\infty} I_n(f_n(t, x)), \]  
(4.2)
where
\[ f_n(t, x; y_1, \ldots, y_n) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P_{t-s_n}(x - y_n) P_{s_n-s_{n-1}}(y_n - y_{n-1}) \cdots P_{s_2-s_1}(y_2 - y_1) P_{s_1}(y_1 - \tilde{y}) f(\tilde{y}) dy_{n}ds \]
(4.3)
and
\[ I_n(f_n(t, x)) = \int_{\mathbb{R}^{nd}} f_n(t, x; y_1, \ldots, y_n) w(y_1) \circ \cdots \circ w(y_n) dy_1 \cdots dy_n. \]  
(4.4)
Theorem 4.1 When $d < 4$, each $I_n(f_n(t, x))$ is well-defined as an element in $L^2$.

1) Let $d < 2$ and let $\alpha = 1 - d/4$. If $|u_0(x)| \leq C < \infty$, then

$$\limsup_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha - 1)}} \log \left\{ \sup_{x \in \mathbb{R}^d} E |u_t(x)|^2 \right\} \leq (U(\alpha))^{1/(2\alpha - 1)},$$

where

$$U(\alpha) := \frac{(2\alpha - 1)^{2\alpha - 1}}{(4\pi)^{d/2}\alpha^{2\alpha}} \Gamma(\alpha)^2.$$

If $u_0(x) \geq C > 0$, then

$$\liminf_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha - 1)}} \log \left\{ \inf_{x \in \mathbb{R}^d} E |u_t(x)|^2 \right\} \geq (L(\alpha))^{1/(2\alpha - 1)},$$

where

$$L(\alpha) := \frac{(2\alpha - 1)^{2\alpha - 1}\Gamma(2\alpha)}{(2\pi)^{d/2}\alpha^{2\alpha}}.$$

2) Let $d = 2$. If $|u_0(x)| \leq C < \infty$, then when $t < 2$, $u_t(x)$ is defined as an element in $L^2$ for all $x \in \mathbb{R}^d$. If $u_0(x) \geq C > 0$, then when $t > 2\pi$,

$$E |u_t(x)|^2 = \infty, \quad \forall \ x \in \mathbb{R}^d. \quad (4.7)$$

3) When $d > 2$, $E |u_t(x)|^2 = \infty$ for all $t > 0$ and $x \in \mathbb{R}^d$ if $u_0(x) \geq C > 0$ for some constant $C > 0$.

Remark 2 When $d = 1$, the upper bound $U(1/2) \approx 0.1381$ and $L(1/2) \approx 0.0145$. In [28], the upper is given by $1/4$ and lower bound is $1/(8\pi e^3) \approx 0.0002$.

Remark 3 To get a better idea about the difference between the two bounds let us divide the above upper bound by the lower bound ((4.5) and (4.6)) and denote this number by Ratio. Thus

$$\text{Ratio} = \left( \frac{2^{2\alpha} \Gamma(\alpha)^2}{2^{d/2} \Gamma(2\alpha)} \right)^{1/(2\alpha - 1)}.$$

When $d = 1$, we have $\alpha = 3/4$, $2\alpha - 1 = 1/2$. Hence

$$\text{Ratio} = \left( \frac{2 \Gamma(3/4)^2}{\Gamma(3/2)} \right)^2.$$

From [1], $\Gamma(3/4) \approx 1.2254$, $\Gamma(3/2) = \frac{1}{2} \Gamma(1/2) = \sqrt{\pi}/2 \approx 0.8862$. Thus $\text{Ratio} \approx 11.4844$. 

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Proof. In this section we denote

\[ \Theta_n := \int_{\mathbb{R}^d} \left\{ \int_{T_n} P_{t-s_n}(x-y_n)P_{s_n-s_{n-1}}(y_n-y_{n-1}) \cdots P_{s_2-s_1}(y_2-y_1) \, ds \right\} \, dy \]  

(4.8)

which is \( \mathbb{E} \left( I_n(f_n(t,x)) \right)^2 / n! \) when \( u_0(x) = 1 \). It is easy to see that

\[ \mathbb{E} \left( I_n(f_n(t,x)) \right)^2 \leq n! \| f \|^2 \Theta_n. \]

Using the semigroup property \( \int_{\mathbb{R}} P_s(x-y)P_t(y-z) \, dy = P_{s+t}(x-z) \), we have

\[
\begin{align*}
\Theta_n &= \int_{T_n \times T_n} \int_{\mathbb{R}^n} P_{t-s_n}(x-y_n)P_{s_n-s_{n-1}}(y_n-y_{n-1}) \cdots P_{s_2-s_1}(y_2-y_1) \\
&= \frac{(2\pi)^{-nd/2}}{T_n \times T_n} \int_{T_n} (s_2 - s_1)^{-d/4} \cdots (s_n - s_{n-1})^{-d/4} \\
&= (4\pi)^{-nd/2} \left\{ \frac{\Gamma(\alpha)^{n+1}}{\Gamma(n\alpha + 1)} \right\}^2.
\end{align*}
\]

(4.9)

This shows that \( \mathbb{E} u_k(x)^2 \) is independent of \( x \) if \( u_0(x) = 1 \). We shall bound \( \Theta_n \) from above and from below. Let us bound \( \Theta_n \) from above first. From \( 2\sqrt{ab} \leq a + b \), it follows that \( (a + b)^{-d/2} \leq 2^{-d/2} a^{-d/4} b^{-d/4} \). Thus

\[
\begin{align*}
\Theta_n &\leq 2^{-nd/2} (2\pi)^{-nd/2} \int_{T_n} (s_2 - s_1)^{-d/4} \cdots (s_n - s_{n-1})^{-d/4} \\
&= (4\pi)^{-nd/2} \left\{ \frac{\Gamma(\alpha)^{n+1}}{\Gamma(n\alpha + 1)} \right\}^2.
\end{align*}
\]

(4.10)

This shows that when \( d < 4 \), \( I_n(f_n(t,x)) \) is a well-defined square integrable random variable. Throughout this paper \( C_x \) denotes a generic function of \( x \) such that there are constants \( \rho_1, \rho_2, C_1 > 0 \), and \( C_2 > 0 \) such that \( C_1 x^{\rho_1} \leq C_x \leq C_2 x^{\rho_2} \) for all \( x \geq 1 \), where \( \rho_1, \rho_2, C_1 \), and \( C_2 > 0 \) may depend on the dimension \( d \) (i.e. \( \alpha \)). \( C_x \) may be different in different appearances. From the Stirling’s formula, i.e. there is a constant \( C_x \) such that \( \Gamma(x + 1) = C_x x^e e^{-x} \), it follows

\[
\frac{n!}{\Gamma(\alpha n + 1)} = C_n \frac{n^\alpha e^{-n}}{(\alpha n)^{2\alpha n} e^{-2\alpha n}} = \left( \frac{(2\alpha - 1)^{2\alpha-1}}{\alpha^{2\alpha}} \right)^n \frac{C_n}{((2\alpha - 1)n)^{2\alpha-1)n} e^{-(2\alpha-1)n} = \left( \frac{(2\alpha - 1)^{2\alpha-1}}{\alpha^{2\alpha}} \right)^n \frac{C_n}{\Gamma((2\alpha - 1)n + 1)}.
\]

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For all \( \rho \) and all \( \varepsilon > 1 \), there are constants \( C_1 \) and \( C_2 \) such that \( n^{\rho} \leq C_1 + C_2 \varepsilon^n \). Thus for all \( \varepsilon > 1 \), there are constants \( C_0, C_1, C_2 \), and \( N \) depending on \( \varepsilon \), dimension, and sup norm of the initial condition, such that the \( L_2 \) norm of the solution is bounded as follows

\[
\mathbb{E} |u_t(x)|^2 \leq \|u_0\|_2^2 \sum_{n=0}^{\infty} n! \Theta_n
\]

\[
= \|u_0\|_2^2 \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n\alpha + 1)^2} \frac{1}{4\pi}^{-nd/2} \Gamma(\alpha)^{2n} t^{2\alpha n}
\]

\[
\leq \sum_{n=0}^{\infty} \frac{C_n}{\Gamma((2\alpha - 1)n + 1)} \left( \frac{2\alpha - 1}{\alpha^{2\alpha}} \right)^{\varepsilon} \frac{1}{4\pi}^{-nd/2} \Gamma(\alpha)^{2n} t^{2\alpha n}
\]

\[
\leq C_0 + C_1 t^N + C_2 \sum_{n=0}^{\infty} \frac{1}{\Gamma((2\alpha - 1)n + 1)} \left( \frac{\varepsilon(2\alpha - 1)^{2\alpha-1}}{\alpha^{2\alpha}} \right)^{\varepsilon} \frac{1}{4\pi}^{-nd/2} \Gamma(\alpha)^{2n} t^{2\alpha n}
\]

\[
= C_0 + C_1 t^N + C_2 E_{2\alpha-1} \left( \frac{\varepsilon(2\alpha - 1)^{2\alpha-1}\Gamma(\alpha)^2 t^{2\alpha}}{(4\pi)^{d/2}\alpha^{2\alpha}} \right)^{1/(2\alpha-1)}
\]

where \( E_{2\alpha-1} \) is the Mittag-Leffler function.

Thus we see that \( t \to \infty \),

\[
\mathbb{E} |u_t(x)|^2 \leq C_0 + C_1 t^N + C_2 \exp \left( \frac{\varepsilon(2\alpha - 1)^{2\alpha-1}\Gamma(\alpha)^2 t^{2\alpha}}{(4\pi)^{d/2}\alpha^{2\alpha}} \right)^{1/(2\alpha-1)}
\]

Consequently we have

\[
\limsup_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log \left( \sup_{x \in \mathbb{R}^d} \mathbb{E} |u_t(x)|^2 \right) \leq \left( \frac{\varepsilon(2\alpha - 1)^{2\alpha-1}\Gamma(\alpha)^2}{(4\pi)^{d/2}\alpha^{2\alpha}} \right)^{1/(2\alpha-1)}
\]

Since \( \varepsilon > 1 \) is arbitrary, we have

\[
\limsup_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log \left( \sup_{x \in \mathbb{R}^d} \mathbb{E} |u_t(x)|^2 \right) \leq \left( \frac{(2\alpha - 1)^{2\alpha-1}\Gamma(\alpha)^2}{(4\pi)^{d/2}\alpha^{2\alpha}} \right)^{1/(2\alpha-1)}
\]

proving (4.5).

Now let us bound \( \Theta_n \) from below. This will be slightly more complicated. Let \( \beta = 2 - d/2 = 2\alpha \). By a simple substitution,

\[
\Theta_n = (2\pi)^{-nd/2} t^{\beta n} \int_{T_n(1)} \left( s_2 + r_2 - s_1 - r_1 \right)^{-d/2} \cdots \left( s_n + r_n - s_{n-1} - r_{n-1} \right)^{-d/2} (2 - s_n - r_n)^{-d/2} dsdr,
\]

where

\[
T_n(1) := \{(s_1, \cdots, s_n); \quad 0 < s_1 < \cdots < s_n < 1\}.
\]
To bound $\Theta_n$ from below, let us make the following substitution,

$$
x_1 = r_1 + s_1, \quad x_2 = r_2 + s_2 - r_1 - s_1, \quad \cdots, \quad x_n = r_n + s_n - r_{n-1} - s_{n-1}
$$

$$
y_1 = s_1, \quad y_2 = s_2 - s_1, \quad \cdots, \quad y_n = s_n - s_{n-1}
$$

It is easy to notice that if $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ are in

$$
\mathcal{Y}_n = \{(x_1, \cdots, x_n, y_1, \cdots, y_n); \ y_i < x_i, \ x_i \geq 0, \ i = 1, 2, \cdots, n, \ x_1 + x_2 + \cdots + x_n < 1\},
$$

then $0 < r_1 < \cdots < r_n < 1$ and $0 < s_1 < \cdots < s_n < 1$. Namely, if $x, y \in \mathcal{Y}_n$, then $(s_1, \cdots, s_n)$ and $(r_1, \cdots, r_n)$ are in $T_n(1)$. It is also easy to check that the Jacobi determinant of the above substitution is 1. Thus

$$
\Theta_n \geq (2\pi)^{-nd/4} t^\beta n \int_{\mathcal{Y}_n} x_2^{\beta-2} x_3^{\beta-2} \cdots x_n^{\beta-2} (2 - x_1 - x_2 - \cdots - x_n)^{\beta-2} dy_1 \cdots dy_n dx_1 \cdots dx_n
$$

$$
\geq 2^{-d/2} (2\pi)^{-nd/4} t^\beta n \int_{\mathcal{Y}_n} x_2^{\beta-2} x_3^{\beta-2} \cdots x_n^{\beta-2} dy_1 \cdots dy_n dx_1 \cdots dx_n
$$

$$
\geq 2^{-d/2} (2\pi)^{-nd/4} t^\beta n \int_{1 < 0, \cdots, n < 0, \ x_1 + x_2 + \cdots + x_n < 1} x_1 x_2^{\beta-1} x_3^{\beta-1} \cdots x_n^{\beta-1} dx_1 \cdots dx_n
$$

It follows then

$$
\Theta_n \geq 2^{-d/2} (2\pi)^{-nd/2} \frac{\Gamma(\beta)^{n-1}}{\Gamma((n - 1)\beta + 3)} t^\beta n. \quad (4.11)
$$

Thus when $u_0 \geq c > 0$,

$$
E |u_t(x)|^2 \geq c^2 \sum_{n=0}^{\infty} n! \Theta_n
$$

$$
\geq c^2 \sum_{n=0}^{\infty} (2\pi)^{-nd/2} \frac{n! \Gamma(\beta)^{n-1}}{\Gamma((n - 1)\beta + 3)} t^\beta n. \quad (4.12)
$$

On the other hand, by the Stirling’s formula, we have (using the generic notation $C_2$ introduced earlier),

$$
\frac{n!}{\Gamma(\beta n + 3 - \beta)} = C_n \frac{n!}{\Gamma(\beta n)}
$$

$$
= C_n \frac{n^n e^{-n}}{(\beta n)^{\beta n} e^{-\beta n}}
$$

$$
\geq \frac{(\beta - 1)^{\beta - 1}}{\beta} \left(\frac{C_n}{((\beta - 1)n)^{(\beta - 1)n} e^{-(\beta - 1)n}}\right)^n
$$

$$
\geq \frac{(\beta - 1)^{\beta - 1}}{\beta} \frac{C_n}{\Gamma((\beta - 1)n + 1)}.
$$
Thus for all \( \varepsilon < 1 \), there is an \( N \) such that when \( n \geq N \),

\[
\frac{n!}{\Gamma(\beta n + 3 - \beta)} \geq \left( \frac{\varepsilon(\beta - 1)^{\beta-1}}{\beta^\beta} \right)^n \frac{1}{\Gamma((\beta - 1)n + 1)}.
\]

Thus when \( u_0 \geq c \) for some constant \( c > 0 \),

\[
\mathbb{E} |u_t(x)|^2 \geq C_0 + C_1 t^N + C_2 \sum_{n=0}^{\infty} \frac{1}{\Gamma((\beta - 1)n + 1)} \left( \frac{\varepsilon t^\beta(\beta - 1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2} \beta^\beta} \right)^n
\]

\[
= C_0 + C_1 t^N + C_2 E_{\beta-1} \left( \frac{\varepsilon t^\beta(\beta - 1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2} \beta^\beta} \right)
\]

\[
\geq C_0 + C_1 t^N + C_2 \exp \left\{ \left( \frac{\varepsilon t^\beta(\beta - 1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2} \beta^\beta} \right)^{1/(\beta-1)} \right\}
\]

Thus when \( t \to \infty \), it follows

\[
\liminf_{t \to \infty} \frac{1}{t^{\beta/(\beta-1)}} \log \left\{ \inf_{x \in \mathbb{R}^d} \mathbb{E} |u_t(x)|^2 \right\} \geq \left( \frac{\varepsilon(\beta - 1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2} \beta^\beta} \right)^{1/(\beta-1)}.
\]

Let \( \varepsilon \to 1 \), it follows

\[
\liminf_{t \to \infty} \frac{1}{t^{\beta/(\beta-1)}} \log \left\{ \inf_{x \in \mathbb{R}^d} \mathbb{E} |u_t(x)|^2 \right\} \geq \left( \frac{(\beta - 1)^{\beta-1}\Gamma(\beta)}{(2\pi)^{d/2} \beta^\beta} \right)^{1/(\beta-1)}.
\]

Replacing \( \beta \) by \( 2\alpha \) we show (4.6).

From the Stirling’s formula as shown previously, it follows

\[
\frac{n!}{\Gamma(n/2 + 1)^2} \leq C_n 2^n
\]

for some constant \( C_n \) as defined earlier this section. Thus by (4.10) we have when \( d = 2 \) and

\[
|u_0(x)| \leq C < \infty
\]

\[
\mathbb{E} |u_t(x)|^2 \leq C^2 \sum_{n=0}^{\infty} (4\pi)^{-n} n! \Gamma(1/2)^{2n} t^n \leq \sum_{n=0}^{\infty} C_n \left( \frac{t}{2} \right)^n
\]

This series is convergent when \( t < 2 \), which proves the first part of 2) in Theorem 4.1.

On the other hand, \( \beta = 1 \) when \( d = 2 \). By (4.12), when \( u_0 \geq C \) for some constant \( C > 0 \),

\[
\mathbb{E} |u_t(x)|^2 \geq \sum_{n=0}^{\infty} C_n \frac{t^n}{(2\pi)^n}.
\]

This shows that when \( t > 2\pi \), \( \mathbb{E} |u_t(x)|^2 = \infty \), proving (4.7).
It is also easy to see from the estimate that when \( d > 2 \), \( \mathbb{E} \ |u_t(x)|^2 = \infty \) for all \( t > 0 \) if \( u_0 \geq C \) for some constant \( C > 0 \). This completes the proof of Theorem 4.1. \( \square \)

Let
\[
  u_t^N(x) = P_t f(x) + \sum_{n=1}^N I_n(f_n(t, x)).
\]

The above estimate can be used to find the exact rate of convergence of approximating \( u_t(x) \) by the finite sum \( u_t^N(x) \).

**Theorem 4.2** There are constants \( C_1(N) \) and \( C_2(N) \) of polynomial growth or polynomial decay such that
\[
  C_1(N) \frac{U(\alpha)^N T^{2\alpha N}}{\Gamma((2\alpha - 1)n + 1)} \leq \inf_{x \in \mathbb{R}^d} \mathbb{E} \ |u_T(x) - u_T^N(x)|^2 \leq \sup_{0 \leq t \leq T, x \in \mathbb{R}^d} \mathbb{E} \ |u_t(x) - u_t^N(x)|^2 \leq C_2(N) \frac{U(\alpha)^N T^{2\alpha N}}{\Gamma((2\alpha - 1)n + 1)}.
\]

**Proof** Since \( \mathbb{E} \ |u_T(x) - u_T^N(x)|^2 \geq (N + 1)! \Theta_{N+1} \). The lower bound follows easily from (4.11). On the other hand, we have
\[
  \mathbb{E} \ |u_t(x) - u_t^N(x)|^2 = \sum_{n=N+1}^{\infty} c(n) \frac{U(\alpha)^n t^{2\alpha n}}{\Gamma((2\alpha - 1)n + 1)} \leq C_2(N) \frac{U(\alpha)^N t^{2\alpha N}}{\Gamma((2\alpha - 1)N + 1)},
\]
proving the theorem. \( \square \)

**Remark 4** The above theorem implies in particular that when \( d = 1 \), \( \mathbb{E} \ |u_t(x) - u_t^N(x)|^2 \) is of order \( N^{-N/2} \). This is a very good approximation.

Let us recall the distribution and test functional spaces of Meyer-Watanabe type.

Let \( F \in \mathcal{L}_2 \). Then \( F \) admits an Itô-Wiener chaos expansion
\[
  F = \sum_{n=0}^{\infty} F_n. \tag{4.13}
\]

Let us denote by \( \mathcal{G}_f \) the set of all finite sum of chaos, i.e. \( \mathcal{G}_f = \{ F : F = \sum_{n=0}^{k} F_n \} \). \( \mathcal{G}_f \) is a dense subset of \( \mathcal{L}_2 \). We denote by \( D_\gamma \) the completion of \( \mathcal{G}_f \) under the norm
\[
  |F|_\gamma^2 := \sum_{n=0}^{k} (n + 1)\gamma \mathbb{E} (F_n^2), \quad \text{where} \quad F = \sum_{n=0}^{k} F_n.
\]

Since the radius of convergence and the asymptotics as \( t \to \infty \) of the series \( \sum_{n=0}^{\infty} a_n n^{\gamma} t^n \) do not depend on \( \gamma \), the asymptotics of \( U_t(x) \) defined by (4.2) in the Meyer-Watanabe spaces will be the same as in \( \mathcal{L}_2 \). Namely, we can extend Theorem 4.1 to the following theorem.
Theorem 4.3 Let \( d < 4 \).

1) Let \( d < 2 \), \( \gamma \in \mathbb{R} \) and let \( \alpha = 1 - d/4 \). If \( |u_0(x)| \leq C < \infty \), then \( u_t(x) \in D_\gamma \) for all \( t > 0 \) and

\[
\limsup_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log \left\{ \sup_{x \in \mathbb{R}^d} |u_t(x)|^2_\gamma \right\} \leq \left( \frac{(2\alpha - 1)^{2\alpha - 1 - \gamma}}{(4\pi)^{d/2} \alpha^{2\alpha}} \Gamma(\alpha)^{2 \gamma} \right)^{1/(2\alpha-1)}.
\]

(4.14)

If \( u_0(x) \geq C > 0 \), then

\[
\liminf_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log \left\{ \inf_{x \in \mathbb{R}^d} |u_t(x)|^2_\gamma \right\} \geq \left( \frac{(2\alpha - 1)^{2\alpha - 1 - \gamma}}{(4\pi)^{d/2} \alpha^{2\alpha}} \Gamma(\alpha)^{2 \gamma} \right)^{1/(2\alpha-1)}.
\]

(4.15)

2) Let \( d = 2 \). If \( |u_0(x)| \leq C < \infty \), then when \( t < 2 \), \( u_t(x) \) is defined as an element in \( D_\gamma \). If \( u_0(x) \geq C > 0 \), then when \( t > 2\pi \),

\[
|u_t(x)|_\gamma = \infty.
\]

(4.16)

3) When \( d > 2 \), \( |u_t(x)|_\gamma = \infty \) for all \( t > 0 \) if \( u_0(x) \geq c > 0 \) for some constant \( c > 0 \).

Now we introduce a class of “flat” \( L_2 \) type of spaces.

Let \( \mathcal{S}_\gamma \) denote the completion of \( \mathcal{G}_f \) under the norm

\[
|||F|||^2_\gamma := \sum_{n=0}^{k} (n!)^\gamma \mathbb{E} \left( F_n^2 \right), \quad \text{where} \quad F = \sum_{n=0}^{k} F_n.
\]

These spaces are not weighted (compare these with the distribution spaces introduced in [21], [22]). For each element in this space, its \( n \)-th chaos is in \( L_2 \).

The following theorem deals with the solutions of (4.1) in \( \mathcal{S}_\gamma \).

Theorem 4.4 Let \( d < 4 \). Then

1) Let \( \gamma < 2\alpha - 1 = 1 - d/2 \). Then \( u_t(x) \in \mathcal{S}_\gamma \) for all \( t \in [0, \infty) \) and \( x \in \mathbb{R}^d \). Moreover, when \( |u_0(x)| \leq C < \infty \),

\[
\limsup_{t \to \infty} \frac{1}{t^{2\alpha - 1 - \gamma}} \log \left\{ \sup_{x \in \mathbb{R}^d} ||u_t(x)||^2_\gamma \right\} \leq (2\alpha - 1 - \gamma) \left( \frac{\Gamma(\alpha)^2}{(4\pi)^{d/2} \alpha^{2\alpha}} \right)^{1/(2\alpha-1)}.
\]

(4.17)

When \( u_0(x) \geq C > 0 \),

\[
\liminf_{t \to \infty} \frac{1}{t^{2\alpha - 1 - \gamma}} \log \left\{ \inf_{x \in \mathbb{R}^d} ||u_t(x)||^2_\gamma \right\} \geq (2\alpha - 1 - \gamma) \left( \frac{\Gamma(\alpha)}{(2\pi)^{d/2} \alpha^{2\alpha}} \right)^{1/(2\alpha-1)}.
\]

(4.18)

2) Let \( \gamma = 2\alpha - 1 = 1 - d/2 \). Then when \( |u_0(x)| \leq C < \infty \) and

\[
t < \frac{\alpha(4\pi)^{d/(2\alpha)}}{\Gamma(\alpha)^{1/\alpha}},
\]

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the solution \(u_t(x)\) exists in \(S_\gamma\) for all \(x \in \mathbb{R}^d\). When \(u_0(x) \geq C > 0\) and

\[ t > \frac{2\alpha(2\pi)^{d/(4\alpha)}}{\Gamma(2\alpha)^{1/(2\alpha)}}, \]

\(u_t(x)\) is not an element of \(S_\gamma\) for any \(x \in \mathbb{R}^d\).

3) Let \(\gamma > 2\alpha - 1 = 1 - d/2\). If \(u_0(x) \geq C > 0\), then \(u_t(x)\) is not in \(S_\gamma\) for any \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\).

Proof The proof is similar to that of Theorem 4.1 and is omitted. \(\Box\)

5 Space White Noise: \(d \geq 4\)

When \(d \geq 4\), it is easy to see that \(f_n(t, x)\) defined by (4.3) is not in \(L^2(T^2)\) if \(u_0(x) \geq C > 0\). Thus none of the chaos \(I_n(f_n(t, x))\) of \(u_t(x)\) defined by (4.2) is in \(L_2\).

Therefore we shall introduce a renormalization procedure. Let

\[ w^\varepsilon(x) = \int_{\mathbb{R}^d} P_\varepsilon(x - y)w(y)dy. \]

Consider the following approximation of the stochastic heat equation

\[ \frac{\partial u_t}{\partial t} = \frac{1}{2} \Delta u_t + w^\varepsilon(x) \circ u_t. \tag{5.1} \]

The initial condition \(u_0(x)\) is given and is deterministic.

For all \(\varepsilon > 0\), the solution to the above approximated equation exists in some \(S_\gamma\). Its Wiener chaos expansion is given by

\[ u^\varepsilon_t(x) = P_t u_0(x) + \sum_{n=1}^\infty I_n(f^\varepsilon_n(t, x)), \tag{5.2} \]

where

\[ f^\varepsilon_n(t, x; z_1, \ldots, z_n) = \text{Sym}\left[ \int_{T^n \times \mathbb{R}^d} P_{t-s_n}(x - y_n)P_{s_n-s_{n-1}}(y_n - y_{n-1}) \cdots P_{s_2-s_1}(y_2 - y_1) P_\varepsilon(y_1 - z_1) \cdots P_\varepsilon(y_n - z_n)P_s(y_1 - \tilde{y})u_0(\tilde{y})dydyds \right] \tag{5.3} \]

and

\[ I_n(f^\varepsilon_n(t, x)) = \int_{\mathbb{R}^d} f^\varepsilon_n(t, x; z_1, \ldots, z_n)w(z_1) \circ \cdots \circ w(z_n)dz_1 \cdots dz_n. \tag{5.4} \]

Theorem 5.1 1) Let \(d > 4\). Let \(u_0\) be bounded on \(\mathbb{R}^d\). Then for all \(\lambda < 0\), \(\Gamma\left(\lambda, \sqrt{\varepsilon^{d/2-1}}\right) u^\varepsilon_t\) is a uniformly bounded sequence in \(L^2\) with respect to \(\varepsilon \in (0, e^{-1}], t \in (0, \infty)\) and \(x \in \mathbb{R}^d\).

2) Let \(d = 4\). Let \(u_0\) be bounded on \(\mathbb{R}^d\). Then for all \(\lambda < 0\), \(T \in (0, \infty)\) and \(x \in \mathbb{R}^d\), \(\Gamma\left(\lambda, \sqrt{-\log \varepsilon}\right) u^\varepsilon_t\) is a uniformly bounded sequence in \(L^2\) with respect to \(\varepsilon \in (0, e^{-1}], t \in (0, T]\) and \(x \in \mathbb{R}^d\).
Proof. Let \( \kappa_i = 2\varepsilon \) and let
\[
\kappa_{i+1} = \frac{2\varepsilon (\kappa_i + s_{i+1} - s_i + r_{i+1} - r_i)}{2\varepsilon + \kappa_i + s_{i+1} - s_i + r_{i+1} - r_i}, \quad i = 1, 2, \ldots, n - 1.
\]
Similar to (3.5), we have
\[
\Theta_n = \int_{\mathbb{T}^d_n} \prod_{i=1}^{n-1} (2\pi (\kappa_i + 2\varepsilon + s_{i+1} - s_i + r_{i+1} - r_i))^{-d/2} (2\pi (\kappa_n + s_{i+1} - s_i + r_{i+1} - r_i))^{-d/2} dsdr.
\]
Similar to (3.7),
\[
\Theta_n \leq \begin{cases} 
C^n \varepsilon^{d-n} & \text{when } d > 4 \\
C^n (-\log \varepsilon)^n \log(e + T)^n & \text{when } d = 4
\end{cases}
\]
The theorem then follows. \( \square \)

An theorem analogous to Theorem 3.3 can be also stated and proved.

6 General Stochastic Heat Equations

In this section we will discuss the stochastic Heat equations when the Laplacian \( \Delta \) is replaced by a general elliptic operator. For simplicity we assume that the operator is of divergence form.

Let \( a_{ij}(x), 1 \leq i, j \leq d \) be measurable functions on \( \mathbb{R}^d \) and let \( A(x) = (a_{ij}(x))_{1 \leq i, j \leq d} \) be symmetric positive matrices for all \( x \in \mathbb{R}^d \) such that
\[
\lambda I \leq A(x) \leq \mu I, \quad \forall \ x \in \mathbb{R}^d,
\]
where \( I \) is the \( d \) dimensional unit matrix; \( \lambda \) and \( \mu \) are two positive numbers. Let \( L = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) \) be the second order elliptic operator of divergence form with coefficient \( A \). It is known that the semigroup associated with \( L \) is given by a kernel of the form \( Q_t(x-y) \). Moreover the following Nash’s estimate holds (see [8, 27])
\[
C_1 P_{\mu_1 t}(x) \leq Q_t(x) \leq C_2 P_{\mu_2 t}(x), \quad (6.1)
\]
where \( \mu_1, C_1, \mu_2, \) and \( C_2 \) are positive constants; \( P_t \) is defined as in the beginning of Section 2.

Consider the following stochastic heat equation
\[
\frac{\partial u_t}{\partial t} = Lu_t + w \circ u_t, \quad (6.2)
\]
where \( w \) is a space white or space-time white noise, \( u_0(x) \) is a given deterministic function.

All results obtained for Eq. (1.1) can be extended to (6.2) by (6.1). For instance, we can state the following theorem
Theorem 6.1 (1) If \( w_s(x) \) is a space-time white noise, if \( d = 1 \) and if there is positive constant \( \mu > 0 \) such that \( |u_0(x)| \leq \mu \) for all \( x \in \mathbb{R}^d \), then

\[
\limsup_{t \to \infty} \frac{\log \mathbb{E} (|u_t(x)|^2)}{t} \leq C, \quad \forall \ x \in \mathbb{R}. \tag{6.3}
\]

If there is positive constant \( \mu_1 \) such that \( u_0(x) \geq \mu_1 \) for all \( x \in \mathbb{R}^d \), then

\[
\liminf_{t \to \infty} \frac{\log \mathbb{E} (|u_t(x)|^2)}{t} \geq C, \quad \forall \ x \in \mathbb{R}. \tag{6.4}
\]

(2) Let \( w_s(x) \) be a space white noise.

i) Let \( d < 2 \) and let \( \alpha = 1 - d/4 \). If \( |u_0(x)| \leq C < \infty \), then

\[
\limsup_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log \left\{ \sup_{x \in \mathbb{R}^d} \mathbb{E} |u_t(x)|^2 \right\} \leq C. \tag{6.5}
\]

If \( u_0(x) \geq C > 0 \), then

\[
\liminf_{t \to \infty} \frac{1}{t^{2\alpha/(2\alpha-1)}} \log \left\{ \inf_{x \in \mathbb{R}^d} \mathbb{E} |u_t(x)|^2 \right\} \geq C. \tag{6.6}
\]

ii) Let \( d = 2 \). If \( |u_0(x)| \leq C_1 < \infty \), then there is a constant \( \alpha > 0 \) when \( t < \alpha \), \( u_t(x) \) is defined as an element in \( L^2 \). If \( u_0(x) \geq C_2 > 0 \), then when \( t > \beta \),

\[
\mathbb{E} |u_t(x)|^2 = \infty. \tag{6.7}
\]

iii) When \( d > 2 \), \( \mathbb{E} |u_t(x)|^2 = \infty \) for all \( t > 0 \) if \( u_0(x) \geq C > 0 \) for some constant \( C > 0 \).

Other results may be stated in a similar way.

References


